Notes on BEGS^N

1 Outline (After May 17th meeting)

1.1 The premise

Representative agent economies with non-negativity restriction on transfers as way of capturing some implicit redistribution motives. These models predict that in the long run distortionary taxes are smooth (in fact zero if the government trades only a bond) and the government uses transfers /assets to completely hedge aggregate shocks. We study the long run dynamics of assets and taxes in an economy with a)heterogeneous agents and b)no restriction on transfers. Concerns for redistribution make fluctuations in transfers costly. In the long run optimal policy trade offs costs of fluctuating taxes with that of transfers. In the note below we discuss how the implications for the ergodic distribution taxes and assets are altered in this setting.

1.2 The ingredients

Agents are heterogeneous w.r.t,

a) Initial productivities. This heterogeneity may vary over time with uninsurable innovations to their productivities,

b) Initial Pareto weights, and

c) Initial assets.

A government, under commitment, finances a stochastic expenditure stream with an affine tax system. We model the restriction on fiscal hedging induced by limited state contingency by allowing arbitrary correlation between payoffs and government expenditures.

1.3 The findings

Absent borrowing constraints, Ricardian equivalence holds. Using the normalization that the minimum assets are zero we can pinn down the total debt of the government. We have two main results

1. The support of the invariant distribution of taxes and aggregate debt is wide

2. The mean and variance of taxes and aggregate debt are driven by the interaction of two "covariances": a) between payoffs and aggregate expenditures b) covariance (cross-sectional) between consumption and labor earnings. These two covariances summarize the implications of the limits of market incompleteness on public and the private sector.

1.4 The results

Quasilinear- IID example

- Taxes are increasing in debt and approach $-\infty$ as $b_t \to \infty$ and $\frac{\gamma}{1+\gamma}$ as b_t approaches the natural debt limit.
- As the $\|\text{cov}(P(s), g(s))\|$ goes to zero, the support of assets and taxes is unbounded to the first order. Imposing arbitrary limits on b_t will generate an ergodic distribution of taxes with support given by limits on b_t that spreads out but located at a mean corresponding to zero debt (or a relatively equal distribution of assets). To first order approximation assets follow a random walk without a drift with reflecting boundaries. Relative to the representative agent models, the costs of transfers eliminates first best as an absorbing point.
- For the other extreme when the payoffs are strictly aligned with expenditure shocks, we can prove a stronger result - The limiting distribution is degenerate even with respect to the 'global' policy rules. The level is pinned down by the payoff vector. More precisely we approach a complete market economy with a particular distribution of initial assets.
- For $\omega < \bar{\omega}$ there is a range of initial conditions (distribution of assets) for which the allocation is immediately in "steady state". In these steady states the non-negativity constraint on agents consumption will always be slack. This range of steady states assets may depend on the payoff vector $\mathbb{P}$. As ω approaches zero, the range of initial conditions such that the non-negativity constraint is slack approaches all feasible initial distributions of debt.

More general cases

- The insights extend to risk aversion, and persistent expenditure shocks. Conceptually there are two modifications - First there is an endogenous component to the returns as the price of the risky bond fluctuates. This induces endogenous covariance between expenditure shocks and asset returns. Secondly, the costs of transfers are additionally driven by the spread in marginal utilities.
- Absent idiosyncratic shocks, the dynamics of wealth distribution are driven by the differences in precautionary motives of productive and unproductive agents. With incomplete markets, the volatility in transfers accentuates the precautionary motives of the unproductive agents making them more accumulate assets in the long run.

  In the numerical exercise we add persistent but un-insurable idiosyncratic productivity shocks to get a "realistic" co-movement of consumption, assets and labor earnings. This also has implications for the invariant distribution of taxes. A higher steady state covariance between consumption and earnings raises the mean level of taxes. This covariance will be driven by the persistence of idiosyncratic shocks. The variance of taxes will be governed by the strength of the hedging motive against aggregate shocks.

