1. Appendix

1. Proof of Theorem ??

We prove a slight more general version of our result. Consider an infinite horizon, incomplete markets economy in which an agent maximizes utility function $U : \mathbb{R}^n_+ \to \mathbb{R}$ subject to an infinite sequence of budget constraints. We assume that U is concave and differentiable. Let $\mathbf{x}(s^t)$ be a vector of n goods and let $\mathbf{p}(s^t)$ be a price vector in state s^t with $p_i(s^t)$ denoting the price of good i. We use a normalization $p_1\left(s^t\right) = 1$ for all s^t. Let $b(s^t)$ be the agent's asset holdings, and let $\mathbf{e}\left(s^t\right)$ be a stochastic vector of endowments.

Consumer maximization problem

$$\max_{\mathbf{x}_t, b_t} \sum_{t=0}^{\infty} \beta^t \Pr\left(s^t\right) U(\mathbf{x}\left(s^t\right)) \tag{1}$$

subject to

$$\mathbf{p}\left(s^t\right) \mathbf{x}\left(s^t\right) + q(s^t) b\left(s^t\right) = \mathbf{p}\left(s^t\right) \mathbf{e}\left(s^t\right) + P(s_t) b\left(s^{t-1}\right) \tag{2}$$

and $\{b\left(s^t\right)\}$ is bounded and $\{q(s^t)\}$ is the price of the risk-free bond.

The Euler conditions are

$$\begin{aligned} \mathbf{U}_x(s^t) &= U_1(s^t)\mathbf{p}(s^t) \\ \Pr\left(s^t\right) U_1\left(s^t\right) q(s^t) &= \beta \sum_{s^{t+1} > s^t} \Pr\left(s^{t+1}\right) U_1\left(s^{t+1}\right). \end{aligned} \tag{3}$$

Proposition 1 *Consider an allocation $\{\mathbf{x}_t, b_t\}$ that satisfies (2), (3) and $\{b_t\}_t$ is bounded. Then $\{\mathbf{x}_t, b_t\}$ is a solution to (1).*

Proof. The proof follows closely Constantinides and Duffie (1996). Suppose there is another budget feasible allocation $\mathbf{x} + \mathbf{h}$ that maximizes (1). Since U is strictly concave,

$$\begin{aligned} & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(\mathbf{x}_t + \mathbf{h}_t) - \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(\mathbf{x}_t) \\ \leq \quad & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \mathbf{U}_x(\mathbf{x}_t)\mathbf{h}_t \end{aligned} \tag{4}$$

To attain $\mathbf{x}+\mathbf{h}$, the agent must deviate by φ_t from his original portfolio b_t such that $\{\varphi_t\}_t$ is bounded, $\varphi_{-1} = 0$ and

$$\mathbf{p}(s^t)\mathbf{h}\left(s^t\right) = P(s_t)\varphi(s^{t-1}) - q(s^t)\varphi(s^t)$$

Multiply by $\beta^t \Pr(s^t) U_1(s^t)$ to get:

$$\begin{aligned}\beta^t \Pr\left(s^t\right) U_1(s^t)\mathbf{p}(s^t)\mathbf{h}\left(s^t\right) &= \beta^t \Pr\left(s^t\right) U_1(s^t)\varphi(s^{t-1}) - q(s^t)\beta^t \Pr\left(s^t\right) U_1(s^t)\varphi(s^t) \\ &= \beta^t \Pr\left(s^t\right) U_1(s^t)\varphi(s^{t-1}) - \beta^{t+1} \sum_{s^{t+1}>s^t} \Pr\left(s^{t+1}\right) U_1\left(s^{t+1}\right)\varphi(s^t)\end{aligned}$$

where we used the second part of (3) in the second equality. Sum over the first T periods and use the first part of (3) to eliminate $\mathbf{U}_x(\mathbf{x}_t) = U_1(s^t)\mathbf{p}(s^t)$

$$\sum_{t=0}^{T} \beta^t \Pr\left(s^t\right) \mathbf{U}_x(\mathbf{x}_t)\mathbf{h}\left(s^t\right) = -\sum_{s^{T+1}>s^T} \beta^{T+1} \Pr\left(s^{T+1}\right) U_1\left(s^{T+1}\right)\varphi(s^T).$$

Since $\{\varphi_t\}_t$ is bounded there must exist $\bar{\varphi}$ s.t. $|\varphi_t| \leq \bar{\varphi}$ for all t. By Theorem 5.2 of Magill and Quinzii (1994), this equilibrium with debt constraints implies a transversality condition on the right hand side of the last equation, so by transitivity we have

$$\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \Pr\left(s^t\right) \mathbf{U}_x(\mathbf{x}_t)\mathbf{h}\left(s^t\right) = 0.$$

Substitute this into (4) to show that $\mathbf{h}$ does not improve utility of consumer. ■

2. Proof of Theorem ??

Proof. The optimal Ramsey plan solves the following Bellman equation. Let $V(b_{\_})$ be the maximum ex-ante value the government can achieve with debt $b_{\_}$.

$$V(b_{\_}) = \max_{c(s),l(s),b(s)} \sum_s \pi(s) \left\{ c(s) - \frac{l(s)^{1+\gamma}}{1+\gamma} + \beta V(b(s)) \right\} \tag{5}$$

subject to

$$c(s) + b(s) = l(s)^{1+\gamma} + \beta^{-1} P(s) b_{\_} \tag{6a}$$

$$c(s) + g(s) \leq \theta l(s) \tag{6b}$$

Let $\bar{b} = -\underline{B}$

$$\underline{b} \leq b(s) \leq \bar{b} \tag{6c}$$

Part 1 of Theorem ??

Lemma 1 *There exists a $\bar{b}$ such that $b_t \leq \bar{b}$. This is the natural debt limit for the government.*

Proof. As we drive μ to $-\infty$, the tax rate approaches a maximum limit, $\bar{\tau} = \frac{\gamma}{1+\gamma}$. In state s, the government surplus,

$$S(s,\tau) = \theta^{\frac{\gamma}{1+\gamma}}(1-\tau)^{\frac{1}{\gamma}}\tau - g(s),$$

which is maximized at $\tau = \frac{\gamma}{1+\gamma}$ when $(1-\tau)^{\frac{1}{\gamma}}\tau$ is also maximized. This would impose a natural borrowing limit for the government.

■

From now we assume that $\bar{b}$ represents the natural borrowing limit. We begin with some useful lemmas

let $L \equiv l^{1+\gamma}$., To make this problem convex,

Substitute for $c(s)$

$$V(b_{-}) = \max_{L(s),b(s)} \sum_{s\in S} \pi(s)\left[\frac{1}{1+\gamma}L(s) + \frac{1}{\beta}P(s)\,b_{-} - b(s) + \beta V(b(s))\right]$$

s.t.

$$\begin{aligned}
\frac{1}{\beta}P(s)\,b - b(s) + g(s) &\leq \theta L^{\frac{1}{1+\gamma}}(s) - L(s) \\
b(s) &\leq \bar{b} \\
L(s) &\geq 0.
\end{aligned}$$

Lemma 2 *$V(b)$ is stictly concave, continuous, differentiable and $V(b) < \beta^{-1}$ for all $b < \bar{b}$. The feasibility constraint binds for all $b \in (-\infty, \bar{b}]$, $s \in S$ and $(L^*(s))^{1-\frac{1}{1+\gamma}} \geq \frac{1}{1+\gamma}$.*[1]

Proof. *Concavity*

$V(b)$ is concave because we maximize linear objective function over convex set.

Binding feasibility

Suppose that feasibility does not bind for some b, s. Then the optimal $L(s)$ solve $\max_{L(s)\geq 0} \pi(s)\frac{\gamma}{1+\gamma}L(s)$ which sets $L(s) = \infty$. This violates feasility for any finite $b, b(s)$.

