

Optimal Taxation with Incomplete Markets

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September 17, 2014

Abstract

This paper characterizes tax and debt dynamics in Ramsey plans for incomplete markets economies that generalize an ? economy by allowing a single asset traded by the government to be risky. Long run debt and tax dynamics can be attracted not only to the first-best continuation allocations discovered by ? for quasi-linear preferences, but instead to a continuation allocation associated with a level of (marginal-utility-scaled) government debt that would prevail in a Lucas-Stokey economy that starts from a particular initial level of government debt. The paper formulates, analyzes, and numerically solves Bellman equations for two value functions for a Ramsey planner, one for $t \geq 1$, the other for $t = 0$.

KEYWORDS: Complete markets, incomplete markets, Ramsey plan, taxes, debt, competitive equilibrium, implementability constraints, dynamic programming squared

1 Introduction

“... the option to issue state-contingent debt is important: tax policies that are optimal under uncertainty have an essential ‘insurance’ aspect to them.” ?,
p. 88

Controversies sparked by ? motivate us to reassess what we know and don’t know about two elementary questions. How much government debt is optimal? And is government debt even a pertinent state variable? ? and ? offer different answers to these two questions in the context of economic environments that are identical in all respects but one: ? let the government to issue a complete set of Arrow securities, while ? restrict the government to issue only a one-period risk-free government bond. For ?, under an optimal tax and debt policy, government debt is not an independent state variable but instead is an exact function of the Markov state variable that drives government expenditures. In ?’s model, the optimal state-by-state levels of government debt depend on the initial level of government debt. By way of contrast, for ? government debt is an independent state variable with a limiting distribution that does not depend on the initial government debt. The quote by ? pinpoints the source of these differences: the government’s purchase of *insurance* from the private sector through explicit state-contingent securities underlies ?’s answers to our two questions; while a government’s *self-insurance* achieved through its accumulation of a risk-free asset underlies ?’s answers.

This paper revisits our two questions in the context of a generalization of the ? environment. We continue to restrict the government to issue only a single security, but allow that security to be risky. The government manages purchases and sales of that single security as best it can. We study how the design of that single security affects optimal government debt dynamics. We use this generalization of ?’s setup to attack questions left unanswered by ? and also to say some new things about alternative ways that the government purchases insurance in an equilibrium of the original ? model. Our analysis exploits new, or at least previously unstated, connections between the ? and ? economies.

? obtained their sharpest results for an economy with a quasi-linear one-period household utility function. Linearity of utility in consumption tied down the risk-free one-period interest rate and enabled them to show that in the long run the government accumulates a big enough stock of the risk-free asset to finance its expenditures entirely from interest earnings; so the tail of that ? Ramsey plan features a zero distorting tax on labor and a first-best allocation. ? are able to say much less about outcomes for preferences that

exhibit risk-aversion in consumption because then the Lagrange multiplier on the key incomplete markets implementability constraint becomes a risk-adjusted martingale rather than the pure martingale that it is under quasi-linearity. Here we are able to say much more than ?. We accomplish this by recognizing connections to limits of (our generalization of) their economy and the allocation associated with a ? economy for a particular initial level of government debt. With preferences that exhibit risk aversion in consumption, an attractor for the limiting debt dynamics of our economy is not associated with the first-best continuation allocation found in the quasi-linear economy of ?, but rather a continuation allocation associated with a Lucas-Stokey economy, or one close to it.

Our analysis sheds light on the risk-sharing described in the quotation with which we begin this paper. We exploit insights about exactly *how* the Ramsey planner in a Lucas-Stokey economy delivers the insurance through state-contingent debt that Lucas and Stokey stress is part and parcel of an optimal tax plan: fluctuations in equilibrium interest rates do part of the job. We can construct examples in which the Lucas-Stokey Ramsey planner chooses to issue risk-free debt and to achieve the required state-contingencies entirely through equilibrium fluctuations in the risk-free interest rate.

It is enlightening to compare our work with related but conceptually distinct inquiries of ?, ?, and ?, who, like us, want to understand links among the design of government securities, interest rate fluctuations, and an optimal tax and debt management plan. Our basic strategy is first to find the tail of an incomplete markets Ramsey allocation, then to ask whether that continuation allocation coincides with a Lucas-Stokey complete markets Ramsey allocation for *some* initial government debt. We describe conditions under which the answer is ‘yes’ or ‘almost yes’.

Unlike us, ?, ?, and ? all *start* with a Lucas-Stokey complete market Ramsey allocation and then construct conditions under which it can be supported by a limited collection of non-contingent debts of different maturities.¹ Equilibrium interest rate fluctuations play a big role in their settings, as they do in some of ours.

In addition to the intrinsic interest adhering to the two questions with which we began, this paper can be viewed as a prolegomenon to an analysis of debt dynamics in the richer economic environment featured in ?. There, a Ramsey planner levies a distorting tax on labor partly to finance exogenous government and partly to redistribute goods among heterogeneously skilled households. Forces similar to those present in the simpler environment of this paper drive debt dynamics there, but those forces are obscured by the presence of

¹In contrast, we *start* with an incomplete markets Ramsey allocation.

additional ones. We find it enlightening to isolate underlying forces by studying them in the simpler setting of this paper.

2 Environment

We analyze economies that share the following features. Government expenditures at time t , $g_t = g(s_t)$, and a productivity shock $\theta_t = \theta(s_t)$ are both functions of a Markov shock $s_t \in \mathcal{S}$ having $S \times S$ transition matrix Π and initial condition s_{-1} . We will denote time t histories with s^t and z_t will refer to a generic random variable measurable with respect to s^t . Sometimes we will denote $z_t(s^t)$ indicate a particular realization of z_t . An infinitely lived representative consumer has preferences over allocations $\{c_t, l_t\}_{t=0}^{\infty}$ of consumption and labor supply that are ordered by

$$\mathbb{E}_{-1} \sum_{t=0}^{\infty} \beta^t U(c_t, l_t), \quad (1)$$

where U is the period utility function for consumption and labor. For most of the paper, we shall assume that U separable in consumption and labor. We describe additional assumptions later. Labor produces output via the linear technology

$$y_t = \theta_t l_t$$

The representative consumer's tax bill at time $t \geq 0$ is

$$-T_t + \tau_t \theta_t l_t, \quad T_t \geq 0,$$

where $\tau_t(s^t, \cdot)$ is a flat rate tax on labor income and T_t is a nonnegative transfer. Often, we'll set $T_t = 0$. The government and consumer trade a single possibly risky asset whose time t payoff p_t is described by an $S \times S$ matrix \mathbb{P} :

$$p_t = \mathbb{P}(s_t, s_{t-1})$$

Let B_t denote the government's holdings of the asset and b_t be the consumer's holdings. Let $q_t = q_t(s^t)$ be the price of the single asset at time t . At $t \geq 0$, the household's time

budget constraint is

$$c_t + b_t = (1 - \tau_t) \theta_t l_t + \frac{p_t}{q_{t-1}} b_{t-1} + T_t \quad (2)$$

and the government's is

$$g_t + B_t + T_t = \tau_t \theta_t l_t + \frac{p_t}{q_{t-1}} B_{t-1}. \quad (3)$$

Feasible allocations satisfy

$$c_t + g_t = \theta_t l_t, \quad \forall t \geq 0 \quad (4)$$

Clearing in the time $t \geq 0$ market for the single asset requires

$$b_t + B_t = 0. \quad (5)$$

Initial assets satisfy $b_{-1} = -B_{-1}$ ² An initial value of the exogenous state s_{-1} is given. Equilibrium objects including $\{c_t, l_t, \tau_t\}_{t=0}^{\infty}$ will come in the form of sequences of functions of initial government debt b_{-1} and $s^t = [s_t, s_{t-1}, \dots, s_0, s_{-1}]$.

Borrowing from a standard boilerplate, we use the following:

Definition 2.1. An **allocation** is a sequence $\{c_t, l_t\}_{t=0}^{\infty}$ for consumption and labor. An **asset profile** is a sequence $\{b_t, B_t\}_{t=0}^{\infty}$. A **price system** is a sequence of asset prices $\{q_t\}_{t=0}^{\infty}$. A **budget-feasible government policy** is a sequence of taxes and transfers $\{\tau_t, T_t\}_{t=0}^{\infty}$.

Definition 2.2. Given $(b_{-1} = -B_{-1}, s_{-1})$ and a government policy, a **competitive equilibrium with distorting taxes** is a price system, an asset profile, a government policy, and an allocation such that (a) the allocation maximizes (??) subject to (??), (b) given prices, $\{b_t\}_{t=0}^{\infty}$ is bounded; and (c) equations (??), (??) and (??) are satisfied.

Definition 2.3. Given (b_{-1}, B_{-1}, s_{-1}) , a **Ramsey plan** is a welfare-maximizing competitive equilibrium with distorting taxes.

3 Two Ramsey problems

Following ? and ?, we use a “primal approach.” To encode a government policy and price system as a restriction on an allocation, we first obtain the representative household's first

²We assume that b_{-1} are obligations with accrued interest. This is equivalent to setting $q_{-1} = 1$.

order conditions³

$$U_{c,t}q_t = \beta \mathbb{E}_t p_{t+1} U_{c,t+1} \quad (6a)$$

$$(1 - \tau_t)\theta_t U_{c,t} = -U_{l,t} \quad (6b)$$

We substitute these into the household's budget constraint to get a difference equation that we solve forward at every history for every $t \geq 0$. That yields *implementability constraints* on a Ramsey allocation that fall into two categories: (1) the time $t = 0$ version is identical with the *single* implementability constraint imposed by ?; (2) the time $t \geq 1$ implementability constraints are counterparts of the additional *measurability restrictions* that ? impose on a Ramsey allocation.

We first state our Ramsey problem, then Lucas and Stokey's.

Problem 3.1. *The Ramsey problem is to choose an allocation and an bounded government debt sequence $\{b_t\}_{t=0}^\infty$ that attain:*

$$\max_{\{c_t, l_t, b_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t, l_t) \quad (7)$$

subject to

$$c_t + g_t = \theta_t l_t, \quad t \geq 0 \quad (8a)$$

$$b_{-1} = \frac{1}{U_{c,0}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (U_{c,t} c_t + U_{l,t} l_t) \quad (8b)$$

$$\frac{b_{t-1} U_{c,t-1}}{\beta} = \frac{\mathbb{E}_{t-1} p_t U_{c,t}}{p_t U_{c,t}} \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j (U_{c,t+j} c_{t+j} + U_{l,t+j} l_{t+j}) \quad \text{for } t \geq 1 \quad (8c)$$

Problem 3.2. *Lucas and Stokey's Ramsey problem is to choose an allocation that attains*

$$\max_{\{c_t, l_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t, l_t) \quad (9)$$

subject to the single implementability constraint (??) and feasibility (??) for all t, s^t .

