

Taxes, debts, and redistributions with aggregate shocks*

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Abstract

This paper studies how taxes and debt respond to aggregate shocks in the presence of incomplete markets and redistribution concerns. A planner sets a lump sum transfer and a linear tax on labor income in an economy with heterogeneous agents, aggregate uncertainty, and markets restricted to a single asset whose payoffs can vary with aggregate states. Two forces shape long-run outcomes: the planner's desire to minimize the welfare costs of fluctuating transfers, which calls for a negative correlation between the distribution of net assets and agents' skills; and the planner's desire to use fluctuations in the real interest rate to compensate for missing state-contingent securities. In a model parameterized to match stylized facts about US booms and recessions, distributional concerns mainly determine optimal policies over business cycle frequencies. These features of optimal policy differ markedly from ones that emerge from representative agent Ramsey models

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1 Introduction

The paper is motivated by three policy questions: what are the welfare costs of public debt? In particular should governments payback high initial debts and if so how quickly? Further how should tax rates and transfers respond aggregate shocks especially when markets are incomplete and consequently perfect smoothing of distortionary costs of taxes may not be possible?

We develop the answers to these questions in an economy with heterogeneous agents and incomplete markets and a government that decides proportional labor taxes, lump sum transfers and asset purchases in response to aggregate shocks. Agents differ in their productivities and assets and we impose no restrictions on the sign of transfers. If some agents are sufficiently poor or if the government wants enough redistribution, the government always chooses positive transfers. Figure ?? shows that an affine structure better approximates the US tax-transfer system than just proportional labor taxes. Finally we introduce a flexible way of capturing incomplete markets by allowing agents and the government to trade a single financial security whose payoffs can potentially depend on aggregate shocks.

In this economy the main tradeoffs arise from the interaction of two features: limits to redistribution and incompleteness of markets. The government decides a policy mix of tax rates, transfers and asset purchases as a response to aggregate shocks that influence its need for revenues. Each of these instruments has different welfare consequences. Firstly depending on how returns on assets co-move with aggregate shocks a given distribution of asset introduces financial flows between the private agents and the government that may differ from the what the government desires in order to minimize efficiency costs associated with labor taxes while achieving both its financing and redistribution needs. Although unrestricted, fluctuating transfers is costly too as a decrease in transfers in response to adverse aggregate shocks disproportionately affects agents having low present values of earnings.

The paper attempts to disentangle these considerations by building up analytical results in a setup with quasilinear preferences and then validate the findings in a more quantitative exercise where the model is calibrated to US data.

We begin by briefly noting a Ricardian property that holds in our setting. Gross asset positions (in particular the level of government debt) do not affect the set of feasible allocations that can be implemented in competitive equilibria with taxes and transfers. We use this result to reduce the dimension of the state space that characterizes the optimal allocation and settle on a normalization that helps us interpret transfers and public debt separately.

To isolate the hedging concerns from redistribution we analyse a representative agent setting

with quasilinear preferences and no transfers. Besides being informative about the more general case with multiple agents when the costs of transfers are high, it complements results in literature with single agent that focus on risk free bond market structure. The main finding here is that for a large class of payoff structures, there is a tendency of aggregate debt to drift towards a level where the government's hedging abilities are maximized. This level primarily depends on how the payoffs correlate with needs for revenue. In particular if payoffs are high when the government needs revenue, optimal hedging requires it to issue positive debt and vice versa. The magnitude of debt (or assets) is decreasing and the speed at which the debt converges is increasing in the strength of this co-movement. For special cases where payoffs are affine in expenditure shocks, we can show that the ergodic distribution is degenerate. For other cases we develop tools to approximate the ergodic distribution that clarify how the spread of debt and taxes increase with how far the payoffs are from perfect spanning.

Next we illustrate how concerns for redistribution matter when transfers are unrestricted and we have multiple agents. As before in a extended setting with quasilinear preferences, we establish that the target level of assets is decreasing in the governments redistributive concerns. This comes from the fact that a more redistributive government relies more on transfers and has less needs to accumulate assets for precautionary reasons. These insights extend to economies with more general preferences featuring risk aversion.

The final set of results study optimal government policy in booms and recessions at shorter horizons. What we have to say about this comes from a version of our model calibrated to US data that captures (1) the initial heterogeneity wages and assets (1) the fact that left tail of the cross-section distribution of labor income falls by more than right tail in recessions; and (2) how inflation and interest rate risk co-move with labor productivity. When we calibrate to those patterns, we find that during recessions accompanied by higher inequality, it is optimal to increase taxes and transfers and to issue government debt. These outcomes differ substantially both qualitatively and quantitatively from those in either a representative agent model or in a version of our model in which a recession is modelled as a pure TFP shock that leaves the distribution of skills unchanged. Simulating this model we also verify that the analytical insights about the ergodic distribution developed in tractable versions are preserved.

1.1 Relationships to literatures

At a fundamental level, our paper descends from both Barro (1974), who showed Ricardian equivalence in a representative agent economy with lump sum taxes, and Barro (1979), who studied optimal taxation when lump sum taxes are ruled out. In our environment with incom-

plete markets and heterogeneous workers, both of the forces discovered by Barro play large roles, although the distributive motives that we include lead to richer policy prescriptions.

A large literature on Ramsey problems exogenously rules out transfers in the context of representative agent, general equilibrium models. Lucas and Stokey (1983), Chari et al. (1994), and Aiyagari et al. (2002) (henceforth called AMSS) are leading examples. In contrast to those papers, our Ramsey planner cares about redistribution among agents with different skills and wealths. Other than not allowing them to depend on agents' personal identities, we leave transfers unrestricted and let the Ramsey planner set them optimally. Nevertheless, we find that some of the same general principles that emerge from that representative agent, no-transfers literature continue to hold, in particular, the prescription to smooth distortions across time and states. However, it is also true that allowing the government to set transfers optimally substantially changes qualitative and quantitative insights about the optimal policy in important respects.¹

Several recent papers impute distributive concerns to a Ramsey planner. Three papers that are perhaps most closely related to ours are Bassetto (1999), Shin (2006), and Werning (2007). Like us, those authors allow heterogeneity and study distributional consequences of alternative tax and borrowing policies. Bassetto (1999) extends the Lucas and Stokey (1983) environment to include I types of agents who are heterogeneous in their time-invariant labor productivities. There are complete markets and a Ramsey planner has access only to proportional taxes on labor income and history-contingent borrowing and lending. Bassetto studies how the Ramsey planner's vector of Pareto weights influences how he responds to government expenditures and other shocks by adjusting the proportional labor tax and government borrowing to cover expenses while manipulating prices in ways that redistribute wealth between 'rentiers' (who have low productivities and whose main income is from their asset holdings) and 'workers' (who have high productivities) whose main income source is their labor.

Shin (2006) extends the AMSS (Aiyagari et al. (2002)) economy to have two risk-averse households who face idiosyncratic income risk. When idiosyncratic income risk is big enough relative to aggregate government expenditure risk, the Ramsey planner chooses to issue debt in order to help households engage in precautionary saving, thereby overturning the AMSS result

¹ There is also a more recent strand of literature that focuses on the optimal policy in settings with heterogeneous agents when a government can impose arbitrary taxes subject only to explicit informational constraints (see Golosov et al. (2007) for a review). A striking result from that literature is that when agent's asset holdings are perfectly observable, the distribution of assets among agents is irrelevant and an optimal allocation can be achieved purely through taxation (see, e.g. Bassetto and Kocherlakota (2004)). In the previous version of the paper we showed that a mechanism design version of the model with unobservable assets generates some of the similar predictions to the model with affine taxes that we study, in particular, the relevance of net assets and history dependence of taxes. We leave further analysis along this direction to the future.

that in their quasilinear case a Ramsey planner eventually sets taxes to zero and lives off its earnings from assets forevermore. Shin emphasizes that the government does this at the cost of imposing tax distortions. While being constrained to use proportional labor income taxes and nonnegative transfers, Shin’s Ramsey planner balances two competing self-insurance motives: aggregate tax smoothing and individual consumption smoothing.

Werning (2007) studies a complete markets economy with heterogeneous agents and transfers that are unrestricted in sign. He obtains counterparts to our results about net versus gross asset positions, including that government assets can be set to zero in all periods. Because he allows unrestricted taxation of initial assets, the initial distribution of assets plays no role. Theorem 1 and its corollaries substantially generalize Werning’s results by showing that all allocations of assets among agents and the government that imply the same net asset position lead to the same optimal allocation, a conclusion that holds for market structures beyond complete markets. Werning (2007) provides an extensive characterization of optimal allocations and distortions in complete market economies, while we focus on precautionary savings motives for private agents and the government that are not present when markets are complete.^{2,3}

Finally, our numerical analysis in Section 7 is related to McKay and Reis (2013). While our focus differs from theirs – McKay and Reis study the effect of calibrated US tax and transfer system on stabilization of output, while we focus on optimal policy in a simpler economy – both papers confirm the importance of transfers and redistribution over business-cycle frequencies.

2 Environment

Exogenous fundamentals include a cross section distribution of skills $\{\theta_{i,t}\}$ and government expenditures $\{g_t\}$ and **payoffs p_t on the asset that agents including the government trade**. These are all functions of a shock s_t that is governed by an irreducible Markov process, where $s_t \in S$ and S is a finite set. We let $s^t = (s_0, \dots, s_t)$ denote a history of shocks with joint density $\pi(s^t)$.⁴

There is a mass n_i of a type $i \in I$ agents, with $\sum_{i=1}^I n_i = 1$. Types differ in skills indexed by $\{\theta_{i,t}\}_t$. Preferences of an agent of type i over stochastic processes for consumption $\{c_{i,t}\}_t$ and

²Werning (2012) studies optimal taxation with incomplete markets and explores conditions under which optimal taxes depend only on the aggregate state.

³More recent closely related papers are Azzimonti et al. (2008a,b) and Correia (2010). While these authors study optimal policy in economies in which agents are heterogeneous in skills and initial assets, they do not allow aggregate shocks.

⁴To save on notation, mostly we use z_t to denote a random variable with a time t conditional distribution that is a function of the history s^t . Occasionally, we use the more explicit notion $z(s^t)$ to denote a realization at a particular history s^t .

labor supply $\{l_{i,t}\}_t$ are ordered by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U^i(c_{i,t}, l_{i,t}), \quad (1)$$

where \mathbb{E}_t is a mathematical expectations operator conditioned on time t information and $\beta \in (0, 1)$ is a time discount factor. Except in section 2.1, we assume that $U^i : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ is concave in $(c, -l)$ and twice continuously differentiable. We let $U_{x,t}^i$ or $U_{xy,t}^i$ denote first and second derivatives of U^i with respect to $x, y \in \{c, l\}$ in period t and assume $\lim_{x \rightarrow 0} U_l^i(c, x) = 0$ for all c and i .⁵

An agent of type i who supplies l_i units of labor produces $\theta_i(s_t) l_i$ units of output, where $\theta_i(s_t) \in \Theta$ is a nonnegative state-dependent scalar. Feasible allocations satisfy

$$\sum_{i=1}^I n_i c_{i,t} + g_t = \sum_{i=1}^I n_i \theta_{i,t} l_{i,t} \quad (2)$$

where g_t denotes exogenous government expenditures in state s_t .

The government and agents trade a single, possibly risky, asset. The time t payoff p_t on the single asset is described by an $S \times S$ matrix \mathbb{P}

$$p_t = \mathbb{P}(s_t | s_{t-1}),$$

satisfying the normalizations $\mathbb{E}_t p_{t+1} = 1$. Specifying the random asset payoffs in this way is a convenient way for us to investigate impacts of the correlation between asset returns, on the one hand, and government expenditures or shocks to the skill distribution, skills, and government purchases, on the other hand.

Private agents and the government begin with assets $\{b_{i,-1}\}_{i=1}^I$ and B_{-1} , respectively. These asset holdings satisfy the market clearing condition

$$\sum_{i=1}^I n_i b_{i,t} + B_t = 0 \text{ for all } t \geq -1. \quad (3)$$

The price of the single asset at time t is $q_t = q_t(s^t)$, so and $R_t = \frac{p_t}{q_{t-1}}$ is the one-period return on the asset.

The government imposes an affine tax with proportional labor tax rate τ_t and common lump transfers component T_t . The tax bill of an agent with wage earnings $l_{i,t} \theta_{i,t}$ is

$$-T_t + \tau_t \theta_{i,t} l_{i,t}.$$

⁵Results in section 2.1 hold even if we weaken assumptions like differentiability and convexity of U^i .

We leave the sign of T_t unrestricted.

A type i agent's budget constraint at $t \geq 0$ is

$$c_{i,t} + b_{i,t} = (1 - \tau_t) \theta_{i,t} l_{i,t} + R_t b_{i,t-1} + T_t. \quad (4)$$

The government budget constraint is

$$g_t + B_t = \tau_t \sum_{i=1}^I n_i \theta_{i,t} l_{i,t} - T_t + R_t B_{t-1}. \quad (5)$$

Definition 1 An allocation is a sequence $\{c_{i,t}, l_{i,t}\}_{i,t}$. An asset profile is a sequence $\{b_{i,t}\}_i, B_t\}_t$. A returns process is a sequence $\{R_t\}_t$. A tax policy is a sequence $\{\tau_t, T_t\}_t$.

Remark 1 It is necessary to impose debt limits on the asset profile. For households, we shall impose natural debt limits that will depend on the tax policy.⁶

Definition 2 For a given initial asset distribution $(\{b_{i,-1}\}_i, B_{-1})$, a competitive equilibrium with affine taxes is a sequence $\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, B_t, R_t\}_t$ and a tax policy $\{\tau_t, T_t\}_t$, such that $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ maximize (1) subject to (4) $\{b_{i,t}\}_{i,t}$ satisfies the borrowing limits; and constraints (2), (5) and (3) are satisfied.

A Ramsey planner's preferences over a vector of competitive equilibrium stochastic processes for consumption and labor supply are ordered by

$$\mathbb{E}_0 \sum_{i=1}^I \omega_i \sum_{t=0}^{\infty} \beta^t U_t^i(c_{i,t}, l_{i,t}), \quad (6)$$

where the Pareto weights satisfy $\omega_i \geq 0$, $\sum_{i=1}^I \omega_i = 1$.

The Ramsey planner chooses the following object:

Definition 3 Given $(\{b_{i,-1}\}_i, B_{-1})$, an optimal competitive equilibrium with affine taxes is a tax policy $\{\tau_t^*, T_t^*\}_t$, an allocation $\{c_{i,t}^*, l_{i,t}^*\}_{i,t}$, an asset profile $\{b_{i,t}^*\}_i, B_t^*\}_t$, and a return process $\{R_t^*\}_t$ such that (i) given $(\{b_{i,-1}\}_i, B_{-1})$, the tax policy, return process, and allocation constitute a competitive equilibrium, (ii) B_t satisfies the borrowing constraints; and (iii) there is no other tax policy $\{\tau_t, T_t\}_t$ such that a competitive equilibrium given $(\{b_{i,-1}\}_i, B_{-1})$ and $\{\tau_t, T_t\}_t$ has a strictly higher value of (6).

We call $\{\tau_t^*, T_t^*\}_t$ an optimal tax policy, $\{c_{i,t}^*, l_{i,t}^*\}_{i,t}$ an optimal allocation, and $\{b_{i,t}^*\}_i, B_t^*\}_t$ an optimal asset profile.

⁶An alternative is to impose ad-hoc debt limits in the form of exogenous history-contingent bounds for each agent. Appendix A.1 discusses how restricting attention to natural debt limits for the households only shrinks the set of allocations that can be implemented as competitive equilibria.

2.1 State dimension reduction

The arithmetic of budget constraints and market clearing instructs us how to cut out some frills in formulating the optimal policy problem. The key is to note that an equivalence class of tax policies and asset profiles support the same competitive equilibrium allocation.

Theorem 1 *Given $(\{b_{i,-1}\}_i, B_{-1})$, let $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, B_t, R_t\}_t$ and $\{\tau_t, T_t\}_t$ be a competitive equilibrium. For any bounded sequences $\{\hat{b}_{i,t}\}_{i,t \geq -1}$ that satisfy*

$$\hat{b}_{i,t} - \hat{b}_{1,t} = \tilde{b}_{i,t} \equiv b_{i,t} - b_{1,t} \text{ for all } t \geq -1, i \geq 2,$$

there exist sequences $\{\hat{T}_t\}_t$ and $\{\hat{B}_t\}_{t \geq -1}$ that satisfy (3) and that make $\{\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\}_i, \hat{B}_t, R_t\}_t$ and $\{\tau_t, \hat{T}_t\}_t$ constitute a competitive equilibrium given $(\{\hat{b}_{i,-1}\}_i, \hat{B}_{-1})$.

In the spirit of Barro (1974), this is to be interpreted as a Ricardian equivalence result for our environment. We relegate the proof to appendix A.2 .

Theorem 1 implies that the tax policy and asset profile contain some redundant information that it is convenient for us to eliminate.⁷

XXX Tom and David: The state reduction strictly comes from thm 1 that allows normalization along all histories and not just the initial one as highlighted in the corollary.

The following corollary teaches us how to reduce the dimension of the information to be encoded in a description of government policy to be used in a concise formulation of a Ramsey problem.

Corollary 1 *For any pair B'_{-1}, B''_{-1} , there are asset profiles $\{b'_{i,-1}\}_i$ and $\{b''_{i,-1}\}_i$ such that equilibrium allocations starting from $(\{b'_{i,-1}\}_i, B'_{-1})$ and from $(\{b''_{i,-1}\}_i, B''_{-1})$ are the same. These asset profiles satisfy*

$$b'_{i,-1} - b'_{1,-1} = b''_{i,-1} - b''_{1,-1} \quad \forall i.$$

Thus, total government debt is not what matters, who owns it does.

XXX Tom: Is there a way to salvage this discussion? It has some good insights but I am not sure it fits in the new vision. I put it just for discussion and we can take it out later

To appreciate how these Ricardian irrelevance results affect optimal equilibria, suppose that we increase an initial level of government debt from 0 to some arbitrary level $B'_{-1} > 0$. If the

⁷This result holds in more general environments. For example, we could allow agents to trade all conceivable Arrow securities and still show that equilibrium allocations depend only on agents' net assets positions.

government were to hold transfers $\{T_t\}_t$ fixed, it would have to increase tax rates $\{\tau_t\}_t$ enough to collect a present value of revenues sufficient to repay B'_{-1} . Since deadweight losses are convex in τ , higher levels of debt financed with bigger distorting taxes $\{\tau_t\}$ impose larger distortions on the economy, thereby degrading the equilibrium allocation. But this would not happen if the government were instead to adjust transfers in response to a higher initial debt. To determine optimal transfers, we need to know who owns the initial government debt B'_{-1} . For example, suppose that agents hold equal amounts of it. Then each unit of debt repayment achieves the same redistribution as one unit of transfers. If the original tax policy at $B'_{-1} = 0$ were optimal, then the best policy for a government with initial debt $B'_{-1} > 0$ would be to reduce the present value transfers by exactly the amount of the increase in per capita debt, because then distorting taxes $\{\tau_t\}$ and the allocation would both remain unchanged.⁸

But the situation would be different if holdings of government debt were not equal across agents. For example, suppose that richer people owned disproportionately more government debt than poorer people. That would mean that inequality is effectively initially higher in an economy with higher initial government debt. As a result, a government with Pareto weights $\{\alpha_i\}$ that favor equality would want to increase both distorting tax rates $\{\tau_t\}$ and transfers $\{T_t\}$ to offset the increase in inequality associated with the increase in government debt. The conclusion would be the opposite if government debt were to be owned mostly by poorer households.

This logic shows how important it is to know the distribution of government debt across people. Government debt that is widely distributed across households (e.g., implicit Social Security debt) is less distorting than government debt owned mostly by people whose incomes are at the top of the income distribution (e.g., government debt held by hedge funds).⁹

Throughout this paper we avail ourselves of theorem 1 to impose a normalization on asset profiles that will determine what we mean we say “public debt.” We assume that productivities are ordered as $\theta_{1,t} \geq \theta_{2,t} \dots \geq \theta_{N,t}$. We set $b_{1,t} = 0$, a normalization that tells us to interpret $-B_t = \sum_{i>1} n_i b_{i,t}$ as public debt.¹⁰ With this normalization, imposing limit on B_t become the counterparts to imposing debt limits in a representative agent settings.

⁸This example illustrates principles proclaimed by Simon ?, p. 85 in the quotation with which we began this paper.

⁹It is possible to extend our analysis to open economy with foreign holdings of domestic debt. The more government debt is owned by the foreigners, the higher are the distorting taxes that the government needs to impose.

¹⁰If for some type i , $\theta_{i,t} = 0$, $b_{i,-1} = 0$ and U^i is defined only on \mathcal{R}_+^2 , the type i agent’s budget constraint will imply that all allocations feasible for the planner have nonnegative present values of transfers, since transfers are the sole source of a type i agent’s wealth and consumption.

3 Optimal equilibria with affine taxes

We use the state dimension reduction from subsection 2.1 to formulate the Ramsey problem. Following Lucas and Stokey (1983) and Aiyagari et al. (2002), we use households' first-order necessary conditions to describe restrictions on competitive equilibrium allocations.

With natural borrowing limits for the households, first-order necessary conditions for the consumers' problems are

$$(1 - \tau_t) \theta_{i,t} U_{c,t}^i = -U_{l,t}^i, \quad (7)$$

and

$$U_{c,t}^i = \beta \mathbb{E}_t R_{t+1} U_{c,t+1}^i. \quad (8)$$

To help characterize an equilibrium, we use

Theorem 2 *A sequence $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, R_t, \tau_t, T_t\}_t$ is part of a competitive equilibrium with affine taxes if and only if it satisfies (2), (4), (7), and (8) and $b_{i,t}$ is bounded for all i and t .*

Proof. Necessity is obvious. In appendix A.3, we use arguments of Magill and Quinzii (1994) and Constantinides and Duffie (1996) to show that any $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ that satisfies (4), (7), and (8) is a solution to consumer i 's problem. Equilibrium $\{B_t\}_t$ is determined by (3) and constraint (5) is then implied by Walras' Law ■

To find an optimal equilibrium, by Theorem 2 we can choose $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, R_t, \tau_t, T_t\}_t$ to maximize (6) subject to (2), (4), (7), and (8). We apply a first-order approach and follow steps similar to ones taken by Lucas and Stokey (1983) and Aiyagari et al. (2002). Substituting consumers' first-order conditions (7) and (8) into the budget constraints (4) yields implementability constraints

$$c_{i,t} + b_{i,t} = -\frac{U_{l,t}^i}{U_{c,t}^i} l_{i,t} + T_t + \frac{p_t U_{c,t-1}^i}{\beta \mathbb{E}_{t-1} p_t U_{c,t}^i} b_{i,t-1} \text{ for all } i, t. \quad (9)$$

For $I \geq 2$, we can use constraint (9) for $i = 1$ to eliminate T_t from (9) for $i > 1$. Letting $\tilde{b}_{i,t} \equiv b_{i,t} - b_{1,t}$, we can represent the implementability constraints as

$$\begin{aligned} & (c_{i,t} - c_{1,t}) + \tilde{b}_{i,t} \\ &= -\frac{U_{l,t}^i}{U_{c,t}^i} l_{i,t} + \frac{U_{l,t}^1}{U_{c,t}^1} l_{1,t} + \frac{p_t U_{c,t-1}^i}{\beta \mathbb{E}_{t-1} p_t U_{c,t}^i} \tilde{b}_{i,t-1} \text{ for } i > 1 \text{ and } t \geq 0, \end{aligned} \quad (10)$$

so that the planner's maximization problem involves only on the $I - 1$ variables $\tilde{b}_{i,t-1}$. The reduction of the dimensionality from I to $I - 1$ is a consequence of corollary 1 of theorem 1.