Bounds on L

Let $\phi(s) > 0$ be a Lagrange multiplier on the feasibility. The FOC for $L(s)$ is

$$\frac{1}{1+\gamma} + \phi(s)\left(\frac{1}{1+\gamma}L(s)^{\frac{1}{1+\gamma}-} - \theta\right) = 0.$$

This gives

$$\frac{1}{1+\gamma}L^{\frac{1}{1+\gamma}-1} - \theta = -\frac{1}{\lambda}\frac{\gamma}{1+\gamma} < 0$$

[1] This last condition simply means that we do not tax to the right of the peak of the Laffer curve. The revenue maximizing tax is $1-\bar{\tau} = \frac{1}{1+\gamma}$. At the same time $1-\tau = l^{\gamma}$ so if taxes are always to the left of the peak, $\frac{1}{1+\gamma} \leq l^{\gamma} = \left(L^{\frac{1}{1+\gamma}}\right)^{\gamma} = L^{1-\frac{1}{1+\gamma}}$.

or

$$L^{1-\frac{1}{1+\gamma}} \geq \frac{\theta}{1+\gamma}.$$

Continuity

For any L that satisfy $L^{1-\overline{1+\gamma}} \geq \frac{\theta}{1+\gamma}$, define function Ψ that satisfies $\Psi\left(L^{\frac{1}{1+\gamma}} - \theta L\right) = L$. Since $L^{\frac{1}{1+\gamma}} - L$ is strictly decreasing in L for $L^{1-\frac{1}{1+\gamma}} \geq \frac{1}{1+\gamma}$, this function is well defined. Note that $\Psi\left(\right) \underbrace{\left(\frac{1}{1+\gamma} L^{\frac{1}{1+\gamma}-1} - \theta\right)}_{<0} =$ 1 (so that $\Psi > 0$, i.e. Ψ is strictly decreasing) and $\Psi'' \underbrace{\left(\frac{1}{1+\gamma} L^{\frac{1}{1+\gamma}-1} - 1\right)^2}_{>0} + \underbrace{\Psi}_{<0} \underbrace{\frac{1}{1+\gamma} \frac{\gamma}{1+\gamma} L^{\frac{1}{1+\gamma}-2}}_{<0} = 0$ (so that $\Psi'' \geq 0$, $\Psi'' > 0$, i.e. Ψ is strictly concave on the interior). Ψ is also continuous. When $L^{1-\frac{1}{1+\gamma}} = \frac{1}{1+\gamma}$, $L = (1+\gamma)^{-\frac{(1+\gamma)}{(\gamma)}}$. Let $D \equiv (1+\gamma)^{\frac{-1}{\gamma} - (1+\gamma)^{-\frac{1+\gamma}{(\gamma)}}}$. Then the objective is

$$V\left(b_{-}\right) = \max_{b(s)} \sum_{s \in S} \pi\left(s\right) \left[\Psi\left(\frac{1}{\beta} P\left(s\right) b - b\left(s\right) + g\left(s\right)\right) + \frac{1}{\beta} P\left(s\right) b_{-} - b\left(s\right) + \beta V\left(b\left(s\right)\right)\right]$$

s.t.

$$\begin{aligned} b\left(s\right) &\leq \bar{b} \\ \frac{1}{\beta} P\left(s\right) b_{-} - b\left(s\right) + g\left(s\right) &\leq D. \end{aligned}$$

This function is continuous so V is also continuous.

Differentiability

Continuity and convexity implies differentiability everywhere, including the boundaries.

Strict concavity

Ψ is strictly concave, so on the interior V is strictly concave.

■

Next we characterize policy functions

Lemma 3 *$b\left(b_{-}, s\right)$ is an increasing function of b for all s for all $\left(b_{-}, s\right)$ where $b\left(s\right)$ is interior.*

Proof. Take the FOCs for $b\left(s\right)$ from the condition in the previous problem. If $b\left(s\right)$ is interior

$$\Psi\left(\frac{1}{\beta} P\left(s\right) b_{-} - b\left(s\right) + g\left(s\right)\right) = \beta V\left(b\left(s\right)\right).$$

Suppose $b_1 < b_2$ but $b_2\left(s\right) < b_1\left(s\right)$. Then from stict concavity

$$\begin{aligned} V\left(b_2\left(s\right)\right) &< V'\left(b_1\left(s\right)\right) \\ \Psi\left(\frac{1}{\beta} P\left(s\right) b_2 - b_2\left(s\right) + g\left(s\right)\right) &> \Psi\left(\frac{1}{\beta} P\left(s\right) b_1 - b_1\left(s\right) + g\left(s\right)\right). \end{aligned}$$

■

Lemma 4 *There exists an invariant distribution of the stochastic process* $b_{t+1} = b(s_{t+1}, b_t)$

Proof. The state spaces for b_t and s_t are compact. Further the transition function on $s_{t+1}|s_t$ is trivially increasing under i.i.d shocks. We can apply standard arguments as in **?**(see corollary 3) to argue that there exists invariant distribution of assets. ∎

Now we characterize the support of this distribution using further properties of the policy rules for $b(s|b_{\_})$

Lemma 5 *For any* $b_{\_} \in (\underline{b}, \bar{b})$, *there are* s, s'' *s.t.* $b\,(s) \geq b_{\_} \geq b\,(s'')$. *Moreover, if there are any states* s'', s''' *s.t.* $b\,(s'') \neq b\,(s''')$, *those inequalities are strict.*

Proof. The FOCs together with the envelope theorem imply that $\mathbb{E}P(s)V'(b(s)) = V'(b_{\_}) + \kappa(s)$ We can rewrite this as $\tilde{\mathbb{E}}V'(b(s)) = b + \kappa(s)$ with $\tilde{\pi}(s) = P(s)\pi(s)$

Now if there is at least one $b\,(s)$ s.t. $b\,(s) > b_{\_}$, by strict concavity of V there must be some s'' s.t. $b\,(s'') < b$.

If there is at least one $b\,(s)$ s.t. $b\,(s) < b_{\_}$, the inequality above is strictly only if $b\,(s''') = \bar{b}$ for some s'''. But $V\left(\bar{b}\right) < V\,(b)$ so there must be some s'' s.t. $b\,(s'') > b$. Equality is possible only if $b_{\_} = b\,(s)$ for all s. ∎

Lemma 6 *Let* $\mu(b,s)$ *be the optimal policy function for the Lagrange multiplier* $\mu(s)$. *If* $P(s') > P(s'')$ *then there exists a* $b^*_{s',s''}$ *such that for all* $b < (>)\ b_{1,s',s''}$ *we have* $\mu(b,s') > (<)\ \mu(b,s'')$. *If* $\underline{b} < b^*_{s',s''} < \bar{b}$ *then* $\mu(b^*_{s',s''}, s') = \mu(b^*_{s',s''}, s'')$.

Proof. Suppose that $\mu(b,s') \leq \mu(b,s'')$. Subtracting the implementability for s'' from the implementability constraint for s' we have

$$\begin{aligned}
\frac{P(s') - P(s'')}{\beta} b &= S_{s'}(\mu(b,s')) - S_{s''}(\mu(b,s'')) + b'(b,s') - b'(b,s'') \\
&\geq S_{s'}(\mu(b,s')) - S_{s''}(\mu(b,s')) + b'(b,s') - b'(b,s'') \\
&\geq S_{s'}(\mu(b,s')) - S_{s''}(\mu(b,s')) = g(s'') - g(s')
\end{aligned}$$

We get the first inequality from noting that $S_s(\mu') \geq S_s(\mu'')$ if $\mu' \leq \mu''$. We obtain the second inequality by noting that $\mu(b,s') \leq \mu(b,s'')$ implies $b'(b,s') \geq b'(b,s'')$ (which comes directly from the concavity of V). Thus, $\mu(b,s') \leq \mu(b,s'')$ implies that

$$b \geq \frac{\beta(g(s'') - g(s'))}{P(s') - P(s'')} = b^*_{s',s''} \tag{7}$$

The converse of this statement is that if $b < b^*_{s',s''}$ then $\mu(b,s') > \mu(b,s'')$. The reverse statement that $\mu(b,s') \geq \mu(b,s'')$ implies $b \leq b^*_{s,s'}$ follows by symmetry. Again, the converse implies that if $b > b^*_{s',s''}$ then $\mu(b,s') < \mu(b,s'')$. Finally, if $\underline{b} < b^*_{s',s''} < \bar{b}$ then continuity of the policy functions implies that there must exist a root of $\mu(b,s') - \mu(b,s'')$ and that root can only be at $b^*_{s',s''}$. ∎

Lemma 7 *$P \in \mathcal{P}^*$ is necessary and sufficient for existence of b^* such that $b(s, b^*) = b*$ for all ss*

Proof. The necessary part follows from taking differences of the (6a) for s',s''. We have

$$[P(s) - P(s'')]\frac{b^*}{\beta} = g(s) - g(s'')$$

Thus $P \in \mathcal{P}^*$. The sufficient part follows from the Lemma 6. If $P \notin \mathcal{P}^*$, equation 7 that defines $b^*_{s',s''}$ will not be same across all pairs. Thus b^* that satisfies $b(s; b^*)$ independent of s will not exist. ■

Lemma 7 implies that under the hypothesis of part 1 of the Theorem **??** there cannot exist an interior absorbing point for the dynamics of debt. This allows us to construct a sequences $\{b_t\}_t$ such that $b_t < b_{t+1}$ with the property that $\lim_t b_t = \underline{b}$. Thus, for any $\epsilon > 0$, there exists a finite history of shocks that can take us arbitrarily close to $\underline{b}$. Since the shocks are i.i.d this finite sequence will repeat i.o. With a symmetric argument we can show that b_t will come arbitrarily close to its upper limit i.o too

Part 2 of Theorem **??**

In this first section we will show that there exists b_1, and if p is sufficiently volatile a b_2, such that if $b_t \leq b_1$ then

$$\mu_t \geq \mathbb{E}_t \mu_{t+1}$$

and if $b_t \geq b_2$ then

$$\mu_t \leq \mathbb{E}_t \mu_{t+1}.$$

Recall that b is decreasing in μ, so this implies that if b_t is low (large) enough then there will exist a drift away from the lower (upper) limit of government debt.

