Taxes, debts, and redistributions with aggregate shocks*

Anmol Bhandari apb296@nyu.edu

David Evans dgevans@nyu.edu

Mikhail Golosov golosov@princeton.edu

Thomas J. Sargent thomas.sargent@nyu.edu

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Abstract

This paper studies how taxes and debt respond to aggregate shocks in the presence of incomplete markets and redistribution concerns. A planner sets a lump sum transfer and a linear tax on labor income in an economy with heterogeneous agents, aggregate uncertainty, and markets restricted to a single asset whose payoffs can wary with aggregate states. Two forces shape long-run outcomes: the planner's desire to minimize the welfare costs of fluctuating transfers, which calls for a negative correlation between the distribution of net assets and agents' skills; and the planner's desire to use fluctuations in the real interest rate to adjust for missing state-contingent securities. In a model parameterized to match stylized facts about US booms and recessions, distributional concerns mainly determine optimal policies over business cycle frequencies. These features of optimal policy differ markedly from ones that emerge from representative agent Ramsey models

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1. Introduction

2. Environment

Exogenous fundamentals of the economy are functions of a shock s_t that follows an irreducible Markov process, where $s_t \in S$ and S is a finite set. We let $s^t = (s_0, ..., s_t)$ denote a history of shocks with marginal densities denoted by $Pr(s^t)$. [1]

There is a mass n_i of a type $i \in I$ agent, with $\sum_{i=1}^{I} n_i = 1$. Types differ ~~by their~~ in skills, indexed by $\{\theta_i\}_i$. Preferences of an agent of type i over stochastic processes for consumption $\{c_{i,t}\}_t$ and labor supply $\{l_{i,t}\}_t$ are ordered by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U^i \left(c_{i,t}, l_{i,t}\right), \tag{1}$$

where $\mathbb{E}_t$ is a mathematical expectations operator conditioned on time t information and $\beta \in (0,1)$ is ~~the~~ time discount factor. Results in section 2.1. require no additional assumptions on U^i like differentiability or convexity,[2] but results in later sections do.

An agent of type i who supplies l_i units of labor produces $\theta_i\left(s_t\right) l_i$ units of output, where $\theta_i(s_t) \in \Theta$ is a nonnegative state-dependent scalar. Feasible allocations satisfy

$$\sum_{i=1}^{I} n_i c_{i,t} + g_t = \sum_{i=1}^{I} \pi_i \theta_{i,t} l_{i,t}, \tag{2}$$

where g_t denotes exogenous government expenditures in state s_t.

A Ramsey planner's preferences over a vector of stochastic processes for consumption and labor supply are ordered by

$$\mathbb{E}_0 \sum_{i=1}^{I} \omega_i \sum_{t=0}^{\infty} \beta^t U_t^i \left(c_{i,t}, l_{i,t}\right), \tag{3}$$

introduce amounts of assets here

move assumptions about U here

where the Pareto weights satisfy $\omega_i \geq 0, \sum_{i=1}^{I} \omega_i = 1$ and

The government and agents trade a single ~~but~~ possibly risky asset whose time t payoff p_t is described by an $S \times S$ matrix $\mathbb{P}$

$$p_t = \mathbb{P}(s_t | s_{t-1}),$$

~~with~~ satisfying the normalization ~~that~~ $\mathbb{E}_t p_{t+1} = 1$. The payoff shocks ~~are~~ is a parsimonious way to capture variations in holding period returns on ~~portfolios~~ portfolio of the government in ways that resemble due inflation risk, interest rate risk ~~for~~ longer maturity bonds, or defaults risk.

rewrite

[1] To save on notation, mostly we use z_t to denote a random variable with a time t conditional distribution that is a function of the history s^t. Occasionally, we use the more explicit notion $z\left(s^t\right)$ to denote a realization at a particular history s^t.

[2] Consequently our setup allows both extensive and intensive ~~responses of~~ labor.

intensive variations — supply

desire first and treatment of assets here

We assume that the government imposes an affine tax. We denote proportional labor taxes by τ and lump sump transfers by T. With this the tax bill of an agent with wage earnings $l_{i,t}\theta_{i,t}$ is given by

$$-T_t + \tau_t \theta_{i,t} l_{i,t}.$$

We do not restrict the sign of T_t at any t or s^t. If for some type i, $\theta_{i,t} = 0$, $b_{i,-1} = 0$ and U^i is defined only on $\mathcal{R}^2_+$, his budget constraint will imply that the all allocations feasible for the planner have nonnegative present values of transfers, since transfers are the sole source of a type i agent's ~~wealth and consumption~~.

Let $q_t = q_t(s^t)$ be the price of the single asset at time t and $R_t = \frac{p_t}{q_{t-1}}$ be the one period return from holding the asset from $t-1$ to t. Under the affine tax system, agent i's budget constraint at t is

$$c_{i,t} + b_{i,t} = (1-\tau_t)\,\theta_{i,t} l_{i,t} + R_t b_{i,t-1} + T_t, \tag{4}$$

where $b_{i,t}$ denotes asset holdings of a type i agent at time $t \geq 0$.

The government budget constraint is

$$g_t + B_t = \tau_t \sum_{i=1}^{I} n_i \theta_{i,t} l_{i,t} - T_t + R_t B_{t-1}, \tag{5}$$

where B_t denotes the government's assets at time t.

We assume that private agents and the government start with assets $\{b_{i,-1}\}_{i=1}^{I}$ and B_{-1}, respectively. Asset holdings satisfy the market clearing condition

$$\sum_{i=1}^{I} n_i b_{i,t} + B_t = 0 \text{ for all } t \geq -1. \tag{6}$$

Components of competitive equilibria are described below

Definition 1 *An* allocation *is a sequence* $\{c_{i,t}, l_{i,t}\}_{i,t}$. *An* asset profile *is a sequence* $\{\{b_{i,t}\}_i, B_t\}_t$. *A* price system *is an interest rate sequence* $\{R_t\}_t$. *A* tax policy *is a sequence* $\{\tau_t, T_t\}_t$.

Representative agent models usually impose debt limits on the government. Section XX describes that in our setting with affine taxes and heterogeneous agents, competitive allocations only pin down net asset positions. Thus $\tilde{D}_t = -(B_t - b_{i,t})$ for some i is comparable to the notion of *public debt* and we will assume that it is bounded. As far the households we will impose natural debt limits. These typically depend on the tax and transfer policies adopted by the government.[3]

[3] An alternative is to use ad-hoc debt limits which are exogenous history contingent bounds for each agent. Appendix XX discusses how restricting attention to natural debt limits for the households only shrinks the set of allocations that can be implemented as competitive equilibria.

(1)

Definition 2 *For a given initial asset distribution* $\left(\{b_{i,-1}\}_i, B_{-1}\right)$, *a competitive equilibrium with affine taxes is a sequence* $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, B_t, R_t\}_t$ *and a tax policy* $\{\tau_t, T_t\}_t$, *such that* $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ *maximize (1) subject to (4) and* $\{b_{i,t}\}_{i,t}$ *satisfies the borrowing limits; and constraints (2), (5) and (6) are satisfied.*

on

Next

Lastly we define optimal competitive equilibria.

Definition 3 *Given* $(\{b_{i,-1}\}_i, B_{-1})$, *an optimal competitive equilibrium with affine taxes is a tax policy* $\{\tau_t^*, T_t^*\}_t$, *an allocation* $\{c_{i,t}^*, l_{i,t}^*\})_{i,t}$, *an asset profile* $\left\{\{b_{i,t}^*\}_i, B_t^*\right\}_t$, *and a price system* $\{R_t^*\}_t$ *such that (i) given* $\left(\{b_{i,-1}\}_i, B_{-1}\right)$, *the tax policy, ~~the~~ price system, and ~~the~~ allocation constitute a competitive equilibrium, (ii)* B_t *satisfies the borrowing constraints; and (iii) there is no other tax policy* $\{\tau_t, T_t\}_t$ *such that a competitive equilibrium given* $\left(\{b_{i,-1}\}_i, B_{-1}\right)$ *and* $\{\tau_t, T_t\}_t$ *has a strictly higher value of (3).* clumsy

We call $\{\tau_t^*, T_t^*\}_t$ an *optimal tax policy*, $\{c_{i,t}^*, l_{i,t}^*\}_{i,t}$ an *optimal allocation*, and $\left\{\{b_{i,t}^*\}_i, B_t^*\right\}_t$ an *optimal asset profile*.

2.1. Ricardian equivalence

In this section we recover a *Ricardian equivalence* result in the spirit of Barro (1974). We use this result to highlight that the level of government debt is not a state variable in our setting. ~~The reason being~~ that there is an equivalence class of tax policies and asset profiles that support the same competitive equilibrium allocation and as such pin down only net asset positions. Here's how the same

Theorem 1 *Given* $\left(\{b_{i,-1}\}_i, B_{-1}\right)$, *let* $\left\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, B_t, R_t\right\}_t$ *and* $\{\tau_t, T_t\}_t$ *be a competitive equilibrium. For any bounded sequences* $\left\{\hat{b}_{i,t}\right\}_{i,t\geq -1}$ *that satisfy*

and that make

$$\hat{b}_{i,t} - \hat{b}_{1,t} = \tilde{b}_{i,t} \equiv b_{i,t} - b_{1,t} \text{ for all } t \geq -1, i \geq 2,$$

there exist sequences $\left\{\hat{T}_t\right\}_t$ *and* $\left\{\hat{B}_t\right\}_{t\geq -1}$ *that satisfy (6) such that* $\left\{\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_i, \hat{B}_t, R_t\right\}_t$ *and* $\left\{\tau_t, \hat{T}_t\right\}_t$ *constitute a competitive equilibrium given* $\left(\left\{\hat{b}_{i,-1}\right\}_i, \hat{B}_{-1}\right)$.

?

clumsy!

Proof. Let

$$\hat{T}_t = T_t + \left(\hat{b}_{1,t} - b_{1,t}\right) - R_{t-1}\left(\hat{b}_{1,t-1} - b_{1,t-1}\right) \text{ for all } t \geq 0. \tag{7}$$

Given a tax policy $\left\{\tau_t, \hat{T}_t\right\}_t$, the allocation $\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_{i,t}$ is a feasible choice for consumer i since it satisfies

$$\begin{aligned}
c_{i,t} &= (1-\tau_t)\,\theta_{i,t}l_{i,t} + R_{t-1}b_{i,t-1} - b_{i,t} + T_t \\
&= (1-\tau_t)\,\theta_{i,t}l_{i,t} + R_{t-1}\left(b_{i,t-1} - b_{1,t-1}\right) - \left(b_{i,t} - b_{1,t}\right) + T_t + R_{t-1}b_{1,t-1} - b_{1,t} \\
&= (1-\tau_t)\,\theta_{i,t}l_{i,t} + R_{t-1}\left(\hat{b}_{i,t-1} - \hat{b}_{1,t-1}\right) - \left(\hat{b}_{i,t} - \hat{b}_{1,t}\right) + T_t + R_{t-1}b_{1,t-1} - b_{1,t} \\
&= (1-\tau_t)\,\theta_{i,t}l_{i,t} + R_{t-1}\hat{b}_{i,t-1} - \hat{b}_{i,t} + \hat{T}_t.
\end{aligned}$$

Suppose that $\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_{i,t}$ is not the optimal choice for consumer i, in the sense that there exists some other sequence $\left\{\hat{c}_{i,t}, \hat{l}_{i,t}, \hat{b}_{i,t}\right\}_t$ that gives strictly higher utility. Then the choice $\left\{\hat{c}_{i,t}, \hat{l}_{i,t}, b_{i,t}\right\}_t$ is feasible given the tax rates $\{\tau_t, T_t\}_t$, which contradicts the assumption that $\{c_{i,t}, l_{i,t}, b_{i,t}\}_t$ is the optimal choice for the consumer given taxes $\{\tau_t, T_t\}_t$. The new allocation satisfies all other constraints and therefore is an equilibrium. ■

An immediate corollary is that it is not total government debt but rather who owns it that affects equilibrium allocations.

