

# SIMULATING GAUSSIAN BOSON SAMPLING ON GRAPHS IN POLYNOMIAL TIME

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**ABSTRACT.** We show that a distribution related to Gaussian Boson Sampling (GBS) on graphs can be sampled classically in polynomial time. Graphical applications of GBS typically sample from this distribution, and thus quantum algorithms do not provide exponential speedup for these applications. We also show that another distribution related to Boson sampling can be sampled classically in polynomial time.

## 1. INTRODUCTION

Bosons are subatomic particles with quantum spins. Boson sampling (BS) is a model of quantum computation based on interacting bosons that can be implemented using an optical network. It was proposed by Aaronson and Arkhipov [AA13] as a model that may have quantum advantage in the sense that sampling approximately from the output distribution can be done on a Boson computer. However, assuming two conjectures - the permanent anti-concentration conjecture (concerning the permanent of a matrix of i.i.d. Gaussians) - and the permanent of Gaussian conjecture (that estimating this permanent is  $\#P$ -hard), they show that classically sampling from the same distribution would collapse the polynomial hierarchy. Later, Gaussian boson sampling (GBS), a variant of BS which avoids some of the costs of producing photon sources, was proposed by Hamilton, Kruse, Sansoni, Barkhofen, Silberhorn, and Jex [HKS<sup>+</sup>17]. GBS has gained popularity quickly, and various teams have made considerable efforts to build photonic quantum computers based on it [ZWD<sup>+</sup>20, DGG<sup>+</sup>23, MLA<sup>+</sup>22, RAA<sup>+</sup>25].

The main proposed applications of GBS are to solve hard graph problems, such as counting perfect matchings [BDR<sup>+</sup>18], finding densest subgraphs [AB18], finding graph isomorphisms [BFI<sup>+</sup>21], and finding planted bipartite cliques [CMT25]. The implementation in [DGG<sup>+</sup>23] uses samples from GBS for related graph search problems.

We next use the notation of [ZZW<sup>+</sup>25] to describe the output distribution of GBS in this graph algorithm case (where the input matrix is non-negative). Generalising to weighted graphs does not present difficulties — we will do this later in the paper, but for clarity we start with the unweighted case. Given a graph  $G = (V, E)$  and a subset  $S \subseteq V$ , let  $|S|$  denote the size of  $S$  and let  $PM(G)$  denote the number of perfect matchings in the induced subgraph  $G[S]$ . There is a parameter  $c \in (0, 1)$  — specifically  $c$  is the inverse of maximum norm of an eigenvalue of the

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adjacency matrix of  $G$ . Then the GBS distribution corresponding to  $G$  is defined as follows:

$$(1) \quad \mu_{GBS,G}(S) \propto c^{2|S|} PM(S)^2.$$

While there are certainly fast quantum algorithms for sampling from the GBS distribution — this is the whole point of GBS — it was an open question whether there is a fast classical algorithm, even in the non-negative (graph algorithm) case. Zhang, Zhou, Wang, Wang, Yang, Yang, Xue, and Li [ZZW<sup>+</sup>25] showed recently that there is a fast algorithm for very dense graphs. In particular, they [ZZW<sup>+</sup>25, Theorem 5] showed that for any real number  $\xi \geq 1$  there is an algorithm which samples from  $\mu_{GBS,G}$  in time  $\tilde{O}(n^{2\xi+18})$  for  $n$ -vertex graphs with minimum degree at least  $n - \xi$ . Note that this is a very severe restriction on the density of the graph.

A related result which is interesting for graph applications is the quantum-inspired algorithm by Oh, Fefferman, Jiang, and Quesada [OFJQ24]. Motivated by the application of GBS to graph problems and by the desire to develop efficient classical algorithms, they proposed sampling from a distribution that is a little different from  $\mu_{GBS,G}$ , in the sense that the probability of the output  $S$  is proportional to  $PM(S)$  rather than to  $PM(S)^2$  as in (1). They found that, in practice, their classical algorithm (from the different distribution) did not perform much worse in experiments than GBS. Their distribution can be sampled by two-photon boson sampling, which can be simulated classically (or can be directly sampled classically using the algorithm of Jerrum and Sinclair; see Proposition 2.1). Nevertheless, their work left open the question of whether there is an efficient classical algorithm for GBS in the non-negative case.

Our main result is that there is indeed a classical polynomial-time algorithm to sample from  $\mu_{GBS,G}$  (for all graphs).

**Theorem 1.1.** *There is an algorithm that, given a graph  $G = (V, E)$  and positive real numbers  $c$  and  $\varepsilon$ , samples from a distribution that is  $\varepsilon$ -close to  $\mu_{GBS,G}$  in total variation distance, in time  $O(\bar{c}mn^4 \log^2(n\bar{c}/\varepsilon))$  where  $m = |E|$ ,  $n = |V|$ , and  $\bar{c} = \max\{1, c\}$ .*

*Remark 1.2.* As a consequence of Theorem 1.1, we have shown that there is no exponential-time quantum speedup for any application based on sampling from the distribution  $\mu_{GBS,G}$ .