Remark 3.3. *Equation (??) imposes feasibility, while equation (??) is the single implementability constraint present in ?. Equations (??) express additional implementability*

³We thus focus on interior equilibria. Arguments by ? and ? can be used to show that $\{c_t, l_t, b_t\}_{t=0}^\infty$ with bounded $\{b_t\}$ that also satisfy equations (??) and (??) solve the consumers problem.

constraints at every node from time $t \geq 1$. These generalize the ? measurability constraints on a Ramsey allocation to our more general payoff structure \mathbb{P} for the single asset. The measurability constraints (??) are cast in terms of the date, history $(t-1, s^{t-1})$ measurable state variable b_{t-1} that for $t \geq 1$ is absent from Lucas and Stokey's complete markets Ramsey problem. Evidently, Ramsey allocation for our incomplete markets economy automatically satisfies the single implementability constraint imposed by ?.

Remark 3.4. State-contingent, but not history-dependent, values of consumption, labor supply, and continuation government debt $\check{b}(s)$ solve the ? Ramsey problem ??. As intermediated by the Lagrange multiplier on the implementability constraint (??), consumption, labor supply, and $\check{b}(s)$ are functions of initial government debt b_{-1} and the current state s_t , but not past history s^{t-1} .

3.1 Motivation for quasi-linear U

Asymptotic properties of a Ramsey plan for our incomplete markets economy vary with asset returns $R_{t-1,t} \equiv \frac{\mathbb{P}(s_t|s_{t-1})}{q_{t-1}}$. We see that \mathbb{P} affects these returns directly through the ex-post exogenous payoffs and indirectly through prices q_{t-1} . To focus exclusively on the exogenous \mathbb{P} part of returns, we begin by studying an economy with quasi-linear utility function:

$$U(c, l) = c - \frac{l^{1+\gamma}}{1+\gamma}, \quad (10)$$

which sets $U_{c,t} = 1$. Asymptotic properties of a Ramsey plan for our incomplete markets economy vary with asset returns that reflect properties of equilibrium prices $\{q_t(s^t|B_{-1}, s_{-1})\}_t$ and the exogenous asset payoff matrix \mathbb{P} . At an interior solution, quasi-linear preferences and the Euler equation (??) pins down $q_t = \beta \mathbb{E}_t \mathbb{P}(s_{t+1}|s_t)$. After studying the consequences of quasi-linear utility, we shall solve for Ramsey plans for utility functions that express risk aversion with respect to consumption and so activate endogenous fluctuations in q_t .

4 Quasi-linear preferences

Throughout this section, we assume that U is quasi-linear and use an indirect three step approach to characterize the asymptotic behavior of government debt and the tax rate.

(1) Construct an optimal payoff matrix:

We pose the following problem:

Problem 4.1. *Given arbitrary initial government debt b_{-1} , what is an optimal asset payoff matrix?*

Let \mathcal{P} be the set of all $\mathcal{S} \times \mathcal{S}$ real matrices. Define the indirect utility function $\mathcal{W}(\mathbb{P}; b_{-1})$ as the solution to problem ?? for $\mathbb{P} \in \mathcal{P}$ and initial debt b_{-1} . This induces an operator \mathbb{P}^* that maps initial government debt into an optimal payoff matrix ⁴

$$\mathbb{P}^*(b_{-1}) \in \arg \max_{\mathbb{P} \in \mathcal{P}} W(\mathbb{P}; b_{-1})$$

(2) **Apply the inverse of the operator \mathbb{P}^* .**

For an arbitrary payoff matrix \mathbb{P} , let

$$\mathbb{P}^{*-1}(\mathbb{P}) = \min_b \|\mathbb{P} - \mathbb{P}^*(b^*)\|, \quad (11)$$

where $\|\cdot\|$ is the Frobenius matrix norm. For initial government debt b_{-1} such that $\mathbb{P}^*(b_{-1}) = \mathbb{P}$, we shall show that a Ramsey plan for the incomplete markets economy has $b_t = b^*$ for all $t \geq 0$.

(3) **Long run assets**

Starting from an arbitrary initial government b_{-1} and an arbitrary payoff matrix \mathbb{P} , establish conditions under which $b_t \rightarrow b^*$ under a Ramsey plan.

In particular, where $S = 2$ and shocks s_t are IID, we describe a large set of \mathbb{P} 's for which government debt b_t under a Ramsey plan converges to b^* . For more general shock processes, we numerically find an ergodic set of b_t 's hovering around the debt level b^* . We execute steps (1), (2) and (3) in sections ??, ??, and ??.

4.1 The Optimal Payoff Matrix

We construct an optimal payoff matrix by first solving problem ?? for a Lucas-Stokey Ramsey allocation associated with a given b_{-1} . Next we construct a sequence $\{p_t\}_t$ that satisfies the implementability constraints imposed in (??). Note that these implementability constraints are invariant to scaling of p_t by a constant k_{t-1} that can depend on s^{t-1} . From this equivalence class of $\{p_t\}_t$'s we select a $\{p_t\}_t$ that satisfies a normalization $\mathbb{E}_{t-1} p_t = 1$

⁴We will demonstrate existence of a maximizer that is unique up to a constant factor along each row of the matrix.

and also satisfies

$$p_t = \frac{\beta}{b_{t-1}} \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j (c_{t+j} + U_{l,t+j} l_{t+j}), \quad (12)$$

where

$$b_{t-1} = \beta \mathbb{E}_{t-1} \sum_{j=0}^{\infty} \beta^j (c_{t+j} + U_{l,t+j} l_{t+j}). \quad (13)$$

The term $c_{t+j} + U_{l,t+j} l_{t+j} = (1 - \tau_{t+j}) l_{t+j} - g_{t+j}$ is the net-of-interest government surplus at time $t + j$. From equations (??) and (??), note that $\frac{1}{p_t} - 1$ is the percentage innovation in the present value government surplus at time t .

Note that by construction, p_t disarms the time $t \geq 1$ measurability constraints⁵. Using the remark ?? fact that the Lucas-Stokey Ramsey allocation is not history-dependent, construct the optimal payoff matrix as

$$\mathbb{P}^*(s_t, s_{t-1} | b_{-1}) = p_t.$$

Thus, given initial government debt b_{-1} , let $\mu(b_{-1})$ be the Lagrange multiplier on the Lucas-Stokey implementability constraint (??) at the Lucas-Stokey Ramsey allocation. The tax rate in the Ramsey allocation is $\tau(\mu) = \frac{\gamma\mu}{(1+\gamma)\mu-1}$, which implies a net-of-interest government surplus $S(s, \tau)$ that satisfies

$$S(s, \tau) = \theta(s)^{\frac{\gamma}{1+\gamma}} (1 - \tau)^{\frac{1}{\gamma}} \tau - g(s)$$

If the aggregate state process s_t is i.i.d. then the ‘disarm-the-measurability-constraints’ equation (??) implies that the optimal payoff matrix is

$$\mathbb{P}^*(s, s_- | b_{-1}) = \beta \frac{S(s, \tau)}{b_{-1}} + \beta = (1 - \beta) \frac{S(s, \tau)}{\mathbb{E}S(s, \tau)} + \beta, \quad (14)$$

which is independent of s_- .

Equation (??) lets us depict an optimal payoff matrix as a function of initial government debt. Figure ?? plots the optimal payoff in both states of the world when either government

⁵Although we assume quasi-linear preferences throughout this particular construction, please note that equation (??) can be generalized to preferences with curvature via

$$p_t = \frac{\beta}{U_{c,t-1} b_{t-1} U_{c,t}} \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j (U_{c,t+j} c_{t+j} + U_{l,t+j} l_{t+j})$$

with the normalization $\mathbb{E}_{t-1} U_{c,t} p_t = 1$

expenditures or TFP follows a 2 state i.i.d. process. In both cases, we see that the ordering of the payoff flips on either side of zero government debt.

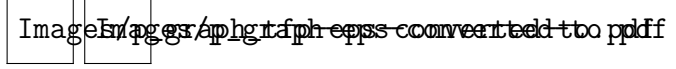


Figure 1: Optimal asset payoff structure as a function of initial government debt when TFP follows a 2 shock i.i.d. process (left) and when government expenditures follow a 2 shock i.i.d process (right).

To appreciate how the initial government debt level influences the optimal asset payoff structure via formula (??), call a state s “adverse” if it implies either “high” government expenditures or “low ” TFP; formally, say that s is “adverse” if

$$g(s)\mathbb{E}\theta^{\frac{\gamma}{1+\gamma}} - \theta(s)^{\frac{\gamma}{1+\gamma}}\mathbb{E}g > 0$$

A “good” state is the opposite of an “adverse” state. “Adverse” states have the property that for wide range of initial government debts, the net-of-interest government surplus is lower than in “good” states. When initial government assets are positive, (??) implies that \mathbb{P}^* pays *more* in “adverse” states, while when initial government assets are negative, \mathbb{P}^* pays *less* in “adverse” states.

4.2 The Inverse of \mathbb{P}^* Again

Temporarily assume that s_t is i.i.d and $S = 2$. In this case, note that (??) implies that the optimal payoff matrix \mathbb{P}^* has identical rows. This lets us restrict our attention to $\mathbb{P}(s, s_-)$ that have payoffs that are independent of s_- . This in turn lets us summarize \mathbb{P} with a vector. Under the normalization $\mathbb{E}\mathbb{P}(s) = 1$, payoffs on the single asset are determined by a scalar \mathbf{p} , the payoff in state 1. A risk-free bond is then a security for which $\mathbf{p} = 1$. Without loss of generality, we shall assume that $g(1)\mathbb{E}\theta^{\frac{\gamma}{1+\gamma}} - \theta(1)^{\frac{\gamma}{1+\gamma}}\mathbb{E}g < 0$, and thus, \mathbf{p} is the payoff in the “good” state of the world. Because the optimal payoff matrix can be summarized by a single scalar variable, we can recast the optimal matrix map $\mathbb{P}^*(b)$ as a single scalar function $\mathbf{p}^*(b)$. The steady state level of debt associated with an exogenous payoff \mathbf{p} is then

$$b^* = \mathbf{p}^{*-1}(\mathbf{p}). \tag{15}$$

Proposition 4.2. *There exists $0 \geq \alpha_2 \geq \alpha_1 \geq 1$ such that*

a. If $\mathbf{p} \leq \alpha_1$, then $b^* < 0$

b. If $\mathbf{p} \geq \alpha_2$, then $b^* > 0$

c. If $\alpha_1 > \mathbf{p} > \alpha_2$, then b^* solving (??) does not exist

Proof. Let g_1 and θ_1 denote government expenditures and TFP, respectively in the “good” state of the world. In state s , the government surplus is

$$S(s, \tau) = \theta(s)^{\frac{\gamma}{1+\gamma}} (1 - \tau)^{\frac{1}{\gamma}} \tau - g(s),$$

which is maximized at $\tau = \frac{\gamma}{1+\gamma}$ when $(1 - \tau)^{\frac{1}{\gamma}} \tau$ is also maximized. Furthermore, in the region $(-\infty, \frac{\gamma}{1+\gamma}]$, $S(\cdot, \tau)$ is an increasing function of τ . In an i.i.d. world with complete markets, government debt at a constant tax rate τ would be

$$\frac{\beta}{1 - \beta} \sum_s \Pi(s) S(s, \tau), \quad (16)$$

which is an increasing function of τ . The maximal initial government debt sustainable with *complete* markets is then

$$\bar{b}^n = \frac{1}{1 - \beta} \sum_s \Pi(s) \theta(s)^{\frac{\gamma}{1+\gamma}} \left(\frac{1}{1 + \gamma} \right)^{\frac{1}{\gamma}} \frac{\gamma}{1 + \gamma} - g(s).$$

Inverting the equation (??) mapping from the tax rate into government debt gives us a function $\tau(b)$ that maps initial government debt into an optimal tax rate. The function $\tau(b)$ is an increasing function of b on the domain of possible complete markets initial debts $(-\infty, \bar{b}^n]$, with $\tau((-\infty, \bar{b}^n]) = \left(-\infty, \frac{\gamma}{1+\gamma}\right]$.

