Taxes, debts, and redistributions with aggregate shocks*

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Abstract

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If, indeed, the debt were distributed in exact proportion to the taxes to be paid so that every one should pay out in taxes as much as he received in interest, it would cease to be a burden.. . . if it were possible, there would be [no] need of incurring the debt. For if a man has money to loan the Government, he certainly has money to pay the Government what he owes it. Simon **?**, p.85

1. Introduction

2. Environment

Exogenous fundamentals of the economy are functions of a shock s_t that follows an irreducible Markov process, where $s_t \in S$ and S is a finite set. We let $s^t = (s_0, ..., s_t)$ denote a history of shocks.

There is a mass n_i of a type $i \in I$ agent, with $\sum_{i=1}^{I} n_i = 1$. Types differ by their skills. Preferences of an agent of type i over stochastic processes for consumption $\{c_{i,t}\}_t$ and labor supply $\{l_{i,t}\}_t$ are ordered by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U^i \left(c_i(s^t), l_i(s^t)\right), \tag{1}$$

where $\mathbb{E}_t$ is a mathematical expectations operator conditioned on time t information and $\beta \in (0,1)$ is the time discount factor. We assume that $l_i \in \left[0, \bar{l}_i\right]$ for some $\bar{l}_i < \infty$. Results in section 3. require no additional assumptions on U^i like differentiability or convexity,[1] but results in later sections do.

An agent of type i who supplies l_i units of labor produces $\theta_i\left(s_t\right) l_i$ units of output, where $\theta_i(s_t) \in \Theta$ is a nonnegative state-dependent scalar. Feasible allocations satisfy

$$\sum_{i=1}^{I} n_i c_i(s^t) + g\left(s_t\right) = \sum_{i=1}^{I} \pi_i \theta_i\left(s_t\right) l_i(s^t), \tag{2}$$

where $g\left(s_t\right)$ denotes exogenous government expenditures in state s_t. We allow s_t to affect government expenditures $g(s_t)$, and the type-specific productivities $\theta_i(s_t)$.

To save on notation, mostly we use z_t to denote a random variable with a time t conditional distribution that is a function of the history s^t. Occasionally, we use the more explicit notion $z\left(s^t\right)$ to denote a realization at a particular history s^t.

A Ramsey planner's preferences over a vector of stochastic processes for consumption and labor supply are ordered by

$$\mathbb{E}_0 \sum_{i=1}^{I} \omega_i \sum_{t=0}^{\infty} \beta_t U_t^i\left(c_{i,t}, l_{i,t}\right), \tag{3}$$

[1] Consequently our setup allows both extensive and intensive responses of labor.

where the Pareto weights satisfy $\omega_i \geq 0$, $\sum_{i=1}^{I} \omega_i = 1$

In most of this paper, we study an optimal government policy when agents can trade only a one-period risk-free bond. We assume that the government imposes an affine tax. We denote proportional labor taxes by τ and lump sump transfers by T. With this the tax bill of an agent with wage earnings $l_{i,t}\theta_{i,t}$ is given by

$$-T_t + \tau_t \theta_{i,t} l_{i,t}.$$

We do not restrict the sign of T_t at any t or s^t. If for some type i, $\theta_{i,t} = 0$, $b_{i,-1} = 0$ and U^i is defined only on $\mathcal{R}_+^2$, his budget constraint will imply that the all allocations feasible for the planner have non-negative present values of transfers, since transfers are the sole source of a type i agent's wealth and consumption.

The government and agents trade a single possibly risky asset whose time t payoff p_t is described by an $S \times S$ matrix $\mathbb{P}$:

$$p_t = \mathbb{P}(s_t, s_{t-1})$$

We normalize the payoffs such that $\mathbb{E}_t p_{t+1} = 1$.

Let $q_t = q_t(s^t)$ be the price of the single asset at time t and $R_t = \frac{p_t}{q_{t-1}}$ be the one period returns from holding the asset from $t-1$ to t.

Under an affine tax system, agent i's budget constraint at t is

$$c_{i,t} + b_{i,t} = (1 - \tau_t)\,\theta_{i,t} l_{i,t} + R_t b_{i,t-1} + T_t, \tag{4}$$

where $b_{i,t}$ denotes asset holdings of a type i agent at time $t \geq 0$, R_t is a gross one-period return rate from $t-1$ to t for $t \geq 1$, and $R_{-1} \equiv 1$. To rule out Ponzi schemes, we assume that $b_{i,t}$ must be bounded from below. Except in subsection 3.1., we impose no further constraints on agents' borrowing and lending. Subsection 3.1. briefly studies economies with arbitrary borrowing constraints.

The government budget constraint is

$$g_t + B_t = \tau_t \sum_{i=1}^{I} \pi_i \theta_{i,t} l_{i,t} - T_t + R_t B_{t-1}, \tag{5}$$

where B_t denotes the government's assets at time t, which we assume are bounded from below. Our assumptions about preferences imply that the government can collect only finite revenues in each period, so this restriction rules out government-run Ponzi schemes.

We assume that private agents and the government start with assets $\{b_{i,-1}\}_{i=1}^{I}$ and B_{-1}, respectively. Asset holdings satisfy the market clearing condition

$$\sum_{i=1}^{I} n_i b_{i,t} + B_t = 0 \text{ for all } t \geq -1. \tag{6}$$

Since B_t and all $b_{i,t}$ are bounded from below, equation (6) implies that they are also bounded from above.

Components of competitive equilibria are described below

Definition 1 *An* allocation *is a sequence* $\{c_{i,t}, l_{i,t}\}_{i,t}$. *An* asset profile *is a sequence* $\{\{b_{i,t}\}_i, B_t\}_t$. *A* price system *is an interest rate sequence* $\{R_t\}_t$. *A* tax policy *is a sequence* $\{\tau_t, T_t\}_t$.

Definition 2 *For a given initial asset distribution* $(\{b_{i,-1}\}_i, B_{-1})$*, a competitive equilibrium with affine taxes is a sequence* $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, B_t, R_t\}_t$ *and a tax policy* $\{\tau_t, T_t\}_t$*, such that* $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ *maximize (1) subject to (4) and* $\{b_{i,t}\}_{i,t}$ *is bounded; and constraints (2), (5) and (6) are satisfied.*

Lastly we define optimal competitive equilibria.

Definition 3 *Given* $(\{b_{i,-1}\}_i, B_{-1})$*, an optimal competitive equilibrium with affine taxes is a tax policy* $\{\tau_t^*, T_t^*\}_t$*, an allocation* $\{c_{i,t}^*, l_{i,t}^*\})_{i,t}$*, an asset profile* $\left\{\{b_{i,t}^*\}_i, B_t^*\right\}_t$*, and a price system* $\{R_t^*\}_t$ *such that (i) given* $(\{b_{i,-1}\}_i, B_{-1})$*, the tax policy, the price system, and the allocation constitute a competitive equilibrium; and (ii) there is no other tax policy* $\{\tau_t, T_t\}_t$ *such that a competitive equilibrium given* $(\{b_{i,-1}\}_i, B_{-1})$ *and* $\{\tau_t, T_t\}_t$ *has a strictly higher value of (3).*

We call $\{\tau_t^*, T_t^*\}_t$ an *optimal tax policy*, $\{c_{i,t}^*, l_{i,t}^*\}_{i,t}$ an *optimal allocation*, and $\left\{\{b_{i,t}^*\}_i, B_t^*\right\}_t$ an *optimal asset profile*.

3. Ricardian equivalence

This section sets forth a result that underlies much of this paper, namely, that the level of government debt is not a state variable for our economy. The reason is that there is an equivalence class of tax policies and asset profiles that support the same competitive equilibrium allocation. A competitive equilibrium allocation pins down only net asset positions. The assertions in this section apply to all competitive equilibria, not just the optimal ones that will be our focus in subsequent sections.

Theorem 1 *Given* $(\{b_{i,-1}\}_i, B_{-1})$*, let* $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, B_t, R_t\}_t$ *and* $\{\tau_t, T_t\}_t$ *be a competitive equilibrium. For any bounded sequences* $\left\{\hat{b}_{i,t}\right\}_{i,t\geq -1}$ *that satisfy*

$$\hat{b}_{i,t} - \hat{b}_{1,t} = \tilde{b}_{i,t} \equiv b_{i,t} - b_{1,t} \textit{ for all } t \geq -1, i \geq 2,$$

there exist sequences $\left\{\hat{T}_t\right\}_t$ *and* $\left\{\hat{B}_t\right\}_{t\geq -1}$ *that satisfy* (6) *such that* $\left\{\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_i, \hat{B}_t, R_t\right\}_t$ *and* $\left\{\tau_t, \hat{T}_t\right\}_t$ *constitute a competitive equilibrium given* $\left(\left\{\hat{b}_{i,-1}\right\}_i, \hat{B}_{-1}\right)$.