QL economy: More details

Consider an economy with two types of agents with productivities $\theta_1 = 0$ and $\theta_2 > 0$. These agents value consumption and leisure $(1-l)$ using preferences $u(c,l) = c - \frac{l^{1+\gamma}}{1+\gamma}$. Let ω be the Pareto weight of the Planner on the productive agent 1 and n be his mass. The government expenditures are iid denoted by $g(s)$. The payoffs on the traded assets are given by $P(s)$. The exogenous state is drawn with probabilities $\pi(s)$

Let $b_{\_}$ be the debt of the government under the normalization that agent 1 hold no assets. The Ramsey allocation solves the following Bellman equation.

$$V(b_{\_}) = \max_{c_1(s),c_2(s),b'(s)} \sum_s \pi(s) \left\{ \omega \left[u(c_1(s), l_1(s)) \right] + (1-\omega) \left[c_2(s) \right] + \beta V(b(s)) \right\} \tag{1}$$

subject to

$$c_1(s) - c_2(s) + b(s) = l(s)^{1+\gamma} + \beta^{-1} P(s) b_{\_} \tag{2a}$$

$$nc_1(s) + (1-n)c_2(s) + g(s) \leq \theta_2 l(s) n \tag{2b}$$

$$c_2(s) \geq 0 \tag{2c}$$

Let $\mu(s), \phi(s), \lambda(s)$ be the Lagrange multipliers on the respective constraints. The FOC and the envelope conditions are summarized below

$$\frac{\lambda(s)}{2} = \frac{\omega - \mu}{n} - 1 \tag{3a}$$

$$\mu(s) = \frac{\omega \left[\frac{\theta_2}{l^\gamma(s)} - 1 \right]}{\left[\frac{\theta_2}{l^\gamma(s)} - 1 - \gamma \right]} \tag{3b}$$

$$\mu_{\_} = \mathbb{E}\mu(s) - \operatorname{cov}(P(s), \mu(s)) \tag{3c}$$

We summarize a few properties of allocation using the following lemmas

Lemma 1 *The value function of the planner is strictly concave and differentiable*

Proof. concavity of V can be shown by observing that objective function and the implementability constraint is linear in $\tilde{l} = l_1^\gamma$. The resource constraint is a weak inequality with the RHS concave in $\tilde{l}$. Thus the constraint set is convex. Further the value function is bounded, we can apply analogues of theorem 4.6-4.8 in SLP to prove that V is concave ■

Lemma 2 *The multiplier on the budget constraint $\mu(s)$ is bounded above*

$$\mu(s) \leq \min \left\{ \omega - n, \frac{\omega}{1+\gamma} \right\}$$

Similiarly the multiplier of the resource constraint is bounded below,

$$\phi(s) \geq \max \left\{ 1, \frac{\omega}{n} \left[\frac{\gamma}{1+\gamma} \right] \right\}$$

Proof.

Notice that the labor choice of the productive household implies $\frac{1}{1-\tau} = \frac{\theta_2}{l^\gamma(s)}$.

As taxes go to $-\infty$ (**??**) implies that $\mu(s)$ approaches $\frac{\omega}{1+\gamma}$ from below. Similiarly the non-negativity of $c_2(s)$ imposes a lower bound of 1 on $\phi(s)$. This translates into an upper bound of $\omega - n$ on μ.

■

Lemma 3 *There exists a $\bar{b}$ such that $b_t \leq \bar{b}_n$. This is the natural debt limit for the government.*

Proof. As we drive μ to $-\infty$, the tax rate approaches a maximum limit, $\bar{\tau} = \frac{\gamma}{1+\gamma}$. In state s, the government surplus is

$$S(s,\tau) = \theta_2^{\frac{\gamma}{1+\gamma}}(1-\tau)^{\frac{1}{\gamma}}\tau - g(s),$$

which is maximized at $\tau = \frac{\gamma}{1+\gamma}$ when $(1-\tau)^{\frac{1}{\gamma}}\tau$ is also maximized. This would impose a natural borrowing limit for the government. ■

Lemma 4 *There exists a $\bar{\omega}$ such that $\omega > \bar{\omega}$ implies $c_2(s) = 0$ for all b*

By the KKT conditions $c_2(s) = 0$ if $\lambda(s) > 0$. Now (**??**) implies this is true if $\mu(s) < \omega - n$. The previous lemma bounds $\mu(s)$ by $\frac{\omega}{1+\gamma}$.