Substituting the formula for $S(s, \tau)$ into equation (??), we obtain

$$\mathbf{p}^*(\tau) = (1 - \beta) \frac{\theta_1^{\frac{\gamma}{1+\gamma}} (1 - \tau)^{\frac{1}{\gamma}} \tau - g_1}{\mathbb{E} \theta^{\frac{\gamma}{1+\gamma}} (1 - \tau)^{\frac{1}{\gamma}} \tau - \mathbb{E} g} + \beta.$$

Solving for $(1 - \tau)^{\frac{1}{\gamma}} \tau$ gives

$$(1 - \tau)^{\frac{1}{\gamma}} \tau = \frac{(\mathbf{p}^* - \beta) \mathbb{E} g - (1 - \beta) g_1}{(\mathbf{p}^* - \beta) \mathbb{E} \theta^{\frac{\gamma}{1+\gamma}} - (1 - \beta) \theta_1^{\frac{\gamma}{1+\gamma}}}.$$

The set of complete market optimal tax rates is $(-\infty, \frac{\gamma}{1+\gamma}]$. Since the mapping $(1 - \tau)^{\frac{1}{\gamma}} \tau$ is one to one and $b(\tau)$ is increasing on this domain, we conclude that $\mathbf{p}^*(b)$ is one to one. Differentiating $\mathbf{p}^*(\tau)$ with respect to τ yields

$$\frac{d}{d\tau} \mathbf{p}^*(\tau) = (1 - \beta)(1 - \tau)^{\frac{1}{\gamma}-1} [\gamma - (1 + \gamma)\tau] \frac{g_1 \mathbb{E} \theta^{\frac{\gamma}{1+\gamma}} - \theta_1^{\frac{\gamma}{1+\gamma}} \mathbb{E} g}{(\mathbb{E} \theta^{\frac{\gamma}{1+\gamma}} (1 - \tau)^{\frac{1}{\gamma}} \tau - \mathbb{E} g)^2} < 0,$$

implying that $\mathbf{p}^*(b)$ is decreasing in b . Since $b = 0$ implies that $\mathbb{E} S(\tau(b)) = 0$, the function $\mathbf{p}^*(b)$ has a pole at $b = 0$. That $\mathbf{p}^*(b)$ decreasing in b must therefore imply that $\lim_{b \rightarrow 0^-} \mathbf{p}^*(b) = -\infty$ and $\lim_{b \rightarrow 0^+} \mathbf{p}^*(b) = \infty$. We conclude that

$$\mathbf{p}^*((-\infty, \bar{b}]) = \mathbf{p}^*((-\infty, 0)) \cup \mathbf{p}^*((0, \bar{b}]) = (-\infty, \alpha_1) \cup [\alpha_2, \infty).$$

We compute the bounds α_1 and α_2 by taking the limits of \mathbf{p}^* as b approaches $-\infty$ and the upper bound for government debt under complete markets \bar{b} , or equivalently as τ approaches $-\infty$ and $\frac{\gamma}{1+\gamma}$, respectively. \square

With only government expenditure shocks, we compute

$$\alpha_1 = 1 \text{ and } \alpha_2 = (1 - \beta) \frac{\theta^{\frac{\gamma}{1+\gamma}} \left(\frac{1}{1+\gamma} \right)^{\frac{1}{\gamma}} \frac{\gamma}{1+\gamma} - g(s_1)}{\theta^{\frac{\gamma}{1+\gamma}} \left(\frac{1}{1+\gamma} \right)^{\frac{1}{\gamma}} \frac{\gamma}{1+\gamma} - \mathbb{E} g} + \beta > 1$$

With only TFP shocks, we compute

$$\alpha_1 = (1 - \beta) \frac{\theta(s_1)^{\frac{\gamma}{1+\gamma}}}{\mathbb{E} \theta^{\frac{\gamma}{1+\gamma}}} + \beta > 1$$

and

$$\alpha_2 = (1 - \beta) \frac{\theta(s_1)^{\frac{\gamma}{1+\gamma}} \left(\frac{1}{1+\gamma} \right)^{\frac{1}{\gamma}} \frac{\gamma}{1+\gamma} - g}{\mathbb{E} \theta^{\frac{\gamma}{1+\gamma}} \left(\frac{1}{1+\gamma} \right)^{\frac{1}{\gamma}} \frac{\gamma}{1+\gamma} - g} + \beta > \alpha_1$$

Remark 4.3. *With only TFP shocks, the bond payoff has the special property that it is associated with a steady state asset level that supports the first-best allocation, $\mathbf{p}^{*-1}(1) = b_{fb}$. At the first-best taxes are zero, so the net-of-interest government surplus is constant across states.*

We illustrate Proposition ?? in figure ?. The blue curve is the inverse map \mathbf{p}^{*-1} . Two

constants α_1 and α_2 divide possible payoff structures into three regions: one in which a steady state exists with the government holding assets, another in which a steady state exists with the government owing debt, and yet another in which where a steady state does not exist.

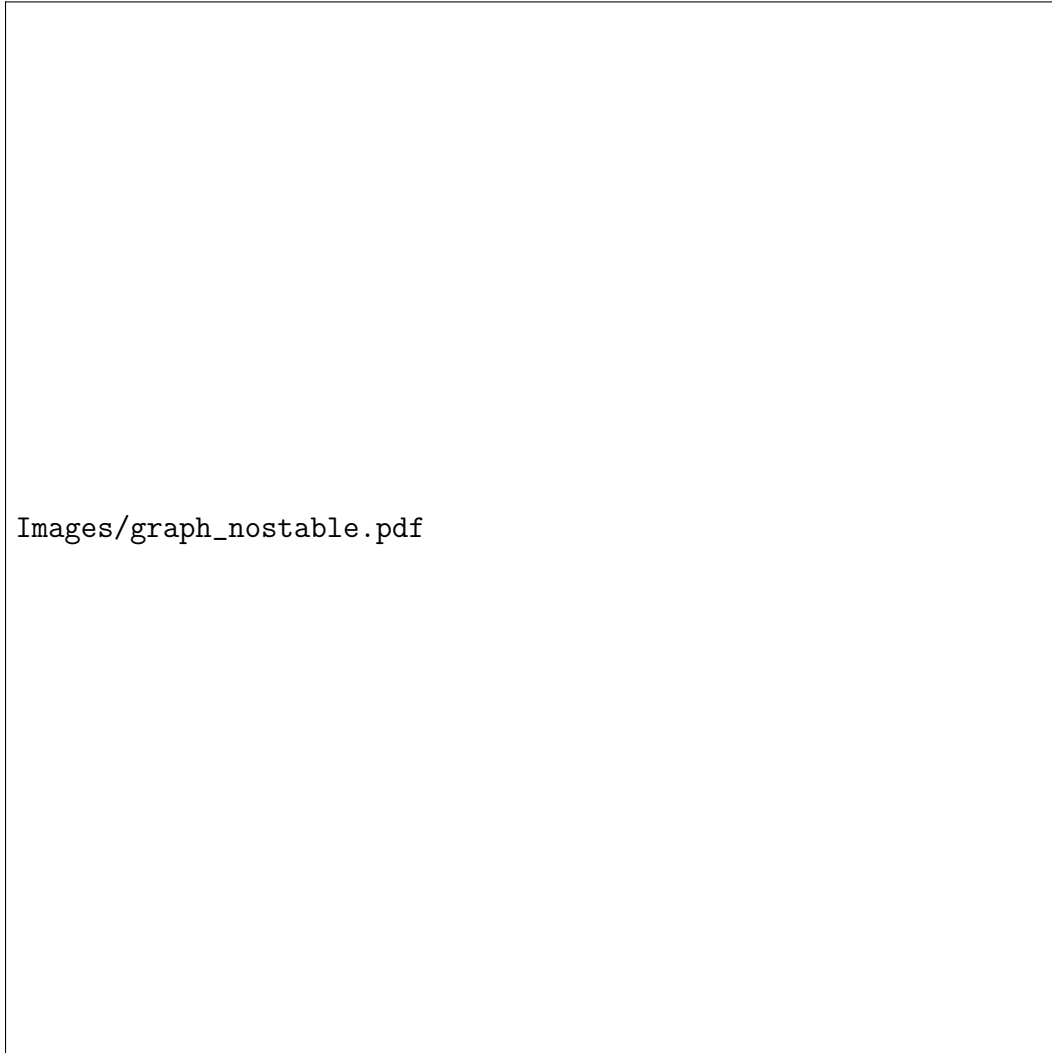


Figure 2: Three regions in \mathbf{p} space.

4.3 Long Run Assets

In subsection ??, we provided conditions under which there exists b^* such $\mathbf{p}^*(b^*) = \mathbf{p}$. By construction, if $b_{-1} = b^*$ then the allocation that solves complete markets Ramsey problem

?? for initial condition b^* automatically satisfies the measurability constraints (??). That allocation therefore solves the incomplete markets Ramsey problem ???. This implies that if $b_{-1} = b^*$, then $b_t = b^*$ for all t . Thus, $p^{*-1}(\mathbf{p})$ corresponds to a “steady state”. It remains to be determined whether the incomplete markets Ramsey b_t converges to b^* for arbitrary b_{-1} . Theorem ?? provides sufficient conditions for convergence.

Theorem 4.4. *Let b_{fb} denote the level of government debt associated with the first-best allocation with complete markets. Then*

- a. *If $\mathbf{p} \leq \min(\alpha_1, 1)$, then $b_{fb} < b^* < 0$ and $b_t \rightarrow b^*$ with probability 1.*
- b. *If $\mathbf{p} \geq \alpha_2$, then $0 < b^*$ and $b_t \rightarrow b^*$ with probability 1.*
- c. *If $\min(\alpha_1, 1) < \mathbf{p} < \alpha_2$, b^* either does not exist or is unstable.*

For \mathbf{p} in region (c), the government run up debt over time.