Proof. Let

$$\hat{T}_t = T_t + \left(\hat{b}_{1,t} - b_{1,t}\right) - R_t\left(\hat{b}_{1,t-1} - b_{1,t-1}\right) \text{ for all } t \geq 0. \tag{7}$$

Given a tax policy $\left\{\tau_t, \hat{T}_t\right\}_t$, the allocation $\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_{i,t}$ is a feasible choice for consumer i since it satisfies

$$\begin{aligned}
c_{i,t} &= (1-\tau_t)\,\theta_{i,t} l_{i,t} + R_t b_{i,t-1} - b_{i,t} + T_t \\
&= (1-\tau_t)\,\theta_{i,t} l_{i,t} + R_t\left(b_{i,t-1} - b_{1,t-1}\right) - \left(b_{i,t} - b_{1,t}\right) + T_t + R_t b_{1,t-1} - b_{1,t} \\
&= (1-\tau_t)\,\theta_{i,t} l_{i,t} + R_t\left(\hat{b}_{i,t-1} - \hat{b}_{1,t-1}\right) - \left(\hat{b}_{i,t} - \hat{b}_{1,t}\right) + T_t + R_t b_{1,t-1} - b_{1,t} \\
&= (1-\tau_t)\,\theta_{i,t} l_{i,t} + R_t \hat{b}_{i,t-1} - \hat{b}_{i,t} + \hat{T}_t.
\end{aligned}$$

Suppose that $\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_{i,t}$ is not the optimal choice for consumer i, in the sense that there exists some other sequence $\left\{\hat{c}_{i,t}, \hat{l}_{i,t}, \hat{b}_{i,t}\right\}_t$ that gives strictly higher utility. Then the choice $\left\{\hat{c}_{i,t}, \hat{l}_{i,t}, b_{i,t}\right\}_t$ is feasible given the tax rates $\{\tau_t, T_t\}_t$, which contradicts the assumption that $\{c_{i,t}, l_{i,t}, b_{i,t}\}_t$ is the optimal choice for the consumer given taxes $\{\tau_t, T_t\}_t$. The new allocation satisfies all other constraints and therefore is an equilibrium. ■

An immediate corollary is that it is not total government debt but rather who owns it that affects equilibrium allocations.

Corollary 1 *For any pair* B'_{-1}, B''_{-1}, *there are asset profiles* $\{b'_{i,-1}\}_i$ *and* $\{b''_{i,-1}\}_i$ *such that equilibrium allocations starting from* $\left(\{b'_{i,-1}\}_i, B'_{-1}\right)$ *and from* $\left(\{b''_{i,-1}\}_i, B''_{-1}\right)$ *are the same. These asset profiles satisfy*

$$b'_{i,-1} - b'_{1,-1} = b''_{i,-1} - b''_{1,-1} \textit{ for all } i.$$

This result is closely related to Ricardian Equivalence in **?**. There are, however, some important distinctions. In Barro's representative agent model, lump sum taxes are not distortionary. In our economy, since the planner does not have person-specific taxes, a lump sum transfer introduces distortions in inequality, a force that has a significant effect on optimal policy, as we will see in following sections. Despite this, Ricardian equivalence continues to hold.[2] Theorem 1 shows that many transfer sequences $\{T_t\}_t$ and asset profiles $\{b_{i,t}, B_t\}_{i,t}$ support the same equilibrium allocation. For example, one can set government assets $B_{i,t} = 0$ without loss of generality. Alternatively, we can normalize by setting assets $b_{i,t}$ of any type i to zero.

[2]**?**'s Modigliani-Miller theorem for a class of government open market operations has a similar flavor. **?** describes the structure of a set of related Modigliani-Miller theorems for government finance.

Theorem 1 continues to hold in more general environments. For example, we could allow agents to trade all conceivable Arrow securities and still show that equilibrium allocations depend only on agents' net assets positions. Similarly, our results hold in economies with capital.

3.1. Extension: Borrowing constraints

Representative agent models rule out Ricardian equivalence either by assuming distorting taxes or by imposing ad hoc borrowing constraints. By way of contrast, we have verified that Ricardian equivalence holds in our economy even though there are distorting taxes. Imposing ad-hoc borrowing limits also leaves Ricardian equivalence intact in our economy.[3] In economies with exogenous borrowing constraints, agents' maximization problems include the additional constraints

$$b_{i,t} \geq \underline{b}_i \tag{8}$$

for some exogenously given $\{\underline{b}_i\}_i$.

Definition 4 *For given $\left(\{b_{i,-1}, \underline{b}_i\}_i, B_{-1}\right)$ and $\{\tau_t, T_t\}_t$, a competitive equilibrium with affine taxes and exogenous borrowing constraints is a sequence $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, B_t, R_t\}_t$ such that $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ maximizes (1) subject to (4) and (8), $\{b_{i,t}\}_{i,t}$ are bounded, and constraints (2), (5) and (6) are satisfied.*

We can define an *optimal* competitive equilibrium with exogenous borrowing constraints by extending Definition 3.

The introduction of the ad-hoc debt limits leaves unaltered the conclusions of Corollary 1 and the role of the initial distribution of assets across agents. The next proposition asserts that ad-hoc borrowing limits do not limit a government's ability to respond to aggregate shocks.[4]

Proposition 1 *Given an initial asset distribution $\left(\{b_{i,-1}\}_i, B_{-1}\right)$, let $\{c_{i,t}, l_{i,t}\}_{i,t}$ and $\{R_t\}_t$ be a competitive equilibrium allocation and interest rate sequence in an economy without exogenous borrowing constraints. Then for any exogenous constraints $\{\underline{b}_i\}_i$, there is a government tax policy $\{\tau_t, T_t\}_t$ such that $\{c_{i,t}, l_{i,t}\}_{i,t}$ is a competitive equilibrium allocation in an economy with exogenous borrowing constraints $\left(\{b_{i,-1}, \underline{b}_i\}_i, B_{-1}\right)$ and $\{\tau_t, T_t\}_t$.*

Proof. Let $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ be a competitive equilibrium allocation without exogenous borrowing constraints. Let $\Delta_t \equiv \max_i \{\underline{b}_i - b_{i,t}\}$. Define $\hat{b}_{i,t} \equiv b_{i,t} + \Delta_t$ for all $t \geq 0$ and $\hat{b}_{i,-1} = b_{-1}$. By Theorem 1, $\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_{i,t}$ is also a competitive equilibrium allocation without exogenous borrowing constraints.

[3] **?** describe how a government can use borrowing constraints as part of a welfare-improving policy to finance exogenous government expenditures. **?** describe Modigliani-Miller theorems for government finance in a collection of economies in which borrowing constraints on classes of agents produce the kind of rate of return discrepancies that Bryant and Wallace manipulate.

[4] See **??** who shows a closely related result.

Moreover, by construction $\hat{b}_{i,t} - \underline{b}_i = b_{i,t} + \Delta_t - \underline{b}_i \geq 0$. Therefore, $\hat{b}_{i,t}$ satisfies (8). Since agents' budget sets are smaller in the economy with exogenous borrowing constraints, and $\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_{i,t}$ are feasible at interest rate process $\{R_t\}_t$, then $\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_{i,t}$ is also an optimal choice for agents in the economy with exogenous borrowing constraints $\{\underline{b}_i\}_i$. Since all market clearing conditions are satisfied, $\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_{i,t}$ is a competitive equilibrium allocation and asset profile. ■

To provide some intuition for Proposition 1, suppose to the contrary that the exogenous borrowing constraints restricted a government's ability to achieve a desired allocation. That means that the government would want to increase its borrowing and to repay agents later, which the borrowing constraints prevent. But the government can just reduce transfers today and increase them tomorrow. That would achieve the desired allocation without violating the exogenous borrowing constraints.

