Taxation, debt, and redistribution[*]

Anmol Bhandari, David Evans, Mikhail Golosov, Thomas Sargent

June 2013

Abstract

We study optimal income taxes and transfers in an economy with heterogeneous agents and aggregate shocks. The net distribution of debt holdings across agents influences optimal allocations, transfers, and tax rates, but the level of government debt does not. Higher cross-section correlations of debt holdings and labor incomes imply more distortions and lower welfare. In incomplete markets economies, setting taxes and transfers optimally substantially alters the character of the government's precautionary incentive to accumulate assets relative to representative agent Ramsey models like Aiyagari et al. (2002). We analyze how the government's long-run asset or debt position emerges from possibly countervailing incentives that confront the Ramsey planner. Its distributional motives make the Ramsey planner want to smooth transfers and also possibly to manipulate the correlation between the interest rate and the government's net asset position vis a vis high skilled workers. Related forces also shape higher frequency optimal government responses to recessions accompanied by shifts in the cross-section distribution of skills, responses that differ markedly from those emerging from models with a representative consumer.

Key words: Finding the state is an art. Distorting taxes. Transfers. Redistribution. Government debt. Interest rate risk.

[*]We thank Mark Aguiar, Stefania Albanesi, Manuel Amador, Andrew Atkeson, V.V. Chari, Harold L. Cole, Guy Laroque, Robert E. Lucas, Jr., Ali Shourideh, and seminar participants at Bocconi, Chicago, EIEF, the Federal Reserve Bank of Minneapolis, IES, Princeton, Stanford, UCL, Universidade Católica, 2012 Minnesota macro conference, Monetary Policy Workshop NY Fed for helpful comments.

If, indeed, the debt were distributed in exact proportion to the taxes to be paid so that every one should pay out in taxes as much as he received in interest, it would cease to be a burden.... if it were possible, there would be [no] need of incurring the debt. For if a man has money to loan the Government, he certainly has money to pay the Government what he owes it. Simon **?**, p.85

1 Introduction

This paper studies an economy with agents who differ in their productivities and a benevolent government that imposes an affine income tax that consists of a distortionary proportional tax on labor income and a lump-sum tax or transfer. We impose no restrictions on the sign of transfers. If some agents are sufficiently poor or if the government wants enough redistribution, the government always chooses positive transfers. For most of the paper, we study an economy without capital in which a one-period risk-free bond is the only financial asset traded.

We obtain three sets of findings, one that is purely mechanical but that nevertheless reveals basic principles that underly the other two more novel and interesting collections of results. The first set of findings describes the irrelevance of government debt, an extension of Ricardian irrelevance reasoning that identifies the distribution of net assets across agents as a component of a state vector that restrains the time t decisions of a Ramsey planner.

The second set of results identifies key forces that affect the long-run behavior of the distribution of assets in an incomplete markets economy in which private agent's and government portfolios are confined to risk-free one-period claims. A driving force here is that decreasing transfers in response to adverse aggregate shocks affects low-income workers disproportionately. This gives a Ramsey planner who cares about low-skilled workers an incentive to smooth fluctuations in transfers. That makes the Ramsey planner want to impart a negative correlation between fluctuations in labor income and a low-income agent's net assets. Properly recognizing the relevance of net but not gross asset positions stressed in our first set of findings, this can be accomplished by having the government effectively accumulate risk-free claims on high-skilled workers. But another, possibly countervailing or possibly reinforcing, force comes from the Ramsey planner's incentive to use fluctuations in the interest rate to compensate for missing asset markets. If interest rates are high (low) when revenue needs are high, it creates an incentive for the government to accumulate assets (debt) vis-a-vis high-skilled worker. Thus, depending on the comovement of the interest rate with other fundamentals, these two forces may either reinforce each other (for example, in response to a pure TFP shock where the implied interest rates are countercyclical) or go in opposite direction (for example, in the case where recessions are accompanied by a lower interest rate due to correlated discount factor shocks).

A third set of results concerns higher frequency implications, in particular, the nature of optimal government policy in booms and recessions. What we have to say about this comes from a version of our model calibrated to capture what we take to be key stylized facts that during recessions (1) interest rates fall, and (2) the left tail of the cross-section distribution of labor income falls by more than right tail. When we calibrate to fit those targets, we find that in recessions, it is optimal to increase taxes, transfers, and issues of government debt. These effects differ substantially both qualitatively and quantitatively from what would flow from either a representative agent model or our model were a recession is modeled to be a pure TFP shock that leaves the distribution of skills unchanged.

Here is how we construct these results. After section 2 describes preferences, technologies, endowments, and information flows, section 3 sets the stage for our subsequent results. Theorem 1 exploits the fact that the set of feasible allocations is invariant with respect to changes in transfers and *gross* asset positions that leave unaltered *net* asset positions held by agents and the government. Theorem 1 and its corollaries imply (a) that Ricardian equivalence holds in the presence of distorting taxes; (b) that ad-hoc borrowing limits do not restrict the government's ability to respond to shocks; and (c) that government debt is not among the state variables relevant for formulating optimal policy problems recursively, while a vector expressing the net distribution of assets among all agents is. Implication (c) also applies to structures with complete or incomplete asset markets and more general structures of taxes, with and without physical capital. For example, as subsection 3.2 indicates, it applies to structures typical in the New Dynamic Public Finance (NDPF).

Sections 4, 5, 6, and 7 incorporate theorem 1's insights about the state variables that are appropriate for formulating an optimal policy problem. Section 4 formulates a pair of Bellman equations, one for $t \geq 1$, another for $t = 0$, that we use to characterize an optimal policy with affine taxes recursively. The policy functions that satisfy the $t \geq 1$ Bellman equation capture history dependence in the optimal allocation, the tax rate, and transfers through their dependence on two components of the state vector: a vector of marginal utility adjusted net asset positions and a vector of pairwise ratios of marginal utilities.

Before analyzing and numerically solving these Bellman equations, section 5 studies a special case where much can be said analytically, namely, the quasi-linear preferences (they are linear in consumption) featured in a leading example of **?** (henceforth AMSS). We characterize some important differences that emerge in our heterogeneous agent economy with affine taxes vis a vis the representative agent economy with linear taxes of AMSS. First, in our heterogeneous agent economy, there exists a unique set that contains the ergodic distribution of net assets, one that is approached starting from all initial distributions of net asset positions. Second, to ameliorate non-negativity constraints on the consumption of an agent whose Pareto weight is low, the optimal tax-transfer-debt policy distributes assets toward that agent until a point is eventually reached after which all fluctuations in government expenditures can be absorbed by adjusting transfers. The tax rate and output are constant thereafter. Third, unlike the AMSS outcome, the tail allocation is not first-best and the tail of the distorting tax rate on labor income is constant but not zero. These outcomes reflect how very different 'precautionary savings' motives operate in our economy versus the AMSS economy.

The section 5 quasi-linear preference specification is useful for isolating key forces influencing limiting outcomes, but it acquires its analytical tractability by eliminating endogenous interest rate fluctuations. Interest rate fluctuations can be correlated with fundamentals (i.e., government expenditures and shocks to the distribution of skills). Section 6 studies how preference specifications featuring aversion to consumption risk lead to endogenous fluctuations in interest rates that, via their correlations with government expenditures, transfers or fluctuations in the labor tax base (coming from endogenous responses to different types of shocks to skill distributions), influence the long run distribution of net assets. Here a key avenue is that interest rate risk induces risk in the government's earnings on its holdings of risk-free bonds. We show that there exists an ergodic set in the space of marginal utility adjusted net asset positions and pairwise ratios of marginal utilities to which outcomes converge under an optimal policy. Within that ergodic set, fluctuations of the state vector are typically quite small relative to its variation along transient paths approaching that set. When the state space of the exogenous shocks is of dimension 2, the ergodic set is a single point. In that special case, history dependence of allocations, the tax rate, and transfers vanish after that point is attained. The endogenous correlation of interest rates with the fundamental shocks that influence the government's need for revenues to finance transfers and government purchases is a key determinant of the ergodic set for the distribution of net asset positions.

For a calibrated version of our economy, section 7 studies both asymptotic and transient responses to shocks that in different ways shape the endogenous correlation between interest rates and the government's revenue needs. We allow correlated shocks to the skill distribution, to discount factors, and to government expenditures. We find low frequency components that ultimately determine ergodic sets for the two state vectors, and through them, the dynamics of tail allocations and tail government policies. The origin of the shock is also crucial for determining the policy mix over higher frequencies. For example, if recessions are accompanied with increases in inequality, in a recession the optimal policy increases transfers, the tax rate, and government debt, under a plausible normalization.

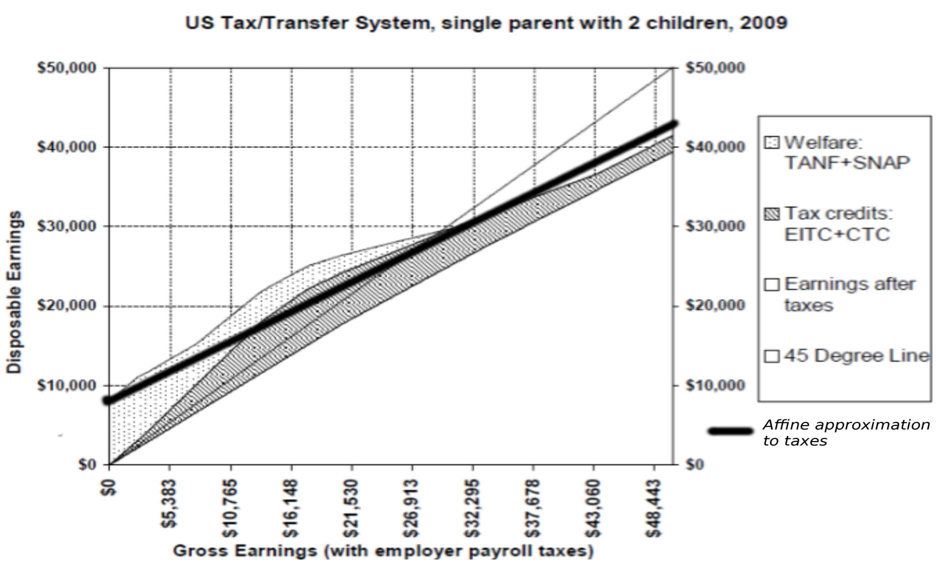

Figure 1: The U.S. tax-transfer system is poorly approximated by a linear function, better by an afine function.

1.1 Relationships to literatures

Two literatures study optimal fiscal policy in response to aggregate shocks. A large literature on Ramsey problems exogenously rules out transfers in the context of representative agent models. **?**, **?**, **?**, and AMSS are leading examples of this approach. In contrast to those papers, our Ramsey planner cares about redistribution among agents with different skills and wealths. Other than prohibiting them from depending explicitly on agent's personal identity, we leave transfers unrestricted and have the Ramsey planner set them optimally. Nevertheless, we find that some of the general principles that emerge from that representative agent, no-transfers literature continue to hold, in particular, the prescription to smooth distortions across time and states. However, it is also true that allowing the government to set transfers optimally substantially changes qualitative and quantitative insights about the optimal policy in important respects.

A second strand of literature focuses on the optimal policy in settings with heterogeneous agents when a government can impose arbitrary taxes subject only to explicit informational constraints (see **?** for a review). A striking result from that literature is that when agent's asset holdings are perfectly observable, the distribution of assets among agents is irrelevant and an optimal allocation can be achieved purely through taxation (see, e.g. **?**). We are able to show that when agents' assets are not observable, the mechanism design problem has some of the same general features as the more simplified Ramsey problem that we study (for example, history dependence of optimal allocations). But because characterization of dynamic mechanism design models with unobservable asset trading is an intricate topic in itself, we leave further analysis along this direction to the future.

Several recent papers impute distributive concerns to a Ramsey planner. Three papers that are perhaps most closely related to ours are **?**, **?**, and **?**. Like us, those authors depart from a representative agent assumption by allowing heterogeneity and considering distributional consequences of alternative tax and borrowing policies. The first paper by Bassetto and Kocherlakota extends the **?** environment to include I types of agents who are heterogeneous in their time-invariant labor productivities w_i. There are complete markets and a Ramsey planner who has access only to proportional taxes on labor income and history-contingent borrowing and lending. The authors study how the Ramsey planner's vector of Pareto weights influences how he responds to government expenditures and other shocks by adjusting the proportional labor tax and government borrowing to cover expenses while manipulating prices in ways that redistribute wealth between low-w_i type 'rentiers' whose main income is from their asset holdings and high-w_i type 'workers' whose main income source is their labor.

? extends the AMSS economy to have two risk-averse households who face idiosyncratic income risk. When idiosyncratic income risk is big enough relative to aggregate government expenditure risk, the Ramsey planner chooses to issue debt in order to help households engage in precautionary saving, thereby overturning the AMSS result that in their quasi-linear case a Ramsey planner eventually sets taxes to zero and lives off its earnings from assets forevermore. Shin emphasizes that the government does this at the cost of imposing tax distortions. While being confined to proportional labor income taxes and nonnegative transfers, Shin's Ramsey planner balances two competing self-insurance motives: aggregate tax smoothing and individual consumption smoothing.

? studies a complete markets economy with heterogeneous agents and transfers that are unrestricted in sign. He obtains counterparts to our results about net versus gross asset positions, including that government assets can be set to zero in all periods. Because he allows unrestricted taxation of initial assets, the initial distribution of assets plays no role. Theorem 1 and its corollaries substantially generalize Werning's results by showing that all allocations of assets among agents and the government that imply the same optimal net asset position lead to the same optimal allocation, a conclusion that holds for market structures beyond complete markets. **?** provides an extensive characterization of optimal allocations and distortions in complete market economies, while we focus on precautionary savings motives for private agents and the government that are not present when markets are complete.,[1] [2]

2 Environment

Exogenous fundamentals of the economy are functions of a shock s_t that follows an irreducible Markov process, where $s_t \in S$ and S is a finite set. We let $s^t = (s_0, ..., s_t)$ denote a history of shocks.

There is a mass π_i of a type $i \in I$ agent, with $\sum_{i=1}^{I} \pi_i = 1$. Types differ by their skills. Preferences of an agent of type i over stochastic processes for consumption $\{c_{i,t}\}_t$ and labor supply $\{l_{i,t}\}_t$ are ordered by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \left[\Pi_{j=0}^{t} \beta(s_j)\right] U^i \left(c_{i,t}, l_{i,t}\right) \tag{1}$$

where $\mathbb{E}_t$ is a mathematical expectations operator conditioned on time t information and $\beta(s_t) \in (0,1)$ is a state-dependent discount factor[3]. We assume that $l_i \in \left[0, \bar{l}_i\right]$ for some $\bar{l}_i < \infty$. Results in section 3 require no additional assumptions on U^i like differentiability or convexity, but results in later sections do.[4]

An agent of type i who supplies l_i units of labor produces $\theta_i\left(s_t\right) l_i$ units of output, where $\theta_i(s_t) \in \Theta$ is a nonnegative state-dependent scalar. Feasible allocations satisfy

$$\sum_{i=1}^{I} \pi_i c_{i,t} + g\left(s_t\right) = \sum_{i=1}^{I} \pi_i \theta_i\left(s_t\right) l_{i,t}, \tag{2}$$

where $g\left(s_t\right)$ denotes exogenous government expenditures in state s_t. We allow s_t to affect $\beta(s_t)$, government expenditures $g(s_t)$, and the type-specific productivities $\theta_i(s_t)$.

To save on notation, mostly we use z_t to denote a random variable with a time t conditional distribution that is a function of the history s^t. Occasionally, we use the more explicit notion $z\left(s^t\right)$ to denote a realization at a particular history s^t.

[1] **?** studies optimal taxation with incomplete markets and explores conditions under which optimal taxes depend only on the aggregate state.

[2] More recent closely related papers are **??** and **?**. While these authors study optimal policy in economies in which agents are heterogeneous in skills and initial assets, they do not allow aggregate shocks.

[3] We allow the discount factor to depend on the Markov state s to generate flexible comovement patterns between real interest rates and fundamentals

[4] Consequently our setup allows both extensive and intensive responses of labor.

A Ramsey planner's preferences over a vector of stochastic processes for consumption and work are ordered by

$$\mathbb{E}_0 \sum_{i=1}^{I} \pi_i \alpha_i \sum_{t=0}^{\infty} \left[\Pi_{j=0}^{t} \beta(s_j)\right] U_t^i \left(c_{i,t}, l_{i,t}\right) \tag{3}$$

where the Pareto weights satisfy $\alpha_i \geq 0$, $\sum_{i=1}^{I} \alpha_i = 1$.

In most of this paper, we study an optimal government policy when agents can trade only a one-period risk-free bond. Except in section 3.2, we assume that the government imposes an affine tax

$$T_t + \tau_t \theta_{i,t} l_{i,t}.$$

We do not restrict the sign of T_t at any t or s^t. If for some type i, $\alpha_i > 0, \theta_{i,t} = 0$, and $b_{i,-1} = 0$, then the planner will choose a nonnegative present value of transfers, since that is the sole source of a type i agent's wealth and consumption.

