A Ramsey planner redistributes and finances government expenditures by using proportionate labor taxes, transfers, and borrowing or lending. The only way to borrow or lend is to exchange a risk-free bond with consumer-workers. Restrictions on transfers limit the government's ability to redistribute across agents and also to hedge government expenditure shocks. Therefore, those restrictions affect the debt dynamics under a Ramsey plan.

It is instructive to disentangle the consequences of the government's motives to redistribute from its motives to hedge fiscal shocks. So we begin by abstracting from redistribution in order to isolate how restrictions on transfers affect optimal fiscal policy. Our tool for doing this is the representative agent economy of section 1 that generalizes a model of AMSS by considering alternatives to the nonnegativity restriction that AMSS imposed on transfers. We describe how changing that restriction has big consequences for debt dynamics and therefore for the dynamics of the labor tax rate. To activate interactions between preferences about redistribution and with motives for fiscal hedging, section 2 turns to an environment with two types of agents. The possible presence of nonnegativity restrictions on one of the agent type's consumption together with particular sets of Pareto weights in effect endogenizes the restrictions on on transfers in ways that mimic some of those imposed exogenously in the section 1 representative agent analysis. Section 2 reinterprets one of the section 1 economy as a special case of a two-agent economy.

1 Representative agent

A representative agent has a one-period quasi linear utility function $u(c,l) = c - \frac{l^{1+\gamma}}{1+\gamma}$ and trades a one-period risk-free bond. Using the primal approach, we use the agent's first-order conditions for its choices of labor and consumption to express the labor tax rate and the one-period risk-free interest rate in terms of the allocation, and then substitute them into the consumer's budget constraint to get a sequence of implementability conditions that, in addition to feasibility of an allocation, restrict the Ramsey planner's choice of an allocation and government debt sequence $\{b_t\}_{t=0}^{\infty}$. The Ramsey allocation solves following problem given initial government debt (which equals the consumer's initial assets) b_{-1}:

$$W(b_{-1}) = \max_{\{c_t, l_t, b_t, T_t\}_t,} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[c_t - \frac{l_t^{1+\gamma}}{1+\gamma} \right] \tag{1}$$

subject to the following constraints at $t \geq 0$

$$c_t + b_t = l_t^{1+\gamma} + \beta^{-1} b_{t-1} + T_t, \tag{2}$$

$$c_t + g_t \leq \theta l_t \tag{3}$$

$$\underline{b} \leq b_t \leq \bar{b} \tag{4}$$

We impose arbitrary (sometimes called *ad hoc*) limits on government debt, though it is not difficult to extend the analysis to allow these to be 'natural debt limits' that are themselves functions of the government's policy.

The optimal allocation depends restrictions imposed on the domain of transfers. Theorem 1 describes outcomes under three cases that progressively restrict transfers $\{T_t\}$ more and more.

Theorem 1 *For the representative agent with quasilinear utility,*

1. ***[First Best]:*** *If T_t are unrestricted*

$$\tau_t = 0, l_t = \theta^{1/\gamma}, c_t = \theta^{1+\frac{1}{\gamma}} - g_t, T_t = b_{-1}(\beta^{-1}-1) - g_t, b_t = b_{-1}$$

2. ***[AMSS]:*** *If $T_t \geq 0$ and $\underline{b} < \frac{-\max_s g(s)}{\beta^{-1}-1}$*

$$\lim_t \tau_t = 0, \lim_t l_t = \theta^{\frac{1}{\gamma}}, \lim_t c_t = \theta^{1+\frac{1}{\gamma}} - g_t, \lim_t b_t = \frac{-\max_s g(s)}{\beta^{-1}-1}, T_t > 0 \quad i.o$$

3. ***[BEGS]:****If $T_t \equiv 0$, there is an invariant distribution of government assets such that*

$$\forall \epsilon > 0, \quad \Pr\{b_t < \underline{b} + \epsilon \quad \textit{or } b_t > \bar{b} - \epsilon \quad i.o\} = 1$$

Furthermore, across economies as $\underline{b} \to -\infty$,

$$\lim_t b_t = -\infty, \lim_t \tau_t = -\infty$$

For cases 2 and 3 we can show that labor taxes, $\tau_t = \tau(s_t|b_{t-1})$ with $\frac{\partial \tau(s_t|b_{t-1})}{\partial b_{t-1}} > 0$. Suppose $\bar{b}$ is the natural debt limit for the government, then

$$\lim_{b_- \to \bar{b}} \tau(s|b_-) = \frac{\gamma}{1+\gamma} \textit{ and } \lim_{b_- \to -\infty} \tau(s|b_-) = -\infty$$

Whether and when the implementability constraint (2) binds is the symptom that drives diverse outcomes across the three cases set forth in theorem 1. The sequence of Lagrange multipliers on this sequence of constraints is the intermediating object through which shocks and the Ramsey planner's purposes affect debt dynamics. Outcomes differ from the first best (i.e the allocation with $\tau_t = 0, \forall t \geq 0$) only if implementability constraint (2) ever binds. The Lagrange multipliers on these constraints at any date t measure the marginal welfare costs of raising revenues through distorting taxes. The multiplier's initial value depends on the government's initial debt b_{-1}. Restrictions on T_t affect the multiplier's movement over time.