Our algorithm to sample from  $\mu_{GBS,G}$  is different from the approach in [ZZW<sup>+</sup>25]. In fact, our algorithm is simpler, and faster, and it also applies to all graphs. We first construct the Cartesian product  $G \square K_2$  of  $G$  with an edge. We then note that the distribution  $\mu_{GBS,G}$  can be obtained by first sampling a perfect matching from  $G \square K_2$  with an appropriate weight function on edges and then projecting this perfect matching onto the vertex set matched by the original edges of  $G$ . To sample weighted perfect matchings from  $G \square K_2$ , we use the Markov chain of Jerrum and Sinclair [JS89]. The key property which guarantees that the Jerrum–Sinclair (JS) chain generates a perfect matching in polynomial time is that the ratio between the number of near-perfect matchings of  $G \square K_2$  (matchings with two unmatched vertices) and the number of perfect matchings of  $G \square K_2$  is bounded above by a polynomial in the size of the graph. We show that, for any graph  $G$ , this property holds for  $G \square K_2$ .

In addition to Gaussian boson sampling, we consider the more original boson sampling model [AA13]. Here (using the notation of [CC18]), the input is an  $m \times n$  matrix  $A$  with  $n \leq m$ , where  $A$  is the first  $n$  columns of a random  $m \times m$  unitary matrix. Let  $\Phi_{m,n}$  denote the set of vectors  $\mathbf{z} = \{z_1, \dots, z_m\}$  such that  $\sum_{i=1}^m z_i = n$ . The boson sampling distribution is

$$(2) \quad \forall \mathbf{z} \in \Phi_{m,n}, \quad \mu_{BS}(\mathbf{z}) = \frac{|\text{Perm}(A_{\mathbf{z}})|^2}{\prod_{i=1}^m z_i!},$$

where  $\text{Perm}(\cdot)$  is the permanent function. Exponential time classical simulation algorithms [CC18, CC24] are known for this distribution, but it is conjectured that no polynomial-time classical

approximate sampler exists [AA13]. Our second main result shows that, if the input matrix is non-negative instead, then one can efficiently sample from a similar distribution, where we modify the definition in (2) by renormalising appropriately, but retaining the fact that  $\forall \mathbf{z} \in \Phi_{m,n}$ ,  $\mu_{BS}(\mathbf{z}) \propto \frac{|\text{Perm}(A_{\mathbf{z}})|^2}{\prod_{i=1}^m z_i!}$ .

**Theorem 1.3.** *There is an algorithm that, given a non-negative  $m \times n$  matrix  $A$  and a real number  $\varepsilon \in (0, 1)$ , samples from a distribution that is  $\varepsilon$ -close to  $\mu_{BS}$  in total variation distance, in time  $O\left(\frac{m^7 n^{14}}{\varepsilon^7} \log^4\left(\frac{mn}{\varepsilon}\right)\right)$ .*

Our technique for proving Theorem 1.3 is a combinatorial reduction similar to the proof of Theorem 1.1. Given the matrix  $A$ , we construct a bipartite graph  $G$  such that a certain distribution on weighted perfect matchings of  $G$  induces a distribution that is close to  $\mu_{BS}$ . We then use the algorithm of Jerrum, Sinclair, and Vigoda [JSV04] to efficiently sample from the distribution on perfect matchings.

When  $A$  is not a non-negative matrix, the Jerrum–Sinclair–Vigoda algorithm (JSV) is no longer applicable. However our construction would still work, if  $|\text{Perm}(A_{\mathbf{z}})|$  could be efficiently approximated for each  $\mathbf{z}$ . The task of approximating the norm of these permanents is thus the main source of difficulty for classically simulating boson sampling. The problem of approximating the permanent is **NP**-hard in general, even for real positive semi-definite matrices [Mei23].

In Section 2 we define the terminology and review some known efficient classical algorithms for sampling matchings in graphs. We first prove Theorem 1.3 in Section 3 as a warmup, and then prove Theorem 1.1 in Section 4.

