Capability Indices

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Introduction

Process capability analysis is an important technique that gives the chance to locate the problems and fix them before the manufacturing process has began.

The most commonly used initial tool for this purpose is the set of Process Capability Indices (PCIs). Quality improvement can not be imagined without the PCIs. Today, it is one of the most used and well-known tools of Statistical Process Control(SPC).

Process capability indices as a tool is defined as a combination of materials, methods, equipment and people engaged in producing a measurable output.(Saha and Maiti 2015) The idea behind these indices is simple: they caption the connection between the process performance and its specification limits. So, these indices in the end measure and show us whether the predicted level of production or performance was met or not. That is why they so broadly used in the process assessment.

In this paper firstly will be reviewed the five basic indices:C_p, C_{pk}, k, C_{pu} and C_{pl}. These indices combined together since they all came from the Japanese manufacturing facilities.(Palmer and Tsui 1999) They are closely related to the Six Sigma theory. The PCIs played a key role in popularizing this methodology. With their help, the idea behind Six Sigma - that the reduction of variability is essential - became the often practical application.

The main goal of Six Sigma methodology is the reduction of the process variance. In the perfect scenario the interval of the Six Sigma should contain 99,73% of the whole population. To implement this concept we use PCIs as our tools through that we can actually see the natural variation in the given limitations. (Li and Xie 2024)

This methodology came from the manufacturing industry and then was implemented in the wide range of other fields. This new approach later came into the management theory and helped to not only consider the quality of the product but also the starting point of quality control. This made it the mix of quality management and statistics.

Six Sigma concept implies that the variability of the process is included into this Six Sigma limit. Which gives us these limits:

$$\begin{aligned}\text{UNTL} &= \mu + 3\sigma \quad \text{(Upper Natural Tolerance Limit)}\\ \text{LNTL} &= \mu - 3\sigma \quad \text{(Lower Natural Tolerance Limit)}\end{aligned}$$

Under the normality assumption these bounds include 99.73% of the variability.(Montgomery 2009)

By Taguchi and others in 1985 further the C_{pm} index was developed.(Chen, Lai, and Nien 2010) The new index introduced the target value, which helped to navigate how far the mean value of the process deviates from the desired result. The third-generation index - C_{pmk} was introduces by Pearn and others in 1992.(Pearn and Lin 2003) This index is one of the most useful index for processes with two-sided specification limits. It combines the centering from the C_{pk} and target leveling from C_{pm}.

There are a lot of literature about the capability indices and their practical use. To build a theoretical foundation about the capability analysis and the role of PCIs in it, the books of next authors were reviewed: Montgomery (2009), Kotz and Lovelace (1998). These books helped with breaking the main role of the PCIs. The articles of Wu, Pearn, and Kotz (2009) and Saha and Maiti (2015) helped to better understand the role of the capability analysis in SPC. In the further classification of indices and their development the work of Palmer and Tsui (1999) and Wu, Pearn, and Kotz (2009) were critical to create a structured and meaningful foundation for the exploration of the second and third-generation indices. Further, Wu and Darmawan (2025), Pearn and Lin (2003), Spiring (1991), Wang, and and Li (2025) contributed to the theoretical part of the C_{pm} and C_{pmk} indices and the estimation techniques.

Theoretical part

C_p index

The Six Sigma spread represents the natural tolerance limits of the process. This method walks us to the process capability ratio - C_p - the most basic way to measure the process capability within the specification limits:

$$C_p = \frac{\text{USL} - \text{LSL}}{6\sigma} = \frac{d}{3\sigma}$$

$$\begin{aligned} \text{where:} \quad \text{USL} &- \text{upper specification limit,} \\ \text{LSL} &- \text{lower specification limit,} \\ \sigma &- \text{process standard deviation.} \\ d &= \frac{\text{USL} - \text{LSL}}{2} \quad \text{(half-length of the specifications)} \end{aligned}$$

Under the assumption of unknown σ the estimation of C_p would look like this:

$$\hat{C_p} = \frac{\text{USL} - \text{LSL}}{\hat{\sigma}}$$

where the $\hat{\sigma}$ calculated as:

$$\hat{\sigma} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

We also can apply the same concept to the one-sided specification limits and use one-sided process-capability indices C_{pu} and C_{pl}:

$$C_{pu} = \frac{\text{USL} - \mu}{3\sigma} \quad \text{(upper specification only)}$$
$$C_{pl} = \frac{\mu - \text{LSL}}{3\sigma} \quad \text{(lower specification only)}$$

where:

μ – true process mean.