Welfare can be strictly higher in an economy with exogenous borrowing constraints relative to an economy without borrowing constraints because a government might want to push some agents against their borrowing limits. When agents' borrowing constraints bind, their shadow interest rates differ from the common interest rate that unconstrained agents face. When the government rearranges tax policies to affect the interest rate, it affects constrained and unconstrained agents differently. By facilitating redistribution, this can improve welfare. In appendix **??**, we construct an example without any shocks in which the government can achieve higher welfare by using borrowing constraints to improve its ability to redistribute.

4. Optimal equilibria with affine taxes

We return to the more general problem formulated in section 2.. We further assume that $U^i : \mathbb{R}^2_+ \to \mathbb{R}$ is concave in $(c, -l)$ and twice continuously differentiable. We let $U^i_{x,t}$ or $U^i_{xy,t}$ denote first and second derivatives of U^i with respect to $x, y \in \{c, l\}$ in period t and assume that $\lim_{x \to \bar{l}_i} U^i_l(c, x) = \infty$ and $\lim_{x \to 0} U^i_l(c, x) = 0$ for all c and i.

We focus on interior equilibria. First-order necessary conditions for the consumer's problem are

$$(1 - \tau_t)\, \theta_{i,t} U^i_{c,t} = -U^i_{l,t}, \tag{9}$$

and

$$U^i_{c,t} = \beta \mathbb{E}_t R_{t+1} U^i_{c,t+1}. \tag{10}$$

To help characterize an equilibrium, we use

Proposition 2 *A sequence $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, R_t, \tau_t, T_t\}_t$ is part of a competitive equilibrium with affine taxes if and only if it satisfies (2), (4), (9), and (10) and $b_{i,t}$ is bounded for all i and t.*

Proof. Necessity is obvious. In appendix **??**, we use arguments of **?** and **?** to show that any $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ that satisfies (4), (9), and (10) is a solution to consumer i's problem. Equilibrium $\{B_t\}_t$ is determined by (6) and constraint (5) is then implied by Walras' Law ■

To find an optimal equilibrium, by Proposition 2 we can choose $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, R_t, \tau_t, T_t\}_t$ to maximize (3) subject to (2), (4), (9), and (10). We apply a first-order approach and follow steps similar to ones taken by **?** and AMSS. Substituting consumers' first-order conditions (9) and (10) into the budget constraints (4) yields implementability constraints

$$c_{i,t} + b_{i,t} = -\frac{U^i_{l,t}}{U^i_{c,t}} l_{i,t} + T_t + \frac{p_t U^i_{c,t-1}}{\beta \mathbb{E}_{t-1} p_t U^i_{c,t}} b_{i,t-1} \text{ for all } i, t. \tag{11}$$

For $I \geq 2$, we can use constraint (11) for $i = 1$ to eliminate T_t from (11) for $i > 1$. Letting $\tilde{b}_{i,t} \equiv b_{i,t} - b_{1,t}$, we can represent the implementability constraints as

$$\begin{aligned} & (c_{i,t} - c_{1,t}) + \tilde{b}_{i,t} \\ = \quad & -\frac{U^i_{l,t}}{U^i_{c,t}} l_{i,t} + \frac{U^1_{l,t}}{U^1_{c,t}} l_{1,t} + \frac{p_t U^i_{c,t-1}}{\beta \mathbb{E}_{t-1} p_t U^i_{c,t}} \tilde{b}_{i,t-1} \text{ for all } i > 1, t. \end{aligned} \tag{12}$$

With this representation of the implementability constraints, the planner's maximization problem depends only on the $I - 1$ variables $\tilde{b}_{i,t-1}$. The reduction of the dimensionality from I to $I - 1$ is another consequence of theorem 1.

Denote $Z^i_t = (c_{i,t} - c_{1,t}) + \tilde{b}_{i,t} + \frac{U^i_{l,t}}{U^i_{c,t}} l_{i,t} - \frac{U^1_{l,t}}{U^1_{c,t}} l_{1,t}$. Formulated in a space of sequences, the optimal policy problem is:

$$\max_{c_{i,t}, l_{i,t}, \tilde{b}_{i,t}} \mathbb{E}_0 \sum_{i=1}^{I} \omega_i \sum_{t=0}^{\infty} \bar{\beta}_t U^i_t (c_{i,t}, l_{i,t}), \tag{13}$$

subject to

$$\tilde{b}_{i,t-1} \frac{p_t U^i_{c,t-1}}{\mathbb{E}_{t-1} p_t U^i_{c,t}} = \mathbb{E}_t \sum_{k=t}^{\infty} \beta^{k-t} \left(\frac{U^i_{c,k}}{U^i_{c,t}} \right) Z^i_k \quad \forall t \geq 1 \tag{14a}$$

$$\tilde{b}_{i,-1} = \mathbb{E}_{-1} \sum_{k=0}^{\infty} \beta^k \left(\frac{U^i_{c,k}}{U^i_{c,t}} \right) Z^i_k \tag{14b}$$

$$\frac{\mathbb{E}_t p_{t+1} U^i_{c,t+1}}{U^i_{c,t}} = \frac{\mathbb{E}_t p_{t+1} U^j_{c,t+1}}{U^j_{c,t}} \tag{14c}$$

$$\sum_{i=1}^{I} n_i c_i(s^t) + g(s_t) = \sum_{i=1}^{I} \pi_i \theta_i (s_t) l_i(s^t), \tag{14d}$$

$$\frac{U^i_{l,t}}{\theta_{i,t} U^i_{c,t}} = \frac{U^1_{l,t}}{\theta_{1,t} U^1_{c,t}} \tag{14e}$$

$$\tilde{b}_{t-1}\frac{U^i_{c,t-1}}{\beta_{t-1}} \text{ is bounded} \tag{14f}$$

Constraint (14a) is a measurablity restriction on allocations that requires that the right side is determined at time $t-1$. This condition is inherited from the restriction that only risk-free bonds are traded.

For both computational and educational purposes, it is convenient to represent the optimal policy problem recursively. For the purpose of constructing a recursive representation, let $\boldsymbol{x} = \beta^{-1}\left(U^2_c\tilde{b}_2, ..., U^I_c\tilde{b}_I\right)$, $\boldsymbol{\rho} = \left(U^2_c/U^1_c, ..., U^I_c/U^1_c\right)$, and denote an allocation $a = \{c_i, l_i\}^I_{i=1}$. In the spirit of **?** and **?**, we split the Ramsey problem into a time-0 problem that takes $(\{\tilde{b}_{i,-1}\}^I_{i=2}, s_0)$ as given and a time $t \geq 1$ continuation problem that takes $\boldsymbol{x}, \boldsymbol{\rho}, s_-$ as given. We formulate two Bellman equations and two value functions, one that pertains to $t \geq 1$, another to $t = 0$. The time inconsistency of an optimal policy manifests itself in there being distinct value functions and Bellman equations at $t = 0$ and $t \geq 1$.

For $t \geq 1$, let $V(\boldsymbol{x}, \boldsymbol{\rho}, s_-)$ be the planner's continuation value given $\boldsymbol{x}_{t-1} = \boldsymbol{x}, \boldsymbol{\rho}_{t-1} = \boldsymbol{\rho}, s_{t-1} = s_-$. It satisfies the Bellman equation

$$V(\boldsymbol{x}, \boldsymbol{\rho}, s_-) = \max_{a(s), x'(s), \rho'(s)} \sum_s \pi(s|s_-)\left(\left[\sum_i \omega_i U^i(s)\right] + \beta V(\boldsymbol{x}'(s), \boldsymbol{\rho}'(s), s)\right) \tag{15}$$

where the maximization is subject to

$$U^i_c(s)\left[c_i(s) - c_1(s)\right] + x'_i(s) + \left(U^i_l(s)l_i(s) - U^i_c(s)\frac{U^1_l(s)}{U^1_c(s)}l_1(s)\right) = \frac{xP(s|s_-)U^i_c(s)}{\beta\mathbb{E}_{s_-}P\boldsymbol{U}^i_c} \text{ for all } s, i \geq 2 \tag{16a}$$

$$\frac{\mathbb{E}_{s_-}P\boldsymbol{U}^i_c}{\mathbb{E}_{s_-}P\boldsymbol{U}^1_c} = \rho_i \text{ for all } i \geq 2 \tag{16b}$$

$$\frac{U^i_l(s)}{\theta_i(s)U^i_c(s)} = \frac{U^1_l(s)}{\theta_1(s)U^1_c(s)} \text{ for all } s, i \geq 2 \tag{16c}$$

$$\sum_i n_i c_i(s) + g(s) = \sum_i n_i(s)l_i(s) \;\; \forall s \tag{16d}$$

$$\rho'_i(s) = \frac{U^i_c(s)}{U^1_c(s)} \text{ for all } s, i \geq 2 \tag{16e}$$

$$\underline{x}_i(s; \boldsymbol{x}, \boldsymbol{\rho}, s_-) \leq x_i(s) \leq \bar{x}_i(s; \boldsymbol{x}, \boldsymbol{\rho}, s_-) \tag{16f}$$