Denote $Z_t^i = (c_{i,t} - c_{1,t}) + \tilde{b}_{i,t} + \frac{U_{l,t}^i}{U_{c,t}^i} l_{i,t} - \frac{U_{l,t}^1}{U_{c,t}^1} l_{1,t}$. The Ramsey problem is:

$$\max_{c_{i,t}, l_{i,t}, \tilde{b}_{i,t}} \mathbb{E}_0 \sum_{i=1}^I \omega_i \sum_{t=0}^{\infty} \bar{\beta}_t U_t^i(c_{i,t}, l_{i,t}), \quad (11)$$

subject to

$$\tilde{b}_{i,t-1} \frac{p_t U_{c,t-1}^i}{\mathbb{E}_{t-1} p_t U_{c,t}^i} = \mathbb{E}_t \sum_{k=t}^{\infty} \beta^{k-t} \left(\frac{U_{c,k}^i}{U_{c,t}^i} \right) Z_k^i \quad \forall t \geq 1 \quad (12a)$$

$$\tilde{b}_{i,-1} = \mathbb{E}_{-1} \sum_{k=0}^{\infty} \beta^k \left(\frac{U_{c,k}^i}{U_{c,t}^i} \right) Z_k^i \quad (12b)$$

$$\frac{\mathbb{E}_t p_{t+1} U_{c,t+1}^i}{U_{c,t}^i} = \frac{\mathbb{E}_t p_{t+1} U_{c,t+1}^j}{U_{c,t}^j} \quad (12c)$$

$$\sum_{i=1}^I n_i c_i(s^t) + g(s^t) = \sum_{i=1}^I n_i \theta_i(s^t) l_i(s^t), \quad (12d)$$

$$\frac{U_{l,t}^i}{\theta_{i,t} U_{c,t}^i} = \frac{U_{l,t}^1}{\theta_{1,t} U_{c,t}^1} \quad (12e)$$

$$\sum_{i=1}^N \tilde{b}_{i,t-1} \text{ is bounded} \quad (12f)$$

Constraint (12a) requires that the conditional expectation on the right side, a conditional expectation at time t , be an exact function of information at time $t - 1$, the same type of measurability condition present in Aiyagari et al. (2002). This condition is inherited from the restriction that only one asset is traded between the private and the public sector, and also within the private sector, and that it has payoffs p_t .

It is convenient and informative to represent the Ramsey problem recursively. Let $\mathbf{x} = (U_c^2 \tilde{b}_2, \dots, U_c^I \tilde{b}_I)$, $\boldsymbol{\rho} = (U_c^2/U_c^1, \dots, U_c^I/U_c^1)$, and denote an allocation $a = \{c_i, l_i\}_{i=1}^I$. In the spirit of Kydland and Prescott (1980) and Farhi (2010), we split the Ramsey problem into a time-0 problem that takes $(\{\tilde{b}_{i,-1}\}_{i=2}^I, s_0)$ as state variables and a time $t \geq 1$ continuation problem that takes $\mathbf{x}, \boldsymbol{\rho}, s_-$ as state variables. There are two value functions, one that pertains to $t \geq 1$, another to $t = 0$.¹¹ As usual, we work backwards and describe the $t \geq 1$ Bellman equations first, and then the $t = 0$ Bellman equation.

For $t \geq 1$, let $V(\mathbf{x}, \boldsymbol{\rho}, s_-)$ be the planner's continuation value given $\mathbf{x}_{t-1} = \mathbf{x}$, $\boldsymbol{\rho}_{t-1} = \boldsymbol{\rho}$, $s_{t-1} = s_-$. It satisfies the Bellman equation

¹¹The time inconsistency of an optimal policy manifests itself in there being distinct value functions and Bellman equations at $t = 0$ and $t \geq 1$.

$$V(\mathbf{x}, \boldsymbol{\rho}, s_-) = \max_{a(s), \mathbf{x}'(s), \boldsymbol{\rho}'(s)} \sum_s \pi(s|s_-) \left(\left[\sum_i \omega_i U^i(s) \right] + \beta V(\mathbf{x}'(s), \boldsymbol{\rho}'(s), s) \right) \quad (13)$$

where the maximization is subject to

$$U_c^i(s) [c_i(s) - c_1(s)] + x'_i(s) + \left(U_l^i(s) l_i(s) - U_c^i(s) \frac{U_l^1(s)}{U_c^1(s)} l_1(s) \right) = \frac{x P(s|s_-) U_c^i(s)}{\beta \mathbb{E}_{s_-} P U_c^i} \text{ for all } s, i \geq 2 \quad (14a)$$

$$\frac{\mathbb{E}_{s_-} P U_c^i}{\mathbb{E}_{s_-} P U_c^1} = \rho_i \text{ for all } i \geq 2 \quad (14b)$$

$$\frac{U_l^i(s)}{\theta_i(s) U_c^i(s)} = \frac{U_l^1(s)}{\theta_1(s) U_c^1(s)} \text{ for all } s, i \geq 2 \quad (14c)$$

$$\sum_i n_i c_i(s) + g(s) = \sum_i n_i(s) l_i(s) \quad \forall s \quad (14d)$$

$$\rho'_i(s) = \frac{U_c^i(s)}{U_c^1(s)} \text{ for all } s, i \geq 2 \quad (14e)$$

$$\sum_{i>1} x_i(s) \frac{\beta}{U_c^i(s)} \text{ is bounded} \quad (14f)$$

Constraints (14b) and (14e) imply (8). The definition of x_t and constraints (14a) together imply equation (10) scaled by U_c^i .

Next we describe the Bellman equation pertinent for $t = 0$. Let $V_0 \left(\{\tilde{b}_{i,-1}\}_{i=2}^I, s_0 \right)$ be the value to the planner at $t = 0$, where $\tilde{b}_{i,-1}$ denotes initial debt inclusive of accrued interest. It satisfies the Bellman equation

$$V_0 \left(\{\tilde{b}_{i,-1}\}_{i=2}^I, s_0 \right) = \max_{a_0, x_0, \rho_0} \sum_i \omega_i U^i(c_{i,0}, l_{i,0}) + \beta V(x_0, \rho_0, s_0) \quad (15)$$

where the maximization is subject to

$$U_{c,0}^i [c_{i,0} - c_{1,0}] + x_{i,0} + \left(U_{l,0}^i l_{i,0} - U_{c,0}^i \frac{U_{l,0}^1}{U_{c,0}^1} l_{1,0} \right) = U_{c,0}^i \tilde{b}_{i,-1} \text{ for all } i \geq 2 \quad (16a)$$

$$\frac{U_{l,0}^i}{\theta_{i,0} U_{c,0}^i} = \frac{U_{l,0}^1}{\theta_{1,0} U_{c,0}^1} \text{ for all } i \geq 2 \quad (16b)$$

$$\sum_i \pi_i c_{i,0} + g_0 = \sum_i \pi_i \theta_{i,0} l_{i,0} \quad (16c)$$

$$\rho_{i,0} = \frac{U_{c,0}^i}{U_{c,0}^1} \quad \forall i \geq 2 \quad (16d)$$

A tell-tale sign of the time consistency of the optimal tax plan is that (14b), which constrains the time $t \geq 1$ Bellman equations, is absent from the time 0 problem.

4 Asymptotic properties of optimal allocations

The Ramsey plan induces an ergodic distribution of debt, transfers, and the labor tax rate that we shall describe in sections 5 and 6. We shall verify that the long-run levels and spreads in debt and the tax rate are determined by two factors: a) the ability of the government to hedge aggregate shocks by taking advantage of fluctuations in the returns on the single asset that it trades; and b) the Ramsey planner's preferences for redistribution. In particular, the government wants to accumulate debt if interest rates are lower when the its needs for revenues are higher and *vice versa*. The long run variances of debt and the tax rate are lower and rates of convergence to the ergodic distribution are higher in economies where the magnitude of this co movement is larger. Governments that want more redistribution eventually issue more debt.

XXXXX Check if the previous sentence is still confusing.

To extract these implications, we use the following strategy. In section 5, we begin by studying a simplified economy with quasilinear household preferences and i.i.d aggregate shocks. This setting is tractable enough to allow us to get some sharp analytic results that isolate the main forces that drive the outcomes summarized in the preceding paragraph. In section 6, we economies that are more general in terms of their heterogeneity, preferences, and shock structures. Then in section 7, we numerically verify that the forces isolated in the simpler models extend to a version of the model with several types of agents calibrated to match US data.

5 Quasilinear preferences

We specialize the problem described in section 3 by imposing the following assumptions to be maintained throughout this section.

Assumption 1 *IID aggregate shocks: s_t is i.i.d over time*

Assumption 2 *Quasilinear preference: $u(c, l) = c - \frac{l^{1+\gamma}}{1+\gamma}$* ¹²

With i.i.d shocks we can restrict our attention to payoff matrices \mathbb{P} that have identical rows denoted by a vector $P(s)$ with a normalization that $\mathbb{E}P(s) = 1$. We collect a particular set of these vectors that are perfectly correlated with expenditure shocks $g(s)$ in a set

$$\mathcal{P}^* = \left\{ P(s) : P(s) = 1 + \frac{\beta}{B^*} (g(s) - \mathbb{E}g) \text{ for some } B^* \in [\bar{B}, \underline{B}] \right\},$$

¹²Aiyagari et al. (2002) assume quasilinear preferences in an important part of their analysis.

where \bar{B} and \underline{B} are upper and lower bounds for government assets.

Before characterizing a Ramsey allocation for an economy with heterogeneous agents and no restrictions on transfers, it is instructive to study a simpler representative agent economy in which the government *cannot* use transfers. Later we shall show that Ramsey allocations for this economy are also Ramsey allocations for a multiple-agent economy under some interesting restrictions on Pareto weights.

XXXXX Should we say more here?

5.1 Representative agent

Environment

This section describes a representative agent environment with risky debt and no transfers.¹³ Given a tax, asset policy $\{\tau_t, B_t\}$, the household solves,

$$\max_{\{c_t, l_t, b_t\}_t} \mathbb{E}_0 \sum_t \beta^t \left[c_t - \frac{l_t^{1+\gamma}}{1+\gamma} \right] = W_0(b_{-1}) \quad (17)$$

subject to

$$c_t + b_t = (1 - \tau_t)\theta l_t + R_t b_{t-1}. \quad (18)$$

Using the first-order conditions for labor and savings, the implementability constraints become

$$b_{t-1} \frac{P_t}{E_{t-1} P_t} = \mathbb{E}_t \sum_j \beta^{t+j} [c_t - l_t^{1+\gamma}] \quad \forall t \geq 0 \quad (19)$$

We also have the feasibility constraints

$$c_t + g_t \leq \theta l_t, \quad \forall t \geq 0 \quad (20a)$$

and the market-clearing conditions for bonds,

$$b_t + B_t = 0, \quad \forall t \geq 0. \quad (20b)$$

The Ramsey allocation solves $\max_{\{c_t, l_t\}_t} W_0(b_{-1})$ subject to (19), feasibility (20a), market clearing for bonds (20b), and debt limits \underline{B}, \bar{B} on government assets.¹⁴

Let $V(B_-)$ be the ex-ante value of a Ramsey plan starting with initial government assets B_- . It satisfies the Bellman equation

¹³This economy differs from the Aiyagari et al. (2002) economy in two ways: first, the government trades a possibly risky instead of a risk-free bond, and second, the government is prohibited from using transfer at all, whereas Aiyagari et al. restrict transfers to be non negative. Both of these features have important implications for Aiyagari et al. result that in the long run the tax rate on labor is zero.

¹⁴In some calculations, we will impose a natural debt limit for the government.

$$V(B_-) = \max_{c(s), l(s), B(s)} \sum_s \pi(s) \left\{ c(s) - \frac{l(s)^{1+\gamma}}{1+\gamma} + \beta V(B(s)) \right\} \quad (21)$$

where the maximization is subject to

$$c(s) - B(s) = l(s)^{1+\gamma} - \beta^{-1} P(s) B_- \quad (22a)$$

$$c(s) + g(s) \leq \theta l(s) \quad (22b)$$

$$\underline{B} \leq B(s) \leq \bar{B} \quad (22c)$$

Results

Theorems 3 and 4 assert our main results about this economy. The first result describes some general properties of the invariant distribution of debt for a large class of payoffs. The second result approximates the mean and variance of the invariant distribution of debt when payoffs are close to the set \mathcal{P}^* .

When payoffs do not belong to \mathcal{P}^* , the support of the invariant distribution of debt is wide in the sense that (almost surely) paths of debt sequences recurrently revisit small neighborhoods of any arbitrary lower and upper bounds on government debt. The labor tax rate is increasing in government debt, so tax rates vary too. These outcomes contrast sharply with those both in a corresponding complete market benchmark like Lucas and Stokey's where both debt and tax rates would be constant sequences, and in the incomplete markets economy of Aiyagari et al., where assets approach levels that allow the limiting tax rate to be zero and the tail allocation to be first-best..

With more structure on the payoffs, we show that there is an average inward drift to government assets. More precisely, the sequence of Lagrange multipliers on the sequence of implementability constraints forms a sub (or super) martingale in regions with low (or high) debt. The envelope theorem links the dynamics of the multiplier to the dynamics of government debts. The concavity of the value function implies mean reversion for government debt. Mean reversion is particularly stark when $P(s) \in \mathcal{P}^*(s)$: here government debt converges to a constant.¹⁵

To acquire more insights about the invariant distribution, we linearize the law of motion for the evolution of government debt with respect to both government debt and payoffs. Here we artfully choose the point about which to take a linear approximation, namely, a closest (in

¹⁵Thus, the limiting allocation is a particular Lucas and Stokey economy with constant government debt and taxes; however, the level and the sign of long-run government debt is determined by the joint properties of shocks and payoffs rather than by the initial government debt, as it is in the Lucas and Stokey model.

l_2 sense) complete market economy that serves as the steady state of an economy for some $P(s) \in \mathcal{P}^*(s)$. Exploiting the structure of these approximate laws of motion allows us to obtain bounds on the standard deviation of government debt in the ergodic distribution and also the rate at which the mean debt level converges, a rate that can be expressed in terms of primitives in the form of the joint distribution of shocks and payoffs.

Theorem 3 *In a representative agent economy satisfying assumptions 1 and 2, the behavior of government assets under a Ramsey plan can be characterized as follows:*

1. Suppose $P(s) \notin \mathcal{P}^*$. There is an invariant distribution of government assets such that

$$\forall \epsilon > 0, \quad \Pr\{B_t < \underline{B} + \epsilon \text{ or } B_t > \overline{B} - \epsilon \text{ i.o.}\} = 1$$

2. Suppose $P(s) - P(s') > \beta \frac{g(s) - g(s')}{B} \quad \forall s, s'$. For large enough government assets (or debt), there is a drift towards an interior region. The value function $V(B_-)$ is strictly concave and there exists $B_1 < B_2$ such that

$$\mathbb{E}V'(B(s)) > V'(B_-) \quad B_- > B_2$$

and

$$\mathbb{E}V'(B(s)) < V'(B_-) \quad B_- < B_1$$

3. Suppose $P(s) \in \mathcal{P}^*$. Government assets converge to a degenerate steady state

$$\lim_t B_t = B^* \quad a.s. \quad \forall B_{-1}$$

where B^* is the object appearing in the definition of \mathcal{P}^* .

Furthermore, when $P(s) \in \mathcal{P}^*$, in the long-run government assets are

$$B^* = \beta \frac{\text{var}(g(s))}{\text{cov}(P(s), g(s))} \tag{23}$$

When $P(s) \in \mathcal{P}^*$ and when government debt has converged to B^* , the government perfectly hedges fluctuations in its expenditures and its net-of-interest deficit. Whether the government holds assets or owes debt is determined by the sign of the covariance of $P(s)$ with $g(s)$.

Keeping the tax rate and therefore tax revenues constant, the government must finance a higher primary deficit when it gets positive expenditure shock. If the asset returns more when government expenditures are high, the asset is a good hedge. The government optimally holds

positive assets and uses high returns on its portfolio to finance its net-of-interest deficit. On the other hand, if payoffs on the asset are lower when government expenditures at its net-of-interest deficit is high, then owing debt is useful because of how it lowers the interest burden when government expenditures are high.

The long-run tax rate is inversely related to B^* and satisfies:

$$\lim_{B^* \rightarrow \underline{B}} \tau^* = \frac{\gamma}{1 + \gamma} \quad \lim_{B^* \rightarrow \infty} \tau^* = -\infty$$

We now say some more things about the invariant distribution of government debt and the tax rate when $P(s) \notin \mathcal{P}^*$. Because there is no closed form solution for law of motion of government debt, we use an approximation based on the following orthogonal decomposition of an arbitrary $P(s)$:

$$P(s) = \hat{P}(s) + P^*(s)$$

where $P^*(s) \in \mathcal{P}^*$ and $\hat{P}(s)$ is orthogonal to $g(s)$. We expand a Ramsey plan around the steady state of the $P^*(s)$ to obtain the next theorem.¹⁶

Theorem 4 *Under a first order approximation of dynamics around $P^*(s)$, an approximation to the ergodic distribution of government debt has the following properties:*

- **Mean:** *The ergodic mean is B^* ; it equals the steady state level of debt of an economy with payoff vector $P^*(s)$*
- **Variance:** *The ergodic coefficient of variation is*

$$\frac{\sigma(B)}{\mathbb{E}(B)} = \sqrt{\frac{\text{var}(P(s)) - |\text{cov}(g(s), P(s))|}{(1 + |\text{cov}(g(s), P(s))|)|\text{cov}(g(s), P(s))|}} \leq \sqrt{\frac{\text{var}(\hat{P}(s))}{\text{var}(P^*(s))}}$$

- **Convergence rate:** *The speed of convergence to the ergodic distribution is*

$$\frac{\mathbb{E}_{t-1}(B_t - B^*)}{(B_{t-1} - B^*)} = \frac{1}{1 + |\text{cov}(P(s), g(s))|}$$

Recall that when payoffs on the assets are $P^*(s)$, the government can keep the tax rate constant and perfectly hedge fluctuations in its net-of-interest surplus by using total income $P^*(s)B^*$ from the government portfolio. When $B_t \neq B^*$, the incompleteness of markets prevents complete

¹⁶We construct P^* as the projection of P onto the space spanned by \mathcal{P}^* . We then take a first-order Taylor approximation to the decision rules and laws of motion for the state variables of our economy around complete market counterparts associated with $P^* \in \mathcal{P}^*$. Note that the point of approximation is not a deterministic steady state. Appendix A.5 contains more details of the approximation method.

hedging, so shocks are hedged with a combination of changes in the tax rate and the level of government debt. The theorem asserts how deviations from complete spanning map into larger variances for government debt and the tax rate under the ergodic distribution. Figure 1 shows how the ergodic distribution of debt and taxes spread as we exogenously alter the covariance $P(s)$ with $g(s)$.

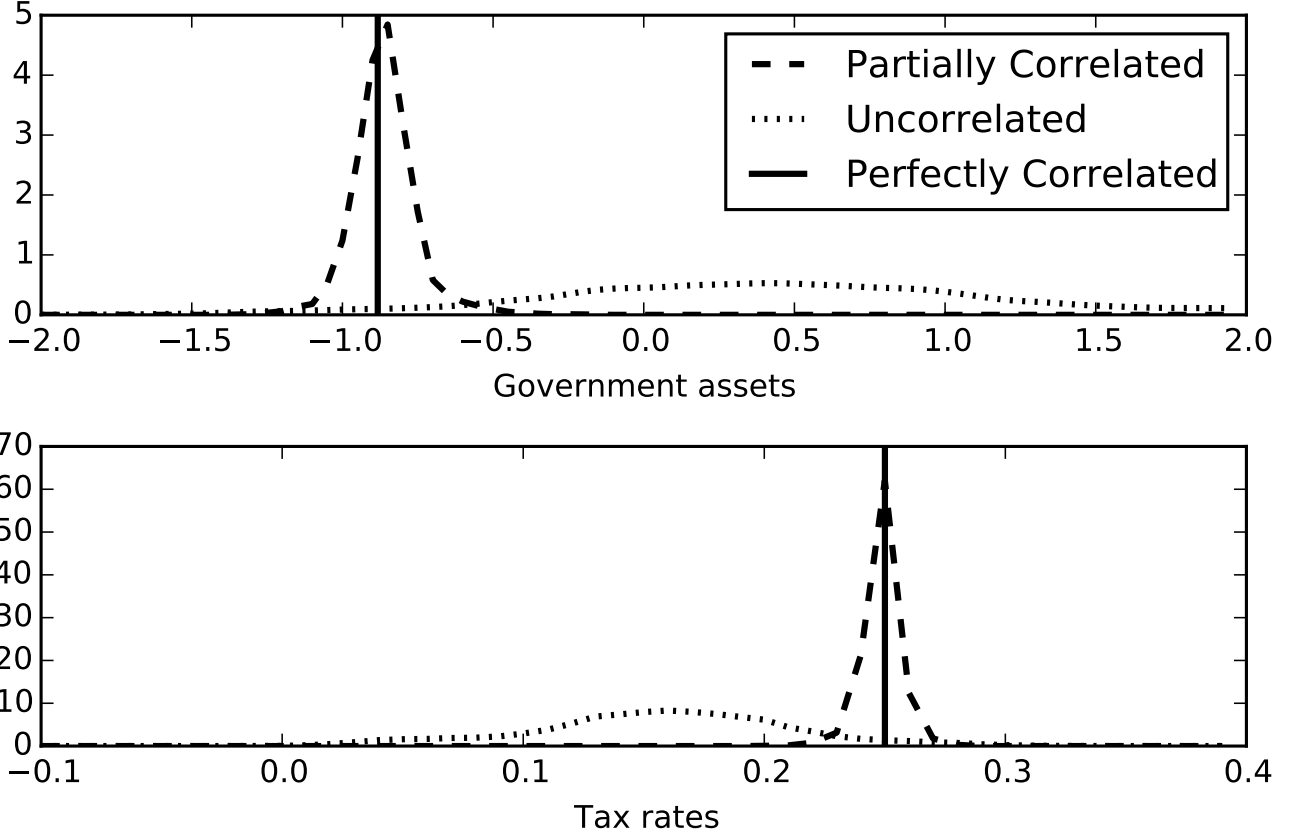


Figure 1: Ergodic distribution for assets B_t and labor taxes τ_t in the representative agent quasilinear economy for three choices $P(s)$.

5.2 Heterogeneous agent economy with quasilinear preferences

We now turn to a more still restricted special economy, but one that now features both heterogeneous agents and transfers. We add to the section 5.1 representative agent economy a second agent who has zero productivity and require that his consumption be nonnegative. Given the Ricardian equivalence result discussed in section 2.1, we maintain a normalization that assets of the unproductive agent are zero throughout this section.

Assumption 3 $\theta_1 > \theta_2 = 0$ and $c_{2,t} \geq 0$.

The assumption that $\theta_2 = 0$ allows us to characterize how the Ramsey plan depends on the Pareto weights. The nonnegativity constraint on the unskilled agent 2's consumption adds enough curvature to the Ramsy problem to unleash main forces that will also prevail in more general settings. In these more general settings, risk aversion and Inada restrictions generate the same forces.