## 2. PRELIMINARIES

**2.1. Classical algorithm to sample (perfect) matchings.** Let  $G = (V, E)$  be a graph. A *matching*  $M$  of  $G$  is a subset of edges such that no two edges of  $M$  share an endpoint. Let  $V_M$  denote the vertex set of  $M$ . If  $V_M = V$ , then  $M$  is a *perfect matching*. If  $|V_M| = |V| - 2$  then  $M$  is a *near-perfect matching*. Denote by  $\mathcal{M}$  the set of all matchings of  $G$ , and by  $\mathcal{M}_k$  the set of matchings of size  $k$ . Then, when  $|V|$  is even, the set of perfect matchings is  $\mathcal{M}_{|V|/2}$ . Let  $(\lambda_e)_{e \in E}$  be a collection of (non-negative) weights on the edges in  $E$ . Define the following distribution over the matchings of  $G$ :

$$(3) \quad \forall M \in \mathcal{M}, \quad \mu_{\text{matching}, \lambda}(M) \propto \prod_{e \in M} \lambda_e.$$

Jerrum and Sinclair [JS89] showed that there is a polynomial-time algorithm to approximately sample from the distribution (3). To explain this more precisely, we need the following notation. The total variation (TV) distance between two distributions  $\mu$  and  $\nu$  over some discrete space  $\Omega$  is defined as

$$\text{dist}_{TV}(\mu, \nu) := \frac{1}{2} \sum_{\omega \in \Omega} |\mu(\omega) - \nu(\omega)|.$$

**Proposition 2.1.** (Jerrum and Sinclair [JS89]) *There is an algorithm that, given  $G = (V, E)$ , weights  $(\lambda_e)_{e \in E}$ , and a real number  $\varepsilon \in (0, 1)$ , samples from a distribution that is  $\varepsilon$ -close to  $\mu_{\text{matching}, \lambda}$  in TV distance, in time  $O(\bar{\lambda} m n^2 \log \frac{n\bar{\lambda}}{\varepsilon})$ , where  $m = |E|$ ,  $n = |V|$ , and  $\bar{\lambda} = \max_{e \in E} \{1, \lambda_e\}$ .*

This particular running time is derived from [JS96, Proposition 12.4].

Let  $m_k = |\mathcal{M}_k|$  be the number of matchings of size  $k$ . Jerrum and Sinclair [JS89, Theorem 5.1] also showed that the sequence  $(m_k)$  is log-concave.

**Proposition 2.2.** *For any graph  $G = (V, E)$  and  $1 \leq k \leq |V|/2$ ,  $m_{k-1} m_{k+1} \leq m_k^2$ .*

In addition, Jerrum, Sinclair, and Vigoda [JSV04] showed that in bipartite graphs, one can efficiently sample from the distribution (3) restricting to perfect matchings. The running time of their algorithm was then improved by Bezáková, Štefankovič, Vazirani, and Vigoda [BŠVV08]. More precisely, for any graph  $G = (V, E)$  and any  $M \in \mathcal{M}_{|V|/2}$ , let

$$(4) \quad \mu_{PM,\lambda}(M) \propto \prod_{e \in M} \lambda_e.$$

**Proposition 2.3.** (*Jerrum, Sinclair, Vigoda [JSV04]; Bezáková, Štefankovič, Vazirani, and Vigoda [BŠVV08]*) *There is an algorithm that, given a bipartite  $G = (V, E)$ , weights  $(\lambda_e)_{e \in E}$ , and a real number  $\varepsilon \in (0, 1)$ , samples from a distribution that is  $\varepsilon$ -close to  $\mu_{PM,\lambda}$  in TV distance, in time  $O(n^7 \log^4 n + n^5 \log \frac{n}{\varepsilon})$ , where  $n = |V|$ .*

This running time comes from [BŠVV08]. One needs to first spend  $O(n^7 \log^4 n)$  time to estimate certain weights, and then each subsequent sampling takes  $O(n^5 \log \frac{n}{\varepsilon})$  time. See [BŠVV08, Theorem 1.1 and Theorem 4.1].

**2.2. Permanent and Hafnian.** Two matrix functions will be useful throughout this paper — the permanent and the hafnian. Let  $A$  be an  $n \times n$  matrix. The permanent is defined as follows

$$\text{Perm}(A) := \sum_{\sigma \in S_n} \prod_{i=1}^n A_{i,\sigma(i)},$$

where  $S_n$  is the symmetric group of order  $n$ . If  $A$  is the biadjacency matrix of some bipartite graph  $G$ , then  $\text{Perm}(A)$  is the number of perfect matchings in  $G$ .

Similarly, for a  $2n \times 2n$  symmetric matrix  $A$ ,

$$\text{Haf}(A) := \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \prod_{i=1}^n A_{\sigma(2i-1), \sigma(2i)}.$$

If  $A$  is the adjacency matrix of some graph  $G$  with  $2n$  vertices, then  $\text{Haf}(A)$  is the number of perfect matchings of  $G$ .