We can thus define $\bar{\omega} = n\left(\frac{1+\gamma}{\gamma}\right)$ as the required threshold Pareto weight to ensure that the unproductive agent has zero consumption forever.

We are interested in the ergodic distribution of taxes as we change the "extent" of market incompleteness for the government by varying $P(s)$. Formally we split the results into three parts:

1. A risk-free bond economy where $P(s) = 1$
2. A risky bond economy with where $P(s)$ is perfectly correlated to expenditure shocks or

$$P(s) = 1 - \frac{\beta}{b}(g(s) - \mathbb{E}g).$$

   for some $b \in (-\infty, \bar{b})$

3. A risky bond economy where $P(s)$ is imperfectly correlated to expenditure shocks

Theorem 1 *Let $\omega > \bar{\omega}$ and suppose $P(s) = 1$, $\lim_t \tau_t = -\infty$, $\lim_t b_t = -\infty$ a.s*

Proof. This comes from the super-martingale converge theorem applied to μ_t. We then argue that μ_∞ cannot converge to any value except its upper bound as this will violate the natural debt limit. ■

Corollary 1 *Suppose we augment our problem with a constraint $b_t \geq \underline{b}$. Then with risk-free debt there is an invariant distribution ψ. Morever, for any $\hat{b} \in (\underline{b}, \bar{b}_n)$, $\psi\left(\left[\underline{b}, \hat{b}\right]\right) > 0$ and $\psi\left(\left[\hat{b}, \bar{b}_n\right]\right) > 0$..*

Remark 1 *The set of ω satisfying this condition is empty if $n > \frac{\gamma}{1+\gamma}$. In this case taxes and debt approach a finite value. The limiting tax rate is given by*

$$\tau(\mu) = \frac{\gamma(\omega - n)}{(1+\gamma)(\omega - n) - 1}$$

and consumption of agent 2 or transfers are non zero i.o .

Conjecture 1 *Suppose we do not have a bond economy but* $\text{cov}(P(s), g(s)) = 0$, *we might have a wide distribution of assets and taxes without imposing the upper bound. The planner value is concave in debt. As assets diverge, if* $\text{var}P(s) > 0$, *this maginifies the fluctuations in assets. This can provide a force towards interior. We dont have a proof for this yet and the comments are based on numerical results.*

Theorem 2 *Let $\omega > \bar{\omega}$ and suppose $\overline{b} < \overline{b}_n$ and payoffs satisfy*

$$P(s) = 1 - \frac{\beta}{\overline{b}}(g(s) - \mathbb{E}g)$$

then for all b_{-1}, $b_t \to \overline{b}$ with probability 1.

Proof. This comes from showing a sequence of steps:

- We first establish that the restriction on $P(s)$ is sufficient for existence of fixed point $\bar{\mu}$ that satisfies $\mu_t = \bar{\mu}$ implies $\mu_{t+s} = \bar{\mu}$ for all t
- We show that $\mu(s|\mu_{\text{-}})$ is continuous and monotonic in μ
- Then show that $\text{cov}(\mu(s), P(s))$ is uniformly negative for $\mu < \bar{\mu}$ and uniformly positive for $\mu > \bar{\mu}$.
- This shows that μ_t is a sub martingale bounded above by the steady state value for all $\mu < \bar{\mu}$ and super martingale bounded below in the region $\mu > \bar{\mu}$
- The last step is to establish that the only limit point it converges to is $\bar{\mu}$

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Corollary 2 *Suppose we augment our problem with a constraint $b_t \geq \underline{b} < \bar{b}$. Then the invariant distribution ψ is degenerate. Moreover the allocation corresponds to a Lucas Stokey allocation (i.e with complete markets and a representative agent) with initial debt given by* $\overline{b} = -\beta \frac{\text{var}(g(s))}{\text{cov}(P(s), g(s))}$. *and constant taxes.*

For more general payoff structures $P(s)$ we appeal to an approximation around the payoff that are perfectly aligned with g. In particular, we obtain a orthogonal decomposition of $P(s)$ as follows,

$$P(s) = \hat{P}(s) + \bar{P}(s)$$

where

$$\overline{P}(s) = 1 - \frac{\beta}{\overline{b}}(g - \mathbb{E}g).$$

and $\hat{P}(s)$ is orthogonal to $g(s)$.