Constraints (16b) and (16e) imply (10). The definition of x_t and constraints (16a) together imply equation (12) scaled by U^i_c. Let $V_0\left(\{\tilde{b}_{i,-1}\}^I_{i=2}, s_0\right)$ be the value to the planner at $t = 0$, where $\tilde{b}_{i,-1}$ denotes initial debt inclusive of accrued interest. It satisfies the Bellman equation

$$V_0\left(\{\tilde{b}_{i,-1}\}^I_{i=2}, s_0\right) = \max_{a_0, x_0, \rho_0} \sum_i \omega_i U^i(c_{i,0}, l_{i,0}) + \beta V\left(x_0, \rho_0, s_0\right) \tag{17}$$

where the maximization is subject to

$$U^i_{c,0}\left[c_{i,0}-c_{1,0}\right]+x_{i,0}+\left(U^i_{l,0}l_{i,0}-U^i_{c,0}\frac{U^1_{l,0}}{U^1_{c,0}}l_{1,0}\right)=U^i_{c,0}\tilde{b}_{i,-1}\text{ for all } i\geq 2 \tag{18a}$$

$$\frac{U^i_{l,0}}{\theta_{i,0}U^i_{c,0}}=\frac{U^1_{l,0}}{\theta_{1,0}U^{1,0}_c}\text{ for all } i\geq 2 \tag{18b}$$

$$\sum_i \pi_i c_{i,0}+g_0=\sum_i \pi_i\theta_{i,0}l_{i,0} \tag{18c}$$

$$\rho_{i,0}=\frac{U^i_{c,0}}{U^1_{c,0}}\text{ for all } i\geq 2 \tag{18d}$$

Because constraint (16b) is absent from the time 0 problem, the time 0 problem differs from the time $t\geq 1$ problem, a source of the time consistency of the optimal tax plan.

5. Characterization: Optimal allocation

In sections 6. and 7. we study the long run properties of public debt and taxes. The main finding is that the levels and spreads in debt and tax rates are determined by two factors: a) the ability of the government to span aggregate shocks through the returns on the asset it trades and b)its redistributive preferences. In particular, the government accumulates debt if interest rates are lower when the its need for revenue are higher and vice versa. The long run variance of debt and taxes along with the rates of rates of convergence to the ergodic distribution are higher in economies where the magnitude of this co movement is larger. Lastly more redistributive governments issue more debt.

To study these implications, in section 6. we first examine a simple economy with quasilinear preferences and i.i.d aggregate shocks. This allows us adequate tractability to formally demonstrate and clarify the main driving forces for the results mentioned above. In section 7. we study more general economies (in terms of heterogeneity, preferences and shocks) numerically and show that all the insights go through.

6. Quasilinear economy

We specialize the problem described in section 4. by imposing the following assumptions that are maintained throughout in this section.

Assumption 1 *IID aggregate shocks: s_t is i.i.d over time*

Assumption 2 *Quasi linear preference: $u(c,l)=c-\frac{l^{1+\gamma}}{1+\gamma}$*

With i.i.d shocks we can restrict our attention to payoff matrices $\mathbb{P}$ that have identical rows denoted by the vector $P(s)$ with a corresponding normalization that $\mathbb{E}P(s) = 1$. We collect a particular set of these vectors that are perfectly correlated with expenditure shocks $g(s)$ in the set $\mathcal{P}^*$ defined below,

$$\mathcal{P}^* = \left\{ P(s) : P(s) = 1 + \frac{\beta}{B^*}(g(s) - \mathbb{E}g) \text{ for some } B^* \in [\overline{B}, \underline{B}] \right\},$$

where $\overline{B}$ and $\underline{B}$ are upper and lower bounds for government assets.

Before characterizing the properties of Ramsey allocation for the economy with heterogeneous agents and no restrictions on transfers, we develop some results in a representative agent economy where the government *cannot* use transfers. We later show that the allocations in this economy are obtained under certain limits on the Pareto weights for the setting with heterogeneous agents.

6.1. Representative agent

Environment

This section describes the representative agent environment with risky debt and no transfers.[5] Given a tax, asset policy $\{\tau_t, B_t\}$, the household solves,

$$W_0(b_{-1}) \max_{\{c_t, l_t, b_t\}_t} \mathbb{E}_0 \sum_t \beta^t \left[c_t - \frac{l_t^{1+\gamma}}{1+\gamma} \right] \tag{19}$$

subject to

$$c_t + b_t = (1 - \tau_t)\theta l_t + R_t P_t b_{t-t} \tag{20}$$

Using the optimality condition for labor and savings we can summarize the set of implementability constraints for the government as follows

$$b_{t-1} \frac{P_t}{E_{t-1} P_t} = \mathbb{E}_t \sum_j \beta^{t+j} [c_t - l_t^{1+\gamma}] \quad \forall t \tag{21}$$

In addition we have also have the feasibility constraint

$$c_t + g_t \leq \theta l_t, \tag{22a}$$

and the market clearing for bonds,

$$b_t + B_t = 0. \tag{22b}$$

[5] This differs from the model studied in AMSS in two ways: first, the government trades a "risky" bond instead of a risk free bond and second, the government is prohibited from using transfer where as AMSS restrict transfers to be non negative. Both of them have critical implications on the long run zero tax results that AMSS obtained. We discuss this later in the section.

The optimal Ramsey allocation solves $\max_{\{c_t, l_t\}_t} W_0(b_{-1})$ subject to (21), feasibility (22a), market clearing for bonds (22b) and natural debt limit for the government $\underline{B}$.[6]

Results

The main results are organized in Theorems 2 and 3. The first result obtains some general properties about the invariant distribution of debt for a large class of payoffs. The second result uses a novel expansion method to get an approximation to the mean and variance of the invariant distribution of debt when payoffs are close the set $\mathcal{P}^*$

When payoffs are not perfectly aligned, the support of the invariant distribution of debt is wide in the sense that (almost surely) the paths of debt sequences approach any arbitrary lower and upper bounds. Note that tax rates are increasing in debt and the variation in debt is analogously reflected in variation in tax rates. This can be contrasted with both, a complete market benchmark as in Lucas Stokey (1983) where both debt and tax rates will be constant sequences and AMSS (2002) which allows for non-negative transfers and risk-free debt where assets approach the first best and limiting tax rates are zero under these preferences.

With some more structure on the payoffs, we show that there is an average inward drift to government assets. More precisely, the multiplier on the implementability constraint is sub (or super) martingale in the region with low (or high) debt. The envelope theorem links the dynamics of the multiplier to that of debts and in turn the concavity of the value function implies mean reversion for debt. This is particularly stark when $P(s) \in \mathcal{P}^*(s)$ where debt converges to a constant. [7]

Next to gain more insights about the invariant distribution we linearize the law of motion for the evolution of debt with respect to both the endogenous state variable that is debt today and payoffs. The point of approximation is the closest (in l_2 sense) complete market economy corresponding to the steady state of some $P(s) \in \mathcal{P}^*(s)$. Exploiting the structure of these approximate laws of motion allows us to obtain bounds on the standard deviation of debt and also rates at which the mean debt level converges that can be expressed in terms of primitive: shocks and payoffs.

Theorem 2 *In the representative agent economy satisfying assumptions 1 and 2, the long run assets under the optimal Ramsey allocation are characterized as follows*

1. *Suppose $P \notin \mathcal{P}^*$, there is an invariant distribution of government such that*

[6] These will be explicitly derived for the cases we solve in this section. We also impose an upper bound $\overline{B}$ on government assets.

