This is a simple environment to explore the long run properties of the economies with incomplete markets and quasi-linear preferences.

We have a prepresentation agent with preferences $c - \frac{1}{\gamma} l^{\gamma}$ and discount factor β. The government imposes linear taxes to finance exogenous government expenditures. The expenditures have to be at least g_t in each period, but can be a bigger number. I do it this way because otherwise I do not know how to show concavity and differentiability of the value function.

Each state $s \in S$ (for now s only stands for g shocks). Shocks are iid with probability $\pi(s)$. I will do the general structure of payoffs: an assets pays off $p(s)$ and the price of asset is q_t. We have

$$\beta \sum_{s \in S} \pi(s) p(s) = q_t$$

so q_t is fixed over time. I will normalize $\sum_{s \in S} \pi(s) p(s) = 1$, so that $q_t = \beta^{-1}$ for all t.

Agents cannot have more than $\bar{b}$ units of assets where $\bar{b}$ is the natural debt limit for the government. Agent can pay out arbitrary amounts, so $\in (-\infty, \bar{b}]$.

We have the Ramsey problem

$$\max_{\{c_t, l_t, b_t\}} E_0 \sum \beta^t \left[c_t - \frac{1}{\gamma} l_t^{\gamma} \right]$$

s.t.

$$c_t + b_t = l_t^{\gamma} + p_t b_{t-1}$$
$$c_t + g_t \leq l_t$$

To make this problem convex, let $L \equiv l^{\gamma}$. In this case this problem is

$$\max_{\{c_t, l_t, b_t\}} E_0 \sum \beta^t \left[c_t - \frac{1}{\gamma} L_t \right]$$

s.t.

$$c_t + b_t = L_t + p_t b_{t-1}$$
$$c_t + g_t \leq L_t^{1/\gamma}$$

There are two ways to write Bellman equations for this problem. One is ex-ante Bellman equation

$$V_{e.a.}(b) = \max_{c, L, b'} \sum_{s \in S} \pi(s) \left[c(s) - \frac{1}{\gamma} L(s) + \beta V_{e.a.}(b(s)) \right]$$

s.t.

$$\begin{aligned} c(s) + b(s) &= L(s) + \frac{1}{\beta} p(s) b \\ c(s) + g(s) &\leq L^{1/\gamma}(s) \\ b(s) &\leq \bar{b} \\ L(s) &\geq 0. \end{aligned}$$

Substitute for $c(s)$

$$V_{e.a.}(b) = \max_{L,b'} \sum_{s\in S} \pi(s) \left[\frac{\gamma-1}{\gamma} L(s) + \frac{1}{\beta} p(s)\, b - b(s) + \beta V_{e.a.}(b(s)) \right]$$

s.t.

$$\begin{aligned} \frac{1}{\beta} p(s)\, b - b(s) + g(s) &\leq L^{1/\gamma}(s) - L(s) \\ b(s) &\leq \bar{b} \\ L(s) &\geq 0. \end{aligned}$$

Lemma 1 *$V_{e.a}(b)$ is stictly concave, continuous, differentiable and $V'_{e.a}(b) < \beta^{-1}$ for all $b < \bar{b}$. The feasibility constraint binds for all $b \in (-\infty, \bar{b}]$, $s \in S$ and $(L^*(s))^{1-1/\gamma} \geq 1/\gamma$.*[1] *Also, when $\bar{b}$ is the natural debt limit, I think that*

$$\begin{aligned} \lim_{b\to-\infty} V'_{e.a.}(b) &= \frac{1}{\beta} - 1 \\ \lim_{b\to\bar{b}} V'_{e.a.}(b) &= -\infty. \end{aligned}$$

Proof. *Concavity*

$V_{e.a}(b)$ is concave because we maximize linear objective function over convex set.

Binding feasibility

Suppose that feasibility does not bind for some b, s. Then the optimal $L(s)$ solve $\max_{L(s)\geq 0} \pi(s) \frac{\gamma-1}{\gamma} L(s)$ which sets $L(s) = \infty$. This violates feasility for any finite $b, b(s)$.

Bounds on L

Let $\lambda(s) > 0$ be a Lagrange multiplier on the feasibility. The FOC for $L(s)$ is

$$\frac{\gamma-1}{\gamma} + \lambda \left(\frac{1}{\gamma} L^{1/\gamma-1} - 1 \right) = 0.$$

This gives

$$\frac{1}{\gamma} L^{1/\gamma-1} - 1 = -\frac{1}{\lambda} \frac{\gamma-1}{\gamma} < 0$$

or

$$L^{1-1/\gamma} \geq \frac{1}{\gamma}.$$

Continuity

For any L that satisfy $L^{1-1/\gamma} \geq 1/\gamma$, define function Ψ that satisfies $\Psi\left(L^{1/\gamma} - L\right) = L$. Since $L^{1/\gamma} - L$ is strictly decreasing in L for $L^{1-1/\gamma} \geq 1/\gamma$, this function