Proof. The optimal allocation can be represented recursively in terms of functions $c_t(\mu_t)$, $l_t(\mu_t)$, $b_t(\mu_t)$ together with a law of motion for $\mu' = \mu'(\mu, s)$ for μ . We shall establish show global stability under the assumption that $\mu'(\mu, s)$ an increasing function of μ . The heart of the proof revolves around the twisted-martingale equation for μ :

$$\mu_t = \sum_s \Pi(s) p_s \mu'(\mu_t, s) = \mathbb{E}_t p_{t+1} \mu_{t+1}.$$

We have shown that there is at most one μ^* such that $\mu'(\mu^*, s) = \mu^*$ for all s . Here we focus on showing global stability for $\mu < \mu^*$. The twisted-martingale equation can be decomposed as follows

$$\mu_t = \mathbb{E}_t \mu_{t+1} + Cov_t(p_{t+1}, \mu_{t+1}).$$

By signing $Cov_t(p_{t+1}, \mu_{t+1})$, we can determine whether μ_t follows a sub or super-martingale. Given that μ_t is bounded from above,⁶ we can verify global convergence to the steady state if μ_t is a supermartingale. As in the statement of the theorem, we will split the proof up into three cases. Recall that \mathbf{p} is the payoff in the “good” state 1.

1. $\mathbf{p} < \min\{1, \alpha_1\}$: Let \bar{b}_s^n be maximal debt with which the government could enter a period and be able to pay off, assuming that it receives shock s from this period onward.

⁶Since $\mu'(\mu^*, s) = \mu^*$ and $\mu'(\mu, s)$ is increasing in μ , we know that if $\mu_t < \mu^*$, then $\mu_{t+j} < \mu^*$ for all histories s^{t+j} .

Then

$$\bar{b}_s^n = \left(\frac{p_s}{\beta} - 1 \right)^{-1} \left(\theta_s^{\frac{\gamma}{1+\gamma}} \left(\frac{1}{1+\gamma} \right)^{\frac{1}{\gamma}} \frac{\gamma}{1+\gamma} - g_s \right)$$

because the government maximizes tax revenue by setting τ to $\frac{\gamma}{1+\gamma}$. For $\mathbf{p} < \alpha_2$, it is possible to show that $\bar{b}_1^n > \bar{b}_2^n$, and thus the natural debt limit is attained under repetition of the “adverse” state. This implies that $\lim_{\mu \rightarrow -\infty} b(\mu) = \bar{b}_2^n$ and $\lim_{\mu \rightarrow -\infty} \mu'(\mu, 2) = -\infty$. In order for the period-by-period budget constraint

$$\frac{p_s}{\beta} b(\mu) = S(\mu'(\mu, s)) + b(\mu'(\mu, s))$$

to be satisfied for all s , it must be true that $\lim_{\mu \rightarrow -\infty} \mu'(\mu, 1) > -\infty$ (as $\bar{b}_1^n > \bar{b}_2^n$). Continuity of μ together with the uniqueness of the steady state μ^* then implies that $\mu'(\mu, 1) > \mu'(\mu, 2)$ for all $\mu < \mu^*$. $\mathbf{p} < 1$ implies that $p_1 < p_2$, allowing us to conclude that $Cov_t(p_{t+1}, \mu_{t+1}) < 0$. We then have that

$$\mu_t < \mathbb{E}_t \mu_{t+1}$$

for $\mu_t < \mu^*$. Since $\mu'(\mu, s)$ is increasing and continuous, and since $\mu'(\mu^*, s) = \mu^*$, we can iterate on the policy functions to show that if $\mu_t < \mu^*$, then for all $j > 0$, we must have $\mu_{t+j} < \mu^*$. Thus, if $\mu_t < \mu^*$, then μ_t is a supermartingale bounded from below. That implies that $\mu_t \rightarrow \tilde{\mu}$ for some constant $\tilde{\mu}$ with probability 1. Then we can use the continuity of $\mu'(\mu, s)$ to show that

$$\mu'(\tilde{\mu}, s) = \tilde{\mu},$$

implying that $\tilde{\mu} = \mu^*$, as μ^* is the unique steady state. The steady state is then globally stable since $\mu_t \rightarrow \mu^*$ with probability 1.

2. $\mathbf{p} \geq \alpha_2$: Following the same approach used in for case 1, we know for $\mathbf{p} > \alpha_2$ that $\bar{b}_1^n < \bar{b}_2^n$, implying that the natural debt limit is attained under repetition of the “good” state. As in case 1, by taking limits we obtain $\lim_{\mu \rightarrow -\infty} \mu'(\mu, 1) = -\infty$ and $\lim_{\mu \rightarrow -\infty} \mu'(\mu, 2) > -\infty$. This implies that $\mu'(\mu, 1) < \mu'(\mu, 2)$, which along with $p_1 > p_2$ implies $Cov_t(p_{t+1}, \mu_{t+1}) < 0$. As in case 1, we then have global stability of the steady state for $\mu_t < \mu^*$.

3. $\min(\alpha_1, 1) < \mathbf{p} < \alpha_2$: In this case, either there exists a steady state if $1 < \mathbf{p} \leq \alpha_1$ or

there does not exist a steady state. In either case the analysis for case 1 implies that $\mu'(\mu, 1) > \mu'(\mu, 2)$ for $\mu < \mu^*$.⁷ Since $\mathbf{p} > 1$ implies that $p_1 > p_2$, we can conclude that $Cov_t(p_{t+1}, \mu_{t+1}) > 0$, implying that

$$\mu_t > \mathbb{E}_t \mu_{t+1}.$$

We thus cannot apply the martingale convergence theorem, leaving open the possibility that the steady state is not stable.

□

Remark 4.5. *Figure ?? illustrates Theorem ??. In addition to depicting values of \mathbf{p} for which a steady state exists, it also highlights regions where a steady state is stable. The theorem asserts that there exist \mathbf{p} for which a steady state exists but is unstable.*

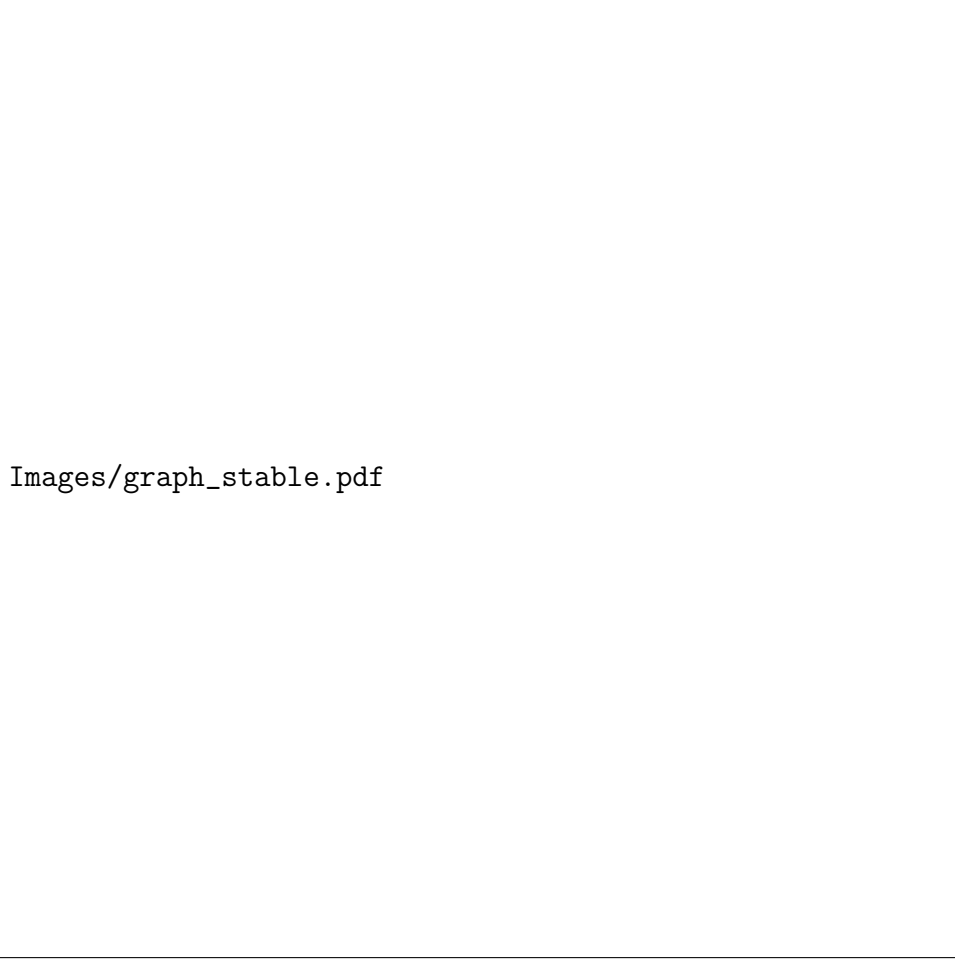
An important aspect of this model is that small changes in primitives (specifically \mathbf{p}) can lead to major differences in long run allocations. To illustrate this, in Figure ?? we plot two sample paths where the only difference is the asset restriction \mathbf{p} .

4.4 Economic forces driving convergence

In summary, when the aggregate state follows a 2-state i.i.d. process, government debt b_t often converges to b^* , while the tail of the allocation equals Ramsey allocation for an economy with complete markets and initial government debt b^* . The level and sign of b^* depend on the asset payoff structure, which we have expressed in terms of a scalar \mathbf{p} that concisely captures what in more general settings we represented with the asset payoff matrix \mathbb{P} .

Facing incomplete markets, the Ramsey planner recognizes that the government's debt *level* combines with the payoff structure on its debt instrument to affect the welfare costs associated with varying the distorting labor tax rate across states. When the instrument is a risk-free bond, the government's marginal cost of raising funds μ_t is a martingale. In this situation, *changes* in debt levels help smooth tax distortions across time. However, if the payoff on the debt instrument varies across states, then by affecting its state-contingent revenues, the *level* of government debt can help smooth tax distortions across states. For our two state, iid shock process, the steady state debt level b^* , when it exists, is the unique

⁷When a steady state does not exist, take μ^* to be ∞ .



Images/graph_stable.pdf

Figure 3: Stability regions in \mathbf{p} space.

amount of government debt that provides just enough “state contingency” completely to fill the void left by missing assets markets. The Ramsey planner takes into account the additional benefits from tax smoothing as the government debt approaches b^* ; that puts a risk-adjustment into the martingale governing μ and leads the government either to accumulate or decumulate debt. Although accumulating government assets requires raising distorting taxes, locally the welfare costs of higher taxes are second-order and so are dominated by the welfare gains from approaching b^* , which are first-order.



Figure 4: A sample path with $p > 1$ (left) $p < 1$ (right).