Case 1 [**First best**] describes how with no restriction on T_t, the implementability constraints in equations (2) are always slack (the multipliers are zero for all $t \geq 0$). This enables the planner to use lump sum taxes to implement the first best allocation for $t \geq 0$.

In case 2 [**AMSS**], transfers are restricted to be non negative, as assumed by AMSS. Here, under a Ramsey plan the multipliers form a martingale. The non negativity constraint on transfers implies that the martingale has an upper bound of zero. A martingale convergence theorem implies that the martingale converges to its upper bound almost surely, so eventually the implementability constraint (2) will become slack. The dynamics of asset holdings reflect the government's incomplete ability permanently to hedge government expenditure shocks with transfers. The incompleteness of markets imparts to the government a precautionary motive that causes it to accumulate assets. Eventually, the Ramsey plan sends the tail of allocation to the first best; associated with this tail allocation is a level of government assets sufficiently large that the government's interest earnings are big enough to finance the biggest possible expenditure realization. After government assets have attained this level, excess government revenues are returned to the consumer in the form of positive transfers. A zero labor tax rate supports the first-best tail allocation.

Case 3 [**BEGS**] eliminates transfers altogether by setting T_t to zero at all histories. Effectively the government looses the ability completely to do without the distorting labor tax that it has immediately in case 1 and that it acquires eventually in case 2 after accumulating sufficient assets. An argument by counterexample shows how a Ramsey plan can't set a constant rate on labor. For any given level of initial government assets, assume to the contrary that the tax rate is constant across all values of shocks that can occur next period. This implies fluctuations in the net of interest deficit that in the absence of transfers lead to fluctuations in the government's asset holdings. Thus, depending on the realized shock the government will have more or less assets next period. Since the tightness of the implementability constraint depends on the assets with which the government enters a period, marginal costs of raising resources through taxes are *not* constant. This contradicts the optimality of policy keeps labor taxes constant. The optimal plan features fluctuating debt/taxes. Depending on the sequences of shocks, debt and taxes can vary widely along an outcome path. The ergodic distribution of debt visits all neighborhoods of the bounds $\underline{b}, \bar{b}$ on government debt in inequalities (4) with probability one. Since the labor tax rate depends on government debt, it varies accordingly.

2 Two QL agents

To study how restrictions on transfers interfere with a Ramsey planner's ability to redistribute and how that affects debt dynamics, we alter the section 1 economy to include a positive mass of unproductive agents who also have quasilinear preferences. The Ricardian equivalence property lets us normalize the assets of these unproductive agents to zero so that the budget constraint of the second type of agent becomes simply $c_{2t} = T_t$ (because the agent is unproductive and has zero assets). In this section, we allow transfers themselves to be unrestricted. Instead we might or might not impose a nonnegativity constraint on c_{2t} (but not on c_{1t}). But of course, since the type 2 agent's budget constraint makes $c_{2t} = T_t$, a nonnegativity constraint on c_{2t} immediately translates into a nonnegativity constraint on T_t. Further, depending on the Pareto weights that express preferences for redistribution, it is possible for outcomes to make the section 1 T_t restricted to zero case 3 emerge as a choice by the Ramsey planner in the present setting.

A Ramsey planner solves

$$W(b_{1,-1}) = \max_{\{c_{1,t},c_{2,t},l_{1,t},b_{1,t}\}_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\omega \left(c_{1,t} - \frac{l_{1,t}^{1+\gamma}}{1+\gamma} \right) + (1-\omega) c_{2,t} \right] \tag{5}$$

subject to the following two constraints for $t \geq 0$

$$c_{1,t} - c_{2,t} + b_{1,t} = l_{1,t}^{1+\gamma} + \beta^{-1} b_{1,t-1} \tag{6}$$

$$nc_{1,t} + (1-n)c_{2_t} \leq n\theta l_{1,t} \tag{7}$$

$$\underline{b} \leq b_t \leq \bar{b} \tag{8}$$

The allocation depends on constraints on $c_{2,t}$. Given the normalization that assets of agent 2 are zero we have transfers $T_t = c_{2,t}$

