

Optimal Fiscal policies in some economies with incomplete asset markets

Anmol Bhandari, David Evans, Mikhail Golosov, Thomas J.
Sargent

September 2013

What do we do?

We study optimal taxation under commitment with

- ▶ **Representative agent**
- ▶ **Incomplete markets**
 - assets with more general payoff structures
- ▶ **Linear tax schedules**
 - Government levies a proportional tax on labor earnings
- ▶ **Aggregate shocks**
 - To productivities, government expenditure etc.

What are we after?

1. Existing literature studies two extremes : economies with complete markets (LS) and economies with risk free bonds (AMSS)
2. We trace out intermediate cases where the asset traded provides partial spanning
3. In particular study how the payoff structure affect long run properties of optimal government policies and equilibrium allocations?

Environment

- ▶ **Uncertainty:** Markov aggregate shocks s_t
- ▶ **Demography:** infinitely lived representative agent plus a benevolent planner
- ▶ **Technology:** Output $\theta_t l_t$ is linear in labor supply
- ▶ **Preferences (Households)**

$$E_0 \sum_{t=0}^{\infty} \beta^t U^i(c_i(s^t), l_i(s^t))$$

Environment, II

- ▶ **Asset markets:** Private sector has complete markets;
Government trades are restricted
- ▶ **Linear Taxes:** Agent i 's tax bill

$$-T_t + \tau_t \theta_{i,t} b_{i,t}$$

▶ **Budget constraints**

- ▶ Agent i : $c_{i,t} + b_{i,t} = (1 - \tau_t) \theta_{i,t} l_{i,t} + R_{t-1} b_{i,t-1} + T_t$
- ▶ Government: $g_t + B_t + T_t = \pi_t \sum_{i=1}^I \pi_i \theta_{i,t} b_{i,t} + R_{t-1} B_{t-1}$

▶ **Market Clearing**

- ▶ Goods: $\sum_{i=1}^I \pi_i c_{i,t} + g_t = \sum_{i=1}^I \pi_i \theta_{i,t} b_{i,t}$
- ▶ Assets: $\sum_{i=1}^I \pi_i b_{i,t} + B_t = 0$

- ▶ **Initial conditions:** Distribution of assets $\{b_{i,-1}\}_i$ and B_{-1}

Ramsey Problem

Definition

Allocation, price system, government policy: Standard

Definition

Competitive equilibrium: Given $(\{b_{i,-1}\}_i, B_{-1})$ and $\{\tau_t, T_t\}_{t=0}^{\infty}$ all allocations are chosen optimally, markets clear¹

Definition

Optimal competitive equilibrium: A welfare-maximizing competitive equilibrium for a given $(\{b_{i,-1}\}_i, B_{-1})$

¹Usually, we impose only "natural" debt limits.

Sequential problem

Optimal payoff structures

Study comparative statics in LS 1] quasi linear LS

1. FOC
2. Theorem for quasi linear
3. Extentions for CES

Graphs

- a graph that explains the main parts of the theorem
 - payoff structures and taxes as a function of initial debt

Recursive AMSS

Steady states

Existence

- Impose SS on FOC of AMSS = FOC of LS
- apply the LS theorem
- mention that SS is just the selection of a particular initial condition for a LS economy

Characterization

- Re-examine payoffs and sign of portfolio

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- future implementability constraints are relaxed

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Convergence

- theorem (1) quasilinear
- theorem (2) extensions to risk aversion
- remarks on more general shock structures

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for realistic calibrations - speed of convergence - nature of convergence when there are no steady states

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Sequential problem

$$\max_{\{c_t, l_t, b_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t, l_t)$$

subject to

$$\frac{b_{t-1} U_{c,t-1}}{\beta} = \frac{\mathbb{E}_{t-1} p_t U_{c,t}}{\mathbb{E}_t U_{c,t}} \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j (U_{c,t+j} c_{t+j} + U_{l,t+j} l_{t+j}) \text{ for } t \geq 1$$

$$b_{-1} = \frac{1}{U_{c,0}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (U_{c,t} c_t + U_{l,t} l_t)$$

$$c_t + g_t = \theta_t k$$

Where p_t is an iid random variable, representing an exogenous restriction on payoff structure of government assets.

Optimal payoff structures

- ▶ The planner's problem with state contingent debt can be represented as the problem above without the $t \geq 1$ measurability restrictions.
- ▶ It can be thought of then as the planner choosing a payoff structure p_t , such that the measurability constraints for $t \geq 1$ are always satisfied.
- ▶ The optimal allocation then defines the optimal payoff structure chosen to be

$$p_t = \frac{\beta}{U_{c,t} b_{t-1}} \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j (U_{c,t+j} c_{t+j} + U_{l,t+j} l_{t+j})$$

where

$$b_{t-1} = \frac{\beta}{U_{c,t-1}} \mathbb{E}_{t-1} \sum_{j=0}^{\infty} \beta^j (U_{c,t+j} c_{t+j} + U_{l,t+j} l_{t+j})$$

under the normalization $\frac{\mathbb{E}_{t-1} p_t U_{c,t}}{U_{c,t-1}} = 1$

Quasilinear FOC

The familiar first order conditions for the planner's problem where preferences are given by $U(c, l) = c - \frac{l^{1+\gamma}}{1+\gamma}$ are

$$1 = \mu + \xi_t$$
$$-l_t^\gamma + \mu(1 + \gamma)l_t^\gamma + \theta_t \xi_t = 0$$

Where μ is the Lagrange multiplier on the implementability constraint and $\beta^r \xi_t$ is the multiplier of the resource constraint. These equations give the tax rate τ as a function of μ

$$\tau(\mu) = \frac{\gamma\mu}{(1 + \gamma)\mu - 1}$$

and the period government surplus

$$S_t(\tau) = \theta_t^{\frac{\gamma}{1+\gamma}} (1 + \tau)^{\frac{1}{\gamma}} \tau - g_t$$

Quasilinear Asset Structure

Under the i.i.d. assumption, the optimal payoff structure chosen by the planner is given by the equation

$$p = (1 - \beta) \frac{St(\tau)}{\mathbb{E}S(\tau)} + \beta$$

where τ is the optimal tax rate given government debt b .

Lemma

Let $p_{\theta,g}(b)$ be the payoff chosen by the planner in aggregate state (θ, g) when government debt is given by b . Then $p_{\theta,g}(b)$ will, generically, be a one-to-one function of b . If $g\mathbb{E}\theta^{\frac{\gamma}{1+\gamma}} - \theta^{\frac{\gamma}{1+\gamma}}\mathbb{E}g > (<)0$ then $p_{\theta,g}(b)$ will be an increasing (decreasing) function of b .

Graphs

- a graph that explains the main parts of the theorem
 - payoff structures and taxes as a function of initial debt

Incomplete Markets Solution

With quasilinear preferences the measurability constraints for the incomplete markets problem can be written as the period by period constraint

$$\frac{p_t b_{t-1}}{\beta} = c_t - l_t^{1+\gamma} + b_t$$

under the normalization of $E_{t-1} p_t = 1$ The first order conditions associated with this problem are

$$1 = \mu_t + \xi_t$$

$$-l_t^\gamma + \mu_t(1 + \gamma)l_t^\gamma + \theta_t \xi_t = 0$$

$$\mu_t = E_t p_{t+1} \mu_{t+1}$$

Where $\beta^r \mu_t$ is the multiplier on the period by period budget constraint and $\beta^r \xi_t$ is the multiplier on the resource constraint.

Steady states

- ▶ We will restrict ourselves to an iid economy.
- ▶ If there existed a steady state then $\mu_t = \mu$ for all periods and $b_t = b$.
- ▶ The first order conditions for the incomplete markets economy then simplify to

$$\begin{aligned}1 &= \mu + \xi \\-\hat{l}_t^\gamma + \mu_t(1 + \gamma)\hat{l}_t^\gamma + \theta_t\xi &= 0\end{aligned}$$

- ▶ Note these are the exact same first order conditions governing the complete markets solution.
- ▶ Thus an incomplete markets steady state allocation replicates a complete markets solution.
- ▶ Finding a steady state amounts to finding a complete markets optimal allocation who's optimal payoff structure matches p_t

Existence

- Impose SS on FOC of AMSS = FOC of LS
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Incomplet Markets Solution

By choosing a new state variable $x_{t-1} = b_{t-1} U_{c,t-1}$ the measurability constraints can be written as a period by period constraint

$$\frac{x_{t-1} p_t U_{c,t}}{\beta E_{t-1} p_t U_{c,t}} = U_{c,t} c_t + U_{l,t} l_t + x_t$$

Which allows us to write the problem recursively as follows V

numerical section

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Taxes, debts, and redistributions with aggregate shocks*

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Abstract

A planner sets a lump sum transfer and a linear tax on labor income in an economy with incomplete markets, heterogeneous agents, and aggregate shocks. The planner's concerns about redistribution impart a welfare cost to fluctuating transfers. The distribution of net asset holdings across agents affects optimal allocations, transfers, and tax rates, but the level of government debt does not. Two forces shape long-run outcomes: the planner's desire to minimize the welfare costs of fluctuating transfers, which calls for a negative correlation between the distribution of net assets and agents' skills; and the planner's desire to use fluctuations in the real interest rate to adjust for missing state-contingent securities. In a model parameterized to match stylized facts about US booms and recessions, distributional concerns mainly determine optimal policies over business cycle frequencies. These features of optimal policy differ markedly from ones that emerge from representative agent Ramsey models like Aiyagari et al. (2002).

KEY WORDS: Distorting taxes. Transfers. Redistribution. Government debt. Interest rate risk.

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If, indeed, the debt were distributed in exact proportion to the taxes to be paid so that every one should pay out in taxes as much as he received in interest, it would cease to be a burden.... if it were possible, there would be [no] need of incurring the debt. For if a man has money to loan the Government, he certainly has money to pay the Government what he owes it. Simon Newcomb (1865, p.85)

1 Introduction

This paper studies an economy with agents who differ in their productivities and a benevolent government that imposes an affine income tax that consists of a distortionary proportional tax on labor income and a lump-sum tax or transfer. Figure 1 shows that an affine structure better approximates the US tax-transfer system than just proportional labor taxes. We impose no restrictions on the sign of transfers. If some agents are sufficiently poor or if the government wants enough redistribution, the government always chooses positive transfers. For most of the paper, we study an economy without capital in which a one-period risk-free bond is the only financial asset traded.

In this economy, concerns for redistribution impart a welfare cost to fluctuating transfers. A decrease in transfers in response to adverse aggregate shocks disproportionately affects agents having low present values of earnings. We show that the welfare cost of such a decrease depends on the distribution of net asset holdings across households with different productivities, where by net asset we mean the assets of an agent minus the assets of some benchmark agent.

Our first set of results extends a Ricardian equivalence type of reasoning to argue that gross asset positions (in particular the level of government debt) do not affect the set of feasible allocations that can be implemented in competitive equilibria with taxes and transfers. For example, an increase in government debt that is shared equally by all agents and thus leaves net assets unchanged has no welfare consequences, in line with the principles proclaimed by Simon Newcomb (1865) in the quotation above. This result also implies (a) that Ricardian equivalence holds in the presence of distorting taxes; and (b) that ad-hoc borrowing limits do not restrict the government's ability to respond to shocks. This logic applies to structures with and without physical capital and with complete or incomplete asset markets and more general structures of taxes.

Our second set of results characterizes the long run behavior of taxes, transfers, and the distribution of assets. In a special case of our economy in which aggregate shocks are iid and take two values, we show that there generally exists a steady state in which marginal utility-adjusted net asset positions are constant and taxes and transfers depend only on current realization of shocks. We also identify conditions under which this steady state is stable. In the steady state, the magnitude of fluctuations in taxes is low for many standard preferences and zero when preferences are CES. For more general shock processes, we show numerically that while a steady state does not exist, similar principles apply - there is an ergodic set to which net asset position converge; in that set, fluctuations of taxes and transfers are substantially diminished.

Two forces determine the properties of the ergodic set of net asset positions. The first force comes from a planner's desire to minimize the welfare cost of fluctuating transfers, which transmits to the Ramsey planner incentives to set taxes and transfers to generate a negative correlation between households' productivities and their net assets. In light of the relevance of net but not gross asset positions stressed in our first set of findings, this can be accomplished by having the government effectively accumulate risk-free claims on high-skilled workers. But another, possibly countervailing or possibly reinforcing, force comes from the Ramsey planner's incentive to use interest rate fluctuations to compensate for missing asset markets. If interest rates are high (low) when government revenue needs are high, the government has an incentive to accumulate assets (debt) vis-a-vis high-skilled worker. Thus, depending on comovements of the interest rate with other fundamentals, these two forces may either reinforce each other (for example, in response to a pure TFP shock where the implied interest rates are countercyclical) or oppose each other (for example, when interest rates are procyclical because TFP shocks are negatively correlated with discount factor shocks).

Optimal policy is particularly striking when agents have quasi-linear preferences and aggregate shocks affect only exogenous government expenditures. In this case, both forces are absent and for any initial asset distribution economy is immediately in a steady state in which assets and taxes remain constant forever. This outcome provides a stark contrast with normative predictions from the representative agent models studied by Aiyagari et al. (2002), in which long-run dynamics are driven by exogenous restrictions on the ability of the government to set transfers optimally. This example also indicates that incorporating a redistributive motive can substantially modify implications about optimal responses to aggregate shocks.

A third set of results concerns higher frequencies, in particular, optimal government policy in booms and recessions. What we have to say about this comes from a version of our model calibrated to patterns observed during recent US recessions: (1) the left tail of the cross-section distribution of labor income falls by more than right tail; and (2) interest rates fall. When we calibrate to those patterns, we find that during recessions accompanied by higher inequality, it is optimal to increase taxes and transfers and to issue government debt. These outcomes differ substantially both qualitatively and quantitatively from those in either a representative agent model or in a version of our model in which a recession is modeled as a pure TFP shock that leaves the distribution of skills unchanged. Furthermore recessions with low interest rates, thought not critical for producing our short-run outcomes, nevertheless affect the transient dynamics and long run properties of optimal policy. Starting from a net asset position (government vis-a-vis the high-skilled agent) that implies a 60% debt - to - GDP ratio, a benchmark economy with procyclical interest rates shows no discernable trend in net asset positions over a time span as long as 5000 years. But in economies with countercyclical interest rates (generated by shutting down discount factor shocks), the government repays this debt and eventually accumulates assets, although it takes about 2500 years to do so.

We assemble these results as follows. After section 2 describes preferences, technologies, endowments, and information flows, section 3 sets the stage for our subsequent results. Theorem 1 exploits the fact

that the set of feasible allocations is invariant to changes in transfers and *gross* asset positions to obtain Ricardian equivalence and the importance of net assets. Section 4 analyzes outcomes when households' preferences are quasi linear. Sections 5 characterizes the planner's maximization problem and formulates a recursive version that features two Bellman equations. Section 6 characterizes long run allocations, while Section 7 provides numerical simulations of optimal policy responses to business cycle-like shocks. Section 8 offers concluding remarks linking our model to recent discussions of government debt-GDP ratios.

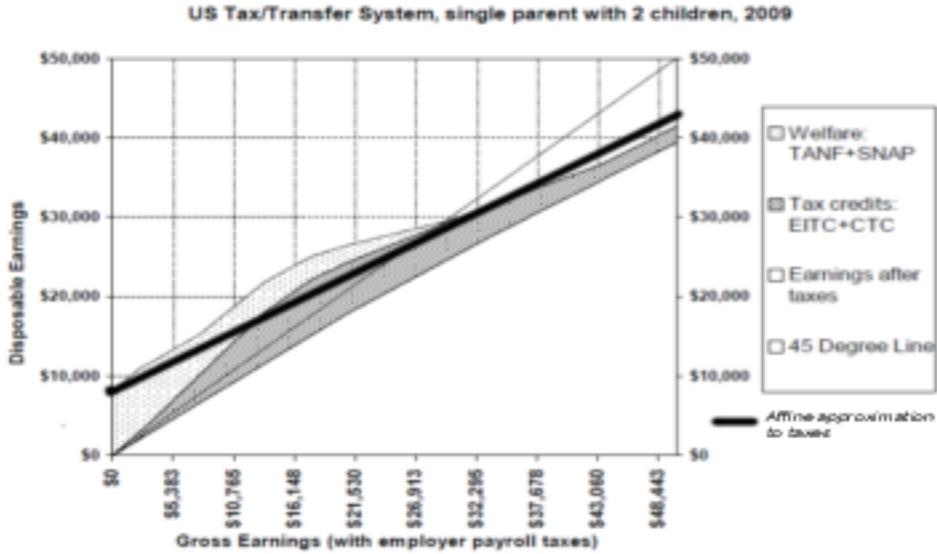


Figure 1: The U.S. tax-transfer system is poorly approximated by a linear function, better by an affine function.

1.1 Relationships to literatures

At a fundamental level, our paper descends from both Barro (1974), who showed Ricardian equivalence in a representative agent economy with lump sum taxes, and Barro (1979), who studied optimal taxation when lump sum taxes are ruled out. In our environment with incomplete markets and heterogeneous workers, both of the forces discovered by Barro play large roles, although the distributive motives that we include lead to richer policy prescriptions.

A large literature on Ramsey problems exogenously rules out transfers in the context of representative agent, general equilibrium models. Lucas and Stokey (1983), Chari et al. (1994), and Aiyagari et al. (2002) (henceforth called AMSS) are leading examples. In contrast to those papers, our Ramsey planner cares about redistribution among agents with different skills and wealths. Other than not allowing them to depend on agents' personal identities, we leave transfers unrestricted and let the Ramsey planner set them optimally. Nevertheless, we find that some of the same general principles that emerge from that representative agent, no-transfers literature continue to hold, in particular, the prescription to smooth distortions across time and states. However, it is also true that allowing the government to set transfers optimally substantially changes qualitative and quantitative insights about the optimal policy

in important respects.¹

Several recent papers impute distributive concerns to a Ramsey planner. Three papers that are perhaps most closely related to ours are Bassetto (1999), Shin (2006), and Werning (2007). Like us, those authors allow heterogeneity and study distributional consequences of alternative tax and borrowing policies. Bassetto (1999) extends the Lucas and Stokey (1983) environment to include I types of agents who are heterogeneous in their time-invariant labor productivities. There are complete markets and a Ramsey planner has access only to proportional taxes on labor income and history-contingent borrowing and lending. Bassetto studies how the Ramsey planner's vector of Pareto weights influences how he responds to government expenditures and other shocks by adjusting the proportional labor tax and government borrowing to cover expenses while manipulating prices in ways that redistribute wealth between 'rentiers' (who have low productivities and whose main income is from their asset holdings) and 'workers' (who have high productivities) whose main income source is their labor.

Shin (2006) extends the AMSS (Aiyagari et al. (2002)) economy to have two risk-averse households who face idiosyncratic income risk. When idiosyncratic income risk is big enough relative to aggregate government expenditure risk, the Ramsey planner chooses to issue debt in order to help households engage in precautionary saving, thereby overturning the AMSS result that in their quasi-linear case a Ramsey planner eventually sets taxes to zero and lives off its earnings from assets forevermore. Shin emphasizes that the government does this at the cost of imposing tax distortions. While being constrained to use proportional labor income taxes and nonnegative transfers, Shin's Ramsey planner balances two competing self-insurance motives: aggregate tax smoothing and individual consumption smoothing.

Werning (2007) studies a complete markets economy with heterogeneous agents and transfers that are unrestricted in sign. He obtains counterparts to our results about net versus gross asset positions, including that government assets can be set to zero in all periods. Because he allows unrestricted taxation of initial assets, the initial distribution of assets plays no role. Theorem 1 and its corollaries substantially generalize Werning's results by showing that all allocations of assets among agents and the government that imply the same net asset position lead to the same optimal allocation, a conclusion that holds for market structures beyond complete markets. Werning (2007) provides an extensive characterization of optimal allocations and distortions in complete market economies, while we focus on precautionary savings motives for private agents and the government that are not present when markets are complete.^{2,3}

Finally, our numerical analysis in Section 7 is related to McKay and Reis (2013). While our focus

¹ There is also a more recent strand of literature that focuses on the optimal policy in settings with heterogeneous agents when a government can impose arbitrary taxes subject only to explicit informational constraints (see Golosov et al. (2007) for a review). A striking result from that literature is that when agent's asset holdings are perfectly observable, the distribution of assets among agents is irrelevant and an optimal allocation can be achieved purely through taxation (see, e.g. Bassetto and Kochenakota (2004)). In the previous version of the paper we showed that a mechanism design version of the model with unobservable assets generates some of the similar predictions to the model with affine taxes that we study, in particular, the relevance of net assets and history dependence of taxes. We leave further analysis along this direction to the future.

² Werning (2012) studies optimal taxation with incomplete markets and explores conditions under which optimal taxes depend only on the aggregate state.

³ More recent closely related papers are Azzimonti et al. (2008a,b) and Correia (2010). While these authors study optimal policy in economies in which agents are heterogeneous in skills and initial assets, they do not allow aggregate shocks.

differs from theirs – McKay and Reis study the effect of calibrated US tax and transfer system on stabilization of output, while we focus on optimal policy in a simpler economy – both papers confirm the importance of transfers and redistribution over business-cycle frequencies.

2 Environment

Exogenous fundamentals of the economy are functions of a shock s_t that follows an irreducible Markov process, where $s_t \in S$ and S is a finite set. We let $s^t = (s_0, \dots, s_t)$ denote a history of shocks.

There is a mass π_i of a type $i \in I$ agent, with $\sum_{i=1}^I \pi_i = 1$. Types differ by their skills. Preferences of an agent of type i over stochastic processes for consumption $\{c_{i,t}\}_t$ and labor supply $\{l_{i,t}\}_t$ are ordered by

$$E_0 \sum_{t=0}^{\infty} [\Pi_{j=0}^{t-1} \beta(s_j)] U^i(c_i(s^t), l_i(s^t)), \quad (1)$$

where E_t is a mathematical expectations operator conditioned on time t information and $\beta(s_t) \in (0, 1)$ is a state-dependent discount factor⁴. We assume that $l_i \in [0, \bar{l}_i]$ for some $\bar{l}_i < \infty$. Results in section 3 require no additional assumptions on U^i like differentiability or convexity,⁵ but results in later sections do.

An agent of type i who supplies l_i units of labor produces $\theta_i(s_t) l_i$ units of output, where $\theta_i(s_t) \in \Theta$ is a nonnegative state-dependent scalar. Feasible allocations satisfy

$$\sum_{i=1}^I \pi_i c_i(s^t) + g(s_t) = \sum_{i=1}^I \pi_i \theta_i(s_t) l_i(s^t), \quad (2)$$

where $g(s_t)$ denotes exogenous government expenditures in state s_t . We allow s_t to affect $\beta(s_t)$, government expenditures $g(s_t)$, and the type-specific productivities $\theta_i(s_t)$.

To save on notation, mostly we use z_t to denote a random variable with a time t conditional distribution that is a function of the history s^t . Occasionally, we use the more explicit notion $z(s^t)$ to denote a realization at a particular history s^t .

A Ramsey planner's preferences over a vector of stochastic processes for consumption and labor supply are ordered by

$$E_0 \sum_{i=1}^I \pi_i \alpha_i \sum_{t=0}^{\infty} \bar{\beta}_t U_t^i(c_{i,t}, l_{i,t}), \quad (3)$$

where the Pareto weights satisfy $\alpha_i \geq 0$, $\sum_{i=1}^I \alpha_i = 1$, and $\bar{\beta}_t = [\Pi_{j=0}^{t-1} \beta_j]$

In most of this paper, we study an optimal government policy when agents can trade only a one-period risk-free bond. We assume that the government imposes an affine tax. We denote proportional labor taxes by τ and lump sum transfers by T . With this the tax bill of an agent with wage earnings $l_{i,t} \theta_{i,t}$ is given by

$$-T_t + \tau_t \theta_{i,t} l_{i,t}.$$

⁴We let the discount factor at time t to depend on the history of shocks s^{t-1} . This allows us to generate flexible comovements between real interest rates and fundamentals which we exploit later in section 6 and 7.

⁵Consequently our setup allows both extensive and intensive responses of labor.

We do not restrict the sign of T_t at any t or s^t . If for some type i , $\theta_{i,t} = 0$, $b_{i,-1} = 0$ and U^i is defined only on \mathcal{R}_+^2 , his budget constraint will imply that the all allocations feasible for the planner have nonnegative present values of transfers, since transfers are the sole source of a type i agent's wealth and consumption.

While results in sections 4, 5, 6, and 7 depend on these assumptions about an affine tax system and incomplete markets, key results of section 3 apply under more general tax functions and market structures.

Under an affine tax system, agent i 's budget constraint at t is

$$c_{i,t} + b_{i,t} = (1 - \tau_t) \theta_{i,t} l_{i,t} + R_{t-1} b_{i,t-1} + T_t, \quad (4)$$

where $b_{i,t}$ denotes asset holdings of a type i agent at time $t \geq 0$, R_{t-1} is a gross risk-free one-period interest rate from $t-1$ to t for $t \geq 1$, and $R_{-1} \equiv 1$. For $t \geq 0$, R_t is measurable with respect to s^t . To rule out Ponzi schemes, we assume that $b_{i,t}$ must be bounded from below. Except in subsection 3.1, we impose no further constraints on agents' borrowing and lending. Subsection 3.1 briefly studies economies with arbitrary borrowing constraints.

The government budget constraint is

$$g_t + B_t = \tau_t \sum_{i=1}^I \pi_i \theta_{i,t} l_{i,t} - T_t + R_{t-1} B_{t-1}, \quad (5)$$

where B_t denotes the government's assets at time t , which we assume are bounded from below. Our assumptions about preferences imply that the government can collect only finite revenues in each period, so this restriction rules out government-run Ponzi schemes.

We assume that private agents and the government start with assets $\{b_{i,-1}\}_{i=1}^I$ and B_{-1} , respectively. Asset holdings satisfy the market clearing condition

$$\sum_{i=1}^I \pi_i b_{i,t} + B_t = 0 \text{ for all } t \geq -1. \quad (6)$$

Since B_t and all $b_{i,t}$ are bounded from below, equation (6) implies that they are also bounded from above.

Components of competitive equilibria are described below

Definition 1 An allocation is a sequence $\{c_{i,t}, l_{i,t}\}_{i,t}$. An asset profile is a sequence $\{\{b_{i,t}\}_i, B_t\}_t$. A price system is an interest rate sequence $\{R_t\}_t$. A tax policy is a sequence $\{\tau_t, T_t\}_t$.

Definition 2 For a given initial asset distribution $(\{b_{i,-1}\}_i, B_{-1})$, a competitive equilibrium with affine taxes is a sequence $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, B_t, R_t\}_t$ and a tax policy $\{\tau_t, T_t\}_t$, such that $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ maximize (1) subject to (4) and $\{b_{i,t}\}_{i,t}$ is bounded; and constraints (2), (5) and (6) are satisfied.

Lastly we define optimal competitive equilibria.

Definition 3 Given $(\{b_{i,-1}\}_i, B_{-1})$, an optimal competitive equilibrium with affine taxes is a tax policy $\{\tau_t^*, T_t^*\}_t$, an allocation $\{c_{i,t}^*, l_{i,t}^*\}_{i,t}$, an asset profile $\{\{b_{i,t}^*\}_i, B_t^*\}_t$, and a price system $\{R_t^*\}_t$ such

that (i) given $(\{b_{i,-1}\}_i, B_{-1})$, the tax policy, the price system, and the allocation constitute a competitive equilibrium; and (ii) there is no other tax policy $\{\tau_t, T_t\}_t$ such that a competitive equilibrium given $(\{b_{i,-1}\}_i, B_{-1})$ and $\{\tau_t, T_t\}_t$ has a strictly higher value of (3).

We call $\{\tau_t^*, T_t^*\}_t$ an *optimal tax policy*, $\{c_{i,t}^*, l_{i,t}^*\}_{i,t}$ an *optimal allocation*, and $\left\{\left\{b_{i,t}^*\right\}_i, B_t^*\right\}_t$ an *optimal asset profile*.

3 Relevant and Irrelevant Aspects of the Distribution of Government Debt

This section sets forth a result that underlies much of this paper, namely, that the level of government debt is not a state variable for our economy. The reason is that there is an equivalence class of tax policies and asset profiles that support the same competitive equilibrium allocation. A competitive equilibrium allocation pins down only net asset positions. The assertions in this section apply to all competitive equilibria, not just the optimal ones that will be our focus in subsequent sections.

Theorem 1 Given $(\{b_{i,-1}\}_i, B_{-1})$, let $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, B_t, R_t\}_t$ and $\{\tau_t, T_t\}_t$ be a competitive equilibrium. For any bounded sequences $\{\hat{b}_{i,t}\}_{i,t \geq -1}$ that satisfy

$$\hat{b}_{i,t} - \hat{b}_{1,t} = \tilde{b}_{i,t} \equiv b_{i,t} - b_{1,t} \text{ for all } t \geq -1, i \geq 2,$$

there exist sequences $\{\hat{T}_t\}_t$ and $\{\hat{B}_t\}_{t \geq -1}$ that satisfy (6) such that $\{\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\}_i, \hat{B}_t, R_t\}_t$ and $\{\tau_t, \hat{T}_t\}_t$ constitute a competitive equilibrium given $(\{\hat{b}_{i,-1}\}_i, \hat{B}_{-1})$.

Proof. Let

$$\hat{T}_t = T_t + (\hat{b}_{1,t} - b_{1,t}) - R_{t-1}(\hat{b}_{1,t-1} - b_{1,t-1}) \text{ for all } t \geq 0. \quad (7)$$

Given a tax policy $\{\tau_t, \hat{T}_t\}_t$, the allocation $\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\}_{i,t}$ is a feasible choice for consumer i since it satisfies

$$\begin{aligned} c_{i,t} &= (1 - \tau_t) \theta_{i,t} l_{i,t} + R_{t-1} b_{i,t-1} - b_{i,t} + T_t \\ &= (1 - \tau_t) \theta_{i,t} l_{i,t} + R_{t-1} (b_{i,t-1} - b_{1,t-1}) - (b_{i,t} - b_{1,t}) + T_t + R_{t-1} b_{1,t-1} - b_{1,t} \\ &= (1 - \tau_t) \theta_{i,t} l_{i,t} + R_{t-1} (\hat{b}_{i,t-1} - b_{1,t-1}) - (\hat{b}_{i,t} - b_{1,t}) + T_t + R_{t-1} b_{1,t-1} - b_{1,t} \\ &= (1 - \tau_t) \theta_{i,t} l_{i,t} + R_{t-1} \hat{b}_{1,t-1} - \hat{b}_{i,t} + \hat{T}_t. \end{aligned}$$

Suppose that $\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\}_{i,t}$ is not the optimal choice for consumer i , in the sense that there exists some other sequence $\{\hat{c}_{i,t}, \hat{l}_{i,t}, \hat{b}_{i,t}\}_t$ that gives strictly higher utility. Then the choice $\{\hat{c}_{i,t}, \hat{l}_{i,t}, b_{i,t}\}_t$ is feasible given the tax rates $\{\tau_t, T_t\}_t$, which contradicts the assumption that $\{c_{i,t}, l_{i,t}, b_{i,t}\}_t$ is the optimal choice for the consumer given taxes $\{\tau_t, T_t\}_t$. The new allocation satisfies all other constraints and therefore is an equilibrium. ■

An immediate corollary is that it is not total government debt but rather who owns it that affects equilibrium allocations.

Corollary 1 For any pair B'_{-1}, B''_{-1} , there are asset profiles $\{b'_{i,-1}\}_i$ and $\{b''_{i,-1}\}_i$ such that equilibrium allocations starting from $(\{b'_{i,-1}\}_i, B'_{-1})$ and from $(\{b''_{i,-1}\}_i, B''_{-1})$ are the same. These asset profiles satisfy

$$b'_{i,-1} - b'_{1,-1} = b''_{i,-1} - b''_{1,-1} \text{ for all } i.$$

This result is closely related to Ricardian Equivalence in Barro (1974). There are, however, some important distinctions. In Barro's representative agent model, lump sum taxes are not distortionary. In our economy, since the planner does not have person-specific taxes, a lump sum transfer introduces distortions in inequality, a force that has a significant effect on optimal policy, as we will see in following sections. Despite this, Ricardian equivalence continues to hold.⁶ Theorem 1 shows that many transfer sequences $\{T_t\}_t$ and asset profiles $\{b_{i,t}, B_t\}_{i,t}$ support the same equilibrium allocation. For example, one can set government assets $B_{i,t} = 0$ without loss of generality. Alternatively, we can normalize by setting assets $b_{i,t}$ of any type i to zero.

Theorem 1 continues to hold in more general environments. For example, we could allow agents to trade all conceivable Arrow securities and still show that equilibrium allocations depend only on agents' net assets positions. Similarly, our results hold in economies with capital.

3.1 Extension: Borrowing constraints

Representative agent models rule out Ricardian equivalence either by assuming distorting taxes or by imposing ad hoc borrowing constraints. By way of contrast, we have verified that Ricardian equivalence holds in our economy even though there are distorting taxes. Imposing ad-hoc borrowing limits also leaves Ricardian equivalence intact in our economy.⁷ In economies with exogenous borrowing constraints, agents' maximization problems include the additional constraints

$$b_{i,t} \geq \underline{b}_i \tag{8}$$

for some exogenously given $\{\underline{b}_i\}_i$.

Definition 4 For given $(\{b_{i,-1}, \underline{b}_i\}_i, B_{-1})$ and $\{\tau_t, T_t\}_t$, a competitive equilibrium with affine taxes and exogenous borrowing constraints is a sequence $\{(c_{i,t}, l_{i,t}, b_{i,t})_i, B_t, R_t\}_t$ such that $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ maximizes (1) subject to (4) and (8), $\{b_{i,t}\}_{i,t}$ are bounded, and constraints (2), (5) and (6) are satisfied.

We can define an *optimal* competitive equilibrium with exogenous borrowing constraints by extending Definition 3.

⁶Wallace (1981)'s Modigliani-Miller theorem for a class of government open market operations has a similar flavor. Sargent (1987) describes the structure of a set of related Modigliani-Miller theorems for government finance.

⁷Bryant and Wallace (1984) describe how a government can use borrowing constraints as part of a welfare-improving policy to finance exogenous government expenditures. Sargent and Smith (1987) describe Modigliani-Miller theorems for government finance in a collection of economies in which borrowing constraints on classes of agents produce the kind of rate of return discrepancies that Bryant and Wallace manipulate.

The introduction of the ad-hoc debt limits leaves unaltered the conclusions of Corollary 1 and the role of the initial distribution of assets across agents. The next proposition asserts that ad-hoc borrowing limits do not limit a government's ability to respond to aggregate shocks.⁸

Proposition 1 *Given an initial asset distribution $(\{b_{i,-1}\}_i, B_{-1})$, let $\{c_{i,t}, l_{i,t}\}_{i,t}$ and $\{R_t\}_t$ be a competitive equilibrium allocation and interest rate sequence in an economy without exogenous borrowing constraints. Then for any exogenous constraints $\{\underline{b}_i\}_i$, there is a government tax policy $\{\tau_t, T_t\}_t$ such that $\{c_{i,t}, l_{i,t}\}_{i,t}$ is a competitive equilibrium allocation in an economy with exogenous borrowing constraints $(\{b_{i,-1}, \underline{b}_i\}_i, B_{-1})$ and $\{\tau_t, T_t\}_t$.*

Proof. Let $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ be a competitive equilibrium allocation without exogenous borrowing constraints. Let $\Delta_t \equiv \max_i \{b_i - b_{i,t}\}$. Define $\hat{b}_{i,t} \equiv b_{i,t} + \Delta_t$ for all $t \geq 0$ and $\hat{b}_{i,-1} = b_{i,-1}$. By Theorem 1, $\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\}_{i,t}$ is also a competitive equilibrium allocation without exogenous borrowing constraints. Moreover, by construction $\hat{b}_{i,t} - \underline{b}_i = b_{i,t} + \Delta_t - \underline{b}_i \geq 0$. Therefore, $\hat{b}_{i,t}$ satisfies (8). Since agents' budget sets are smaller in the economy with exogenous borrowing constraints, and $\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\}_{i,t}$ are feasible at interest rate process $\{R_t\}_t$, then $\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\}_{i,t}$ is also an optimal choice for agents in the economy with exogenous borrowing constraints $\{\underline{b}_i\}_i$. Since all market clearing conditions are satisfied, $\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\}_{i,t}$ is a competitive equilibrium allocation and asset profile. ■

To provide some intuition for Proposition 1, suppose to the contrary that the exogenous borrowing constraints restricted a government's ability to achieve a desired allocation. That means that the government would want to increase its borrowing and to repay agents later, which the borrowing constraints prevent. But the government can just reduce transfers today and increase them tomorrow. That would achieve the desired allocation without violating the exogenous borrowing constraints.

Welfare can be strictly higher in an economy with exogenous borrowing constraints relative to an economy without borrowing constraints because a government might want to push some agents against their borrowing limits. When agents' borrowing constraints bind, their shadow interest rates differ from the common interest rate that unconstrained agents face. When the government rearranges tax policies to affect the interest rate, it affects constrained and unconstrained agents differently. By facilitating redistribution, this can improve welfare. In appendix A.1, we construct an example without any shocks in which the government can achieve higher welfare by using borrowing constraints to improve its ability to redistribute.

3.2 Ricardian irrelevance and optimal equilibria

Our statements about Ricardian irrelevance apply to all competitive equilibrium allocations, not just the optimal ones that are the main focus of this paper. To appreciate how these Ricardian irrelevance results affect optimal equilibria, suppose that we increase an initial level of government debt from 0 to some arbitrary level $B'_{-1} > 0$. If the government were to hold transfers $\{T_t\}_t$ fixed, it would have to increase

⁸See Yared (2012, 2013) who shows a closely related result.

tax rates $\{\tau_t\}_t$ enough to collect a present value of revenues sufficient to repay B'_{-1} . Since dead weight losses are convex in τ , higher levels of debt financed with bigger distorting taxes $\{\tau_t\}$ impose larger distortions on the economy, thereby degrading the equilibrium allocation. But this would not happen if the government were instead to adjust transfers in response to a higher initial debt. To determine optimal transfers, we need to know who owns the initial government debt B'_{-1} . For example, suppose that agents hold equal amounts of it. Then each unit of debt repayment achieves the same redistribution as one unit of transfers. If the original tax policy at $B'_{-1} = 0$ were optimal, then the best policy for a government with initial debt $B'_{-1} > 0$ would be to reduce the present value transfers by exactly the amount of the increase in per capita debt, because then distorting taxes $\{\tau_t\}$ and the allocation would both remain unchanged.⁹

But the situation would be different if holdings of government debt were not equal across agents. For example, suppose that richer people owned disproportionately more government debt than poorer people. That would mean that inequality is effectively initially higher in an economy with higher initial government debt. As a result, a government with Pareto weights $\{\alpha_i\}$ that favor equality would want to increase both distorting tax rates $\{\tau_t\}$ and transfers $\{T_t\}$ to offset the increase in inequality associated with the increase in government debt. The conclusion would be the opposite if government debt were to be owned mostly by poorer households.

This logic shows how important it is to know the distribution of government debt across people. Government debt that is widely distributed across households (e.g., implicit Social Security debt) is less distorting than government debt owned mostly by people whose incomes are at the top of the income distribution (e.g., government debt held by hedge funds).¹⁰

4 Quasi-linear preferences

Before we present a more general characterization in section 5, in this section we study a special case of our economy that allows us to get a long way analytically and to identify some forces that drive outcomes. Here we assume that the only source of aggregate shocks is g_t and that preferences are given by $U^i(c, l) = c - h_i(l)$ where h_i is an increasing differentiable function with $h'_i(0) = 0$ and $h'_i(\bar{l}_i) = \infty$. Such quasi-linear preferences have been assumed in the context of representative agent economies (see, e.g., AMSS, Farhi (2010), Battaglini and Coate (2007, 2008), Yared (2010), Faraglia et al. (2012)). We demonstrate two important things in this section. First, optimal policies in our heterogeneous agent economy in which transfers are chosen optimally differ in interesting ways from those in a representative agent economy that exogenously restricts transfers. Second, the special set up under study switches off two forces present more generally, namely, that the marginal utilities of agents are differentially affected by changes in transfers and that with other preference specifications the interest rate is not constant.

⁹This example illustrates principles proclaimed by Simon Newcomb (1865, p. 85) in the quotation with which we began this paper.

¹⁰It is possible to extend our analysis to open economy with foreign holdings of domestic debt. The more government debt is owned by the foreigners, the higher are the distorting taxes that the government needs to impose.

These two forces play important roles in determining the long run outcomes described in Section 6

To simplify notation, we now assume that the initial debt is $\{\beta^{-1}b_{i,-1}\}_i$.

Proposition 2 Suppose that preferences are quasi-linear and that g_t is the only aggregate variable subject to shocks. Then the optimal tax rate τ_i^* satisfies $\tau_i^* = \tau^*$. An optimum asset profile $\{b_{i,t}^*, B_t^*\}_{i,t}$ can be chosen to satisfy $b_{i,t}^* = b_{i,-1}$ for all i , $t \geq 0$ and $B_t^* = B_{-1}$ for all $t \geq 0$.¹¹

Proof. When preferences are quasilinear, the interest rate $R_t = \beta^{-1}$ for all t and $(1 - \tau_t)\theta_i = h'_i(l_{i,t})$ for all t . For our purposes, it is more convenient to express the labor supply component of the allocation as a function of $(1 - \tau)$ and optimize with respect to τ rather than $\{l_i\}_i$. We invert $h'_i(\cdot)$ to express labor supply l_i as a function of $(1 - \tau)$. Call this function $H_i(1 - \tau)$. Use the budget constraint (4) to obtain

$$c_{i,t} + b_{i,t} - (1 - \tau_t)H_i(1 - \tau_t) = T_t + \beta^{-1}b_{i,t-1}. \quad (9)$$

The optimal allocation solves

$$\max_{\{c_{i,t}, b_{i,t}, \pi_i, T_t\}_{i,t}} \mathbb{E}_0 \sum_{t=0}^{\infty} \sum_{i=1}^I \alpha_i \pi_i \beta^t [c_{i,t} - h_i(H_i(1 - \tau_t))] \quad (10)$$

subject to $\{b_{i,t}\}_{i,t}$ being bounded, (9), and

$$\sum_{i=1}^I \pi_i c_{i,t} + g_t = \sum_{i=1}^I \pi_i H_i(1 - \tau_t) \theta_i.$$

Note that since $\{b_{i,t}\}_{i,t}$ is bounded,

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t [\beta^{-1}b_{i,t-1} - b_{i,t}] = \beta^{-1}b_{i,-1} + \lim_{T \rightarrow \infty} \mathbb{E}_0 \left(\sum_{t=0}^T \beta^t [b_{i,t} - b_{i,t}] - \beta^{T+1}b_{i,T+1} \right) = \beta^{-1}b_{i,-1}.$$

Use (9) to eliminate $c_{i,t}$ and then use the preceding expression to get

$$\max_{\{b_{i,t}, \tau_t, T_t\}_{i,t}} \mathbb{E}_0 \sum_{t=0}^{\infty} \sum_{i=1}^I \alpha_i \pi_i \beta^t [T_t + (1 - \tau_t)H_i(1 - \tau_t) - h_i(H_i(1 - \tau_t))] + \beta^{-1} \sum_{i=1}^I b_{i,-1}. \quad (11)$$

subject to

$$\sum_{i=1}^I \pi_i [T_t + \beta^{-1}b_{i,t-1} - b_{i,t} + (1 - \tau_t)H_i(1 - \tau_t)\theta_i] + g_t = \sum_{i=1}^I \pi_i H_i(1 - \tau_t)\theta_i. \quad (12)$$

Let $\beta^t \lambda_t$ be the Lagrange multiplier on the time t feasibility constraint (12). The first-order condition with respect to T_t implies that $\lambda_t = \sum_{i=1}^I \alpha_i \pi_i$ is constant and independent of t . Therefore, optimal taxes $\tau_t = \tau^*$ are also constant and independent of t . Using τ^* , equation (12) pins down $\sum_{i=1}^I \pi_i [T_t + \beta^{-1}b_{i,t-1} - b_{i,t}]$. Without loss of generality we can set $b_{i,t}^* = b_{i,-1}$ and T_t^* to satisfy (12).

■

¹¹We thank Guy Laroque for suggesting the idea for this proof.

In the optimal equilibria for the quasi-linear economy described in Proposition 2, fluctuations in lump-sum taxes and transfers “do all the work”. In period 0, the government chooses an optimal present value of transfers and a constant tax rate that pays for it. Tax rates and transfers depend on the Pareto weights $\{\alpha_i\}$: higher Pareto weights on low-skilled agents imply higher transfers and tax rates. In response to a shock g_t , the government adjusts transfers in period t by the amount of the shock. Since all agents are risk-neutral, welfare is unaffected by fluctuations in transfers. This allows the government to perfectly smooth distorting tax rates.

Comparison with representative agent economies

The Lucas and Stokey (1983) and AMSS representative agent models impose $T_t \geq 0$. An informal justification for this restriction is the desire of the government not to hurt poor agents who can't afford lump-sum taxes. This constraint always binds in the Lucas and Stokey model and it binds in the AMSS model until the government has acquired enough assets to finance all future expenditures from earnings on those assets. In those representative agent models, the government would like to impose lump-sum *taxes*, not transfers. We explicitly model redistributive concerns by attaching Pareto weights to heterogeneous agents and obtain the very different dynamics shown in Proposition 2. In this section, we argue that differences in the dynamics come from the presence of this arbitrary restrictions on transfers and not from redistributive motives.