The problem in both two and one-sided ratios is that the true σ is almost always unknown and has to be replaced by the estimation of it:

$$\hat{C}_{pu} = \frac{\text{USL} - \hat{\mu}}{3\hat{\sigma}} \quad \text{(upper specification only)}$$
$$\hat{C}_{pl} = \frac{\hat{\mu} - \text{LSL}}{3\hat{\sigma}} \quad \text{(lower specification only)}$$

where:

$$\hat{\mu} = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$$
$$\hat{\sigma} = \hat{S} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{X})^2}$$

The measurement of this index represented through the parts per million(ppm). This ppm value displays the number of the possible mistakes in the process. For example:

- $C_p = 1$; represents 2700 ppm,
- $C_p = 1.33$; represents 63 ppm,
- $C_p = 2$; represents less than 0.1 ppm.

It follows that as closer the C_p index is to 2, the better the process capability. The $C_p < 1$ tells us that the process spread does not fit into the tolerance limits. The value of 1.33 only gives the idea that the process has the potential to perform so that the 75% of the specification range will be used. This is only possible when the process is centered at the mid-point level.

This index is pretty problematic and suffers from the main problem. It is not taking into account the process mean because of its simplicity. It is only able to consider the overall variability based on the six sigma spread.

C_{pk} index

To fix this problem and center the process the C_{pk} index was found. This index is based on the C_p but with a little change.

$$C_{pk} = \min\left\{\frac{\text{USL} - \mu}{3\sigma}, \frac{\mu - \text{LSL}}{3\sigma}\right\}$$

As the C_{pk} takes the centering into account, compared to the C_p index, the next formulation is correct:

$$C_{pk} = \min(C_{pu}, C_{pl})$$

The alternative form of this index might be useful to show its important feature:

$$C_{pk} = C_p \times C_a = \frac{\text{USL} - \text{LSL}}{6\sigma} \times \left(1 - \frac{|\mu - m|}{d}\right)$$

where:

$$m = \frac{\text{USL} + \text{LSL}}{2} \quad \text{(mid-point of the specifications)}$$

This formula comes from the combination of the C_p index and C_a index(“accuracy” index) where the second one´s purpose is to measure the level centering.

With C_{pk} index we can clearly see that if it goes down, than the mean differs from the mid point of the specification limits:

$$C_{pk} = \frac{d - |\mu - m|}{3\sigma}$$

The process yield in case of the C_{pk} is not that easily interpreted as it was with the C_p. Since this index takes only the shortest tail of the specification spread, the next is true:(Kotz and Lovelace 1998)

$$C_{pk} \leq C_{pu} \Rightarrow \text{USL} \geq \mu + 3\sigma C_{pk}$$
$$C_{pk} \geq C_{pl} \Rightarrow \text{LSL} \leq \mu - 3\sigma C_{pk}$$

The possible outcomes of the index might be perceived as it reflected in the table:(Kotz and Lovelace 1998)

C_{pk}	Parts Outside Tolerance Limits	Approximate ppm
0.25	16 out of 100	160,000
0.5	7 out of 100	70,000
1.0	13 out of 10,000	1,300
1.33	3 out of 100,000	30
1.67	1 out of 1,000,000	1
2.0	1 out of 1,000,000,000	0.001

The standard value for C_{pk} is around 1.33. This value shows that minimum 75% of the given specification rabge is used.

The observation of these indices gives us the understanding of the main difference between C_p and C_{pk}.Under the assumption of $C_p = C_{pk}$, the process is perfectly centered. The relationship between these two indices is the marker of how off-center the process is.

As we could see, these indices should be used carefully. Even though both - mean and variability – in C_{pk} considered, the assumption of normality is important. Also, the use of single C_{pk} index might be inadequate. Only the comparison with the C_p might say whether the process is off-center or not. The value of C_{pk} does not give any information about the location of the mean. If we compare these two indices there are two possible scenarios:

$$
\begin{aligned}
C_p &= C_{pk} \text{ (centered process)} \\
C_p &> C_{pk} \text{ (off-centered process)}
\end{aligned}
$$

There is no possibility of $C_p < C_{pk}$ since C_p by default assumes that the process mean is perfectly centered. If it is not so, then the C_{pk} can only become smaller. This simply means that: $C_p \ C_{pk}$.

