Taxes, debts, and redistributions with aggregate shocks*

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Abstract

This paper studies how taxes and debt respond to aggregate shocks in the presence of incomplete markets and redistribution concerns. A planner sets a lump sum transfer and a linear tax on labor income in an economy with heterogeneous agents, aggregate uncertainty, and markets restricted to a single asset whose payoffs can wary with aggregate states. Two forces shape long-run outcomes: the planner's desire to minimize the welfare costs of fluctuating transfers, which calls for a negative correlation between the distribution of net assets and agents' skills; and the planner's desire to use fluctuations in the real interest rate to adjust for missing state-contingent securities. In a model parameterized to match stylized facts about US booms and recessions, distributional concerns mainly determine optimal policies over business cycle frequencies. These features of optimal policy differ markedly from ones that emerge from representative agent Ramsey models

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1. Introduction

2. Environment

2.1. Ricardian equivalence

In this section we recover a *Ricardian equivalence* result in the spirit of **?**. We use this result to highlight that the level of government debt is not a state variable in our setting. The reason being that there is an equivalence class of tax policies and asset profiles that support the same competitive equilibrium allocation and as such pin down only net asset positions.

Theorem 1 *Given* $\left(\{b_{i,-1}\}_i, B_{-1}\right)$*, let* $\left\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, B_t, R_t\right\}_t$ *and* $\{\tau_t, T_t\}_t$ *be a competitive equilibrium. For any bounded sequences* $\left\{\hat{b}_{i,t}\right\}_{i,t\geq -1}$ *that satisfy*

$$\hat{b}_{i,t} - \hat{b}_{1,t} = \tilde{b}_{i,t} \equiv b_{i,t} - b_{1,t} \text{ for all } t \geq -1, i \geq 2,$$

there exist sequences $\left\{\hat{T}_t\right\}_t$ *and* $\left\{\hat{B}_t\right\}_{t\geq -1}$ *that satisfy* (**??**) *such that* $\left\{\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_i, \hat{B}_t, R_t\right\}_t$ *and* $\left\{\tau_t, \hat{T}_t\right\}_t$ *constitute a competitive equilibrium given* $\left(\left\{\hat{b}_{i,-1}\right\}_i, \hat{B}_{-1}\right)$.

Proof. Let

$$\hat{T}_t = T_t + \left(\hat{b}_{1,t} - b_{1,t}\right) - R_{t-1}\left(\hat{b}_{1,t-1} - b_{1,t-1}\right) \text{ for all } t \geq 0. \tag{1}$$

Given a tax policy $\left\{\tau_t, \hat{T}_t\right\}_t$, the allocation $\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_{i,t}$ is a feasible choice for consumer i since it satisfies

$$\begin{aligned}
c_{i,t} &= (1-\tau_t)\,\theta_{i,t} l_{i,t} + R_{t-1} b_{i,t-1} - b_{i,t} + T_t \\
&= (1-\tau_t)\,\theta_{i,t} l_{i,t} + R_{t-1}\left(b_{i,t-1} - b_{1,t-1}\right) - \left(b_{i,t} - b_{1,t}\right) + T_t + R_{t-1} b_{1,t-1} - b_{1,t} \\
&= (1-\tau_t)\,\theta_{i,t} l_{i,t} + R_{t-1}\left(\hat{b}_{i,t-1} - \hat{b}_{1,t-1}\right) - \left(\hat{b}_{i,t} - \hat{b}_{1,t}\right) + T_t + R_{t-1} b_{1,t-1} - b_{1,t} \\
&= (1-\tau_t)\,\theta_{i,t} l_{i,t} + R_{t-1}\hat{b}_{i,t-1} - \hat{b}_{i,t} + \hat{T}_t.
\end{aligned}$$

Suppose that $\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_{i,t}$ is not the optimal choice for consumer i, in the sense that there exists some other sequence $\left\{\hat{c}_{i,t}, \hat{l}_{i,t}, \hat{b}_{i,t}\right\}_t$ that gives strictly higher utility. Then the choice $\left\{\hat{c}_{i,t}, \hat{l}_{i,t}, b_{i,t}\right\}_t$ is feasible given the tax rates $\{\tau_t, T_t\}_t$, which contradicts the assumption that $\{c_{i,t}, l_{i,t}, b_{i,t}\}_t$ is the optimal choice for the consumer given taxes $\{\tau_t, T_t\}_t$. The new allocation satisfies all other constraints and therefore is an equilibrium. ■

An immediate corollary is that it is not total government debt but rather who owns it that affects equilibrium allocations.

Corollary 1 *For any pair B'_{-1}, B''_{-1}, there are asset profiles $\left\{b'_{i,-1}\right\}_i$ and $\left\{b''_{i,-1}\right\}_i$ such that equilibrium allocations starting from $\left(\left\{b'_{i,-1}\right\}_i, B'_{-1}\right)$ and from $\left(\left\{b''_{i,-1}\right\}_i, B''_{-1}\right)$ are the same. These asset profiles satisfy*

$$b'_{i,-1} - b'_{1,-1} = b''_{i,-1} - b''_{1,-1} \text{ for all } i.$$

We note that the result continues to hold in more general environments. For example, we could allow agents to trade all conceivable Arrow securities or allow for capital accumulation and still show that equilibrium allocations depend only on agents' net assets positions.

Proposition 1 shows that many transfer sequences $\{T_t\}_t$ and asset profiles $\{b_{i,t}, B_t\}_{i,t}$ support the same equilibrium allocation. As such we use a normalization to define the notion of *public debt* for our setting. Assume that productivities are ordered with $\theta_{1,t} \leq \theta_{2,t} \ldots \leq \theta_{N,t}$. Using proposition 1 to set $b_{1,t} = 0$, we will interpret $-B_t = \sum_{i>1} n_i b_{i,t}$ as public debt. This inturn explains why imposing limits on $B_t - b_{1,t}$ are comparable to debt limits in a representative agent settings.

3. Optimal equilibria with affine taxes

We now focus on the characterization of optimal plans by applying the primal approach. This involves using household optimality conditions to obtain a set of restrictions on allocations chosen by the government to ensure that they are 'implementable' as competitive equilibria. In the last section we describe a recursive formulation of the Ramsey plan.

Assume that $U^i : \mathbb{R}^2_+ \to \mathbb{R}$ is concave in $(c, -l)$ and twice continuously differentiable. We let $U^i_{x,t}$ or $U^i_{xy,t}$ denote first and second derivatives of U^i with respect to $x, y \in \{c, l\}$ in period t and assume $\lim_{x\to 0} U^i_l(c, x) = 0$ for all c and i.

With natural borrowing limits for the households, first-order necessary conditions for the consumer's problem are

$$(1 - \tau_t)\,\theta_{i,t} U^i_{c,t} = -U^i_{l,t}, \tag{2}$$

and

$$U^i_{c,t} = \beta \mathbb{E}_t R_{t+1} U^i_{c,t+1}. \tag{3}$$

To help characterize an equilibrium, we use

Proposition 1 *A sequence $\left\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, R_t, \tau_t, T_t\right\}_t$ is part of a competitive equilibrium with affine taxes if and only if it satisfies (**??**), (**??**), (2), and (3) and $b_{i,t}$ is bounded for all i and t.*

Proof. Necessity is obvious. In appendix **??**, we use arguments of **?** and **?** to show that any $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ that satisfies (**??**), (2), and (3) is a solution to consumer i's problem. Equilibrium $\{B_t\}_t$ is determined by (**??**) and constraint (**??**) is then implied by Walras' Law ■

To find an optimal equilibrium, by Proposition 1 we can choose $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, R_t, \tau_t, T_t\}_t$ to maximize (**??**) subject to (**??**), (**??**), (2), and (3). We apply a first-order approach and follow steps similar to ones taken by **?** and AMSS. Substituting consumers' first-order conditions (2) and (3) into the budget constraints (**??**) yields implementability constraints

$$c_{i,t} + b_{i,t} = -\frac{U^i_{l,t}}{U^i_{c,t}} l_{i,t} + T_t + \frac{p_t U^i_{c,t-1}}{\beta \mathbb{E}_{t-1} p_t U^i_{c,t}} b_{i,t-1} \text{ for all } i, t. \tag{4}$$

For $I \geq 2$, we can use constraint (4) for $i = 1$ to eliminate T_t from (4) for $i > 1$. Letting $\tilde{b}_{i,t} \equiv b_{i,t} - b_{1,t}$, we can represent the implementability constraints as

$$\begin{aligned} & (c_{i,t} - c_{1,t}) + \tilde{b}_{i,t} \\ = \quad & -\frac{U^i_{l,t}}{U^i_{c,t}} l_{i,t} + \frac{U^1_{l,t}}{U^1_{c,t}} l_{1,t} + \frac{p_t U^i_{c,t-1}}{\beta \mathbb{E}_{t-1} p_t U^i_{c,t}} \tilde{b}_{i,t-1} \text{ for all } i > 1, t. \end{aligned} \tag{5}$$

With this representation of the implementability constraints, the planner's maximization problem depends only on the $I - 1$ variables $\tilde{b}_{i,t-1}$. The reduction of the dimensionality from I to $I - 1$ is another consequence of theorem 1.

Denote $Z^i_t = (c_{i,t} - c_{1,t}) + \tilde{b}_{i,t} + \frac{U^i_{l,t}}{U^i_{c,t}} l_{i,t} - \frac{U^1_{l,t}}{U^1_{c,t}} l_{1,t}$. Formulated in a space of sequences, the optimal policy problem is:

$$\max_{c_{i,t}, l_{i,t}, \tilde{b}_{i,t}} \mathbb{E}_0 \sum_{i=1}^{I} \omega_i \sum_{t=0}^{\infty} \bar{\beta}_t U^i_t (c_{i,t}, l_{i,t}), \tag{6}$$

subject to

$$\tilde{b}_{i,t-1} \frac{p_t U^i_{c,t-1}}{\mathbb{E}_{t-1} p_t U^i_{c,t}} = \mathbb{E}_t \sum_{k=t}^{\infty} \beta^{k-t} \left(\frac{U^i_{c,k}}{U^i_{c,t}} \right) Z^i_k \quad \forall t \geq 1 \tag{7a}$$

$$\tilde{b}_{i,-1} = \mathbb{E}_{-1} \sum_{k=0}^{\infty} \beta^k \left(\frac{U^i_{c,k}}{U^i_{c,t}} \right) Z^i_k \tag{7b}$$

$$\frac{\mathbb{E}_t p_{t+1} U^i_{c,t+1}}{U^i_{c,t}} = \frac{\mathbb{E}_t p_{t+1} U^j_{c,t+1}}{U^j_{c,t}} \tag{7c}$$

$$\sum_{i=1}^{I} n_i c_i(s^t) + g(s_t) = \sum_{i=1}^{I} \pi_i \theta_i(s_t) l_i(s^t), \tag{7d}$$

$$\frac{U^i_{l,t}}{\theta_{i,t} U^i_{c,t}} = \frac{U^1_{l,t}}{\theta_{1,t} U^1_{c,t}} \tag{7e}$$

$$\sum_{i=1}^{N} \tilde{b}_{i,t-1} \text{ is bounded} \tag{7f}$$

Constraint (7a) is a measurablity restriction on allocations that requires that the right side is determined at time $t-1$. This condition is inherited from the restriction that there is only one only asset with payoffs given by p_t that is traded between the private and the public sector.

For both computational and educational purposes, it is convenient to represent the optimal policy problem recursively. For the purpose of constructing a recursive representation, let $\boldsymbol{x} = \beta^{-1}\left(U_c^2\tilde{b}_2, ..., U_c^I\tilde{b}_I\right)$, $\boldsymbol{\rho} = \left(U_c^2/U_c^1, ..., U_c^I/U_c^1\right)$, and denote an allocation $a = \{c_i, l_i\}_{i=1}^I$. In the spirit of **?** and **?**, we split the Ramsey problem into a time-0 problem that takes $(\{\tilde{b}_{i,-1}\}_{i=2}^I, s_0)$ as given and a time $t \geq 1$ continuation problem that takes $\boldsymbol{x}, \boldsymbol{\rho}, s_{\_}$ as given. We formulate two Bellman equations and two value functions, one that pertains to $t \geq 1$, another to $t = 0$. The time inconsistency of an optimal policy manifests itself in there being distinct value functions and Bellman equations at $t = 0$ and $t \geq 1$.

For $t \geq 1$, let $V(\boldsymbol{x}, \boldsymbol{\rho}, s_{\_})$ be the planner's continuation value given $\boldsymbol{x}_{t-1} = \boldsymbol{x}, \boldsymbol{\rho}_{t-1} = \boldsymbol{\rho}, s_{t-1} = s_{\_}$. It satisfies the Bellman equation

$$V(\boldsymbol{x}, \boldsymbol{\rho}, s_{\_}) = \max_{a(s), x'(s), \rho'(s)} \sum_s \pi(s|s_{\_}) \left(\left[\sum_i \omega_i U^i(s) \right] + \beta V(\boldsymbol{x}'(s), \boldsymbol{\rho}'(s), s) \right) \tag{8}$$

where the maximization is subject to

$$U_c^i(s)\left[c_i(s) - c_1(s)\right] + x_i'(s) + \left(U_l^i(s) l_i(s) - U_c^i(s) \frac{U_l^1(s)}{U_c^1(s)} l_1(s) \right) = \frac{x P(s|s_{\_}) U_c^i(s)}{\beta \mathbb{E}_{s_{\_}} P \boldsymbol{U}_c^i} \text{ for all } s, i \geq 2 \tag{9a}$$

$$\frac{\mathbb{E}_{s_{\_}} P \boldsymbol{U}_c^i}{\mathbb{E}_{s_{\_}} P \boldsymbol{U}_c^1} = \rho_i \text{ for all } i \geq 2 \tag{9b}$$

$$\frac{U_l^i(s)}{\theta_i(s) U_c^i(s)} = \frac{U_l^1(s)}{\theta_1(s) U_c^1(s)} \text{ for all } s, i \geq 2 \tag{9c}$$

$$\sum_i n_i c_i(s) + g(s) = \sum_i n_i(s) l_i(s) \ \ \forall s \tag{9d}$$

$$\rho_i'(s) = \frac{U_c^i(s)}{U_c^1(s)} \text{ for all } s, i \geq 2 \tag{9e}$$

$$\sum_{i>1} x_i(s) \frac{\beta}{U_c^i(s)} \text{is bounded} \tag{9f}$$

Constraints (9b) and (9e) imply (3). The definition of x_t and constraints (9a) together imply equation (5) scaled by U_c^i. Let $V_0\left(\{\tilde{b}_{i,-1}\}_{i=2}^I, s_0\right)$ be the value to the planner at $t = 0$, where $\tilde{b}_{i,-1}$ denotes initial debt inclusive of accrued interest. It satisfies the Bellman equation

$$V_0\left(\{\tilde{b}_{i,-1}\}_{i=2}^I, s_0\right) = \max_{a_0, x_0, \rho_0} \sum_i \omega_i U^i(c_{i,0}, l_{i,0}) + \beta V\left(x_0, \rho_0, s_0\right) \tag{10}$$

where the maximization is subject to

$$U^i_{c,0}\left[c_{i,0}-c_{1,0}\right]+x_{i,0}+\left(U^i_{l,0}l_{i,0}-U^i_{c,0}\frac{U^1_{l,0}}{U^1_{c,0}}l_{1,0}\right)=U^i_{c,0}\tilde{b}_{i,-1}\text{ for all } i\geq 2 \tag{11a}$$

$$\frac{U^i_{l,0}}{\theta_{i,0}U^i_{c,0}}=\frac{U^1_{l,0}}{\theta_{1,0}U^{1,0}_c}\text{ for all } i\geq 2 \tag{11b}$$

$$\sum_i \pi_i c_{i,0}+g_0=\sum_i \pi_i\theta_{i,0}l_{i,0} \tag{11c}$$

$$\rho_{i,0}=\frac{U^i_{c,0}}{U^1_{c,0}}\text{ for all } i\geq 2 \tag{11d}$$

Because constraint (9b) is absent from the time 0 problem, the time 0 problem differs from the time $t\geq 1$ problem, a source of the time consistency of the optimal tax plan. The next section characterizes the properties of optimal plans.