With Lemma 6 we can order the policy functions $\mu(b, \cdot)$ for particular regions of the state space. Take b_1 to be

$$b_1 = \min \left\{ b^*_{s',s''} \right\}$$

and WLOG choose $\underline{b} < b_1$. For all $b < b_1$ we have shown that $P(s) > P(s')$ implies that $\mu(b, s) > \mu(b, s')$. The FOC for the problem imply,

$$\mu_t = \mathbb{E}_t p_{t+1} \mu_{t+1} + \underline{\kappa}_t \tag{8}$$

The inequality in the resource constraint implies that $\xi(s) \geq 0$ implying that $\mu(s) \leq 1$. With some minor algebra algebra we obtain

By decomposing $\mathbb{E}\mu_{t+1} p_{t+1}$ in equation (8), we obtain (using $\mathbb{E}_t p_{t+1} = 1$)

$$\mu_t = \mathbb{E}\mu_{t+1} + \text{cov}_t(\mu_{t+1}, p_{t+1}) + \kappa_t \tag{9}$$

Our analysis has just shown that for $b_t < b_1$ we have $\text{cov}_t(\mu_{t+1}, p_{t+1}) > 0$ so

$$\mu_t > \mathbb{E}_t \mu_{t+1}.$$

If p is sufficiently volatile:

$$P(s') - Ps'' > \frac{\beta(g_{s''} - g_{s'})}{\bar{b}}$$

then

$$b_2 = \max\left\{b^*_{s',s''}\right\} < \bar{b}$$

and through a similar argument we can conclude that $\mathrm{cov}_t(\mu_{t+1}, p_{t+1}) < 0$

$$\mu_t < \mathbb{E}_t \mu_{t+1}$$

for $b_t > b_2$ (note $b_t > \underline{b}$ implies $\kappa_t = 0$) which gives us a drift away from the upper-bound.

Part 3 of Theorem **??**

When $P \in \mathcal{P}^*$, Lemma 7 implies existence of b^* as the steady state debt level.

Lemma 8 *There exists μ^* such that μ_t is a sub-martingale bounded above in the region $(-\infty, \mu^*)$ and super-martingale bounded below in the region $(\mu^*, \frac{1}{1+\gamma})$*

Proof. Let μ^* be the associated multiplier, i.e $V_b(b^*) = \mu^*$. Using the results of the previous section, we have that $b_1 = b_2 = b^*$, implying that $\mu_t < (>)\mathbb{E}_t\mu_{t+1}$ for $b_t < (>)b^*$. ■

Lastly we show that $\lim_t \mu_t = \mu^*$. Suppose $b_t < b^*$, we know that $\mu_t > \mu^*$. The previous lemma implies that in this region, μ_t is a super martingale. The lemma 3 shows that $b(b_-, s)$ is continuous and increasing. This translates into $\mu(\mu(b_-), s)$ to be continuous and increasing as well. Thus

$$\mu_t > \mu^* \implies \mu(\mu_t, s_{t+1}) > \mu(\mu^*, s_{t+1})$$

or

$$\mu_{t+1} > \mu^*$$

Thus $\mu*$ provides a lower bound to this super martingale. Using standard martingale convergence theorem converges. The uniqueness of steady state implies that it can only converge to μ^*. For $\mu < \mu^*$, the argument is symmetric.

■

3. Proof of Theorem ??

Working with the first order conditions of problem 5, we obtain

$$l(s)^{\gamma} = \frac{\mu(s) - 1}{(1+\gamma)\mu(s) - 1} = 1 - \tau(\mu(s)),$$

implying the relationship between tax rate τ and multiplier μ given by

$$\tau(\mu) = \frac{\gamma\mu}{(1+\gamma)\mu - 1} \tag{10}$$

The rest of the first order conditions are summarized below

$$\begin{aligned}\frac{b_{-}P(s)}{\beta} &= S(\mu(s), s) + b(s) \\ V'(b) &= \mathbb{E}P(s)\mu(s) \\ \mu(s) &= V'(b(s))\end{aligned}$$

where $S(\mu, s)$ is the government surplus in state s given by

$$S(\mu, s) = (1 - \tau(\mu))^{\frac{1}{\gamma}}\tau(\mu) - g(s) = I(\mu) - g(s)$$

The proof of the theorem will have four steps:

Step 1: Obtaining a recursive representation of the optimal allocation in the incomplete markets economy with payoffs $P(s)$ with state variable μ_{-}

Given a pair $\{P(s), g(s)\}$, since $V'(b)$ is one-to-one, so we can re-characterize these equations as searching for a function $b(\mu)$ and $\mu(s|\mu_{-})$such that the following two equations can be solved for all μ_{-}.

$$\frac{b(\mu_{-})P(s)}{\beta} = I(\mu(s)) - g(s) + b(\mu(s)) \tag{11}$$

$$\mu = \mathbb{E}\mu(s)P(s) \tag{12}$$

Step 2: Describe how the policy rules are approximated

Usually perturbation approaches to solve equilibrium conditions as above look for the solutions to $\{\mu(s|\mu_{-})\}$ and $b(\mu_{-})$ around deterministic steady state of the model. Thus for any b^{ss}, there exists a μ^{ss} that will solve

$$\frac{b^{SS}}{\beta} = I(\mu^{SS}) - \bar{g} + b^{SS}$$

For example if we set the perturbation parameter q to scale the shocks, $g(s) = \mathbb{E}g(s) + q\Delta_g(s)$ and $P(s) = 1 + q\Delta_P(s)$, the first order expansion of $\mu(s|\mu_{-})$ will imply that it is a martingale. Such approximations are not informative about the ergodic distribution. [2]

In contrast we will approximate the functions $\mu(s|\mu_{-})$ around around economy with payoffs in $\bar{P} \in \mathcal{P}^*$.

In contrast we a) explicitly recognize that policy rules depend on payoffs: $\mu(s|\mu_{-}, \{P(s)\}_s)$ and $b(\mu_{-}, \{P(s)\}_s)$ and then take take the first order expansion with respect to both μ_{-} and $\{P(s)\}$ around the vector $(\bar{\mu}, \{\bar{P}(s)\}_s)$ where $\bar{P}(s) \in \mathcal{P}^*$: these payoffs support an allocation such that limiting distribution of debt is degenerate around the some value $\bar{b}$; and $\overline{\mu}$ is the corresponding limiting value of multiplier. The next two expression make the link between $\bar{\mu}$ and $\bar{b}$ explicit.

$$\bar{b} = \frac{\beta}{1-\beta}\left(I(\overline{\mu}) - \overline{g}\right) \tag{13a}$$

where $\overline{g} = \mathbb{E}g$ and $\overline{p}$ as

$$\overline{P}(s) = 1 + \frac{\beta}{\overline{b}}(g(s) - \overline{g}) \tag{13b}$$

As noted before $b(\overline{\mu}; \overline{p}) = \overline{b}$ solves the the system of equations (11-12) for $\mu'(s) = \overline{\mu}$ when the payoffs are $\overline{P}(s)$

We next obtain the expressions that characterize the linear approximation of $\mu(s|\mu_{-}, \{P(s)\})$ and $(\mu_{-}, \{P(s)\})$ around some arbitrary point $(\bar{\mu}, \{\bar{P}(s)\}_s)$ where $\bar{P}(s) \in \mathcal{P}^*$. We will use these expressions to compute the mean and variance of the ergodic distribution associated with the approximated policy rules. Finally as a last step we propose a particular choice of the point of approximation.