Even though these two indices are working way better side by side, the C_{pk} alone can not offer the adequate assumption about the process mean or about the natural width.

Plus, it is important not to forget that the C_{pk} is the combination of C_{pu} and C_{pl}, which might also lead to some misleading results. The value of this index is almost always skewed to one of the specification limits. Therefor the result might be enormously underestimated, since we take the minimum value of C_{pu} and C_{pl}. That results in such problem, that samples close to one of the specification limits might not be considered at all.

k index

The exclusive use of C_p or C_{pk} is not describing the whole process. Both have their own weaknesses. To use the strong sides of these indices the k index is used. It conveys the underlying connection between these first-generation indices. The index defined like this:(Wu, Pearn, and Kotz 2009)

$$k = \frac{|\mu - m|}{d}$$

When the process mid-point is not equal to the target value, the k index defined as follows:

$$k' = \frac{|T - \mu|}{\min(T - \mathrm{LSL},\ \mathrm{USL} - T)}$$

This index describes process capability only in terms of process location. It is just the scaled measure of process centeredness. Its goal is not in comparing C_p and C_{pk}. Interpretation of the result for k:

k = 0 $\rightarrow$the process is centered on the target value;

k = 1 $\rightarrow$the process mean is located at one of the specification limits;

$0<k<1$ $\rightarrow$the process mean is located between the target value and a specification limit by an amount indicated by the k value.

But still the k index describes the process from the process centering point and C_p only in terms of variation.

C_{pm} index

C_p and C_{pk} work already good together with process centering and the spread. But what if we want to also check the specific mean value? It might be useful if we for example need to have the specific mean size of the water bottles, we produce. The reasons for it might be different: design goals or even safety of the product. For these purposes we have to introduce the new parameter – T (target value).

Here Taguchi's Quality Philosophy and Practice (TQPP) must be mentioned. It gives us the idea that any deviation from the target value causes the overall process loss for the customer and (probably) for the society. Taguchi, with his approach, sort of challenged the old quality control approach. He went beyond the interval limitations and moved our focus to other important aspects of the process performance.(Gamage, Jayamaha, and Grigg 2017)

We will now examine the formula of the Cp_m index and define the difference from previous PCIs:

$$C_{pm} = \frac{\text{USL} - \text{LSL}}{6\sqrt{\sigma^2 + (\mu - T)^2}}$$

As with the previous indices, the assumption of unknown μ and σ^2 results into this formula:

$$\hat{C}_{pm} = \frac{\text{USL} - \text{LSL}}{6\sqrt{\hat{\sigma}^2 + (\hat{\mu} - T)^2}}$$

Denominator in these both formulas is still our $\hat{\sigma}$, but with the adjustment in the form of penalty. Worth noticing that if the process is fully unbiased (i.e. $\mu = T$), then the C_{pm} equals to C_p:

$$C_{pm} = \frac{\text{USL} - \text{LSL}}{6\sqrt{\text{Var}(X)}} = C_p$$

In C_{pm} formula the overall loss, described above, is the squared difference between the process mean and the target value. Because of the use of the target value, any deviation (even the slightest one) will always be penalised by this index. This is how it works. The more mean value would move away from the target value, the more C_{pm} index would go down due to the square penalization $(\mu - T)^2$. This will happen even under assumption of the $\hat{\sigma}$ being constant through out the whole process.

The novelty of this index is not only in the introduction of the target value. Taguchi also introduced the loss function curve. This curve has a quadratic form and shows the lowest loss at the target value. The more process deviates from is, the more increases the loss value.

Another interesting part of this method is that the asymmetry stops being a problem. Target value can be located closer to upper or lower specification limits. It happens because no matter where T is located, we still use the whole allowable width in the numerator. The denominator will expand only if either bias or the variance is increased.