Theorem 2 • *If $c_{2,t}$ is unconstrained,*

1. *The problem is concave only if* $\omega < n\left(\frac{1+\gamma}{\gamma}\right)$. *In this case the supremum* $W(b_{-1})$ *is finite and the (unique) optimal government debt satisfies* $b_t = b_{-1}$. *The Ramsey allocation is*

$$l_{1,t} = \left(\frac{n\theta_1}{\omega - (\omega - n)(1+\gamma)}\right)^{\frac{1}{\gamma}},$$

$$c_{2,t} = nl_{1,t}(\theta_1 - l_{1,t}^{\gamma}) - n(\beta^{-1} - 1)b_{-1} - g_t,$$

$$c_{1,t} = (1-n)(\beta^{-1} - 1)b_{-1} - g_t + l_{1,t}^{1+\gamma} - n(\theta_1 - l_{1,t}^{\gamma}).$$

2. *For* $\omega \geq n\left(\frac{1+\gamma}{\gamma}\right)$, *the Ramsey planner is maximizing a convex function over an unbounded choice set, so the problem is ill posed and no solution exists. However, we can construct another Ramsey problem that is well posed and take a limit that in a sense approximates and tells us something about that ill posed Ramsey problem. Suppose* $\tau_t \geq \underline{\tau}$. *The optimal policy will have a corner solution for taxes with* $\tau_t = \underline{\tau}, b_t = b_{-1}, T_t = \underline{\tau} n\theta_1 l_1(\underline{\tau}) - (\beta^{-1} - 1)b_{-1} - g_t)$ *for all* $t \geq 0$ *and as* $\underline{\tau} \to -\infty$, $W(b_{-1})$ *approaches* ∞.

- *If* $c_{2,t} \geq 0$

1. *For* $\omega \geq n\left(\frac{1+\gamma}{\gamma}\right)$, $c_{2,t} = T_t = 0$. *The optimal policy* $\{\tau_t, b_{1,t}\}$ *is identical with that for the section 1 representative agent economy without transfers provided that we set* $b_{1,t} = b_t$.
2. *For* $\omega < n\left(\frac{1+\gamma}{\gamma}\right)$ *there exist a* $\mathcal{B}(\omega)$ *satisfying* $\mathcal{B}'(\omega) < 0$ *and a* $\tau^*(\omega)$ *such that*

   (a) *If* $b_{1,-1-} \leq \mathcal{B}(\omega)$

   $$T_t > 0, \quad \tau_t = \tau^*(\omega), \text{ and } b_{1,t} = b_{1,-1} \quad \forall t \geq 0$$

   (b) *If* $b_{1,-1} > \mathcal{B}(\omega)$ *then*

   $$T_t > 0 \text{ i.o.}, \quad \lim_t \tau_t = \tau^*(\omega) \text{ and } \lim_t b_{1,t} = \mathcal{B}(\omega) \quad a.s$$

With two types of agents, using transfers to hedge aggregate shocks entails also giving resources to agents about whom the planner may care too little. The welfare costs of transfers mainly depend on Pareto weights attached to an agent's type relative to its mass. These costs are low when the Pareto weight ω relative to its mass n of a productive agent is low. If the welfare costs of using transfers to finance government expenditures is high, the government will instead use labor income taxes despite their distortions.

But in our first case, in which when the consumption of agent 2 can be negative, for low enough values of ω, the planner hedges all fiscal shocks with transfers and keeps both government debt and the labor tax rate constant. The tax rate is decreasing ω and approaches $-\infty$ when ω reaches the threshold $n\left(\frac{1+\gamma}{\gamma}\right)$.

By way of contrast, when we impose a non-negativity constraint on $c_{2,t}$, the Ramsey planner can't use negative transfers in that way. In this case, the optimal policy gradually reduces government debt (or builds up government assets) until the debt holdings reach a threshold where the planner can finance fluctuations in the net-of interest deficit entirely with positive transfers and keep the labor tax rate constant. This threshold level of government debt is higher for a more redistributive planner. This is because such low ω planners collect higher revenues from labor taxes, making the earnings on asset holdings required to finance government expenditure shocks be smaller.

For large enough ω, for any level of initial government debt, transfers are too costly and hence the non-negativity constraint implies that they are equal to zero at all dates. In this case, our two-types-of-agent economy here leads to a Ramsey plan and allocation that is equivalent to the representative agent economy allocation from section 1 in which the restriction $T_t = 0$ was exogenously imposed.[1]

[1] Here is what would happen if we were to alter the structure in the following way. Alter the type 2 agent's one period utility function to be the power function $\frac{c_2^{1-\gamma}}{1-\gamma}$ with $\gamma > 0$, and retain all other features of the specification. Now if $\underline{b} = \infty$,

- If $n > \frac{\gamma}{1+\gamma}$

$$\tau_t = \tau^*(n, \gamma), T_t = \infty, b_t \to -\infty$$

  and

- If $n < \frac{\gamma}{1+\gamma}$

$$\tau_t = -\infty, T_t = T^* < \infty, b_t \to -\infty$$

Thus, government assets diverge for all Pareto weights $\omega \in (0, 1)$. Adding a finite lower bound to b_t implies that $\{b_t\}$ converges to an invariant distribution whose properties are difficult to characterize analytically.