4. Long run properties of optimal allocations

In sections 5. and 6. we characterize the long run properties of aggregate debt and taxes. The main finding is that the levels and spreads in debt and tax rates are determined by two factors: a) the ability of the government to span aggregate shocks through the returns on the asset it trades and b)its redistributive preferences. In particular, the government accumulates debt if interest rates are lower when the its need for revenue are higher and vice versa. The long run variance of debt and taxes along with the rates of rates of convergence to the ergodic distribution are higher in economies where the magnitude of this co movement is larger. And lastly more redistributive governments issue more debt.

To study these implications, in section 5. we first examine a simple economy with quasilinear preferences and i.i.d aggregate shocks. This allows us adequate tractability to formally demonstrate and clarify the main driving forces for the results mentioned above. In section 6. we study more general economies (in terms of heterogeneity, preferences and shocks) finally in section **??**, we numerically verify that all the insights go through in a version of the model calibrated to US data.

5. Quasilinear economy

We specialize the problem described in section 3. by imposing the following assumptions that are maintained throughout in this section.

Assumption 1 *IID aggregate shocks: s_t is i.i.d over time*

Assumption 2 *Quasi linear preference: $u(c,l)=c-\frac{l^{1+\gamma}}{1+\gamma}$*

With i.i.d shocks we can restrict our attention to payoff matrices $\mathbb{P}$ that have identical rows denoted by the vector $P(s)$ with a corresponding normalization that $\mathbb{E}P(s) = 1$. We collect a particular set of these vectors that are perfectly correlated with expenditure shocks $g(s)$ in the set $\mathcal{P}^*$ defined below,

$$\mathcal{P}^* = \left\{ P(s) : P(s) = 1 + \frac{\beta}{B^*}(g(s) - \mathbb{E}g) \text{ for some } B^* \in [\overline{B}, \underline{B}] \right\},$$

where $\overline{B}$ and $\underline{B}$ are upper and lower bounds for government assets.

Before characterizing the properties of Ramsey allocation for the economy with heterogeneous agents and no restrictions on transfers, we develop some results in a representative agent economy where the government *cannot* use transfers. We later show that the allocations in this economy are obtained under certain limits on the Pareto weights for the setting with heterogeneous agents.

5.1. Representative agent

Environment

This section describes the representative agent environment with risky debt and no transfers.[1] Given a tax, asset policy $\{\tau_t, B_t\}$, the household solves,

$$W_0(b_{-1}) \max_{\{c_t, l_t, b_t\}_t} \mathbb{E}_0 \sum_t \beta^t \left[c_t - \frac{l_t^{1+\gamma}}{1+\gamma} \right] \tag{12}$$

subject to

$$c_t + b_t = (1 - \tau_t)\theta l_t + R_t P_t b_{t-t} \tag{13}$$

Using the optimality condition for labor and savings we can summarize the set of implementability constraints for the government as follows

$$b_{t-1} \frac{P_t}{E_{t-1} P_t} = \mathbb{E}_t \sum_j \beta^{t+j} [c_t - l_t^{1+\gamma}] \quad \forall t \tag{14}$$

In addition we have also have the feasibility constraint

$$c_t + g_t \leq \theta l_t, \tag{15a}$$

and the market clearing for bonds,

$$b_t + B_t = 0. \tag{15b}$$

[1] This differs from the model studied in AMSS in two ways: first, the government trades a "risky" bond instead of a risk free bond and second, the government is prohibited from using transfer where as AMSS restrict transfers to be non negative. Both of them have critical implications on the long run zero tax results that AMSS obtained. We discuss this later in the section.

The optimal Ramsey allocation solves $\max_{\{c_t,l_t\}_t} W_0(b_{-1})$ subject to (29), feasibility (30), market clearing for bonds (15b) and debt limits for the government $\underline{B}, \overline{B}$.[2]

Results

Theorem 2 *In the representative agent economy satisfying assumptions 1 and 2, the long run assets under the optimal Ramsey allocation are characterized as follows*

1. *Suppose $P \notin \mathcal{P}^*$, there is an invariant distribution of government such that*

$$\forall \epsilon > 0, \quad \Pr\{B_t < \underline{B} + \epsilon \textit{ and } B_t > \overline{B} - \epsilon \quad i.o\} = 1$$

2. *Suppose $P(s) - P(s') > \beta \frac{g(s)-g(s')}{\underline{B}} \quad \forall s, s'$, then for large enough assets (or debt) there is a drift towards the interior region. In particular the value function $V(B)$ is strictly concave and there exists $B_1 < B_2$ such that*

$$\mathbb{E}V'(B(s)) > V'(B_{\_}) \quad B_{\_} > B_2$$

and

$$\mathbb{E}V'(B(s)) < V'(B_{\_}) \quad B_{\_} < B_1$$

3. *Suppose $P(s) \in \mathcal{P}^*$, then the long run assets converge to a degenerate steady state*

$$\lim_t B_t = B^* \quad a.s \quad \forall B_{-1}$$

In the case where $P(s) \in \mathcal{P}^*$, we can express the long run assets

$$B^* = \beta \frac{\mathrm{var}(g(s))}{\mathrm{cov}(P(s), g(s))} \tag{16}$$

Theorem 3 *The ergodic distribution of debt (using the first order approximation of dynamics near $P^*(s)$) has the following properties,*

- ***Mean:*** *The ergodic mean is B^* which corresponds to the steady state level of debt of an economy with payoff vector $P^*(s)$*

- ***Variance:*** *The coefficient of variation is given by*

$$\frac{\sigma(B)}{\mathbb{E}(B)} = \sqrt{\frac{\mathrm{var}(P(s)) - |\mathrm{cov}(g(s), P(s))|}{(1 + |\mathrm{cov}(g(s), P(s))|)|\mathrm{cov}(g(s), P(s))|}} \leq \sqrt{\frac{\mathrm{var}(\hat{P}(s))}{\mathrm{var}(P^*(s))}}$$

[2] In some calculations we will use the natural debt limit for the government. In this case one can explicitly derived.

- ***Convergence rate:*** *The speed of convergence to the ergodic distribution described by*

$$\frac{\mathbb{E}_{t-1}(B_t - B^*)}{(B_{t-1} - B^*)} = \frac{1}{1 + |\text{cov}(P(s), g(s))|}$$

Notice that when payoffs are equal to $P^*(s)$, the government can keep taxes constant and perfectly offset the fluctuations in its surplus with returns $P^*(s)B^*$. Away from this, the incompleteness of markets is binding and shocks are hedged with a combination of changes in tax rates and debt levels. These theorem shows exactly how the deviations from perfect spanning map into larger variances for debt (and taxes) in the long run. Figure 1 shows how the ergodic distribution of debt and taxes spread as we vary the covariance $P(s)$ with $g(s)$.

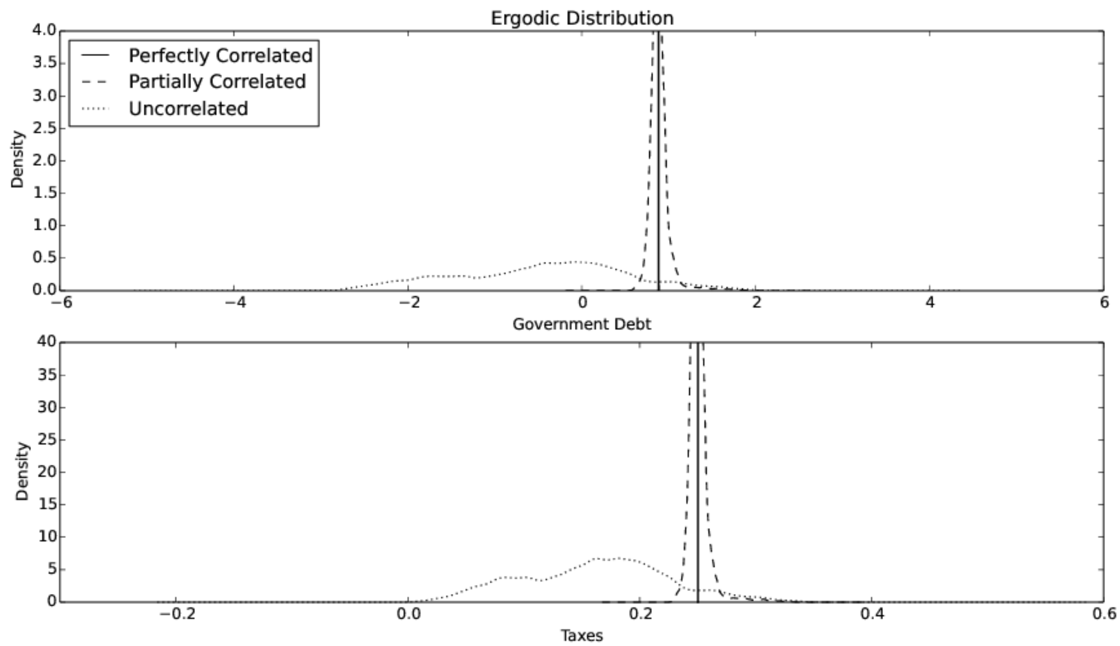

Figure 1: Ergodic distribution for debt and taxes in the representative agent quasilinear economy for three choices $P(s)$.

5.2. Heterogeneous agents

Assumption 3 *The productivity of agents are ordered,* $\theta_1 > \theta_2 = 0$ *and* $c_{2,t} \geq 0$.

Theorem 4 *Let* ω, n *be the Pareto weight and mass of the productive agent with* $n < \frac{\gamma}{1+\gamma}$. *The optimal tax, transfer and asset policies* $\{\tau_t, T_t, B_t\}$ *are characterized as follows,*

1. *For* $\omega \geq n\left(\frac{1+\gamma}{\gamma}\right)$ *we have* $T_t = 0$ *and the optimal policy is same as in a representative agent economy studied in Theorems 2,and 3*

2. *For* $\omega < n\left(\frac{1+\gamma}{\gamma}\right)$*, suppose we further assume that* $\min_s\{P(s)\} > \beta$*. We have two parts:*

   There exists $\mathcal{B}(\omega)$ *and* $\tau^*(\omega)$ *with* $\mathcal{B}'(\omega) > 0$ *and* $\lim_{\omega\to 0}\mathcal{B}(\omega) < 0$ *such that*

   (a) $B_{\_} > \mathcal{B}(\omega)$

   $$T_t > 0, \quad \tau_t = \tau^*(\omega), \text{ and } B_t = B_{\_} \quad \forall t$$

   (b) $B_{\_} \leq \mathcal{B}(\omega)$*, the policies depend on the structure of* $P(s)$*.*

      i. For $P(s) \notin \mathcal{P}^*$

   $$\lim_t T_t > 0 \text{ i.o.}, \quad \lim_t \tau_t = \tau^*(\omega) \text{ and } \lim_t B_t = \mathcal{B}(\omega) \quad a.s$$

      ii. For $P(s) \in \mathcal{P}^*$ *we have two cases depending on* $B_{\_}$

         A. For $B_{\_} \leq B^*$

   $$T_t = 0, \quad \lim_t \tau_t = \tau^{**}(\omega), \text{ and } \lim_t B_t = B^* \quad a.s$$

         B. For $\mathcal{B}(\omega) > B_{\_} > B^*$

   $$\Pr\{\lim_t T_t = 0, \lim_t \tau_t = \tau^{**}(\omega), \lim_t B_t = B^* \text{ or } \lim_t T_t > 0 \text{ i.o.}, \lim_t \tau_t = \tau^*(\omega), \lim_t B_t = \mathcal{B}(\omega)\} > 0$$

6. More general economies

6.1. Spanning with binary shocks

Let $\Psi(s; \boldsymbol{x}, \boldsymbol{\rho}, s_{\_})$ be an optimal law of motion for the state variables for the $t \geq 1$ recursive problem, i.e., $\Psi(s; \boldsymbol{x}, \boldsymbol{\rho}, s_{\_}) = (x'(s), \rho'(s))$ solves (8) given state $(\boldsymbol{x}, \boldsymbol{\rho}, s_{\_})$.

Definition 1 *A steady state* $(\boldsymbol{x}^{SS}, \boldsymbol{\rho}^{SS})$ *satisfies* $(\boldsymbol{x}^{SS}, \boldsymbol{\rho}^{SS}) = \Psi(s; \boldsymbol{x}^{SS}, \boldsymbol{\rho}^{SS}, s_{-})$ *for all* $s, s_{\_}$*.*

Since in this steady state $\rho_i = U_c^i(s)/U_c^1(s)$ does not depend on the realization of shock s, the ratios of marginal utilities of all agents are constant. The continuation allocation depends only on s_t and not on the history s^{t-1}.

We begin by noting that a competitive equilibrium fixes an allocation $\{c_i(s), l_i(s)\}_i$ given a choice for $\{\tau(s), \boldsymbol{\rho}(s)\}$ using equations (9c), (9d) and (9e). Let us denote $U(\tau, \boldsymbol{\rho}, s)$ as the value for the planner from the implied allocation using Pareto weights $\{\omega_i\}_i$,

$$U(\tau, \boldsymbol{\rho}, s) = \sum_i \omega_i U^i(s).$$

As before define $Z_i(\tau, \rho, s)$ as

$$Z_i(\tau, \boldsymbol{\rho}, s) = U_c^i(s)c_i(s) + U_l^i(s)l_i(s) - \rho_i(s)\left[U_c^1(s)c_1(s) + U_l^1(s)l_1(s)\right].$$

For the IID case, the optimal policy solves the following Bellman equation for $\boldsymbol{x}(s^{t-1}) = \boldsymbol{x}$, $\boldsymbol{\rho}(s^{t-1}) = \boldsymbol{\rho}$

$$V(\boldsymbol{x}, \boldsymbol{\rho}) = \max_{\tau(s), \boldsymbol{\rho}'(s), \boldsymbol{x}'(s)} \sum_s \pi(s)\left[U(\tau(s), \boldsymbol{\rho}'(s), s) + \beta(s)V(\boldsymbol{x}'(s), \boldsymbol{\rho}'(s))\right] \tag{17}$$

subject to the constraints

$$Z_i(\tau(s), \boldsymbol{\rho}'(s), s) + x_i'(s) = \frac{x_i\beta^{-1}P(s)U_c^i(\tau(s), \boldsymbol{\rho}'(s), s)}{\mathbb{E}U_c^i(\tau, \rho)} \text{ for all } s, i \geq 2, \tag{18}$$

$$\sum_s \pi(s)P(s)U_c^1(\tau(s), \boldsymbol{\rho}'(s), s)(\rho_i'(s) - \rho_i) = 0 \text{ for } i \geq 2. \tag{19}$$

Constraint (19) is obtained by rearranging constraint (9b). It implies that $\rho(s)$ is a risk-adjusted martingale. We next check if the first-order necessary conditions are consistent with stationary policies for some $(\boldsymbol{x}, \boldsymbol{\rho})$.[3]

Lemma 1 *With risk aversion $\|S\| = 2$ is necessary for a steady state to exist*

Proof.