Theorem 5 *Let $(\omega, n) \in [0, 1] \times [0, 1]$ be a Pareto weight and mass assigned to the productive type 1 agent. Assume that $n < \frac{\gamma}{1+\gamma}$. The optimal tax rate, transfer, and government asset policies $\{\tau_t, T_t, B_t\}$ are characterized as follows:*

1. *For $\omega \geq n \left(\frac{1+\gamma}{\gamma} \right)$ we have $T_t = 0$ and the optimal policy is same as in the representative agent economy studied in Theorems 3, and 4*
2. *For $\omega < n \left(\frac{1+\gamma}{\gamma} \right)$, suppose that $\min_s \{P(s)\} > \beta$.*

There exist a $\mathcal{B}(\omega)$ satisfying $\mathcal{B}'(\omega) > 0$ and a $\tau^(\omega)$ such that*

(a) $B_- > \mathcal{B}(\omega)$

$$T_t > 0, \quad \tau_t = \tau^*(\omega), \quad \text{and } B_t = B_- \quad \forall t \geq 0$$

(b) $B_- \leq \mathcal{B}(\omega)$, the policies depend on the structure of $P(s)$.

i. *For $P(s) \notin \mathcal{P}^*$*

$$T_t > 0 \text{ i.o.}, \quad \lim_t \tau_t = \tau^*(\omega) \text{ and } \lim_t B_t = \mathcal{B}(\omega) \quad \text{a.s.}$$

ii. *For $P(s) \in \mathcal{P}^*$, we have two cases depending on B_-*

A. *For $B_- \leq B^*$*

$$T_t = 0, \quad \lim_t \tau_t = \tau^{**}(\omega), \quad \text{and } \lim_t B_t = B^* \quad \text{a.s.}$$

B. *For $\mathcal{B}(\omega) > B_- > B^*$*

$$\Pr\{\lim_t T_t = 0, \lim_t \tau_t = \tau^{**}(\omega), \lim_t B_t = B^* \text{ or } T_t > 0 \text{ i.o.}, \lim_t \tau_t = \tau^*(\omega), \lim_t B_t = \mathcal{B}(\omega)\} > 0$$

In the theorem 5 two types economy, Ramsey planner bears costs of using fluctuating transfers to hedge aggregate shocks. The environment is simple enough to allow us to pinpoint how these costs depend on the Pareto weights. For a “regressive” planner who cares more about the productive type 1 agents, using transfers is especially costly. For a high ω Pareto planner,

increasing transfers entails subsidizing the unproductive type 2 agents whose consumption he values little. A Ramsey planner who assigns a Pareto weight ω to the productive type 1 agent above a threshold $\bar{\omega} = n \left(\frac{1+\gamma}{\gamma} \right)$ sets transfers to zero always. This makes the Ramsey plan in the theorem 5 two-type of agents economy be identical to the plan for the representative agent economy of theorem 3: when $\omega \geq \bar{\omega}$, the type type 1 agent in effect becomes the representative agent of the theorem 3 economy. However, for a less regressive $\omega < \bar{\omega}$ Ramsey planner, transfers remain a useful tool for subsidizing the unproductive agent. To finance these transfers, the Ramsey planner chooses to tax the labor income of the productive agent and need not accumulate a large buffer stock of assets. Thus, the limiting stock of government assets is lower and tax rates are larger for sufficiently more redistributive Ramsey planners.

XXXX Tom: Can we discuss the couple of paragraphs below?

The intuition is that if the government can hedge shocks with transfers then assets need not fluctuate as in the case when markets are complete. The only reason the government does would not do it is because there are costs of using transfers.¹⁷

Suppose for given Pareto weights, the government accumulates some level of assets such that the marginal costs of transfers is sufficiently low to keep taxes fixed at a constant level and finance all fluctuations in expenditure through transfers.

Now perturb the Pareto weights such that the government cares more about the unproductive agent, what is true is that marginal costs of transfers will be lower, hence transfers can be used much earlier and the government optimally stays with higher constant taxes (associated with lower steady state assets)

6 More general economies

To facilitate analysis, the section 5.2 economy simplifies things along several dimensions: there is no curvature in the utility from consumption, shocks lack persistence shocks, and there are only two agents. These simplifications make the return on debt be exogenous and equal to $\beta^{-1}P(s)$. Adding curvature to utility from consumption makes the returns on debt endogenous even for a standard risk-free bond having a payoff vector $P(s) = 1$. Adding curvature requires keeping track of relative marginal utilities of consumption in order to characterize how the Ramsey planner makes the tax rate, government debt, and transfers respond to shocks. This confounds the effects of the planner's motives to redistribute and to use the level of government debt in conjunction with fluctuations in asset returns to hedge shocks to government expenditure and

¹⁷For example in (Aiyagari et al., 2002), for precautionary reasons with incomplete markets the government accumulates assets until the point it can use transfers in a costless manner and keep tax rates at zero.

productivity.

In the next subsection, we first show that with curvature in the utility from consumption, there exists a generic class of economies that eventually feature complete hedging. We construct them in setting with IID shocks that take two values. We also show that how the comovement of the asset returns with exogenous shocks governs the governments' incentives to accumulate assets, an outcome reminiscent of outcomes in the section 5.2 economy with quasilinear preferences.

6.1 Eventual complete hedging with binary shocks

For a given state $(\mathbf{x}, \boldsymbol{\rho}, s_-)$, let $\Psi(s; \mathbf{x}, \boldsymbol{\rho}, s_-) = (x'(s), \rho'(s))$ solve (13) so that $\Psi(s; \mathbf{x}, \boldsymbol{\rho}, s_-)$ is an optimal law of motion for the state variables under a Ramsey plan at $t \geq 1$.

Definition 4 A steady state satisfies $(\mathbf{x}^{SS}, \boldsymbol{\rho}^{SS}) = \Psi(s; \mathbf{x}^{SS}, \boldsymbol{\rho}^{SS}, s_-)$ for all s, s_- .

In a steady state, the ratios of marginal utilities $\rho_i = U_c^i(s)/U_c^1(s)$ are constant and so do not depend on s . This means that in a steady state, the continuation allocation depends only on s_t and not on the history s^{t-1} .

A competitive equilibrium allocation $\{c_i(s), l_i(s)\}_i$ associated with a choice for $\{\tau(s), \boldsymbol{\rho}(s)\}$ is determined by equations (14c), (14d) and (14e).

We now show how to construct a generic set of economies where such a steady state and hence perfect hedging occurs with risk aversion.

Denote $U(\tau, \boldsymbol{\rho}, s)$ as the value of that competitive equilibrium allocation to a planner with Pareto weights $\{\omega_i\}_i$:

$$U(\tau, \boldsymbol{\rho}, s | \{\omega_i\}_i) = \sum_i \omega_i U^i(s).$$

As before define $Z_i(\tau, \boldsymbol{\rho}, s)$ as

$$Z_i(\tau, \boldsymbol{\rho}, s) = U_c^i(s)c_i(s) + U_l^i(s)l_i(s) - \rho_i(s) [U_c^1(s)c_1(s) + U_l^1(s)l_1(s)].$$

When shocks are IID, the Ramsey optimal policy solves the following Bellman equation in $\mathbf{x}(s^{t-1}) = \mathbf{x}, \boldsymbol{\rho}(s^{t-1}) = \boldsymbol{\rho}$

$$V(\mathbf{x}, \boldsymbol{\rho}) = \max_{\tau(s), \boldsymbol{\rho}'(s), \mathbf{x}'(s)} \sum_s \pi(s) [U(\tau(s), \boldsymbol{\rho}'(s), s) + \beta V(\mathbf{x}'(s), \boldsymbol{\rho}'(s))] \quad (24)$$

where the maximization is subject to the constraints subject to the constraints

$$Z_i(\tau(s), \boldsymbol{\rho}'(s), s) + x'_i(s) = \frac{x_i \beta^{-1} P(s) U_c^i(\tau(s), \boldsymbol{\rho}'(s), s)}{\mathbb{E} P U_c^i(\tau, \boldsymbol{\rho})} \text{ for all } s, i \geq 2, \quad (25)$$

$$\sum_s \pi(s) P(s) U_c^1(\tau(s), \boldsymbol{\rho}'(s), s) (\rho'_i(s) - \rho_i) = 0 \text{ for } i \geq 2. \quad (26)$$

Constraint (26), which rearranges constraint (14b), implies that $\rho(s)$ is a risk-adjusted martingale.

Our next job is to study conditions that allow the first-order necessary conditions and feasibility as compressed into (25) and (26) to be consistent with the steady state restrictions imposed on policies.¹⁸

Lemma 1 *With the utility is strictly concave in consumption, $\|S\| = 2$ is necessary for a steady state to exist generically.*

Proof. Let $\pi(s)\mu_i(s)$ and λ_i be Lagrange multipliers on constraints (25) and (26). Imposing the restrictions $x'_i(s) = x_i$ and $\rho'_i(s) = \rho_i$, at a steady state $\{\mu_i, \lambda_i, x_i, \rho_i\}_{i=2}^N$ and $\{\tau(s)\}_s$ are determined by the following equations:

$$Z_i(\tau(s), \boldsymbol{\rho}, s) + x_i = \frac{\beta^{-1} P(s) x_i U_c^i(\tau(s), \boldsymbol{\rho}, s)}{\mathbb{E} U_c^i(\tau, \boldsymbol{\rho})} \text{ for all } s, i \geq 2, \quad (27a)$$

$$U_\tau(\tau(s), \boldsymbol{\rho}, s) - \sum_i \mu_i Z_{i,\tau}(\tau(s), \boldsymbol{\rho}, s) = 0 \text{ for all } s, \quad (27b)$$

$$U_{\rho_i}(\tau(s), \boldsymbol{\rho}, s) - \sum_j \mu_j Z_{j,\rho_i}(\tau(s), \boldsymbol{\rho}, s) + \lambda_i P(s) U_c^1(\tau(s), \boldsymbol{\rho}, s) - \lambda_i \beta \mathbb{E} P(s) U_c^1(\tau(s), \boldsymbol{\rho}(s), s) = 0. \text{ for all } s, i \geq 2 \quad (27c)$$

When the shock s takes only two values, (27) is a square system in $4(N-1) + 2$ unknowns $\{\mu_i^{SS}, \lambda_i^{SS}, x_i^{SS}, \rho_i^{SS}\}_{i=2}^N$ and $\{\tau^{SS}(s)\}_s$. For $|S| \geq 3$, there are more equations than unknowns. So even if one can satisfy them for a given set of parameters, a generic solution does not exist.

■

XXX Tom and David: Lets discuss this and see if we can make it more precise.

At a steady state, outcomes resemble those in the complete market economy of Werning (2007). The tax rate and transfers both depend only on the current realization of shock s_t . Furthermore, arguments of Werning can be adapted to show that the tax rate is constant when preferences have the CES form $c^{1-\sigma}/(1-\sigma) - l^{1+\gamma}/(1-\gamma)$, and also that fluctuations in the tax rate are very small when preferences take forms consistent with the existence of balanced growth. We return to this point after we discuss convergence to a steady state.

To verify existence of a steady state for a particular set of parameter values requires checking that there exists a solution of system (27). Since (27) is a non-linear system, existence can be verified only numerically in general. However, sometimes more can be established. Thus, we

¹⁸Appendix A.8 discusses second-order conditions that ensure these policies are optimal.

now provide a simple example with risk averse agents in which the existence of a root of (27) can be established analytically. The analytical characterization of the steady state in this special case will help us develop some comparative statics and some connections between the quasilinear economy of section 5.2 and the more general economies to be analyzed quantitatively in section 7.

A two-agent example

Consider an economy consisting of two types of households with $\theta_{1,t} > \theta_{2,t} = 0$ and common one-period utilities $\ln c - \frac{1}{2}l^2$. The shock s takes two values $\{s_L, s_H\}$ that are i.i.d across time. We assume that $g(s) = g$ for all s , and $\theta_1(s_H) > \theta_1(s_L)$.

Theorem 6 *Suppose that $g < \theta(s)$ for all s . Let $R(s|s_-)$ be the gross return and $x = U_c^2(s) [b_2(s) - b_1(s)]$*

1. **Countercyclical returns.** *If $P(s_H) = P(s_L)$, then there exists a steady state (x^{SS}, ρ^{SS}) such that $x^{SS} > 0$, $R^{SS}(s_H|s_-) < R^{SS}(s_L|s_-)$.*
2. **Procyclical interest rate.** *There exists a pair $\{P(s_H), P(s_L)\}$ such that there exists a steady state with $x^{SS} < 0$ and $R^{SS}(s_H|s_-) > R^{SS}(s_L|s_-)$.*

In both cases, the tax rate $\tau(s) = \tau^{SS}$ is independent of s .

By setting the assets of the unproductive agent to zero, which theorem 1 tells us amounts only to a normalization, we can interpret x as the marginal-utility-adjusted assets of the government.

Besides establishing existence of a steady state, theorem 6 emphasizes the cyclical properties of the real interest rate as a determinant of the the sign of government assets under a Ramsey plan.

Theorem 6 highlights two main forces that determine the dynamics of the tax rate and government assets: fluctuations in inequality and fluctuations asset returns. Keeping the asset returns fixed for the moment, the government can in principle adjust two instruments in response to an adverse shock (i.e., a fall in θ_1): it can either increase the tax rate τ or it can decrease transfers T . Both responses are distorting, but for different reasons. Increasing the tax rate increases distortions because the deadweight loss is convex in the tax rate, as in Barro (1979). The Ramsey planner copes with this distortion in the present economy in the same way that it does in representative agent economies. **But in a heterogeneous agent economy like ours, adjusting transfers T is also costly. Starting from $x = 0$ or (when agents' asset holdings are**

identical) a decrease in transfers disproportionately affects a low-skilled agent, so his marginal utility falls by more than does the marginal utility of a high-skilled agent. Consequently, a decrease in transfers increases inequality, giving rise to a cost not present in representative agent economies.

The government can reduce the costs of inequality distortions by choosing tax rate policies that make the net asset positions of the high-skilled agent decrease over time. That makes the two agents' after-tax and after-interest income become closer, allowing decreases in transfers to have smaller effects on inequality in marginal utilities. If the net asset position of a high-skilled agent is sufficiently low, then a change in transfers has no effect on inequality and all distortions from fluctuations in transfers are eliminated.¹⁹ This pushes x to be positive in the long run.

Turning now to the second force, asset returns generally fluctuates with shocks. Parts 1 and 2 of theorem 6 isolates forces that drive those fluctuations. Consider again the example of a decrease in the productivity of high-skilled agents. If the tax rate τ is left unchanged, since g is constant, the government requires extra revenues. But suppose that the interest rate increases whenever θ_1 decreases, as happens, for example, with a risk free bond. If the government holds positive assets, its earnings from those assets increase. So holding assets allows higher interest income to offset some of the government's revenue losses from taxes on labor. The situation reverses if the returns falls at times of increased need for government revenues, as in part 2 of theorem 6, so the steady state allocation features the government's owning debt.

The net effect on long run assets depends on the balance of the two forces: inequality distortions that push x to be positive and hedging motives can either go in the same direction as in part 1 or for sufficiently procyclical returns push the long run x to be negative as in part 2 of the theorem 6

These outcomes have counterparts in the representative agent quasilinear economy studied in section 5. There, exploiting linearity allowed us to provide a sharper characterization of how the covariance of the asset returns and exogenous shocks affected the sign (and level) of debt through expression (23). In parts 1 and 2 of theorem 6, with binary shocks, altering the gap $P(s_H) - P(s_L)$ allows us to obtain a corresponding variation in asset returns. The reasoning and underling forces are the same.

6.2 Stability

We extend the Theorem 4 approximation methods to more general economies with strictly concave utility functions. Unlike the quasilinear case where we could obtain an analytical char-

¹⁹This convergence outcome has a similar flavor to "back-loading" results of Ray (2002) and Albanesi and Armenter (2012) that reflect the optimality of structuring policies intertemporally eventually to disarm distortions.

acterization, here we present a numerical convergence criterion and use it to show local stability of a steady state for a wide range of parameters.

As before, let $\pi(s)\mu_i(s)$ and λ_i be Lagrange multipliers on constraints (25) and (26). In Appendix A.8 we show that the history-dependent optimal policies (they are sequences of functions of s^t) can be represented recursively in terms of $\{\boldsymbol{\mu}(s^{t-1}), \boldsymbol{\rho}(s^{t-1})\}$ and s_t . A recursive representation of an optimal policy can be linearized around steady state values of the state variables $(\boldsymbol{\mu}, \boldsymbol{\rho})$.²⁰ Let $\hat{\Psi}_t = \begin{bmatrix} \boldsymbol{\mu}_t - \boldsymbol{\mu}^{SS} \\ \boldsymbol{\rho}_t - \boldsymbol{\rho}^{SS} \end{bmatrix}$ be deviations from a steady state. Construct a linear approximation

$$\hat{\Psi}_{t+1} = B(s_{t+1})\hat{\Psi}_t. \quad (28)$$

This linearized system has coefficients $B(s)$ that are functions of the shock. The next theorem describes a simple numerical test that allows us to determine whether this linear system converges to zero in probability.

Theorem 7 *If the (real parts) of the eigenvalues of $\mathbb{E}B(s)$ are less than 1, system (28) converges to zero in mean. Further for large t , the conditional variance of $\hat{\Psi}$, denoted by $\Sigma_{\Psi,t}$, is governed by*

$$vec(\Sigma_{\Psi,t}) = \hat{B}vec(\Sigma_{\Psi,t-1}),$$

where \hat{B} is a square matrix of dimension $(2I - 2)^2$. In addition, if the (real parts) of the eigenvalues of \hat{B} are less than 1, system (28) converges in probability.

The dominant eigenvalue is informative not only about whether the system is locally stable but also about how quickly the steady state is reached. The half-life of convergence to the steady state is $\frac{\log(0.5)}{\|\iota\|}$, where $\|\iota\|$ is the absolute value of the dominant eigenvalue. Thus, the closer the dominant eigenvalue is to one, the slower is the speed of convergence.

We have applied Theorem 7 to verify local stability for a wide range of examples. Since the parameter space is high dimensional, we relegate the comparative statics to Appendix A.9. **The typical finding there is that the steady state is stable but that convergence is slow. The rates of convergence are increasing in the strength of covariance of interest rates and aggregate shocks that affect the government's need for revenue. We return to this feature in section 7 where we study low frequency components of government debt.**

²⁰One could in principle look for a solution in state variables $(\boldsymbol{x}(s^{t-1}), \boldsymbol{\rho}(s^{t-1}))$. For $I = 2$ with $\{\theta_i(s)\}$ different across agents, this would give identical policies and a map that is (locally) invertible between \boldsymbol{x} and $\boldsymbol{\mu}$ for a given $\boldsymbol{\rho}$. However in other cases, it turns out there are unique linear policies in $(\boldsymbol{\mu}, \boldsymbol{\rho})$ and not necessarily in $(\boldsymbol{x}, \boldsymbol{\rho})$. This comes from the fact that the set of feasible $(\boldsymbol{x}, \boldsymbol{\rho})$ are restricted at time 0 and may not contain an open set around the steady state values. When we linearize using $(\boldsymbol{\mu}, \boldsymbol{\rho})$ as state variables, the optimal policies for $\boldsymbol{x}(s^t), \boldsymbol{\rho}(s^t)$ converge to their steady state levels for all perturbations in $(\boldsymbol{\mu}, \boldsymbol{\rho})$.

7 Numerical example

In sections 5 and 6, we studied steady states as a way of summarizing the asymptotic behavior of Ramsey allocations and the forces that shape the asymptotic level of government and private assets. In this section, we use a calibrated version of the economy a) to revisit the magnitude of these forces; and b) to study optimal policy responses at business cycle frequencies when the economy is possibly far away from a (stochastic) steady state. We choose shocks and initial conditions to match stylized facts from the recent recession in US. The numerical calculations use methods adapted from Evans (2014) and described in the Appendix ???. The next section describes how we set parameters and initial conditions.

7.1 Calibration

We assume five types of agents²¹ of equal measures with preferences $u(c, l) = \frac{c^{1-\sigma}}{1-\sigma} - \frac{l^{1+\gamma}}{1+\gamma}$. These agents stand in for the 90th, 75th, 50th, 25th, and 10th quantiles of the US wage distribution.

Let $\mathcal{Q}(i)$ be the quantile of agent i . We assume i.i.d aggregate shocks ϵ_t that affect both the labor productivities of all agents $\{\theta_{i,t}\}_{i=1}^N$ and the payoff p_t of the single asset:

$$\log \theta_{i,t} = \log \bar{\theta}_i + \epsilon_t [1 + (.9 - \mathcal{Q}(i))m]$$

$$p_t = 1 + \chi \epsilon_t$$

Following Autor et al. (2008), we set average productivities $\{\bar{\theta}_i\}_{i=1}^N$ to match quantiles of average weekly earnings of full time wage and salary earners from the Current Population Survey (CPS).

The parameter m allows us to generate recessions associated with different falls in income for different types of agents. We calibrate m to match facts reported by Guvenen et al. (2014). Figure 7.1 (adapted from Guvenen et al.) reports that in the latest US recession the fall in income for agents in the first decile of earnings was about three times that experienced by the 90th percentile. Furthermore, between the 10th and the 90th percentiles, the change in the percentage drop in earnings was almost linear. From these facts we infer a slope $m = \frac{1.5}{0.8}$.

The parameter χ captures the ex-post comovement in returns on government assets and aggregate shocks. Our model is silent about the source of these comovements. In the data, they could come from variations in real payoffs due inflation, interest rate risk for longer maturity bonds, or defaults. For the purpose of our numerical exercise, we use US data on inflation and

²¹We report the results for $N = 5$ to capture sufficient heterogeneity in wealth and earnings. Our methods let solve for arbitrary number of agents. We have verified that the main qualitative and quantitative insights are unchanged when we have more than five types.

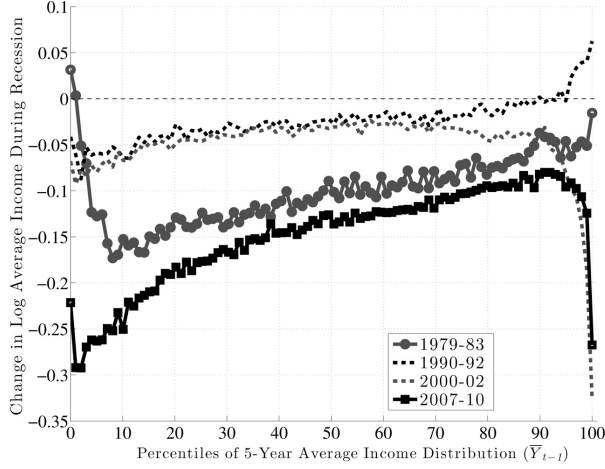


Figure 2: Change in log average earnings during recessions, prime-age males from Guvenen et al. (2014)

interest rates of longer maturities bonds to calibrate χ . We calibrate the comovement in the following way. Let $q_t^{(n)}$ be the log price of a nominal bond of maturity n . We can define real holding period returns $r_{t,t+1}^{(n)}$ as

$$r_{t,t+1}^{(n)} = q_{t+1}^{(n-1)} - q_t^{(n)} - \pi_{t+1}$$

With the transformation $y_t^{(n)} : -\frac{1}{n}q_t^{(n)}$ we can express $r_{t,t+1}^{(n)}$ as follows:

$$r_{t,t+1}^{(n)} = \underbrace{y_t^{(n)}}_{\text{Ex-ante part}} - (n-1) \left[\underbrace{\left(y_{t+1}^{(n)} - y_t^{(n)} \right)}_{\text{Interest rate risk given } n} + \underbrace{\left(y_{t+1}^{(n-1)} - y_{t+1}^{(n)} \right)}_{\text{Term structure risk}} \right] - \underbrace{\pi_{t+1}}_{\text{Inflation risk}}$$

In our model, the holding period returns are given by $\log \left[\frac{p_{t+1}}{q_t} \right]$ and $q_t = \frac{\beta \mathbb{E}_t u_{c,t+1} P_{t+1}}{u_{c,t}}$. Note that p_{t+1} allows us to capture ex-post fluctuations in returns to the government's debt portfolio coming from maturity and inflation.

Table 1 summarizes the comovement between labor productivity $\{\epsilon_t\}$ and bond prices $\{q_t^n\}$ for different maturities inferred from quarterly US data for the period 1952 to 2003. The table's first line reports the correlation between the ex post returns and labor productivities. In our baseline calibration, ϵ_t is i.i.d over time. Hence the parameter $\chi = \frac{\sigma_r}{\sigma_\epsilon} \text{Corr}(r, \epsilon)$. By averaging over different maturities we infer a value of $\chi = -0.06$.²² Thus, payoffs are weakly

²²The second line of table 1 computes the correlation of labor productivity with the ex-post component of returns. For the shortest maturity, 3 month real tbbill returns $\text{Corr}(\epsilon_{t+1}, y_t^{1qtr} - \pi_{t+1}) = -0.11$. These results together give us a range for χ of zero to negative -0.09 . The numerical results are not sensitive to values of χ in this range.

countercyclical for US. Besides the results for the benchmark value of $\chi = -0.06$, the long simulations in section 7.2 include outcomes for a range of χ 's from -1.0 to 1.0 .