Corollary 1 *For any pair B'_{-1}, B''_{-1}, there are asset profiles $\left\{b'_{i,-1}\right\}_i$ and $\left\{b''_{i,-1}\right\}_i$ such that equilibrium allocations starting from $\left(\left\{b'_{i,-1}\right\}_i, B'_{-1}\right)$ and from $\left(\left\{b''_{i,-1}\right\}_i, B''_{-1}\right)$ are the same. These asset profiles satisfy*

$$b'_{i,-1} - b'_{1,-1} = b''_{i,-1} - b''_{1,-1} \text{ for all } i.$$

We note that the result continues to hold in more general environments. For example, we could allow agents to trade all conceivable Arrow securities or allow for capital accumulation and still show that equilibrium allocations depend only on agents' net assets positions.

Proposition 1 shows that many transfer sequences $\{T_t\}_t$ and asset profiles $\{b_{i,t}, B_t\}_{i,t}$ support the same equilibrium allocation. As such we use a normalization to define the notion of *public debt* for our setting. Assume that productivities are ordered with $\theta_{1,t} \leq \theta_{2,t} \ldots \leq \theta_{N,t}$. Using proposition 1 to set $b_{1,t} = 0$, we will interpret $-B_t = \sum_{i>1} n_i b_{i,t}$ as public debt. This inturn explains why imposing limits on $B_t - b_{1,t}$ are comparable to debt limits in a representative agent settings.

3. Optimal equilibria with affine taxes

We now focus on the characterization of optimal plans by applying the primal approach. This involves using household optimality conditions to obtain a set of restrictions on allocations chosen by the government to ensure that they are 'implementable' as competitive equilibria. In the last section we describe a recursive formulation of the Ramsey plan.

Assume that $U^i : \mathbb{R}^2_+ \to \mathbb{R}$ is concave in $(c, -l)$ and twice continuously differentiable. We let $U^i_{x,t}$ or $U^i_{xy,t}$ denote first and second derivatives of U^i with respect to $x, y \in \{c, l\}$ in period t and assume $\lim_{x\to 0} U^i_l(c, x) = 0$ for all c and i.

With natural borrowing limits for the households, first-order necessary conditions for the consumer's problem are

natural borrowing limits

$$(1 - \tau_t)\, \theta_{i,t} U^i_{c,t} = -U^i_{l,t}, \tag{8}$$

and

$$U^i_{c,t} = \beta \mathbb{E}_t R_{t+1} U^i_{c,t+1}. \tag{9}$$

To help characterize an equilibrium, we use

Proposition 1 *A sequence $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, R_t, \tau_t, T_t\}_t$ is part of a competitive equilibrium with affine taxes if and only if it satisfies (2), (4), (8), and (9) and $b_{i,t}$ is bounded for all i and t.*

Proof. Necessity is obvious. In appendix **??**, we use arguments of Magill and Quinzii (1994) and Constantinides and Duffie (1996) to show that any $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ that satisfies (4), (8), and (9) is a solution to consumer i's problem. Equilibrium $\{B_t\}_t$ is determined by (6) and constraint (5) is then implied by Walras' Law ■

To find an optimal equilibrium, by Proposition 1 we can choose $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, R_t, \tau_t, T_t\}_t$ to maximize (3) subject to (2), (4), (8), and (9). We apply a first-order approach and follow steps similar to ones taken by Lucas and Stokey (1983) and AMSS. Substituting consumers' first-order conditions (8) and (9) into the budget constraints (4) yields implementability constraints

$$c_{i,t} + b_{i,t} = -\frac{U^i_{l,t}}{U^i_{c,t}} l_{i,t} + T_t + \frac{p_t U^i_{c,t-1}}{\beta \mathbb{E}_{t-1} p_t U^i_{c,t}} b_{i,t-1} \text{ for all } i, t. \tag{10}$$

For $I \geq 2$, we can use constraint (10) for $i = 1$ to eliminate T_t from (10) for $i > 1$. Letting $\tilde{b}_{i,t} \equiv b_{i,t} - b_{1,t}$, we can represent the implementability constraints as

$$\begin{aligned} & (c_{i,t} - c_{1,t}) + \tilde{b}_{i,t} \\ = \quad & -\frac{U^i_{l,t}}{U^i_{c,t}} l_{i,t} + \frac{U^1_{l,t}}{U^1_{c,t}} l_{1,t} + \frac{p_t U^i_{c,t-1}}{\beta \mathbb{E}_{t-1} p_t U^i_{c,t}} \tilde{b}_{i,t-1} \text{ for all } i > 1, t. \end{aligned} \tag{11}$$

With this representation of the implementability constraints, the planner's maximization problem depends only on the $I - 1$ variables $\tilde{b}_{i,t-1}$. The reduction of the dimensionality from I to $I - 1$ is another consequence of theorem 1.

Denote $Z^i_t = (c_{i,t} - c_{1,t}) + \tilde{b}_{i,t} + \frac{U^i_{l,t}}{U^i_{c,t}} l_{i,t} - \frac{U^1_{l,t}}{U^1_{c,t}} l_{1,t}$. Formulated in a space of sequences, the optimal policy problem is:

$$\max_{c_{i,t},l_{i,t},\tilde{b}_{i,t}} \mathbb{E}_0 \sum_{i=1}^{I} \omega_i \sum_{t=0}^{\infty} \bar{\beta}_t U_t^i \left(c_{i,t}, l_{i,t}\right), \tag{12}$$

subject to

$$\tilde{b}_{i,t-1} \frac{p_t U^i_{c,t-1}}{\mathbb{E}_{t-1} p_t U^i_{c,t}} = \mathbb{E}_t \sum_{k=t}^{\infty} \beta^{k-t} \left(\frac{U^i_{c,k}}{U^i_{c,t}}\right) Z^i_k \quad \forall t \geq 1 \tag{13a}$$

$$\tilde{b}_{i,-1} = \mathbb{E}_{-1} \sum_{k=0}^{\infty} \beta^{k} \left(\frac{U^i_{c,k}}{U^i_{c,t}}\right) Z^i_k \tag{13b}$$

$$\frac{\mathbb{E}_t p_{t+1} U^i_{c,t+1}}{U^i_{c,t}} = \frac{\mathbb{E}_t p_{t+1} U^j_{c,t+1}}{U^j_{c,t}} \tag{13c}$$

$$\sum_{i=1}^{I} n_i c_i(s^t) + g\left(s_t\right) = \sum_{i=1}^{I} \pi_i \theta_i \left(s_t\right) l_i(s^t), \tag{13d}$$

$$\frac{U^i_{l,t}}{\theta_{i,t} U^i_{c,t}} = \frac{U^1_{l,t}}{\theta_{1,t} U^1_{c,t}} \tag{13e}$$

$$\sum_{i=1}^{N} \tilde{b}_{i,t-1} \text{ is bounded} \tag{13f}$$

Constraint (13a) is a measurablity restriction on allocations that requires that the right side is determined at time $t-1$. This condition is inherited from the restriction that there is only one only asset with payoffs given by p_t that is traded between the private and the public sector.

For both computational and educational purposes, it is convenient to represent the optimal policy problem recursively. For the purpose of constructing a recursive representation, let $\boldsymbol{x} = \beta^{-1}\left(U_c^2 \tilde{b}_2, ..., U_c^I \tilde{b}_I\right)$, $\boldsymbol{\rho} = \left(U_c^2/U_c^1, ..., U_c^I/U_c^1\right)$, and denote an allocation $a = \{c_i, l_i\}_{i=1}^I$. In the spirit of Kydland and Prescott (1980) and Farhi (2010), we split the Ramsey problem into a time-0 problem that takes $(\{\tilde{b}_{i,-1}\}_{i=2}^I, s_0)$ as given and a time $t \geq 1$ continuation problem that takes $\boldsymbol{x}, \boldsymbol{\rho}, s_{\_}$ as given. We formulate two Bellman equations and two value functions, one that pertains to $t \geq 1$, another to $t = 0$. The time inconsistency of an optimal policy manifests itself in there being distinct value functions and Bellman equations at $t = 0$ and $t \geq 1$.

For $t \geq 1$, let $V(\boldsymbol{x}, \boldsymbol{\rho}, s_{\_})$ be the planner's continuation value given $\boldsymbol{x}_{t-1} = \boldsymbol{x}, \boldsymbol{\rho}_{t-1} = \boldsymbol{\rho}, s_{t-1} = s_{\_}$. It satisfies the Bellman equation

$$V(\boldsymbol{x}, \boldsymbol{\rho}, s_{\_}) = \max_{a(s), x'(s), \rho'(s)} \sum_{s} \pi(s|s_{\_}) \left(\left[\sum_i \omega_i U^i(s)\right] + \beta V(\boldsymbol{x}'(s), \boldsymbol{\rho}'(s), s)\right) \tag{14}$$

where the maximization is subject to

$$U_c^i(s)\left[c_i(s)-c_1(s)\right]+x_i'(s)+\left(U_l^i(s)l_i(s)-U_c^i(s)\frac{U_l^1(s)}{U_c^1(s)}l_1(s)\right)=\frac{xP(s|s_{-})U_c^i(s)}{\beta\mathbb{E}_{s_{-}}P\boldsymbol{U}_c^i}\text{ for all } s, i\geq 2 \tag{15a}$$

$$\frac{\mathbb{E}_{s_{-}}P\boldsymbol{U}_c^i}{\mathbb{E}_{s_{-}}P\boldsymbol{U}_c^1}=\rho_i\text{ for all } i\geq 2 \tag{15b}$$

$$\frac{U_l^i(s)}{\theta_i(s)U_c^i(s)}=\frac{U_l^1(s)}{\theta_1(s)U_c^1(s)}\text{ for all } s, i\geq 2 \tag{15c}$$

$$\sum_i n_i c_i(s)+g(s)=\sum_i n_i(s)l_i(s)\ \ \forall s \tag{15d}$$

$$\rho_i'(s)=\frac{U_c^i(s)}{U_c^1(s)}\text{ for all } s, i\geq 2 \tag{15e}$$

$$\sum_{i>1}x_i(s)\frac{\beta}{U_c^i(s)}\text{is bounded} \tag{15f}$$

Constraints (15b) and (15e) imply (9). The definition of x_t and constraints (15a) together imply equation (11) scaled by U_c^i. Let $V_0\left(\{\tilde{b}_{i,-1}\}_{i=2}^I, s_0\right)$ be the value to the planner at $t=0$, where $\tilde{b}_{i,-1}$ denotes initial debt inclusive of accrued interest. It satisfies the Bellman equation

$$V_0\left(\{\tilde{b}_{i,-1}\}_{i=2}^I, s_0\right)=\max_{a_0,x_0,\rho_0}\sum_i\omega_i U^i(c_{i,0},l_{i,0})+\beta V\left(x_0,\rho_0,s_0\right) \tag{16}$$

where the maximization is subject to

$$U_{c,0}^i\left[c_{i,0}-c_{1,0}\right]+x_{i,0}+\left(U_{l,0}^i l_{i,0}-U_{c,0}^i\frac{U_{l,0}^1}{U_{c,0}^1}l_{1,0}\right)=U_{c,0}^i\tilde{b}_{i,-1}\text{ for all } i\geq 2 \tag{17a}$$

$$\frac{U_{l,0}^i}{\theta_{i,0}U_{c,0}^i}=\frac{U_{l,0}^1}{\theta_{1,0}U_c^{1,0}}\text{ for all } i\geq 2 \tag{17b}$$

$$\sum_i\pi_i c_{i,0}+g_0=\sum_i\pi_i\theta_{i,0}l_{i,0} \tag{17c}$$

$$\rho_{i,0}=\frac{U_{c,0}^i}{U_{c,0}^1}\text{ for all } i\geq 2 \tag{17d}$$

Because constraint (15b) is absent from the time 0 problem, the time 0 problem differs from the time $t\geq 1$ problem, a ~~source~~ of the time consistency of the optimal tax plan. The next section characterizes the properties of optimal plans.