Let $\pi(s)\mu_i(s)$ and λ_i be the multipliers on constraints (18) and (19). Imposing the restrictions $x_i'(s) = x_i$ and $\rho_i'(s) = \rho_i$, at a steady state $\{\mu_i, \lambda_i, x_i, \rho_i\}_{i=2}^N$ and $\{\tau(s)\}_s$ are determined by the following equations

$$Z_i(\tau(s), \boldsymbol{\rho}, s) + x_i = \frac{\beta^{-1}P(s)x_iU_c^i(\tau(s), \boldsymbol{\rho}, s)}{\mathbb{E}U_c^i(\tau, \rho)} \text{ for all } s, i \geq 2, \tag{20a}$$

$$U_\tau(\tau(s), \boldsymbol{\rho}, s) - \sum_i \mu_i Z_{i,\tau}(\tau(s), \boldsymbol{\rho}, s) = 0 \text{ for all } s, \tag{20b}$$

$$U_{\rho_i}(\tau(s), \boldsymbol{\rho}, s) - \sum_j \mu_j Z_{j,\rho_i}(\tau(s), \boldsymbol{\rho}, s) + \lambda_i P(s)U_c^1(\tau(s), \boldsymbol{\rho}, s) - \lambda_i\beta\mathbb{E}P(s)U_c^1(\tau(s), \boldsymbol{\rho}(s), s) = 0. \text{ for all } s, i \geq 2 \tag{20c}$$

Since the shock s can take only two values, (20) is a square system in $4(N-1)+2$ unknowns $\{\mu_i^{SS}, \lambda_i^{SS}, x_i^{SS}, \rho_i^{SS}\}_{i=2}^N$ and $\{\tau^{SS}(s)\}_s$. ■

The behavior of the economy in the steady state is similar to the behavior of the complete market economy characterized by Werning (2007). Both taxes and transfers depend only on the current realization of shock s_t. Moreover, the arguments of Werning (2007) can be adapted to show that taxes are constant when preferences have a CES form $c^{1-\sigma}/(1-\sigma) - l^{1+\gamma}/(1-\gamma)$ and fluctuations in tax rates are

[3] Appendix **??** discuses the associated second order conditions that ensure these policies are optimal

very small when preferences take forms consistent with the existence of balanced growth. We return to this point after we discuss convergence properties.

The previous calculations provides a simple way to verify existence of a steady state for wide range of parameter values by checking that there exists a root for system (20). Since the system of equations (20) is non-linear, existence can generally be verified only numerically. Next, we provide a simple example with risk averse agents in which we can show existence of the root of (20) analytically. The analytical characterization of the steady state will help us develop some comparative statics and build a connection from the quasilinear economy to the quantitative analysis to appear in section **??**.

A two-agent example

Consider an economy consisting of two types of households with $\theta_{1,t} > \theta_{2,t} = 0$. One period utilities are $\ln c - \frac{1}{2}l^2$. The shock s takes two values, $s \in \{s_L, s_H\}$ with probabilities $\Pr(s|s_{\_})$ that are independent of s_{-}. We assume that $g(s) = g$ for all s, and $\theta_1(s_H) > \theta_1(s_L)$. We allow the discount factor $\beta(s)$ to depend on s.

Theorem 5 *Suppose that $g < \theta(s)$ for all s. Let $R(s)$ be the gross interest rates and $x = U_c^2(s)\left[b_2(s) - b_1(s)\right]$*

1. ***Countercyclical interest rates.*** *If $P(s_H) = P(s_L)$, then there exists a steady state $\left(x^{SS}, \rho^{SS}\right)$ such that $x^{SS} > 0,\ R^{SS}(s_H) < R^{SS}(s_L)$.*

2. ***Acyclical interest rates.*** *There exists a pair $\{P(s_H), P(s_L)\}$ such that there exists a steady state with $x^{SS} > 0$ and $R^{SS}(s_H) = R^{SS}(s_L)$.*

3. ***Procyclical interest rates.*** *There exists a pair $\{P(s_H), P(s_L)\}$ such that there exists a steady state with $x^{SS} < 0$ and $R^{SS}(s_H) > R^{SS}(s_L)$.*

In all cases, taxes $\tau(s) = \tau^{SS}$ are independent of the realized state.

6.2. Stability

In this section we extend the approximation methods used to characterize outcomes in Theorem 3 to the general problem with risk aversion. Unlike of obtaining an the quasilinear case where we could obtain analytical characterization, we present a test for convergence show local stability of a steady state for a wide range of parameters.

As before, let assume that $\pi(s)\mu_i(s)$ and λ_i be the multipliers on constraints (18) and (19). In Appendix **??** we show that the history-dependent optimal policies (they are sequences of functions of s^t)

can be represented recursively in terms of $\{\boldsymbol{\mu}(s^{t-1}), \boldsymbol{\rho}(s^{t-1})\}$ and s_t. A recursive representation of an optimal policy can be linearized around the steady state using $(\boldsymbol{\mu}, \boldsymbol{\rho})$ as state variables.[4]

Formally, let $\hat{\Psi}_t = \begin{bmatrix} \boldsymbol{\mu}_t - \boldsymbol{\mu}^{SS} \\ \boldsymbol{\rho}_t - \boldsymbol{\rho}^{SS} \end{bmatrix}$ be deviations from a steady state. From a linear approximation, one can obtain $B(s)$ such that

$$\hat{\Psi}_{t+1} = B(s_{t+1})\hat{\Psi}_t. \tag{21}$$

This linearized system has coefficients that are functions of the shock. The next proposition describes a simple numerical test that allows us to determine whether this linear system converges to zero in probability.

Theorem 6 *If the (real part) of eigenvalues of $\mathbb{E}B(s)$ are less than 1, system (21) converges to zero in mean. Further for large t, the conditional variance of $\hat{\Psi}$, denoted by $\Sigma_{\Psi,t}$, follows a deterministic process governed by*

$$vec(\Sigma_{\Psi,t}) = \hat{B}\,vec(\Sigma_{\Psi,t-1}),$$

where $\hat{B}$ is a square matrix of dimension $(2I-2)^2$. In addition, if the (real part) of eigenvalues of $\hat{B}$ are less than 1, the system converges in probability.

The eigenvalues (in particular the largest or the dominant one) are instructive not only for whether the system is locally stable but also how quickly the steady state is reached. In particular, the half-life of convergence to the steady state is given by $\frac{\log(0.5)}{\|\iota\|}$, where $\|\iota\|$ is the absolute value of the dominant eigenvalue. Thus, the closer the dominant eigenvalue is to one, the slower is the speed of convergence.

We used Theorem 6 to verify local stability of a wide range of examples. Since the parameters space is high dimensional we relegate the comparative statics to Appendix **??**. The typical finding is that the steady state is stable and that convergence is slow. The rates of convergence are increasing in the covariance of interest rates and governments needs for revenue.

7. Numerical example

8. Conclusion

[4]One could in principle look for a solution in state variables $\left(\boldsymbol{x}(s^{t-1}), \boldsymbol{\rho}(s^{t-1})\right)$. For $I = 2$ with $\{\theta_i(s)\}$ different across agents, this would give identical policies and a map which is (locally) invertible between $\boldsymbol{x}$ and $\boldsymbol{\mu}$ for a given $\boldsymbol{\rho}$. However in other cases, it turns out there are unique linear policies in $(\ bm\mu, \boldsymbol{\rho})$ and not necessarily in $(\boldsymbol{x}, \boldsymbol{\rho})$. This comes from the fact that the set of feasible $(\boldsymbol{x}, \boldsymbol{\rho})$ are restricted at time 0 and may not contain an open set around the steady state values. When we linearize using $(\boldsymbol{\mu}, \boldsymbol{\rho})$ as state variables, the optimal policies for $\boldsymbol{x}(s^t), \boldsymbol{\rho}(s^t)$ converge to their steady state levels for all perturbations in $(\boldsymbol{\mu}, \boldsymbol{\rho})$.

A Appendix

A1. Extension: Borrowing constraints

Proposition 2 *Given an initial asset distribution $\left(\{b_{i,-1}\}_i, B_{-1}\right)$, let $\{c_{i,t}, l_{i,t}\}_{i,t}$ and $\{R_t\}_t$ be a competitive equilibrium allocation and interest rate sequence in an economy without exogenous borrowing constraints. Then for any exogenous constraints $\{\underline{b}_i\}_i$, there is a government tax policy $\{\tau_t, T_t\}_t$ such that $\{c_{i,t}, l_{i,t}\}_{i,t}$ is a competitive equilibrium allocation in an economy with exogenous borrowing constraints $\left(\{b_{i,-1}, \underline{b}_i\}_i, B_{-1}\right)$ and $\{\tau_t, T_t\}_t$.*

Proof. Let $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ be a competitive equilibrium allocation without exogenous borrowing constraints. Let $\Delta_t \equiv \max_i \{\underline{b}_i - b_{i,t}\}$. Define $\hat{b}_{i,t} \equiv b_{i,t} + \Delta_t$ for all $t \geq 0$ and $\hat{b}_{i,-1} = b_{-1}$. By Theorem 1, $\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_{i,t}$ is also a competitive equilibrium allocation without exogenous borrowing constraints. Moreover, by construction $\hat{b}_{i,t} - \underline{b}_i = b_{i,t} + \Delta_t - \underline{b}_i \geq 0$. Therefore, $\hat{b}_{i,t}$ satisfies (**??**). Since agents' budget sets are smaller in the economy with exogenous borrowing constraints, and $\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_{i,t}$ are feasible at interest rate process $\{R_t\}_t$, then $\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_{i,t}$ is also an optimal choice for agents in the economy with exogenous borrowing constraints $\{\underline{b}_i\}_i$. Since all market clearing conditions are satisfied, $\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_{i,t}$ is a competitive equilibrium allocation and asset profile. ■

In this section we construct an example in which the government can achieve higher welfare in the economy with ad-hoc borrowing limits. We restrict ourselves to a deterministic economy with $g_t = 0$, $\beta_t = \beta$ and $I = 2$. Further the utility function over consumption and labor supply $U(c, l)$ is separable in the arguments and satisfies the Inada conditions. The planners problem can then be written as the following sequence problem

$$\max_{\{c_{i,t}, l_{i,t}, b_{i,t}, R_t\}_t} \sum_{t=0}^{\infty} \beta^t \left[\alpha_1 U(c_{1,t}, l_{1,t}) + \alpha_2 U(c_{2,t}, l_{2,t})\right] \tag{22}$$

subject to

$$c_{2,t} + \frac{U_{l2,t} l_{2,t}}{U_{c2,t}} - \left(c_{1,t} + \frac{U_{l1,t} l_{1,t}}{U_{c1,t}}\right) + \frac{1}{R_t}\left(b_{2,t} - b_{1,t}\right) = b_{2,t-1} - b_{1,t-1} \tag{23a}$$

$$\frac{U_{l1,t}}{\theta_1 U_{c1,t}} = \frac{U_{l2,t}}{\theta_2 U_{c2,t}} \tag{23b}$$

$$c_{1,t} + c_{2,t} \leq \theta_1 l_{1,t} + \theta_2 l_{2,t} \tag{23c}$$

$$\left(\frac{U_{ci,t}}{U_{ci,t+1}} - \beta R_t\right)\left(b_{i,t} - \underline{b}_i\right) = 0 \tag{23d}$$

$$\frac{U_{ci,t}}{U_{ci,t+1}} \geq \beta R_t \tag{23e}$$

$$b_{i,t} \geq \underline{b}_i \tag{23f}$$

Where $\underline{b}_i$ is the exogenous borrowing constraint for agent i. We obtain equation (23a) by eliminating transfers from the budget equations of the households and using the optimality for labor supply decision. Equations (23d) and (23e) capture the inter-temporal optimality conditions modified for possbily binding constraints.

Let c_i^{fb} and l_i^{fb} be the allocation that solves the first best problem, that is maximizing equation (22) subject to (23c), and define

$$Z^{fb} = c_2^{fb} + \frac{U_{l2}^{fb} l_2^{fb}}{U_{c2}^{fb}} - \left(c_1^{fb} + \frac{U_{l1}^{fb} l_1^{fb}}{U_{c1}^{fb}} \right) \tag{24}$$

and

$$\tilde{b}_2^{fb} = \frac{Z^{fb}}{\frac{1}{\beta} - 1} \tag{25}$$

We will assume that the exogenous borrowing constraints satisfy $\underline{b}_2 = \underline{b}_1 + \tilde{b}_2^{fb}$. We then have the following lemma

Lemma 2 *If* $\tilde{b}_2^{fb} > (<)0$ *and* $b_{2,-1} - b_{1,-1} > (<)\tilde{b}_2^{fb}$ *then the planner can implement the first best.*

Proof. We will consider the candidate allocation where $c_{i,t} = c_i^{fb}$, $l_{i,t} = l_i^{fb}$, $b_{i,t} = \underline{b}_i$ and interest rates are given by $R_t = \frac{1}{\beta}$ for $t \geq 1$. It should be clear then that equations (23b) and (23c) are satisfied as a property of the first best allocation. Equation (23d) is trivially satisfied since the agents are at their borrowing constraints. For $t \geq 1$ equations (23a) and (23e) are both satisfied by the choice of $R_t = \frac{1}{\beta}$ and the first best allocations. It remains to check that equation (23a) is satisfied at time $t = 0$ for an interest rate $R_0 < \frac{1}{\beta}$. At time zero the constraint is give by

$$Z^{fb} + \frac{1}{R_0}\tilde{b}_2^{fb} = b_{2,-1} - b_{1,-1} \tag{26}$$

The assumption that $b_{2,-1} - b_{1,-1} > (<)\tilde{b}_2^{fb}$ if $\tilde{b}_2^{fb} > (<)0$ then implies that

$$R_0 = \frac{\tilde{b}_2^{fb}}{b_{2,-1} - b_{1,-1} - Z^{fb}} < \frac{1}{\beta}$$

as desired. ■

This will improve upon the planners problem without exogenous borrowing constraints, as first best can only be achieved in this scenario when $b_{2,-1} - b_{1,-1} = \tilde{b}_2^{fb}$.