[1] This last condition simply means that we do not tax to the right of the peak of the Laffer curve. The revenue maximizing tax is $1 - \bar{\tau} = \frac{1}{\gamma}$. At the same time $1 - \tau = l^{\gamma-1}$ so if taxes are always to the left of the peak, $\frac{1}{\gamma} \leq l^{\gamma-1} = \left(L^{1/\gamma}\right)^{\gamma-1} = L^{1-1/\gamma}$.

is well defined. Note that $\Psi'\left(\right)\underbrace{\left(\frac{1}{\gamma}L^{1/\gamma-1}-1\right)}_{<0}=1$ (so that $\Psi'>0$, i.e. Ψ is strictly decreasing) and $\Psi''\underbrace{\left(\frac{1}{\gamma}L^{1/\gamma-1}-1\right)^2}_{>0}+\underbrace{\Psi'}_{<0}\underbrace{\frac{1}{\gamma}\frac{1-\gamma}{\gamma}L^{1/\gamma-2}}_{<0}=0$ (so that $\Psi''\geq 0$, $\Psi''>0$, i.e. Ψ is strictly concave on the interior). Ψ is also continuous. When $L^{1-1/\gamma}=1/\gamma$, $L=\gamma^{-\gamma/(\gamma-1)}$. Let $D\equiv\gamma^{-1/(\gamma-1)}-\gamma^{-\gamma/(\gamma-1)}$. Then the objective is

$$V_{e.a.}\left(b\right)=\max_{b(s)}\sum_{s\in S}\pi\left(s\right)\left[\Psi\left(\frac{1}{\beta}p\left(s\right)b-b\left(s\right)+g\left(s\right)\right)+\frac{1}{\beta}p\left(s\right)b-b\left(s\right)+\beta V_{e.a.}\left(b\left(s\right)\right)\right]$$

s.t.

$$\begin{aligned} b\left(s\right) &\leq \bar{b} \\ \frac{1}{\beta}p\left(s\right)b-b\left(s\right)+g\left(s\right) &\leq D. \end{aligned}$$

This function is continuous so $V_{e.a.}$ is also continuous.

Differentiability

Continuity and convexity implies differentiability everywhere, including the boundaries.

Strict concavity

Ψ is strictly concave, so on the interior $V_{e.a}$ is strictly concave.

Value of derivative

Away from the boundary $V'_{e.a.}\left(b\right)=\frac{1}{\beta}+\sum_{s\in S}\pi\left(s\right)\Psi'<\frac{1}{\beta}$. I think we can also show that the boundary conditions. As $b\to\bar{b}$, $\Psi'\to\Psi\left(0\right)$, I think. From $\Psi'\left(\right)\left(\frac{1}{\gamma}L^{1/\gamma-1}-1\right)=1$, as $b\to\bar{b}$ if $\bar{b}$ is a natural debt limit, then $L\to\gamma^{-\gamma/(\gamma-1)}$ and $\left(\frac{1}{\gamma}L^{1/\gamma-1}-1\right)\to 0$ so this equation can be satisfied only if $\Psi'\to-\infty$. On the other hand, as $b\to-\infty$, $L\to\infty$ and $\left(\frac{1}{\gamma}L^{1/\gamma-1}-1\right)\to-1$. Hence $\Psi'\left(\right)\to-1$. If this analysis is correct, then $\lim_{b\to-\infty}V'_{e.a.}\left(b\right)=\frac{1}{\beta}-1>0$ and $\lim_{b\to\bar{b}}V'_{e.a.}\left(b\right)=-\infty$. ■

Next we characterize policy functions

Lemma 2 *$b\left(s\right)$ is an increasing function of b for all s for all $\left(b,s\right)$ where $b\left(s\right)$ is interior.*

Proof. Take the FOCs for $b\left(s\right)$ from the condition in the previous problem. If $b\left(s\right)$ is interior

$$\Psi'\left(\frac{1}{\beta}p\left(s\right)b-b\left(s\right)+g\left(s\right)\right)=\beta V'_{e.a.}\left(b\left(s\right)\right).$$

Suppose $b_1 < b_2$ but $b_2(s) < b_1(s)$. Then from stict concavity

$$\begin{aligned} V'_{e.a.}(b_2(s)) &< V'_{e.a.}(b_1(s)) \\ \Psi'\left(\frac{1}{\beta}p(s)b_2 - b_2(s) + g(s)\right) &> \Psi'\left(\frac{1}{\beta}p(s)b_1 - b_1(s) + g(s)\right). \end{aligned}$$

■

This analysis builds up to the following result

Proposition 3 *For any $b \in (-\infty, \bar{b})$, there are s', s'' s.t. $b(s') \geq b \geq b(s'')$. Moreover, if there are any states s'', s''' s.t. $b(s'') \neq b(s''')$, those inequalities are strict.*