As the C_{pk} can be described as a product of C_p and C_a, Taguchi´s index C_{pm} can also be expressed this way. Although the C_p and C_a indices would have a non-proportional relationship to each other:

$$C_{pm} = \frac{C_p}{\sqrt{1 + (3C_pC_a)^2}}.$$

Summing up all said above about the C_{pm} index, next can be said. No matter what target value is set to, until it is equal to the process mean, the C_{pm} will continue to be equal to the value of the C_p (Not the C_{pk}, since it coincide with C_p only if the mean located at exactly the mid-point of the specification interval.) The moment this assumption changes (i.e. $\mu \neq T$), the C_{pm} value will decrease. The perfect industrial solution would be $C_{pm} = 2$, which would be the Six Sigma standard. But the C_{pm} higher than 1.33 would be already a very good result, considered to be sufficient for the representation of the process capability. Our goal is to increase the C_{pm} value. To do so we need to either decrease the variance of the process or move the mean closer to the target value. In some scenario, both of these modifications should be done. The approach should be based on the idea, that the denominator of the fraction should be minimized. It is unclear what might be done in the specific case. This secon-generation index was eventually improved and the C_{pmk} index was developed.

C_{pmk} index

We already observed three main process capability indices: C_p, C_{pk}, C_{pm}. All of them combine within themselves different important aspects. Every next index, as we could see, had been improved with some new feature. For example: -

C_p´s main focus is only on the general process spread. This index would not go any further. - C_{pk}, in turn, is already able to notice weather the process is off-center or not. However only in the comparison to C_p´s value - C_{pm} is not only considering the spread and the lack of centering, but additionally tracks if any shift from the target is present.

C_{pm} might be already a very good index, that would show the capability of the manufacturing process. Although the assumption of centering in the process between the specification limits is utterly important for the use of C_{pm}. Without the presence of assumption $T = m$, the C_{pm} is not applicable.

In cases, where this assumption can not be met, we might turn to the C_{pmk} index. In some literature the C^*_{pm} notation might be used.

The theoretical formula of C_{pmk} looks like this:

$$C_{pmk} = \min\left\{\frac{\text{USL} - \mu}{3\sqrt{\sigma^2 + (\mu - T)^2}}, \frac{\mu - \text{LSL}}{3\sqrt{\sigma^2 + (\mu - T)^2}}\right\}$$

As with all other indices, under assumption of unknown mean and standard deviation unknown (which is almost always the case), the parameters can be estimated. And the formula on practice will look this way:

$$\hat{C}_{pmk} = \min\left\{\frac{\text{USL} - \bar{X}}{3\sqrt{S^2 + (\bar{X} - T)^2}}, \frac{\bar{X} - \text{LSL}}{3\sqrt{S^2 + (\bar{X} - T)^2}}\right\}$$

As in all other PCIs, the process spread represented in the numerator. The denominator represents how far the process mean from the target value. The capability of the process goes up together with the value of the C_{pmk}.

Previously, for estimating population μ and population σ, the unbiased estimators were used. Although Wu and Darmawan (2025), in their article, offering the maximum likelihood estimator (MLEs) as the prediction of $\hat{\sigma}$. I will choose to stick to this estimator for C_{pmk} index:

$$S_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

The symmetric definition would look like this(Wu, Pearn, and Kotz 2009):

$$C_{pmk} = \frac{C_{pk} \times C_{pm}}{C_p} = \frac{d - |\mu - m|}{3\sqrt{\sigma^2 + (\mu - T)^2}}$$

This PCI combines in itself all other´s indexes advantages and adds the focus on the lack of centering. This means, that the usage of it is possible, even without the centering of the process mean and target value.

This leads us to the next connection of the indeces:(Kotz and Lovelace 1998)

$$\text{If} \quad T = \mu \Rightarrow C_{pm} = C_p$$
$$\text{If} \quad \mu = m \Rightarrow C_{pm} = C_{pmk} = C_{pk}$$

Despite the familiarity of this index with the C_{pm}, it is still closer to the C_{pk} index. Not only based on the form of it, but also in the mathematical sense. C_{pk} plays the role of the base index here. When the C_{pm} introduces the penalty for the C_{pmk}.

To this index also refer as to the third-generation index. Compared to all other indices, the C_{pmk} is the most comprehensive one. It offers a decent quality assurance. Although, the sensitivity to the variance and the bias might be the weak side of this index.