We impose the following in our maximization problem (10), namely,¹²,

$$T_t \geq 0 \text{ for all } t. \quad (13)$$

The following proposition states it is optimal for the planner to set policy in such a way that constraint (13) eventually becomes slack.¹³

Proposition 3 *Assume that $I \geq 1$ and g_t takes more than one value. Let $\beta^t \chi_t$ be the Lagrange multiplier on constraint (13) in a version of maximization problem (10) that is augmented with constraint (13). Then $\chi_t \rightarrow 0$ a.s.*

Proof. Our augmented version of the maximization problem (10) can be expressed as maximization problem (11) with an additional constraint (13). The first-order conditions for T_t yield $\sum_{i=1}^I \alpha_i \pi_i = \mu_t + \chi_t$, while the first-order condition for $b_{i,t}$ implies $\mu_t = E_t \mu_{t+1}$. Since $\chi_t \geq 0$, these two conditions imply that χ_t is a nonnegative martingale and therefore χ_t must almost surely converge to a constant. This in turn implies that μ_t must almost surely converge to a constant. Then the first-order conditions for τ_t also imply that τ_t must converge a.s. to some τ^* .

Suppose $\chi_t \rightarrow \chi^* > 0$. This implies that $T_t \rightarrow 0$ and (12) becomes

$$-\beta^{-1} B_{i,t-1} + B_{i,t} + \sum_{i=1}^I \pi_i (1 - \tau^*) H_i (1 - \tau^*) \theta_i + g_t = \sum_{i=1}^I \pi_i H_i (1 - \tau^*) \theta_i,$$

¹²This makes AMSS a special case of our economy.

¹³It can be shown that if g is not too high and a government is sufficiently redistributive (i.e., α_i is sufficiently high for low productivity agents), constraints (13) is always slack.

where we used (6) to substitute for $\sum_{t=1}^T \pi_t b_{t,t}$. If g_t can take more than one value and follows an irreducible Markov process, then for any bound on B_t , we can find a sequence of government expenditures g_t for which this bound will eventually be violated, leading to a contradiction. This implies that $\chi_t \rightarrow 0$. ■

Proposition 2 emphasized that in absence of restriction (13) the government uses fluctuations in transfers to finance all fluctuations in expenditures. Constraint (13) imposes an asymmetry in using transfers to smooth fluctuations in expenditures. Around zero it is costless to *increase* transfers by a small amount but infinitely costly to *decrease* them by the same amount. This induces the optimal policy to engineer tax rates, transfers, and asset distributions that allow the economy eventually to disarm this constraint.

For quasi-linear preferences, figure 2 compares equilibrium dynamics in a representative agent (AMSS) economy and an economy with two agents, one who is not productive, and Pareto weights chosen to make transfers be positive at all times and along all histories. The sequences of s_t shocks are identical across the two economies. While tax rates converge to zero for the AMSS economy, they are constant for the heterogeneous agent economy.¹⁴

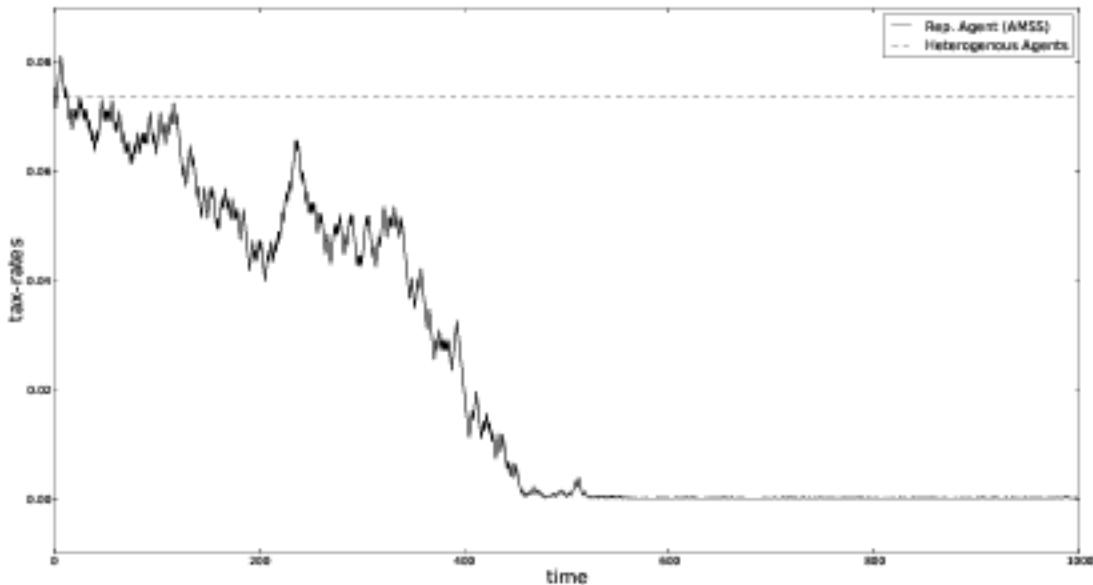


Figure 2: Taxes in AMSS (solid line) and heterogeneous agent economy (dotted line) with quasi-linear preferences.

¹⁴The plots for the AMSS and the heterogeneous agent economies are both for quasi-linear preferences with a Frisch elasticity of labor equals 0.5 and a discount factor $\beta = 0.95$. In the AMSS economy, the agent's initial assets are zero and government expenditure shocks $g(s_t) \in \{.1, .3\}$ are generated using an IID process with equally likely outcomes. For the heterogeneous agent economy, we set $\alpha_2 = .54$ so that the initial labor taxes are similar to those for the AMSS economy

5 Optimal equilibria with affine taxes

We return to the more general problem formulated in section 2. We further assume that $U^i : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ is concave in $(c, -l)$ and twice continuously differentiable. We let $U_{x,t}^i$ or $U_{xy,t}^i$ denote first and second derivatives of U^i with respect to $x, y \in \{c, l\}$ in period t and assume that $\lim_{x \rightarrow L_i} U_l^i(c, x) = \infty$ and $\lim_{x \rightarrow 0} U_l^i(c, x) = 0$ for all c and i .

We focus on interior equilibria. First-order necessary conditions for the consumer's problem are

$$(1 - \tau_t) \theta_{i,t} U_{c,t}^i = -U_{l,t}^i, \quad (14)$$

and

$$U_{c,t}^i = \beta_t R_t E_t U_{c,t+1}^i. \quad (15)$$

To help characterize an equilibrium, we use

Proposition 4 *A sequence $\{(c_{i,t}, l_{i,t}, b_{i,t})_i, R_t, \tau_t, T_t\}_t$ is part of a competitive equilibrium with affine taxes if and only if it satisfies (2), (4), (14), and (15) and $b_{i,t}$ is bounded for all i and t .*

Proof. Necessity is obvious. In appendix A.2, we use arguments of Magill and Quinzii (1994) and Constantinides and Duffie (1996) to show that any $\{(c_{i,t}, l_{i,t}, b_{i,t})_i\}_t$ that satisfies (4), (14), and (15) is a solution to consumer i 's problem. Equilibrium $\{B_t\}_t$ is determined by (6) and constraint (5) is then implied by Walras' Law ■

To find an optimal equilibrium, by Proposition 4 we can choose $\{(c_{i,t}, l_{i,t}, b_{i,t})_i, R_t, \tau_t, T_t\}_t$ to maximize (3) subject to (2), (4), (14), and (15). We apply a first-order approach and follow steps similar to ones taken by Lucas and Stokey (1983) and AMSS. Substituting consumers' first-order conditions (14) and (15) into the budget constraints (4) yields implementability constraints

$$c_{i,t} + b_{i,t} = -\frac{U_{l,t}^i}{U_{c,t}^i} l_{i,t} + T_t + \frac{U_{c,t-1}^i}{\beta_{t-1} E_{t-1} U_{c,t}^i} b_{i,t-1} \text{ for all } i, t. \quad (16)$$

For $I \geq 2$, we can use constraint (16) for $i = 1$ to eliminate T_t from (16) for $i > 1$. Letting $\tilde{b}_{i,t} \equiv b_{i,t} - b_{1,t}$, we can represent the implementability constraints as

$$\begin{aligned} & (c_{i,t} - c_{1,t}) + \tilde{b}_{i,t} \\ &= -\frac{U_{l,t}^i}{U_{c,t}^i} l_{i,t} + \frac{U_{l,t}^1}{U_{c,t}^1} l_{1,t} + \frac{U_{c,t-1}^i}{\beta_{t-1} E_{t-1} U_{c,t}^i} \tilde{b}_{i,t-1} \text{ for all } i > 1, t. \end{aligned} \quad (17)$$

With this representation of the implementability constraints, the planner's maximization problem depends only on the $I-1$ variables $\tilde{b}_{i,t-1}$. The reduction of the dimensionality from I to $I-1$ is another consequence of theorem 1.

Denote $Z_t^i = U_{c,t}^i c_{i,t} + U_{l,t}^i l_{i,t} - \frac{U_{c,t}^i}{U_{c,t}^1} [U_{c,t}^1 c_{1,t} + U_{l,t}^1 l_{1,t}]$. Formulated in a space of sequences, the optimal policy problem is:

$$\max_{c_{i,t}, l_{i,t}, \tilde{b}_{i,t}} E_0 \sum_{i=1}^I \pi_i \alpha_i \sum_{t=0}^{\infty} \bar{\beta}_t Z_t^i (c_{i,t}, l_{i,t}), \quad (18)$$

subject to

$$\tilde{b}_{t-1} \frac{U_{c,t-1}^i}{\beta_{t-1}} = \left(\frac{\mathbb{E}_{t-1} U_{c,t}^i}{U_{c,t}^i} \right) \mathbb{E}_t \sum_{k=t}^{\infty} \left[\prod_{j=t}^{k-1} \beta_j \right] Z_k^i \quad \forall t \geq 1 \quad (19a)$$

$$\tilde{b}_{t-1} = \mathbb{E}_{-1} \sum_{k=0}^{\infty} \left[\prod_{j=0}^{k-1} \beta_j \right] Z_k^i \quad (19b)$$

$$\frac{\mathbb{E}_t U_{c,t+1}^i}{U_{c,t}^i} = \frac{\mathbb{E}_t U_{c,t+1}^j}{U_{c,t}^j} \quad (19c)$$

$$\sum_{i=1}^I \pi_i c_i(s^t) + g(s_t) = \sum_{i=1}^I \pi_i \theta_i(s_t) l_i(s^t), \quad (19d)$$

$$\frac{U_{l,t}^i}{\theta_{l,t} U_{c,t}^i} = \frac{U_{l,t}^1}{\theta_{l,t} U_{c,t}^1} \quad (19e)$$

$$\tilde{b}_{t-1} \frac{U_{c,t-1}^i}{\beta_{t-1}} \text{ is bounded} \quad (19f)$$

Constraint (19a) is a measurability restriction on allocations that requires that the right side is determined at time $t - 1$. This condition is inherited from the restriction that only risk-free bonds are traded.

For both computational and educational purposes, it is convenient to represent the optimal policy problem recursively. For the purpose of constructing a recursive representation, let $\boldsymbol{x} = \beta^{-1} (U_e^2 \tilde{b}_2, \dots, U_e^I \tilde{b}_I)$, $\boldsymbol{\rho} = (U_e^2/U_e^1, \dots, U_e^I/U_e^1)$, and denote an allocation $a = \{c_i, l_i\}_{i=1}^I$. In the spirit of Kydland and Prescott (1980) and Farhi (2010), we split the Ramsey problem into a time-0 problem that takes $(\{\tilde{b}_{i,-1}\}_{i=2}^I, s_0)$ as given and a time $t \geq 1$ continuation problem that takes $\boldsymbol{x}, \boldsymbol{\rho}, s_-$ as given. We formulate two Bellman equations and two value functions, one that pertains to $t \geq 1$, another to $t = 0$. The time inconsistency of an optimal policy manifests itself in there being distinct value functions and Bellman equations at $t = 0$ and $t \geq 1$.

For $t \geq 1$, let $V(\boldsymbol{x}, \boldsymbol{\rho}, s_-)$ be the planner's continuation value given $\boldsymbol{x}_{t-1} = \boldsymbol{x}$, $\boldsymbol{\rho}_{t-1} = \boldsymbol{\rho}$, $s_{t-1} = s_-$. It satisfies the Bellman equation

$$V(\boldsymbol{x}, \boldsymbol{\rho}, s_-) = \max_{a(s), \boldsymbol{x}'(s), \boldsymbol{\rho}'(s)} \sum_s \Pr(s|s_-) \left(\left[\sum_i \pi_i \alpha_i U^i(s) \right] + \beta(s) V(\boldsymbol{x}'(s), \boldsymbol{\rho}'(s), s) \right) \quad (20)$$

where the maximization is subject to

$$U_e^i(s) [\alpha_i(s) - c_i(s)] + \beta(s) x'_i(s) + \left(U_l^i(s) l_i(s) - U_e^i(s) \frac{U_l^1(s)}{U_e^1(s)} l_1(s) \right) = \frac{x U_e^i(s)}{\mathbb{E}_{s_-} U_e^i} \text{ for all } s, i \geq 2 \quad (21a)$$

$$\frac{\mathbb{E}_{s_-} U_e^i}{\mathbb{E}_{s_-} U_e^1} = \rho_i \text{ for all } i \geq 2 \quad (21b)$$

$$\frac{U_l^i(s)}{\theta_i(s) U_e^i(s)} = \frac{U_l^1(s)}{\theta_1(s) U_e^1(s)} \text{ for all } s, i \geq 2 \quad (21c)$$

$$\sum_i \pi_i c_i(s) + g(s) = \sum_i \pi_i \theta_i(s) l_i(s) \quad \forall s \quad (21d)$$

$$\rho'_i(s) = \frac{U_e^i(s)}{U_e^1(s)} \text{ for all } s, i \geq 2 \quad (21e)$$

$$\underline{x}_i(s; \boldsymbol{x}, \boldsymbol{\rho}, s_-) \leq x_i(s) \leq \bar{x}_i(s; \boldsymbol{x}, \boldsymbol{\rho}, s_-) \quad (21f)$$

Constraints (21b) and (21e) imply (15). The definition of x_t and constraints (21a) together imply equation (17) scaled by U_e^i . Let $V_0(\{\tilde{b}_{t-1}\}_{t=2}^T, s_0)$ be the value to the planner at $t = 0$, where \tilde{b}_{t-1} denotes initial debt inclusive of accrued interest. It satisfies the Bellman equation

$$V_0(\{\tilde{b}_{t-1}\}_{t=2}^T, s_0) = \max_{a_0, x_0, \rho_0} \sum_i \pi_i \alpha_i U^i(c_{i,0}, l_{i,0}) + \beta(s_0) V(x_0, \rho_0, s_0) \quad (22)$$

where the maximization is subject to

$$U_{e,0}^i [c_{i,0} - c_{i,0}] + \beta(s_0) x_{i,0} + \left(U_{l,0}^i l_{i,0} - U_{e,0}^i \frac{U_{l,0}^1}{U_{e,0}^1} l_{1,0} \right) = U_{e,0}^i \tilde{b}_{i-1} \text{ for all } i \geq 2 \quad (23a)$$

$$\frac{U_{l,0}^i}{\theta_{i,0} U_{e,0}^i} = \frac{U_{l,0}^1}{\theta_{1,0} U_e^1} \text{ for all } i \geq 2 \quad (23b)$$

$$\sum_i \pi_i c_{i,0} + g_0 = \sum_i \pi_i \theta_{i,0} l_{i,0} \quad (23c)$$

$$\rho_{i,0} = \frac{U_{e,0}^i}{U_{e,0}^1} \text{ for all } i \geq 2 \quad (23d)$$

Because constraint (21b) is absent from the time 0 problem, the time 0 problem differs from the time $t \geq 1$ problem, a source of the time consistency of the optimal tax plan.

6 Ergodic distribution and policies in the long run

In this section, we describe an ergodic set to which state variables converge. We start with the case in which aggregate shocks are iid and can take two values. We show that for this shock structure there generally exists a pair $(\boldsymbol{x}^{SS}, \boldsymbol{\rho}^{SS})$ such that if economy ever reaches a state within this set, it stays there. Tax rates and transfers in this steady state depend only on the current realization of the shock; for commonly used preferences, fluctuations of tax rates are small. We also describe properties of the steady state as well conditions under which economy converges to it and the speed of convergence. Section 6.3 then extends the analysis to more general shocks and shows numerically that while a fixed non-random steady state generally does not exist, asymptotic outcomes inside the ergodic set are very similar to those that prevail in the two shock iid case in which there is a fixed steady state. Throughout this section we assume that preferences are separable in consumption and labor.

6.1 IID shocks with two values

Let $\Psi(s; \mathbf{x}, \boldsymbol{\rho}, s_-)$ be an optimal law of motion for the state variables for the $t \geq 1$ recursive problem, i.e., $\Psi(s; \mathbf{x}, \boldsymbol{\rho}, s_-) = (\mathbf{x}'(s), \boldsymbol{\rho}'(s))$ solves (20) given state $(\mathbf{x}, \boldsymbol{\rho}, s_-)$.

Definition 5 A steady state $(\mathbf{x}^{SS}, \boldsymbol{\rho}^{SS})$ satisfies $(\mathbf{x}^{SS}, \boldsymbol{\rho}^{SS}) = \Psi(s; \mathbf{x}^{SS}, \boldsymbol{\rho}^{SS}, s_-)$ for all s, s_- .

Since in this steady state $\rho_i = U_e^i(s)/U_e^1(s)$ does not depend on the realization of shock s , the ratios of marginal utilities of all agents are constant. The continuation allocation depends only on s_1 and not on the history s^{t-1} .

We begin by noting that a competitive equilibrium fixes an allocation $\{c_i(s), l_i(s)\}_i$ given a choice for $\{\tau(s), \boldsymbol{\rho}(s)\}$ using equations (21c), (21d) and (21e). Let us denote $U(\tau, \boldsymbol{\rho}, s)$ as the value for the planner from the implied allocation using Pareto weights $\{\alpha_i\}_i$,

$$U(\tau, \boldsymbol{\rho}, s) = \sum_i \alpha_i U^i(s).$$

As before define $Z_i(\tau, \boldsymbol{\rho}, s)$ as

$$Z_i(\tau, \boldsymbol{\rho}, s) = U_e^i(s)c_i(s) - U_l^i(s)l_i(s) - \rho_i(s) [U_e^1(s)c_1(s) - U_l^1(s)l_1(s)].$$

For the IID case, the optimal policy solves the following Bellman equation for $\mathbf{x}(s^{t-1}) = \mathbf{x}, \boldsymbol{\rho}(s^{t-1}) = \boldsymbol{\rho}$

$$V(\mathbf{x}, \boldsymbol{\rho}) = \max_{\tau(s), \boldsymbol{\rho}'(s), \mathbf{x}'(s)} \sum_s P(s) [U(\tau(s), \boldsymbol{\rho}'(s), s) + \beta(s)V(\mathbf{x}'(s), \boldsymbol{\rho}'(s))] \quad (24)$$

subject to the constraints

$$Z_i(\tau(s), \boldsymbol{\rho}'(s), s) + \beta(s)x'_i(s) = \frac{x_i U_e^i(\tau(s), \boldsymbol{\rho}'(s), s)}{\mathbb{E} U_e^i(\tau, \boldsymbol{\rho})} \text{ for all } s, i \geq 2, \quad (25)$$

$$\sum_s P(s) U_e^1(\tau(s), \boldsymbol{\rho}'(s), s) (\rho'_i(s) - \rho_i) = 0 \text{ for } i \geq 2. \quad (26)$$

Constraint (26) is obtained by rearranging constraint (21b). It implies that $\boldsymbol{\rho}(s)$ is a risk-adjusted martingale. We next check if the first-order necessary conditions are consistent with stationary policies for some $(\mathbf{x}, \boldsymbol{\rho})$.¹⁵

Lemma 1 Let $\Pr(s)\mu_i(s)$ and λ_i be the multipliers on constraints (25) and (26). Imposing the restrictions $x'_i(s) = x_i$ and $\rho'_i(s) = \rho_i$, at a steady state $\{\mu_i, \lambda_i, x_i, \rho_i\}_{i=2}^N$ and $\{\tau(s)\}_s$ are determined by the following equations

$$Z_i(\tau(s), \boldsymbol{\rho}, s) + \beta(s)x_i = \frac{x_i U_e^i(\tau(s), \boldsymbol{\rho}, s)}{\mathbb{E} U_e^i(\tau, \boldsymbol{\rho})} \text{ for all } s, i \geq 2, \quad (27a)$$

$$U_\tau(\tau(s), \boldsymbol{\rho}, s) - \sum_i \mu_i Z_{i,\tau}(\tau(s), \boldsymbol{\rho}, s) = 0 \text{ for all } s, \quad (27b)$$

$$U_{\rho_i}(\tau(s), \boldsymbol{\rho}, s) - \sum_j \mu_j Z_{j,\rho_i}(\tau(s), \boldsymbol{\rho}, s) + \lambda_i U_e^i(\tau(s), \boldsymbol{\rho}', s) - \lambda_i \beta(s) \mathbb{E} U_e^i(\tau, \boldsymbol{\rho}) = 0 \text{ for all } s, i \geq 2 \quad (27c)$$

¹⁵Appendix A.5 discusses the associated second order conditions that ensure these policies are optimal

Since the shock s can take only two values, (27) is a square system in $4(N - 1) + 2$ unknowns $\{\mu_i^{SS}, \lambda_i^{SS}, x_i^{SS}, \rho_i^{SS}\}_{i=2}^N$ and $\{\tau^{SS}(s)\}_s$. For wide range of primitives, we can verify numerically that this system has a solution. In the next section we formally establish this for a class of two-agent economies that, while special, illustrate general forces that affect outcomes. The example will help us develop some comparative statics and interpret outcomes from a quantitative analysis to appear in section 7.

Lemma 1 also highlights the tradeoffs that the planner faces. Defining $\tilde{\lambda} = -\lambda \mathbb{E}U_c(\tau, \rho)$ and taking expectations in equation (27c), we get

$$\mathbb{E}U_{\rho_i}(\tau(s), \rho, s) = \mathbb{E} \sum_j \mu_j(s) Z_{j,\rho_i}(\tau(s), \rho, s) + (1 - \mathbb{E}\beta(s)) \tilde{\lambda}_i \quad (28)$$

The multiplier on the implementability constraint for i can be interpreted as the marginal cost of extracting funds from i and $\tilde{\lambda}_i$ is proportional to the multiplier on the constraint $\frac{\mathbb{E}U_i}{\mathbb{E}U_c} = \rho$. This constraint ensures that at the optimal allocation, agent i has no incentive to change his bond portfolio. The left side of (28) captures the cost for the planner if inequality (measured by the ratios of marginal utilities of consumption) deviates from his ideal point, given by α_1/α_i . In the absence of any constraints, at the first best the planner would set $\mathbb{E}U_{\rho_i}(\tau(s), \rho, s) = 0$, which implies that $\alpha_i U_c^i = \alpha_1 U_c^1$ for all i . The right side of equation (28) captures the cost of approaching the planner's ideal point, which come from the costs of raising taxes (the first term on the left hand side) and the ability of agents to trade (the second term).

The behavior of the economy in the steady state is similar to the behavior of the complete market economy characterized by Werning (2007). Both taxes and transfers depend only on the current realization of shock s_t . Moreover, the arguments of Werning (2007) can be adapted to show that taxes are constant when preferences have a CES form $c^{1-\sigma}/(1-\sigma) - l^{1+\gamma}/(1-\gamma)$ and fluctuations in tax rates are very small when preferences take forms consistent with the existence of balanced growth. We return to this point after we discuss convergence properties.

A two-agent example

Lemma 4 provides a simple way to verify existence of a steady state for wide range of parameter values by checking that there exists a root for system (27). Since the system of equations (27) is non-linear, existence can generally be verified only numerically. In this section, we provide a simple example with risk averse agents in which we can show existence of the root of (27) analytically. The analytical characterization of the steady state will allow us to show two main forces that determine the steady state asset distribution. These forces will also help to understand the long run behavior of the calibrated economy that we study in section 7.

Consider an economy consisting of two types of households with $\theta_{1,t} > \theta_{2,t} = 0$. One period utilities are $\ln c - \frac{1}{2}l^2$. The shock s takes two values, $s \in \{s_L, s_H\}$ with probabilities $\Pr(s|s_-)$ that are independent of s_- . We assume that $g(s) = g$ for all s , and $\theta_1(s_H) > \theta_1(s_L)$. We allow the discount factor $\beta(s)$ to depend on s .

Proposition 5 Suppose that $g < \theta(s)$ for all s . Let $R(s)$ be the gross interest rates and $x = U_c^2(s) [b_2(s) - b_1(s)]$

1. **Countercyclical interest rates.** If $\beta(s_H) = \beta(s_L)$, then there exists a steady state (x^{SS}, ρ^{SS}) such that $x^{SS} > 0$, $R^{SS}(s_H) < R^{SS}(s_L)$.
2. **Acyclical interest rates.** There exists a pair $\{\beta(s_H), \beta(s_L)\}$ such that there exists a steady state with $x^{SS} > 0$ and $R^{SS}(s_H) = R^{SS}(s_L)$.
3. **Procylical interest rates.** There exists a pair $\{\beta(s_H), \beta(s_L)\}$ such that there exists a steady state with $x^{SS} < 0$ and $R^{SS}(s_H) > R^{SS}(s_L)$.

In all cases, taxes $\tau(s) = \tau^{SS}$ are independent of the realized state.

In this two-agent case, by normalizing assets of the unproductive agent (using theorem 1) we can interpret x as the marginal utility adjusted assets of the government. Besides establishing existence, the proposition identifies the importance of cyclical properties of real interest rates in determining the sign of these assets.

Proposition 5 shows two main forces that determine the dynamics of taxes and assets: fluctuations in inequality and fluctuations in the interest rates. Let's start with part 2 of proposition 5, which turns off the second force. When interest rates are fixed, the government can adjust two instruments in response to an adverse shock (i.e., a fall in θ_1): it can either increase the tax rate τ or it can decrease transfers T . Both responses are distorting, but for different reasons. Increasing the tax rate increases distortions because the deadweight loss is convex in the tax rate, as in Barro (1979). This force operates in our economy just as it does in representative agent economies. But in a heterogeneous agent economy like ours, adjusting transfers T is also costly. When agents' asset holdings are identical, a decrease in transfers disproportionately affects a low-skilled agent, so his marginal utility falls by more than does the marginal utility of a high-skilled agent. Consequently, a decrease in transfers increases inequality, giving rise to a cost not present in representative agent economies.

The government can reduce the costs of inequality distortions by choosing tax rate policies that make the net asset positions of the high-skilled agent decrease over time. That makes the two agents' after-tax and after-interest income become closer, allowing decreases in transfers to have smaller effects on inequality in marginal utilities. If the net asset position of a high-skilled agent is sufficiently low, then a change in transfers has no effect on inequality and all distortions from fluctuations in transfers are eliminated.¹⁶

Turning now to the second force, interest rates generally fluctuate with shocks. Parts 1 and 3 of proposition 5 indicate what drives those fluctuations. Consider again the example of a decrease in productivity of high-skilled agent. If the tax rate τ is left unchanged, the government faces a shortfall of

¹⁶This convergence outcome has a similar flavor to "back-loading" results of Ray (2002) and Albanesi and Armenter (2012) that reflect the optimality of structuring policies intertemporally eventually to disarm distortions.

revenues. Since g is constant, the government requires extra sources of revenues. But suppose that the interest rate increases whenever θ_1 decreases, as happens, for example, when discount factors are constant and θ_1 is the only source of shocks. If the government holds positive assets, its earnings from those assets increase. So holding assets allows higher interest income to offset some of the government's revenue losses from taxes on labor. The situation reverses if interest rates fall at times of increased need for government revenues, as in part 3 of proposition 5, and the steady state allocation features the government's owning debt.

What matters for our second force is the comovement of the interest rate with fundamental shocks. States with low average TFP (and therefore a lower base for labor taxes), high g , or a high spread of productivities that threatens to induce higher inequality (and therefore higher transfers and thirst for more government revenues to finance them) are "adverse" from the point of view of current government finance. The government can cope with such adverse states in less distorting ways if it finds itself holding positive (negative) assets if interest rates are high (low).¹⁷

Depending on details of shock processes, these two forces can either reinforce each other (as happens in Part 1 of proposition 5) or oppose each other (as in Part 3 of proposition 5). In the latter case, whether the government ends up with assets or debt in the long run depends on the relative strengths of the two forces.

Besides discount factor shocks, the level of net assets in the steady state depends on other primitives such as preferences for redistribution. An interesting comparative static exercise involves shutting off discount factor shocks and increasing α_1 , the Pareto weight attached to the high-skilled agent. This implies that the planner taxes the high-skilled agent less and redistributes less income to the low-skilled agent. Since the after-tax income of a low-skilled agent is lower, fluctuations in transfers affect the low-skilled agent more adversely than the high-skilled agent. To smooth those fluctuations, the government needs to accumulate more claims on the high skilled, implying a positive relationship between the steady state level of government assets and Pareto weight on high-skilled agents. Thus the planner substitutes the shortfall in revenues from taxes with higher earnings from his assets. Figure 3 plots how taxes and assets of the government vary as we change the Pareto weight α_1 on a high-skilled worker.

Correlation of net assets and productivities

For $I = 2$, proposition 5 signs the marginal utility adjusted net assets that we denoted x . This implies a particular ordering of net assets across the agents in the steady state. These implications generalize to settings with $I \geq 3$. In general, one can verify that without discount factor shocks (as in part 1 of proposition 5 when interest rates are countercyclical), steady state net asset levels (scaled by marginal utilities) are ordered inversely to productivities. By making net asset positions be negatively correlated with labor earnings, the planner can minimize the costs of fluctuating transfers. Further for similar

¹⁷The results of proposition 5 can be substantially generalized to an economy with no heterogeneity. In Bhandari et al. (2013), we study a representative agent economy with a more general incomplete market structure and distorting taxes. We show that for a wide range of preferences, a steady state exists, that it is globally stable, and that the sign of long run asset position of the government is determined by the co-movement of returns on assets and shocks.

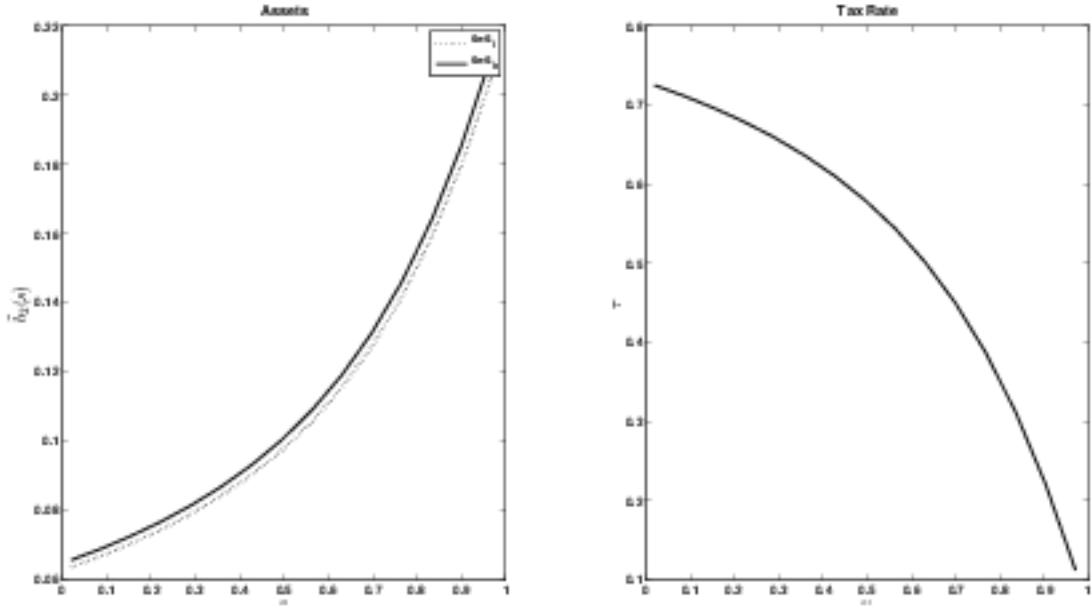


Figure 3: Steady state assets : $\tilde{b}_2(s) = \frac{\beta x^{SS}}{U_c'(s)}$ and taxes : τ^{SS} as a function of Agent 1's (high skilled) Pareto weight

reasons as in the $I = 2$ case, with exogenous variations in discount factors large enough to make interest rates be procyclical, the planner makes net asset positions be positively correlated with productivities. In a recession the government faces a short fall of tax revenues that it can make up from revenues of its assets. If interest rates are sufficiently lower, issuing debt implies a lower interest liability and frees up some resources to make up the low tax revenues. Furthermore, by borrowing a larger amount from the more productive agents the government can reduce the welfare costs from lowering transfers in adverse times. This implies a positive correlation between assets and productivities.

6.2 Stability

In this section, we return to the general section 2 formulation of the Ramsey problem to study convergence to a steady state. We begin by describing a test for local convergence cast in terms of a linear approximation of optimal policies at a steady state. We apply this test to show local stability of a steady state for a wide range of parameters. From these examples it emerges that slow convergence to the steady state occurs for commonly calibrated parameter values.

To study convergence, we return to the maximization problem (24) and assume that it admits a steady state. As before, let assume that $\Pr(s)\mu_i(s)$ and λ_i be the multipliers on constraints (25) and (26). In Appendix A.5 we show that the history-dependent optimal policies (they are sequences of functions of s^t) can be represented recursively in terms of $\{\mu(s^{t-1}), \rho(s^{t-1})\}$ and s_t . A recursive representation of an optimal policy can be linearized around the steady state using (μ, ρ) as state variables.¹⁸

¹⁸One could in principle look for a solution in state variables $(\omega(s^{t-1}), \rho(s^{t-1}))$. For $I = 2$ with $\{\theta_i(s)\}$ different across agents, this would give identical policies and a map which is (locally) invertible between ω and μ for a given ρ . However

Formally, let $\hat{\Psi}_t = \begin{bmatrix} \mu_t - \mu^{ss} \\ \rho_t - \rho^{ss} \end{bmatrix}$ be deviations from a steady state. From a linear approximation, one can obtain $B(s)$ such that

$$\hat{\Psi}_{t+1} = B(s_{t+1})\hat{\Psi}_t. \quad (29)$$

This linearized system has coefficients that are functions of the shock. The next proposition describes a simple numerical test that allows us to determine whether this linear system converges to zero in probability.

Proposition 6 *If the (real part) of eigenvalues of $EB(s)$ are less than 1, system (29) converges to zero in mean. Further for large t , the conditional variance of $\hat{\Psi}_t$, denoted by $\Sigma_{\Psi,t}$, follows a deterministic process governed by*

$$\text{vec}(\Sigma_{\Psi,t}) = \hat{B}\text{vec}(\Sigma_{\Psi,t-1}),$$

where \hat{B} is a square matrix of dimension $(2I - 2)^2$. In addition, if the (real part) of eigenvalues of \hat{B} are less than 1, the system converges in probability.

The eigenvalues (in particular the largest or the dominant one) are instructive not only for whether the system is locally stable but also how quickly the steady state is reached. In particular, the half-life of convergence to the steady state is given by $\frac{\log(0.5)}{\|\lambda\|}$, where $\|\lambda\|$ is the absolute value of the dominant eigenvalue. Thus, the closer the dominant eigenvalue is to one, the slower is the speed of convergence.

We used proposition 6 to verify local stability of a wide range of examples. The typical finding is that the steady state is generically stable and that convergence is slow. In figure 4 we plot the comparative statics for the dominant eigenvalue and the associated half-life for a two-agent economy with CES preferences. We set the other parameter to match a Frisch elasticity of 0.5, a real interest rate of 2%, marginal tax rates around 20%, and a 90-10 percentile ratio of wage earnings of 4. In the first exercise, we vary the size of the expenditure shock keeping risk aversion σ at one. The x -axis plots the spread in expenditure normalized by the undistorted GDP and reported in percentages. In the bottom panel, we fix the size of shock such that it produces a 5% fall in expenditure fall at risk aversion of one and vary σ from 0.8 to 7. We see that the dominant eigenvalue is everywhere less than one but very close to one, so that the steady state is stable but convergence is slow for reasonable values of curvatures and shocks. We return to this feature in section 7 where we study low frequency components of government debt. Both increasing the size of the shock or risk aversion increases the volatility of the interest rates, speeding up the transition towards the steady state.

6.3 More general shocks

The results on existence and convergence to a steady state relied on a special binary-IID restriction. When there are more than two possible values for the shocks or when shocks are persistent, the time-invariant

in other cases, it turns out there are unique linear policies in $(bm\mu, \rho)$ and not necessarily in (x, ρ) . This comes from the fact that the set of feasible (x, ρ) are restricted at time 0 and may not contain an open set around the steady state values. When we linearize using (μ, ρ) as state variables, the optimal policies for $x(s^t), \rho(s^t)$ converge to their steady state levels for all perturbations in (μ, ρ) .

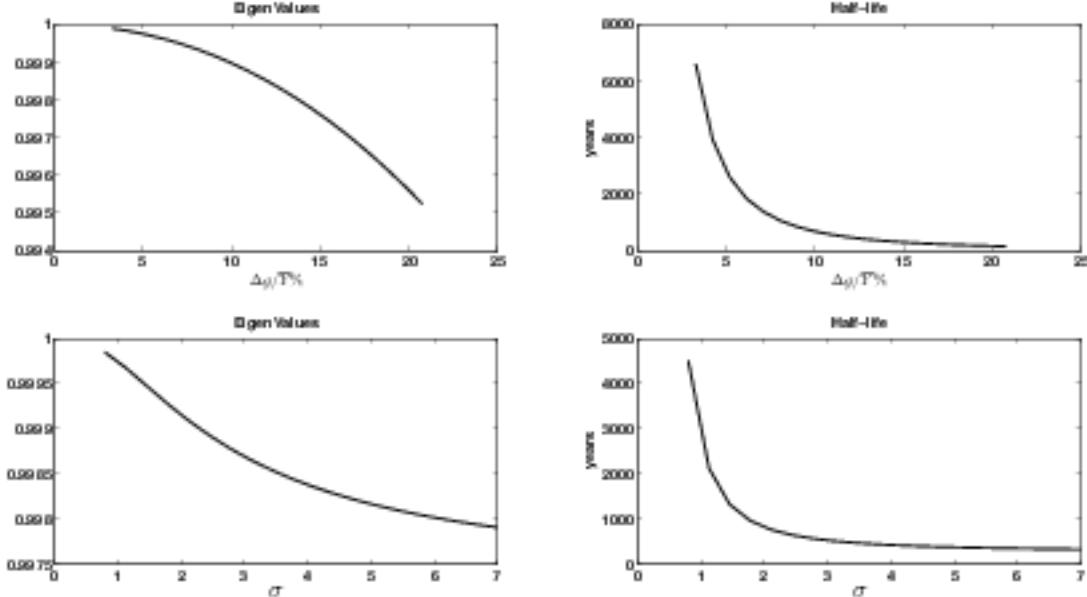


Figure 4: The top (bottom) panel plots the dominant eigenvalue of \hat{B} and the associated half life as we increase the spread between the expenditure levels (risk aversion).

steady state will no longer exist. Mathematically, this occurs because one asset and one risk-free rate of return cannot span all possible needs for government revenues. With richer shock structures, there exists an attraction region in the (x, ρ) space to which the dynamic system converges. Although (x, ρ) are no longer constant in such region, their fluctuations tend to be markedly reduced relative to the transient fluctuations that occur away from that region, and general properties of x and ρ are the same as those described in Proposition 5. Figure 5 shows long sample paths for economies hit by more general TFP shocks. The top panel has IID shocks with 2 (bold) and 3 (dotted) possible values and the bottom panel has persistent shocks with 2 (bold) and 3 (dotted) possible values.

7 Optimal policy in booms and recessions

In section 6 we used steady states to characterize the long-run behavior of optimal allocations and forces that guide the asymptotic level of net assets. In this section, we use a calibrated version of the economy to a) revisit the magnitude of these forces and b) study optimal policy responses at business cycle frequencies when the economy is possibly far away from the steady state. We choose shocks to match stylized facts about recent recessions in US.

We consider an economy with two types of agents of equal measures with preferences¹⁹

$$U(c, l) = \psi \ln c + (1 - \psi) \ln(1 - l).$$

¹⁹We restrict our attention to the economy with two agents for computational tractability. We want to understand both short-run and long-run responses to shocks. For some of our computations, it is important to allow our dynamic systems to travel over a large subset of state space, including regions encountered infrequently in the invariant distribution. With more agents, it seems possible to apply other methods, for example those of Judd et al. (2011), to study dynamics of our economy within its invariant distribution. We hope to pursue such extensions in future work.

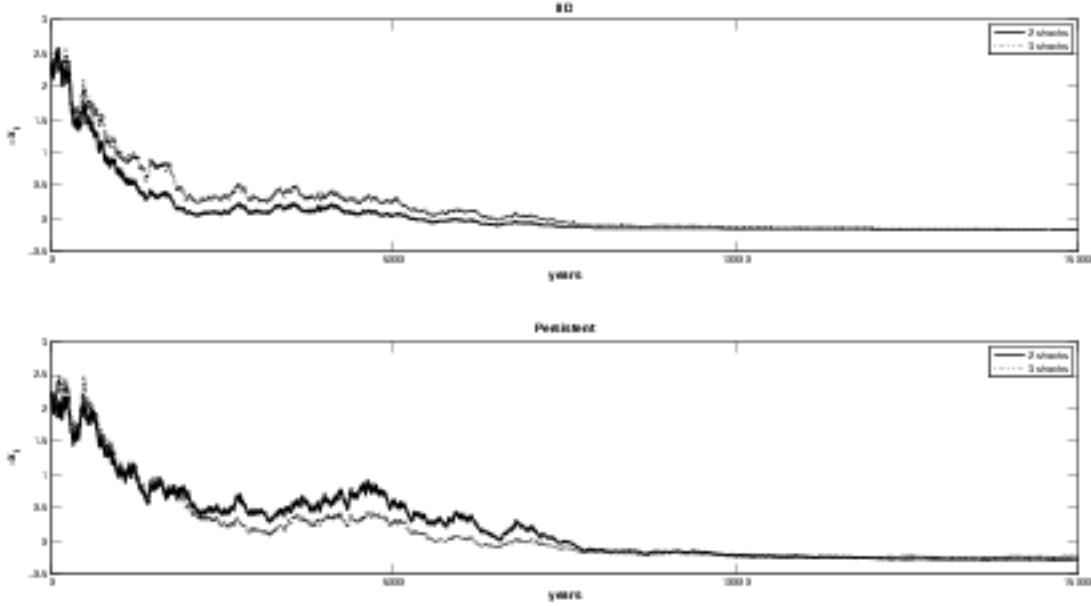


Figure 5: The figure depicts sample paths of marginal utility adjusted debt of the government i.e $-x_t$. The top panel has IID shocks with 2 (bold) and 3 (dotted) possible values and the bottom panel has persistent shocks with 2 (bold) and 3 (dotted) possible values

The shock s takes two values, s_H and s_L , and follows a persistent process. We allow $\beta, \theta_i, i = 1, 2$, and g to be functions of s . We first pick $\bar{\theta}_i, \bar{g}$ and $\bar{\beta}$ for a deterministic economy without shocks and calibrate (ψ, α) to some low frequency data moments. Then to match some business cycle moments we pick shocks according to

$$\begin{aligned}\theta_i(s) &= \bar{\theta}_i[1 + \hat{\theta}_i(s)], \\ \beta(s) &= \bar{\beta}[1 + \hat{\beta}(s)], \\ g(s) &= \bar{g}[1 + \hat{g}(s)],\end{aligned}\tag{30}$$

where $\hat{\theta}_i(s) \in \{-e_{i,\theta}, e_{i,\theta}\}$, $\hat{\beta}(s) \in \{-e_\beta, e_\beta\}$, and $\hat{g}(s) \in \{-e_g, e_g\}$. Throughout our experiments, we normalize $b_{2,t} = 0$ for all $t \geq -1$. From market clearing, $B_t = -b_{1,t}$. We refer to B_t as government debt (when negative) and assets (when positive).

7.1 Calibration

We calibrate the model in two steps. We first choose baseline parameters that govern preferences and technology so that an optimal equilibrium for a no-shocks version of the economy matches some sample moments in post war US data. In the second step, we adjusted other parameters to make the amplitudes of fluctuations equal to average peak-trough spreads observed in the three most recent recessions (1991-92, 2001-02 and 2008-10).

We first discuss calibration of $(\psi, \alpha, \bar{\theta}_i, \bar{g}, \bar{\beta})$. Although these parameters jointly determine the relevant moments, it is helpful to explain which moment in the data mainly influences each parameter. We

normalize $\bar{\theta}_2 = 1$ and pick $\bar{\theta}_1$ to match a log wage ratio of 90 wage percentile to 10 wage percentile of 4 from Autor et al. (2008). We set the discount factor $\bar{\beta}$ to match an (annual) interest rate of 2%. We set the parameter ψ to match a Frisch elasticity of labor supply equal to 0.5. In our model, \bar{g} corresponds to non-transfer government expenditures, which in the U.S. varied from 7% and 11% in the post WWII period and were above 20% during WWII. We set \bar{g} equal to 12% of GDP. Finally, we set Pareto weights α to match the average marginal tax rate in the US of about 20% as in Chari et al. (1994).²⁰

Next we turn to business cycle targets. We calibrate $\{e_{4\theta}, e_\beta, \Pr(s|s_-)\}$ to match the following four facts about booms and recessions (using NBER dates, for the last 3 recessions, i.e., 1991-92, 2001-02 and 2008-10): the log of incomes individuals at both the 10th and the 90th percentile falls the recessions; 10th percentile income falls by more than 90th percentile; an inflation-adjusted interest rate on government debt is generally lower in recessions; and booms last longer than recessions. We calibrate the average spread in labor productivity to match the average 3% loss in output observed in the last three recessions. The inequality shock is designed to match facts documented in Guvenen et al. (2012) that the fall in earnings of the 10-percentile is about 2.5 times of 90-percentile. The discount factor shocks match the average boom-recession difference of about 1.96% in the real risk-free interest rate (3 month T bill rate - inflation rate) seen in the last three recessions.²¹ We calibrate the transition matrix to match the average duration of booms and recessions. For comparison, we also report optimal responses to a drop in government expenditure that leads to an output drop of similar magnitude.

