Taxes, debts, and redistributions with aggregate shocks*

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Abstract

This paper studies how taxes and debt respond to aggregate shocks in the presence of incomplete markets and redistribution concerns. A planner sets a lump sum transfer and a linear tax on labor income in an economy with heterogeneous agents, aggregate uncertainty, and markets restricted to a single asset whose payoffs can wary with aggregate states. Two forces shape long-run outcomes: the planner's desire to minimize the welfare costs of fluctuating transfers, which calls for a negative correlation between the distribution of net assets and agents' skills; and the planner's desire to use fluctuations in the real interest rate to adjust for missing state-contingent securities. In a model parameterized to match stylized facts about US booms and recessions, distributional concerns mainly determine optimal policies over business cycle frequencies. These features of optimal policy differ markedly from ones that emerge from representative agent Ramsey models

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1 Introduction

2 Environment

Exogenous fundamentals include a cross section distribution of skills $\{\theta_{i,t}\}$ and government expenditures $\{g_t\}$. These are all functions of a shock s_t that is governed by an irreducible Markov process, where $s_t \in S$ and S is a finite set. We let $s^t = (s_0, ..., s_t)$ denote a history of shocks with joint density $Pr(s^t)$. [1]

There is a mass n_i of a type $i \in I$ agents, with $\sum_{i=1}^{I} n_i = 1$. Types differ in skills indexed by $\{\theta_{i,t}\}_t$. Preferences of an agent of type i over stochastic processes for consumption $\{c_{i,t}\}_t$ and labor supply $\{l_{i,t}\}_t$ are ordered by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U^i \left(c_{i,t}, l_{i,t}\right), \tag{1}$$

where $\mathbb{E}_t$ is a mathematical expectations operator conditioned on time t information and $\beta \in (0,1)$ is a time discount factor. Except in section 2.1, we assume that $U^i : \mathbb{R}_+^2 \to \mathbb{R}$ is concave in $(c, -l)$ and twice continuously differentiable. We let $U_{x,t}^i$ or $U_{xy,t}^i$ denote first and second derivatives of U^i with respect to $x, y \in \{c, l\}$ in period t and assume $\lim_{x\to 0} U_l^i\left(c, x\right) = 0$ for all c and i.[2]

An agent of type i who supplies l_i units of labor produces $\theta_i\left(s_t\right) l_i$ units of output, where $\theta_i(s_t) \in \Theta$ is a nonnegative state-dependent scalar. Feasible allocations satisfy

$$\sum_{i=1}^{I} n_i c_{i,t} + g_t = \sum_{i=1}^{I} \pi_i \theta_{i,t} l_{i,t} \tag{2}$$

where g_t denotes exogenous government expenditures in state s_t.

The government and agents trade a single, possibly risky, asset. Private agents and the government begin with assets $\{b_{i,-1}\}_{i=1}^{I}$ and B_{-1}, respectively. These asset holdings satisfy the market clearing condition

$$\sum_{i=1}^{I} n_i b_{i,t} + B_t = 0 \text{ for all } t \geq -1. \tag{3}$$

The time t payoff p_t on the single asset is described by an $S \times S$ matrix $\mathbb{P}$

$$p_t = \mathbb{P}(s_t | s_{t-1}),$$

[1] To save on notation, mostly we use z_t to denote a random variable with a time t conditional distribution that is a function of the history s^t. Occasionally, we use the more explicit notion $z\left(s^t\right)$ to denote a realization at a particular history s^t.

[2] Results in section 2.1 hold even if we weaken assumptions like differentiability and convexity of U^i.

satisfying the normalizations $\mathbb{E}_t p_{t+1} = 1$. Specifying the random asset payoffs in this way is a convenient way for us to investigate impacts of the correlation between asset returns, on the one hand, and government expenditures or shocks to the skill distribution, skills, and government purchases, on the other hand. The price of the single asset at time t is $q_t = q_t(s^t)$, so and $R_t = \frac{p_t}{q_{t-1}}$ is the one-period return on the asset.

The government imposes an affine tax with proportional labor tax rate τ_t and common lump lump transfers component T_t. The tax bill of an agent with wage earnings $l_{i,t}\theta_{i,t}$ is

$$-T_t + \tau_t \theta_{i,t} l_{i,t}.$$

We leave the sign of T_t unrestricted.

A type i agent's budget constraint at $t \geq 0$ is

$$c_{i,t} + b_{i,t} = (1 - \tau_t)\, \theta_{i,t} l_{i,t} + R_t b_{i,t-1} + T_t. \tag{4}$$

The government budget constraint is

$$g_t + B_t = \tau_t \sum_{i=1}^{I} n_i \theta_{i,t} l_{i,t} - T_t + R_t B_{t-1}. \tag{5}$$

Definition 1 *An* allocation *is a sequence* $\{c_{i,t}, l_{i,t}\}_{i,t}$. *An* asset profile *is a sequence* $\{\{b_{i,t}\}_i, B_t\}_t$. *A* returns process *is a sequence* $\{R_t\}_t$. *A* tax policy *is a sequence* $\{\tau_t, T_t\}_t$.

Remark 1 *It is necessary to impose debt limits on the asset profile. For households, we shall impose natural debt limits that will depend on the tax policy.*[3]

Definition 2 *For a given initial asset distribution* $\left(\{b_{i,-1}\}_i, B_{-1}\right)$, *a competitive equilibrium with affine taxes is a sequence* $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, B_t, R_t\}_t$ *and a tax policy* $\{\tau_t, T_t\}_t$, *such that* $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ *maximize (1) subject to (4) and* $\{b_{i,t}\}_{i,t}$ *satisfies the borrowing limits; and constraints (**??**), (5) and (3) are satisfied.*

A Ramsey planner's preferences over a vector of competitive equilibrium stochastic processes for consumption and labor supply are ordered by

$$\mathbb{E}_0 \sum_{i=1}^{I} \omega_i \sum_{t=0}^{\infty} \beta^t U_t^i \left(c_{i,t}, l_{i,t}\right), \tag{6}$$

where the Pareto weights satisfy $\omega_i \geq 0$, $\sum_{i=1}^{I} \omega_i = 1$.

The Ramsey planner chooses the following object:

[3] An alternative is to impose ad-hoc debt limits in the form of exogenous history-contingent bounds for each agent. Appendix XX discusses how restricting attention to natural debt limits for the households only shrinks the set of allocations that can be implemented as competitive equilibria.

Definition 3 *Given* $(\{b_{i,-1}\}_i, B_{-1})$*, an optimal competitive equilibrium with affine taxes is a tax policy* $\{\tau_t^*, T_t^*\}_t$*, an allocation* $\left\{c_{i,t}^*, l_{i,t}^*\right\})_{i,t}$*, an asset profile* $\left\{\left\{b_{i,t}^*\right\}_i, B_t^*\right\}_t$*, and a return process* $\{R_t^*\}_t$ *such that (i) given* $\left(\{b_{i,-1}\}_i, B_{-1}\right)$*, the tax policy, return process, and allocation constitute a competitive equilibrium, (ii)* B_t *satisfies the borrowing constraints; and (iii) there is no other tax policy* $\{\tau_t, T_t\}_t$ *such that a competitive equilibrium given* $\left(\{b_{i,-1}\}_i, B_{-1}\right)$ *and* $\{\tau_t, T_t\}_t$ *has a strictly higher value of (6).*

We call $\{\tau_t^*, T_t^*\}_t$ an *optimal tax policy*, $\{c_{i,t}^*, l_{i,t}^*\}_{i,t}$ an *optimal allocation*, and $\left\{\left\{b_{i,t}^*\right\}_i, B_t^*\right\}_t$ an *optimal asset profile.*

2.1 State dimension reduction

The arithmetic of budget constraints and market clearing instructs us how to cut out some frills in formulating the optimal policy problem. The key is to note that an equivalence class of tax policies and asset profiles support the same competitive equilibrium allocation.

Theorem 1 *Given* $\left(\{b_{i,-1}\}_i, B_{-1}\right)$*, let* $\left\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, B_t, R_t\right\}_t$ *and* $\{\tau_t, T_t\}_t$ *be a competitive equilibrium. For any bounded sequences* $\left\{\hat{b}_{i,t}\right\}_{i,t\geq -1}$ *that satisfy*

$$\hat{b}_{i,t} - \hat{b}_{1,t} = \tilde{b}_{i,t} \equiv b_{i,t} - b_{1,t} \text{ for all } t \geq -1, i \geq 2,$$

there exist sequences $\left\{\hat{T}_t\right\}_t$ *and* $\left\{\hat{B}_t\right\}_{t\geq -1}$ *that satisfy (3) and that make* $\left\{\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_i, \hat{B}_t, R_t\right\}_t$ *and* $\left\{\tau_t, \hat{T}_t\right\}_t$ *constitute a competitive equilibrium given* $\left(\left\{\hat{b}_{i,-1}\right\}_i, \hat{B}_{-1}\right)$.

In the spirit of Barro (1974), this is to be interpreted as a Ricardian equivalence result for our environment. We relegate the proof to appendix XXXXX.

Theorem 1 implies that the tax policy and asset profile contain some redundant information that it is convenient for us to eliminate. The following corollary teaches us how to reduce the dimension of the information to be encoded in a description of government policy to be used in a concise formulation of a Ramsey problem.[4]

Corollary 1 *For any pair* B'_{-1}, B''_{-1}*, there are asset profiles* $\left\{b'_{i,-1}\right\}_i$ *and* $\left\{b''_{i,-1}\right\}_i$ *such that equilibrium allocations starting from* $\left(\left\{b'_{i,-1}\right\}_i, B'_{-1}\right)$ *and from* $\left(\left\{b''_{i,-1}\right\}_i, B''_{-1}\right)$ *are the same. These asset profiles satisfy*

$$b'_{i,-1} - b'_{1,-1} = b''_{i,-1} - b''_{1,-1} \ \forall i.$$

[4]This result holds in more general environments. For example, we could allow agents to trade all conceivable Arrow securities and still show that equilibrium allocations depend only on agents' net assets positions.

Thus, total government debt is not what matters, who owns it does.

Throughout this paper we avail ourselves of theorem 1 to impose a normalization on asset profiles that will determine what we mean we say “public debt.” We assume that productivities are ordered as $\theta_{1,t} \geq \theta_{2,t} \ldots \geq \theta_{N,t}$. **Anmol XXXXXX: it seems that we can move and/or change the preceding sentence about ordering of θ_i's anywhere we want – here it is only used in the following footnote. I flag the ordering issue for you below too.** We set $b_{1,t} = 0$, a normalization that tells us to interpret $-B_t = \sum_{i>1} n_i b_{i,t}$ as public debt.[5] This in turn explains why imposing limits on $B_t - b_{N,t}$ are comparable to debt limits in a representative agent settings. **Anmol XXXXXX: 1. I am not sure what the last sentence means and think it could be disposed of or else should be sharpened. 2. We should decide on the “‘ordering” of θ_i's soon and set them for the entire paper.**

Tom and David XXXX: Lets go with the ordering where Agent N is the least productive. This is consistent with our theorems. I will change the numerical section to reflect this.

Tom and David XXXX: Regarding the debt limit, what I had in mind was connecting to the fact that we are imposing debt limits on $B_t - b_{N,t}$. We mentioned that we impose limits on this object hinting at Ricardian equivalence when we setup the problem.

3 Optimal equilibria with affine taxes

We use the reduced dimension description of government policy from subsection 2.1 to formulate the Ramsey problem. Following Lucas and Stokey (1983) and Aiyagari et al. (2002), we use households' first-order necessary conditions to describe restrictions on competitive equilibrium allocations.

With natural borrowing limits for the households, first-order necessary conditions for the consumers' problems are

$$(1 - \tau_t)\, \theta_{i,t} U^i_{c,t} = -U^i_{l,t}, \tag{7}$$

and

$$U^i_{c,t} = \beta \mathbb{E}_t R_{t+1} U^i_{c,t+1}. \tag{8}$$

To help characterize an equilibrium, we use

Theorem 1 *A sequence $\left\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, R_t, \tau_t, T_t\right\}_t$ is part of a competitive equilibrium with affine taxes if and only if it satisfies (2), (4), (7), and (8) and $b_{i,t}$ is bounded for all i and t.*

[5] If for some type i, $\theta_{i,t} = 0$, $b_{i,-1} = 0$ and U^i is defined only on $\mathcal{R}^2_+$, the type i agent's budget constraint will imply that all allocations feasible for the planner have nonnegative present values of transfers, since transfers are the sole source of a type i agent's wealth and consumption.

Proof. Necessity is obvious. In appendix A.2, we use arguments of Magill and Quinzii (1994) and Constantinides and Duffie (1996) to show that any $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ that satisfies (4), (7), and (8) is a solution to consumer i's problem. Equilibrium $\{B_t\}_t$ is determined by (3) and constraint (5) is then implied by Walras' Law ■

To find an optimal equilibrium, by Theorem 1 we can choose $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, R_t, \tau_t, T_t\}_t$ to maximize (6) subject to (2), (4), (7), and (8). We apply a first-order approach and follow steps similar to ones taken by Lucas and Stokey (1983) and Aiyagari et al. (2002). Substituting consumers' first-order conditions (7) and (8) into the budget constraints (4) yields implementability constraints

$$c_{i,t} + b_{i,t} = -\frac{U^i_{l,t}}{U^i_{c,t}} l_{i,t} + T_t + \frac{p_t U^i_{c,t-1}}{\beta \mathbb{E}_{t-1} p_t U^i_{c,t}} b_{i,t-1} \text{ for all } i, t. \tag{9}$$

For $I \geq 2$, we can use constraint (9) for $i = 1$ to eliminate T_t from (9) for $i > 1$. Letting $\tilde{b}_{i,t} \equiv b_{i,t} - b_{1,t}$, we can represent the implementability constraints as

$$\begin{aligned} & (c_{i,t} - c_{1,t}) + \tilde{b}_{i,t} \\ = \; & -\frac{U^i_{l,t}}{U^i_{c,t}} l_{i,t} + \frac{U^1_{l,t}}{U^1_{c,t}} l_{1,t} + \frac{p_t U^i_{c,t-1}}{\beta \mathbb{E}_{t-1} p_t U^i_{c,t}} \tilde{b}_{i,t-1} \text{ for } \; i > 1 \text{ and } \; t \geq 0, \end{aligned} \tag{10}$$

so that the planner's maximization problem involves only on the $I - 1$ variables $\tilde{b}_{i,t-1}$. The reduction of the dimensionality from I to $I - 1$ is a consequence of corollary 1 of theorem 1.

Denote $Z^i_t = (c_{i,t} - c_{1,t}) + \tilde{b}_{i,t} + \frac{U^i_{l,t}}{U^i_{c,t}} l_{i,t} - \frac{U^1_{l,t}}{U^1_{c,t}} l_{1,t}$. The Ramsey problem is:

$$\max_{c_{i,t}, l_{i,t}, \tilde{b}_{i,t}} \mathbb{E}_0 \sum_{i=1}^{I} \omega_i \sum_{t=0}^{\infty} \bar{\beta}_t U^i_t (c_{i,t}, l_{i,t}), \tag{11}$$

subject to

$$\tilde{b}_{i,t-1} \frac{p_t U^i_{c,t-1}}{\mathbb{E}_{t-1} p_t U^i_{c,t}} = \mathbb{E}_t \sum_{k=t}^{\infty} \beta^{k-t} \left(\frac{U^i_{c,k}}{U^i_{c,t}} \right) Z^i_k \quad \forall t \geq 1 \tag{12a}$$

$$\tilde{b}_{i,-1} = \mathbb{E}_{-1} \sum_{k=0}^{\infty} \beta^k \left(\frac{U^i_{c,k}}{U^i_{c,t}} \right) Z^i_k \tag{12b}$$

$$\frac{\mathbb{E}_t p_{t+1} U^i_{c,t+1}}{U^i_{c,t}} = \frac{\mathbb{E}_t p_{t+1} U^j_{c,t+1}}{U^j_{c,t}} \tag{12c}$$

$$\sum_{i=1}^{I} n_i c_i(s^t) + g(s_t) = \sum_{i=1}^{I} \pi_i \theta_i(s_t) l_i(s^t), \tag{12d}$$

$$\frac{U^i_{l,t}}{\theta_{i,t} U^i_{c,t}} = \frac{U^1_{l,t}}{\theta_{1,t} U^1_{c,t}} \tag{12e}$$

$$\sum_{i=1}^{N} \tilde{b}_{i,t-1} \text{ is bounded} \tag{12f}$$

Constraint (12a) requires that the conditional expectation on the right side, a conditional expectation at time t, be an exact function of information at time $t-1$, the same type of measurability condition present in Aiyagari et al. (2002). This condition is inherited from the restriction that only one only asset with payoffs p_t is traded between the private and the public sector.

It is convenient and informative to represent the Ramsey problem recursively. Let $\boldsymbol{x} = \beta^{-1}\left(U_c^2\tilde{b}_2, ..., U_c^I\tilde{b}_I\right)$, $\boldsymbol{\rho} = \left(U_c^2/U_c^1, ..., U_c^I/U_c^1\right)$, and denote an allocation $a = \{c_i, l_i\}_{i=1}^I$. In the spirit of Kydland and Prescott (1980) and Farhi (2010), we split the Ramsey problem into a time-0 problem that takes $(\{\tilde{b}_{i,-1}\}_{i=2}^I, s_0)$ as state variables and a time $t \geq 1$ continuation problem that takes $\boldsymbol{x}, \boldsymbol{\rho}, s_-$ as state variables. We formulate two Bellman equations and two value functions, one that pertains to $t \geq 1$, another to $t = 0$.[6] As usual, we work backwards and describe the $t \geq 1$ Bellman equations first, and then the $t = 0$ Bellman equation.

For $t \geq 1$, let $V(\boldsymbol{x}, \boldsymbol{\rho}, s_-)$ be the planner's continuation value given $\boldsymbol{x}_{t-1} = \boldsymbol{x}$, $\boldsymbol{\rho}_{t-1} = \boldsymbol{\rho}$, $s_{t-1} = s_-$. It satisfies the Bellman equation

$$V(\boldsymbol{x}, \boldsymbol{\rho}, s_-) = \max_{a(s), x'(s), \rho'(s)} \sum_s \pi(s|s_-) \left(\left[\sum_i \omega_i U^i(s) \right] + \beta V(\boldsymbol{x}'(s), \boldsymbol{\rho}'(s), s) \right) \tag{13}$$

where the maximization is subject to

$$U_c^i(s)\left[c_i(s) - c_1(s)\right] + x_i'(s) + \left(U_l^i(s) l_i(s) - U_c^i(s) \frac{U_l^1(s)}{U_c^1(s)} l_1(s) \right) = \frac{x P(s|s_-) U_c^i(s)}{\beta \mathbb{E}_{s_-} P \boldsymbol{U}_c^i} \text{ for all } s, i \geq 2 \tag{14a}$$

$$\frac{\mathbb{E}_{s_-} P \boldsymbol{U}_c^i}{\mathbb{E}_{s_-} P \boldsymbol{U}_c^1} = \rho_i \text{ for all } i \geq 2 \tag{14b}$$

$$\frac{U_l^i(s)}{\theta_i(s) U_c^i(s)} = \frac{U_l^1(s)}{\theta_1(s) U_c^1(s)} \text{ for all } s, i \geq 2 \tag{14c}$$

$$\sum_i n_i c_i(s) + g(s) = \sum_i n_i(s) l_i(s) \quad \forall s \tag{14d}$$

$$\rho_i'(s) = \frac{U_c^i(s)}{U_c^1(s)} \text{ for all } s, i \geq 2 \tag{14e}$$

$$\sum_{i>1} x_i(s) \frac{\beta}{U_c^i(s)} \text{is bounded} \tag{14f}$$

[6] The time inconsistency of an optimal policy manifests itself in there being distinct value functions and Bellman equations at $t = 0$ and $t \geq 1$.

Constraints (14b) and (14e) imply (8). The definition of x_t and constraints (14a) together imply equation (10) scaled by U_c^i.

Now forthe Bellman equation pertinent for $t = 0$. Let $V_0\left(\{\tilde{b}_{i,-1}\}_{i=2}^{I}, s_0\right)$ be the value to the planner at $t = 0$, where $\tilde{b}_{i,-1}$ denotes initial debt inclusive of accrued interest. It satisfies the Bellman equation

$$V_0\left(\{\tilde{b}_{i,-1}\}_{i=2}^{I}, s_0\right) = \max_{a_0, x_0, \rho_0} \sum_i \omega_i U^i(c_{i,0}, l_{i,0}) + \beta V\left(x_0, \rho_0, s_0\right) \tag{15}$$

where the maximization is subject to

$$U_{c,0}^i\left[c_{i,0} - c_{1,0}\right] + x_{i,0} + \left(U_{l,0}^i l_{i,0} - U_{c,0}^i \frac{U_{l,0}^1}{U_{c,0}^1} l_{1,0}\right) = U_{c,0}^i \tilde{b}_{i,-1} \text{ for all } i \geq 2 \tag{16a}$$

$$\frac{U_{l,0}^i}{\theta_{i,0} U_{c,0}^i} = \frac{U_{l,0}^1}{\theta_{1,0} U_c^{1,0}} \text{ for all } i \geq 2 \tag{16b}$$

$$\sum_i \pi_i c_{i,0} + g_0 = \sum_i \pi_i \theta_{i,0} l_{i,0} \tag{16c}$$

$$\rho_{i,0} = \frac{U_{c,0}^i}{U_{c,0}^1} \ \forall\ i \geq 2 \tag{16d}$$

A tell-tale sign of the time consistency of the optimal tax plan is that (14b), which constrains the time $t \geq 1$ Bellman equations, is absent from the time 0 problem.

Anmol and David XXXXXX: at 8:05 pm Nov 19, I have edited sections 2, 3, and 4. I am stopping here because for now because before I continue I want the committee to decide about the convention with respect to the ordering of the θ_i's. We want a common convention throughout the paper. What shall we do?