While results in sections 4, 5, 6, and 7 depend on these assumptions about an affine tax system and incomplete markets, key results of section 3 apply under more general tax functions and market structures.

Under an affine tax system, agent i's budget constraint at t is

$$c_{i,t} + b_{i,t} = \left(1 - \tau_t\right) \theta_{i,t} l_{i,t} + R_{t-1} b_{i,t-1} + T_t, \tag{4}$$

where $b_{i,t}$ denotes asset holdings of a type i agent at time $t \geq 0$, R_{t-1} is a gross risk-free one-period interest rate from $t-1$ to t for $t \geq 1$, and $R_{-1} \equiv 1$. For $t \geq 0$, R_t is measurable with respect to s^t. To rule out Ponzi schemes, we assume that $b_{i,t}$ must be bounded from below. Except in subsection 3.1, we impose no further constraints on agents' borrowing and lending. Subsection 3.1 briefly studies economies with arbitrary borrowing constraints.

The government budget constraint is

$$g_t + B_t = \tau_t \sum_{i=1}^{I} \pi_i \theta_{i,t} l_{i,t} - T_t + R_{t-1} B_{t-1}, \tag{5}$$

where B_t denotes the government's assets at time t, which we assume are bounded from below. Our assumptions about preferences imply that the government can collect only finite revenues in each period, so this restriction rules out government-run Ponzi schemes.

We assume that private agents and the government start with assets $\{b_{i,-1}\}_{i=1}^{I}$ and B_{-1}, respectively. Asset holdings satisfy the market clearing condition

$$\sum_{i=1}^{I} \pi_i b_{i,t} + B_t = 0 \text{ for all } t \geq -1. \tag{6}$$

Since B_t and all $b_{i,t}$ are bounded from below, (6) implies that they are also bounded from above.

Components of competitive equilibria are described below

Definition 1 *An* allocation *is a sequence* $\{c_{i,t}, l_{i,t}\}_{i,t}$. *An* asset profile *is a sequence* $\{\{b_{i,t}\}_i, B_t\}_t$. *A* price system *is an interest rate sequence* $\{R_t\}_t$. *A* tax policy *is a sequence* $\{\tau_t, T_t\}_t$.

Definition 2 *For a given initial asset distribution* $\left(\{b_{i,-1}\}_i, B_{-1}\right)$*, a competitive equilibrium with affine taxes is a sequence* $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, B_t, R_t\}_t$ *and a tax policy* $\{\tau_t, T_t\}_t$*, such that the allocation and the private components* $\{b_{i,t}\}_t$ *of the asset profile maximize (1) subject to (4); the asset profile* $\{\{b_{i,t}\}_i, B_t\}_t$ *is bounded; and constraints (2), (5) and (6) are satisfied.*

Lastly we define optimal competitive equilibria.

Definition 3 *Given* $(\{b_{i,-1}\}_i, B_{-1})$*, an optimal competitive equilibrium with affine taxes is a tax policy* $\{\tau_t^*, T_t^*\}_t$*, an allocation* $\left\{c_{i,t}^*, l_{i,t}^*\right\}$*, an asset profile* $\left\{\left\{b_{i,t}^*\right\}_i, B_t^*\right\}_t$*, and a price system* $\{R_t^*\}_t$ *such that (i) given* $\left(\{b_{i,-1}\}_i, B_{-1}\right)$*, the tax policy, the price system, and the allocation constitute a competitive equilibrium; and (ii) there is no other tax policy* $\{\tau_t, T_t\}_t$ *such that a competitive equilibrium given* $\left(\{b_{i,-1}\}_i, B_{-1}\right)$ *and* $\{\tau_t, T_t\}_t$ *has a strictly higher value of (3).*

We call $\{\tau_t^*, T_t^*\}_t$ an *optimal tax policy*, $\{c_{i,t}^*, l_{i,t}^*\}_{i,t}$ an *optimal allocation*, and $\left\{\left\{b_{i,t}^*\right\}_i, B_t^*\right\}_t$ an *optimal asset profile*.

3 Relevant and Irrelevant Aspects of the Distribution of Government Debt

This section sets forth a result that underlies much of the analysis in this paper, namely, that the level of government debt is not a state variable for our economy. The reason is that there is an equivalence class of tax policies and asset profiles that support the same competitive equilibrium allocation. A competitive equilibrium allocation pins down only net asset positions. The assertions in this section apply to all competitive equilibria, not just the optimal ones that will be our focus in subsequent sections.

Theorem 1 *Given* $\left(\{b_{i,-1}\}_i, B_{-1}\right)$*, let* $\left\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, B_t, R_t\right\}_t$ *and* $\{\tau_t, T_t\}_t$ *be a competitive equilibrium. For any bounded sequences* $\left\{\hat{b}_{i,t}\right\}_{i,t\geq -1}$ *that satisfy*

$$\hat{b}_{i,t} - \hat{b}_{1,t} = \tilde{b}_{i,t} \equiv b_{i,t} - b_{1,t} \text{ for all } t \geq -1, i \geq 2,$$

there exist sequences $\left\{\hat{T}_t\right\}_t$ *and* $\left\{\hat{B}_t\right\}_{t\geq -1}$ *that satisfy (6) such that* $\left\{\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_i, \hat{B}_t, R_t\right\}_t$ *and* $\left\{\tau_t, \hat{T}_t\right\}_t$ *constitute a competitive equilibrium given* $\left(\left\{\hat{b}_{i,-1}\right\}_i, \hat{B}_{-1}\right)$.

Proof. Let

$$\hat{T}_t = T_t + \left(\hat{b}_{1,t} - b_{1,t}\right) - R_{t-1}\left(\hat{b}_{1,t-1} - b_{1,t-1}\right) \text{ for all } t \geq 0. \tag{7}$$

Given a tax policy $\left\{\tau_t, \hat{T}_t\right\}_t$, the allocation $\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_t$ is a feasible choice for consumer i since it satisfies

$$\begin{aligned}
c_{i,t} &= (1-\tau_t)\,\theta_{i,t}l_{i,t} + R_{t-1}b_{i,t-1} - b_{i,t} + T_t. \\
&= (1-\tau_t)\,\theta_{i,t}l_{i,t} + R_{t-1}\left(b_{i,t-1} - b_{1,t-1}\right) - \left(b_{i,t} - b_{1,t}\right) + T_t + R_{t-1}b_{1,t-1} - b_{1,t} \\
&= (1-\tau_t)\,\theta_{i,t}l_{i,t} + R_{t-1}\left(\hat{b}_{i,t-1} - \hat{b}_{1,t-1}\right) - \left(\hat{b}_{i,t} - \hat{b}_{1,t}\right) + T_t + R_{t-1}b_{1,t-1} - b_{1,t} \\
&= (1-\tau_t)\,\theta_{i,t}l_{i,t} + R_{t-1}\hat{b}_{i,t-1} - \hat{b}_{i,t} + \hat{T}_t
\end{aligned}$$

Suppose that $\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_t$ is not the optimal choice for consumer i, in the sense that there exists some other sequence $\left\{\hat{c}_{i,t}, \hat{l}_{i,t}, \hat{b}_{i,t}\right\}_t$ that gives strictly higher utility. Then the choice $\left\{\hat{c}_{i,t}, \hat{l}_{i,t}, b_{i,t}\right\}_t$ is feasible given the tax rates $\{\tau_t, T_t\}_t$, which contradicts the assumption that $\{c_{i,t}, l_{i,t}, b_{i,t}\}_t$ is the optimal choice for the consumer given taxes $\{\tau_t, T_t\}_t$. The new allocation satisfies all other constraints and therefore is an equilibrium. ■

An immediate corollary is that it is not total government debt but rather who owns it that affects equilibrium allocations.

Corollary 1 *For any pair B'_{-1}, B''_{-1}, there are asset profiles $\left\{b'_{i,-1}\right\}_i$ and $\left\{b''_{i,-1}\right\}_i$ such that equilibrium allocations starting from $\left(\left\{b'_{i,-1}\right\}_i, B'_{-1}\right)$ and from $\left(\left\{b''_{i,-1}\right\}_i, B''_{-1}\right)$ are the same. These asset profiles satisfy*

$$b'_{i,-1} - b'_{1,-1} = b''_{i,-1} - b''_{1,-1} \text{ for all } i.$$

This version of Ricardian equivalence holds despite the fact that the government can use distorting taxes.[5] Theorem 1 shows that many different transfer sequences $\{T_t\}_t$ and asset profiles $\{b_{i,t}, B_t\}_{i,t}$ support the same equilibrium allocation. For example, one can set government assets $B_{i,t} = 0$ without loss of generality. Alternatively, we can normalize assets $b_{i,t}$ of any type i.

Theorem 1 continues to hold in more general environments. For example, we could allow agents to trade all Arrow securities and still show that equilibrium allocations depend only on agents' net assets positions. Similarly, our results hold in economies with capital. In section 3.2, we explore outcomes when a government has access to more general taxes, for example, of a form $T_t(y_t)$ or $T_t(y_t, ..., y_0)$. While those richer tax systems allow better redistribution and smaller distortions, they do not affect the conclusions of theorem 1.

3.1 Extension 1: Borrowing constraints

Representative agent models rule out Ricardian equivalence either by assuming distorting taxes or by imposing ad hoc borrowing constraints. By way of contrast, we have verified that Ricardian equivalence holds in our economy even though there are distorting taxes. Imposing ad-hoc borrowing limits also leaves Ricardian equivalence intact in our economy.[6] In economies with exogenous borrowing constraints, agents' maximization problems include the additional constraints

$$b_{i,t} \geq \underline{b}_i \tag{8}$$

for some exogenously given $\{\underline{b}_i\}_i$.

[5] **?**'s Modigliani-Miller theorem for a class of government open market operations has a similar flavor. **?** describes the structure of a set of related Modigliani-Miller theorems for government finance.

[6] **?** describe how a government can use borrowing constraints as part of a welfare-improving policy to finance exogenous government expenditures.**?** describe Modigliani-Miller theorems for government finance in a collection of economies in which borrowing constraints on classes of agents produce the kind of rate of return discrepancies that Bryant and Wallace manipulate.

Definition 4 *For given* $\left(\{b_{i,-1}, \underline{b}_i\}_i, B_{-1}\right)$ *and* $\{\tau_t, T_t\}_t$, *a competitive equilibrium with affine taxes and exogenous borrowing constraints is a sequence* $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, B_t, R_t\}_t$ *such that* $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ *maximizes (1) subject to (4) and (8),* $\{\{b_{i,t}\}_i, B_t\}_t$ *are bounded, and constraints (2), (5) and (6) are satisfied.*

We can define an *optimal* competitive equilibrium with exogenous borrowing constraints by extending Definition 3.

The introduction of the ad-hoc debt limits leaves unaltered the conclusions of Corollary 1 and the role of the initial distribution of assets across agents. The next proposition asserts that ad-hoc borrowing limits do not limit a government's ability to respond to aggregate shocks.[7]

Proposition 1 *Given an initial asset distribution* $\left(\{b_{i,-1}\}_i, B_{-1}\right)$, *let* $\{c_{i,t}, l_{i,t}\}_{i,t}$ *and* $\{R_t\}_t$ *be a competitive equilibrium allocation and interest rate sequence in an economy without exogenous borrowing constraints. Then for any exogenous constraints* $\{\underline{b}_i\}_i$, *there is a government tax policy* $\{\tau_t, T_t\}_t$ *such that* $\{c_{i,t}, l_{i,t}\}_{i,t}$ *is a competitive equilibrium allocation in an economy with exogenous borrowing constraints* $\left(\{b_{i,-1}, \underline{b}_i\}_i, B_{-1}\right)$ *and* $\{\tau_t, T_t\}_t$.

Proof. Let $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ be a competitive equilibrium allocation without exogenous borrowing constraints. Let $\Delta_t \equiv \max_i \{\underline{b}_i - b_{i,t}\}$. Define $\hat{b}_{i,t} \equiv b_{i,t} + \Delta_t$ for all $t \geq 0$ and $\hat{b}_{i,-1} = b_{-1}$. By Theorem 1, $\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_{i,t}$ is also a competitive equilibrium allocation without exogenous borrowing constraints. Moreover, by construction $\hat{b}_{i,t} - \underline{b}_i = b_{i,t} + \Delta_t - \underline{b}_i \geq 0$. Therefore, $\hat{b}_{i,t}$ satisfies (8). Since agents' budget sets are smaller in the economy with exogenous borrowing constraints, and $\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_{i,t}$ are feasible at interest rate process $\{R_t\}_t$, then $\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_{i,t}$ is also an optimal choice for agents in the economy with exogenous borrowing constraints $\{\underline{b}_i\}_i$. Since all market clearing conditions are satisfied, $\left\{c_{i,t}, l_{i,t}, \hat{b}_{i,t}\right\}_{i,t}$ is a competitive equilibrium allocation and asset profile. ■

To explore the intuition underlying Proposition 1, suppose to the contrary that the exogenous borrowing constraints restricted a government's ability to achieve a desired allocation. That means that the government would want to increase its borrowing and to repay agents later, which the borrowing constraints prevent. But the government can just reduce transfers today and increase them tomorrow. That would achieve the desired allocation without violating the exogenous borrowing constraints.

Actually, welfare can be strictly higher in an economy with exogenous borrowing constraints because a government might want to push some agents against their borrowing limits. When some agents' borrowing constraints bind, their shadow interest rates differ from the common interest rate that unconstrained agents face. When the government rearranges tax policies to affect the interest rate, it affects constrained and unconstrained agents differently. By facilitating redistribution, this can improve welfare. In appendix 9.1, we construct an example without any shocks in which the government can achieve higher welfare by using borrowing constraints to improve its ability to redistribute.

[7] See **??** shows a closely related result.

3.2 Extension 2: More general taxes

The conclusions of theorem 1 also apply in settings typical in a New Dynamic Public Finance (NDPF) literature that studies taxes and other arrangements that decentralize an optimal allocation that respects information gaps between agents and a government.[8] Thus, consider an economy where agents' skills θ are private information (and dont change over time). Suppose that the government observes output $y \equiv \theta l$ and c for each agent. Constrained optimal allocations solve the mechanism design problem

$$\max_{\{c_{i,t},y_{i,t}\}} \mathbb{E}_0 \sum_{i=1}^{I} \alpha_i \pi_i \sum_{t=0}^{\infty} \left[\Pi_{j=0}^{t} \beta(s_j)\right] U^i \left(c_{i,t}, \frac{y_{i,t}}{\theta_i}\right) \tag{9}$$

subject to incentive constraints

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \left[\Pi_{j=0}^{t} \beta(s_j)\right] U^i \left(c_{i,t}, \frac{y_{i,t}}{\theta_i}\right) \geq \mathbb{E}_0 \sum_{t=0}^{\infty} \left[\Pi_{j=0}^{t} \beta(s_j)\right] \left(c_{j,t}, \frac{y_{j,t}}{\theta_i}\right) \text{ for all } i,j \tag{10}$$

and the feasibility constraint

$$\sum_{i=1}^{I} \pi_i c_{i,t} + g_t = \sum_{i=1}^{I} \pi_i y_{i,t}. \tag{11}$$

Let $\left\{c_{i,t}^{sp}, y_{i,t}^{sp}\right\}_{i,t}$ be the optimal allocation. To implement the optimal allocation in a competitive equilibrium, we allow a general non-linear tax schedule $T_t\left(y_t, X_t\right)$, where X_t is a vector of agent-specific variables like past labor earnings.[9]

We assume that agents begin with initial debt holdings $\{b_{i,-1}\}_i$ and that each period they can trade a one-period risk-free bond with gross return $\{R\}_t$. An agent's budget constraint is the same as (4), except that now $-T_t + \tau y_t$ is replaced by $T_t\left(y_t, X_t\right)$. We modify the definition of a competitive equilibrium with these more general taxes in the natural way.

Debt plays no significant role, because if his debt differs from zero, the government can tax away *all* of an agent's income from assets by setting $T_t\left(y_t, b_{t-1}, F\left(\{y_s\}_{s=0}^{t-1}\right)\right) = y_t + \max\{R_t b_t, 0\}$ if $b_t \neq 0$.

Thus, in this economy, a much stronger version of theorem 1 emerges.[10]

Corollary 2 *Any sequence of assets $\{b_{i,t}\}_{i,t}$ is a part of an optimal competitive equilibrium allocation for some optimal non-linear tax $T_t\left(y_t, b_{t-1}, F\left(\{y_s\}_{s=0}^{t-1}\right)\right)$.*

This conclusion is sensitive to the presumption that the private information pertains only to agent's skills. Appendix 9.5 shows that if in addition, agents assets are also unobservable, then theorem 1 can still be recovered in the sense that only *net* positions matter.

[8] For surveys, see **?** and **?**.