Maturity (n)	2yr	3yr	4yr	5yr
$Corr(\epsilon_{t+1}, r_{t,t+1}^{(n)})$	-0.11	-0.093	-0.083	-0.072
$Corr(\epsilon_{t+1}, r_{t,t+1}^{(n)} - ny_t^{(n)})$	0.00	-0.0463	-0.080	-0.091
$Corr(\epsilon_{t+1}, y_t^{(n)} - \pi_{t+1})$	-0.097	-0.086	-0.080	-0.073
$\frac{\sigma(r_{t+1}^n)}{\sigma(\epsilon_{t+1})}$	0.820	0.835	0.843	0.845

Table 1: Correlation between holding period returns and productivity

As for parameters of household preferences, we set $\sigma = 1$, $\gamma = 2$, which imply Frisch elasticity of labor supply of 0.5. We set the time discount factor $\beta = 0.98$, which implies the annual interest rate in an economy without shocks would be 2% per year.

We assume that the initial wealth is perfectly correlated with wages and calibrate the wealth distribution to get the relative quantiles as in Kuhn (2014) and Quadrini and Rios-Rull (2014). These papers document the quantiles of net worth for US households computed up to 2010 Survey of Consumer Finances.

For the Pareto weights and government expenditures, we use an optimal allocation in an economy without shocks to target a (pre-transfers, federal) expenditure output ratio of 12%, a tax rate of 23%, a ratio of transfers to gdp of 10%, and a government debt to gdp of 100%.

7.2 Long run outcomes

Figure 7.2 simulations of 2000 periods for the government debt to output ratio, the labor tax rate, and the transfers to output ratio for three values of $\chi \in \{-1.0, -0.06, 1.0\}$ in red, black, and blue, respectively. The three simulations start from the same initial conditions and all share the same sequence of underlying shocks.

Two features emerge. Different values χ give rise to different locations of the long-run marginal distribution of government assets and also to different rates of convergence to that long-run distribution. A sufficiently positive χ generates lower payoffs in recessions relative to booms. In line with assertions of theorems 3 and 6, we see from the blue line that the government does not repay its initial debt during these 2000 periods. On the other hand, under the benchmark χ (black line) or when χ is negative (red line), the government accumulates assets.

In order to get a clearer picture of the speed of convergence, we plot paths of the conditional means for debt and the tax rate in figure 7.2. To explain how we generated these plots, let

Parameter	Value	Description
$\{\theta_i\}$	$\{4.9, 3.24, 2.1, 1.4, 1\}$	Wages dispersion for $\{90, 75, 50, 25, 10\}$ percentiles
γ	2	Average Frisch elasticity of labor supply of 0.5
β	0.98	Average (annual) risk free interest rate of 2%
m	$\frac{1.5}{.8}$	Heterogeneity in wage growth over business cycles
χ	-0.06	Covariance between holding period returns and labor productivity%
σ_e	0.03	vol of labor productivity
g	.13 %	Average pre-transfer expenditure-output ratio of 12 %

Table 2: Benchmark calibration

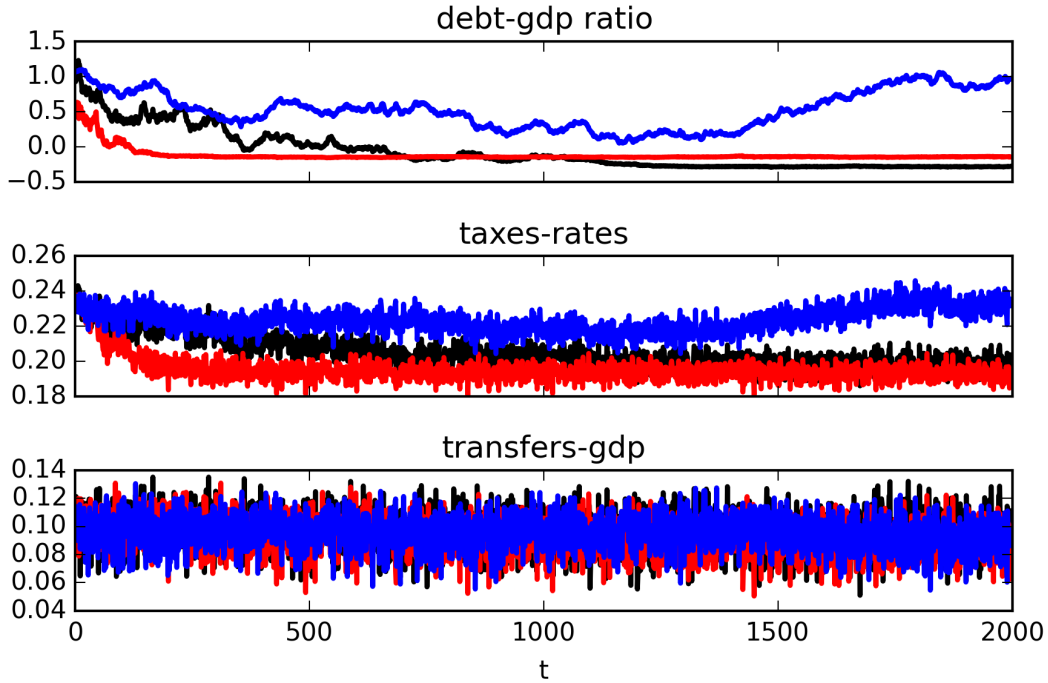


Figure 3: The red, black and blue lines plot simulations for a common sequence of shocks for values of $\chi = -1.0, -0.06, 1.0$ respectively

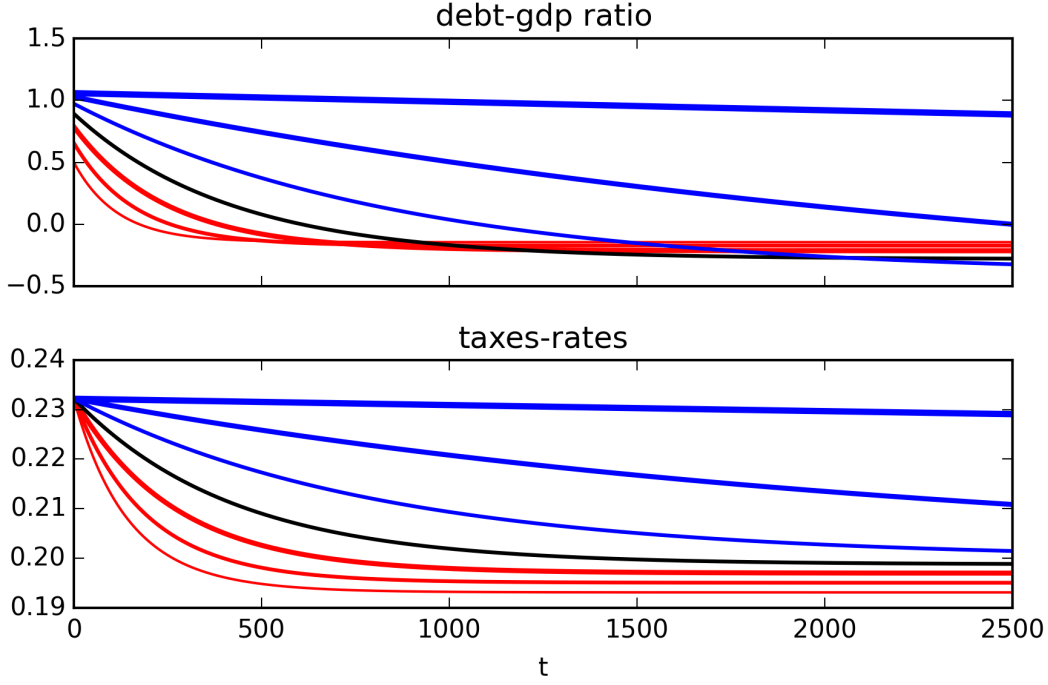


Figure 4: The plot shows conditional mean paths for different values of χ . The red (blue) lines have $\chi < 0$ ($\chi > 0$). The thicker lines represent larger values.

$B(s_{t+1}, \mathbf{x}_t, \boldsymbol{\rho}_t)$ be the Ramsey decision rules that generate the assets B of the government and let $\Psi(s_{t+1}; \mathbf{x}_t, \boldsymbol{\rho}_t)$ be the law of motion for the state variables for the Ramsey plan. For a given history, the conditional mean of government assets is:

$$B_{t+1}^{cm} = \mathbb{E}B(s_{t+1}, \mathbf{x}_t^{cm}, \boldsymbol{\rho}_t^{cm}) \quad (30a)$$

$$\mathbf{x}_t^{cm}, \boldsymbol{\rho}_t^{cm} = \mathbb{E}\Psi(s_t, \mathbf{x}_{t-1}^{cm}, \boldsymbol{\rho}_{t-1}^{cm}) \quad (30b)$$

Note how these conditional mean paths smooth the high frequency movements in the dynamics of the state variables but retain the low frequency drifts. As before, different lines correspond to different values of χ between -1.0 and 1.0 with the blue (red) lines representing positive (negative) values of χ . Thicker lines depict outcomes associated with larger values of χ . The figure shows that the speed of convergence is increasing and the magnitude of the limiting assets is decreasing in the strength of correlation between productivities and payoffs. This pattern confirms the approximation results characterized in theorem 4.

To verify the wide support of the ergodic distributions, we take the initial conditions at the end of the long simulation and subject the economy to a sequence of 100 periods of ϵ_t shocks that are 2 standard deviations below the mean. In figure 7.2 we see that given a sufficiently long

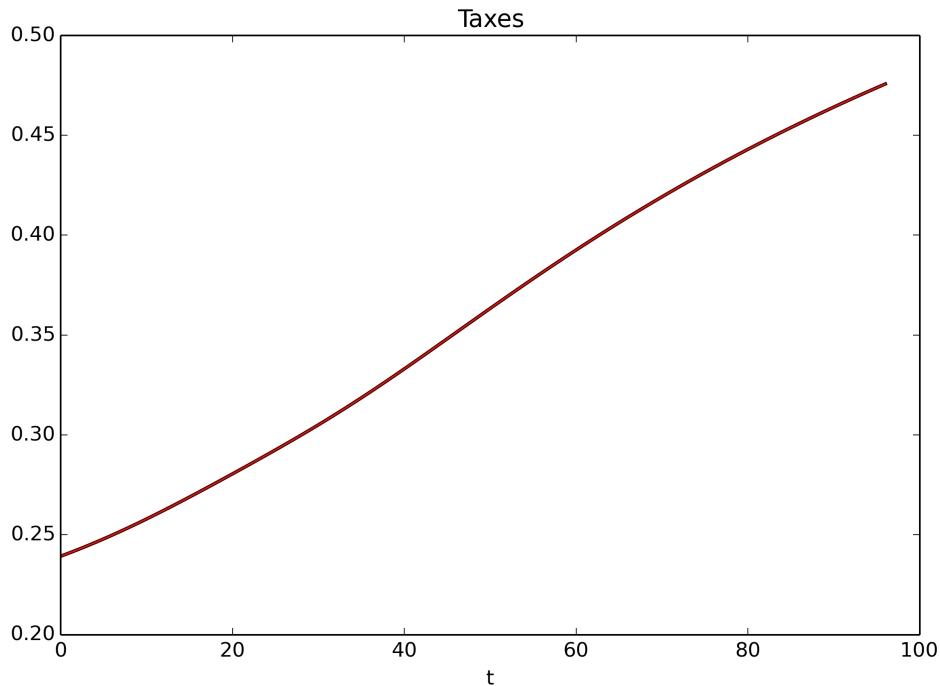


Figure 5: Taxes for a sequence of -1 s.d shocks to aggregate productivity of length 100

sequence of negative productivity shocks the economy will eventually deviate significantly from its ergodic mean.

A further inference from the analysis of earlier sections was that government assets B in the steady state are decreasing in the redistributive motive of the government. We check this numerically here by changing Pareto weights. We parametrize the redistributive motive using α . The planner places evenly spaced Pareto weights from $0.2 + \alpha$ on the lowest productivity agent to $0.2 - \alpha$ on the highest productivity agent. Increasing α lowers concerns for redistribution. In figure 7.2 we plot mean of the government assets in the ergodic distribution as a function of α .

7.3 Short run

The analysis of the previous subsection studied very low frequency components of a Ramsey plan. Here we focus on business cycle frequencies. In our setting, these higher frequency responses can conveniently be classified in terms of magnitudes of changes as we switch from “boom” to “recession,” and the dynamics during periods when a recession or boom state persists. A recession is a negative -1.0 standard deviation realization for the ϵ_t process. Given the initial conditions and the benchmark calibration, the plots below trace the paths for debt, the tax rate, and transfers for a sequence of shocks that feature a recession of four periods from $t = 3$. Before

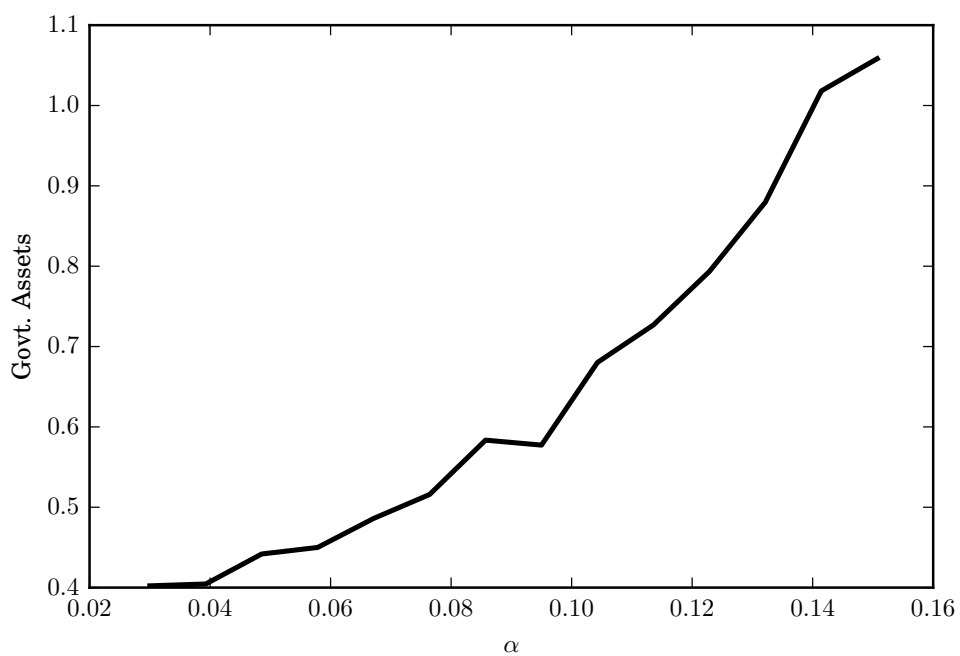


Figure 6: This plot shows long run assets of the government as a function of α which parametrizes the redistributive concern. Higher α represent planner's with relatively higher weights on productive agents.

and after this recession, the economy receives $\epsilon_t = 0$.

The main exercise here is to compute how the Ramsey tax rate, transfers, and government debt in recessions accompanied by larger inequality differ from those in a recession that affects all agents alike. Under the benchmark calibration, log wages for agent i are given by $\log \theta_i = \log \bar{\theta}_i + \epsilon[1 + (.9 - \mathcal{Q}(i))m]$. We decompose the total responses into a TFP only component by setting $m = 0$ and an inequality only component as follows:

$$\log \theta_i^{tfp} = \log \bar{\theta}_i + \epsilon$$

$$\log \theta_i^{ineq} = \log \bar{\theta}_i + \epsilon[(.9 - \mathcal{Q}(i))m]$$

Figure 7.3 plots impulse responses. The shaded region is the induced recession and the bold line captures the benchmark (total) response. The dashed (dotted) line reflects the TFP only (inequality) effect. In the benchmark, the government responds to an adverse shock by a making big increases in transfers, the tax rate, and government debt. However, without inequality shocks (dotted line), the government responds by decreasing transfers and increasing both debt and the tax rate, but by amounts an order of magnitude smaller than in the benchmark.

Next we average over sample paths of length 100 periods and report the volatility, autocorrelation, and correlation with exogenous shocks for the tax rate and transfers in table 3. We see that taxes are twice as volatile and that the correlation between transfers and productivities switches sign. This indicates how ignoring redistributive goals affect prescriptions for government policy in recessions.

Moments	Tfp	Tfp+Ineq
vol. of taxes	0.003	0.006
vol. of transfers	0.01	0.02
autocorr. in taxes	0.93	0.66
autocorr. in transfers	0.17	0.18
corr. of taxes with tfp	0.15	-0.63
corr. of transfers with tfp	0.99	-0.98

Table 3: Sample moments for taxes and transfers averaged across simulations of 100 periods

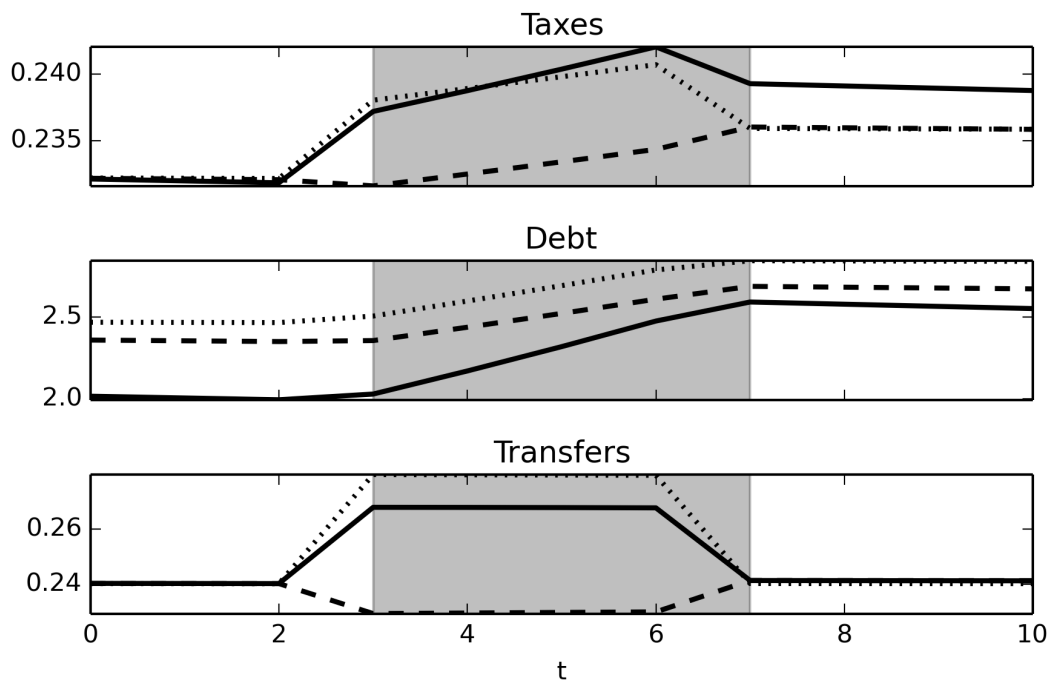


Figure 7: The bold line is the total response. The dashed (dotted) line reflects the only TFP (inequality) effect. The shaded region is the recession

8 Conclusion

A Appendix

A.1 Extension: Borrowing constraints

Representative agent models rule out Ricardian equivalence either by assuming distorting taxes or by imposing ad hoc borrowing constraints. By way of contrast, we have verified that Ricardian equivalence holds in our economy even though there are distorting taxes. Imposing ad-hoc borrowing limits also leaves Ricardian equivalence intact in our economy.²³ In economies with exogenous borrowing constraints, agents' maximization problems include the additional constraints

$$b_{i,t} \geq \underline{b}_i \quad (31)$$

for some exogenously given $\{\underline{b}_i\}_i$.

²³Bryant and Wallace (1984) describe how a government can use borrowing constraints as part of a welfare-improving policy to finance exogenous government expenditures. Sargent and Smith (1987) describe Modigliani-Miller theorems for government finance in a collection of economies in which borrowing constraints on classes of agents produce the kind of rate of return discrepancies that Bryant and Wallace manipulate.

Definition 5 For given $(\{b_{i,-1}, \underline{b}_i\}_i, B_{-1})$ and $\{\tau_t, T_t\}_t$, a competitive equilibrium with affine taxes and exogenous borrowing constraints is a sequence $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, B_t, R_t\}_t$ such that $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ maximizes (1) subject to (4) and (31), $\{b_{i,t}\}_{i,t}$ are bounded, and constraints (??), (5) and (3) are satisfied.

We can define an *optimal* competitive equilibrium with exogenous borrowing constraints by extending Definition 3.

The introduction of the ad-hoc debt limits leaves unaltered the conclusions of Corollary 1 and the role of the initial distribution of assets across agents. The next theorem asserts that ad-hoc borrowing limits do not limit a government's ability to respond to aggregate shocks.²⁴

Theorem 8 Given an initial asset distribution $(\{b_{i,-1}\}_i, B_{-1})$, let $\{c_{i,t}, l_{i,t}\}_{i,t}$ and $\{R_t\}_t$ be a competitive equilibrium allocation and interest rate sequence in an economy without exogenous borrowing constraints. Then for any exogenous constraints $\{\underline{b}_i\}_i$, there is a government tax policy $\{\tau_t, T_t\}_t$ such that $\{c_{i,t}, l_{i,t}\}_{i,t}$ is a competitive equilibrium allocation in an economy with exogenous borrowing constraints $(\{b_{i,-1}, \underline{b}_i\}_i, B_{-1})$ and $\{\tau_t, T_t\}_t$.

Proof. Let $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ be a competitive equilibrium allocation without exogenous borrowing constraints. Let $\Delta_t \equiv \max_i \{\underline{b}_i - b_{i,t}\}$. Define $\hat{b}_{i,t} \equiv b_{i,t} + \Delta_t$ for all $t \geq 0$ and $\hat{b}_{i,-1} = b_{-1}$. By Theorem 1, $\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\}_{i,t}$ is also a competitive equilibrium allocation without exogenous borrowing constraints. Moreover, by construction $\hat{b}_{i,t} - \underline{b}_i = b_{i,t} + \Delta_t - \underline{b}_i \geq 0$. Therefore, $\hat{b}_{i,t}$ satisfies (31). Since agents' budget sets are smaller in the economy with exogenous borrowing constraints, and $\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\}_{i,t}$ are feasible at interest rate process $\{R_t\}_t$, then $\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\}_{i,t}$ is also an optimal choice for agents in the economy with exogenous borrowing constraints $\{\underline{b}_i\}_i$. Since all market clearing conditions are satisfied, $\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\}_{i,t}$ is a competitive equilibrium allocation and asset profile. ■

To provide some intuition for Theorem 8, suppose to the contrary that the exogenous borrowing constraints restricted a government's ability to achieve a desired allocation. That means that the government would want to increase its borrowing and to repay agents later, which the borrowing constraints prevent. But the government can just reduce transfers today and increase them tomorrow. That would achieve the desired allocation without violating the exogenous borrowing constraints.