Note that because each is an exact function of s_t , government expenditures, the discount factor, and productivities are perfectly correlated: a recession is an episode in which TFP falls, inequality rises, and the discount factor is high. We set initial government debt to be 60% of GDP, roughly to match the ratio of federal debt held by public at the beginning of 2010.

Table 1 summarizes some details about our calibration.

Parameter	Value	Description	Target
ψ	0.6994	Frisch elasticity of labor supply	0.5
$\bar{\theta}_1$	4	Log 90-10 wage ratio (Autor et all)	4
$\bar{\theta}_2$	1	Normalize to 1	1
β	0.98	Average (annual) risk free interest rate	2%
α_1	0.69	Marginal tax rate in the economy with no shocks	20%
g	12%	Average pre-transfer expenditure- output ratio	12 %
$\frac{\bar{\theta}_2}{\bar{\theta}_1}$	2.5	Relative drop in wage income of 10th percentile as compared to 90th percentile	2.5
$\bar{\theta}_1$	1.2%	Average output loss	3%
$\hat{\beta}(s)$	1.96%	Difference in real interest rates between booms and recession	1.96%
$P(r r)$	0.63	Duration of recessions	2.33 years
$P(b b)$	0.84	Duration of booms	7 years

Table 1: Benchmark calibration

²⁰We use federal government expenditures (excluding current transfers) since the labor tax rate of 20% in Chari et al. (1994) is calibrated to federal marginal taxes.

²¹It has long been noticed that the standard RBC model predicts counter-factual negative correlation between real interest rates and output (e.g. Boklin et al. (2001)). In the data HP filtered output is roughly uncorrelated with real interest rates, but this relationship turn positive if we look at peak vs troughs. We report the optimal responses for both economies with positive and zero correlation of interest rates and output and contrast with a response to a pure TFP shock.

7.2 Outcomes

We discuss separately long run and short run implications for optimal policy. In particular, we study an economy with the calibration discussed above (“Benchmark”) and a few variants that successively turn off particular sources of variation.

1. **Acyclical interest rates:** In the first variant, we recalibrate the discount factor shocks to make the risk-free rate be uncorrelated with output.
2. **Countercyclical interest rates:** Here we shut off discount factor shocks by setting $\hat{\beta}(s) = 0$ in (30). Note that this makes interest rates countercyclical.
3. **No inequality:** This variant modifies the “Benchmark” by setting $\hat{\beta}(s) = 0$ and $\hat{\theta}_1(s) = \hat{\theta}_2(s) = 3\%$ in (30). This corresponds to a case in which the only source of business cycle fluctuations is a TFP shock that affects all agents equally. This case more closely matches the experiments in the RBC literature such as Chari et al. (1994).
4. **Government expenditure shocks:** The last variant compares optimal responses to shocks to government expenditures. In this experiment, we set $\hat{\theta}(s) = \hat{\beta}(s) = 0$ and choose $\hat{g}(s)$ to produce a drop in output of a similar magnitude to that in the first three experiments. This compares to the studies of responses to government shocks by AMSS and Faraglia et al. (2012).

Long run

Figure 6 plots government debt. All experiments start with an initial government debt to GDP ratio of 60%. Several features emerge from this figure.

As expected from the analysis in Section 6, in all four economies the state (x, ρ) converges to some long run ergodic set, while government debt and the tax rate converge to associated sets. When there are no discount factor shocks (See lines with \diamond , \square in figure 6) or small discount factor shocks that produce acyclical interest rates (see the line with $+$ in figure 6) the government holds claims on the private sector when the state is in the ergodic set. Consistent with Proposition 5 the optimal policy adjusts net asset positions to ameliorate the two key constraints impinging on the government policy, namely, the inability to award agent-specific transfers (the restriction to *affine taxes*) and the absence of state-contingent assets (the restriction to *risk-free debt*). Starting from a point when the relative assets of the low-skilled agent (or the government if we use a normalization that sets $b_{2,t} = 0$) are low, extracting resources through lower transfers exacerbates inequality. This is costly since the government has to use higher taxes in future to redistribute. On the margin, the optimal policy requires the government (or low-skill agent) to accumulate assets. But interest rate fluctuations interact with net asset positions to generate state-contingent earnings from assets. If interest rate are high when the government needs additional revenues, accumulating assets relaxes the restriction imposed by absence of state contingent assets. Thus, with

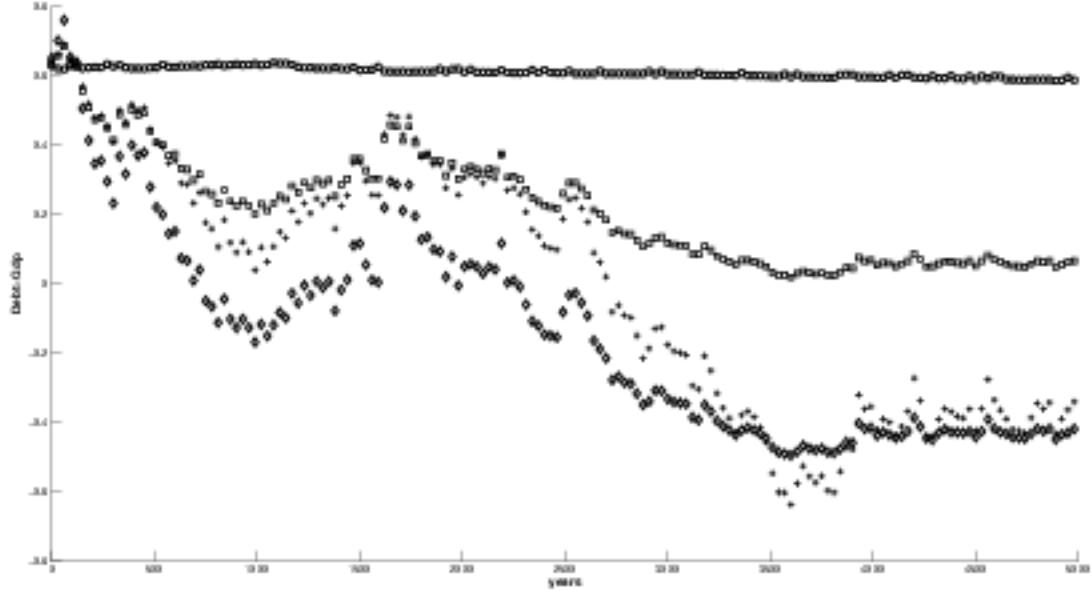


Figure 6: Debt benchmark (o), acyclical interest rates (+), countercyclical interest rates (\diamond) and no inequality shocks (\square)

countercyclical interest rates, these forces reinforce each other, making the government's long run asset position become positive.

In data, however, interest rates generally decline in recessions. Procylical interest rates mean that the two forces outlined in the previous paragraph now oppose each other. For large enough interest rate fluctuations, this means that the government may want to accumulate debt. In figure 6, the line with (o) represents the benchmark with discount factor shocks rigged to replicate procylical fluctuations in interest rates. For a particular initial condition for government debt, the planner can refrain from varying debt for a very long time.²²

Convergence to the ergodic region is very slow. With persistent shocks and an initial 60% debt-GDP ratio, it takes about 3,000 years for the government to want to pay off all that debt and then start accumulating assets. With discount factor shocks, it takes even longer to repay the debt. It is still indebted after 5000 years. These outcomes confirm the comparative statics of the eigenvalues of linearized system in proposition 6.

Thus, the covariance of interest rates with fundamentals as emphasized in proposition 5 substantially influences the ergodic distribution of government assets.

²²Like the finding in proposition 5 for large discount factor shocks (in a way that interest rates are procylical) there exist regions where x_t, ρ_t have low volatility and the government is *not* accumulating assets. But these regions are typically unstable. The two forces highlighted before that guide accumulation of assets now work in opposite direction and the net effect depends on the relative strengths. In particular the sample paths from different initial conditions $\bar{b}_{2,-1}$ (which would imply different choices for the initial x_0, ρ_0) may display larger fluctuations in assets. However, at the calibrated initial conditions (60% debt-gdp ratio), the uncertainty associated with the mean path is very low for the first 5000 periods.

	Δg	ΔB	ΔT	$\Delta[\tau\theta_1l_1]$	$\Delta[\tau\theta_2l_2]$	ΔY	$\Delta \tau$
Benchmark	0.0000	-1.1561	0.6871	-0.1593	-0.3096	-2.8536	0.3732
Acyclical Interest Rates	0.0000	-1.1126	0.6591	-0.1497	-0.3038	-2.8613	0.3879
Countercyclical Interest Rates	0.0000	-1.0794	0.6387	-0.1415	-0.2992	-2.8677	0.3997
No Inequality	0.0000	-0.1380	-0.5459	-0.5635	-0.1204	-2.6294	0.0622
Expenditure Shocks	-7.5037	2.9137	2.8612	-1.3759	-0.3530	-2.3443	-1.1598

Table 2: The tables summarizes the changes in the different components of the government budget as we transit from “boom” to a “recession”. All numbers are normalized by un-distorted GDP except τ and reported in percentages.

Short run

The analysis of the previous subsection studied aspects of very low frequency components of the optimal policy. Here we focus on business cycle frequencies. In our setting, these higher frequency responses can conveniently be divided into the magnitudes of changes as we switch from “boom” to “recession,” and the dynamics during periods when a recession or boom state persists.

We set the exogenous state s_0 so that we are at the outset of a recession. Then we solve the time 0 problem with identical initial conditions across different settings. This pins down the initial state vector x_0, ρ_0 that appears in our time 0 Bellman equation. We then use the policy rules to compute fluctuations of different components in the government budget constraint across states. These responses are summarized in Table 2. For each variable z in the table we report in the form $\Delta z \equiv (z(s_l|x_0, \rho_0, s_0) - z(s_h|x_0, \rho_0, s_0)) / \bar{Y}$ where \bar{Y} is average undistorted GDP.²³

The source of shocks is very important. Three different types of shocks that produce similar drops in GDP have very different consequences for optimal policies, both qualitatively and quantitatively. In the benchmark, the government responds to an adverse shock by making big increases in transfers, the tax rate, and government debt. However, without inequality shocks (row 4), the government responds by decreasing transfers and increasing both debt and the tax rate, but by an amount an order of magnitude smaller than in the benchmark. This indicates that ignoring distributional goals can produce a misleading prescriptions for government policy in recessions.

Discount factor shocks have minor effects on impact and matter more for transient dynamics that ultimately have big long run effects. Figures 7 and 8 show how the transient dynamics for prolonged booms (or recessions) differ with and without discount factor shocks. The four panels show tax rates, transfers, debt and interest rate movements for a path of 25 years. The bold lines in figures 7 and 8 refer to the benchmark (with procyclical interest rates) and the version with acyclical interest rates, respectively. The dotted line in both the figures is the version with countercyclical interest rates. The shaded regions are periods with low output. We see that in a prolonged booms, the government accumulates assets and that it lowers the tax rate when there are no discount factor shocks.

²³Note that predetermined variables like repayment on existing debt drop out of the accounting and we have

$$\Delta[g] + \Delta[T] + \Delta[B] = \Delta[\tau\theta_1l_1] + \Delta[\tau\theta_2l_2]$$

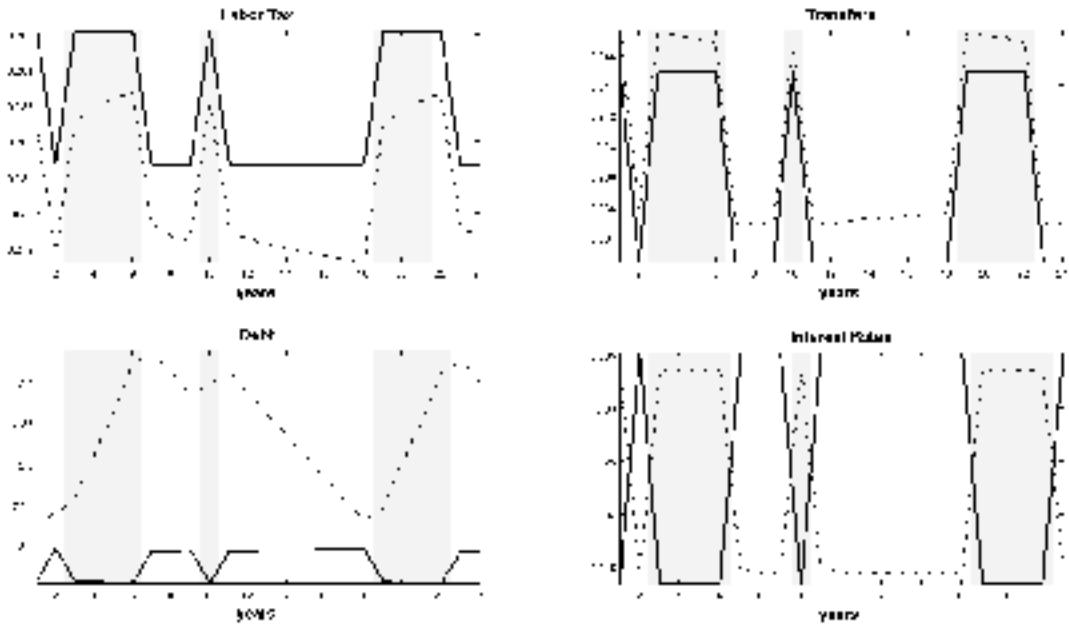


Figure 7: This plots a typical sample path taxes, transfers, debt and interest rates. The bold lines are with benchmark calibration and the dotted lines refer to the variant with countercyclical interest rates. The shaded regions are recessions.

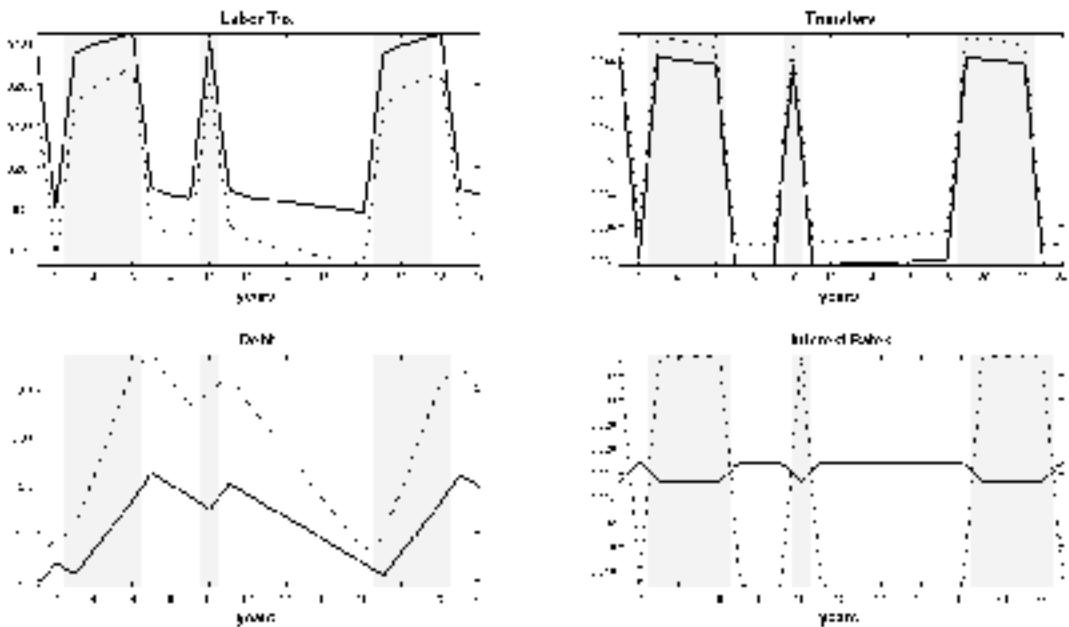


Figure 8: This plots a typical sample path taxes, transfers, debt and interest rates. The bold lines are with acyclical interest rates calibration and the dotted lines correspond to the case with countercyclical interest rates. The shaded regions are recessions.

8 Concluding remarks

The spring of 2013 witnessed a heated debate in newspapers and economic magazines about the accuracy and meaning of empirical correlations between output growth rates and ratios of government debt to GDP inferred from data sets assembled by Reinhart and Rogoff (2010). From the perspectives of our paper and of Werning (2007), those correlations and those debates are difficult to interpret because in our settings, total government debt is not a relevant state variable that affects allocations, government transfers, or tax rates. The principal message of our paper is that without exogenous restrictions on transfers, the level of government debt doesn't matter. What does matter is how government debt is distributed among people relative to society's attitudes toward unequal allocations of consumption and work. Using a recursive representation that employ correct state variables — a vector of marginal utility adjusted net asset positions and a vector of pairwise ratios of marginal utilities — we have presented a sequence of examples designed to show how agents net positions affect optimal government policies for choosing distorting tax rates, transfers, and government issues or holdings of risk-free bonds. We find that a significant determinant of an optimal asymptotic government debt or government debt-GDP ratio is how interest rate risks are correlated with risks to fundamentals that threaten to widen or narrow inequality in after-tax and after-transfer incomes. To interpret those Reinhart-Rogoff facts country-by-country, we would want to know much more about the distributions of net assets across people across countries and how they interact with risks to interest rates and to fundamental sources of inequality.

A Appendix

A.1 Additional details for Section 3.1

In this section we construct an example in which the government can achieve higher welfare in the economy with ad-hoc borrowing limits. We restrict ourselves to a deterministic economy with $g_t = 0$, $\beta_t = \beta$ and $I = 2$. Further the utility function over consumption and labor supply $U(c, l)$ is separable in the arguments and satisfies the Inada conditions. The planners problem can then be written as the following sequence problem

$$\max_{\{c_{i,t}, l_{i,t}, b_{i,t}, R_t\}_t} \sum_{t=0}^{\infty} \beta^t [\alpha_1 U(c_{1,t}, l_{1,t}) + \alpha_2 U(c_{2,t}, l_{2,t})] \quad (31)$$

subject to

$$c_{2,t} + \frac{U_{l2,t} l_{2,t}}{U_{c2,t}} - \left(c_{1,t} + \frac{U_{l1,t} l_{1,t}}{U_{c1,t}} \right) + \frac{1}{R_t} (b_{2,t} - b_{1,t}) = b_{2,t-1} - b_{1,t-1} \quad (32a)$$

$$\frac{U_{l1,t}}{\theta_1 U_{c1,t}} = \frac{U_{l2,t}}{\theta_2 U_{c2,t}} \quad (32b)$$

$$c_{1,t} + c_{2,t} \leq \theta_1 l_{1,t} + \theta_2 l_{2,t} \quad (32c)$$

$$\left(\frac{U_{c1,t}}{U_{c,t+1}} - \beta R_t \right) (b_{1,t} - \underline{b}_i) = 0 \quad (32d)$$

$$\frac{U_{c1,t}}{U_{c,t+1}} \geq \beta R_t \quad (32e)$$

$$b_{1,t} \geq \underline{b}_i \quad (32f)$$

Where \underline{b}_i is the exogenous borrowing constraint for agent i . We obtain equation (32a) by eliminating transfers from the budget equations of the households and using the optimality for labor supply decision. Equations (32d) and (32e) capture the inter-temporal optimality conditions modified for possibly binding constraints.

Let c_i^{fb} and l_i^{fb} be the allocation that solves the first best problem, that is maximizing equation (31) subject to (32c), and define

$$Z^{fb} = c_2^{fb} + \frac{U_{l2}^{fb} l_2^{fb}}{U_{c2}^{fb}} - \left(c_1^{fb} + \frac{U_{l1}^{fb} l_1^{fb}}{U_{c1}^{fb}} \right) \quad (33)$$

and

$$\tilde{b}_2^{fb} = \frac{Z^{fb}}{\frac{1}{\beta} - 1} \quad (34)$$

We will assume that the exogenous borrowing constraints satisfy $b_2 = \underline{b}_1 + \tilde{b}_2^{fb}$. We then have the following lemma

Lemma 2 *If $\tilde{b}_2^{fb} > (<)0$ and $b_{2,-1} - b_{1,-1} > (<)\tilde{b}_2^{fb}$ then the planner can implement the first best.*

Proof. We will consider the candidate allocation where $c_{i,t} = c_i^{fb}$, $l_{i,t} = l_i^{fb}$, $b_{i,t} = \underline{b}_i$ and interest rates are given by $R_t = \frac{1}{\beta}$ for $t \geq 1$. It should be clear then that equations (32b) and (32c) are satisfied as a property of the first best allocation. Equation (32d) is trivially satisfied since the agents are at their

borrowing constraints. For $t \geq 1$ equations (32a) and (32e) are both satisfied by the choice of $R_t = \frac{1}{\beta}$ and the first best allocations. It remains to check that equation (32a) is satisfied at time $t = 0$ for an interest rate $R_0 < \frac{1}{\beta}$. At time zero the constraint is give by

$$Z^{fb} + \frac{1}{R_0} \tilde{b}_2^{fb} = b_{2,-1} - b_{1,-1} \quad (35)$$

The assumption that $b_{2,-1} - b_{1,-1} > (<) \tilde{b}_2^{fb}$ if $\tilde{b}_2^{fb} > (<) 0$ then implies that

$$R_0 = \frac{\tilde{b}_2^{fb}}{b_{2,-1} - b_{1,-1} - Z^{fb}} < \frac{1}{\beta}$$

as desired. ■

This will improve upon the planners problem without exogenous borrowing constraints, as first best can only be achieved in this scenario when $b_{2,-1} - b_{1,-1} = \tilde{b}_2^{fb}$.

A.2 Proof of Proposition 4

We prove a slight more general version of our result. Consider an infinite horizon, incomplete markets economy in which an agent maximizes utility function $U : \mathbb{R}_+^n \rightarrow \mathbb{R}$ subject to an infinite sequence of budget constraints. We assume that U is concave and differentiable. Let $\mathbf{a}(s^t)$ be a vector of n goods and let $\mathbf{p}(s^t)$ be a price vector in state s^t with $p_i(s^t)$ denoting the price of good i . We use a normalization $p_1(s^t) = 1$ for all s^t . There is a risk-free bond.

Let $b(s^t)$ be the agent's bond holdings, and let $\mathbf{e}(s^t)$ be a stochastic vector of endowments.

Consumer maximization problem

$$\max_{\mathbf{a}_t, b_t} \sum_{t=0}^{\infty} [\Pi_{j=0}^{t-1} \beta(s_j)] \Pr(s^t) U(\mathbf{a}(s^t)) \quad (36)$$

subject to

$$\mathbf{p}(s^t) \mathbf{a}(s^t) + q(s^t)b(s^t) = \mathbf{p}(s^t) \mathbf{e}(s^t) + b(s^{t-1}) \quad (37)$$

and $\{b(s^t)\}$ is bounded and $\{q(s^t)\}$ is the price of the risk-free bond.

The Euler conditions are

$$\begin{aligned} U_a(s^t) &= U_1(s^t) \mathbf{p}(s^t) \\ \Pr(s^t) U_1(s^t) q(s^t) &= \beta(s_t) \sum_{s^{t+1} > s^t} \Pr(s^{t+1}) U_1(s^{t+1}). \end{aligned} \quad (38)$$

Lemma 3 Consider an allocation $\{\mathbf{a}_t, b_t\}$ that satisfies (37), (38) and $\{b_t\}_t$ is bounded. Then $\{\mathbf{a}_t, b_t\}$ is a solution to (36).

Proof. The proof follows closely Constantinides and Duffie (1996). Suppose there is another budget feasible allocation $\mathbf{a} + \mathbf{h}$ that maximizes (36). Since U is strictly concave,

$$\begin{aligned} &\mathbb{E}_0 \sum_{t=0}^{\infty} [\Pi_{j=0}^{t-1} \beta(s_j)] U(\mathbf{a}_t + \mathbf{h}_t) - \mathbb{E}_0 \sum_{t=0}^{\infty} [\Pi_{j=0}^{t-1} \beta(s_j)] U(\mathbf{a}_t) \\ &\leq \mathbb{E}_0 \sum_{t=0}^{\infty} [\Pi_{j=0}^{t-1} \beta(s_j)] \mathbf{U}_a(\mathbf{a}_t) \mathbf{h}_t \end{aligned} \quad (39)$$

To attain $\mathbf{a} + \mathbf{h}$, the agent must deviate by φ_t from his original portfolio b_t such that $\{\varphi_t\}_t$ is bounded, $\varphi_{-1} = 0$ and

$$\mathbf{p}(s^t)\mathbf{h}(s^t) = \varphi(s^{t-1}) - q(s^t)\varphi(s^t)$$

Multiply by $[\Pi_{j=0}^{t-1}\beta(s_j)] \Pr(s^t) U_1(s^t)$ to get:

$$\begin{aligned} [\Pi_{j=0}^{t-1}\beta(s_j)] \Pr(s^t) U_1(s^t) \mathbf{p}(s^t) \mathbf{h}(s^t) &= [\Pi_{j=0}^{t-1}\beta(s_j)] \Pr(s^t) U_1(s^t) \varphi(s^{t-1}) - q(s^t) [\Pi_{j=0}^{t-1}\beta(s_j)] \Pr(s^t) U_1(s^t) \varphi(s^t) \\ &= [\Pi_{j=0}^{t-1}\beta(s_j)] \Pr(s^t) U_1(s^t) \varphi(s^{t-1}) - [\Pi_{j=0}^{t-1}\beta(s_j)] \beta(s_t) \sum_{s^{t+1} > s^t} \Pr(s^{t+1}) U_1(s^{t+1}) \varphi(s^t) \end{aligned}$$

where we used the second part of (38) in the second equality. Sum over the first T periods (pathwise) and use the first part of (38) to eliminate $\mathbf{U}_a(\mathbf{a}_t) = U_1(s^t) \mathbf{p}(s^t)$

$$\sum_{t=0}^T [\Pi_{j=0}^{t-1}\beta(s_j)] \Pr(s^t) \mathbf{U}_a(\mathbf{a}_t) \mathbf{h}(s^t) = -[\Pi_{j=0}^T \beta(s_j)] \sum_{s^{T+1} > s^T} \Pr(s^{T+1}) U_1(s^{T+1}) \varphi(s^T).$$

Since $\{\varphi_t\}_t$ is bounded there must exist $\bar{\varphi}$ s.t. $|\varphi_t| \leq \bar{\varphi}$ for all t . By Theorem 5.2 of Magill and Quinzi (1994), this equilibrium with debt constraints implies a transversality condition on the right hand side of the last equation, so by transitivity we have

$$\lim_{T \rightarrow \infty} \sum_{t=0}^T [\Pi_{j=0}^{t-1}\beta(s_j)] \Pr(s^t) \mathbf{U}_a(\mathbf{a}_t) \mathbf{h}(s^t) = 0.$$

Substitute this into (39) to show that \mathbf{h} does not improve utility of consumer. ■

A.3 Additional details for Lemma 1

Given $P(s)\mu_i(s)$ and λ_i be the multipliers on constraints (25) and (26). The first order conditions are then as follows

$x'_i(s) :$

$$\beta(s)V_{x_i}(\mathbf{x}'(s), \boldsymbol{\rho}'(s)) - \beta(s)\mu_i(s) = 0 \quad (40)$$

$\tau(s) :$

$$\begin{aligned} U\tau(\tau(s), \boldsymbol{\rho}'(s), s) + \sum_i \left(\frac{x_i U_{c\tau}^i(\tau(s), \boldsymbol{\rho}'(s), s)}{E U_c^i(\tau, \boldsymbol{\rho}', s)} \left[\mu_i(s) - \frac{E\mu_i(s)U_c^i(\tau, \boldsymbol{\rho}', s)}{E U_c^i(\tau, \boldsymbol{\rho}', s)} \right] - \mu_i(s)Z_{i,\tau}(\tau(s), \boldsymbol{\rho}'(s), s) \right) \\ + \sum_i [\lambda_i U_{c,\tau}^i(\tau(s), \boldsymbol{\rho}'(s), s)(\rho'_i(s) - \rho_i)] = 0 \end{aligned} \quad (41)$$

$\rho_i(s) :$

$$\begin{aligned} U_{\rho_i}(\tau(s), \boldsymbol{\rho}'(s), s) + \sum_j \left(\frac{x_j u_{c,1\rho_i}(\tau(s), \boldsymbol{\rho}'(s), s)}{E u_{c,1}(\tau, \boldsymbol{\rho}', s)} \left[\mu_j(s) - \frac{E\mu_j(s)u_{c,1}(\tau, \boldsymbol{\rho}', s)}{E u_{c,1}(\tau, \boldsymbol{\rho}', s)} \right] - \mu_j(s)Z_{j,\rho_i}(\tau(s), \boldsymbol{\rho}'(s), s) \right) \\ + \sum_j [\lambda_j u_{c,j\rho_i}(\tau(s), \boldsymbol{\rho}'(s), s)(\rho'_j(s) - \rho_j)] + \lambda_i u_{c,i}(\tau(s), \boldsymbol{\rho}'(s), s) + \beta(s)V_{\rho_i}(\mathbf{x}'(s), \boldsymbol{\rho}'(s)) = 0. \end{aligned} \quad (42)$$

Finally the envelope conditions are

$$V_{x_i}(\boldsymbol{x}, \boldsymbol{\rho}) = \frac{\mathbb{E}\mu_i(s)U_c^i(\tau, \boldsymbol{\rho}', s)}{\mathbb{E}U_c^i(\tau, \boldsymbol{\rho}', s)}, \quad (43)$$

and

$$V_{\rho_i}(\boldsymbol{x}, \boldsymbol{\rho}) = -\lambda_i \mathbb{E}U_c^i(\tau, \boldsymbol{\rho}', s). \quad (44)$$

The equations for the steady state can then be obtained by imposing $x'_i(s) = x_i$ and $\rho'_i(s) = \rho_i$. It is then readily noted that equations (40) and (43) are satisfied when $\mu_i(s) = \mu_i = \beta V_{x_i}(\boldsymbol{x}, \boldsymbol{\rho})$. Further equation (26) drops out and the equation (25) simplifies to

$$Z_i(\tau(s), \boldsymbol{\rho}, s) + \beta(s)x'_i = \frac{x_i U_c^i(\tau(s), \boldsymbol{\rho}, s)}{\mathbb{E}U_c^i(\tau, \boldsymbol{\rho})}.$$

Imposing the steady state restrictions, equations (41) and (42) reduce to,

$$U_\tau(\tau(s), \boldsymbol{\rho}, s) - \sum_i \mu_i Z_{i,\tau}(\tau(s), \boldsymbol{\rho}, s) = 0 \quad (45a)$$

$$U_{\rho_i}(\tau(s), \boldsymbol{\rho}, s) - \sum_j \mu_j Z_{j,\rho_i}(\tau(s), \boldsymbol{\rho}, s) + \lambda_i U_c^i(\tau(s), \boldsymbol{\rho}', s) - \lambda_i \beta(s) \mathbb{E}U_c^i(\tau, \boldsymbol{\rho}) = 0. \quad (45b)$$

A.4 Proof of Proposition 5

The Bellman equation for the optimal planners problem with log quadratic preferences and IID shocks can be written as

$$V(x, \rho) = \max_{c_1, c_2, l_1, x', \rho'} \sum_s \Pr(s) \left[\alpha_1 \left(\log c_1(s) - \frac{l_1(s)^2}{2} \right) + \alpha_2 \log c_2(s) + \beta(s) V(x'(s), \rho'(s)) \right]$$

subject to the constraints

$$1 + \rho'(s)[l_1(s)^2 - 1] + \beta(s)x'(s) - \frac{x \frac{1}{c_2(s)}}{\mathbb{E}[\frac{1}{c_2}]} = 0 \quad (46)$$

$$\sum_s \frac{\Pr(s)}{c_1(s)} (\rho'(s) - \rho) = 0 \quad (47)$$

$$\theta_1(s)l_1(s) - c_1(s) - c_2(s) - g = 0 \quad (48)$$

$$\rho'(s)c_2(s) - c_1(s) = 0 \quad (49)$$

where the $\Pr(s)$ is the probability distribution of the aggregate state s . If we let $\Pr(s)\mu(s)$, λ , $\Pr(s)\xi(s)$ and $\Pr(s)\phi(s)$ be the Lagrange multipliers for the constraints (46)-(49) respectively then we obtain the following FONC for the planners problem ²⁴

$c_1(s) :$

$$\frac{\alpha_1 \Pr(s)}{c_1(s)} - \frac{\lambda \Pr(s)}{c_1(s)^2} (\rho'(s) - \rho) - \Pr(s)\xi(s) - \Pr(s)\phi(s) = 0 \quad (50)$$

²⁴Appendix A.5 discusses the associated second order conditions that ensure these policies are optimal

$$c_2(s) :$$

$$\frac{\alpha_2 \Pr(s)}{c_2(s)} + \frac{x \Pr(s)}{c_2(s)^2 \mathbb{E}[\frac{1}{c_2}]} \left[\mu(s) - \frac{\mathbb{E}[\mu \frac{1}{c_2}]}{\mathbb{E}[\frac{1}{c_2}]} \right] - \Pr(s)\xi(s) + \Pr(s)\rho'(s)\phi(s) = 0 \quad (51)$$

$$l_1(s) :$$

$$-\alpha_1 \Pr(s)l_1(s) + 2\mu(s) \Pr(s)\rho'(s)l_1(s) + \theta_1(s) \Pr(s)\xi(s) = 0 \quad (52)$$

$$x'(s) :$$

$$\beta(s) \Pr(s)V_x(x'(s), \rho'(s)) + \beta(s) \Pr(s)\mu(s) = 0 \quad (53)$$

$$\rho'(s) :$$

$$\beta(s) \Pr(s)V_\rho(x'(s), \rho'(s)) + \frac{\lambda \Pr(s)}{c_1(s)} + \mu(s) \Pr(s)[l_1(s)^2 - 1] + \Pr(s)\phi(s)c_2(s) = 0 \quad (54)$$

In addition there are two envelope conditions given by

$$V_x(x, \rho) = - \sum_{s'} \frac{\mu(s') \Pr(s') \frac{1}{c_2(s')}}{\mathbb{E}[\frac{1}{c_2}]} = - \frac{\mathbb{E}[\mu \frac{1}{c_2}]}{\mathbb{E}[\frac{1}{c_2}]} \quad (55)$$

$$V_\rho(x, \rho) = -\lambda \mathbb{E}[\frac{1}{c_1}] \quad (56)$$

In the steady state, we need to solve for a collection of allocations, initial conditions and Lagrange multipliers $\{c_1(s), c_2(s), l_1(s), x, \rho, \mu(s), \lambda, \xi(s), \phi(s)\}$ such that equations (46)-(56) are satisfied when $\rho'(s) = \rho$ and $x'(s) = x$. It should be clear that if we replace $\mu(s) = \mu$, equation (53) and the envelope condition with respect to x is always satisfied. Additionally under this assumption equation (51) simplifies significantly, since

$$\frac{x \Pr(s)}{c_2(s)^2 \mathbb{E}[\frac{1}{c_2}]} \left[\mu(s) - \frac{\mathbb{E}[\mu \frac{1}{c_2}]}{\mathbb{E}[\frac{1}{c_2}]} \right] = 0$$

The first order conditions for a steady can then be written simply as

$$1 + \rho[l_1(s)^2 - 1] + \beta(s)x - \frac{x}{c_2(s) \mathbb{E}[\frac{1}{c_2}]} = 0 \quad (57)$$

$$\theta_1(s)l_1(s) - c_1(s) - c_2(s) - g = 0 \quad (58)$$

$$\rho c_2(s) - c_1(s) = 0 \quad (59)$$

$$\frac{\alpha_1}{c_1(s)} - \xi(s) - \phi(s) = 0 \quad (60)$$

$$\frac{\alpha_2}{c_2(s)} - \xi(s) + \rho\phi(s) = 0 \quad (61)$$

$$[2\mu\rho - \alpha_1]l_1(s) + \theta_1(s)\xi(s) = 0 \quad (62)$$

$$\lambda \left[\frac{1}{c_1(s)} - \beta(s) \mathbb{E}[\frac{1}{c_1}] \right] + \mu[l_1(s)^2 - 1] + \phi(s)c_2(s) = 0 \quad (63)$$

We can rewrite equation (60) as

$$\frac{\alpha_1}{c_2(s)} - \rho\xi(s) - \rho\phi(s) = 0$$

by substituting $c_1(s) = \rho c_2(s)$. Adding this to equation (61) and normalizing $\alpha_1 + \alpha_2 = 1$ we obtain

$$\xi(s) = \frac{1}{(1+\rho)c_2(s)} \quad (64)$$

which we can use to solve for $\phi(s)$ as

$$\phi(s) = \frac{\alpha_1 - \rho\alpha_2}{(\rho(1+\rho))c_2(s)} \quad (65)$$

From equation (57) we can solve for $l_1(s)^2 - 1$ as

$$l_1(s)^2 - 1 = \frac{x}{\rho E[\frac{1}{c_2}]} \left(\frac{1}{c_2(s)} - \beta(s)E[\frac{1}{c_2}] \right) - \frac{1}{\rho}$$

This can be used along with equations (63) and (65) to obtain

$$\left(\frac{\lambda}{\rho} + \frac{\mu x}{\rho E[\frac{1}{c_2}]} \right) \left(\frac{1}{c_2(s)} - \beta(s)E[\frac{1}{c_2}] \right) = \frac{\mu}{\rho} + \frac{\rho\alpha_2 - \alpha_1}{\rho(1+\rho)}$$

Note that the LHS depends on s while the RHS does not, hence the solution to this equation is

$$\lambda = -\frac{\mu x}{E[\frac{1}{c_2}]} \quad (66)$$

and

$$\mu = \frac{\alpha_1 - \rho\alpha_2}{1+\rho} \quad (67)$$

Combining these with equation (62) we quickly obtain that

$$\left[2\rho \frac{\alpha_1 - \rho\alpha_2}{1+\rho} - \alpha_1 \right] l_1(s) + \frac{\theta_1(s)}{(1+\rho)c_2(s)} = 0$$

Then solving for $l_1(s)$ gives

$$l_1(s) = \frac{\theta_1(s)}{(\alpha_1(1-\rho) + 2\rho^2\alpha_2)c_2(s)}$$

Remark 1 Note that the labor tax rate is given by $1 - \frac{\alpha_1(s)l_1(s)}{\theta_1(s)}$. The previous expression shows that labor taxes are constant at the steady state. This property holds generally for CES preferences separable in consumption and leisure

This we can plug into the aggregate resource constraint (58) to obtain

$$l_1(s) = \left(\frac{1+\rho}{\alpha_1(1-\rho) + 2\rho^2\alpha_2} \right) \frac{1}{l_1(s)} + \frac{g}{\theta_1(s)}$$

letting $C(\rho) = \frac{1+\rho}{\alpha_1(1-\rho) + 2\rho^2\alpha_2}$ we can then solve for $l_1(s)$ as

$$l_1(s) = \frac{g \pm \sqrt{g^2 + 4C(\rho)\theta_1(s)^2}}{2\theta_1(s)}$$

The marginal utility of agent 2 is then

$$\frac{1}{c_2(s)} = \left(\frac{1+\rho}{C(\rho)} \right) \left(\frac{g \pm \sqrt{g^2 + 4C(\rho)\theta_1(s)^2}}{2\theta_1(s)^2} \right)$$

Note that in order for either of these terms to be positive we need $C(\rho) \geq 0$ implying that there is only one economically meaningful root. Thus

$$l_1(s) = \frac{g + \sqrt{g^2 + 4C(\rho)\theta_1(s)^2}}{2\theta_1(s)} \quad (68)$$

and

$$\frac{1}{c_2(s)} = \left(\frac{1+\rho}{C(\rho)} \right) \left(\frac{g + \sqrt{g^2 + 4C(\rho)\theta_1(s)^2}}{2\theta_1(s)^2} \right) \quad (69)$$

A steady state is then a value of ρ such that

$$x(s) = \frac{1 + \rho[l_1(\rho, s)^2 - 1]}{\frac{1/c_2(\rho, s)}{E[\frac{1}{c_2}](\rho)} - \beta(s)} \quad (70)$$

s independent of s .

The following lemma, which orders consumption and labor across states, will be useful in proving the parts of proposition 5. As a notational aside we will often use $\theta_{1,l}$ and $\theta_{1,h}$ to refer to $\theta_1(s_l)$ and $\theta_1(s_h)$ respectively. Where s_l refers to the low TFP state and s_h refers to the high TFP state.

Lemma 4 Suppose that $\theta_1(s_l) < \theta_2(s_h)$ and ρ such that $C(\rho) > 0$ then

$$l_{1,l} = \frac{g + \sqrt{g^2 + 4C(\rho)\theta_{1,l}^2}}{2\theta_{1,l}} > \frac{g + \sqrt{g^2 + 4C(\rho)\theta_{1,h}^2}}{2\theta_{1,h}} = l_{1,h}$$

and

$$\frac{1}{c_{2,l}} = \frac{1+\rho}{C(\rho)} \frac{g + \sqrt{g^2 + 4C(\rho)\theta_{1,l}^2}}{2\theta_{1,l}^2} > \frac{1+\rho}{C(\rho)} \frac{g + \sqrt{g^2 + 4C(\rho)\theta_{1,h}^2}}{2\theta_{1,h}^2} = \frac{1}{c_{2,h}}$$

Proof. The results should follow directly from showing that the function

$$l_1(\theta) = \frac{g + \sqrt{g^2 + 4C(\rho)\theta^2}}{2\theta}$$

is decreasing in θ . Taking the derivative with respect to θ

$$\begin{aligned} \frac{dl_1}{d\theta}(\theta) &= -\frac{g}{2\theta^2} - \frac{\sqrt{g^2 + 4C(\rho)\theta^2}}{2\theta^2} + \frac{4C(\rho)\theta}{2\theta\sqrt{g^2 + 4C(\rho)\theta^2}} \\ &= -\frac{g}{2\theta^2} - \frac{g + 4C(\rho)\theta^2 - 4C(\rho)\theta^2}{2\theta^2\sqrt{g^2 + 4C(\rho)\theta^2}} \\ &= -\frac{g}{2\theta^2} - \frac{g}{2\theta^2\sqrt{g^2 + 4C(\rho)\theta^2}} < 0 \end{aligned}$$

That $\frac{1}{c_{2,l}} > \frac{1}{c_{2,h}}$ follows directly. ■

Proof of Proposition 5.

Part 1. In order for there to exist a ρ such that equation (70) is independent of the state (and hence have a steady state) we need the existence of root for the following function

$$f(\rho) = \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} - \frac{\frac{1/c_2(\rho, s_h)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta}{\frac{1/c_2(\rho, s_l)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta}$$

From lemma 4 we can conclude that

$$1 + \rho[l_1(\rho, s_l)^2 - 1] > 1 + \rho[l_1(\rho, s_h)^2 - 1] \quad (71)$$

and

$$\frac{1/c_2(\rho, s_l)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta > \frac{1/c_2(\rho, s_h)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta \quad (72)$$

for all $\rho > 0$ such that $C(\rho) \geq 0$. To begin with we will define $\underline{\rho}$ such that $C(\rho) > 0$ for all $\rho > \underline{\rho}$. Note that we will have to deal with two different cases.

$\alpha_1(1 - \rho) + 2\rho^2\alpha_2 > 0$ **for all** $\rho \geq 0$: In this case we know that $C(\rho) \geq 0$ for all ρ and is bounded above and thus we will let $\underline{\rho} = 0$.

$\alpha_1(1 - \rho) + 2\rho^2\alpha_2 = 0$ **for some** $\rho > 0$: In this case let $\underline{\rho}$ be the largest positive root of $\alpha_1(1 - \rho) + 2\rho^2\alpha_2$. Note that $\lim_{\rho \rightarrow \underline{\rho}^+} C(\rho) = \infty$

With this we note that²⁵

$$\lim_{\rho \rightarrow \underline{\rho}^+} \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} = 1$$

We can also show that

$$\lim_{\rho \rightarrow \underline{\rho}^+} \frac{\frac{1/c_2(\rho, s_h)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta}{\frac{1/c_2(\rho, s_l)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta} < 1$$

which implies that $\lim_{\rho \rightarrow \underline{\rho}^+} f(\rho) > 0$.

Taking the limit as $\rho \rightarrow \infty$ we see that $C(\rho) \rightarrow 0$, given that $\frac{\theta}{\theta(s)} < 1$, we can then conclude that

$$\lim_{\rho \rightarrow \infty} 1 + \rho[l_1(\rho, s)^2 - 1] = -\infty$$

Thus, there exists $\bar{\rho}$ such that $1 + \bar{\rho}[l_1(\bar{\rho}, s_l)^2 - 1] = 0$.²⁶ From equation (71), we know that

$$0 = 1 + \bar{\rho}[l_1(\bar{\rho}, s_l)^2 - 1] > 1 + \bar{\rho}[l_1(\bar{\rho}, s_h)^2 - 1]$$

which implies in the limit

$$\lim_{\rho \rightarrow \bar{\rho}^-} \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} = -\infty$$

²⁵In the first case $\underline{\rho} = 0$ and in the second case $l_1(\rho, s_l) = l_1(\rho, s_h)$ as $\rho \rightarrow \underline{\rho}^+$

²⁶This can be seen from the fact $\lim_{\rho \rightarrow \underline{\rho}^+} 1 + \rho[l_1(\rho, s_l)^2 - 1] > 0$ and $\lim_{\rho \rightarrow \infty} 1 + \rho[l_1(\rho, s_l)^2 - 1] > -\infty$, thus $\bar{\rho}$ exists in $(\underline{\rho}, \infty)$

which along with

$$\frac{\frac{1/c_2(\rho, s_h)}{\mathbb{E}[\frac{1}{c_2}]} - \beta}{\frac{1/c_2(\rho, s_l)}{\mathbb{E}[\frac{1}{c_2}]} - \beta} \geq -1$$

allows us to conclude that $\lim_{\rho \rightarrow \bar{\rho}^-} f(\rho) = -\infty$. The intermediate value theorem then implies that there exists ρ_{SS} such that $f(\rho_{SS}) = 0$ and hence that ρ_{SS} is a steady state.