4 Long run properties of optimal allocations

In sections 5 and 6 we characterize the long run properties of aggregate debt and taxes. The main finding is that the levels and spreads in debt and tax rates are determined by two factors: a) the ability of the government to span aggregate shocks through the returns on the asset it trades and b)its redistributive preferences. In particular, the government accumulates debt if interest rates are lower when the its need for revenue are higher and vice versa. The long run variance of debt and taxes along with the rates of rates of convergence to the ergodic distribution are higher in economies where the magnitude of this co movement is larger. And lastly more redistributive governments issue more debt.

To study these implications, in section 5 we first examine a simple economy with quasilinear preferences and i.i.d aggregate shocks. This allows us adequate tractability to formally demonstrate and clarify the main driving forces for the results mentioned above. In section 6 we study more general economies (in terms of heterogeneity, preferences and shocks) finally in section 7, we numerically verify that all the insights go through in a version of the model calibrated to US data.

5 Quasilinear economy

We specialize the problem described in section 3 by imposing the following assumptions that are maintained throughout in this section.

Assumption 1 *IID aggregate shocks:* s_t *is i.i.d over time*

Assumption 2 *Quasi linear preference:* $u(c,l) = c - \frac{l^{1+\gamma}}{1+\gamma}$

With i.i.d shocks we can restrict our attention to payoff matrices $\mathbb{P}$ that have identical rows denoted by the vector $P(s)$ with a corresponding normalization that $\mathbb{E}P(s) = 1$. We collect a particular set of these vectors that are perfectly correlated with expenditure shocks $g(s)$ in the set $\mathcal{P}^*$ defined below,

$$\mathcal{P}^* = \left\{ P(s) : P(s) = 1 + \frac{\beta}{B^*}(g(s) - \mathbb{E}g) \text{ for some } B^* \in [\overline{B}, \underline{B}] \right\},$$

where $\overline{B}$ and $\underline{B}$ are upper and lower bounds for government assets.

Before characterizing the properties of Ramsey allocation for the economy with heterogeneous agents and no restrictions on transfers, we develop some results in a representative agent economy where the government *cannot* use transfers. We later show that the allocations in this economy are obtained under certain limits on the Pareto weights for the setting with heterogeneous agents.

5.1 Representative agent

Environment

This section describes the representative agent environment with risky debt and no transfers.[7] Given a tax, asset policy $\{\tau_t, B_t\}$, the household solves,

$$W_0(b_{-1}) \max_{\{c_t, l_t, b_t\}_t} \mathbb{E}_0 \sum_t \beta^t \left[c_t - \frac{l_t^{1+\gamma}}{1+\gamma} \right] \tag{17}$$

[7] This differs from the model studied in AMSS in two ways: first, the government trades a "risky" bond instead of a risk free bond and second, the government is prohibited from using transfer where as AMSS restrict transfers to be non negative. Both of them have critical implications on the long run zero tax results that AMSS obtained. We discuss this later in the section.

subject to

$$c_t + b_t = (1 - \tau_t)\theta l_t + R_t P_t b_{t-t} \tag{18}$$

Using the optimality condition for labor and savings we can summarize the set of implementability constraints for the government as follows

$$b_{t-1} \frac{P_t}{E_{t-1} P_t} = \mathbb{E}_t \sum_j \beta^{t+j} [c_t - l_t^{1+\gamma}] \quad \forall t \tag{19}$$

In addition we have also have the feasibility constraint

$$c_t + g_t \leq \theta l_t, \tag{20a}$$

and the market clearing for bonds,

$$b_t + B_t = 0. \tag{20b}$$

The optimal Ramsey allocation solves $\max_{\{c_t, l_t\}_t} W_0(b_{-1})$ subject to (19), feasibility (20a), market clearing for bonds (20b) and debt limits for the government $\underline{B}, \overline{B}$.[8]

Results

The main results are organized in Theorems 2 and 3. The first result obtains some general properties about the invariant distribution of debt for a large class of payoffs. The second result uses a novel expansion method to get an approximation to the mean and variance of the invariant distribution of debt when payoffs are close the set $\mathcal{P}^*$

When payoffs are not perfectly aligned, the support of the invariant distribution of debt is wide in the sense that (almost surely) the paths of debt sequences approach any arbitrary lower and upper bounds. Note that tax rates are increasing in debt and the variation in debt is analogously reflected in variation in tax rates. This can be contrasted with both, a complete market benchmark as in Lucas Stokey (1983) where both debt and tax rates will be constant sequences and AMSS (2002) which allows for non-negative transfers and risk-free debt where assets approach the first best and limiting tax rates are zero under these preferences.

With some more structure on the payoffs, we show that there is an average inward drift to government assets. More precisely, the multiplier on the implementability constraint is sub (or super) martingale in the region with low (or high) debt. The envelope theorem links the dynamics of the multiplier to that of debts and in turn the concavity of the value function implies

[8] In some calculations we will use the natural debt limit for the government. In this case one can explicitly derived.

mean reversion for debt. This is particularly stark when $P(s) \in \mathcal{P}^*(s)$ where debt converges to a constant. [9]

Next to gain more insights about the invariant distribution we linearize the law of motion for the evolution of debt with respect to both the endogenous state variable that is debt today and payoffs. The point of approximation is the closest (in l_2 sense) complete market economy corresponding to the steady state of some $P(s) \in \mathcal{P}^*(s)$. Exploiting the structure of these approximate laws of motion allows us to obtain bounds on the standard deviation of debt and also rates at which the mean debt level converges that can be expressed in terms of primitive: shocks and payoffs.

Theorem 2 *In the representative agent economy satisfying assumptions 1 and 2, the long run assets under the optimal Ramsey allocation are characterized as follows*

1. *Suppose $P \notin \mathcal{P}^*$, there is an invariant distribution of government such that*

$$\forall \epsilon > 0, \quad \Pr\{B_t < \underline{B} + \epsilon \textit{ and } B_t > \overline{B} - \epsilon \quad i.o\} = 1$$

2. *Suppose $P(s) - P(s') > \beta \frac{g(s)-g(s')}{\underline{B}} \quad \forall s, s'$, then for large enough assets (or debt) there is a drift towards the interior region. In particular the value function $V(B)$ is strictly concave and there exists $B_1 < B_2$ such that*

$$\mathbb{E}V'(B(s)) > V'(B_-) \quad B_- > B_2$$

and

$$\mathbb{E}V'(B(s)) < V'(B_-) \quad B_- < B_1$$

3. *Suppose $P(s) \in \mathcal{P}^*$, then the long run assets converge to a degenerate steady state*

$$\lim_t B_t = B^* \quad a.s \quad \forall B_{-1}$$

In the case where $P(s) \in \mathcal{P}^*$, we can express the long run assets

$$B^* = \beta \frac{\mathrm{var}(g(s))}{\mathrm{cov}(P(s), g(s))} \tag{21}$$

Keeping tax rates (and hence tax revenues in this case) the government needs to finance a higher primary deficit when it gets positive expenditure shock. If in such states the assets pays

[9] Thus the limiting allocation is a particular Lucas Stokey (1983) with stationary debt and taxes, however the level and sign of the long run debt is determined by the joint properties of shocks and payoffs rather than initial condition as would be the case in Lucas Stokey(1983)

off more, then optimally the government holds positive assets and uses the these high returns to finance this deficit. On the other hand if payoff are lower in times when the government needs resources, holding debt is valuable since it lowers the interest burden. Thus using the level of its assets B^* it can perfectly span the fluctuations in deficits and the sign is given by the sign of the covariance of $P(s)$ with $g(s)$.

The long run tax rate is inversely related to B^* with the following limits,

$$\lim_{B^* \to \underline{B}} \tau^* = \frac{\gamma}{1+\gamma} \qquad \lim_{B^* \to \infty} \tau^* = -\infty$$

Now we proceed to obtain a sharper characterization of the invariant distribution. In general, there is no closed form solution for law of motion of government debt and hence we resort to an approximation. We begin with a orthogonal decomposition for an arbitrary $P(s)$,

$$P(s) = \hat{P}(s) + P^*(s)$$

where $P^*(s) \in \mathcal{P}^*$ and $\hat{P}(s)$ is orthogonal to $g(s)$. Expanding the policy rules around the steady state of the $P^*(s)$ economy we have the next theorem.[10]

Theorem 3 *The ergodic distribution of debt (using the first order approximation of dynamics near $P^*(s)$) has the following properties,*

- ***Mean:*** *The ergodic mean is B^* which corresponds to the steady state level of debt of an economy with payoff vector $P^*(s)$*

- ***Variance:*** *The coefficient of variation is given by*

$$\frac{\sigma(B)}{\mathbb{E}(B)} = \sqrt{\frac{\text{var}(P(s)) - |\text{cov}(g(s), P(s))|}{(1 + |\text{cov}(g(s), P(s))|)|\text{cov}(g(s), P(s))|}} \leq \sqrt{\frac{\text{var}(\hat{P}(s))}{\text{var}(P^*(s))}}$$

- ***Convergence rate:*** *The speed of convergence to the ergodic distribution described by*

$$\frac{\mathbb{E}_{t-1}(B_t - B^*)}{(B_{t-1} - B^*)} = \frac{1}{1 + |\text{cov}(P(s), g(s))|}$$

Notice that when payoffs are equal to $P^*(s)$, the government can keep taxes constant and perfectly offset the fluctuations in its surplus with returns $P^*(s)B^*$. Away from this, the incompleteness of markets is binding and shocks are hedged with a combination of changes in tax rates and debt levels. These theorem shows exactly how the deviations from perfect spanning map into larger variances for debt (and taxes) in the long run. Figure I shows how the ergodic distribution of debt and taxes spread as we vary the covariance $P(s)$ with $g(s)$.

[10] Formally $P^*(s)$ is obtained by projecting $P(s)$ on the space spanned by $\mathcal{P}^*$. These approximations differ from

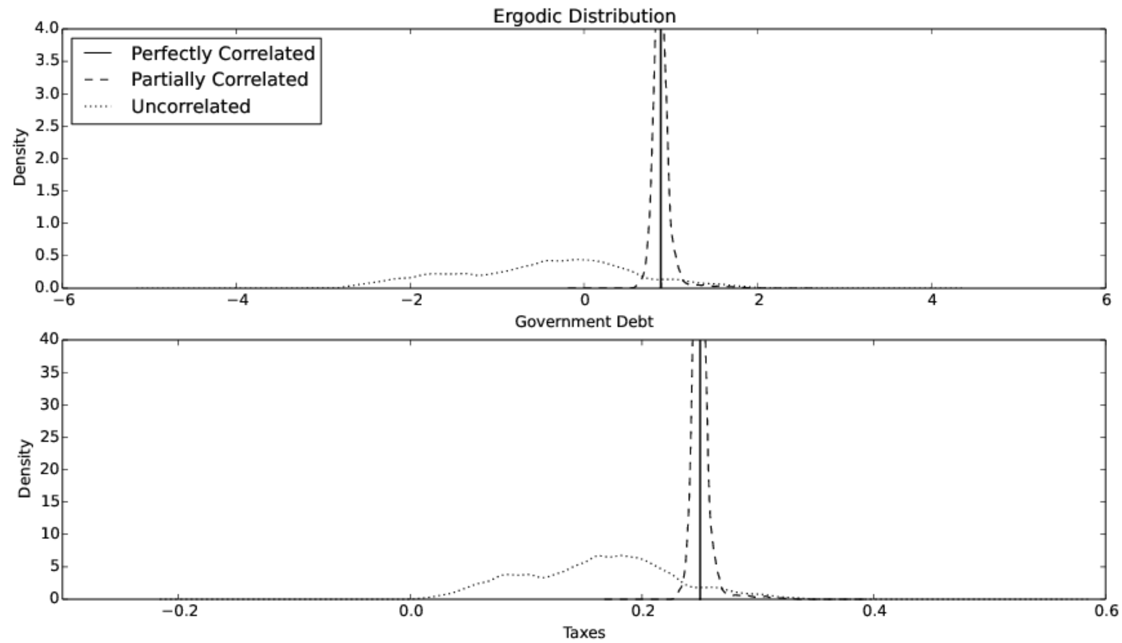

Figure I: Ergodic distribution for debt and taxes in the representative agent quasilinear economy for three choices $P(s)$.

5.2 Heterogeneous agents

In this section we turn to general problem of characterizing outcomes in an economy with heterogeneous agents and transfers. In particular we introduce a second agent who has zero productivity and impose a non negativity constraint on his consumption. Given the Ricardian equivalnce result discussed in section **??**, we maintain a normalization that assets of the unproductive agent are zero throughout this section.

Assumption 3 *The productivity of agents are ordered,* $\theta_1 > \theta_2 = 0$ *and* $c_{2,t} \geq 0$.

Before we discuss the results, a few words on the assumptions are pertinent. The assumption that $\theta_2 = 0$ makes the problem tractable and allows us to obtain a complete characterization of the problem as we wary the Pareto weights. The restriction on consumption is necessary to add curvature to the problem. In more general settings risk aversion will impose Inada restrictions that play a similar role. We now state the theorem and then discuss its implications.

those obtained in Woodford (XXX) where the point of approximation is a deterministic steady state. Appendix **??** contains more details of the approximation method including comparing outcomes to those obtained numerically solving the same economy.

Theorem 4 *Let ω, n be the Pareto weight and mass of the productive agent with $n < \frac{\gamma}{1+\gamma}$. The optimal tax, transfer and asset policies $\{\tau_t, T_t, B_t\}$ are characterized as follows,*

1. *For $\omega \geq n\left(\frac{1+\gamma}{\gamma}\right)$ we have $T_t = 0$ and the optimal policy is same as in a representative agent economy studied in Theorems 2,and 3*

2. *For $\omega < n\left(\frac{1+\gamma}{\gamma}\right)$, suppose we further assume that $\min_s\{P(s)\} > \beta$. We have two parts: There exists $\mathcal{B}(\omega)$ and $\tau^*(\omega)$ with $\mathcal{B}'(\omega) > 0$ and $\lim_{\omega\to 0}\mathcal{B}(\omega) < 0$ such that*

   (a) $B_- > \mathcal{B}(\omega)$

   $$T_t > 0, \quad \tau_t = \tau^*(\omega), \text{ and } B_t = B_- \quad \forall t$$

   (b) $B_- \leq \mathcal{B}(\omega)$, the policies depend on the structure of $P(s)$.

      i. For $P(s) \notin \mathcal{P}^*$

      $$\lim_t T_t > 0 \text{ i.o.}, \quad \lim_t \tau_t = \tau^*(\omega) \text{ and } \lim_t B_t = \mathcal{B}(\omega) \quad a.s$$

      ii. For $P(s) \in \mathcal{P}^*$we have two cases depending on B_-

         A. For $B_- \leq B^*$

         $$T_t = 0, \quad \lim_t \tau_t = \tau^{**}(\omega), \text{ and } \lim_t B_t = B^* \quad a.s$$

         B. For $\mathcal{B}(\omega) > B_- > B^*$

         $$\Pr\{\lim_t T_t = 0, \lim_t \tau_t = \tau^{**}(\omega), \lim_t B_t = B^* \text{ or } \lim_t T_t > 0 \text{ i.o.}, \lim_t \tau_t = \tau^*(\omega), \lim_t B_t = \mathcal{B}(\omega)\} > 0$$

The main concern in this setting with heterogeneous agents is that costs of fluctuating transfers to hedge aggregate shocks are endogenous. The simplifications in the environment allow us to highlight how these depend on the Pareto weights (relative to the mass) of the Planner: $\{\omega, 1-\omega\}$ corresponding to Agents 1 and 2 respectively. A regressive planner who cares a lot about the productive agents in effect faces high costs of using transfers. For such a planner (with a high ω), increasing transfers also means giving resources to the unproductive agent whose consumption he does not value as much. In fact the threshold $\bar{\omega} = n\left(\frac{1+\gamma}{\gamma}\right)$ is such that above this, transfers are never used and thus the allocations are identical to the representative agent economy studied before.

For a less regressive planner (such that $\omega < \bar{\omega}$) transfers are an important tool for redistributing resources to the unproductive agent in adverse times. To finance these these it taxes the productive agent and does not need to accumulates a large buffer stock of assets. Thus limiting assets are lower and tax rates are larger for more redistributive planners.

6 More general economies

The analysis in the previous section was simplified in many dimensions - no curvature on the utility from consumption, lack of persistence in shocks and restricting heterogeneity to two agents. The key implication we could thereby exploit was that the return on debt was exogenous, given by $\beta^{-1}P(s)$. Adding curvature makes these returns endogenous even for standard risk free bond with a payoff vector $P(s) = 1$. Technically, it requires us to additionally keep track relative marginal utilities to characterize how taxes, debt, and transfers react to shocks. This makes it harder to separate spanning and redistribution concerns.

In the next section we first show that with risk aversion, exact spanning can occur only if shocks are binary and IID. For instance with CES preferences, the limiting allocation has constant relative consumptions and taxes rates. We show that how similar to the quasilinear case, the co movement of interest rates and exogenous shocks govern the governments' incentives to accumulate assets.

6.1 Spanning with binary shocks

Let $\Psi(s;\boldsymbol{x},\boldsymbol{\rho},s_{\text{-}})$ be an optimal law of motion for the state variables for the $t \geq 1$ recursive problem, i.e., $\Psi(s;\boldsymbol{x},\boldsymbol{\rho},s_{\text{-}}) = (x'(s), \rho'(s))$ solves (13) given state $(\boldsymbol{x},\boldsymbol{\rho},s_{\text{-}})$.

Definition 4 *A steady state* $\left(\boldsymbol{x}^{SS},\boldsymbol{\rho}^{SS}\right)$ *satisfies* $\left(\boldsymbol{x}^{SS},\boldsymbol{\rho}^{SS}\right) = \Psi\left(s;\boldsymbol{x}^{SS},\boldsymbol{\rho}^{SS},s_{-}\right)$ *for all* s, s_{-}.

Since in this steady state $\rho_i = U^i_c(s)/U^1_c(s)$ does not depend on the realization of shock s, the ratios of marginal utilities of all agents are constant. The continuation allocation depends only on s_t and not on the history s^{t-1}.

We begin by noting that a competitive equilibrium fixes an allocation $\{c_i(s), l_i(s)\}_i$ given a choice for $\{\tau(s), \boldsymbol{\rho}(s)\}$ using equations (14c), (14d) and (14e). Let us denote $U(\tau,\boldsymbol{\rho},s)$ as the value for the planner from the implied allocation using Pareto weights $\{\omega_i\}_i$,

$$U(\tau,\boldsymbol{\rho},s) = \sum_i \omega_i U^i(s).$$

As before define $Z_i(\tau,\rho,s)$ as

$$Z_i(\tau,\boldsymbol{\rho},s) = U^i_c(s)c_i(s) + U^i_l(s)l_i(s) - \rho_i(s)\left[U^1_c(s)c_1(s) + U^1_l(s)l_1(s)\right].$$

For the IID case, the optimal policy solves the following Bellman equation for $\boldsymbol{x}(s^{t-1}) = \boldsymbol{x}, \boldsymbol{\rho}(s^{t-1}) = \boldsymbol{\rho}$

$$V(\boldsymbol{x},\boldsymbol{\rho}) = \max_{\tau(s),\boldsymbol{\rho}'(s),\boldsymbol{x}'(s)} \sum_s \pi(s)\left[U(\tau(s),\boldsymbol{\rho}'(s),s) + \beta(s)V(\boldsymbol{x}'(s),\boldsymbol{\rho}'(s))\right] \tag{22}$$

subject to the constraints

$$Z_i(\tau(s), \boldsymbol{\rho}'(s), s) + x_i'(s) = \frac{x_i \beta^{-1} P(s) U_c^i(\tau(s), \boldsymbol{\rho}'(s), s)}{\mathbb{E} U_c^i(\tau, \rho)} \text{ for all } s, i \geq 2, \tag{23}$$

$$\sum_s \pi(s) P(s) U_c^1(\tau(s), \boldsymbol{\rho}'(s), s)(\rho_i'(s) - \rho_i) = 0 \text{ for } i \geq 2. \tag{24}$$

Constraint (24) is obtained by rearranging constraint (14b). It implies that $\rho(s)$ is a risk-adjusted martingale. We next check if the first-order necessary conditions are consistent with stationary policies for some $(\boldsymbol{x}, \boldsymbol{\rho})$.[11]

Lemma 1 *With risk aversion $\|S\| = 2$ is necessary for a steady state to exist*

Proof.

Let $\pi(s)\mu_i(s)$ and λ_i be the multipliers on constraints (23) and (24). Imposing the restrictions $x_i'(s) = x_i$ and $\rho_i'(s) = \rho_i$, at a steady state $\{\mu_i, \lambda_i, x_i, \rho_i\}_{i=2}^N$ and $\{\tau(s)\}_s$ are determined by the following equations

$$Z_i(\tau(s), \boldsymbol{\rho}, s) + x_i = \frac{\beta^{-1} P(s) x_i U_c^i(\tau(s), \boldsymbol{\rho}, s)}{\mathbb{E} U_c^i(\tau, \rho)} \text{ for all } s, i \geq 2, \tag{25a}$$

$$U_\tau(\tau(s), \boldsymbol{\rho}, s) - \sum_i \mu_i Z_{i,\tau}(\tau(s), \boldsymbol{\rho}, s) = 0 \text{ for all } s, \tag{25b}$$

$$U_{\rho_i}(\tau(s), \boldsymbol{\rho}, s) - \sum_j \mu_j Z_{j,\rho_i}(\tau(s), \boldsymbol{\rho}, s) + \lambda_i P(s) U_c^1(\tau(s), \boldsymbol{\rho}, s) - \lambda_i \beta \mathbb{E} P(s) U_c^1(\tau(s), \boldsymbol{\rho}(s), s) = 0. \text{ for all } s, i \geq 2 \tag{25c}$$

Since the shock s can take only two values, (25) is a square system in $4(N-1)+2$ unknowns $\{\mu_i^{SS}, \lambda_i^{SS}, x_i^{SS}, \rho_i^{SS}\}_{i=2}^N$ and $\{\tau^{SS}(s)\}_s$. ■

The behavior of the economy in the steady state is similar to the behavior of the complete market economy characterized by Werning (2007). Both taxes and transfers depend only on the current realization of shock s_t. Moreover, the arguments of Werning (2007) can be adapted to show that taxes are constant when preferences have a CES form $c^{1-\sigma}/(1-\sigma) - l^{1+\gamma}/(1-\gamma)$ and fluctuations in tax rates are very small when preferences take forms consistent with the existence of balanced growth. We return to this point after we discuss convergence properties.