Proof. The envelope theorem gives $V'(b) = \frac{1}{\beta}\sum_{s\in S}\mu(s)p(s)$. The focs for $b'(s)$ are $\pi(s)V'(b'(s)) = \frac{1}{\beta}\mu(s) + \frac{1}{\beta}\zeta(s)$. Mutliply by $p(s)$:

$$\begin{aligned} \sum_{s\in S}\pi(s)p(s)V'_{e.a.}(b(s)) &= \frac{1}{\beta}\sum_{s\in S}\mu(s)p(s) + \frac{1}{\beta}\sum_{s\in S}\zeta(s)p(s) \\ &\geq \frac{1}{\beta}\sum_{s\in S}\mu(s)p(s) \\ &= V'_{e.a.}(b)\sum_{s\in S}\pi(s)p(s). \end{aligned}$$

Let $\tilde{\pi}(s) \equiv \pi(s)p(s)/\sum_{s\in S}\pi(s)p(s)$. Then it becomes

$$\sum_{s\in S}\tilde{\pi}(s)V'_{e.a.}(b(s)) \geq V'_{e.a.}(b).$$

If there is at least one $b(s')$ s.t. $b(s') > b$, by strict concavity of V there must be some s'' s.t. $b(s'') < b$.

If there is at least one $b(s')$ s.t. $b(s') < b$, the inequality above is strictly only if $b(s''') = \bar{b}$ for some s'''. But $V'_{e.a.}\left(\bar{b}\right) < V'_{e.a.}(b)$ so there must be some s'' s.t. $b(s'') > b$. Equality is possible only if $b = b(s)$ for all s. ■

Now we can use these insights to prove some results. I will order states such that $g(s_1) < ... < g\left(s_{|S|}\right)$.

0.1 Risk-free debt

Proposition 4 *Suppose agents trade a risk-free bond, $p(s) = 1$ for all s. Then for any b, $b(s_1) < ... < b\left(s_{|S|}\right)$ if $b(s)$ are interior. Moreover, if $\bar{b}$ is the natural debt limit,*

$$b(s_1) < b \leq b\left(s_{|S|}\right)$$

Proof. Follows again from

$$\Psi'\left(\frac{1}{\beta}b - b(s) + g(s)\right) = \beta V'_{e.a.}(b(s))$$

and same steps as the previous proof. If we hit the natural debt limit, we can only hit it for the worst possible shock, $g\left(s_{|S|}\right)$, for all others we should be below. At $b = \bar{b}$, we stay at $\bar{b}$ only for the worst shock, otherwise we deaccumulate assets. ■

This shows that if we get the worst shock, the government accumulates debt, if we get the best shock it deaccumulates.

Corollary 5 *When debt is risk-free, $b_t \to -\infty$ w.p. 1*

Proof. When $p(s) = 1$,

$$\sum_{s \in S} \pi(s) V'_{e.a.}(b(s)) \geq V'_{e.a.}(b)$$

and so $V'_{e.a.}(b(s))$ is a submartingale which is bounded above by $\beta^{-1} - 1$. Hence it need to converge, but the only place it can converge in light of the previous lemma is to the point $\beta^{-1} - 1$, i.e. for $b \to -\infty$. ■

When $b_t \to -\infty$ we violate some other contraints about which we were sloppy, i.e. the ones that rule out Ponzi schemes. Suppose we add an additional constraint that $b_t \geq \underline{b}$ for all t. Then we have the following result, which I think is more indicative of the general forces. I write it as a conjecture since some technical conditions need to be checked for it.

Conjecture 6 *Suppose we augment our problem with a constraint $b_t \geq \underline{b}$. Then with risk-free debt there is an invariant distribution ψ. Morever, for any $\hat{b} \in \left(\underline{b}, \bar{b}\right)$, $\psi\left(\left[\underline{b}, \hat{b}\right]\right) > 0$ and $\psi\left(\left[\hat{b}, \bar{b}\right]\right) > 0$.*

Proof. Our b lies in a compact set, so some invariant distribuiton exists under quite mild condition. For long sequence of bad shocks, b keeps strictly decreasing, for a long sequence of good shocks b is increasing. Under some technical conditions, this should imply that there is a positive mass both in the neighborhood of $\underline{b}$ and $\bar{b}$. ■

0.2 Risky debt

Now consider risky debt. Let $b(s; b)$ be the policy functions given b. The key result is the following

Proposition 7 *For any s', s'', $b(s'; \cdot)$ and $b(s''; \cdot)$ can only cross once.*

Proof. Our optimality condition

$$\Psi'\left(\frac{1}{\beta} p(s) b - b(s; b) + g(s)\right) = \beta V'_{e.a.}(b(s; b)).$$