[7] Thus the limiting allocation is a particular Lucas Stokey (1983) with stationary debt and taxes, however the level and sign of the long run debt is determined by the joint properties of shocks and payoffs rather than initial condition as would be the case in Lucas Stokey(1983)

$$\forall \epsilon > 0, \quad \Pr\{B_t < \underline{B} + \epsilon \textit{ and } B_t > \overline{B} - \epsilon \quad i.o\} = 1$$

2. *Suppose $P(s) - P(s') > \beta \frac{g(s)-g(s')}{\underline{B}} \quad \forall s, s'$, then for large enough assets (or debt) there is a drift towards the interior region. In particular the value function $V(B)$ is strictly concave and there exists $B_1 < B_2$ such that*

$$\mathbb{E}V'(B(s)) > V'(B_{\text{-}}) \quad B_{\text{-}} > B_2$$

and

$$\mathbb{E}V'(B(s)) < V'(B_{\text{-}}) \quad B_{\text{-}} < B_1$$

3. *Suppose $P(s) \in \mathcal{P}^*$, then the long run assets converge to a degenerate steady state*

$$\lim_t B_t = B^* \quad a.s \quad \forall B_{-1}$$

In the case where $P(s) \in \mathcal{P}^*$, we can express the long run assets

$$B^* = \beta \frac{\text{var}(g(s))}{\text{cov}(P(s), g(s))}$$

Keeping tax rates (and hence tax revenues in this case) the government needs to finance a higher primary deficit when it gets positive expenditure shock. If in such states the assets pays off more, then optimally the government holds positive assets and uses the these high returns to finance this deficit. On the other hand if payoff are lower in times when the government needs resources, holding debt is valuable since it lowers the interest burden. Thus using the level of its assets B^* it can perfectly span the fluctuations in deficits and the sign is given by the sign of the covariance of $P(s)$ with $g(s)$.

The long run tax rate is inversely related to B^* with the following limits,

$$\lim_{B^* \to \underline{B}} \tau^* = \frac{\gamma}{1+\gamma} \qquad \lim_{B^* \to \infty} \tau^* = -\infty$$

Now we proceed to obtain a sharper characterization of the invariant distribution. In general, there is no closed form solution for law of motion of government debt and hence we resort to an approximation. We begin with a orthogonal decomposition for an arbitrary $P(s)$,

$$P(s) = \hat{P}(s) + P^*(s)$$

where $P^*(s) \in \mathcal{P}^*$ and $\hat{P}(s)$ is orthogonal to $g(s)$. Expanding the policy rules around the steady state of the $P^*(s)$ economy we have the next theorem.[8]

Theorem 3 *The ergodic distribution of debt (using the first order approximation of dynamics near $P^*(s)$) has the following properties,*

- ***Mean:*** *The ergodic mean is B^* which corresponds to the steady state level of debt of an economy with payoff vector $P^*(s)$*
- ***Variance:*** *The coefficient of variation is given by*

$$\frac{\sigma(B)}{\mathbb{E}(B)} = \sqrt{\frac{\operatorname{var}(P(s)) - |\operatorname{cov}(g(s), P(s))|}{(1 + |\operatorname{cov}(g(s), P(s))|)|\operatorname{cov}(g(s), P(s))|}} \leq \sqrt{\frac{\operatorname{var}(\hat{P}(s))}{\operatorname{var}(P^*(s))}}$$

- ***Convergence rate:*** *The speed of convergence to the ergodic distribution described by*

$$\frac{\mathbb{E}_{t-1}(B_t - B^*)}{(B_{t-1} - B^*)} = \frac{1}{1 + |\operatorname{cov}(P(s), g(s))|}$$

Notice that when payoffs are equal to $P^*(s)$, the government can keep taxes constant and perfectly offset the fluctuations in its surplus with returns $P^*(s)B^*$. Away from this, the incompleteness of markets is binding and shocks are hedged with a combination of changes in tax rates and debt levels. These theorem shows exactly how the deviations from perfect spanning map into larger variances for debt (and taxes) in the long run. Figure 1 shows how the ergodic distribution of debt and taxes spread as we vary the covariance $P(s)$ with $g(s)$.

6.2. Heterogeneous agents

In this section we turn to general problem of characterizing outcomes in an economy with heterogeneous agents and transfers. In particular we introduce a second agent who has zero productivity and impose a non negativity constraint on his consumption. Given the Ricardian equivalnce result discussed in section **??**, we maintain a normalization that assets of the unproductive agent are zero throughout this section.

Assumption 3 *The productivity of agents are ordered, $\theta_1 > \theta_2 = 0$ and $c_{2,t} \geq 0$.*

Before we discuss the results, a few words on the assumptions are pertinent. The assumption that $\theta_2 = 0$ makes the problem tractable and allows us to obtain a complete characterization of the problem as we wary the Pareto weights. The restriction on consumption is necessary to add curvature to the

[8]Formally $P^*(s)$ is obtained by projecting $P(s)$ on the space spanned by $\mathcal{P}^*$. These approximations differ from those obtained in Woodford (XXX) where the point of approximation is a deterministic steady state. Appendix **??** contains more details of the approximation method including comparing outcomes to those obtained numerically solving the same economy.

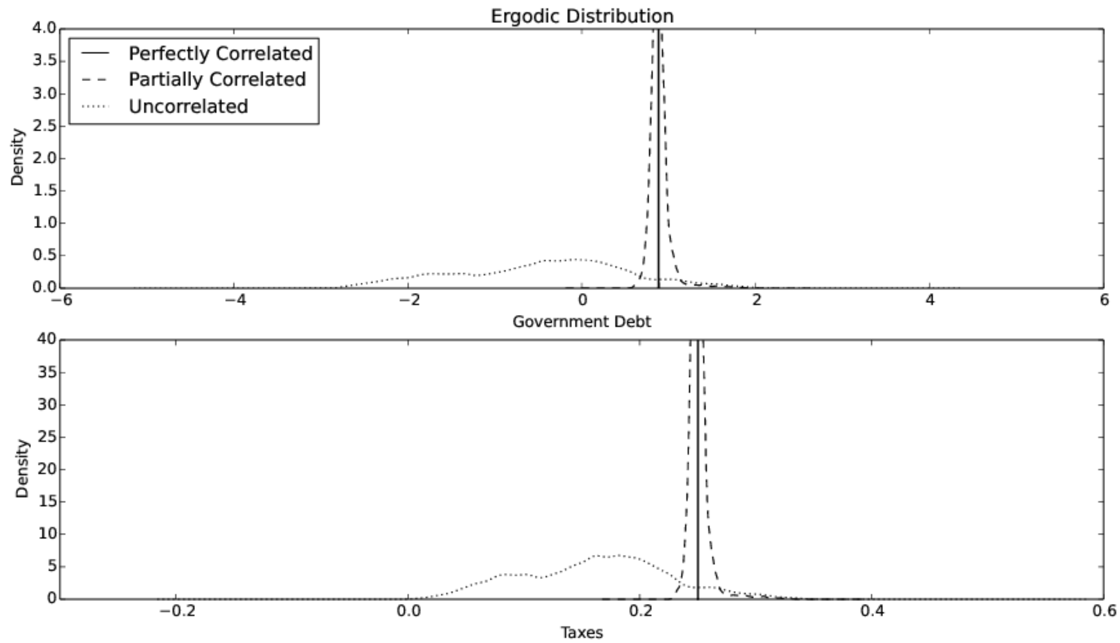

Figure 1: Ergodic distribution for debt and taxes in the representative agent quasilinear economy for three choices $P(s)$.

problem. In more general settings risk aversion will impose Inada restrictions that play a similar role. We now state the theorem and then discuss its implications.

