Taxes, Debts, and Redistribution

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Motivation

- How costly are high levels of government debt? What determines welfare cost of debt?
- Should the gov't try to reduce its initial high debt? If so, how quickly?
- How should tax rates, transfers, and government debt respond to aggregate shocks, especially if markets are incomplete?

Motivation

- Analysis with complete markets is well known:
  - Smooth distortionary costs of raising revenue
  - Labor taxes are (approximately) constant
  - Arrow securities used to finance all expenditure needs
- Another extreme is where the government has a “rich” enough set of tax instruments.

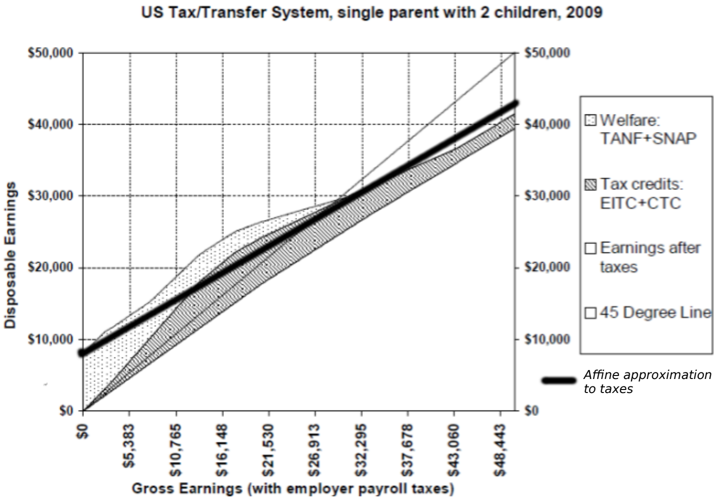
Our focus: Markets less than fully complete and there are limits to redistribution

Key ingredients

- **Heterogeneity:** Agents are heterogeneous in productivities and assets
- **Instruments:** A tax system that is linear in labor income and an intercept that is uniform across agents
- **Markets:** All agents trade a *single* security whose payoff might depend on aggregate shocks

Characterize optimal tax rate, transfers and asset purchases

US taxes: Affine taxes



Findings I

- **Welfare cost** of debt is determined by distribution of asset positions across agents
  - Ricardian logic: Increasing all agents' assets and reducing transfers keeps budget sets unaltered
  - Costs are lower when debt is more equally distributed
  - Credit constraints (if present) may weakly improve welfare
- **Ergodic distribution** of debts and taxes, in particular mean, variance and speed of convergence depend on
  - **Spanning ability:** correlation of returns on the traded asset with govt's needs for revenue, and
  - **Redistribution concerns:** Welfare weights relative to "market" weights that depend on wealth and productivities

Findings II

- What mechanisms drives **long run debt and tax rates**?
  - If interest rate co-moves with revenue needs: issue positive debt
  - Larger the correlation: lower the magnitude debt and higher is the speed of convergence
  - More redistributive governments: larger transfers and less incentives to accumulate assets
- Analytical results for quasilinear preferences and some extensions to more general preferences

Findings III

- **Calibration:** In the US data,
  - Correlation of interest rates and business cycles is small
  - In recent recessions, low income agents faced much larger drops in income than high income agents
- **Optimal responses over business cycle**
  - For short run responses, nature of shock matters
  - In recessions with high inequality: big increase in transfers and debt, moderate increase in tax rates
  - Normative predictions are very different from representative agent RBC models

Related literature

- Representative agent incomplete market economies
  - Barro (1974, 1979), Aiyagari et al (2002), Faraglia-Marcet-Scott (2012), Farhi (2010), etc
- Representative agent complete market economies
  - Lucas-Stokey (1983), Chari-Kehoe (1999), etc
- Heterogeneous agents with complete markets
  - Werning (2007), Azzimonti-Francisco-Krusell (2008)

Environment

- **Uncertainty**: Markov aggregate shocks s_t
- **Demography**: N types of infinitely lived agents (mass n_i) plus a benevolent planner
- **Technology**: Output $\sum_i n_i \theta_{i,t} l_{i,t}$ is linear in labor supplies.
- **Preferences** (Households)

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U^i \left(c_{i,t}, l_{i,t}\right)$$

- **Preferences** (Planner): Given Pareto weights $\{\omega_i\}$

$$\mathbb{E}_0 \sum_i \omega_i \sum_{t=0}^{\infty} \beta^t U_t^i \left(c_{i,t}, l_{i,t}\right)$$