reflection

4. Long run properties of optimal allocations

In sections 5. and 6. we characterize the long run properties of aggregate debt and taxes. The main finding is that the levels and spreads in debt and tax rates are determined by two factors: a) the ability of the government to span aggregate shocks through the returns on the asset it trades and b)its redistributive preferences. In particular, the government accumulates debt if interest rates are lower when the its need for revenue are higher and vice versa. The long run variance of debt and taxes along with the rates of rates of convergence to the ergodic distribution are higher in economies where the magnitude of this co movement is larger. And lastly more redistributive governments issue more debt.

To study these implications, in section 5. we first examine a simple economy with quasilinear preferences and i.i.d aggregate shocks. This allows us adequate tractability to formally demonstrate and clarify the main driving forces for the results mentioned above. In section 6. we study more general economies (in terms of heterogeneity, preferences and shocks) finally in section 7., we numerically verify that all the insights go through in a version of the model calibrated to US data.

5. Quasilinear economy

We specialize the problem described in section 3. by imposing the following assumptions that are maintained throughout in this section.

Assumption 1 *IID aggregate shocks: s_t is i.i.d over time*

Assumption 2 *Quasi linear preference:* $u(c,l) = c - \frac{l^{1+\gamma}}{1+\gamma}$

With i.i.d shocks we can restrict our attention to payoff matrices $\mathbb{P}$ that have identical rows denoted by the vector $P(s)$ with a corresponding normalization that $\mathbb{E}P(s) = 1$. We collect a particular set of these vectors that are perfectly correlated with expenditure shocks $g(s)$ in the set $\mathcal{P}^*$ defined below,

$$\mathcal{P}^* = \left\{ P(s) : P(s) = 1 + \frac{\beta}{B^*}(g(s) - \mathbb{E}g) \text{ for some } B^* \in [\overline{B}, \underline{B}] \right\},$$

where $\overline{B}$ and $\underline{B}$ are upper and lower bounds for government assets.

Before characterizing the properties of Ramsey allocation for the economy with heterogeneous agents and no restrictions on transfers, we develop some results in a representative agent economy where the government *cannot* use transfers. We later show that the allocations in this economy are obtained under certain limits on the Pareto weights for the setting with heterogeneous agents.

5.1. Representative agent

Environment

This section describes the representative agent environment with risky debt and no transfers.[4] Given a tax, asset policy $\{\tau_t, B_t\}$, the household solves,

$$W_0(b_{-1}) \max_{\{c_t,l_t,b_t\}_t} \mathbb{E}_0 \sum_t \beta^t \left[c_t - \frac{l_t^{1+\gamma}}{1+\gamma}\right] \tag{18}$$

subject to

$$c_t + b_t = (1-\tau_t)\theta l_t + R_t P_t b_{t-t} \tag{19}$$

Using the optimality condition for labor and savings we can summarize the set of implementability constraints for the government as follows

$$b_{t-1} \frac{P_t}{E_{t-1}P_t} = \mathbb{E}_t \sum_j \beta^{t+j}[c_t - l_t^{1+\gamma}] \quad \forall t \tag{20}$$

In addition we have also have the feasibility constraint

$$c_t + g_t \leq \theta l_t, \tag{21a}$$

and the market clearing for bonds,

$$b_t + B_t = 0. \tag{21b}$$

The optimal Ramsey allocation solves $\max_{\{c_t,l_t\}_t} W_0(b_{-1})$ subject to (20), feasibility (21a), market clearing for bonds (21b) and debt limits for the government $\underline{B}, \overline{B}$.[5]

Results

The main results are organized in Theorems 2 and 3. The first result obtains some general properties about the invariant distribution of debt for a large class of payoffs. The second result uses a novel expansion method to get an approximation to the mean and variance of the invariant distribution of debt when payoffs are close the set $\mathcal{P}^*$

When payoffs are not perfectly aligned, the support of the invariant distribution of debt is wide in the sense that (almost surely) the paths of debt sequences approach any arbitrary lower and upper bounds. Note that tax rates are increasing in debt and the variation in debt is analogously reflected in variation in tax rates. This can be contrasted with both, a complete market benchmark as in Lucas

[4]This differs from the model studied in AMSS in two ways: first, the government trades a “risky” bond instead of a risk free bond and second, the government is prohibited from using transfer where as AMSS restrict transfers to be non negative. Both of them have critical implications on the long run zero tax results that AMSS obtained. We discuss this later in the section.

[5]In some calculations we will use the natural debt limit for the government. In this case one can explicitly derived.

Stokey (1983) where both debt and tax rates will be constant sequences and AMSS (2002) which allows for non-negative transfers and risk-free debt where assets approach the first best and limiting tax rates are zero under these preferences.

With some more structure on the payoffs, we show that there is an average inward drift to government assets. More precisely, the multiplier on the implementability constraint is sub (or super) martingale in the region with low (or high) debt. The envelope theorem links the dynamics of the multiplier to that of debts and in turn the concavity of the value function implies mean reversion for debt. This is particularly stark when $P(s) \in \mathcal{P}^*(s)$ where debt converges to a constant. [6]

Next to gain more insights about the invariant distribution we linearize the law of motion for the evolution of debt with respect to both the endogenous state variable that is debt today and payoffs. The point of approximation is the closest (in l_2 sense) complete market economy corresponding to the steady state of some $P(s) \in \mathcal{P}^*(s)$. Exploiting the structure of these approximate laws of motion allows us to obtain bounds on the standard deviation of debt and also rates at which the mean debt level converges that can be expressed in terms of primitive: shocks and payoffs.

Theorem 2 *In the representative agent economy satisfying assumptions 1 and 2, the long run assets under the optimal Ramsey allocation are characterized as follows*

1. *Suppose $P \notin \mathcal{P}^*$, there is an invariant distribution of government such that*

$$\forall \epsilon > 0, \quad \Pr\{B_t < \underline{B} + \epsilon \textit{ and } B_t > \overline{B} - \epsilon \quad i.o\} = 1$$

2. *Suppose $P(s) - P(s') > \beta \frac{g(s)-g(s')}{\underline{B}} \quad \forall s, s'$, then for large enough assets (or debt) there is a drift towards the interior region. In particular the value function $V(B)$ is strictly concave and there exists $B_1 < B_2$ such that*

$$\mathbb{E}V'(B(s)) > V'(B_-) \quad B_- > B_2$$

and

$$\mathbb{E}V'(B(s)) < V'(B_-) \quad B_- < B_1$$

3. *Suppose $P(s) \in \mathcal{P}^*$, then the long run assets converge to a degenerate steady state*

$$\lim_t B_t = B^* \quad a.s \quad \forall B_{-1}$$

[6]Thus the limiting allocation is a particular Lucas Stokey (1983) with stationary debt and taxes, however the level and sign of the long run debt is determined by the joint properties of shocks and payoffs rather than initial condition as would be the case in Lucas Stokey(1983)

In the case where $P(s) \in \mathcal{P}^*$, we can express the long run assets

$$B^* = \beta \frac{\text{var}(g(s))}{\text{cov}(P(s), g(s))} \tag{22}$$

Keeping tax rates (and hence tax revenues in this case) the government needs to finance a higher primary deficit when it gets positive expenditure shock. If in such states the assets pays off more, then optimally the government holds positive assets and uses the these high returns to finance this deficit. On the other hand if payoff are lower in times when the government needs resources, holding debt is valuable since it lowers the interest burden. Thus using the level of its assets B^* it can perfectly span the fluctuations in deficits and the sign is given by the sign of the covariance of $P(s)$ with $g(s)$.

The long run tax rate is inversely related to B^* with the following limits,

$$\lim_{B^* \to \underline{B}} \tau^* = \frac{\gamma}{1+\gamma} \qquad \lim_{B^* \to \infty} \tau^* = -\infty$$

Now we proceed to obtain a sharper characterization of the invariant distribution. In general, there is no closed form solution for law of motion of government debt and hence we resort to an approximation. We begin with a orthogonal decomposition for an arbitrary $P(s)$,

$$P(s) = \hat{P}(s) + P^*(s)$$

where $P^*(s) \in \mathcal{P}^*$ and $\hat{P}(s)$ is orthogonal to $g(s)$. Expanding the policy rules around the steady state of the $P^*(s)$ economy we have the next theorem.[7]

Theorem 3 *The ergodic distribution of debt (using the first order approximation of dynamics near $P^*(s)$) has the following properties,*

- ***Mean:*** *The ergodic mean is B^* which corresponds to the steady state level of debt of an economy with payoff vector $P^*(s)$*

- ***Variance:*** *The coefficient of variation is given by*

$$\frac{\sigma(B)}{\mathbb{E}(B)} = \sqrt{\frac{\text{var}(P(s)) - |\text{cov}(g(s), P(s))|}{(1 + |\text{cov}(g(s), P(s))|)|\text{cov}(g(s), P(s))|}} \leq \sqrt{\frac{\text{var}(\hat{P}(s))}{\text{var}(P^*(s))}}$$

- ***Convergence rate:*** *The speed of convergence to the ergodic distribution described by*

$$\frac{\mathbb{E}_{t-1}(B_t - B^*)}{(B_{t-1} - B^*)} = \frac{1}{1 + |\text{cov}(P(s), g(s))|}$$

[7] Formally $P^*(s)$ is obtained by projecting $P(s)$ on the space spanned by $\mathcal{P}^*$. These approximations differ from those obtained in Woodford (XXX) where the point of approximation is a deterministic steady state. Appendix **??** contains more details of the approximation method including comparing outcomes to those obtained numerically solving the same economy.

Notice that when payoffs are equal to $P^*(s)$, the government can keep taxes constant and perfectly offset the fluctuations in its surplus with returns $P^*(s)B^*$. Away from this, the incompleteness of markets is binding and shocks are hedged with a combination of changes in tax rates and debt levels. These theorem shows exactly how the deviations from perfect spanning map into larger variances for debt (and taxes) in the long run. Figure 1 shows how the ergodic distribution of debt and taxes spread as we vary the covariance $P(s)$ with $g(s)$.