The previous calculations provides a simple way to verify existence of a steady state for wide range of parameter values by checking that there exists a root for system (25). Since the system of equations (25) is non-linear, existence can generally be verified only numerically. Next, we provide a simple example with risk averse agents in which we can show existence of the root of (25) analytically. The analytical characterization of the steady state will help us develop some

[11] Appendix A.7 discuses the associated second order conditions that ensure these policies are optimal

comparative statics and build a connection from the quasilinear economy to the quantitative analysis to appear in section 7.

A two-agent example

Consider an economy consisting of two types of households with $\theta_{1,t} > \theta_{2,t} = 0$. One period utilities are $\ln c - \frac{1}{2}l^2$. The shock s takes two values, $s \in \{s_L, s_H\}$ with probabilities $\Pr(s|s_{-})$ that are independent of s_{-}. We assume that $g(s) = g$ for all s, and $\theta_1(s_H) > \theta_1(s_L)$. We allow the discount factor $\beta(s)$ to depend on s.

Theorem 5 *Suppose that $g < \theta(s)$ for all s. Let $R(s)$ be the gross interest rates and $x = U_c^2(s)[b_2(s) - b_1(s)]$*

1. ***Countercyclical interest rates.*** *If $P(s_H) = P(s_L)$, then there exists a steady state (x^{SS}, ρ^{SS}) such that $x^{SS} > 0$, $R^{SS}(s_H) < R^{SS}(s_L)$.*
2. ***Procyclical interest rates.*** *There exists a pair $\{P(s_H), P(s_L)\}$ such that there exists a steady state with $x^{SS} < 0$ and $R^{SS}(s_H) > R^{SS}(s_L)$.*

In both cases, taxes $\tau(s) = \tau^{SS}$ are independent of the realized state.

In this two-agent case, by normalizing assets of the unproductive agent (using theorem 1) we can interpret x as the marginal utility adjusted assets of the government. Besides establishing existence, the proposition identifies the importance of cyclical properties of real interest rates in determining the sign of these assets.

Theorem 5 shows two main forces that determine the dynamics of taxes and assets: fluctuations in inequality and fluctuations in the interest rates. Keeping interest rates fixed for the moment, the government can in principle adjust two instruments in response to an adverse shock (i.e., a fall in θ_1): it can either increase the tax rate τ or it can decrease transfers T. Both responses are distorting, but for different reasons. Increasing the tax rate increases distortions because the deadweight loss is convex in the tax rate, as in Barro (1979). This force operates in our economy just as it does in representative agent economies. But in a heterogeneous agent economy like ours, adjusting transfers T is also costly. When agents' asset holdings are identical, a decrease in transfers disproportionately affects a low-skilled agent, so his marginal utility falls by more than does the marginal utility of a high-skilled agent. Consequently, a decrease in transfers increases inequality, giving rise to a cost not present in representative agent economies.

The government can reduce the costs of inequality distortions by choosing tax rate policies that make the net asset positions of the high-skilled agent decrease over time. That makes the two agents' after-tax and after-interest income become closer, allowing decreases in transfers to have smaller effects on inequality in marginal utilities. If the net asset position of a high-skilled agent is sufficiently low, then a change in transfers has no effect on inequality and all distortions from fluctuations in transfers are eliminated.[12]

Turning now to the second force, interest rates generally fluctuate with shocks. Parts 1 and 2 of proposition 5 indicate what drives those fluctuations. Consider again the example of a decrease in productivity of high-skilled agent. If the tax rate τ is left unchanged, the government faces a shortfall of revenues. Since g is constant, the government requires extra sources of revenues. But suppose that the interest rate increases whenever θ_1 decreases, as happens, for example, with a risk free bond. If the government holds positive assets, its earnings from those assets increase. So holding assets allows higher interest income to offset some of the government's revenue losses from taxes on labor. The situation reverses if interest rates fall at times of increased need for government revenues, as in part 2 of theorem 5, and the steady state allocation features the government's owning debt.

One can see the parallel with the representative agent quasilinear economy studied in section 5. There, exploiting linearity allowed us to provide a sharper characterization of how co-variance of interest rates with exogenous shocks affected the sign (and level) of debt through expression (21). In parts 1 and 2 of theorem 5, with binary shocks, altering the gap $P(s_H) - P(s_L)$ allows us to obtain the corresponding variation in interest rates. The reasoning and the underling forces are exactly the same.

6.2 Stability

In this section we extend the approximation methods used to characterize outcomes in Theorem 3 to the general problem with risk aversion. Unlike of obtaining an the quasilinear case where we could obtain analytical characterization, we present a test for convergence show local stability of a steady state for a wide range of parameters.

As before, let assume that $\pi(s)\mu_i(s)$ and λ_i be the multipliers on constraints (23) and (24). In Appendix A.7 we show that the history-dependent optimal policies (they are sequences of functions of s^t) can be represented recursively in terms of $\{\boldsymbol{\mu}(s^{t-1}), \boldsymbol{\rho}(s^{t-1})\}$ and s_t. A recursive representation of an optimal policy can be linearized around the steady state using $(\boldsymbol{\mu}, \boldsymbol{\rho})$ as state variables.[13]

[12] This convergence outcome has a similar flavor to "back-loading" results of Ray (2002) and Albanesi and Armenter (2012) that reflect the optimality of structuring policies intertemporally eventually to disarm distortions.

[13] One could in principle look for a solution in state variables $\left(\boldsymbol{x}(s^{t-1}), \boldsymbol{\rho}(s^{t-1})\right)$. For $I = 2$ with $\{\theta_i(s)\}$ different across agents, this would give identical policies and a map which is (locally) invertible between $\boldsymbol{x}$ and $\boldsymbol{\mu}$ for a given $\boldsymbol{\rho}$. However in other cases, it turns out there are unique linear policies in ($bm\mu, \boldsymbol{\rho}$) and not necessarily in

Formally, let $\hat{\Psi}_t = \begin{bmatrix} \boldsymbol{\mu}_t - \boldsymbol{\mu}^{SS} \\ \boldsymbol{\rho}_t - \boldsymbol{\rho}^{SS} \end{bmatrix}$ be deviations from a steady state. From a linear approximation, one can obtain $B(s)$ such that

$$\hat{\Psi}_{t+1} = B(s_{t+1})\hat{\Psi}_t. \tag{26}$$

This linearized system has coefficients that are functions of the shock. The next theorem describes a simple numerical test that allows us to determine whether this linear system converges to zero in probability.

Theorem 6 *If the (real part) of eigenvalues of $\mathbb{E}B(s)$ are less than 1, system (26) converges to zero in mean. Further for large t, the conditional variance of $\hat{\Psi}$, denoted by $\Sigma_{\Psi,t}$, follows a deterministic process governed by*

$$vec(\Sigma_{\Psi,t}) = \hat{B}vec(\Sigma_{\Psi,t-1}),$$

where $\hat{B}$ is a square matrix of dimension $(2I-2)^2$. In addition, if the (real part) of eigenvalues of $\hat{B}$ are less than 1, the system converges in probability.

The eigenvalues (in particular the largest or the dominant one) are instructive not only for whether the system is locally stable but also how quickly the steady state is reached. In particular, the half-life of convergence to the steady state is given by $\frac{\log(0.5)}{\|\iota\|}$, where $\|\iota\|$ is the absolute value of the dominant eigenvalue. Thus, the closer the dominant eigenvalue is to one, the slower is the speed of convergence.

We used Theorem 6 to verify local stability of a wide range of examples. Since the parameters space is high dimensional we relegate the comparative statics to Appendix A.8 typical finding is that the steady state is stable and that convergence is slow. The rates of convergence are increasing in the covariance of interest rates and governments needs for revenue.

7 Numerical example

In sections 5 and 6 we used steady states to characterize the long-run behavior of optimal allocations and forces that guide the asymptotic level of net assets. In this section, we use a calibrated version of the economy to a) revisit the magnitude of these forces and and b) study optimal policy responses at business cycle frequencies when the economy is possibly far away from the steady state. We choose shocks and initial conditions to match stylized facts about

$(\boldsymbol{x}, \boldsymbol{\rho})$. This comes from the fact that the set of feasible $(\boldsymbol{x}, \boldsymbol{\rho})$ are restricted at time 0 and may not contain an open set around the steady state values. When we linearize using $(\boldsymbol{\mu}, \boldsymbol{\rho})$ as state variables, the optimal policies for $\boldsymbol{x}(s^t), \boldsymbol{\rho}(s^t)$ converge to their steady state levels for all perturbations in $(\boldsymbol{\mu}, \boldsymbol{\rho})$.

recent recession in US. The numerical calculations use methods adapted from Evans (2014) and described more in the Appendix **??**. The next section outlines how we choose the parameters and initial conditions.

7.1 Calibration

We start with five types of agents[14] of equal measures with preferences $u(c,l) = \frac{c^{1-\sigma}}{1-\sigma} - \frac{l^{1+\gamma}}{1+\gamma}$. These agents will represent the $90^{\text{th}}, 75^{\text{th}}, 50^{\text{th}}, 25^{\text{th}}$, and 10^{th} percentile of US wage distribution.

We assume i.i.d aggregate shocks ϵ_t that affect labor productivities of each agent $\{\theta_{i,t}\}_{i=1}^{N}$ and payoff of the asset p_t as follows,

$$\log \theta_{i,t} = \bar{\theta}_i \epsilon_t [1 + (.9 - i)m]$$

$$p_t = 1 + \chi \epsilon_t$$

Following Autor et al. (2008), the average productivities $\{\bar{\theta}_i\}_{i=1}^{N}$ are set to match the respective earnings using Current Population Survey (CPS) that reports weekly earnings of full time wage and salary earners.

The parameter m allows us to generate recessions that lead to heterogeneous falls in income for different agents. We calibrate m to match the facts reported Guvenen et al. (2014b). Figure 7.1 (adapted from their paper) reports that in the last recession the fall in income for the agents in the first decile of earnings was about three times that of the 90th percentile. Moreover between the 10th and the 90th change in the percentage drop in earnings was almost linear. This gives us a slope $m = \frac{1.5}{0.8}$.

The parameter χ captures the ex-post co-movement in returns to government holdings and aggregate shocks. Our model takes no explicit stand where these come from. In principle they could capture variations in payoffs due inflation, interest rate risk (for longer maturity bonds) or defaults in some countries. For the purpose of the numerical exercise we use US data on inflation and interest rates of longer maturities bonds to calibrate χ. This is described below:

Let $q_t^{(n)}$ be the log price of a nominal bond of maturity n. We can define the real holding period returns $r_{t,t+1}^{(n)}$ as follows

$$r_{t,t+1}^{(n)} = q_{t+1}^{(n-1)} - q_t^{(n)} - \pi_{t+1}$$

[14] We report the results for $N = 5$ to capture sufficient heterogeneity in wealth and earnings. Our methods allow us to solve for arbitrary number of agents and the qualitative/quantitative insights are unchanged by adding more intermediate types.

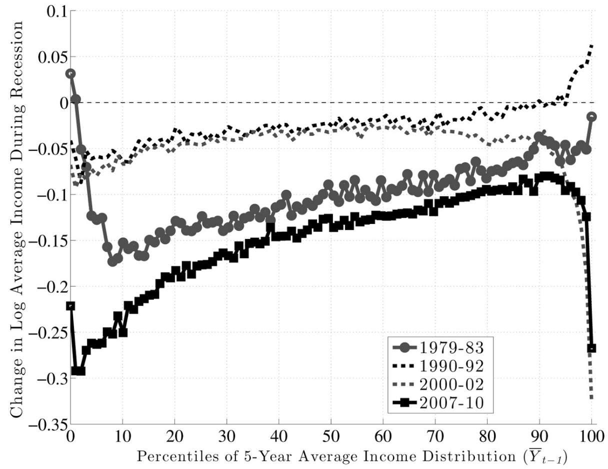

Figure II: Change in log average earnings during recessions, prime-age males from Guvenen et al. (2014b)

With the transfromation $y_t^{(n)} : -\frac{1}{n}q_t^{(n)}$ we can express $r_{t,t+1}^{(n)}$ as follows:

$$r_{t,t+1}^{(n)} = \underbrace{y_t^{(n)}}_{\text{Ex-ante part}} -(n-1)\left[\underbrace{\left(y_{t+1}^{(n)} - y_t^{(n)}\right)}_{\text{Interest rate risk given } n} + \underbrace{\left(y_{t+1}^{(n-1)} - y_{t+1}^{(n)}\right)}_{\text{Term structure risk}}\right] - \underbrace{\pi_{t+1}}_{\text{Inflation risk}}$$

In our model the holding period returns are given by $\log\left[\frac{p_{t+1}}{q_t}\right]$ and $q_t = \frac{\beta \mathbb{E}_t u_{c,t+1} P_{t+1}}{u_{c,t}}$. Note that p_{t+1} allows us to captures ex-post fluctuations in returns to the government's debt portfolio coming from maturity and inflation.

The next table summarizes the co-movement between labor productivity $\{\epsilon_t\}$ and bond prices $\{q_t^n\}_n$ for different maturities (using quarterly data for the period XXXX to XXXX)

Maturity (n)	2yr	3yr	4yr	5yr
$Corr(\epsilon_{t+1}, r_{t,t+1}^{(n)})$	-0.11	-0.093	-0.083	-0.072
$Corr(\epsilon_{t+1}, r_{t,t+1}^{(n)} - ny_t^{(n)})$	0.00	-0.0463	-0.080	-0.091
$Corr(\epsilon_{t+1}, y_t^{(n)} - \pi_{t+1})$	-0.097	-0.086	-0.080	-0.073
$\frac{\sigma(r_{t+1}^n)}{\sigma(\epsilon_{t+1})}$	0.820	0.835	0.843	0.845

Table I:

The first line computes the correlation between the ex post returns and labor productivities. In our baseline calibration, ϵ_t is i.i.d over time. Hence the parameter $\chi = \frac{\sigma_r}{\sigma_\epsilon} Corr(r, \epsilon)$. Averaging over different maturities gives us a value of $\chi = -0.06$. [15] Thus payoffs are weakly

[15]The second line of table III computes only the correltaion of labor productivity with ex-post component of returns in the data. For the shortest maturity, 3 month real tbill returns $Corr(\epsilon_{t+1}, y_t^{1qtr} - \pi_{t+1}) = -0.11$. These

countercyclical. Besides the results for the benchmark value of χ, the long run simulations in the next section we will report the results a large range of χ's from -1.0 to 1.0.

We next turn to the parameters that describe the household preferences : We set risk aversion, $\sigma = 1$, $\gamma = 2$. This yields a Frisch elasticity of labor supply of 0.5 and time discount factor, $\beta = 0.98$ such that the annual interest rate in an economy without shocks is 2% per year.

We assume that the initial wealth is perfectly correlated with wages and calibrate the wealth distribution to get the relative quantiles as in Kuh (2014a) and Quadrini and Rios-Rull [2014]. These papers document the quantiles of net worth for US households computed up to 2010 Survey of Consumer Finances.

For the rest of parameters, namely Pareto weights and government expenditure, we use outcomes of the optimal allocation in the economy without shocks to target a (pre-trasnfers, federal) expenditure output ratio of 12%, tax rate of 23%, transfers to gdp of 10% and debt to gdp of 100%.

Parameter	Value	Description
$\{\bar{\theta}_i\}$	{4.9,3.24,2.1,1.4,1}	Wages dispersion for {90,75,50,25,10} percentiles
γ	2	Average Frisch elasticity of labor supply of 0.5
β	0.98	Average (annual) risk free interest rate of 2%
m	$\frac{1.5}{.8}$	Heterogeneity in wage growth over business cycles
χ	-0.06	Covariance between holding period returns and labor productivity%
σ_e	0.03	vol of labor productivity
g	.13 %	Average pre-transfer expenditure- output ratio of 12 %

Table II: Benchmark calibration

results together give us range for χ of zero to negative -0.09. The numerical results are not sensitive to the value of χ is this range.

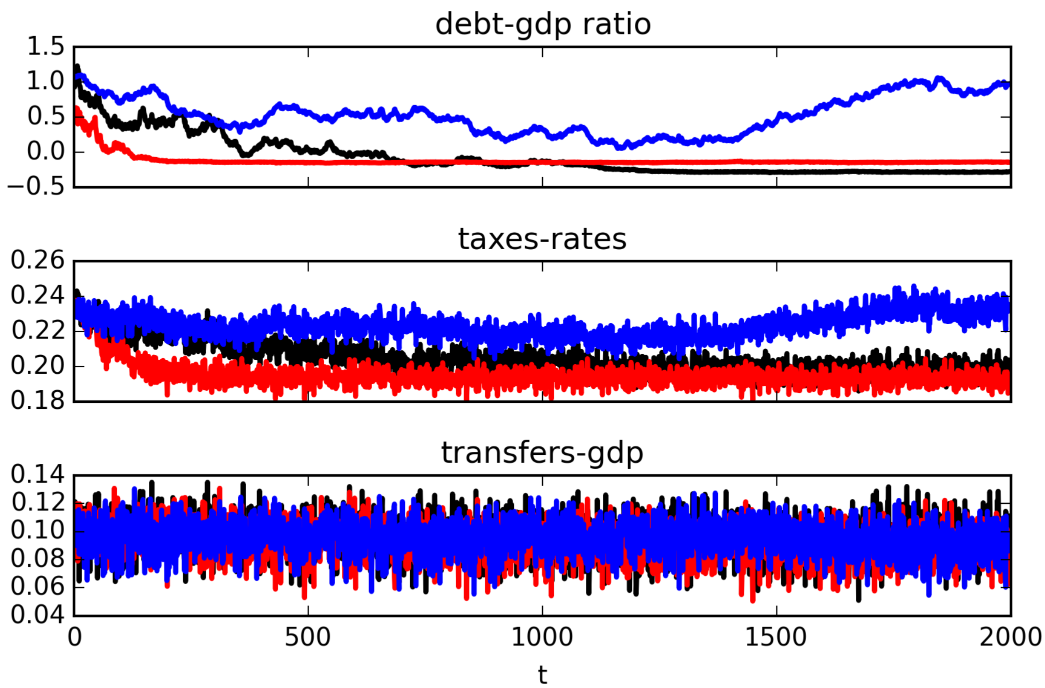

Figure III: The red, black and blue lines plot simulations for a common sequence of shocks for values of $\chi = -1.5, 0, 1.5$ respectively

7.2 Long run outcomes

Figure 7.2 plots the simulated paths (of length 2000 periods) for debt to output ratio, labor tax rates and transfers to output ratio for three values of $\chi \in \{-1.0, -0.06, 1.0\}$ in red, black and blue respectively. The three simulations have same initial conditions and sequence of underlying shocks.

Two features emerge: Different values χ have implications for position and speed of convergence for long run assets of the government. A sufficiently positive χ generates lower payoffs in recessions relative to booms. In line with theorem 2 or theorem 5 , we see (blue line) that that government does not repay its initial debt for 2000 periods. On the other had under the benchmark (black line) or the when χ is negative (red line), the government accumulates assets.

In order to get a clearer picture of the speed of convergence, we plot the paths of the conditional means for debt and taxes in figure 7.2. To explain how we generate these plots, let $B(s_{t+1}, \boldsymbol{x}_t, \boldsymbol{\rho}_t)$ be the policy rules that generate the assets of the government and $\Psi\left(s_{t+1}; \boldsymbol{x}_t, \boldsymbol{\rho}_t\right)$, the law of motion for the state variables. For a given history, the conditional mean of government assets is defined as follows:

$$B_{t+1}^{cm} = \mathbb{E}B(s_{t+1}, \boldsymbol{x}_t^{cm}, \boldsymbol{\rho}_t^{cm}) \tag{28a}$$

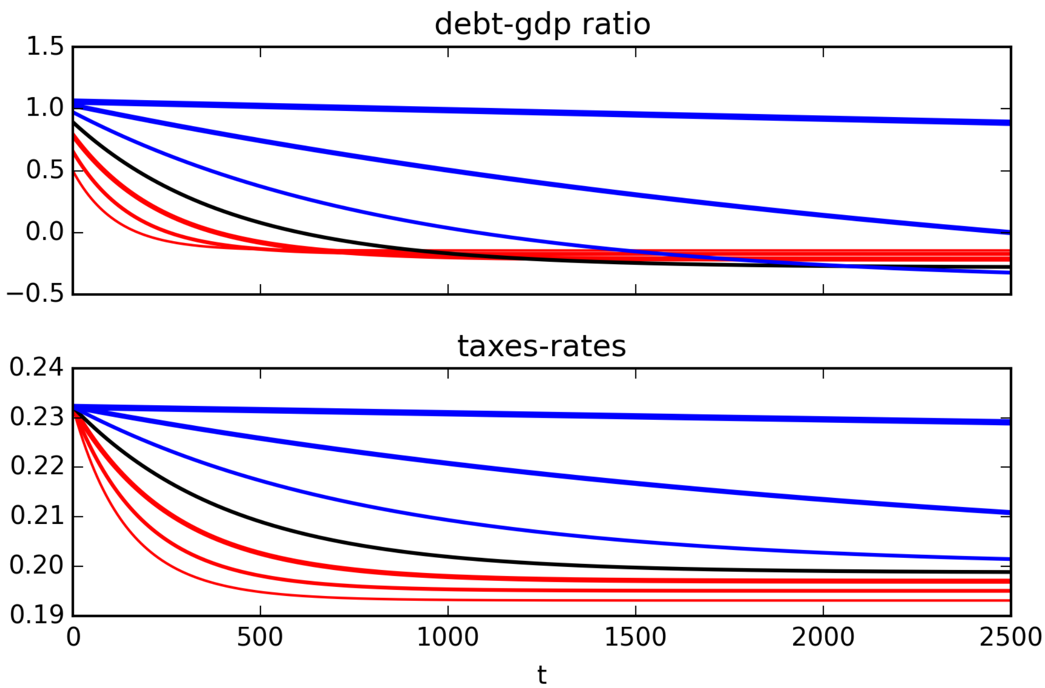

Figure IV: The plot shows conditional mean paths for different values of χ. The red (blue) lines have $\chi < 0$ ($\chi > 0$). The thicker lines represent larger values.

$$\boldsymbol{x}_t^{cm}, \boldsymbol{\rho}_t^{cm} = \mathbb{E}\Psi(s_t, \boldsymbol{x}_{t-1}^{cm}, \boldsymbol{\rho}_{t-1}^{cm}) \tag{28b}$$

Note that these paths smooths the high frequency movements in the dynamics of the state variables but retain the low frequency drifts. The different lines as before represent different values of χ between -1.0 and 1.0 with the blue (red) lines representing positive (negative) values of χ. The thickness of the lines represent larger values of χ. The figure clearly shows the speed of convergence is increasing and the magnitude of the limiting assets in decreasing the strength of correlation between productivities and payoffs. This confirms the approximation results characterized in theorem 3.