The derivatives $\frac{\delta\mu(s|\mu_{-},\{P(s)\}}{\delta\mu_{-}}, \frac{\delta\mu(s|\mu_{-},\{P(s)\}}{\delta P(s)}$ and similarly for $b(\mu_{-}, \{P(s)\}$ are obtained below:

Differentiating equation (11) with respect to μ around $(\overline{\mu}, \overline{P})$ we obtain

$$\frac{\overline{P}(s)}{\beta}\frac{\partial b}{\partial \mu_{-}} = \left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu_{-}}\right]\frac{\partial \mu(s)}{\partial \mu_{-}}.$$

Differentiating equation (12) with respect to μ_{-} we obtain

$$1 = \sum_s \pi(s)\overline{P}(s)\frac{\partial \mu'(s)}{\partial \mu_{-}}$$

combining these two equations we see that

$$\frac{1}{\beta}\left(\sum_s \pi(s)\overline{P}(s)^2\right)\frac{\partial b}{\partial \mu_{-}} = I'(\overline{\mu}) + \frac{\partial b}{\partial \mu_{-}}$$

Noting that $\mathbb{E}\overline{P}^2(s) = 1 + \frac{\beta^2}{\overline{b}^2}\sigma_g^2$ we obtain

$$\frac{\partial b}{\partial \mu_{-}} = \frac{I'(\overline{\mu})}{\frac{\beta}{\overline{b}^2}\sigma_g^2 + \frac{1-\beta}{\beta}} < 0 \tag{14}$$

[2] One can do higher order approximations, but part 3 of theorem **??** hints that for economies with payoffs close to $\mathcal{P}^*$, the stochastic steady state in general is far away from μ^{SS}.

as $I'(\overline{\mu}) < 0$. We then have directly that

$$\frac{\partial \mu'(s)}{\partial \mu} = \frac{\overline{P}(s)}{\frac{\beta^2}{\overline{b}^2}\sigma_g^2 + 1} = \frac{\overline{P}(s)}{\mathbb{E}\overline{P}(s)^2} \tag{15}$$

We can perform the same procedure for $P(s)$. Differentiating equation (11) with respect to $P(s)$ we around $(\overline{\mu}, \overline{P})$ we obtain

$$\frac{\overline{P}(s')}{\beta}\frac{\partial b}{\partial P(s)} + 1_{s,s'}\frac{\overline{b}}{\beta} - \frac{\pi(s)\overline{b}\overline{P}(s')}{\beta} = \left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu}\right]\frac{\partial \mu(s')}{\partial P(s)} \tag{16}$$

Here $1_{s,s'}$ is 1 if $s = s'$ and zero otherwise. Differentiating equation (12) with respect to $P(s)$ we obtain

$$0 = \pi(s)\overline{\mu} - \pi(s)\overline{\mu} + \sum_{s'}\pi(s)\overline{P}(s')\frac{\partial \mu(s')}{\partial P(s)} = \sum_{s'}\pi(s')\overline{P}(s')\frac{\partial \mu(s')}{\partial P(s)}$$

Again we can combine these two equations to give us

$$\frac{\mathbb{E}\overline{P}(s)^2}{\beta}\frac{\partial b}{\partial P(s)} + \frac{\pi(s)\overline{b}}{\beta}(\overline{P}(s) - \mathbb{E}\overline{P}(s)^2) = 0$$

or

$$\frac{\partial b}{\partial P(s)} = \pi(s)\overline{b}\frac{\mathbb{E}\overline{P}^2 - \overline{P}(s)}{\mathbb{E}\overline{P}^2} \tag{17}$$

Going back to equation (16) we have

$$\frac{\partial \mu(s')}{\partial P(s)} = \frac{\overline{b}}{\beta\left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu}\right]}\left(1_{s,s'} - \frac{\pi(s)\overline{P}(s)\overline{P}(s')}{\mathbb{E}\overline{P}^2}\right) \tag{18}$$

Step 3: Getting expressions for the mean and variance of the ergodic distribution around some arbitrary point of approximation

For an arbitrary $\left(\overline{\mu}, \{\overline{P}(s)\}_s\right)$, using the derivatives that we computed, we can characterize the dynamics of $\hat{\mu} \equiv \mu_t - \overline{\mu}$ using our approximated policies.

$$\hat{\mu}_{t+1} = B\hat{\mu}_t + C,$$

where $B(s)$ and $C(s)$ are respective derivatives. Note that both are random variables and let us denote their means $\overline{B}$ and $\overline{C}$, and variances σ_B^2 and σ_C^2 . Suppose that $\hat{\mu}$ is distributed according to the ergodic distribution of this linear system with mean $\mathbb{E}\hat{\mu}$ and variance σ_μ^2. Since

$$B\hat{\mu} + C,$$

has the same distribution we can compute the mean of this distribution as

$$\begin{aligned}
\mathbb{E}\hat{\mu} &= \mathbb{E}\left[B\hat{\mu} + C\right] \\
&= \mathbb{E}\left[\mathbb{E}_{\hat{\mu}}\left[B\hat{\mu} + C\right]\right] \\
&= \mathbb{E}\left[\overline{B}\hat{\mu} + \overline{C}\right] \\
&= \overline{B}\mathbb{E}\hat{\mu} + \overline{C}
\end{aligned}$$

solving for $\mathbb{E}\hat{\mu}$ we get

$$\mathbb{E}\hat{\mu} = \frac{\overline{C}}{1 - \overline{B}} \tag{19}$$

For the variance $\sigma^2_{\hat{\mu}}$ we know that

$$\sigma^2_{\hat{\mu}} = \text{var}(B\hat{\mu} + C) = \text{var}(B\hat{\mu}) + \sigma^2_C + 2\text{cov}(B\hat{\mu}, C)$$

Computing the variance of $B\hat{\mu}$ we have

$$\begin{aligned}
\text{var}(B\hat{\mu}) &= \mathbb{E}\left[(B\hat{\mu} - \overline{B}\mathbb{E}\hat{\mu})^2\right] \\
&= \mathbb{E}\left[(B\hat{\mu} - \overline{B}\hat{\mu} + \overline{B}\hat{\mu} - \overline{B}\mathbb{E}\hat{\mu})^2\right] \\
&= \mathbb{E}\left[\mathbb{E}_{\hat{\mu}}\left[(B - \overline{B})^2\hat{\mu}^2 + 2(B - \overline{B})(\hat{\mu} - \mathbb{E}\hat{\mu})\overline{B}\mathbb{E}\hat{\mu} + (\hat{\mu} - \mathbb{E}\hat{\mu})^2\bar{B}^2\right]\right] \\
&= \mathbb{E}\left[\sigma^2_B\hat{\mu}^2 + (\hat{\mu} - \mathbb{E}\hat{\mu})^2\overline{B}\right] \\
&= \sigma^2_B(\sigma^2_{\hat{\mu}} + (\mathbb{E}\hat{\mu})^2) + \sigma^2_{\hat{\mu}}\overline{B}^2
\end{aligned}$$

while for the covariance of $B\hat{\mu}$ and C

$$\text{cov}(B\hat{\mu}, C) = \sigma_{BC}\mathbb{E}\hat{\mu}$$

Putting this all together we have

$$\sigma^2_{\hat{\mu}} = \frac{\sigma^2_B(\mathbb{E}\hat{\mu})^2 + \sigma_{BC}\mathbb{E}\hat{\mu} + \sigma^2_C}{1 - \overline{B}^2 - \sigma^2_B} \tag{20}$$

Step 4: Choice of the point of approximation

To get the expressions in Theorem **??**, we finally choose a particular $\overline{P} = P^*(s) \in \mathcal{P}^*$. This will be the closest complete market economy to our the given $P(s)$ in L^2 sense. Formally,

$$\min_{\tilde{P} \in \mathcal{P}^*} \sum_s \pi(s)(P(s) - \tilde{P}(s))^2.$$

Since all payoffs in $\mathcal{P}^*$ are associated with some b^* and μ^* via equations (13), we can re write the above problem as choosing $\overline{\mu}$ so as to minimize the variance of the difference between $P(s)$ and the set of steady state payoffs. Let P^* be the solution to this minimization problem. The first order condition for this linearization gives us

$$2\sum_{s'}\pi(P(s')-P^*(s',\mu^*))\frac{\delta P^*(s,\mu^*)}{\delta\mu^*}=0$$

as noted before

$$P^*(s)=1-\frac{\beta}{b^*(\mu^*)}\left(g(s)-\mathbb{E}g\right)$$

thus

$$\frac{\delta P^*}{\delta\mu^*}\propto P^*-1$$

Thus we can see the the optimal choice of $\overline{\mu}$ is equivalent to choosing $\overline{\mu}$ such that

$$\begin{aligned}0&=\sum_{s'}\Pi_{s'}(P(s')-P^*(s',\mu^*))(P^*(s',\mu^*)-1)\\&=-\sum_{s'}\Pi_{s'}(P(s')-P^*(s',\mu^*))+\sum_{s'}\Pi_{s'}(P(s')-P^*(s',\mu^*))P^*(s',\mu^*)\\&=\sum_{s'}\Pi_{s'}(P(s')-P^*(s',\mu^*))P^*(s',\mu^*)\\&=\mathbb{E}\left[(P-P^*)P^*\right]\end{aligned}\tag{21}$$

At these values of $\overline{P}=P^*$ and $\overline{\mu}=\mu^*$ we have that C for our linearized system is

$$C(s')=\sum_s\left\{\frac{b^*}{\beta\left[I'(\overline{\mu})+\frac{\partial b}{\partial\mu}\right]}\left(1_{s,s'}-\frac{\pi(s)P^*(s)P^*(s')}{\mathbb{E}\overline{P}^2}\right)(P(s)-P^*(s))\right\}$$

Taking expectations we have that

$$\begin{aligned}\overline{C}&=\sum_s\left\{\frac{b^*}{\beta\left[I'(\overline{\mu})+\frac{\partial b}{\partial\mu}\right]}\left(\pi(s)-\frac{\pi(s)P^*(s)}{\mathbb{E}\overline{P}^2}\right)(P(s)-P^*(s))\right\}\\&=\frac{b^*}{\beta\left[I'(\overline{\mu})+\frac{\partial b}{\partial\mu}\right]}\left(\mathbb{E}(P-\overline{P})-\frac{\mathbb{E}\left[(P-\overline{P})\overline{P}\right]}{\mathbb{E}\overline{P}^2}\right)\\&=0\end{aligned}\tag{22}$$

Thus the linearized system will have the same mean for μ, $\overline{\mu}$, as the closest approximating steady state payoff structure.