### 3. BOSON SAMPLING

Boson sampling was proposed by Aaronson and Arkhipov [AA13] as a way to demonstrate quantum advantage. For exact classical simulation, see [CC18, CC24]. Mathematically, the distribution to sample from is the following. Let  $A$  be an  $m \times n$  matrix with  $n \leq m$ . Let  $\mathbf{z} = \{z_1, \dots, z_m\}$  be a vector of non-negative integers with  $\sum_{i=1}^m z_i = n$ , and let  $\Phi_{m,n}$  denote the set of these vectors. Then,

$$(5) \quad \forall \mathbf{z} \in \Phi_{m,n}, \quad \mu_{BS}(\mathbf{z}) \propto \frac{|\text{Perm}(A_{\mathbf{z}})|^2}{\prod_{i=1}^m z_i!},$$

where  $A_{\mathbf{z}}$  is the square matrix composed of  $z_i$  copies of row  $i$  of  $A$ . Physically, the matrix  $A$  is usually the first  $n$  columns of an  $m \times m$  Haar random unitary matrix, in which case the normalising factor in (5) is 1.

In the rest of this section, we prove Theorem 1.3, which we re-state for convenience.

**Theorem 1.3.** *There is an algorithm that, given a non-negative  $m \times n$  matrix  $A$  and a real number  $\varepsilon \in (0, 1)$ , samples from a distribution that is  $\varepsilon$ -close to  $\mu_{BS}$  in total variation distance, in time  $O\left(\frac{m^7 n^{14}}{\varepsilon^7} \log^4\left(\frac{mn}{\varepsilon}\right)\right)$ .*

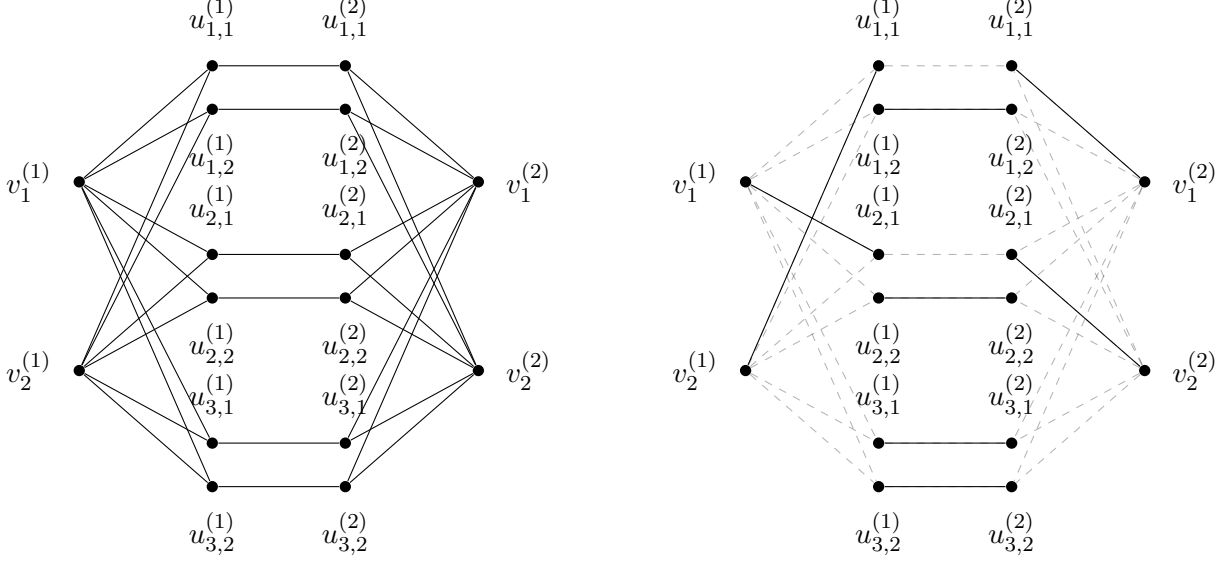


FIGURE 1. On the left we have our graph  $G$  with vertex sets from left to right:  $L_1, R_1, R_2, L_2$ . The corresponding matrix  $A$  is a  $2 \times 3$  matrix with all 1 entries, and  $k = 2$  in this example. On the right, a perfect matching  $M$  selected from  $G$  is highlighted. Here  $S_1(M) = \{u_{1,1}^{(1)}, u_{2,1}^{(1)}\}$  and  $\mathbf{z} = \{1, 1\}$ .

*Proof.* Given the matrix  $A$ , we will construct a bipartite graph  $G$  and weights  $\lambda$  such that its distribution  $\mu_{PM,\lambda}$  in (4) is  $\varepsilon/2$ -close to the distribution  $\mu_{BS}$  in (5). We will then finish by invoking Proposition 2.3 with error  $\varepsilon/2$ .