A2. Representative agent

This section describes the representative agent environment with risky debt and no transfers. The household values consumption and leisure using a quasi-linear utility function and solves

$$W_0(b_{-1}) \max_{\{c_t, l_t, b_t\}_t} \mathbb{E}_0 \beta^t \left\{ c_t - \frac{l_t^{1+\gamma}}{1+\gamma} \right\} \tag{27}$$

subject to

$$c_t + b_t = (1-\tau_t)\theta l_t + R_t P_t b_{t-t} \tag{28}$$

Using the optimality condition for labor and savings we can summarize the set of implementability constraints for the government as follows

$$b_{t-1} \frac{P_t}{E_{t-1} P_t} = \mathbb{E}_t \sum_j \beta^{t+j} [c_{t+j} - l_{t+j}^{1+\gamma}] \quad \forall t \tag{29}$$

We also have the feasibility constraint

$$c_t + g_t \leq \theta l_t, \tag{30}$$

and the market clearing for bonds $b_t + B_t = 0$.

The optimal Ramsey allocation solves $\max_{\{c_t, l_t\}_t} W_0(b_{-1})$ subject to (29), feasibility (30) and natural debt limits for the government $\underline{B}$[5]. For the rest of the note we assume i.i.d exogenous shocks to expenditures denote expenditure and payoffs by $g_t = g(s_t)$and $P_t = P(s_t)$ with the normalization $\mathbb{E}P(s) = 1$

Theorem 7 *Suppose $P(s)$ satisfies*

$$P(s) - P(s') > \beta \frac{g(s) - g(s')}{\underline{B}} \quad \forall \quad s, s,$$

there exists an invariant distribution of assets with unbounded support. Further for large enough assets (or debt) there is a drift towards the interior region.

In particular V is strictly concave and there exists $\underline{B} > B_1 > B_2 > -\infty$ such that

$$\mathbb{E}V'(B(s)) > V'(B_{\text{-}}) \quad B_{\text{-}} > B_2$$

and

$$\mathbb{E}V'(B(s)) < V'(B_{\text{-}}) \quad B_{\text{-}} < B_1$$

Theorem 8 *For the two special payoff structures, the optimal tax and asset policies $\{\tau_t, B_t\}_t$ are characterized as follows,*

[5]These will be explicitly derived for the examples we solve in this section.

1. *If* $P(s) = 1$

$$\lim_t \tau_t = -\infty, \quad \lim_t B_t = \infty \quad a.s$$

2. *If* $P(s) = 1 + \frac{\beta}{B^*}(g(s) - \mathbb{E}g)$ *for some* $B^* \geq \underline{B}$

$$\lim_t \tau_t = \tau^* > \infty, \quad \lim_t B_t = B^* \quad a.s \quad \forall B_{-1}$$

Remark 1 *In the case where payoffs are perfectly correlated with expenditure shocks (case 2), we can express the long run assets*

$$B^* = \beta \frac{\mathrm{var}(g(s))}{\mathrm{cov}(P(s), g(s))}$$

. Keeping tax rates (and hence tax revenues in this case) the government needs to finance a higher primary deficit when it gets positive expenditure shock. If in such states the assets pays off more, then optimally the government holds positive assets and uses the these high returns to finance this deficit. On the other hand if payoff are lower in times when the government needs resources, holding debt is valuable since it lowers the interest burden. Thus using the level of its assets B^* *it can perfectly span the fluctuations in deficits and the sign is given by the sign of the covariance of* $P(s)$ *with* $g(s)$

The long run tax rate is inversely related to B^* *with the following limits,*

$$\lim_{B^* \to \underline{B}} \tau^* = \frac{\gamma}{1+\gamma} \qquad \lim_{B^* \to \infty} \tau^* = -\infty$$

Corollary 2 *Let* $\underline{B} > -\infty$ *be the natural debt limit for the government. Suppose we impose an upper bound on assets* $\overline{B} < \infty$,

1. *If* $P(s) = 1 + \frac{\beta}{B^*}(g(s) - \mathbb{E}g)$ *for some* $\overline{B} > B^* \geq \underline{B}$

$$\exists \epsilon > 0, \quad \Pr\{B_t < \underline{B} + \epsilon \textit{ or } B_t > \overline{B} - \epsilon\} = 0$$

2. *For all other payoffs*

$$\forall \epsilon > 0, \quad \Pr\{B_t < \underline{B} + \epsilon \textit{ and } B_t > \overline{B} - \epsilon\} = 1$$

Theorem 9 *Consider a orthogonal decomposition of* $P(s)$ *as follows*

$$P(s) = \hat{P}(s) + P^*(s)$$

where

$$P^*(s) = 1 + \frac{\beta}{B^*}(g - \mathbb{E}g) \textit{ for some } B^* \geq \underline{B}$$

and $\hat{P}(s)$ is orthogonal to $g(s)$. Expanding the policy rules around the steady state of the $P^*(s)$ economy we have the following characterization,

- *The ergodic distribution of debt of the policy rules linearized around $(B^*, P^*(s))$ will have mean B^*,*

- *The coefficient of variation is given by*

$$\frac{\sigma(B)}{\mathbb{E}(B)} = \sqrt{\frac{\text{var}(P(s)) - |\text{cov}(g(s), P(s))|}{(1 + |\text{cov}(g(s), P(s))|)|\text{cov}(g(s), P(s))|}} \leq \sqrt{\frac{\text{var}(P(s)) - |\text{cov}(g(s), P(s))|}{|\text{cov}(g(s), P(s))|}}$$

- *The speed of convergence to the ergodic distribution described by*

$$\frac{\mathbb{E}_{t-1}(B_t - B^*)}{(B_{t-1} - B^*)} = \frac{1}{1 + |\text{cov}(P(s), g(s))|}$$

A3. Heterogeneous agent: Quasi-linear unproductive agent

Suppose the unproductive agent has quasi-linear preferences and we additionally impose a non negativity constraint on his consumption.

Theorem 10 *Let ω, n be the Pareto weight and mass of the productive agent with $n < \frac{\gamma}{1+\gamma}$. The government trades a risky bond with payoffs $P(s)$ and faces i.i.d expenditure shocks. The optimal tax, transfer and asset policies $\{\tau_t, T_t, B_t\}$ are characterized as follows,*

1. *For $\omega \geq n\left(\frac{1+\gamma}{\gamma}\right)$ we have $T_t = 0$ and the optimal policy is same as in a representative agent economy studied in theorems 7, 8 and 9*

2. *For $\omega < n\left(\frac{1+\gamma}{\gamma}\right)$, suppose we further assume that $\min_s\{P(s)\} > \beta$. We have two parts:*

   There exists $\mathcal{B}(\omega)$ and $\tau^*(\omega)$ with $\mathcal{B}'(\omega) > 0$ and $\lim_{\omega \to 0} \mathcal{B}(\omega) < 0$ such that

   (a) $B_- > \mathcal{B}(\omega)$

   $$T_t > 0, \quad \tau_t = \tau^*(\omega), \text{ and } B_t = B_- \quad \forall t$$

   (b) $B_- \leq \mathcal{B}(\omega)$, the policies depend on the structure of $P(s)$.

   i. For a risky bond with $P(s) = 1 + \frac{\beta}{B^*}(g(s) - \mathbb{E}g)$ for some $B^* < \mathcal{B}(\omega)$ we have two cases depending on B_-

   A. For $B_- \leq B^*$

   $$T_t = 0, \quad \lim_t \tau_t = \tau^{**}(\omega), \text{ and } \lim_t B_t = B^* \quad a.s$$

B. *For* $\mathcal{B}(\omega) > B_{-} > B^*$

$$\Pr\{\lim_t T_t = 0, \lim_t \tau_t = \tau^{**}(\omega), \lim_t B_t = B^* \textit{ or } \lim_t T_t > 0 \textit{ i.o.}, \lim_t \tau_t = \tau^*(\omega), \lim_t B_t = \mathcal{B}(\omega)\} > 0$$

ii. *For all other payoffs*

$$\lim_t T_t = 0 \textit{ i.o.}, \quad \lim_t \tau_t = \tau^*(\omega) \textit{ and } \lim_t B_t = \mathcal{B}(\omega) \quad \textit{a.s}$$

B Appendix

Part 1 of Theorem 2

Proof. The optimal Ramsey plan solves the following Bellman equation. Let $V(b_{\_})$ be the maximum ex-ante value the government can achieve with debt $b_{\_}$.

$$V(b_{\_}) = \max_{c(s),l(s),b(s)} \sum_s \pi(s) \left\{ c(s) - \frac{l(s)^{1+\gamma}}{1+\gamma} + \beta V(b(s)) \right\} \tag{31}$$

subject to

$$c(s) + b(s) = l(s)^{1+\gamma} + \beta^{-1} P(s) b_{\_} \tag{32a}$$

$$c(s) + g(s) \leq \theta l(s) \tag{32b}$$

Let $\bar{b} = -\underline{B}$

$$\underline{b} \leq b(s) \leq \bar{b} \tag{32c}$$

Lemma 3 *There exists a $\bar{b}$ such that $b_t \leq \bar{b}$. This is the natural debt limit for the government.*

Proof. As we drive μ to $-\infty$, the tax rate approaches a maximum limit, $\bar{\tau} = \frac{\gamma}{1+\gamma}$. In state s, the government surplus,

$$S(s,\tau) = \theta^{\frac{\gamma}{1+\gamma}} (1-\tau)^{\frac{1}{\gamma}} \tau - g(s),$$

which is maximized at $\tau = \frac{\gamma}{1+\gamma}$ when $(1-\tau)^{\frac{1}{\gamma}}\tau$ is also maximized. This would impose a natural borrowing limit for the government.

■

From now we assume that $\bar{b}$ represents the natural borrowing limit. We begin with some useful lemmas

let $L \equiv l^{1+\gamma}$., To make this problem convex,

Substitute for $c(s)$

$$V(b_{\_}) = \max_{L(s),b(s)} \sum_{s\in S} \pi(s) \left[\frac{1}{1+\gamma} L(s) + \frac{1}{\beta} P(s) b_{\_} - b(s) + \beta V(b(s)) \right]$$

s.t.

$$
\begin{aligned}
\frac{1}{\beta} P(s) b - b(s) + g(s) &\leq \theta L^{\frac{1}{1+\gamma}}(s) - L(s) \\
b(s) &\leq \bar{b} \\
L(s) &\geq 0.
\end{aligned}
$$

Lemma 4 *$V(b)$ is stictly concave, continuous, differentiable and $V(b) < \beta^{-1}$ for all $b < \bar{b}$. The feasibility constraint binds for all $b \in (-\infty, \bar{b}],\ s \in S$ and $(L^*(s))^{1-\frac{1}{1+\gamma}} \geq \frac{1}{1+\gamma}$.*[6]

Proof. *Concavity*

$V(b)$ is concave because we maximize linear objective function over convex set.

Binding feasibility

Suppose that feasibility does not bind for some b, s. Then the optimal $L(s)$ solve $\max_{L(s)\geq 0} \pi(s) \frac{\gamma}{1+\gamma} L(s)$ which sets $L(s) = \infty$. This violates feasility for any finite $b, b(s)$.

Bounds on L

Let $\phi(s) > 0$ be a Lagrange multiplier on the feasibility. The FOC for $L(s)$ is

$$\frac{1}{1+\gamma} + \phi(s)\left(\frac{1}{1+\gamma} L(s)^{\frac{1}{1+\gamma}-} - \theta\right) = 0.$$

This gives

$$\frac{1}{1+\gamma} L^{\frac{1}{1+\gamma}-1} - \theta = -\frac{1}{\lambda}\frac{\gamma}{1+\gamma} < 0$$

or

$$L^{1-\frac{1}{1+\gamma}} \geq \frac{\theta}{1+\gamma}.$$

Continuity

For any L that satisfy $L^{1-\frac{}{1+\gamma}} \geq \frac{\theta}{1+\gamma}$, define function Ψ that satisfies $\Psi\left(L^{\frac{1}{1+\gamma}} - \theta L\right) = L$. Since $L^{\frac{1}{1+\gamma}} - L$ is strictly decreasing in L for $L^{1-\frac{1}{1+\gamma}} \geq \frac{1}{1+\gamma}$, this function is well defined. Note that $\Psi() \underbrace{\left(\frac{1}{1+\gamma} L^{\frac{1}{1+\gamma}-1} - \theta\right)}_{<0} = 1$ (so that $\Psi > 0$, i.e. Ψ is strictly decreasing) and $\Psi'' \underbrace{\left(\frac{1}{1+\gamma} L^{\frac{1}{1+\gamma}-1} - 1\right)^2}_{>0} + \underbrace{\Psi}_{<0} \underbrace{\frac{1}{1+\gamma}\frac{\gamma}{1+\gamma} L^{\frac{1}{1+\gamma}-2}}_{<0} = 0$ (so that $\Psi'' \geq 0$, $\Psi'' > 0$, i.e. Ψ is strictly concave on the interior). Ψ is also continuous. When

[6] This last condition simply means that we do not tax to the right of the peak of the Laffer curve. The revenue maximizing tax is $1 - \bar{\tau} = \frac{1}{1+\gamma}$. At the same time $1 - \tau = l^\gamma$ so if taxes are always to the left of the peak, $\frac{1}{1+\gamma} \leq l^\gamma = \left(L^{\frac{1}{1+\gamma}}\right)^\gamma = L^{1-\frac{1}{1+\gamma}}$.

$L^{1-\frac{1}{1+\gamma}} = \frac{1}{1+\gamma}, L = (1+\gamma)^{-\frac{(1+\gamma)}{(\gamma)}}$. Let $D \equiv (1+\gamma)^{\frac{-1}{\gamma}-(1+\gamma)^{-\frac{1+\gamma}{(\gamma)}}}$. Then the objective is

$$V(b_{-}) = \max_{b(s)} \sum_{s\in S} \pi(s) \left[\Psi\left(\frac{1}{\beta}P(s)b - b(s) + g(s)\right) + \frac{1}{\beta}P(s)b_{-} - b(s) + \beta V(b(s)) \right]$$

s.t.

$$\begin{aligned} b(s) &\leq \bar{b} \\ \frac{1}{\beta}P(s)b_{-} - b(s) + g(s) &\leq D. \end{aligned}$$

This function is continuous so V is also continuous.

Differentiability

Continuity and convexity implies differentiability everywhere, including the boundaries.

Strict concavity

Ψ is strictly concave, so on the interior V is strictly concave.