- **Asset markets**: A risky bond with payoffs $P_t = \mathbb{P}(s_t | s_{t-1})$

Environment, II

- **Affine Taxes**: Agent i's tax bill

$$-T_t + \tau_t \theta_{i,t} l_{i,t}$$

- **Budget constraints** Let $R_{t-1,t} = \frac{P_t}{q_{t-1}}$
  - Agent i: $c_{i,t} + b_{i,t} = (1 - \tau_t)\, \theta_{i,t} l_{i,t} + R_{t-1,t} b_{i,t-1} + T_t$
  - Government: $g_t + B_t + T_t = \tau_t \sum_i n_i \theta_{i,t} l_{i,t} + R_{t-1,t} B_{t-1}$

- **Market Clearing**
  - Goods: $\sum_i n_i c_{i,t} + g_t = \sum_i n_i \theta_{i,t} l_{i,t}$
  - Assets: $\sum_i n_i b_{i,t} + B_t = 0$

- **Initial conditions**: $(\{b_{i,-1}, B_{-1}\}_i, s_{-1})$

Ramsey Problem

Definition

Allocation, price system, government policy: Standard

Definition

Competitive equilibrium: Given $(\{b_{i,-1}\}_i, B_{-1}, s_{-1})$ and $\{\tau_t, T_t\}_{t=0}^{\infty}$, a competitive equilibrium is an allocation and price system such that households are optimizing and markets clear

Definition

Optimal competitive equilibrium: A welfare-maximizing competitive equilibrium for a given $(\{b_{i,-1}\}_i, B_{-1}, s_{-1})$

Ricardian Equivalence

Result: A **large set** of transfers and asset profiles support the same competitive equilibrium allocation

Notation: $\tilde{b}_{i,t} = b_{i,t} - b_{1,t}$: **relative assets** of Agent i

Theorem

Given $\left(\{b_{i,-1}\}_i, B_{-1}\right)$, let $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, B_t, R_t\}_t$ and $\{\tau_t, T_t\}_t$ be a competitive equilibrium.

For any bounded sequences $\left\{\hat{b}_{i,t}\right\}_{i,t\geq -1}$ that satisfy

$$\hat{b}_{i,t} - \hat{b}_{1,t} = \tilde{b}_{i,t} \text{ for all } t \geq -1, i \geq 2,$$

there exist sequences $\left\{\hat{T}_t\right\}_t$ and $\left\{\hat{B}_t\right\}_{t\geq -1}$ such that $\left\{\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_i, \hat{B}_t, R_t\right\}_t$ and $\left\{\tau_t, \hat{T}_t\right\}_t$ constitute a competitive equilibrium given $\left(\left\{\hat{b}_{i,-1}\right\}_i, \hat{B}_{-1}\right)$.

Ricardian Equivalence: Implications

- Present value of tax revenues and gov't debt is pinned down but not period-by-period transfers
- Can set $b_{i,t} = 0$ for any t, i or government without loss of generality
- Generally, more equally spread debt promised (implicit Social Security promises, debt in Japan) are less distortionary than debt skewed towards highly productive agents or foreigners (debt in Greece)
- **Extension:** Welfare is weakly higher with exogenous borrowing constraints of the form $b_{i,t} > \underline{b}_i$

More details

Characterization of optimal policy: Road map

- Active channels:
  1. Limited hedging ability
  2. Concerns for redistribution
- Analytical results:
  1. Quasi Linear preferences :$u(c, l) = c - \frac{l^{1+\gamma}}{1+\gamma}$
  2. IID aggregate shocks
- 2 step build up:
  1. Assume first that there is one agent and no ability to use T
  2. Use results to characterize outcomes in the more general settings with heterogeneous agents and no restriction on transfers
  3. Allows us to disentangle hedging and redistribution motives
  4. Informative about the setup with multiple agents where transfers are unrestricted but their costs are endogenously high

Single agent quasi-linear economy with $T \equiv 0$

Let $V(B_{-})$ be the maximum ex-ante value the government can achieve with assets B_{-}.