Welfare can be strictly higher in an economy with exogenous borrowing constraints relative to an economy without borrowing constraints because a government might want to push some

²⁴See Yared (2012, 2013) who shows a closely related result.

agents against their borrowing limits. When agents' borrowing constraints bind, their shadow interest rates differ from the common interest rate that unconstrained agents face. When the government rearranges tax policies to affect the interest rate, it affects constrained and unconstrained agents differently. By facilitating redistribution, this can improve welfare. We next construct an example without any shocks in which the government can achieve higher welfare by using borrowing constraints to improve its ability to redistribute. In this section we construct an example in which the government can achieve higher welfare in the economy with ad-hoc borrowing limits. We restrict ourselves to a deterministic economy with $g_t = 0$, $\beta_t = \beta$ and $I = 2$. Further the utility function over consumption and labor supply $U(c, l)$ is separable in the arguments and satisfies the Inada conditions. The planners problem can then be written as the following sequence problem

$$\max_{\{c_{i,t}, l_{i,t}, b_{i,t}, R_t\}_t} \sum_{t=0}^{\infty} \beta^t [\alpha_1 U(c_{1,t}, l_{1,t}) + \alpha_2 U(c_{2,t}, l_{2,t})] \quad (32)$$

subject to

$$c_{2,t} + \frac{U_{l2,t} l_{2,t}}{U_{c2,t}} - \left(c_{1,t} + \frac{U_{l1,t} l_{1,t}}{U_{c1,t}} \right) + \frac{1}{R_t} (b_{2,t} - b_{1,t}) = b_{2,t-1} - b_{1,t-1} \quad (33a)$$

$$\frac{U_{l1,t}}{\theta_1 U_{c1,t}} = \frac{U_{l2,t}}{\theta_2 U_{c2,t}} \quad (33b)$$

$$c_{1,t} + c_{2,t} \leq \theta_1 l_{1,t} + \theta_2 l_{2,t} \quad (33c)$$

$$\left(\frac{U_{ci,t}}{U_{ci,t+1}} - \beta R_t \right) (b_{i,t} - \underline{b}_i) = 0 \quad (33d)$$

$$\frac{U_{ci,t}}{U_{ci,t+1}} \geq \beta R_t \quad (33e)$$

$$b_{i,t} \geq \underline{b}_i \quad (33f)$$

Where \underline{b}_i is the exogenous borrowing constraint for agent i . We obtain equation (33a) by eliminating transfers from the budget equations of the households and using the optimality for labor supply decision. Equations (33d) and (33e) capture the inter-temporal optimality conditions modified for possibly binding constraints.

Let c_i^{fb} and l_i^{fb} be the allocation that solves the first best problem, that is maximizing equation (32) subject to (33c), and define

$$Z^{fb} = c_2^{fb} + \frac{U_{l2}^{fb} l_2^{fb}}{U_{c2}^{fb}} - \left(c_1^{fb} + \frac{U_{l1}^{fb} l_1^{fb}}{U_{c1}^{fb}} \right) \quad (34)$$

and

$$\tilde{b}_2^{fb} = \frac{Z^{fb}}{\frac{1}{\beta} - 1} \quad (35)$$

We will assume that the exogenous borrowing constraints satisfy $\underline{b}_2 = \underline{b}_1 + \tilde{b}_2^{fb}$. We then have the following lemma

Lemma 2 *If $\tilde{b}_2^{fb} > (<)0$ and $b_{2,-1} - b_{1,-1} > (<)\tilde{b}_2^{fb}$ then the planner can implement the first best.*

Proof. We will consider the candidate allocation where $c_{i,t} = c_i^{fb}$, $l_{i,t} = l_i^{fb}$, $b_{i,t} = \underline{b}_i$ and interest rates are given by $R_t = \frac{1}{\beta}$ for $t \geq 1$. It should be clear then that equations (33b) and (33c) are satisfied as a property of the first best allocation. Equation (33d) is trivially satisfied since the agents are at their borrowing constraints. For $t \geq 1$ equations (33a) and (33e) are both satisfied by the choice of $R_t = \frac{1}{\beta}$ and the first best allocations. It remains to check that equation (33a) is satisfied at time $t = 0$ for an interest rate $R_0 < \frac{1}{\beta}$. At time zero the constraint is give by

$$Z^{fb} + \frac{1}{R_0} \tilde{b}_2^{fb} = b_{2,-1} - b_{1,-1} \quad (36)$$

The assumption that $b_{2,-1} - b_{1,-1} > (<)\tilde{b}_2^{fb}$ if $\tilde{b}_2^{fb} > (<)0$ then implies that

$$R_0 = \frac{\tilde{b}_2^{fb}}{b_{2,-1} - b_{1,-1} - Z^{fb}} < \frac{1}{\beta}$$

as desired. ■

This will improve upon the planners problem without exogenous borrowing constraints, as first best can only be achieved in this scenario when $b_{2,-1} - b_{1,-1} = \tilde{b}_2^{fb}$.

A.2 Proof of Theorem 1

Proof. Let

$$\hat{T}_t = T_t + (\hat{b}_{1,t} - b_{1,t}) - R_{t-1} (\hat{b}_{1,t-1} - b_{1,t-1}) \text{ for all } t \geq 0. \quad (37)$$

Given a tax policy $\{\tau_t, \hat{T}_t\}_t$, the allocation $\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\}_{i,t}$ is a feasible choice for consumer i since it satisfies

$$\begin{aligned} c_{i,t} &= (1 - \tau_t) \theta_{i,t} l_{i,t} + R_{t-1} b_{i,t-1} - b_{i,t} + T_t \\ &= (1 - \tau_t) \theta_{i,t} l_{i,t} + R_{t-1} (b_{i,t-1} - b_{1,t-1}) - (b_{i,t} - b_{1,t}) + T_t + R_{t-1} b_{1,t-1} - b_{1,t} \\ &= (1 - \tau_t) \theta_{i,t} l_{i,t} + R_{t-1} (\hat{b}_{i,t-1} - \hat{b}_{1,t-1}) - (\hat{b}_{i,t} - \hat{b}_{1,t}) + T_t + R_{t-1} b_{1,t-1} - b_{1,t} \\ &= (1 - \tau_t) \theta_{i,t} l_{i,t} + R_{t-1} \hat{b}_{i,t-1} - \hat{b}_{i,t} + \hat{T}_t. \end{aligned}$$

Suppose that $\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\}_{i,t}$ is not the optimal choice for consumer i , in the sense that there exists some other sequence $\{\hat{c}_{i,t}, \hat{l}_{i,t}, \hat{b}_{i,t}\}_t$ that gives strictly higher utility. Then the choice $\{\hat{c}_{i,t}, \hat{l}_{i,t}, b_{i,t}\}_t$ is feasible given the tax rates $\{\tau_t, T_t\}_t$, which contradicts the assumption that $\{c_{i,t}, l_{i,t}, b_{i,t}\}_t$ is the optimal choice for the consumer given taxes $\{\tau_t, T_t\}_t$. The new allocation satisfies all other constraints and therefore is an equilibrium. ■

A.3 Proof of Theorem 2

We prove a slight more general version of our result. Consider an infinite horizon, incomplete markets economy in which an agent maximizes utility function $U : \mathbb{R}_+^n \rightarrow \mathbb{R}$ subject to an infinite sequence of budget constraints. We assume that U is concave and differentiable. Let $\mathbf{x}(s^t)$ be a vector of n goods and let $\mathbf{p}(s^t)$ be a price vector in state s^t with $p_i(s^t)$ denoting the price of good i . We use a normalization $p_1(s^t) = 1$ for all s^t . Let $b(s^t)$ be the agent's asset holdings, and let $\mathbf{e}(s^t)$ be a stochastic vector of endowments.

Consumer maximization problem

$$\max_{\mathbf{x}_t, b_t} \sum_{t=0}^{\infty} \beta^t \Pr(s^t) U(\mathbf{x}(s^t)) \quad (38)$$

subject to

$$\mathbf{p}(s^t) \mathbf{x}(s^t) + q(s^t) b(s^t) = \mathbf{p}(s^t) \mathbf{e}(s^t) + P(s_t) b(s^{t-1}) \quad (39)$$

and $\{b(s^t)\}$ is bounded and $\{q(s^t)\}$ is the price of the risk-free bond.

The Euler conditions are

$$\begin{aligned} \mathbf{U}_x(s^t) &= U_1(s^t) \mathbf{p}(s^t) \\ \Pr(s^t) U_1(s^t) q(s^t) &= \beta \sum_{s^{t+1} > s^t} \Pr(s^{t+1}) U_1(s^{t+1}). \end{aligned} \quad (40)$$

Theorem 9 Consider an allocation $\{\mathbf{x}_t, b_t\}$ that satisfies (39), (40) and $\{b_t\}_t$ is bounded. Then $\{\mathbf{x}_t, b_t\}$ is a solution to (38).

Proof. The proof follows closely Constantinides and Duffie (1996). Suppose there is another budget feasible allocation $\mathbf{x} + \mathbf{h}$ that maximizes (38). Since U is strictly concave,

$$\begin{aligned} &\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(\mathbf{x}_t + \mathbf{h}_t) - \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(\mathbf{x}_t) \\ &\leq \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \mathbf{U}_x(\mathbf{x}_t) \mathbf{h}_t \end{aligned} \quad (41)$$

To attain $\mathbf{x} + \mathbf{h}$, the agent must deviate by φ_t from his original portfolio b_t such that $\{\varphi_t\}_t$ is bounded, $\varphi_{-1} = 0$ and

$$\mathbf{p}(s^t) \mathbf{h}(s^t) = P(s_t) \varphi(s^{t-1}) - q(s^t) \varphi(s^t)$$

Multiply by $\beta^t \Pr(s^t) U_1(s^t)$ to get:

$$\begin{aligned} \beta^t \Pr(s^t) U_1(s^t) \mathbf{p}(s^t) \mathbf{h}(s^t) &= \beta^t \Pr(s^t) U_1(s^t) \varphi(s^{t-1}) - q(s^t) \beta^t \Pr(s^t) U_1(s^t) \varphi(s^t) \\ &= \beta^t \Pr(s^t) U_1(s^t) \varphi(s^{t-1}) - \beta^{t+1} \sum_{s^{t+1} > s^t} \Pr(s^{t+1}) U_1(s^{t+1}) \varphi(s^t) \end{aligned}$$

where we used the second part of (40) in the second equality. Sum over the first T periods and use the first part of (40) to eliminate $\mathbf{U}_x(\mathbf{x}_t) = U_1(s^t)\mathbf{p}(s^t)$

$$\sum_{t=0}^T \beta^t \Pr(s^t) \mathbf{U}_x(\mathbf{x}_t) \mathbf{h}(s^t) = - \sum_{s^{T+1} > s^T} \beta^{T+1} \Pr(s^{T+1}) U_1(s^{T+1}) \varphi(s^T).$$

Since $\{\varphi_t\}_t$ is bounded there must exist $\bar{\varphi}$ s.t. $|\varphi_t| \leq \bar{\varphi}$ for all t . By Theorem 5.2 of Magill and Quinzii (1994), this equilibrium with debt constraints implies a transversality condition on the right hand side of the last equation, so by transitivity we have

$$\lim_{T \rightarrow \infty} \sum_{t=0}^T \beta^t \Pr(s^t) \mathbf{U}_x(\mathbf{x}_t) \mathbf{h}(s^t) = 0.$$

Substitute this into (41) to show that \mathbf{h} does not improve utility of consumer. ■

A.4 Proof of Theorem 3

Proof. The optimal Ramsey plan solves the following Bellman equation. Let $V(b_-)$ be the maximum ex-ante value the government can achieve with debt b_- .

$$V(b_-) = \max_{c(s), l(s), b(s)} \sum_s \pi(s) \left\{ c(s) - \frac{l(s)^{1+\gamma}}{1+\gamma} + \beta V(b(s)) \right\} \quad (42)$$

subject to

$$c(s) + b(s) = l(s)^{1+\gamma} + \beta^{-1} P(s) b_- \quad (43a)$$

$$c(s) + g(s) \leq \theta l(s) \quad (43b)$$

Let $\bar{b} = -\underline{B}$

$$\underline{b} \leq b(s) \leq \bar{b} \quad (43c)$$

Let $\mu(s), \phi(s), \kappa(s), \bar{\kappa}(s)$ be the Lagrange multipliers on the respective constraints. Part 1 of Theorem 3

Lemma 3 *There exists a \bar{b} such that $b_t \leq \bar{b}$. This is the natural debt limit for the government.*

Proof. As we drive μ to $-\infty$, the tax rate approaches a maximum limit, $\bar{\tau} = \frac{\gamma}{1+\gamma}$. In state s , the government surplus,

$$S(s, \tau) = \theta^{\frac{\gamma}{1+\gamma}} (1 - \tau)^{\frac{1}{\gamma}} \tau - g(s),$$

which is maximized at $\tau = \frac{\gamma}{1+\gamma}$ when $(1 - \tau)^{\frac{1}{\gamma}} \tau$ is also maximized. This would impose a natural borrowing limit for the government.

■

From now we assume that \bar{b} represents the natural borrowing limit. We begin with some useful lemmas

let $L \equiv l^{1+\gamma}$, to make this problem convex,

Substitute for $c(s)$

$$V(b_-) = \max_{L(s), b(s)} \sum_{s \in S} \pi(s) \left[\frac{1}{1+\gamma} L(s) + \frac{1}{\beta} P(s) b_- - b(s) + \beta V(b(s)) \right]$$

s.t.

$$\begin{aligned} \frac{1}{\beta} P(s) b - b(s) + g(s) &\leq \theta L^{\frac{1}{1+\gamma}}(s) - L(s) \\ b(s) &\leq \bar{b} \\ L(s) &\geq 0. \end{aligned}$$

Lemma 4 $V(b)$ is strictly concave, continuous, differentiable and $V(b) < \beta^{-1}$ for all $b < \bar{b}$. The feasibility constraint binds for all $b \in (-\infty, \bar{b}]$, $s \in S$ and $(L^*(s))^{1-\frac{1}{1+\gamma}} \geq \frac{1}{1+\gamma}$.²⁵

Proof. *Concavity*

$V(b)$ is concave because we maximize linear objective function over convex set.

Binding feasibility

Suppose that feasibility does not bind for some b, s . Then the optimal $L(s)$ solve $\max_{L(s) \geq 0} \pi(s) \frac{\gamma}{1+\gamma} L(s)$ which sets $L(s) = \infty$. This violates feasibility for any finite $b, b(s)$.

Bounds on L

Let $\phi(s) > 0$ be a Lagrange multiplier on the feasibility. The FOC for $L(s)$ is

$$\frac{1}{1+\gamma} + \phi(s) \left(\frac{1}{1+\gamma} L(s)^{\frac{1}{1+\gamma}} - \theta \right) = 0.$$

This gives

$$\frac{1}{1+\gamma} L^{\frac{1}{1+\gamma}-1} - \theta = -\frac{1}{\lambda} \frac{\gamma}{1+\gamma} < 0$$

or

$$L^{1-\frac{1}{1+\gamma}} \geq \frac{\theta}{1+\gamma}.$$

Continuity

For any L that satisfy $L^{1-\frac{1}{1+\gamma}} \geq \frac{\theta}{1+\gamma}$, define function Ψ that satisfies $\Psi \left(L^{\frac{1}{1+\gamma}} - \theta L \right) = L$. Since $L^{\frac{1}{1+\gamma}} - L$ is strictly decreasing in L for $L^{1-\frac{1}{1+\gamma}} \geq \frac{1}{1+\gamma}$, this function is well defined.

Note that $\Psi \left(\underbrace{\left(\frac{1}{1+\gamma} L^{\frac{1}{1+\gamma}-1} - \theta \right)}_{<0} \right) = 1$ (so that $\Psi > 0$, i.e. Ψ is strictly decreasing) and

$$\Psi'' \left(\underbrace{\left(\frac{1}{1+\gamma} L^{\frac{1}{1+\gamma}-1} - 1 \right)^2}_{>0} + \underbrace{\Psi}_{<0} \underbrace{\frac{1}{1+\gamma} \frac{\gamma}{1+\gamma} L^{\frac{1}{1+\gamma}-2}}_{<0} \right) = 0 \text{ (so that } \Psi'' \geq 0, \Psi'' > 0, \text{ i.e. } \Psi \text{ is}$$

strictly concave on the interior). Ψ is also continuous. When $L^{1-\frac{1}{1+\gamma}} = \frac{1}{1+\gamma}$, $L = (1+\gamma)^{-\frac{(1+\gamma)}{(\gamma)}}$.

Let $D \equiv (1+\gamma)^{\frac{-1}{\gamma} - (1+\gamma)^{-\frac{1+\gamma}{(\gamma)}}}$. Then the objective is

$$V(b_-) = \max_{b(s)} \sum_{s \in S} \pi(s) \left[\Psi \left(\frac{1}{\beta} P(s) b - b(s) + g(s) \right) + \frac{1}{\beta} P(s) b_- - b(s) + \beta V(b(s)) \right]$$

s.t.

$$b(s) \leq \bar{b}$$

$$\frac{1}{\beta} P(s) b_- - b(s) + g(s) \leq D.$$

²⁵ This last condition simply means that we do not tax to the right of the peak of the Laffer curve. The revenue maximizing tax is $1 - \bar{\tau} = \frac{1}{1+\gamma}$. At the same time $1 - \tau = l^\gamma$ so if taxes are always to the left of the peak, $\frac{1}{1+\gamma} \leq l^\gamma = \left(L^{\frac{1}{1+\gamma}} \right)^\gamma = L^{1-\frac{1}{1+\gamma}}$.

This function is continuous so V is also continuous.

Differentiability

Continuity and convexity implies differentiability everywhere, including the boundaries.

Strict concavity

Ψ is strictly concave, so on the interior V is strictly concave.

■

Next we characterize policy functions

Lemma 5 $b(s, b_-)$ is an increasing function of b for all s for all s, b_- where $b(s)$ is interior.

Proof. Take the FOCs for $b(s)$ from the condition in the previous problem. If $b(s)$ is interior

$$\Psi\left(\frac{1}{\beta}P(s)b_- - b(s) + g(s)\right) = \beta V(b(s)).$$

Suppose $b_1 < b_2$ but $b_2(s) < b_1(s)$. Then from strict concavity

$$\begin{aligned} V(b_2(s)) &< V(b_1(s)) \\ \Psi\left(\frac{1}{\beta}P(s)b_2 - b_2(s) + g(s)\right) &> \Psi\left(\frac{1}{\beta}P(s)b_1 - b_1(s) + g(s)\right). \end{aligned}$$

■

Lemma 6 There exists an invariant distribution of the stochastic process $b_{t+1} = b(s_{t+1}, b_t)$

Proof. The state spaces for b_t and s_t are compact. Further the transition function on $s_{t+1}|s_t$ is trivially increasing under i.i.d shocks. We can apply standard arguments as in (see corollary 3) to argue that there exists invariant distribution of assets. ■

Now we characterize the support of this distribution using further properties of the policy rules for $b(s|b_-)$

Lemma 7 For any $b_- \in (\underline{b}, \bar{b})$, there are s, s'' s.t. $b(s) \geq b_- \geq b(s'')$. Moreover, if there are any states s'', s''' s.t. $b(s'') \neq b(s''')$, those inequalities are strict.

Proof. The FOCs together with the envelope theorem imply that $\mathbb{E}P(s)V'(b(s)) = V'(b_-) + \kappa(s)$. We can rewrite this as $\tilde{\mathbb{E}}V'(b(s)) = b + \kappa(s)$ with $\tilde{\pi}(s) = P(s)\pi(s)$

Now if there is at least one $b(s)$ s.t. $b(s) > b_-$, by strict concavity of V there must be some s'' s.t. $b(s'') < b$.

If there is at least one $b(s)$ s.t. $b(s) < b_-$, the inequality above is strictly only if $b(s''') = \bar{b}$ for some s''' . But $V(\bar{b}) < V(b)$ so there must be some s'' s.t. $b(s'') > b$. Equality is possible only if $b_- = b(s)$ for all s . ■

Lemma 8 Let $\mu(b, s)$ be the optimal policy function for the Lagrange multiplier $\mu(s)$. If $P(s') > P(s'')$ then there exists a $b_{s', s''}^*$ such that for all $b < (>) b_{1, s', s''}$ we have $\mu(b, s') > (<) \mu(b, s'')$. If $\underline{b} < b_{s', s''}^* < \bar{b}$ then $\mu(b_{s', s''}^*, s') = \mu(b_{s', s''}^*, s'')$.

Proof. Suppose that $\mu(b, s') \leq \mu(b, s'')$. Subtracting the implementability for s'' from the implementability constraint for s' we have

$$\begin{aligned} \frac{P(s') - P(s'')}{\beta} b &= S_{s'}(\mu(b, s')) - S_{s''}(\mu(b, s'')) + b'(b, s') - b'(b, s'') \\ &\geq S_{s'}(\mu(b, s')) - S_{s''}(\mu(b, s')) + b'(b, s') - b'(b, s'') \\ &\geq S_{s'}(\mu(b, s')) - S_{s''}(\mu(b, s')) = g(s'') - g(s') \end{aligned}$$

We get the first inequality from noting that $S_s(\mu') \geq S_s(\mu'')$ if $\mu' \leq \mu''$. We obtain the second inequality by noting that $\mu(b, s') \leq \mu(b, s'')$ implies $b'(b, s') \geq b'(b, s'')$ (which comes directly from the concavity of V). Thus, $\mu(b, s') \leq \mu(b, s'')$ implies that

$$b \geq \frac{\beta(g(s'') - g(s'))}{P(s') - P(s'')} = b_{s', s''}^* \quad (44)$$

The converse of this statement is that if $b < b_{s', s''}^*$ then $\mu(b, s') > \mu(b, s'')$. The reverse statement that $\mu(b, s') \geq \mu(b, s'')$ implies $b \leq b_{s', s''}^*$ follows by symmetry. Again, the converse implies that if $b > b_{s', s''}^*$ then $\mu(b, s') < \mu(b, s'')$. Finally, if $\underline{b} < b_{s', s''}^* < \bar{b}$ then continuity of the policy functions implies that there must exist a root of $\mu(b, s') - \mu(b, s'')$ and that root can only be at $b_{s', s''}^*$. ■

Lemma 9 $P \in \mathcal{P}^*$ is necessary and sufficient for existence of b^* such that $b(s, b^*) = b^*$ for all ss

Proof. The necessary part follows from taking differences of the (43a) for s', s'' . We have

$$[P(s) - P(s'')] \frac{b^*}{\beta} = g(s) - g(s'')$$

Thus $P \in \mathcal{P}^*$. The sufficient part follows from the Lemma 8. If $P \notin \mathcal{P}^*$, equation (44) that defines $b_{s', s''}^*$ will not be same across all pairs. Thus b^* that satisfies $b(s; b^*)$ independent of s will not exist. ■

Lemma 9 implies that under the hypothesis of part 1 of the Theorem 3 there cannot exist an interior absorbing point for the dynamics of debt. This allows us to construct a sequences $\{b_t\}_t$ such that $b_t < b_{t+1}$ with the property that $\lim_t b_t = \underline{b}$. Thus, for any $\epsilon > 0$, there exists a finite history of shocks that can take us arbitrarily close to \underline{b} . Since the shocks are i.i.d this finite

sequence will repeat i.o. With a symmetric argument we can show that b_t will come arbitrarily close to its upper limit i.o too

Part 2 of Theorem 3

In this first section we will show that there exists b_1 , and if $P(s)$ is sufficiently volatile a b_2 , such that if $b_t \leq b_1$ then

$$\mu_t \geq \mathbb{E}_t \mu_{t+1}$$

and if $b_t \geq b_2$ then

$$\mu_t \leq \mathbb{E}_t \mu_{t+1}.$$

Recall that b is decreasing in μ , so this implies that if b_t is low (large) enough then there will exist a drift away from the lower (upper) limit of government debt.

With Lemma 8 we can order the policy functions $\mu(b, \cdot)$ for particular regions of the state space. Take b_1 to be

$$b_1 = \min \{b_{s', s''}^*\}$$

and WLOG choose $\underline{b} < b_1$. For all $b < b_1$ we have shown that $P(s) > P(s')$ implies that $\mu(b, s) > \mu(b, s')$. The FOC for the problem imply,

$$\mu_- = \mathbb{E}P(s)\mu(s) + \underline{\kappa}(s) \tag{45}$$

The inequality in the resource constraint implies that $\phi(s) \geq 0$ implying that $\mu(s) \leq 1$. With some minor algebra algebra we obtain

By decomposing $\mathbb{E}\mu(s)P(s)$ in equation (45), we obtain (using $\mathbb{E}P(s) = 1$)

$$\mu_- = \mathbb{E}\mu(s) + \text{cov}(\mu(s), P(s)) + \underline{\kappa}(s) \tag{46}$$

Our analysis has just shown that for $b_- < b_1$ we have $\text{cov}_t(\mu(s), P(s)) > 0$ so

$$\mu_- > \mathbb{E}\mu(s).$$

If p is sufficiently volatile:

$$P(s') - P(s'') > \frac{\beta(g(s'') - g(s'))}{\bar{b}}$$

then

$$b_2 = \max \{b_{s', s''}^*\} < \bar{b}$$

and through a similar argument we can conclude that $\text{cov}(\mu(s), P(s)) < 0$

$$\mu_- < \mathbb{E}\mu(s)$$

for $b_- > b_2$ (note $b_- > \underline{b}$ implies $\underline{\kappa}(s) = 0$) which gives us a drift away from the upper-bound.