Suppose they cross more than once. That is, there are $\tilde{b}, \hat{b}$ and s'' and s' such that $b\left(s'; \hat{b}\right) = b\left(s''; \hat{b}\right)$ and $b\left(s'; \tilde{b}\right) = b\left(s''; \tilde{b}\right)$. Then

$$\begin{aligned} \frac{1}{\beta} p(s') \hat{b} - b\left(s'; \hat{b}\right) + g(s') &= \frac{1}{\beta} p(s'') \hat{b} - b\left(s''; \hat{b}\right) + g(s'') \\ \frac{1}{\beta} p(s') \tilde{b} - b\left(s'; \tilde{b}\right) + g(s') &= \frac{1}{\beta} p(s'') \hat{b} - b\left(s''; \tilde{b}\right) + g(s'') \end{aligned}$$

Take out equalities

$$\begin{aligned} \frac{1}{\beta} p(s') \hat{b} + g(s') &= \frac{1}{\beta} p(s'') \hat{b} + g(s'') \\ \frac{1}{\beta} p(s') \tilde{b} + g(s') &= \frac{1}{\beta} p(s'') \hat{b} + g(s'') \end{aligned}$$

Substract

$$p(s')\left(\hat{b} - \tilde{b}\right) = p(s'')\left(\hat{b} - \tilde{b}\right)$$

It implies that $\hat{b} \neq \tilde{b}$ only if $p(s') = p(s'')$. But if $p(s') = p(s'')$, our optimality condition

$$\frac{1}{\beta} p(s') \hat{b} - b\left(s'; \hat{b}\right) + g(s') = \frac{1}{\beta} p(s'') \hat{b} - b\left(s''; \hat{b}\right) + g(s'')$$

becomes

$$g(s') = g(s''),$$

which is a contradiction. ■

This proposition does not say that the policy function must necessarily cross. They must cross for level $b^*_{s',s''}$ that solves

$$\frac{1}{\beta} p(s') b + g(s') = \frac{1}{\beta} p(s'') b + g(s'').$$

This equation always has a solution, but we need to ensure that it is feasible, i.e. $b^*_{s',s''} \leq \bar{b}$ (if we have constraints to rule out Ponzi schemes, those can be chosen ex-post to not bind).

Assumption A. $\{p(s), g(s)\}$ are such that $b^*_{s',s''} \leq \bar{b}$ for all s', s''.

This assumption can easy to make in terms of the primitives.

The two shock case easily follows

Proposition 8 *Suppose $S = 2$ and Assumption A is satisfied. Then $b_t \to b^*_{s_1,s_2}$ w.p. 1.*

Proof. Since assumption A is satisfied, there exists $b^*_{s_1,s_2}$ s.t. $b\left(s_1, b^*_{s_1,s_2}\right) = b\left(s_2, b^*_{s_1,s_2}\right)$. By Proposition 3, $b\left(s_1, b^*_{s_1,s_2}\right) = b\left(s_2, b^*_{s_1,s_2}\right) = b^*_{s_1,s_2}$, so that it lies on the 45 degree line. Since policy functions can cross only once, no other state is absorbing and thhe result follows from the following picture. If we are to the left of the point of b^*, we cannot ever cross to the right of it due to monotonicity and continuity. There is a positive probability of reaching b^* starting from any $b > b^*$, and there are no other absorbing points in $\left[b^*, \bar{b}\right]$, so we must get to b^* eventually. Symmetric logic works for the left of b^*. ■

However, this result in some sense seems unrepresentative of the general forces. The risk-free example seems to provide a better description of the economic forces.

Conjecture 9 *Suppose that $|S| > 2$ and assumption A holds. Moreover, supposed to consider an augmented economy from Conjecture 6, i.e. bounded b from below (without violating Assumption A). Then conclusion of Conjecture 6 still holds generically, i.e. for almost all specifications of $\{p(s), g(s)\}$, there is an invariant distriution ψ and for any $\hat{b} \in \left(\underline{b}, \bar{b}\right)$, $\psi\left(\left[\underline{b}, \hat{b}\right]\right) > 0$ and $\psi\left(\left[\hat{b}, \bar{b}\right]\right) > 0$.*

Proof. Generically, with more than 2 shocks all policy functions $b(s; \cdot)$ cannot intersect in the same point. When this is the case, Proposition 3 implies that we can get some crossings above the 45 degree line, others below. But if this is the case, we can always find a sequence of good/bad shocks to take us arbitrarily close to either $\underline{b}$ or $\bar{b}$ starting from any arbitrary point $b \in \left[\underline{b}, \bar{b}\right]$. See the picture. ■

Finally, a side note. I did not allow lump-sum transfers anywhere. If I did, then there would be another absorbing state – the first best. I conjecture with both with risk-free debt and with $|S| > 2$ generically we get there.