Let  $k = \lceil 4n^2/\varepsilon \rceil$ . We define  $G$  as follows For each  $\ell \in \{1, 2\}$  let  $L_\ell = \{v_i^{(\ell)} : i \in [n]\}$ . Let  $R_\ell = \{u_{j,t}^{(\ell)} : j \in [m], t \in [k]\}$ . The vertex set of  $G$  is  $L_1 \cup R_1 \cup L_2 \cup R_2$ . The edge set of  $G$  is defined as follows. For each  $\ell \in \{1, 2\}$  let  $E_\ell = \{(v_i^{(\ell)}, u_{j,t}^{(\ell)}) : i \in [n], j \in [m], t \in [k], A_{i,j} \neq 0\}$ . For each  $e = (v_i^{(\ell)}, u_{j,t}^{(\ell)}) \in E_\ell$ ,  $\lambda_e = A_{i,j}$ . Let  $E_R = \{(u_{j,t}^{(1)}, u_{j,t}^{(2)}) : j \in [m], t \in [k]\}$ . For each  $e \in E_R$ ,  $\lambda_e = 1$ . The edge set of  $G$  is  $E_1 \cup E_2 \cup E_R$ . An example is given in Figure 1.

Let  $\mu_{PM,\lambda}$  from (4) be the distribution over perfect matchings in  $G$  with weights given by  $\lambda$ . Let  $M \sim \mu_{PM,\lambda}$  be a sample from this distribution. Let  $S_1(M)$  be the set of vertices in  $R_1$  that are matched to vertices in  $L_1$  by  $M$  (rather than to vertices in  $R_2$ ). Note that, because every vertex in  $L_1$  needs to be matched with some vertex in  $R_1$ ,  $|S_1(M)| = |L_1| = n$ . We identify  $S_1(M)$  with a vector  $\mathbf{z} \in \Phi_{m,n}$ , by defining  $z_i$  as follows for each  $i \in m$ :  $z_i = |S_1(M) \cap \{u_{i,1}^{(1)}, \dots, u_{i,k}^{(1)}\}|$ . Let  $\nu$  be the resulting distribution over  $\Phi_{m,n}$ . We claim that, for  $k = \lceil \frac{4n^2}{\varepsilon} \rceil$ ,

$$(6) \quad \text{dist}_{TV}(\nu, \mu_{BS}) \leq \frac{\varepsilon}{2}.$$

The theorem follows from the claim and Proposition 2.3.

In the rest of the proof we show (6). For any particular  $\mathbf{z} \in \Phi_{m,n}$ , the way to obtain the sample  $\mathbf{z}$  is to first choose  $z_i$  copies of each  $u_i$  from  $\{u_{i,1}^{(1)}, \dots, u_{i,k}^{(1)}\}$  - denote the set of these by  $S_1$  - and then match  $S_1$  with  $L_1$  perfectly using edges in  $E_1$ . Moreover, the vertices in  $R_1 \setminus S_1$  must be matched through  $E_R$  edges (which have weight 1) to  $R_2$ . As a result, the vertices in  $R_2$  with the same indices as those in  $S_1$  must be matched via edges in  $E_2$  to  $L_2$ . Let  $S_2$  denote the set of these vertices in  $R_2$ . The total weight of matching  $L_1$  with  $S_1$  perfectly is indeed  $\text{Perm}(A_{\mathbf{z}})$ , and so is

the total weight of matching  $L_2$  with  $S_2$ . To summarise,

$$\begin{aligned}
\nu(\mathbf{z}) &\propto k^{-n} \nu(\mathbf{z}) \propto k^{-n} \prod_{i=1}^m \binom{k}{z_i} \text{Perm}(A_{\mathbf{z}})^2 = \frac{|\text{Perm}(A_{\mathbf{z}})|^2}{\prod_{i=1}^m z_i!} \cdot k^{-n} \prod_{i=1}^m \frac{k!}{(k-z_i)!} \\
(7) \quad &= \frac{|\text{Perm}(A_{\mathbf{z}})|^2}{\prod_{i=1}^m z_i!} \cdot k^{-n} \prod_{i=1}^m \prod_{j=0}^{z_i-1} (k-j) = \frac{|\text{Perm}(A_{\mathbf{z}})|^2}{\prod_{i=1}^m z_i!} \cdot \prod_{i=1}^m \prod_{j=0}^{z_i-1} \left(1 - \frac{j}{k}\right).
\end{aligned}$$

Furthermore, using the facts that  $k = \lceil \frac{4n^2}{\varepsilon} \rceil$  and  $z_i \leq k$  and  $\sum_{i=1}^m z_i = n$ ,

$$1 \geq \prod_{i=1}^m \prod_{j=0}^{z_i-1} \left(1 - \frac{j}{k}\right) \geq \left(1 - \frac{n}{k}\right)^n \geq e^{-2n^2/k} = e^{-\varepsilon/2}.$$

Plugging the estimate above into (7), we have that the multiplicative error for each  $\mathbf{z}$  between  $\nu$  and  $\mu_{BS}$  is between 1 and  $e^{-\varepsilon/2}$ , which implies (6).  $\square$