We can also compute the variance of the ergodic distribution for μ. Note

$$\begin{aligned}
C(s') &= \sum_s \left\{ \frac{b^*}{\beta\left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu}\right]} \left(1_{s,s'} - \frac{\pi(s)P^*(s)P^*(s')}{\mathbb{E}P^{*2}}\right)(P(s) - P^*(s)) \right\} \\
&= \frac{b^*}{\beta\left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu}\right]} \left(P(s') - P^*(s') - P^*(s') \frac{\sum_s \pi(s)P^*(s)(p_s - P^*(s))}{\mathbb{E}P^{*2}}\right) \\
&= \frac{b^*}{\beta\left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu}\right]} (P(s') - P^*(s))
\end{aligned}$$

As noted before

$$\sigma_\mu^2 = \frac{b^{*2}}{\beta^2\left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu}\right]^2 \left(1 - \overline{B}^2 - \sigma_B^2\right)} \|P - P^*\|^2$$

The variance of government debt in the linearized system is

$$\sigma_b^2 = \frac{b^{*2}\left(\frac{\partial b}{\partial \mu}\right)^2}{\beta^2\left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu}\right]^2 \left(1 - \overline{B}^2 - \sigma_B^2\right)} \|P - P^*\|^2$$

This can be simplified using the following expressions:

$$I'(\overline{\mu}) + \frac{\partial b}{\partial \mu} = \frac{\mathbb{E}P^{*2}}{\beta}\frac{\partial b}{\partial \mu},$$

$$\overline{B} = \frac{1}{\mathbb{E}P^{*2}}$$

and

$$\sigma_B^2 = \frac{\mathrm{var}(P^*)}{(\mathbb{E}P^{*2})^2}$$

to

$$\sigma_b^2 = \frac{b^{*2}}{\mathbb{E}P^{*2}\mathrm{var}(P^*)} \|P - P^*\|^2 \tag{23}$$

Noting that $\mathbb{E}P^{*2} = 1 + \mathrm{var}(P^*) > 1$, we have immediately that up to first order the relative spread of debt is bounded by

$$\frac{\sigma_b}{b^*} \leq \sqrt{\frac{\|P - P^*\|^2}{\mathrm{var}(P^*)}} \tag{24}$$

4. Proof of Theorem ??

Proof.

Using Theorem **??** let $\tilde{b} = b_1 - b_2$. Under the normalization that $b_2 = 0$, the variable $\tilde{b}$ represents public debt government or the assets of the productive agent. Specializing the formulations in section **??** we have the optimal plan solves the following Bellman equation.

$$V(\tilde{b}_{\text{-}}) = \max_{c_1(s), c_2(s), b'(s)} \sum_s \pi(s) \left\{ \omega \left[u(c_1(s), l_1(s)) \right] + (1 - \omega) \left[c_2(s) \right] + \beta V(\tilde{b}(s)) \right\} \tag{25}$$

subject to

$$c_1(s) - c_2(s) + \tilde{b}(s) = l(s)^{1+\gamma} + \beta^{-1} P(s) \tilde{b}_{\text{-}} \tag{26a}$$

$$nc_1(s) + (1 - n)c_2(s) + g(s) \leq \theta_2 l(s) n \tag{26b}$$

$$c_2(s) \geq 0 \tag{26c}$$

$$\bar{b} \geq \tilde{b}(s) \geq \underline{b} \tag{26d}$$

Let $\mu(s), \phi(s), \lambda(s), \underline{\kappa}(s), \overline{\kappa}(s)$ be the Lagrange multipliers on the respective constraints. The FOC are summarized below

$$\omega - \mu(s) = n\phi(s) \tag{27a}$$

$$1 - \omega + \mu(s) - \phi(s)(1 - n) + \lambda(s) = 0 \tag{27b}$$

$$-\omega l^{\gamma}(s) + \mu(s)(1 + \gamma) l^{\gamma}(s) + n\phi(s)\theta = 0 \tag{27c}$$

$$\beta V'(\tilde{b}(s)) - \mu(s) - \overline{\kappa}(s) + \underline{\kappa}(s) = 0 \tag{27d}$$

and the envelope condition

$$V'(\tilde{b}_{\text{-}}) = \sum_s \pi(s) \mu(s) \beta^{-1} P(s) \tag{27e}$$

To show part 1 of Theorem **??**, we show that $\frac{\omega}{n} > \frac{1+\gamma}{\gamma}$ is sufficient for the Lagrange multiplier $\lambda(s)$ on the non-negativity constraint to bind.

Lemma 9 *The multiplier on the budget constraint $\mu(s)$ is bounded above*

$$\mu(s) \leq \min\left\{\omega - n, \frac{\omega}{1+\gamma}\right\}$$

Similiarly the multiplier of the resource constraint is bounded below,

$$\phi(s) \geq \max\left\{1, \frac{\omega}{n}\left[\frac{\gamma}{1+\gamma}\right]\right\}$$

Proof.

Notice that the labor choice of the productive household implies $\frac{1}{1-\tau} = \frac{\theta_2}{l^\gamma(s)}$.

As taxes go to $-\infty$ (27c) implies that $\mu(s)$ approaches $\frac{\omega}{1+\gamma}$ from below. Similiarly the non-negativity of $c_2(s)$ imposes a lower bound of 1 on $\phi(s)$. This translates into an upper bound of $\omega - n$ on μ. ■

Lemma 10 *There exists a $\bar{\omega}$ such that $\omega > \bar{\omega}$ implies $c_2(s) = 0$ for all b*

Proof.

By the KKT conditions $c_2(s) = 0$ if $\lambda(s) > 0$. Now (27b) implies this is true if $\mu(s) < \omega - n$. The previous lemma bounds $\mu(s)$ by $\frac{\omega}{1+\gamma}$.

We can thus define $\bar{\omega} = n\left(\frac{1+\gamma}{\gamma}\right)$ as the required threshold Pareto weight to ensure that the unproductive agent has zero consumption forever.

■

Now for the rest of the parts $\omega < n\frac{1+\gamma}{\gamma}$, we can have postive transfers for low enough public debt. In particular, we can define a maximum level of debt $\mathcal{B}$ that is consistent with an interior solution for the unproductive agents' consumption.

Guess an interior solution $c_{2,t} > 0$ or $\lambda_t = 0$ for all t. This gives us $l(s) = l^*$ defined below:

$$l^* = \left[\frac{n\theta}{\omega - (\omega - n)(1+\gamma)}\right]^{\frac{1}{\gamma}} \tag{28}$$

As long as $\omega < n\left(\frac{1+\gamma}{\gamma}\right)$ At the interior solution $\tilde{b}(s) = \tilde{b}_{-}$ and using the implementability constraint and resource constraints (26a) and (26b) respectively, we can obtain the expression for $c_2(s)$

$$c_2(s) = n\theta l^* - n{l^*}^{1+\gamma} - \tilde{b}_{-}(1 - P(s)\beta^{-1}) - g(s)$$

Non-negativity of c_2 implies,

$$\tilde{b}_{-} \leq \frac{g(s) - n\theta l^* + n{l^*}^{1+\gamma}}{\beta^{-1}P(s) - 1}$$

We can also express this as

$$\tilde{b}_{-} \leq \frac{g(s) - \tau^* y^*}{\beta^{-1}P(s) - 1},$$

where the right hand side of the previous equation is just the present discounted value of the primary deficit of the government at the constant taxes τ^* associated with l^* defined in (28). As long as $\beta^{-1}P(s) - 1 > 0$, this object is well defined, we define $\mathcal{B} = \min_s \left[\frac{g(s) - n\theta l^* + n{l^*}^{1+\gamma}}{\beta^{-1}P(s) - 1}\right]$. Thus for $\tilde{b}_- < \mathcal{B}$ the optimal allocation has constant taxes given by τ^* and debt $\tilde{b}_-$, while transfers are given by

$$T(s) = n\theta l^* - n{l^*}^{1+\gamma} - \tilde{b}_-(1 - P(s)\beta^{-1}) - g(s),$$

and are strictly positive.

For initial debt greater than $\mathcal{B}$, we distinguish cases when payoffs are perfectly aligned with $g(s)$ i.e belong to the set $\mathcal{P}^*$ and when they are not. For part 2 case b, let $P \notin \mathcal{P}^*$.