[9] It is easy to see that the optimal allocation is *history independent.* We can rewrite the mechanism design problem as

$$\max_{\{c_{i,t},y_{i,t}\}} \min_{\{\eta_{ij}\}} \mathbb{E}_0 \sum_{i=1}^{I} \sum_{t=0}^{\infty} \left[\Pi_{j=0}^{t} \beta(s_j)\right] \left\{ \left(\alpha_i \pi_i + \eta_{i,i}\right) U^i \left(c_{i,t}, \frac{y_{i,t}}{\theta_i}\right) - \sum_{j \neq i} \eta_{j,i} U^j \left(c_{j,t}, \frac{y_{j,t}}{\theta_i}\right) \right\}$$

subject to (11). This problem is equivalent to a sequence of static problems for each realization of g_t. Appendix 9.5 constructs an example that shows if, in addition, agents' assets are also unobservable, then optimal allocations are history dependent

[10] See also a related result of **?**, who showed that when taxes can depend on past labor income, the path of government debt is indeterminate.

3.3 Ricardian irrelevance and optimal equilibria

Our statements about Ricardian irrelevance apply to all competitive equilibrium allocations, not just the optimal ones that are the main focus of this paper. To appreciate how these Ricardian irrelevance results affect optimal equilibria, suppose that we increase an initial level of government debt from 0 to some arbitrary level $B'_{-1} > 0$. If the government were to hold transfers $\{T_t\}_t$ fixed, it would have to increase tax rates $\{\tau_t\}_t$ enough to collect a present value of revenues sufficient to repay B'_{-1}. Since deadweight losses are convex in τ, higher levels of debt financed with bigger distorting taxes $\{\tau_t\}$ impose larger distortions on the economy, thereby degrading the equilibrium allocation. But this would not happen if the government were instead to adjust transfers in response to a higher initial debt. To determine optimal transfers, we need to know who owns the initial government debt B'_{-1}. For example, suppose that agents hold equal amounts of it. Then each unit of debt repayment achieves the same redistribution as one unit of transfers. If the original tax policy at $B'_{-1} = 0$ were optimal, then the best policy for a government with initial debt $B'_{-1} > 0$ would be to reduce the present value transfers by exactly the amount of the increase in per capita debt, because then distorting taxes $\{\tau_t\}$ and the allocation would both remain unchanged.[11]

But the situation would be different if holdings of government debt were not equal across agents. For example, suppose that richer people owned disproportionately more government debt than poorer people. That would mean that inequality is effectively initially higher in an economy with higher initial government debt. As a result, a government with Pareto weights $\{\alpha_i\}$ that favor equality would want to increase both distorting tax rates $\{\tau_t\}$ and transfers $\{T_t\}$ to offset the increase in inequality associated with the increase in government debt. The conclusion would be the opposite if government debt were to be owned mostly by poorer households.

This logic shows how important it is to know the distribution of government debt across people. Government debt that is widely distributed across households (e.g., implicit Social Security debt) is less distorting than government debt owned mostly by people whose incomes are at the top of the income distribution (e.g., government debt held by hedge funds).[12]

4 Optimal equilibria with affine taxes

This section prepares a recursive formulation of a Ramsey planner's problem with affine taxes in a risk-free-bond only economy. We display implementability conditions and two Bellman equations associated with the planning problem. In section 5, we use these Bellman equations to characterize an optimal equilibrium in an economy with quasilinear preferences. In section 6, we use them to study asymptotic outcomes under more general preferences. In section 7, we use numerical solutions of these Bellman equations to study a calibrated economy.

We now assume that $U^i : \mathbb{R}^2_+ \to \mathbb{R}$ is concave in $(c, -l)$ and twice continuously differentiable. We let

[11] This example illustrates principles proclaimed by Simon **?**, p. 85 in the quotation with which we began this paper.

[12] It is straightforward to extend our analysis to open economy with foreign holdings of domestic debt. The more government debt is owned by the foreigners, the higher are the distorting taxes that the government needs to impose.

$U_{x,t}^i$ or $U_{xy,t}^i$ denote first and second derivatives of U^i with respect to $x, y \in \{c, l\}$ in period t and assume that $\lim_{x \to \bar{l}_i} U_l^i(c, x) = \infty$, $\lim_{x \to 0} U_l^i(c, x) = 0$ for all c and i.

4.1 Implementability conditions

We focus on interior equilibria. First-order necessary conditions for the consumer's problem are

$$(1 - \tau_t)\, \theta_{i,t} U_{c,t}^i = -U_{l,t}^i \tag{12}$$

and

$$U_{c,t}^i = \beta(s_t) R_t \mathbb{E}_t U_{c,t+1}^i. \tag{13}$$

To help characterize an equilibrium, we use

Proposition 2 *A sequence $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, R_t, \tau_t, T_t\}_t$ is part of a competitive equilibrium with affine taxes if and only if it satisfies (2), (4), (12), and (13) and $b_{i,t}$ is bounded for all i and t.*

Proof. Necessity is obvious. In the appendix 9.2, we use arguments of **?** and **?** to show that any $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ that satisfies (4), (12), and (13) is a solution to consumer i's problem. Equilibrium $\{B_t\}$ is determined by (6) and constraint (5) is then implied by Walras' Law ■

To find an optimal equilibrium, by Proposition 2 we can choose $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, R_t, \tau_t, T_t\}_t$ to maximize (3) subject to (2), (12), and (13). We apply a first-order approach and follow steps similar to ones taken by **?** and AMSS. Substituting consumers' first-order conditions (12) and (13) into the budget constraints (4) yields implementability constraints

$$c_{i,t} + b_{i,t} = -\frac{U_{l,t}^i}{U_{c,t}^i} l_{i,t} + T_t + \frac{U_{c,t-1}^i}{\beta(s_{t-1}) \mathbb{E}_{t-1} U_{c,t}^i} b_{i,t-1} \text{ for all } i, t. \tag{14}$$

For $I \geq 2$, we can use constraint (14) for $i = 1$ to eliminate T_t from (14) for $i > 1$. Define $\tilde{b}_{i,t} \equiv b_{i,t} - b_{1,t}$ and represent the implementability constraints as

$$\begin{aligned} & (c_{i,t} - c_{1,t}) + \tilde{b}_{i,t} \\ = \quad & -\frac{U_{l,t}^i}{\theta_{i,t} U_{c,t}^i} (\theta_{i,t} l_{i,t} - \theta_{1,t} l_{1,t}) + \frac{U_{c,t-1}^i}{\beta(s_{t-1}) \mathbb{E}_{t-1} U_{c,t}^i} \tilde{b}_{i,t-1} \text{ for all } i > 1, t. \end{aligned} \tag{15}$$

With this representation of the implementability constraints, the planner's maximization problem depends only on the $I-1$ variables $\tilde{b}_{i,t-1}$. The reduction of the dimensionality from I to $I-1$ is another consequence of theorem 1.

Let $\mathbf{x} = \beta^{-1} \left(U_c^2 \tilde{b}_2, ..., U_c^I \tilde{b}_I \right)$, $\boldsymbol{\rho} = \left(U_c^2 / U_c^1, ..., U_c^I / U_c^1 \right)$, and denote an allocation $a = \{c_i, l_i\}_{i=1}^I$. In the spirit of **?** and **?**, we split the problem into a time-0 problem that takes $(\{\tilde{b}_{i,-1}\}_{i=2}^I, s_0)$ as given and a time $t \geq 1$ continuation problem that takes x, ρ, s_{-} as given. We formulate two Bellman equations and two value functions, one that pertains to $t \geq 1$, another for $t = 0$.

For $t \geq 1$, let $V(\mathbf{x}, \boldsymbol{\rho}, s_{-})$ be the continuation value to the planner given $x_{t-1} = x, \rho_{t-1} = \rho, s_{t-1} = s_{-}$. It satisfies the Bellman equation

$$V(\mathbf{x}, \boldsymbol{\rho}, s_{-}) = \max_{a(s), x'(s), \rho'(s)} \sum_s \Pr(s|s_{-}) \left(\left[\sum_i \pi_i \alpha_i U^i(s) \right] + \beta(s) V(\mathbf{x}'(s), \boldsymbol{\rho}'(s), s) \right) \tag{16}$$

subject to

$$U_c^i(s)\left[c_i(s) - c_1(s)\right] + \beta(s) x_i'(s) + \left(U_l^i(s) l_i(s) - U_c^i(s) \frac{U_l^1(s)}{U_c^1(s)} l_1(s) \right) = \frac{x U_c^i(s)}{\mathbb{E}_{s_{-}} \mathbf{U}_c^i} \text{ for all } s, i \geq 2 \tag{17a}$$

$$\frac{\mathbb{E}_{s_{-}} \mathbf{U}_c^i}{\mathbb{E}_{s_{-}} \mathbf{U}_c^1} = \rho_i \text{ for all } i \geq 2 \tag{17b}$$

$$\frac{U_l^i(s)}{\theta_i(s) U_c^i(s)} = \frac{U_l^1(s)}{\theta_1(s) U_c^1(s)} \text{ for all } s, i \geq 2 \tag{17c}$$

$$\sum_i \pi_i c_i(s) + g(s) = \sum_i \pi_i \theta_i(s) l_i(s) \quad \forall s \tag{17d}$$

$$\rho_i'(s) = \frac{U_c^i(s)}{U_c^1(s)} \text{ for all } s, i \geq 2 \tag{17e}$$

Let $V_0\left(\{\tilde{b}_{i,-1}\}_{i=2}^I, s_0\right)$ be the value to the planner at $t = 0$, where $\tilde{b}_{i,-1}$ denotes initial debt inclusive of accrued interest. It satisfies the Bellman equation

$$V_0\left(\{\tilde{b}_{i,-1}\}_{i=2}^I, s_0\right) = \max_{a_0, x_0, \rho_0} \sum_i \pi_i \alpha_i U^i(c_{i,0}, l_{i,0}) + \beta(s_0) V\left(x_0, \rho_0, s_0\right) \tag{18}$$

subject to

$$U_{c,0}^i\left[c_{i,0} - c_{1,0}\right] + \beta(s_0) x_{i,0} + \left(U_{l,0}^i l_{i,0} - U_{c,0}^i \frac{U_{l,0}^1}{U_{c,0}^1} l_{1,0} \right) = U_{c,0}^i \tilde{b}_{i,-1} \text{ for all } i \geq 2 \tag{19a}$$

$$\frac{U_{l,0}^i}{\theta_{i,0} U_{c,0}^i} = \frac{U_{l,0}^1}{\theta_{1,0} U_c^{1,0}} \text{ for all } i \geq 2 \tag{19b}$$

$$\sum_i \pi_i c_{i,0} + g_0 = \sum_i \pi_i \theta_{i,0} l_{i,0} \tag{19c}$$

$$\rho_{i,0} = \frac{U_{c,0}^i}{U_{c,0}^1} \text{ for all } i \geq 2 \tag{19d}$$

The time 0 problem differs from the time $t \geq 1$ problem since constraint (17b) is absent from the time 0 problem.

5 Quasi-linear preferences

Before we use the section 4 Bellman equations, this section restricts preferences in a way that allows us to get a long way analytically and to identify key forces that drive outcomes and that distinguish them from what emerge in representative agent economies. In particular, we use quasi-linear preferences[13]

$$U^i(c, l) = c - h_i(l), \tag{20}$$

[13]These preferences without the nonnegativity constraint on consumption have been studied in the context of representative agent economies. See AMSS, **?**, **??**, **?**, **?**.

where h_i is an increasing differentiable function with $h_i'(0) = 0$ and $h_i'\left(\bar{l}_i\right) = \infty$. We restrict consumption to be non-negative

$$c \geq 0. \tag{21}$$

We divide our presentation into two parts. First, we consider the optimal response to expenditure shocks in our heterogenous agent economy in which optimal equilibria are interior in the sense that constraint (21) does not bind. We show that while equilibrium dynamics in AMSS are driven by the exogenous restriction that transfers are non-negative, with a sufficiently redistributive government in a heterogenous agent economy like ours, such constraints are not present. Furthermore, if they were present, they might not bind. That leads to substantially different optimal policies.

In the second part of this section, we study optimal policy when an optimal equilibrium is not interior, meaning that constraint (21) binds in some states. We describe a force present in economies with heterogeneous agents, namely, that a reduction in transfers can lead to an increase in inequality and a decrease in welfare. This type of welfare loss is not present in representative agent economies.

5.1 The optimal response to government expenditure shocks

To contrast outcomes in our environment with those in AMSS's, we temporarily restrict our attention to economies with shocks only to g_t. So for now, β and $\{\theta_i\}_i$ do not vary with s_t. We assume that parameters and initial conditions are such that constraint (21) does not bind for any type i. This will be true if the government puts a sufficiently high Pareto weight on low productivity types and the shocks to g are not too high.[14] To simplify notation, we now assume that the initial debt is $\left\{\beta^{-1} b_{i,-1}\right\}_i$.

Proposition 3 *Suppose that preferences are quasi-linear for all i and that an equilibrium is interior. Then the optimal tax rate τ_t^* satisfies $\tau_t^* = \tau^*$. An optimum asset profile $\left\{b_{i,t}^*, B_t^*\right\}_{i,t}$ can be chosen to satisfy $b_{i,t}^* = b_{i,-1}$ for all i and $B_t^* = B_{-1}$ for all $t \geq 0$.*[15]

Proof. When an equilibrium allocation is interior, the interest rate $R_t = \beta^{-1}$ for all t and $(1-\tau_t)\,\theta_i = h_i'(l_{i,t})$ for all t. For our purposes, it is more convenient to express the labor supply component of the allocation as a function of $(1-\tau)$ and to optimize with respect to τ rather than $\{l_i\}_i$. We invert $h_i'(\cdot)$ to express labor supply l_i as a function of $(1-\tau)$. Call this function $H_i(1-\tau)$. Using this we can rewrite the implementability constraint (15) as

$$c_{i,t} - c_{1,t} + \tilde{b}_{i,t} - (1-\tau_t)\left[\theta_i H_i(1-\tau_t) - \theta_1 H_1(1-\tau_t)\right] = \beta^{-1}\tilde{b}_{i,t-1}. \tag{22}$$

Given $\{\tilde{b}_{i,-1}\}_{i\geq 2}$, the optimal policy solves

$$\max_{\left\{c_{i,t}, \tilde{b}_{i,t}, tau_t, T_t\right\}_{i,t}} \mathbb{E}_0 \sum_{t=0}^{\infty} \sum_{i=1}^{I} \alpha_i \pi_i \beta^t \left[c_{i,t} - h_i\left(H_i(1-\tau)\right)\right] \tag{23}$$

[14] In section 5.2, we show these conditions explicitly when agents have CES disutility from labor.

[15] We thank Guy Laroque for suggesting the idea for this proof.

subject to $\{b_{i,t}\}_{i,t}$ being bounded, (22), and

$$\sum_{i=1}^{I} \pi_i c_{i,t} + g_t = \sum_{i=1}^{I} \pi_i \theta_i H_i\left(1-\tau_t\right) \tag{24}$$

Note that since $\left\{\tilde{b}_{i,t}\right\}_{i,t}$ is bounded,

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\beta^{-1}\tilde{b}_{i,t-1} - \tilde{b}_{i,t}\right] = \beta^{-1}\tilde{b}_{i,-1} + \lim_{\mathcal{T}\to\infty} \mathbb{E}_0 \left(\sum_{t=0}^{\mathcal{T}} \beta^t \left[\tilde{b}_{i,t} - \tilde{b}_{i,t}\right] - \beta^{\mathcal{T}+1}\tilde{b}_{i,\mathcal{T}+1}\right) = \beta^{-1}\tilde{b}_{i,-1}.$$

By theorem 1, we can normalize $b_{1,t} = 0 \quad t \geq -1$. This gives

$$c_{1,t} = \theta_i(1-\tau_t)H_i(1-\tau_t) + T_t. \tag{25}$$

Furthermore, multiplying by β^t and summing (22) over all t, we can re-write the objective as

$$\mathbb{E}_0 \sum_{t=0}^{\infty}\sum_{i=2}^{I} \alpha_i \pi_i \beta^t \left[(1-\tau_t)\left\{\theta_i H_i(1-\tau_t) - \theta_1 H_1(1-\tau_t)\right\} - h_i\left(H_i(1-\tau_t)\right) + h_1\left(H_1(1-\tau_t)\right)\right] \tag{26}$$
$$+\, \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[(1-\tau_t)\theta_1 H_1(1-\tau_t) + T_t - h_1\left(H_1(1-\tau_t)\right)\right] + \beta^{-1}\sum_{i=2}^{I} \beta^{-1}\tilde{b}_{i,-1}.$$

Similarly, we can eliminate $c_{i,t}$'s from the implementability constraint (22) and obtain

$$\sum_i \pi_i \theta_i H_i(\tau_t) - g_t - (1-\tau_t)\theta_1 H_1(1-\tau_t) - T_t - \sum_{i=2}^{I}\left[(1-\tau)\left\{\theta_i H_i(1-\tau_t) - \theta_1 H_1(1-\tau_t)\right\} - \tilde{b}_{i,t} + \beta^{-1}\tilde{b}_{i,t-1}\right] = 0 \tag{27}$$

Note that the revised objective (26) and the implementability constraint (27) are both linear in T_t. The first order condition with respect to T_t implies that the time t multiplier $\mu_t\beta^t = \mu\beta^t \quad \forall t$. Therefore, optimal tax rates τ_t^* are also constant and independent of t. Given τ_t^*, constraint (27) pins down $T_t + \sum_{i=2}^{I} \tilde{b}_{i,t} - \beta^{-1}\tilde{b}_{i,t-1}$. Without loss of generality, we can set the asset profiles to be constant and choose T_t^* to satisfy equation (27).