Finally, as $\rho_{SS} < \bar{\rho}$ we know that

$$1 + \rho_{SS}[l_1(\rho_{SS}, s_l) - 1] > 0$$

as $\frac{1/c_2(\rho, s_l)}{\mathbb{E}[\frac{1}{c_2}]} > 1$ we can conclude

$$x_{SS} = \frac{1 + \rho_{SS}[l_1(\rho_{SS}, s_l) - 1]}{\frac{1/c_2(\rho, s_l)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta} > 0$$

implying that the government will hold assets in the steady state (under the normalization that agent 2 holds no assets).

Part 2. The condition that $R(s_h) = R(s_l)$ implies that

$$\frac{1/c_2(\rho, s_l)}{\beta(s_l)\mathbb{E}[\frac{1}{c_2}]} = \frac{1/c_2(\rho, s_h)}{\beta(s_h)\mathbb{E}[\frac{1}{c_2}]}$$

which simplifies to

$$\frac{\beta(s_h)}{\beta(s_l)} = \frac{1/c_2(\rho, s_h)}{1/c_2(\rho, s_l)} \quad (73)$$

In order for a steady state to exist with constant interest rates there must be a root of the following function

$$\begin{aligned} f(\rho) &= \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} - \frac{\frac{1/c_2(\rho, s_h)}{\mathbb{E}[\frac{1}{c_2}]} - \beta(s_h)}{\frac{1/c_2(\rho, s_l)}{\mathbb{E}[\frac{1}{c_2}]} - \beta(s_l)} \\ &= \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} - \frac{\frac{1/c_2(\rho, s_h)}{\beta(s_h)\mathbb{E}[\frac{1}{c_2}]} - 1}{\frac{1/c_2(\rho, s_l)}{\beta(s_l)\mathbb{E}[\frac{1}{c_2}]} - 1} \frac{\beta(s_h)}{\beta(s_l)} \\ &= \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} - \frac{1/c_2(\rho, s_h)}{1/c_2(\rho, s_l)} \end{aligned}$$

Taking limits of $f(\rho)$ as ρ approaches $\underline{\rho}$ from the positive side we already demonstrated

$$\lim_{\rho \rightarrow \underline{\rho}^+} \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} = 1$$

From equation (69) and Lemma 4 it is straightforward to see that

$$\lim_{\rho \rightarrow \underline{\rho}^+} \frac{1/c_2(\rho, s_h)}{1/c_2(\rho, s_l)} < 1$$

which allows us to conclude that

$$\lim_{\rho \rightarrow \underline{\rho}^+} f(\rho) > 0$$

Taking limits as ρ approaches $\bar{\rho}$ from the negative direction we know that

$$\lim_{\rho \rightarrow \bar{\rho}^-} \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} = -\infty$$

As $\frac{1/c_2(\rho, s_h)}{1/c_2(\rho, s_l)} > 0$ for all ρ it is straightforward to conclude that

$$\lim_{\rho \rightarrow \bar{\rho}^-} f(\rho) = -\infty$$

Continuity then implies the existence of a ρ^{SS} such that $f(\rho^{SS}) = 0$, and thus there exists a $\beta(s_l)$ and $\beta(s_h)$ such that $R(s_l) = R(s_h)$ in steady state. From Lemma 4

$$l(\rho, s_l) > l(\rho, s_h).$$

In order for

$$\frac{1 + \rho^{SS}[l_1(\rho^{SS}, s_h)^2 - 1]}{1 + \rho^{SS}[l_1(\rho^{SS}, s_l)^2 - 1]} = \frac{1/c_2(\rho^{SS}, s_h)}{1/c_2(\rho^{SS}, s_l)} < 1$$

it is necessary that

$$1 + \rho^{SS}[l_1(\rho^{SS}, s_l)^2 - 1] > 1 + \rho^{SS}[l_1(\rho^{SS}, s_h)^2 - 1] > 0$$

implying that the steady state asset level

$$x_{SS} = \frac{1 + \rho_{SS}[l_1(\rho_{SS}, s_l) - 1]}{\frac{1/c_2(\rho_{SS}, s_l)}{\mathbb{E}[\frac{1}{c_2}]} - \beta(s_l)} > 0$$

Part 3 As noted before, since $g/\theta(s) < 1$ for all s we have

$$\lim_{\rho \rightarrow \infty} 1 + \rho[l_1(\rho, s)^2 - 1] = -\infty$$

Thus, there exists ρ_{SS} such that

$$0 > 1 + \rho_{SS}[l_1(\rho_{SS}, s_l)^2 - 1] > 1 + \rho_{SS}[l_1(\rho_{SS}, s_h)^2 - 1]$$

It is then possible to choose $\beta(s) < \frac{1/c_2(\rho_{SS}, s)}{\mathbb{E}[\frac{1}{c_2}]}$ such that

$$1 > \frac{1 + \rho_{SS}[l_1(\rho_{SS}, s_l)^2 - 1]}{1 + \rho_{SS}[l_1(\rho_{SS}, s_h)^2 - 1]} = \frac{\frac{1/c_2(\rho_{SS}, s_l)}{\mathbb{E}[\frac{1}{c_2}]} - \beta(s_l)}{\frac{1/c_2(\rho_{SS}, s_h)}{\mathbb{E}[\frac{1}{c_2}]} - \beta(s_h)} \quad (74)$$

Implying that for discount factor shocks $\beta(s)$, ρ_{SS} is a steady state level for the ratio of marginal utilities, with steady state marginal utility weighted government debt

$$x_{SS} = \frac{1 + \rho_{SS}[l_1(\rho_{SS}, s_l)^2 - 1]}{\frac{1/c_2(\rho_{SS}, s_l)}{\mathbb{E}[\frac{1}{c_2}]} - \beta(s_l)} < 0$$

Thus, in the steady state, the government is holding debt, under the normalization that the unproductive worker holds no assets. As $\frac{1/c_2(\rho, s_l)}{\mathbb{E}[\frac{1}{c_2}]} > \frac{1/c_2(\rho, s_h)}{\mathbb{E}[\frac{1}{c_2}]}$, in order for equation (74) to hold we need $\beta_l > \beta_h$. We can then rewrite equation (74) as

$$1 > \frac{\beta(s_h)}{\beta(s_l)} > \frac{\frac{1/c_2(\rho_{SS}, s_l)}{\beta_l \mathbb{E}[\frac{1}{c_2}]} - 1}{\frac{1/c_2(\rho_{SS}, s_h)}{\beta(s_h) \mathbb{E}[\frac{1}{c_2}]} - 1}$$

Thus

$$R(s_l) \frac{1/c_2(\rho_{SS}, s_l)}{\beta(s_l) \mathbb{E}[\frac{1}{c_2}]} < \frac{1/c_2(\rho_{SS}, s_h)}{\beta(s_h) \mathbb{E}[\frac{1}{c_2}]} = R(s_h) \quad (75)$$

in the steady state interest rates are positively correlated with TFP.

■

A.5 Linearization Algorithm

This section will outline our numerical methods used to solve for and linearize around the steady state in the case of a 2 state iid process for the aggregate state.

$$V(\mathbf{x}, \boldsymbol{\rho}) = \max_{c_i(s), l_i(s), \mathbf{x}'(s), \boldsymbol{\rho}'(s)} \sum_s P(s) \left(\left[\sum_i \pi_i \alpha_i U(c_i(s), l_i(s)) \right] + \beta(s) V(\mathbf{x}'(s), \boldsymbol{\rho}'(s)) \right) \quad (76)$$

$$U_{c,i}(s)c_i(s) + U_{l,i}(s)l_i(s) - \rho'_i(s) [U_{c,1}(s)c_1(s) + U_{l,1}(s)l_1(s)] + \beta(s)x'_i(s) = \frac{x_i U_{c,i}(s)}{\mathbb{E} U_{c,i}} \quad (77a)$$

$$\sum_s \Pr(s) U_{c,1}(s) (\rho_i(s) - \rho_i) = 0 \quad (77b)$$

$$\frac{\rho'(s)}{\theta_1(s)} U_{l,1}(s) = \frac{1}{\theta_i(s)} U_{l,i}(s) \quad (77c)$$

$$\sum_{j=0}^N \pi_j c_j(s) + g(s) = \sum_{j=0}^N \pi_j \theta_j(s) l_j(s) \quad (77d)$$

$$U_{c,i}(s) = \rho'_i(s) U_{c,1}(s) \quad (77e)$$

For $i = 2, \dots, N$. Note that some of the constraints have been modified a little for ease of differentiation. Associated with these constraints we have the Lagrange multipliers $\Pr(s)\mu'_i(s)$, $\lambda_i \Pr(s)\phi_i(s)$, $\Pr(s)\xi(s)$, and $P(s)\zeta_i(s)$.

The first order conditions with respect to the choice variables are as follows (note we will be using the notation $\mathbb{E} z$ to represent $\sum_s \Pr(s)z(s)$ for some variable z)

$c_1(s)$:

$$\begin{aligned} \pi_1 \alpha_1 U_{c,1}(s) + \sum_{i=2}^N (\mu'_i(s) \rho'_i(s)) [U_{c,1}(s)c_1(s) + U_{c,1}(s)] \\ + \lambda U_{cc,1}(s) \sum_{i=2}^N (\rho'_i(s) - \rho_1) - \pi_1 \xi(s) + \sum_{i=2}^N \zeta_i(s) \rho'_i(s) U_{cc,1}(s) = 0 \end{aligned} \quad (78a)$$

$c_i(s)$: for $i \geq 2$

$$\pi_i \alpha_i U_{c,i}(s) - \mu'_i(s) [U_{cc,i}(s)c_i(s) + U_{c,i}(s)] + \frac{x_i U_{cc,i}(s)}{\mathbb{E} U_{c,i}} \left(\mu'_i(s) - \frac{\mathbb{E} \mu'_i U_{c,i}}{\mathbb{E} U_{c,i}} \right) - \pi_i \xi(s) - \zeta_i(s) U_{cc,i}(s) = 0 \quad (78b)$$

$l_1(s)$:

$$\pi_1 \alpha_1 U_{l,1}(s) + \sum_{i=2}^N \mu'_i(s) \rho_i(s) [U_{ll,1}(s) l_1(s) + U_{l,1}(s)] - \sum_{i=2}^N \frac{\rho'_i(s) \phi_i(s)}{\theta_i(s)} U_{ll,1}(s) + \pi_1 \theta_1(s) \xi(s) = 0 \quad (78c)$$

$l_2(s)$:

$$\pi_i \alpha_i U_{l,i}(s) - \mu'_i(s) [U_{ll,i}(s) l_i(s) + U_{l,i}(s)] + \frac{\phi_i(s)}{\theta_i(s)} U_{ll,i}(s) + \pi_i \theta_i(s) \xi(s) = 0 \quad (78d)$$

$\rho'_i(s)$:

$$\beta(s) V_{\rho_i}(\mathbf{x}'(s), \boldsymbol{\rho}'(s)) + \mu'_i(s) [U_{c,1}(s)c_1(s) + U_{l,1}(s)l_1(s)] + \lambda_i U_{c,1}(s) - \phi_i(s) \frac{U_{l,1}(s)}{\theta_i(s)} + U_{c,1}(s) \zeta_i(s) = 0 \quad (78e)$$

$x'_i(s)$:

$$V_{x_i}(\mathbf{x}'(s), \boldsymbol{\rho}'(s)) - \mu'_i(s) = 0. \quad (78f)$$

Equations (77a)-(77e) and (78a)-(78e) then define the necessary conditions for an interior maximization of the planners problem for the state $(\mathbf{x}, \boldsymbol{\rho})$. In addition to these we have the two envelop conditions

$$V_{x_i}(\mathbf{x}, \boldsymbol{\rho}) = \frac{\sum_s P(s) \mu'_i(s) U_{c,i}(s)}{\mathbb{E} U_{c,i}(s)} = \frac{\mathbb{E} \mu'_i U_{c,i}}{\mathbb{E} U_{c,i}}, \quad (79a)$$

and

$$V_{\rho_i}(\mathbf{x}, \boldsymbol{\rho}) = -\lambda_i \mathbb{E} U_{c,1}. \quad (79b)$$

In order to check local stability we linearize locally around the steady state. Furthermore we find that the policy functions have better numerical properties when the state variables are chosen to be $(\boldsymbol{\mu}, \boldsymbol{\rho})$ rather than $(\mathbf{x}, \boldsymbol{\rho})$, and thus, we will proceed with the linearization procedure using $(\boldsymbol{\mu}, \boldsymbol{\rho})$ as the endogenous state vector. The evolution of the state variable $\boldsymbol{\mu}$ must follow the weighted martingale

$$\mu_i - \frac{\sum_s P(s) \mu'_i(s) U_{c,i}(s)}{\sum_s P(s) U_{c,i}(s)} = 0. \quad (80)$$

The optimal policy function, which we will denote as $z(\boldsymbol{\mu}, \boldsymbol{\rho})$, must satisfy $F(z, y, g(z)) = 0$ where F represents the system of equations (77a)-(78e) and (80), y is the state vector $(\mathbf{x}, \boldsymbol{\rho})$, and g is the mapping of the policies into functions of future variables, namely $\mathbf{x}'(s)$ and $V_{\rho}(\boldsymbol{\mu}'(s), \boldsymbol{\rho}'(s))$. In other words

$$g(z) = \begin{pmatrix} \mathbf{x}(\boldsymbol{\mu}'(1), \boldsymbol{\rho}'(1)) \\ V_{\rho}(\boldsymbol{\mu}'(1), \boldsymbol{\rho}'(1)) \\ \mathbf{x}(\boldsymbol{\mu}'(2), \boldsymbol{\rho}'(2)) \\ V_{\rho}(\boldsymbol{\mu}'(2), \boldsymbol{\rho}'(2)) \end{pmatrix}.$$

Finally $z(\mu, \rho)$ are the stacked variables $\{c_1(s), c_1(s), l_1(s), l_1(s), \mathbf{x}, \rho'(s), \mu'(s), \lambda, \phi(s), \xi(s), \zeta(s)\}$. The optimal policy function is then a function $z(y)$ that satisfies the relationship $F(z(y), y, g(z(y))) = 0$. Taking total derivatives around the steady state \bar{y} and $\bar{z} = z(\bar{y})$

$$D_z F(\bar{z}, \bar{y}, g(\bar{z})) D_y z(\bar{y}) + D_y F(\bar{z}, \bar{y}, g(\bar{z})) + D_g F(\bar{z}, \bar{y}, g(\bar{z})) Dg(\bar{z}) D_y z(\bar{z}) = 0$$

In order to linearize $z(y)$ around the steady state \bar{y} we need to compute $D_y z(\bar{y})$. The envelope condition (79b) tell us that V_ρ can be computed from the optimal policies, i.e.

$$\begin{pmatrix} \mathbf{x}(\mu, \rho) \\ V_\rho(\mu, \rho) \end{pmatrix} = w(z(\mu, \rho)) = \begin{pmatrix} \mathbf{x} \\ -\lambda \mathbb{E}[U_{c,1}] \end{pmatrix}$$

If we let Φ_s be the matrix that maps $z(\mu, \rho)$ into $\begin{pmatrix} \mu'(s) \\ \rho'(s) \end{pmatrix}$ then we can write $g(\mu, \rho)$ using z and w as follows

$$g(z) = \begin{pmatrix} w(z(\Phi_1 z)) \\ w(z(\Phi_2 z)) \end{pmatrix}$$

taking derivatives we quickly obtain that

$$\begin{aligned} D_z g(\bar{z}) &= \begin{pmatrix} Dw(z(\Phi_1 \bar{z})) & 0 \\ 0 & Dw(z(\Phi_2 \bar{z})) \end{pmatrix} \begin{pmatrix} D_y z(\Phi_1 \bar{z}) & 0 \\ 0 & D_y z(\Phi_2 \bar{z}) \end{pmatrix} \underbrace{\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}}_{\Phi} \\ &= \begin{pmatrix} Dw(\bar{z}) & 0 \\ 0 & Dw(\bar{z}) \end{pmatrix} \begin{pmatrix} D_y z(\bar{y}) & 0 \\ 0 & D_y z(\bar{y}) \end{pmatrix} \Phi \\ &= \begin{pmatrix} Dw(\bar{z}) D_y z(\bar{y}) & 0 \\ 0 & Dw(\bar{z}) D_y z(\bar{y}) \end{pmatrix} \Phi \end{aligned}$$

We can then go back to our original matrix equation to obtain

$$D_z F(\bar{z}, \bar{y}, \bar{w}) D_y z(\bar{y}) + D_y F(\bar{z}, \bar{y}, \bar{w}) + D_w F(\bar{z}, \bar{y}, \bar{w}) \begin{pmatrix} Dw(\bar{z}) D_y z(\bar{y}) & 0 \\ 0 & Dw(\bar{z}) D_y z(\bar{y}) \end{pmatrix} \Phi D_y z(\bar{z}) = 0, \quad (81)$$

where $\bar{w} = g(\bar{z}) = w(\bar{z})$. This is now a non-linear matrix equation for $D_y z(\bar{y})$, where all the other terms can be computed using the steady state values \bar{z} and \bar{y} (note $g(\bar{z})$ is known from the envelope conditions at the steady state). Furthermore, $D_y z(\bar{y})$ gives us the linearization of the policy rules since to first order

$$z \approx \bar{z} + D_y z(\bar{y})(y - \bar{y})$$

Our procedure for computing the linearization proceeds as follows

1. Find the steady state by solving the system of equations (27). Numerically, we have found that this is very robust to the parameters of the model.
2. Compute $D_z F(\bar{z}, \bar{y}, g(\bar{z}))$, $D_y F(\bar{z}, \bar{y}, g(\bar{z}))$ and $D_w F(\bar{z}, \bar{y}, g(\bar{z}))$ by numerically differentiating F . This is straightforward using auto-differentiation.
3. Compute $Dw(\bar{z})$ using auto-differentiation.

4. Construct a matrix equation as follows. Given policies $A = Dw(\bar{z})D_y z(\bar{y})$ (these are the linearized policies of \boldsymbol{x} and V_ρ with respect to (μ, ρ)), it is possible to solve for $D_y z(\bar{y})$ from

$$D_y z(\bar{z}) = - \left(D_z F(\bar{z}, \bar{y}, \bar{w}) + D_w F(\bar{z}, \bar{y}, \bar{w}) \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \Phi \right)^{-1} D_y F(\bar{z}, \bar{y}, \bar{w})$$

We wish to find an A such that

$$A = Dw(\bar{z})D_y z(\bar{z})$$

Given the linearized policy rules it is then possible to evaluate the local stability of the steady state. We find that in the absence of discount factor shocks the steady state is stable generically across the parameter space.

This linearization can be used to construct the bordered hessian of the problem (24) at the steady state. We can then apply second order tests to verify that the first order necessary conditions are sufficient.

A.6 Proof for Proposition 6

Proof.

The state at time t can be written as

$$\hat{\Psi}_t = B_t B_{t-1} \cdots B_1 \hat{\Psi}_0.$$

where the B_i are all random variables being $B(s)$ with probability $\Pr(s)$. Taking expectations and applying independence we then obtain

$$\mathbb{E}_0[\hat{\Psi}_t] = \mathbb{E}_0[B_t B_{t-1} \cdots B_1] \hat{\Psi}_0 \quad (82)$$

$$= \mathbb{E}[B_t] \mathbb{E}[B_{t-1}] \cdots \mathbb{E}[B_1] \hat{\Psi}_0 \quad (83)$$

$$= \bar{B}^t \hat{\Psi}_0 \quad (84)$$

where $\bar{B} = \mathbb{E}B(s)$. If eigenvalues of \bar{B} are positive and strictly less than 1, at least, in expectation the linearized system converges that is

$$\bar{\hat{\Psi}}_{t|0} \equiv \mathbb{E}_0[\hat{\Psi}_t] = \bar{B}^t \hat{\Psi}_0 \rightarrow \mathbf{0}. \quad (85)$$

It should be noted that the conditional expectation actually captures a significant portion of the linearized dynamics. The remaining question is does the distribution converge to $\mathbf{0}$. This can be done by analyzing the variance. Let

$$\Sigma_{\Psi,t|0} = \mathbb{E}_0 \left[(\hat{\Psi}_t - \bar{\hat{\Psi}}_t)(\hat{\Psi}_{t|0} - \bar{\hat{\Psi}}_{t|0})' \right]$$

or

$$\Sigma_{\Psi,t|0} = \mathbb{E}_0 \hat{\Psi}_t \hat{\Psi}'_t - \bar{\hat{\Psi}}_{t|0} \bar{\hat{\Psi}}'_{t|0}. \quad (86)$$

Note that if eigenvalues of \bar{B} are positive and strictly less than 1, $\bar{\hat{\Psi}}_{t|0}$ converges to 0. Using the independence of $\hat{\Psi}_{t-1}$ and B_t , and $\hat{\Psi}_t = B_t \hat{\Psi}_{t-1}$, we quickly obtain that for large t

$$\Sigma_{\Psi,t|0} \approx \mathbb{E}[B\Sigma_{\Psi,t-1|0}B'] \quad (87)$$

Showing that $\hat{\Psi}_{t|0} \rightarrow \mathbf{0}$ in distribution, amounts to showing that $\Sigma_{\Psi,t|0} \rightarrow 0$ for any starting point Σ_Ψ and following the process in equation (87). One can obtain a necessary condition for $\|\Sigma_{\Psi,t|0}\| \rightarrow 0$ under the process in equation (87). That process can be rewritten as follows

$$\Sigma_{\Psi,t|0} = \mathbb{E}[B\Sigma_{\Psi,t-1|0}B'] \quad (88)$$

$$= \sum_s \Pr(s) B(s) \Sigma_{\Psi,t-1|0} B(s)' \quad (89)$$

$$= \sum_s \Pr(s) (\bar{B} + (B(s) - \bar{B})) \Sigma_{\Psi,t-1|0} (\bar{B} + (B(s) - \bar{B}))' \quad (90)$$

$$= \bar{B} \Sigma_{\Psi,t-1|0} \bar{B}' + \sum_s \Pr(s) (B(s) - \bar{B}) \Sigma_{\Psi,t-1|0} (B(s) - \bar{B})'. \quad (91)$$

This is a deterministic linear system in $\Sigma_{\Psi,t|0}$. Suppose we reshape $\Sigma_{\Psi,t|0}$ as a vector (denoted by $\text{vec}(\Sigma_{\Psi,t|0})$) and let \hat{B} be a (square) matrix such that equation 91 is written as

$$\text{vec}(\Sigma_{\Psi,t|0}) = \hat{B} \text{vec}(\Sigma_{\Psi,t-1|0}).$$

The stability of this system is guaranteed if the (real part) of eigenvalues of \hat{B} are less than 1. ■

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Taxes, debts, and redistributions with aggregate shocks*

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Abstract

A planner sets a lump sum transfer and a linear tax on labor income in an economy with incomplete markets, heterogeneous agents, and aggregate shocks. The planner's concerns about redistribution impart a welfare cost to fluctuating transfers. The distribution of net asset holdings across agents affects optimal allocations, transfers, and tax rates, but the level of government debt does not. Two forces shape long-run outcomes: the planner's desire to minimize the welfare costs of fluctuating transfers, which calls for a negative correlation between the distribution of net assets and agents' skills; and the planner's desire to use fluctuations in the real interest rate to adjust for missing state-contingent securities. In a model parameterized to match stylized facts about US booms and recessions, distributional concerns mainly determine optimal policies over business cycle frequencies. These features of optimal policy differ markedly from ones that emerge from representative agent Ramsey models like Aiyagari et al. (2002).

KEY WORDS: Distorting taxes. Transfers. Redistribution. Government debt. Interest rate risk.

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If, indeed, the debt were distributed in exact proportion to the taxes to be paid so that every one should pay out in taxes as much as he received in interest, it would cease to be a burden.... if it were possible, there would be [no] need of incurring the debt. For if a man has money to loan the Government, he certainly has money to pay the Government what he owes it. Simon Newcomb (1865, p.85)

1 Introduction

This paper studies an economy with agents who differ in their productivities and a benevolent government that imposes an affine income tax that consists of a distortionary proportional tax on labor income and a lump-sum tax or transfer. Figure I shows that an affine structure better approximates the US tax-transfer system than just proportional labor taxes. We impose no restrictions on the sign of transfers. If some agents are sufficiently poor or if the government wants enough redistribution, the government always chooses positive transfers. For most of the paper, we study an economy without capital in which a one-period risk-free bond is the only financial asset traded.

In this economy, concerns for redistribution impart a welfare cost to fluctuating transfers. A decrease in transfers in response to adverse aggregate shocks disproportionately affects agents having low present values of earnings. We show that the welfare cost of such a decrease depends on the distribution of net asset holdings across households with different productivities, where by net asset we mean the assets of an agent minus the assets of some benchmark agent.

Our first set of results extends a Ricardian equivalence type of reasoning to argue that gross asset positions (in particular the level of government debt) do not affect the set of feasible allocations that can be implemented in competitive equilibria with taxes and transfers. For example, an increase in government debt that is shared equally by all agents and thus leaves net assets unchanged has no welfare consequences, in line with the principles proclaimed by Simon Newcomb (1865) in the quotation above. This result also implies (a) that Ricardian equivalence holds in the presence of distorting taxes; and (b) that ad-hoc borrowing limits do not restrict the government's ability to respond to shocks. This logic applies to structures with and without physical capital and with complete or incomplete asset markets and more general structures of taxes.

Our second set of results characterizes the long run behavior of taxes, transfers, and the distribution of assets. In a special case of our economy in which aggregate shocks are iid and take two values, we show that there generally exists a steady state in which marginal utility-adjusted net asset positions are constant and taxes and transfers depend only on current realization of shocks. We also identify conditions under which this steady state is stable. In the steady state, the magnitude of fluctuations in taxes is low for many standard preferences and zero when preferences are CES. For more general shock processes, we show numerically that while a steady state does not exist, similar principles apply - there is an ergodic set to which net asset position converge; in that set, fluctuations of taxes and transfers are substantially diminished.

Two forces determine the properties of the ergodic set of net asset positions. The first force comes from a planner's desire to minimize the welfare cost of fluctuating transfers, which transmits to the Ramsey planner incentives to set taxes and transfers to generate a negative correlation between households' productivities and their net assets. In light of the relevance of net but not gross asset positions stressed in our first set of findings, this can be accomplished by having the government effectively accumulate risk-free claims on high-skilled workers. But another, possibly countervailing or possibly reinforcing, force comes from the Ramsey planner's incentive to use interest rate fluctuations to compensate for missing asset markets. If interest rates are high (low) when government revenue needs are high, the government has an incentive to accumulate assets (debt) vis-a-vis high-skilled worker. Thus, depending on comovements of the interest rate with other fundamentals, these two forces may either reinforce each other (for example, in response to a pure TFP shock where the implied interest rates are countercyclical) or oppose each other (for example, when interest rates are procyclical because TFP shocks are negatively correlated with discount factor shocks).

Optimal policy is particularly striking when agents have quasi-linear preferences and aggregate shocks affect only exogenous government expenditures. In this case, both forces are absent and for any initial asset distribution economy is immediately in a steady state in which assets and taxes remain constant forever. This outcome provides a stark contrast with normative predictions from the representative agent models studied by Aiyagari et al. (2002), in which long-run dynamics are driven by exogenous restrictions on the ability of the government to set transfers optimally. This example also indicates that incorporating a redistributive motive can substantially modify implications about optimal responses to aggregate shocks.

A third set of results concerns higher frequencies, in particular, optimal government policy in booms and recessions. What we have to say about this comes from a version of our model calibrated to patterns observed during recent US recessions: (1) the left tail of the cross-section distribution of labor income falls by more than right tail; and (2) interest rates fall. When we calibrate to those patterns, we find that during recessions accompanied by higher inequality, it is optimal to increase taxes and transfers and to issue government debt. These outcomes differ substantially both qualitatively and quantitatively from those in either a representative agent model or in a version of our model in which a recession is modeled as a pure TFP shock that leaves the distribution of skills unchanged. Furthermore recessions with low interest rates, thought not critical for producing our short-run outcomes, nevertheless affect the transient dynamics and long run properties of optimal policy. Starting from a net asset position (government vis-a-vis the high-skilled agent) that implies a 60% debt - to - GDP ratio, a benchmark economy with procyclical interest rates shows no discernable trend in net asset positions over a time span as long as 5000 years. But in economies with countercyclical interest rates (generated by shutting down discount factor shocks), the government repays this debt and eventually accumulates assets, although it takes about 2500 years to do so.

We assemble these results as follows. After section 2 describes preferences, technologies, endowments, and information flows, section 3 sets the stage for our subsequent results. Theorem 1 exploits the fact

that the set of feasible allocations is invariant to changes in transfers and *gross* asset positions to obtain Ricardian equivalence and the importance of net assets. Section 4 analyzes outcomes when households' preferences are quasi linear. Sections 5 characterizes the planner's maximization problem and formulates a recursive version that features two Bellman equations. Section 6 characterizes long run allocations, while Section 7 provides numerical simulations of optimal policy responses to business cycle-like shocks. Section 8 offers concluding remarks linking our model to recent discussions of government debt-GDP ratios.

1.1 Relationships to literatures

At a fundamental level, our paper descends from both Barro (1974), who showed Ricardian equivalence in a representative agent economy with lump sum taxes, and Barro (1979), who studied optimal taxation when lump sum taxes are ruled out. In our environment with incomplete markets and heterogeneous workers, both of the forces discovered by Barro play large roles, although the distributive motives that we include lead to richer policy prescriptions.

A large literature on Ramsey problems exogenously rules out transfers in the context of representative agent, general equilibrium models. Lucas and Stokey (1983), Chari et al. (1994), and Aiyagari et al. (2002) (henceforth called AMSS) are leading examples. In contrast to those papers, our Ramsey planner cares about redistribution among agents with different skills and wealths. Other than not allowing them to depend on agents' personal identities, we leave transfers unrestricted and let the Ramsey planner set them optimally. Nevertheless, we find that some of the same general principles that emerge from that representative agent, no-transfers literature continue to hold, in particular, the prescription to smooth distortions across time and states. However, it is also true that allowing the government to set transfers optimally substantially changes qualitative and quantitative insights about the optimal policy in important respects.¹

Several recent papers impute distributive concerns to a Ramsey planner. Three papers that are perhaps most closely related to ours are Bassetto (1999), Shin (2006), and Werning (2007). Like us, those authors allow heterogeneity and study distributional consequences of alternative tax and borrowing policies. Bassetto (1999) extends the Lucas and Stokey (1983) environment to include I types of agents who are heterogeneous in their time-invariant labor productivities. There are complete markets and a Ramsey planner has access only to proportional taxes on labor income and history-contingent borrowing and lending. Bassetto studies how the Ramsey planner's vector of Pareto weights influences how he responds to government expenditures and other shocks by adjusting the proportional labor tax and government borrowing to cover expenses while manipulating prices in ways that redistribute wealth

¹ There is also a more recent strand of literature that focuses on the optimal policy in settings with heterogeneous agents when a government can impose arbitrary taxes subject only to explicit informational constraints (see Golosov et al. (2007) for a review). A striking result from that literature is that when agent's asset holdings are perfectly observable, the distribution of assets among agents is irrelevant and an optimal allocation can be achieved purely through taxation (see, e.g. Bassetto and Kocherlakota (2004)). In the previous version of the paper we showed that a mechanism design version of the model with unobservable assets generates some of the similar predictions to the model with affine taxes that we study, in particular, the relevance of net assets and history dependence of taxes. We leave further analysis along this direction to the future.

between ‘rentiers’ (who have low productivities and whose main income is from their asset holdings) and ‘workers’ (who have high productivities) whose main income source is their labor.

Shin (2006) extends the AMSS (Aiyagari et al. (2002)) economy to have two risk-averse households who face idiosyncratic income risk. When idiosyncratic income risk is big enough relative to aggregate government expenditure risk, the Ramsey planner chooses to issue debt in order to help households engage in precautionary saving, thereby overturning the AMSS result that in their quasi-linear case a Ramsey planner eventually sets taxes to zero and lives off its earnings from assets forevermore. Shin emphasizes that the government does this at the cost of imposing tax distortions. While being constrained to use proportional labor income taxes and nonnegative transfers, Shin’s Ramsey planner balances two competing self-insurance motives: aggregate tax smoothing and individual consumption smoothing.

Werning (2007) studies a complete markets economy with heterogeneous agents and transfers that are unrestricted in sign. He obtains counterparts to our results about net versus gross asset positions, including that government assets can be set to zero in all periods. Because he allows unrestricted taxation of initial assets, the initial distribution of assets plays no role. Theorem 1 and its corollaries substantially generalize Werning’s results by showing that all allocations of assets among agents and the government that imply the same net asset position lead to the same optimal allocation, a conclusion that holds for market structures beyond complete markets. Werning (2007) provides an extensive characterization of optimal allocations and distortions in complete market economies, while we focus on precautionary savings motives for private agents and the government that are not present when markets are complete.^{2,3}

Finally, our numerical analysis in Section 7 is related to McKay and Reis (2013). While our focus differs from theirs – McKay and Reis study the effect of calibrated US tax and transfer system on stabilization of output, while we focus on optimal policy in a simpler economy – both papers confirm the importance of transfers and redistribution over business-cycle frequencies.

2 Environment

Exogenous fundamentals of the economy are functions of a shock s_t that follows an irreducible Markov process, where $s_t \in S$ and S is a finite set. We let $s^t = (s_0, \dots, s_t)$ denote a history of shocks.

There is a mass π_i of a type $i \in I$ agent, with $\sum_{i=1}^I \pi_i = 1$. Types differ by their skills. Preferences of an agent of type i over stochastic processes for consumption $\{c_{i,t}\}_t$ and labor supply $\{l_{i,t}\}_t$ are ordered by

$$E_0 \sum_{t=0}^{\infty} [\Pi_{j=0}^{t-1} \beta(s_j)] U^i(c_i(s^t), l_i(s^t)), \quad (1)$$

where E_t is a mathematical expectations operator conditioned on time t information and $\beta(s_t) \in (0, 1)$ is a state-dependent discount factor⁴. We assume that $l_i \in [0, \bar{l}_i]$ for some $\bar{l}_i < \infty$. Results in section 3

²Werning (2012) studies optimal taxation with incomplete markets and explores conditions under which optimal taxes depend only on the aggregate state.

³More recent closely related papers are Azzimonti et al. (2008a,b) and Correia (2010). While these authors study optimal policy in economies in which agents are heterogeneous in skills and initial assets, they do not allow aggregate shocks.

⁴We let the discount factor at time t to depend on the history of shocks s^{t-1} . This allows us to generate flexible

require no additional assumptions on U^i like differentiability or convexity,⁵ but results in later sections do.

An agent of type i who supplies l_i units of labor produces $\theta_i(s_t) l_i$ units of output, where $\theta_i(s_t) \in \Theta$ is a nonnegative state-dependent scalar. Feasible allocations satisfy

$$\sum_{i=1}^I \pi_i c_i(s^t) + g(s_t) = \sum_{i=1}^I \pi_i \theta_i(s_t) l_i(s^t), \quad (2)$$

where $g(s_t)$ denotes exogenous government expenditures in state s_t . We allow s_t to affect $\beta(s_t)$, government expenditures $g(s_t)$, and the type-specific productivities $\theta_i(s_t)$.

To save on notation, mostly we use z_t to denote a random variable with a time t conditional distribution that is a function of the history s^t . Occasionally, we use the more explicit notion $z(s^t)$ to denote a realization at a particular history s^t .

A Ramsey planner's preferences over a vector of stochastic processes for consumption and labor supply are ordered by

$$E_0 \sum_{i=1}^I \pi_i \alpha_i \sum_{t=0}^{\infty} \bar{\beta}_t U_t^i(c_{i,t}, l_{i,t}), \quad (3)$$

where the Pareto weights satisfy $\alpha_i \geq 0$, $\sum_{i=1}^I \alpha_i = 1$, and $\bar{\beta}_t = [\Pi_{j=0}^{t-1} \beta_j]$

In most of this paper, we study an optimal government policy when agents can trade only a one-period risk-free bond. We assume that the government imposes an affine tax. We denote proportional labor taxes by τ and lump sum transfers by T . With this the tax bill of an agent with wage earnings $l_{i,t} \theta_{i,t}$ is given by

$$-T_t + \tau_t \theta_{i,t} l_{i,t}.$$

We do not restrict the sign of T_t at any t or s^t . If for some type i , $\theta_{i,t} = 0$, $b_{i,-1} = 0$ and U^i is defined only on \mathcal{R}_+^2 , his budget constraint will imply that the all allocations feasible for the planner have nonnegative present values of transfers, since transfers are the sole source of a type i agent's wealth and consumption.

While results in sections 4, 5, 6, and 7 depend on these assumptions about an affine tax system and incomplete markets, key results of section 3 apply under more general tax functions and market structures.

Under an affine tax system, agent i 's budget constraint at t is

$$c_{i,t} + b_{i,t} = (1 - \tau_t) \theta_{i,t} l_{i,t} + R_{t-1} b_{i,t-1} + T_t, \quad (4)$$

where $b_{i,t}$ denotes asset holdings of a type i agent at time $t \geq 0$, R_{t-1} is a gross risk-free one-period interest rate from $t-1$ to t for $t \geq 1$, and $R_{-1} \equiv 1$. For $t \geq 0$, R_t is measurable with respect to s^t . To rule out Ponzi schemes, we assume that $b_{i,t}$ must be bounded from below. Except in subsection 3.1, we impose no further constraints on agents' borrowing and lending. Subsection 3.1 briefly studies economies with arbitrary borrowing constraints.

⁵movements between real interest rates and fundamentals which we exploit later in section 6 and 7.

⁶Consequently our setup allows both extensive and intensive responses of labor.

The government budget constraint is

$$g_t + B_t = \tau_t \sum_{i=1}^I \pi_i \theta_{i,t} l_{i,t} - T_t + R_{t-1} B_{t-1}, \quad (5)$$

where B_t denotes the government's assets at time t , which we assume are bounded from below. Our assumptions about preferences imply that the government can collect only finite revenues in each period, so this restriction rules out government-run Ponzi schemes.

We assume that private agents and the government start with assets $\{b_{i,-1}\}_{i=1}^I$ and B_{-1} , respectively. Asset holdings satisfy the market clearing condition

$$\sum_{i=1}^I \pi_i b_{i,t} + B_t = 0 \text{ for all } t \geq -1. \quad (6)$$

Since B_t and all $b_{i,t}$ are bounded from below, equation (6) implies that they are also bounded from above.

Components of competitive equilibria are described below

Definition 1 An allocation is a sequence $\{c_{i,t}, l_{i,t}\}_{i,t}$. An asset profile is a sequence $\{\{b_{i,t}\}_i, B_t\}_t$. A price system is an interest rate sequence $\{R_t\}_t$. A tax policy is a sequence $\{\tau_t, T_t\}_t$.

Definition 2 For a given initial asset distribution $(\{b_{i,-1}\}_i, B_{-1})$, a competitive equilibrium with affine taxes is a sequence $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, B_t, R_t\}_t$ and a tax policy $\{\tau_t, T_t\}_t$, such that $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ maximize (1) subject to (4) and $\{b_{i,t}\}_{i,t}$ is bounded; and constraints (2), (5) and (6) are satisfied.

Lastly we define optimal competitive equilibria.

Definition 3 Given $(\{b_{i,-1}\}_i, B_{-1})$, an optimal competitive equilibrium with affine taxes is a tax policy $\{\tau_t^*, T_t^*\}_t$, an allocation $\{\{c_{i,t}^*, l_{i,t}^*\}_i\}_t$, an asset profile $\{\{b_{i,t}^*\}_i, B_t^*\}_t$, and a price system $\{R_t^*\}_t$ such that (i) given $(\{b_{i,-1}\}_i, B_{-1})$, the tax policy, the price system, and the allocation constitute a competitive equilibrium; and (ii) there is no other tax policy $\{\tau_t, T_t\}_t$ such that a competitive equilibrium given $(\{b_{i,-1}\}_i, B_{-1})$ and $\{\tau_t, T_t\}_t$ has a strictly higher value of (3).

We call $\{\tau_t^*, T_t^*\}_t$ an optimal tax policy, $\{c_{i,t}^*, l_{i,t}^*\}_i$ an optimal allocation, and $\{\{b_{i,t}^*\}_i, B_t^*\}_t$ an optimal asset profile.

3 Relevant and Irrelevant Aspects of the Distribution of Government Debt

This section sets forth a result that underlies much of this paper, namely, that the level of government debt is not a state variable for our economy. The reason is that there is an equivalence class of tax policies and asset profiles that support the same competitive equilibrium allocation. A competitive equilibrium allocation pins down only net asset positions. The assertions in this section apply to all competitive equilibria, not just the optimal ones that will be our focus in subsequent sections.

Theorem 1 Given $(\{b_{i,-1}\}_i, B_{-1})$, let $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, B_t, R_t\}_t$ and $\{\tau_t, T_t\}_t$ be a competitive equilibrium. For any bounded sequences $\{\hat{b}_{i,t}\}_{i,t \geq -1}$ that satisfy

$$\hat{b}_{i,t} - \hat{b}_{1,t} = \tilde{b}_{i,t} \equiv b_{i,t} - b_{1,t} \text{ for all } t \geq -1, i \geq 2,$$

there exist sequences $\{\hat{T}_t\}_t$ and $\{\hat{B}_t\}_{t \geq -1}$ that satisfy (6) such that $\{\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\}_i, \hat{B}_t, R_t\}_t$ and $\{\tau_t, \hat{T}_t\}_t$ constitute a competitive equilibrium given $(\{\hat{b}_{i,-1}\}_i, \hat{B}_{-1})$.

Proof. Let

$$\hat{T}_t = T_t + (\hat{b}_{1,t} - b_{1,t}) - R_{t-1}(\hat{b}_{1,t-1} - b_{1,t-1}) \text{ for all } t \geq 0. \quad (7)$$

Given a tax policy $\{\tau_t, \hat{T}_t\}_t$, the allocation $\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\}_{i,t}$ is a feasible choice for consumer i since it satisfies

$$\begin{aligned} c_{i,t} &= (1 - \tau_t) \theta_{i,t} l_{i,t} + R_{t-1} b_{i,t-1} - b_{i,t} + T_t \\ &= (1 - \tau_t) \theta_{i,t} l_{i,t} + R_{t-1} (b_{i,t-1} - b_{1,t-1}) - (b_{i,t} - b_{1,t}) + T_t + R_{t-1} b_{1,t-1} - b_{1,t} \\ &= (1 - \tau_t) \theta_{i,t} l_{i,t} + R_{t-1} (\hat{b}_{i,t-1} - \hat{b}_{1,t-1}) - (\hat{b}_{i,t} - \hat{b}_{1,t}) + T_t + R_{t-1} b_{1,t-1} - b_{1,t} \\ &= (1 - \tau_t) \theta_{i,t} l_{i,t} + R_{t-1} \hat{b}_{i,t-1} - \hat{b}_{i,t} + \hat{T}_t. \end{aligned}$$

Suppose that $\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\}_{i,t}$ is not the optimal choice for consumer i , in the sense that there exists some other sequence $\{\hat{c}_{i,t}, \hat{l}_{i,t}, \hat{b}_{i,t}\}_t$ that gives strictly higher utility. Then the choice $\{\hat{c}_{i,t}, \hat{l}_{i,t}, \hat{b}_{i,t}\}_t$ is feasible given the tax rates $\{\tau_t, T_t\}_t$, which contradicts the assumption that $\{c_{i,t}, l_{i,t}, b_{i,t}\}_t$ is the optimal choice for the consumer given taxes $\{\tau_t, T_t\}_t$. The new allocation satisfies all other constraints and therefore is an equilibrium. ■

An immediate corollary is that it is not total government debt but rather who owns it that affects equilibrium allocations.

Corollary 1 For any pair B'_{-1}, B''_{-1} , there are asset profiles $\{b'_{i,-1}\}_i$ and $\{b''_{i,-1}\}_i$ such that equilibrium allocations starting from $(\{b'_{i,-1}\}_i, B'_{-1})$ and from $(\{b''_{i,-1}\}_i, B''_{-1})$ are the same. These asset profiles satisfy

$$b'_{i,-1} - b'_{1,-1} = b''_{i,-1} - b''_{1,-1} \text{ for all } i.$$

This result is closely related to Ricardian Equivalence in Barro (1974). There are, however, some important distinctions. In Barro's representative agent model, lump sum taxes are not distortionary. In our economy, since the planner does not have person-specific taxes, a lump sum transfer introduces distortions in inequality, a force that has a significant effect on optimal policy, as we will see in following sections. Despite this, Ricardian equivalence continues to hold.⁶ Theorem 1 shows that many transfer sequences $\{T_t\}_t$ and asset profiles $\{b_{i,t}, B_t\}_{i,t}$ support the same equilibrium allocation. For example, one

⁶Wallace (1981)'s Modigliani-Miller theorem for a class of government open market operations has a similar flavor. Sargent (1987) describes the structure of a set of related Modigliani-Miller theorems for government finance.

can set government assets $B_{i,-1} = 0$ without loss of generality. Alternatively, we can normalize by setting assets $b_{i,t}$ of any type i to zero.