$$V(B_{-}) = \max_{c(s),l(s),B(s)} \sum_{s} \pi(s) \left\{ c(s) - \frac{l(s)^{1+\gamma}}{1+\gamma} + \beta V(B(s)) \right\}$$

subject to

$$c(s) - B(s) = l(s)^{1+\gamma} - \beta^{-1} P(s) B_{-}$$

$$c(s) + g(s) \leq \theta l(s)$$

$$\underline{B} \leq B(s) \leq \bar{B}$$

Single agent quasi-linear economy with $T \equiv 0$

- Decompose the set of payoffs:

$$\mathcal{P}^* = \left\{ P(s) : P(s) = 1 + \frac{\beta}{B^*}(g(s) - \mathbb{E}g) \text{ for some } B^* \in [\overline{B}, \underline{B}] \right\}$$

- Spanning condition that supports complete market allocations

Invariant distribution

Theorem

1. *Suppose $P \notin \mathcal{P}^*$, there is an invariant distribution of government assets such that*

$$\forall \epsilon > 0, \quad \Pr\{B_t < \underline{B} + \epsilon \text{ or } B_t > \overline{B} - \epsilon \quad i.o\} = 1$$

2. *Suppose $P(s) - P(s') > \beta \frac{g(s)-g(s')}{\underline{B}} \quad \forall s, s'$, then for large enough government assets (or debt) there is a drift towards the interior region. In particular the value function $V(B)$ is strictly concave and there exists $B_1 < B_2$ such that*

$$\mathbb{E}V'(B(s)) > V'(B_{\_}) \quad B_{\_} > B_2$$

and

$$\mathbb{E}V'(B(s)) < V'(B_{\_}) \quad B_{\_} < B_1$$

3. *Suppose $P(s) \in \mathcal{P}^*$, then the long run government assets converge to a degenerate steady state*

$$\lim_t B_t = B^* \quad a.s \quad \forall B_{-1}$$

Perfect spanning

- For $P(s) \in \mathcal{P}^*$, we can replicate complete markets perfectly asymptotically
- Target assets
$$B^* = \beta \frac{\mathrm{var}(g(s))}{\mathrm{cov}(P(s), g(s))}$$
- Tax rate is constant in long run and inversely related to B^*.
- Use this to construct an approximation for the ergodic distribution of debt and taxes of an economy with $P(s)$ "close" enough to $\mathcal{P}^*$. In particular split $P(s)$
$$P(s) = \hat{P}(s) + P^*(s)$$
where $P^*(s) \in \mathcal{P}^*$ and $\hat{P}(s)$ is orthogonal to $g(s)$. More details

Imperfect spanning

Theorem

The ergodic distribution of debt (under the first order approximation of dynamics near $P^*(s)$) has the following properties,

- **Mean:** *The ergodic mean is B^* which corresponds to the steady state level of govt. assets of an economy with payoff vector $P^*(s)$*
- **Variance:** *The coefficient of variation of assets satisfies*

$$\frac{\sigma(B)}{\mathbb{E}(B)} = \sqrt{\frac{\mathrm{var}(P(s)) - |\mathrm{cov}(g(s), P(s))|}{(1 + |\mathrm{cov}(g(s), P(s))|)|\mathrm{cov}(g(s), P(s))|}} \leq \sqrt{\frac{\mathrm{var}(\hat{P}(s))}{\mathrm{var}(P^*(s))}}$$

- **Convergence rate:** *The speed of convergence to the ergodic distribution is*

$$\frac{\mathbb{E}_{t-1}(B_t - B^*)}{(B_{t-1} - B^*)} = \frac{1}{1 + |\mathrm{cov}(P(s), g(s))|}$$

Ergodic distribution

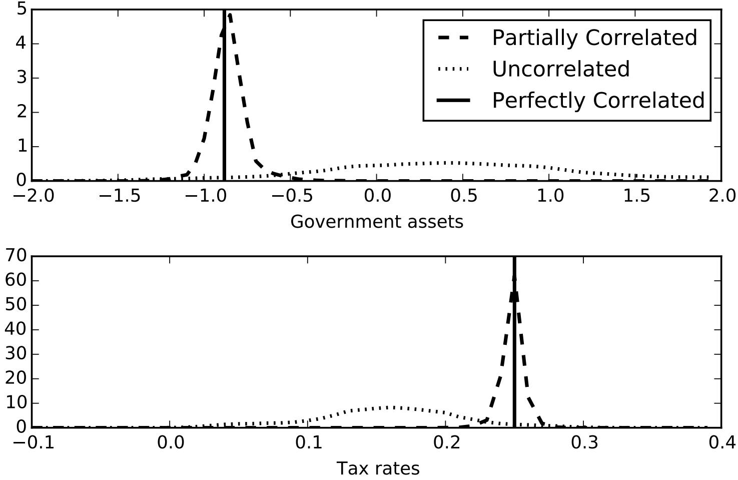


Figure : Ergodic distribution for debt and taxes in the representative agent quasilinear economy for three choices $P(s)$.