To verify the wide support of the ergodic distribution we take the initial conditions at the end of the long simulation and subject the economy to a sequence of 100 periods of ϵ_t shocks which are 2 standard deviations below the mean. In figure 7.2 we see that given a sufficiently long sequence of negative productivity shocks the economy will eventually deviate significantly from its ergodic mean.

Our last conclusion was that the assets held by the government in the steady state was decreasing in the re-distributive motive of the government. We check this last part by changing the Pareto weights of the government. In our baseline case the government places equal Pareto

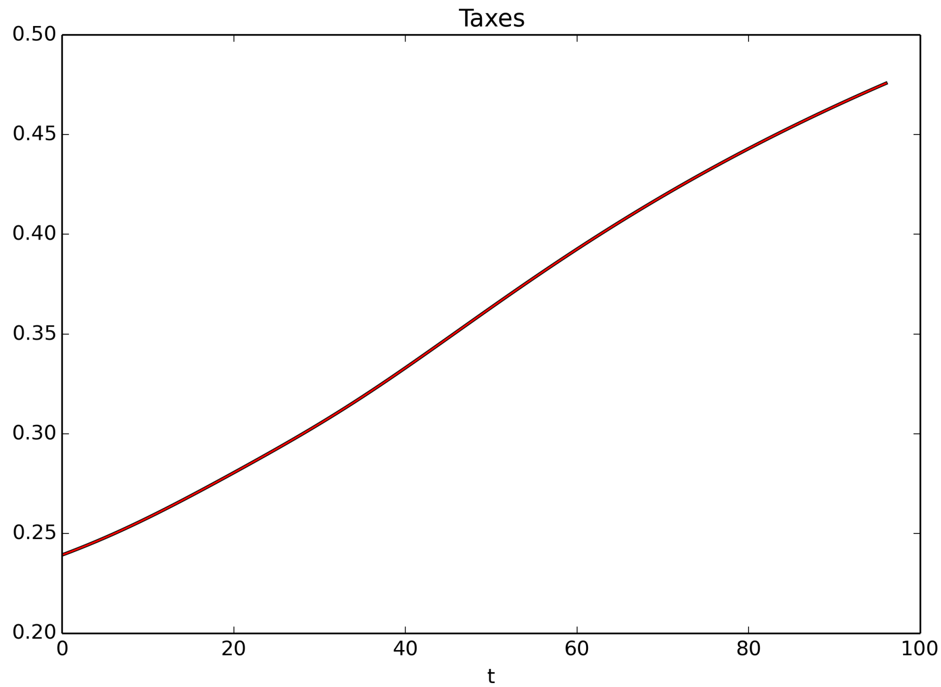

Figure V: Taxes for a sequence of -1 s.d shocks to aggregate productivity of length 100

weights on all agents. We introduce a re-distributive motive through a parameter α. The planner places evenly spaced Pareto weights from $0.2-\alpha$ on the lowest productivity agent to $0.2+\alpha$ on the highest productivity agent. Increases α decreases the concerns for redistribution. We plot total assets of the government in steady state as a function of alpha in figure **??** and see that the relationship does indeed hold.

7.3 Short run

The analysis of the previous subsection studied aspects of very low frequency components of the optimal policy. Here we focus on business cycle frequencies. In our setting, these higher frequency responses can conveniently be divided into the magnitudes of changes as we switch from "boom" to "recession," and the dynamics during periods when a recession or boom state persists. A recession is a negative -1.0 standard deviation realization for the ϵ_t process. Given the initial conditions and the benchmark calibration, the plots below trace the paths for debt, taxes and transfers for sequence of shocks that feature a recession of four periods from $t=3$. Before and after this recession, the economy receives $\epsilon_t = 0$.

The main exercise here is to compute how optimal taxes, transfers and debt in recessions accompanied by larger inequality are different in a recession that affects all agents alike. Under

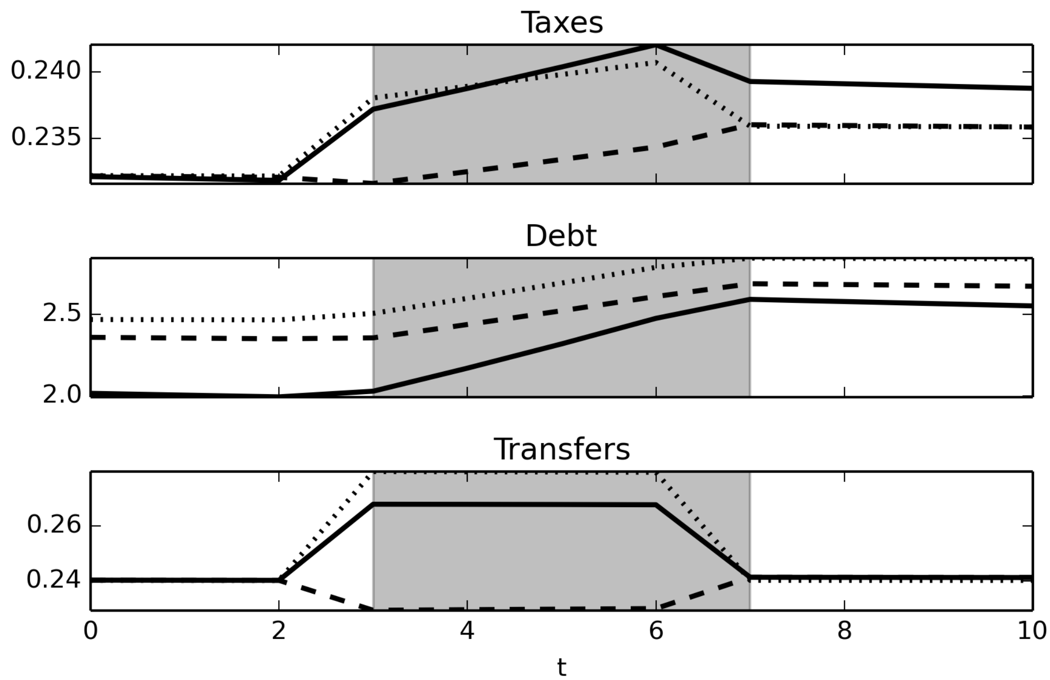

Figure VI: The bold line is the total response. The dashed (dotted) line reflects the only TFP (inequality) effect. The shaded region is the recession

the benchmark calibration, log wages for agent i are given by $\log \theta_i = \epsilon[1 + (.9 - i)m]$. We decompose the total responses into an only TFP component by (by setting $m = 0$) and an inequality component as follows:

$$\log \theta_i^{tfp} = \epsilon$$

$$\log \theta_i^{ineq} = \epsilon[(.9 - i)m]$$

Figure 7.3 plots the short run impulse responses. The shaded region is the induced recession and the bold line captures the the benchmark (total) response. The dashed (dotted) line reflects the only TFP (inequality) effect. In the benchmark, the government responds to an adverse shock by a making big increases in transfers, the tax rate, and government debt. However, without inequality shocks (dotted line), the government responds by decreasing transfers and increasing both debt and the tax rate, but by an amount an order of magnitude smaller than in the benchmark.

Next we average over sample paths of length 100 periods and report the volatility, autocorrelation and correlation with exogenous shocks for taxes and transfers in table III. We see that taxes are twice as volatile and correlation between transfers and productivities switches sign. This indicates that ignoring distributional goals can produce a misleading prescriptions for government policy in recessions.

Moments	Tfp	Tfp+Ineq
vol. of taxes	0.003	0.006
vol. of transfers	0.01	0.02
autocorr. in taxes	0.93	0.66
autocorr. in transfers	0.17	0.18
corr. of taxes with tfp	0.15	-0.63
corr. of transfers with tfp	0.99	-0.98

Table III: Sample moments for taxes and transfers averaged across simulations of 100 periods

8 Conclusion

A Appendix

A.1 Extension: Borrowing constraints

Representative agent models rule out Ricardian equivalence either by assuming distorting taxes or by imposing ad hoc borrowing constraints. By way of contrast, we have verified that Ricardian equivalence holds in our economy even though there are distorting taxes. Imposing ad-hoc borrowing limits also leaves Ricardian equivalence intact in our economy.[16] In economies with exogenous borrowing constraints, agents' maximization problems include the additional constraints

$$b_{i,t} \geq \underline{b}_i \tag{29}$$

for some exogenously given $\{\underline{b}_i\}_i$.

Definition 5 *For given $\left(\{b_{i,-1}, \underline{b}_i\}_i, B_{-1}\right)$ and $\{\tau_t, T_t\}_t$, a competitive equilibrium with affine taxes and exogenous borrowing constraints is a sequence $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, B_t, R_t\}_t$ such that $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ maximizes (1) subject to (4) and (29), $\{b_{i,t}\}_{i,t}$ are bounded, and constraints (**??**), (5) and (3) are satisfied.*

We can define an *optimal* competitive equilibrium with exogenous borrowing constraints by extending Definition 3.

[16]Bryant and Wallace (1984) describe how a government can use borrowing constraints as part of a welfare-improving policy to finance exogenous government expenditures. Sargent and Smith (1987) describe Modigliani-Miller theorems for government finance in a collection of economies in which borrowing constraints on classes of agents produce the kind of rate of return discrepancies that Bryant and Wallace manipulate.

The introduction of the ad-hoc debt limits leaves unaltered the conclusions of Corollary 1 and the role of the initial distribution of assets across agents. The next theorem asserts that ad-hoc borrowing limits do not limit a government's ability to respond to aggregate shocks.[17]

Theorem 7 *Given an initial asset distribution* $\left(\{b_{i,-1}\}_i, B_{-1}\right)$*, let* $\{c_{i,t}, l_{i,t}\}_{i,t}$ *and* $\{R_t\}_t$ *be a competitive equilibrium allocation and interest rate sequence in an economy without exogenous borrowing constraints. Then for any exogenous constraints* $\{\underline{b}_i\}_i$*, there is a government tax policy* $\{\tau_t, T_t\}_t$ *such that* $\{c_{i,t}, l_{i,t}\}_{i,t}$ *is a competitive equilibrium allocation in an economy with exogenous borrowing constraints* $\left(\{b_{i,-1}, \underline{b}_i\}_i, B_{-1}\right)$ *and* $\{\tau_t, T_t\}_t$*.*

Proof. Let $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ be a competitive equilibrium allocation without exogenous borrowing constraints. Let $\Delta_t \equiv \max_i \{\underline{b}_i - b_{i,t}\}$. Define $\hat{b}_{i,t} \equiv b_{i,t} + \Delta_t$ for all $t \geq 0$ and $\hat{b}_{i,-1} = b_{-1}$. By Theorem 1, $\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_{i,t}$ is also a competitive equilibrium allocation without exogenous borrowing constraints. Moreover, by construction $\hat{b}_{i,t} - \underline{b}_i = b_{i,t} + \Delta_t - \underline{b}_i \geq 0$. Therefore, $\hat{b}_{i,t}$ satisfies (29). Since agents' budget sets are smaller in the economy with exogenous borrowing constraints, and $\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_{i,t}$ are feasible at interest rate process $\{R_t\}_t$, then $\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_{i,t}$ is also an optimal choice for agents in the economy with exogenous borrowing constraints $\{\underline{b}_i\}_i$. Since all market clearing conditions are satisfied, $\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_{i,t}$ is a competitive equilibrium allocation and asset profile. ■

To provide some intuition for Theorem 7, suppose to the contrary that the exogenous borrowing constraints restricted a government's ability to achieve a desired allocation. That means that the government would want to increase its borrowing and to repay agents later, which the borrowing constraints prevent. But the government can just reduce transfers today and increase them tomorrow. That would achieve the desired allocation without violating the exogenous borrowing constraints.

Welfare can be strictly higher in an economy with exogenous borrowing constraints relative to an economy without borrowing constraints because a government might want to push some agents against their borrowing limits. When agents' borrowing constraints bind, their shadow interest rates differ from the common interest rate that unconstrained agents face. When the government rearranges tax policies to affect the interest rate, it affects constrained and unconstrained agents differently. By facilitating redistribution, this can improve welfare. We next construct an example without any shocks in which the government can achieve higher welfare by using borrowing constraints to improve its ability to redistribute. In this section we construct an example in which the government can achieve higher welfare in the economy with ad-hoc

[17] See Yared (2012, 2013) who shows a closely related result.

borrowing limits. We restrict ourselves to a deterministic economy with $g_t = 0$, $\beta_t = \beta$ and $I = 2$. Further the utility function over consumption and labor supply $U(c, l)$ is separable in the arguments and satisfies the Inada conditions. The planners problem can then be written as the following sequence problem

$$\max_{\{c_{i,t}, l_{i,t}, b_{i,t}, R_t\}_t} \sum_{t=0}^{\infty} \beta^t \left[\alpha_1 U(c_{1,t}, l_{1,t}) + \alpha_2 U(c_{2,t}, l_{2,t})\right] \tag{30}$$

subject to

$$c_{2,t} + \frac{U_{l2,t} l_{2,t}}{U_{c2,t}} - \left(c_{1,t} + \frac{U_{l1,t} l_{1,t}}{U_{c1,t}}\right) + \frac{1}{R_t}(b_{2,t} - b_{1,t}) = b_{2,t-1} - b_{1,t-1} \tag{31a}$$

$$\frac{U_{l1,t}}{\theta_1 U_{c1,t}} = \frac{U_{l2,t}}{\theta_2 U_{c2,t}} \tag{31b}$$

$$c_{1,t} + c_{2,t} \leq \theta_1 l_{1,t} + \theta_2 l_{2,t} \tag{31c}$$

$$\left(\frac{U_{ci,t}}{U_{ci,t+1}} - \beta R_t\right)(b_{i,t} - \underline{b}_i) = 0 \tag{31d}$$

$$\frac{U_{ci,t}}{U_{ci,t+1}} \geq \beta R_t \tag{31e}$$

$$b_{i,t} \geq \underline{b}_i \tag{31f}$$

Where $\underline{b}_i$ is the exogenous borrowing constraint for agent i. We obtain equation (31a) by eliminating transfers from the budget equations of the households and using the optimality for labor supply decision. Equations (31d) and (31e) capture the inter-temporal optimality conditions modified for possbily binding constraints.

Let c_i^{fb} and l_i^{fb} be the allocation that solves the first best problem, that is maximizing equation (30) subject to (31c), and define

$$Z^{fb} = c_2^{fb} + \frac{U_{l2}^{fb} l_2^{fb}}{U_{c2}^{fb}} - \left(c_1^{fb} + \frac{U_{l1}^{fb} l_1^{fb}}{U_{c1}^{fb}}\right) \tag{32}$$

and

$$\tilde{b}_2^{fb} = \frac{Z^{fb}}{\frac{1}{\beta} - 1} \tag{33}$$

We will assume that the exogenous borrowing constraints satisfy $\underline{b}_2 = \underline{b}_1 + \tilde{b}_2^{fb}$. We then have the following lemma

Lemma 2 *If $\tilde{b}_2^{fb} > (<)0$ and $b_{2,-1} - b_{1,-1} > (<)\tilde{b}_2^{fb}$ then the planner can implement the first best.*

Proof. We will consider the candidate allocation where $c_{i,t} = c_i^{fb}$, $l_{i,t} = l_i^{fb}$, $b_{i,t} = \underline{b}_i$ and interest rates are given by $R_t = \frac{1}{\beta}$ for $t \geq 1$. It should be clear then that equations (31b) and (31c) are satisfied as a property of the first best allocation. Equation (31d) is trivially satisfied since the agents are at their borrowing constraints. For $t \geq 1$ equations (31a) and (31e) are both satisfied by the choice of $R_t = \frac{1}{\beta}$ and the first best allocations. It remains to check that equation (31a) is satisfied at time $t = 0$ for an interest rate $R_0 < \frac{1}{\beta}$. At time zero the constraint is give by

$$Z^{fb} + \frac{1}{R_0}\tilde{b}_2^{fb} = b_{2,-1} - b_{1,-1} \tag{34}$$

The assumption that $b_{2,-1} - b_{1,-1} > (<)\tilde{b}_2^{fb}$ if $\tilde{b}_2^{fb} > (<)0$ then implies that

$$R_0 = \frac{\tilde{b}_2^{fb}}{b_{2,-1} - b_{1,-1} - Z^{fb}} < \frac{1}{\beta}$$

as desired. ■

This will improve upon the planners problem without exogenous borrowing constraints, as first best can only be achieved in this scenario when $b_{2,-1} - b_{1,-1} = \tilde{b}_2^{fb}$.

A.2 Proof of Theorem 1

We prove a slight more general version of our result. Consider an infinite horizon, incomplete markets economy in which an agent maximizes utility function $U : \mathbb{R}^n_+ \to \mathbb{R}$ subject to an infinite sequence of budget constraints. We assume that U is concave and differentiable. Let $\mathbf{x}(s^t)$ be a vector of n goods and let $\mathbf{p}(s^t)$ be a price vector in state s^t with $p_i(s^t)$ denoting the price of good i. We use a normalization $p_1\left(s^t\right) = 1$ for all s^t. Let $b(s^t)$ be the agent's asset holdings, and let $\mathbf{e}\left(s^t\right)$ be a stochastic vector of endowments.

Consumer maximization problem

$$\max_{\mathbf{x}_t, b_t} \sum_{t=0}^{\infty} \beta^t \Pr\left(s^t\right) U(\mathbf{x}\left(s^t\right)) \tag{35}$$

subject to

$$\mathbf{p}\left(s^t\right)\mathbf{x}\left(s^t\right) + q(s^t)b\left(s^t\right) = \mathbf{p}\left(s^t\right)\mathbf{e}\left(s^t\right) + P(s_t)b\left(s^{t-1}\right) \tag{36}$$

and $\left\{b\left(s^t\right)\right\}$ is bounded and $\left\{q(s^t)\right\}$ is the price of the risk-free bond.