5 More than Two Shocks

Our analysis thus far has been limited to environments where there are two aggregate states of the world which are independently and identically distributed over time. While we cannot solve analytically for the optimal policies for more general stochastic processes, we will show in this section how it is possible to obtain first order approximations to the optimal policy rules and compute the mean and variance of the ergodic distribution of the linearize system. For this section we will restrict our attention to government expenditure shocks. We set aggregate productivity, θ , to be constant and normalize to one. We allow exogenous government expenditure to take S possible values which are i.i.d. over time with probability of government expenditure g_s being \mathbb{P}_s .

In order to approximate the optimal policies we need to linearize around a steady state. The standard approach is to linearize around the non-stochastic steady state as the size of the shocks approaches zero. We have found for this planning problem this approach does not accurately characterize the decision rules of the non-linear system⁸. Instead, we propose an alternative method utilizing the complete market steady states of the previous

⁸When linearizing around the steady state as the size of the shocks approaches zero, we find that the marginal value of debt μ_t is a martingale. A key property of the policy rules for μ_t that produces the degenerate distribution of debt is that μ_t is a sub-martingale for sufficiently small μ_t and super-martingale for sufficiently large μ_t .

section.

We will show that if the payoff vector $\bar{\mathbf{p}}$ is perfectly correlated with government debt then there exists a complete markets Ramsey allocation that is a solution to the incomplete markets planning problem with payoff vector $\bar{\mathbf{p}}$. The sign and magnitude of the covariance of $\bar{\mathbf{p}}$ with the government expenditure shock will determine the sign and magnitude of government debt in this allocation. Given that the state variable \bar{b} , government debt, is constant under this allocation we can linearize the policy rules with respect to both government debt and payoff vector, allowing us to approximate the incomplete market policy rules at some pair (b, \mathbf{p}) near $(\bar{b}, \bar{\mathbf{p}})$. In fact, we will show that the best choice of $(\bar{b}, \bar{\mathbf{p}})$ is to minimize the distance

$$\|\mathbf{p} - \bar{\mathbf{p}}\|^2 = \sum_s \mathbb{P}_s (p_s - \bar{p}_s)^2 \quad (17)$$

as then ergodic distribution of the linearized system will be centered around \bar{b} .

5.1 Steady State Payoff Vectors

Our first task is to find the set of payoff vectors, $\bar{\mathbf{p}}$ for which a complete markets allocation is a solution to the incomplete markets planning problem. In Lemma ?? we show that these payoff vectors must be perfectly correlated with g :

Lemma 5.1. *The payoff vector $\bar{\mathbf{p}}$, normalized so that $\mathbb{E}\bar{\mathbf{p}} = 1$, associated with complete markets Ramsey allocation with government debt $\bar{b} \leq \bar{b}^n$ (\bar{b}^n is the natural debt limit) must satisfy*

$$\bar{p}_s = 1 - \frac{\beta}{\bar{b}} (g_s - \mathbb{E}g). \quad (18)$$

Moreover, the steady state \bar{b} is globally stable: $b_t \rightarrow \bar{b}$ with probability 1.

The proof of this Lemma is included in the Appendix, but the intuition is straightforward. The complete markets allocation must have a constant tax rate $\bar{\tau}$. The government surplus in state s , $S(\bar{\tau}, s)$, is then made up of two components: tax income $I(\bar{\tau})$, which is independent of s , and government expenditure $-g_s$. Income from government debt $\frac{\bar{b}}{\beta} \bar{p}_s$, must perfectly account for fluctuations in government surplus: $-g_s$. Thus, $\bar{\mathbf{p}}$ must be perfectly correlated with the exogenous government expenditure process g_s .

Rearranging equation (??), and taking the covariance with respect to g we see that if $\bar{\mathbf{p}}$ is perfectly correlated with government expenditures the associated level of steady state

debt is

$$\bar{b} = -\beta \frac{\text{var}(\mathbf{g})}{\text{cov}(\bar{\mathbf{p}}, \mathbf{g})}$$

The covariance of $\bar{\mathbf{p}}$ with \mathbf{g} relative to the variance of government expenditure determines the level of steady state government debt, and the sign of the covariance determines if the government holds debt or assets in the steady state. As before, the government is using fluctuating payments on debt to smooth tax rates. If payoffs are perfectly correlated with good fiscal times (low g) then the government will hold debt in long run. If payoffs are correlated with hard fiscal (high g) times then the government will hold assets.

5.2 Ergodic Distribution of Linearized Policy Rules

With the previous section we are able to complete characterize the ergodic distribution of global policy rules for a one-dimensional subspace of the space of all payoff vectors \mathbb{R}^{S-1} (dimension $S - 1$ after normalizing $\mathbb{E}\mathbf{p} = 1$). For \mathbf{p} which are not perfectly correlated with \mathbf{g} we can compute the mean and variance of the ergodic distribution of the linearized policy rules. Any payoff vector \mathbf{p} , with $\mathbb{E}\mathbf{p} = 1$, can be uniquely decomposed into two components:

$$\mathbf{p} = \hat{\mathbf{p}} + \bar{\mathbf{p}}$$

where $\bar{\mathbf{p}}$ is perfectly correlated to \mathbf{g} and $\hat{\mathbf{p}}$ is orthogonal to \mathbf{g} . This is equivalent to finding $\bar{\mathbf{p}}$, perfectly correlated with \mathbf{g} , that minimizes $\|\mathbf{p} - \bar{\mathbf{p}}\|^2$ in equation (??). Proposition ?? below then tells us that the ergodic distribution of government debt of the linearized policy rules will be centered around the steady state level of debt, \bar{b} , associated with $\bar{\mathbf{p}}$

Proposition 5.2. *Suppose \mathbf{p} admits a decomposition $\mathbf{p} = \hat{\mathbf{p}} + \bar{\mathbf{p}}$ with $\hat{\mathbf{p}}$ orthogonal to \mathbf{g} and*

$$\bar{\mathbf{p}} = 1 - \frac{\beta}{\bar{b}}(\mathbf{g} - \mathbb{E}g).$$

with $\bar{b} \leq \bar{b}^n$. Then the ergodic distribution of debt of the policy rules linearized around $(\bar{b}, \bar{\mathbf{p}})$ will have mean \bar{b} and variance

$$\frac{\bar{b}^2 \text{var}(\hat{\mathbf{p}})}{\mathbb{E}[\bar{\mathbf{p}}^2] \text{var}(\bar{\mathbf{p}})}. \quad (19)$$

The proof of this proposition is included in the appendix. From Proposition ?? we see that the location of the ergodic distribution of debt for the incomplete markets Ramsey allocation will be determined by the covariance of \mathbf{p} and \mathbf{g} . Specifically, since $\text{cov}(\mathbf{p}, \mathbf{g}) =$

$\text{cov}(\bar{\mathbf{p}}, \mathbf{g})$, the ergodic distribution of the linearized policy rules will be centered around

$$-\beta \frac{\text{var}(\mathbf{g})}{\text{cov}(\bar{\mathbf{p}}, \mathbf{g})} = -\beta \frac{\text{var}(\mathbf{g})}{\text{cov}(\mathbf{p}, \mathbf{g})}.$$

We can also use equation (??) to quickly bound the variation of linearized system. Rearranging terms and noting that $\mathbb{E}[\bar{\mathbf{p}}^2] \geq 1$ we can bound the coefficient of variation of the ergodic distribution of the linearized policy rules

$$\frac{\sigma_b}{\bar{b}} \leq \sqrt{\frac{\text{var}(\hat{\mathbf{p}})}{\text{var}(\bar{\mathbf{p}})}} \quad (20)$$

The spread of the ergodic distribution relative to its mean is bounded by the loading of the payoff vector \mathbf{p} on the component orthogonal to \mathbf{g} relative to the loading on the component parallel to \mathbf{g} .

6 Turning on risk-aversion

We now depart from quasi-linearity of $U(c, l)$ and thus activate an additional source of return fluctuations coming from endogenous fluctuations in prices of the asset q_t . To obtain a recursive representation of a Ramsey plan, we employ the endogenous state variable

$$x_t = u_{c,t} b_t,$$

and study how long-run properties of x_t depend on equilibrium returns $R_{t,t+1} = \frac{\mathbb{P}(s_t, s_{t+1})}{q_t(s^t)}$. Activating risk aversion in consumption makes q_t vary in interesting ways.

Commitment to a Ramsey plan implies that government actions at $t \geq 1$ are constrained by the household's anticipations about them at $s < t$. Following ?, we use the marginal utility of consumption that the Ramsey planner promises to the household to account for that 'forward looking' restriction on the Ramsey planner. That comes from the fact that the Euler equation restricts allocations such that expected marginal utility in time t is constrained by consumption choices in time $t - 1$. It is convenient for us that scaling the household's budget constraint by the marginal utility of consumption makes Ramsey problem recursive in $x = U_c b$. In particular, implementability constraints (??) can be

represented as

$$\frac{x_{t-1}\mathbb{P}(s_t, s_{t-1})U_{c,t}}{\beta\mathbb{E}_{t-1}\mathbb{P}U_{c,t}} = U_{c,t}c_t + U_{l,t}l_t + x_t, \quad t \geq 1 \quad (21)$$

Problem 6.1. Let $V(x, s_{-1})$ be the expected continuation value of the Ramsey plan at $t \geq 1$ given promised marginal utility government debt inherited from the past $x = U_{c,t}b_t$ and time $t - 1$ Markov state s_{-1} . After the realization of time 0 Markov shock s_0 , let $W(b_{-1}, s_0)$ be the value of the Ramsey plan when initial government debt is b_{-1} . The (ex ante) Bellman equation for $t \geq 1$ is

$$V(x, s_-) = \max_{c(s), l(s), x'(s)} \sum_s \Pi(s, s_-) \left(U(c(s), l(s)) + \beta V(x'(s), s) \right) \quad (22)$$

subject to $x'(s) \in [\underline{x}, \bar{x}]$ and

$$\frac{x\mathbb{P}(s, s_-)U_c(s)}{\beta\mathbb{E}_{s_-}\mathbb{P}U_c} = U_c(s)c(s) + U_l(s)l(s) + x'(s) \quad (23)$$

$$c(s) + g(s) = \theta(s)l(s) \quad (24)$$

Equation (??) is the implementability constraint and (??) is feasibility. Given an initial debt b_{-1} , time 0 Markov state s_0 , and continuation value function $V(x, s_-)$, the (ex post) time 0 Bellman equation is

$$W(b_{-1}, s_0) = \max_{c_0, l_0, x_0} U(c, l) + \beta V(x_0, s_0) \quad (25)$$

subject to time zero implementability constraint

$$U_c(c_0, l_0)c + U_l(c_0, l_0)l_0 + x_0 = U_c(c_0, l_0)b_{-1}$$

and the resource constraint

$$c_0 + g(s_0) = \theta(s_0)l_0$$

and

$$x_0 \in [\underline{x}, \bar{x}]$$

Lemma 6.2. Let V, W be the optimal value functions for problem ??. The allocation given by the corresponding optimal policy function solves problem ??.