Theorem 4 *Let ω, n be the Pareto weight and mass of the productive agent with $n < \frac{\gamma}{1+\gamma}$. The optimal tax, transfer and asset policies $\{\tau_t, T_t, B_t\}$ are characterized as follows,*

1. *For $\omega \geq n\left(\frac{1+\gamma}{\gamma}\right)$ we have $T_t = 0$ and the optimal policy is same as in a representative agent economy studied in Theorems 2,and 3*

2. *For $\omega < n\left(\frac{1+\gamma}{\gamma}\right)$, suppose we further assume that $\min_s\{P(s)\} > \beta$. We have two parts: There exists $\mathcal{B}(\omega)$ and $\tau^*(\omega)$ with $\mathcal{B}'(\omega) > 0$ and $\lim_{\omega \to 0} \mathcal{B}(\omega) < 0$ such that*

   (a) *$B_- > \mathcal{B}(\omega)$*

   $$T_t > 0, \quad \tau_t = \tau^*(\omega), \text{ and } B_t = B_- \quad \forall t$$

   (b) *$B_- \leq \mathcal{B}(\omega)$, the policies depend on the structure of $P(s)$.*

      i. *For $P(s) \notin \mathcal{P}^*$*

   $$\lim_t T_t > 0 \text{ i.o.}, \quad \lim_t \tau_t = \tau^*(\omega) \text{ and } \lim_t B_t = \mathcal{B}(\omega) \quad a.s$$

ii. For $P(s) \in \mathcal{P}^*$ we have two cases depending on B_-

A. For $B_- \leq B^*$

$$T_t = 0, \quad \lim_t \tau_t = \tau^{**}(\omega), \textit{ and } \lim_t B_t = B^* \quad a.s$$

B. For $\mathcal{B}(\omega) > B_- > B^*$

$$\Pr\{\lim_t T_t = 0, \lim_t \tau_t = \tau^{**}(\omega), \lim_t B_t = B^* \textit{ or } \lim_t T_t > 0 \textit{ i.o.}, \lim_t \tau_t = \tau^*(\omega), \lim_t B_t = \mathcal{B}(\omega)\} > 0$$

The main concern in this setting with heterogeneous agents is that costs of fluctuating transfers to hedge aggregate shocks are endogenous. The simplifications in the environment allow us to highlight how these depend on the Pareto weights (relative to the mass) of the Planner: $\{\omega, 1-\omega\}$ corresponding to Agents 1 and 2 respectively. A regressive planner who cares a lot about the productive agents in effect faces high costs of using transfers. For such a planner (with a high ω), increasing transfers also means giving resources to the unproductive agent whose consumption he does not value as much. In fact the threshold $\bar{\omega} = n\left(\frac{1+\gamma}{\gamma}\right)$ is such that above this, transfers are never used and thus the allocations are identical to the representative agent economy studied before.

For a less regressive planner (such that $\omega < \bar{\omega}$) transfers are an important tool for redistributing resources to the unproductive agent in adverse times. To finance these these it taxes the productive agent and does not need to accumulates a large buffer stock of assets. Thus limiting assets are lower and tax rates are larger for more redistributive planners.

7. More general economies

The analysis in the previous section was simplified in many dimensions - no curvature on the utility from consumption, lack of persistence in shocks and restricting heterogeneity to two agents. The key implication we could thereby exploit was that the return on debt was exogenous, given by $\beta^{-1}P(s)$. Adding curvature makes these returns endogenous even for standard risk free bond with a payoff vector $P(s) = 1$. Technically, it requires us to additionally keep track relative marginal utilities to characterize how taxes, debt, and transfers react to shocks. This makes it harder to separate spanning and redistribution concerns.

In the next section we first show that with risk aversion, exact spanning can occur only if shocks are binary and IID. For instance with CES preferences, the limiting allocation has constant relative consumptions and taxes rates. We show that how similar to the quasilinear case, the co movement of interest rates and exogenous shocks govern the governments' incentives to accumulate assets.

In the last section we numerically solve for a economy with realistic heterogeneity and more general shocks. Using simulations from this economy we verify the long properties of taxes and debt for economies that differ in spanning and re-distributive preferences of the planner.

7.1. Spanning with binary shocks

Let $\Psi\left(s;\boldsymbol{x},\boldsymbol{\rho},s_{-}\right)$ be an optimal law of motion for the state variables for the $t\geq 1$ recursive problem, i.e., $\Psi\left(s;\boldsymbol{x},\boldsymbol{\rho},s_{-}\right)=\left(x'\left(s\right),\rho'\left(s\right)\right)$ solves (15) given state $\left(\boldsymbol{x},\boldsymbol{\rho},s_{-}\right)$.

Definition 5 *A steady state* $\left(\boldsymbol{x}^{SS},\boldsymbol{\rho}^{SS}\right)$ *satisfies* $\left(\boldsymbol{x}^{SS},\boldsymbol{\rho}^{SS}\right)=\Psi\left(s;\boldsymbol{x}^{SS},\boldsymbol{\rho}^{SS},s_{-}\right)$ *for all* s,s_{-}.

Since in this steady state $\rho_i=U_c^i(s)/U_c^1(s)$ does not depend on the realization of shock s, the ratios of marginal utilities of all agents are constant. The continuation allocation depends only on s_t and not on the history s^{t-1}.

We begin by noting that a competitive equilibrium fixes an allocation $\{c_i(s),l_i(s)\}_i$ given a choice for $\{\tau(s),\boldsymbol{\rho}(s)\}$ using equations (16c), (16d) and (16e). Let us denote $U(\tau,\boldsymbol{\rho},s)$ as the value for the planner from the implied allocation using Pareto weights $\{\omega_i\}_i$,

$$U(\tau,\boldsymbol{\rho},s)=\sum_i\omega_iU^i(s).$$

As before define $Z_i(\tau,\rho,s)$ as

$$Z_i(\tau,\boldsymbol{\rho},s)=U_c^i(s)c_i(s)+U_l^i(s)l_i(s)-\rho_i(s)\left[U_c^1(s)c_1(s)+U_l^1(s)l_1(s)\right].$$

For the IID case, the optimal policy solves the following Bellman equation for $\boldsymbol{x}(s^{t-1})=\boldsymbol{x},\boldsymbol{\rho}(s^{t-1})=\boldsymbol{\rho}$

$$V(\boldsymbol{x},\boldsymbol{\rho})=\max_{\tau(s),\boldsymbol{\rho}'(s),\boldsymbol{x}'(s)}\sum_s\pi(s)\left[U(\tau(s),\boldsymbol{\rho}'(s),s)+\beta(s)V(\boldsymbol{x}'(s),\boldsymbol{\rho}'(s))\right]\tag{23}$$

subject to the constraints

$$Z_i(\tau(s),\boldsymbol{\rho}'(s),s)+x_i'(s)=\frac{x_i\beta^{-1}P(s)U_c^i(\tau(s),\boldsymbol{\rho}'(s),s)}{\mathbb{E}U_c^i(\tau,\rho)}\text{ for all }s,i\geq 2,\tag{24}$$

$$\sum_s\pi(s)P(s)U_c^1(\tau(s),\boldsymbol{\rho}'(s),s)(\rho_i'(s)-\rho_i)=0\text{ for }i\geq 2.\tag{25}$$

Constraint (25) is obtained by rearranging constraint (16b). It implies that $\rho(s)$ is a risk-adjusted martingale. We next check if the first-order necessary conditions are consistent with stationary policies for some $(\boldsymbol{x},\boldsymbol{\rho})$.[9]

Lemma 1 *With risk aversion* $\|S\|=2$ *is necessary for a steady state to exist*

Proof.

[9]Appendix **??** discuses the associated second order conditions that ensure these policies are optimal

Let $\pi(s)\mu_i(s)$ and λ_i be the multipliers on constraints (24) and (25). Imposing the restrictions $x_i'(s) = x_i$ and $\rho_i'(s) = \rho_i$, at a steady state $\{\mu_i, \lambda_i, x_i, \rho_i\}_{i=2}^N$ and $\{\tau(s)\}_s$ are determined by the following equations

$$Z_i(\tau(s), \boldsymbol{\rho}, s) + x_i = \frac{\beta^{-1} P(s) x_i U_c^i(\tau(s), \boldsymbol{\rho}, s)}{\mathbb{E} U_c^i(\tau, \rho)} \text{ for all } s, i \geq 2, \tag{26a}$$

$$U_\tau(\tau(s), \boldsymbol{\rho}, s) - \sum_i \mu_i Z_{i,\tau}(\tau(s), \boldsymbol{\rho}, s) = 0 \text{ for all } s, \tag{26b}$$

$$U_{\rho_i}(\tau(s), \boldsymbol{\rho}, s) - \sum_j \mu_j Z_{j,\rho_i}(\tau(s), \boldsymbol{\rho}, s) + \lambda_i P(s) U_c^1(\tau(s), \boldsymbol{\rho}, s) - \lambda_i \beta \mathbb{E} P(s) U_c^1(\tau(s), \boldsymbol{\rho}(s), s) = 0. \text{ for all } s, i \geq 2 \tag{26c}$$

Since the shock s can take only two values, (26) is a square system in $4(N-1)+2$ unknowns $\{\mu_i^{SS}, \lambda_i^{SS}, x_i^{SS}, \rho_i^{SS}\}_{i=2}^N$ and $\{\tau^{SS}(s)\}_s$. ■

The behavior of the economy in the steady state is similar to the behavior of the complete market economy characterized by Werning (2007). Both taxes and transfers depend only on the current realization of shock s_t. Moreover, the arguments of Werning (2007) can be adapted to show that taxes are constant when preferences have a CES form $c^{1-\sigma}/(1-\sigma) - l^{1+\gamma}/(1-\gamma)$ and fluctuations in tax rates are very small when preferences take forms consistent with the existence of balanced growth. We return to this point after we discuss convergence properties.