■

In the optimal equilibria for the quasi-linear economy described in Proposition 3, fluctuations in lump-sum taxes and transfers "do all the work". In period 0, the government chooses an optimal present value of transfers and a constant tax rate that pays for it. Tax rates and transfers depend on the Pareto weights $\{\alpha_i\}$: higher Pareto weights on low skilled agents imply higher transfers and tax rates. In response to a shock g_t, the government adjusts transfers in period t by the amount of the shock. Since all agents are risk-neutral, welfare is unaffected by fluctuations in transfers. This allows the government perfectly to smooth distorting tax rates.

Comparison with representative agent economies

The **?** and AMSS representative agent models impose $T_t \geq 0$, a constraint that always binds in the Lucas and Stokey model and that binds in the AMSS model until the government has acquired enough assets to finance all future expenditures from earnings on those assets. In those representative agent models, the government would like to impose lump-sum *taxes*, not transfers. Distributional motives not present in those models, but present here, make things different.

Proposition 3 shows that with quasi-linear preferences, the dynamics of optimal tax rates, transfers, and allocations in our economy differ starkly from those in representative agent Ramsey models like AMSS. To indicate how these differences are explained by concerns for redistribution and not the absence of exogenous restrictions on transfers, we impose an additional constraint in our maximization problem (23), namely,

$$T_t \geq 0 \text{ for all } t. \tag{28}$$

The following proposition states that our economy converges to an allocation similar to that characterized in proposition 3 where constraint (28) was slack [16].

Proposition 4 *Assume that $I > 1$. Let $\beta^t \chi_t$ be the Lagrange multiplier on constraint (28) in a version of maximization problem (23) that is augmented with constraint (28). Then $\chi_t \to 0$ a.s.*

Proof. Our augmented version of the maximization problem (23) can be expressed as maximization problem with an additional constraint (28). The first-order conditions for T_t yield $\sum_{i=1}^{I} \alpha_i \pi_i = \mu_t - \chi_t$, while the first-order condition for $b_{i,t}$ implies $\mu_t = \mathbb{E}_t \mu_{t+1}$. Since $\chi_t \geq 0$, these two conditions imply that χ_t is a nonnegative martingale and therefore χ_t must almost surely converge to a constant. This, in turn, implies that μ_t must almost surely converge to a constant. Then the first-order conditions for τ_t also imply that τ_t must converge a.s. to some τ^*.

Suppose $\chi_t \to \chi^* > 0$. This implies that $T_t \to 0$ and (27) becomes

$$\sum_i \pi_i \theta_i H_i(\tau^*) - g_t - (1-\tau)\theta_1 H_1(1-\tau^*) - \sum_{i=2}^{I} \left[(1-\tau^*) \left\{ \theta_i H_i(1-\tau^*) - \theta_1 H_1(1-\tau^*) \right\} + B_t - \beta^{-1} B_{t-1} \right] = 0 \tag{29}$$

where we used (6) to substitute for $\sum_{i=2}^{I} \pi_i b_{i,t}$. If g_t can take more than one value and follows an irreducible Markov process, then for any bound on B_t, we can find a sequence of government expenditures g_t for which this bound will eventually be violated, leading to a contradiction. This implies that $\chi_t \to 0$. ∎

Proposition 3 and 4 shed light on the key force that drives asymptotic outcomes in AMSS for quasi-linear preferences. Although constraint (28) need not bind in $I > 1$ economies like ours, when $I = 1$ it almost *always* binds. Since the risk-free interest rate equals the discount rate, a Ramsey planner saves partly to relax future constraints (28), a motive that endures until the planner has saved enough to render slack all future constraints (28). Distortionary taxes τ_t^* are asymptotically zero.

In contrast, if the restriction that $T_t \geq 0$ is imposed when agents are heterogenous and the government cares about redistribution, the government will reallocate assets across heterogeneous agents until the

[16] In some settings with heterogenous agents, for example, one with $\alpha_i > 0$ for all i and $\theta_1 = b_{1,-1} = 0$, constraints (28) is *always* slack.

constraint $T_t \geq 0$ never again binds. After that, the tax rate will typically be positive and the continuation allocation will not be first-best, in contradiction to the AMSS limiting outcome with quasi-linear preferences.

For quasi-linear preferences, figure 2 compares equilibrium dynamics in a representative agent (AMSS) economy and an economy with two agents, one who is not productive, and Pareto weights chosen to make transfers be positive at all times and along all histories. The sequences of s_t shocks are identical across the two economics. While tax rates converge to zero for the AMSS economy, they are constant for the heterogeneous agent economy.[17].

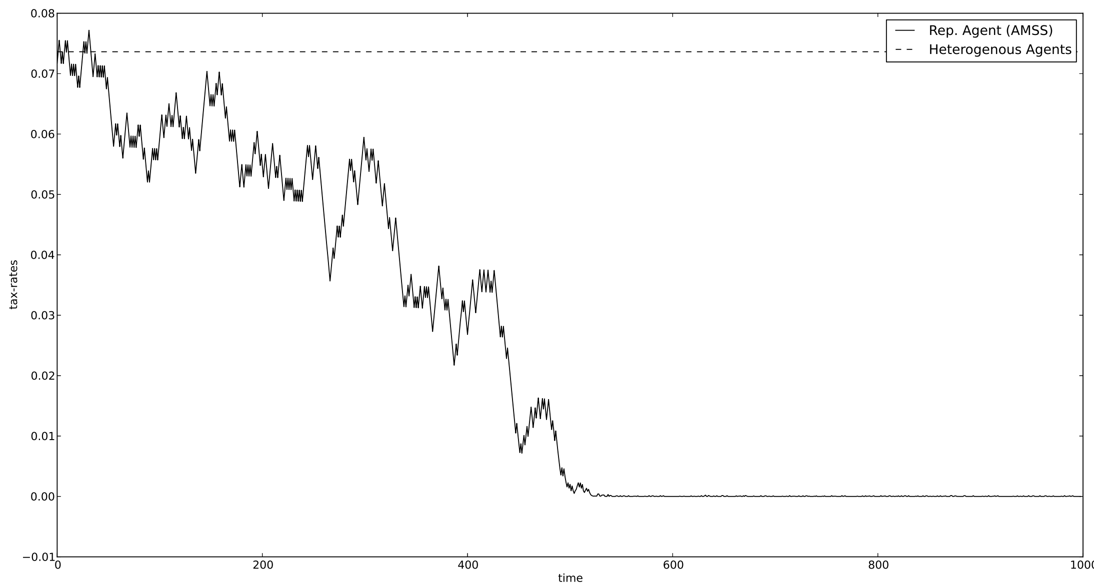

Figure 2: Taxes in AMSS (solid line) and heterogenous agent economy (dotted line) with quasi-linear preferences

5.2 Non-interior equilibria

In this subsection, we specify fundamentals of our economy in ways designed to illustrate the role played by the Pareto planner's preferences for equality. We restrict ourselves to the case where $I = 2$, $\theta_1 > \theta_2 = 0$ and CES disutility of labor $h_1(l) = \frac{1}{1+\gamma} l^{1+\gamma}$. We also maintain

Assumption 1

$$\alpha_2 \geq \frac{\gamma - 1}{\gamma}$$

[17]The plots for the AMSS and the heterogeneous agent economies are both for quasi-linear preferences with a Frisch elasticity of labor equals 0.5 and a discount factor $\beta = 0.95$. In the AMSS economy, the agent's initial assets are zero and government expenditure shocks $g(s_t) \in \{.1, .3\}$ are generated using an IID process with equally likely outcomes. For the heterogeneous agent economy, we set $\alpha_2 = .54$ so that the initial labor taxes are similar to those for the AMSS economy

Assumption 2

$$\theta_1 \left(\frac{\theta_1}{2\left[\left(\alpha_2 - \frac{1}{2}\right)(1+\gamma) + (1-\alpha_2)\right]} \right)^{\frac{1}{\gamma}} > \max_s g(s)$$

Assumption 3

$$\max_s g(s) < \gamma \left(\frac{\theta_1}{1+\gamma} \right)^{\frac{1}{1+\gamma}}$$

The following proposition characterizes the optimal policy in this environment

Proposition 5 *Let $\tilde{b}_{2,t} = b_{2,t} - b_{1,t}$, under assumptions 1 and 2,*

1. *There exist $\left(\underline{\mathcal{B}}_2(\alpha_2), \bar{\mathcal{B}}_2(\alpha_2)\right)$ such that the optimal equilibrium is interior if $\tilde{b}_{2,-1} \in \left[\underline{\mathcal{B}}_2(\alpha_2), \bar{\mathcal{B}}_2(\alpha_2)\right]$.*

2. *$\underline{\mathcal{B}}_2(\alpha_2), \bar{\mathcal{B}}_2(\alpha_2)$ are decreasing in α_2; if assumption 3 is also satisfied, then $\lim_{\alpha_2 \to 1} \underline{\mathcal{B}}(\alpha_2) < 0$.*

3. *If $\tilde{b}_{2,-1} < \underline{\mathcal{B}}_2(\alpha_2)$, then $\tilde{b}_{2,t} \to \underline{\mathcal{B}}_2(\alpha_2)$.*

4. *If $\tilde{b}_{2,-1} > \bar{\mathcal{B}}_2(\alpha_2)$, then $\tilde{b}_{2,t} \to \bar{\mathcal{B}}_2(\alpha_2)$.*

5. *The labor supplied by the productive type 1 agent satisfies $h_1'(l_{1,t}) \leq \mathbb{E}_t h_1'(l_{1,t+1})$.*

This proposition highlights the link between the Pareto planner's redistributive motive and the evolution of (net) distribution of assets in a quasi-linear environment that shuts down interest rate fluctuations. By theorem 1, we can always normalize $b_{2,t} = 0$ for all t, while market clearing implies that $B_t = -b_{1,t} = \tilde{b}_{2,t}$, i.e., government assets equal minus the assets owned by high skilled agents. Start from an arbitrary Pareto weight α_2 on the low skilled agent that measures how much the government cares about redistribution. The proposition states that there is a non-empty interval such that if initial assets are within it, sample paths of debt, the tax rate, and output are each constant sequences, and fluctuations in $g(s_t)$ are completely absorbed by transfers (see figure 2 for instance). If initial assets are outside this interval, then assets eventually reach the boundary of the interval to which initial assets are the closest, then stay there. Furthermore, as the government gets more redistributive, the interval $\left(\underline{\mathcal{B}}_2(\alpha_2), \bar{\mathcal{B}}_2(\alpha_2)\right)$ shifts leftward; for large α_2, the interval can contain zero. The quasilinear setup thus sets the stage for two features of optimal policy to emerge in economies with more general preferences: i) an inverse relationship between the level of government assets and the government's redistributive concerns; and ii) the government's incentives to accumulate or decumulate assets.

To reduce the adverse effects of the non-negativity constraints on consumption, the government acquires claims against the productive agent. In response to adverse government expenditure shocks, the government raises the tax rate and reduces transfers. The welfare costs of reducing transfers depend on inequality in the asset distribution vis-a-vis the government's re-distributive motives. When a government that cares about the unproductive agent but has few claims on the productive agents[18] reduces transfers,

[18] For example, think of a a situation in which $\tilde{b}_2 \approx \underline{\mathcal{B}}_2(\alpha_2)$.

it increases the probability of hitting the unproductive agent's lower bound on consumption and faces a high marginal cost. To avoid this, the optimal plan is for the government to accumulate assets until it reaches a safe zone where all shocks to government expenditures can be financed with fluctuations in transfers and constant tax rates. In this safety zone, the distortionary costs of transfers are zero.

For all initial conditions, the long run asset level of the government is bounded by $\left(\underline{\mathcal{B}}_2\left(\alpha_2\right), \bar{\mathcal{B}}_2\left(\alpha_2\right)\right)$.Alongwith Part 2 of proposition 5, this implies an inverse relationship between the government's redistributive motive and its incentive to accumulate assets. The intuition here is that when the re-distributive motive is large (i.e., as $\alpha_2 \to 1$), the constraint that threatens to bind is the productive agent's consumption must be nonnegative. Since $\tilde{b}_2$ are the government's claims against the productive $i = 1$ agent, a smaller value of these claims is required to assure that this nonnegativity constraint is satisfied. In this way, the non-negativity constraint on $c_1(s)$ puts limits on $\bar{\mathcal{B}}_2(\alpha_2)$.

In summary, to ameliorate non-negativity constraints on the consumption of an agent whose Pareto weight is low, the optimal tax-transfer-debt policy distributes assets toward that agent until a point is eventually reached after which all fluctuations in government expenditures can be absorbed by adjusting transfers and keeping distortionary taxes constant.

The quasi-linear setup also allows us to compare the forces for accumulating assets in our model with those in representative agent models like AMSS. In section 5.1, we argued that the presence of a non-negativity constraint on transfers motivates the government eventually to accumulate assets in the quasi-linear AMSS model. The AMSS model approximates a version of our heterogeneous agent economy in which $\theta_1 > \theta_2 = 0$ as the planner's weight on the high skill type, α_1, approaches 1. Activating redistributive concerns by increasing the Pareto weight α_2 on the low skilled type 2 agent disarms the government's motive to accumulate assets unless initial government debt is very high. In the absence of nonnegativity constraints on transfers, the dynamics of our heterogeneous agent model with quasi-linear preferences are driven by the non-negativity constraint (21) on consumption.

In next section, we show that when agents are risk averse with respect to consumption fluctuations, similar dynamics arise naturally from the fact that a decrease in transfers generally affects low skilled agents more than high skilled agents, leading to an increase in the difference between their marginal utilities of consumption. As we shall study in detail below, when households are risk averse, another force also affects equilibrium dynamics, namely, endogenous interest rate fluctuations.

6 Long-run allocations in the two-type economy

In this section, we analytically solve a simple example economy that, while special, illustrates general forces that affect outcomes when agents are risk averse. The example will help us interpret outcomes from numerical solutions of the section 4 Bellman equations that we shall report in section 7.

There are two types of households with $\theta_{1,t} > \theta_{2,t} = 0$. One period utilities are $\ln c - \frac{1}{2}l^2$. The shock s takes two values, $s \in \{s_L, s_H\}$ with probabilities $\Pr\left(s|s_{\_}\right)$ that are independent of s_{-}. We assume that $g\left(s\right) = g$ for all s, and $\theta_1\left(s_H\right) > \theta_1\left(s_L\right)$. We allow the discount factor β to depend on s.

Let $\Psi(s; \mathbf{x}, \boldsymbol{\rho}, s_{\_})$ be an optimal law of motion for the state variables for the $t \geq 1$ recursive problem, i.e. $\Psi(s; \mathbf{x}, \boldsymbol{\rho}, s_{\_}) = (x'(s), \rho'(s))$ that solve (16) given state $(\mathbf{x}, \boldsymbol{\rho}, s_{\_})$.

The next proposition shows the existence of a *steady state* $\left(x^{SS}, \rho^{SS}\right)$ that satisfies $\left(x^{SS}, \rho^{SS}\right) = \Psi\left(s; x^{SS}, \rho^{SS}, s_{-}\right)$ for all s, s_{-}. Since in this steady state $\rho = U_c^i(s)/U_c^1(s)$ does not depend on the realization of shock s, the ratio of marginal utilities of the two agents is constant. The allocation depends only on s_t and not on the history s^{t-1}. Further, for CES preferences, separable in consumption and leisure, the tax rate on labor income is constant. These outcomes are reminiscent though not quite identical to those in a complete market economy (see Werning 2007).[19]

Proposition 6 *Suppose that $g < \theta(s)$ for all s.*

1. ***Countercyclical interest rates.*** *If $\beta(s_H) = \beta(s_L)$, then there exists a steady state $\left(x^{SS}, \rho^{SS}\right)$ such that $x^{SS} > 0$, $R^{SS}(s_H) < R^{SS}(s_L)$.*

2. ***Acyclical interest rates.*** *There exists a pair $\{\beta(s_H), \beta(s_L)\}$ such that there exists a steady state with $x^{SS} > 0$ and $R^{SS}(s_H) = R^{SS}(s_L)$.*

3. ***Procyclical interest rates.*** *There exists a pair $\{\beta(s_H), \beta(s_L)\}$ such that there exists a steady state with $x^{SS} < 0$ and $R^{SS}(s_H) > R^{SS}(s_L)$.*

The key step of the proof, which appears in the appendix, is to show that there exists (x, ρ) such that condition (17a) is satisfied for all s.