Part 3 of Theorem 3

When $P \in \mathcal{P}^*$, Lemma 9 implies existence of b^* as the steady state debt level.

Lemma 10 *There exists μ^* such that μ_t is a sub-martingale bounded above in the region $(-\infty, \mu^*)$ and super-martingale bounded below in the region $(\mu^*, \frac{1}{1+\gamma})$*

Proof. Let μ^* be the associated multiplier, i.e $V_b(b^*) = \mu^*$. Using the results of the previous section, we have that $b_1 = b_2 = b^*$, implying that $\mu_t < (>) \mathbb{E}_t \mu_{t+1}$ for $b_t < (>) b^*$. ■

Lastly we show that $\lim_t \mu_t = \mu^*$. Suppose $b_t < b^*$, we know that $\mu_t > \mu^*$. The previous lemma implies that in this region, μ_t is a super martingale. The lemma 5 shows that $b(s, b_-)$ is continuous and increasing. This translates into $\mu(\mu(b_-), s)$ to be continuous and increasing as well. Thus

$$\mu_t > \mu^* \implies \mu(\mu_t, s_{t+1}) > \mu(\mu^*, s_{t+1})$$

or

$$\mu_{t+1} > \mu^*$$

Thus μ^* provides a lower bound to this super martingale. Using standard martingale convergence theorem converges. The uniqueness of steady state implies that it can only converge to μ^* . For $\mu < \mu^*$, the argument is symmetric.

■

A.5 Proof of Theorem 4

Working with the first order conditions of problem 42, we obtain

$$l(s)^\gamma = \frac{\mu(s) - 1}{(1 + \gamma)\mu(s) - 1} = 1 - \tau(\mu(s)),$$

implying the relationship between tax rate τ and multiplier μ given by

$$\tau(\mu) = \frac{\gamma\mu}{(1 + \gamma)\mu - 1} \quad (47)$$

At the interior, the rest of the first order conditions and the implementability constraints are summarized below

$$\begin{aligned} \frac{b_- P(s)}{\beta} &= S(\mu(s), s) + b(s) \\ \mu(b_-) &= \mathbb{E}P(s)\mu(s) \end{aligned}$$

where $S(\mu, s)$ is the government surplus in state s given by

$$S(\mu, s) = (1 - \tau(\mu))^{\frac{1}{\gamma}} \tau(\mu) - g(s) = I(\mu) - g(s)$$

The proof of the theorem will have four steps:

Step 1: Obtaining a recursive representation of the optimal allocation in the incomplete markets economy with payoffs $P(s)$ with state variable μ_-

Given a pair $\{P(s), g(s)\}$, since $V'(b)$ is one-to-one, so we can re-characterize these equations as searching for a function $b(\mu)$ and $\mu(s|\mu_-)$ such that the following two equations can be solved for all μ_- .

$$\frac{b(\mu_-)P(s)}{\beta} = I(\mu(s)) - g(s) + b(\mu(s)) \quad (48)$$

$$\mu_- = \mathbb{E}\mu(s)P(s) \quad (49)$$

Step 2: Describe how the policy rules are approximated

Usually perturbation approaches to solve equilibrium conditions as above look for the solutions to $\{\mu(s|\mu_-)\}$ and $b(\mu_-)$ around deterministic steady state of the model. Thus for any b^{ss} , there exists a μ^{ss} that will solve

$$\frac{b^{ss}}{\beta} = I(\mu^{ss}) - \bar{g} + b^{ss}$$

For example if we set the perturbation parameter q to scale the shocks, $g(s) = \mathbb{E}g(s) + q\Delta_g(s)$ and $P(s) = 1 + q\Delta_P(s)$, the first order expansion of $\mu(s|\mu_-)$ will imply that it is a martingale. Such approximations are not informative about the ergodic distribution.²⁶

In contrast we will approximate the functions $\mu(s|\mu_-)$ around around economy with payoffs in $\bar{P} \in \mathcal{P}^*$.

In contrast we a) explicitly recognize that policy rules depend on payoffs: $\mu(s|\mu_-, \{P(s)\}_s)$ and $b(\mu_-, \{P(s)\}_s)$ and then take the first order expansion with respect to both μ_- and $\{P(s)\}_s$ around the vector $(\bar{\mu}, \{\bar{P}(s)\}_s)$ where $\bar{P}(s) \in \mathcal{P}^*$: these payoffs support an allocation such that limiting distribution of debt is degenerate around the some value \bar{b} ; and $\bar{\mu}$ is the corresponding limiting value of multiplier. The next two expression make the link between $\bar{\mu}$ and \bar{b} explicit.

$$\bar{b} = \frac{\beta}{1 - \beta} (I(\bar{\mu}) - \bar{g}) \quad (50a)$$

where $\bar{g} = \mathbb{E}g$ and \bar{p} as

$$\bar{P}(s) = 1 - \frac{\beta}{\bar{b}} (g(s) - \bar{g}) \quad (50b)$$

As noted before $b(\bar{\mu}; \bar{p}) = \bar{b}$ solves the the system of equations (48-49) for $\mu'(s) = \bar{\mu}$ when the payoffs are $\bar{P}(s)$

We next obtain the expressions that characterize the linear approximation of $\mu(s|\mu_-, \{P(s)\}_s)$ and $(\mu_-, \{P(s)\}_s)$ around some arbitrary point $(\bar{\mu}, \{\bar{P}(s)\}_s)$ where $\bar{P}(s) \in \mathcal{P}^*$. We will use these expressions to compute the mean and variance of the ergodic distribution associated with the approximated policy rules. Finally as a last step we propose a particular choice of the point of approximation.

The derivatives $\frac{\delta\mu(s|\mu_-, \{P(s)\}_s)}{\delta\mu_-}$, $\frac{\delta\mu(s|\mu_-, \{P(s)\}_s)}{\delta P(s)}$ and similarly for $b(\mu_-, \{P(s)\}_s)$ are obtained below:

Differentiating equation (48) with respect to μ around $(\bar{\mu}, \bar{P})$ we obtain

$$\frac{\bar{P}(s)}{\beta} \frac{\partial b}{\partial \mu_-} = \left[I'(\bar{\mu}) + \frac{\partial b}{\partial \mu_-} \right] \frac{\partial \mu(s)}{\partial \mu_-}.$$

Differentiating equation (49) with respect to μ_- we obtain

$$1 = \sum_s \pi(s) \bar{P}(s) \frac{\partial \mu'(s)}{\partial \mu_-}$$

combining these two equations we see that

$$\frac{1}{\beta} \left(\sum_s \pi(s) \bar{P}(s)^2 \right) \frac{\partial b}{\partial \mu_-} = I'(\bar{\mu}) + \frac{\partial b}{\partial \mu_-}$$

²⁶One can do higher order approximations, but part 3 of theorem 3 hints that for economies with payoffs close to \mathcal{P}^* , the stochastic steady state in general is far away from μ^{SS} .

Noting that $\mathbb{E}\bar{P}^2(s) = 1 + \frac{\beta^2}{\bar{b}^2}\sigma_g^2$ we obtain

$$\frac{\partial b}{\partial \mu_-} = \frac{I'(\bar{\mu})}{\frac{\beta}{\bar{b}^2}\sigma_g^2 + \frac{1-\beta}{\beta}} < 0 \quad (51)$$

as $I'(\bar{\mu}) < 0$. We then have directly that

$$\frac{\partial \mu'(s)}{\partial \mu} = \frac{\bar{P}(s)}{\frac{\beta^2}{\bar{b}^2}\sigma_g^2 + 1} = \frac{\bar{P}(s)}{\mathbb{E}\bar{P}(s)^2} \quad (52)$$

We can perform the same procedure for $P(s)$. Differentiating equation (48) with respect to $P(s)$ we around $(\bar{\mu}, \bar{p})$ we obtain

$$\frac{\bar{p}(s')}{\beta} \frac{\partial b}{\partial P(s)} + 1_{s,s'} \frac{\bar{b}}{\beta} - \frac{\pi(s)\bar{b}\bar{p}(s')}{\beta} = \left[I'(\bar{\mu}) + \frac{\partial b}{\partial \mu} \right] \frac{\partial \mu(s')}{\partial P(s)} \quad (53)$$

Here $1_{s,s'}$ is 1 if $s = s'$ and zero otherwise. Differentiating equation (49) with respect to $P(s)$ we obtain

$$0 = \pi(s)\bar{\mu} - \pi(s)\bar{\mu} + \sum_{s'} \pi(s)\bar{p}(s') \frac{\partial \mu(s')}{\partial P(s)} = \sum_{s'} \pi(s')\bar{p}(s') \frac{\partial \mu(s')}{\partial P(s)}$$

Again we can combine these two equations to give us

$$\frac{\mathbb{E}\bar{p}(s)^2}{\beta} \frac{\partial b}{\partial P(s)} + \frac{\pi(s)\bar{b}}{\beta} (\bar{p}(s) - \mathbb{E}\bar{p}(s)^2) = 0$$

or

$$\frac{\partial b}{\partial P(s)} = \pi(s)\bar{b} \frac{\mathbb{E}\bar{p}^2 - \bar{p}(s)}{\mathbb{E}\bar{p}^2} \quad (54)$$

Going back to equation (53) we have

$$\frac{\partial \mu(s')}{\partial P(s)} = \frac{\bar{b}}{\beta \left[I'(\bar{\mu}) + \frac{\partial b}{\partial \mu} \right]} \left(1_{s,s'} - \frac{\pi(s)\bar{p}(s)\bar{p}(s')}{\mathbb{E}\bar{p}^2} \right) \quad (55)$$

Step 3: Getting expressions for the mean and variance of the ergodic distribution around some arbitrary point of approximation

For an arbitrary $(\bar{\mu}, \{\bar{P}(s)\}_s)$, using the derivatives that we computed, we can characterize the dynamics of $\hat{\mu} \equiv \mu_t - \bar{\mu}$ using our approximated policies.

$$\hat{\mu}_{t+1} = B(s_{t+1})\hat{\mu}_t + C(s_{t+1}),$$

where $B(s)$ and $C(s)$ are respective derivatives. Note that both are random variables and let us denote their means \bar{B} and \bar{C} , and variances σ_B^2 and σ_C^2 . Suppose that $\hat{\mu}$ is distributed according to the ergodic distribution of this linear system with mean $\mathbb{E}\hat{\mu}$ and variance σ_μ^2 . Since

$$B\hat{\mu} + C,$$

has the same distribution we can compute the mean of this distribution as

$$\begin{aligned}
\mathbb{E}\hat{\mu} &= \mathbb{E}[B\hat{\mu} + C] \\
&= \mathbb{E}[\mathbb{E}_{\hat{\mu}}[B\hat{\mu} + C]] \\
&= \mathbb{E}[\overline{B}\hat{\mu} + \overline{C}] \\
&= \overline{B}\mathbb{E}\hat{\mu} + \overline{C}
\end{aligned}$$

solving for $\mathbb{E}\hat{\mu}$ we get

$$\mathbb{E}\hat{\mu} = \frac{\overline{C}}{1 - \overline{B}} \quad (56)$$

For the variance $\sigma_{\hat{\mu}}^2$ we know that

$$\sigma_{\hat{\mu}}^2 = \text{var}(B\hat{\mu} + C) = \text{var}(B\hat{\mu}) + \sigma_C^2 + 2\text{cov}(B\hat{\mu}, C)$$

Computing the variance of $B\hat{\mu}$ we have

$$\begin{aligned}
\text{var}(B\hat{\mu}) &= \mathbb{E}[(B\hat{\mu} - \overline{B}\mathbb{E}\hat{\mu})^2] \\
&= \mathbb{E}[(B\hat{\mu} - \overline{B}\hat{\mu} + \overline{B}\hat{\mu} - \overline{B}\mathbb{E}\hat{\mu})^2] \\
&= \mathbb{E}[\mathbb{E}_{\hat{\mu}}[(B - \overline{B})^2\hat{\mu}^2 + 2(B - \overline{B})(\hat{\mu} - \mathbb{E}\hat{\mu})\overline{B}\mathbb{E}\hat{\mu} + (\hat{\mu} - \mathbb{E}\hat{\mu})^2\overline{B}^2]] \\
&= \mathbb{E}[\sigma_B^2\hat{\mu}^2 + (\hat{\mu} - \mathbb{E}\hat{\mu})^2\overline{B}] \\
&= \sigma_B^2(\sigma_{\hat{\mu}}^2 + (\mathbb{E}\hat{\mu})^2) + \sigma_{\hat{\mu}}^2\overline{B}^2
\end{aligned}$$

while for the covariance of $B\hat{\mu}$ and C

$$\text{cov}(B\hat{\mu}, C) = \sigma_{BC}\mathbb{E}\hat{\mu}$$

Putting this all together we have

$$\sigma_{\hat{\mu}}^2 = \frac{\sigma_B^2(\mathbb{E}\hat{\mu})^2 + \sigma_{BC}\mathbb{E}\hat{\mu} + \sigma_C^2}{1 - \overline{B}^2 - \sigma_B^2} \quad (57)$$

Step 4: Choice of the point of approximation

To get the expressions in Theorem 3, we finally choose a particular $\overline{P} = P^*(s) \in \mathcal{P}^*$. This will be the closest complete market economy to our the given $P(s)$ in L^2 sense. Formally,

$$\min_{\tilde{P} \in \mathcal{P}^*} \sum_s \pi(s)(P(s) - \tilde{P}(s))^2.$$

Since all payoffs in \mathcal{P}^* are associated with some b^* and μ^* via equations (50), we can re write the above problem as choosing $\bar{\mu}$ so as to minimize the variance of the difference between $P(s)$

and the set of steady state payoffs. Let P^* be the solution to this minimization problem. The first order condition for this linearization gives us

$$2 \sum_{s'} \pi(P(s') - P^*(s', \mu^*)) \frac{\delta P^*(s, \mu^*)}{\delta \mu^*} = 0$$

as noted before

$$P^*(s) = 1 - \frac{\beta}{b^*(\mu^*)} (g(s) - \mathbb{E}g)$$

thus

$$\frac{\delta P^*}{\delta \mu^*} \propto P^* - 1$$

Thus we can see the the optimal choice of $\bar{\mu}$ is equivalent to choosing $\bar{\mu}$ such that

$$\begin{aligned} 0 &= \sum_{s'} \pi(s') (P(s') - P^*(s', \mu^*)) (P^*(s', \mu^*) - 1) \\ &= - \sum_{s'} \pi(s') (P(s') - P^*(s', \mu^*)) + \sum_{s'} \pi(s') (P(s') - P^*(s', \mu^*)) P^*(s', \mu^*) \\ &= \sum_{s'} \pi(s') (P(s') - P^*(s', \mu^*)) P^*(s', \mu^*) \\ &= \mathbb{E}[(P - P^*)P^*] \end{aligned} \tag{58}$$

At these values of $\bar{p} = P^*$ and $\bar{\mu} = \mu^*$ we have that C for our linearized system is

$$C(s') = \sum_s \left\{ \frac{b^*}{\beta \left[I'(\bar{\mu}) + \frac{\partial b}{\partial \mu} \right]} \left(1_{s,s'} - \frac{\pi(s) P^*(s) P^*(s')}{\mathbb{E} \bar{p}^2} \right) (P(s) - P^*(s)) \right\}$$

Taking expectations we have that

$$\begin{aligned} \bar{C} &= \sum_s \left\{ \frac{b^*}{\beta \left[I'(\bar{\mu}) + \frac{\partial b}{\partial \mu} \right]} \left(\pi(s) - \frac{\pi(s) P^*(s)}{\mathbb{E} \bar{p}^2} \right) (P(s) - P^*(s)) \right\} \\ &= \frac{b^*}{\beta \left[I'(\bar{\mu}) + \frac{\partial b}{\partial \mu} \right]} \left(\mathbb{E}(P - \bar{p}) - \frac{\mathbb{E}[(P - \bar{p})\bar{p}]}{\mathbb{E} \bar{p}^2} \right) \\ &= 0 \end{aligned} \tag{59}$$

Thus the linearized system will have the same mean for μ , $\bar{\mu}$, as the closest approximating steady state payoff structure.

We can also compute the variance of the ergodic distribution for μ . Note

$$\begin{aligned}
C(s') &= \sum_s \left\{ \frac{b^*}{\beta \left[I'(\bar{\mu}) + \frac{\partial b}{\partial \mu} \right]} \left(1_{s,s'} - \frac{\pi(s)P^*(s)P^*(s')}{\mathbb{E}P^{*2}} \right) (P(s) - P^*(s)) \right\} \\
&= \frac{b^*}{\beta \left[I'(\bar{\mu}) + \frac{\partial b}{\partial \mu} \right]} \left(P(s') - P^*(s') - P^*(s') \frac{\sum_s \pi(s)P^*(s)(p_s - P^*(s))}{\mathbb{E}P^{*2}} \right) \\
&= \frac{b^*}{\beta \left[I'(\bar{\mu}) + \frac{\partial b}{\partial \mu} \right]} (P(s') - P^*(s))
\end{aligned}$$

As noted before

$$\sigma_\mu^2 = \frac{b^{*2}}{\beta^2 \left[I'(\bar{\mu}) + \frac{\partial b}{\partial \mu} \right]^2 \left(1 - \bar{B}^2 - \sigma_B^2 \right)} \|P - P^*\|^2$$

The variance of government debt in the linearized system is

$$\sigma_b^2 = \frac{b^{*2} \left(\frac{\partial b}{\partial \mu} \right)^2}{\beta^2 \left[I'(\bar{\mu}) + \frac{\partial b}{\partial \mu} \right]^2 \left(1 - \bar{B}^2 - \sigma_B^2 \right)} \|P - P^*\|^2$$

This can be simplified using the following expressions:

$$I'(\bar{\mu}) + \frac{\partial b}{\partial \mu} = \frac{\mathbb{E}P^{*2}}{\beta} \frac{\partial b}{\partial \mu},$$

$$\bar{B} = \frac{1}{\mathbb{E}P^{*2}}$$

and

$$\sigma_B^2 = \frac{\text{var}(P^*)}{(\mathbb{E}P^{*2})^2}$$

to

$$\sigma_b^2 = \frac{b^{*2}}{\mathbb{E}P^{*2} \text{var}(P^*)} \|P - P^*\|^2 \tag{60}$$

Noting that $\mathbb{E}P^{*2} = 1 + \text{var}(P^*) > 1$, we have immediately that up to first order the relative spread of debt is bounded by

$$\frac{\sigma_b}{b^*} \leq \sqrt{\frac{\|P - P^*\|^2}{\text{var}(P^*)}} \tag{61}$$

A.6 Proof of Theorem 5

Proof.

Using Theorem 1 let $\tilde{b} = b_1 - b_2$. Under the normalization that $b_2 = 0$, the variable \tilde{b} represents public debt government or the assets of the productive agent. The optimal plan solves the following Bellman equation,

$$V(\tilde{b}_-) = \max_{c_1(s), c_2(s), b'(s)} \sum_s \pi(s) \left\{ \omega \left[c_1(s) - \frac{l_1^{1+\gamma}(s)}{1+\gamma} \right] + (1-\omega)c_2(s) + \beta V(\tilde{b}(s)) \right\} \quad (62)$$

subject to

$$c_1(s) - c_2(s) + \tilde{b}(s) = l(s)^{1+\gamma} + \beta^{-1}P(s)\tilde{b}_- \quad (63a)$$

$$nc_1(s) + (1-n)c_2(s) + g(s) \leq \theta_2 l(s)n \quad (63b)$$

$$c_2(s) \geq 0 \quad (63c)$$

$$\bar{b} \geq \tilde{b}(s) \geq \underline{b} \quad (63d)$$

Let $\mu(s), \phi(s), \lambda(s), \underline{\kappa}(s), \bar{\kappa}(s)$ be the Lagrange multipliers on the respective constraints. The FOC are summarized below

$$\omega - \mu(s) = n\phi(s) \quad (64a)$$

$$1 - \omega + \mu(s) - \phi(s)(1-n) + \lambda(s) = 0 \quad (64b)$$

$$-\omega l^\gamma(s) + \mu(s)(1+\gamma)l^\gamma(s) + n\phi(s)\theta = 0 \quad (64c)$$

$$\beta V'(\tilde{b}(s)) - \mu(s) - \bar{\kappa}(s) + \underline{\kappa}(s) = 0 \quad (64d)$$

and the envelope condition

$$V'(\tilde{b}_-) = \sum_s \pi(s)\mu(s)\beta^{-1}P(s) \quad (64e)$$

To show part 1 of Theorem 5, we show that $\frac{\omega}{n} > \frac{1+\gamma}{\gamma}$ is sufficient for the Lagrange multiplier $\lambda(s)$ on the non-negativity constraint to bind.

Lemma 11 *The multiplier on the budget constraint $\mu(s)$ is bounded above*

$$\mu(s) \leq \min \left\{ \omega - n, \frac{\omega}{1+\gamma} \right\}$$

Similarly the multiplier of the resource constraint is bounded below,

$$\phi(s) \geq \max \left\{ 1, \frac{\omega}{n} \left\lceil \frac{\gamma}{1+\gamma} \right\rceil \right\}$$

Proof.

Notice that the labor choice of the productive household implies $\frac{1}{1-\tau} = \frac{\theta_2}{l^\gamma(s)}$.

As taxes go to $-\infty$ (64c) implies that $\mu(s)$ approaches $\frac{\omega}{1+\gamma}$ from below. Similarly the non-negativity of $c_2(s)$ imposes a lower bound of 1 on $\phi(s)$. This translates into an upper bound of $\omega - n$ on μ . ■

Lemma 12 *There exists a $\bar{\omega}$ such that $\omega > \bar{\omega}$ implies $c_2(s) = 0$ for all b*

Proof.

By the KKT conditions $c_2(s) = 0$ if $\lambda(s) > 0$. Now (64b) implies this is true if $\mu(s) < \omega - n$. The previous lemma bounds $\mu(s)$ by $\frac{\omega}{1+\gamma}$.

We can thus define $\bar{\omega} = n \left(\frac{1+\gamma}{\gamma} \right)$ as the required threshold Pareto weight to ensure that the unproductive agent has zero consumption forever.

■

Now for the rest of the parts $\omega < n \left(\frac{1+\gamma}{\gamma} \right)$, we can have positive transfers for low enough public debt. In particular, we can define a maximum level of debt \mathcal{B} that is consistent with an interior solution for the unproductive agents' consumption.

Guess an interior solution $c_{2,t} > 0$ or $\lambda_t = 0$ for all t . This gives us $l_t = l^*$ defined below:

$$l^* = \left[\frac{n\theta}{\omega - (\omega - n)(1 + \gamma)} \right]^{\frac{1}{\gamma}} \quad (65)$$

As long as $\omega < n \left(\frac{1+\gamma}{\gamma} \right)$ At the interior solution $\tilde{b}(s) = \tilde{b}_-$ and using the implementability constraint and resource constraints (63a) and (63b) respectively, we can obtain the expression for $c_2(s)$

$$c_2(s) = n\theta l^* - n l^{*1+\gamma} - \tilde{b}_-(1 - P(s)\beta^{-1}) - g(s)$$

Non-negativity of c_2 implies,

$$\tilde{b}_- \leq \frac{g(s) - n\theta l^* + nl^{*1+\gamma}}{\beta^{-1}P(s) - 1}$$

We can also express this as

$$\tilde{b}_- \leq \frac{g(s) - \tau^* y^*}{\beta^{-1}P(s) - 1},$$

where the right hand side of the previous equation is just the present discounted value of the primary deficit of the government at the constant taxes τ^* associated with l^* defined in (65). As long as $\beta^{-1}P(s) - 1 > 0$, this object is well defined, we define $\mathcal{B} = \min_s \left[\frac{g(s) - n\theta l^* + nl^{*1+\gamma}}{\beta^{-1}P(s) - 1} \right]$. Thus for $\tilde{b}_- < \mathcal{B}$ the optimal allocation has constant taxes given by τ^* and debt \tilde{b}_- , while transfers are given by

$$T(s) = n\theta l^* - nl^{*1+\gamma} - \tilde{b}_-(1 - P(s)\beta^{-1}) - g(s),$$

and are strictly positive.