■

Next we characterize policy functions

Lemma 5 $b(b_{-}, s)$ *is an increasing function of* b *for all* s *for all* (b_{-}, s) *where* $b(s)$ *is interior.*

Proof. Take the FOCs for $b(s)$ from the condition in the previous problem. If $b(s)$ is interior

$$\Psi\left(\frac{1}{\beta}P(s)b_{-} - b(s) + g(s)\right) = \beta V(b(s)).$$

Suppose $b_1 < b_2$ but $b_2(s) < b_1(s)$. Then from stict concavity

$$\begin{aligned} V(b_2(s)) &< V'(b_1(s)) \\ \Psi\left(\frac{1}{\beta}P(s)b_2 - b_2(s) + g(s)\right) &> \Psi\left(\frac{1}{\beta}P(s)b_1 - b_1(s) + g(s)\right). \end{aligned}$$

■

Lemma 6 *There exists an invariant distribution of the stochastic process* $b_{t+1} = b(s_{t+1}, b_t)$

Proof. The state spaces for b_t and s_t are compact. Further the transition function on $s_{t+1}|s_t$ is trivially increasing under i.i.d shocks. We can apply standard arguments as in **?**(see corollary 3) to argue that there exists invariant distribution of assets. ■

Now we characterize the support of this distribution using further properties of the bolicy rules for $b(s|b_{-})$

Lemma 7 *For any $b_{-} \in (\underline{b}, \bar{b})$, there are s, s'' s.t. $b(s) \geq b_{-} \geq b(s'')$. Moreover, if there are any states s'', s''' s.t. $b(s'') \neq b(s''')$, those inequalities are strict.*

Proof. The FOCs together with the envelope theorem imply that $\mathbb{E}P(s)V'(b(s)) = V'(b_{-}) + \kappa(s)$ We can rewrite this as $\tilde{\mathbb{E}}V'(b(s)) = b + \kappa(s)$ with $\tilde{\pi}(s) = P(s)\pi(s)$

Now if there is at least one $b(s)$ s.t. $b(s) > b_{-}$, by strict concavity of V there must be some s'' s.t. $b(s'') < b$.

If there is at least one $b(s)$ s.t. $b(s) < b_{-}$, the inequality above is strictly only if $b(s''') = \bar{b}$ for some s'''. But $V(\bar{b}) < V(b)$ so there must be some s'' s.t. $b(s'') > b$. Equality is possible only if $b_{-} = b(s)$ for all s. ■

Lemma 8 *Let $\mu(b, s)$ be the optimal policy function for the Lagrange multiplier $\mu(s)$. If $P(s') > P(s'')$ then there exists a $b^*_{s',s''}$ such that for all $b < (>)\ b_{1,s',s''}$ we have $\mu(b, s') > (<)\ \mu(b, s'')$. If $\underline{b} < b^*_{s',s''} < \bar{b}$ then $\mu(b^*_{s',s''}, s') = \mu(b^*_{s',s''}, s'')$.*

Proof. Suppose that $\mu(b, s') \leq \mu(b, s'')$. Subtracting the implementability for s'' from the implementability constraint for s' we have

$$\begin{aligned}
\frac{P(s') - P(s'')}{\beta} b &= S_{s'}(\mu(b, s')) - S_{s''}(\mu(b, s'')) + b'(b, s') - b'(b, s'') \\
&\geq S_{s'}(\mu(b, s')) - S_{s''}(\mu(b, s')) + b'(b, s') - b'(b, s'') \\
&\geq S_{s'}(\mu(b, s')) - S_{s''}(\mu(b, s')) = g(s'') - g(s')
\end{aligned}$$

We get the first inequality from noting that $S_s(\mu') \geq S_s(\mu'')$ if $\mu' \leq \mu''$. We obtain the second inequality by noting that $\mu(b, s') \leq \mu(b, s'')$ implies $b'(b, s') \geq b'(b, s'')$ (which comes directly from the concavity of V). Thus, $\mu(b, s') \leq \mu(b, s'')$ implies that

$$b \geq \frac{\beta(g(s'') - g(s'))}{P(s') - P(s'')} = b^*_{s',s''} \tag{33}$$

The converse of this statement is that if $b < b^*_{s',s''}$ then $\mu(b, s') > \mu(b, s'')$. The reverse statement that $\mu(b, s') \geq \mu(b, s'')$ implies $b \leq b^*_{s,s'}$ follows by symmetry. Again, the converse implies that if $b > b^*_{s',s''}$ then $\mu(b, s') < \mu(b, s'')$. Finally, if $\underline{b} < b^*_{s',s''} < \bar{b}$ then continuity of the policy functions implies that there must exist a root of $\mu(b, s') - \mu(b, s'')$ and that root can only be at $b^*_{s',s''}$. ■

Lemma 9 *$P \in \mathcal{P}^*$ is necessary and sufficient for existence of b^* such that $b(s, b^*) = b*$ for all ss*

Proof. The necessary part follows from taking differences of the (32a) for s',s''. We have

$$[P(s) - P(s'')]\frac{b^*}{\beta} = g(s) - g(s'')$$

Thus $P \in \mathcal{P}^*$. The sufficient part follows from the Lemma 8. If $P \notin \mathcal{P}^*$, equation 33 that defines $b^*_{s',s''}$ will not be same across all pairs. Thus b^* that satisfies $b(s; b^*)$ independent of s will not exist. ■

Lemma 9 implies that under the hypothesis of part 1 of the Theorem 2 there cannot exist an interior absorbing point for the dynamics of debt. This allows us to construct a sequences $\{b_t\}_t$ such that $b_t < b_{t+1}$ with the property that $\lim_t b_t = \underline{b}$. Thus, for any $\epsilon > 0$, there exists a finite history of shocks that can take us arbitrarily close to $\underline{b}$. Since the shocks are i.i.d this finite sequence will repeat i.o. With a symmetric argument we can show that b_t will come arbitrarily close to its upper limit i.o too

Part 2 of Theorem **2**

In this first section we will show that there exists b_1, and if p is sufficiently volatile a b_2, such that if $b_t \leq b_1$ then

$$\mu_t \geq \mathbb{E}_t \mu_{t+1}$$

and if $b_t \geq b_2$ then

$$\mu_t \leq \mathbb{E}_t \mu_{t+1}.$$

Recall that b is decreasing in μ, so this implies that if b_t is low (large) enough then there will exist a drift away from the lower (upper) limit of government debt.

With Lemma 8 we can order the policy functions $\mu(b, \cdot)$ for particular regions of the state space. Take b_1 to be

$$b_1 = \min \left\{ b^*_{s',s''} \right\}$$

and WLOG choose $\underline{b} < b_1$. For all $b < b_1$ we have shown that $P(s) > P(s')$ implies that $\mu(b, s) > \mu(b, s')$. The FOC for the problem imply,

$$\mu_t = \mathbb{E}_t p_{t+1} \mu_{t+1} + \underline{\kappa}_t \tag{34}$$

The inequality in the resource constraint implies that $\xi(s) \geq 0$ implying that $\mu(s) \leq 1$. With some minor algebra algebra we obtain

By decomposing $\mathbb{E}\mu_{t+1} p_{t+1}$ in equation (34), we obtain (using $\mathbb{E}_t p_{t+1} = 1$)

$$\mu_t = \mathbb{E}\mu_{t+1} + \text{cov}_t(\mu_{t+1}, p_{t+1}) + \kappa_t \tag{35}$$

Our analysis has just shown that for $b_t < b_1$ we have $\text{cov}_t(\mu_{t+1}, p_{t+1}) > 0$ so

$$\mu_t > \mathbb{E}_t \mu_{t+1}.$$

If p is sufficiently volatile:

$$P(s') - Ps'' > \frac{\beta(g_{s''} - g_{s'})}{\bar{b}}$$

then

$$b_2 = \max\left\{b^*_{s',s''}\right\} < \bar{b}$$

and through a similar argument we can conclude that $\text{cov}_t(\mu_{t+1}, p_{t+1}) < 0$

$$\mu_t < \mathbb{E}_t \mu_{t+1}$$

for $b_t > b_2$ (note $b_t > \underline{b}$ implies $\kappa_t = 0$) which gives us a drift away from the upper-bound.

Part 3 of Theorem 2

When $P \in \mathcal{P}^*$, Lemma 9 implies existence of b^* as the steady state debt level.

Lemma 10 *There exists μ^* such that μ_t is a sub-martingale bounded above in the region $(-\infty, \mu^*)$ and super-martingale bounded below in the region $(\mu^*, \frac{1}{1+\gamma})$*

Proof. Let μ^* be the associated multiplier, i.e $V_b(b^*) = \mu^*$. Using the results of the previous section, we have that $b_1 = b_2 = b^*$, implying that $\mu_t < (>)\mathbb{E}_t\mu_{t+1}$ for $b_t < (>)b^*$. ■

Lastly we show that $\lim_t \mu_t = \mu^*$. Suppose $b_t < b^*$, we know that $\mu_t > \mu^*$. The previous lemma implies that in this region, μ_t is a super martingale. The lemma 5 shows that $b(b_-, s)$ is continuous and increasing. This translates into $\mu(\mu(b_-), s)$ to be continuous and increasing as well. Thus

$$\mu_t > \mu^* \implies \mu(\mu_t, s_{t+1}) > \mu(\mu^*, s_{t+1})$$

or

$$\mu_{t+1} > \mu^*$$

Thus $\mu*$ provides a lower bound to this super martingale. Using standard martingale convergence theorem converges. The uniqueness of steady state implies that it can only converge to μ^*. For $\mu < \mu^*$, the argument is symmetric.

■

Theorem 2

Working with the first order conditions of problem 31, we obtain

$$l(s)^{\gamma} = \frac{\mu(s) - 1}{(1+\gamma)\mu(s) - 1} = 1 - \tau(\mu(s)),$$

implying the relationship between tax rate τ and multiplier μ given by

$$\tau(\mu) = \frac{\gamma\mu}{(1+\gamma)\mu - 1} \tag{36}$$

The rest of the first order conditions are summarized below

$$\begin{aligned} \frac{b_{-}P(s)}{\beta} &= S(\mu(s), s) + b(s) \\ V'(b) &= \mathbb{E}P(s)\mu(s) \\ \mu(s) &= V'(b(s)) \end{aligned}$$

where $S(\mu, s)$ is the government surplus in state s given by

$$S(\mu, s) = (1 - \tau(\mu))^{\frac{1}{\gamma}}\tau(\mu) - g(s) = I(\mu) - g(s)$$

Given a pair $\{P(s), g(s)\}$, since $V'(b)$ is one-to-one, so we can re-characterize these equations as searching for a function $b(\mu)$ and $\mu(s|\mu_{-})$such that the following two equations can be solved for all μ_{-}.

$$\frac{b(\mu_{-})P(s)}{\beta} = I(\mu(s)) - g(s) + b(\mu(s)) \tag{37}$$

$$\mu = \mathbb{E}\mu(s)P(s) \tag{38}$$

Usually perturbation approaches to solve equilibrium conditions as above look for the solutions to $\{\mu(s|\mu_{-})\}$ and $b(\mu_{-})$ around deterministic steady state of the model. Thus for any b^{ss}, there exists a μ^{ss} that will solve

$$\frac{b^{SS}}{\beta} = I(\mu^{SS}) - \bar{g} + b^{SS}$$

For example if we set the perturbation parameter q to scale the shocks, $g(s) = \mathbb{E}g(s) + q\Delta_g(s)$ and $P(s) = 1 + q\Delta_P(s)$, the first order expansion of $\mu(s|\mu_{-})$ will imply that it is a martingale. Such approximations are not informative about the ergodic distribution. [7]

In contrast we will approximate the functions $\mu(s|\mu_{-})$ around economy with payoffs in $\bar{P} \in \mathcal{P}^*$.

[7] One can do higher order approximations, but part 3 of theorem 2 hints that for economies with payoffs close to $\mathcal{P}^*$, the stochastic steady state in general is far away from μ^{SS}.

These payoffs support some complete market allocation as described in the text.

To proceed we a) explicitly recognize that policy rules depend on payoffs: $\mu(s|\mu_{-}, \{P(s)\}_s)$ and $b(\mu_{-}, \{P(s)\}_s)$ and then take take the first order expansion with respect to both μ_{-} and $\{P(s)\}$ around the vector $(\bar{\mu}, \{\bar{P}(s)\}_s)$. Note that $\overline{\mu}$, or the complete markets multiplier on the implementability constraint solves

$$\bar{b} = \frac{\beta}{1-\beta}\left(I(\overline{\mu}) - \overline{g}\right) \tag{39a}$$

where $\overline{g} = \mathbb{E}g$ and $\overline{p}$ as

$$\overline{P}(s) = 1 + \frac{\beta}{\bar{b}}(g(s) - \overline{g}) \tag{39b}$$

As noted before $b(\overline{\mu}; \overline{p}) = \bar{b}$ solves the the system of equations (37-38) for $\mu'(s) = \overline{\mu}$.

We next describe the details of the linearization and then the choice of a particular complete market economy as the point of approximation. In particular we will solve for derivatives, $\frac{\delta\mu(s|\mu_{-},\{P(s)\}}{\delta\mu_{-}}, \frac{\delta\mu(s|\mu_{-},\{P(s)\}}{\delta P(s)}$ and similarly for $b(\mu_{-}, \{P(s)\}$.

Differentiating equation (37) with respect to μ around $(\overline{\mu}, \overline{P})$ we obtain

$$\frac{\overline{P}(s)}{\beta}\frac{\partial b}{\partial \mu_{-}} = \left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu_{-}}\right]\frac{\partial \mu(s)}{\partial \mu_{-}}.$$

Differentiating equation (38) with respect to μ_{-} we obtain

$$1 = \sum_s \pi(s)\overline{P}(s)\frac{\partial \mu'(s)}{\partial \mu_{-}}$$

combining these two equations we see that

$$\frac{1}{\beta}\left(\sum_s \pi(s)\overline{P}(s)^2\right)\frac{\partial b}{\partial \mu_{-}} = I'(\overline{\mu}) + \frac{\partial b}{\partial \mu_{-}}$$

Noting that $\mathbb{E}\overline{P}^2(s) = 1 + \frac{\beta^2}{\bar{b}^2}\sigma_g^2$ we obtain

$$\frac{\partial b}{\partial \mu_{-}} = \frac{I'(\overline{\mu})}{\frac{\beta}{\bar{b}^2}\sigma_g^2 + \frac{1-\beta}{\beta}} < 0 \tag{40}$$

as $I'(\overline{\mu}) < 0$. We then have directly that

$$\frac{\partial \mu'(s)}{\partial \mu} = \frac{\overline{P}(s)}{\frac{\beta^2}{\bar{b}^2}\sigma_g^2 + 1} = \frac{\overline{P}(s)}{\mathbb{E}\overline{P}(s)^2} \tag{41}$$

We can perform the same procedure for $P(s)$. Differentiating equation (37) with respect to $P(s)$ we around $(\overline{\mu}, \overline{P})$ we obtain

$$\frac{\overline{P}(s')}{\beta}\frac{\partial b}{\partial P(s)} + 1_{s,s'}\frac{\overline{b}}{\beta} - \frac{\pi(s)\overline{b}\overline{P}(s')}{\beta} = \left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu}\right]\frac{\partial \mu(s')}{\partial P(s)} \tag{42}$$

Here $1_{s,s'}$ is 1 if $s = s'$ and zero otherwise. Differentiating equation (38) with respect to $P(s)$ we obtain

$$0 = \pi(s)\overline{\mu} - \pi(s)\overline{\mu} + \sum_{s'}\pi(s)\overline{P}(s')\frac{\partial \mu(s')}{\partial P(s)} = \sum_{s'}\pi(s')\overline{P}(s')\frac{\partial \mu(s')}{\partial P(s)}$$