Proposition 6 shows two main forces that determine the dynamics of taxes and assets. To describe the first force, it is useful to start from the second part of the proposition, which turns off what we shall call the second force, namely, fluctuations in interest rates. When interest rates are fixed, the government can adjust two instruments in response to an adverse shock (i.e., a fall in θ_1): it can either increase the tax rate τ or it can decrease transfers T. Both responses are distortionary, but for different reasons. Increasing the tax rate increases distortions because the deadweight loss is convex in the tax rate, as in **?**. This force operates in our economy just as it does in representative agent economies. But in a heterogeneous agent economy like ours, adjusting transfers T is also costly. When agents' asset holdings are identical, a decrease in transfers disproportionately affects a low-skilled agent, so his marginal utility falls by more than does the marginal utility of a high-skilled agent. Consequently, a decrease in transfers increases inequality, a cost not present in representative agent economies.

The government can reduce the costs of inequality distortions by choosing tax rate policies that make the net asset positions of the high skilled agent decrease over time. That makes the two agents' after-tax and after-interest income get closer together, allowing decreases in transfers to have smaller effects on inequality in marginal utilities. If the net asset position of a high skilled agent is sufficiently low, then

[19] The similarity is limited to the absence of history dependence in the allocation and constancy of the ratio of marginal utilities (and also the labor tax rate for CES-separable preferences) that prevails in the stationary allocation. Typically, there does not exist an initial distribution of assets and list of Pareto weights for which the optimal allocation with complete markets will *coincide* with the aforementioned stationary allocation with incomplete markets.

a change in transfers has no effect on inequality and all distortions from fluctuations in transfers are eliminated.

It is possible to show numerically[20] that the steady state described in part two of proposition 6 is stable, so the economy converges to it. This convergence outcome has a similar flavor to "back-loading" results of **?** and **?** that reflect the optimality of structuring policies intertemporally eventually to disarm distortions.

Theorem 1 provides a useful way to compare our results with representative agent economies. By that theorem, we can normalize $b_{2,t} = 0$ for all t, in which case the negative of net assets of high-skilled agents can be interpreted as assets of the government. Because $x^{SS} > 0$, the government accumulates assets over time, as in AMSS.

Turning now to the second force, interest rates generally fluctuate with shocks. Parts 1 and 3 of proposition 6 indicate what drives those fluctuations. Consider again the example of a decrease in productivity of high skilled agent. If the tax rate τ is left unchanged, the government faces a shortfall of revenues. Since g is constant, the government requires extra sources of revenues. But suppose that the interest rate increases whenever θ_1 decreases, as happens, for example, when discount factors are constant and θ_1 is the only source of shocks. If the government holds positive assets, its earnings from those assets increase. So holding assets allows higher interest income to offset some of the government's revenue losses from taxes on labor. The situation reverses if interest rates fall at times of increased need for government revenues, as in part 3 of proposition 6 and the steady state allocation features government holding debt.

What matters for our second force is the comovement of the interest rate with fundamentals shocks. States with low average TFP (and therefore a lower base for labor taxes), high g, or a high spread of productivities that threatens to induce higher inequality (and therefore higher transfers and thirst for more government revenues to finance them) are "adverse" from the point of view of current government finance. The government can cope with such adverse states in less distorting ways if finds itself holding positive (negative) assets if interest rates are high (low).

Depending on details of shock processes, these two forces can either reinforce each other (as happens in the Part 1 of proposition 6) or work in the opposite direction (as in Part 3 of proposition 6). In the latter case, whether the government ends up with assets or debt in the long run depends on the relative strengths of the two forces. In section 7, we calibrate shock processes to observed recessions to identify the long run asset dynamics for the government.

Although we proved Proposition 6 for a particular specification of preferences and shocks, the same messages continue to hold more generally. The only place in the proof where we used the particular form of utility functions was to prove an existence of a solution to a particular equation that with our log-quadratic specification is a quadratic that is easy to solve analytically. For other types of preferences, existence of a fixed point of the pertinent equation can be verified numerically.[21] When there are more

[20] Appendix 9.6 describes a numerical test for existence and local stability cast in terms of primitives.
[21] Appendix 9.6 describes a numerical test for existence and local stability cast in terms of primitives.

than two possible values for the shocks or when shocks are persistent, the time-invariant steady state will no longer exist. Mathematically, this occurs because one asset and one risk-free rate of return cannot span all possible needs for government revenues. With richer shock structures, there exists an attraction region in the (x, ρ) space to which the dynamic system converges. Although (x, ρ) are no longer constant in such region, their fluctuations tend to be markedly reduced relative to the transients that occur away from that region, and general properties of x and ρ are the same as those described in Proposition 6. We will discuss these issues in more details in the section 7.

7 Optimal policy in booms and recessions

Section 6 showed that the long run policy response of the government depends on the nature of the shocks, in particular, whether interest rates are high or low in periods which are "adverse" (low TFP, high inequality or high expenditure) from the point of view of governments budget constraint.

In this section, we further investigate this issue. We focus on optimal policy responses when the economy starts away from the long-run steady state. We discuss quantitative and qualitative features of the response as well as the speed of convergence to the long-run allocation by calibrating shocks to match stylized facts about recent recessions.

We consider an economy with two types of agents[22] of equal measures with preferences

$$U(c, l) = \psi \ln c + (1 - \psi) \ln (1 - l).$$

The shock s takes two values, s_H and s_L, and follows a persistent process. We allow β, θ_i and g to be functions of s. We first pick $\bar{\theta}_i, \bar{g}$ and $\bar{\beta}$ for a deterministic economy without shocks and calibrate (ψ, α) to some low frequency data moments. Then to match some business cycle moments we pick shocks according to

$$\begin{aligned} \theta_i(s) &= \bar{\theta}_i[1 + \hat{\theta}_i(s)], \\ \beta(s) &= \bar{\beta}\left[1 + \hat{\beta}(s)\right], \\ g(s) &= \bar{g}\left[1 + \hat{g}(s)\right], \end{aligned} \tag{30}$$

where $\hat{\theta}_i(s) \in \{-e_{i,\theta}, e_{i,\theta}\}$, $\hat{\beta}(s) \in \{-e_\beta, e_\beta\}$ and $\hat{g}(s) \in \{-e_g, e_g\}$. Throughout our experiments, we normalize $b_{2,t} = 0$ for all $t \geq -1$. From market clearing, $B_t = -b_{1,t}$. We refer to B_t as government debt (when negative) and assets (when positive).

[22]We restrict our attention to the economy with two agents for computational tractability. We want to understand both short-run and long-run responses to shocks. For some of our computations, it is important to allow our dynamic systems to travel over a large subset of state space, including regions encountered infrequently. With more agents, it seems possible to apply other methods, for example those of **?**, to study dynamics of our economy within its invariant distribution. We hope to pursue such extensions in future work.

7.1 Calibration

We calibrate the model in two steps. We first chose baseline parameters that govern preferences and technology so that an optimal equilibrium for the static[23] version of the economy matches some sample moments in post war US data. In the second step, we adjusted other parameters to make the amplitudes of fluctuations equal to average peak-trough spreads observed in the three most recent recessions (1991-92, 2001-02 and 2008-10).

We first discuss calibration of $\left(\psi, \alpha, \bar{\theta}_i, \bar{g}, \bar{\beta}\right)$. Although these parameters jointly determine the relevant moments, it is helpful to explain which moment in the data mainly influences each parameter. We normalize $\bar{\theta}_2 = 1$ and pick $\bar{\theta}_1$ to match log wage ratio of 90 wage percentile to 10 wage percentile of 4 from **?**. We set the discount factor $\bar{\beta}$ to match an (annual) interest rate of 2%. We set the parameter ψ to match Frisch elasticity of labor supply equal to 0.5. In our model, $\bar{g}$ corresponds to non-transfer government expenditures, which in the U.S. varied from 7% and 11% in the post WWII period and were above 20% during the war. We set $\bar{g}$ to 12% of GDP. Finally, we set Pareto weights α to match the average marginal tax rate in the US of about 20% as in **?**. [24]

Next we turn to some business cycle targets. We calibrate $\{e_{i,\theta}, e_\beta, \Pr(s|s_-)\}$ to match the following four facts about booms and recessions: the log of the incomes individuals at both the 10th and the 90th percentile falls the recessions; 10th percentile income falls by more than 90th percentile; an inflation-adjusted interest rate on government debt is generally lower in recessions; and booms last longer than recessions. We calibrate the average spread in labor productivity to match the average 3% loss in output seen in the last three recessions.[25] The inequality shock is designed to match the facts documented in **?** that the fall in earnings of the 10-percentile is about 2.5 times of 90-percentile. The discount factor shocks match the average boom-recession difference of about 1.96% in the real risk-free interest rate (3 month T bill rate - inflation rate) seen in the last three recessions. We calibrate the transition matrix to get match the average duration of booms vs recessions. For comparison, we also calibrate a drop in government expenditure to get a drop in output similar to recessions induced by the productivity shocks

Note that because each of them is an exact function of s_t, government expenditures, the discount factor, and productivities are perfectly correlated: a recession is an episode in which TFP falls, inequality rises, and the discount factor is high. We set the initial level of government debt to be 60%, roughly to match the ratio of federal debt held by public at the beginning of 2010.

Table 1 summarizes some details about our calibration.

[23]Formally, an equilibrium in an economy where all shocks are forever equal to their mean value

[24]We use federal government expenditures (excluding current transfers) since the labor tax rate of 20% in **?** is calibrated to federal marginal taxes.

[25]It has long been noticed that the standard RBC model predicts counterfactual negative correlation between real interest rates and output (e.g. **?**). In the data HP filtered output is roughly uncorrelated with real interest rates, but this relationship turn positive if we look at peak vs troughs. We report the optimal responses for both economies with positive and zero correlation of interest rates and output and contrast with a response to a pure TFP shock.

Parameter	Target		
ψ	Frisch elasticity of labor supply	0.6994	
$\bar{\theta}_1$	Log 90-10 wage ratio (Autor et all)	4.0552	
$\bar{\theta}_2$	Normalize to 1	1	
$\hat{\theta}_1$	Business cycle fluctuations in the earnings of 90th percentile	1.2 %	
$\hat{\theta}_2$	Business cycle fluctuations in the earnings of 10th percentile	3%	
β	Average (annual) risk free interest rate	0.98	
$\hat{\beta}(s)$	Business cycle fluctuations	.02	
α_1	marginal tax rate in the economy with no shocks	0.69	
g	average pre-transfer expenditure- output ratio	12 %	
$P(r\|r)$	Recession-Boom episodes	0.63	
$P(b\|b)$	Recession-Boom episodes	0.84	

Table 1: Benchmark calibration

7.2 Outcomes

We discuss separately long run and short run implications for optimal policy. In particular, we study the economy (**"Benchmark"**) with the calibration discussed above and a few variants that successively turn off particular sources of variation.

1. **Acyclical Interest Rates**: In the first variant, we recalibrate the discount factor shocks to make the risk-free rate be uncorrelated with output.

2. **Countercyclical Interest Rates**: Here we shut off discount factor shocks by setting $\hat{\beta}(s) = 0$ in (30). Note that under this assumption, interest rates are countercyclical.

3. **No Inequality**: This variant modifies the "Benchmark" by setting $\hat{\beta}(s) = 0$ and $\hat{\theta}_1(s) = \hat{\theta}_2(s) =$ 3% in (30). This corresponds to a case when the only source of business cycle fluctuations is a TFP shock that affects all agents equally. This case more closely matches the experiments in the RBC literature such as **?**.

4. **Government expenditure Shocks**: The last variant compares optimal responses to shocks to government expenditures. In this experiment, we set $\hat{\theta}(s) = \hat{\beta}(s) = 0$ and choose $\hat{g}(s)$ to produce a drop in output of a similar magnitude to that in the first three experiments. This compares to the studies of responses to government shocks by AMSS and **?**.

Long run

Figure 3 plots government debt . All experiments start with government debt to GDP ratio of 60%. Several features emerge from this figure.

In line with Section 6, all four economies the state (x, ρ) converges to some long run ergodic set, so that government debt and the tax rate converge to associated sets. When there are no discount factor shocks (See lines with $\diamond, \square$ in figure 3), the government has accumulated assets in this ergodic set. As alluded to in section 6 the optimal policy adjusts net asset positions to ameliorate the two key constraints impinging on the government policy, namely, the inability to award agent-specific transfers (the restriction to *affine*

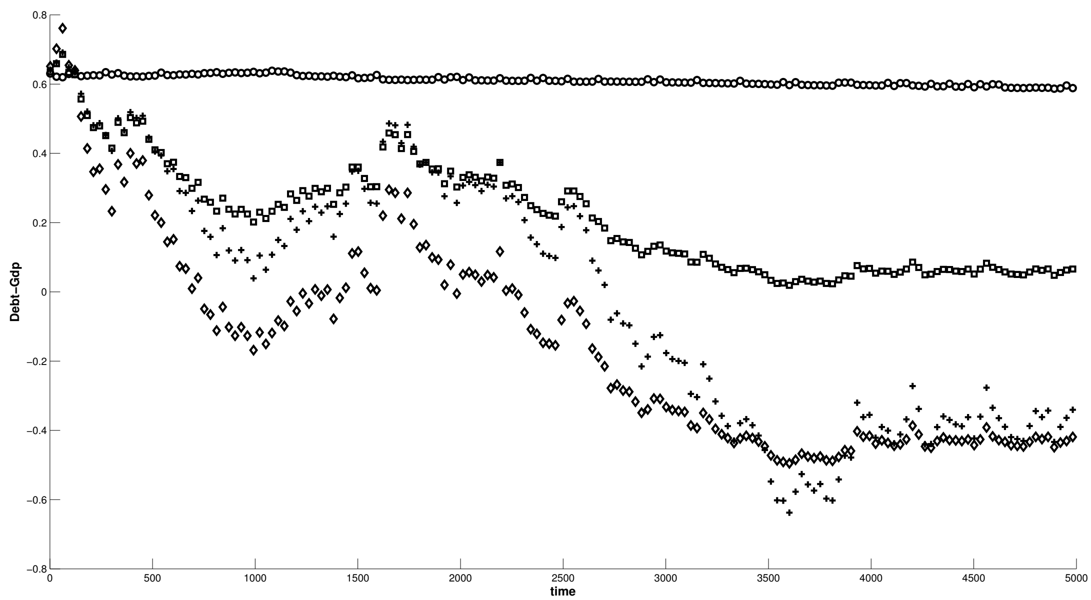

Figure 3: Debt benchmark (o), acyclical interest rates (+), countercyclical interest rates ($\diamond$) and no inequality shocks ($\square$)

taxes) and the absence of state-contingent assets (the restriction to *risk-free debt*). Starting from a point when the relative assets of the unproductive agent (or the government if we use the normalization that sets $b_{2,t} = 0$) are low, extracting resources through lower transfers exacerbates inequality. This is costly since the government has to use higher taxes in future to redistribute. On the margin, the government wants to accumulate assets. But interest rate fluctuations interact with net asset positions to generate state-contingent earnings from assets. If interest rate are high when the government needs additional revenues, accumulating assets relaxes the restriction imposed by absence of state contingent assets. Thus, with countercyclical interest rates, these forces reinforce each other, making the government's long run asset position be positive.

In data, however, interest rates generally decline in recessions. Procyclical interest rates mean that the two forces outlined in the previous paragraph now oppose each other. For large enough interest rate fluctuations, this means that the government may want to accumulate debt. In Figure 3, the black line represents the benchmark with discount factor shocks rigged to replicate procylical fluctuations in interest rates. For a particular initial condition for government debt, the planner can refrain from varying debt for a very long time.[26] The blue and the red lines, respectively are the acyclical and countercyclical interest rate cases in Proposition 6 and we observe a trend towards a region in which the government accumulates assets.

[26] Like the finding in Proposition 6 for large discount factor shocks (in a way that interest rates are procylical) there exist regions where x_t, ρ_t have low volatility and the government is *not* accumulating assets. But these regions are typically unstable. The two forces highlighted before that guide accumulation of assets now work in opposite direction and the net effect depends on the relative strengths. In particular the sample paths from different initial conditions $\tilde{b}_{2,-1}$ (which would imply different choices for the initial x_0, ρ_0) may display larger fluctuations in assets. However, at the calibrated initial conditions (60% debt-gdp ratio), the uncertainty associated with the mean path is very low for the first 5000 periods.