6.1 Computational and analytic strategy

The analysis in this section is based on two pillars: (1) a suite of `python` computer programs that solves Bellman equations (??) and (??); and (2) some mathematical analysis of first-order conditions satisfied by the optimal policy function that attain the right sides of these Bellman equations.

We attacked Ramsey Problem ?? with two weapons: (1) a suite of `python` computer programs that solve Bellman equations (??) and (??); and (2) mathematical analysis of the first-order conditions satisfied by the optimal policy functions that attain U and V . In our computational approach, we solved the Bellman equations via policy iteration on first-order conditions. Appendix ?? tells how we take $V_x(x, s_-) = \tilde{\mu}$ as a state variable and approximate the optimal policy $x(\tilde{\mu}, s_-)$ with quadratic splines.⁹ As with the quasilinear section, our analytical approach is confined to environments with a two state i.i.d. process for the aggregate state s_t . We use our numerically computed optimal policy functions to confirm that much of the intuition acquired from our formal mathematical analysis of the two-state, i.i.d. process extends to general stochastic processes for s_t .

6.2 Motivation to focus on risk-free bond economy

As mentioned in section ??, properties of a Ramsey plan for our incomplete markets economy vary sensitively with asset returns that reflect properties of equilibrium prices $\{q_t(s^t|B_{-1}, s_{-1})\}_t$ and the exogenous asset payoff matrix \mathbb{P} . By studying quasi-linear preferences, we eliminated fluctuations in returns coming from prices. Here we turn the table and by studying an economy with a risk-free bond, we eliminate fluctuations in returns coming from the exogenous asset payoff matrix \mathbb{P} . Thus, we set $\mathbb{P}(s|s_-) = 1 \ \forall (s, s_-)$.

Let $x'(s; x, s_-)$ be the decision rule for x' that attains the right side of the $t \geq 1$ Bellman equation (??). A steady state x^* satisfies $x^* = x'(s; x^*, s_-)$ for all s, s_- . A steady state is a node at which the continuation allocation and tax rate have no further history dependence.

Proposition 6.3. *Assume that U is separable and iso-elastic, $U(c, l) = \frac{c^{1-\sigma}}{1-\sigma} - \frac{l^{1+\gamma}}{1+\gamma}$. Assume that*

the Markov state s take two values is i.i.d with s_b being the “adverse” state (either low TFP or high govt. expenditures) and s_g begin the good state. Let x_{fb} be the discounted present value of marginal utility weighted government surpluses associated with the first

⁹We also solved the problem numerically using value function iteration. Numerical experiments showed that policy function iteration provided more accurate and stable solutions.

best allocation. Let $q_{fb}(s)$ be the shadow price of government debt in state s at the first best allocation. If

$$\frac{1 - q_{fb}(s_b)}{1 - q_{fb}(s_g)} > \frac{g(s_b)}{g(s_g)} \geq 1, \quad (26)$$

then there exists a steady state with $x_{fb} > x^* > 0$

Proof. As in the quasi-linear case, a steady state is associated with a continuation allocation of a complete markets allocation starting from some initial debt level. We can index such continuation allocations by their associated multiplier μ on the implementability constraint. Letting $S(\mu, s)$ be the government surplus at state s and multiplier μ , a steady state has a multiplier μ^* at which the budget constraint in both states of the world is satisfied:

$$\frac{S(\mu^*, s_g)}{\frac{c(\mu^*, s_g)^{-\sigma}}{\beta \mathbb{E}c(\mu^*)^{-\sigma}} - 1} = \frac{S(\mu^*, s_b)}{\frac{c(\mu^*, s_b)^{-\sigma}}{\beta \mathbb{E}c(\mu^*)^{-\sigma}} - 1}.$$

By choosing μ_1 so that $S(\mu_1, s_g) = 0$, we conclude that

$$0 = \frac{S(\mu_1, s_g)}{\frac{c(\mu_1, s_g)^{-\sigma}}{\beta \mathbb{E}c(\mu_1)^{-\sigma}} - 1} > \frac{S(\mu_1, s_b)}{\frac{c(\mu_1, s_b)^{-\sigma}}{\beta \mathbb{E}c(\mu_1)^{-\sigma}} - 1}.$$

We derived this equation directly from $S(\mu, s_g) < S(\mu, s_b)$ for all μ and $c(\mu, s_g) > c(\mu, s_b)$ for all μ .

Eliminating q_{fb} , equation (??) can expressed as

$$\frac{g(s_g)}{1 - \frac{\beta \mathbb{E}c_{fb}^{-\sigma}}{c_{fb}(s_g)^{-\sigma}}} > \frac{g(s_b)}{1 - \frac{\beta \mathbb{E}c_{fb}^{-\sigma}}{c_{fb}(s_b)^{-\sigma}}}.$$

Multiplying both sides by -1 and factoring out $\beta \mathbb{E}c_{fb}^{-\sigma}$, this equation simplifies to

$$\frac{-c_{fb}(s_g)^{-\sigma}g(s_g)}{\frac{c_{fb}(s_g)^{-\sigma}}{\beta \mathbb{E}c_{fb}^{-\sigma}} - 1} < \frac{-c_{fb}(s_b)^{-\sigma}g(s_b)}{\frac{c_{fb}(s_b)^{-\sigma}}{\beta \mathbb{E}c_{fb}^{-\sigma}} - 1}$$

or

$$\frac{S(0, s_g)}{\frac{c(0, s_g)^{-\sigma}}{\beta \mathbb{E}c(0)^{-\sigma}} - 1} < \frac{S(0, s_b)}{\frac{c(0, s_b)^{-\sigma}}{\beta \mathbb{E}c(0)^{-\sigma}} - 1}.$$

Existence of μ^* follows directly from the Intermediate Value Theorem. \square

Proposition 6.4. *There exist $\underline{x} < x^*$ and $\bar{x} > 0$ such that if $\{c_t(s^t), l_t(s^t), x_t(s^{t-1})\}$ solves*

the incomplete markets Ramsey problem ?? with bounds \underline{x} and \bar{x} , then $x_t(s^t) \rightarrow x^*$ as $t \rightarrow \infty$ with probability 1.

Proof. The proof relies on the concavity of the value function V and two lemmas that describe the structure of the policy functions. Proofs of the lemmas appear in the appendix.

Lemma 6.5. *Consumption is ordered by the state of the world. In particular, there exist \underline{x} and \bar{x} such that for all $x \in [\underline{x}, \bar{x}]$, the policy function for consumption satisfies $c(x, s_g) > c(x, s_b)$.*

This lemma assures that for the same level of marginal utility weighted government debt, consumption is larger in “good” states of the world than in “adverse” states of the world.

Lemma 6.6. *There exist \underline{x} and \bar{x} such that the optimal government savings policy $x'(x, s)$ satisfies*

1. *For $x \in (x^*, \bar{x}]$, $x'(x, s_g) < x'(x, s_b)$*
2. *For $x \in [\underline{x}, x^*)$, $x'(x, s_g) > x'(x, s_b)$*

Furthermore, $x'(x, \cdot)$ is increasing in x .

Property 1 states that if government debt exceeds its steady state value, then the government issues more debt in bad states of the world than in good states of the world. Property 2 states that if government debt is smaller than its steady state amount, then the government has accumulated enough assets that the lower interest rate in the “adverse” state of the world allow it to purchase more assets (issue less debt) than in the “good” states of the world.¹⁰ The last part of the lemma guarantees that if the government enters with more debt, it will pass on more debt to future periods. We can now prove global convergence. We will focus on the case where $x_t \geq x^*$, since the analysis of the other case is symmetric. Since $x'(x, \cdot)$ is increasing in x , we can iterate the policy functions forward to conclude that $x_{t+j} > x^*$ for all j as long as $x_t > x^*$. Letting $\mu_t = V'(x_t)$ be the multiplier on the implementability constraint and $\bar{\lambda}_t$ be the multiplier on the constraint $x_t \leq \bar{x}$, we have

$$\mu_t = \frac{1}{\mathbb{E}_t[c_{t+1}^{-\sigma}]} \mathbb{E}_t[\mu_{t+1} c_{t+1}^{-\sigma}] - \bar{\lambda}_t$$

¹⁰Remember that in the steady state, the government owns a positive amount of the risk-free asset.

Lemma ?? along with concavity of V allows us to conclude that $\mu_{t+1}(s_g) > \mu_{t+1}(s_b)$. From Lemma ??, we know that $c_{t+1}(s_g) > c_{t+1}(s_b)$, which implies that $Cov_t(\mu_{t+1}, c_{t+1}^{-\sigma}) < 0$, so

$$\frac{1}{\mathbb{E}_t[c_{t+1}^{-\sigma}]} \mathbb{E}_t[\mu_{t+1} c_{t+1}^{-\sigma}] < \mathbb{E}_t[\mu_{t+1}].$$

Since $\bar{\lambda}_t \geq 0$, we conclude that

$$\mu_t < \mathbb{E}_t[\mu_{t+1}].$$

Moreover $\mu_t < V'(x^*) = \mu^*$, so μ_t is a submartingale that is bounded from above. Applying the martingale convergence theorem, we conclude that $\mu_t \rightarrow \mu^*$ with probability 1. Continuity of the policy functions and uniqueness of the steady state in the region $[x^*, \bar{x}]$ implies that $x_t \rightarrow x^*$ with probability 1. \square

Remark 6.7. *In this economy, fluctuations in the risk-free interest rate come from fluctuations in marginal utility of consumption. The interest rate is low in “good” states (i.e., when TFP is high or government expenditures are low). In a steady state, the government holds claims against the private sector, an outcome that resembles those in economies with quasi-linear utility and low \mathbf{p} . For all admissible initial levels of government debt, an incomplete markets Ramsey allocation converges to a particular Lucas-Stokey Ramsey allocation.*

Remark 6.8. *Propositions ?? and ?? should be interpreted approximately as supplying a converse to Lemma 3 from section 5 of ?, which provided sufficient conditions for their incomplete markets Ramsey plan economy to fail to converge to a complete markets continuation allocation. Our propositions ?? and ?? provide sufficient conditions for a complete markets steady state continuation allocation to exist, and for the incomplete market Ramsey allocation to converge to that steady state continuation allocation. Note that propositions ?? and ?? assume a very special stochastic process for s . For more general stochastic processes, a steady state does not exist. But in simulations, we have found that the outcomes described in Propositions ?? and ?? do a good job of approximating long run dynamics of incomplete markets Ramsey plans for richer shock stochastic processes, in the sense that they converge to regions of low volatility.*

Figure ?? plots a simulation of the Ramsey plan. The path of marginal utility weighted government debt resembles the path of government debt for the quasilinear economy with low \mathbf{p} plotted earlier in Figure ??.



Figure 5: Sample path of x_t for an economy with risk aversion and a 2 state i.i.d. process for TFP.