Again we can combine these two equations to give us

$$\frac{\mathbb{E}\overline{P}(s)^2}{\beta}\frac{\partial b}{\partial P(s)} + \frac{\pi(s)\overline{b}}{\beta}(\overline{P}(s) - \mathbb{E}\overline{P}(s)^2) = 0$$

or

$$\frac{\partial b}{\partial P(s)} = \pi(s)\overline{b}\frac{\mathbb{E}\overline{P}^2 - \overline{P}(s)}{\mathbb{E}\overline{P}^2} \tag{43}$$

Going back to equation (42) we have

$$\frac{\partial \mu(s')}{\partial P(s)} = \frac{\overline{b}}{\beta\left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu}\right]}\left(1_{s,s'} - \frac{\pi(s)\overline{P}(s)\overline{P}(s')}{\mathbb{E}\overline{P}^2}\right) \tag{44}$$

For an arbitrary $\left(\overline{\mu}, \{\overline{P}(s)\}_s\right)$, using the derivatives that we computed, we can characterize the dynamics of $\hat{\mu} \equiv \mu_t - \overline{\mu}$ using our approximated policies.

$$\hat{\mu}_{t+1} = B\hat{\mu}_t + C,$$

where $B(s)$ and $C(s)$ are respective derivatives. Note that both are random variables and let us denote their means $\overline{B}$ and $\overline{C}$, and variances σ_B^2 and σ_C^2 . Suppose that $\hat{\mu}$ is distributed according to the ergodic distribution of this linear system with mean $\mathbb{E}\hat{\mu}$ and variance σ_μ^2. Since

$$B\hat{\mu} + C,$$

has the same distribution we can compute the mean of this distribution as

$$\begin{aligned}
\mathbb{E}\hat{\mu} &= \mathbb{E}\left[B\hat{\mu} + C\right] \\
&= \mathbb{E}\left[\mathbb{E}_{\hat{\mu}}\left[B\hat{\mu} + C\right]\right] \\
&= \mathbb{E}\left[\overline{B}\hat{\mu} + \overline{C}\right] \\
&= \overline{B}\mathbb{E}\hat{\mu} + \overline{C}
\end{aligned}$$

solving for $\mathbb{E}\hat{\mu}$ we get

$$\mathbb{E}\hat{\mu} = \frac{\overline{C}}{1-\overline{B}} \tag{45}$$

For the variance $\sigma^2_{\hat{\mu}}$ we know that

$$\sigma^2_{\hat{\mu}} = \text{var}(B\hat{\mu} + C) = \text{var}(B\hat{\mu}) + \sigma^2_C + 2\text{cov}(B\hat{\mu}, C)$$

Computing the variance of $B\hat{\mu}$ we have

$$\begin{aligned}
\text{var}(B\hat{\mu}) &= \mathbb{E}\left[(B\hat{\mu} - \overline{B}\mathbb{E}\hat{\mu})^2\right] \\
&= \mathbb{E}\left[(B\hat{\mu} - \overline{B}\hat{\mu} + \overline{B}\hat{\mu} - \overline{B}\mathbb{E}\hat{\mu})^2\right] \\
&= \mathbb{E}\left[\mathbb{E}_{\hat{\mu}}\left[(B-\overline{B})^2\hat{\mu}^2 + 2(B-\overline{B})(\hat{\mu} - \mathbb{E}\hat{\mu})\overline{B}\mathbb{E}\hat{\mu} + (\hat{\mu} - \mathbb{E}\hat{\mu})^2\bar{B}^2\right]\right] \\
&= \mathbb{E}\left[\sigma^2_B\hat{\mu}^2 + (\hat{\mu} - \mathbb{E}\hat{\mu})^2\overline{B}\right] \\
&= \sigma^2_B(\sigma^2_{\hat{\mu}} + (\mathbb{E}\hat{\mu})^2) + \sigma^2_{\hat{\mu}}\overline{B}^2
\end{aligned}$$

while for the covariance of $B\hat{\mu}$ and C

$$\text{cov}(B\hat{\mu}, C) = \sigma_{BC}\mathbb{E}\hat{\mu}$$

Putting this all together we have

$$\sigma^2_{\hat{\mu}} = \frac{\sigma^2_B(\mathbb{E}\hat{\mu})^2 + \sigma_{BC}\mathbb{E}\hat{\mu} + \sigma^2_C}{1 - \overline{B}^2 - \sigma^2_B} \tag{46}$$

To get the expressions in Theorem 2, we finally choose a particular $\overline{P} = P^*(s) \in \mathcal{P}^*$. This will be the closest complete market economy to our the given $P(s)$ in L^2 sense. Formally,

$$\min_{\tilde{P} \in \mathcal{P}^*} \sum_s \pi(s)(P(s) - \tilde{P}(s))^2.$$

Since all payoffs in $\mathcal{P}^*$ are associated with some b^* and μ^* via equations (39), we can re write the above problem as choosing $\overline{\mu}$ so as to minimize the variance of the difference between $P(s)$ and the set of steady state payoffs. Let P^* be the solution to this minimization problem. The first order condition for this linearization gives us

$$2\sum_{s'} \pi(P(s') - P^*(s', \mu^*))\frac{\delta P^*(s, \mu^*)}{\delta \mu^*} = 0$$

as noted before

$$P^*(s) = 1 - \frac{\beta}{b^*(\mu^*)}\left(g(s) - \mathbb{E}g\right)$$

thus

$$\frac{\delta P^*}{\delta \mu^*} \propto P^* - 1$$

Thus we can see the the optimal choice of $\overline{\mu}$ is equivalent to choosing $\overline{\mu}$ such that

$$\begin{aligned}
0 &= \sum_{s'} \Pi_{s'}(P(s') - P^*(s',\mu^*))(P^*(s',\mu^*) - 1) \\
&= -\sum_{s'} \Pi_{s'}(P(s') - P^*(s',\mu^*)) + \sum_{s'} \Pi_{s'}(P(s') - P^*(s',\mu^*))P^*(s',\mu^*) \\
&= \sum_{s'} \Pi_{s'}(P(s') - P^*(s',\mu^*))P^*(s',\mu^*) \\
&= \mathbb{E}\left[(P - P^*)P^*\right]
\end{aligned} \tag{47}$$

At these values of $\overline{P} = P^*$ and $\overline{\mu} = \mu^*$ we have that C for our linearized system is

$$C(s') = \sum_s \left\{ \frac{b^*}{\beta\left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu}\right]} \left(1_{s,s'} - \frac{\pi(s)P^*(s)P^*(s')}{\mathbb{E}\overline{P}^2}\right)(P(s) - P^*(s)) \right\}$$

Taking expectations we have that

$$\begin{aligned}
\overline{C} &= \sum_s \left\{ \frac{b^*}{\beta\left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu}\right]} \left(\pi(s) - \frac{\pi(s)P^*(s)}{\mathbb{E}\overline{P}^2}\right)(P(s) - P^*(s)) \right\} \\
&= \frac{b^*}{\beta\left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu}\right]} \left(\mathbb{E}(P - \overline{P}) - \frac{\mathbb{E}\left[(P - \overline{P})\overline{P}\right]}{\mathbb{E}\overline{P}^2}\right) \\
&= 0
\end{aligned} \tag{48}$$

Thus the linearized system will have the same mean for μ, $\overline{\mu}$, as the closest approximating steady state payoff structure.

We can also compute the variance of the ergodic distribution for μ. Note

$$\begin{aligned}
C(s') &= \sum_s \left\{ \frac{b^*}{\beta\left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu}\right]} \left(1_{s,s'} - \frac{\pi(s)P^*(s)P^*(s')}{\mathbb{E}{P^*}^2}\right)(P(s) - P^*(s)) \right\} \\
&= \frac{b^*}{\beta\left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu}\right]} \left(P(s') - P^*(s') - P^*(s')\frac{\sum_s \pi(s)P^*(s)(p_s - P^*(s))}{\mathbb{E}{P^*}^2}\right) \\
&= \frac{b^*}{\beta\left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu}\right]}(P(s') - P^*(s))
\end{aligned}$$

As noted before

$$\sigma_\mu^2 = \frac{{b^*}^2}{\beta^2\left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu}\right]^2\left(1 - \overline{B}^2 - \sigma_B^2\right)} \|P - P^*\|^2$$

The variance of government debt in the linearized system is

$$\sigma_b^2 = \frac{{b^*}^2 \left(\frac{\partial b}{\partial \mu}\right)^2}{\beta^2 \left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu}\right]^2 \left(1 - \overline{B}^2 - \sigma_B^2\right)} \|P - P^*\|^2$$

This can be simplified using the following expressions:

$$I'(\overline{\mu}) + \frac{\partial b}{\partial \mu} = \frac{\mathbb{E}{P^*}^2}{\beta} \frac{\partial b}{\partial \mu},$$

$$\overline{B} = \frac{1}{\mathbb{E}{P^*}^2}$$

and

$$\sigma_B^2 = \frac{\mathrm{var}(P^*)}{(\mathbb{E}{P^*}^2)^2}$$

to

$$\sigma_b^2 = \frac{{b^*}^2}{\mathbb{E}{P^*}^2 \mathrm{var}(P^*)} \|P - P^*\|^2 \tag{49}$$

Noting that $\mathbb{E}{P^*}^2 = 1 + \mathrm{var}(P^*) > 1$, we have immediately that up to first order the relative spread of debt is bounded by

$$\frac{\sigma_b}{b^*} \leq \sqrt{\frac{\|P - P^*\|^2}{\mathrm{var}(P^*)}} \tag{50}$$

Theorem 4 Proof.

Using Theorem 1 let $\tilde{b} = b_1 - b_2$. Under the normalization that $b_2 = 0$, the variable $\tilde{b}$ represents public debt government or the assets of the productive agent. Specializing the formulations in section **??** we have the optimal plan solves the following Bellman equation.

$$V(\tilde{b}_{\text{-}}) = \max_{c_1(s),c_2(s),b'(s)} \sum_s \pi(s) \left\{ \omega \left[u(c_1(s), l_1(s)) \right] + (1-\omega) \left[c_2(s) \right] + \beta V(\tilde{b}(s)) \right\} \tag{51}$$

subject to

$$c_1(s) - c_2(s) + \tilde{b}(s) = l(s)^{1+\gamma} + \beta^{-1} P(s) \tilde{b}_{\text{-}} \tag{52a}$$

$$nc_1(s) + (1-n)c_2(s) + g(s) \leq \theta_2 l(s) n \tag{52b}$$

$$c_2(s) \geq 0 \tag{52c}$$

$$\bar{b} \geq \tilde{b}(s) \geq \underline{b} \tag{52d}$$

Let $\mu(s), \phi(s), \lambda(s), \underline{\kappa}(s), \overline{\kappa}(s)$ be the Lagrange multipliers on the respective constraints. The FOC are summarized below

$$\omega - \mu(s) = n\phi(s) \tag{53a}$$

$$1 - \omega + \mu(s) - \phi(s)(1-n) + \lambda(s) = 0 \tag{53b}$$

$$-\omega l^\gamma(s) + \mu(s)(1+\gamma) l^\gamma(s) + n\phi(s)\theta = 0 \tag{53c}$$

$$\beta V'(\tilde{b}(s)) - \mu(s) - \overline{\kappa}(s) + \underline{\kappa}(s) = 0 \tag{53d}$$

and the envelope condition

$$V'(\tilde{b}_{\text{-}}) = \sum_s \pi(s) \mu(s) \beta^{-1} P(s) \tag{53e}$$

To show part 1 of Theorem **??**, we show that $\frac{\omega}{n} > \frac{1+\gamma}{\gamma}$ is sufficient for the Lagrange multiplier $\lambda(s)$

on the non-negativity constraint to bind.

Lemma 11 *The multiplier on the budget constraint $\mu(s)$ is bounded above*

$$\mu(s) \leq \min\left\{\omega - n, \frac{\omega}{1+\gamma}\right\}$$

Similiarly the multiplier of the resource constraint is bounded below,

$$\phi(s) \geq \max\left\{1, \frac{\omega}{n}\left[\frac{\gamma}{1+\gamma}\right]\right\}$$

Proof.

Notice that the labor choice of the productive household implies $\frac{1}{1-\tau} = \frac{\theta_2}{l^\gamma(s)}$.

As taxes go to $-\infty$ (53c) implies that $\mu(s)$ approaches $\frac{\omega}{1+\gamma}$ from below. Similiarly the non-negativity of $c_2(s)$ imposes a lower bound of 1 on $\phi(s)$. This translates into an upper bound of $\omega - n$ on μ. ∎

Lemma 12 *There exists a $\bar{\omega}$ such that $\omega > \bar{\omega}$ implies $c_2(s) = 0$ for all b*

Proof.

By the KKT conditions $c_2(s) = 0$ if $\lambda(s) > 0$. Now (53b) implies this is true if $\mu(s) < \omega - n$. The previous lemma bounds $\mu(s)$ by $\frac{\omega}{1+\gamma}$.

We can thus define $\bar{\omega} = n\left(\frac{1+\gamma}{\gamma}\right)$ as the required threshold Pareto weight to ensure that the unproductive agent has zero consumption forever.

∎

Now for the rest of the parts $\omega < n\frac{1+\gamma}{\gamma}$, we can have postive transfers for low enough public debt. In particular, we can define a maximum level of debt $\mathcal{B}$ that is consistent with an interior solution for the unproductive agents' consumption.

Guess an interior solution $c_{2,t} > 0$ or $\lambda_t = 0$ for all t. This gives us $l(s) = l^*$ defined below:

$$l^* = \left[\frac{n\theta}{\omega - (\omega - n)(1+\gamma)}\right]^{\frac{1}{\gamma}} \tag{54}$$

As long as $\omega < n\left(\frac{1+\gamma}{\gamma}\right)$ At the interior solution $\tilde{b}(s) = \tilde{b}_{\text{-}}$ and using the implementability constraint and resource constraints (52a) and (52b) respectively, we can obtain the expression for $c_2(s)$

$$c_2(s) = n\theta l^* - n{l^*}^{1+\gamma} - \tilde{b}_{\text{-}}(1 - P(s)\beta^{-1}) - g(s)$$

Non-negativity of c_2 implies,

$$\tilde{b}_{\text{-}} \leq \frac{g(s) - n\theta l^* + n{l^*}^{1+\gamma}}{\beta^{-1}P(s) - 1}$$

We can also express this as

$$\tilde{b}_{-} \leq \frac{g(s) - \tau^* y^*}{\beta^{-1} P(s) - 1},$$

where the right hand side of the previous equation is just the present discounted value of the primary deficit of the government at the constant taxes τ^* associated with l^* defined in (54). As long as $\beta^{-1}P(s) - 1 > 0$, this object is well defined, we define $\mathcal{B} = \min_s \left[\frac{g(s) - n\theta l^* + n{l^*}^{1+\gamma}}{\beta^{-1}P(s)-1}\right]$. Thus for $\tilde{b}_{-} < \mathcal{B}$ the optimal allocation has constant taxes given by τ^* and debt $\tilde{b}_{-}$, while transfers are given by

$$T(s) = n\theta l^* - n{l^*}^{1+\gamma} - \tilde{b}_{-}(1 - P(s)\beta^{-1}) - g(s),$$

and are strictly positive.