	Δg	ΔB	ΔT	$\Delta[\tau\theta_1 l_1]$	$\Delta[\tau\theta_2 l_2]$	ΔY	$\Delta\tau$
Benchmark	0.0000	-1.1561	0.6871	-0.1593	-0.3096	-2.8536	0.3732
Acyclical Interest Rates	0.0000	-1.1126	0.6591	-0.1497	-0.3038	-2.8613	0.3879
Countercyclical Interest Rates	0.0000	-1.0794	0.6387	-0.1415	-0.2992	-2.8677	0.3997
No Inequality	0.0000	-0.1380	-0.5459	-0.5635	-0.1204	-2.6294	0.0622
Expenditure Shocks	-7.5037	2.9137	2.8612	-1.3759	-0.3530	-2.3443	-1.1598

Table 2: The tables summarizes the changes in the different components of the government budget as we transit from "boom" to a "recession". All numbers are normalized by un-distorted GDP except τ

Convergence to the ergodic region is very slow. With persistent shocks and an initial 60% debt-GDP ratio, it takes about 3,000 years for the government to want to pay off all that debt and then start accumulating assets. With discount factor shocks, it takes even longer to repay the debt. It is still indebted after 5000 years.

Thus, the covariance of interest rates with fundamentals as emphasized in proposition 6 substantially influences the ergodic distribution of government assets.

Short run

The analysis of the previous subsection studied aspects of very low frequency components of the optimal policy. Here we focus on business cycle frequencies. In our setting, these higher frequency responses can conveniently be divided into the magnitudes of changes as we switch from "booms" to "recession," and the dynamics during periods when recession or boom state persist.

We set the exogenous state s_0 so that we are in an outset of a recession. Then we solve the time 0 problem with identical initial conditions across different settings. This pins down the initial state vector x_0, ρ_0 that appears in our time 0 Bellman equation. We then use the policy rules to compute fluctuations of different components in the government budget constraint across states. These responses are summarized in Table 2. For each variable z in the table we report in the form $\Delta z \equiv \left(z\left(s_l|x_0,\rho_0,s_0\right) - z\left(s_h|x_0,\rho_0,s_0\right)\right)/\bar{Y}$ where $\bar{Y}$ is average undistorted-GDP [27].

The source of shocks is very important. Three different types of shocks that produce drops in GDP have very different consequences for optimal policies, both qualitatively and quantitatively. In the benchmark, the government responds to a shock by a making big increases in transfers, the tax rate, and government debt. However, without inequality shocks (row 4), the government responds by decreasing transfers and increasing both debt and the tax rate, but by an amount an order of magnitude smaller than the benchmark. This indicates that ignoring distributional goals can produce a misleading view about optimal government policy in recessions.

Discount factor shocks have minor effects on impact and matter more for transient dynamics that ultimately have big long run effects. Figures 4 and 5 show how the transient dynamics for prolonged booms

[27] Note that predetermined variables like repayment on existing debt drop out of the accounting and we have

$$\Delta[g] + \Delta[T] + \Delta[B] = \Delta[\tau\theta_1 l_1] + \Delta[\tau\theta_2 l_2]$$

(or recessions) differ with and without discount factor shocks. The four panels have taxes, transfers, debt and interest rate movements for a path of 25 years. The bold lines in figures 4 and 5 refer to the benchmark (with procylical interest rates) and the version with acyclical interest rates, respectively. The dotted line in both the figures is the version with countercyclical interest rates. The shaded regions are periods with low output. We see that in a prolonged booms, the government accumulates assets and that it lowers the tax rate when there are no discount factor shocks.

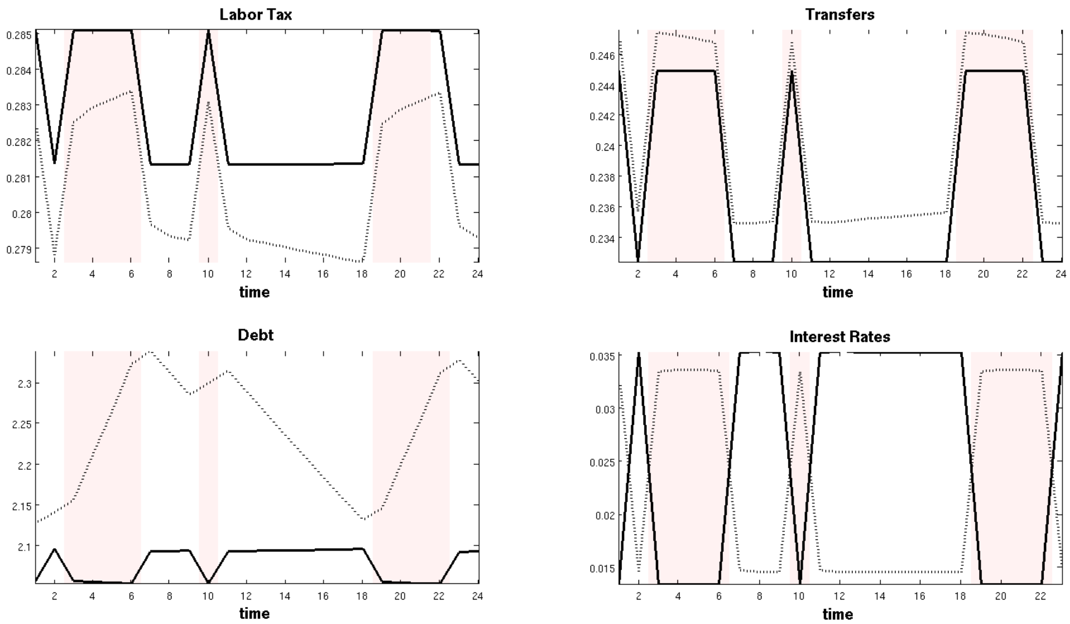

Figure 4: This plots a typical sample path taxes, transfers,debt and interest rates. The bold lines are with benchmark calibration and the dotted lines refer to the variant with countercyclical interest rates. The shaded regions are recessions.

8 Concluding remarks

The spring of 2013 witnessed a heated debate in newspapers and economic magazines about the accuracy and meaning of empirical correlations between output growth rates and ratios of government debt to GDP and in data sets assembled by **?**. From the perspectives of our paper and of **?**, those correlations and those debates are especially difficult to interpret because in our settings, total government debt is not a relevant state variable that affects allocations, government transfers, or tax rates. The principal message of our paper is that without exogenous restrictions on transfers, the level of government debt is not what matters. What does matter is how government debt is distributed among people relative to society's attitudes toward unequal allocations of consumption and labor. Using a recursive representation that works with a correct state variables — a vector of marginal utility adjusted net asset positions and a vector of pairwise ratios of marginal utilities – we have presented a sequence of examples designed to show how agents net positions affect optimal government policies for choosing distorting tax rates, transfers,

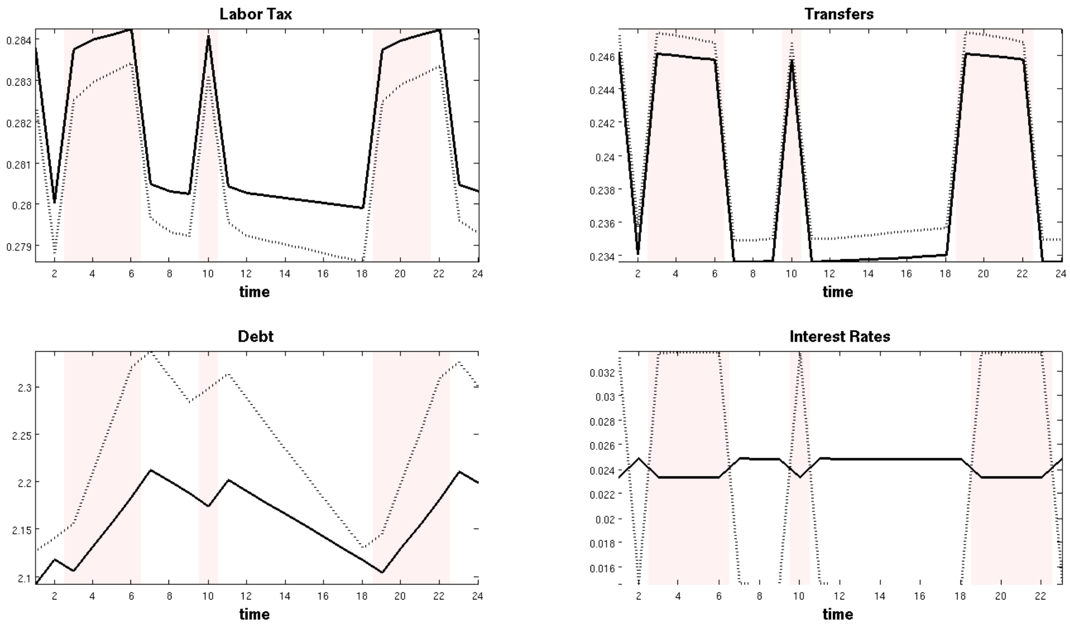

Figure 5: This plots a typical sample path taxes, transfers, debt and interest rates. The bold lines are with acyclical interest rates calibration and the dotted lines correspond to the case with countercyclical interest rates. The shaded regions are recessions.

and government issues or holdings of risk-free bonds. We find that a significant determinant of an optimal asymptotic government debt or government debt-GDP ratio is how interest rate risk is correlated with risks to fundamentals that threaten to widen or narrow inequality in after-tax and after-transfer incomes. To interpret those Reinhart-Rogoff facts country-by-country, we would want to know much more about the distribution of net assets across people within each country and how they interact with interest rate risks and other risks.

9 Appendix

9.1 Additional details for Section 3.1

In this section we construct an example in which the government can achieve higher welfare in the economy with ad-hoc borrowing limits. We restrict ourselves to $I = 2$, $\theta_1 > \theta_2 = 0$, quasilinear preferences for Agent 1 : $U^1(c,l) = c - h(l)$ and a concave utility function (that satisfies Inada conditions) for Agent 2

Suppose that $g_t = 0$ for all t, so that the economy is deterministic. In addition, assume that $\underline{b}_1 = 0$ and $\underline{b}_2 = -\infty$. Given $\left(\beta^{-1}b_{1,-1}, \beta^{-1}b_{2,-1}, \beta^{-1}B_{-1}\right)$, the optimal policy solves :

$$\max_{\{c_{1,t},c_{2,t},l_{1,t},b_{1,t},b_{2,t},R_t\}_t} \sum_{t=0}^{\infty} \beta^t \left[\alpha_1 \left(c_{1,t} - h(l_{1,t})\right) + \alpha_2 U^2(c_{2,t})\right] \tag{31}$$

subject to

$$c_{2,t} - c_{1,t} + b_{2,t} - b_{1,t} + h'(l_{1,t})l_{1,t} = R_{t-1}\left(b_{2,t-1} - b_{1,t-1}\right) \tag{32a}$$

$$c_{1,t} + c_{2,t} \leq \theta l_{1,t} \tag{32b}$$

$$1 \geq \beta R_t \tag{32c}$$

$$(1 - \beta R_t) b_{1,t} = 0 \tag{32d}$$

$$U^2_{c,t} = \beta R_t U^2_{c,t+1} \tag{32e}$$

$$b_{1,t} \geq 0 \tag{32f}$$

We solve this maximization problem in two stages. First, drop constraint (32c) and solve the resulting relaxed problem subject to the remaining constraints for a fixed sequence of $\{R_t\}_t$. Denote the value of the objective function for the reduced problem by $W\left(\{R_t\}_t\right)$. Second, solve the original problem by choosing $\{R_t\}_t$ to maximize

$$W\left(\{R_t\}_t\right)$$

subject to (32c).

Let $\mu_t \beta^t \Pr(s^t)$ is Lagrange multiplier associated with the constraint (32a)

Lemma 1 *For $R_t = \beta^{-1}$, we can choose a time invariant solution $c_{1,t} = \bar{c}_1, c_{2,t} = \bar{c}_2, l_{1,t} = \bar{l}_1, b_{1,t} = \bar{b}_1, b_{2,t} = \bar{b}_2$ to the relaxed problem that satisfies*

$$\bar{b}_2 - \bar{b}_1 = \bar{b}_2 = \bar{b}_{2,-1} - \bar{b}_{1,-1}.$$

Proof. By Theorem 1, we can set $b_{1,t} = 0$ and ignore constraint (32f). Further ignoring constraint (32e), a stationary interior solution is given by

$$U^2_c[\bar{c}_2] = \frac{2\bar{\mu} + \alpha_1}{\alpha_2} \tag{33a}$$

$$\alpha_1 h'(\bar{l}_1) = \theta_1 \alpha_1 + \bar{\mu}[\theta_1 - h''(\bar{l}_1)\bar{l}_1 - h'(\bar{l}_1)] \tag{33b}$$

$$b_{2,-1} = \frac{2\bar{c}_2 + \bar{l}_1\left[h'(\bar{l}_1) - \theta_1\right]}{\beta^{-1} - 1} \tag{33c}$$

Note that if a solution to the above set of equations exists, (32e) is naturally satisfied.

We first establish some comparative statics with respect to $\bar{\mu}$. It is easy to see that concavity of U^2implies $\frac{\partial \bar{c}_2}{\partial \mu} < 0$. Further equation (33b) can be rearranged to get

$$\frac{h'(\bar{l}_1)}{\theta_1} = \frac{\alpha_1 + \bar{\mu}}{\left[\alpha_1 + \bar{\mu} + \bar{\mu}\left(\frac{h''(\bar{l}_1)\bar{l}_1}{h'(\bar{l}_1)}\right)\right]}$$

Using convexity of h we have $\frac{\partial \bar{l}_1}{\partial \mu} < 0$.

As $\bar{\mu} \to -\frac{\alpha_1}{2}$, the RHS of (33c) approaches $+\infty$ and $\bar{\mu} \to +\infty$ it approaches some $\underline{b}_2 < 0$. Thus we have a stationary solution for a range of $b_{2,-1}$. [28]

■

Lemma 2 *For $R_t = \beta^{-1}$, if $b_{2,-1} < b_{1,-1}$ then $\bar{\mu} > 0$*

Proof. Suppose $\bar{\mu} \leq 0$ when $b_{2,-1} < b_{1,-1}$. At $\mu = 0$, $h'(\bar{l}_1) = \theta_1$ and the RHS of (33c) is positive. At $\mu < 0$ we have $h'(\bar{l}_1) > \theta_1$. The observations above imply that the RHS of (33c) is increasing in μ and this clearly violates equation (33c). Thus we have a contradiction. ■

When $R_t = 1/\beta$ for all t, the solution of the reduced problem is an optimal allocation for an economy in which agents face no borrowing constraints

Let $\frac{\partial}{\partial R_1} W\left(\{R_t\}_t\right)\Big|_{\{R_t\}=\boldsymbol{\beta}^{-1}}$ be the derivative of $W\left(\{R_t\}_t\right)$ with respect to R_1 evaluated at $R_t = \beta^{-1}$ for all t. Our observations above imply that

$$\frac{\partial}{\partial R_1} W\left(\{R_t\}_t\right)\Big|_{\{R_t\}=\boldsymbol{\beta}^{-1}} = \bar{\mu}\bar{b}_{2,t} = \bar{\mu}\left(\bar{b}_{2,-1} - \bar{b}_{1,-1}\right) < 0$$

and therefore $R_t = \beta^{-1}$ for all t is not the optimal equilibrium sequence. Therefore, welfare in the economy with exogenous borrowing constraints is strictly higher than in the economy without exogenous borrowing constraints.[29]

The outcome that welfare can be strictly higher with exogenous borrowing constraints depends on our assumption that agents dont face idiosyncratic risk . If agents were also subject to idiosyncratic shocks, exogenous borrowing constraints would have the additional effect of limiting agents' ability to self-insure against those shocks.[30] Nevertheless, the insight from the example carries through that even though exogenous borrowing constraints can hurt agents' to insure against idiosyncratic shocks, they can help a government smooth distortions with respect to aggregate shocks like government expenditure shocks.

[28] We may need additional restrictions for non-negativity of c_1. Alternatively we could allow Agent 1's consumption to take negative values

[29] The mechanism in this example is similar to a finding of **?**, who showed that relaxing agents' borrowing constraints can be suboptimal in an economy with idiosyncratic shocks. Our analysis shows that this insight is more general and holds even in economies with no shocks.

[30] See **?** and **?** for details.

9.2 Proof of Proposition 2

We prove a slight more general version of our result. Consider an infinite horizon, incomplete markets economy in which an agent maximizes utility function $U : \mathbb{R}^n_+ \rightarrow \mathbb{R}$ subject to an infinite sequence of budget constraints. We assume that U is concave and differentiable. Let $\mathbf{a}(s^t)$ be a vector of n goods and let $\mathbf{p}(s^t)$ be a price vector in state s^t with $p_i(s^t)$ denoting the price of good i. We use a normalization $p_1\left(s^t\right) = 1$ for all s^t. There is a risk-free bond.

Let $b(s^t)$ be the agent's bond holdings, and let $\mathbf{e}\left(s^t\right)$ be a stochastic vector of endowments.

Consumer maximization problem

$$\max_{\mathbf{a}_t, b_t} \sum_{t=0}^{\infty} \left[\Pi_{j=0}^{t} \beta(s_j)\right] \Pr\left(s^t\right) U(\mathbf{a}\left(s^t\right)) \tag{34}$$

subject to

$$\mathbf{p}\left(s^t\right) \mathbf{a}\left(s^t\right) + q(s^t) b\left(s^t\right) = \mathbf{p}\left(s^t\right) \mathbf{e}\left(s^t\right) + b\left(s^{t-1}\right) \tag{35}$$

and $\left\{b\left(s^t\right)\right\}$ is bounded and $\left\{q(s^t)\right\}$ is the price of the risk-free bond.