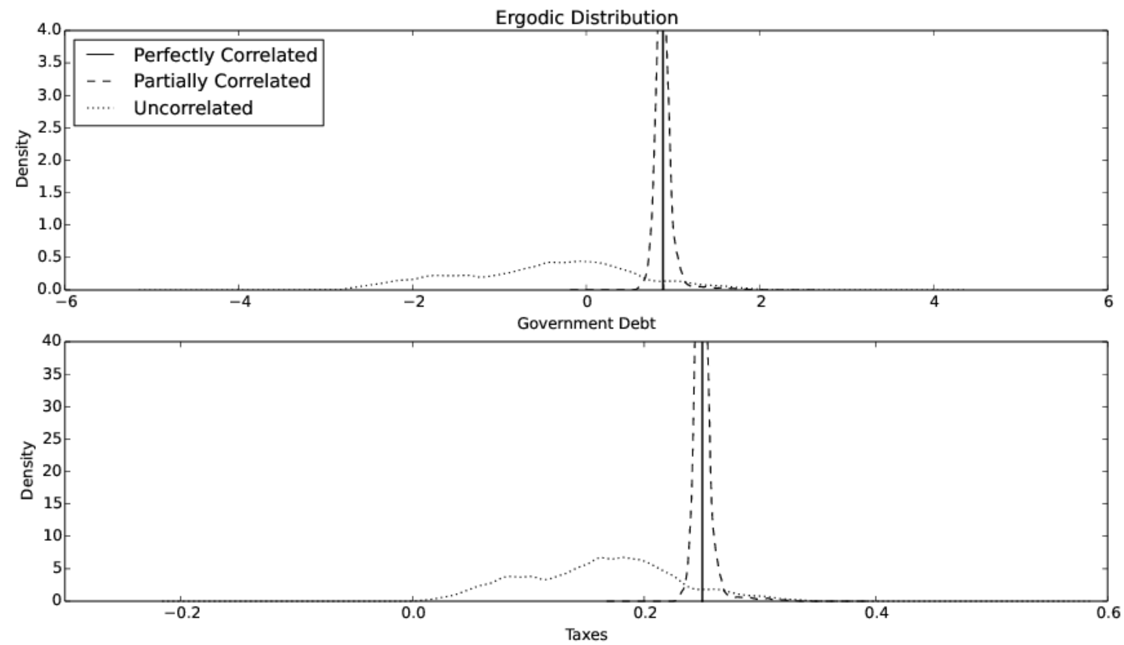

Figure 1: Ergodic distribution for debt and taxes in the representative agent quasilinear economy for three choices $P(s)$.

5.2. Heterogeneous agents

In this section we turn to general problem of characterizing outcomes in an economy with heterogeneous agents and transfers. In particular we introduce a second agent who has zero productivity and impose a non negativity constraint on his consumption. Given the Ricardian equivalnce result discussed in section **??**, we maintain a normalization that assets of the unproductive agent are zero throughout this section.

Assumption 3 *The productivity of agents are ordered,* $\theta_1 > \theta_2 = 0$ *and* $c_{2,t} \geq 0$.

Before we discuss the results, a few words on the assumptions are pertinent. The assumption that $\theta_2 = 0$ makes the problem tractable and allows us to obtain a complete characterization of the problem as we wary the Pareto weights. The restriction on consumption is necessary to add curvature to the problem. In more general settings risk aversion will impose Inada restrictions that play a similar role. We now state the theorem and then discuss its implications.

Theorem 4 *Let ω, n be the Pareto weight and mass of the productive agent with $n < \frac{\gamma}{1+\gamma}$. The optimal tax, transfer and asset policies $\{\tau_t, T_t, B_t\}$ are characterized as follows,*

1. *For $\omega \geq n\left(\frac{1+\gamma}{\gamma}\right)$ we have $T_t = 0$ and the optimal policy is same as in a representative agent economy studied in Theorems 2,and 3*

2. *For $\omega < n\left(\frac{1+\gamma}{\gamma}\right)$, suppose we further assume that $\min_s\{P(s)\} > \beta$. We have two parts:*

   There exists $\mathcal{B}(\omega)$ and $\tau^*(\omega)$ with $\mathcal{B}'(\omega) > 0$ and $\lim_{\omega\to 0}\mathcal{B}(\omega) < 0$ such that

   (a) $B_- > \mathcal{B}(\omega)$

   $$T_t > 0, \quad \tau_t = \tau^*(\omega), \text{ and } B_t = B_- \quad \forall t$$

   (b) $B_- \leq \mathcal{B}(\omega)$*, the policies depend on the structure of* $P(s)$.

   i. For $P(s) \notin \mathcal{P}^*$

   $$\lim_t T_t > 0 \textit{ i.o.}, \quad \lim_t \tau_t = \tau^*(\omega) \text{ and } \lim_t B_t = \mathcal{B}(\omega) \quad \textit{a.s}$$

   ii. For $P(s) \in \mathcal{P}^*$ *we have two cases depending on* B_-

   A. For $B_- \leq B^*$

   $$T_t = 0, \quad \lim_t \tau_t = \tau^{**}(\omega), \text{ and } \lim_t B_t = B^* \quad a.s$$

   B. For $\mathcal{B}(\omega) > B_- > B^*$

   $$\Pr\{\lim_t T_t = 0, \lim_t \tau_t = \tau^{**}(\omega), \lim_t B_t = B^* \text{ or } \lim_t T_t > 0 \textit{ i.o.}, \lim_t \tau_t = \tau^*(\omega), \lim_t B_t = \mathcal{B}(\omega)\} > 0$$

The main concern in this setting with heterogeneous agents is that costs of fluctuating transfers to hedge aggregate shocks are endogenous. The simplifications in the environment allow us to highlight how these depend on the Pareto weights (relative to the mass) of the Planner: $\{\omega, 1-\omega\}$ corresponding to Agents 1 and 2 respectively. A regressive planner who cares a lot about the productive agents in effect faces high costs of using transfers. For such a planner (with a high ω), increasing transfers also means giving resources to the unproductive agent whose consumption he does not value as much. In fact the threshold $\bar{\omega} = n\left(\frac{1+\gamma}{\gamma}\right)$ is such that above this, transfers are never used and thus the allocations are identical to the representative agent economy studied before.

For a less regressive planner (such that $\omega < \bar{\omega}$) transfers are an important tool for redistributing resources to the unproductive agent in adverse times. To finance these these it taxes the productive agent and does not need to accumulates a large buffer stock of assets. Thus limiting assets are lower and tax rates are larger for more redistributive planners.

6. More general economies

The analysis in the previous section was simplified in many dimensions - no curvature on the utility from consumption, lack of persistence in shocks and restricting heterogeneity to two agents. The key implication we could thereby exploit was that the return on debt was exogenous, given by $\beta^{-1}P(s)$. Adding curvature makes these returns endogenous even for standard risk free bond with a payoff vector $P(s) = 1$. Technically, it requires us to additionally keep track relative marginal utilities to characterize how taxes, debt, and transfers react to shocks. This makes it harder to separate spanning and redistribution concerns.

In the next section we first show that with risk aversion, exact spanning can occur only if shocks are binary and IID. For instance with CES preferences, the limiting allocation has constant relative consumptions and taxes rates. We show that how similar to the quasilinear case, the co movement of interest rates and exogenous shocks govern the governments' incentives to accumulate assets.

6.1. Spanning with binary shocks

Let $\Psi\left(s;\boldsymbol{x},\boldsymbol{\rho},s_{\_}\right)$ be an optimal law of motion for the state variables for the $t \geq 1$ recursive problem, i.e., $\Psi\left(s;\boldsymbol{x},\boldsymbol{\rho},s_{\_}\right) = \left(x'\left(s\right),\rho'\left(s\right)\right)$ solves (14) given state $\left(\boldsymbol{x},\boldsymbol{\rho},s_{\_}\right)$.

Definition 4 *A steady state* $\left(\boldsymbol{x}^{SS},\boldsymbol{\rho}^{SS}\right)$ *satisfies* $\left(\boldsymbol{x}^{SS},\boldsymbol{\rho}^{SS}\right) = \Psi\left(s;\boldsymbol{x}^{SS},\boldsymbol{\rho}^{SS},s_{-}\right)$ *for all* s, s_{-}.

Since in this steady state $\rho_i = U_c^i(s)/U_c^1(s)$ does not depend on the realization of shock s, the ratios of marginal utilities of all agents are constant. The continuation allocation depends only on s_t and not on the history s^{t-1}.

We begin by noting that a competitive equilibrium fixes an allocation $\{c_i(s), l_i(s)\}_i$ given a choice for $\{\tau(s), \boldsymbol{\rho}(s)\}$ using equations (15c), (15d) and (15e). Let us denote $U(\tau,\boldsymbol{\rho},s)$ as the value for the planner from the implied allocation using Pareto weights $\{\omega_i\}_i$,

$$U(\tau,\boldsymbol{\rho},s) = \sum_i \omega_i U^i(s).$$

As before define $Z_i(\tau, \rho, s)$ as

$$Z_i(\tau, \boldsymbol{\rho}, s) = U_c^i(s)c_i(s) + U_l^i(s)l_i(s) - \rho_i(s)\left[U_c^1(s)c_1(s) + U_l^1(s)l_1(s)\right].$$

For the IID case, the optimal policy solves the following Bellman equation for $\boldsymbol{x}(s^{t-1}) = \boldsymbol{x}, \boldsymbol{\rho}(s^{t-1}) = \boldsymbol{\rho}$

$$V(\boldsymbol{x}, \boldsymbol{\rho}) = \max_{\tau(s), \boldsymbol{\rho}'(s), \boldsymbol{x}'(s)} \sum_s \pi(s)\left[U(\tau(s), \boldsymbol{\rho}'(s), s) + \beta(s)V(\boldsymbol{x}'(s), \boldsymbol{\rho}'(s))\right] \tag{23}$$

subject to the constraints

$$Z_i(\tau(s), \boldsymbol{\rho}'(s), s) + x_i'(s) = \frac{x_i\beta^{-1}P(s)U_c^i(\tau(s), \boldsymbol{\rho}'(s), s)}{\mathbb{E}U_c^i(\tau, \rho)} \text{ for all } s, i \geq 2, \tag{24}$$

$$\sum_s \pi(s)P(s)U_c^1(\tau(s), \boldsymbol{\rho}'(s), s)(\rho_i'(s) - \rho_i) = 0 \text{ for } i \geq 2. \tag{25}$$

Constraint (25) is obtained by rearranging constraint (15b). It implies that $\rho(s)$ is a risk-adjusted martingale. We next check if the first-order necessary conditions are consistent with stationary policies for some $(\boldsymbol{x}, \boldsymbol{\rho})$.[8]

Lemma 1 *With risk aversion $\|S\| = 2$ is necessary for a steady state to exist*

Proof.