#### 4. GAUSSIAN BOSON SAMPLING

Gaussian boson sampling (GBS) is a variant proposed by Hamilton, Kruse, Sansoni, Barkhofen, Silberhorn, and Jex [HKS<sup>+</sup>17]. For  $n$  photons and  $m$  modes, the input is a Gaussian state characterised by a  $2m \times 2m$  covariance matrix  $\sigma$ . Let  $A$  be the *sampling matrix* given by  $A = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix} (I_{2m} - \sigma_Q^{-1})$ , where  $I_m$  denotes the  $m \times m$  identity matrix and  $\sigma_Q := \sigma + I_{2m}/2$ . For any  $\mathbf{z} \in \Phi_{m,n}$ , the probability of the output pattern  $\mathbf{z}$  is

$$(8) \quad \mu_{GBS}(\mathbf{z}) := \frac{\text{Haf}(A_{\mathbf{z}})}{\sqrt{\det(\sigma_Q)} \prod_{i=1}^m z_i!},$$

where  $A_{\mathbf{z}}$  is the square matrix corresponding to  $\mathbf{z}$  as given in [KHS<sup>+</sup>19, Section III]. When  $n = O(\sqrt{m})$ , with high probability  $z_i \leq 1$  (see [KHS<sup>+</sup>19, Section B]), and in this case the matrix  $A_{\mathbf{z}}$  is the principal submatrix of  $A$  formed by keeping rows and columns  $i$  and  $m+i$  for each  $i \in [m]$  with  $z_i = 1$ .

Bradler, Dallaire-Demers, Rebentrost, Su, and Weedbrook [BDR<sup>+</sup>18] proposed using GBS to solve graph problems. They showed that one can choose the covariance matrix  $\sigma$  and  $c > 0$  such that, for any graph  $G$  with adjacency matrix  $A$ , the distribution in (8) can be interpreted as a distribution over  $S \subseteq V$ :

$$(9) \quad \mu_{GBS,G}(S) \propto c^{2|S|} \text{Haf}(A_S)^2,$$

where  $A_S$  is the principal submatrix of the adjacency matrix  $A$  corresponding to  $S$ . Note that  $\text{Haf}(A_S)$  counts the number of perfect matchings in the induced subgraph  $G[S]$ . The distribution induced by GBS in fact has the parameter  $0 < c < 1$ .

In the rest of this section, we prove our main result, Theorem 1.1, which shows that the distribution in (9) can be (classically) sampled in polynomial time for any graph  $G$  and  $c > 0$ .

The sampling algorithm that we use to prove Theorem 1.1 takes the input graph  $G = (V, E)$  and constructs the graph  $G \square K_2$ , which the Cartesian product of  $G$  and an edge. In order to construct  $G \square K_2$ , we start with a copy  $G' = (V', E')$  of  $G$ . Then  $E_0 = \{(v, v') : v \in V\}$  and  $G \square K_2 = (V \cup V', E \cup E' \cup E_0)$ . See Figure 2 for an example. Moreover, for each edge  $e \in E \cup E'$ , let  $\lambda_e = c^2$ , and for each  $e \in E_0$ , let  $\lambda_e = 1$ .

Suppose that  $M$  is a sample from the perfect matching distribution (4) on the graph  $G \square K_2$  with this weight function  $\lambda$ . Let  $S_M \subset V$  be the set of vertices that are endpoints of some edge in  $M \cap E$ . We have the following lemma, which shows that the induced distribution of  $S_M$  is given by Equation (9).

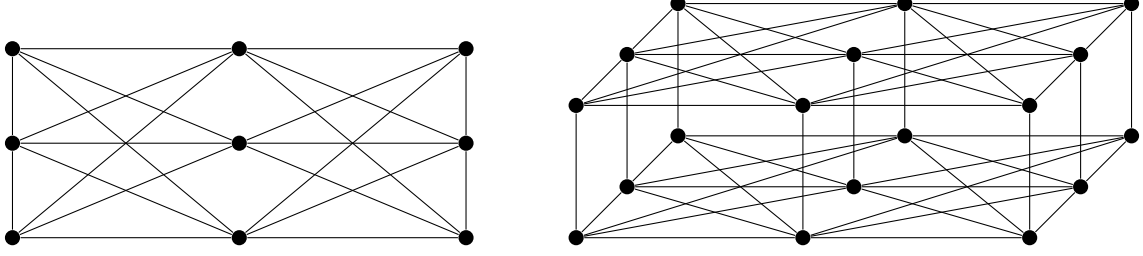


FIGURE 2. On the left we have our original graph  $G$ , while on the right we have  $G \square K_2$ .