Summary and next steps

So far: In a single agent - quasilinear - no transfers economy we saw that,

- Target level of assets maximizes spanning
  - taxes are constant when perfect spanning is achieved
- When markets are imperfect, can be far away from the target
  - invariant distribution of taxes also has large support
- Speed of moving to the target debt level depends on covariance of asset payoff and shocks
  - low covariance $\Longrightarrow$ slow speed

Next: A version with heterogeneous agents and no restrictions on transfers

- This adds a new instrument to hedge shocks but welfare cost of using transfers is endogenous
- The single agent results are informative about cases where these costs are large

Heterogeneous agents

Suppose we have 2 agents

- Quasi-linear preferences as before
- Productivities: $\theta_1 > \theta_2 = 0$
- Pareto weights, mass of Agent 1 and 2: $\{\omega, 1-\omega\}$ and $\{n, 1-n\}$ respectively
- Non-negative consumption: $c_2 \geq 0$

Normalize $b_{2,t} = 0$, thus $B_t = -nb_{1,t}$ are interpreted to be government assets

Heterogeneous Agents

Theorem

Let ω, n be the Pareto weight and mass of the productive agent with $n < \frac{\gamma}{1+\gamma}$. The optimal tax, transfer and asset policies $\{\tau_t, T_t, B_t\}$ are characterized as follows,

1. *For $\omega \geq n\left(\frac{1+\gamma}{\gamma}\right)$ we have $T_t = 0$ and the optimal policy is same as in our representative agent economy studied*
2. *For $\omega < n\left(\frac{1+\gamma}{\gamma}\right)$, suppose we assume that $P(s) \notin \mathcal{P}^*$ and $\min_s\{P(s)\} > \beta$. There exists $\mathcal{B}(\omega)$ and $\tau^*(\omega)$ with $\mathcal{B}'(\omega) > 0$ such that*

   2.1 $B_{-} > \mathcal{B}(\omega)$

   $$T_t > 0, \quad \tau_t = \tau^*(\omega), \text{ and } B_t = B_{-} \quad \forall t$$

   2.2 $B_{-} \leq \mathcal{B}(\omega)$

   $$T_t > 0 \text{ i.o.}, \quad \lim_t \tau_t = \tau^*(\omega) \text{ and } \lim_t B_t = \mathcal{B}(\omega) \quad a.s$$

Concerns for redistribution

- Balancing costs of fluctuations in tax rates and transfer
  - fluctuations in taxes is costly: deadweight loss
  - fluctuations in transfers is costly: deviations from target level of redistribution
- For large ω transfers are costly as the planner gives resources to unproductive agents
- For low ω, transfers are used:
  - For low initial debt, interior solution: All shocks hedged by transfers
  - For high debt, accumulate assets until costs of transfers are equalized to costs of collecting labor taxes
- The more redistributory the planner is:
  - bigger average tax rates and transfers
  - less need to accumulate assets for precautionary reasons

Risk aversion

- With risk aversion: for a (generic) set of parameters there is asset allocation replicating complete market economy
  - arguments harder since "real" interest rates $\mathbb{E}_t U'(c_{t+1}) R(s_{t+1}) / U'(c_t)$ is endogenous
- Same general flavor as quasi-linear economy
  - cost of fluctuations in transfers comes from cost of fluctuation in $U_c \iff$ similar to multiplier on constraint $c \geq 0$ in quasi-linear case
  - If real payoffs are positively correlated with g : accumulate assets
  - If real payoffs are (sufficiently) negatively correlated with g : accumulate debt
  - absolute amount of asset/debt is decreasing in redistributive objective

More details

Numerical exercise

Solve $N = 5$ agent economy with realistic level and movements in wage dispersion across booms and recessions

- Long run dynamics: Study settings that differ in covariance of interest rates and output
- Transient dynamics: Study outcomes in recessions that are accompanied by higher inequality