Let $\pi(s)\mu_i(s)$ and λ_i be the multipliers on constraints (24) and (25). Imposing the restrictions $x_i'(s) = x_i$ and $\rho_i'(s) = \rho_i$, at a steady state $\{\mu_i, \lambda_i, x_i, \rho_i\}_{i=2}^N$ and $\{\tau(s)\}_s$ are determined by the following equations

$$Z_i(\tau(s), \boldsymbol{\rho}, s) + x_i = \frac{\beta^{-1}P(s)x_iU_c^i(\tau(s), \boldsymbol{\rho}, s)}{\mathbb{E}U_c^i(\tau, \rho)} \text{ for all } s, i \geq 2, \tag{26a}$$

$$U_\tau(\tau(s), \boldsymbol{\rho}, s) - \sum_i \mu_i Z_{i,\tau}(\tau(s), \boldsymbol{\rho}, s) = 0 \text{ for all } s, \tag{26b}$$

$$U_{\rho_i}(\tau(s), \boldsymbol{\rho}, s) - \sum_j \mu_j Z_{j,\rho_i}(\tau(s), \boldsymbol{\rho}, s) + \lambda_i P(s)U_c^1(\tau(s), \boldsymbol{\rho}, s) - \lambda_i\beta\mathbb{E}P(s)U_c^1(\tau(s), \boldsymbol{\rho}(s), s) = 0. \text{ for all } s, i \geq 2 \tag{26c}$$

Since the shock s can take only two values, (26) is a square system in $4(N-1)+2$ unknowns $\{\mu_i^{SS}, \lambda_i^{SS}, x_i^{SS}, \rho_i^{SS}\}_{i=2}^N$ and $\{\tau^{SS}(s)\}_s$. ■

The behavior of the economy in the steady state is similar to the behavior of the complete market economy characterized by Werning (2007). Both taxes and transfers depend only on the current realization of shock s_t. Moreover, the arguments of Werning (2007) can be adapted to show that taxes are constant when preferences have a CES form $c^{1-\sigma}/(1-\sigma) - l^{1+\gamma}/(1-\gamma)$ and fluctuations in tax rates are

[8] Appendix **??** discuses the associated second order conditions that ensure these policies are optimal

very small when preferences take forms consistent with the existence of balanced growth. We return to this point after we discuss convergence properties.

The previous calculations provides a simple way to verify existence of a steady state for wide range of parameter values by checking that there exists a root for system (26). Since the system of equations (26) is non-linear, existence can generally be verified only numerically. Next, we provide a simple example with risk averse agents in which we can show existence of the root of (26) analytically. The analytical characterization of the steady state will help us develop some comparative statics and build a connection from the quasilinear economy to the quantitative analysis to appear in section 7..

A two-agent example

Consider an economy consisting of two types of households with $\theta_{1,t} > \theta_{2,t} = 0$. One period utilities are $\ln c - \frac{1}{2} l^2$. The shock s takes two values, $s \in \{s_L, s_H\}$ with probabilities $\Pr(s|s_{-})$ that are independent of s_{-}. We assume that $g(s) = g$ for all s, and $\theta_1(s_H) > \theta_1(s_L)$. We allow the discount factor $\beta(s)$ to depend on s.

Theorem 5 *Suppose that $g < \theta(s)$ for all s. Let $R(s)$ be the gross interest rates and $x = U_c^2(s)[b_2(s) - b_1(s)]$*

1. ***Countercyclical interest rates.*** *If $P(s_H) = P(s_L)$, then there exists a steady state $\left(x^{SS}, \rho^{SS}\right)$ such that $x^{SS} > 0,\ R^{SS}(s_H) < R^{SS}(s_L)$.*
2. ***Acyclical interest rates.*** *There exists a pair $\{P(s_H), P(s_L)\}$ such that there exists a steady state with $x^{SS} > 0$ and $R^{SS}(s_H) = R^{SS}(s_L)$.*
3. ***Procyclical interest rates.*** *There exists a pair $\{P(s_H), P(s_L)\}$ such that there exists a steady state with $x^{SS} < 0$ and $R^{SS}(s_H) > R^{SS}(s_L)$.*

In all cases, taxes $\tau(s) = \tau^{SS}$ are independent of the realized state.

In this two-agent case, by normalizing assets of the unproductive agent (using theorem 1) we can interpret x as the marginal utility adjusted assets of the government. Besides establishing existence, the proposition identifies the importance of cyclical properties of real interest rates in determining the sign of these assets.

Proposition 5 shows two main forces that determine the dynamics of taxes and assets: fluctuations in inequality and fluctuations in the interest rates. Let's start with part 2 of proposition 5, which turns off the second force. When interest rates are fixed, the government can adjust two instruments in response to an adverse shock (i.e., a fall in θ_1): it can either increase the tax rate τ or it can decrease transfers T. Both responses are distorting, but for different reasons. Increasing the tax rate increases distortions because the deadweight loss is convex in the tax rate, as in Barro (1979). This force operates in our economy just as it does in representative agent economies. But in a heterogeneous agent economy like ours, adjusting transfers T is also costly. When agents' asset holdings are identical, a decrease in transfers disproportionately affects a low-skilled agent, so his marginal utility falls by more than does the marginal utility of a high-skilled agent. Consequently, a decrease in transfers increases inequality, giving rise to a cost not present in representative agent economies.

The government can reduce the costs of inequality distortions by choosing tax rate policies that make the net asset positions of the high-skilled agent decrease over time. That makes the two agents' after-tax and after-interest income become closer, allowing decreases in transfers to have smaller effects on inequality in marginal utilities. If the net asset position of a high-skilled agent is sufficiently low, then a change in transfers has no effect on inequality and all distortions from fluctuations in transfers are eliminated.[9]

Turning now to the second force, interest rates generally fluctuate with shocks. Parts 1 and 3 of proposition **??** indicate what drives those fluctuations. Consider again the example of a decrease in productivity of high-skilled agent. If the tax rate τ is left unchanged, the government faces a shortfall of revenues. Since g is constant, the government requires extra sources of revenues. But suppose that the interest rate increases whenever θ_1 decreases, as happens, for example, with a risk free bond. If the government holds positive assets, its earnings from those assets increase. So holding assets allows higher interest income to offset some of the government's revenue losses from taxes on labor. The situation reverses if interest rates fall at times of increased need for government revenues, as in part 3 of proposition **??**, and the steady state allocation features the government's owning debt.

One can see the parallel with the representative agent quasilinear economy studied in section 5.. There, exploiting linearity allowed us to provide a sharper characterization of how co-variance of interest rates with exogenous shocks affected the sign (and level) of debt through expression 22. In parts 1 and 3 of the propositon, with binary shocks, altering the gap $P(s_H) - P(s_L)$ allows us to obtain the corresponding variation in interest rates. The resoning and the underling forces are exactly the same.

6.2. Stability

In this section we extend the approximation methods used to characterize outcomes in Theorem 3 to the general problem with risk aversion. Unlike of obtaining an the quasilinear case where we could obtain analytical characterization, we present a test for convergence show local stability of a steady state for a wide range of parameters.

[9] This convergence outcome has a similar flavor to "back-loading" results of Ray (2002) and Albanesi and Armenter (2012) that reflect the optimality of structuring policies intertemporally eventually to disarm distortions.

As before, let assume that $\pi(s)\mu_i(s)$ and λ_i be the multipliers on constraints (24) and (25). In Appendix **??** we show that the history-dependent optimal policies (they are sequences of functions of s^t) can be represented recursively in terms of $\{\boldsymbol{\mu}(s^{t-1}), \boldsymbol{\rho}(s^{t-1})\}$ and s_t. A recursive representation of an optimal policy can be linearized around the steady state using $(\boldsymbol{\mu}, \boldsymbol{\rho})$ as state variables.[10]

Formally, let $\hat{\Psi}_t = \begin{bmatrix} \boldsymbol{\mu}_t - \boldsymbol{\mu}^{SS} \\ \boldsymbol{\rho}_t - \boldsymbol{\rho}^{SS} \end{bmatrix}$ be deviations from a steady state. From a linear approximation, one can obtain $B(s)$ such that

$$\hat{\Psi}_{t+1} = B(s_{t+1})\hat{\Psi}_t. \tag{27}$$

This linearized system has coefficients that are functions of the shock. The next proposition describes a simple numerical test that allows us to determine whether this linear system converges to zero in probability.

Theorem 6 *If the (real part) of eigenvalues of $\mathbb{E}B(s)$ are less than 1, system (27) converges to zero in mean. Further for large t, the conditional variance of $\hat{\Psi}$, denoted by $\Sigma_{\Psi,t}$, follows a deterministic process governed by*

$$\textit{vec}(\Sigma_{\Psi,t}) = \hat{B}\,\textit{vec}(\Sigma_{\Psi,t-1}),$$

where $\hat{B}$ is a square matrix of dimension $(2I-2)^2$. In addition, if the (real part) of eigenvalues of $\hat{B}$ are less than 1, the system converges in probability.

The eigenvalues (in particular the largest or the dominant one) are instructive not only for whether the system is locally stable but also how quickly the steady state is reached. In particular, the half-life of convergence to the steady state is given by $\frac{\log(0.5)}{\|\iota\|}$, where $\|\iota\|$ is the absolute value of the dominant eigenvalue. Thus, the closer the dominant eigenvalue is to one, the slower is the speed of convergence.

We used Theorem 6 to verify local stability of a wide range of examples. Since the parameters space is high dimensional we relegate the comparative statics to Appendix **??**. The typical finding is that the steady state is stable and that convergence is slow. The rates of convergence are increasing in the covariance of interest rates and governments needs for revenue.

[10] One could in principle look for a solution in state variables $\left(\boldsymbol{x}(s^{t-1}), \boldsymbol{\rho}(s^{t-1})\right)$. For $I = 2$ with $\{\theta_i(s)\}$ different across agents, this would give identical policies and a map which is (locally) invertible between $\boldsymbol{x}$ and $\boldsymbol{\mu}$ for a given $\boldsymbol{\rho}$. However in other cases, it turns out there are unique linear policies in $(\ bm\mu, \boldsymbol{\rho})$ and not necessarily in $(\boldsymbol{x}, \boldsymbol{\rho})$. This comes from the fact that the set of feasible $(\boldsymbol{x}, \boldsymbol{\rho})$ are restricted at time 0 and may not contain an open set around the steady state values. When we linearize using $(\boldsymbol{\mu}, \boldsymbol{\rho})$ as state variables, the optimal policies for $\boldsymbol{x}(s^t), \boldsymbol{\rho}(s^t)$ converge to their steady state levels for all perturbations in $(\boldsymbol{\mu}, \boldsymbol{\rho})$.

7. Numerical example

In section **??** we used steady states to characterize the long-run behavior of optimal allocations and forces that guide the asymptotic level of net assets. In this section, we use a calibrated version of the economy to a) revisit the magnitude of these forces and and b) study optimal policy responses at business cycle frequencies when the economy is possibly far away from the steady state. We choose shocks and initial conditions to match stylized facts about recent recession in US. The numerical calculations use methods adapted from Evans (2014) and described more in the Appendix **??**. The next section outlines how we choose the parameters and initial conditions.

7.1. Calibration

We start with five types of agents[11] of equal measures with preferences $u(c,l) = \frac{c^{1-\sigma}}{1-\sigma} - \frac{l^{1+\gamma}}{1+\gamma}$. These agents will represent the $10^{\text{th}}, 25^{\text{th}}, 50^{\text{th}}, 75^{\text{th}}$, and 90^{th} percentile of US wage distribution.

We assume i.i.d aggregate shocks ϵ_t that affect labor productivities of each agent $\{\theta_{i,t}\}_{i=1}^N$ and payoff of the asset p_t as follows,

$$\log \theta_{i,t} = \bar{\theta}_i \epsilon_t [1 + (.9 - i)m]$$

$$p_t = 1 + \chi \epsilon_t$$

Following Autor et al. (2008), the average productivities $\{\bar{\theta}_i\}_{i=1}^N$ are set to match the respective earnings using Current Population Survey (CPS) that reports weekly earnings of full time wage and salary earners.