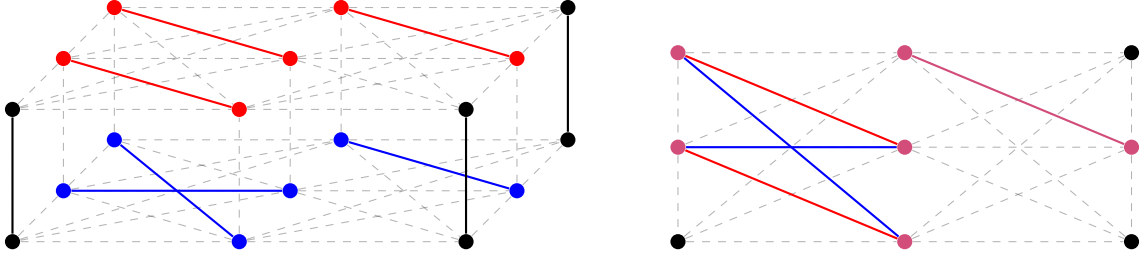


FIGURE 3. On the left, we have a perfect matching  $M$  in  $G \square K_2$ , with  $M \cap E$  highlighted in red in the top copy,  $M \cap E'$  blue in the bottom copy, and  $M \cap E_0$  black. On the right, we have our underlying graph  $G$  with the set  $S_M$  and the corresponding edges chosen by  $M$  (in either or both copies) highlighted.

**Lemma 4.1.** *The induced distribution of  $S_M$  is exactly  $\mu_{GBS,G}(S)$ .*

*Proof.* For  $M \sim \mu_{PM,\lambda}$ , as it is a perfect matching of  $G \square K_2$ , each vertex  $v$  in  $V$  is either matched with an edge in  $M \cap E$  or  $M \cap E_0$ . If  $v$  is matched in  $E_0$  then it is matched with its copy  $v'$ . Thus the set of endpoints of edges in  $M \cap E'$ , denoted  $S'$ , is the set of copies of vertices in  $S_M$ . See Figure 3 for an illustration.

To summarise, the probability of outputting a particular  $S \subset V$  is

$$\begin{aligned} \Pr_{M \sim \mu_{PM,\lambda}} [S_M = S] &\propto c^{2|S|} \text{Haf}[A_S] \text{Haf}[A_{S'}] \\ &= c^{2|S|} \text{Haf}[A_S]^2 \end{aligned} \quad \square$$

Our problem now is to sample from  $\mu_{PM,\lambda}$  for  $G \square K_2$ . For this, we use Proposition 2.1, which can be used to efficiently sample perfect matchings in only if the ratio between the number of perfect and near-perfect matchings is bounded from above by a polynomial. We will show that this is indeed the case for  $G \square K_2$ .

To be more specific, define a new weight function  $\lambda'$  for  $G \square K_2$  as  $\lambda'_e = 4n^2\lambda_e$ . This weight function favours matchings of larger sizes, and will eventually be the weight function that we will use when invoking the Jerrum–Sinclair chain [JS89]. For a matching  $M$  of  $G \square K_2$ , let  $w(M) := \prod_{e \in M} \lambda_e$  and  $w'(M) := \prod_{e \in M} \lambda'_e$ . Let  $M_k$  denote the set of matchings with  $k$  edges in  $G \square K_2$ , where  $0 \leq k \leq n$ , and  $m_k = |M_k|$ . Let  $Z_k$  be the total weight from  $k$ -edge matchings under  $\lambda$ , namely,

$$Z_k := \sum_{M \in M_k} w(M),$$

and  $Z := \sum_{i=1}^k Z_k$  be the partition function of  $\mu_{\text{matching}, \lambda}$ . Similarly, define  $Z'_k$  and  $Z'$  for  $\lambda'$ . Note that the proof of Lemma 4.1 implies that

$$(10) \quad Z_n = \sum_{S \subseteq V} c^{2|S|} \text{Haf}(A_S)^2.$$

**Lemma 4.2.**  $Z_{n-1} < 2n^2 Z_n$ .

*Proof.* We write  $\text{Haf}(S)$  for  $\text{Haf}(A_S)$ . Note that by choosing the two vertices to remove, we have

$$(11) \quad \begin{aligned} Z_{n-1} &= 2 \sum_{(u_1, u_2) \in \binom{V}{2}} \sum_{S \subseteq V \setminus \{u_1, u_2\}} c^{2|S|+2} \text{Haf}(S) \cdot \text{Haf}(S \cup \{u_1, u_2\}) \\ &\quad + \sum_{u_1, u_2 \in V} \sum_{S \subseteq V \setminus \{u_1, u_2\}} c^{2|S|+2} \text{Haf}(S \cup \{u_1\}) \cdot \text{Haf}(S \cup \{u_2\}). \end{aligned}$$