Aggregate shocks affect,

1. Wages:
$$\log \theta_i = \epsilon[1 + (.9 - d(i))m]$$
2. Payoffs:
$$P = 1 + \chi\epsilon$$

Calibrating m: Inequality over business cycles

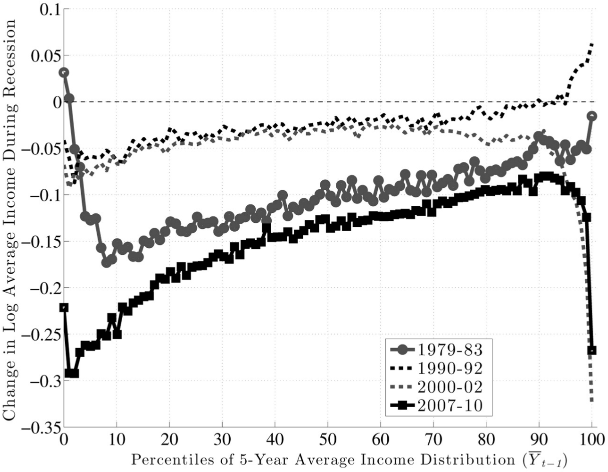


Figure : Change in log average earnings during recessions, prime-age males from Guvenen et all [2014]

Calibrating χ: Ex post variation in Payoffs

Let $q_t^{(n)}$ be the log price of a nominal bond of maturity n. We can define the real holding period returns $r_{t,t+1}^{(n)}$ as follows

$$r_{t,t+1}^{(n)} = q_{t+1}^{(n-1)} - q_t^{(n)} - \pi_{t+1}$$

With the transformation $y_t^{(n)} : -\frac{1}{n} q_t^{(n)}$ we can express $r_{t,t+1}^{(n)}$ as follows:

$$r_{t,t+1}^{(n)} = \underbrace{y_t^{(n)}}_{\text{Ex-ante part}} - (n-1)\left[\underbrace{\left(y_{t+1}^{(n)} - y_t^{(n)}\right)}_{\text{Interest rate risk given } n} + \underbrace{\left(y_{t+1}^{(n-1)} - y_{t+1}^{(n)}\right)}_{\text{Term structure risk}}\right] - \underbrace{\pi_{t+1}}_{\text{Inflation risk}}$$

Interest rates and TFP

- In the model the holding period returns are given by $\log \left[\frac{P_{t+1}}{q_t^1}\right]$ and $q_t^1 = \frac{\beta \mathbb{E}_t u_{c,t+1} P_{t+1}}{u_{c,t}}$.
- P_{t+1} allows us to captures ex-post fluctuations in returns to the government's debt portfolio coming from maturity and inflation.
- Since ϵ_t is i.i.d over time in our calibration $\chi = \frac{\sigma_r}{\sigma_\epsilon} Corr(r, \epsilon)$

Using data on labor productivity ϵ_t and $\{q_t^n\}_n$:

Maturity (n)	2yr	3yr	4yr	5yr
$Corr(\epsilon_{t+1}, r_{t,t+1}^{(n)})$	-0.11	-0.093	-0.083	-0.072
$Corr(\epsilon_{t+1}, r_{t,t+1}^{(n)} - ny_t^{(n)})$	0.00	-0.0463	-0.080	-0.091
$Corr(\epsilon_{t+1}, y_t^{(n)} - \pi_{t+1})$	-0.097	-0.086	-0.080	-0.073
$\frac{\sigma(r_{t+1}^n)}{\sigma(\epsilon_{t+1})}$	0.820	0.835	0.843	0.845

Table

Calibration

Parameter	Value	Description
$\{\bar{\theta}_i\}$	$\{1$, 1.4, 2.1, 3.24, 4.9$\}$	Wages dispersion for $\{$10,25,50,75,90$\}$ percentiles
γ	2	Average Frisch elasticity of labor supply of 0.5
β	0.98	Average (annual) risk free interest rate of 2%
m	$\frac{1.5}{.8}$	Changes in dispersion
χ	-0.06	covariance between holding period returns and labor productivity
σ_e	0.03	vol of labor productivity
g	.13 %	Average pre-transfer expenditure- output ratio of 12 %

Table : Benchmark calibration

The Pareto weights and initial distribution of wealth are chosen to match an average tax rate of 20%, and debt to gdp ratio of 100%, transfers to gdp ratio of 10%, and deciles of US wealth distribution