Lemma **??** provides a simple way to verify existence of a steady state for wide range of parameter values by checking that there exists a root for system (26). Since the system of equations (26) is nonlinear, existence can generally be verified only numerically. Next, we provide a simple example with risk averse agents in which we can show existence of the root of (26) analytically. The analytical characterization of the steady state will help us develop some comparative statics and build a connection from the quasilinear economy to the quantitative analysis to appear in section **??**.

A two-agent example

Consider an economy consisting of two types of households with $\theta_{1,t} > \theta_{2,t} = 0$. One period utilities are $\ln c - \frac{1}{2} l^2$. The shock s takes two values, $s \in \{s_L, s_H\}$ with probabilities $\Pr(s|s_-)$ that are independent of s_-. We assume that $g(s) = g$ for all s, and $\theta_1(s_H) > \theta_1(s_L)$. We allow the discount factor $\beta(s)$ to depend on s.

Theorem 5 *Suppose that $g < \theta(s)$ for all s. Let $R(s)$ be the gross interest rates and $x = U_c^2(s)[b_2(s) - b_1(s)]$*

1. ***Countercyclical interest rates.*** *If $P(s_H) = P(s_L)$, then there exists a steady state $\left(x^{SS}, \rho^{SS}\right)$ such that $x^{SS} > 0,\ R^{SS}(s_H) < R^{SS}(s_L)$.*

2. ***Acyclical interest rates.*** *There exists a pair* $\{P(s_H), P(s_L)\}$ *such that there exists a steady state with* $x^{SS} > 0$ *and* $R^{SS}(s_H) = R^{SS}(s_L)$.

3. ***Procyclical interest rates.*** *There exists a pair* $\{P(s_H), P(s_L)\}$ *such that there exists a steady state with* $x^{SS} < 0$ *and* $R^{SS}(s_H) > R^{SS}(s_L)$.

In all cases, taxes $\tau(s) = \tau^{SS}$ *are independent of the realized state.*

In this two-agent case, by normalizing assets of the unproductive agent (using theorem 1) we can interpret x as the marginal utility adjusted assets of the government. Besides establishing existence, the proposition identifies the importance of cyclical properties of real interest rates in determining the sign of these assets.

Proposition 5 shows two main forces that determine the dynamics of taxes and assets: fluctuations in inequality and fluctuations in the interest rates. Let's start with part 2 of proposition 5, which turns off the second force. When interest rates are fixed, the government can adjust two instruments in response to an adverse shock (i.e., a fall in θ_1): it can either increase the tax rate τ or it can decrease transfers T. Both responses are distorting, but for different reasons. Increasing the tax rate increases distortions because the deadweight loss is convex in the tax rate, as in **?**. This force operates in our economy just as it does in representative agent economies. But in a heterogeneous agent economy like ours, adjusting transfers T is also costly. When agents' asset holdings are identical, a decrease in transfers disproportionately affects a low-skilled agent, so his marginal utility falls by more than does the marginal utility of a high-skilled agent. Consequently, a decrease in transfers increases inequality, giving rise to a cost not present in representative agent economies.

The government can reduce the costs of inequality distortions by choosing tax rate policies that make the net asset positions of the high-skilled agent decrease over time. That makes the two agents' after-tax and after-interest income become closer, allowing decreases in transfers to have smaller effects on inequality in marginal utilities. If the net asset position of a high-skilled agent is sufficiently low, then a change in transfers has no effect on inequality and all distortions from fluctuations in transfers are eliminated.[10]

Turning now to the second force, interest rates generally fluctuate with shocks. Parts 1 and 3 of proposition 5 indicate what drives those fluctuations. This is the same force that was operating in the representative agent quasilinear economy studied in section 6.. Exploiting linearity allowed us to get sufficient conditions for existence of steady state and provide a sharper characterization of how co-movement of interest rates matter. Here in the case with binary shocks we can sign the debt as flip the ordering of interest rates by changing the gap $P(s_L) - P(s_H)$.

[10]This convergence outcome has a similar flavor to "back-loading" results of **?** and **?** that reflect the optimality of structuring policies intertemporally eventually to disarm distortions.

7.2. Stability

In this section we extend the approximation methods used to characterize outcomes in Theorem 3 to the general problem with risk aversion. Instead of obtaining an analytical characterization as in the quasilinear case, we present a test for convergence show local stability of a steady state for a wide range of parameters.

As before, let assume that $\pi(s)\mu_i(s)$ and λ_i be the multipliers on constraints (24) and (25). In Appendix **??** we show that the history-dependent optimal policies (they are sequences of functions of s^t) can be represented recursively in terms of $\{\boldsymbol{\mu}(s^{t-1}), \boldsymbol{\rho}(s^{t-1})\}$ and s_t. A recursive representation of an optimal policy can be linearized around the steady state using $(\boldsymbol{\mu}, \boldsymbol{\rho})$ as state variables.[11]

Formally, let $\hat{\Psi}_t = \begin{bmatrix} \boldsymbol{\mu}_t - \boldsymbol{\mu}^{SS} \\ \boldsymbol{\rho}_t - \boldsymbol{\rho}^{SS} \end{bmatrix}$ be deviations from a steady state. From a linear approximation, one can obtain $B(s)$ such that

$$\hat{\Psi}_{t+1} = B(s_{t+1})\hat{\Psi}_t. \tag{27}$$

This linearized system has coefficients that are functions of the shock. The next proposition describes a simple numerical test that allows us to determine whether this linear system converges to zero in probability.

Theorem 6 *If the (real part) of eigenvalues of $\mathbb{E}B(s)$ are less than 1, system (27) converges to zero in mean. Further for large t, the conditional variance of $\hat{\Psi}$, denoted by $\Sigma_{\Psi,t}$, follows a deterministic process governed by*

$$\textit{vec}(\Sigma_{\Psi,t}) = \hat{B}\textit{vec}(\Sigma_{\Psi,t-1}),$$

where $\hat{B}$ is a square matrix of dimension $(2I-2)^2$. In addition, if the (real part) of eigenvalues of $\hat{B}$ are less than 1, the system converges in probability.

The eigenvalues (in particular the largest or the dominant one) are instructive not only for whether the system is locally stable but also how quickly the steady state is reached. In particular, the half-life of convergence to the steady state is given by $\frac{\log(0.5)}{\|\iota\|}$, where $\|\iota\|$ is the absolute value of the dominant eigenvalue. Thus, the closer the dominant eigenvalue is to one, the slower is the speed of convergence.

We used Theorem 6 to verify local stability of a wide range of examples. Since the parameters space is high dimensional we relegate the comparative statics to Appendix **??**. The typical finding is that the

[11]One could in principle look for a solution in state variables $(\boldsymbol{x}(s^{t-1}), \boldsymbol{\rho}(s^{t-1}))$. For $I = 2$ with $\{\theta_i(s)\}$ different across agents, this would give identical policies and a map which is (locally) invertible between $\boldsymbol{x}$ and $\boldsymbol{\mu}$ for a given $\boldsymbol{\rho}$. However in other cases, it turns out there are unique linear policies in $(\ bm\mu, \boldsymbol{\rho})$ and not necessarily in $(\boldsymbol{x}, \boldsymbol{\rho})$. This comes from the fact that the set of feasible $(\boldsymbol{x}, \boldsymbol{\rho})$ are restricted at time 0 and may not contain an open set around the steady state values. When we linearize using $(\boldsymbol{\mu}, \boldsymbol{\rho})$ as state variables, the optimal policies for $\boldsymbol{x}(s^t)$, $\boldsymbol{\rho}(s^t)$ converge to their steady state levels for all perturbations in $(\boldsymbol{\mu}, \boldsymbol{\rho})$.

steady state is stable and that convergence is slow. The rates of convergence are increasing in the covariance of interest rates and governments needs for revenue.

7.3. Numerical example

As a last step in this analysis we finally turn to numerical methods and solve for more general economies in economies where a degenerate steady state does not exist. Using simulations we verify the key predictions of the theoretical analysis in the previous sections regarding the properties of ergodic distribution of debt and taxes. We frst briefly discuss the calibration and then move on to the results.