Lemma 11 *There exists a $\check{b} > \mathcal{B}$ such that there are two shocks $\underline{s}$ and $\overline{s}$ and the optimal choice of debt starting from $\tilde{b}_- \leq \check{b}$ satisfies the following two inequalities:*

$$\tilde{b}(\underline{s}, \tilde{b}_-) > \mathcal{B}$$

$$\tilde{b}(\overline{s}, \tilde{b}_-) \leq \mathcal{B}$$

Proof. At $\mathcal{B}$, there exist some $\overline{s}$ such that $T(\overline{s}, \mathcal{B}) = \epsilon > 0$. Now define $\check{b}$ as follows:

$$\check{b} = \mathcal{B} + \frac{\epsilon\beta}{2P(\overline{s})}$$

Now suppose to the contrary $\tilde{b}(\overline{s}, \tilde{b}_-) > \mathcal{B}$ for some $\tilde{b}_- \leq \check{b}$. This implies that $\tau(s, \tilde{b}_-) > \tau^*$ and $T(\overline{s}, \tilde{b}_-) = 0$.[3]

The government budget constraint implies

$$\frac{P(\overline{s})\tilde{b}_-}{\beta} + g(s) = \tilde{b}(\overline{s}, \tilde{b}_-) + (1 - \tau(\overline{s}, \tilde{b})_-)l(\overline{s}, \tilde{b}_-).$$

As,

$$\frac{P(\overline{s})\tilde{b}_-}{\beta} + g(\overline{s}) \leq \frac{P(\overline{s})\mathcal{B}}{\beta} + g(\overline{s}) + \frac{\epsilon}{2} < \frac{P(\overline{s})\mathcal{B}}{\beta} + g(\overline{s}) + \epsilon$$

This further implies,

$$\tilde{b}(\overline{s}, \tilde{b}_-) + (1 - \tau(\overline{s}, \tilde{b}_-))l(\tau(\overline{s}, \tilde{b}_-)) > [\tilde{b}(\overline{s}, \tilde{b}_-) + (1 - \tau^*)l^* > \mathcal{B} + (1 - \tau^*)l^* > \frac{P(\overline{s})\tilde{b}_-}{\beta} + g(\overline{s}) + T(\overline{s}, \tilde{b}_-) = \frac{P(\overline{s})\tilde{b}_-}{\beta} + g(\overline{s}) + \epsilon.$$

Combining the previous two inequalities yields a contradiction. The second inequality, $\tilde{b}(\underline{s}, \tilde{b}_-) > \mathcal{B}$ follows from the definition of $\mathcal{B}$.

Now define $\overline{\mu}(\tilde{b}(s, \tilde{b}_-))$ as $\max_s \mu(s, \tilde{b}_-)$ and $\hat{s}(\tilde{b}_-)$ as the shock that achieves this maximum. Now we show that $\hat{\mu}(\tilde{b}(s, \tilde{b}_-))$ is finite for all $b_- \leq \overline{b}$. We show the claim for the natural debt limit.

Let $b^n(s) = (\beta^{-1}P(s) - 1)^{-1} \left[\theta^{\frac{\gamma}{1+\gamma}} \left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}} \left(\frac{\gamma}{1+\gamma}\right) - g(s)\right]$ be the maximum debt supported by a particular shock s. The natural debt limit is defined as $\overline{b}^n = \min_s b^n(s)$. Note that $\lim_{b \to \overline{b}^n} \mu(\tilde{b}_-) = \infty$

[3] Explain why

Now choose s such that $b^n(s) > \overline{b}^n$ and consider the debt choice next period for the same shock s when it comes in with debt $\overline{b}^n$.

Suppose it chooses $\tilde{b}(s, \overline{b}^n) = \overline{b}^n$, then taxes will have to be set to $\frac{\gamma}{1+\gamma}$ and the tax income will be $\frac{\gamma}{1+\gamma} l(\frac{\gamma}{1+\gamma}) = \theta^{\frac{\gamma}{1+\gamma}} \left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}} \left(\frac{\gamma}{1+\gamma}\right)$. The budget constraint will then imply that,

$$\frac{\overline{b}^n P(s)}{\beta} + g(s) = \theta^{\frac{\gamma}{1+\gamma}} \left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}} \left(\frac{\gamma}{1+\gamma}\right) + \overline{b}^n$$

$$\overline{b}^n = (P(s)\beta^{-1} - 1)^{-1} \left(\theta^{\frac{\gamma}{1+\gamma}} \left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}} \left(\frac{\gamma}{1+\gamma}\right) - g(s)\right)$$

However the right hand side is the definition of $b^n(s)$ and,

$$b^n(s) > \overline{b}^n.$$

Thus we have a contradiction and the optimal choice of debt at the natural debt limit $\tilde{b}(s, \overline{b}^n) < \overline{b}^n$.

This inturn means that $\lim_{\tilde{b} \to \overline{b}^n} \overline{\mu}(\tilde{b}) < \infty$.

Now note that $\overline{\mu}(\tilde{b}_{-}) - \mu(\tilde{b}_{-})$ is continuous on $[\check{b}, \quad \overline{b}^n]$ and is bounded below by zero, therefore attains a minimum at $\tilde{b}^{min}$. Let $\delta = \hat{\mu}(\tilde{b}^{min}) - \mu(\tilde{b}^{min}) > \eta > 0$. If this was not true then $P(s) \in \mathcal{P}^*$ as μ will have an absorbing state.

Let $\mu(\omega, n) = \omega - n$. This is the value of μ when debt falls below $\mathcal{B}$.

Now consider any initial $\tilde{b}_{-} \in [\mathcal{B}, \overline{b}^n]$. If $\tilde{b}_{-} \le \check{b}$, then by lemma 11, we know that $\mathcal{B}$ will be reached in one shock. Otherwise if $\tilde{b}_{-} > \check{b}$, we can construct a sequence of shocks $s_t = \hat{s}(\tilde{b}_{t-1})$ of length $N = \frac{\mu(\omega,n) - \mu(\tilde{b}_{-})}{\delta}$. There exits $t < N$ such that $\tilde{b}_t < \check{b}$, otherwise,

$$\mu_t > \mu(\tilde{b}_{-}) + N\delta > \mu(\omega, n)$$

Thus we can reach $\mathcal{B}$ in finite steps. Since shocks are i.i.d, this is an almost sure statement. At $\mathcal{B}$, transfers are strictly positive for some shocks $T_t > 0$ a.s. and taxes are given by τ^*.

Now consider the payoffs $P \in \mathcal{P}^*$ such that the associated steady state debt $b^* > \mathcal{B}$. Under the guess $T_t = 0$, the same algebra as in Theorem **??** goes through and we can show that $\tilde{b}_{-} = b^*$ is a steady state for the heterogeneous agent economy. Thus the heterogeneous agent economy for a given $P \in \mathcal{P}^*$ has a continuum of steady states given by the set $[\overline{b}, \quad \mathcal{B}] \cup \{b^*\}$.

In the region $\tilde{b}_{-} > b^*$, as before μ_t is supermartingale bounded below by b^*. Since there is a unique fixed point in the region $\tilde{b}_{-} \in [b^*, \overline{b}^n]$, μ_t converges to μ^* associated with b^*. Transfers are zero and taxes are given by τ^{**}

$$\tau^{**} = \frac{\gamma\mu^*}{(1+\gamma)\mu^* - 1} \tag{29}$$

In the region $[\mathcal{B}, \quad b^*]$ the outcomes depend on the exact sequence of shocks we can show that

μ_t is a submartingale. This follows from the observation that for all $\tilde{b}_{-} \geq \mathcal{B}$, we have $T(s) = 0$ and the outcomes from the representative agent economy allow us to order $\mu(s)$ relative $P(s)$. At $\tilde{b}_{-} = \mathcal{B}$, $\mu(s) = \omega - n$ and is constant. Thus in the region $[\mathcal{B}, \quad B^*]$, μ_t is sub martingale and it converges. However f $\tilde{b}_t$ gets sufficiently close to $\check{b}$, then it can converge to $\mathcal{B}$ and if it gets sufficiently close to b^*, it can converge to b^*. Either of this can happen with strictly positive probability. ■

5. Proof of Theorem ??

The Bellman equation for the optimal planners problem with log quadratic preferences and IID shocks can be written as

$$V(x,\rho) = \max_{c_1,c_2,l_1,x',\rho'} \sum_s \pi(s)\left[\alpha_1\left(\log c_1(s) - \frac{l_1(s)^2}{2}\right) + \alpha_2 \log c_2(s) + \beta V(x'(s),\rho'(s))\right]$$

subject to the constraints

$$1 + \rho'(s)[l_1(s)^2 - 1] + \beta x'(s) - \frac{x\frac{P(s)}{c_2(s)}}{\mathbb{E}[\frac{P(s)}{c_2(s)}]} = 0 \tag{30}$$

$$\mathbb{E}\frac{P(s)}{c_1(s)}(\rho'(s) - \rho) = 0 \tag{31}$$

$$\theta_1(s)l_1(s) - c_1(s) - c_2(s) - g = 0 \tag{32}$$

$$\rho'(s)c_2(s) - c_1(s) = 0 \tag{33}$$

where the $\pi(s)$ is the probability distribution of the aggregate state s. If we let $\pi(s)\mu(s)$, λ, $\pi(s)\xi(s)$ and $\pi(s)\phi(s)$ be the Lagrange multipliers for the constraints (30)-(33) respectively then we obtain the following FONC for the planners problem [4]