The Euler conditions are

$$\begin{aligned} \mathbf{U}_a(s^t) &= U_1(s^t)\mathbf{p}(s^t) \\ \Pr\left(s^t\right) U_1\left(s^t\right) q(s^t) &= \beta(s_t) \sum_{s^{t+1} > s^t} \Pr\left(s^{t+1}\right) U_1\left(s^{t+1}\right). \end{aligned} \tag{36}$$

Lemma 3 *Consider an allocation $\{\mathbf{a}_t, b_t\}$ that satisfies (35), (36) and $\{b_t\}_t$ is bounded. Then $\{\mathbf{a}_t, b_t\}$ is a solution to (34).*

Proof. The proof follows closely **?**. Suppose there is another budget feasible allocation $\mathbf{a} + \mathbf{h}$ that maximizes (34). Since U is strictly concave,

$$\begin{aligned} & \mathbb{E}_0 \sum_{t=0}^{\infty} \left[\Pi_{j=0}^{t} \beta(s_j)\right] U(\mathbf{a}_t + \mathbf{h}_t) - \mathbb{E}_0 \sum_{t=0}^{\infty} \left[\Pi_{j=0}^{t} \beta(s_j)\right] U(\mathbf{a}_t) \\ \leq\ & \mathbb{E}_0 \sum_{t=0}^{\infty} \left[\Pi_{j=0}^{t} \beta(s_j)\right] \mathbf{U}_a(\mathbf{a}_t)\mathbf{h}_t \end{aligned} \tag{37}$$

To attain $\mathbf{a}+\mathbf{h}$, the agent must deviate by φ_t from his original portfolio b_t such that $\{\varphi_t\}_t$ is bounded, $\varphi_{-1} = 0$ and

$$\mathbf{p}(s^t)\mathbf{h}\left(s^t\right) = \varphi(s^{t-1}) - q(s^t)\varphi(s^t)$$

Multiply by $\left[\Pi_{j=0}^{t-1} \beta(s_j)\right] \Pr\left(s^t\right) U_1(s^t)$ to get:

$$\begin{aligned} \left[\Pi_{j=0}^{t-1} \beta(s_j)\right] \Pr\left(s^t\right) U_1(s^t)\mathbf{p}(s^t)\mathbf{h}\left(s^t\right) &= \left[\Pi_{j=0}^{t-1} \beta(s_j)\right] \Pr\left(s^t\right) U_1(s^t)\varphi(s^{t-1}) - q(s^t)\left[\Pi_{j=0}^{t-1} \beta(s_j)\right] \Pr\left(s^t\right) U_1(s^t)\varphi(s^t) \\ &= \left[\Pi_{j=0}^{t-1} \beta(s_j)\right] \Pr\left(s^t\right) U_1(s^t)\varphi(s^{t-1}) - \left[\Pi_{j=0}^{t-1} \beta(s_j)\right] \beta(s_t) \sum_{s^{t+1} > s^t} \Pr\left(s^{t+1}\right) U_1\left(s^{t+1}\right) \varphi(s^t) \end{aligned}$$

where we used the second part of (36) in the second equality. Sum over the first T periods (pathwise) and use the first part of (36) to eliminate $\mathbf{U}_a(\mathbf{a}_t) = U_1(s^t)\mathbf{p}(s^t)$

$$\sum_{t=0}^{T} \left[\Pi_{j=0}^{t-1} \beta(s_j)\right] \Pr\left(s^t\right) \mathbf{U}_a(\mathbf{a}_t)\mathbf{h}\left(s^t\right) = -\left[\Pi_{j=0}^{T} \beta(s_j)\right] \sum_{s^{T+1} > s^T} \Pr\left(s^{T+1}\right) U_1\left(s^{T+1}\right) \varphi(s^T).$$

Since $\{\varphi_t\}_t$ is bounded there must exist $\bar{\varphi}$ s.t. $|\varphi_t| \leq \bar{\varphi}$ for all t. By Theorem 5.2 of Magill and Quinzii (1994), this equilibrium with debt constraints implies a transversality condition on the right hand side of the last equation, so by transitivity we have

$$\lim_{T\to\infty} \sum_{t=0}^{T} \left[\Pi_{j=0}^{t-1}\beta(s_j)\right] \Pr\left(s^t\right) \mathbf{U}_a(\mathbf{a}_t)\mathbf{h}\left(s^t\right) = 0.$$

Substitute this into (37) to show that $\mathbf{h}$ does not improve utility of consumer. ■

9.3 Proof of Proposition 5

Let α_2 be the Pareto weight on the unproductive agent. Given a (initial) net distribution of assets $\tilde{b}_2 = b_2 - b_1$ the optimal solution solves the following Bellman equation.

$$V(\beta^{-1}\tilde{b}_2, s_-) = \max_{c_1(s),c_2(s),l_1(s),\tilde{b}_2'(s)} \sum_s \Pr(s|s_-) \Big\{ (1-\alpha_2)\left(c_1(s) - h[l_1(s)]\right) + \alpha_2 c_2(s) + \beta V(\beta^{-1}\tilde{b}_2'(s), s) \Big\} \tag{38}$$

subject to

$$c_2(s) - c_1(s) + \tilde{b}_2'(s) + l_1(s)h'[l_1(s)] = \frac{\tilde{b}_2}{\beta} \tag{39a}$$

$$c_2(s) + c_1(s) + g(s) \leq \theta_1 l_1(s) \tag{39b}$$

$$c_1(s) \geq 0 \quad c_2(s) \geq 0 \tag{39c}$$

$$\tilde{b}_2'(s) \in [\underline{b}, \bar{b}] \tag{39d}$$

Let $\mu(s)\Pr(s|s_-), \xi(s)\Pr(s|s_-)$, $\lambda^{c_i}(s)\Pr(s|s_-)$ be the multipliers on the implementability (39a), resource constraint (39b) and the non-negativity constraints (39c) for the respective consumption. For now we assume that $\underline{b}$ and $\bar{b}$ are natural debt limits and ignore them on the equilibrium path. The FONCs of the problem are

$$1 - \alpha_2 + \mu(s) + \lambda^{c_1}(s) - \xi(s) = 0 \tag{40a}$$

$$\alpha_2 - \mu(s) + \lambda^{c_2}(s) - \xi(s) = 0 \tag{40b}$$

Let $g(l) = lh''(l) + h'(l)$

$$-(1-\alpha_2)h'[l_1(s)] - \mu(s)g[l_1(s)] + \theta_1\xi(s) = 0 \tag{40c}$$

$$\mu(s) = V_{\tilde{b}_2}[\tilde{b}_2'(s), s] \tag{40d}$$

and lastly the Envelope theorem gives us

$$\mathbb{E}_{s_-} V_{\tilde{b}_2}[\tilde{b}_2'(s), s] = V_{\tilde{b}_2}[\tilde{b}_2, s_-] \tag{40e}$$

For an interior solution,i.e $c_1(s) > 0, c_2(s) > 0$, we have constant labor supply that solves the following equation

$$-(1-\alpha_2)h'(l_1) - (\alpha_2 - \frac{1}{2})g(l_1) + \frac{\theta_1}{2} = 0 \tag{41}$$

For CES labor disutility, $h(l) = \frac{l^{1+\gamma}}{1+\gamma}$, this yields

$$l_1^*(\alpha_2) = \left(\frac{\theta_1}{2\left[\left(\alpha_2 - \frac{1}{2}\right)(1+\gamma) + (1-\alpha_2)\right]} \right)^{\frac{1}{\gamma}} \tag{42}$$

The condition $\alpha_2 \geq \frac{(\gamma-1)}{2\gamma}$ ensures that $l_1^*(\alpha_2) \geq 0$

We can use the implementability constraint (39a) with the guess $\tilde{b}_2'(s) = \tilde{b}_2$ and the resource constraint (39b) to back out consumptions for each agent.

$$c_1(s) = \frac{1}{2}\left[\theta_1 l_1^*(\alpha_2) + l_1^*(\alpha_2) h'(l_1^*(\alpha_2)) - \tilde{b}_2\left(\frac{1}{\beta} - 1\right) - g(s)\right] \tag{43a}$$

$$c_2(s) = \frac{1}{2}\left[\theta_1 l_1^*(\alpha_2) - l_1^*(\alpha_2) h'(l_1^*(\alpha_2)) + \tilde{b}_2\left(\frac{1}{\beta} - 1\right) - g(s)\right] \tag{43b}$$

To be a valid interior solution we need $\tilde{b}_2 \in \left[\underline{\mathcal{B}}_2(\alpha_2), \bar{\mathcal{B}}_2(\alpha_2)\right]$ where this interval is given by

$$\underline{\mathcal{B}}_2(\alpha_2) = \frac{\max_s g(s) - \theta_1 l_1^*(\alpha_2) + l_1^*(\alpha_2) h'(l_1^*(\alpha_2))}{\frac{1}{\beta} - 1} \tag{44a}$$

$$\bar{\mathcal{B}}_2(\alpha_2) = \frac{-\max_s g(s) + \theta_1 l_1^*(\alpha_2) + l_1^*(\alpha_2) h'(l_1^*(\alpha_2))}{\frac{1}{\beta} - 1} \tag{44b}$$

The condition for non-empty interior is a very natural one $-\max_s g(s) + \theta_1 l_1^*(\alpha_2) > 0$, that says that output sufficient to cover the worst possible realization of government consumption.

Lemma 4 $\underline{\mathcal{B}}_2(\alpha_2)$ *and* $\bar{\mathcal{B}}_2(\alpha_2)$ *are decreasing in* α_2

Using the expression 42 for $l_1^*(\alpha_2)$, we have

$$\underline{\mathcal{B}}_2'(\alpha_2) = \frac{\partial l_1^*(\alpha_2)}{\partial \alpha_2}\left[(1+\gamma) l_1^{*\gamma}(\alpha_2) - \theta_1\right]$$

$$\bar{\mathcal{B}}_2'(\alpha_2) = \frac{\partial l_1^*(\alpha_2)}{\partial \alpha_2}\left[(1+\gamma) l_1^{*\gamma}(\alpha_2) + \theta_1\right]$$

Its easy to see that $l_1^*(\alpha_2)$ is decreasing in α_2. For the range $\alpha_2 \geq \frac{\gamma-1}{2\gamma}$ it is also non-negative. It suffices to show that the term $[(1+\gamma) l_1^{*\gamma}(\alpha_2) - \theta_1]$ is non-negative. Note that

$$\max_{\alpha_2 \in [0,1]} \left(\alpha_2 - \frac{1}{2}\right)(1+\gamma) + (1-\alpha_2)] = 1+\gamma$$

Thus $l_1^{*\gamma}(\alpha_2)(1+\gamma) \geq \theta_1$.

Now we show that for large α_2, the interval $\left[\underline{\mathcal{B}}_2(\alpha_2), \bar{\mathcal{B}}_2(\alpha_2)\right]$ can contain zero.

Lemma 5 *If* $\max_s g(s) < \theta_1^{\frac{1+\gamma}{\gamma}} \left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}} \left(\frac{\gamma}{1+\gamma}\right)$, *then*

$$\lim_{\alpha_2 \to 1} \underline{\mathcal{B}}(\alpha_2) < 0$$

Proof. Note that $\lim_{\alpha_2 \to 1} l_1^*(\alpha_2) = \left(\frac{\theta_1}{1+\gamma}\right)^{\frac{1}{\gamma}}$. Using equation (44b)

$$\lim_{\alpha_2 \to 1} \underline{\mathcal{B}}(\alpha_2) = \max_s g(s) - \theta_1 \left(\frac{\theta_1}{1+\gamma}\right)^{\frac{1}{\gamma}} + \left(\frac{\theta_1}{1+\gamma}\right)^{\frac{1+\gamma}{\gamma}}$$

Under assumption 3,

$$\lim_{\alpha_2 \to 1} \underline{\mathcal{B}}(\alpha_2) = \max_s g(s) - \theta_1^{\frac{1+\gamma}{\gamma}} \left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}} \left(\frac{\gamma}{1+\gamma}\right) < 0$$

■

Next we discuss what happens when the assets are outside the interval that supports the interior solution. WLOG, consider the case $\tilde{b}_2 > \bar{\mathcal{B}}_2(\alpha_2)$ where the non-negativity constraint is binding on Agent 1's consumption in at least one state and $\lambda^{c_1}(s) > 0$. We will use the following three lemmas

Lemma 6 $\lim_t \lambda^{c_1}(s^t) = 0$

Proof. First note that, the Envelope condition is still valid and the multiplier on the implementability constraint, $\mu(s^t)$ is a martingale. The FOC imply that $\lambda^{c_1}(s^t)$ and $\mu(s^t)$ are related by

$$\mu(s^t) = \alpha_2 - \frac{1}{2} - \frac{\lambda^{c_1}(s^t)}{2}$$

Thus $\lambda^{c_1}(s^t)$ is also a martingale. Further the complementary slackness conditions provide a lower bound of zero i.e $\lambda^{c_1}(s^t) \geq 0$. We can now apply the standard martingale convergence theorems to argue that $\lambda^{c_1}(s^t) \to \lambda^*$. We now argue that this limit is always 0. Suppose this is not the case and $\lambda^* > 0$

Since the FOC of l_1 only depends on λ^{c_1} (and other constants), $l_1 \to l_1^*$. Since $\lambda^* > 0$, Agent 1's consumption $c_1(s^t) \to 0$ and the resource constraint solves for Agent'2 consumption to be

$$c_2(s^t) \to \theta_1 l_1^* - g(s^t)$$

[31].

The implementability constraint now can be written as

$$\theta_1 l_1(s^t) + \tilde{b}_2(s^t) + l(s^t)h'(l_1(s^t)) - g(s^t) = \frac{\tilde{b_2(s^t)}}{\beta}$$

Iterating and taking limits we see that

$$\lim_j \sum_j \beta^j \theta_1 l_1(s^{t+j}) + \lim_j \beta^j \tilde{b}_2(s^{t+j}) + \lim_j \sum_j \beta^j l(s^t)h'(l_1(s^{t+j})) - \sum_j \beta^j g(s^{t+j}) = \frac{\tilde{b_2(s^t)}}{\beta}$$

[31]Under the regularity assumption that $-\max_s g(s) + \theta_1 l_1^*(\alpha_2) > 0$, we can verify the guess that the non-negativity constraint on c_2 is slack

Note that all terms except $\lim_j \beta^j \tilde{b}_2(s^{t+j})$ and $\sum_j \beta^j g(s^{t+j})$ approach a constant. Further since $g(s^t)$ is stochastic, $\lim_j \beta^j \tilde{b}_2(s^{t+j})$ is different from zero with strictly positive probability, implying that assets will violate any finite bounds. Thus we have $\lambda^* = 0$, or the constraint is eventually slack. ■

Now we show that the long run assets are constant at $\bar{\mathcal{B}}_2(\alpha_2)$. This will follow from the fact that $\tilde{b}_2'(\tilde{b}_2, s)$ is continuous and weakly increasing in $\tilde{b}_2$.

Lemma 7 $\tilde{b}_2'(\tilde{b}_2, s)$*is continuous and weakly increasing in* $\tilde{b}_2$

Proof. First note that $\lambda^{c1}(\tilde{b}_2, s)$ is increasing in $\tilde{b}_2$ as a higher value of $\tilde{b}_2$ tightens the non-negativity constraint on c_1. Since $\mu(s) = \alpha_2 - \frac{1}{2} - \frac{\lambda^{c1}(s)}{2}$, it is decreasing in $\tilde{b}_2$. Weak concavity of V [32] and the envelope theorem imply that $\tilde{b}_2'(\tilde{b}_2, s)$ is weakly increasing. Continuity comes from applying the Maximum theorem to the problem ■

Lemma 8 *if* $\tilde{b}_{2,-1} > \bar{\mathcal{B}}_2(\alpha_2)$ *then* $\lim_t \tilde{b}_{2,t} = \bar{\mathcal{B}}_2(\alpha_2)$

Proof. Suppose not, then there exist a t such that $\tilde{b}_t > \bar{\mathcal{B}}(\alpha_2)$ and $\tilde{b}_{2,t+1} < \bar{\mathcal{B}}(\alpha_2)$. This implies that

$$\tilde{b}_{2,t+1} = \tilde{b}_2'[\tilde{b}_{2,t}, s_t] < \tilde{b}_2'[\bar{\mathcal{B}}_2(\alpha_2), s_t]$$

Since the previous lemma shows $\tilde{b}_2'[\tilde{b}_2, s]$ is increasing, we have a contradiction. ■

With $\tilde{b}_2 < \underline{\mathcal{B}}_2(\alpha_2)$, we have a symmetric argument and long run assets settle down at the lower limit.

Lastly we show that marginal disutility of labor is a sub-martingale.