Next expanding the policy rules around the steady state of the $\bar{P}(s)$ economy we have the following characterization,

Theorem 3 *For $\omega > \bar{\omega}$, the ergodic distribution of debt of the policy rules linearized around $(\bar{b}, \overline{P}(s))$ will have mean $\bar{b}$ and and coefficient of variation*

$$\frac{\sigma_b}{\bar{b}} = \sqrt{\frac{\mathrm{var}(P(s)) - |\mathrm{cov}(g(s), P(s))|}{(1 + |\mathrm{cov}(g(s), P(s))|)|\mathrm{cov}(g(s), P(s))|}} \leq \sqrt{\frac{\mathrm{var}(P(s)) - |\mathrm{cov}(g(s), P(s))|}{|\mathrm{cov}(g(s), P(s))|}}$$

The speed of convergence to the ergodic distribution given by

$$\frac{\mathbb{E}_{t-1}(b_t - \bar{b})}{(b_{t-1} - \bar{b})} = \frac{1}{1 + |\mathrm{cov}(P(s), g(s))|}$$

Lastly we discuss the case when $\omega < \bar{\omega}$. In this case long run taxes are constant. The next theorem characterizes the dynamics for the risk free bond.

Theorem 4 *Suppose $P(s) = 1$ and $\omega < \bar{\omega}$, there exists a threshold $\mathcal{B}(\omega)$, such that $b_- < \mathcal{B}(\omega)$ such that taxes are always constant to $\tau^*(\omega)$. For $b_- > \mathcal{B}(\omega)$, taxes are eventually limit to $\tau^*(\omega)$*

Proof. Guess an interior solution. Labor supply solves

$$l_1^*(\omega) = \left(\frac{\theta_1}{2\left[\left(\frac{1}{2} - \omega\right)(1+\gamma) + \omega\right]}\right)^{\frac{1}{\gamma}} \tag{4}$$

We can use the budget constraint (**??**) with the guess $b(b_- s) = b_-$ and the resource constraint (**??**) to back out consumptions for each agent.

$$c_1(s) = \frac{1}{2}\left[\theta_1 l_1^*(\omega) + l_1^*(\omega)^{1+\gamma} + b\left(\beta^{-1}P(s) - 1\right) - g(s)\right] \tag{5a}$$

$$c_2(s) = \frac{1}{2}\left[\theta_1 l_1^*(\omega) - l_1^*(\omega)^{1+\gamma} + b\left(\beta^{-1}P(s) - 1\right) - g(s)\right] \tag{5b}$$

To be a valid interior solution we need $c_2(s) > 0$. This implies $b < \mathcal{B}(\omega)$ where this threshold is given by

$$\mathcal{B}(\omega) = \max_s \left\{\frac{g(s) - \theta_1 l_1^*(\omega) + l_1^*(\omega)\, h'(l_1^*(\omega))}{\frac{P(s)}{\beta} - 1}\right\} \tag{6}$$

For $b_- > \mathcal{B}(\omega)$ we can show that

Lemma 5 $\lim_t \lambda_t = 0$

Proof. First note that, the Envelope condition is still valid and the multiplier on the implementability constraint, μ_t is a martingale. The FOC imply $\lambda^{c_1}(s^t)$ is also a martingale. Further the complementary slackness conditions provide a lower bound of zero i.e $\lambda^{c_1}(s^t) \geq 0$. We can now apply the standard martingale convergence theorems to argue that $\lambda^{c_1}(s^t) \to \lambda^*$. We now argue that this limit is always 0. ∎

Lastly we show that the long run assets are constant at $\mathcal{B}(\omega)$. This will follow from the fact that $b(b_-, s)$ is continuous and weakly increasing in b_-.