In the next lemma we show how \mathcal{B} varies with ω .

Lemma 13 For $\omega \leq n\frac{1+\gamma}{\gamma}$, we have $\frac{\partial \mathcal{B}}{\partial \omega} > 0$.

Proof. The sign of the derivative of \mathcal{B} with respect to ω is the same as the sign of the following derivative:

$$\frac{\partial [l^{*1+\gamma} - \theta l^*]}{\partial \omega}$$

Note that (65) implies that l^* is increasing in ω . Note that,

$$\frac{\partial [l^{*1+\gamma} - \theta l^*]}{\partial \omega} = \frac{\partial l^*}{\partial \omega} [(1 + \gamma)l^{*\gamma} - \theta]$$

So the sign of the required derivative depends on $[(1 + \gamma)l^{*\gamma} - \theta]$. We now argue that this expression is positive over the range $\omega \leq n\frac{1+\gamma}{\gamma}$.

Again from the expression for l^* , we see that

$$\min_{\omega \leq n\frac{1+\gamma}{\gamma}} l^{*\gamma} = \frac{\theta}{1 + \gamma}$$

Thus we can see that \mathcal{B} is increasing in ω

■

For initial debt greater than \mathcal{B} , we distinguish cases when payoffs are perfectly aligned with $g(s)$ i.e belong to the set \mathcal{P}^* and when they are not. For part 2 case b, let $P \notin \mathcal{P}^*$.

Lemma 14 *There exists a $\check{b} > \mathcal{B}$ such that there are two shocks \underline{s} and \bar{s} and the optimal choice of debt starting from $\tilde{b}_- \leq \check{b}$ satisfies the following two inequalities:*

$$\tilde{b}(\underline{s}, \tilde{b}_-) > \mathcal{B}$$

$$\tilde{b}(\bar{s}, \tilde{b}_-) \leq \mathcal{B}$$

Proof. At \mathcal{B} , there exist some \bar{s} such that $T(\bar{s}, \mathcal{B}) = \epsilon > 0$. Now define \check{b} as follows:

$$\check{b} = \mathcal{B} + \frac{\epsilon\beta}{2P(\bar{s})}$$

Now suppose to the contrary $\tilde{b}(\bar{s}, \tilde{b}_-) > \mathcal{B}$ for some $\tilde{b}_- \leq \check{b}$. This implies that $\tau(s, \tilde{b}_-) > \tau^*$ and $T(\bar{s}, \tilde{b}_-) = 0$.

The government budget constraint implies

$$\frac{P(\bar{s})\tilde{b}_-}{\beta} + g(s) = \tilde{b}(\bar{s}, \tilde{b}_-) + (1 - \tau(\bar{s}, \tilde{b}_-))l(\bar{s}, \tilde{b}_-).$$

As,

$$\frac{P(\bar{s})\tilde{b}_-}{\beta} + g(\bar{s}) \leq \frac{P(\bar{s})\mathcal{B}}{\beta} + g(\bar{s}) + \frac{\epsilon}{2} < \frac{P(\bar{s})\mathcal{B}}{\beta} + g(\bar{s}) + \epsilon$$

This further implies,

$$\tilde{b}(\bar{s}, \tilde{b}_-) + (1 - \tau(\bar{s}, \tilde{b}_-))l(\bar{s}, \tilde{b}_-) > [\tilde{b}(\bar{s}, \tilde{b}_-) + (1 - \tau^*)l^* > \mathcal{B} + (1 - \tau^*)l^* > \frac{P(\bar{s})\tilde{b}_-}{\beta} + g(\bar{s}) + T(\bar{s}, \tilde{b}_-) = \frac{P(\bar{s})\tilde{b}_-}{\beta} + g(\bar{s}) + \epsilon.$$

Combining the previous two inequalities yields a contradiction. The other inequality, $\tilde{b}(\underline{s}, \tilde{b}_-) > \mathcal{B}$ follows from the definition of \mathcal{B} . This is because if it was not true then $\tilde{b}(s, \tilde{b}_-) \leq \mathcal{B}$ for all shocks. This implies that the solution is interior. However the only initial conditions that have this property are less than equal to \mathcal{B} .

Now define $\bar{\mu}(\tilde{b}(s, \tilde{b}_-))$ as $\max_s \mu(s, \tilde{b}_-)$ and $\hat{s}(\tilde{b}_-)$ as the shock that achieves this maximum. Now we show that $\hat{\mu}(\tilde{b}(s, \tilde{b}_-))$ is finite for all $b_- \leq \bar{b}$. We show the claim for the natural debt limit.

Let $b^n(s) = (\beta^{-1}P(s) - 1)^{-1} \left[\theta^{\frac{\gamma}{1+\gamma}} \left(\frac{1}{1+\gamma} \right)^{\frac{1}{\gamma}} \left(\frac{\gamma}{1+\gamma} \right) - g(s) \right]$ be the maximum debt supported by a particular shock s . The natural debt limit is defined as $\bar{b}^n = \min_s b^n(s)$. Note that $\lim_{b \rightarrow \bar{b}^n} \mu(\tilde{b}_-) = \infty$

Now choose s such that $b^n(s) > \bar{b}^n$ and consider the debt choice next period for the same shock s when it comes in with debt \bar{b}^n .

Suppose it chooses $\tilde{b}(s, \bar{b}^n) = \bar{b}^n$, then taxes will have to be set to $\frac{\gamma}{1+\gamma}$ and the tax income will be $\frac{\gamma}{1+\gamma}l(\frac{\gamma}{1+\gamma}) = \theta^{\frac{\gamma}{1+\gamma}} \left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}} \left(\frac{\gamma}{1+\gamma}\right)$. The budget constraint will then imply that,

$$\frac{\bar{b}^n P(s)}{\beta} + g(s) = \theta^{\frac{\gamma}{1+\gamma}} \left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}} \left(\frac{\gamma}{1+\gamma}\right) + \bar{b}^n$$

$$\bar{b}^n = (P(s)\beta^{-1} - 1)^{-1} \left(\theta^{\frac{\gamma}{1+\gamma}} \left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}} \left(\frac{\gamma}{1+\gamma}\right) - g(s) \right)$$

However the right hand side is the definition of $b^n(s)$ and,

$$b^n(s) > \bar{b}^n.$$

Thus we have a contradiction and the optimal choice of debt at the natural debt limit $\tilde{b}(s, \bar{b}^n) < \bar{b}^n$.

This inturn means that $\lim_{\tilde{b} \rightarrow \bar{b}^n} \bar{\mu}(\tilde{b}) < \infty$.

Now note that $\bar{\mu}(\tilde{b}_-) - \mu(\tilde{b}_-)$ is continuous on $[\check{b}, \bar{b}^n]$ and is bounded below by zero, therefore attains a minimum at \tilde{b}^{min} . Let $\delta = \hat{\mu}(\tilde{b}^{min}) - \mu(\tilde{b}^{min}) > \eta > 0$. If this was not true then $P(s) \in \mathcal{P}^*$ as μ will have an absorbing state.

Let $\mu(\omega, n) = \omega - n$. This is the value of μ when debt falls below \mathcal{B} .

Now consider any initial $\tilde{b}_- \in [\mathcal{B}, \bar{b}^n]$. If $\tilde{b}_- \leq \check{b}$, then by lemma 14, we know that \mathcal{B} will be reached in one shock. Otherwise if $\tilde{b}_- > \check{b}$, we can construct a sequence of shocks $s_t = \hat{s}(\tilde{b}_{t-1})$ of length $N = \frac{\mu(\omega, n) - \mu(\tilde{b}_-)}{\delta}$. There exists $t < N$ such that $\tilde{b}_t < \check{b}$, otherwise,

$$\mu_t > \mu(\tilde{b}_-) + N\delta > \mu(\omega, n)$$

Thus we can reach \mathcal{B} in finite steps. Since shocks are i.i.d, this is an almost sure statement. At \mathcal{B} , transfers are strictly positive for some shocks $T_t > 0$ a.s. and taxes are given by τ^* .

Now consider the payoffs $P \in \mathcal{P}^*$ such that the associated steady state debt $b^* > \mathcal{B}$. Under the guess $T_t = 0$, the same algebra as in Theorem 3 goes through and we can show that $\tilde{b}_- = b^*$ is a steady state for the heterogeneous agent economy. Thus the heterogeneous agent economy for a given $P \in \mathcal{P}^*$ has a continuum of steady states given by the set $[\bar{b}, \mathcal{B}] \cup \{b^*\}$.

In the region $\tilde{b}_- > b^*$, as before μ_t is supermartingale bounded below by b^* . Since there is a unique fixed point in the region $\tilde{b}_- \in [b^*, \bar{b}^n]$, μ_t converges to μ^* associated with b^* . Transfers are zero and taxes are given by τ^{**}

$$\tau^{**} = \frac{\gamma \mu^*}{(1 + \gamma) \mu^* - 1} \tag{66}$$

In the region $[\mathcal{B}, \quad b^*]$ the outcomes depend on the exact sequence of shocks we can show that μ_t is a submartingale. This follows from the observation that for all $\tilde{b}_- > \mathcal{B}$, we have $T(s) = 0$ and the outcomes from the representative agent economy allow us to order $\mu(s)$ relative $P(s)$. At $\tilde{b}_- = \mathcal{B}$, $\mu(s) = \omega - n$ and is constant. Thus in the region $[\mathcal{B}, \quad B^*]$, μ_t is sub martingale and it converges. However if \tilde{b}_t gets sufficiently close to \check{b} , then it can converge to \mathcal{B} and if it gets sufficiently close to b^* , it can converge to b^* . Either of this can happen with strictly positive probability. ■

A.7 Proof of Theorem ??

The Bellman equation for the optimal planners problem with log quadratic preferences and IID shocks can be written as

$$V(x, \rho) = \max_{c_1, c_2, l_1, x', \rho'} \sum_s \pi(s) \left[\alpha_1 \left(\log c_1(s) - \frac{l_1(s)^2}{2} \right) + \alpha_2 \log c_2(s) + \beta V(x'(s), \rho'(s)) \right]$$

subject to the constraints

$$1 + \rho'(s)[l_1(s)^2 - 1] + \beta x'(s) - \frac{x \frac{P(s)}{c_2(s)}}{\mathbb{E}[\frac{P(s)}{c_2(s)}]} = 0 \quad (67)$$

$$\mathbb{E} \frac{P(s)}{c_1(s)} (\rho'(s) - \rho) = 0 \quad (68)$$

$$\theta_1(s)l_1(s) - c_1(s) - c_2(s) - g = 0 \quad (69)$$

$$\rho'(s)c_2(s) - c_1(s) = 0 \quad (70)$$

where the $\pi(s)$ is the probability distribution of the aggregate state s . If we let $\pi(s)\mu(s)$, λ , $\pi(s)\xi(s)$ and $\pi(s)\phi(s)$ be the Lagrange multipliers for the constraints (67)-(70) respectively then we obtain the following FONC for the planners problem ²⁷

$$c_1(s) : \quad \frac{\alpha_1 \pi(s)}{c_1(s)} - \frac{\lambda \pi(s)}{c_1(s)^2} (\rho'(s) - \rho) - \pi(s)\xi(s) - \pi(s)\phi(s) = 0 \quad (71)$$

$$c_2(s) : \quad \frac{\alpha_2 \pi(s)}{c_2(s)} + \frac{x P(s) \pi(s)}{c_2(s)^2 \mathbb{E}[\frac{P}{c_2}]} \left[\mu(s) - \frac{\mathbb{E}[\mu \frac{P}{c_2}]}{\mathbb{E}[\frac{P}{c_2}]} \right] - \pi(s)\xi(s) + \pi(s)\rho'(s)\phi(s) = 0 \quad (72)$$

$$l_1(s) : \quad -\alpha_1 \pi(s)l_1(s) + 2\mu(s)\pi(s)\rho'(s)l_1(s) + \theta_1(s)\pi(s)\xi(s) = 0 \quad (73)$$

$$x'(s) : \quad V_x(x'(s), \rho'(s)) + \mu(s) = 0 \quad (74)$$

$$\rho'(s) : \quad \beta V_\rho(x'(s), \rho'(s)) + \frac{\lambda \pi(s)}{c_1(s)} + \mu(s)[l_1(s)^2 - 1] + \pi(s)\phi(s)c_2(s) = 0 \quad (75)$$

²⁷Appendix A.8 discusses the associated second order conditions that ensure these policies are optimal

In addition there are two envelope conditions given by

$$V_x(x, \rho) = - \sum_{s'} \frac{\mu(s') \pi(s') \frac{P(s')}{c_2(s')}}{\mathbb{E}[\frac{P}{c_2}]} = - \frac{\mathbb{E}[\mu \frac{P}{c_2}]}{\mathbb{E}[\frac{P}{c_2}]} \quad (76)$$

$$V_\rho(x, \rho) = -\lambda \mathbb{E}[\frac{P}{c_1}] \quad (77)$$

In the steady state, we need to solve for a collection of allocations, initial conditions and Lagrange multipliers $\{c_1(s), c_2(s), l_1(s), x, \rho, \mu(s), \lambda, \xi(s), \phi(s)\}$ such that equations (67)-(77) are satisfied when $\rho'(s) = \rho$ and $x'(s) = x$. It should be clear that if we replace $\mu(s) = \mu$, equation (74) and the envelope condition with respect to x is always satisfied. Additionally under this assumption equation (72) simplifies significantly, since

$$\frac{xP(s)\pi(s)}{c_2(s)^2 \mathbb{E}[\frac{P}{c_2}]} \left[\mu(s) - \frac{\mathbb{E}[\mu \frac{P}{c_2}]}{\mathbb{E}[\frac{P}{c_2}]} \right] = 0$$

The first order conditions for a steady can then be written simply as

$$1 + \rho[l_1(s)^2 - 1] + \beta x - \frac{xP(s)}{c_2(s) \mathbb{E}[\frac{P}{c_2}]} = 0 \quad (78)$$

$$\theta_1(s)l_1(s) - c_1(s) - c_2(s) - g = 0 \quad (79)$$

$$\rho c_2(s) - c_1(s) = 0 \quad (80)$$

$$\frac{\alpha_1}{c_1(s)} - \xi(s) - \phi(s) = 0 \quad (81)$$

$$\frac{\alpha_2}{c_2(s)} - \xi(s) + \rho\phi(s) = 0 \quad (82)$$

$$[2\mu\rho - \alpha_1]l_1(s) + \theta_1(s)\xi(s) = 0 \quad (83)$$

$$\lambda \left(\frac{P(s)}{c_1(s)} - \beta \mathbb{E} \left[\frac{P}{c_1} \right] \right) + \mu[l_1(s)^2 - 1] + \phi(s)c_2(s) = 0 \quad (84)$$

We can rewrite equation (81) as

$$\frac{\alpha_1}{c_2(s)} - \rho\xi(s) - \rho\phi(s) = 0$$

by substituting $c_1(s) = \rho c_2(s)$. Adding this to equation (82) and normalizing $\alpha_1 + \alpha_2 = 1$ we obtain

$$\xi(s) = \frac{1}{(1 + \rho) c_2(s)} \quad (85)$$

which we can use to solve for $\phi(s)$ as

$$\phi(s) = \frac{\alpha_1 - \rho\alpha_2}{(\rho(1 + \rho)) c_2(s)} \quad (86)$$

From equation (78) we can solve for $l_1(s)^2 - 1$ as

$$l_1(s)^2 - 1 = \frac{x}{\rho \mathbb{E}[\frac{P}{c_2}]} \left(\frac{P(s)}{c_2(s)} - \beta \mathbb{E} \left[\frac{P}{c_2} \right] \right) - \frac{1}{\rho}$$

This can be used along with equations (84) and (86) to obtain

$$\left(\frac{\lambda}{\rho} + \frac{\mu x}{\rho \mathbb{E}[\frac{P}{c_2}]} \right) \left(\frac{P(s)}{c_2(s)} - \beta \mathbb{E} \left[\frac{P}{c_2} \right] \right) = \frac{\mu}{\rho} + \frac{\rho \alpha_2 - \alpha_1}{\rho(1 + \rho)}$$

Note that the LHS depends on s while the RHS does not, hence the solution to this equation is

$$\lambda = -\frac{\mu x}{\mathbb{E}[\frac{P}{c_2}]} \quad (87)$$

and

$$\mu = \frac{\alpha_1 - \rho \alpha_2}{1 + \rho} \quad (88)$$

Combining these with equation (83) we quickly obtain that

$$\left[2\rho \frac{\alpha_1 - \rho \alpha_2}{1 + \rho} - \alpha_1 \right] l_1(s) + \frac{\theta_1(s)}{(1 + \rho) c_2(s)} = 0$$

Then solving for $l_1(s)$ gives

$$l_1(s) = \frac{\theta_1(s)}{(\alpha_1(1 - \rho) + 2\rho^2 \alpha_2) c_2(s)}$$

Remark 2 Note that the labor tax rate is given by $1 - \frac{c_1(s)l_1(s)}{\theta(s)}$. The previous expression shows that labor taxes are constant at the steady state. This property holds generally for CES preferences separable in consumption and leisure

This we can plug into the aggregate resource constraint (79) to obtain

$$l_1(s) = \left(\frac{1 + \rho}{\alpha_1(1 - \rho) + 2\rho^2 \alpha_2} \right) \frac{1}{l_1(s)} + \frac{g}{\theta_1(s)}$$

letting $C(\rho) = \frac{1 + \rho}{\alpha_1(1 - \rho) + 2\rho^2 \alpha_2}$ we can then solve for $l_1(s)$ as

$$l_1(s) = \frac{g \pm \sqrt{g^2 + 4C(\rho)\theta_1(s)^2}}{2\theta_1(s)}$$

The marginal utility of agent 2 is then

$$\frac{1}{c_2(s)} = \left(\frac{1 + \rho}{C(\rho)} \right) \left(\frac{g \pm \sqrt{g^2 + 4C(\rho)\theta_1(s)^2}}{2\theta_1(s)^2} \right)$$

Note that in order for either of these terms to be positive we need $C(\rho) \geq 0$ implying that there is only one economically meaningful root. Thus

$$l_1(s) = \frac{g + \sqrt{g^2 + 4C(\rho)\theta_1(s)^2}}{2\theta_1(s)} \quad (89)$$

and

$$\frac{1}{c_2(s)} = \left(\frac{1+\rho}{C(\rho)} \right) \left(\frac{g + \sqrt{g^2 + 4C(\rho)\theta_1(s)^2}}{2\theta_1(s)^2} \right) \quad (90)$$

A steady state is then a value of ρ such that

$$x(s) = \frac{1 + \rho[l_1(\rho, s)^2 - 1]}{\frac{P(s)/c_2(\rho, s)}{\mathbb{E}[\frac{P}{c_2}](\rho)} - \beta} \quad (91)$$

s independent of s .

The following lemma, which orders consumption and labor across states, will be useful in proving the parts of theorem ???. As a notational aside we will often use $\theta_{1,l}$ and $\theta_{1,h}$ to refer to $\theta_1(s_l)$ and $\theta_1(s_h)$ respectively. Where s_l refers to the low TFP state and s_h refers to the high TFP state.

Lemma 15 *Suppose that $\theta_1(s_l) < \theta_2(s_h)$ and ρ such that $C(\rho) > 0$ then*

$$l_{1,l} = \frac{g + \sqrt{g^2 + 4C(\rho)\theta_{1,l}^2}}{2\theta_{1,l}} > \frac{g + \sqrt{g^2 + 4C(\rho)\theta_{1,h}^2}}{2\theta_{1,h}} = l_{1,h}$$

and

$$\frac{1}{c_{2,l}} = \frac{1+\rho}{C(\rho)} \frac{g + \sqrt{g^2 + 4C(\rho)\theta_{1,l}^2}}{2\theta_{1,l}^2} > \frac{1+\rho}{C(\rho)} \frac{g + \sqrt{g^2 + 4C(\rho)\theta_{1,h}^2}}{2\theta_{1,h}^2} = \frac{1}{c_{2,h}}$$

Proof. The results should follow directly from showing that the function

$$l_1(\theta) = \frac{g + \sqrt{g^2 + 4C(\rho)\theta^2}}{2\theta}$$

is decreasing in θ . Taking the derivative with respect to θ

$$\begin{aligned} \frac{dl_1}{d\theta}(\theta) &= -\frac{g}{2\theta^2} - \frac{\sqrt{g^2 + 4C(\rho)\theta^2}}{2\theta^2} + \frac{4C(\rho)\theta}{2\theta\sqrt{g^2 + 4C(\rho)\theta^2}} \\ &= -\frac{g}{2\theta^2} - \frac{g + 4C(\rho)\theta^2 - 4C(\rho)\theta^2}{2\theta^2\sqrt{g^2 + 4C(\rho)\theta^2}} \\ &= -\frac{g}{2\theta^2} - \frac{g}{2\theta^2\sqrt{g^2 + 4C(\rho)\theta^2}} < 0 \end{aligned}$$

That $\frac{1}{c_{2,l}} > \frac{1}{c_{2,h}}$ follows directly. ■

Now we use these lemma to prove the part 1 and part 2 of theorem 6

Proof.

[**Part 1.**] For a riskfree bond when $P(s) = 1$. In order for there to exist a ρ such that equation (91) is independent of the state (and hence have a steady state) we need the existence of root for the following function

$$f(\rho) = \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} - \frac{\frac{1/c_2(\rho, s_h)}{\mathbb{E}[\frac{P}{c_2}](\rho)} - \beta}{\frac{1/c_2(\rho, s_l)}{\mathbb{E}[\frac{P}{c_2}](\rho)} - \beta}$$

From lemma 15 we can conclude that

$$1 + \rho[l_1(\rho, s_l)^2 - 1] > 1 + \rho[l_1(\rho, s_h)^2 - 1] \quad (92)$$

and

$$\frac{1/c_2(\rho, s_l)}{\mathbb{E}[\frac{P}{c_2}](\rho)} - \beta > \frac{1/c_2(\rho, s_h)}{\mathbb{E}[\frac{P}{c_2}](\rho)} - \beta \quad (93)$$

for all $\rho > 0$ such that $C(\rho) \geq 0$. To begin with we will define $\underline{\rho}$ such that $C(\rho) > 0$ for all $\rho > \underline{\rho}$. Note that we will have to deal with two different cases.

$\alpha_1(1 - \rho) + 2\rho^2\alpha_2 > 0$ **for all** $\rho \geq 0$: In this case we know that $C(\rho) \geq 0$ for all ρ and is bounded above and thus we will let $\underline{\rho} = 0$.

$\alpha_1(1 - \rho) + 2\rho^2\alpha_2 = 0$ **for some** $\rho > 0$: In this case let $\underline{\rho}$ be the largest positive root of $\alpha_1(1 - \rho) + 2\rho^2\alpha_2$. Note that $\lim_{\rho \rightarrow \underline{\rho}^+} C(\rho) = \infty$

With this we note that²⁸

$$\lim_{\rho \rightarrow \underline{\rho}^+} \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} = 1$$

We can also show that

$$\lim_{\rho \rightarrow \underline{\rho}^+} \frac{\frac{1/c_2(\rho, s_h)}{\mathbb{E}[\frac{P}{c_2}](\rho)} - \beta}{\frac{1/c_2(\rho, s_l)}{\mathbb{E}[\frac{P}{c_2}](\rho)} - \beta} < 1$$

which implies that $\lim_{\rho \rightarrow \underline{\rho}^+} f(\rho) > 0$.