Theorem 1 continues to hold in more general environments. For example, we could allow agents to trade all conceivable Arrow securities and still show that equilibrium allocations depend only on agents' net assets positions. Similarly, our results hold in economies with capital.

3.1 Extension: Borrowing constraints

Representative agent models rule out Ricardian equivalence either by assuming distorting taxes or by imposing ad hoc borrowing constraints. By way of contrast, we have verified that Ricardian equivalence holds in our economy even though there are distorting taxes. Imposing ad-hoc borrowing limits also leaves Ricardian equivalence intact in our economy.⁷ In economies with exogenous borrowing constraints, agents' maximization problems include the additional constraints

$$b_{i,t} \geq \underline{b}_i \quad (8)$$

for some exogenously given $\{\underline{b}_i\}_i$.

Definition 4 For given $(\{b_{i,-1}, \underline{b}_i\}_i, B_{-1})$ and $\{\tau_t, T_t\}_t$, a competitive equilibrium with affine taxes and exogenous borrowing constraints is a sequence $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, B_t, R_t\}_t$ such that $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ maximizes (1) subject to (4) and (8), $\{b_{i,t}\}_{i,t}$ are bounded, and constraints (2), (5) and (6) are satisfied.

We can define an *optimal* competitive equilibrium with exogenous borrowing constraints by extending Definition 3.

The introduction of the ad-hoc debt limits leaves unaltered the conclusions of Corollary 1 and the role of the initial distribution of assets across agents. The next proposition asserts that ad-hoc borrowing limits do not limit a government's ability to respond to aggregate shocks.⁸

Proposition 1 Given an initial asset distribution $(\{b_{i,-1}\}_i, B_{-1})$, let $\{c_{i,t}, l_{i,t}\}_{i,t}$ and $\{R_t\}_t$ be a competitive equilibrium allocation and interest rate sequence in an economy without exogenous borrowing constraints. Then for any exogenous constraints $\{\underline{b}_i\}_i$, there is a government tax policy $\{\tau_t, T_t\}_t$ such that $\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\}_{i,t}$ is a competitive equilibrium allocation in an economy with exogenous borrowing constraints $(\{b_{i,-1}, \underline{b}_i\}_i, B_{-1})$ and $\{\tau_t, T_t\}_t$.

Proof. Let $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ be a competitive equilibrium allocation without exogenous borrowing constraints. Let $\Delta_t \equiv \max_i \{b_i - b_{i,t}\}$. Define $\hat{b}_{i,t} \equiv b_{i,t} + \Delta_t$ for all $t \geq 0$ and $\hat{b}_{i,-1} = b_{-1}$. By Theorem 1, $\{\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\}\}_{i,t}$ is also a competitive equilibrium allocation without exogenous borrowing constraints. Moreover, by construction $\hat{b}_{i,t} - \underline{b}_i = b_{i,t} + \Delta_t - \underline{b}_i \geq 0$. Therefore, $\hat{b}_{i,t}$ satisfies (8). Since agents' budget

⁷Bryant and Wallace (1984) describe how a government can use borrowing constraints as part of a welfare-improving policy to finance exogenous government expenditures. Sargent and Smith (1987) describe Modigliani-Miller theorems for government finance in a collection of economies in which borrowing constraints on classes of agents produce the kind of rate of return discrepancies that Bryant and Wallace manipulate.

⁸See Yared (2012, 2013) who shows a closely related result.

sets are smaller in the economy with exogenous borrowing constraints, and $\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\}_{i,t}$ are feasible at interest rate process $\{R_t\}_t$, then $\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\}_{i,t}$ is also an optimal choice for agents in the economy with exogenous borrowing constraints $\{\bar{b}_i\}_i$. Since all market clearing conditions are satisfied, $\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\}_{i,t}$ is a competitive equilibrium allocation and asset profile. ■

To provide some intuition for Proposition 1, suppose to the contrary that the exogenous borrowing constraints restricted a government's ability to achieve a desired allocation. That means that the government would want to increase its borrowing and to repay agents later, which the borrowing constraints prevent. But the government can just reduce transfers today and increase them tomorrow. That would achieve the desired allocation without violating the exogenous borrowing constraints.

Welfare can be strictly higher in an economy with exogenous borrowing constraints relative to an economy without borrowing constraints because a government might want to push some agents against their borrowing limits. When agents' borrowing constraints bind, their shadow interest rates differ from the common interest rate that unconstrained agents face. When the government rearranges tax policies to affect the interest rate, it affects constrained and unconstrained agents differently. By facilitating redistribution, this can improve welfare. In appendix A.1, we construct an example without any shocks in which the government can achieve higher welfare by using borrowing constraints to improve its ability to redistribute.

3.2 Ricardian irrelevance and optimal equilibria

Our statements about Ricardian irrelevance apply to all competitive equilibrium allocations, not just the optimal ones that are the main focus of this paper. To appreciate how these Ricardian irrelevance results affect optimal equilibria, suppose that we increase an initial level of government debt from 0 to some arbitrary level $B'_{-1} > 0$. If the government were to hold transfers $\{T_t\}_t$ fixed, it would have to increase tax rates $\{\tau_t\}_t$ enough to collect a present value of revenues sufficient to repay B'_{-1} . Since deadweight losses are convex in τ , higher levels of debt financed with bigger distorting taxes $\{\tau_t\}$ impose larger distortions on the economy, thereby degrading the equilibrium allocation. But this would not happen if the government were instead to adjust transfers in response to a higher initial debt. To determine optimal transfers, we need to know who owns the initial government debt B'_{-1} . For example, suppose that agents hold equal amounts of it. Then each unit of debt repayment achieves the same redistribution as one unit of transfers. If the original tax policy at $B'_{-1} = 0$ were optimal, then the best policy for a government with initial debt $B'_{-1} > 0$ would be to reduce the present value transfers by exactly the amount of the increase in per capita debt, because then distorting taxes $\{\tau_t\}$ and the allocation would both remain unchanged.⁹

But the situation would be different if holdings of government debt were not equal across agents. For example, suppose that richer people owned disproportionately more government debt than poorer people. That would mean that inequality is effectively initially higher in an economy with higher initial

⁹This example illustrates principles proclaimed by Simon Newcomb (1865, p. 85) in the quotation with which we began this paper.

government debt. As a result, a government with Pareto weights $\{\alpha_i\}$ that favor equality would want to increase both distorting tax rates $\{\tau_t\}$ and transfers $\{T_t\}$ to offset the increase in inequality associated with the increase in government debt. The conclusion would be the opposite if government debt were to be owned mostly by poorer households.

This logic shows how important it is to know the distribution of government debt across people. Government debt that is widely distributed across households (e.g., implicit Social Security debt) is less distorting than government debt owned mostly by people whose incomes are at the top of the income distribution (e.g., government debt held by hedge funds).¹⁰

4 Quasi-linear preferences

Before we present a more general characterization in section 5, in this section we study a special case of our economy that allows us to get a long way analytically and to identify some forces that drive outcomes. Here we assume that the only source of aggregate shocks is g_t and that preferences are given by $U^i(c, l) = c - h_i(l)$ where h_i is an increasing differentiable function with $h'_i(0) = 0$ and $h'_i(\bar{l}_i) = \infty$. Such quasi-linear preferences have been assumed in the context of representative agent economies (see, e.g., AMSS, Farhi (2010), Battaglini and Coate (2007, 2008), Yared (2010), Faraglia et al. (2012)). We demonstrate two important things in this section. First, optimal policies in our heterogenous agent economy in which transfers are chosen optimally differ in interesting ways from those in a representative agent economy that exogenously restricts transfers. Second, the special set up under study switches off two forces present more generally, namely, that the marginal utilities of agents are differentially affected by changes in transfers and that with other preference specifications the interest rate is not constant. These two forces play important roles in determining the long run outcomes described in Section 6.

To simplify notation, we now assume that the initial debt is $\{\beta^{-1}b_{i,-1}\}_i$.

Proposition 2 Suppose that preferences are quasi-linear and that g_t is the only aggregate variable subject to shocks. Then the optimal tax rate τ_t^* satisfies $\tau_t^* = \tau^*$. An optimum asset profile $\{b_{i,t}^*, B_t^*\}_{i,t}$ can be chosen to satisfy $b_{i,t}^* = b_{i,-1}$ for all i , $t \geq 0$ and $B_t^* = B_{-1}$ for all $t \geq 0$.¹¹

Proof. When preferences are quasilinear, the interest rate $R_t = \beta^{-1}$ for all t and $(1 - \tau_t)\theta_i = h'_i(l_{i,t})$ for all t . For our purposes, it is more convenient to express the labor supply component of the allocation as a function of $(1 - \tau)$ and optimize with respect to τ rather than $\{l_i\}_i$. We invert $h'_i(\cdot)$ to express labor supply l_i as a function of $(1 - \tau)$. Call this function $H_i(1 - \tau)$. Use the budget constraint (4) to obtain

$$c_{i,t} + b_{i,t} - (1 - \pi_i)H_i(1 - \tau_t)\theta_i = T_t + \beta^{-1}b_{i,t-1}. \quad (9)$$

The optimal allocation solves

$$\max_{\{c_{i,t}, b_{i,t}, \pi_i, T_t\}_{i,t}} \mathbb{E}_0 \sum_{t=0}^{\infty} \sum_{i=1}^I \alpha_i \pi_i \beta^t [c_{i,t} - h_i(H_i(1 - \pi_i))] \quad (10)$$

¹⁰It is possible to extend our analysis to open economy with foreign holdings of domestic debt. The more government debt is owned by the foreigners, the higher are the distorting taxes that the government needs to impose.

¹¹We thank Guy Laroque for suggesting the idea for this proof.

subject to $\{b_{i,t}\}_{i,t}$ being bounded, (9), and

$$\sum_{i=1}^I \pi_i c_{i,t} + g_t = \sum_{i=1}^I \pi_i H_i (1 - \tau_t) \theta_i.$$

Note that since $\{b_{i,t}\}_{i,t}$ is bounded,

$$E_0 \sum_{t=0}^{\infty} \beta^t [\beta^{-1} b_{i,t-1} - b_{i,t}] = \beta^{-1} b_{i,-1} + \lim_{T \rightarrow \infty} E_0 \left(\sum_{t=0}^T \beta^t [b_{i,t} - b_{i,t}] - \beta^{T+1} b_{i,T+1} \right) = \beta^{-1} b_{i,-1}.$$

Use (9) to eliminate $c_{i,t}$ and then use the preceding expression to get

$$\max_{\{b_{i,t}, \tau_t, T_t\}_{i,t}} E_0 \sum_{t=0}^{\infty} \sum_{i=1}^I \alpha_i \pi_i \beta^t [T_t + (1 - \tau_t) H_i (1 - \tau_t) \theta_i - h_i (H_i (1 - \eta))] + \beta^{-1} \sum_{i=1}^I b_{i,-1}. \quad (11)$$

subject to

$$\sum_{i=1}^I \pi_i [T_t + \beta^{-1} b_{i,t-1} - b_{i,t} + (1 - \tau_t) H_i (1 - \tau_t) \theta_i] + g_t = \sum_{i=1}^I \pi_i H_i (1 - \eta) \theta_i. \quad (12)$$

Let $\beta^t \lambda_t$ be the Lagrange multiplier on the time t feasibility constraint (12). The first-order condition with respect to T_t implies that $\lambda_t = \sum_{i=1}^I \alpha_i \pi_i$ is constant and independent of t . Therefore, optimal taxes $\tau_t = \tau^*$ are also constant and independent of t . Using τ^* , equation (12) pins down $\sum_{i=1}^I \pi_i [T_t + \beta^{-1} b_{i,t-1} - b_{i,t}]$. Without loss of generality we can set $b_{i,t}^* = b_{i,-1}$ and T_t^* to satisfy (12).

■

In the optimal equilibria for the quasi-linear economy described in Proposition 2, fluctuations in lump-sum taxes and transfers "do all the work". In period 0, the government chooses an optimal present value of transfers and a constant tax rate that pays for it. Tax rates and transfers depend on the Pareto weights $\{\alpha_i\}$: higher Pareto weights on low-skilled agents imply higher transfers and tax rates. In response to a shock g_t , the government adjusts transfers in period t by the amount of the shock. Since all agents are risk-neutral, welfare is unaffected by fluctuations in transfers. This allows the government to perfectly smooth distorting tax rates.

Comparison with representative agent economies

The Lucas and Stokey (1983) and AMSS representative agent models impose $T_t \geq 0$. An informal justification for this restriction is the desire of the government not to hurt poor agents who can't afford lump-sum taxes. This constraint always binds in the Lucas and Stokey model and it binds in the AMSS model until the government has acquired enough assets to finance all future expenditures from earnings on those assets. In those representative agent models, the government would like to impose lump-sum *taxes*, not transfers. We explicitly model redistributive concerns by attaching Pareto weights to heterogeneous agents and obtain the very different dynamics shown in Proposition 2. In this section, we argue that differences in the dynamics come from the presence of this arbitrary restrictions on transfers and not from redistributive motives.

We impose the following in our maximization problem (10), namely,¹²,

$$T_t \geq 0 \text{ for all } t. \quad (13)$$

The following proposition states it is optimal for the planner to set policy in such a way that constraint (13) eventually becomes slack.¹³

Proposition 3 *Assume that $I \geq 1$ and g_t takes more than one value. Let $\beta^t \chi_t$ be the Lagrange multiplier on constraint (13) in a version of maximization problem (10) that is augmented with constraint (13). Then $\chi_t \rightarrow 0$ a.s.*

Proof. Our augmented version of the maximization problem (10) can be expressed as maximization problem (11) with an additional constraint (13). The first-order conditions for T_t yield $\sum_{i=1}^I \alpha_i \pi_i = \mu_t + \chi_t$, while the first-order condition for $b_{i,t}$ implies $\mu_t = E_t \mu_{t+1}$. Since $\chi_t \geq 0$, these two conditions imply that χ_t is a nonnegative martingale and therefore χ_t must almost surely converge to a constant. This in turn implies that μ_t must almost surely converge to a constant. Then the first-order conditions for τ_t also imply that τ_t must converge a.s. to some τ^* .

Suppose $\chi_t \rightarrow \chi^* > 0$. This implies that $T_t \rightarrow 0$ and (12) becomes

$$-\beta^{-1} B_{i,t-1} + B_{i,t} + \sum_{i=1}^I \pi_i (1 - \tau^*) H_i (1 - \tau^*) \theta_i + g_t = \sum_{i=1}^I \pi_i H_i (1 - \tau^*) \theta_i,$$

where we used (6) to substitute for $\sum_{i=1}^I \pi_i b_{i,t}$. If g_t can take more than one value and follows an irreducible Markov process, then for any bound on B_t , we can find a sequence of government expenditures g_t for which this bound will eventually be violated, leading to a contradiction. This implies that $\chi_t \rightarrow 0$. ■

Proposition 2 emphasized that in absence of restriction (13) the government uses fluctuations in transfers to finance all fluctuations in expenditures. Constraint (13) imposes an asymmetry in using transfers to smooth fluctuations in expenditures. Around zero it is costless to *increase* transfers by a small amount but infinitely costly to *decrease* them by the same amount. This induces the optimal policy to engineer tax rates, transfers, and asset distributions that allow the economy eventually to disarm this constraint.

For quasi-linear preferences, figure II compares equilibrium dynamics in a representative agent (AMSS) economy and an economy with two agents, one who is not productive, and Pareto weights chosen to make transfers be positive at all times and along all histories. The sequences of s_t shocks are identical across the two economies. While tax rates converge to zero for the AMSS economy, they are constant for the heterogeneous agent economy.¹⁴

¹²This makes AMSS a special case of our economy.

¹³It can be shown that if g is not too high and a government is sufficiently redistributive (i.e., α_i is sufficiently high for low productivity agents), constraints (13) is always slack.

¹⁴The plots for the AMSS and the heterogeneous agent economies are both for quasi-linear preferences with a Frisch elasticity of labor equals 0.5 and a discount factor $\beta = 0.95$. In the AMSS economy, the agent's initial assets are zero and government expenditure shocks $g(s_t) \in \{.1, .3\}$ are generated using an IID process with equally likely outcomes. For the heterogeneous agent economy, we set $\alpha_2 = .54$ so that the initial labor taxes are similar to those for the AMSS economy.

5 Optimal equilibria with affine taxes

We return to the more general problem formulated in section 2. We further assume that $U^i : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ is concave in $(c, -l)$ and twice continuously differentiable. We let $U_{x,t}^i$ or $U_{xy,t}^i$ denote first and second derivatives of U^i with respect to $x, y \in \{c, l\}$ in period t and assume that $\lim_{x \rightarrow L_i} U_l^i(c, x) = \infty$ and $\lim_{x \rightarrow 0} U_l^i(c, x) = 0$ for all c and i .

We focus on interior equilibria. First-order necessary conditions for the consumer's problem are

$$(1 - \tau_t) \theta_{i,t} U_{c,t}^i = -U_{l,t}^i, \quad (14)$$

and

$$U_{c,t}^i = \beta_t R_t E_t U_{c,t+1}^i. \quad (15)$$

To help characterize an equilibrium, we use

Proposition 4 *A sequence $\{(c_{i,t}, l_{i,t}, b_{i,t})_i, R_t, \tau_t, T_t\}_t$ is part of a competitive equilibrium with affine taxes if and only if it satisfies (2), (4), (14), and (15) and $b_{i,t}$ is bounded for all i and t .*

Proof. Necessity is obvious. In appendix A.2, we use arguments of Magill and Quinzii (1994) and Constantinides and Duffie (1996) to show that any $\{(c_{i,t}, l_{i,t}, b_{i,t})_i\}_t$ that satisfies (4), (14), and (15) is a solution to consumer i 's problem. Equilibrium $\{B_t\}_t$ is determined by (6) and constraint (5) is then implied by Walras' Law ■

To find an optimal equilibrium, by Proposition 4 we can choose $\{(c_{i,t}, l_{i,t}, b_{i,t})_i, R_t, \tau_t, T_t\}_t$ to maximize (3) subject to (2), (4), (14), and (15). We apply a first-order approach and follow steps similar to ones taken by Lucas and Stokey (1983) and AMSS. Substituting consumers' first-order conditions (14) and (15) into the budget constraints (4) yields implementability constraints

$$c_{i,t} + b_{i,t} = -\frac{U_{l,t}^i}{U_{c,t}^i} l_{i,t} + T_t + \frac{U_{c,t-1}^i}{\beta_{t-1} E_{t-1} U_{c,t}^i} b_{i,t-1} \text{ for all } i, t. \quad (16)$$

For $I \geq 2$, we can use constraint (16) for $i = 1$ to eliminate T_t from (16) for $i > 1$. Letting $\tilde{b}_{i,t} \equiv b_{i,t} - b_{1,t}$, we can represent the implementability constraints as

$$\begin{aligned} & (c_{i,t} - c_{1,t}) + \tilde{b}_{i,t} \\ &= -\frac{U_{l,t}^i}{U_{c,t}^i} l_{i,t} + \frac{U_{l,t}^1}{U_{c,t}^1} l_{1,t} + \frac{U_{c,t-1}^i}{\beta_{t-1} E_{t-1} U_{c,t}^i} \tilde{b}_{i,t-1} \text{ for all } i > 1, t. \end{aligned} \quad (17)$$

With this representation of the implementability constraints, the planner's maximization problem depends only on the $I-1$ variables $\tilde{b}_{i,t-1}$. The reduction of the dimensionality from I to $I-1$ is another consequence of theorem 1.

Denote $Z_t^i = U_{c,t}^i c_{i,t} + U_{l,t}^i l_{i,t} - \frac{U_{c,t}^i}{U_{c,t}^1} [U_{c,t}^1 c_{1,t} + U_{l,t}^1 l_{1,t}]$. Formulated in a space of sequences, the optimal policy problem is:

$$\max_{c_{i,t}, l_{i,t}, b_{i,t}} E_0 \sum_{i=1}^I \pi_i \alpha_i \sum_{t=0}^{\infty} \bar{\beta}_t Z_t^i (c_{i,t}, l_{i,t}), \quad (18)$$

subject to

$$\tilde{b}_{t-1} \frac{U_{c,t-1}^i}{\beta_{t-1}} = \left(\frac{\mathbb{E}_{t-1} U_{c,t}^i}{U_{c,t}^i} \right) \mathbb{E}_t \sum_{k=t}^{\infty} \left[\prod_{j=t}^{k-1} \beta_j \right] Z_k^i \quad \forall t \geq 1 \quad (19a)$$

$$\tilde{b}_{t-1} = \mathbb{E}_{-1} \sum_{k=0}^{\infty} \left[\prod_{j=0}^{k-1} \beta_j \right] Z_k^i \quad (19b)$$

$$\frac{\mathbb{E}_t U_{c,t+1}^i}{U_{c,t}^i} = \frac{\mathbb{E}_t U_{c,t+1}^j}{U_{c,t}^j} \quad (19c)$$

$$\sum_{i=1}^I \pi_i c_i(s^t) + g(s_t) = \sum_{i=1}^I \pi_i \theta_i(s_t) l_i(s^t), \quad (19d)$$

$$\frac{U_{l,t}^i}{\theta_{l,t} U_{c,t}^i} = \frac{U_{l,t}^1}{\theta_{l,t} U_{c,t}^1} \quad (19e)$$

$$\tilde{b}_{t-1} \frac{U_{c,t-1}^i}{\beta_{t-1}} \text{ is bounded} \quad (19f)$$

Constraint (19a) is a measurability restriction on allocations that requires that the right side is determined at time $t - 1$. This condition is inherited from the restriction that only risk-free bonds are traded.

For both computational and educational purposes, it is convenient to represent the optimal policy problem recursively. For the purpose of constructing a recursive representation, let $\boldsymbol{x} = \beta^{-1} (U_e^2 \tilde{b}_2, \dots, U_e^I \tilde{b}_I)$, $\boldsymbol{\rho} = (U_e^2/U_e^1, \dots, U_e^I/U_e^1)$, and denote an allocation $a = \{c_i, l_i\}_{i=1}^I$. In the spirit of Kydland and Prescott (1980) and Farhi (2010), we split the Ramsey problem into a time-0 problem that takes $(\{\tilde{b}_{i,-1}\}_{i=2}^I, s_0)$ as given and a time $t \geq 1$ continuation problem that takes $\boldsymbol{x}, \boldsymbol{\rho}, s_-$ as given. We formulate two Bellman equations and two value functions, one that pertains to $t \geq 1$, another to $t = 0$. The time inconsistency of an optimal policy manifests itself in there being distinct value functions and Bellman equations at $t = 0$ and $t \geq 1$.

For $t \geq 1$, let $V(\boldsymbol{x}, \boldsymbol{\rho}, s_-)$ be the planner's continuation value given $\boldsymbol{x}_{t-1} = \boldsymbol{x}$, $\boldsymbol{\rho}_{t-1} = \boldsymbol{\rho}$, $s_{t-1} = s_-$. It satisfies the Bellman equation

$$V(\boldsymbol{x}, \boldsymbol{\rho}, s_-) = \max_{a(s), \boldsymbol{x}'(s), \boldsymbol{\rho}'(s)} \sum_s \Pr(s|s_-) \left(\left[\sum_i \pi_i \alpha_i U^i(s) \right] + \beta(s) V(\boldsymbol{x}'(s), \boldsymbol{\rho}'(s), s) \right) \quad (20)$$

where the maximization is subject to

$$U_e^i(s) [\alpha_i(s) - c_i(s)] + \beta(s) x'_i(s) + \left(U_l^i(s) l_i(s) - U_e^i(s) \frac{U_l^1(s)}{U_e^1(s)} l_1(s) \right) = \frac{x U_e^i(s)}{\mathbb{E}_{s_-} U_e^i} \text{ for all } s, i \geq 2 \quad (21a)$$

$$\frac{\mathbb{E}_{s_-} U_e^i}{\mathbb{E}_{s_-} U_e^1} = \rho_i \text{ for all } i \geq 2 \quad (21b)$$

$$\frac{U_l^i(s)}{\theta_i(s) U_e^i(s)} = \frac{U_l^1(s)}{\theta_1(s) U_e^1(s)} \text{ for all } s, i \geq 2 \quad (21c)$$

$$\sum_i \pi_i c_i(s) + g(s) = \sum_i \pi_i \theta_i(s) l_i(s) \quad \forall s \quad (21d)$$

$$\rho'_i(s) = \frac{U_e^i(s)}{U_e^1(s)} \text{ for all } s, i \geq 2 \quad (21e)$$

$$\underline{x}_i(s; \boldsymbol{x}, \boldsymbol{\rho}, s_-) \leq x_i(s) \leq \bar{x}_i(s; \boldsymbol{x}, \boldsymbol{\rho}, s_-) \quad (21f)$$

Constraints (21b) and (21e) imply (15). The definition of x_t and constraints (21a) together imply equation (17) scaled by U_e^i . Let $V_0(\{\tilde{b}_{t-1}\}_{t=2}^T, s_0)$ be the value to the planner at $t = 0$, where \tilde{b}_{t-1} denotes initial debt inclusive of accrued interest. It satisfies the Bellman equation

$$V_0(\{\tilde{b}_{t-1}\}_{t=2}^T, s_0) = \max_{a_0, x_0, \rho_0} \sum_i \pi_i \alpha_i U^i(c_{i,0}, l_{i,0}) + \beta(s_0) V(x_0, \rho_0, s_0) \quad (22)$$

where the maximization is subject to

$$U_{e,0}^i [c_{i,0} - c_{i,0}] + \beta(s_0) x_{i,0} + \left(U_{l,0}^i l_{i,0} - U_{e,0}^i \frac{U_{l,0}^1}{U_{e,0}^1} l_{1,0} \right) = U_{e,0}^i \tilde{b}_{i-1} \text{ for all } i \geq 2 \quad (23a)$$

$$\frac{U_{l,0}^i}{\theta_{i,0} U_{e,0}^i} = \frac{U_{l,0}^1}{\theta_{1,0} U_e^1} \text{ for all } i \geq 2 \quad (23b)$$

$$\sum_i \pi_i c_{i,0} + g_0 = \sum_i \pi_i \theta_{i,0} l_{i,0} \quad (23c)$$

$$\rho_{i,0} = \frac{U_{e,0}^i}{U_{e,0}^1} \text{ for all } i \geq 2 \quad (23d)$$

Because constraint (21b) is absent from the time 0 problem, the time 0 problem differs from the time $t \geq 1$ problem, a source of the time consistency of the optimal tax plan.

6 Ergodic distribution and policies in the long run

In this section, we describe an ergodic set to which state variables converge. We start with the case in which aggregate shocks are iid and can take two values. We show that for this shock structure there generally exists a pair $(\boldsymbol{x}^{SS}, \boldsymbol{\rho}^{SS})$ such that if economy ever reaches a state within this set, it stays there. Tax rates and transfers in this steady state depend only on the current realization of the shock; for commonly used preferences, fluctuations of tax rates are small. We also describe properties of the steady state as well conditions under which economy converges to it and the speed of convergence. Section 6.3 then extends the analysis to more general shocks and shows numerically that while a fixed non-random steady state generally does not exist, asymptotic outcomes inside the ergodic set are very similar to those that prevail in the two shock iid case in which there is a fixed steady state. Throughout this section we assume that preferences are separable in consumption and labor.

6.1 IID shocks with two values

Let $\Psi(s; \mathbf{x}, \boldsymbol{\rho}, s_-)$ be an optimal law of motion for the state variables for the $t \geq 1$ recursive problem, i.e., $\Psi(s; \mathbf{x}, \boldsymbol{\rho}, s_-) = (\mathbf{x}'(s), \boldsymbol{\rho}'(s))$ solves (20) given state $(\mathbf{x}, \boldsymbol{\rho}, s_-)$.

Definition 5 A steady state $(\mathbf{x}^{SS}, \boldsymbol{\rho}^{SS})$ satisfies $(\mathbf{x}^{SS}, \boldsymbol{\rho}^{SS}) = \Psi(s; \mathbf{x}^{SS}, \boldsymbol{\rho}^{SS}, s_-)$ for all s, s_- .

Since in this steady state $\rho_i = U_e^i(s)/U_e^1(s)$ does not depend on the realization of shock s , the ratios of marginal utilities of all agents are constant. The continuation allocation depends only on s_1 and not on the history s^{t-1} .

We begin by noting that a competitive equilibrium fixes an allocation $\{c_i(s), l_i(s)\}_i$ given a choice for $\{\tau(s), \boldsymbol{\rho}(s)\}$ using equations (21c), (21d) and (21e). Let us denote $U(\tau, \boldsymbol{\rho}, s)$ as the value for the planner from the implied allocation using Pareto weights $\{\alpha_i\}_i$,

$$U(\tau, \boldsymbol{\rho}, s) = \sum_i \alpha_i U^i(s).$$

As before define $Z_i(\tau, \boldsymbol{\rho}, s)$ as

$$Z_i(\tau, \boldsymbol{\rho}, s) = U_e^i(s)c_i(s) + U_l^i(s)l_i(s) - \rho_i(s)[U_e^1(s)c_1(s) + U_l^1(s)l_1(s)].$$

For the IID case, the optimal policy solves the following Bellman equation for $\mathbf{x}(s^{t-1}) = \mathbf{x}, \boldsymbol{\rho}(s^{t-1}) = \boldsymbol{\rho}$

$$V(\mathbf{x}, \boldsymbol{\rho}) = \max_{\tau(s), \boldsymbol{\rho}'(s), \mathbf{x}'(s)} \sum_s P(s) [U(\tau(s), \boldsymbol{\rho}'(s), s) + \beta(s)V(\mathbf{x}'(s), \boldsymbol{\rho}'(s))] \quad (24)$$

subject to the constraints

$$Z_i(\tau(s), \boldsymbol{\rho}'(s), s) + \beta(s)x'_i(s) = \frac{x_i U_e^i(\tau(s), \boldsymbol{\rho}'(s), s)}{\mathbb{E} U_e^i(\tau, \boldsymbol{\rho})} \text{ for all } s, i \geq 2, \quad (25)$$

$$\sum_s P(s) U_e^1(\tau(s), \boldsymbol{\rho}'(s), s) (\rho'_i(s) - \rho_i) = 0 \text{ for } i \geq 2. \quad (26)$$

Constraint (26) is obtained by rearranging constraint (21b). It implies that $\boldsymbol{\rho}(s)$ is a risk-adjusted martingale. We next check if the first-order necessary conditions are consistent with stationary policies for some $(\mathbf{x}, \boldsymbol{\rho})$.¹⁵

Lemma 1 Let $\Pr(s)\mu_i(s)$ and λ_i be the multipliers on constraints (25) and (26). Imposing the restrictions $x'_i(s) = x_i$ and $\rho'_i(s) = \rho_i$, at a steady state $\{\mu_i, \lambda_i, x_i, \rho_i\}_{i=2}^N$ and $\{\tau(s)\}_s$ are determined by the following equations

$$Z_i(\tau(s), \boldsymbol{\rho}, s) + \beta(s)x_i = \frac{x_i U_e^i(\tau(s), \boldsymbol{\rho}, s)}{\mathbb{E} U_e^i(\tau, \boldsymbol{\rho})} \text{ for all } s, i \geq 2, \quad (27a)$$

$$U_\tau(\tau(s), \boldsymbol{\rho}, s) - \sum_i \mu_i Z_{i,\tau}(\tau(s), \boldsymbol{\rho}, s) = 0 \text{ for all } s, \quad (27b)$$

$$U_{\rho_i}(\tau(s), \boldsymbol{\rho}, s) - \sum_j \mu_j Z_{j,\rho_i}(\tau(s), \boldsymbol{\rho}, s) + \lambda_i U_e^i(\tau(s), \boldsymbol{\rho}', s) - \lambda_i \beta(s) \mathbb{E} U_e^i(\tau, \boldsymbol{\rho}) = 0 \text{ for all } s, i \geq 2 \quad (27c)$$

¹⁵Appendix A.5 discusses the associated second order conditions that ensure these policies are optimal

Since the shock s can take only two values, (27) is a square system in $4(N - 1) + 2$ unknowns $\{\mu_i^{SS}, \lambda_i^{SS}, x_i^{SS}, \rho_i^{SS}\}_{i=2}^N$ and $\{\tau^{SS}(s)\}_s$. For wide range of primitives, we can verify numerically that this system has a solution. In the next section we formally establish this for a class of two-agent economies that, while special, illustrate general forces that affect outcomes. The example will help us develop some comparative statics and interpret outcomes from a quantitative analysis to appear in section 7.

Lemma 1 also highlights the tradeoffs that the planner faces. Defining $\tilde{\lambda} = -\lambda \mathbb{E}U_c(\tau, \rho)$ and taking expectations in equation (27c), we get

$$\mathbb{E}U_{\rho_i}(\tau(s), \rho, s) = \mathbb{E} \sum_j \mu_j(s) Z_{j,\rho_i}(\tau(s), \rho, s) + (1 - \mathbb{E}\beta(s)) \tilde{\lambda}_i \quad (28)$$

The multiplier on the implementability constraint for i can be interpreted as the marginal cost of extracting funds from i and $\tilde{\lambda}_i$ is proportional to the multiplier on the constraint $\frac{\mathbb{E}U_i}{\mathbb{E}U_c} = \rho$. This constraint ensures that at the optimal allocation, agent i has no incentive to change his bond portfolio. The left side of (28) captures the cost for the planner if inequality (measured by the ratios of marginal utilities of consumption) deviates from his ideal point, given by α_1/α_i . In the absence of any constraints, at the first best the planner would set $\mathbb{E}U_{\rho_i}(\tau(s), \rho, s) = 0$, which implies that $\alpha_i U_c^i = \alpha_1 U_c^1$ for all i . The right side of equation (28) captures the cost of approaching the planner's ideal point, which come from the costs of raising taxes (the first term on the left hand side) and the ability of agents to trade (the second term).

The behavior of the economy in the steady state is similar to the behavior of the complete market economy characterized by Werning (2007). Both taxes and transfers depend only on the current realization of shock s_t . Moreover, the arguments of Werning (2007) can be adapted to show that taxes are constant when preferences have a CES form $c^{1-\sigma}/(1-\sigma) - l^{1+\gamma}/(1-\gamma)$ and fluctuations in tax rates are very small when preferences take forms consistent with the existence of balanced growth. We return to this point after we discuss convergence properties.

A two-agent example

Lemma 4 provides a simple way to verify existence of a steady state for wide range of parameter values by checking that there exists a root for system (27). Since the system of equations (27) is non-linear, existence can generally be verified only numerically. In this section, we provide a simple example with risk averse agents in which we can show existence of the root of (27) analytically. The analytical characterization of the steady state will allow us to show two main forces that determine the steady state asset distribution. These forces will also help to understand the long run behavior of the calibrated economy that we study in section 7.

Consider an economy consisting of two types of households with $\theta_{1,t} > \theta_{2,t} = 0$. One period utilities are $\ln c - \frac{1}{2}l^2$. The shock s takes two values, $s \in \{s_L, s_H\}$ with probabilities $\Pr(s|s_-)$ that are independent of s_- . We assume that $g(s) = g$ for all s , and $\theta_1(s_H) > \theta_1(s_L)$. We allow the discount factor $\beta(s)$ to depend on s .

Proposition 5 Suppose that $g < \theta(s)$ for all s . Let $R(s)$ be the gross interest rates and $x = U_e^2(s) [b_2(s) - b_1(s)]$

1. **Countercyclical interest rates.** If $\beta(s_H) = \beta(s_L)$, then there exists a steady state (x^{SS}, ρ^{SS}) such that $x^{SS} > 0$, $R^{SS}(s_H) < R^{SS}(s_L)$.
2. **Acyclical interest rates.** There exists a pair $\{\beta(s_H), \beta(s_L)\}$ such that there exists a steady state with $x^{SS} > 0$ and $R^{SS}(s_H) = R^{SS}(s_L)$.
3. **Procylical interest rates.** There exists a pair $\{\beta(s_H), \beta(s_L)\}$ such that there exists a steady state with $x^{SS} < 0$ and $R^{SS}(s_H) > R^{SS}(s_L)$.

In all cases, taxes $\tau(s) = \tau^{SS}$ are independent of the realized state.

In this two-agent case, by normalizing assets of the unproductive agent (using theorem 1) we can interpret x as the marginal utility adjusted assets of the government. Besides establishing existence, the proposition identifies the importance of cyclical properties of real interest rates in determining the sign of these assets.

Proposition 5 shows two main forces that determine the dynamics of taxes and assets: fluctuations in inequality and fluctuations in the interest rates. Let's start with part 2 of proposition 5, which turns off the second force. When interest rates are fixed, the government can adjust two instruments in response to an adverse shock (i.e., a fall in θ_1): it can either increase the tax rate τ or it can decrease transfers T . Both responses are distorting, but for different reasons. Increasing the tax rate increases distortions because the deadweight loss is convex in the tax rate, as in Barro (1979). This force operates in our economy just as it does in representative agent economies. But in a heterogeneous agent economy like ours, adjusting transfers T is also costly. When agents' asset holdings are identical, a decrease in transfers disproportionately affects a low-skilled agent, so his marginal utility falls by more than does the marginal utility of a high-skilled agent. Consequently, a decrease in transfers increases inequality, giving rise to a cost not present in representative agent economies.

The government can reduce the costs of inequality distortions by choosing tax rate policies that make the net asset positions of the high-skilled agent decrease over time. That makes the two agents' after-tax and after-interest income become closer, allowing decreases in transfers to have smaller effects on inequality in marginal utilities. If the net asset position of a high-skilled agent is sufficiently low, then a change in transfers has no effect on inequality and all distortions from fluctuations in transfers are eliminated.¹⁶

Turning now to the second force, interest rates generally fluctuate with shocks. Parts 1 and 3 of proposition 5 indicate what drives those fluctuations. Consider again the example of a decrease in productivity of high-skilled agent. If the tax rate τ is left unchanged, the government faces a shortfall of

¹⁶This convergence outcome has a similar flavor to "back-loading" results of Ray (2002) and Albanesi and Armenter (2012) that reflect the optimality of structuring policies intertemporally eventually to disarm distortions.

revenues. Since g is constant, the government requires extra sources of revenues. But suppose that the interest rate increases whenever θ_1 decreases, as happens, for example, when discount factors are constant and θ_1 is the only source of shocks. If the government holds positive assets, its earnings from those assets increase. So holding assets allows higher interest income to offset some of the government's revenue losses from taxes on labor. The situation reverses if interest rates fall at times of increased need for government revenues, as in part 3 of proposition 5, and the steady state allocation features the government's owning debt.

What matters for our second force is the co-movement of the interest rate with fundamental shocks. States with low average TFP (and therefore a lower base for labor taxes), high g , or a high spread of productivities that threatens to induce higher inequality (and therefore higher transfers and thirst for more government revenues to finance them) are "adverse" from the point of view of current government finance. The government can cope with such adverse states in less distorting ways if it finds itself holding positive (negative) assets if interest rates are high (low).¹⁷

Depending on details of shock processes, these two forces can either reinforce each other (as happens in Part 1 of proposition 5) or oppose each other (as in Part 3 of proposition 5). In the latter case, whether the government ends up with assets or debt in the long run depends on the relative strengths of the two forces.

Besides discount factor shocks, the level of net assets in the steady state depends on other primitives such as preferences for redistribution. An interesting comparative static exercise involves shutting off discount factor shocks and increasing α_1 , the Pareto weight attached to the high-skilled agent. This implies that the planner taxes the high-skilled agent less and redistributes less income to the low-skilled agent. Since the after-tax income of a low-skilled agent is lower, fluctuations in transfers affect the low-skilled agent more adversely than the high-skilled agent. To smooth those fluctuations, the government needs to accumulate more claims on the high skilled, implying a positive relationship between the steady state level of government assets and Pareto weight on high-skilled agents. Thus the planner substitutes the shortfall in revenues from taxes with higher earnings from his assets. Figure III plots how taxes and assets of the government vary as we change the Pareto weight α_1 on a high-skilled worker.

Correlation of net assets and productivities

For $I = 2$, proposition 5 signs the marginal utility adjusted net assets that we denoted x . This implies a particular ordering of net assets across the agents in the steady state. These implications generalize to settings with $I \geq 3$. In general, one can verify that without discount factor shocks (as in part 1 of proposition 5 when interest rates are countercyclical), steady state net asset levels (scaled by marginal utilities) are ordered inversely to productivities. By making net asset positions be negatively correlated with labor earnings, the planner can minimize the costs of fluctuating transfers. Further for similar

¹⁷The results of proposition 5 can be substantially generalized to an economy with no heterogeneity. In Bhandari et al. (2013), we study a representative agent economy with a more general incomplete market structure and distorting taxes. We show that for a wide range of preferences, a steady state exists, that it is globally stable, and that the sign of long run asset position of the government is determined by the co-movement of returns on assets and shocks.

reasons as in the $I = 2$ case, with exogenous variations in discount factors large enough to make interest rates be procyclical, the planner makes net asset positions be positively correlated with productivities. In a recession the government faces a short fall of tax revenues that it can make up from revenues of its assets. If interest rates are sufficiently lower, issuing debt implies a lower interest liability and frees up some resources to make up the low tax revenues. Furthermore, by borrowing a larger amount from the more productive agents the government can reduce the welfare costs from lowering transfers in adverse times. This implies a positive correlation between assets and productivities.

6.2 Stability

In this section, we return to the general section 2 formulation of the Ramsey problem to study convergence to a steady state. We begin by describing a test for local convergence cast in terms of a linear approximation of optimal policies at a steady state. We apply this test to show local stability of a steady state for a wide range of parameters. From these examples it emerges that slow convergence to the steady state occurs for commonly calibrated parameter values.

To study convergence, we return to the maximization problem (24) and assume that it admits a steady state. As before, let assume that $\Pr(s)\mu_i(s)$ and λ_i be the multipliers on constraints (25) and (26). In Appendix A.5 we show that the history-dependent optimal policies (they are sequences of functions of s^t) can be represented recursively in terms of $\{\mu(s^{t-1}), \rho(s^{t-1})\}$ and s_t . A recursive representation of an optimal policy can be linearized around the steady state using (μ, ρ) as state variables.¹⁸

Formally, let $\hat{\Psi}_t = \begin{bmatrix} \mu_t - \mu^{SS} \\ \rho_t - \rho^{SS} \end{bmatrix}$ be deviations from a steady state. From a linear approximation, one can obtain $B(s)$ such that

$$\hat{\Psi}_{t+1} = B(s_{t+1})\hat{\Psi}_t. \quad (29)$$

This linearized system has coefficients that are functions of the shock. The next proposition describes a simple numerical test that allows us to determine whether this linear system converges to zero in probability.

Proposition 6 *If the (real part) of eigenvalues of $EB(s)$ are less than 1, system (29) converges to zero in mean. Further for large t , the conditional variance of $\hat{\Psi}_t$, denoted by $\Sigma_{\hat{\Psi},t}$, follows a deterministic process governed by*

$$\text{vec}(\Sigma_{\hat{\Psi},t}) = \hat{B} \text{vec}(\Sigma_{\hat{\Psi},t-1}),$$

where \hat{B} is a square matrix of dimension $(2I - 2)^2$. In addition, if the (real part) of eigenvalues of \hat{B} are less than 1, the system converges in probability.

¹⁸One could in principle look for a solution in state variables $(\alpha(s^{t-1}), \rho(s^{t-1}))$. For $I = 2$ with $\{\theta_i(s)\}$ different across agents, this would give identical policies and a map which is (locally) invertible between α and μ for a given ρ . However in other cases, it turns out there are unique linear policies in (α, ρ) and not necessarily in (α, μ) . This comes from the fact that the set of feasible (α, ρ) are restricted at time 0 and may not contain an open set around the steady state values. When we linearize using (μ, ρ) as state variables, the optimal policies for $\alpha(s^t), \rho(s^t)$ converge to their steady state levels for all perturbations in (μ, ρ) .

The eigenvalues (in particular the largest or the dominant one) are instructive not only for whether the system is locally stable but also how quickly the steady state is reached. In particular, the half-life of convergence to the steady state is given by $\frac{\log(0.5)}{\|\lambda\|}$, where $\|\lambda\|$ is the absolute value of the dominant eigenvalue. Thus, the closer the dominant eigenvalue is to one, the slower is the speed of convergence.

We used proposition 6 to verify local stability of a wide range of examples. The typical finding is that the steady state is generically stable and that convergence is slow. In figure IV we plot the comparative statics for the dominant eigenvalue and the associated half-life for a two-agent economy with CES preferences. We set the other parameter to match a Frisch elasticity of 0.5, a real interest rate of 2%, marginal tax rates around 20%, and a 90-10 percentile ratio of wage earnings of 4. In the first exercise, we vary the size of the expenditure shock keeping risk aversion σ at one. The x -axis plots the spread in expenditure normalized by the undistorted GDP and reported in percentages. In the bottom panel, we fix the size of shock such that it produces a 5% fall in expenditure fall at risk aversion of one and vary σ from 0.8 to 7. We see that the dominant eigenvalue is everywhere less than one but very close to one, so that the steady state is stable but convergence is slow for reasonable values of curvatures and shocks. We return to this feature in section 7 where we study low frequency components of government debt. Both increasing the size of the shock or risk aversion increases the volatility of the interest rates, speeding up the transition towards the steady state.

6.3 More general shocks

The results on existence and convergence to a steady state relied on a special binary-IID restriction. When there are more than two possible values for the shocks or when shocks are persistent, the time-invariant steady state will no longer exist. Mathematically, this occurs because one asset and one risk-free rate of return cannot span all possible needs for government revenues. With richer shock structures, there exists an attraction region in the (x, ρ) space to which the dynamic system converges. Although (x, ρ) are no longer constant in such region, their fluctuations tend to be markedly reduced relative to the transient fluctuations that occur away from that region, and general properties of x and ρ are the same as those described in Proposition 5. Figure V shows long sample paths for economies hit by more general TFP shocks. The top panel has IID shocks with 2 (bold) and 3 (dotted) possible values and the bottom panel has persistent shocks with 2 (bold) and 3 (dotted) possible values.