Long run

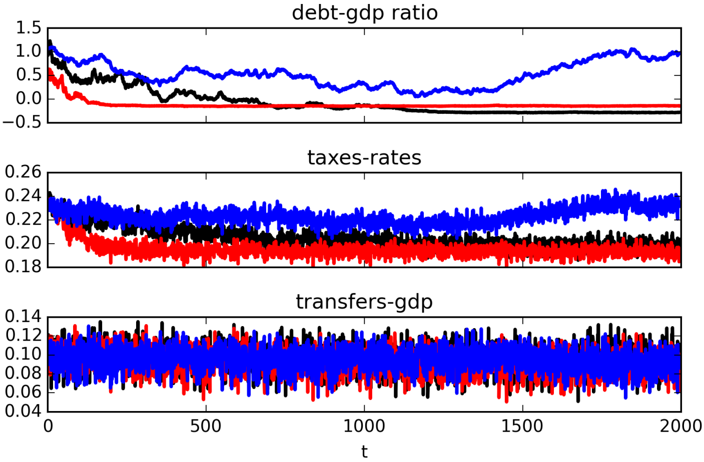


Figure : The red, black and blue lines plot simulations for a common sequence of shocks for values of $\chi = -1.0, 0, 1.0$ respectively

Long run: Speed of convergence

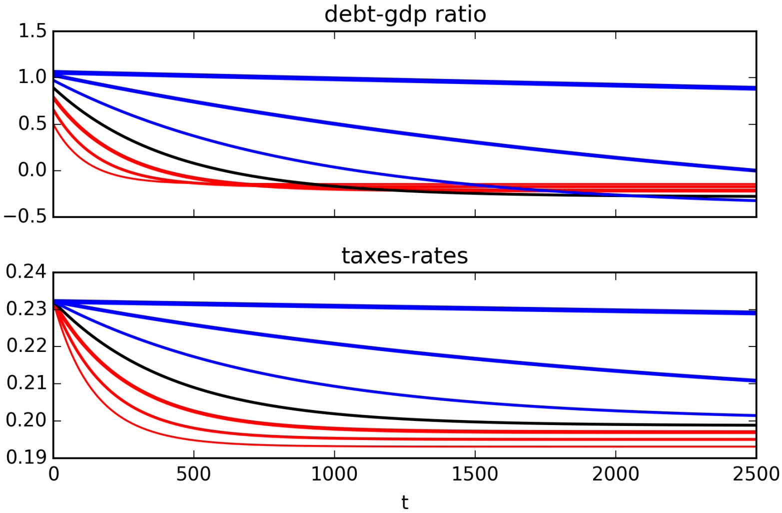


Figure : The plot shows conditional mean paths for different values of χ. The red (blue) lines have $\chi < 0$ ($\chi > 0$). The thicker lines represent larger values

Spreading of tax rates

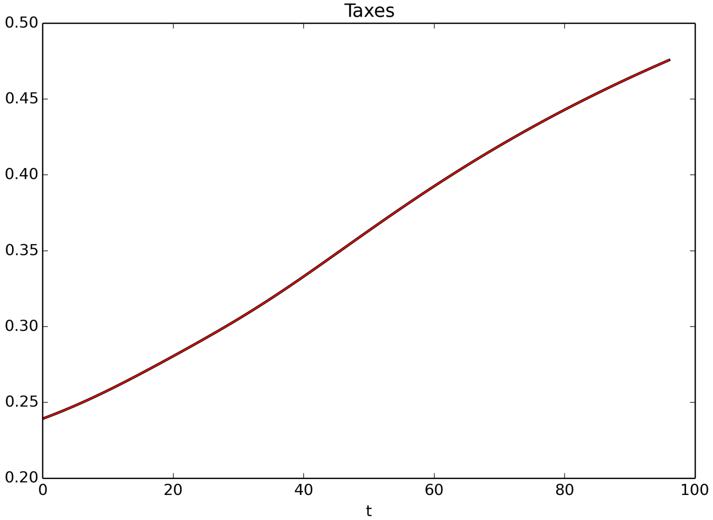


Figure : Tax rate for a sequence of -1 s.d shocks to aggregate productivity

Short run

Let us denote consecutive period of negative (positive) one s.d ϵ shocks a "recession" (boom)