The parameter m allows us to generate recessions that lead to heterogeneous falls in income for different agents. We calibrate m to match the facts reported Guvenen et al. (2014). Figure 2 (adapted from their paper) reports that in the last recession the fall in income for the agents in the first decile of earnings was about three times that of the 90th percentile. Moreover between the 10th and the 90th change in the percentage drop in earnings was almost linear. This gives us a slope $m = \frac{1.5}{0.8}$.

The parameter χ captures the ex-post co-movement in returns to government holdings and aggregate shocks. Our model takes no explicit stand where these come from. In principle they could capture variations in payoffs due inflation, interest rate risk (for longer maturity bonds) or defaults in some countries. For the purpose of the numerical exercise we use US data on inflation and interest rates of longer maturities bonds to calibrate χ. This is described below:

Let $q_t^{(n)}$ be the log price of a nominal bond of maturity n. We can define the real holding period returns $r_{t,t+1}^{(n)}$ as follows

[11] We report the results for $N = 5$ to capture sufficient heterogeneity in wealth and earnings. Our methods allow us to solve for arbitrary number of agents and the qualitative/quantitative insights are unchanged by adding more intermediate types.

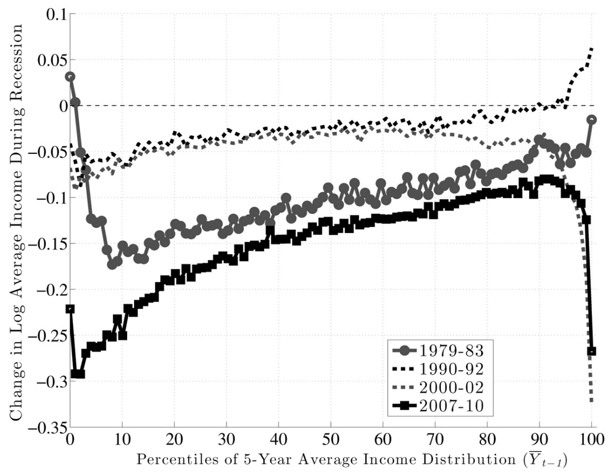

Figure 2: Change in log average earnings during recessions, prime-age males from Guvenen et al. (2014)

$$r_{t,t+1}^{(n)} = q_{t+1}^{(n-1)} - q_t^{(n)} - \pi_{t+1}$$

With the transfromation $y_t^{(n)} : -\frac{1}{n} q_t^{(n)}$ we can express $r_{t,t+1}^{(n)}$ as follows:

$$r_{t,t+1}^{(n)} = \underbrace{y_t^{(n)}}_{\text{Ex-ante part}} - (n-1)\left[\underbrace{\left(y_{t+1}^{(n)} - y_t^{(n)}\right)}_{\text{Interest rate risk given } n} + \underbrace{\left(y_{t+1}^{(n-1)} - y_{t+1}^{(n)}\right)}_{\text{Term structure risk}}\right] - \underbrace{\pi_{t+1}}_{\text{Inflation risk}}$$

In our model the holding period returns are given by $\log\left[\frac{p_{t+1}}{q_t}\right]$ and $q_t = \frac{\beta \mathbb{E}_t u_{c,t+1} P_{t+1}}{u_{c,t}}$. Note that p_{t+1} allows us to captures ex-post fluctuations in returns to the government's debt portfolio coming from maturity and inflation.

The next table summarizes the co-movement between labor productivity $\{\epsilon_t\}$ and bond prices $\{q_t^n\}_n$ for different maturities (using quarterly data for the period XXXX to XXXX)

Maturity (n)	2yr	3yr	4yr	5yr
$Corr(\epsilon_{t+1}, r_{t,t+1}^{(n)})$	-0.11	-0.093	-0.083	-0.072
$Corr(\epsilon_{t+1}, r_{t,t+1}^{(n)} - ny_t^{(n)})$	0.00	-0.0463	-0.080	-0.091
$Corr(\epsilon_{t+1}, y_t^{(n)} - \pi_{t+1})$	-0.097	-0.086	-0.080	-0.073
$\frac{\sigma(r_{t+1}^n)}{\sigma(\epsilon_{t+1})}$	0.820	0.835	0.843	0.845

Table 1:

The first line computes the correlation between the ex post returns and labor productivities. In our baseline calibration, ϵ_t is i.i.d over time. Hence the parameter $\chi = \frac{\sigma_r}{\sigma_\epsilon} Corr(r, \epsilon)$. Averaging over different maturities gives us a value of $\chi = -0.06$. [12] Thus payoffs are weakly countercyclical. Besides the results for the benchmark value of χ, the long run simulations in the next section we will report the results a large range of χ's from -1.0 to 1.0.

We next turn to the parameters that describe the household preferences : We set risk aversion, $\sigma = 1$, $\gamma = 2$. This yields a Frisch elasticity of labor supply of 0.5 and time discount factor, $\beta = 0.98$ such that the annual interest rate in an economy without shocks is 2% per year.

We assume that the initial wealth is perfectly correlated with wages and calibrate the wealth distribution to get the relative quantiles as in Kuh (2199) and Quadrini and Rios-Rull [2014]. These papers document the quantiles of net worth for US households computed up to 2010 Survey of Consumer Finances.

For the rest of parameters, namely Pareto weights and government expenditure, we use outcomes of the optimal allocation in the economy without shocks to target a (pre-trasnfers, federal) expenditure output ratio of 12%, tax rate of 23%, transfers to gdp of 10% and debt to gdp of 100%.

Parameter	Value	Description
$\{\bar{\theta}_i\}$	{1 , 1.4, 2.1, 3.24, 4.9}	Wages dispersion for {10,25,50,75,90} percentiles
γ	2	Average Frisch elasticity of labor supply of 0.5
β	0.98	Average (annual) risk free interest rate of 2%
m	$\frac{1.5}{.8}$	Heterogeneity in wage growth over business cycles
χ	-0.06	Covariance between holding period returns and labor productivity%
σ_e	0.03	vol of labor productivity
g	.13 %	Average pre-transfer expenditure-output ratio of 12 %

Table 2: Benchmark calibration

[12]The second line of table 3 computes only the correltaion of labor productivity with ex-post component of returns in the data. For the shortest maturity, 3 month real tbill returns $Corr(\epsilon_{t+1}, y_t^{1qtr} - \pi_{t+1}) = -0.11$. These results together give us range for χ of zero to negative -0.09. The numerical results are not sensitive to the value of χ is this range.

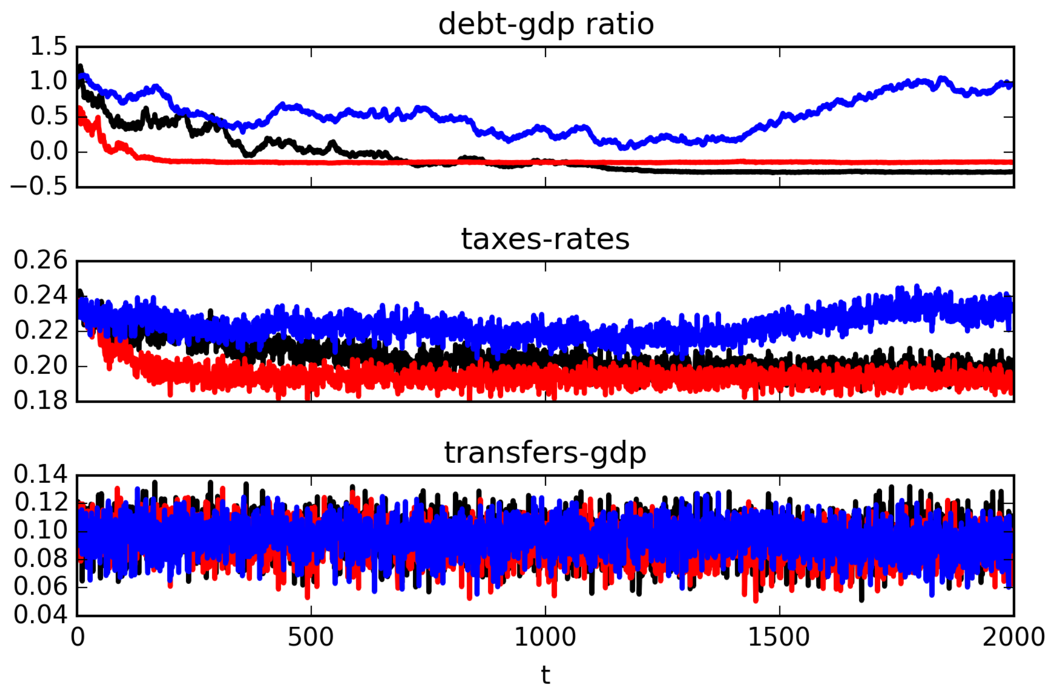

Figure 3: The red, black and blue lines plot simulations for a common sequence of shocks for values of $\chi = -1.5, 0, 1.5$ respectively

7.2. Long run outcomes

Figure 3 plots the simulated paths (of length 2000 periods) for debt to output ratio, labor tax rates and transfers to output ratio for three values of $\chi \in \{-1.0, -0.06, 1.0\}$ in red, black and blue respectively. The three simulations have same initial conditions and sequence of underlying shocks.

Two features emerge: Different values χ have implications for position and speed of convergence for long run assets of the government. A sufficiently positive χ generates lower payoffs in recessions relative to booms. In line with theorem 2 or theorem 5 , we see (blue line) that that government does not repay its initial debt for 2000 periods. On the other had under the benchmark (black line) or the when χ is negative (red line), the government accumulates assets.