7 Optimal policy in booms and recessions

In section 6 we used steady states to characterize the long-run behavior of optimal allocations and forces that guide the asymptotic level of net assets. In this section, we use a calibrated version of the economy to a) revisit the magnitude of these forces and b) study optimal policy responses at business cycle frequencies when the economy is possibly far away from the steady state. We choose shocks to match stylized facts about recent recessions in US.

We consider an economy with two types of agents of equal measures with preferences¹⁹

$$U(c, l) = \psi \ln c + (1 - \psi) \ln (1 - l).$$

The shock s takes two values, s_H and s_L , and follows a persistent process. We allow $\beta, \theta_i, i = 1, 2$, and g to be functions of s . We first pick $\bar{\theta}_i, \bar{g}$ and $\bar{\beta}$ for a deterministic economy without shocks and calibrate (ψ, α) to some low frequency data moments. Then to match some business cycle moments we pick shocks according to

$$\begin{aligned}\theta_i(s) &= \bar{\theta}_i[1 + \hat{\theta}_i(s)], \\ \beta(s) &= \bar{\beta}[1 + \hat{\beta}(s)], \\ g(s) &= \bar{g}[1 + \hat{g}(s)],\end{aligned}\tag{30}$$

where $\hat{\theta}_i(s) \in \{-e_{i,\theta}, e_{i,\theta}\}$, $\hat{\beta}(s) \in \{-e_\beta, e_\beta\}$, and $\hat{g}(s) \in \{-e_g, e_g\}$. Throughout our experiments, we normalize $b_{2,t} = 0$ for all $t \geq -1$. From market clearing, $B_t = -b_{1,t}$. We refer to B_t as government debt (when negative) and assets (when positive).

7.1 Calibration

We calibrate the model in two steps. We first choose baseline parameters that govern preferences and technology so that an optimal equilibrium for a no-shocks version of the economy matches some sample moments in post war US data. In the second step, we adjusted other parameters to make the amplitudes of fluctuations equal to average peak-trough spreads observed in the three most recent recessions (1991-92, 2001-02 and 2008-10).

We first discuss calibration of $(\psi, \alpha, \bar{\theta}_i, \bar{g}, \bar{\beta})$. Although these parameters jointly determine the relevant moments, it is helpful to explain which moment in the data mainly influences each parameter. We normalize $\bar{\theta}_2 = 1$ and pick $\bar{\theta}_1$ to match a log wage ratio of 90 wage percentile to 10 wage percentile of 4 from Autor et al. (2008). We set the discount factor $\bar{\beta}$ to match an (annual) interest rate of 2%. We set the parameter ψ to match a Frisch elasticity of labor supply equal to 0.5. In our model, \bar{g} corresponds to non-transfer government expenditures, which in the U.S. varied from 7% and 11% in the post WWII period and were above 20% during WWII. We set \bar{g} equal to 12% of GDP. Finally, we set Pareto weights α to match the average marginal tax rate in the US of about 20% as in Chari et al. (1994).²⁰

Next we turn to business cycle targets. We calibrate $\{e_{i,\theta}, e_\beta, \Pr(s|s_-)\}$ to match the following four facts about booms and recessions (using NBER dates, for the last 3 recessions, i.e., 1991-92, 2001-02 and 2008-10): the log of incomes individuals at both the 10th and the 90th percentile falls the recessions; 10th

¹⁹We restrict our attention to the economy with two agents for computational tractability. We want to understand both short-run and long-run responses to shocks. For some of our computations, it is important to allow our dynamic systems to travel over a large subset of state space, including regions encountered infrequently in the invariant distribution. With more agents, it seems possible to apply other methods, for example those of Judd et al. (2011), to study dynamics of our economy within its invariant distribution. We hope to pursue such extensions in future work.

²⁰We use federal government expenditures (excluding current transfers) since the labor tax rate of 20% in Chari et al. (1994) is calibrated to federal marginal taxes.

percentile income falls by more than 90th percentile; an inflation-adjusted interest rate on government debt is generally lower in recessions; and booms last longer than recessions. We calibrate the average spread in labor productivity to match the average 3% loss in output observed in the last three recessions. The inequality shock is designed to match facts documented in Guvenen et al. (2012) that the fall in earnings of the 10-percentile is about 2.5 times of 90-percentile. The discount factor shocks match the average boom-recession difference of about 1.96% in the real risk-free interest rate (3 month T bill rate - inflation rate) seen in the last three recessions.²¹ We calibrate the transition matrix to match the average duration of booms and recessions. For comparison, we also report optimal responses to a drop in government expenditure that leads to an output drop of similar magnitude.

Note that because each is an exact function of s_t , government expenditures, the discount factor, and productivities are perfectly correlated: a recession is an episode in which TFP falls, inequality rises, and the discount factor is high. We set initial government debt to be 60% of GDP, roughly to match the ratio of federal debt held by public at the beginning of 2010.

Table I summarizes some details about our calibration.

7.2 Outcomes

We discuss separately long run and short run implications for optimal policy. In particular, we study an economy with the calibration discussed above (“Benchmark”) and a few variants that successively turn off particular sources of variation.

1. **Acyclical interest rates:** In the first variant, we recalibrate the discount factor shocks to make the risk-free rate be uncorrelated with output.
2. **Countercyclical interest rates:** Here we shut off discount factor shocks by setting $\hat{\beta}(s) = 0$ in (30). Note that this makes interest rates countercyclical.
3. **No inequality:** This variant modifies the “Benchmark” by setting $\hat{\beta}(s) = 0$ and $\hat{\theta}_1(s) = \hat{\theta}_2(s) = 3\%$ in (30). This corresponds to a case in which the only source of business cycle fluctuations is a TFP shock that affects all agents equally. This case more closely matches the experiments in the RBC literature such as Chari et al. (1994).
4. **Government expenditure shocks:** The last variant compares optimal responses to shocks to government expenditures. In this experiment, we set $\hat{\theta}(s) = \hat{\beta}(s) = 0$ and choose $\hat{g}(s)$ to produce a drop in output of a similar magnitude to that in the first three experiments. This compares to the studies of responses to government shocks by AMSS and Faraglia et al. (2012).

²¹It has long been noticed that the standard RBC model predicts counter-factual negative correlation between real interest rates and output (e.g. Bokdrin et al. (2001)). In the data HP filtered output is roughly uncorrelated with real interest rates, but this relationship turns positive if we look at peak vs troughs. We report the optimal responses for both economies with positive and zero correlation of interest rates and output and contrast with a response to a pure TFP shock.

Long run

Figure VI plots government debt. All experiments start with an initial government debt to GDP ratio of 60%. Several features emerge from this figure.

As expected from the analysis in Section 6, in all four economies the state (x, ρ) converges to some long run ergodic set, while government debt and the tax rate converge to associated sets. When there are no discount factor shocks (See lines with \diamond , \square in figure VI) or small discount factor shocks that produce acyclical interest rates (see the line with $+$ in figure VI) the government holds claims on the private sector when the state is in the ergodic set. Consistent with Proposition 5 the optimal policy adjusts net asset positions to ameliorate the two key constraints impinging on the government policy, namely, the inability to award agent-specific transfers (the restriction to *affine taxes*) and the absence of state-contingent assets (the restriction to *risk-free debt*). Starting from a point when the relative assets of the low-skilled agent (or the government if we use a normalization that sets $b_{2,t} = 0$) are low, extracting resources through lower transfers exacerbates inequality. This is costly since the government has to use higher taxes in future to redistribute. On the margin, the optimal policy requires the government (or low-skill agent) to accumulate assets. But interest rate fluctuations interact with net asset positions to generate state-contingent earnings from assets. If interest rate are high when the government needs additional revenues, accumulating assets relaxes the restriction imposed by absence of state contingent assets. Thus, with countercyclical interest rates, these forces reinforce each other, making the government's long run asset position become positive.

In data, however, interest rates generally decline in recessions. Procylical interest rates mean that the two forces outlined in the previous paragraph now oppose each other. For large enough interest rate fluctuations, this means that the government may want to accumulate debt. In figure VI, the line with (o) represents the benchmark with discount factor shocks rigged to replicate procylical fluctuations in interest rates. For a particular initial condition for government debt, the planner can refrain from varying debt for a very long time.²²

Convergence to the ergodic region is very slow. With persistent shocks and an initial 60% debt-GDP ratio, it takes about 3,000 years for the government to want to pay off all that debt and then start accumulating assets. With discount factor shocks, it takes even longer to repay the debt. It is still indebted after 5000 years. These outcomes confirm the comparative statics of the eigenvalues of linearized system in proposition 6.

Thus, the covariance of interest rates with fundamentals as emphasized in proposition 5 substantially influences the ergodic distribution of government assets.

²²Like the finding in proposition 5 for large discount factor shocks (in a way that interest rates are procylical) there exist regions where x_t, ρ_t have low volatility and the government is *not* accumulating assets. But these regions are typically unstable. The two forces highlighted before that guide accumulation of assets now work in opposite direction and the net effect depends on the relative strengths. In particular the sample paths from different initial conditions $\bar{b}_{2,-1}$ (which would imply different choices for the initial x_0, ρ_0) may display larger fluctuations in assets. However, at the calibrated initial conditions (60% debt-gdp ratio), the uncertainty associated with the mean path is very low for the first 5000 periods.

Short run

The analysis of the previous subsection studied aspects of very low frequency components of the optimal policy. Here we focus on business cycle frequencies. In our setting, these higher frequency responses can conveniently be divided into the magnitudes of changes as we switch from “boom” to “recession,” and the dynamics during periods when a recession or boom state persists.

We set the exogenous state s_0 so that we are at the outset of a recession. Then we solve the time 0 problem with identical initial conditions across different settings. This pins down the initial state vector x_0, ρ_0 that appears in our time 0 Bellman equation. We then use the policy rules to compute fluctuations of different components in the government budget constraint across states. These responses are summarized in Table II. For each variable z in the table we report in the form $\Delta z \equiv (z(s_l|x_0, \rho_0, s_0) - z(s_h|x_0, \rho_0, s_0)) / \bar{Y}$ where \bar{Y} is average undistorted GDP.²³

The source of shocks is very important. Three different types of shocks that produce similar drops in GDP have very different consequences for optimal policies, both qualitatively and quantitatively. In the benchmark, the government responds to an adverse shock by making big increases in transfers, the tax rate, and government debt. However, without inequality shocks (row 4), the government responds by decreasing transfers and increasing both debt and the tax rate, but by an amount an order of magnitude smaller than in the benchmark. This indicates that ignoring distributional goals can produce a misleading prescriptions for government policy in recessions.

Discount factor shocks have minor effects on impact and matter more for transient dynamics that ultimately have big long run effects. Figures VII and VIII show how the transient dynamics for prolonged booms (or recessions) differ with and without discount factor shocks. The four panels show tax rates, transfers, debt and interest rate movements for a path of 25 years. The bold lines in figures VII and VIII refer to the benchmark (with procyclical interest rates) and the version with acyclical interest rates, respectively. The dotted line in both the figures is the version with countercyclical interest rates. The shaded regions are periods with low output. We see that in a prolonged booms, the government accumulates assets and that it lowers the tax rate when there are no discount factor shocks.

8 Concluding remarks

The spring of 2013 witnessed a heated debate in newspapers and economic magazines about the accuracy and meaning of empirical correlations between output growth rates and ratios of government debt to GDP inferred from data sets assembled by Reinhart and Rogoff (2010). From the perspectives of our paper and of Werning (2007), those correlations and those debates are difficult to interpret because in our settings, total government debt is not a relevant state variable that affects allocations, government

²³Note that predetermined variables like repayment on existing debt drop out of the accounting and we have

$$\Delta[g] + \Delta[T] + \Delta[B] = \Delta[\tau\theta_1 l_1] + \Delta[\tau\theta_2 l_2]$$

transfers, or tax rates. The principal message of our paper is that without exogenous restrictions on transfers, the level of government debt doesn't matter. What does matter is how government debt is distributed among people relative to society's attitudes toward unequal allocations of consumption and work. Using a recursive representation that employ correct state variables — a vector of marginal utility adjusted net asset positions and a vector of pairwise ratios of marginal utilities — we have presented a sequence of examples designed to show how agents' net positions affect optimal government policies for choosing distorting tax rates, transfers, and government issues or holdings of risk-free bonds. We find that a significant determinant of an optimal asymptotic government debt or government debt-GDP ratio is how interest rate risks are correlated with risks to fundamentals that threaten to widen or narrow inequality in after-tax and after-transfer incomes. To interpret those Reinhart-Rogoff facts country-by-country, we would want to know much more about the distributions of net assets across people across countries and how they interact with risks to interest rates and to fundamental sources of inequality.

A Appendix

A.1 Additional details for Section 3.1

In this section we construct an example in which the government can achieve higher welfare in the economy with ad-hoc borrowing limits. We restrict ourselves to a deterministic economy with $g_t = 0$, $\beta_t = \beta$ and $I = 2$. Further the utility function over consumption and labor supply $U(c, l)$ is separable in the arguments and satisfies the Inada conditions. The planners problem can then be written as the following sequence problem

$$\max_{\{c_{i,t}, l_{i,t}, b_{i,t}, R_t\}_t} \sum_{t=0}^{\infty} \beta^t [\alpha_1 U(c_{1,t}, l_{1,t}) + \alpha_2 U(c_{2,t}, l_{2,t})] \quad (31)$$

subject to

$$c_{2,t} + \frac{U_{l2,t} l_{2,t}}{U_{c2,t}} - \left(c_{1,t} + \frac{U_{l1,t} l_{1,t}}{U_{c1,t}} \right) + \frac{1}{R_t} (b_{2,t} - b_{1,t}) = b_{2,t-1} - b_{1,t-1} \quad (32a)$$

$$\frac{U_{l1,t}}{\theta_1 U_{c1,t}} = \frac{U_{l2,t}}{\theta_2 U_{c2,t}} \quad (32b)$$

$$c_{1,t} + c_{2,t} \leq \theta_1 l_{1,t} + \theta_2 l_{2,t} \quad (32c)$$

$$\left(\frac{U_{c1,t}}{U_{c,t+1}} - \beta R_t \right) (b_{1,t} - \underline{b}_i) = 0 \quad (32d)$$

$$\frac{U_{c1,t}}{U_{c,t+1}} \geq \beta R_t \quad (32e)$$

$$b_{1,t} \geq \underline{b}_i \quad (32f)$$

Where \underline{b}_i is the exogenous borrowing constraint for agent i . We obtain equation (32a) by eliminating transfers from the budget equations of the households and using the optimality for labor supply decision. Equations (32d) and (32e) capture the inter-temporal optimality conditions modified for possibly binding constraints.

Let c_i^{fb} and l_i^{fb} be the allocation that solves the first best problem, that is maximizing equation (31) subject to (32c), and define

$$Z^{fb} = c_2^{fb} + \frac{U_{l2}^{fb} l_2^{fb}}{U_{c2}^{fb}} - \left(c_1^{fb} + \frac{U_{l1}^{fb} l_1^{fb}}{U_{c1}^{fb}} \right) \quad (33)$$

and

$$\tilde{b}_2^{fb} = \frac{Z^{fb}}{\frac{1}{\beta} - 1} \quad (34)$$

We will assume that the exogenous borrowing constraints satisfy $b_2 = \underline{b}_1 + \tilde{b}_2^{fb}$. We then have the following lemma

Lemma 2 *If $\tilde{b}_2^{fb} > (<)0$ and $b_{2,-1} - b_{1,-1} > (<)\tilde{b}_2^{fb}$ then the planner can implement the first best.*

Proof. We will consider the candidate allocation where $c_{i,t} = c_i^{fb}$, $l_{i,t} = l_i^{fb}$, $b_{i,t} = \underline{b}_i$ and interest rates are given by $R_t = \frac{1}{\beta}$ for $t \geq 1$. It should be clear then that equations (32b) and (32c) are satisfied as a property of the first best allocation. Equation (32d) is trivially satisfied since the agents are at their

borrowing constraints. For $t \geq 1$ equations (32a) and (32e) are both satisfied by the choice of $R_t = \frac{1}{\beta}$ and the first best allocations. It remains to check that equation (32a) is satisfied at time $t = 0$ for an interest rate $R_0 < \frac{1}{\beta}$. At time zero the constraint is give by

$$Z^{fb} + \frac{1}{R_0} \tilde{b}_2^{fb} = b_{2,-1} - b_{1,-1} \quad (35)$$

The assumption that $b_{2,-1} - b_{1,-1} > (<) \tilde{b}_2^{fb}$ if $\tilde{b}_2^{fb} > (<) 0$ then implies that

$$R_0 = \frac{\tilde{b}_2^{fb}}{b_{2,-1} - b_{1,-1} - Z^{fb}} < \frac{1}{\beta}$$

as desired. ■

This will improve upon the planners problem without exogenous borrowing constraints, as first best can only be achieved in this scenario when $b_{2,-1} - b_{1,-1} = \tilde{b}_2^{fb}$.

A.2 Proof of Proposition 4

We prove a slight more general version of our result. Consider an infinite horizon, incomplete markets economy in which an agent maximizes utility function $U : \mathbb{R}_+^n \rightarrow \mathbb{R}$ subject to an infinite sequence of budget constraints. We assume that U is concave and differentiable. Let $\mathbf{a}(s^t)$ be a vector of n goods and let $\mathbf{p}(s^t)$ be a price vector in state s^t with $p_i(s^t)$ denoting the price of good i . We use a normalization $p_1(s^t) = 1$ for all s^t . There is a risk-free bond.

Let $b(s^t)$ be the agent's bond holdings, and let $\mathbf{e}(s^t)$ be a stochastic vector of endowments.

Consumer maximization problem

$$\max_{\mathbf{a}_t, b_t} \sum_{t=0}^{\infty} [\Pi_{j=0}^{t-1} \beta(s_j)] \Pr(s^t) U(\mathbf{a}(s^t)) \quad (36)$$

subject to

$$\mathbf{p}(s^t) \mathbf{a}(s^t) + q(s^t)b(s^t) = \mathbf{p}(s^t) \mathbf{e}(s^t) + b(s^{t-1}) \quad (37)$$

and $\{b(s^t)\}$ is bounded and $\{q(s^t)\}$ is the price of the risk-free bond.

The Euler conditions are

$$\begin{aligned} U_a(s^t) &= U_1(s^t) \mathbf{p}(s^t) \\ \Pr(s^t) U_1(s^t) q(s^t) &= \beta(s_t) \sum_{s^{t+1} > s^t} \Pr(s^{t+1}) U_1(s^{t+1}). \end{aligned} \quad (38)$$

Lemma 3 Consider an allocation $\{\mathbf{a}_t, b_t\}$ that satisfies (37), (38) and $\{b_t\}_t$ is bounded. Then $\{\mathbf{a}_t, b_t\}$ is a solution to (36).

Proof. The proof follows closely Constantinides and Duffie (1996). Suppose there is another budget feasible allocation $\mathbf{a} + \mathbf{h}$ that maximizes (36). Since U is strictly concave,

$$\begin{aligned} &\mathbb{E}_0 \sum_{t=0}^{\infty} [\Pi_{j=0}^{t-1} \beta(s_j)] U(\mathbf{a}_t + \mathbf{h}_t) - \mathbb{E}_0 \sum_{t=0}^{\infty} [\Pi_{j=0}^{t-1} \beta(s_j)] U(\mathbf{a}_t) \\ &\leq \mathbb{E}_0 \sum_{t=0}^{\infty} [\Pi_{j=0}^{t-1} \beta(s_j)] \mathbf{U}_a(\mathbf{a}_t) \mathbf{h}_t \end{aligned} \quad (39)$$

To attain $\mathbf{a} + \mathbf{h}$, the agent must deviate by φ_t from his original portfolio b_t such that $\{\varphi_t\}_t$ is bounded, $\varphi_{-1} = 0$ and

$$\mathbf{p}(s^t)\mathbf{h}(s^t) = \varphi(s^{t-1}) - q(s^t)\varphi(s^t)$$

Multiply by $[\Pi_{j=0}^{t-1}\beta(s_j)] \Pr(s^t) U_1(s^t)$ to get:

$$\begin{aligned} [\Pi_{j=0}^{t-1}\beta(s_j)] \Pr(s^t) U_1(s^t) \mathbf{p}(s^t) \mathbf{h}(s^t) &= [\Pi_{j=0}^{t-1}\beta(s_j)] \Pr(s^t) U_1(s^t) \varphi(s^{t-1}) - q(s^t) [\Pi_{j=0}^{t-1}\beta(s_j)] \Pr(s^t) U_1(s^t) \varphi(s^t) \\ &= [\Pi_{j=0}^{t-1}\beta(s_j)] \Pr(s^t) U_1(s^t) \varphi(s^{t-1}) - [\Pi_{j=0}^{t-1}\beta(s_j)] \beta(s_t) \sum_{s^{t+1} > s^t} \Pr(s^{t+1}) U_1(s^{t+1}) \varphi(s^{t+1}) \end{aligned}$$

where we used the second part of (38) in the second equality. Sum over the first T periods (pathwise) and use the first part of (38) to eliminate $\mathbf{U}_a(\mathbf{a}_t) = U_1(s^t) \mathbf{p}(s^t)$

$$\sum_{t=0}^T [\Pi_{j=0}^{t-1}\beta(s_j)] \Pr(s^t) \mathbf{U}_a(\mathbf{a}_t) \mathbf{h}(s^t) = -[\Pi_{j=0}^T \beta(s_j)] \sum_{s^{T+1} > s^T} \Pr(s^{T+1}) U_1(s^{T+1}) \varphi(s^T).$$

Since $\{\varphi_t\}_t$ is bounded there must exist $\bar{\varphi}$ s.t. $|\varphi_t| \leq \bar{\varphi}$ for all t . By Theorem 5.2 of Magill and Quinzi (1994), this equilibrium with debt constraints implies a transversality condition on the right hand side of the last equation, so by transitivity we have

$$\lim_{T \rightarrow \infty} \sum_{t=0}^T [\Pi_{j=0}^{t-1}\beta(s_j)] \Pr(s^t) \mathbf{U}_a(\mathbf{a}_t) \mathbf{h}(s^t) = 0.$$

Substitute this into (39) to show that \mathbf{h} does not improve utility of consumer. ■

A.3 Additional details for Lemma 1

Given $P(s)\mu_i(s)$ and λ_i be the multipliers on constraints (25) and (26). The first order conditions are then as follows

$x'_i(s) :$

$$\beta(s)V_{x_i}(\mathbf{x}'(s), \boldsymbol{\rho}'(s)) - \beta(s)\mu_i(s) = 0 \quad (40)$$

$\tau(s) :$

$$\begin{aligned} U\tau(\tau(s), \boldsymbol{\rho}'(s), s) + \sum_i \left(\frac{x_i U_{c\tau}^i(\tau(s), \boldsymbol{\rho}'(s), s)}{E U_c^i(\tau, \boldsymbol{\rho}', s)} \left[\mu_i(s) - \frac{E\mu_i(s)U_c^i(\tau, \boldsymbol{\rho}', s)}{E U_c^i(\tau, \boldsymbol{\rho}', s)} \right] - \mu_i(s)Z_{i,\tau}(\tau(s), \boldsymbol{\rho}'(s), s) \right) \\ + \sum_i [\lambda_i U_{c,\tau}^i(\tau(s), \boldsymbol{\rho}'(s), s)(\rho'_i(s) - \rho_i)] = 0 \end{aligned} \quad (41)$$

$\rho_i(s) :$

$$\begin{aligned} U_{\rho_i}(\tau(s), \boldsymbol{\rho}'(s), s) + \sum_j \left(\frac{x_j u_{c,1\rho_i}(\tau(s), \boldsymbol{\rho}'(s), s)}{E u_{c,1}(\tau, \boldsymbol{\rho}', s)} \left[\mu_j(s) - \frac{E\mu_j(s)u_{c,1}(\tau, \boldsymbol{\rho}', s)}{E u_{c,1}(\tau, \boldsymbol{\rho}', s)} \right] - \mu_j(s)Z_{j,\rho_i}(\tau(s), \boldsymbol{\rho}'(s), s) \right) \\ + \sum_j [\lambda_j u_{c,j\rho_i}(\tau(s), \boldsymbol{\rho}'(s), s)(\rho'_j(s) - \rho_j)] + \lambda_i u_{c,i}(\tau(s), \boldsymbol{\rho}'(s), s) + \beta(s)V_{\rho_i}(\mathbf{x}'(s), \boldsymbol{\rho}'(s)) = 0. \end{aligned} \quad (42)$$

Finally the envelope conditions are

$$V_{x_i}(\boldsymbol{x}, \boldsymbol{\rho}) = \frac{\mathbb{E}\mu_i(s)U_c^i(\tau, \boldsymbol{\rho}', s)}{\mathbb{E}U_c^i(\tau, \boldsymbol{\rho}', s)}, \quad (43)$$

and

$$V_{\rho_i}(\boldsymbol{x}, \boldsymbol{\rho}) = -\lambda_i \mathbb{E}U_c^i(\tau, \boldsymbol{\rho}', s). \quad (44)$$

The equations for the steady state can then be obtained by imposing $x'_i(s) = x_i$ and $\rho'_i(s) = \rho_i$. It is then readily noted that equations (40) and (43) are satisfied when $\mu_i(s) = \mu_i = \beta V_{x_i}(\boldsymbol{x}, \boldsymbol{\rho})$. Further equation (26) drops out and the equation (25) simplifies to

$$Z_i(\tau(s), \boldsymbol{\rho}, s) + \beta(s)x'_i = \frac{x_i U_c^i(\tau(s), \boldsymbol{\rho}, s)}{\mathbb{E}U_c^i(\tau, \boldsymbol{\rho})}.$$

Imposing the steady state restrictions, equations (41) and (42) reduce to,

$$U_\tau(\tau(s), \boldsymbol{\rho}, s) - \sum_i \mu_i Z_{i,\tau}(\tau(s), \boldsymbol{\rho}, s) = 0 \quad (45a)$$

$$U_{\rho_i}(\tau(s), \boldsymbol{\rho}, s) - \sum_j \mu_j Z_{j,\rho_i}(\tau(s), \boldsymbol{\rho}, s) + \lambda_i U_c^i(\tau(s), \boldsymbol{\rho}', s) - \lambda_i \beta(s) \mathbb{E}U_c^i(\tau, \boldsymbol{\rho}) = 0. \quad (45b)$$

A.4 Proof of Proposition 5

The Bellman equation for the optimal planners problem with log quadratic preferences and IID shocks can be written as

$$V(x, \rho) = \max_{c_1, c_2, l_1, x', \rho'} \sum_s \Pr(s) \left[\alpha_1 \left(\log c_1(s) - \frac{l_1(s)^2}{2} \right) + \alpha_2 \log c_2(s) + \beta(s) V(x'(s), \rho'(s)) \right]$$

subject to the constraints

$$1 + \rho'(s)[l_1(s)^2 - 1] + \beta(s)x'(s) - \frac{x \frac{1}{c_2(s)}}{\mathbb{E}[\frac{1}{c_2}]} = 0 \quad (46)$$

$$\sum_s \frac{\Pr(s)}{c_1(s)} (\rho'(s) - \rho) = 0 \quad (47)$$

$$\theta_1(s)l_1(s) - c_1(s) - c_2(s) - g = 0 \quad (48)$$

$$\rho'(s)c_2(s) - c_1(s) = 0 \quad (49)$$

where the $\Pr(s)$ is the probability distribution of the aggregate state s . If we let $\Pr(s)\mu(s)$, λ , $\Pr(s)\xi(s)$ and $\Pr(s)\phi(s)$ be the Lagrange multipliers for the constraints (46)-(49) respectively then we obtain the following FONC for the planners problem ²⁴

$c_1(s) :$

$$\frac{\alpha_1 \Pr(s)}{c_1(s)} - \frac{\lambda \Pr(s)}{c_1(s)^2} (\rho'(s) - \rho) - \Pr(s)\xi(s) - \Pr(s)\phi(s) = 0 \quad (50)$$

²⁴Appendix A.5 discusses the associated second order conditions that ensure these policies are optimal

$$c_2(s) :$$

$$\frac{\alpha_2 \Pr(s)}{c_2(s)} + \frac{x \Pr(s)}{c_2(s)^2 \mathbb{E}[\frac{1}{c_2}]} \left[\mu(s) - \frac{\mathbb{E}[\mu \frac{1}{c_2}]}{\mathbb{E}[\frac{1}{c_2}]} \right] - \Pr(s)\xi(s) + \Pr(s)\rho'(s)\phi(s) = 0 \quad (51)$$

$$l_1(s) :$$

$$-\alpha_1 \Pr(s)l_1(s) + 2\mu(s) \Pr(s)\rho'(s)l_1(s) + \theta_1(s) \Pr(s)\xi(s) = 0 \quad (52)$$

$$x'(s) :$$

$$\beta(s) \Pr(s)V_x(x'(s), \rho'(s)) + \beta(s) \Pr(s)\mu(s) = 0 \quad (53)$$

$$\rho'(s) :$$

$$\beta(s) \Pr(s)V_\rho(x'(s), \rho'(s)) + \frac{\lambda \Pr(s)}{c_1(s)} + \mu(s) \Pr(s)[l_1(s)^2 - 1] + \Pr(s)\phi(s)c_2(s) = 0 \quad (54)$$

In addition there are two envelope conditions given by

$$V_x(x, \rho) = - \sum_{s'} \frac{\mu(s') \Pr(s') \frac{1}{c_2(s')}}{\mathbb{E}[\frac{1}{c_2}]} = - \frac{\mathbb{E}[\mu \frac{1}{c_2}]}{\mathbb{E}[\frac{1}{c_2}]} \quad (55)$$

$$V_\rho(x, \rho) = -\lambda \mathbb{E}[\frac{1}{c_1}] \quad (56)$$

In the steady state, we need to solve for a collection of allocations, initial conditions and Lagrange multipliers $\{c_1(s), c_2(s), l_1(s), x, \rho, \mu(s), \lambda, \xi(s), \phi(s)\}$ such that equations (46)-(56) are satisfied when $\rho'(s) = \rho$ and $x'(s) = x$. It should be clear that if we replace $\mu(s) = \mu$, equation (53) and the envelope condition with respect to x is always satisfied. Additionally under this assumption equation (51) simplifies significantly, since

$$\frac{x \Pr(s)}{c_2(s)^2 \mathbb{E}[\frac{1}{c_2}]} \left[\mu(s) - \frac{\mathbb{E}[\mu \frac{1}{c_2}]}{\mathbb{E}[\frac{1}{c_2}]} \right] = 0$$

The first order conditions for a steady can then be written simply as

$$1 + \rho[l_1(s)^2 - 1] + \beta(s)x - \frac{x}{c_2(s) \mathbb{E}[\frac{1}{c_2}]} = 0 \quad (57)$$

$$\theta_1(s)l_1(s) - c_1(s) - c_2(s) - g = 0 \quad (58)$$

$$\rho c_2(s) - c_1(s) = 0 \quad (59)$$

$$\frac{\alpha_1}{c_1(s)} - \xi(s) - \phi(s) = 0 \quad (60)$$

$$\frac{\alpha_2}{c_2(s)} - \xi(s) + \rho\phi(s) = 0 \quad (61)$$

$$[2\mu\rho - \alpha_1]l_1(s) + \theta_1(s)\xi(s) = 0 \quad (62)$$

$$\lambda \left[\frac{1}{c_1(s)} - \beta(s) \mathbb{E}[\frac{1}{c_1}] \right] + \mu[l_1(s)^2 - 1] + \phi(s)c_2(s) = 0 \quad (63)$$

We can rewrite equation (60) as

$$\frac{\alpha_1}{c_2(s)} - \rho\xi(s) - \rho\phi(s) = 0$$

by substituting $c_1(s) = \rho c_2(s)$. Adding this to equation (61) and normalizing $\alpha_1 + \alpha_2 = 1$ we obtain

$$\xi(s) = \frac{1}{(1+\rho)c_2(s)} \quad (64)$$

which we can use to solve for $\phi(s)$ as

$$\phi(s) = \frac{\alpha_1 - \rho\alpha_2}{(\rho(1+\rho))c_2(s)} \quad (65)$$

From equation (57) we can solve for $l_1(s)^2 - 1$ as

$$l_1(s)^2 - 1 = \frac{x}{\rho E[\frac{1}{c_2}]} \left(\frac{1}{c_2(s)} - \beta(s)E[\frac{1}{c_2}] \right) - \frac{1}{\rho}$$

This can be used along with equations (63) and (65) to obtain

$$\left(\frac{\lambda}{\rho} + \frac{\mu x}{\rho E[\frac{1}{c_2}]} \right) \left(\frac{1}{c_2(s)} - \beta(s)E[\frac{1}{c_2}] \right) = \frac{\mu}{\rho} + \frac{\rho\alpha_2 - \alpha_1}{\rho(1+\rho)}$$

Note that the LHS depends on s while the RHS does not, hence the solution to this equation is

$$\lambda = -\frac{\mu x}{E[\frac{1}{c_2}]} \quad (66)$$

and

$$\mu = \frac{\alpha_1 - \rho\alpha_2}{1+\rho} \quad (67)$$

Combining these with equation (62) we quickly obtain that

$$\left[2\rho \frac{\alpha_1 - \rho\alpha_2}{1+\rho} - \alpha_1 \right] l_1(s) + \frac{\theta_1(s)}{(1+\rho)c_2(s)} = 0$$

Then solving for $l_1(s)$ gives

$$l_1(s) = \frac{\theta_1(s)}{(\alpha_1(1-\rho) + 2\rho^2\alpha_2)c_2(s)}$$

Remark 1 Note that the labor tax rate is given by $1 - \frac{\alpha_1(s)l_1(s)}{\theta_1(s)}$. The previous expression shows that labor taxes are constant at the steady state. This property holds generally for CES preferences separable in consumption and leisure

This we can plug into the aggregate resource constraint (58) to obtain

$$l_1(s) = \left(\frac{1+\rho}{\alpha_1(1-\rho) + 2\rho^2\alpha_2} \right) \frac{1}{l_1(s)} + \frac{g}{\theta_1(s)}$$

letting $C(\rho) = \frac{1+\rho}{\alpha_1(1-\rho) + 2\rho^2\alpha_2}$ we can then solve for $l_1(s)$ as

$$l_1(s) = \frac{g \pm \sqrt{g^2 + 4C(\rho)\theta_1(s)^2}}{2\theta_1(s)}$$

The marginal utility of agent 2 is then

$$\frac{1}{c_2(s)} = \left(\frac{1+\rho}{C(\rho)} \right) \left(\frac{g \pm \sqrt{g^2 + 4C(\rho)\theta_1(s)^2}}{2\theta_1(s)^2} \right)$$

Note that in order for either of these terms to be positive we need $C(\rho) \geq 0$ implying that there is only one economically meaningful root. Thus

$$l_1(s) = \frac{g + \sqrt{g^2 + 4C(\rho)\theta_1(s)^2}}{2\theta_1(s)} \quad (68)$$

and

$$\frac{1}{c_2(s)} = \left(\frac{1+\rho}{C(\rho)} \right) \left(\frac{g + \sqrt{g^2 + 4C(\rho)\theta_1(s)^2}}{2\theta_1(s)^2} \right) \quad (69)$$

A steady state is then a value of ρ such that

$$x(s) = \frac{1 + \rho[l_1(\rho, s)^2 - 1]}{\frac{1/c_2(\rho, s)}{E[\frac{1}{c_2}](\rho)} - \beta(s)} \quad (70)$$

s independent of s .

The following lemma, which orders consumption and labor across states, will be useful in proving the parts of proposition 5. As a notational aside we will often use $\theta_{1,l}$ and $\theta_{1,h}$ to refer to $\theta_1(s_l)$ and $\theta_1(s_h)$ respectively. Where s_l refers to the low TFP state and s_h refers to the high TFP state.

Lemma 4 Suppose that $\theta_1(s_l) < \theta_2(s_h)$ and ρ such that $C(\rho) > 0$ then

$$l_{1,l} = \frac{g + \sqrt{g^2 + 4C(\rho)\theta_{1,l}^2}}{2\theta_{1,l}} > \frac{g + \sqrt{g^2 + 4C(\rho)\theta_{1,h}^2}}{2\theta_{1,h}} = l_{1,h}$$

and

$$\frac{1}{c_{2,l}} = \frac{1+\rho}{C(\rho)} \frac{g + \sqrt{g^2 + 4C(\rho)\theta_{1,l}^2}}{2\theta_{1,l}^2} > \frac{1+\rho}{C(\rho)} \frac{g + \sqrt{g^2 + 4C(\rho)\theta_{1,h}^2}}{2\theta_{1,h}^2} = \frac{1}{c_{2,h}}$$

Proof. The results should follow directly from showing that the function

$$l_1(\theta) = \frac{g + \sqrt{g^2 + 4C(\rho)\theta^2}}{2\theta}$$

is decreasing in θ . Taking the derivative with respect to θ

$$\begin{aligned} \frac{dl_1}{d\theta}(\theta) &= -\frac{g}{2\theta^2} - \frac{\sqrt{g^2 + 4C(\rho)\theta^2}}{2\theta^2} + \frac{4C(\rho)\theta}{2\theta\sqrt{g^2 + 4C(\rho)\theta^2}} \\ &= -\frac{g}{2\theta^2} - \frac{g + 4C(\rho)\theta^2 - 4C(\rho)\theta^2}{2\theta^2\sqrt{g^2 + 4C(\rho)\theta^2}} \\ &= -\frac{g}{2\theta^2} - \frac{g}{2\theta^2\sqrt{g^2 + 4C(\rho)\theta^2}} < 0 \end{aligned}$$

That $\frac{1}{c_{2,l}} > \frac{1}{c_{2,h}}$ follows directly. ■

Proof of Proposition 5.

Part 1. In order for there to exist a ρ such that equation (70) is independent of the state (and hence have a steady state) we need the existence of root for the following function

$$f(\rho) = \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} - \frac{\frac{1/c_2(\rho, s_h)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta}{\frac{1/c_2(\rho, s_l)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta}$$

From lemma 4 we can conclude that

$$1 + \rho[l_1(\rho, s_l)^2 - 1] > 1 + \rho[l_1(\rho, s_h)^2 - 1] \quad (71)$$

and

$$\frac{1/c_2(\rho, s_l)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta > \frac{1/c_2(\rho, s_h)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta \quad (72)$$

for all $\rho > 0$ such that $C(\rho) \geq 0$. To begin with we will define $\underline{\rho}$ such that $C(\rho) > 0$ for all $\rho > \underline{\rho}$. Note that we will have to deal with two different cases.

$\alpha_1(1 - \rho) + 2\rho^2\alpha_2 > 0$ **for all** $\rho \geq 0$: In this case we know that $C(\rho) \geq 0$ for all ρ and is bounded above and thus we will let $\underline{\rho} = 0$.

$\alpha_1(1 - \rho) + 2\rho^2\alpha_2 = 0$ **for some** $\rho > 0$: In this case let $\underline{\rho}$ be the largest positive root of $\alpha_1(1 - \rho) + 2\rho^2\alpha_2$. Note that $\lim_{\rho \rightarrow \underline{\rho}^+} C(\rho) = \infty$

With this we note that²⁵

$$\lim_{\rho \rightarrow \underline{\rho}^+} \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} = 1$$

We can also show that

$$\lim_{\rho \rightarrow \underline{\rho}^+} \frac{\frac{1/c_2(\rho, s_h)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta}{\frac{1/c_2(\rho, s_l)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta} < 1$$

which implies that $\lim_{\rho \rightarrow \underline{\rho}^+} f(\rho) > 0$.

Taking the limit as $\rho \rightarrow \infty$ we see that $C(\rho) \rightarrow 0$, given that $\frac{\theta}{\theta(s)} < 1$, we can then conclude that

$$\lim_{\rho \rightarrow \infty} 1 + \rho[l_1(\rho, s)^2 - 1] = -\infty$$

Thus, there exists $\bar{\rho}$ such that $1 + \bar{\rho}[l_1(\bar{\rho}, s_l)^2 - 1] = 0$.²⁶ From equation (71), we know that

$$0 = 1 + \bar{\rho}[l_1(\bar{\rho}, s_l)^2 - 1] > 1 + \bar{\rho}[l_1(\bar{\rho}, s_h)^2 - 1]$$

which implies in the limit

$$\lim_{\rho \rightarrow \bar{\rho}^-} \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} = -\infty$$

²⁵In the first case $\underline{\rho} = 0$ and in the second case $l_1(\rho, s_l) = l_1(\rho, s_h)$ as $\rho \rightarrow \underline{\rho}^+$

²⁶This can be seen from the fact $\lim_{\rho \rightarrow \underline{\rho}^+} 1 + \rho[l_1(\rho, s_l)^2 - 1] > 0$ and $\lim_{\rho \rightarrow \infty} 1 + \rho[l_1(\rho, s_l)^2 - 1] > -\infty$, thus $\bar{\rho}$ exists in $(\underline{\rho}, \infty)$

which along with

$$\frac{\frac{1/c_2(\rho, s_h)}{\mathbb{E}[\frac{1}{c_2}]} - \beta}{\frac{1/c_2(\rho, s_l)}{\mathbb{E}[\frac{1}{c_2}]} - \beta} \geq -1$$

allows us to conclude that $\lim_{\rho \rightarrow \bar{\rho}^-} f(\rho) = -\infty$. The intermediate value theorem then implies that there exists ρ_{SS} such that $f(\rho_{SS}) = 0$ and hence that ρ_{SS} is a steady state.

Finally, as $\rho_{SS} < \bar{\rho}$ we know that

$$1 + \rho_{SS}[l_1(\rho_{SS}, s_l) - 1] > 0$$

as $\frac{1/c_2(\rho, s_l)}{\mathbb{E}[\frac{1}{c_2}]} > 1$ we can conclude

$$x_{SS} = \frac{1 + \rho_{SS}[l_1(\rho_{SS}, s_l) - 1]}{\frac{1/c_2(\rho, s_l)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta} > 0$$

implying that the government will hold assets in the steady state (under the normalization that agent 2 holds no assets).

Part 2. The condition that $R(s_h) = R(s_l)$ implies that

$$\frac{1/c_2(\rho, s_l)}{\beta(s_l)\mathbb{E}[\frac{1}{c_2}]} = \frac{1/c_2(\rho, s_h)}{\beta(s_h)\mathbb{E}[\frac{1}{c_2}]}$$

which simplifies to

$$\frac{\beta(s_h)}{\beta(s_l)} = \frac{1/c_2(\rho, s_h)}{1/c_2(\rho, s_l)} \quad (73)$$

In order for a steady state to exist with constant interest rates there must be a root of the following function

$$\begin{aligned} f(\rho) &= \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} - \frac{\frac{1/c_2(\rho, s_h)}{\mathbb{E}[\frac{1}{c_2}]} - \beta(s_h)}{\frac{1/c_2(\rho, s_l)}{\mathbb{E}[\frac{1}{c_2}]} - \beta(s_l)} \\ &= \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} - \frac{\frac{1/c_2(\rho, s_h)}{\beta(s_h)\mathbb{E}[\frac{1}{c_2}]} - 1}{\frac{1/c_2(\rho, s_l)}{\beta(s_l)\mathbb{E}[\frac{1}{c_2}]} - 1} \frac{\beta(s_h)}{\beta(s_l)} \\ &= \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} - \frac{1/c_2(\rho, s_h)}{1/c_2(\rho, s_l)} \end{aligned}$$

Taking limits of $f(\rho)$ as ρ approaches $\underline{\rho}$ from the positive side we already demonstrated

$$\lim_{\rho \rightarrow \underline{\rho}^+} \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} = 1$$

From equation (69) and Lemma 4 it is straightforward to see that

$$\lim_{\rho \rightarrow \underline{\rho}^+} \frac{1/c_2(\rho, s_h)}{1/c_2(\rho, s_l)} < 1$$

which allows us to conclude that

$$\lim_{\rho \rightarrow \underline{\rho}^+} f(\rho) > 0$$

Taking limits as ρ approaches $\bar{\rho}$ from the negative direction we know that

$$\lim_{\rho \rightarrow \bar{\rho}^-} \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} = -\infty$$

As $\frac{1/c_2(\rho, s_h)}{1/c_2(\rho, s_l)} > 0$ for all ρ it is straightforward to conclude that

$$\lim_{\rho \rightarrow \bar{\rho}^-} f(\rho) = -\infty$$

Continuity then implies the existence of a ρ^{SS} such that $f(\rho^{SS}) = 0$, and thus there exists a $\beta(s_l)$ and $\beta(s_h)$ such that $R(s_l) = R(s_h)$ in steady state. From Lemma 4

$$l(\rho, s_l) > l(\rho, s_h).$$

In order for

$$\frac{1 + \rho^{SS}[l_1(\rho^{SS}, s_h)^2 - 1]}{1 + \rho^{SS}[l_1(\rho^{SS}, s_l)^2 - 1]} = \frac{1/c_2(\rho^{SS}, s_h)}{1/c_2(\rho^{SS}, s_l)} < 1$$

it is necessary that

$$1 + \rho^{SS}[l_1(\rho^{SS}, s_l)^2 - 1] > 1 + \rho^{SS}[l_1(\rho^{SS}, s_h)^2 - 1] > 0$$

implying that the steady state asset level

$$x_{SS} = \frac{1 + \rho_{SS}[l_1(\rho_{SS}, s_l) - 1]}{\frac{1/c_2(\rho_{SS}, s_l)}{\mathbb{E}[\frac{1}{c_2}]} - \beta(s_l)} > 0$$

Part 3 As noted before, since $g/\theta(s) < 1$ for all s we have

$$\lim_{\rho \rightarrow \infty} 1 + \rho[l_1(\rho, s)^2 - 1] = -\infty$$

Thus, there exists ρ_{SS} such that

$$0 > 1 + \rho_{SS}[l_1(\rho_{SS}, s_l)^2 - 1] > 1 + \rho_{SS}[l_1(\rho_{SS}, s_h)^2 - 1]$$

It is then possible to choose $\beta(s) < \frac{1/c_2(\rho_{SS}, s)}{\mathbb{E}[\frac{1}{c_2}]}$ such that

$$1 > \frac{1 + \rho_{SS}[l_1(\rho_{SS}, s_l)^2 - 1]}{1 + \rho_{SS}[l_1(\rho_{SS}, s_h)^2 - 1]} = \frac{\frac{1/c_2(\rho_{SS}, s_l)}{\mathbb{E}[\frac{1}{c_2}]} - \beta(s_l)}{\frac{1/c_2(\rho_{SS}, s_h)}{\mathbb{E}[\frac{1}{c_2}]} - \beta(s_h)} \quad (74)$$

Implying that for discount factor shocks $\beta(s)$, ρ_{SS} is a steady state level for the ratio of marginal utilities, with steady state marginal utility weighted government debt

$$x_{SS} = \frac{1 + \rho_{SS}[l_1(\rho_{SS}, s_l)^2 - 1]}{\frac{1/c_2(\rho_{SS}, s_l)}{\mathbb{E}[\frac{1}{c_2}]} - \beta(s_l)} < 0$$

Thus, in the steady state, the government is holding debt, under the normalization that the unproductive worker holds no assets. As $\frac{1/c_2(\rho, s_l)}{\mathbb{E}[\frac{1}{c_2}]} > \frac{1/c_2(\rho, s_h)}{\mathbb{E}[\frac{1}{c_2}]}$, in order for equation (74) to hold we need $\beta_l > \beta_h$. We can then rewrite equation (74) as

$$1 > \frac{\beta(s_h)}{\beta(s_l)} > \frac{\frac{1/c_2(\rho_{SS}, s_l)}{\beta_l \mathbb{E}[\frac{1}{c_2}]} - 1}{\frac{1/c_2(\rho_{SS}, s_h)}{\beta(s_h) \mathbb{E}[\frac{1}{c_2}]} - 1}$$

Thus

$$R(s_l) \frac{1/c_2(\rho_{SS}, s_l)}{\beta(s_l) \mathbb{E}[\frac{1}{c_2}]} < \frac{1/c_2(\rho_{SS}, s_h)}{\beta(s_h) \mathbb{E}[\frac{1}{c_2}]} = R(s_h) \quad (75)$$

in the steady state interest rates are positively correlated with TFP.