- Engineer a recession of four periods from $t = 3$. Before and after this recession, the economy receives $\epsilon_t = 0$.
- Decompose responses into TFP component and inequality component:

$$\textbf{Baseline:}\ \log \theta_i = \epsilon[1 + (.9 - d(i))m]$$

  - Only TFP:

$$\log \theta_i = \epsilon$$

  - Only Ineq:

$$\log \theta_i = \epsilon[(.9 - d(i))m]$$

Recessions with higher inequality

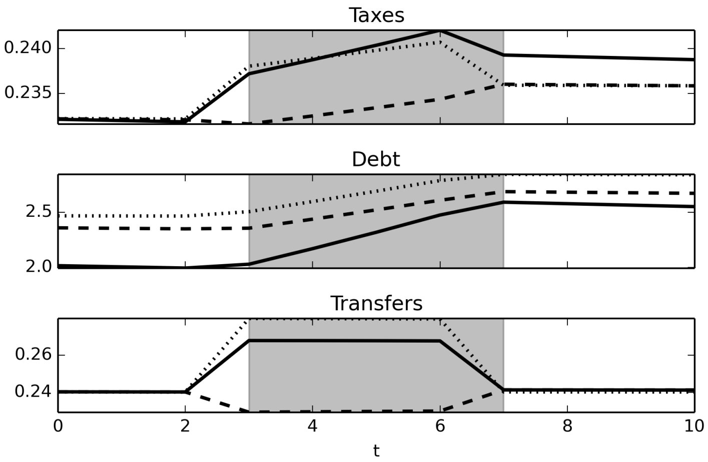


Figure : The bold line is the total response. The dashed (dotted) line reflects the only TFP (inequality) effect. The shaded region is the recession

Tfp and Tfp+Ineq recessions: Sample moments

Moments	Tfp	Tfp+Ineq
vol. of tax rates	0.003	0.006
vol. of transfers	0.01	0.02
autocorr. in tax rates	0.93	0.66
autocorr. in transfers	0.17	0.18
corr. of taxes with tfp	0.15	-0.63
corr. of transfers with tfp	0.99	-0.98

Table : These are sample moments averaged acrosss simulations of 100 periods

Conclusion

- Size of government debt alone is not informative $\Longrightarrow$ need to know the net distribution of assets in the economy
- Ignoring heterogeneity produces misleading results about size and direction of the optimal policy response
- The better ability we have to tax assets, the less debt matters and can approximate complete markets closer

Credit limits

Impose $b_{i,t} \geq \underline{b}_i$. and extend the definition of competitive eqb. in the obvious way. Now we have

Theorem

Given an initial asset distribution $\left(\{b_{i,-1}\}_i, B_{-1}\right)$, *let* $\{c_{i,t}, l_{i,t}\}_{i,t}$ *and* $\{R_t\}_t$ *be a competitive equilibrium allocation and interest rate sequence in an economy without exogenous borrowing constraints. Then for any exogenous constraints* $\{\underline{b}_i\}_i$, *there is a government tax policy* $\{\tau_t, T_t\}_t$ *such that* $\{c_{i,t}, l_{i,t}\}_{i,t}$ *is a competitive equilibrium allocation in an economy with exogenous borrowing constraints* $\left(\{b_{i,-1}, \underline{b}_i\}_i, B_{-1}\right)$ *and* $\{\tau_t, T_t\}_t$.

back

Ergodic distribution: Linear approximation

- For a given $P(s), g(s)$, we can compress the equilibrium conditions to two functions $b(\mu_{-})$ and a law of motion $\mu(s|\mu_{-})$
- Instead of approximating near a deterministic steady state we,
  - explicitly recognize that policy rules depend on payoffs: $\mu(s|\mu_{-}, \{P(s)\}_s)$ and $b(\mu_{-}, \{P(s)\}_s)$
  - take the first order expansion with respect to both μ_{-} and $\{P(s)\}$ around the vector $(\bar{\mu}, \{\bar{P}(s)\}_s)$ where $\bar{P}(s) \in \mathcal{P}^*$:
- The choice of $\bar{P}(s)$ is pinned down by

$$\min_{\tilde{P} \in \mathcal{P}^*} \sum_s \pi(s)(P(s) - \tilde{P}(s))^2.$$

- The law of motion approximated by

$$\mu_t - \mu^* = (\mu_{t-1} - \mu^*)B(s_t) + C(s_t)$$

back

More details on cases with risk aversion

- With risk aversion $\|S\| = 2$ is necessary for a steady state to exist
- Existence: Consider an economy consisting of two types of households with only one productive agent and i.i.d binary shocks to his productivity

Theorem

Suppose $u(c, l) = \ln c - \frac{1}{2} l^2$ *and* $g < \theta(s)$ *for all* s. *Let* $x = U_c^2(s) \left[b_2(s) - b_1(s) \right]$

1. **Countercyclical interest rates.** *If* $P(s_H) = P(s_L)$, *then there exists a steady state* $\left(x^{SS}, \rho^{SS} \right)$ *such that* $x^{SS} > 0, \; R^{SS}(s_H) < R^{SS}(s_L)$.
2. **Procyclical interest rates.** *There exists a pair* $\{P(s_H), P(s_L)\}$ *such that there exists a steady state with* $x^{SS} < 0$ *and* $R^{SS}(s_H) > R^{SS}(s_L)$.