The Euler conditions are

$$\begin{aligned} \mathbf{U}_x(s^t) &= U_1(s^t)\mathbf{p}(s^t) \\ \Pr\left(s^t\right) U_1\left(s^t\right) q(s^t) &= \beta \sum_{s^{t+1} > s^t} \Pr\left(s^{t+1}\right) U_1\left(s^{t+1}\right). \end{aligned} \tag{37}$$

Theorem 8 *Consider an allocation $\{\mathbf{x}_t, b_t\}$ that satisfies (36), (37) and $\{b_t\}_t$ is bounded. Then $\{\mathbf{x}_t, b_t\}$ is a solution to (35).*

Proof. The proof follows closely Constantinides and Duffie (1996). Suppose there is another budget feasible allocation $\mathbf{x} + \mathbf{h}$ that maximizes (35). Since U is strictly concave,

$$\begin{aligned} & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(\mathbf{x}_t + \mathbf{h}_t) - \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(\mathbf{x}_t) \\ \leq \ & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \mathbf{U}_x(\mathbf{x}_t)\mathbf{h}_t \end{aligned} \tag{38}$$

To attain $\mathbf{x} + \mathbf{h}$, the agent must deviate by φ_t from his original portfolio b_t such that $\{\varphi_t\}_t$ is bounded, $\varphi_{-1} = 0$ and

$$\mathbf{p}(s^t)\mathbf{h}\left(s^t\right) = P(s_t)\varphi(s^{t-1}) - q(s^t)\varphi(s^t)$$

Multiply by $\beta^t \Pr\left(s^t\right) U_1(s^t)$ to get:

$$\begin{aligned} \beta^t \Pr\left(s^t\right) U_1(s^t)\mathbf{p}(s^t)\mathbf{h}\left(s^t\right) &= \beta^t \Pr\left(s^t\right) U_1(s^t)\varphi(s^{t-1}) - q(s^t)\beta^t \Pr\left(s^t\right) U_1(s^t)\varphi(s^t) \\ &= \beta^t \Pr\left(s^t\right) U_1(s^t)\varphi(s^{t-1}) - \beta^{t+1} \sum_{s^{t+1} > s^t} \Pr\left(s^{t+1}\right) U_1\left(s^{t+1}\right) \varphi(s^t) \end{aligned}$$

where we used the second part of (37) in the second equality. Sum over the first T periods and use the first part of (37) to eliminate $\mathbf{U}_x(\mathbf{x}_t) = U_1(s^t)\mathbf{p}(s^t)$

$$\sum_{t=0}^{T} \beta^t \Pr\left(s^t\right) \mathbf{U}_x(\mathbf{x}_t)\mathbf{h}\left(s^t\right) = -\sum_{s^{T+1}>s^T} \beta^{T+1} \Pr\left(s^{T+1}\right) U_1\left(s^{T+1}\right) \varphi(s^T).$$

Since $\{\varphi_t\}_t$ is bounded there must exist $\bar{\varphi}$ s.t. $|\varphi_t| \leq \bar{\varphi}$ for all t. By Theorem 5.2 of Magill and Quinzii (1994), this equilibrium with debt constraints implies a transversality condition on the right hand side of the last equation, so by transitivity we have

$$\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \Pr\left(s^t\right) \mathbf{U}_x(\mathbf{x}_t)\mathbf{h}\left(s^t\right) = 0.$$

Substitute this into (38) to show that $\mathbf{h}$ does not improve utility of consumer. ■

A.3 Proof of Theorem 2

Proof. The optimal Ramsey plan solves the following Bellman equation. Let $V(b_{-})$ be the maximum ex-ante value the government can achieve with debt b_{-}.

$$V(b_{-}) = \max_{c(s),l(s),b(s)} \sum_s \pi(s) \left\{ c(s) - \frac{l(s)^{1+\gamma}}{1+\gamma} + \beta V(b(s)) \right\} \tag{39}$$

subject to

$$c(s) + b(s) = l(s)^{1+\gamma} + \beta^{-1} P(s) b_{-} \tag{40a}$$

$$c(s) + g(s) \leq \theta l(s) \tag{40b}$$

Let $\bar{b} = -\underline{B}$

$$\underline{b} \leq b(s) \leq \bar{b} \tag{40c}$$

Let $\mu(s), \phi(s), \underline{\kappa}(s), \overline{\kappa}(s)$ be the Lagrange multipliers on the respective constraints. Part 1 of Theorem **2**

Lemma 3 *There exists a $\overline{b}$ such that $b_t \leq \overline{b}$. This is the natural debt limit for the government.*

Proof. As we drive μ to $-\infty$, the tax rate approaches a maximum limit, $\bar{\tau} = \frac{\gamma}{1+\gamma}$. In state s, the government surplus,

$$S(s,\tau) = \theta^{\frac{\gamma}{1+\gamma}} (1-\tau)^{\frac{1}{\gamma}} \tau - g(s),$$

which is maximized at $\tau = \frac{\gamma}{1+\gamma}$ when $(1-\tau)^{\frac{1}{\gamma}} \tau$ is also maximized. This would impose a natural borrowing limit for the government.

■

From now we assume that $\bar{b}$ represents the natural borrowing limit. We begin with some useful lemmas

let $L \equiv l^{1+\gamma}$, to make this problem convex,

Substitute for $c(s)$

$$V(b_{-}) = \max_{L(s),b(s)} \sum_{s \in S} \pi(s) \left[\frac{1}{1+\gamma} L(s) + \frac{1}{\beta} P(s) b_{-} - b(s) + \beta V(b(s)) \right]$$

s.t.

$$\begin{aligned} \frac{1}{\beta} P(s) b - b(s) + g(s) &\leq \theta L^{\frac{1}{1+\gamma}}(s) - L(s) \\ b(s) &\leq \bar{b} \\ L(s) &\geq 0. \end{aligned}$$

Lemma 4 *$V(b)$ is stictly concave, continuous, differentiable and $V(b) < \beta^{-1}$ for all $b < \bar{b}$. The feasibility constraint binds for all $b \in (-\infty, \bar{b}],\ s \in S$ and $(L^*(s))^{1-\frac{1}{1+\gamma}} \geq \frac{1}{1+\gamma}$.*[18]

Proof. *Concavity*

$V(b)$ is concave because we maximize linear objective function over convex set.

Binding feasibility

Suppose that feasibility does not bind for some b, s. Then the optimal $L(s)$ solve $\max_{L(s)\geq 0} \pi(s) \frac{\gamma}{1+\gamma} L(s)$ which sets $L(s) = \infty$. This violates feasility for any finite $b, b(s)$.

Bounds on L

Let $\phi(s) > 0$ be a Lagrange multiplier on the feasibility. The FOC for $L(s)$ is

$$\frac{1}{1+\gamma} + \phi(s)\left(\frac{1}{1+\gamma} L(s)^{\frac{1}{1+\gamma}-} - \theta\right) = 0.$$

This gives

$$\frac{1}{1+\gamma} L^{\frac{1}{1+\gamma}-1} - \theta = -\frac{1}{\lambda}\frac{\gamma}{1+\gamma} < 0$$

or

$$L^{1-\frac{1}{1+\gamma}} \geq \frac{\theta}{1+\gamma}.$$

Continuity

For any L that satisfy $L^{1-\overline{1+\gamma}} \geq \frac{\theta}{1+\gamma}$, define function Ψ that satisfies $\Psi\left(L^{\frac{1}{1+\gamma}} - \theta L\right) = L$. Since $L^{\frac{1}{1+\gamma}} - L$ is strictly decreasing in L for $L^{1-\frac{1}{1+\gamma}} \geq \frac{1}{1+\gamma}$, this function is well defined. Note that $\Psi()\underbrace{\left(\frac{1}{1+\gamma} L^{\frac{1}{1+\gamma}-1} - \theta\right)}_{<0} = 1$ (so that $\Psi > 0$, i.e. Ψ is strictly decreasing) and $\Psi''\underbrace{\left(\frac{1}{1+\gamma} L^{\frac{1}{1+\gamma}-1} - 1\right)^2}_{>0} + \underbrace{\Psi}_{<0}\underbrace{\frac{1}{1+\gamma}\frac{\gamma}{1+\gamma} L^{\frac{1}{1+\gamma}-2}}_{<0} = 0$ (so that $\Psi'' \geq 0$, $\Psi'' > 0$, i.e. Ψ is strictly concave on the interior). Ψ is also continuous. When $L^{1-\frac{1}{1+\gamma}} = \frac{1}{1+\gamma}$, $L = (1+\gamma)^{-\frac{(1+\gamma)}{(\gamma)}}$. Let $D \equiv (1+\gamma)^{\frac{-1}{\gamma} - (1+\gamma)^{-\frac{1+\gamma}{(\gamma)}}}$. Then the objective is

$$V(b_-) = \max_{b(s)} \sum_{s\in S} \pi(s)\left[\Psi\left(\frac{1}{\beta}P(s)\, b - b(s) + g(s)\right) + \frac{1}{\beta}P(s)\, b_- - b(s) + \beta V(b(s))\right]$$

s.t.

$$\begin{aligned} b(s) &\leq \bar{b} \\ \frac{1}{\beta}P(s)\, b_- - b(s) + g(s) &\leq D. \end{aligned}$$

[18] This last condition simply means that we do not tax to the right of the peak of the Laffer curve. The revenue maximizing tax is $1 - \bar{\tau} = \frac{1}{1+\gamma}$. At the same time $1 - \tau = l^\gamma$ so if taxes are always to the left of the peak, $\frac{1}{1+\gamma} \leq l^\gamma = \left(L^{\frac{1}{1+\gamma}}\right)^\gamma = L^{1-\frac{1}{1+\gamma}}$.

This function is continuous so V is also continuous.

Differentiability

Continuity and convexity implies differentiability everywhere, including the boundaries.

Strict concavity

Ψ is strictly concave, so on the interior V is strictly concave.

■

Next we characterize policy functions

Lemma 5 *$b(s, b_{-})$ is an increasing function of b for all s for all s, b_{-} where $b(s)$ is interior.*

Proof. Take the FOCs for $b(s)$ from the condition in the previous problem. If $b(s)$ is interior

$$\Psi\left(\frac{1}{\beta}P(s)b_{-} - b(s) + g(s)\right) = \beta V(b(s)).$$

Suppose $b_1 < b_2$ but $b_2(s) < b_1(s)$. Then from stict concavity

$$\begin{aligned} V(b_2(s)) &< V'(b_1(s)) \\ \Psi\left(\frac{1}{\beta}P(s)b_2 - b_2(s) + g(s)\right) &> \Psi\left(\frac{1}{\beta}P(s)b_1 - b_1(s) + g(s)\right). \end{aligned}$$

■

Lemma 6 *There exists an invariant distribution of the stochastic process $b_{t+1} = b(s_{t+1}, b_t)$*

Proof. The state spaces for b_t and s_t are compact. Further the transition function on $s_{t+1}|s_t$ is trivially increasing under i.i.d shocks. We can apply standard arguments as in **?**(see corollary 3) to argue that there exists invariant distribution of assets. ■

Now we characterize the support of this distribution using further properties of the policy rules for $b(s|b_{-})$

Lemma 7 *For any $b_{-} \in (\underline{b}, \bar{b})$, there are s, s'' s.t. $b(s) \geq b_{-} \geq b(s'')$. Moreover, if there are any states s'', s''' s.t. $b(s'') \neq b(s''')$, those inequalities are strict.*

Proof. The FOCs together with the envelope theorem imply that $\mathbb{E}P(s)V'(b(s)) = V'(b_{-}) + \kappa(s)$ We can rewrite this as $\tilde{\mathbb{E}}V'(b(s)) = b + \kappa(s)$ with $\tilde{\pi}(s) = P(s)\pi(s)$

Now if there is at least one $b(s)$ s.t. $b(s) > b_{-}$, by strict concavity of V there must be some s'' s.t. $b(s'') < b$.

If there is at least one $b(s)$ s.t. $b(s) < b_{-}$, the inequality above is strictly only if $b(s''') = \bar{b}$ for some s'''. But $V(\bar{b}) < V(b)$ so there must be some s'' s.t. $b(s'') > b$. Equality is possible only if $b_{-} = b(s)$ for all s. ■

Lemma 8 *Let $\mu(b,s)$ be the optimal policy function for the Lagrange multiplier $\mu(s)$. If $P(s') > P(s'')$ then there exists a $b^*_{s',s''}$ such that for all $b < (>)\ b_{1,s',s''}$ we have $\mu(b,s') > (<)\ \mu(b,s'')$. If $\underline{b} < b^*_{s',s''} < \overline{b}$ then $\mu(b^*_{s',s''}, s') = \mu(b^*_{s',s''}, s'')$.*

Proof. Suppose that $\mu(b,s') \leq \mu(b,s'')$. Subtracting the implementability for s'' from the implementability constraint for s' we have

$$\begin{aligned}\frac{P(s') - P(s'')}{\beta} b &= S_{s'}(\mu(b,s')) - S_{s''}(\mu(b,s'')) + b'(b,s') - b'(b,s'') \\ &\geq S_{s'}(\mu(b,s')) - S_{s''}(\mu(b,s')) + b'(b,s') - b'(b,s'') \\ &\geq S_{s'}(\mu(b,s')) - S_{s''}(\mu(b,s')) = g(s'') - g(s')\end{aligned}$$

We get the first inequality from noting that $S_s(\mu') \geq S_s(\mu'')$ if $\mu' \leq \mu''$. We obtain the second inequality by noting that $\mu(b,s') \leq \mu(b,s'')$ implies $b'(b,s') \geq b'(b,s'')$ (which comes directly from the concavity of V). Thus, $\mu(b,s') \leq \mu(b,s'')$ implies that

$$b \geq \frac{\beta(g(s'') - g(s'))}{P(s') - P(s'')} = b^*_{s',s''} \tag{41}$$

The converse of this statement is that if $b < b^*_{s',s''}$ then $\mu(b,s') > \mu(b,s'')$. The reverse statement that $\mu(b,s') \geq \mu(b,s'')$ implies $b \leq b^*_{s,s'}$ follows by symmetry. Again, the converse implies that if $b > b^*_{s',s''}$ then $\mu(b,s') < \mu(b,s'')$. Finally, if $\underline{b} < b^*_{s',s''} < \overline{b}$ then continuity of the policy functions implies that there must exist a root of $\mu(b,s') - \mu(b,s'')$ and that root can only be at $b^*_{s',s''}$. ■

Lemma 9 *$P \in \mathcal{P}^*$ is necessary and sufficient for existence of b^* such that $b(s,b^*) = b^*$ for all ss*

Proof. The necessary part follows from taking differences of the (40a) for s',s''. We have

$$[P(s) - P(s'')]\frac{b^*}{\beta} = g(s) - g(s'')$$

Thus $P \in \mathcal{P}^*$. The sufficient part follows from the Lemma 8. If $P \notin \mathcal{P}^*$, equation (41) that defines $b^*_{s',s''}$ will not be same across all pairs. Thus b^* that satisfies $b(s;b^*)$ independent of s will not exist. ■

Lemma 9 implies that under the hypothesis of part 1 of the Theorem 2 there cannot exist an interior absorbing point for the dynamics of debt. This allows us to construct a sequences $\{b_t\}_t$ such that $b_t < b_{t+1}$ with the property that $\lim_t b_t = \underline{b}$. Thus, for any $\epsilon > 0$, there exists a finite history of shocks that can take us arbitrarily close to $\underline{b}$. Since the shocks are i.i.d this finite sequence will repeat i.o. With a symmetric argument we can show that b_t will come arbitrarily close to its upper limit i.o too

Part 2 of Theorem **2**

In this first section we will show that there exists b_1, and if $P(s)$ is sufficiently volatile a b_2, such that if $b_t \leq b_1$ then

$$\mu_t \geq \mathbb{E}_t \mu_{t+1}$$

and if $b_t \geq b_2$ then

$$\mu_t \leq \mathbb{E}_t \mu_{t+1}.$$

Recall that b is decreasing in μ, so this implies that if b_t is low (large) enough then there will exist a drift away from the lower (upper) limit of government debt.

With Lemma 8 we can order the policy functions $\mu(b, \cdot)$ for particular regions of the state space. Take b_1 to be

$$b_1 = \min \left\{ b^*_{s',s''} \right\}$$

and WLOG choose $\underline{b} < b_1$. For all $b < b_1$ we have shown that $P(s) > P(s')$ implies that $\mu(b, s) > \mu(b, s')$. The FOC for the problem imply,

$$\mu_- = \mathbb{E}P(s)\mu(s) + \underline{\kappa}(s) \tag{42}$$

The inequality in the resource constraint implies that $\phi(s) \geq 0$ implying that $\mu(s) \leq 1$. With some minor algebra algebra we obtain

By decomposing $\mathbb{E}\mu(s)P(s)$ in equation (42), we obtain (using $\mathbb{E}P(s) = 1$)

$$\mu_- = \mathbb{E}\mu(s) + \text{cov}(\mu(s), P(s)) + \underline{\kappa}(s) \tag{43}$$

Our analysis has just shown that for $b_- < b_1$ we have $\text{cov}_t(\mu(s), P(s)) > 0$ so

$$\mu_- > \mathbb{E}\mu(s).$$

If p is sufficiently volatile:

$$P(s') - P(s'') > \frac{\beta(g(s'') - g(s'))}{\bar{b}}$$

then

$$b_2 = \max \left\{ b^*_{s',s''} \right\} < \bar{b}$$

and through a similar argument we can conclude that $\text{cov}(\mu(s), P(s)) < 0$

$$\mu_- < \mathbb{E}\mu(s)$$

for $b_- > b_2$ (note $b_- > \underline{b}$ implies $\underline{\kappa}(s) = 0$) which gives us a drift away from the upper-bound.

Part 3 of Theorem **2**

When $P \in \mathcal{P}^*$, Lemma 9 implies existence of b^* as the steady state debt level.

Lemma 10 *There exists μ^* such that μ_t is a sub-martingale bounded above in the region $(-\infty, \mu^*)$ and super-martingale bounded below in the region $(\mu^*, \frac{1}{1+\gamma})$*

Proof. Let μ^* be the associated multiplier, i.e $V_b(b^*) = \mu^*$. Using the results of the previous section, we have that $b_1 = b_2 = b^*$, implying that $\mu_t < (>)\mathbb{E}_t\mu_{t+1}$ for $b_t < (>)b^*$. ■

Lastly we show that $\lim_t \mu_t = \mu^*$. Suppose $b_t < b^*$, we know that $\mu_t > \mu^*$. The previous lemma implies that in this region, μ_t is a super martingale. The lemma 5 shows that $b(s, b_-)$ is continuous and increasing. This translates into $\mu(\mu(b_-), s)$ to be continuous and increasing as well. Thus

$$\mu_t > \mu^* \implies \mu(\mu_t, s_{t+1}) > \mu(\mu^*, s_{t+1})$$

or

$$\mu_{t+1} > \mu^*$$

Thus $\mu*$ provides a lower bound to this super martingale. Using standard martingale convergence theorem converges. The uniqueness of steady state implies that it can only converge to μ^*. For $\mu < \mu^*$, the argument is symmetric.

■

A.4 Proof of Theorem 3

Working with the first order conditions of problem 39, we obtain

$$l(s)^{\gamma} = \frac{\mu(s) - 1}{(1+\gamma)\mu(s) - 1} = 1 - \tau(\mu(s)),$$

implying the relationship between tax rate τ and multiplier μ given by

$$\tau(\mu) = \frac{\gamma\mu}{(1+\gamma)\mu - 1} \tag{44}$$

At the interior, the rest of the first order conditions and the implementability constraints are summarized below

$$\frac{b_{-}P(s)}{\beta} = S(\mu(s), s) + b(s)$$
$$\mu(b_{-}) = \mathbb{E}P(s)\mu(s)$$

where $S(\mu, s)$ is the government surplus in state s given by

$$S(\mu, s) = (1 - \tau(\mu))^{\frac{1}{\gamma}}\tau(\mu) - g(s) = I(\mu) - g(s)$$

The proof of the theorem will have four steps:

Step 1: Obtaining a recursive representation of the optimal allocation in the incomplete markets economy with payoffs $P(s)$ with state variable μ_{-}

Given a pair $\{P(s), g(s)\}$, since $V'(b)$ is one-to-one, so we can re-characterize these equations as searching for a function $b(\mu)$ and $\mu(s|\mu_{-})$such that the following two equations can be solved for all μ_{-}.

$$\frac{b(\mu_{-})P(s)}{\beta} = I(\mu(s)) - g(s) + b(\mu(s)) \tag{45}$$
$$\mu_{-} = \mathbb{E}\mu(s)P(s) \tag{46}$$

Step 2: Describe how the policy rules are approximated

Usually perturbation approaches to solve equilibrium conditions as above look for the solutions to $\{\mu(s|\mu_{-})\}$ and $b(\mu_{-})$ around deterministic steady state of the model. Thus for any b^{ss}, there exists a μ^{ss} that will solve

$$\frac{b^{SS}}{\beta} = I(\mu^{SS}) - \bar{g} + b^{SS}$$

For example if we set the perturbation parameter q to scale the shocks, $g(s) = \mathbb{E}g(s) + q\Delta_g(s)$ and $P(s) = 1 + q\Delta_P(s)$, the first order expansion of $\mu(s|\mu_-)$ will imply that it is a martingale. Such approximations are not informative about the ergodic distribution. [19]

In contrast we will approximate the functions $\mu(s|\mu_-)$ around around economy with payoffs in $\bar{P} \in \mathcal{P}^*$.

In contrast we a) explicitly recognize that policy rules depend on payoffs: $\mu(s|\mu_-, \{P(s)\}_s)$ and $b(\mu_-, \{P(s)\}_s)$ and then take take the first order expansion with respect to both μ_- and $\{P(s)\}$ around the vector $(\bar{\mu}, \{\bar{P}(s)\}_s)$ where $\bar{P}(s) \in \mathcal{P}^*$: these payoffs support an allocation such that limiting distribution of debt is degenerate around the some value $\bar{b}$; and $\overline{\mu}$ is the corresponding limiting value of multiplier. The next two expression make the link between $\bar{\mu}$ and $\bar{b}$ explicit.

$$\bar{b} = \frac{\beta}{1-\beta}\left(I(\overline{\mu}) - \overline{g}\right) \tag{47a}$$

where $\overline{g} = \mathbb{E}g$ and $\overline{p}$ as

$$\overline{P}(s) = 1 - \frac{\beta}{\bar{b}}(g(s) - \overline{g}) \tag{47b}$$

As noted before $b(\overline{\mu}; \overline{p}) = \bar{b}$ solves the the system of equations (45-46) for $\mu'(s) = \overline{\mu}$ when the payoffs are $\overline{P}(s)$

We next obtain the expressions that characterize the linear approximation of $\mu(s|\mu_-, \{P(s)\})$ and $\underline{(}\mu_-, \{P(s)\})$ around some arbitrary point $(\bar{\mu}, \{\bar{P}(s)\}_s)$ where $\bar{P}(s) \in \mathcal{P}^*$. We will use these expressions to compute the mean and variance of the ergodic distribution associated with the approximated policy rules. Finally as a last step we propose a particular choice of the point of approximation.

The derivatives $\frac{\delta\mu(s|\mu_-,\{P(s)\}}{\delta\mu_-}, \frac{\delta\mu(s|\mu_-,\{P(s)\}}{\delta P(s)}$ and similarly for $b(\mu_-, \{P(s)\}$ are obtained below:

Differentiating equation (45) with respect to μ around $(\overline{\mu}, \overline{P})$ we obtain

$$\frac{\overline{P}(s)}{\beta}\frac{\partial b}{\partial \mu_-} = \left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu_-}\right]\frac{\partial \mu(s)}{\partial \mu_-}.$$

Differentiating equation (46) with respect to μ_- we obtain

$$1 = \sum_s \pi(s)\overline{P}(s)\frac{\partial \mu'(s)}{\partial \mu_-}$$

combining these two equations we see that

$$\frac{1}{\beta}\left(\sum_s \pi(s)\overline{P}(s)^2\right)\frac{\partial b}{\partial \mu_-} = I'(\overline{\mu}) + \frac{\partial b}{\partial \mu_-}$$

[19] One can do higher order approximations, but part 3 of theorem 2 hints that for economies with payoffs close to $\mathcal{P}^*$, the stochastic steady state in general is far away from μ^{SS}.