■

A.5 Linearization Algorithm

This section will outline our numerical methods used to solve for and linearize around the steady state in the case of a 2 state iid process for the aggregate state.

$$V(\mathbf{x}, \boldsymbol{\rho}) = \max_{c_i(s), l_i(s), \mathbf{x}'(s), \boldsymbol{\rho}'(s)} \sum_s P(s) \left(\left[\sum_i \pi_i \alpha_i U(c_i(s), l_i(s)) \right] + \beta(s) V(\mathbf{x}'(s), \boldsymbol{\rho}'(s)) \right) \quad (76)$$

$$U_{c,i}(s)c_i(s) + U_{l,i}(s)l_i(s) - \rho'_i(s) [U_{c,1}(s)c_1(s) + U_{l,1}(s)l_1(s)] + \beta(s)x'_i(s) = \frac{x_i U_{c,i}(s)}{\mathbb{E} U_{c,i}} \quad (77a)$$

$$\sum_s \Pr(s) U_{c,1}(s) (\rho_i(s) - \rho_i) = 0 \quad (77b)$$

$$\frac{\rho'(s)}{\theta_1(s)} U_{l,1}(s) = \frac{1}{\theta_1(s)} U_{l,i}(s) \quad (77c)$$

$$\sum_{j=0}^I \pi_j c_j(s) + g(s) = \sum_{j=0}^I \pi_j \theta_j(s) l_j(s) \quad (77d)$$

$$U_{c,i}(s) = \rho'_i(s) U_{c,1}(s) \quad (77e)$$

For $i = 2, \dots, I$. Note that some of the constraints have been modified a little for ease of differentiation. Associated with these constraints we have the Lagrange multipliers $\Pr(s)\mu'_i(s)$, $\lambda_i \Pr(s)\phi_i(s)$, $\Pr(s)\xi(s)$, and $P(s)\zeta_i(s)$.

The first order conditions with respect to the choice variables are as follows (note we will be using the notation $\mathbb{E} z$ to represent $\sum_s \Pr(s)z(s)$ for some variable z)

$c_1(s)$:

$$\begin{aligned} \pi_1 \alpha_1 U_{c,1}(s) + \sum_{i=2}^I (\mu'_i(s) \rho'_i(s)) [U_{c,1}(s)c_1(s) + U_{c,i}(s)] \\ + \lambda U_{cc,1}(s) \sum_{i=2}^I (\rho'_i(s) - \rho_1) - \pi_1 \xi(s) + \sum_{i=2}^N \zeta_i(s) \rho'_i(s) U_{cc,1}(s) = 0 \end{aligned} \quad (78a)$$

$c_i(s)$: for $i \geq 2$

$$\pi_i \alpha_i U_{c,i}(s) - \mu'_i(s) [U_{cc,i}(s)c_i(s) + U_{c,i}(s)] + \frac{x_i U_{cc,i}(s)}{\mathbb{E} U_{c,i}} \left(\mu'_i(s) - \frac{\mathbb{E} \mu'_i U_{c,i}}{\mathbb{E} U_{c,i}} \right) - \pi_i \xi(s) - \zeta_i(s) U_{cc,i}(s) = 0 \quad (78b)$$

$l_1(s)$:

$$\pi_1 \alpha_1 U_{l,1}(s) + \sum_{i=2}^I \mu'_i(s) \rho_i(s) [U_{ll,1}(s) l_1(s) + U_{l,1}(s)] - \sum_{i=2}^N \frac{\rho'_i(s) \phi_i(s)}{\theta_i(s)} U_{ll,1}(s) + \pi_1 \theta_1(s) \xi(s) = 0 \quad (78c)$$

$l_2(s)$:

$$\pi_i \alpha_i U_{l,i}(s) - \mu'_i(s) [U_{ll,i}(s) l_i(s) + U_{l,i}(s)] + \frac{\phi_i(s)}{\theta_i(s)} U_{ll,i}(s) + \pi_i \theta_i(s) \xi(s) = 0 \quad (78d)$$

$\rho'_i(s)$:

$$\beta(s) V_{\rho_i}(\mathbf{x}'(s), \boldsymbol{\rho}'(s)) + \mu'_i(s) [U_{c,1}(s)c_1(s) + U_{l,1}(s)l_1(s)] + \lambda_i U_{c,1}(s) - \phi_i(s) \frac{U_{l,1}(s)}{\theta_i(s)} + U_{c,1}(s) \zeta_i(s) = 0 \quad (78e)$$

$x'_i(s)$:

$$V_{x_i}(\mathbf{x}'(s), \boldsymbol{\rho}'(s)) - \mu'_i(s) = 0. \quad (78f)$$

Equations (77a)-(77e) and (78a)-(78e) then define the necessary conditions for an interior maximization of the planners problem for the state $(\mathbf{x}, \boldsymbol{\rho})$. In addition to these we have the two envelop conditions

$$V_{x_i}(\mathbf{x}, \boldsymbol{\rho}) = \frac{\sum_s P(s) \mu'_i(s) U_{c,i}(s)}{\mathbb{E} U_{c,i}(s)} = \frac{\mathbb{E} \mu'_i U_{c,i}}{\mathbb{E} U_{c,i}}, \quad (79a)$$

and

$$V_{\rho_i}(\mathbf{x}, \boldsymbol{\rho}) = -\lambda_i \mathbb{E} U_{c,1}. \quad (79b)$$

In order to check local stability we linearize locally around the steady state. Furthermore we find that the policy functions have better numerical properties when the state variables are chosen to be $(\boldsymbol{\mu}, \boldsymbol{\rho})$ rather than $(\mathbf{x}, \boldsymbol{\rho})$, and thus, we will proceed with the linearization procedure using $(\boldsymbol{\mu}, \boldsymbol{\rho})$ as the endogenous state vector. The evolution of the state variable $\boldsymbol{\mu}$ must follow the weighted martingale

$$\mu_i - \frac{\sum_s P(s) \mu'_i(s) U_{c,i}(s)}{\sum_s P(s) U_{c,i}(s)} = 0. \quad (80)$$

The optimal policy function, which we will denote as $z(\boldsymbol{\mu}, \boldsymbol{\rho})$, must satisfy $F(z, y, g(z)) = 0$ where F represents the system of equations (77a)-(78e) and (80), y is the state vector $(\mathbf{x}, \boldsymbol{\rho})$, and g is the mapping of the policies into functions of future variables, namely $\mathbf{x}'(s)$ and $V_{\rho}(\boldsymbol{\mu}'(s), \boldsymbol{\rho}'(s))$. In other words

$$g(z) = \begin{pmatrix} \mathbf{x}(\boldsymbol{\mu}'(1), \boldsymbol{\rho}'(1)) \\ V_{\rho}(\boldsymbol{\mu}'(1), \boldsymbol{\rho}'(1)) \\ \mathbf{x}(\boldsymbol{\mu}'(2), \boldsymbol{\rho}'(2)) \\ V_{\rho}(\boldsymbol{\mu}'(2), \boldsymbol{\rho}'(2)) \end{pmatrix}.$$

Finally $z(\mu, \rho)$ are the stacked variables $\{c_1(s), c_1(s), l_1(s), l_1(s), \mathbf{x}, \rho'(s), \mu'(s), \lambda, \phi(s), \xi(s), \zeta(s)\}$. The optimal policy function is then a function $z(y)$ that satisfies the relationship $F(z(y), y, g(z(y))) = 0$. Taking total derivatives around the steady state \bar{y} and $\bar{z} = z(\bar{y})$

$$D_z F(\bar{z}, \bar{y}, g(\bar{z})) D_y z(\bar{y}) + D_y F(\bar{z}, \bar{y}, g(\bar{z})) + D_g F(\bar{z}, \bar{y}, g(\bar{z})) Dg(\bar{z}) D_y z(\bar{z}) = 0$$

In order to linearize $z(y)$ around the steady state \bar{y} we need to compute $D_y z(\bar{y})$. The envelope condition (79b) tell us that V_ρ can be computed from the optimal policies, i.e.

$$\begin{pmatrix} \mathbf{x}(\mu, \rho) \\ V_\rho(\mu, \rho) \end{pmatrix} = w(z(\mu, \rho)) = \begin{pmatrix} \mathbf{x} \\ -\lambda \mathbb{E}[U_{c,1}] \end{pmatrix}$$

If we let Φ_s be the matrix that maps $z(\mu, \rho)$ into $\begin{pmatrix} \mu'(s) \\ \rho'(s) \end{pmatrix}$ then we can write $g(\mu, \rho)$ using z and w as follows

$$g(z) = \begin{pmatrix} w(z(\Phi_1 z)) \\ w(z(\Phi_2 z)) \end{pmatrix}$$

taking derivatives we quickly obtain that

$$\begin{aligned} D_z g(\bar{z}) &= \begin{pmatrix} Dw(z(\Phi_1 \bar{z})) & 0 \\ 0 & Dw(z(\Phi_2 \bar{z})) \end{pmatrix} \begin{pmatrix} D_y z(\Phi_1 \bar{z}) & 0 \\ 0 & D_y z(\Phi_2 \bar{z}) \end{pmatrix} \underbrace{\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}}_{\Phi} \\ &= \begin{pmatrix} Dw(\bar{z}) & 0 \\ 0 & Dw(\bar{z}) \end{pmatrix} \begin{pmatrix} D_y z(\bar{y}) & 0 \\ 0 & D_y z(\bar{y}) \end{pmatrix} \Phi \\ &= \begin{pmatrix} Dw(\bar{z}) D_y z(\bar{y}) & 0 \\ 0 & Dw(\bar{z}) D_y z(\bar{y}) \end{pmatrix} \Phi \end{aligned}$$

We can then go back to our original matrix equation to obtain

$$D_z F(\bar{z}, \bar{y}, \bar{w}) D_y z(\bar{y}) + D_y F(\bar{z}, \bar{y}, \bar{w}) + D_w F(\bar{z}, \bar{y}, \bar{w}) \begin{pmatrix} Dw(\bar{z}) D_y z(\bar{y}) & 0 \\ 0 & Dw(\bar{z}) D_y z(\bar{y}) \end{pmatrix} \Phi D_y z(\bar{z}) = 0, \quad (81)$$

where $\bar{w} = g(\bar{z}) = w(\bar{z})$. This is now a non-linear matrix equation for $D_y z(\bar{y})$, where all the other terms can be computed using the steady state values \bar{z} and \bar{y} (note $g(\bar{z})$ is known from the envelope conditions at the steady state). Furthermore, $D_y z(\bar{y})$ gives us the linearization of the policy rules since to first order

$$z \approx \bar{z} + D_y z(\bar{y})(y - \bar{y})$$

Our procedure for computing the linearization proceeds as follows

1. Find the steady state by solving the system of equations (27). Numerically, we have found that this is very robust to the parameters of the model.
2. Compute $D_z F(\bar{z}, \bar{y}, g(\bar{z}))$, $D_y F(\bar{z}, \bar{y}, g(\bar{z}))$ and $D_w F(\bar{z}, \bar{y}, g(\bar{z}))$ by numerically differentiating F . This is straightforward using auto-differentiation.
3. Compute $Dw(\bar{z})$ using auto-differentiation.

4. Construct a matrix equation as follows. Given policies $A = Dw(\bar{z})D_y z(\bar{y})$ (these are the linearized policies of \boldsymbol{x} and V_ρ with respect to (μ, ρ)), it is possible to solve for $D_y z(\bar{y})$ from

$$D_y z(\bar{z}) = - \left(D_z F(\bar{z}, \bar{y}, \bar{w}) + D_w F(\bar{z}, \bar{y}, \bar{w}) \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \Phi \right)^{-1} D_y F(\bar{z}, \bar{y}, \bar{w})$$

We wish to find an A such that

$$A = Dw(\bar{z})D_y z(\bar{z})$$

Given the linearized policy rules it is then possible to evaluate the local stability of the steady state. We find that in the absence of discount factor shocks the steady state is stable generically across the parameter space.

This linearization can be used to construct the bordered hessian of the problem (24) at the steady state. We can then apply second order tests to verify that the first order necessary conditions are sufficient.

A.6 Proof for Proposition 6

Proof.

The state at time t can be written as

$$\hat{\Psi}_t = B_t B_{t-1} \cdots B_1 \hat{\Psi}_0.$$

where the B_i are all random variables being $B(s)$ with probability $\Pr(s)$. Taking expectations and applying independence we then obtain

$$\mathbb{E}_0[\hat{\Psi}_t] = \mathbb{E}_0[B_t B_{t-1} \cdots B_1] \hat{\Psi}_0 \quad (82)$$

$$= \mathbb{E}[B_t] \mathbb{E}[B_{t-1}] \cdots \mathbb{E}[B_1] \hat{\Psi}_0 \quad (83)$$

$$= \bar{B}^t \hat{\Psi}_0 \quad (84)$$

where $\bar{B} = \mathbb{E}B(s)$. If eigenvalues of \bar{B} are positive and strictly less than 1, at least, in expectation the linearized system converges that is

$$\bar{\hat{\Psi}}_{t|0} \equiv \mathbb{E}_0[\hat{\Psi}_t] = \bar{B}^t \hat{\Psi}_0 \rightarrow \mathbf{0}. \quad (85)$$

It should be noted that the conditional expectation actually captures a significant portion of the linearized dynamics. The remaining question is does the distribution converge to $\mathbf{0}$. This can be done by analyzing the variance. Let

$$\Sigma_{\Psi,t|0} = \mathbb{E}_0 \left[(\hat{\Psi}_t - \bar{\hat{\Psi}}_t)(\hat{\Psi}_{t|0} - \bar{\hat{\Psi}}_{t|0})' \right]$$

or

$$\Sigma_{\Psi,t|0} = \mathbb{E}_0 \hat{\Psi}_t \hat{\Psi}'_t - \bar{\hat{\Psi}}_{t|0} \bar{\hat{\Psi}}'_{t|0}. \quad (86)$$

Note that if eigenvalues of \bar{B} are positive and strictly less than 1, $\bar{\hat{\Psi}}_{t|0}$ converges to 0. Using the independence of $\hat{\Psi}_{t-1}$ and B_t , and $\hat{\Psi}_t = B_t \hat{\Psi}_{t-1}$, we quickly obtain that for large t

$$\Sigma_{\Psi,t|0} \approx \mathbb{E}[B\Sigma_{\Psi,t-1|0}B'] \quad (87)$$

Showing that $\hat{\Psi}_{t|0} \rightarrow \mathbf{0}$ in distribution, amounts to showing that $\Sigma_{\Psi,t|0} \rightarrow 0$ for any starting point Σ_Ψ and following the process in equation (87). One can obtain a necessary condition for $\|\Sigma_{\Psi,t|0}\| \rightarrow 0$ under the process in equation (87). That process can be rewritten as follows

$$\Sigma_{\Psi,t|0} = \mathbb{E}[B\Sigma_{\Psi,t-1|0}B'] \quad (88)$$

$$= \sum_s \Pr(s)B(s)\Sigma_{\Psi,t-1|0}B(s)' \quad (89)$$

$$= \sum_s \Pr(s)(\bar{B} + (B(s) - \bar{B}))\Sigma_{\Psi,t-1|0}(\bar{B} + (B(s) - \bar{B}))' \quad (90)$$

$$= \bar{B}\Sigma_{\Psi,t-1|0}\bar{B}' + \sum_s \Pr(s)(B(s) - \bar{B})\Sigma_{\Psi,t-1|0}(B(s) - \bar{B})'. \quad (91)$$

This is a deterministic linear system in $\Sigma_{\Psi,t|0}$. Suppose we reshape $\Sigma_{\Psi,t|0}$ as a vector (denoted by $\text{vec}(\Sigma_{\Psi,t|0})$) and let \hat{B} be a (square) matrix such that equation 91 is written as

$$\text{vec}(\Sigma_{\Psi,t|0}) = \hat{B}\text{vec}(\Sigma_{\Psi,t-1|0}).$$

The stability of this system is guaranteed if the (real part) of eigenvalues of \hat{B} are less than 1. ■

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Parameter	Value	Description	Target
ψ	0.6994	Frisch elasticity of labor supply	0.5
θ_1	4	Log 90-10 wage ratio (Autor et all)	4
$\bar{\theta}_2$	1	Normalize to 1	1
β	0.98	Average (annual) risk free interest rate	2%
α_1	0.69	Marginal tax rate in the economy with no shocks	20%
g	12%	Average pre-transfer expenditure- output ratio	12 %
$\frac{\theta_2}{\theta_1}$	2.5	Relative drop in wage income of 10th percentile as compared to 90th percentile	2.5
θ_1	1.2%	Average output loss	3%
$\hat{\beta}(s)$	1.96%	Difference in real interest rates between booms and recession	1.96%
$P(r r)$	0.63	Duration of recessions	2.33 years
$P(b b)$	0.84	Duration of booms	7 years

Table I: Benchmark calibration

	Δg	ΔB	ΔT	$\Delta[\tau\theta_1l_1]$	$\Delta[\tau\theta_2l_2]$	ΔY	$\Delta\tau$
Benchmark	0.0000	-1.1561	0.6871	-0.1593	-0.3096	-2.8536	0.3732
Acyclical Interest Rates	0.0000	-1.1126	0.6591	-0.1497	-0.3038	-2.8613	0.3879
Countercyclical Interest Rates	0.0000	-1.0794	0.6387	-0.1415	-0.2992	-2.8677	0.3997
No Inequality	0.0000	-0.1380	-0.5459	-0.5635	-0.1204	-2.6294	0.0622
Expenditure Shocks	-7.5037	2.9137	2.8612	-1.3759	-0.3530	-2.3443	-1.1598

Table II: The tables summarizes the changes in the different components of the government budget as we transit from “boom” to a “recession”. All numbers are normalized by un-distorted GDP except τ and reported in percentages.

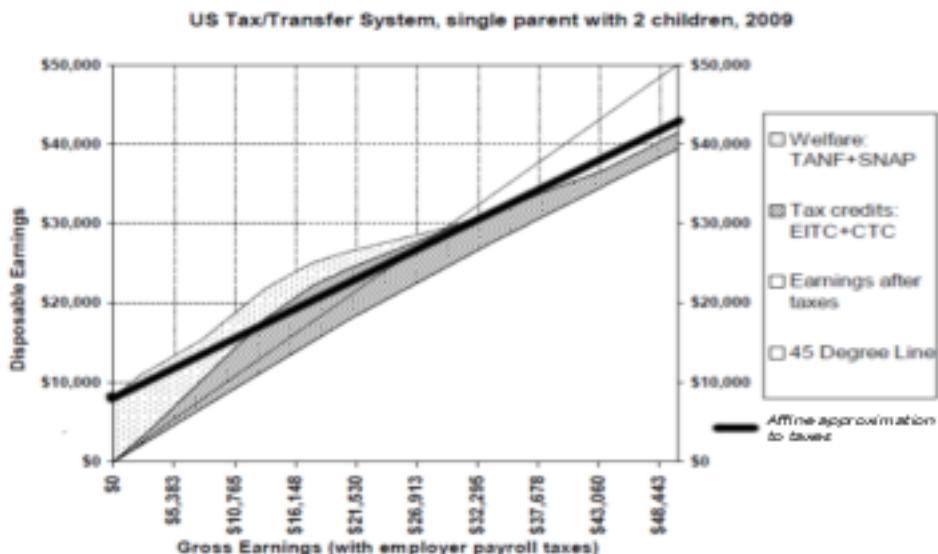


Figure I: The U.S. tax-transfer system is poorly approximated by a linear function, better by an affine function.

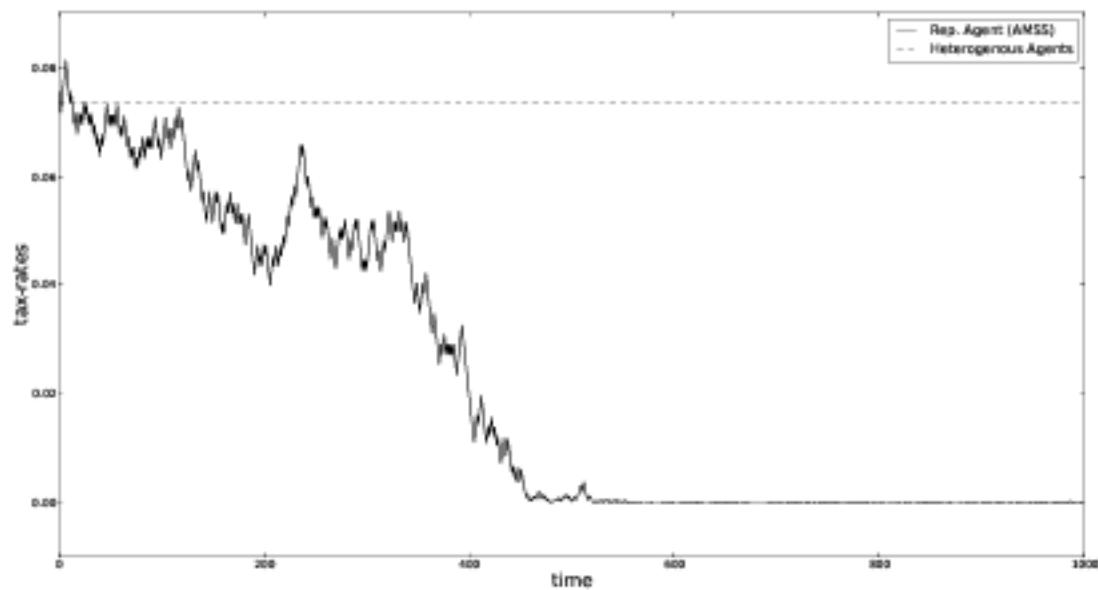


Figure II: Taxes in AMSS (solid line) and heterogeneous agent economy (dotted line) with quasi-linear preferences.

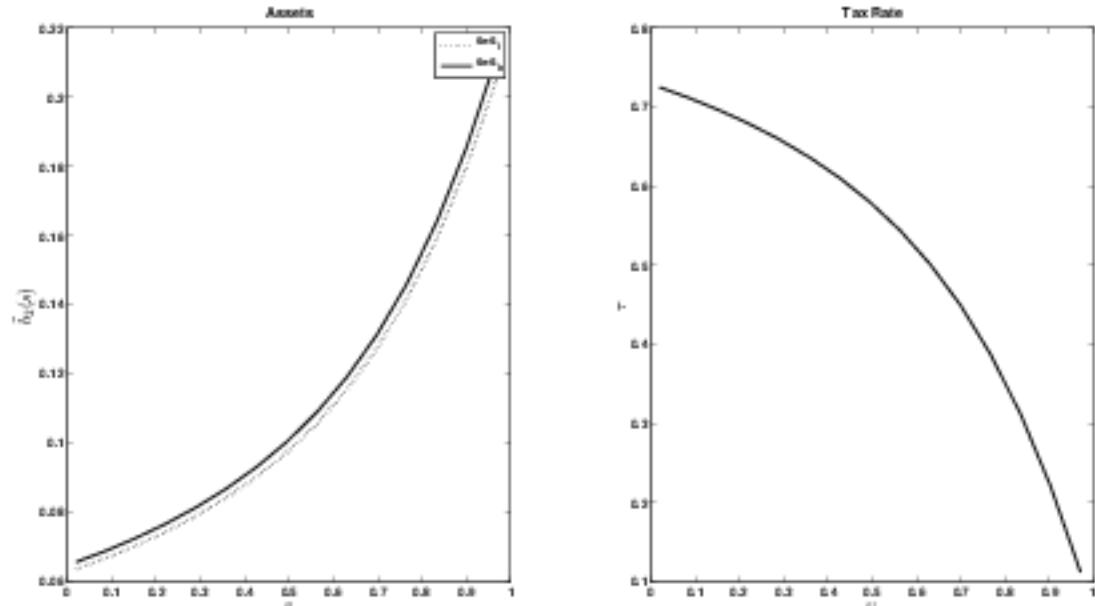


Figure III: Steady state assets : $\tilde{b}_2(s) = \frac{\beta x^{\text{SS}}}{U_c^2(s)}$ and taxes : τ^{SS} as a function of Agent 1's (high skilled) Pareto weight

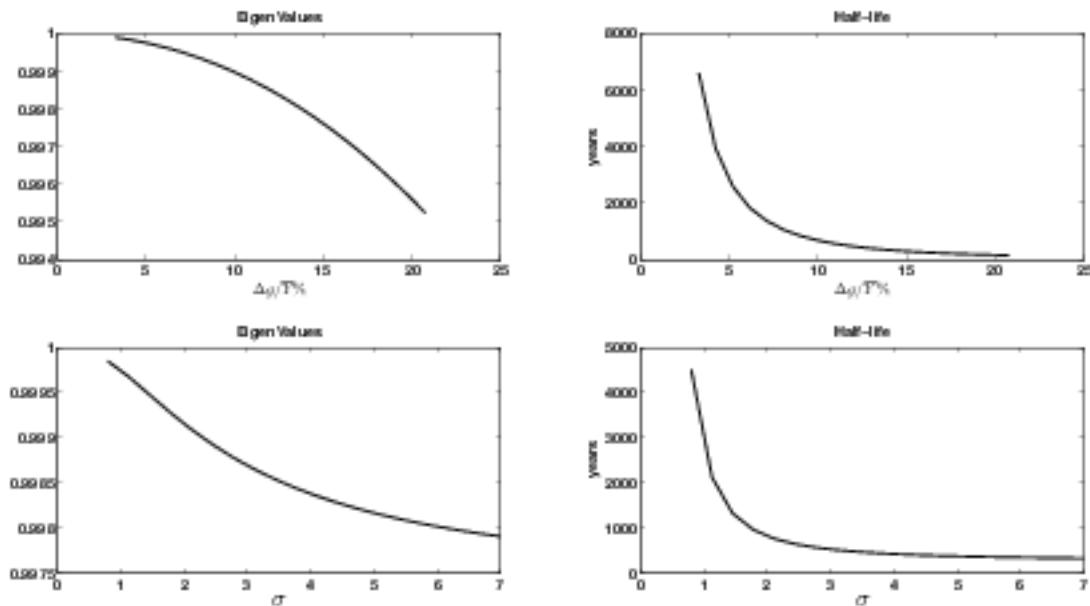


Figure IV: The top (bottom) panel plots the dominant eigenvalue of \hat{B} and the associated half life as we increase the spread between the expenditure levels (risk aversion).

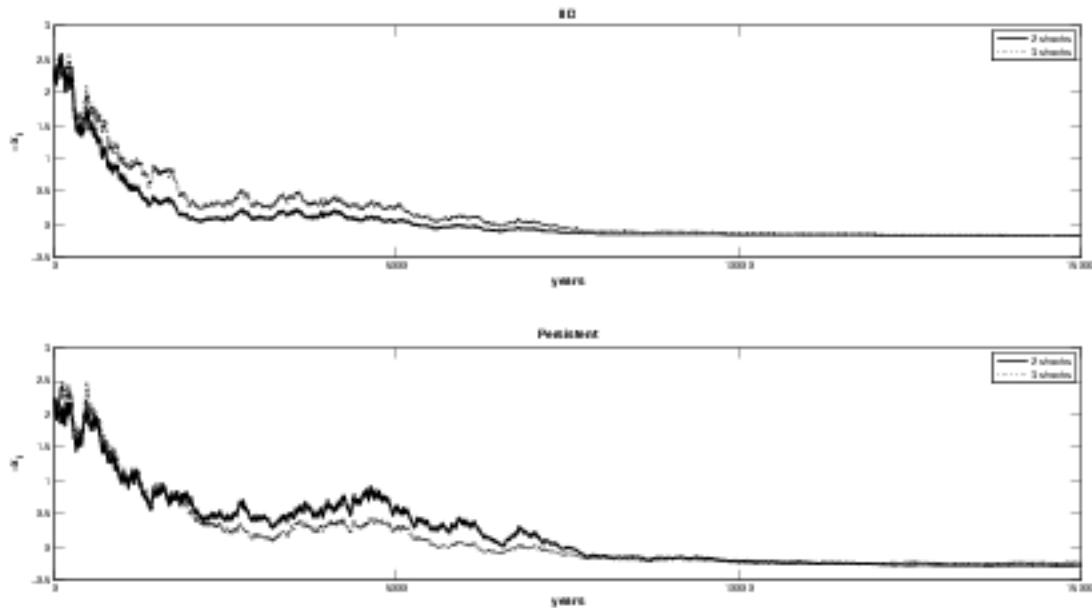


Figure V: The figure depicts sample paths of marginal utility adjusted debt of the government i.e $-x_t$. The top panel has IID shocks with 2 (bold) and 3 (dotted) possible values and the bottom panel has persistent shocks with 2 (bold) and 3 (dotted) possible values

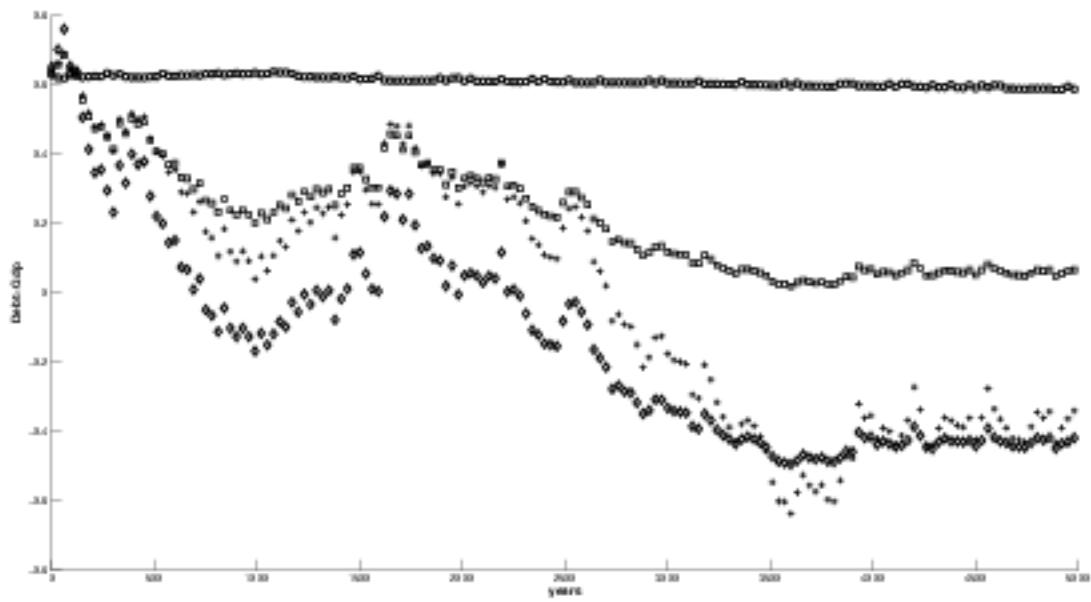


Figure VI: Debt benchmark (\circ), acyclical interest rates (+), countercyclical interest rates (\diamond) and no inequality shocks (\square)

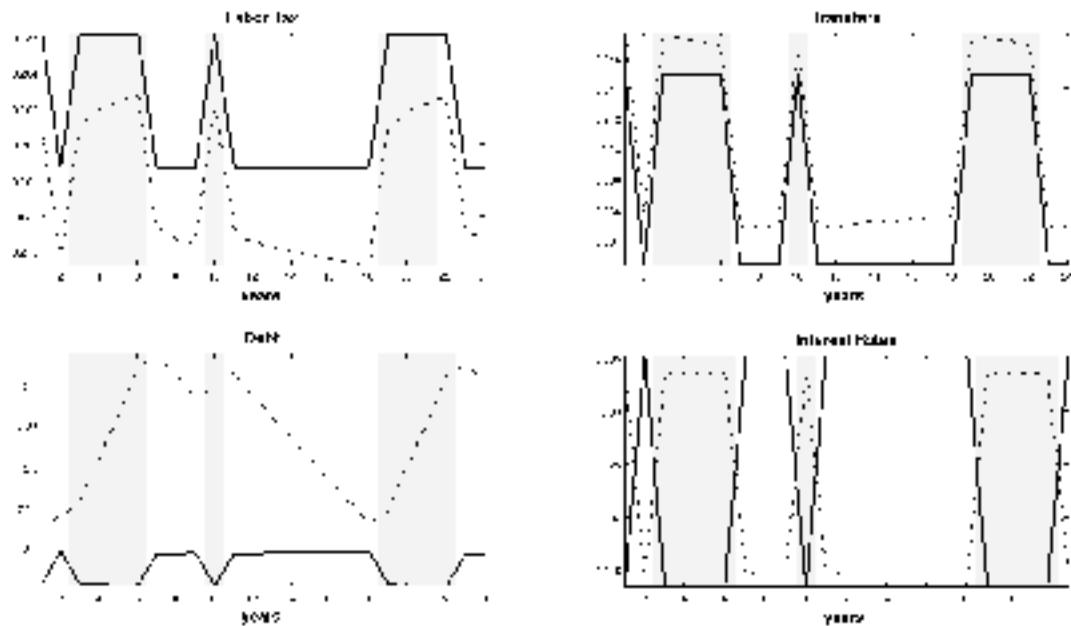


Figure VII: This plots a typical sample path taxes, transfers, debt and interest rates. The bold lines are with benchmark calibration and the dotted lines refer to the variant with countercyclical interest rates. The shaded regions are recessions.

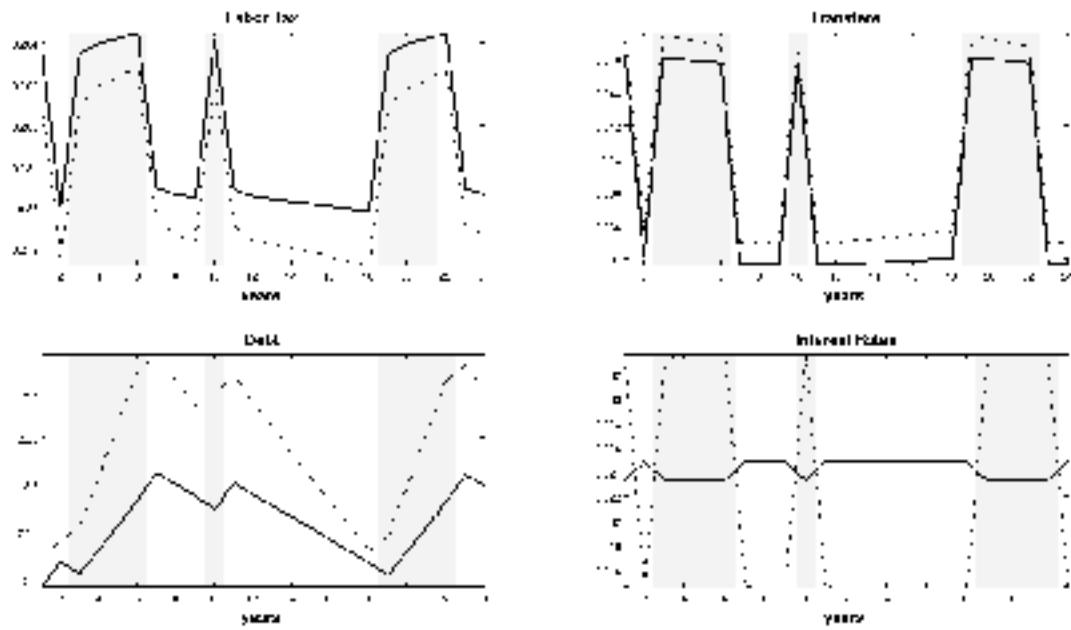


Figure VIII: This plots a typical sample path taxes, transfers, debt and interest rates. The bold lines are with acyclical interest rates calibration and the dotted lines correspond to the case with countercyclical interest rates. The shaded regions are recessions.

Taxes, Debts, and Redistributions

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3 Questions

1. How costly are government debts?
2. How do concerns for redistribution affect tax smoothing motives?
3. How should tax, transfer, debt policies respond to aggregate shocks that might also change inequality?

Representative agent models (I)

Representative agent models

- ▶ Higher levels of debt are "more" distortionary.
- ▶ LS: With complete markets tax rates should be smooth.
- ▶ Werning: Extends the LS insights to heterogeneous agents by establishing an aggregation result.
- ▶ AMSS: With a risk free bond tax rates are eventually smooth and all aggregate shocks are financed using transfers.

Representative agent models (II)

- ▶ Redistribution is implicitly modeled as a non-negativity constraint on lump sum transfers.
- ▶ Hardwires a discontinuity in the costs of fluctuating transfers around zero.
 1. The Ramsey planner either uses state-contingent securities to hedge aggregate shocks.

or
 2. Accumulates a war chest of assets big enough to finance expenditures using returns on assets plus only positive transfers.

We begin with explicit redistribution motives and let the government set transfers optimally.

Key ingredients

- ▶ **Heterogeneity:** Agents are heterogeneous in
 - a) Productivities
 - b) Pareto weights, and
 - c) Assets.
- ▶ **Instruments:** Affine tax system
- ▶ **Markets:** All agents trade a *single* security whose payoffs can depend on aggregate shocks

Answers to the 3 questions

1. Invariance of debt level:

Absent borrowing constraints, Ricardian equivalence holds.

Borrowing constraints only increase welfare

2. Invariant distribution of tax rates:

- ▶ Has wide support
- ▶ The mean and variance of tax rates are driven by market incompleteness. The two "covariances" that matter are
 - + Between returns on the asset and aggregate shocks (Public sector)
 - + Between consumption and (pre-tax) labor earnings (Private sector)

3. Business cycle dynamics: (Preliminary) Recessions that are accompanied by higher inequality call for increase in both taxes and transfers

Environment

- ▶ **Uncertainty:** Markov aggregate shocks s_t
- ▶ **Demography:** Continuum of infinitely lived agents plus a benevolent planner
- ▶ **Technology:** Output $\int \theta_{i,t} l_{i,t} di$ is linear in labor supplies.
- ▶ **Preferences (Households)**

$$E_0 \sum_{t=0}^{\infty} \beta^t U^i(c_{i,t}, l_{i,t})$$

- ▶ **Preferences (Planner):** Given Pareto weights $\{\omega_i\}$

$$E_0 \int \omega_i \sum_{t=0}^{\infty} \beta^t U_t^i(c_{i,t}, l_{i,t}) di$$

- ▶ **Asset markets:** A "risky" bond with payoffs $P = P(s|s_-)$

Environment, II

- ▶ **Affine Taxes:** Agent i 's tax bill

$$-T_t + \tau_t \theta_{i,t} b_{i,t}$$

- ▶ **Budget constraints** Let $R_{t-1,t} = \frac{P_t}{q_{t-1}}$

- ▶ Agent i : $c_{i,t} + b_{i,t} = (1 - \tau_t) \theta_{i,t} b_{i,t} + R_{t-1,t} b_{i,t-1} + T_t$
- ▶ Government: $g_t + B_t + T_t = \tau_t \int \theta_{i,t} b_{i,t} di + R_{t-1,t} B_{t-1}$

- ▶ **Market Clearing**

- ▶ Goods: $\int c_{i,t} di + g_t = \int \theta_{i,t} b_{i,t} di$
- ▶ Assets: $\int b_{i,t} di + B_t = 0$

- ▶ **Initial conditions:** Distribution of assets
 $(\Psi_0(b_{i,-1}), B_{-1}, s_{-1})$

Ramsey Problem

Definition

Allocation, price system, government policy: Standard

Definition

Competitive equilibrium: Given $(\Psi_0(b_{i,-1})_i, B_{-1}, s_{-1})$ and $\{\tau_t, T_t\}_{t=0}^{\infty}$ all allocations are chosen optimally, markets clear¹

Definition

Optimal competitive equilibrium: A welfare-maximizing competitive equilibrium for a given $(\Psi_0(b_{i,-1}), B_{-1}, s_{-1})$

¹Usually, we impose only "natural" debt limits.

Ramsey problem: Recursive formulation

Split into two parts

1. $t \geq 1$: Ex-ante continuation problem with state variables $\{\Gamma_{t-1}(x_{i,t-1}, m_{i,t-1}, \theta_{i,t-1}), s_{t-1}\}$, where
 - ▶ Scaled assets: $x_{i,t-1} = u_{c,i,t-1} b_{i,t}$
 - ▶ Scaled "market weights": $m_{i,t-1} \propto \frac{1}{u_{b,i,t-1}}$
2. $t = 0$: Ex-post initial problem with state variables $(\Psi_0(b_{i,-1}), s_0)$

Bellman Equation for $t \geq 1$

$$V(\Gamma_-, s_-) = \max_{\substack{c_j(s), l_j(s), x_j(s), m_j(s) \\ \tau(s), T(s), \alpha_1, \alpha_2}} \sum_s P(s|s_-) \left[\int \omega_i u(c_i(s), l_i(s)) di + \beta V(\Gamma(s), s) \right]$$

where the maximization is subject to

$$\frac{\partial \mathbb{E}_{s_-}^j u_{j,c}(s) P(s|s_-)}{\partial \mathbb{E}_{s_-}^j u_{j,c}(s) P(s|s_-)} = u_{j,c}(s)[c_j(s) - T(s)] + u_{j,l}(s)l_j(s) + x_j(s)$$

$$\alpha^1 = m_j \mathbb{E}_{s_-}^j u_{j,c}(s)$$

$$\alpha^2(s) = m_j(s) u_{j,c}(s)$$

$$= u_{j,l}(s) - [(1 - \tau(s))u_{j,c}(s)\theta_j(s)]$$

$$\int m_j(s) di = 1$$

$$\int l_j(s)\theta_j(s) di = \int c_j(s) di + g(s)$$

Bellman equation for $t = 0$

$$V_0(\Psi_0(b_{i,-1}), s_0) = \max_{\substack{c_{i,0}, b_{i,0}, x_{i,0}, m_{i,0} \\ \tau_0, T_0}} \int \omega_i U^i(s_0) + \beta V(\Gamma_0, s_0)$$

where the maximization is subject to

$$b_{i,-1} u_{i,c,0} = u_{i,c,0}[c_{i,0} - T_0] + u_{i,l,0} l_{i,0} + x_{i,0}$$

$$-u_{i,l,0} = (1 - \tau_0) u_{i,c,0} \theta_{i,0}$$

$$\int m_{i,0} di = 1$$

$$\int l_{i,0} \theta_{i,0} di = \int c_{i,0} di + g_0$$

A review of LS and AMSS

- ▶ Both impose $T_t \geq 0$
- ▶ The multiplier on the implementability constraint μ_t ,
 - ▶ LS: Constant

$$\mu_t = \mu_0$$

With CES preferences tax rates are smooth.

- ▶ AMSS: Positive risk adjusted martingale

$$\mu_t = \mathbb{E}_t u_{c,t+1} \mu_{t+1}$$

With QL preferences μ_t converges to zero and tax rates converge to zero

- ▶ This makes the constraint on transfers either always slack or eventually slack.