For initial debt greater than $\mathcal{B}$, we distinguish cases when payoffs are perfectly aligned with $g(s)$ i.e belong to the set $\mathcal{P}^*$ and when they are not. For part 2 case b, let $P \notin \mathcal{P}^*$.

Lemma 13 *There exists a $\check{b} > \mathcal{B}$ such that there are two shocks $\underline{s}$ and $\overline{s}$ and the optimal choice of debt starting from $\tilde{b}_{-} \leq \check{b}$ satisfies the following two inequalities:*

$$\tilde{b}(\underline{s}, \tilde{b}_{-}) > \mathcal{B}$$

$$\tilde{b}(\overline{s}, \tilde{b}_{-}) \leq \mathcal{B}$$

Proof. At $\mathcal{B}$, there exist some $\overline{s}$ such that $T(\overline{s}, \mathcal{B}) = \epsilon > 0$. Now define $\check{b}$ as follows:

$$\check{b} = \mathcal{B} + \frac{\epsilon\beta}{2P(\overline{s})}$$

Now suppose to the contrary $\tilde{b}(\overline{s}, \tilde{b}_{-}) > \mathcal{B}$ for some $\tilde{b}_{-} \leq \check{b}$. This implies that $\tau(s, \tilde{b}_{-}) > \tau^*$ and $T(\overline{s}, \tilde{b}_{-}) = 0$.[8]

The government budget constraint implies

$$\frac{P(\overline{s})\tilde{b}_{-}}{\beta} + g(s) = \tilde{b}(\overline{s}, \tilde{b}_{-}) + (1 - \tau(\overline{s}, \tilde{b})_{-}) l(\overline{s}, \tilde{b}_{-}).$$

As,

$$\frac{P(\overline{s})\tilde{b}_{-}}{\beta} + g(\overline{s}) \leq \frac{P(\overline{s})\mathcal{B}}{\beta} + g(\overline{s}) + \frac{\epsilon}{2} < \frac{P(\overline{s})\mathcal{B}}{\beta} + g(\overline{s}) + \epsilon$$

This further implies,

$$\tilde{b}(\overline{s}, \tilde{b}_{-}) + (1 - \tau(\overline{s}, \tilde{b}_{-})) l(\tau(\overline{s}, \tilde{b}_{-})) > [\tilde{b}(\overline{s}, \tilde{b}_{-}) + (1 - \tau^*) l^* > \mathcal{B} + (1 - \tau^*) l^* > \frac{P(\overline{s})\tilde{b}_{-}}{\beta} + g(\overline{s}) + T(\overline{s}, \tilde{b}_{-}) = \frac{P(\overline{s})\tilde{b}_{-}}{\beta} + g(\overline{s}) + \epsilon.$$

[8]Explain why

Combining the previous two inequalities yields a contradiction. The second inequality, $\tilde{b}(\underline{s}, \tilde{b}_{-}) > \mathcal{B}$ follows from the definition of $\mathcal{B}$.

Now define $\overline{\mu}(\tilde{b}(s, \tilde{b}_{-}))$ as $\max_s \mu(s, \tilde{b}_{-})$ and $\hat{s}(\tilde{b}_{-})$ as the shock that achieves this maximum. Now we show that $\hat{\mu}(\tilde{b}(s, \tilde{b}_{-}))$ is finite for all $b_{-} \leq \overline{b}$. We show the claim for the natural debt limit.

Let $b^n(s) = (\beta^{-1}P(s) - 1)^{-1}\left[\theta^{\frac{\gamma}{1+\gamma}}\left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}}\left(\frac{\gamma}{1+\gamma}\right) - g(s)\right]$ be the maximum debt supported by a particular shock s. The natural debt limit is defined as $\overline{b}^n = \min_s b^n(s)$. Note that $\lim_{b \to \overline{b}^n} \mu(\tilde{b}_{-}) = \infty$

Now choose s such that $b^n(s) > \overline{b}^n$ and consider the debt choice next period for the same shock s when it comes in with debt $\overline{b}^n$.

Suppose it chooses $\tilde{b}(s, \overline{b}^n) = \overline{b}^n$, then taxes will have to be set to $\frac{\gamma}{1+\gamma}$ and the tax income will be $\frac{\gamma}{1+\gamma} l(\frac{\gamma}{1+\gamma}) = \theta^{\frac{\gamma}{1+\gamma}}\left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}}\left(\frac{\gamma}{1+\gamma}\right)$. The budget constraint will then imply that,

$$\frac{\overline{b}^n P(s)}{\beta} + g(s) = \theta^{\frac{\gamma}{1+\gamma}}\left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}}\left(\frac{\gamma}{1+\gamma}\right) + \overline{b}^n$$

$$\overline{b}^n = (P(s)\beta^{-1} - 1)^{-1}\left(\theta^{\frac{\gamma}{1+\gamma}}\left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}}\left(\frac{\gamma}{1+\gamma}\right) - g(s)\right)$$

However the right hand side is the definition of $b^n(s)$ and,

$$b^n(s) > \overline{b}^n.$$

Thus we have a contradiction and the optimal choice of debt at the natural debt limit $\tilde{b}(s, \overline{b}^n) < \overline{b}^n$.

This inturn means that $\lim_{\tilde{b} \to \overline{b}^n} \overline{\mu}(\tilde{b}) < \infty$.

Now note that $\overline{\mu}(\tilde{b}_{-}) - \mu(\tilde{b}_{-})$ is continuous on $[\check{b}, \quad \overline{b}^n]$ and is bounded below by zero, therefore attains a minimum at $\tilde{b}^{min}$. Let $\delta = \hat{\mu}(\tilde{b}^{min}) - \mu(\tilde{b}^{min}) > \eta > 0$. If this was not true then $P(s) \in \mathcal{P}^*$ as μ will have an absorbing state.

Let $\mu(\omega, n) = \omega - n$. This is the value of μ when debt falls below $\mathcal{B}$.

Now consider any initial $\tilde{b}_{-} \in [\mathcal{B}, \overline{b}^n]$. If $\tilde{b}_{-} \leq \check{b}$, then by lemma 13, we know that $\mathcal{B}$ will be reached in one shock. Otherwise if $\tilde{b}_{-} > \check{b}$, we can construct a sequence of shocks $s_t = \hat{s}(\tilde{b}_{t-1})$ of length $N = \frac{\mu(\omega,n) - \mu(\tilde{b}_{-})}{\delta}$. There exits $t < N$ such that $\tilde{b}_t < \check{b}$, otherwise,

$$\mu_t > \mu(\tilde{b}_{-}) + N\delta > \mu(\omega, n)$$

Thus we can reach $\mathcal{B}$ in finite steps. Since shocks are i.i.d, this is an almost sure statement. At $\mathcal{B}$, transfers are strictly positive for some shocks $T_t > 0$ a.s. and taxes are given by τ^*.

Now consider the payoffs $P \in \mathcal{P}^*$ such that the associated steady state debt $b^* > \mathcal{B}$. Under the guess $T_t = 0$, the same algebra as in Theorem 2 goes through and we can show that $\tilde{b}_{-} = b^*$ is a steady state for the heterogeneous agent economy. Thus the heterogeneous agent economy for a given $P \in \mathcal{P}^*$ has a continuum of steady states given by the set $[\bar{b}, \quad] \cup \{b^*\}$.

In the region $\tilde{b}_- > b^*$, as before μ_t is supermartingale bounded below by b^*. Since there is a unique fixed point in the region $\tilde{b}_- \in [b^*, \bar{b}^n]$, μ_t converges to μ^* associated with b^*. Transfers are zero and taxes are given by τ^{**}

$$\tau^{**} = \frac{\gamma\mu^*}{(1+\gamma)\mu^* - 1} \tag{55}$$

In the region $[\mathcal{B}, \quad b^*]$ the outcomes depend on the exact sequence of shocks. If b_t gets sufficiently close to $\check{b}$, then it can converge to $\mathcal{B}$ and if it gets sufficiently close to b^*, it can converge to b^*. Either of this can happen with strictly positive probability. ■

1. Proof of Proposition ??

The Bellman equation for the optimal planners problem with log quadratic preferences and IID shocks can be written as

$$V(x,\rho) = \max_{c_1,c_2,l_1,x',\rho'} \sum_s \pi(s)\left[\alpha_1\left(\log c_1(s) - \frac{l_1(s)^2}{2}\right) + \alpha_2 \log c_2(s) + \beta V(x'(s),\rho'(s))\right]$$

subject to the constraints

$$1 + \rho'(s)[l_1(s)^2 - 1] + \beta x'(s) - \frac{x\frac{P(s)}{c_2(s)}}{\mathbb{E}[\frac{P}{c_2}]} = 0 \tag{56}$$

$$\sum_s \frac{\pi(s)P(s)}{c_1(s)}(\rho'(s) - \rho) = 0 \tag{57}$$

$$\theta_1(s)l_1(s) - c_1(s) - c_2(s) - g = 0 \tag{58}$$

$$\rho'(s)c_2(s) - c_1(s) = 0 \tag{59}$$

where the $\pi(s)$ is the probability distribution of the aggregate state s. If we let $\pi(s)\mu(s)$, λ, $\pi(s)\xi(s)$ and $\pi(s)\phi(s)$ be the Lagrange multipliers for the constraints (56)-(59) respectively then we obtain the following FONC for the planners problem [9]

$c_1(s)$:

$$\frac{\alpha_1\pi(s)}{c_1(s)} - \frac{\lambda\pi(s)}{c_1(s)^2}(\rho'(s) - \rho) - \pi(s)\xi(s) - \pi(s)\phi(s) = 0 \tag{60}$$

[9]Appendix **??** discuses the associated second order conditions that ensure these policies are optimal

$c_2(s)$:

$$\frac{\alpha_2\pi(s)}{c_2(s)} + \frac{x\pi(s)}{c_2(s)^2\mathbb{E}[\frac{1}{c_2}]}\left[\mu(s) - \frac{\mathbb{E}[\mu\frac{1}{c_2}]}{\mathbb{E}[\frac{1}{c_2}]}\right] - \pi(s)\xi(s) + \pi(s)\rho'(s)\phi(s) = 0 \tag{61}$$

$l_1(s)$:

$$-\alpha_1\pi(s)l_1(s) + 2\mu(s)\pi(s)\rho'(s)l_1(s) + \theta_1(s)\pi(s)\xi(s) = 0 \tag{62}$$

$x'(s)$:

$$\beta(s)\pi(s)V_x(x'(s),\rho'(s)) + \beta(s)\pi(s)\mu(s) = 0 \tag{63}$$

$\rho'(s)$:

$$\beta(s)\pi(s)V_\rho(x'(s),\rho'(s)) + \frac{\lambda\pi(s)}{c_1(s)} + \mu(s)\pi(s)[l_1(s)^2 - 1] + \pi(s)\phi(s)c_2(s) = 0 \tag{64}$$

In addition there are two envelope conditions given by

$$V_x(x,\rho) = -\sum_{s'}\frac{\mu(s')\Pr(s')\frac{1}{c_2(s')}}{\mathbb{E}[\frac{1}{c_2}]} = -\frac{\mathbb{E}[\mu\frac{1}{c_2}]}{\mathbb{E}[\frac{1}{c_2}]} \tag{65}$$

$$V_\rho(x,\rho) = -\lambda\mathbb{E}[\frac{1}{c_1}] \tag{66}$$

In the steady state, we need to solve for a collection of allocations, initial conditions and Lagrange multipliers $\{c_1(s), c_2(s), l_1(s), x, \rho, \mu(s), \lambda, \xi(s), \phi(s)\}$ such that equations (56)-(66) are satisfied when $\rho'(s) = \rho$ and $x'(s) = x$. It should be clear that if we replace $\mu(s) = \mu$, equation (63) and the envelope condition with respect to x is always satisfied. Additionally under this assumption equation (61) simplifies significantly,since

$$\frac{x\pi(s)}{c_2(s)^2\mathbb{E}[\frac{1}{c_2}]}\left[\mu(s) - \frac{\mathbb{E}[\mu\frac{1}{c_2}]}{\mathbb{E}[\frac{1}{c_2}]}\right] = 0$$

The first order conditions for a steady can then be written simply as

$$1+\rho[l_1(s)^2-1]+\beta(s)x-\frac{x}{c_2(s)\mathbb{E}[\frac{1}{c_2}]}=0 \tag{67}$$

$$\theta_1(s)l_1(s)-c_1(s)-c_2(s)-g=0 \tag{68}$$

$$\rho c_2(s)-c_1(s)=0 \tag{69}$$

$$\frac{\alpha_1}{c_1(s)}-\xi(s)-\phi(s)=0 \tag{70}$$

$$\frac{\alpha_2}{c_2(s)}-\xi(s)+\rho\phi(s)=0 \tag{71}$$

$$[2\mu\rho-\alpha_1]l_1(s)+\theta_1(s)\xi(s)=0 \tag{72}$$

$$\lambda\left[\frac{1}{c_1(s)}-\beta(s)\mathbb{E}[\frac{1}{c_1}]\right]+\mu[l_1(s)^2-1]+\phi(s)c_2(s)=0 \tag{73}$$

We can rewrite equation (70) as

$$\frac{\alpha_1}{c_2(s)}-\rho\xi(s)-\rho\phi(s)=0$$

by substituting $c_1(s)=\rho c_2(s)$. Adding this to equation (71) and normalizing $\alpha_1+\alpha_2=1$ we obtain

$$\xi(s)=\frac{1}{(1+\rho)\,c_2(s)} \tag{74}$$

which we can use to solve for $\phi(s)$ as

$$\phi(s)=\frac{\alpha_1-\rho\alpha_2}{(\rho(1+\rho))\,c_2(s)} \tag{75}$$

From equation (67) we can solve for $l_1(s)^2-1$ as

$$l_1(s)^2-1=\frac{x}{\rho\mathbb{E}[\frac{1}{c_2}]}\left(\frac{1}{c_2(s})-\beta(s)\mathbb{E}[\frac{1}{c_2}]\right)-\frac{1}{\rho}$$

This can be used along with equations (73) and (75) to obtain

$$\left(\frac{\lambda}{\rho}+\frac{\mu x}{\rho\mathbb{E}[\frac{1}{c_2}]}\right)\left(\frac{1}{c_2(s)}-\beta(s)\mathbb{E}\left[\frac{1}{c_2}\right]\right)=\frac{\mu}{\rho}+\frac{\rho\alpha_2-\alpha_1}{\rho(1+\rho)}$$

Note that the LHS depends on s while the RHS does not, hence the solution to this equation is

$$\lambda=-\frac{\mu x}{\mathbb{E}[\frac{1}{c_2}]} \tag{76}$$

and

$$\mu=\frac{\alpha_1-\rho\alpha_2}{1+\rho} \tag{77}$$

Combining these with equation (72) we quickly obtain that

$$\left[2\rho\frac{\alpha_1-\rho\alpha_2}{1+\rho}-\alpha_1\right]l_1(s)+\frac{\theta_1(s)}{(1+\rho)\,c_2(s)}=0$$

Then solving for $l_1(s)$ gives

$$l_1(s)=\frac{\theta_1(s)}{\left(\alpha_1(1-\rho)+2\rho^2\alpha_2\right)c_2(s)}$$

Remark 2 *Note that the labor tax rate is given by* $1-\frac{c_1(s)l_1(s)}{\theta(s)}$. *The previous expression shows that labor taxes are constant at the steady state. This property holds generally for CES preferences separable in consumption and leisure*

This we can plug into the aggregate resource constraint (68) to obtain

$$l_1(s)=\left(\frac{1+\rho}{\alpha_1(1-\rho)+2\rho^2\alpha_2}\right)\frac{1}{l_1(s)}+\frac{g}{\theta_1(s)}$$

letting $C(\rho)=\frac{1+\rho}{\alpha_1(1-\rho)+2\rho^2\alpha_2}$ we can then solve for $l_1(s)$ as

$$l_1(s)=\frac{g\pm\sqrt{g^2+4C(\rho)\theta_1(s)^2}}{2\theta_1(s)}$$

The marginal utility of agent 2 is then

$$\frac{1}{c_2(s)}=\left(\frac{1+\rho}{C(\rho)}\right)\left(\frac{g\pm\sqrt{g^2+4C(\rho)\theta_1(s)^2}}{2\theta_1(s)^2}\right)$$

Note that in order for either of these terms to be positive we need $C(\rho)\geq 0$ implying that there is only one economically meaningful root. Thus

$$l_1(s)=\frac{g+\sqrt{g^2+4C(\rho)\theta_1(s)^2}}{2\theta_1(s)} \tag{78}$$

and

$$\frac{1}{c_2(s)}=\left(\frac{1+\rho}{C(\rho)}\right)\left(\frac{g+\sqrt{g^2+4C(\rho)\theta_1(s)^2}}{2\theta_1(s)^2}\right) \tag{79}$$

A steady state is then a value of ρ such that

$$x(s)=\frac{1+\rho[l_1(\rho,s)^2-1]}{\frac{1/c_2(\rho,s)}{\mathbb{E}[\frac{1}{c_2}](\rho)}-\beta(s)} \tag{80}$$

s independent of s.