By the AM-GM inequality, it holds that

$$\begin{aligned} &\sum_{(u_1, u_2) \in \binom{V}{2}} \sum_{S \subseteq V \setminus \{u_1, u_2\}} c^{2|S|+2} \text{Haf}(S) \cdot \text{Haf}(S \cup \{u_1, u_2\}) \\ &\leq \frac{1}{2} \sum_{(u_1, u_2) \in \binom{V}{2}} \sum_{S \subseteq V \setminus \{u_1, u_2\}} \left( c^{2|S|} \text{Haf}(S)^2 + c^{2|S|+4} \text{Haf}(S \cup \{u_1, u_2\})^2 \right) \\ &\leq \binom{n}{2} \sum_{S \subseteq V} c^{2|S|} \text{Haf}(S)^2 = \binom{n}{2} Z_n, \end{aligned}$$

where we used (10) in the last line. Similarly,

$$\begin{aligned} &\sum_{u_1, u_2 \in V} \sum_{S \subseteq V \setminus \{u_1, u_2\}} c^{2|S|+2} \text{Haf}(S \cup \{u_1\}) \cdot \text{Haf}(S \cup \{u_2\}) \\ &\leq \frac{1}{2} \sum_{u_1, u_2 \in V} \sum_{S \subseteq V \setminus \{u_1, u_2\}} \left( c^{2|S|+2} \text{Haf}(S \cup \{u_1\})^2 + c^{2|S|+2} \text{Haf}(S \cup \{u_2\})^2 \right) \\ &\leq n^2 \cdot \sum_{S \subseteq V} c^{2|S|} \text{Haf}(S)^2 = n^2 Z_n. \end{aligned}$$

Combining with (11), it follows that

$$Z_{n-1} < 2n^2 Z_n. \quad \square$$

Since  $Z'_k = (4n^2)^k Z_k$ , Lemma 4.2 implies that  $Z'_{n-1} < Z'_n/2$ . Moreover, the proof of Proposition 2.2 is via an injection from  $M_{k-1} \times M_{k+1}$  to  $M_k \times M_k$ . It can be verified that the log concavity still holds for weighted matchings. Namely, for any  $1 \leq k \leq n-1$ ,  $\frac{Z_{k-1}}{Z_k} \leq \frac{Z_k}{Z_{k+1}}$ . Together with Lemma 4.2, we have that  $Z_{k-1} \leq 2n^2 Z_k$  for any  $1 \leq k \leq n$ . Thus,  $Z'_{k-1} \leq Z'_k/2$  for any  $1 \leq k \leq n$ . This implies that

$$(12) \quad Z' = \sum_{i=0}^n Z'_i \leq \sum_{i=0}^n 2^{i-n} Z'_n < 2Z'_n.$$

We can now prove Theorem 1.1.

**Theorem 1.1.** *There is an algorithm that, given a graph  $G = (V, E)$  and positive real numbers  $c$  and  $\varepsilon$ , samples from a distribution that is  $\varepsilon$ -close to  $\mu_{GBS, G}$  in total variation distance, in time  $O(\bar{c} m n^4 \log^2(n\bar{c}/\varepsilon))$  where  $m = |E|$ ,  $n = |V|$ , and  $\bar{c} = \max\{1, c\}$ .*

*Proof.* By Lemma 4.1, we just need to sample from  $\mu_{PM,\lambda}$  over perfect matchings in  $G \square K_2$ . To this end, we use Proposition 2.1 to approximately sample from the distribution  $\mu_{matching,\lambda'}$  in (3) for  $G \square K_2$  with the weight function  $\lambda'$  as described above. We set the error as  $\varepsilon' = \min\{1/4, \varepsilon/2\}$  in Proposition 2.1. By (12), there is  $1/2 - \varepsilon' \geq 1/4$  probability that the algorithm outputs a perfect matching. We keep rejecting until we see a perfect matching. The distribution  $\mu_{matching,\lambda'}$  conditioned on outputting a perfect matching is exactly  $\mu_{PM,\lambda}$ . Thus we can approximately sample from  $\mu_{PM,\lambda}$  with  $\varepsilon$  error after at most  $O(\log(1/\varepsilon))$  rejections. As for the running time, we plug in  $\lambda = O(n^2)$  in Proposition 2.1, which finishes the proof.  $\square$

## 5. CONCLUDING REMARKS

In this paper we showed that there are polynomial-time classical algorithms to (approximately) sample from the distribution (9), related to Gaussian Boson Sampling on graphs, and from the distribution (5) when the input matrix  $A$  is non-negative. It would be interesting to know whether the non-negative restriction can be relaxed. While the algorithm on which our method is based works only for non-negative matrices, in the more general case, our construction actually enables a reduction from sampling to approximating permanents of some complex weighted matrices. This is because, as can be seen from the proof of Theorem 1.3, the marginal probability of choosing a row is the ratio between the permanents of two complex weighted matrices, even when conditioned on previous choices. Thus, one can sample from the (conditional) marginal distributions of the rows, one by one, using an oracle that approximates the permanent of complex weighted matrices. However, the latter task is **NP**-hard in general [Mei23], and it is an interesting open question to what extent this strategy can be useful.

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