In order to get a clearer picture of the speed of convergence, we plot the paths of the conditional means for debt and taxes in figure 4. To explain how we generate these plots, let $B(s_{t+1}, \boldsymbol{x}_t, \boldsymbol{\rho}_t)$ be the policy rules that generate the assets of the government and $\Psi\left(s_{t+1}; \boldsymbol{x}_t, \boldsymbol{\rho}_t\right)$, the law of motion for the state variables. For a given history, the conditional mean of government assets is defined as follows:

$$B_{t+1}^{cm} = \mathbb{E}B(s_{t+1}, \boldsymbol{x}_t^{cm}, \boldsymbol{\rho}_t^{cm}) \tag{29a}$$

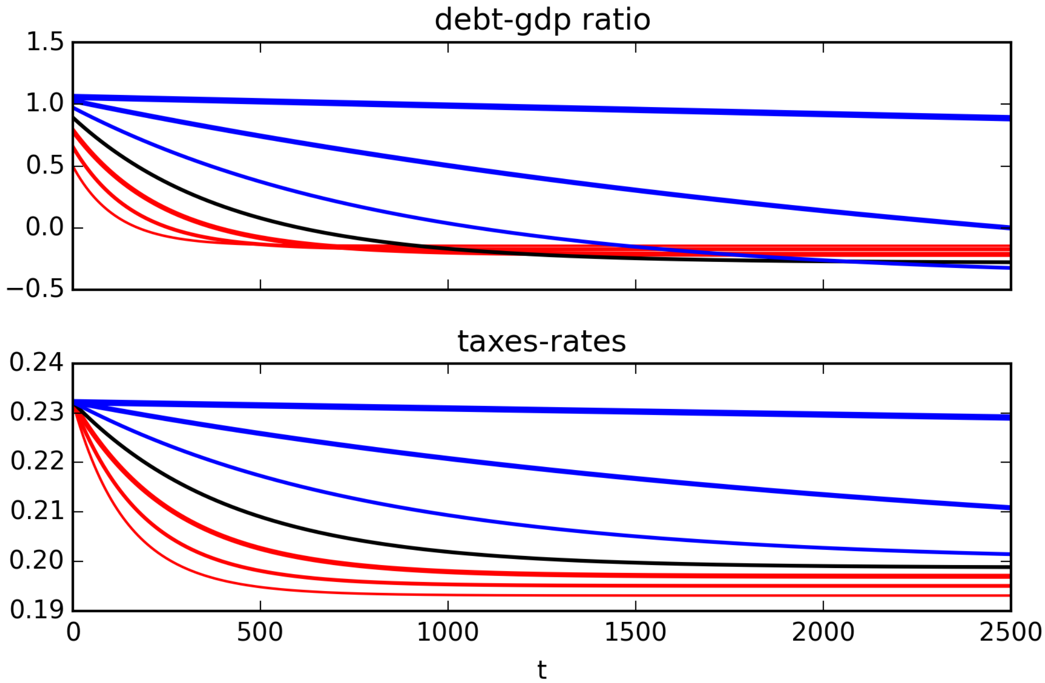

Figure 4: The plot shows conditional mean paths for different values of χ. The red (blue) lines have $\chi < 0$ ($\chi > 0$). The thicker lines represent larger values.

$$\boldsymbol{x}_t^{cm}, \boldsymbol{\rho}_t^{cm} = \mathbb{E}\Psi(s_t, \boldsymbol{x}_{t-1}^{cm}, \boldsymbol{\rho}_{t-1}^{cm}) \tag{29b}$$

Note that these paths smooths the high frequency movements in the dynamics of the state variables but retain the low frequency drifts. The different lines as before represent different values of χ between -1.0 and 1.0 with the blue (red) lines representing positive (negative) values of χ. The thickness of the lines represent larger values of χ. The figure clearly shows the speed of convergence is increasing and the magnitude of the limiting assets in decreasing the strength of correlation between productivities and payoffs. This confirms the approximation results characterized in Theorem 3.

To verify the wide support of the ergodic distribution we take the initial conditions at the end of the long simulation and subject the economy to a sequence of 100 periods of ϵ_t shocks which are 2 standard deviations below the mean. In figure 5 we see that given a sufficiently long sequence of negative productivity shocks the economy will eventually deviate significantly from its ergodic mean.

Our last conclusion was that the assets held by the government in the steady state was decreasing in the re-distributive motive of the government. We check this last part by changing the Pareto weights of the government. In our baseline case the government places equal Pareto weights on all agents. We introduce a re-distributive motive through a parameter α. The planner places evenly spaced Pareto

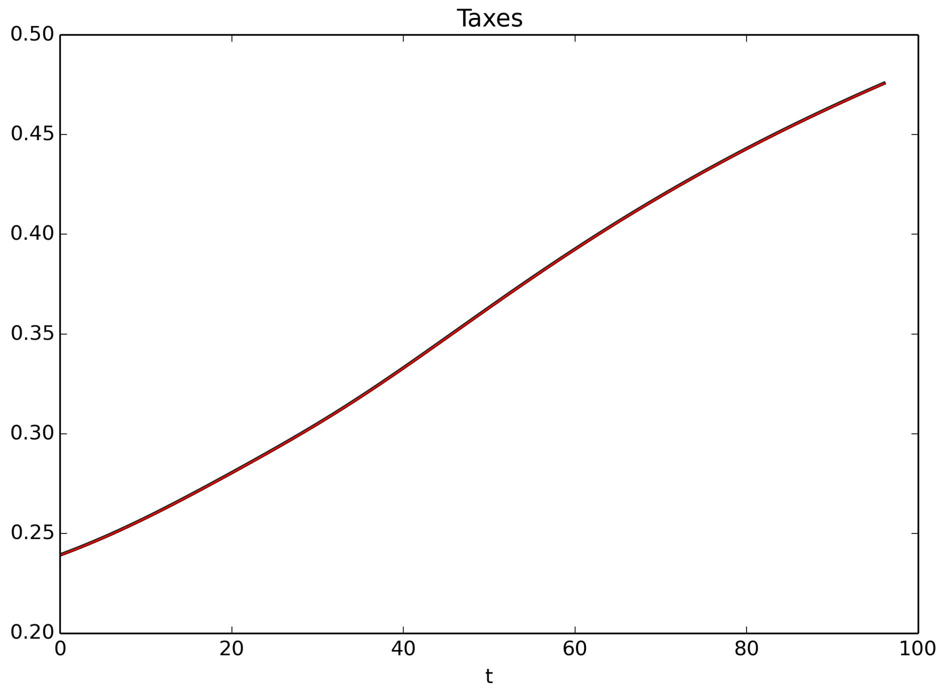

Figure 5: Taxes for a sequence of -1 s.d shocks to aggregate productivity of length 100

weights from $0.2-\alpha$ on the lowest productivity agent to $0.2+\alpha$ on the highest productivity agent. Increases α decreases the concerns for redistribution. We plot total assets of the government in steady state as a function of alpha in figure **??** and see that the relationship does indeed hold.

7.3. Short run

The analysis of the previous subsection studied aspects of very low frequency components of the optimal policy. Here we focus on business cycle frequencies. In our setting, these higher frequency responses can conveniently be divided into the magnitudes of changes as we switch from “boom” to “recession,” and the dynamics during periods when a recession or boom state persists. A recession is a negative -1.0 standard deviation realization for the ϵ_t process. Given the initial conditions and the benchmark calibration, the plots below trace the paths for debt, taxes and transfers for sequence of shocks that feature a recession of four periods from $t=3$. Before and after this recession, the economy receives $\epsilon_t=0$.

The main exercise here is to compute how optimal taxes, transfers and debt in recessions accompanied by larger inequality are different in a recession that affects all agents alike. Under the benchmark calibration, log wages for agent i are given by $\log\theta_i=\epsilon[1+(.9-i)m]$. We decompose the total responses

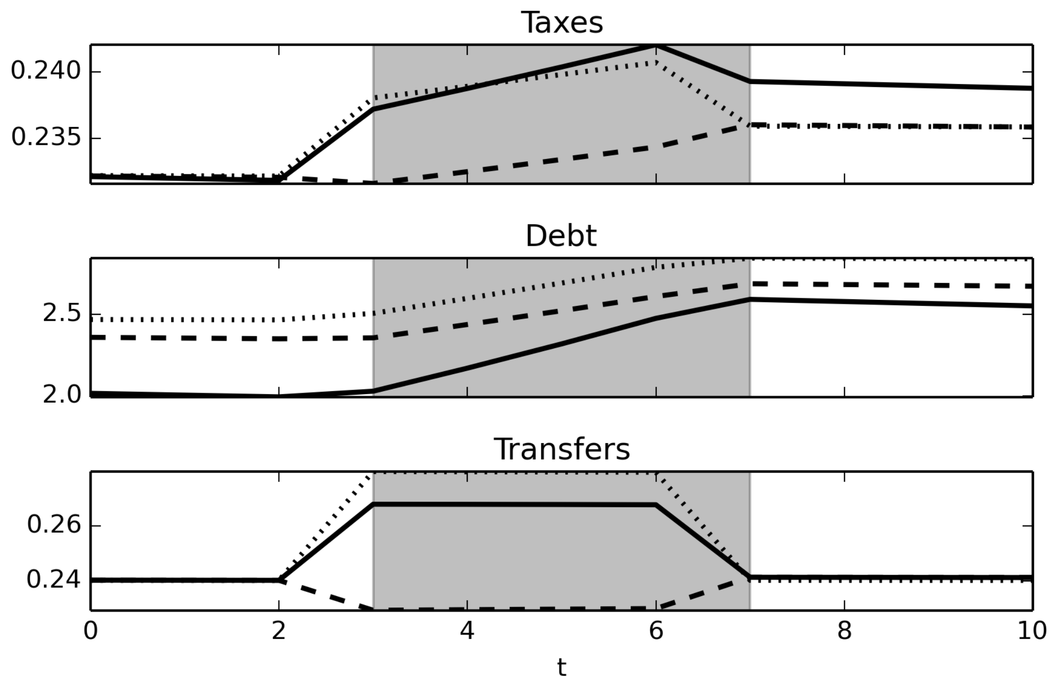

Figure 6: The bold line is the total response. The dashed (dotted) line reflects the only TFP (inequality) effect. The shaded region is the recession

into an only TFP component by (by setting $m = 0$) and an inequality component as follows:

$$\log \theta_i^{tfp} = \epsilon$$

$$\log \theta_i^{ineq} = \epsilon[(.9 - i)m]$$

Figure 6 plots the short run impulse responses. The shaded region is the induced recession and the bold line captures the the benchmark (total) response. The dashed (dotted) line reflects the only TFP (inequality) effect. In the benchmark, the government responds to an adverse shock by a making big increases in transfers, the tax rate, and government debt. However, without inequality shocks (dotted line), the government responds by decreasing transfers and increasing both debt and the tax rate, but by an amount an order of magnitude smaller than in the benchmark.

Next we average over sample paths of length 100 periods and report the volatility, autocorrelation and correlation with exogenous shocks for taxes and transfers in table 3. We see that taxes are twice as volatile and correlation between transfers and productivities switches sign. This indicates that ignoring distributional goals can produce a misleading prescriptions for government policy in recessions.

Moments	Tfp	Tfp+Ineq
vol. of taxes	0.003	0.006
vol. of transfers	0.01	0.02
autocorr. in taxes	0.93	0.66
autocorr. in transfers	0.17	0.18
corr. of taxes with tfp	0.15	-0.63
corr. of transfers with tfp	0.99	-0.98

Table 3: Sample moments for taxes and transfers averaged across simulations of 100 periods

8. Conclusion

A Appendix

A1. Extension: Borrowing constraints

Representative agent models rule out Ricardian equivalence either by assuming distorting taxes or by imposing ad hoc borrowing constraints. By way of contrast, we have verified that Ricardian equivalence holds in our economy even though there are distorting taxes. Imposing ad-hoc borrowing limits also leaves Ricardian equivalence intact in our economy.[13] In economies with exogenous borrowing constraints, agents' maximization problems include the additional constraints

$$b_{i,t} \geq \underline{b}_i \tag{30}$$

for some exogenously given $\{\underline{b}_i\}_i$.

Definition 5 *For given $\left(\{b_{i,-1}, \underline{b}_i\}_i, B_{-1}\right)$ and $\{\tau_t, T_t\}_t$, a competitive equilibrium with affine taxes and exogenous borrowing constraints is a sequence $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, B_t, R_t\}_t$ such that $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ maximizes (1) subject to (4) and (30), $\{b_{i,t}\}_{i,t}$ are bounded, and constraints (2), (5) and (6) are satisfied.*

We can define an *optimal* competitive equilibrium with exogenous borrowing constraints by extending Definition 3.