Noting that $\mathbb{E}\overline{P}^2(s) = 1 + \frac{\beta^2}{\bar{b}^2}\sigma_g^2$ we obtain

$$\frac{\partial b}{\partial \mu_-} = \frac{I'(\overline{\mu})}{\frac{\beta}{\bar{b}^2}\sigma_g^2 + \frac{1-\beta}{\beta}} < 0 \tag{48}$$

as $I'(\overline{\mu}) < 0$. We then have directly that

$$\frac{\partial \mu'(s)}{\partial \mu} = \frac{\overline{P}(s)}{\frac{\beta^2}{\bar{b}^2}\sigma_g^2 + 1} = \frac{\overline{P}(s)}{\mathbb{E}\overline{P}(s)^2} \tag{49}$$

We can perform the same procedure for $P(s)$. Differentiating equation (45) with respect to $P(s)$ we around $(\overline{\mu}, \overline{p})$ we obtain

$$\frac{\overline{p}(s')}{\beta}\frac{\partial b}{\partial P(s)} + 1_{s,s'}\frac{\bar{b}}{\beta} - \frac{\pi(s)\bar{b}\overline{p}(s')}{\beta} = \left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu}\right]\frac{\partial \mu(s')}{\partial P(s)} \tag{50}$$

Here $1_{s,s'}$ is 1 if $s = s'$ and zero otherwise. Differentiating equation (46) with respect to $P(s)$ we obtain

$$0 = \pi(s)\overline{\mu} - \pi(s)\overline{\mu} + \sum_{s'}\pi(s)\overline{p}(s')\frac{\partial \mu(s')}{\partial P(s)} = \sum_{s'}\pi(s')\overline{p}(s')\frac{\partial \mu(s')}{\partial P(s)}$$

Again we can combine these two equations to give us

$$\frac{\mathbb{E}\overline{p}(s)^2}{\beta}\frac{\partial b}{\partial P(s)} + \frac{\pi(s)\bar{b}}{\beta}(\overline{p}(s) - \mathbb{E}\overline{p}(s)^2) = 0$$

or

$$\frac{\partial b}{\partial P(s)} = \pi(s)\bar{b}\frac{\mathbb{E}\overline{p}^2 - \overline{p}(s)}{\mathbb{E}\overline{p}^2} \tag{51}$$

Going back to equation (50) we have

$$\frac{\partial \mu(s')}{\partial P(s)} = \frac{\bar{b}}{\beta\left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu}\right]}\left(1_{s,s'} - \frac{\pi(s)\overline{p}(s)\overline{p}(s')}{\mathbb{E}\overline{p}^2}\right) \tag{52}$$

Step 3: Getting expressions for the mean and variance of the ergodic distribution around some arbitrary point of approximation

For an arbitrary $\left(\overline{\mu}, \{\overline{P}(s)\}_s\right)$, using the derivatives that we computed, we can characterize the dynamics of $\hat{\mu} \equiv \mu_t - \overline{\mu}$ using our approximated policies.

$$\hat{\mu}_{t+1} = B(s_{t+1})\hat{\mu}_t + C(s_{t+1}),$$

where $B(s)$ and $C(s)$ are respective derivatives. Note that both are random variables and let us denote their means $\overline{B}$ and $\overline{C}$, and variances σ_B^2 and σ_C^2 . Suppose that $\hat{\mu}$ is distributed according to the ergodic distribution of this linear system with mean $\mathbb{E}\hat{\mu}$ and variance σ_μ^2. Since

$$B\hat{\mu} + C,$$

has the same distribution we can compute the mean of this distribution as

$$\begin{aligned}
\mathbb{E}\hat{\mu} &= \mathbb{E}\left[B\hat{\mu} + C\right] \\
&= \mathbb{E}\left[\mathbb{E}_{\hat{\mu}}\left[B\hat{\mu} + C\right]\right] \\
&= \mathbb{E}\left[\overline{B}\hat{\mu} + \overline{C}\right] \\
&= \overline{B}\mathbb{E}\hat{\mu} + \overline{C}
\end{aligned}$$

solving for $\mathbb{E}\hat{\mu}$ we get

$$\mathbb{E}\hat{\mu} = \frac{\overline{C}}{1 - \overline{B}} \tag{53}$$

For the variance $\sigma^2_{\hat{\mu}}$ we know that

$$\sigma^2_{\hat{\mu}} = \text{var}(B\hat{\mu} + C) = \text{var}(B\hat{\mu}) + \sigma^2_C + 2\text{cov}(B\hat{\mu}, C)$$

Computing the variance of $B\hat{\mu}$ we have

$$\begin{aligned}
\text{var}(B\hat{\mu}) &= \mathbb{E}\left[(B\hat{\mu} - \overline{B}\mathbb{E}\hat{\mu})^2\right] \\
&= \mathbb{E}\left[(B\hat{\mu} - \overline{B}\hat{\mu} + \overline{B}\hat{\mu} - \overline{B}\mathbb{E}\hat{\mu})^2\right] \\
&= \mathbb{E}\left[\mathbb{E}_{\hat{\mu}}\left[(B - \overline{B})^2\hat{\mu}^2 + 2(B - \overline{B})(\hat{\mu} - \mathbb{E}\hat{\mu})\overline{B}\mathbb{E}\hat{\mu} + (\hat{\mu} - \mathbb{E}\hat{\mu})^2\bar{B}^2\right]\right] \\
&= \mathbb{E}\left[\sigma^2_B\hat{\mu}^2 + (\hat{\mu} - \mathbb{E}\hat{\mu})^2\overline{B}\right] \\
&= \sigma^2_B(\sigma^2_{\hat{\mu}} + (\mathbb{E}\hat{\mu})^2) + \sigma^2_{\hat{\mu}}\overline{B}^2
\end{aligned}$$

while for the covariance of $B\hat{\mu}$ and C

$$\text{cov}(B\hat{\mu}, C) = \sigma_{BC}\mathbb{E}\hat{\mu}$$

Putting this all together we have

$$\sigma^2_{\hat{\mu}} = \frac{\sigma^2_B(\mathbb{E}\hat{\mu})^2 + \sigma_{BC}\mathbb{E}\hat{\mu} + \sigma^2_C}{1 - \overline{B}^2 - \sigma^2_B} \tag{54}$$

Step 4: Choice of the point of approximation

To get the expressions in Theorem 2, we finally choose a particular $\overline{P} = P^*(s) \in \mathcal{P}^*$. This will be the closest complete market economy to our the given $P(s)$ in L^2 sense. Formally,

$$\min_{\tilde{P} \in \mathcal{P}^*} \sum_s \pi(s)(P(s) - \tilde{P}(s))^2.$$

Since all payoffs in $\mathcal{P}^*$ are associated with some b^* and μ^* via equations (47), we can re write the above problem as choosing $\overline{\mu}$ so as to minimize the variance of the difference between $P(s)$

and the set of steady state payoffs. Let P^* be the solution to this minimization problem. The first order condition for this linearization gives us

$$2\sum_{s'}\pi(P(s') - P^*(s',\mu^*))\frac{\delta P^*(s,\mu^*)}{\delta\mu^*} = 0$$

as noted before

$$P^*(s) = 1 - \frac{\beta}{b^*(\mu^*)}\left(g(s) - \mathbb{E}g\right)$$

thus

$$\frac{\delta P^*}{\delta\mu^*} \propto P^* - 1$$

Thus we can see the the optimal choice of $\overline{\mu}$ is equivalent to choosing $\overline{\mu}$ such that

$$\begin{aligned}
0 &= \sum_{s'}\pi(s')(P(s') - P^*(s',\mu^*))(P^*(s',\mu^*) - 1) \\
&= -\sum_{s'}\pi(s')(P(s') - P^*(s',\mu^*)) + \sum_{s'}\pi(s')(P(s') - P^*(s',\mu^*))P^*(s',\mu^*) \\
&= \sum_{s'}\pi(s')(P(s') - P^*(s',\mu^*))P^*(s',\mu^*) \\
&= \mathbb{E}\left[(P - P^*)P^*\right]
\end{aligned} \tag{55}$$

At these values of $\overline{p} = P^*$ and $\overline{\mu} = \mu^*$ we have that C for our linearized system is

$$C(s') = \sum_s\left\{\frac{b^*}{\beta\left[I'(\overline{\mu}) + \frac{\partial b}{\partial\mu}\right]}\left(1_{s,s'} - \frac{\pi(s)P^*(s)P^*(s')}{\mathbb{E}\overline{p}^2}\right)(P(s) - P^*(s))\right\}$$

Taking expectations we have that

$$\begin{aligned}
\overline{C} &= \sum_s\left\{\frac{b^*}{\beta\left[I'(\overline{\mu}) + \frac{\partial b}{\partial\mu}\right]}\left(\pi(s) - \frac{\pi(s)P^*(s)}{\mathbb{E}\overline{p}^2}\right)(P(s) - P^*(s))\right\} \\
&= \frac{b^*}{\beta\left[I'(\overline{\mu}) + \frac{\partial b}{\partial\mu}\right]}\left(\mathbb{E}(P - \overline{p}) - \frac{\mathbb{E}\left[(P - \overline{p})\overline{p}\right]}{\mathbb{E}\overline{p}^2}\right) \\
&= 0
\end{aligned} \tag{56}$$

Thus the linearized system will have the same mean for μ, $\overline{\mu}$, as the closest approximating steady state payoff structure.

We can also compute the variance of the ergodic distribution for μ. Note

$$
\begin{aligned}
C(s') &= \sum_s \left\{ \frac{b^*}{\beta\left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu}\right]} \left(1_{s,s'} - \frac{\pi(s)P^*(s)P^*(s')}{\mathbb{E}P^{*2}}\right)(P(s) - P^*(s)) \right\} \\
&= \frac{b^*}{\beta\left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu}\right]} \left(P(s') - P^*(s') - P^*(s')\frac{\sum_s \pi(s)P^*(s)(p_s - P^*(s))}{\mathbb{E}P^{*2}}\right) \\
&= \frac{b^*}{\beta\left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu}\right]}(P(s') - P^*(s))
\end{aligned}
$$

As noted before

$$\sigma_\mu^2 = \frac{b^{*2}}{\beta^2\left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu}\right]^2\left(1 - \overline{B}^2 - \sigma_B^2\right)}\|P - P^*\|^2$$

The variance of government debt in the linearized system is

$$\sigma_b^2 = \frac{b^{*2}\left(\frac{\partial b}{\partial \mu}\right)^2}{\beta^2\left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu}\right]^2\left(1 - \overline{B}^2 - \sigma_B^2\right)}\|P - P^*\|^2$$

This can be simplified using the following expressions:

$$I'(\overline{\mu}) + \frac{\partial b}{\partial \mu} = \frac{\mathbb{E}P^{*2}}{\beta}\frac{\partial b}{\partial \mu},$$

$$\overline{B} = \frac{1}{\mathbb{E}P^{*2}}$$

and

$$\sigma_B^2 = \frac{\text{var}(P^*)}{(\mathbb{E}P^{*2})^2}$$

to

$$\sigma_b^2 = \frac{b^{*2}}{\mathbb{E}P^{*2}\text{var}(P^*)}\|P - P^*\|^2 \tag{57}$$

Noting that $\mathbb{E}P^{*2} = 1 + \text{var}(P^*) > 1$, we have immediately that up to first order the relative spread of debt is bounded by

$$\frac{\sigma_b}{b^*} \leq \sqrt{\frac{\|P - P^*\|^2}{\text{var}(P^*)}} \tag{58}$$

A.5 Proof of Theorem 4

Proof.

Using Theorem 1 let $\tilde{b} = b_1 - b_2$. Under the normalization that $b_2 = 0$, the variable $\tilde{b}$ represents public debt government or the assets of the productive agent. The optimal plan solves the following Bellman equation,

$$V(\tilde{b}_{\text{-}}) = \max_{c_1(s),c_2(s),b'(s)} \sum_s \pi(s) \left\{ \omega \left[c_1(s) - \frac{l_1^{1+\gamma}(s)}{1+\gamma} \right] + (1-\omega)c_2(s) + \beta V(\tilde{b}(s)) \right\} \tag{59}$$

subject to

$$c_1(s) - c_2(s) + \tilde{b}(s) = l(s)^{1+\gamma} + \beta^{-1} P(s)\tilde{b}_{\text{-}} \tag{60a}$$

$$nc_1(s) + (1-n)c_2(s) + g(s) \leq \theta_2 l(s) n \tag{60b}$$

$$c_2(s) \geq 0 \tag{60c}$$

$$\bar{b} \geq \tilde{b}(s) \geq \underline{b} \tag{60d}$$

Let $\mu(s), \phi(s), \lambda(s), \underline{\kappa}(s), \overline{\kappa}(s)$ be the Lagrange multipliers on the respective constraints. The FOC are summarized below

$$\omega - \mu(s) = n\phi(s) \tag{61a}$$

$$1 - \omega + \mu(s) - \phi(s)(1-n) + \lambda(s) = 0 \tag{61b}$$

$$-\omega l^{\gamma}(s) + \mu(s)(1+\gamma)l^{\gamma}(s) + n\phi(s)\theta = 0 \tag{61c}$$

$$\beta V'(\tilde{b}(s)) - \mu(s) - \overline{\kappa}(s) + \underline{\kappa}(s) = 0 \tag{61d}$$

and the envelope condition

$$V'(\tilde{b}_{\text{-}}) = \sum_s \pi(s)\mu(s)\beta^{-1}P(s) \tag{61e}$$

To show part 1 of Theorem 4, we show that $\frac{\omega}{n} > \frac{1+\gamma}{\gamma}$ is sufficient for the Lagrange multiplier $\lambda(s)$ on the non-negativity constraint to bind.

Lemma 11 *The multiplier on the budget constraint $\mu(s)$ is bounded above*

$$\mu(s) \leq \min\left\{\omega - n, \frac{\omega}{1+\gamma}\right\}$$

Similiarly the multiplier of the resource constraint is bounded below,

$$\phi(s) \geq \max\left\{1, \frac{\omega}{n}\left[\frac{\gamma}{1+\gamma}\right]\right\}$$

Proof.

Notice that the labor choice of the productive household implies $\frac{1}{1-\tau} = \frac{\theta_2}{l^{\gamma}(s)}$.

As taxes go to $-\infty$ (61c) implies that $\mu(s)$ approaches $\frac{\omega}{1+\gamma}$ from below. Similiarly the non-negativity of $c_2(s)$ imposes a lower bound of 1 on $\phi(s)$. This translates into an upper bound of $\omega - n$ on μ. ■

Lemma 12 *There exists a $\bar{\omega}$ such that $\omega > \bar{\omega}$ implies $c_2(s) = 0$ for all b*

Proof.

By the KKT conditions $c_2(s) = 0$ if $\lambda(s) > 0$. Now (61b) implies this is true if $\mu(s) < \omega - n$. The previous lemma bounds $\mu(s)$ by $\frac{\omega}{1+\gamma}$.

We can thus define $\bar{\omega} = n\left(\frac{1+\gamma}{\gamma}\right)$ as the required threshold Pareto weight to ensure that the unproductive agent has zero consumption forever.

■

Now for the rest of the parts $\omega < n\left(\frac{1+\gamma}{\gamma}\right)$, we can have postive transfers for low enough public debt. In particular, we can define a maximum level of debt $\mathcal{B}$ that is consistent with an interior solution for the unproductive agents' consumption.

Guess an interior solution $c_{2,t} > 0$ or $\lambda_t = 0$ for all t. This gives us $l_t = l^*$ defined below:

$$l^* = \left[\frac{n\theta}{\omega - (\omega - n)(1+\gamma)}\right]^{\frac{1}{\gamma}} \tag{62}$$

As long as $\omega < n\left(\frac{1+\gamma}{\gamma}\right)$ At the interior solution $\tilde{b}(s) = \tilde{b}_{-}$ and using the implementability constraint and resource constraints (60a) and (60b) respectively, we can obtain the expression for $c_2(s)$

$$c_2(s) = n\theta l^* - n{l^*}^{1+\gamma} - \tilde{b}_{-}(1 - P(s)\beta^{-1}) - g(s)$$

Non-negativity of c_2 implies,

$$\tilde{b}_{-} \leq \frac{g(s) - n\theta l^{*} + n l^{*1+\gamma}}{\beta^{-1}P(s) - 1}$$

We can also express this as

$$\tilde{b}_{-} \leq \frac{g(s) - \tau^{*} y^{*}}{\beta^{-1}P(s) - 1},$$

where the right hand side of the previous equation is just the present discounted value of the primary deficit of the government at the constant taxes τ^{*} associated with l^{*} defined in (62). As long as $\beta^{-1}P(s) - 1 > 0$, this object is well defined, we define $\mathcal{B} = \min_s \left[\frac{g(s) - n\theta l^{*} + n l^{*1+\gamma}}{\beta^{-1}P(s) - 1}\right]$. Thus for $\tilde{b}_{-} < \mathcal{B}$ the optimal allocation has constant taxes given by τ^{*} and debt $\tilde{b}_{-}$, while transfers are given by

$$T(s) = n\theta l^{*} - n l^{*1+\gamma} - \tilde{b}_{-}(1 - P(s)\beta^{-1}) - g(s),$$

and are strictly positive.

In the next lemma we show how $\mathcal{B}$ varies with ω.

Lemma 13 *For $\omega \leq n\frac{1+\gamma}{\gamma}$, we have $\frac{\partial \mathcal{B}}{\partial \omega} > 0$.*

Proof. The sign of the derivative of $\mathcal{B}$ with respect to ω is the same as the sign of the following derivative:

$$\frac{\partial \left[l^{*1+\gamma} - \theta l^{*}\right]}{\partial \omega}$$

Note that (62) implies that l^{*} is increasing in ω. Note that,

$$\frac{\partial \left[l^{*1+\gamma} - \theta l^{*}\right]}{\partial \omega} = \frac{\partial l^{*}}{\partial \omega}\left[(1+\gamma)l^{*\gamma} - \theta\right]$$

So the sign of the required derivative depends on $[(1+\gamma)l^{*\gamma} - \theta]$. We now argue that this expression is positive over the range $\omega \leq n\frac{1+\gamma}{\gamma}$.

Again from the expression for l^{*}, we see that

$$\min_{\omega \leq n\frac{1+\gamma}{\gamma}} l^{*\gamma} = \frac{\theta}{1+\gamma}$$

.

Thus we can see that $\mathcal{B}$ is increasing in ω

■

For initial debt greater than $\mathcal{B}$, we distinguish cases when payoffs are perfectly aligned with $g(s)$ i.e belong to the set $\mathcal{P}^{*}$ and when they are not. For part 2 case b, let $P \notin \mathcal{P}^{*}$.

Lemma 14 *There exists a $\check{b} > \mathcal{B}$ such that there are two shocks $\underline{s}$ and $\overline{s}$ and the optimal choice of debt starting from $\tilde{b}_{\text{-}} \leq \check{b}$ satisfies the following two inequalities:*

$$\tilde{b}(\underline{s}, \tilde{b}_{\text{-}}) > \mathcal{B}$$

$$\tilde{b}(\overline{s}, \tilde{b}_{\text{-}}) \leq \mathcal{B}$$

Proof. At $\mathcal{B}$, there exist some $\overline{s}$ such that $T(\overline{s}, \mathcal{B}) = \epsilon > 0$. Now define $\check{b}$ as follows:

$$\check{b} = \mathcal{B} + \frac{\epsilon\beta}{2P(\overline{s})}$$

Now suppose to the contrary $\tilde{b}(\overline{s}, \tilde{b}_{\text{-}}) > \mathcal{B}$ for some $\tilde{b}_{\text{-}} \leq \check{b}$. This implies that $\tau(s, \tilde{b}_{\text{-}}) > \tau^*$ and $T(\overline{s}, \tilde{b}_{\text{-}}) = 0$.

The government budget constraint implies

$$\frac{P(\overline{s})\tilde{b}_{\text{-}}}{\beta} + g(s) = \tilde{b}(\overline{s}, \tilde{b}_{\text{-}}) + (1 - \tau(\overline{s}, \tilde{b})_{\text{-}})l(\overline{s}, \tilde{b}_{\text{-}}).$$

As,

$$\frac{P(\overline{s})\tilde{b}_{\text{-}}}{\beta} + g(\overline{s}) \leq \frac{P(\overline{s})\mathcal{B}}{\beta} + g(\overline{s}) + \frac{\epsilon}{2} < \frac{P(\overline{s})\mathcal{B}}{\beta} + g(\overline{s}) + \epsilon$$

This further implies,

$$\tilde{b}(\overline{s}, \tilde{b}_{\text{-}}) + (1 - \tau(\overline{s}, \tilde{b}_{\text{-}}))l(\tau(\overline{s}, \tilde{b}_{\text{-}})) > [\tilde{b}(\overline{s}, \tilde{b}_{\text{-}}) + (1 - \tau^*)l^* > \mathcal{B} + (1 - \tau^*)l^* > \frac{P(\overline{s})\tilde{b}_{\text{-}}}{\beta} + g(\overline{s}) + T(\overline{s}, \tilde{b}_{\text{-}}) = \frac{P(\overline{s})\tilde{b}_{\text{-}}}{\beta} + g(\overline{s}) + \epsilon.$$

Combining the previous two inequalities yields a contradiction. The other inequality, $\tilde{b}(\underline{s}, \tilde{b}_{\text{-}}) > \mathcal{B}$ follows from the definition of $\mathcal{B}$. This is because if it was not true then $\tilde{b}(s, \tilde{b}_{\text{-}}) \leq \mathcal{B}$ for all shocks. This implies that the solution is interior. However the only initial conditions that have this property are less than equal to $\mathcal{B}$.