$c_1(s)$:

$$\frac{\alpha_1\pi(s)}{c_1(s)} - \frac{\lambda\pi(s)}{c_1(s)^2}(\rho'(s) - \rho) - \pi(s)\xi(s) - \pi(s)\phi(s) = 0 \tag{34}$$

$c_2(s)$:

$$\frac{\alpha_2\pi(s)}{c_2(s)} + \frac{xP(s)\pi(s)}{c_2(s)^2\mathbb{E}[\frac{P}{c_2}]}\left[\mu(s) - \frac{\mathbb{E}[\mu\frac{P}{c_2}]}{\mathbb{E}[\frac{P}{c_2}]}\right] - \pi(s)\xi(s) + \pi(s)\rho'(s)\phi(s) = 0 \tag{35}$$

$l_1(s)$:

$$-\alpha_1\pi(s)l_1(s) + 2\mu(s)\pi(s)\rho'(s)l_1(s) + \theta_1(s)\pi(s)\xi(s) = 0 \tag{36}$$

$x'(s)$:

$$V_x(x'(s),\rho'(s)) + \mu(s) = 0 \tag{37}$$

$\rho'(s)$:

$$\beta V_\rho(x'(s),\rho'(s)) + \frac{\lambda\pi(s)}{c_1(s)} + \mu(s)[l_1(s)^2 - 1] + \pi(s)\phi(s)c_2(s) = 0 \tag{38}$$

[4] Appendix **??** discuses the associated second order conditions that ensure these policies are optimal

In addition there are two envelope conditions given by

$$V_x(x,\rho) = -\sum_{s'} \frac{\mu(s')\pi(s')\frac{P(s)}{c_2(s')}}{\mathbb{E}[\frac{P}{c_2}]} = -\frac{\mathbb{E}[\mu\frac{P}{c_2}]}{\mathbb{E}[\frac{P}{c_2}]} \tag{39}$$

$$V_\rho(x,\rho) = -\lambda\mathbb{E}[\frac{P}{c_1}] \tag{40}$$

In the steady state, we need to solve for a collection of allocations, initial conditions and Lagrange multipliers $\{c_1(s), c_2(s), l_1(s), x, \rho, \mu(s), \lambda, \xi(s), \phi(s)\}$ such that equations (30)-(40) are satisfied when $\rho'(s) = \rho$ and $x'(s) = x$. It should be clear that if we replace $\mu(s) = \mu$, equation (37) and the envelope condition with respect to x is always satisfied. Additionally under this assumption equation (35) simplifies significantly,since

$$\frac{xP(s)\pi(s)}{c_2(s)^2\mathbb{E}[\frac{P}{c_2}]}\left[\mu(s) - \frac{\mathbb{E}[\mu\frac{P}{c_2}]}{\mathbb{E}[\frac{P}{c_2}]}\right] = 0$$

The first order conditions for a steady can then be written simply as

$$1 + \rho[l_1(s)^2 - 1] + \beta x - \frac{xP(s)}{c_2(s)\mathbb{E}[\frac{P}{c_2}]} = 0 \tag{41}$$

$$\theta_1(s)l_1(s) - c_1(s) - c_2(s) - g = 0 \tag{42}$$

$$\rho c_2(s) - c_1(s) = 0 \tag{43}$$

$$\frac{\alpha_1}{c_1(s)} - \xi(s) - \phi(s) = 0 \tag{44}$$

$$\frac{\alpha_2}{c_2(s)} - \xi(s) + \rho\phi(s) = 0 \tag{45}$$

$$[2\mu\rho - \alpha_1]l_1(s) + \theta_1(s)\xi(s) = 0 \tag{46}$$

$$\lambda\left(\frac{P(s)}{c_1(s)} - \beta\mathbb{E}\left[\frac{P}{c_1}\right]\right) + \mu[l_1(s)^2 - 1] + \phi(s)c_2(s) = 0 \tag{47}$$

We can rewrite equation (44) as

$$\frac{\alpha_1}{c_2(s)} - \rho\xi(s) - \rho\phi(s) = 0$$

by substituting $c_1(s) = \rho c_2(s)$. Adding this to equation (45) and normalizing $\alpha_1 + \alpha_2 = 1$ we obtain

$$\xi(s) = \frac{1}{(1+\rho)\,c_2(s)} \tag{48}$$

which we can use to solve for $\phi(s)$ as

$$\phi(s) = \frac{\alpha_1 - \rho\alpha_2}{(\rho(1+\rho))\,c_2(s)} \tag{49}$$

From equation (41) we can solve for $l_1(s)^2 - 1$ as

$$l_1(s)^2 - 1 = \frac{x}{\rho\mathbb{E}[\frac{P}{c_2}]}\left(\frac{P(s)}{c_2(s})) - \beta\mathbb{E}\left[\frac{P}{c_2}\right]\right) - \frac{1}{\rho}$$

This can be used along with equations (47) and (49) to obtain

$$\left(\frac{\lambda}{\rho} + \frac{\mu x}{\rho\mathbb{E}[\frac{P}{c_2}]}\right)\left(\frac{P(s)}{c_2(s)} - \beta\mathbb{E}\left[\frac{P}{c_2}\right]\right) = \frac{\mu}{\rho} + \frac{\rho\alpha_2 - \alpha_1}{\rho(1+\rho)}$$

Note that the LHS depends on s while the RHS does not, hence the solution to this equation is

$$\lambda = -\frac{\mu x}{\mathbb{E}[\frac{P}{c_2}]} \tag{50}$$

and

$$\mu = \frac{\alpha_1 - \rho\alpha_2}{1+\rho} \tag{51}$$

Combining these with equation (46) we quickly obtain that

$$\left[2\rho\frac{\alpha_1 - \rho\alpha_2}{1+\rho} - \alpha_1\right] l_1(s) + \frac{\theta_1(s)}{(1+\rho)\, c_2(s)} = 0$$

Then solving for $l_1(s)$ gives

$$l_1(s) = \frac{\theta_1(s)}{(\alpha_1(1-\rho) + 2\rho^2\alpha_2)\, c_2(s)}$$

Remark 1 *Note that the labor tax rate is given by* $1 - \frac{c_1(s)l_1(s)}{\theta(s)}$*. The previous expression shows that labor taxes are constant at the steady state. This property holds generally for CES preferences separable in consumption and leisure*

This we can plug into the aggregate resource constraint (42) to obtain

$$l_1(s) = \left(\frac{1+\rho}{\alpha_1(1-\rho) + 2\rho^2\alpha_2}\right)\frac{1}{l_1(s)} + \frac{g}{\theta_1(s)}$$

letting $C(\rho) = \frac{1+\rho}{\alpha_1(1-\rho)+2\rho^2\alpha_2}$ we can then solve for $l_1(s)$ as

$$l_1(s) = \frac{g \pm \sqrt{g^2 + 4C(\rho)\theta_1(s)^2}}{2\theta_1(s)}$$

The marginal utility of agent 2 is then

$$\frac{1}{c_2(s)} = \left(\frac{1+\rho}{C(\rho)}\right)\left(\frac{g \pm \sqrt{g^2 + 4C(\rho)\theta_1(s)^2}}{2\theta_1(s)^2}\right)$$

Note that in order for either of these terms to be positive we need $C(\rho) \geq 0$ implying that there is only one economically meaningful root. Thus

$$l_1(s) = \frac{g + \sqrt{g^2 + 4C(\rho)\theta_1(s)^2}}{2\theta_1(s)} \tag{52}$$

and

$$\frac{1}{c_2(s)} = \left(\frac{1+\rho}{C(\rho)}\right)\left(\frac{g + \sqrt{g^2 + 4C(\rho)\theta_1(s)^2}}{2\theta_1(s)^2}\right) \tag{53}$$

A steady state is then a value of ρ such that

$$x(s) = \frac{1 + \rho[l_1(\rho, s)^2 - 1]}{\frac{P(s)/c_2(\rho,s)}{\mathbb{E}[\frac{P}{c_2}](\rho)} - \beta} \tag{54}$$

s independent of s.

The following lemma, which orders consumption and labor across states, will be useful in proving the parts of proposition **??**. As a notational aside we will often use $\theta_{1,l}$ and $\theta_{1,h}$ to refer to $\theta_1(s_l)$ and $\theta_1(s_h)$ respectively. Where s_l refers to the low TFP state and s_h refers to the high TFP state.