Ricardian Equivalence

- ▶ **Result:** A large set of transfers and asset profiles support the same competitive allocation
Taking away a unit of all agents' assets and increasing transfers by a unit leaves budget sets unchanged
- ▶ **Implication:** Ceteris paribus, an economy with higher level of initial government debt but same relative holdings has the same welfare
- ▶ **Corollary:** Exogenous borrowing constraints of the form $b_{it} > \underline{b}_i$ are not restrictive

If some borrowing constraints bind, the planner can change transfers to slacken all of them

Thus, Ricardian equivalence holds with distortionary taxes and ad hoc borrowing limits

Active channels

1. Varying labor taxes imposes dead weight losses.
2. Fluctuating transfers is costly because of concerns for redistribution.
3. Effects of taxes on redistribution depends on cross sectional heterogeneity of consumption and earnings.
4. The benefits of fluctuating transfers depend of limits to fiscal hedging.

Understanding the channels

1. Study a two type QL economy: This will shut down idiosyncratic risk
 - ▶ Costs of transfers will come from Pareto weights
 - ▶ Hedging motive will be controlled using payoffs $P(s)$
2. More generally
 - ▶ Costs of transfers will come from spreads in marginal utilities.
A key determinant will be the nature of idiosyncratic risk.
 - ▶ Hedging motives will depend on covariance of asset returns with aggregate shocks
3. Study a calibrated economy

Simple Example: 2 Agent QL economy

Consider a "seemingly" AMSS economy

1. Two classes of agent with constant productivities
 $\theta_1 = 0, \theta_2 > 0$
2. Preferences given by $U(c, l) = c - \frac{l^{1+\gamma}}{1+\gamma}$
3. Pareto weights $\{\omega, 1 - \omega\}$
4. Only i.i.d aggregate shocks to $g(s)$

Normalization: Let the assets of the productive agent be denoted by b

Case 1: Risk free bond

Theorem

Let $\omega > \bar{\omega}$ and suppose $P(s) = 1$, $\lim_t \tau_t = -\infty$, $\lim_t b_t = -\infty$ a.s

Corollary

Suppose we augment our problem with a constraint $b_t \geq \underline{b}$. Then with risk-free debt there is an invariant distribution ψ . Moreover, for any $\hat{b} \in (\underline{b}, \bar{b}_n)$, $\psi([\underline{b}, \hat{b}]) > 0$ and $\psi([\hat{b}, \bar{b}_n]) > 0$.

The invariant distribution of taxes is wide as fluctuations in transfers are costly

Case 2: Perfect hedging

Theorem

Let $\omega > \bar{\omega}$ and suppose $\bar{b} < \bar{b}_n$ and payoffs satisfy

$$P(s) = 1 - \frac{\beta}{\bar{b}}(g(s) - \mathbb{E}g)$$

then for all $b_{-1}, b_t \rightarrow \bar{b}$ a.s

Corollary

Suppose we augment our problem with a constraint $b_t \geq \underline{b} < \bar{b}$.

Then the invariant distribution of b (and also tax rates) is

$$\text{degenerate with } \bar{b} = -\beta \frac{\text{var}[g(s)]}{\text{cov}[P(s), g(s)]}$$

The limiting allocation corresponds to a complete market Ramsey problem

Case 3: Imperfect hedging

Decompose payoffs

$$P(s) = \hat{P}(s) + \bar{P}(s)$$

where $\bar{P}(s) = 1 - \frac{\beta}{b}(g(s) - \mathbb{E}g)$ and $\hat{P}(s)$ is orthogonal to $g(s)$.

Theorem

For $\omega > \bar{\omega}$, the ergodic distribution of debt of the policy rules linearized around $(\bar{b}, \bar{P}(s))$ will have mean \bar{b} and coefficient of variation

$$\frac{\sigma_b}{\bar{b}} \leq \sqrt{\frac{\text{var}(P(s)) - |\text{cov}(g(s), P(s))|}{|\text{cov}(g(s), P(s))|}}$$

The speed of convergence to the ergodic distribution given by

$$\frac{\mathbb{E}_{t-1}(b_t - \bar{b})}{(b_{t-1} - \bar{b})} = \frac{1}{1 + |\text{cov}(P(s), g(s))|}$$

Adding risk aversion

Modifications:

- ▶ Endogenous component to covariance between payoffs and expenditure needs.
- ▶ Costs of transfers come from spreads in marginal utilities.

Implications:

1. Inequality distortions both call for a negative correlation between productivities and net assets.
2. This is exacerbated with countercyclical returns on fiscal assets

Inequality distortions



TFP: Adjust tax rate τ or transfers T , both are costly

Suppose $b = 0$

1. Present value of earnings of productive agent are higher
2. A reduction in transfers hurts the low productivity agent more

Then



b is same as increasing the debt of the productive agent

This drives the after-tax, after-interest incomes of both agents closer together

Interest rate fluctuations

- ▶ Counter-cyclical interest rates:

 TFP: If the tax rate τ is left unchanged, the government faces a shortfall of revenues.

1. Reminder: b is **debt** of the government
2. By holding positive assets the govt. can use higher interest income to offset some revenue losses from its tax on labor in recessions
3. This force is present in representative agent economies with endogenous fluctuations in interest rates

 The government reduces debt b

Adding idiosyncratic risk

- ▶ Eliminates negative correlation between pre-tax earnings and assets.
- ▶ Correlation between consumption and pre-tax earnings drives the mean level of taxes
- ▶ The spread of the ergodic distribution is mainly a determined by public sector's fiscal hedging abilities

Decomposing the tax rate

Define $w_{i,t} = u_{c,t}^i [\omega^i - \mu_t^i(1 + \gamma)]$ and $\bar{w}_t = \int u_{c,t}^i \omega^i di$ and
 $\bar{w}_{i,t} = \frac{w_{i,t}}{\bar{w}_t}$.

$$\frac{1}{1 - \tau_t} = \underbrace{\frac{\hat{Y}_t}{Y_t}}_{\text{Effectiveness of Taxes}} \underbrace{\frac{\bar{w}_t}{\lambda_t}}_{\text{Benefits of Transfers}}$$

Where $Y_t = \int y_t^i di$ and $\hat{Y}_t = \int y_t^i \bar{w}_{i,t} di$

Two stark cases

1. Start with a representative agent economy with no aggregate shocks.
2. Consider two process for idiosyncratic risk
 - ▶ IID shocks
 - ▶ Persistent (close to unit root) shocks

How do taxes evolve as the distribution fans out?

Tax rates in the IID economy

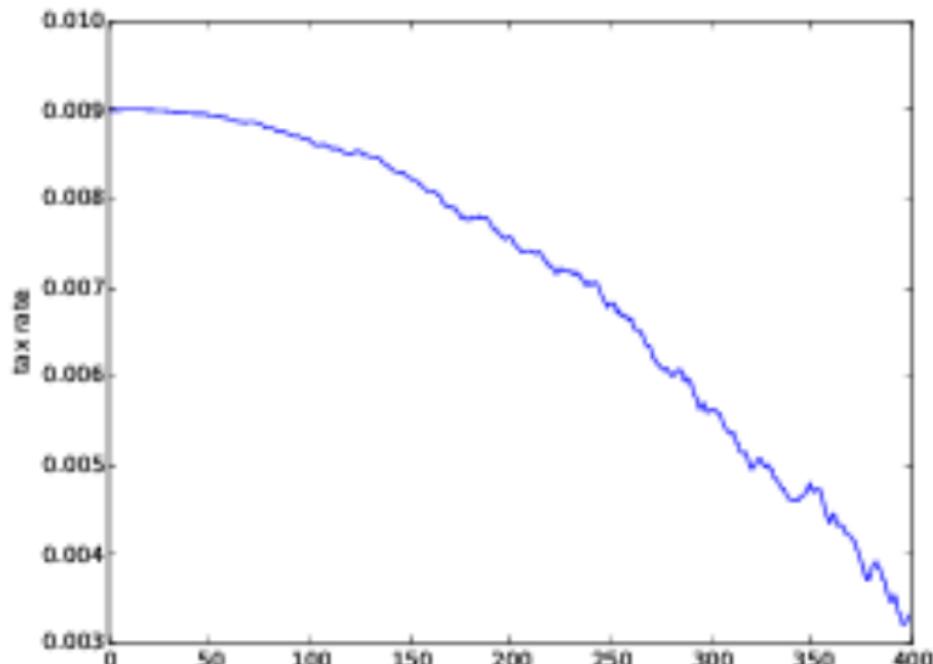


Figure: Taxes in the iid economy.

Tax rates in the close to unit root economy

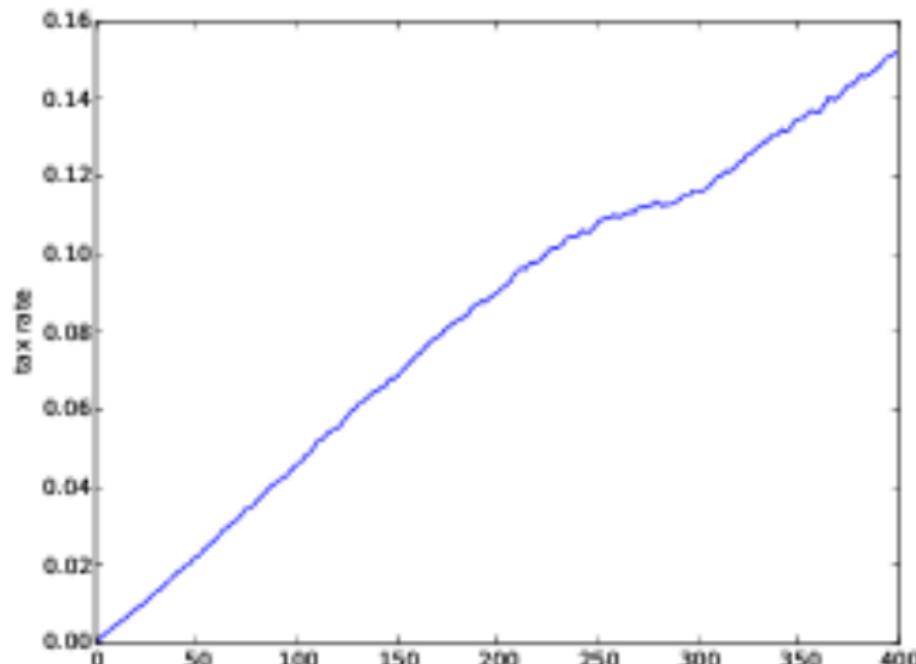


Figure: Taxes in the close to unit root economy.

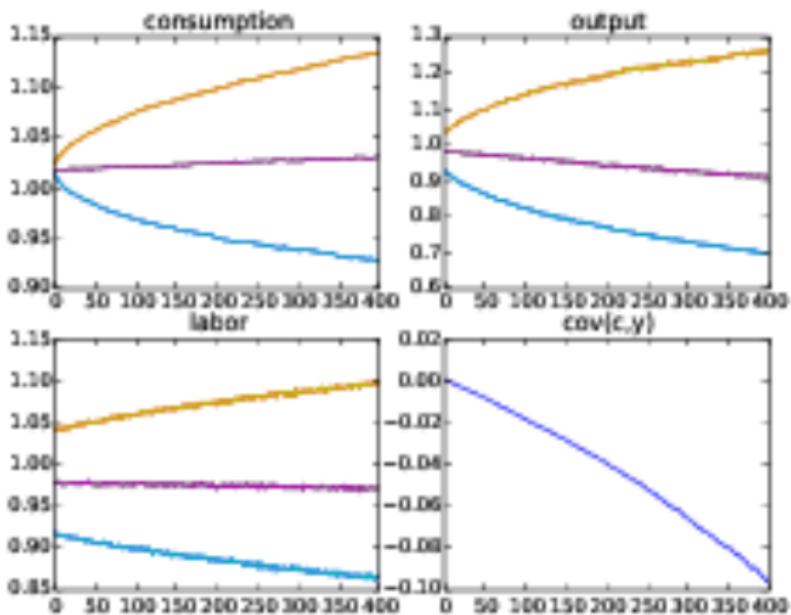


Figure: Inequality in the iid economy. The figure plots the quantiles for consumption, pre-tax labor earnings, labor and the covariance between consumption and pre-tax labor earnings

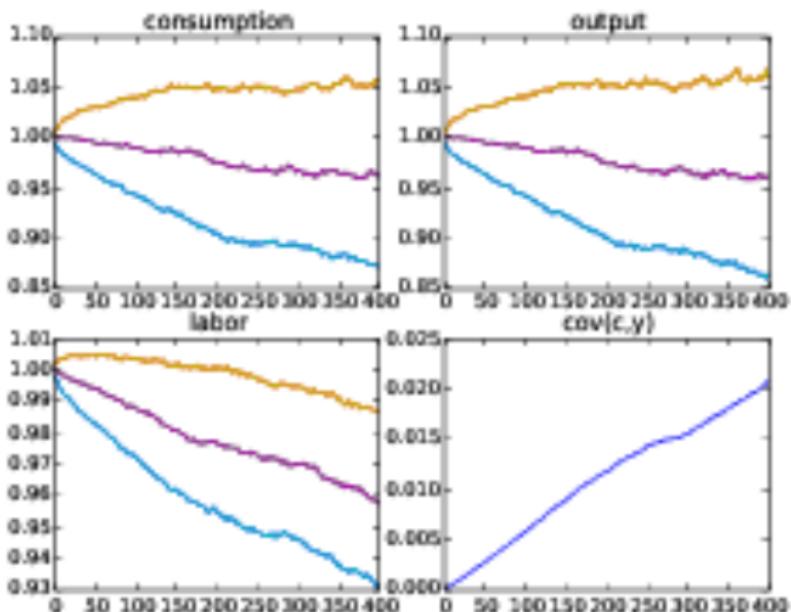


Figure: Inequality in the unit root economy. The figure plots the quantiles for consumption, pre-tax labor earnings, labor and the covariance between consumption and pre-tax labor earnings

Tax rates in the close to unit root economy with aggregate shocks

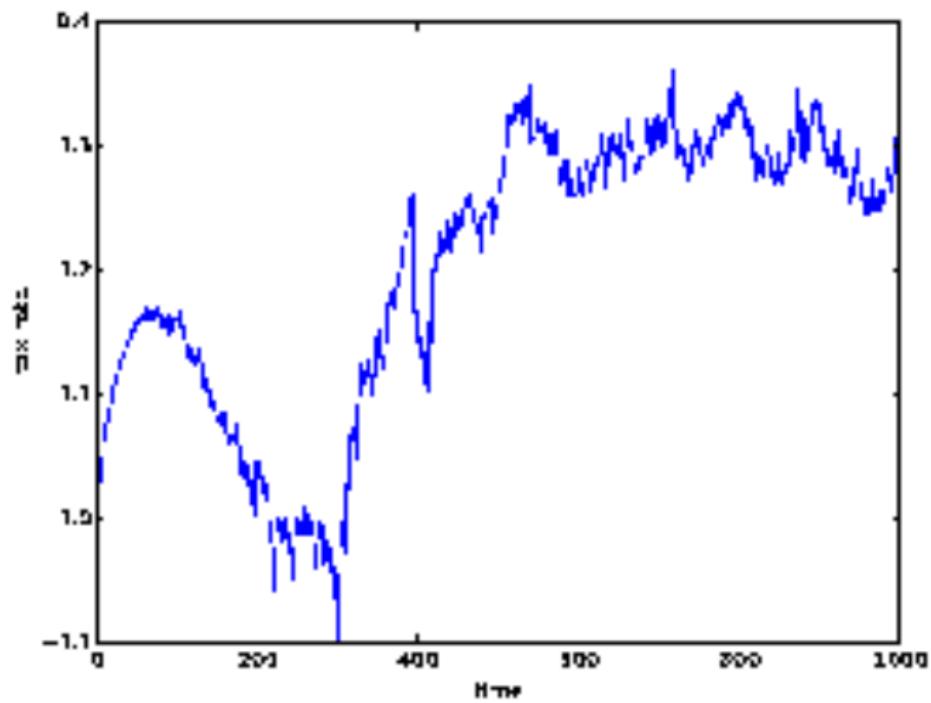


Figure: Taxes in the close to unit root economy with aggregate shocks

Calibrated Example

Take a 2-shock 2-type economy with preferences

$U(c, l) = \psi \log(c) + (1 - \psi) \log(1 - l)$ and with TFP shocks $\theta_i(s)$.

- ▶ Pick baseline parameters to match some low frequency moments
- ▶ Calibrate outcome fluctuations to match three US recessions (i.e., 1991-92, 2001-02 and 2008-10):
 1. The left tail of the cross-section distribution of labor income falls more than right tail
 2. Short term interest rates fall
 3. Booms last longer than recessions

Calibration

Parameter	Value	Description	Target
ψ	0.6994	Friction elasticity of labor supply	0.5
δ_1	4	Log 90-10 wage ratio (Autor et al.)	4
δ_2	1	Normalize to 1	1
β	0.98	Average (annual) risk free interest rate	2%
α_1	0.69	Marginal tax rate in the economy with no shocks	20%
θ	12%	Average pre-transfer expenditure-output ratio	12 %
δ_k	2.5	Relative drop in wage income of 10th percentile	2.5
δ_1	1.2%	Average output loss	3%
$\hat{\delta}(x)$	1.96%	Difference in real interest rates between booms and recession	1.96%
$P(r r)$	0.63	Duration of recessions	2.33 years
$P(b b)$	0.84	Duration of booms	7 years

Table: Benchmark calibration

Initial conditions chosen to make debt to GDP ratio be 60% ²

²We use the same normalization as before i.e, the low productive agent has zero assets

Results: Some variants

We study perturbations of the Benchmark calibration

1. Countercyclical interest rates
2. Acyclical interest rates by adjusting payoffs
3. A case with no inequality where all agents' productivities (TFP shock) fall in parallel

Short Run

To understand the short run responses

- ▶ Solve the time 0 problem with identical initial conditions across different settings.
- ▶ Use optimal policies to compute fluctuations of different components in the government budget constraint as we transition between booms to recessions

Results: Short run

	Δg	ΔB	ΔT	$\Delta[\tau \theta_1 h]$	$\Delta[\tau \theta_2 b]$	ΔY	$\Delta \tau$
Benchmark	0.0000	-1.1981	0.6071	-0.1903	-0.3098	-2.0536	0.3732
Acyclical Interest Rates	0.0000	-1.1126	0.6591	-0.1497	-0.3038	-2.0613	0.3879
Countercyclical Interest Rates	0.0000	-1.0794	0.6387	-0.1415	-0.2992	-2.0677	0.3997
No Inequality	0.0000	-0.1380	-0.5459	-0.5635	-0.1204	-2.6294	0.0622

Table: The table summarizes the changes in the different components of the government budget as the economy transits from "boom" to "recession". All numbers except τ are normalized by un-distorted GDP and reported in percentages.

- For each variable z in the table we report

$$\Delta z \equiv (z(s_l|x_0, m_0, s_0) - z(s_h|x_0, m_0, s_0)) / \bar{Y}$$
 where \bar{Y} is average undistorted GDP in percentages
- Predetermined variables like repayment on existing debt drop out

$$\Delta[g] + \Delta[T] + \Delta[B] = \Delta[\tau \theta_1 h] + \Delta[\tau \theta_2 b]$$

Conclusions

- ▶ Concerns for redistribution imply costs of fluctuating transfers
- ▶ This affects the prescriptions of optimal policy for smoothing tax rates
- ▶ Market incompleteness is a key determinant of the invariant distribution of taxes and debt
- ▶ **Next step:** Calibrate a "realistic" idiosyncratic risk and aggregate risk process to learn about the quantitative implications of the model.

Taxes, Debts, and Redistributions with Aggregate Shocks

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What do we do?

We study optimal taxation under commitment with

- ▶ **Heterogeneous agents**
 - Different productivities
- ▶ **Incomplete markets**
 - All trade a risk-free bond
- ▶ **Affine tax schedules**
 - Government levies a proportional tax on labor earnings
 - + lump sum (tax or transfer)
- ▶ **Aggregate shocks**
 - To productivities, government expenditure etc.

What are we after?

1. How costly are government debts?
2. What are the long run properties of optimal government policies and equilibrium allocations?
3. How should government policies respond to aggregate shocks?

Environment

- ▶ **Uncertainty:** Markov aggregate shocks s_t
- ▶ **Demography:** I types of infinitely lived agents (of mass π_i) plus a benevolent planner
- ▶ **Technology:** Output $\sum_{i=1}^I \theta_i l_{i,t}$ is linear in labor supplies. Productivities $\{\theta_i(s_t)\}_{i,t}$ differ across i .
- ▶ **Preferences (Households)**

$$E_0 \sum_{t=0}^{\infty} \bar{\beta}_t U^i(c_i(s^t), l_i(s^t))$$

where $\bar{\beta}_t = \beta(s_{t-1})\bar{\beta}_{t-1}$, $\beta(s) \in (0, 1)$ and $\beta(s_0) = 1$

- ▶ **Preferences (Planner):** Given Pareto weights $\{\alpha_i\}$

$$E_0 \sum_{i=1}^I \pi_i \alpha_i \sum_{t=0}^{\infty} \bar{\beta}_t U_t^i(c_{i,t}, l_{i,t})$$

- ▶ **Asset markets:** A risk-free bond only

Environment, II

- ▶ **Affine Taxes:** Agent i 's tax bill

$$-T_t + \tau_t \theta_{i,t} l_{i,t}$$

- ▶ **Budget constraints**

- ▶ Agent i : $c_{i,t} + b_{i,t} = (1 - \tau_t) \theta_{i,t} l_{i,t} + R_{t-1} b_{i,t-1} + T_t$
- ▶ Government: $g_t + B_t + T_t = \tau_t \sum_{i=1}^I \pi_i \theta_{i,t} l_{i,t} + R_{t-1} B_{t-1}$

- ▶ **Market Clearing**

- ▶ Goods: $\sum_{j=1}^J \pi_j c_j(s^t) + g(s_t) = \sum_{j=1}^J \pi_j \theta_j(s_t) b(s^t)$
- ▶ Assets: $\sum_{i=1}^I \pi_i b_{i,t} + B_t = 0$

- ▶ **Initial conditions:** Distribution of assets $\{b_{i,-1}\}_i$ and B_{-1}

Ramsey Problem

Definition

Allocation, price system, government policy: Standard

Definition

Competitive equilibrium: Given $(\{b_{i,-1}\}_i, B_{-1})$ and $\{\tau_t, T_t\}_{t=0}^{\infty}$ all allocations are chosen optimally, markets clear¹

Definition

Optimal competitive equilibrium: A welfare-maximizing competitive equilibrium for a given $(\{b_{i,-1}\}_i, B_{-1})$

¹Usually, we impose only "natural" debt limits.

Contrast with representative agent models

Representative agent with linear taxes

- ▶ Higher levels of debt are distortionary
- ▶ With incomplete markets (as in AMSS), the optimal government policy is to accumulate assets

Redistribution and optimal transfers

- ▶ Representative agent models impose restrictions on transfers
 - ▶ These are motivated only implicitly by concerns about redistribution: poor people can't afford lump sum taxes
 - ▶ These constraints *almost* always bind (e.g., Lucas Stokey, AMSS) and drive long run debt dynamics
- ▶ We begin with explicit redistribution motives and let the government set transfers optimally

Prescriptions for optimal tax-transfers differ substantially with explicitly modeled redistribution concerns

Main working parts

- ▶ **Welfare Criterion:** Benevolent planner with explicit redistribution motives
- ▶ **Instruments:** Transfers and a flat tax on labor income
- ▶ **Restrictions:**
 1. The tax on labor income is linear in wage earnings
 2. Transfers are unrestricted in sign and magnitude but not conditioned on agents' identities
 3. Incomplete markets
- ▶ **Trade-offs:**
 1. Varying labor taxes imposes dead weight losses
 2. With explicit redistribution motives come costs of fluctuating transfers. Withdrawing a unit of consumption affects rich and poor people differently

Key forces

- ▶ **Heterogeneity**

1. Two sources of heterogeneity: Productivities and asset holdings
2. Unrestricted transfers → Level of government debt is irrelevant, what matters is its *distribution* across agents
E.G, productive agents holding a lot of government debt is more distortionary

- ▶ **Responses**

Since welfare costs depend on the distribution of assets, optimal policy is affected by and affects the distribution of net assets

1. **Absence of agent specific transfers:** This prompts the govt. to engineer a negative correlation between net assets and labor earnings
2. **Absence of state contingent securities:** This prompts the govt. to exploit endogenous fluctuations in the interest rate

Ricardian Equivalence

- ▶ *Result:* A large set of transfers and asset profiles support the same competitive allocation
- ▶ *Logic:* Taking away a unit of all agents' assets and increasing transfers by a unit leaves budget sets unchanged

Notation: $\tilde{b}_{i,t} = b_{i,t} - b_{1,t}$ be the **relative assets** of Agent i

Theorem

Given $(\{b_{i,-1}\}_i, B_{-1})$, let $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, B_t, R_t\}_t$ and $\{\tau_t, T_t\}_t$ be a competitive equilibrium.

For any bounded sequences $\{\hat{b}_{i,t}\}_{i,t \geq -1}$ that satisfy

$$\hat{b}_{i,t} - \hat{b}_{1,t} = \tilde{b}_{i,t} \text{ for all } t \geq -1, i \geq 2,$$

there exist sequences $\{\hat{T}_t\}_t$ and $\{\hat{B}_t\}_{t \geq -1}$ such that

$\{\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\}_i, \hat{B}_t, R_t\}_t$ and $\{\tau_t, \hat{T}_t\}_t$ constitute a competitive equilibrium given $(\{\hat{b}_{i,-1}\}_i, \hat{B}_{-1})$.

Ricardian Equivalence: Implications

- ▶ Applies to all competitive allocations, not just the optimal one
- ▶ *Ceteris paribus*, an economy with higher level of initial government debt but same relative holdings has the same welfare
- ▶ Exogenous borrowing constraints of the form $b_{it} > \underline{b}_i$ are not restrictive

Logic: If some borrowing constraints bind, the planner can reduce transfers sufficiently to slacken *all* of them

Theorem

For every competitive equilibrium in an economy without exogenous borrowing constraints there is a government tax policy such the same allocation and interest rate sequence is part of a competitive equilibrium in an economy with exogenous borrowing constraints of the form $b_{i,t} > \underline{b}_i$

Thus, Ricardian equivalence holds with distortionary taxes and ad hoc borrowing limits

Optimal allocations: Primal approach

Focus on interior equilibria.

1. Eliminate tax rate τ_t :

$$(1 - \tau_t) \theta_{i,t} U_{c,t}^i = -U_{l,t}^i,$$

2. Eliminate risk free interest rate R_t :

$$U_{c,t}^i = \beta_t R_t E_t U_{c,t+1}^i.$$

3. Eliminate transfers T_t :

$$(c_{i,t} - c_{1,t}) + \tilde{b}_{i,t} = -\frac{U_{j,t}^i}{U_{c,t}^i} b_{i,t} + \frac{U_{j,t}^1}{U_{c,t}^1} b_{1,t} + \frac{U_{c,t-1}^i}{\beta_{t-1} E_{t-1} U_{c,t}^i} \tilde{b}_{i,t-1} \quad \forall i \geq 2, t.$$

This yields "implementability constraints"

Optimal allocations: Sequential formulation

Denote $Z_t^j = U_{c,t}^j c_{j,t} + U_{l,t}^j l_{j,t} = \frac{U_{c,t}^j}{U_{c,t}^1} [U_{c,t}^1 c_{1,t} + U_{l,t}^1 l_{1,t}]$. The optimal policy solves,

$$\max_{c_{j,t}, l_{j,t}, \tilde{b}_{j,t}} \mathbb{E}_0 \sum_{j=1}^J \pi_j \alpha_j \sum_{t=0}^{\infty} \tilde{\beta}_t U_t^j(c_{j,t}, l_{j,t}),$$

subject to

$$\sum_{j=1}^J \pi_j c_j(s^*) + g(s_t) = \sum_{j=1}^J \pi_j \beta_j(s_t) l_j(s^*) \quad (\text{Feas})$$

$$\frac{U_{l,t}^j}{\theta_{l,t} U_{c,t}^j} = \frac{U_{c,t}^1}{\theta_{1,t} U_{c,t}^1} \quad (\text{Wages})$$

$$\frac{\mathbb{E}_{t-1} U_{c,t}^j}{\mathbb{E}_{t-1} U_{c,t}^1} = \frac{U_{c,t-1}^j}{U_{c,t-1}^1} \quad (\text{Bond})$$

$$\tilde{b}_{j,t-1} \frac{U_{c,t-1}^j}{\beta_{t-1}} = \left(\frac{\mathbb{E}_{t-1} U_{c,t}^j}{U_{c,t}^1} \right) \mathbb{E}_t \sum_{k=t}^{\infty} \left[\prod_{j=t}^{k-1} \beta_j \right] Z_k^j \quad (\text{Meas: } t \geq 1)$$

$$\tilde{b}_{j,-1} = \mathbb{E}_{-1} \sum_{k=0}^{\infty} \left[\prod_{j=0}^{k-1} \beta_j \right] Z_k^j \quad (\text{Meas: } t=0)$$

$\tilde{b}_{j,-1} \frac{U_{c,t-1}^j}{\beta_{t-1}}$ is bounded

Ramsey problem: Recursive formulation

Split into two parts

1. $t \geq 1$: Ex-ante continuation problem with state variables (x, ρ, s_{-})

$$x = \beta^{-1} \left(U_{c,t-1}^2 \tilde{b}_{2,t-1}, \dots, U_{c,t-1}^I \tilde{b}_{I,t-1} \right)$$

$$\rho = \left(U_{c,t-1}^2 / U_{c,t-1}^1, \dots, U_{c,t-1}^I / U_{c,t-1}^1 \right)$$

2. $t = 0$: Ex-post initial problem with state variables (\tilde{b}_{-1}, s_0)

Bellman Equation for $t \geq 1$

$$V(x, \rho, s_*) = \max_{c_i(s), l_i(s), x'(s), \rho'(s)} \sum_s \Pr(s|s_*) \left(\left[\sum_j \pi_j c_j U^j(s) \right] + \beta(s) V(x'(s), \rho'(s), s) \right)$$

where the maximization is subject to

$$U_c^i(s) [c_i(s) - c_1(s)] + U_c^i(s) \left(\frac{U_c^i(s)}{U_c^1(s)} l_i(s) - \frac{U_c^1(s)}{U_c^i(s)} l_1(s) \right) + \beta(s) x'_i(s) = \frac{x_i U_c^i(s)}{\mathbb{E}_s U_c^i} \text{ for all } s, i \geq 2$$

$$\frac{\mathbb{E}_s U_c^i}{\mathbb{E}_s U_c^1} = \rho_i \text{ for all } i \geq 2$$

$$\frac{U_c^i(s)}{\theta_i(s) U_c^1(s)} = \frac{U_c^1(s)}{\theta_1(s) U_c^1(s)} \text{ for all } s, i \geq 2$$

$$\sum_j \pi_j c_j(s) + g(s) = \sum_j \pi_j \theta_j(s) l_j(s) \quad \forall s$$

$$\rho'_i(s) = \frac{U_c^i(s)}{U_c^1(s)} \text{ for all } s, i \geq 2$$

$$\underline{x}_i(s; x, \rho, s_*) \leq x_i(s) \leq \bar{x}_i(s; x, \rho, s_*)$$

Bellman equation for $t = 0$

$$V_0 \left(\{\tilde{b}_{i,-1}\}_{i=2}^I, s_0 \right) = \max_{c_{i,0}, b_{i,0}, \rho_0, \pi_0} \sum_j \pi_j \alpha_j U^j(c_{j,0}, b_{j,0}) + \beta(s_0) V(s_0, \rho_0, s_0)$$

where the maximization is subject to

$$U_{c,B}^i [c_{i,0} - c_{1,0}] + U_{c,B}^i \left(\frac{U_{c,B}^i}{U_{c,B}^1} b_{i,0} - \frac{U_{c,B}^1}{U_{c,B}^i} b_{1,0} \right) + \beta(s_0) x_{i,0} = U_{c,B}^i \tilde{b}_{i,-1} \text{ for all } i \geq 2$$

$$\frac{U_{c,B}^i}{\theta_{i,0} U_{c,B}^1} = \frac{U_{c,B}^1}{\theta_{1,0} U_{c,B}^{1,0}} \text{ for all } i \geq 2$$

$$\sum_j \pi_j c_{j,0} + g_0 = \sum_j \pi_j \theta_{j,0} b_{j,0}$$

$$\rho_{i,0} = \frac{U_{c,B}^i}{U_{c,B}^1} \text{ for all } i \geq 2$$

Steady States (SS)

Let $\Psi(s; x, \rho, s_-)$ be an optimal law of motion for the state variables for the $t \geq 1$ recursive problem, i.e.,

$$\Psi(s; x, \rho, s_-) = (x'(s), \rho'(s))$$

attains $t \geq 1$ value function given state (x, ρ, s_-)

Definition

A steady state (x^{ss}, ρ^{ss}) satisfies

$$(x^{ss}, \rho^{ss}) = \Psi(s; x^{ss}, \rho^{ss}, s_-) \text{ for all } s, s_-$$

A steady state is a node at which the continuation allocation and tax schedule has no further history dependence.

Existence

- ▶ **Quasi-linear preferences:**

1. SS exists for a wide range of parameters and shocks
2. The economy reaches a steady state in one period
3. Output, tax rates are constant thereafter and levels are independent of initial conditions

Dynamics of taxes are starkly different than AMSS

- ▶ **General preferences:**

1. *IID shocks with two values:* SS exists and continuation allocation is independent of initial conditions
2. *More general shocks:* There exists an ergodic region in which (x, ρ) is no longer constant, but fluctuations are markedly reduced relative to the transient fluctuations that occur during an approach to a SS

Intuition: A two-agent example

Consider $I=2$ with $\theta_1(s) > \theta_2 = 0$.

Two main forces determine the dynamics of the tax rate, transfers, and assets:

- ▶ **Fluctuations in inequality** measured by spreads in marginal utilities
- ▶ **Fluctuations in interest rate**

For quasi linear preferences both forces are absent

3 Cases: Interest rates and relative assets

- ▶ **Normalization:** By Ricardian equivalence, we can
 1. Normalize $b_2(s) = 0$
 2. Government assets: $B(s) = -b_1(s)$
- ▶ **Interpretation:** The state variable $x \equiv U_c^2 \tilde{b}_2 = U_c^2 [b_2 - b_1]$. Under $b_2(s) = 0$ normalization, x is
 1. Marginal utility scaled **debt** of the productive agent
 2. Marginal utility scaled **assets** held by the government

▶ **Results:**

<i>Interest rates</i>	<i>Discount factors</i>	$x = U_c^2 [b_2 - b_1]$
Countercyclical	$\beta(s_f) = \beta(s_h)$	$x > 0$
Acyclical	$\beta(s_f) > \beta(s_h)$	$x > 0$
Procyclical	$\beta(s_f) \gg \beta(s_h)$	$x < 0$

Inequality distortions

Consider with case with acyclical interest rates

 TFP: Adjust tax rate τ or transfers T , both are costly

Suppose $x = 0$ (or $b_2(s) = b_1(s)$)

1. Present value of earnings of productive agent are higher
2. A reduction in transfers hurts the low productivity agent more

Then

 x is same as increasing the debt of the productive agent

This drives the after-tax, after-interest incomes of both agents closer together

Interest rate fluctuations

- ▶ **Countercyclical interest rates:**

 TFP: If the tax rate τ is left unchanged, the government faces a shortfall of revenues.

1. Reminder: x is marginal utility scaled **assets** of the government
2. By holding positive assets the govt. can use higher interest income to offset some revenue losses from its tax on labor in recessions
3. This force is present in representative agent economies with endogenous fluctuations in interest rates



- ▶ **Pro-cyclical interest rates:** If the interest rate is sufficiently low in a recession, the government may want to hold debt to free resources by lower interest payments.



Remarks on SS

- ▶ Stability:
 1. **Countercyclical interest rates:** Both forces push in the same direction → steady state is stable
 2. **Procydical interest rates:** Both forces push in opposite direction → steady state is unstable
- ▶ For more than 2 agents, we have similar mechanics. In particular
 1. Inequality distortions call for a negative correlation between productivities and (scaled) net assets
 2. Procydical interest rates may flip the sign of the correlation between productivities and net assets to be positive.
 - ▶ Low interest rates in recession prompts the government to hold debt
 - ▶ By borrowing more from agents with higher productivities, the govt. can reduce welfare costs of lowering transfer in adverse times

Numerical Example

Use a calibrated version of the economy to

- ▶ Approximate magnitudes of these forces and
- ▶ Study optimal policy responses at business cycle frequencies when an economy is possibly far away from a steady state

Numerical Example: Calibration

Take a 2-shock 2-type economy with preferences

$U(c, l) = \psi \log(c) + (1 - \psi) \log(1 - l)$ and allow $\theta_i(s)$, $\beta(s)$, $g(s)$ to depend on s .

- ▶ Pick baseline parameters to match some low frequency moments
- ▶ Calibrate outcome fluctuations to match three US recessions (i.e., 1991-92, 2001-02 and 2008-10):
 1. The left tail of the cross-section distribution of labor income falls more than right tail
 2. Short term interest rates fall
 3. Booms last longer than recessions

Calibration

Parameter	Value	Description	Target
ψ	0.6994	French elasticity of labor supply	0.5
δ_1	4	Log 90-10 wage ratio (Autor et al.)	4
δ_2	1	Normalize to 1	1
β	0.98	Average (annual) risk free interest rate	2%
α_1	0.69	Marginal tax rate in the economy with no shocks	20%
g_1	12%	Average pre-transfer expenditure-output ratio	12 %
δ_3	2.5	Relative drop in wage income of 10th percentile	2.5
δ_4	1.2%	Average output loss	3%
$\beta(x)$	1.96%	Difference in real interest rates between booms and recession	1.96%
$P(r r)$	0.63	Duration of recessions	2.33 years
$P(b b)$	0.84	Duration of booms	7 years

Table: Benchmark calibration

Initial conditions chosen to make debt to GDP ratio be 60%

Results: Some variants

We study perturbations of the Benchmark calibration

1. Acyclical interest rates: Smaller spread in discount factor shocks
2. Countercyclical interest rates: No discount factor shocks
3. No inequality: Equal fall in all agents' productivities (TFP shock) and no discount factor shocks
4. Government expenditure shocks: A fall in g that produces a comparable fall in output

Results: Long run

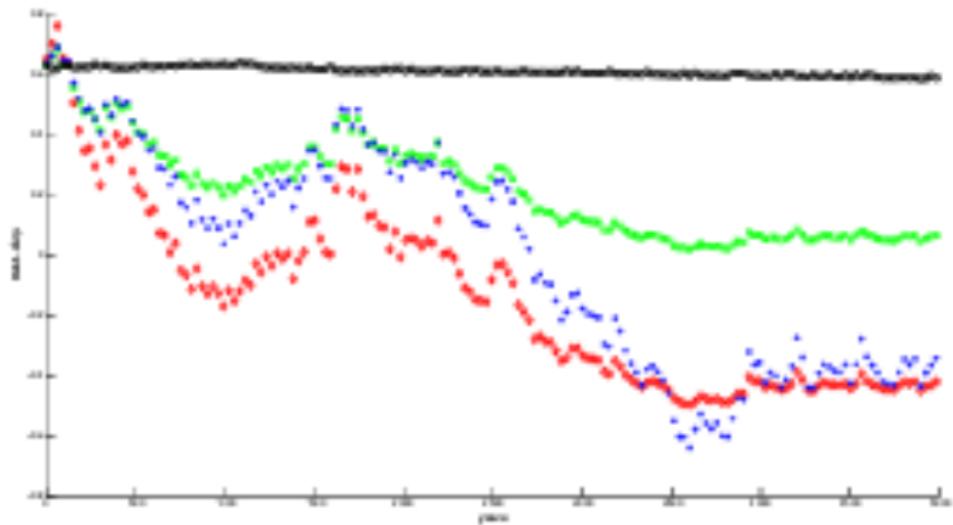


Figure: Govt. debt for several economies: benchmark (o), acyclical interest rates (+), countercyclical interest rates (◊) and no inequality shocks (□)

Observations

- ▶ Long run tendency to converge to some ergodic set. But convergence is very slow [more details on speed of convergence](#).
- ▶ With low discount factor shocks, outcomes approach positive govt. assets
- ▶ With high discount factor shocks that produce procyclical real interest rates, there is no tendency to reduce govt. debt even after 5000 years

Short Run

To understand the short run responses

- ▶ We set the exogenous state s_0 to put us at the onset of a recession
- ▶ We solve the time 0 problem with identical initial conditions across different settings. This pins down the initial state vector x_0, ρ_0 that appears in our time 0 Bellman equation
- ▶ We use optimal policies to compute fluctuations of different components in the government budget constraint across states

For each variable z in the table we report

$$\Delta z \equiv (z(s_l|x_0, \rho_0, s_0) - z(s_h|x_0, \rho_0, s_0)) / \bar{Y} \text{ where } \bar{Y} \text{ is average undistorted GDP in percentages}$$

Results: Short run

	Δg	ΔB	ΔT	$\Delta[\tau\theta_1 h]$	$\Delta[\tau\theta_2 b]$	ΔY	$\Delta \tau$
Benchmark	0.0000	-1.1561	0.6671	-0.1293	-0.3098	-2.8538	0.3732
Acyclical Interest Rates	0.0000	-1.1126	0.6691	-0.1407	-0.3038	-2.8613	0.3879
Countercyclical Interest Rates	0.0000	-1.0794	0.6387	-0.1415	-0.2992	-2.8677	0.3997
No Inequality	0.0000	-0.1380	-0.5459	-0.5635	-0.1204	-2.6294	0.0622
Expenditure Shock	-7.5037	2.9137	2.0612	-1.3159	-0.2630	-2.3443	-1.1598

Table: The tables summarizes the changes in the different components of the government budget as the economy transits from "boom" to "recession". All numbers except τ are normalized by un-distorted GDP and reported in percentages.

Note that predetermined variables like repayment on existing debt drop out of the accounting and we have

$$\Delta[g] + \Delta[T] + \Delta[B] = \Delta[\tau\theta_1 h] + \Delta[\tau\theta_2 b]$$

Conclusions

- ▶ Size of government debt alone is irrelevant \Rightarrow need to know the distribution of net assets
- ▶ Optimal tax and transfer scheme balance
 1. welfare losses from fluctuating taxes
 2. welfare losses from fluctuating transfers
- ▶ Since welfare costs depend on the how debt is distributed, the planner has incentives to move net assets over time
- ▶ With incomplete markets, interest rate fluctuations are a key determinant of long-run correlations between productivities and net assets
- ▶ Ignoring heterogeneity produces misleading results about the size and direction of short run optimal policy responses

Speed of convergence (I)

Suppose we are in the binary-IID world where steady states are deterministic.

- ▶ The optimal policy induces two *risk adjusted martingales*
 1. Multiplier on the implementability constraint : μ_t
 2. The ratio of marginal utilities: ρ_t
- ▶ One can represent the optimal allocation recursively in terms of $\{\mu(s^{t-1}), \rho(s^{t-1})\}$ and s_t .
- ▶ Why (μ, ρ) instead of (x, ρ) ?
- ▶ Linearize optimal policies for each s_t around the constant steady state.
- ▶ Study the eigenvalues of the conditional mean and variance dynamics (these are deterministic linear systems)

Speed of convergence (II)

Let $\hat{\Psi}_t = \begin{bmatrix} \mu_t - \mu^{ss} \\ \rho_t - \rho^{ss} \end{bmatrix}$. Then

$$\hat{\Psi}_{t+1} = B(s_{t+1})\hat{\Psi}_t$$

This linearized system has coefficients that are functions of the shock s .

Proposition

If the (real part) of eigenvalues of $E B(s)$ are less than 1, the system converges to zero in mean. Further for large t , the conditional variance of $\hat{\Psi}$, denoted by $\Sigma_{\Psi,t}$, follows a deterministic process governed by

$$\text{vec}(\Sigma_{\Psi,t}) = \hat{B}\text{vec}(\Sigma_{\Psi,t-1}),$$

where \hat{B} is a square matrix of dimension $(2N - 2)^2$. In addition, if the (real parts) of eigenvalues of \hat{B} are all less than 1, the system converges in probability.

The eigenvalues (in particular the largest one) are instructive not only for whether the system is locally stable but also for how quickly the steady state is reached.

Speed of convergence: Size of shocks and risk aversion

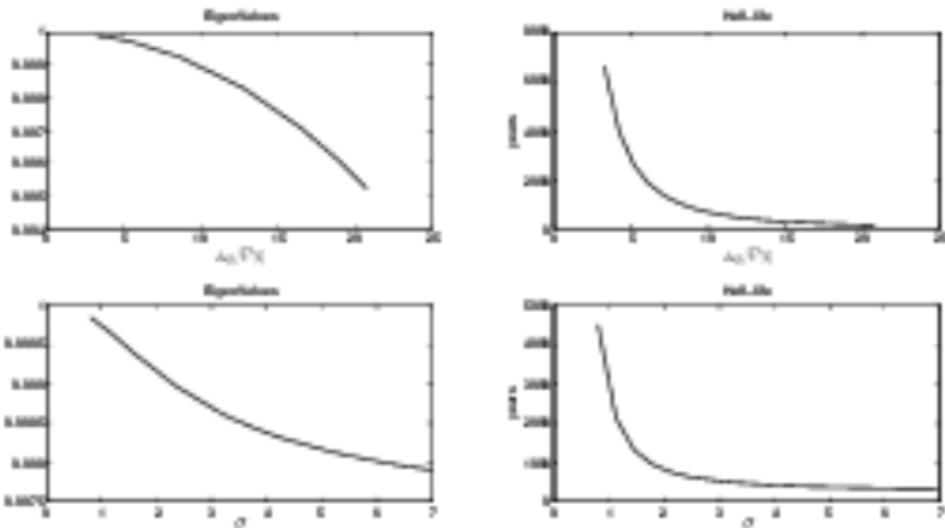


Figure: The top (bottom) panel plots the dominant eigenvalue of \bar{B} and the associated half-life as we increase the spread between the expenditure levels (risk aversion).