The following lemma, which orders consumption and labor across states, will be useful in proving the parts of proposition **??**. As a notational aside we will often use $\theta_{1,l}$ and $\theta_{1,h}$ to refer to $\theta_1(s_l)$ and

$\theta_1(s_h)$ respectively. Where s_l refers to the low TFP state and s_h refers to the high TFP state.

Lemma 14 *Suppose that* $\theta_1(s_l) < \theta_2(s_h)$ *and* ρ *such that* $C(\rho) > 0$ *then*

$$l_{1,l} = \frac{g + \sqrt{g^2 + 4C(\rho)\theta_{1,l}^2}}{2\theta_{1,l}} > \frac{g + \sqrt{g^2 + 4C(\rho)\theta_{1,h}^2}}{2\theta_{1,h}} = l_{1,h}$$

and

$$\frac{1}{c_{2,l}} = \frac{1+\rho}{C(\rho)}\frac{g + \sqrt{g^2 + 4C(\rho)\theta_{1,l}^2}}{2\theta_{1,l}^2} > \frac{1+\rho}{C(\rho)}\frac{g + \sqrt{g^2 + 4C(\rho)\theta_{1,h}^2}}{2\theta_{1,h}^2} = \frac{1}{c_{2.h}}$$

Proof. The results should follow directly from showing that the function

$$l_1(\theta) = \frac{g + \sqrt{g^2 + 4C(\rho)\theta}}{2\theta}$$

is decreasing in θ. Taking the derivative with respect to θ

$$\begin{aligned}
\frac{dl_1}{d\theta}(\theta) &= -\frac{g}{2\theta^2} - \frac{\sqrt{g + 4C(\rho)\theta^2}}{2\theta^2} + \frac{4C(\rho)\theta}{2\theta\sqrt{g^2 + 4C(\rho)\theta^2}} \\
&= -\frac{g}{2\theta^2} - \frac{g + 4C(\rho)\theta^2 - 4C(\rho)\theta^2}{2\theta^2\sqrt{g^2 + 4C(\rho)\theta^2}} \\
&= -\frac{g}{2\theta^2} - \frac{g}{2\theta^2\sqrt{g^2 + 4C(\rho)\theta^2}} < 0
\end{aligned}$$

That $\frac{1}{c_{2,l}} > \frac{1}{c_{2,h}}$ follows directly. ■

Proof of Proposition ??.

Part 1. In order for there to exist a ρ such that equation (80) is independent of the state (and hence have a steady state) we need the existence of root for the following function

$$f(\rho) = \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} - \frac{\frac{1/c_2(\rho, s_h)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta}{\frac{1/c_2(\rho, s_l)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta}$$

From lemma 14 we can conclude that

$$1 + \rho[l_1(\rho, s_l)^2 - 1] > 1 + \rho[l_1(\rho, s_h)^2 - 1] \tag{81}$$

and

$$\frac{1/c_2(\rho, s_l)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta > \frac{1/c_2(\rho, s_h)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta \tag{82}$$

for all $\rho > 0$ such that $C(\rho) \geq 0$. To begin with we will define $\underline{\rho}$ such that $C(\rho) > 0$ for all $\rho > \underline{\rho}$. Note that we will have to deal with two different cases.

$\alpha_1(1-\rho) + 2\rho^2\alpha_2 > 0$ **for all** $\rho \geq 0$**:** In this case we know that $C(\rho) \geq 0$ for all ρ and is bounded above and thus we will let $\underline{\rho} = 0$.

$\alpha_1(1-\rho) + 2\rho^2\alpha_2 = 0$ **for some** $\rho > 0$**:** In this case let $\underline{\rho}$ be the largest positive root of $\alpha_1(1-\rho) + 2\rho^2\alpha_2$. Note that $\lim_{\rho\to\underline{\rho}^+} C(\rho) = \infty$

With this we note that[10]

$$\lim_{\rho\to\underline{\rho}^+} \frac{1+\rho[l_1(\rho,s_h)^2-1]}{1+\rho[l_1(\rho,s_l)^2-1]} = 1$$

We can also show that

$$\lim_{\rho\to\underline{\rho}^+} \frac{\frac{1/c_2(\rho,s_h)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta}{\frac{1/c_2(\rho,s_l)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta} < 1$$

which implies that $\lim_{\rho\to\underline{\rho}^+} f(\underline{\rho}) > 0$.

Taking the limit as $\rho \to \infty$ we see that $C(\rho) \to 0$, given that $\frac{g}{\theta(s)} < 1$, we can then conclude that

$$\lim_{\rho\to\infty} 1 + \rho[l_1(\rho,s)^2 - 1] = -\infty$$

Thus, there exists $\overline{\rho}$ such that $1 + \overline{\rho}[l_1(\overline{\rho},s_l)^2 - 1] = 0$. [11] From equation (81), we know that

$$0 = 1 + \overline{\rho}[l_1(\overline{\rho},s_l)^2 - 1] > 1 + \overline{\rho}[l_1(\overline{\rho},s_h)^2 - 1]$$

which implies in the limit

$$\lim_{\rho\to\overline{\rho}^-} \frac{1+\rho[l_1(\rho,s_h)^2-1]}{1+\rho[l_1(\rho,s_l)^2-1]} = -\infty$$

which along with

$$\frac{\frac{1/c_2(\rho,s_h)}{\mathbb{E}[\frac{1}{c_2}]} - \beta}{\frac{1/c_2(\rho,s_l)}{\mathbb{E}[\frac{1}{c_2}]} - \beta} \geq -1$$

allows us to conclude that $\lim_{\rho\to\overline{\rho}^-} f(\rho) = -\infty$. The intermediate value theorem then implies that there exists ρ_{SS} such that $f(\rho_{SS}) = 0$ and hence that ρ_{SS} is a steady state.

Finally, as $\rho_{SS} < \overline{\rho}$ we know that

$$1 + \rho_{SS}[l_1(\rho_{SS},s_l) - 1] > 0$$

[10] In the first case $\underline{\rho} = 0$ and in the second case $l_1(\rho,s_l) = l_1(\rho,s_h)$ as $\rho \to \underline{\rho}^+$

[11] This can be seen from the fact $\lim_{\rho\to\underline{\rho}^+} 1 + \rho[l_1(\rho,s_l)^2 - 1] > 0$ and $\lim_{\rho\to\infty} 1 + \rho[l_1(\rho,s_l)^2 - 1] > -\infty$, thus $\overline{\rho}$ exists in $(\underline{\rho},\infty)$

as $\frac{1/c_2(\rho,s_l)}{\mathbb{E}[\frac{1}{c_2}]} > 1$ we can conclude

$$x_{SS} = \frac{1 + \rho_{SS}[l_1(\rho_{SS}, s_l) - 1]}{\frac{1/c_2(\rho,s_l)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta} > 0$$

implying that the government will hold assets in the steady state (under the normalization that agent 2 holds no assets).

Part 2. The condition that $R(s_h) = R(s_l)$ implies that

$$\frac{1/c_2(\rho, s_l)}{\beta(s_l)\mathbb{E}[\frac{1}{c_2}]} = \frac{1/c_2(\rho, s_h)}{\beta(s_h)\mathbb{E}[\frac{1}{c_2}]}$$

which simplifies to

$$\frac{\beta(s_h)}{\beta(s_l)} = \frac{1/c_2(\rho, s_h)}{1/c_2(\rho, s_l)} \tag{83}$$

In order for a steady state to exist with constant interest rates there must be a root of the following function

$$\begin{aligned} f(\rho) &= \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} - \frac{\frac{1/c_2(\rho,s_h)}{\mathbb{E}[\frac{1}{c_2}]} - \beta(s_h)}{\frac{1/c_2(\rho,s_l)}{\mathbb{E}[\frac{1}{c_2}]} - \beta(s_l)} \\ &= \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} - \frac{\frac{1/c_2(\rho,s_h)}{\beta(s_h)\mathbb{E}[\frac{1}{c_2}]} - 1}{\frac{1/c_2(\rho,s_l)}{\beta(s_l)\mathbb{E}[\frac{1}{c_2}]} - 1} \frac{\beta(s_h)}{\beta(s_l)} \\ &= \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} - \frac{1/c_2(\rho, s_h)}{1/c_2(\rho, s_l)} \end{aligned}$$

Taking limits of $f(\rho)$ as ρ approaches $\underline{\rho}$ from the positive side we already demonstrated

$$\lim_{\rho \to \underline{\rho}^+} \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} = 1$$

From equation (79) and Lemma 14 it is straightforward to see that

$$\lim_{\rho \to \underline{\rho}^+} \frac{1/c_2(\rho, s_h)}{1/c_2(\rho, s_l)} < 1$$

which allows us to conclude that

$$\lim_{\rho \to \underline{\rho}^+} f(\rho) > 0$$

Taking limits as ρ approaches $\overline{\rho}$ from the negative direction we know that

$$\lim_{\rho \to \overline{\rho}^-} \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} = -\infty$$

As $\frac{1/c_2(\rho,s_h)}{1/c_2(\rho,s_h)} > 0$ for all ρ it is straightforward to conclude that

$$\lim_{\rho\to\overline{\rho}^-} f(\rho) = -\infty$$

Continuity then implies the existence of a ρ^{SS} such that $f(\rho^{SS}) = 0$, and thus there exists a $\beta(s_l)$ and $\beta(s_h)$ such that $R(s_l) = R(s_h)$ in steady state. From Lemma 14

$$l(\rho, s_l) > l(\rho, s_h).$$

In order for

$$\frac{1+\rho^{SS}[l_1(\rho^{SS},s_h)^2-1]}{1+\rho^{SS}[l_1(\rho^{SS},s_l)^2-1]} = \frac{1/c_2(\rho^{SS},s_h)}{1/c_2(\rho^{SS},s_l)} < 1$$

it is necessary that

$$1+\rho^{SS}[l_1(\rho^{SS},s_l)^2-1] > 1+\rho^{SS}[l_1(\rho^{SS},s_h)^2-1] > 0$$

implying that the steady state asset level

$$x_{SS} = \frac{1+\rho_{SS}[l_1(\rho_{SS},s_l)-1]}{\frac{1/c_2(\rho,s_l)}{\mathbb{E}[\frac{1}{c_2}]} - \beta(s_l)} > 0$$

Part 3 As noted before, since $g/\theta(s) < 1$ for all s we have

$$\lim_{\rho\to\infty} 1+\rho[l_1(\rho,s)^2-1] = -\infty$$

Thus, there exists ρ_{SS} such that

$$0 > 1+\rho_{SS}[l_1(\rho_{SS},s_l)^2-1] > 1\rho_{SS}[l_1(\rho_{SS},s_h)^2-1]$$

It is then possible to choose $\beta(s) < \frac{1/c_2(\rho_{SS},s)}{\mathbb{E}[\frac{1}{c_2}]}$ such that

$$1 > \frac{1+\rho_{SS}[l_1(\rho_{SS},s_l)^2-1]}{1+\rho_{SS}[l_1(\rho_{SS},s_h)^2-1]} = \frac{\frac{1/c_2(\rho_{SS},s_l)}{\mathbb{E}[\frac{1}{c_2}]} - \beta(s_l)}{\frac{1/c_2(\rho_{SS},s_h)}{\mathbb{E}[\frac{1}{c_2}]} - \beta(s_h)} \tag{84}$$

Implying that for discount factor shocks $\beta(s)$, ρ_{SS} is a steady state level for the ratio of marginal utilities, with steady state marginal utility weighted government debt

$$x_{SS} = \frac{1+\rho_{SS}[l_1(\rho_{SS},s_l)^2-1]}{\frac{1/c_2(\rho_{SS},s_l)}{\mathbb{E}[\frac{1}{c_2}]} - \beta(s_l)} < 0$$

Thus, in the steady state, the government is holding debt, under the normalization that the unproductive worker holds no assets. As $\frac{1/c_2(\rho,s_l)}{\mathbb{E}[\frac{1}{c_2}]} > \frac{1/c_2(\rho,s_h)}{\mathbb{E}[\frac{1}{c_2}]}$, in order for equation (84) to hold we need $\beta_l > \beta_h$. We can then rewrite equation (84) as

$$1 > \frac{\beta(s_h)}{\beta(s_l)} > \frac{\frac{1/c_2(\rho_{SS},s_l)}{\beta_l \mathbb{E}[\frac{1}{c_2}]} - 1}{\frac{1/c_2(\rho_{SS},s_h)}{\beta(s_h)\mathbb{E}[\frac{1}{c_2}]} - 1}$$

Thus

$$R(s_l)\frac{1/c_2(\rho_{SS}, s_l)}{\beta(s_l)\mathbb{E}[\frac{1}{c_2}]} < \frac{1/c_2(\rho_{SS}, s_h)}{\beta(s_h)\mathbb{E}[\frac{1}{c_2}]} = R(s_h) \tag{85}$$

in the steady state interest rates are positively correlated with TFP.

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