Lemma 12 *Suppose that* $\theta_1(s_l) < \theta_2(s_h)$ *and* ρ *such that* $C(\rho) > 0$ *then*

$$l_{1,l} = \frac{g + \sqrt{g^2 + 4C(\rho)\theta_{1,l}^2}}{2\theta_{1,l}} > \frac{g + \sqrt{g^2 + 4C(\rho)\theta_{1,h}^2}}{2\theta_{1,h}} = l_{1,h}$$

and

$$\frac{1}{c_{2,l}} = \frac{1+\rho}{C(\rho)}\frac{g + \sqrt{g^2 + 4C(\rho)\theta_{1,l}^2}}{2\theta_{1,l}^2} > \frac{1+\rho}{C(\rho)}\frac{g + \sqrt{g^2 + 4C(\rho)\theta_{1,h}^2}}{2\theta_{1,h}^2} = \frac{1}{c_{2.h}}$$

Proof. The results should follow directly from showing that the function

$$l_1(\theta) = \frac{g + \sqrt{g^2 + 4C(\rho)\theta}}{2\theta}$$

is decreasing in θ. Taking the derivative with respect to θ

$$\begin{aligned}
\frac{dl_1}{d\theta}(\theta) &= -\frac{g}{2\theta^2} - \frac{\sqrt{g + 4C(\rho)\theta^2}}{2\theta^2} + \frac{4C(\rho)\theta}{2\theta\sqrt{g^2 + 4C(\rho)\theta^2}} \\
&= -\frac{g}{2\theta^2} - \frac{g + 4C(\rho)\theta^2 - 4C(\rho)\theta^2}{2\theta^2\sqrt{g^2 + 4C(\rho)\theta^2}} \\
&= -\frac{g}{2\theta^2} - \frac{g}{2\theta^2\sqrt{g^2 + 4C(\rho)\theta^2}} < 0
\end{aligned}$$

That $\frac{1}{c_{2,l}} > \frac{1}{c_{2,h}}$ follows directly. ∎

Now we use these lemma to prove the part 1 and part 2 of theorem **??**

Proof.

[Part 1.] For a riskfree bond when $P(s) = 1$. In order for there to exist a ρ such that equation (54) is independent of the state (and hence have a steady state) we need the existence of root for the following function

$$f(\rho) = \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} - \frac{\frac{1/c_2(\rho,s_h)}{\mathbb{E}[\frac{P}{c_2}](\rho)} - \beta}{\frac{1/c_2(\rho,s_l)}{\mathbb{E}[\frac{P}{c_2}](\rho)} - \beta}$$

From lemma 12 we can conclude that

$$1 + \rho[l_1(\rho, s_l)^2 - 1] > 1 + \rho[l_1(\rho, s_h)^2 - 1] \tag{55}$$

and

$$\frac{1/c_2(\rho, s_l)}{\mathbb{E}[\frac{P}{c_2}](\rho)} - \beta > \frac{1/c_2(\rho, s_h)}{\mathbb{E}[\frac{P}{c_2}](\rho)} - \beta \tag{56}$$

for all $\rho > 0$ such that $C(\rho) \geq 0$. To begin with we will define $\underline{\rho}$ such that $C(\rho) > 0$ for all $\rho > \underline{\rho}$. Note that we will have to deal with two different cases.

$\alpha_1(1-\rho) + 2\rho^2\alpha_2 > 0$ **for all** $\rho \geq 0$**:** In this case we know that $C(\rho) \geq 0$ for all ρ and is bounded above and thus we will let $\underline{\rho} = 0$.

$\alpha_1(1-\rho) + 2\rho^2\alpha_2 = 0$ **for some** $\rho > 0$**:** In this case let $\underline{\rho}$ be the largest positive root of $\alpha_1(1-\rho)+2\rho^2\alpha_2$. Note that $\lim_{\rho\to\underline{\rho}^+} C(\rho) = \infty$

With this we note that[5]

$$\lim_{\rho\to\underline{\rho}^+} \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} = 1$$

We can also show that

$$\lim_{\rho\to\underline{\rho}^+} \frac{\frac{1/c_2(\rho,s_h)}{\mathbb{E}[\frac{P}{c_2}](\rho)} - \beta}{\frac{1/c_2(\rho,s_l)}{\mathbb{E}[\frac{P}{c_2}](\rho)} - \beta} < 1$$

which implies that $\lim_{\rho\to\underline{\rho}^+} f(\underline{\rho}) > 0$.

Taking the limit as $\rho \to \infty$ we see that $C(\rho) \to 0$, given that $\frac{g}{\theta(s)} < 1$, we can then conclude that

$$\lim_{\rho\to\infty} 1 + \rho[l_1(\rho, s)^2 - 1] = -\infty$$

Thus, there exists $\overline{\rho}$ such that $1 + \overline{\rho}[l_1(\overline{\rho}, s_l)^2 - 1] = 0$. [6] From equation (55), we know that

$$0 = 1 + \overline{\rho}[l_1(\overline{\rho}, s_l)^2 - 1] > 1 + \overline{\rho}[l_1(\overline{\rho}, s_h)^2 - 1]$$

[5] In the first case $\underline{\rho} = 0$ and in the second case $l_1(\rho, s_l) = l_1(\rho, s_h)$ as $\rho \to \underline{\rho}^+$

[6] This can be seen from the fact $\lim_{\rho\to\underline{\rho}^+} 1 + \rho[l_1(\rho, s_l)^2 - 1] > 0$ and $\lim_{\rho\to\infty} 1 + \rho[l_1(\rho, s_l)^2 - 1] > -\infty$, thus $\overline{\rho}$ exists in $(\underline{\rho}, \infty)$

which implies in the limit

$$\lim_{\rho \to \overline{\rho}^-} \frac{1+\rho[l_1(\rho,s_h)^2-1]}{1+\rho[l_1(\rho,s_l)^2-1]} = -\infty$$

which along with

$$\frac{\frac{1/c_2(\rho,s_h)}{\mathbb{E}[\frac{P}{c_2}]}-\beta}{\frac{1/c_2(\rho,s_l)}{\mathbb{E}[\frac{P}{c_2}]}-\beta} \geq -1$$

allows us to conclude that $\lim_{\rho\to\overline{\rho}^-} f(\rho) = -\infty$. The intermediate value theorem then implies that there exists ρ_{SS} such that $f(\rho_{SS}) = 0$ and hence that ρ_{SS} is a steady state.

Finally, as $\rho_{SS} < \overline{\rho}$ we know that

$$1+\rho_{SS}[l_1(\rho_{SS},s_l)-1] > 0$$

as $\frac{1/c_2(\rho,s_l)}{\mathbb{E}[\frac{P}{c_2}]} > 1$ we can conclude

$$x_{SS} = \frac{1+\rho_{SS}[l_1(\rho_{SS},s_l)-1]}{\frac{1/c_2(\rho,s_l)}{\mathbb{E}[\frac{P}{c_2}](\rho)}-\beta} > 0$$

implying that the government will hold assets in the steady state (under the normalization that agent 2 holds no assets).

[Part 2] As noted before, since $g/\theta(s) < 1$ for all s we have

$$\lim_{\rho\to\infty} 1+\rho[l_1(\rho,s)^2-1] = -\infty$$

Thus, there exists ρ_{SS} such that

$$0 > 1+\rho_{SS}[l_1(\rho_{SS},s_l)^2-1] > 1\rho_{SS}[l_1(\rho_{SS},s_h)^2-1]$$

It is then possible to choose $P(s)$ such that $\beta < \frac{P(s)/c_2(\rho_{SS},s)}{\mathbb{E}[\frac{P}{c_2}]}$ such that

$$1 > \frac{1+\rho_{SS}[l_1(\rho_{SS},s_l)^2-1]}{1+\rho_{SS}[l_1(\rho_{SS},s_h)^2-1]} = \frac{\frac{P(s_l)/c_2(\rho_{SS},s_l)}{\mathbb{E}[\frac{P}{c_2}]}-\beta}{\frac{P(s_h)/c_2(\rho_{SS},s_h)}{\mathbb{E}[\frac{P}{c_2}]}-\beta} \tag{57}$$

Implying that for Payoff shocks $P(s)$, ρ_{SS} is a steady state level for the ratio of marginal utilities, with steady state marginal utility weighted government debt

$$x_{SS} = \frac{1+\rho_{SS}[l_1(\rho_{SS},s_l)^2-1]}{\frac{P(s_l)/c_2(\rho_{SS},s_l)}{\mathbb{E}[\frac{P}{c_2}]}-\beta} < 0$$

Thus, in the steady state, the government is holding debt, under the normalization that the unproductive worker holds no assets. Note this imposes a restriction of $\frac{P(s_l)}{P(s_h)}$.

$$\frac{P(s_l)c_2^{-1}(\rho_{SS}, s_l) - \beta \mathbb{E} P c_2^{-1}}{P(s_h)c_2^{-1}(\rho_{SS}, s_h) - \beta \mathbb{E} P c_2^{-1}} < 1$$

or

$$\frac{P(s_l)}{P(s_h)} < \frac{c_2^{-1}(\rho_{SS}, s_h)}{c_2^{-1}(\rho_{SS}, s_l)} < 1$$

or

Thus $P(s_l) < P(s_h)$ i.e payoffs have to be sufficiently procyclical.

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