Now define $\overline{\mu}(\tilde{b}(s, \tilde{b}_{\text{-}}))$ as $\max_s \mu(s, \tilde{b}_{\text{-}})$ and $\hat{s}(\tilde{b}_{\text{-}})$ as the shock that achieves this maximum. Now we show that $\hat{\mu}(\tilde{b}(s, \tilde{b}_{\text{-}}))$ is finite for all $b_{\text{-}} \leq \overline{b}$. We show the claim for the natural debt limit.

Let $b^n(s) = (\beta^{-1}P(s) - 1)^{-1}\left[\theta^{\frac{\gamma}{1+\gamma}}\left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}}\left(\frac{\gamma}{1+\gamma}\right) - g(s)\right]$ be the maximum debt supported by a particular shock s. The natural debt limit is defined as $\overline{b}^n = \min_s b^n(s)$. Note that $\lim_{b \to \overline{b}^n} \mu(\tilde{b}_{\text{-}}) = \infty$

Now choose s such that $b^n(s) > \overline{b}^n$ and consider the debt choice next period for the same shock s when it comes in with debt $\overline{b}^n$.

Suppose it chooses $\tilde{b}(s,\overline{b}^n) = \overline{b}^n$, then taxes will have to be set to $\frac{\gamma}{1+\gamma}$ and the tax income will be $\frac{\gamma}{1+\gamma} l(\frac{\gamma}{1+\gamma}) = \theta^{\frac{\gamma}{1+\gamma}} \left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}} \left(\frac{\gamma}{1+\gamma}\right)$. The budget constraint will then imply that,

$$\frac{\overline{b}^n P(s)}{\beta} + g(s) = \theta^{\frac{\gamma}{1+\gamma}} \left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}} \left(\frac{\gamma}{1+\gamma}\right) + \overline{b}^n$$

$$\overline{b}^n = (P(s)\beta^{-1} - 1)^{-1} \left(\theta^{\frac{\gamma}{1+\gamma}} \left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}} \left(\frac{\gamma}{1+\gamma}\right) - g(s)\right)$$

However the right hand side is the definition of $b^n(s)$ and,

$$b^n(s) > \overline{b}^n.$$

Thus we have a contradiction and the optimal choice of debt at the natural debt limit $\tilde{b}(s,\overline{b}^n) < \overline{b}^n$.

This inturn means that $\lim_{\tilde{b}\to\overline{b}^n} \overline{\mu}(\tilde{b}) < \infty$.

Now note that $\overline{\mu}(\tilde{b}_{\text{-}}) - \mu(\tilde{b}_{\text{-}})$ is continuous on $[\check{b}, \quad \overline{b}^n]$ and is bounded below by zero, therefore attains a minimum at $\tilde{b}^{min}$. Let $\delta = \hat{\mu}(\tilde{b}^{min}) - \mu(\tilde{b}^{min}) > \eta > 0$. If this was not true then $P(s) \in \mathcal{P}^*$ as μ will have an absorbing state.

Let $\mu(\omega, n) = \omega - n$. This is the value of μ when debt falls below $\mathcal{B}$.

Now consider any initial $\tilde{b}_{\text{-}} \in [\mathcal{B}, \overline{b}^n]$. If $\tilde{b}_{\text{-}} \leq \check{b}$, then by lemma 14, we know that $\mathcal{B}$ will be reached in one shock. Otherwise if $\tilde{b}_{\text{-}} > \check{b}$, we can construct a sequence of shocks $s_t = \hat{s}(\tilde{b}_{t-1})$ of length $N = \frac{\mu(\omega,n) - \mu(\tilde{b}_{\text{-}})}{\delta}$. There exits $t < N$ such that $\tilde{b}_t < \check{b}$, otherwise,

$$\mu_t > \mu(\tilde{b}_{\text{-}}) + N\delta > \mu(\omega, n)$$

Thus we can reach $\mathcal{B}$ in finite steps. Since shocks are i.i.d, this is an almost sure statement. At $\mathcal{B}$, transfers are strictly positive for some shocks $T_t > 0$ a.s. and taxes are given by τ^*.

Now consider the payoffs $P \in \mathcal{P}^*$ such that the associated steady state debt $b^* > \mathcal{B}$. Under the guess $T_t = 0$, the same algebra as in Theorem 2 goes through and we can show that $\tilde{b}_{\text{-}} = b^*$ is a steady state for the heterogeneous agent economy. Thus the heterogeneous agent economy for a given $P \in \mathcal{P}^*$ has a continuum of steady states given by the set $[\overline{b}, \quad \mathcal{B}] \cup \{b^*\}$.

In the region $\tilde{b}_{\text{-}} > b^*$, as before μ_t is supermartingale bounded below by b^*. Since there is a unique fixed point in the region $\tilde{b}_{\text{-}} \in [b^*, \overline{b}^n]$, μ_t converges to μ^* associated with b^*. Transfers are zero and taxes are given by τ^{**}

$$\tau^{**} = \frac{\gamma\mu^*}{(1+\gamma)\mu^* - 1} \tag{63}$$

In the region $[\mathcal{B}, \quad b^*]$ the outcomes depend on the exact sequence of shocks we can show that μ_t is a submartingale. This follows from the observation that for all $\tilde{b}_- > \mathcal{B}$, we have $T(s) = 0$ and the outcomes from the representative agent economy allow us to order $\mu(s)$ relative $P(s)$. At $\tilde{b}_- = \mathcal{B}$, $\mu(s) = \omega - n$ and is constant. Thus in the region $[\mathcal{B}, \quad B^*]$, μ_t is sub martingale and it converges. However if $\tilde{b}_t$ gets sufficiently close to $\check{b}$, then it can converge to $\mathcal{B}$ and if it gets sufficiently close to b^*, it can converge to b^*. Either of this can happen with strictly positive probability. ■

A.6 Proof of Theorem ??

The Bellman equation for the optimal planners problem with log quadratic preferences and IID shocks can be written as

$$V(x,\rho) = \max_{c_1,c_2,l_1,x',\rho'} \sum_s \pi(s)\left[\alpha_1\left(\log c_1(s) - \frac{l_1(s)^2}{2}\right) + \alpha_2 \log c_2(s) + \beta V(x'(s),\rho'(s))\right]$$

subject to the constraints

$$1+\rho'(s)[l_1(s)^2-1] + \beta x'(s) - \frac{x\frac{P(s)}{c_2(s)}}{\mathbb{E}[\frac{P(s)}{c_2(s)}]} = 0 \tag{64}$$

$$\mathbb{E}\frac{P(s)}{c_1(s)}(\rho'(s)-\rho) = 0 \tag{65}$$

$$\theta_1(s)l_1(s) - c_1(s) - c_2(s) - g = 0 \tag{66}$$

$$\rho'(s)c_2(s) - c_1(s) = 0 \tag{67}$$

where the $\pi(s)$ is the probability distribution of the aggregate state s. If we let $\pi(s)\mu(s)$, λ, $\pi(s)\xi(s)$ and $\pi(s)\phi(s)$ be the Lagrange multipliers for the constraints (64)-(67) respectively then we obtain the following FONC for the planners problem [20]

$c_1(s)$:

$$\frac{\alpha_1\pi(s)}{c_1(s)} - \frac{\lambda\pi(s)}{c_1(s)^2}(\rho'(s)-\rho) - \pi(s)\xi(s) - \pi(s)\phi(s) = 0 \tag{68}$$

$c_2(s)$:

$$\frac{\alpha_2\pi(s)}{c_2(s)} + \frac{xP(s)\pi(s)}{c_2(s)^2\mathbb{E}[\frac{P}{c_2}]}\left[\mu(s) - \frac{\mathbb{E}[\mu\frac{P}{c_2}]}{\mathbb{E}[\frac{P}{c_2}]}\right] - \pi(s)\xi(s) + \pi(s)\rho'(s)\phi(s) = 0 \tag{69}$$

$l_1(s)$:

$$-\alpha_1\pi(s)l_1(s) + 2\mu(s)\pi(s)\rho'(s)l_1(s) + \theta_1(s)\pi(s)\xi(s) = 0 \tag{70}$$

$x'(s)$:

$$V_x(x'(s),\rho'(s)) + \mu(s) = 0 \tag{71}$$

$\rho'(s)$:

$$\beta V_\rho(x'(s),\rho'(s)) + \frac{\lambda\pi(s)}{c_1(s)} + \mu(s)[l_1(s)^2-1] + \pi(s)\phi(s)c_2(s) = 0 \tag{72}$$

[20] Appendix A.7 discuses the associated second order conditions that ensure these policies are optimal

In addition there are two envelope conditions given by

$$V_x(x,\rho) = -\sum_{s'} \frac{\mu(s')\pi(s')\frac{P(s)}{c_2(s')}}{\mathbb{E}[\frac{P}{c_2}]} = -\frac{\mathbb{E}[\mu\frac{P}{c_2}]}{\mathbb{E}[\frac{P}{c_2}]} \tag{73}$$

$$V_\rho(x,\rho) = -\lambda\mathbb{E}[\frac{P}{c_1}] \tag{74}$$

In the steady state, we need to solve for a collection of allocations, initial conditions and Lagrange multipliers $\{c_1(s), c_2(s), l_1(s), x, \rho, \mu(s), \lambda, \xi(s), \phi(s)\}$ such that equations (64)-(74) are satisfied when $\rho'(s) = \rho$ and $x'(s) = x$. It should be clear that if we replace $\mu(s) = \mu$, equation (71) and the envelope condition with respect to x is always satisfied. Additionally under this assumption equation (69) simplifies significantly,since

$$\frac{xP(s)\pi(s)}{c_2(s)^2\mathbb{E}[\frac{P}{c_2}]}\left[\mu(s) - \frac{\mathbb{E}[\mu\frac{P}{c_2}]}{\mathbb{E}[\frac{P}{c_2}]}\right] = 0$$

The first order conditions for a steady can then be written simply as

$$1 + \rho[l_1(s)^2 - 1] + \beta x - \frac{xP(s)}{c_2(s)\mathbb{E}[\frac{P}{c_2}]} = 0 \tag{75}$$

$$\theta_1(s)l_1(s) - c_1(s) - c_2(s) - g = 0 \tag{76}$$

$$\rho c_2(s) - c_1(s) = 0 \tag{77}$$

$$\frac{\alpha_1}{c_1(s)} - \xi(s) - \phi(s) = 0 \tag{78}$$

$$\frac{\alpha_2}{c_2(s)} - \xi(s) + \rho\phi(s) = 0 \tag{79}$$

$$[2\mu\rho - \alpha_1]l_1(s) + \theta_1(s)\xi(s) = 0 \tag{80}$$

$$\lambda\left(\frac{P(s)}{c_1(s)} - \beta\mathbb{E}\left[\frac{P}{c_1}\right]\right) + \mu[l_1(s)^2 - 1] + \phi(s)c_2(s) = 0 \tag{81}$$

We can rewrite equation (78) as

$$\frac{\alpha_1}{c_2(s)} - \rho\xi(s) - \rho\phi(s) = 0$$

by substituting $c_1(s) = \rho c_2(s)$. Adding this to equation (79) and normalizing $\alpha_1 + \alpha_2 = 1$ we obtain

$$\xi(s) = \frac{1}{(1+\rho)\, c_2(s)} \tag{82}$$

which we can use to solve for $\phi(s)$ as

$$\phi(s) = \frac{\alpha_1 - \rho\alpha_2}{(\rho(1+\rho))\, c_2(s)} \tag{83}$$

From equation (75) we can solve for $l_1(s)^2 - 1$ as

$$l_1(s)^2 - 1 = \frac{x}{\rho \mathbb{E}[\frac{P}{c_2}]} \left(\frac{P(s)}{c_2(s}) - \beta \mathbb{E}\left[\frac{P}{c_2}\right] \right) - \frac{1}{\rho}$$

This can be used along with equations (81) and (83) to obtain

$$\left(\frac{\lambda}{\rho} + \frac{\mu x}{\rho \mathbb{E}[\frac{P}{c_2}]} \right) \left(\frac{P(s)}{c_2(s)} - \beta \mathbb{E}\left[\frac{P}{c_2}\right] \right) = \frac{\mu}{\rho} + \frac{\rho \alpha_2 - \alpha_1}{\rho(1+\rho)}$$

Note that the LHS depends on s while the RHS does not, hence the solution to this equation is

$$\lambda = -\frac{\mu x}{\mathbb{E}[\frac{P}{c_2}]} \tag{84}$$

and

$$\mu = \frac{\alpha_1 - \rho \alpha_2}{1+\rho} \tag{85}$$

Combining these with equation (80) we quickly obtain that

$$\left[2\rho \frac{\alpha_1 - \rho\alpha_2}{1+\rho} - \alpha_1 \right] l_1(s) + \frac{\theta_1(s)}{(1+\rho)\, c_2(s)} = 0$$

Then solving for $l_1(s)$ gives

$$l_1(s) = \frac{\theta_1(s)}{(\alpha_1(1-\rho) + 2\rho^2 \alpha_2)\, c_2(s)}$$

Remark 2 *Note that the labor tax rate is given by* $1 - \frac{c_1(s) l_1(s)}{\theta(s)}$. *The previous expression shows that labor taxes are constant at the steady state. This property holds generally for CES preferences separable in consumption and leisure*

This we can plug into the aggregate resource constraint (76) to obtain

$$l_1(s) = \left(\frac{1+\rho}{\alpha_1(1-\rho) + 2\rho^2\alpha_2} \right) \frac{1}{l_1(s)} + \frac{g}{\theta_1(s)}$$

letting $C(\rho) = \frac{1+\rho}{\alpha_1(1-\rho)+2\rho^2\alpha_2}$ we can then solve for $l_1(s)$ as

$$l_1(s) = \frac{g \pm \sqrt{g^2 + 4C(\rho)\theta_1(s)^2}}{2\theta_1(s)}$$

The marginal utility of agent 2 is then

$$\frac{1}{c_2(s)} = \left(\frac{1+\rho}{C(\rho)} \right) \left(\frac{g \pm \sqrt{g^2 + 4C(\rho)\theta_1(s)^2}}{2\theta_1(s)^2} \right)$$

Note that in order for either of these terms to be positive we need $C(\rho) \geq 0$ implying that there is only one economically meaningful root. Thus

$$l_1(s) = \frac{g + \sqrt{g^2 + 4C(\rho)\theta_1(s)^2}}{2\theta_1(s)} \tag{86}$$

and

$$\frac{1}{c_2(s)} = \left(\frac{1+\rho}{C(\rho)}\right)\left(\frac{g + \sqrt{g^2 + 4C(\rho)\theta_1(s)^2}}{2\theta_1(s)^2}\right) \tag{87}$$

A steady state is then a value of ρ such that

$$x(s) = \frac{1 + \rho[l_1(\rho, s)^2 - 1]}{\frac{P(s)/c_2(\rho,s)}{\mathbb{E}[\frac{P}{c_2}](\rho)} - \beta} \tag{88}$$

s independent of s.

The following lemma, which orders consumption and labor across states, will be useful in proving the parts of theorem **??**. As a notational aside we will often use $\theta_{1,l}$ and $\theta_{1,h}$ to refer to $\theta_1(s_l)$ and $\theta_1(s_h)$ respectively. Where s_l refers to the low TFP state and s_h refers to the high TFP state.

Lemma 15 *Suppose that $\theta_1(s_l) < \theta_2(s_h)$ and ρ such that $C(\rho) > 0$ then*

$$l_{1,l} = \frac{g + \sqrt{g^2 + 4C(\rho)\theta_{1,l}^2}}{2\theta_{1,l}} > \frac{g + \sqrt{g^2 + 4C(\rho)\theta_{1,h}^2}}{2\theta_{1,h}} = l_{1,h}$$

and

$$\frac{1}{c_{2,l}} = \frac{1+\rho}{C(\rho)}\frac{g + \sqrt{g^2 + 4C(\rho)\theta_{1,l}^2}}{2\theta_{1,l}^2} > \frac{1+\rho}{C(\rho)}\frac{g + \sqrt{g^2 + 4C(\rho)\theta_{1,h}^2}}{2\theta_{1,h}^2} = \frac{1}{c_{2.h}}$$

Proof. The results should follow directly from showing that the function

$$l_1(\theta) = \frac{g + \sqrt{g^2 + 4C(\rho)\theta}}{2\theta}$$

is decreasing in θ. Taking the derivative with respect to θ

$$\begin{aligned}
\frac{dl_1}{d\theta}(\theta) &= -\frac{g}{2\theta^2} - \frac{\sqrt{g + 4C(\rho)\theta^2}}{2\theta^2} + \frac{4C(\rho)\theta}{2\theta\sqrt{g^2 + 4C(\rho)\theta^2}} \\
&= -\frac{g}{2\theta^2} - \frac{g + 4C(\rho)\theta^2 - 4C(\rho)\theta^2}{2\theta^2\sqrt{g^2 + 4C(\rho)\theta^2}} \\
&= -\frac{g}{2\theta^2} - \frac{g}{2\theta^2\sqrt{g^2 + 4C(\rho)\theta^2}} < 0
\end{aligned}$$

That $\frac{1}{c_{2,l}} > \frac{1}{c_{2,h}}$ follows directly. ■

Now we use these lemma to prove the part 1 and part 2 of theorem 5

Proof.

[Part 1.] For a riskfree bond when $P(s) = 1$. In order for there to exist a ρ such that equation (88) is independent of the state (and hence have a steady state) we need the existence of root for the following function

$$f(\rho) = \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} - \frac{\frac{1/c_2(\rho,s_h)}{\mathbb{E}[\frac{P}{c_2}](\rho)} - \beta}{\frac{1/c_2(\rho,s_l)}{\mathbb{E}[\frac{P}{c_2}](\rho)} - \beta}$$

From lemma 15 we can conclude that

$$1 + \rho[l_1(\rho, s_l)^2 - 1] > 1 + \rho[l_1(\rho, s_h)^2 - 1] \tag{89}$$

and

$$\frac{1/c_2(\rho, s_l)}{\mathbb{E}[\frac{P}{c_2}](\rho)} - \beta > \frac{1/c_2(\rho, s_h)}{\mathbb{E}[\frac{P}{c_2}](\rho)} - \beta \tag{90}$$

for all $\rho > 0$ such that $C(\rho) \geq 0$. To begin with we will define $\underline{\rho}$ such that $C(\rho) > 0$ for all $\rho > \underline{\rho}$. Note that we will have to deal with two different cases.

$\alpha_1(1 - \rho) + 2\rho^2\alpha_2 > 0$ **for all** $\rho \geq 0$**:** In this case we know that $C(\rho) \geq 0$ for all ρ and is bounded above and thus we will let $\underline{\rho} = 0$.

$\alpha_1(1 - \rho) + 2\rho^2\alpha_2 = 0$ **for some** $\rho > 0$**:** In this case let $\underline{\rho}$ be the largest positive root of $\alpha_1(1 - \rho) + 2\rho^2\alpha_2$. Note that $\lim_{\rho \to \underline{\rho}^+} C(\rho) = \infty$

With this we note that[21]

$$\lim_{\rho \to \underline{\rho}^+} \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} = 1$$

We can also show that

$$\lim_{\rho \to \underline{\rho}^+} \frac{\frac{1/c_2(\rho,s_h)}{\mathbb{E}[\frac{P}{c_2}](\rho)} - \beta}{\frac{1/c_2(\rho,s_l)}{\mathbb{E}[\frac{P}{c_2}](\rho)} - \beta} < 1$$

which implies that $\lim_{\rho \to \underline{\rho}^+} f(\underline{\rho}) > 0$.

Taking the limit as $\rho \to \infty$ we see that $C(\rho) \to 0$, given that $\frac{g}{\theta(s)} < 1$, we can then conclude that

$$\lim_{\rho \to \infty} 1 + \rho[l_1(\rho, s)^2 - 1] = -\infty$$

[21] In the first case $\underline{\rho} = 0$ and in the second case $l_1(\rho, s_l) = l_1(\rho, s_h)$ as $\rho \to \underline{\rho}^+$

Thus, there exists $\overline{\rho}$ such that $1+\overline{\rho}[l_1(\overline{\rho},s_l)^2-1]=0$. [22] From equation (89), we know that

$$0=1+\overline{\rho}[l_1(\overline{\rho},s_l)^2-1]>1+\overline{\rho}[l_1(\overline{\rho},s_h)^2-1]$$

which implies in the limit

$$\lim_{\rho\to\overline{\rho}^-}\frac{1+\rho[l_1(\rho,s_h)^2-1]}{1+\rho[l_1(\rho,s_l)^2-1]}=-\infty$$

which along with

$$\frac{\frac{1/c_2(\rho,s_h)}{\mathbb{E}[\frac{P}{c_2}]}-\beta}{\frac{1/c_2(\rho,s_l)}{\mathbb{E}[\frac{P}{c_2}]}-\beta}\geq -1$$

allows us to conclude that $\lim_{\rho\to\overline{\rho}^-}f(\rho)=-\infty$. The intermediate value theorem then implies that there exists ρ_{SS} such that $f(\rho_{SS})=0$ and hence that ρ_{SS} is a steady state.

Finally, as $\rho_{SS}<\overline{\rho}$ we know that

$$1+\rho_{SS}[l_1(\rho_{SS},s_l)-1]>0$$

as $\frac{1/c_2(\rho,s_l)}{\mathbb{E}[\frac{P}{c_2}]}>1$ we can conclude

$$x_{SS}=\frac{1+\rho_{SS}[l_1(\rho_{SS},s_l)-1]}{\frac{1/c_2(\rho,s_l)}{\mathbb{E}[\frac{P}{c_2}](\rho)}-\beta}>0$$

implying that the government will hold assets in the steady state (under the normalization that agent 2 holds no assets).