The introduction of the ad-hoc debt limits leaves unaltered the conclusions of Corollary 1 and the role of the initial distribution of assets across agents. The next proposition asserts that ad-hoc

[13] Bryant and Wallace (1984) describe how a government can use borrowing constraints as part of a welfare-improving policy to finance exogenous government expenditures. Sargent and Smith (1987) describe Modigliani-Miller theorems for government finance in a collection of economies in which borrowing constraints on classes of agents produce the kind of rate of return discrepancies that Bryant and Wallace manipulate.

borrowing limits do not limit a government's ability to respond to aggregate shocks.[14]

Proposition 2 *Given an initial asset distribution* $\left(\{b_{i,-1}\}_i, B_{-1}\right)$*, let* $\{c_{i,t}, l_{i,t}\}_{i,t}$ *and* $\{R_t\}_t$ *be a competitive equilibrium allocation and interest rate sequence in an economy without exogenous borrowing constraints. Then for any exogenous constraints* $\{\underline{b}_i\}_i$*, there is a government tax policy* $\{\tau_t, T_t\}_t$ *such that* $\{c_{i,t}, l_{i,t}\}_{i,t}$ *is a competitive equilibrium allocation in an economy with exogenous borrowing constraints* $\left(\{b_{i,-1}, \underline{b}_i\}_i, B_{-1}\right)$ *and* $\{\tau_t, T_t\}_t$.

Proof. Let $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ be a competitive equilibrium allocation without exogenous borrowing constraints. Let $\Delta_t \equiv \max_i \{\underline{b}_i - b_{i,t}\}$. Define $\hat{b}_{i,t} \equiv b_{i,t} + \Delta_t$ for all $t \geq 0$ and $\hat{b}_{i,-1} = b_{-1}$. By Theorem 1, $\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_{i,t}$ is also a competitive equilibrium allocation without exogenous borrowing constraints. Moreover, by construction $\hat{b}_{i,t} - \underline{b}_i = b_{i,t} + \Delta_t - \underline{b}_i \geq 0$. Therefore, $\hat{b}_{i,t}$ satisfies (30). Since agents' budget sets are smaller in the economy with exogenous borrowing constraints, and $\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_{i,t}$ are feasible at interest rate process $\{R_t\}_t$, then $\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_{i,t}$ is also an optimal choice for agents in the economy with exogenous borrowing constraints $\{\underline{b}_i\}_i$. Since all market clearing conditions are satisfied, $\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_{i,t}$ is a competitive equilibrium allocation and asset profile. ■

To provide some intuition for Proposition 2, suppose to the contrary that the exogenous borrowing constraints restricted a government's ability to achieve a desired allocation. That means that the government would want to increase its borrowing and to repay agents later, which the borrowing constraints prevent. But the government can just reduce transfers today and increase them tomorrow. That would achieve the desired allocation without violating the exogenous borrowing constraints.

Welfare can be strictly higher in an economy with exogenous borrowing constraints relative to an economy without borrowing constraints because a government might want to push some agents against their borrowing limits. When agents' borrowing constraints bind, their shadow interest rates differ from the common interest rate that unconstrained agents face. When the government rearranges tax policies to affect the interest rate, it affects constrained and unconstrained agents differently. By facilitating redistribution, this can improve welfare. We next construct an example without any shocks in which the government can achieve higher welfare by using borrowing constraints to improve its ability to redistribute. In this section we construct an example in which the government can achieve higher welfare in the economy with ad-hoc borrowing limits. We restrict ourselves to a deterministic economy with $g_t = 0$, $\beta_t = \beta$ and $I = 2$. Further the utility function over consumption and labor supply $U(c, l)$ is separable in the arguments and satisfies the Inada conditions. The planners

[14] See Yared (2012, 2013) who shows a closely related result.

problem can then be written as the following sequence problem

$$\max_{\{c_{i,t},l_{i,t},b_{i,t},R_t\}_t} \sum_{t=0}^{\infty} \beta^t \left[\alpha_1 U(c_{1,t}, l_{1,t}) + \alpha_2 U(c_{2,t}, l_{2,t})\right] \tag{31}$$

subject to

$$c_{2,t} + \frac{U_{l2,t} l_{2,t}}{U_{c2,t}} - \left(c_{1,t} + \frac{U_{l1,t} l_{1,t}}{U_{c1,t}}\right) + \frac{1}{R_t}\left(b_{2,t} - b_{1,t}\right) = b_{2,t-1} - b_{1,t-1} \tag{32a}$$

$$\frac{U_{l1,t}}{\theta_1 U_{c1,t}} = \frac{U_{l2,t}}{\theta_2 U_{c2,t}} \tag{32b}$$

$$c_{1,t} + c_{2,t} \leq \theta_1 l_{1,t} + \theta_2 l_{2,t} \tag{32c}$$

$$\left(\frac{U_{ci,t}}{U_{ci,t+1}} - \beta R_t\right)(b_{i,t} - \underline{b}_i) = 0 \tag{32d}$$

$$\frac{U_{ci,t}}{U_{ci,t+1}} \geq \beta R_t \tag{32e}$$

$$b_{i,t} \geq \underline{b}_i \tag{32f}$$

Where $\underline{b}_i$ is the exogenous borrowing constraint for agent i. We obtain equation (32a) by eliminating transfers from the budget equations of the households and using the optimality for labor supply decision. Equations (32d) and (32e) capture the inter-temporal optimality conditions modified for possbily binding constraints.

Let c_i^{fb} and l_i^{fb} be the allocation that solves the first best problem, that is maximizing equation (31) subject to (32c), and define

$$Z^{fb} = c_2^{fb} + \frac{U_{l2}^{fb} l_2^{fb}}{U_{c2}^{fb}} - \left(c_1^{fb} + \frac{U_{l1}^{fb} l_1^{fb}}{U_{c1}^{fb}}\right) \tag{33}$$

and

$$\tilde{b}_2^{fb} = \frac{Z^{fb}}{\frac{1}{\beta} - 1} \tag{34}$$

We will assume that the exogenous borrowing constraints satisfy $\underline{b}_2 = \underline{b}_1 + \tilde{b}_2^{fb}$. We then have the following lemma

Lemma 2 *If* $\tilde{b}_2^{fb} > (<) 0$ *and* $b_{2,-1} - b_{1,-1} > (<) \tilde{b}_2^{fb}$ *then the planner can implement the first best.*

Proof. We will consider the candidate allocation where $c_{i,t} = c_i^{fb}$, $l_{i,t} = l_i^{fb}$, $b_{i,t} = \underline{b}_i$ and interest rates are given by $R_t = \frac{1}{\beta}$ for $t \geq 1$. It should be clear then that equations (32b) and (32c) are satisfied as a property of the first best allocation. Equation (32d) is trivially satisfied since the agents are at their borrowing constraints. For $t \geq 1$ equations (32a) and (32e) are both satisfied by the choice of $R_t = \frac{1}{\beta}$ and the first best allocations. It remains to check that equation (32a) is satisfied at time $t = 0$ for an interest rate $R_0 < \frac{1}{\beta}$. At time zero the constraint is give by

$$Z^{fb} + \frac{1}{R_0}\tilde{b}_2^{fb} = b_{2,-1} - b_{1,-1} \tag{35}$$

The assumption that $b_{2,-1} - b_{1,-1} > (<)\tilde{b}_2^{fb}$ if $\tilde{b}_2^{fb} > (<)0$ then implies that

$$R_0 = \frac{\tilde{b}_2^{fb}}{b_{2,-1} - b_{1,-1} - Z^{fb}} < \frac{1}{\beta}$$

as desired. ■

This will improve upon the planners problem without exogenous borrowing constraints, as first best can only be achieved in this scenario when $b_{2,-1} - b_{1,-1} = \tilde{b}_2^{fb}$.

References

Albanesi, S., and R. Armenter. 2012. "Intertemporal Distortions in the Second Best." *The Review of Economic Studies*, 79(4): 1271–1307, URL: http://restud.oxfordjournals.org/lookup/doi/10.1093/restud/rds014, DOI: http://dx.doi.org/10.1093/restud/rds014.

Autor, DH, LF Katz, and MS Kearney. 2008. "Trends in US wage inequality: Revising the revisionists." *The Review of Economics and Statistics*, 90(2): 300–323, URL: http://www.mitpressjournals.org/doi/abs/10.1162/rest.90.2.300, DOI: http://dx.doi.org/10.1162/rest.90.2.300.

Barro, Robert J. 1974. "Are government bonds net wealth?." *The Journal of Political Economy*, 82(6): 1095–1117, URL: http://www.jstor.org/stable/10.2307/1830663.

Barro, Robert J. 1979. "On the determination of the public debt." *The Journal of Political Economy*, 87(5): 940–971, URL: http://www.jstor.org/stable/10.2307/1833077.

Bryant, John, and Neil Wallace. 1984. "A Price Discrimination Analysis of Monetary Policy." *The Review of Economic Studies*, 51(2): , p. 279, URL: http://restud.oxfordjournals.org/lookup/doi/10.2307/2297692, DOI: http://dx.doi.org/10.2307/2297692.

Constantinides, George, and Darrell Duffie. 1996. "Asset pricing with heterogeneous consumers." *Journal of Political economy*, 104(2): 219–240, URL: http://www.jstor.org/stable/10.2307/2138925.

Farhi, Emmanuel. 2010. "Capital Taxation and Ownership When Markets Are Incomplete." *Journal of Political Economy*, 118(5): 908–948, URL: http://www.jstor.org/stable/10.1086/657996.

Guvenen, Fatih, Serdar Ozkan, and Jae Song. 2014. "The Nature of Countercyclical Income Risk." *Journal of Political Economy*, 122(3): pp. 621–660, URL: http://www.jstor.org/stable/10.1086/675535.

, "Trends in income and wealth inequality."

Kydland, Finn E, and Edward C Prescott. 1980. "Dynamic optimal taxation, rational expectations and optimal control." *Journal of Economic Dynamics and Control*, 2(0): 79–91, URL: http://www.sciencedirect.com/science/article/pii/0165188980900524, DOI: http://dx.doi.org/http://dx.doi.org/10.1016/0165-1889(80)90052-4.

Lucas, Robert E, and Nancy L Stokey. 1983. "Optimal fiscal and monetary policy in an economy without capital." *Journal of Monetary Economics*, 12(1): 55–93, URL: http://www.sciencedirect.com/science/article/pii/0304393283900491, DOI: http://dx.doi.org/http://dx.doi.org/10.1016/0304-3932(83)90049-1.

Magill, Michael, and Martine Quinzii. 1994. "Infinite Horizon Incomplete Markets." *Econometrica*, 62(4): 853–880, URL: http://www.jstor.org/stable/2951735, DOI: http://dx.doi.org/10.2307/2951735.

Ray, Debraj. 2002. "The Time Structure of Self-Enforcing Agreements." *Econometrica*, 70(2): 547–582, URL: http://onlinelibrary.wiley.com/doi/10.1111/1468-0262.00295/full.

Sargent, Thomas J., and Bruce D. Smith. 1987. "Irrelevance of open market operations in some economies with government currency being dominated in rate of return." *American Economic Review*, 77(1): 78–92, URL: http://ideas.repec.org/a/aea/aecrev/v77y1987i1p78-92.html.

Yared, Pierre. 2012. "Optimal Fiscal Policy in an Economy with Private Borrowing Limits." DOI: http://dx.doi.org/10.1111/jeea.12010.

Yared, Pierre. 2013. "Public Debt Under Limited Private Credit." *Journal of the European Economic Association*, 11(2): 229–245, URL: http://doi.wiley.com/10.1111/jeea.12010, DOI: http://dx.doi.org/10.1111/jeea.12010.