Calibration

The heterogeneity is calibrated by using 10 agents that captured the deciles of the US earning distribution with labor productivities given by the following table,

$$\theta = [1., 1.6, 2.45, 3.35, 4.4, 5.6, 7., 8.9, 12., 22.]$$

The preference over consumption and labor given by

$$\psi \log(c) + (1 - \psi) \log(1 - l)$$

with ψ set to be 0.6994 to capture a Frisch elasticity of 0.5. The discount factor was set at 0.98 and the level of exogenous government expenditure was assumed to be constant 0.206 to get a pre-trasnfers expenditure to output ratio of XX. Our baseline case has a risk free bond and utilitarian planner who assigns equal weights to all agents. Aggregate TFP is calibrated to have a mean and standard deviation of 3%.

The solution method is adapted from Evans (2014) and described more in the Appendix **??**

Wide support of ergodic distribution

To verify the wide support of the ergodic distribution we take the initial conditions at the end of the long simulation and subject the economy to a sequence of 120 TFP shocks which are 2 standard deviations below the mean. In figure 2 we plot the debt and tax rate during this sequence of bad shocks.

XXX David: Plot the debt sequence also

We see that given a sufficiently long sequence of bad shocks the economy will eventually deviate significantly from its ergodic mean.

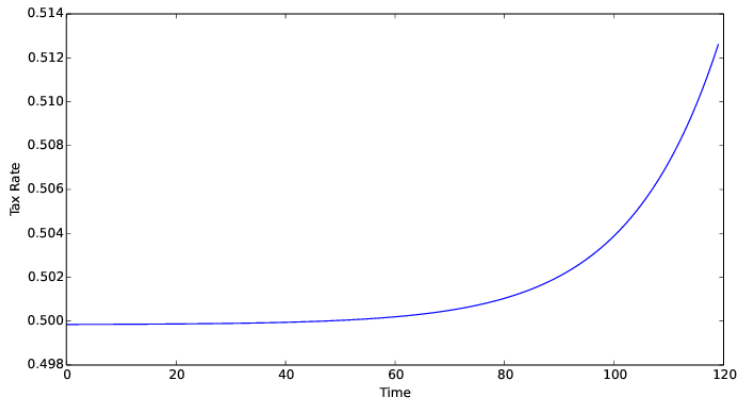

Figure 2: This plots the simulation of tax rates for a sequence of 120 negative (2 s.d below the mean) TFP shocks

Convergence Speed

We next check how the speed of convergence to the ergodic mean varies as we change the correlation of payoffs with the exogenous shock. For this we depart from the baseline risk free bond economy to the payoff structure which pays off more in high TFP states. In particular

$$P_t = 1. + 0.5\epsilon_t$$

where ϵ is the TFP shock. Since marginal utilities are countercyclical, this reduces the correlation of the payoff, $U^i_{c,t}P_t$, with the aggregate shock. In figure 3 we plot a sample path of aggregate marginal utility adjusted debt and the ratio of marginal utilities of the lowest productivity agent to the highest productivity agent for the baseline case (blue) and the modified payoff structure case (green). Both economies start out at the same initial distribution of Pareto weights and multipliers on the implementability constraint and received the same sequence of aggregate shocks. We see that the economy with the lower correlation of payoff with aggregate shock converges to the steady state at a much slower rate.

XXX David: Plot the debt sequence also

Government assets and re-distributive concerns:

Our final conclusion was that the assets held by the government in the steady state was decreasing in the redistributive motive of the government. We check this last part by changing the Pareto weights of the government. In our baseline case the government places equal Pareto weights on all agents. We introduce a redistributive motive through a parameter α. The planner places evenly spaced Pareto

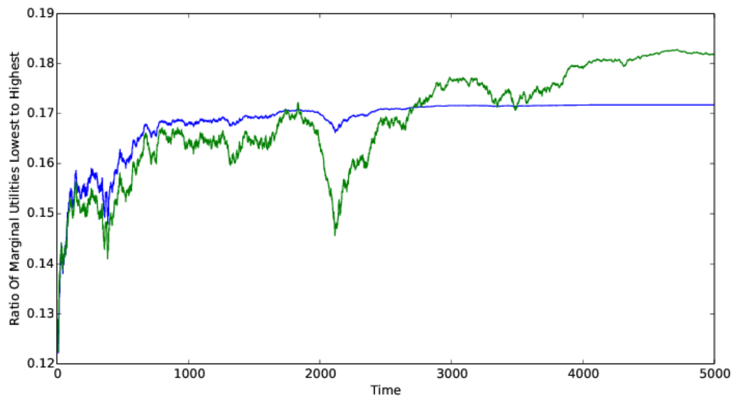

Figure 3: Long sample path for ratio of marginal utilities of Agent 10 to Agent 1 for the bond economy (blue) and the economy with higher correlation of interest rates and TFP(green).

weights from $0.1 - \alpha$ on the lowest productivity agent to $0.1 + \alpha$ on the highest productivity agent. [12]

As alpha increases the redistributive motive decreases. We plot total assets of the government in steady state as a function of alpha in figure 4 and see that the relationship does indeed hold.

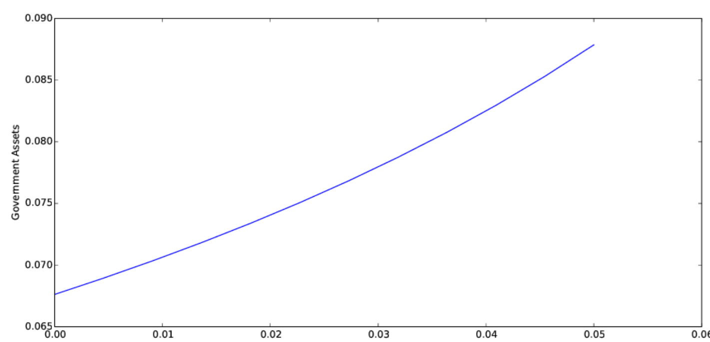

Figure 4: Steady state level of government assets as a function of its redistribute motives. The x-axis plots α with higher α's corresponding to more regressive planners

As a last figure 5 we we plot the asset level of each agent versus the agents productivity.

For $I = 2$, proposition 5 signs the marginal utility adjusted net assets that we denoted x. This implies a particular ordering of net assets across the agents in the steady state. These implications generalize to settings with $I \geq 3$ and more general shocks. In general, one can verify in the baseline with a risk free bond(as in part 1 of proposition 5 when interest rates are countercyclical), mean long

[12]For example when $\alpha = 0.025$ the vector of Pareto weights will be

$$[0.075, 0.0810.086, 0.092, 0.097, 0.1030.1080.114, 0.119, 0.125]$$

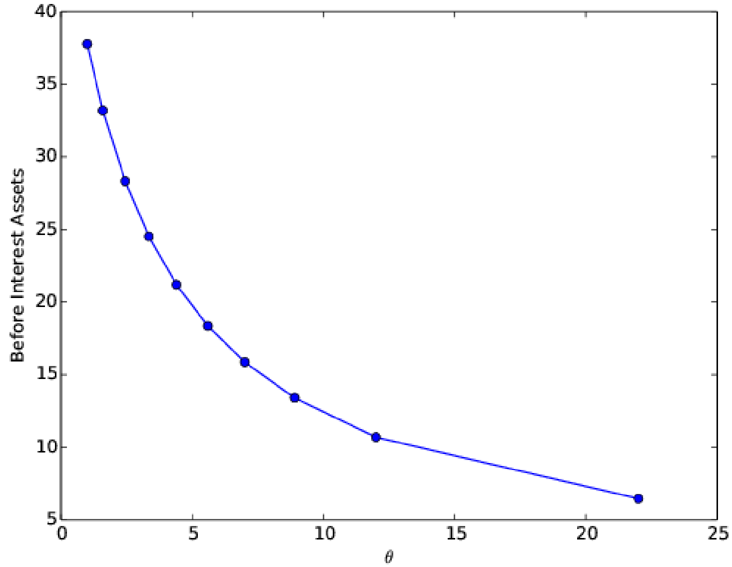

Figure 5: Assets held by agent type in steady state.

run net asset levels (scaled by marginal utilities) are ordered inversely to productivities. By making net asset positions be negatively correlated with labor earnings, the planner can minimize the costs of fluctuating transfers over time.

8. Idiosyncratic risk

9. Conclusion

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