Lemma 6 *$b(b_{-}, s)$ is continuous and weakly increasing in b_{-}.*

Proof. First note that λ is increasing in b_{-} as a higher value of b_{-} tightens the non-negativity constraint on c_1. Since $\mu(s) = \frac{1}{2} - \omega - \frac{\lambda(s)}{2}$, it is decreasing in b_{-}. Weak concavity of V and the envelope theorem imply that $b(b_{-}, s)$ is weakly increasing. Continuity comes from applying the Maximum theorem to the problem ■

Lemma 7 *if $b_{-} > \mathcal{B}(\omega)$ then $\lim_t b_t = \mathcal{B}(\omega)$*

Proof. Suppose not, then there exist a t such that $b_t > \mathcal{B}(\omega)$ and $b_{t+1} < \bar{\mathcal{B}}(\omega)$. This implies that

$$b_{t+1} = b[b_t, s_t] < b[\mathcal{B}(\omega), s_t]$$

Since the previous lemma shows $b[b_{-}, s]$ is increasing(s), we have a contradiction. ■

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2 Another example

In this section we study an example with (some) risk aversion that extends the results of the 2 agent QL economy. The results are not as tight as we cannot yet prove the concavity of the value function.

Suppose the unproductive agent has CRRA utility function with risk aversion σ. We further assume that the unproductive agent does not participate in the bond market. Under these assumption we can formulate the Bellman equation that solves for the Ramsey plan as follows:

$$V(b_{-}) = \max_{c_1(s), c_2(s), b'(s)} \sum_s \pi(s) \left\{ \omega \left[u(c_1(s), l_1(s)) \right] + (1-\omega) \left[\frac{c_2(s)^{1-\sigma}}{1-\sigma} \right] + \beta V(b(s)) \right\} \tag{7}$$

subject to

$$c_1(s) - c_2(s) + b(s) = l(s)^{1+\gamma} + \beta^{-1} P(s) b_{-} \tag{8a}$$

$$nc_1(s) + (1-n)c_2(s) + g(s) \leq \theta_2 n l(s) \tag{8b}$$

Let $\mu(s), \phi(s), \lambda(s)$ be the Lagrange multipliers on the respective constraints. The FOC and the envelope conditions are summarized below

$$\phi(s) = \alpha - \frac{\mu(s)}{n} \tag{9a}$$

$$c_2(s)^{1-\sigma} = \frac{\phi(s) - \omega}{1-\omega} \tag{9b}$$

$$\mu(s) = \frac{\omega \left[\frac{\theta_2}{l^{\gamma}(s)} - 1 \right]}{\left[\frac{\theta_2}{l^{\gamma}(s)} - 1 - \gamma \right]} \tag{9c}$$

$$\mu_{-} = \mathbb{E}\mu(s) - \operatorname{cov}(P(s), \mu(s)) \tag{9d}$$

Lemma 8 *The multipliers $\phi(s)$ and $\mu(s)$ are bounded below and above respectively.*

$$\phi(s) \geq \max\left\{\omega, \frac{\omega\gamma}{n(1+\gamma)}\right\}$$

and

$$\mu(s) \leq \min\left\{\frac{\omega(1-n)}{n}, \frac{\omega}{(1+\gamma)}\right\}$$

Theorem 5 *Suppose $P(s) = 1$ and we have an lower bound on debt $\underline{b} > -\infty$. For $n < 1$ there is an invariant distribution ψ. Morever, for any $\hat{b} \in \left(\underline{b}, \bar{b}_n\right)$, $\psi\left(\left[\underline{b}, \hat{b}\right]\right) > 0$ and $\psi\left(\left[\hat{b}, \bar{b}_n\right]\right)$.*

If $n > \frac{\gamma}{1+\gamma}$, there exists a $\underline{\tau}$ independent of $\underline{b}$ such that

- *$\tau_t \geq \underline{\tau} > -\infty$, and*
- *As $\underline{b} = -\infty$, we have $\tau_t \to \underline{\tau}$ a.s*

Proof. Consider the case when $\underline{b} = -\infty$. Lemma **??** gives us bounds on μ_t. The super-martingale converge theorem implies $\lim_t \mu_t = \mu^*$. First we show that $b_t \to -\infty$

$$\mu^* \leq \min\left\{\frac{\omega(1-n)}{n}, \frac{\omega}{(1+\gamma)}\right\}$$

Suppose not. As $\mu_t \to \mu^*$. Thus taxes, labor supply and output converges to a constant.