Taking the limit as $\rho \rightarrow \infty$ we see that $C(\rho) \rightarrow 0$, given that $\frac{g}{\theta(s)} < 1$, we can then conclude that

$$\lim_{\rho \rightarrow \infty} 1 + \rho[l_1(\rho, s)^2 - 1] = -\infty$$

²⁸In the first case $\underline{\rho} = 0$ and in the second case $l_1(\rho, s_l) = l_1(\rho, s_h)$ as $\rho \rightarrow \underline{\rho}^+$

Thus, there exists $\bar{\rho}$ such that $1 + \bar{\rho}[l_1(\bar{\rho}, s_l)^2 - 1] = 0$.²⁹ From equation (92), we know that

$$0 = 1 + \bar{\rho}[l_1(\bar{\rho}, s_l)^2 - 1] > 1 + \bar{\rho}[l_1(\bar{\rho}, s_h)^2 - 1]$$

which implies in the limit

$$\lim_{\rho \rightarrow \bar{\rho}^-} \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} = -\infty$$

which along with

$$\frac{\frac{1/c_2(\rho, s_h)}{\mathbb{E}[\frac{P}{c_2}]} - \beta}{\frac{1/c_2(\rho, s_l)}{\mathbb{E}[\frac{P}{c_2}]} - \beta} \geq -1$$

allows us to conclude that $\lim_{\rho \rightarrow \bar{\rho}^-} f(\rho) = -\infty$. The intermediate value theorem then implies that there exists ρ_{SS} such that $f(\rho_{SS}) = 0$ and hence that ρ_{SS} is a steady state.

Finally, as $\rho_{SS} < \bar{\rho}$ we know that

$$1 + \rho_{SS}[l_1(\rho_{SS}, s_l) - 1] > 0$$

as $\frac{1/c_2(\rho, s_l)}{\mathbb{E}[\frac{P}{c_2}]} > 1$ we can conclude

$$x_{SS} = \frac{1 + \rho_{SS}[l_1(\rho_{SS}, s_l) - 1]}{\frac{1/c_2(\rho, s_l)}{\mathbb{E}[\frac{P}{c_2}]} - \beta} > 0$$

implying that the government will hold assets in the steady state (under the normalization that agent 2 holds no assets).

[Part 2] As noted before, since $g/\theta(s) < 1$ for all s we have

$$\lim_{\rho \rightarrow \infty} 1 + \rho[l_1(\rho, s)^2 - 1] = -\infty$$

Thus, there exists ρ_{SS} such that

$$0 > 1 + \rho_{SS}[l_1(\rho_{SS}, s_l)^2 - 1] > 1 + \rho_{SS}[l_1(\rho_{SS}, s_h)^2 - 1]$$

It is then possible to choose $P(s)$ such that $\beta < \frac{P(s)/c_2(\rho_{SS}, s)}{\mathbb{E}[\frac{P}{c_2}]}$ such that

$$1 > \frac{1 + \rho_{SS}[l_1(\rho_{SS}, s_l)^2 - 1]}{1 + \rho_{SS}[l_1(\rho_{SS}, s_h)^2 - 1]} = \frac{\frac{P(s_l)/c_2(\rho_{SS}, s_l)}{\mathbb{E}[\frac{P}{c_2}]} - \beta}{\frac{P(s_h)/c_2(\rho_{SS}, s_h)}{\mathbb{E}[\frac{P}{c_2}]} - \beta} \quad (94)$$

²⁹This can be seen from the fact $\lim_{\rho \rightarrow \bar{\rho}^+} 1 + \rho[l_1(\rho, s_l)^2 - 1] > 0$ and $\lim_{\rho \rightarrow \infty} 1 + \rho[l_1(\rho, s_l)^2 - 1] > -\infty$, thus $\bar{\rho}$ exists in (ρ, ∞)

Implying that for Payoff shocks $P(s)$, ρ_{SS} is a steady state level for the ratio of marginal utilities, with steady state marginal utility weighted government debt

$$x_{SS} = \frac{1 + \rho_{SS}[l_1(\rho_{SS}, s_l)^2 - 1]}{\frac{P(s_l)/c_2(\rho_{SS}, s_l)}{\mathbb{E}[\frac{P}{c_2}]} - \beta} < 0$$

Thus, in the steady state, the government is holding debt, under the normalization that the unproductive worker holds no assets. Note this imposes a restriction of $\frac{P(s_l)}{P(s_h)}$.

$$\frac{P(s_l)c_2^{-1}(\rho_{SS}, s_l) - \beta\mathbb{E}Pc_2^{-1}}{P(s_h)c_2^{-1}(\rho_{SS}, s_h) - \beta\mathbb{E}Pc_2^{-1}} < 1$$

or

$$\frac{P(s_l)}{P(s_h)} < \frac{c_2^{-1}(\rho_{SS}, s_h)}{c_2^{-1}(\rho_{SS}, s_l)} < 1$$

or

Thus $P(s_l) < P(s_h)$ i.e payoffs have to be sufficiently procyclical.

■

■

A.8 Linearization Algorithm

This section will outline our numerical methods used to solve for and linearize around the steady state in the case of a 2 state iid process for the aggregate state.

$$V(\mathbf{x}, \boldsymbol{\rho}) = \max_{c_i(s), l_i(s), \mathbf{x}'(s), \boldsymbol{\rho}'(s)} \sum_s P(s) \left(\left[\sum_i \pi_i \alpha_i U(c_i(s), l_i(s)) \right] + \beta(s) V(\mathbf{x}'(s), \boldsymbol{\rho}'(s)) \right) \quad (95)$$

$$U_{c,i}(s)c_i(s) + U_{l,i}(s)l_i(s) - \rho'_i(s) [U_{c,1}(s)c_1(s) + U_{l,1}(s)l_1(s)] + \beta(s)x'_i(s) = \frac{x_i U_{c,i}(s)}{\mathbb{E}U_{c,i}} \quad (96a)$$

$$\sum_s \Pr(s) U_{c,1}(s) (\rho_i(s) - \rho_i) = 0 \quad (96b)$$

$$\frac{\rho'_i(s)}{\theta_1(s)} U_{l,1}(s) = \frac{1}{\theta_i(s)} U_{l,i}(s) \quad (96c)$$

$$\sum_{j=0}^I \pi_j c_j(s) + g(s) = \sum_{j=0}^I \pi_j \theta_j(s) l_j(s) \quad (96d)$$

$$U_{c,i}(s) = \rho'_i(s) U_{c,1}(s) \quad (96e)$$

For $i = 2, \dots, I$. Note that some of the constraints have been modified a little for ease of differentiation. Associated with these constraints we have the Lagrange multipliers $\Pr(s)\mu'_i(s)$, $\lambda_i, \Pr(s)\phi_i(s), \Pr(s)\xi(s)$, and $P(s)\zeta_i(s)$.

The first order conditions with respect to the choice variables are as follows (note we will be using the notation $\mathbb{E}z$ to represent $\sum_s \Pr(s)z(s)$ for some variable z)

$c_1(s)$:

$$\begin{aligned} \pi_1 \alpha_1 U_{c,1}(s) + \sum_{i=2}^I (\mu'_i(s) \rho'_i(s)) [U_{cc,1}(s) c_1(s) + U_{c,1}(s)] \\ + \lambda U_{cc,1}(s) \sum_{i=2}^I (\rho'_i(s) - \rho_i) - \pi_1 \xi(s) + \sum_{i=2}^N \zeta_i(s) \rho'_i(s) U_{cc,1}(s) = 0 \end{aligned} \quad (97a)$$

$c_i(s)$: for $i \geq 2$

$$\pi_i \alpha_i U_{c,i}(s) - \mu'_i(s) [U_{cc,i}(s) c_i(s) + U_{c,i}(s)] + \frac{x_i U_{cc,i}(s)}{\mathbb{E}U_{c,i}} \left(\mu'_i(s) - \frac{\mathbb{E}\mu'_i U_{c,i}}{\mathbb{E}U_{c,i}} \right) - \pi_i \xi(s) - \zeta_i(s) U_{cc,i}(s) = 0 \quad (97b)$$

$l_1(s)$:

$$\pi_1 \alpha_1 U_{l,1}(s) + \sum_{i=2}^I \mu'_i(s) \rho_i(s) [U_{ll,1}(s) l_1(s) + U_{l,1}(s)] - \sum_{i=2}^N \frac{\rho'_i(s) \phi_i(s)}{\theta_1(s)} U_{ll,1}(s) + \pi_1 \theta_1(s) \xi(s) = 0 \quad (97c)$$

$l_2(s)$:

$$\pi_i \alpha_i U_{l,i}(s) - \mu'_i(s) [U_{ll,i}(s) l_i(s) + U_{l,i}(s)] + \frac{\phi_i(s)}{\theta_i(s)} U_{ll,i}(s) + \pi_i \theta_i(s) \xi(s) = 0 \quad (97d)$$

$\rho'_i(s)$:

$$\beta(s) V_{\rho_i}(\mathbf{x}'(s), \boldsymbol{\rho}'_i(s)) + \mu'_i(s) [U_{c,1}(s) c_1(s) + U_{l,1}(s) l_1(s)] + \lambda_i U_{c,1}(s) - \phi_i(s) \frac{U_{l,1}(s)}{\theta_1(s)} + U_{c,1}(s) \zeta_i(s) = 0 \quad (97e)$$

$x'_i(s)$:

$$V_{x_i}(\mathbf{x}'(s), \boldsymbol{\rho}'(s)) - \mu'_i(s) = 0. \quad (97f)$$

Equations (96a)-(96e) and (97a)-(97e) then define the necessary conditions for an interior maximization of the planners problem for the state $(\mathbf{x}, \boldsymbol{\rho})$. In addition to these we have the two envelop conditions

$$V_{x_i}(\mathbf{x}, \boldsymbol{\rho}) = \frac{\sum_s P(s) \mu'_i(s) U_{c,i}(s)}{\mathbb{E}U_{c,i}(s)} = \frac{\mathbb{E}\mu'_i U_{c,i}}{\mathbb{E}U_{c,i}}, \quad (98a)$$

and

$$V_{\rho_i}(\mathbf{x}, \boldsymbol{\rho}) = -\lambda_i \mathbb{E}U_{c,1}. \quad (98b)$$

In order to check local stability we linearize locally around the steady state. Furthermore we find that the policy functions have better numerical properties when the state variables are

chosen to be $(\boldsymbol{\mu}, \boldsymbol{\rho})$ rather than $(\mathbf{x}, \boldsymbol{\rho})$, and thus, we will proceed with the linearization procedure using $(\boldsymbol{\mu}, \boldsymbol{\rho})$ as the endogenous state vector. The evolution of the state variable $\boldsymbol{\mu}$ must follow the weighted martingale

$$\mu_i - \frac{\sum_s P(s) \mu'_i(s) U_{c,i}(s)}{\sum_s P(s) U_{c,i}(s)} = 0. \quad (99)$$

The optimal policy function, which we will denote as $z(\boldsymbol{\mu}, \boldsymbol{\rho})$, must satisfy $F(z, y, g(z)) = 0$ where F represents the system of equations (96a)-(97e) and (99), y is the state vector $(\mathbf{x}, \boldsymbol{\rho})$, and g is the mapping of the policies into functions of future variables, namely $\mathbf{x}'(s)$ and $V_\rho(\boldsymbol{\mu}'(s), \boldsymbol{\rho}(s))$. In other words

$$g(z) = \begin{pmatrix} \mathbf{x}(\boldsymbol{\mu}'(1), \boldsymbol{\rho}'(1)) \\ V_\rho(\boldsymbol{\mu}'(1), \boldsymbol{\rho}'(1)) \\ \mathbf{x}(\boldsymbol{\mu}'(2), \boldsymbol{\rho}'(2)) \\ V_\rho(\boldsymbol{\mu}'(2), \boldsymbol{\rho}'(2)) \end{pmatrix}.$$

Finally $z(\boldsymbol{\mu}, \boldsymbol{\rho})$ are the stacked variables $\{c_1(s), c_i(s), l_1(s), l_i(s), \mathbf{x}, \boldsymbol{\rho}'(s), \boldsymbol{\mu}'(s), \boldsymbol{\lambda}, \phi(s), \xi(s), \zeta(s)\}$. The optimal policy function is then a function $z(y)$ that satisfies the relationship $F(z(y), y, g(z(y))) = 0$. Taking total derivatives around the steady state \bar{y} and $\bar{z} = z(\bar{y})$

$$D_z F(\bar{z}, \bar{y}, g(\bar{z})) D_y z(\bar{y}) + D_y F(\bar{z}, \bar{y}, g(\bar{z})) + D_g F(\bar{z}, \bar{y}, g(\bar{z})) Dg(\bar{z}) D_y z(\bar{z}) = 0$$

In order to linearize $z(y)$ around the steady state \bar{y} we need to compute $D_y z(\bar{y})$. The envelope condition (98b) tell us that V_ρ can be computed from the optimal policies, i.e.

$$\begin{pmatrix} \mathbf{x}(\boldsymbol{\mu}, \boldsymbol{\rho}) \\ V_\rho(\boldsymbol{\mu}, \boldsymbol{\rho}) \end{pmatrix} = w(z(\boldsymbol{\mu}, \boldsymbol{\rho})) = \begin{pmatrix} \mathbf{x} \\ -\boldsymbol{\lambda} \mathbb{E}[U_{c,1}] \end{pmatrix}$$

If we let Φ_s be the matrix that maps $z(\boldsymbol{\mu}, \boldsymbol{\rho})$ into $\begin{pmatrix} \boldsymbol{\mu}'(s) \\ \boldsymbol{\rho}'(s) \end{pmatrix}$ then we can write $g(\boldsymbol{\mu}, \boldsymbol{\rho})$ using z and w as follows

$$g(z) = \begin{pmatrix} w(z(\Phi_1 z)) \\ w(z(\Phi_2 z)) \end{pmatrix}$$

taking derivatives we quickly obtain that

$$\begin{aligned} D_z g(\bar{z}) &= \begin{pmatrix} Dw(z(\Phi_1 \bar{z})) & 0 \\ 0 & Dw(z(\Phi_2 \bar{z})) \end{pmatrix} \begin{pmatrix} D_y z(\Phi_1 \bar{z}) & 0 \\ 0 & D_y z(\Phi_2 \bar{z}) \end{pmatrix} \underbrace{\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}}_{\Phi} \\ &= \begin{pmatrix} Dw(\bar{z}) & 0 \\ 0 & Dw(\bar{z}) \end{pmatrix} \begin{pmatrix} D_y z(\bar{y}) & 0 \\ 0 & D_y z(\bar{y}) \end{pmatrix} \Phi \\ &= \begin{pmatrix} Dw(\bar{z}) D_y z(\bar{y}) & 0 \\ 0 & Dw(\bar{z}) D_y z(\bar{y}) \end{pmatrix} \Phi \end{aligned}$$

We can then go back to our original matrix equation to obtain

$$D_z F(\bar{z}, \bar{y}, \bar{w}) D_y z(\bar{y}) + D_y F(\bar{z}, \bar{y}, \bar{w}) + Dw F(\bar{z}, \bar{y}, \bar{w}) \begin{pmatrix} Dw(\bar{z}) D_y z(\bar{y}) & 0 \\ 0 & Dw(\bar{z}) D_y z(\bar{y}) \end{pmatrix} \Phi D_y z(\bar{z}) = 0, \quad (100)$$

where $\bar{w} = g(\bar{z}) = w(\bar{z})$. This is now a non-linear matrix equation for $D_y z(\bar{y})$, where all the other terms can be computed using the steady state values \bar{z} and \bar{y} (note $g(\bar{z})$ is known from the envelope conditions at the steady state). Furthermore, $D_y z(\bar{y})$ gives us the linearization of the policy rules since to first order

$$z \approx \bar{z} + D_y z(\bar{y})(y - \bar{y})$$

Our procedure for computing the linearization proceeds as follows

1. Find the steady state by solving the system of equations (27). Numerically, we have found that this is very robust to the parameters of the model.
2. Compute $D_z F(\bar{z}, \bar{y}, g(\bar{z}))$, $D_z F(\bar{z}, \bar{y}, g(\bar{z}))$ and $D_v F(\bar{z}, \bar{y}, g(\bar{z}))$ by numerically differentiating F . This is straightforward using auto-differentiation.
3. Compute $Dw(\bar{z})$ using auto-differentiation.
4. Construct a matrix equation as follows. Given policies $A = Dw(\bar{z})D_y z(\bar{y})$ (these are the linearized policies of \mathbf{x} and V_ρ with respect to $(\boldsymbol{\mu}, \boldsymbol{\rho})$), it is possible to solve for $D_y z(\bar{y})$ from

$$D_y z(\bar{z}) = - \left(D_z F(\bar{z}, \bar{y}, \bar{w}) + D_w F(\bar{z}, \bar{y}, \bar{w}) \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \Phi \right)^{-1} D_y F(\bar{z}, \bar{y}, \bar{w})$$

We wish to find an A such that

$$A = Dw(\bar{z})D_y z(\bar{z})$$

Given the linearized policy rules it is then possible to evaluate the local stability of the steady state. We find that in the absence of discount factor shocks the steady state is stable generically across the parameter space.

This linearization can be used to construct the bordered hessian of the problem (24) at the steady state. We can then apply second order tests to verify that the first order necessary conditions are sufficient.

A.9 Proof for Theorem 7

Proof.

The state at time t can be written as

$$\hat{\Psi}_t = B_t B_{t-1} \cdots B_1 \hat{\Psi}_0.$$

where the B_i are all random variables being $B(s)$ with probability $\Pr(s)$. Taking expectations and applying independence we then obtain

$$\mathbb{E}_0[\hat{\Psi}_t] = \mathbb{E}_0[B_t B_{t-1} \cdots B_1] \hat{\Psi}_0 \quad (101)$$

$$= \mathbb{E}[B_t] \mathbb{E}[B_{t-1}] \cdots \mathbb{E}[B_1] \hat{\Psi}_0 \quad (102)$$

$$= \bar{B}^t \hat{\Psi}_0 \quad (103)$$

where $\bar{B} = \mathbb{E}B(s)$. If eigenvalues of \bar{B} are positive and strictly less than 1, at least, in expectation the linearized system converges that is

$$\tilde{\Psi}_{t|0} \equiv \mathbb{E}_0[\hat{\Psi}_t] = \bar{B}^t \hat{\Psi}_0 \rightarrow \mathbf{0}. \quad (104)$$

It should be noted that the conditional expectation actually captures a significant portion of the linearized dynamics. The remaining question is does the distribution converge to $\mathbf{0}$. This can be done by analyzing the variance. Let

$$\Sigma_{\Psi,t|0} = \mathbb{E}_0 \left[(\hat{\Psi}_t - \tilde{\Psi}_t)(\hat{\Psi}_{t|0} - \tilde{\Psi}_{t|0})' \right]$$

or

$$\Sigma_{\Psi,t|0} = \mathbb{E}_0 \hat{\Psi}_t \hat{\Psi}_t' - \tilde{\Psi}_{t|0} \tilde{\Psi}_{t|0}'. \quad (105)$$

Note that if eigenvalues of \bar{B} are positive and strictly less than 1, $\tilde{\Psi}_{t|0}$ converges to 0. Using the independence of $\hat{\Psi}_{t-1}$ and B_t , and $\hat{\Psi}_t = B_t \hat{\Psi}_{t-1}$, we quickly obtain that for large t

$$\Sigma_{\Psi,t|0} \approx \mathbb{E}[B \Sigma_{\Psi,t-1|0} B'] \quad (106)$$

Showing that $\hat{\Psi}_{t|0} \rightarrow \mathbf{0}$ in distribution, amounts to showing that $\Sigma_{\Psi,t|0} \rightarrow 0$ for any starting point Σ_{Ψ} and following the process in equation (106). One can obtain a necessary condition for $\|\Sigma_{\Psi,t|0}\| \rightarrow 0$ under the process in equation (106). That process can be rewritten as follows

$$\Sigma_{\Psi,t|0} = \mathbb{E}[B \Sigma_{\Psi,t-1|0} B'] \quad (107)$$

$$= \sum_s \Pr(s) B(s) \Sigma_{\Psi,t-1|0} B(s)' \quad (108)$$

$$= \sum_s \Pr(s) (\bar{B} + (B(s) - \bar{B})) \Sigma_{\Psi,t-1|0} (\bar{B} + (B(s) - \bar{B}))' \quad (109)$$

$$= \bar{B} \Sigma_{\Psi,t-1|0} \bar{B}' + \sum_s \Pr(s) (B(s) - \bar{B}) \Sigma_{\Psi,t-1|0} (B(s) - \bar{B})'. \quad (110)$$

This is a deterministic linear system in $\Sigma_{\Psi,t|0}$. Suppose we reshape $\Sigma_{\Psi,t|0}$ as a vector (denoted by $\text{vec}(\Sigma_{\Psi,t|0})$) and let \hat{B} be a (square) matrix such that equation 110 is written as

$$\text{vec}(\Sigma_{\Psi,t|0}) = \hat{B} \text{vec}(\Sigma_{\Psi,t-1|0}).$$

The stability of this system is guaranteed if the (real part) of eigenvalues of \hat{B} are less than 1.

■

We used theorem 7 to verify local stability of a wide range of examples. The typical finding is that the steady state is generically stable and that convergence is slow. In figure 8 we plot the comparative statics for the dominant eigenvalue and the associated half-life for a two-agent economy with CES preferences. We set the other parameter to match a Frisch elasticity of 0.5, a real interest rate of 2%, marginal tax rates around 20%, and a 90-10 percentile ratio of wage earnings of 4. In the first exercise, we vary the size of the expenditure shock keeping risk aversion σ at one. The x -axis plots the spread in expenditure normalized by the undistorted GDP and reported in percentages. In the bottom panel, we fix the size of shock such that it produces a 5% fall in expenditure at risk aversion of one and vary σ from 0.8 to 7. We see that the dominant eigenvalue is everywhere less than one but very close to one, so that the steady state is stable but convergence is slow for reasonable values of curvatures and shocks. Both increasing the size of the shock or risk aversion increases the volatility of the interest rates, speeding up the transition towards the steady state.

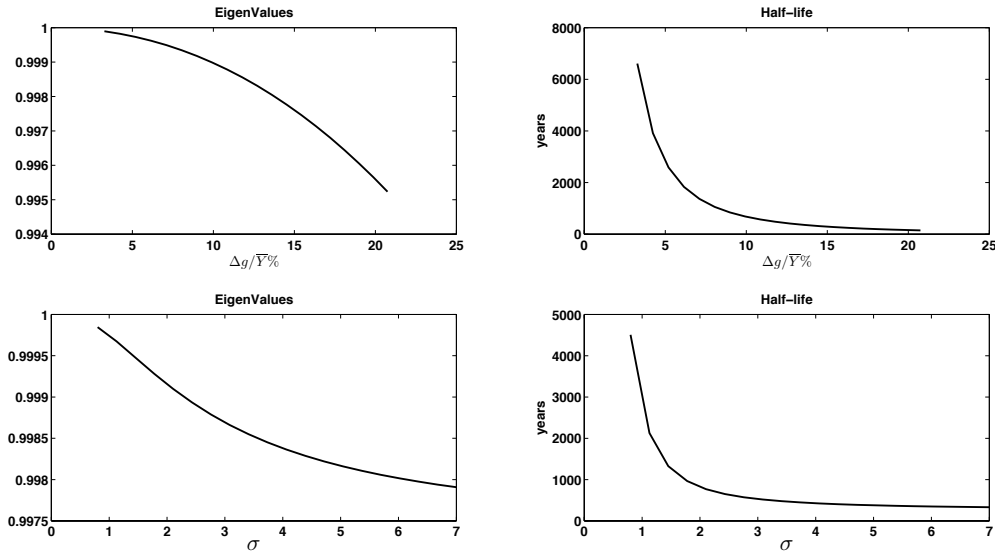


Figure 8: The top (bottom) panel plots the dominant eigenvalue of \hat{B} and the associated half life as we increase the spread between the expenditure levels (risk aversion).

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