6.3 Allowing nonnegative transfers

Figure ?? compares simulations of Ramsey plans for two economies identical in all respects except that one allows the government to award nonnegative lump-sum transfers, while the other doesn't. In both economies, a 2 state i.i.d. shock impinges on government expenditures, but not on TFP and the one-period utility function is quasilinear. With government access to nonnegative lump-sum transfers, it is easy to verify that there exists a fixed level of positive government assets constituting a "steady state" that is associated with a first-best continuation allocation. With nonnegative lump sum transfers, in cases where a steady state exists and is stable, if the initial debt of the government exceeds its steady state level, outcomes converge with probability 1 to the steady state. Thus, counterparts to our earlier results prevail when initial government debt exceeds its steady state value. When initial government debt is less than a steady-state value, then we know somewhat less, but still something useful. In this case, the multiplier on the measurability constraint



Figure 6: Government debt for two otherwise identical economies with quasilinear preferences and a risk-free bond, one economy with nonnegative transfers, the other with zero transfers.

is a bounded sub martingale that therefore converges with probability one. A limit point is associated with a continuation allocation that is either (1) a first-best allocation, or (2) a continuation allocation associated with Lucas-Stokey Ramsey allocations for some initial government debt level. We have constructed simulations of Ramsey plans, all of which display one or the other of these types of limiting behavior. We hope eventually to figure out more about the features of an economy that determine how much of an ergodic distribution is concentrated on those two types of continuation allocations.

7 Concluding remarks

The ? quotation with which we began this paper emphasizes that a Ramsey planner's optimal administration of a flat rate tax on labor depends on its ability to trade a complete set of securities with the public whose payoffs are contingent on possible realizations

of random variables that drive government expenditures. The debt dynamics associated with a Ramsey are central to Lucas and Stokey's message.¹¹ That he implicitly prohibited the government from trading such securities helps account for quite different assertions about optimal debt dynamics made by ? : government debt is a key state variable for Barro, one that that should be governed by a random walk; while government debt is not even an independent state variable for Lucas and Stokey, instead being an exact function of the Markov state driving government expenditures that is influenced by the initial level of government debt. By showing that government debt has a unit-root-like component in a version of ?'s economy restricted to allow the government to issue only risk-free debt, ? went part way, but only part way, toward explaining the striking differences between the debt dynamics in ? and ?. ? obtain analytical results that are both most complete and most consistent with Barro's assertions the special assumption of quasi-linear preferences that lets a fixed discount factor pin down a time-invariant risk-free interest rate. But even in that case, outcomes diverge from what Barro had asserted: ? showed that if the government has access to nonnegative *transfers*, then eventually the government acquires large enough claims on the private sector to set the flat rate tax on labor to zero and to finance all expenditures from earnings on its assets. ? were able to say much less about debt dynamics when preferences were not quasi-linear.

In this paper, we have gotten much further and have discovered that the *limiting* aspects of optimal debt dynamics in an incomplete markets economy approximate outcomes prevailing in a Lucas-Stokey complete markets economy. There exist levels of government debt that let fluctuating returns on government debt – delivered partly through fluctuating interest rates (when preferences show risk-aversion in consumption) and also partly through fluctuations generated by random payoffs in the single risky security that we allow the government to trade – that give the government sufficient access to most of the risk-sharing that Lucas and Stokey stress as an important aspect of optimal taxation. For a wide range of economies, equilibrium dynamics draw government debt (or assets) toward that level, albeit at a rate that can be very slow. That slow rate of convergence that is possibly a descendant of Barro's unit-root intuition. In appendix ??, we present an analysis of the rate of convergence that reveals how it depends on the variability of equilibrium returns on government bonds.

The finding that interest rate fluctuations are a mechanism allowing a fiscal authority

¹¹Lucas and Stokey focus on a deriving an optimal debt management strategy that can render an optimal tax policy time consistent.

to hedge risks is a theme that plays an important role in contributions by ?, ?. A related avenue is also active in ?, though what matters there are not government debt dynamics themselves but rather the dynamics of the debt positions of private agents relative one to another.

A Linearization Methods

In this appendix we go into more detail on the methods surrounding linearizing the product rules. With a little effort the first order conditions for the planning problem can be written as

$$\frac{b_{t-1}p_t}{\beta} = I(\mu_t) - g_t + b_{t+1} \quad (27)$$

$$\mu_t = \mathbb{E}_t p_{t+1} \mu_{t+1} \quad (28)$$

where

$$I(\mu) = (1 - \tau(\mu))^{\frac{1}{\gamma}} \tau(\mu) \quad (29)$$

and

$$\tau(\mu) = \frac{\gamma\mu}{(1 + \gamma)\mu - 1} \quad (30)$$

It is possible to treat μ_t as the state variable, rather than b_t . When this is done the first order conditions can be re-written as searching for a function $b(\mu)$ such that the following equations

$$\frac{b(\mu)p_s}{\beta} = I(\mu'(s)) - g(s) + b(\mu'(s)) \quad (31)$$

$$\mu = \sum_s \Pi_s p_s \mu'(s) \quad (32)$$

can be solved for all μ . We will proceed from here on by treating μ as our state variable and linearizing with respect to it. We begin with a proof of Lemma ??.

Lemma. *The payoff vector \bar{p} , normalized so that $\mathbb{E}\bar{p} = 1$, associated with complete markets Ramsey allocation with government debt $\bar{b} \leq \bar{b}^n$ (\bar{b}^n is the natural debt limit) must satisfy*

$$\bar{p}_s = 1 - \frac{\beta}{\bar{b}}(g_s - \mathbb{E}g). \quad (33)$$

Moreover, the steady state \bar{b} is globally stable: $b_t \rightarrow \bar{b}$ with probability 1.

Proof. There is a multiplier $\bar{\mu}$ associated with each solution to the complete markets Ramsey problem. The level of debt associated with that multiplier is

$$\bar{b}(\bar{\mu}) = \frac{\beta}{1 - \beta}(I(\bar{\mu}) - \bar{g}).$$

In order for $\bar{\mathbf{p}}(\bar{\mu})$ to be the portfolio associated with the complete markets Ramsey allocation the following equations must hold for all s

$$\frac{\bar{p}_s(\bar{\mu})\bar{b}(\bar{\mu})}{\beta} = I(\bar{\mu}) - g_s + \bar{b}(\bar{\mu}).$$

Subtracting equation s' from s we see

$$(\bar{p}_s(\bar{\mu}) - \bar{p}_{s'}(\bar{\mu}))\frac{\bar{b}(\bar{\mu})}{\beta} = g_{s'} - g_s$$

or

$$\bar{p}_s(\bar{\mu}) - \bar{p}_{s'}(\bar{\mu}) = -\frac{\beta}{\bar{b}(\bar{\mu})}(g_s - g_{s'})$$

This combined with $\mathbb{E}\bar{\mathbf{p}} = \sum_s \bar{p}_s \mathbb{P}_s = 1$ immediately gives the desired

$$\bar{\mathbf{p}}(\bar{\mu}) = 1 - \frac{\beta}{\bar{b}(\bar{\mu})}(\mathbf{g} - \mathbb{E}\mathbf{g}) \quad (34)$$

That this steady state is globally stable can be shown using the linearization below. We show that

$$\frac{\partial \mu'(s)}{\partial \mu}(\bar{\mu}) = \frac{\bar{p}_s}{\mathbb{E}[\bar{\mathbf{p}}^2]}$$

This, along with the single crossing of the $\mu'(\mu, s)$ policy functions, implies that $\mu'(\mu, s)$ is positively correlated with \bar{p}_s when $\mu > \bar{\mu}$ and negatively correlated when $\mu < \bar{\mu}$. We then have that $\mu_t \leq (\geq) \mathbb{E}_t \mu_{t+1}$ for $\mu_t \leq (\geq) \bar{\mu}$. The uniqueness of the steady state implies that if $\mu_t \leq (\geq) \bar{\mu}$ then $\mu_{t+j} \leq (\geq) \bar{\mu}$ for all $j > 0$. An application of the Martingale Convergence Theorem then proves $\mu_t \rightarrow \bar{\mu}$ with probability 1. \square

We can expand our policy functions to be functions of μ and \mathbf{p} then the first order

conditions can be written as

$$\frac{b(\mu; \mathbf{p})p_s}{\beta} = I(\mu'(s)) - g(s) + b(\mu'(s); \mathbf{p})$$

$$\mu = \sum_s \Pi_s p_s \mu'(s)$$

We have shown that these first order conditions are satisfied at all pairs $(\bar{\mu}, \bar{\mathbf{p}}(\bar{\mu}))$ by the policy functions $b(\bar{\mu}, \bar{\mathbf{p}}) = \bar{b}(\bar{\mu})$ and $\mu'(\bar{\mu}; \bar{\mathbf{p}}, s) = \bar{\mu}$. We want to approximate the policy function $b(\mu; \mathbf{p})$ and $\mu'(\mu; \mathbf{p}, s)$ by taking the first order approximation around some steady state $(\bar{\mu}, \bar{\mathbf{p}}(\bar{\mu}))$.

Given a payoff vector \mathbf{p} , satisfying $\sum_s \mathbb{P}_s p_s = 1$, we want to choose a steady state $(\bar{\mu}, \bar{\mathbf{p}}(\bar{\mu}))$ to approximate around. A natural choice is to choose $(\bar{\mu}, \bar{\mathbf{p}}(\bar{\mu}))$ so as to minimize

$$\|\mathbf{p} - \bar{\mathbf{p}}(\bar{\mu})\|^2 = \sum_s \mathbb{P}_s (p_s - \bar{p}(\bar{\mu})_s)^2 \quad (35)$$

The first order condition for minimizing equation (??) is

$$2 \sum_{s'} \mathbb{P}'_s (p_{s'} - \bar{p}(\bar{\mu})_{s'}) \bar{p}'(\bar{\mu})_{s'} = 0$$

as noted before

$$\bar{p}(\bar{\mu})_s = 1 - \frac{\beta}{\bar{b}(\bar{\mu})} (g_s - \mathbb{E}g)$$

thus

$$\bar{\mathbf{p}}'(\bar{\mu}) \propto \bar{\mathbf{p}}(\bar{\mu}) - \mathbf{1}$$

Thus we can see the the optimal choice of $\bar{\mu}$ is equivalent to choosing $\bar{\mu}$ such that

$$\begin{aligned} 0 &= \sum_{s'} \mathbb{P}_{s'} (p_{s'} - \bar{p}(\bar{\mu})_{s'}) (\bar{p}(\bar{\mu})_{s'} - 1) \\ &= - \sum_{s'} \mathbb{P}_{s'} (p_{s'} - \bar{p}(\bar{\mu})_{s'}) + \sum_{s'} \mathbb{P}_{s'} (p_{s'} - \bar{p}(\bar{\mu})_{s'}) \bar{p}(\bar{\mu})_{s'} \\ &= \sum_{s'} \mathbb{P}_{s'} (p_{s'} - \bar{p}(\bar{\mu})_{s'}) \bar{p}(\bar{\mu})_{s'} \\ &= \mathbb{E} [(p - \bar{p}(\bar{\mu})) \bar{p}(\bar{\mu})] \end{aligned} \quad (36)$$

Thus, for a given \mathbf{p} we have chosen $\bar{\mathbf{p}}$ such that

$$\hat{\mathbf{p}} = \mathbf{p} - \bar{\mathbf{p}}$$

is orthogonal to $\bar{\mathbf{p}}$ or

$$\mathbb{E}[\hat{\mathbf{p}}\bar{\mathbf{p}}] = \sum_s \mathbb{P}_s \hat{p}_s \bar{p}_s = 0$$

with this choice of $\bar{\mathbf{p}}$ we can proceed with proving Proposition ??