If $n < \frac{\gamma}{1+\gamma}$, $\mu_t \to \frac{\omega}{1+\gamma}$. In this case taxes diverge to $-\infty$. Also$\phi_t \to \phi^* = \frac{\omega\gamma}{n(1+\gamma)} > \omega$. Thus $T_t \to T^* < \infty$ and The fluctuations $g(s)$ are absorbed by $c_1(s)$. With sufficient stochasticity of $g(s)$, the implementability constraint implies that b_t will violate any bounds.

Now if $n > \frac{\gamma}{1+\gamma}$, the multiplier μ_t converges to $\frac{\omega(1-n)}{n}$. The limiting taxes $\underline{\tau}$ can be obtained from the FOC (**??**). In this case $T_t \to \infty$ and $c_{1,t} \to -\infty$.

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Remark 2 *The results for the perfectly aligned payoffs and approximation results are analogously extended*

3 Numerical Results

We begin with a individual wage process that has three components - a) idiosyncratic transitroy, b) idiosyncractic persistent and lastly an aggregate component. For now all these three are independent. Later we will enrich this description such that the higher moments of the individual wage distribution vary systematically with aggregate component. Uninsurable persistent shocks generate a positive comovement between consumption, earnings and assets.

In this environment we ask how taxes, transfers and government debt should respond to aggregate shocks.

The main findings are as follows:

1. The key determinant of taxes is how needs for redistribution vary of business cycle.

2. As before tax rates can drift in the long run. The spread of the invariant distribution is primarily determined by the ability of the government to hedge aggregate shocks as measured by the correlation of payoffs and needs for revenue. However these are low frequency changes and take a long time to accumulate.

3. In the short run what matters is how inequality in consumption, earnings and assets varies over the business cycle. We have two cases depending upon the payoff structure:

   - Countercyclical payoffs: With TFP shocks a typical bond economy generates countercylical payoffs. There we find that taxes, transfers and debt are higher (lower) with positive (negative) aggregate tfp shocks. Normalize transfers to zero. Consider a transitory increase in aggregate tfp. We distingguish two forces that affect relative "asset income (or Transfers)" and "labor income"

     a) The unproductive (or low) productive agents primarily consume out of their asset income (or transfers). As such the lower interest rates hurt them more than the rich agents.

     b) Further the relative increase in the labor supply is larger for productive agents with higher wealth. This is because they suffer less from the wealth effect.The poor agents who have a higher opportunity cost of leisure change their labor supply less.

     Both these lead to higher inequality in assets, earnings and consumption. The government responds by increasing taxes, transfers and debt to induce redistribution.

   - With procylical payoffs: With procyclical payoffs, the higher earnings on assets relatively benifit the unproductive agents.This could flip the effect coming from higher inequality in labor earnings. The government responds by lowering taxes, transfers but increasing debt.

Bond Economy

Procyclical Payoffs

policy_long_sample_bond_economy.png

Figure 1: **Bond Economy:**This plots a simulation for taxes, transfers and debt for 100yrs. The dotted line is the economy with i.i.d aggregate shocks

policy_drfits_bond_economy.png

Figure 2: **Bond Economy:**This plots a simulation for taxes, transfers and debt for a sequence of high tfp (one std. deviation) shocks. The dotted line is the economy with idiosyncractic risk

policy_irf_high_bond_economy.png

Figure 3: **Bond Economy:** This plots a "impulse response" (5 period of high aggregate TFP followed by no aggregate shocks) for taxes, transfers and debt. The dotted line is the economy with idiosyncractic risk

policy_long_sample_procyclical.png

Figure 4: **Procyclical payoffs:** This plots a simulation for taxes, transfers and debt for 100yrs. The dotted line is the economy with i.i.d aggregate shocks

policy_drfits_procyclical.png

Figure 5: **Procyclical payoffs:** This plots a simulation for taxes, transfers and debt for a sequence of high tfp (one std. deviation) shocks. The dotted line is the economy with idiosyncractic risk

policy_irf_high_procyclical.png

Figure 6: **Procyclical payoffs:** This plots a “impulse response” (5 period of high aggregate TFP followed by no aggregate shocks) for taxes, transfers and debt. The dotted line is the economy with idiosyncractic risk