Assumptions

In the theoretical part many times were mentioned the assumptions, underlying the indices. To clarify this important aspect the table was created, which gives the full understanding of the assumptions for each index. The content of this table was based on all the articles and books, that were researched during the work on this paper. These 4 assumptions can be met in each scientific paper, that is dedicated to the capability indices.

Assumption	C_p	C_{pk}	C_{pm}	C_{pmk}
1. Normal distribution of process data	+	+	+	+
2. Process is in statistical control	+	+	+	+
3. Process mean is centered between specification limits ($\mu = m$)	+	−	+	−
4. Target value lies midway between USL and LSL ($T = m$)	−	−	+	−

Practical part

In this part the practical part of the capability indices theory, observed above, will be presented. For the clarity and representativity of the theory, the dataset `bolts` with single continuous value from the library `SixSigma` was chosen. It contains the data frame with 50 observations of the diameter of the bolts manufactured in a production line.

Variables:

- `diameter` a numeric vector with the diameter of the bolts

The structure of the analysis

- verification of the assumptions necessary for the capability analysis
- calculation of the basic capability indices with `qcc` library
- calculation of the C_{pmk} index using the custom function

Assumptions check

1. The normality check: with the function `shapiro.test` from the standard R library the Shapiro-Wilk test was conducted. As the result, the normality of the process was proved.
2. On the next step, with the use of the `qcc` library and a function `qcc`, the process showed itself to be stable and in control. The Shewhart control chart - $\bar{X}$ chart - was a good illustration of it.
3. The process was proved to be centered ($\mu = m$).
4. The target value was set to the mid-point of the process spread ($T = m$).

Initial capability analysis

With the `qcc` library the indices C_p, C_{pl}, C_{pu}, C_{pk} and C_{pm} were calculated.

As the result, all of the indices were equal to ≈ 1, under the assumptions described earlier. This proves the capability of the process, but the value of the indices barely crossed the minimum tolerance.

Exploration of the improvement possibilities

Since $C_p = 1$ is only the minimum acceptable solution, with further analysis the next ways to increase this value were explored:

- reduction of process variance. This would be the best way to do so but it is hard to implement in our case,
- widening of the specification limits is way more realistic.

Therefore, the manual function for this was created. This works this way:

- the spread is determined not based on the 6σ, but on the value of the index we need to achieve,
- as the result, the chance of the defect drops and becomes almost zero,

In the real world, this approach would not be applied. However, for this paper, it is a good way to show the mathematical behavior of the capability indices. The controlled environment offers an excellent chance to demonstrate the theory.

Manual function for calculation of C_{pmk} index

Since none of the libraries of the R language supports the automatic calculation of the C_{pmk} index, a simple custom function was created for this.

As the result, this index also was equal to ≈ 1.

Target value and its role

The importance of target value was shown with the use of a custom function. After a manual calculation of the indices, the penalization of the C_{pm} and C_{pmk} was revealed. It was clearly shown that these indices react to any shift of the target value from the m value.

Summary

This practical part not only displayed the math behind the capability indices, but also highlighted the practical side of it. It supports the theoretical part of this work. Most importantly, it proved the connection between the tolerance limits and process variation.

Conclution

While working on the topic of capability indices, there were several important aspects that stood out repeatedly. The use of this tool is essential at the stage of the process capability analysis. With the combination of indices and control charts, the manufacturing process will only thrive. However, it is immensely important to check for the main assumptions underlying the capability indices. Their use in the non-normal process is possible, but the results in this case may not be helpful to improve the manufacturing capabilities.

Another important issue with the indices is that they are not representative when used alone. Their ability to positively influence the capability of the process exists only when they are used all together or in the specific combination. Though the C_{pmk} is the most powerful index, it is still sensitive to the bias and variance of the process. C_p might give the falsely high value, when the process is off-center. C_{pk} has a tendency to lower the value too much, when the process is even slightly not centered. Taguchi´s index C_{pm} is not representative when the midpoint of the specification limits does not equal to the target value.

Nevertheless, capability indices are powerful tools. Their use goes along with the careful use of process data and its context. This paper could be further extended with the topics of non-normal data and how to apply indices to it. Or the use of indices with fuzzy data.

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