Proposition. Suppose \mathbf{p} admits a decomposition $\mathbf{p} = \hat{\mathbf{p}} + \bar{\mathbf{p}}$ with $\hat{\mathbf{p}}$ orthogonal to \mathbf{g} and

$$\bar{\mathbf{p}} = 1 - \frac{\beta}{\bar{b}}(\mathbf{g} - \mathbb{E}\mathbf{g}).$$

with $\bar{b} \leq \bar{b}^n$. Then the ergodic distribution of debt of the policy rules linearized around $(\bar{b}, \bar{\mathbf{p}})$ will have mean \bar{b} and variance

$$\frac{\bar{b}^2 \text{var}(\hat{\mathbf{p}})}{\mathbb{E}[\bar{\mathbf{p}}^2] \text{var}(\bar{\mathbf{p}})}. \quad (37)$$

Proof. Differentiating the first order conditions with respect to μ we

$$\begin{aligned} \frac{\bar{p}_s}{\beta} \frac{\partial b}{\partial \mu} &= \left[I'(\bar{\mu}) + \frac{\partial b}{\partial \mu} \right] \frac{\partial \mu'(s)}{\partial \mu} \\ 1 &= \sum_{s'} \mathbb{P}_s \bar{p}_s \frac{\partial \mu'(s)}{\partial \mu} \end{aligned}$$

Applying $\sum_s \mathbb{P}_s \bar{p}_s \cdot$ to the first equation we get

$$\frac{\mathbb{E}[\bar{\mathbf{p}}^2]}{\beta} \frac{\partial b}{\partial \mu} = \left[I'(\bar{\mu}) + \frac{\partial b}{\partial \mu} \right]$$

or

$$\frac{\partial b}{\partial \mu} = \frac{\beta I'(\bar{\mu})}{\mathbb{E}[\bar{\mathbf{p}}^2] - \beta} \quad (38)$$

where $\mathbb{E}\bar{\mathbf{p}}^2 = \sum_s \mathbb{P}_s \bar{p}_s^2$ which then quickly gives

$$\frac{\partial \mu'(s)}{\partial \mu} = \frac{\bar{p}_s}{\mathbb{E}[\bar{\mathbf{p}}^2]} \quad (39)$$

We can then differentiate with respect to \mathbf{p} using the shorthand $\frac{\partial b}{\partial \mathbf{p}} = \sum_s \frac{\partial b}{\partial p_s} \hat{p}_s$ and similarly

for $\frac{\partial \mu(s)}{\partial \mathbf{p}}$ to get

$$\begin{aligned} \frac{1}{\beta} \left(\frac{\partial b}{\partial \mathbf{p}} \bar{p}_s + \bar{b} \hat{p}_s \right) &= \left[I'(\bar{\mu}) + \frac{\partial b}{\partial \mu} \right] \frac{\partial \mu'(s)}{\partial \mathbf{p}} + \frac{\partial b}{\partial \mathbf{p}} \\ 0 &= \sum_s \mathbb{P}_s \bar{p}_s \frac{\partial \mu'(s)}{\partial \mathbf{p}} \end{aligned}$$

Applying $\sum_s \mathbb{P}_s \bar{p}_s \cdot$ to the first equation and noting that $\sum_s \hat{p}_s \bar{p}_s = 0$ we quickly see that

$$\frac{\partial b}{\partial \mathbf{p}} = 0 \quad (40)$$

and hence

$$\frac{\partial \mu'(s)}{\partial \mathbf{p}} = \frac{\hat{p}_s \bar{b}}{\beta \left[I'(\bar{\mu}) + \frac{\partial b}{\partial \mu} \right]} \quad (41)$$

The linearized system for μ now follows

$$\hat{\mu}_{t+1} = B_{s_{t+1}} \hat{\mu}_t + C_{s_{t+1}}$$

where $\hat{\mu} = \mu - \bar{\mu}$. Here B and C are both random with means \bar{B} and \bar{C} , and variances σ_B^2 and σ_C^2 . Note we obtained expressions for B and C being

$$B_s = \frac{\partial \mu'(s)}{\partial \mu} = \frac{\bar{p}_s}{\mathbb{E}[\bar{\mathbf{p}}^2]} \quad (42)$$

and

$$C_s = \frac{\partial \mu'(s)}{\partial \mathbf{p}} = \frac{\hat{p}_s \bar{b}}{\beta \left[I'(\bar{\mu}) + \frac{\partial b}{\partial \mu} \right]} \quad (43)$$

Suppose that $\hat{\mu}$ is distributed according to the ergodic distribution of this linear system with mean $\mathbb{E} \hat{\mu}$ and variance σ_μ^2 . Since

$$B \hat{\mu} + C$$

has the same distribution we can compute the mean of this distribution as

$$\begin{aligned}
\mathbb{E}\hat{\mu} &= \mathbb{E}[B\hat{\mu} + C] \\
&= \mathbb{E}[\mathbb{E}_{\hat{\mu}}[B\hat{\mu} + C]] \\
&= \mathbb{E}[\overline{B}\hat{\mu} + \overline{C}] \\
&= \overline{B}\mathbb{E}\hat{\mu} + \overline{C}
\end{aligned}$$

solving for $\mathbb{E}\hat{\mu}$ we get

$$\mathbb{E}\hat{\mu} = \frac{\overline{C}}{1 - \overline{B}} \quad (44)$$

For the variance $\sigma_{\hat{\mu}}^2$ we know that

$$\sigma_{\hat{\mu}}^2 = \text{var}(B\hat{\mu} + C) = \text{var}(B\hat{\mu}) + \sigma_C^2 + 2\text{cov}(B\hat{\mu}, C)$$

Computing the variance of $B\hat{\mu}$ we have

$$\begin{aligned}
\text{var}(B\hat{\mu}) &= \mathbb{E}[(B\hat{\mu} - \overline{B}\mathbb{E}\hat{\mu})^2] \\
&= \mathbb{E}[(B\hat{\mu} - \overline{B}\hat{\mu} + \overline{B}\hat{\mu} - \overline{B}\mathbb{E}\hat{\mu})^2] \\
&= \mathbb{E}[\mathbb{E}_{\hat{\mu}}[(B - \overline{B})^2\hat{\mu}^2 + 2(B - \overline{B})(\hat{\mu} - \mathbb{E}\hat{\mu})\overline{B}\mathbb{E}\hat{\mu} + (\hat{\mu} - \mathbb{E}\hat{\mu})^2\overline{B}^2]] \\
&= \mathbb{E}[\sigma_B^2\hat{\mu}^2 + (\hat{\mu} - \mathbb{E}\hat{\mu})^2\overline{B}] \\
&= \sigma_B^2(\sigma_{\hat{\mu}}^2 + (\mathbb{E}\hat{\mu})^2) + \sigma_{\hat{\mu}}^2\overline{B}^2
\end{aligned}$$

while for the covariance of $B\hat{\mu}$ and C

$$\text{cov}(B\hat{\mu}, C) = \sigma_{BC}\mathbb{E}\hat{\mu}$$

Putting this all together we have

$$\sigma_{\hat{\mu}}^2 = \frac{\sigma_B^2(\mathbb{E}\hat{\mu})^2 + \sigma_{BC}\mathbb{E}\hat{\mu} + \sigma_C^2}{1 - \overline{B}^2 - \sigma_B^2} \quad (45)$$

From equation (??) we have that

$$\overline{C} = 0 \text{ and } \sigma_C^2 = \frac{\text{var}(\hat{p})\overline{b}}{\beta^2 \left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu} \right]^2} \quad (46)$$

and from equation (??) we have

$$\bar{B} = \frac{1}{\mathbb{E}[\bar{\mathbf{p}}^2]} \text{ and } \sigma_B^2 = \frac{\text{var}(\bar{\mathbf{p}})}{(\mathbb{E}[\bar{\mathbf{p}}^2])^2} \quad (47)$$

Which we can use to see that $\mathbb{E}\hat{\mu} = 0$, so the ergodic distribution of the linearized μ is centered around $\bar{\mu}$ with variance

$$\sigma_\mu^2 = \frac{\bar{b}^2 \text{var}(\hat{\mathbf{p}})}{\beta^2 \left[I'(\bar{\mu}) + \frac{\partial b}{\partial \mu} \right]^2 (1 - \bar{B}^2 - \sigma_B^2)} \text{var}(\hat{\mathbf{p}}) \quad (48)$$

For a given $\hat{\mu}$ the associated deviation in b : \hat{b} is given by

$$\hat{b} = \frac{\partial b}{\partial \mu} \hat{\mu} + \frac{\partial b}{\partial \mathbf{p}} = \frac{\partial b}{\partial \mu} \hat{\mu}$$

The ergodic distribution of the linearized system for b is therefore centered around \bar{b} with variance

$$\sigma_b^2 = \frac{\bar{b}^2 \left(\frac{\partial b}{\partial \mu} \right)^2 \text{var}(\hat{\mathbf{p}})}{\beta^2 \left[I'(\bar{\mu}) + \frac{\partial b}{\partial \mu} \right]^2 (1 - \bar{B}^2 - \sigma_B^2)}$$

Noting that $\mathbb{E}\bar{\mathbf{p}}^2 = 1 + \text{var}(\bar{\mathbf{p}})$ and

$$I'(\bar{\mu}) + \frac{\partial b}{\partial \mu} = \frac{\mathbb{E}[\bar{\mathbf{p}}^2]}{\beta} \frac{\partial b}{\partial \mu},$$

this can be simplified to

$$\sigma_b^2 = \frac{\bar{b}^2 \text{var}(\hat{\mathbf{p}})}{\mathbb{E}[\bar{\mathbf{p}}^2] \text{var}(\bar{\mathbf{p}})} \quad (49)$$

which quickly gives us the bound

$$\frac{\sigma_b}{\bar{b}} \leq \sqrt{\frac{\text{var}(\hat{\mathbf{p}})}{\text{var}(\bar{\mathbf{p}})}} \quad (50)$$

□

B Numerical Approximations of Bellman Equations

C Rate of Convergence