In both cases, tax rates $\tau(s) = \tau^{SS}$ *and ratio of consumption* $\frac{c_1}{c_2}$ *are independent of the realized state.*

- We then develop a test for local stability as in the quasilinear case.

back

Ramsey problem: Recursive formulation

Split into two parts

1. $\mathbf{t \geq 1}$: Ex-ante continuation problem with state variables $(\mathbf{x}, \boldsymbol{\rho}, s_{-})$

$$\mathbf{x} = \beta^{-1}\left(U^2_{c,t-1}\tilde{b}_{2,t-1}, ..., U^I_{c,t-1}\tilde{b}_{I,t-1}\right)$$

$$\boldsymbol{\rho} = \left(U^2_{c,t-1}/U^1_{c,t-1}, ..., U^I_{c,t-1}/U^1_{c,t-1}\right)$$

2. $\mathbf{t = 0}$: Ex-post initial problem with state variables $(\tilde{\mathbf{b}}_{-\mathbf{1}}, s_0)$

Bellman Equation for $t \geq 1$

$$V(\mathbf{x}, \boldsymbol{\rho}, s_-) = \max_{c_i(s), l_i(s), x'(s), \rho'(s)} \sum_s \Pr(s|s_-) \left(\left[\sum_i \pi_i \alpha_i U^i(s) \right] + \beta V(\mathbf{x}'(s), \boldsymbol{\rho}'(s), s) \right)$$

where the maximization is subject to

$$U_c^i(s)\left[c_i(s) - c_1(s)\right] + U_c^i(s)\left(\frac{U_l^i(s)}{U_c^i(s)} l_i(s) - \frac{U_l^1(s)}{U_c^1(s)} l_1(s)\right) + \beta x_i'(s) = \frac{x_i P(s|s_-) U_c^i(s)}{\mathbb{E}_{s_-} \mathbf{U}_c^i} \text{ for all } s, i$$

$$\frac{\mathbb{E}_{s_-} P \mathbf{U}_c^i}{\mathbb{E}_{s_-} P \mathbf{U}_c^1} = \rho_i \text{ for all } i \geq 2$$

$$\frac{U_l^i(s)}{\theta_i(s) U_c^i(s)} = \frac{U_l^1(s)}{\theta_1(s) U_c^1(s)} \text{ for all } s, i \geq 2$$

$$\sum_i n_i c_i(s) + g(s) = \sum_i n_i \theta_i(s) l_i(s) \quad \forall s$$

$$\rho_i'(s) = \frac{U_c^i(s)}{U_c^1(s)} \text{ for all } s, i \geq 2$$

$$\underline{x}_i(s; \mathbf{x}, \boldsymbol{\rho}, s_-) \leq x_i(s) \leq \bar{x}_i(s; \mathbf{x}, \boldsymbol{\rho}, s_-)$$

Bellman equation for $t = 0$

$$V_0\left(\{\tilde{b}_{i,-1}\}_{i=2}^{I}, s_0\right) = \max_{c_{i,0}, l_{i,0}, x_0, \rho_0} \sum_i \pi_i \alpha_i U^i(c_{i,0}, l_{i,0}) + \beta V\left(x_0, \rho_0, s_0\right)$$

where the maximization is subject to

$$U^i_{c,0}\left[c_{i,0} - c_{1,0}\right] + U^i_{c,0}\left(\frac{U^i_{l,0}}{U^i_{c,0}} l_{i,0} - \frac{U^1_{l,0}}{U^1_{c,0}} l_{1,0}\right) + \beta x_{i,0} = U^i_{c,0}\tilde{b}_{i,-1}P(s_0) \text{ for all } i \geq 2$$

$$\frac{U^i_{l,0}}{\theta_{i,0}U^i_{c,0}} = \frac{U^1_{l,0}}{\theta_{1,0}U^{1,0}_{c}} \text{ for all } i \geq 2$$

$$\sum_i n_i c_{i,0} + g_0 = \sum_i n_i \theta_{i,0} l_{i,0}$$

$$\rho_{i,0} = \frac{U^i_{c,0}}{U^1_{c,0}} \text{ for all } i \geq 2$$