[Part 2] As noted before, since $g/\theta(s)<1$ for all s we have

$$\lim_{\rho\to\infty}1+\rho[l_1(\rho,s)^2-1]=-\infty$$

Thus, there exists ρ_{SS} such that

$$0>1+\rho_{SS}[l_1(\rho_{SS},s_l)^2-1]>1\rho_{SS}[l_1(\rho_{SS},s_h)^2-1]$$

It is then possible to choose $P(s)$ such that $\beta<\frac{P(s)/c_2(\rho_{SS},s)}{\mathbb{E}[\frac{P}{c_2}]}$ such that

$$1>\frac{1+\rho_{SS}[l_1(\rho_{SS},s_l)^2-1]}{1+\rho_{SS}[l_1(\rho_{SS},s_h)^2-1]}=\frac{\frac{P(s_l)/c_2(\rho_{SS},s_l)}{\mathbb{E}[\frac{P}{c_2}]}-\beta}{\frac{P(s_h)/c_2(\rho_{SS},s_h)}{\mathbb{E}[\frac{P}{c_2}]}-\beta} \tag{91}$$

[22] This can be seen from the fact $\lim_{\rho\to\underline{\rho}^+}1+\rho[l_1(\rho,s_l)^2-1]>0$ and $\lim_{\rho\to\infty}1+\rho[l_1(\rho,s_l)^2-1]>-\infty$, thus $\overline{\rho}$ exists in $(\underline{\rho},\infty)$

Implying that for Payoff shocks $P(s)$, ρ_{SS} is a steady state level for the ratio of marginal utilities, with steady state marginal utility weighted government debt

$$x_{SS} = \frac{1 + \rho_{SS}[l_1(\rho_{SS}, s_l)^2 - 1]}{\frac{P(s_l)/c_2(\rho_{SS}, s_l)}{\mathbb{E}[\frac{P}{c_2}]} - \beta} < 0$$

Thus, in the steady state, the government is holding debt, under the normalization that the unproductive worker holds no assets. Note this imposes a restriction of $\frac{P(s_l)}{P(s_h)}$.

$$\frac{P(s_l)c_2^{-1}(\rho_{SS}, s_l) - \beta \mathbb{E} P c_2^{-1}}{P(s_h)c_2^{-1}(\rho_{SS}, s_h) - \beta \mathbb{E} P c_2^{-1}} < 1$$

or

$$\frac{P(s_l)}{P(s_h)} < \frac{c_2^{-1}(\rho_{SS}, s_h)}{c_2^{-1}(\rho_{SS}, s_l)} < 1$$

or

Thus $P(s_l) < P(s_h)$ i.e payoffs have to be sufficiently procyclical.

■

■

A.7 Linearization Algorithm

This section will outline our numerical methods used to solve for and linearize around the steady state in the case of a 2 state iid process for the aggregate state.

$$V(\boldsymbol{x}, \boldsymbol{\rho}) = \max_{c_i(s), l_i(s), \boldsymbol{x}'(s), \boldsymbol{\rho}'(s)} \sum_s P(s) \left(\left[\sum_i \pi_i \alpha_i U(c_i(s), l_i(s)) \right] + \beta(s) V(\boldsymbol{x}'(s), \boldsymbol{\rho}'(s)) \right) \tag{92}$$

$$U_{c,i}(s)c_i(s) + U_{l,i}(s)l_i(s) - \rho_i'(s)\left[U_{c,1}(s)c_1(s) + U_{l,1}(s)l_1(s)\right] + \beta(s)x_i'(s) = \frac{x_i U_{c,i}(s)}{\mathbb{E}U_{c,i}} \tag{93a}$$

$$\sum_s \Pr(s) U_{c,1}(s)(\rho_i(s) - \rho_i) = 0 \tag{93b}$$

$$\frac{\rho'(s)}{\theta_1(s)} U_{l,1}(s) = \frac{1}{\theta_i(s)} U_{l,i}(s) \tag{93c}$$

$$\sum_{j=0}^{I} \pi_j c_j(s) + g(s) = \sum_{j=0}^{I} \pi_j \theta_j(s) l_j(s) \tag{93d}$$

$$U_{c,i}(s) = \rho_i'(s) U_{c,1}(s) \tag{93e}$$

For $i = 2, \ldots, I$. Note that some of the constraints have been modified a little for ease of differentiation. Associated with these constraints we have the Lagrange multipliers $\Pr(s)\mu_i'(s)$, λ_i,$\Pr(s)\phi_i(s)$,$\Pr(s)\xi(s)$, and $P(s)\zeta_i(s)$.

The first order conditions with respect to the choice variables are as follows (note we will be using the notation $\mathbb{E}z$ to represent $\sum_s \Pr(s)z(s)$ for some variable z)

$c_1(s)$**:**

$$\pi_1\alpha_1 U_{c,1}(s) + \sum_{i=2}^{I}\left(\mu_i'(s)\rho_i'(s)\right)\left[U_{cc,1}(s)c_1(s) + U_{c,1}(s)\right] + \lambda U_{cc,1}(s)\sum_{i=2}^{I}(\rho_i'(s) - \rho_i) - \pi_1\xi(s) + \sum_{i=2}^{N}\zeta_i(s)\rho_i'(s)U_{cc,1}(s) = 0 \tag{94a}$$

$c_i(s)$**:** for $i \geq 2$

$$\pi_i\alpha_i U_{c,i}(s) - \mu_i'(s)\left[U_{cc,i}(s)c_i(s) + U_{c,i}(s)\right] + \frac{x_i U_{cc,i}(s)}{\mathbb{E}U_{c,i}}\left(\mu_i'(s) - \frac{\mathbb{E}\mu_i' U_{c,i}}{\mathbb{E}U_{c,i}}\right) - \pi_i\xi(s) - \zeta_i(s)U_{cc,i}(s) = 0 \tag{94b}$$

$l_1(s)$**:**

$$\pi_1\alpha_1 U_{l,1}(s) + \sum_{i=2}^{I}\mu_i'(s)\rho_i(s)\left[U_{ll,1}(s)l_1(s) + U_{l,1}(s)\right] - \sum_{i=2}^{N}\frac{\rho_i'(s)\phi_i(s)}{\theta_1(s)}U_{ll,1}(s) + \pi_1\theta_1(s)\xi(s) = 0 \tag{94c}$$

$l_2(s)$**:**

$$\pi_i\alpha_i U_{l,i}(s) - \mu_i'(s)\left[U_{ll,i}(s)l_i(s) + U_{l,i}(s)\right] + \frac{\phi_i(s)}{\theta_i(s)}U_{ll,i}(s) + \pi_i\theta_i(s)\xi(s) = 0 \tag{94d}$$

$\rho_i'(s)$**:**

$$\beta(s)V_{\rho_i}(\boldsymbol{x}'(s), \boldsymbol{\rho}_i'(s)) + \mu_i'(s)\left[U_{c,1}(s)c_1(s) + U_{l,1}(s)l_1(s)\right] + \lambda_i U_{c,1}(s) - \phi_i(s)\frac{U_{l,1}(s)}{\theta_1(s)} + U_{c,1}(s)\zeta_i(s) = 0 \tag{94e}$$

$x_i'(s)$**:**

$$V_{x_i}(\boldsymbol{x}'(s), \boldsymbol{\rho}'(s)) - \mu_i'(s) = 0. \tag{94f}$$

Equations (93a)-(93e) and (94a)-(94e) then define the necessary conditions for an interior maximization of the planners problem for the state $(\boldsymbol{x}, \boldsymbol{\rho})$. In addition to these we have the two envelop conditions

$$V_{x_i}(\boldsymbol{x}, \boldsymbol{\rho}) = \frac{\sum_s P(s)\mu_i'(s)U_{c,i}(s)}{\mathbb{E}U_{c,i}(s)} = \frac{\mathbb{E}\mu_i' U_{c,i}}{\mathbb{E}U_{c,i}}, \tag{95a}$$

and

$$V_{\rho_i}(\boldsymbol{x}, \boldsymbol{\rho}) = -\lambda_i \mathbb{E}U_{c,1}. \tag{95b}$$

In order to check local stability we linearize locally around the steady state. Furthermore we find that the policy functions have better numerical properties when the state variables are chosen to be $(\boldsymbol{\mu}, \boldsymbol{\rho})$ rather than $(\boldsymbol{x}, \boldsymbol{\rho})$, and thus, we will proceed with the linearization procedure using $(\boldsymbol{\mu}, \boldsymbol{\rho})$ as the endogenous state vector. The evolution of the state variable $\boldsymbol{\mu}$ must follow the weighted martingale

$$\mu_i - \frac{\sum_s P(s)\mu_i'(s)U_{c,i}(s)}{\sum_s P(s)U_{c,i}(s)} = 0. \tag{96}$$

The optimal policy function, which we will denote as $z(\boldsymbol{\mu}, \boldsymbol{\rho})$, must satisfy $F(z, y, g(z)) = 0$ where F represents the system of equations (93a)-(94e)and (96), y is the state vector $(\boldsymbol{x}, \boldsymbol{\rho})$, and g is the mapping of the policies into functions of future variables, namely $\boldsymbol{x}'(s)$ and $V_{\boldsymbol{\rho}}(\boldsymbol{\mu}'(s), \boldsymbol{\rho}(s))$. In other words

$$g(z) = \begin{pmatrix} \boldsymbol{x}(\boldsymbol{\mu}'(1), \boldsymbol{\rho}'(1)) \\ V_{\boldsymbol{\rho}}(\boldsymbol{\mu}'(1), \boldsymbol{\rho}'(1) \\ \boldsymbol{x}(\boldsymbol{\mu}'(2), \boldsymbol{\rho}'(2)) \\ V_{\boldsymbol{\rho}}(\boldsymbol{\mu}'(2), \boldsymbol{\rho}'(2)) \end{pmatrix}.$$

Finally $z(\boldsymbol{\mu}, \boldsymbol{\rho})$ are the stacked variables $\{c_1(s), c_i(s), l_1(s), l_i(s), \boldsymbol{x}, \boldsymbol{\rho}'(s), \boldsymbol{\mu}'(s), \boldsymbol{\lambda}, \boldsymbol{\phi}(s), \xi(s), \boldsymbol{\zeta}(s)\}$. The optimal policy function is then a function $z(y)$ that satisfies the relationship $F(z(y), y, g(z(y))) = 0$. Taking total derivatives around the steady state $\overline{y}$ and $\overline{z} = z(\overline{y})$

$$D_zF(\overline{z}, \overline{y}, g(\overline{z}))D_yz(\overline{y}) + D_yF(\overline{z}, \overline{y}, g(\overline{z})) + D_gF(\overline{z}, \overline{y}, g(\overline{z}))Dg(\overline{z})D_yz(\overline{z}) = 0$$

In order to linearize $z(y)$ around the steady state $\overline{y}$ we need to compute $D_yz(\overline{y})$. The envelope condition (95b) tell us that $V_{\boldsymbol{\rho}}$ can be computed from the optimal policies, i.e.

$$\begin{pmatrix} \boldsymbol{x}(\boldsymbol{\mu}, \boldsymbol{\rho}) \\ V_{\boldsymbol{\rho}}(\boldsymbol{\mu}, \boldsymbol{\rho}) \end{pmatrix} = w(z(\boldsymbol{\mu}, \boldsymbol{\rho})) = \begin{pmatrix} \boldsymbol{x} \\ -\boldsymbol{\lambda}\mathbb{E}\left[U_{c,1}\right] \end{pmatrix}$$

If we let Φ_s be the matrix that maps $z(\boldsymbol{\mu}, \boldsymbol{\rho})$ into $\begin{pmatrix} \boldsymbol{\mu}'(s) \\ \boldsymbol{\rho}'(s) \end{pmatrix}$ then we can write $g(\boldsymbol{\mu}, \rho)$ using z and w as follows

$$g(z) = \begin{pmatrix} w(z(\Phi_1 z)) \\ w(z(\Phi_2 z)) \end{pmatrix}$$

taking derivatives we quickly obtain that

$$\begin{aligned} D_z g(\overline{z}) &= \begin{pmatrix} Dw(z(\Phi_1\overline{z})) & 0 \\ 0 & Dw(z(\Phi_2\overline{z})) \end{pmatrix} \begin{pmatrix} D_yz(\Phi_1\overline{z}) & 0 \\ 0 & D_yz(\Phi_1\overline{z}) \end{pmatrix} \underbrace{\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}}_{\Phi} \\ &= \begin{pmatrix} Dw(\overline{z}) & 0 \\ 0 & Dw(\overline{z}) \end{pmatrix} \begin{pmatrix} D_yz(\overline{y}) & 0 \\ 0 & D_yz(\overline{y}) \end{pmatrix} \Phi \\ &= \begin{pmatrix} Dw(\overline{z})D_yz(\overline{y}) & 0 \\ 0 & Dw(\overline{z})D_yz(\overline{y}) \end{pmatrix} \Phi \end{aligned}$$

We can then go back to our original matrix equation to obtain

$$D_zF(\overline{z}, \overline{y}, \overline{w})D_yz(\overline{y}) + D_yF(\overline{z}, \overline{y}, \overline{w}) + D_wF(\overline{z}, \overline{y}, \overline{w}) \begin{pmatrix} Dw(\overline{z})D_yz(\overline{y}) & 0 \\ 0 & Dw(\overline{z})D_yz(\overline{y}) \end{pmatrix} \Phi D_yz(\overline{z}) = 0, \tag{97}$$

where $\overline{w} = g(\overline{z}) = w(\overline{z})$. This is now a non-linear matrix equation for $D_y z(\overline{y})$, where all the other terms can be computed using the steady state values $\overline{z}$ and $\overline{y}$ (note $g(\overline{z})$ is known from the envelope conditions at the steady state). Furthermore, $D_y z(\overline{y})$ gives us the linearization of the policy rules since to first order

$$z \approx \overline{z} + D_y z(\overline{y})(y - \overline{y})$$

Our procedure for computing the linearization proceeds as follows

1. Find the steady state by solving the system of equations (25). Numerically, we have found that this is very robust to the parameters of the model.

2. Compute $D_z F(\overline{z}, \overline{y}, g(\overline{z}))$, $D_z F(\overline{z}, \overline{y}, g(\overline{z}))$ and $D_v F(\overline{z}, \overline{y}, g(\overline{z}))$ by numerically differentiating F. This is straightforward using auto-differentiation.

3. Compute $Dw(\overline{z})$ using auto-differentiation.

4. Construct a matrix equation as follows. Given policies $A = Dw(\overline{z}) D_y z(\overline{y})$ (these are the linearized policies of $\boldsymbol{x}$ and $V_{\boldsymbol{\rho}}$ with respect to $(\boldsymbol{\mu}, \boldsymbol{\rho})$), it is possible to solve for $D_y z(\overline{y})$ from
$$D_y z(\overline{z}) = -\left(D_z F(\overline{z}, \overline{y}, \overline{w}) + D_w F(\overline{z}, \overline{y}, \overline{w}) \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \Phi \right)^{-1} D_y F(\overline{z}, \overline{y}, \overline{w})$$
We wish to find an A such that
$$A = Dw(\overline{z}) D_y z(\overline{z})$$

Given the linearized policy rules it is then possible to evaluate the local stability of the steady state. We find that in the absence of discount factor shocks the steady state is stable generically across the parameter space.

This linearization can be used to construct the bordered hessian of the problem (22) at the steady state. We can then apply second order tests to verify that the first order necessary conditions are sufficient.

A.8 Proof for Theorem 6

Proof.

The state at time t can be written as

$$\hat{\Psi}_t = B_t B_{t-1} \cdots B_1 \hat{\Psi}_0.$$

where the B_i are all random variables being $B(s)$ with probability $\Pr(s)$. Taking expectations and applying independence we then obtain

$$\mathbb{E}_0[\hat{\Psi}_t] = \mathbb{E}_0[B_t B_{t-1} \cdots B_1]\hat{\Psi}_0 \tag{98}$$

$$= \mathbb{E}[B_t]\mathbb{E}[B_{t-1}] \cdot \mathbb{E}[B_1]\hat{\Psi}_0 \tag{99}$$

$$= \overline{B}^t \hat{\Psi}_0 \tag{100}$$

where $\overline{B} = \mathbb{E}B(s)$. If eigenvalues of $\overline{B}$ are positive and strictly less than 1, at least, in expectation the linearized system converges that is

$$\bar{\hat{\Psi}}_{t|0} \equiv \mathbb{E}_0[\hat{\Psi}_t] = \overline{B}^t \hat{\Psi}_0 \to \mathbf{0}. \tag{101}$$

It should be noted that the conditional expectation actually captures a significant portion of the linearized dynamics. The remaining question is does the distribution converge to $\mathbf{0}$. This can be done by analyzing the variance. Let

$$\Sigma_{\Psi,t|0} = \mathbb{E}_0\left[(\hat{\Psi}_t - \bar{\hat{\Psi}}_t)(\hat{\Psi}_{t|0} - \bar{\hat{\Psi}}_{t|0})'\right]$$

or

$$\Sigma_{\Psi,t|0} = \mathbb{E}_0 \hat{\Psi}_t \hat{\Psi}_t' - \bar{\hat{\Psi}}_{t|0} \bar{\hat{\Psi}}_{t|0}'. \tag{102}$$

Note that if eigenvalues of $\overline{B}$ are positive and strictly less that 1, $\bar{\hat{\Psi}}_{t|0}$ converges to 0. Using the independence of $\hat{\Psi}_{t-1}$ and B_t, and $\hat{\Psi}_t = B_t \hat{\Psi}_{t-1}$, we quickly obtain that for large t

$$\Sigma_{\Psi,t|0} \approx \mathbb{E}[B\Sigma_{\Psi,t-1|0}B'.] \tag{103}$$

Showing that $\hat{\Psi}_{t|0} \to \mathbf{0}$ in distribution, amounts to showing that $\Sigma_{\Psi,t|0} \to 0$ for any starting point Σ_Ψ and following the process in equation (103). One can obtain a necessary condition for $\|\Sigma_{\Psi,t|0}\| \to 0$ under the process in equation (103). That process can be rewritten as follows

$$\Sigma_{\Psi,t|0} = \mathbb{E}[B\Sigma_{\Psi,t-1|0}B'] \tag{104}$$

$$= \sum_s \Pr(s)B(s)\Sigma_{\Psi,t-1|0}B(s)' \tag{105}$$

$$= \sum_s \Pr(s)(\overline{B} + (B(s) - \overline{B}))\Sigma_{\Psi,t-1|0}(\overline{B} + (B(s) - \overline{B}))' \tag{106}$$

$$= \overline{B}\Sigma_{\Psi,t-1|0}\overline{B}' + \sum_s \Pr(s)(B(s) - \overline{B})\Sigma_{\Psi,t-1|0}(B(s) - \overline{B})'. \tag{107}$$

This is a deterministic linear system in $\Sigma_{\Psi,t|0}$. Suppose we reshape $\Sigma_{\Psi,t|0}$ as a vector (denoted by $\text{vec}(\Sigma_{\Psi,t|0})$) and let $\hat{B}$ be a (square) matrix such that equation 107 is written as

$$\text{vec}(\Sigma_{\Psi,t|0}) = \hat{B}\text{vec}(\Sigma_{\Psi,t-1|0}).$$

The stability of this system is guaranteed if the (real part) of eigenvalues of $\hat{B}$ are less than 1. ■

We used theorem **??** to verify local stability of a wide range of examples. The typical finding is that the steady state is generically stable and that convergence is slow. In figure VII we plot the comparative statics for the dominant eigenvalue and the associated half-life for a two-agent economy with CES preferences. We set the other parameter to match a Frisch elasticity of 0.5, a real interest rate of 2%, marginal tax rates around 20%, and a 90-10 percentile ratio of wage earnings of 4. In the first exercise, we vary the size of the expenditure shock keeping risk aversion σ at one The x- axis plots the spread in expenditure normalized by the undistorted GDP and reported in percentages. In the bottom panel, we fix the size of shock such that it produces a 5% fall in expenditure fall at risk aversion of one and vary σ from 0.8 to 7. We see that the dominant eigenvalue is everywhere less than one but very close to one, so that the steady state is stable but convergence is slow for reasonable values of curvatures and shocks. We return to this feature in section **??** where we study low frequency components of government debt. Both increasing the size of the shock or risk aversion increases the volatility of the interest rates, speeding up the transition towards the steady state.

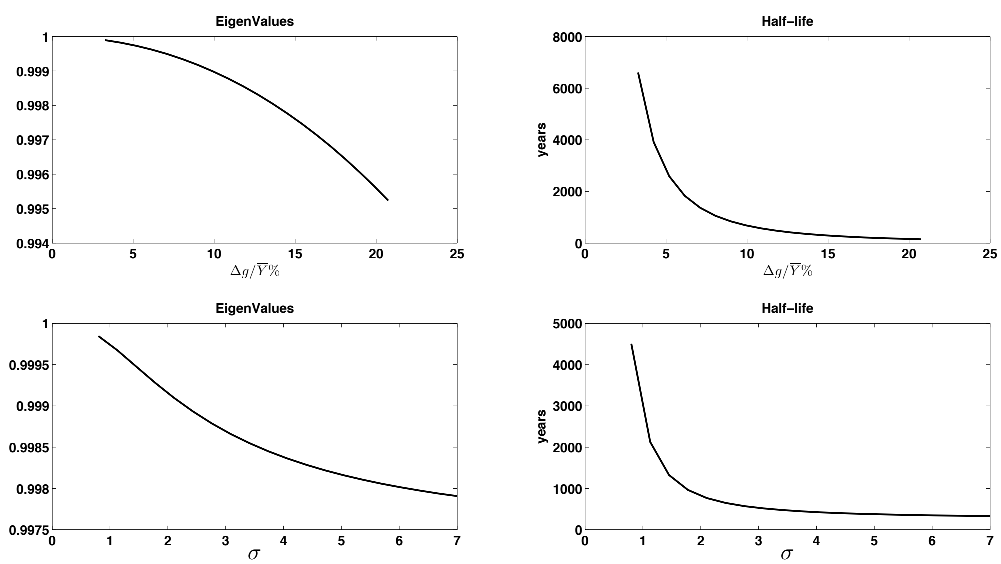

Figure VII: The top (bottom) panel plots the dominant eigenvalue of $\hat{B}$ and the associated half life as we increase the spread between the expenditure levels (risk aversion).

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