Thus we have,

$$l_{1,t}^{\gamma} = \frac{\theta_1(1 + \lambda_t^{c_1} + \lambda_t^{c_2})}{(1-\alpha_2) + (\alpha_2 - \frac{1}{2} + \frac{\lambda_t^{c2}}{2} - \frac{\lambda_t^{c1}}{2})(1+\gamma)} \tag{45}$$

Both $\lambda_t^{c1}, \lambda_t^{c2}$ are martingales and it can be verified that $l_{1,t}^{\gamma}$ is convex in either of the multiplier. Applying Jensen's inequality, we have

$$\mathbb{E} l_{1,t+1}^{\gamma} \geq l_{1,t}^{\gamma}$$

9.4 Proof of Proposition 6

The Bellman equation for the optimal planners problem with log quadratic preferences and IID shocks can be written as

$$V(x, \rho) = \max_{c_1, c_2, l_1, x', \rho'} \sum_s \Pr(s) \left[\alpha_1 \left(\log c_1(s) - \frac{l_1(s)^2}{2} \right) + \alpha_2 \log c_2(s) + \beta(s) V(x'(s), \rho'(s)) \right]$$

[32] In the CES case concavity of V can be shown by observing that objective function and the implementability constraint is linear in $\tilde{l} = l_1^{\gamma}$. The resource constraint is a weak inequality with the RHS concave in $\tilde{l}$. Thus the constraint set is convex. Further the value function is bounded, we can apply analogues of theorem 4.6-4.8 in SLP to prove that V is concave

subject to the constraints

$$1 + \rho'(s)[l_1(s)^2 - 1] + \beta(s)x'(s) - \frac{x\frac{1}{c_2(s)}}{\mathbb{E}[\frac{1}{c_2}]} = 0 \tag{46}$$

$$\sum_s \frac{\Pr(s)}{c_1(s)}(\rho'(s) - \rho) = 0 \tag{47}$$

$$\theta_1(s)l_1(s) - c_1(s) - c_2(s) - g = 0 \tag{48}$$

$$\rho'(s)c_2(s) - c_1(s) = 0 \tag{49}$$

Where the $\Pr(s)$ is the probability distribution of the aggregate state s. If we let $\Pr(s)\mu(s)$, λ, $\Pr(s)\xi(s)$ and $\Pr(s)\phi(s)$ be the Lagrange multipliers for the constraints (46)-(49) respectively then we obtain the following FONC for the planners problem

$c_1(s)$:

$$\frac{\alpha_1 \Pr(s)}{c_1(s)} - \frac{\lambda \Pr(s)}{c_1(s)^2}(\rho'(s) - \rho) - \Pr(s)\xi(s) - \Pr(s)\phi(s) = 0 \tag{50}$$

$c_2(s)$:

$$\frac{\alpha_2 \Pr(s)}{c_2(s)} + \frac{x \Pr(s)}{c_2(s)^2 \mathbb{E}[\frac{1}{c_2}]}\left[\mu(s) - \frac{\mathbb{E}[\mu\frac{1}{c_2}]}{\mathbb{E}[\frac{1}{c_2}]}\right] - \Pr(s)\xi(s) + \Pr(s)\rho'(s)\phi(s) = 0 \tag{51}$$

$l_1(s)$:

$$-\alpha_1 \Pr(s)l_1(s) + 2\mu(s)\Pr(s)\rho'(s)l_1(s) + \theta_1(s)\Pr(s)\xi(s) = 0 \tag{52}$$

$x'(s)$:

$$\beta(s)\Pr(s)V_x(x'(s), \rho'(s)) + \beta(s)\Pr(s)\mu(s) = 0 \tag{53}$$

$\rho'(s)$:

$$\beta(s)\Pr(s)V\rho(x'(s), \rho'(s)) + \frac{\lambda \Pr(s)}{c_1(s)} + \mu(s)\Pr(s)[l_1(s)^2 - 1] + \Pr(s)\phi(s)c_2(s) = 0 \tag{54}$$

In addition there are two envelope conditions given by

$$V_x(x, \rho) = -\sum_{s'} \frac{\mu(s')\Pr(s')\frac{1}{c_2(s')}}{\mathbb{E}[\frac{1}{c_2}]} = -\frac{\mathbb{E}[\mu\frac{1}{c_2}]}{\mathbb{E}[\frac{1}{c_2}]} \tag{55}$$

$$V\rho(x, \rho) = -\lambda\mathbb{E}[\frac{1}{c_1}] \tag{56}$$

A steady state is then a collection of allocations, initial conditions and Lagrange multipliers $\{c_1(s), c_2(s), l_1(s), x, \rho, \mu(s), \lambda, \xi(s), \phi(s)\}$ such that equations (46)-(56) are satisfied when $\rho'(s) = \rho$ and $x'(s) = x$. It should be clear that is that if we replace $\mu(s) = \mu$ then, equation (53) is always satisfied. Additionally under this assumption equation (51) simplifies significantly.

$$\frac{x \Pr(s)}{c_2(s)^2 \mathbb{E}[\frac{1}{c_2}]}\left[\mu(s) - \frac{\mathbb{E}[\mu\frac{1}{c_2}]}{\mathbb{E}[\frac{1}{c_2}]}\right] = 0$$

The first order conditions for a steady can then be written simply as

$$1 + \rho[l_1(s)^2 - 1] + \beta(s)x - \frac{x}{c_2(s)\mathbb{E}[\frac{1}{c_2}]} = 0 \tag{57}$$

$$\theta_1(s)l_1(s) - c_1(s) - c_2(s) - g = 0 \tag{58}$$

$$\rho c_2(s) - c_1(s) = 0 \tag{59}$$

$$\frac{\alpha_1}{c_1(s)} - \xi(s) - \phi(s) = 0 \tag{60}$$

$$\frac{\alpha_2}{c_2(s)} - \xi(s) + \rho\phi(s) = 0 \tag{61}$$

$$[2\mu\rho - \alpha_1]l_1(s) + \theta_1(s)\xi(s) = 0 \tag{62}$$

$$\lambda\left[\frac{1}{c_1(s)} - \beta(s)\mathbb{E}[\frac{1}{c_1}]\right] + \mu[l_1(s)^2 - 1] + \phi(s)c_2(s) = 0 \tag{63}$$

We can rewrite equation (60) as

$$\frac{\alpha_1}{c_2(s)} - \rho\xi(s) - \rho\phi(s) = 0$$

by substituting $c_1(s) = \rho c_2(s)$. Adding this to equation (61) and normalizing $\alpha_1 + \alpha_2 = 1$ we obtain

$$\xi(s) = \frac{1}{(1+\rho)\, c_2(s)} \tag{64}$$

which we can use to solve for $\phi(s)$ as

$$\phi(s) = \frac{\alpha_1 - \rho\alpha_2}{(\rho(1+\rho))\, c_2(s)} \tag{65}$$

From equation (57) we can solve for $l_1(s)^2 - 1$ as

$$l_1(s)^2 - 1 = \frac{x}{\rho\mathbb{E}[\frac{1}{c_2}]}\left(\frac{1}{c_2(s}) - \beta(s)\mathbb{E}[\frac{1}{c_2}]\right) - \frac{1}{\rho}$$

This can be used along with equations (63) and (65) to obtain

$$\left(\frac{\lambda}{\rho} + \frac{\mu x}{\rho\mathbb{E}[\frac{1}{c_2}]}\right)\left(\frac{1}{c_2(s)} - \beta(s)\mathbb{E}\left[\frac{1}{c_2}\right]\right) = \frac{\mu}{\rho} + \frac{\rho\alpha_2 - \alpha_1}{\rho(1+\rho)}$$

Clearly as the LHS depends on s while the RHS does not, Hence the solution to this equation is

$$\lambda = -\frac{\mu x}{\mathbb{E}[\frac{1}{c_2}]} \tag{66}$$

and

$$\mu = \frac{\alpha_1 - \rho\alpha_2}{1+\rho} \tag{67}$$

Combining these with equation (62) we quickly obtain that

$$\left[2\rho\frac{\alpha_1 - \rho\alpha_2}{1+\rho} - \alpha_1\right]l_1(s) + \frac{\theta_1(s)}{(1+\rho)\, c_2(s)} = 0$$

Then solving for $l_1(s)$ gives

$$l_1(s) = \frac{\theta_1(s)}{(\alpha_1(1-\rho) + 2\rho^2\alpha_2)\, c_2(s)}$$

Remark 1 *Note that the labor tax rate is given by* $1-\frac{c_1(s)l_1(s)}{\theta(s)}$*. The previous expression shows that labor taxes are constant at the steady state. This property holds generally for CES preferences separable in consumption and leisure*

This we can plug into the aggregate resource constraint (58) to obtain

$$l_1(s)=\left(\frac{1+\rho}{\alpha_1(1-\rho)+2\rho^2\alpha_2}\right)\frac{1}{l_1(s)}+\frac{g}{\theta_1(s)}$$

letting $C(\rho)=\frac{1+\rho}{\alpha_1(1-\rho)+2\rho^2\alpha_2}$ we can then solve for $l_1(s)$ as

$$l_1(s)=\frac{g\pm\sqrt{g^2+4C(\rho)\theta_1(s)^2}}{2\theta_1(s)}$$

The marginal utility of agent 2 is then

$$\frac{1}{c_2(s)}=\left(\frac{1+\rho}{C(\rho)}\right)\left(\frac{g\pm\sqrt{g^2+4C(\rho)\theta_1(s)^2}}{2\theta_1(s)^2}\right)$$

Note that in order for either of these terms to be positive we need $C(\rho)\geq 0$ implying that there is only one economically meaningful root. Thus

$$l_1(s)=\frac{g+\sqrt{g^2+4C(\rho)\theta_1(s)^2}}{2\theta_1(s)} \tag{68}$$

and

$$\frac{1}{c_2(s)}=\left(\frac{1+\rho}{C(\rho)}\right)\left(\frac{g+\sqrt{g^2+4C(\rho)\theta_1(s)^2}}{2\theta_1(s)^2}\right) \tag{69}$$

A steady state is then a value of ρ such that

$$x(s)=\frac{1+\rho[l_1(\rho,s)^2-1]}{\frac{1/c_2(\rho,s)}{\mathbb{E}[\frac{1}{c_2}](\rho)}-\beta(s)} \tag{70}$$

s independent of s.

The following lemma, which orders consumption and labor across states, will be useful in proving the parts of proposition 6. As a notational aside we will often use $\theta_{1,l}$ and $\theta_{1,h}$ to refer to $\theta_1(s_l)$ and $\theta_1(s_h)$ respectively. Where s_l refers to the low TFP state and s_h refers to the high TFP state.

Lemma 9 *Suppose that* $\theta_1(s_l)<\theta_2(s_h)$ *and* ρ *such that* $C(\rho)>0$ *then*

$$l_{1,l}=\frac{g+\sqrt{g^2+4C(\rho)\theta_{1,l}^2}}{2\theta_{1,l}}>\frac{g+\sqrt{g^2+4C(\rho)\theta_{1,h}^2}}{2\theta_{1,h}}=l_{1,h}$$

and

$$\frac{1}{c_{2,l}}=\frac{1+\rho}{C(\rho)}\frac{g+\sqrt{g^2+4C(\rho)\theta_{1,l}^2}}{2\theta_{1,l}^2}>\frac{1+\rho}{C(\rho)}\frac{g+\sqrt{g^2+4C(\rho)\theta_{1,h}^2}}{2\theta_{1,h}^2}=\frac{1}{c_{2.h}}$$

Proof. The results should follow directly from showing that the function

$$l_1(\theta) = \frac{g + \sqrt{g^2 + 4C(\rho)\theta}}{2\theta}$$

is decreasing in θ. Taking the derivative with respect to θ

$$\begin{aligned}
\frac{dl_1}{d\theta}(\theta) &= -\frac{g}{2\theta^2} - \frac{\sqrt{g + 4C(\rho)\theta^2}}{2\theta^2} + \frac{4C(\rho)\theta}{2\theta\sqrt{g^2 + 4C(\rho)\theta^2}} \\
&= -\frac{g}{2\theta^2} - \frac{g + 4C(\rho)\theta^2 - 4C(\rho)\theta^2}{2\theta^2\sqrt{g^2 + 4C(\rho)\theta^2}} \\
&= -\frac{g}{2\theta^2} - \frac{g}{2\theta^2\sqrt{g^2 + 4C(\rho)\theta^2}} < 0
\end{aligned}$$

That $\frac{1}{c_{2,l}} > \frac{1}{c_{2,h}}$ follows directly. ■

Proof of Proposition 6.

Part 1. In order for there to exist a ρ such that equation (70) is independent of the state (and hence have a steady state) we need the existence of root for the following function

$$f(\rho) = \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} - \frac{\frac{1/c_2(\rho, s_h)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta}{\frac{1/c_2(\rho, s_l)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta}$$

From lemma 9 we can conclude that

$$1 + \rho[l_1(\rho, s_l)^2 - 1] > 1 + \rho[l_1(\rho, s_h)^2 - 1] \tag{71}$$

and

$$\frac{1/c_2(\rho, s_l)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta > \frac{1/c_2(\rho, s_h)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta \tag{72}$$

for all $\rho > 0$ such that $C(\rho) \geq 0$. To begin with we will define $\underline{\rho}$ such that $C(\rho) > 0$ for all $\rho > \underline{\rho}$. Note that we will have to deal with two different cases.

$\alpha_1(1-\rho) + 2\rho^2\alpha_2 > 0$ **for all** $\rho \geq 0$**:** In this case we know that $C(\rho) \geq 0$ for all ρ and is bounded above and thus we will let $\underline{\rho} = 0$.

$\alpha_1(1-\rho) + 2\rho^2\alpha_2 = 0$ **for some** $\rho > 0$**:** In this case let $\underline{\rho}$ be the largest positive root of $\alpha_1(1-\rho) + 2\rho^2\alpha_2$. Note that $\lim_{\rho\to\underline{\rho}^+} C(\rho) = \infty$

With this we note that[33]

$$\lim_{\rho\to\underline{\rho}^+} \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} = 1$$

We can also show that

$$\lim_{\rho\to\underline{\rho}^+} \frac{\frac{1/c_2(\rho, s_h)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta}{\frac{1/c_2(\rho, s_l)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta} < 1$$

[33] In the first case $\underline{\rho} = 0$ and in the second case $l_1(\rho, s_l) = l_1(\rho, s_h)$ as $\rho \to \underline{\rho}^+$

which implies that $\lim_{\rho\to\underline{\rho}^+} f(\underline{\rho}) > 0$.

Taking the limit as $\rho \to \infty$ we see that $C(\rho) \to 0$, given that $\frac{g}{\theta(s)} < 1$, we can then conclude that

$$\lim_{\rho\to\infty} 1 + \rho[l_1(\rho, s)^2 - 1] = -\infty$$

Thus, there exists $\overline{\rho}$ such that $1 + \overline{\rho}[l_1(\overline{\rho}, s_l)^2 - 1] = 0$. [34](to be rigorous we actually want to take the minimum over all roots). From equation (71), we know that

$$0 = 1 + \overline{\rho}[l_1(\overline{\rho}, s_l)^2 - 1] > 1 + \overline{\rho}[l_1(\overline{\rho}, s_h)^2 - 1]$$

which implies in the limit

$$\lim_{\rho\to\overline{\rho}^-} \frac{1 + \rho[l_1(\rho, s_h)^2 - 1]}{1 + \rho[l_1(\rho, s_l)^2 - 1]} = -\infty$$

which along with

$$\frac{\frac{1/c_2(\rho,s_h)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta}{\frac{1/c_2(\rho,s_l)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta} \geq -1$$

allows us to conclude that $\lim_{\rho\to\overline{\rho}^-} f(\rho) = -\infty$. The intermediate value theorem then implies that there exists ρ_{SS} such that $f(\rho_{SS}) = 0$ and hence that ρ_{SS} is a steady state.

Finally, as $\rho_{SS} < \overline{\rho}$ we know that

$$1 + \rho_{SS}[l_1(\rho_{SS}, s_l) - 1] > 0$$

as $\frac{1/c_2(\rho,s_l)}{\mathbb{E}[\frac{1}{c_2}](\rho)} > 1$ we can conclude

$$x_{SS} = \frac{1 + \rho_{SS}[l_1(\rho_{SS}, s_l) - 1]}{\frac{1/c_2(\rho,s_l)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta} > 0$$

implying that the government will hold assets in the steady state (under the normalization that agent 2 holds no assets).

Part 2. The condition that $R(s_h) = R(s_l)$ implies that

$$\frac{1/c_2(\rho, s_l)}{\beta(s_l)\mathbb{E}[\frac{1}{c_2}](\rho)} = \frac{1/c_2(\rho, s_h)}{\beta(s_h)\mathbb{E}[\frac{1}{c_2}](\rho)}$$

which simplifies to

$$\frac{\beta(s_h)}{\beta(s_l)} = \frac{1/c_2(\rho, s_h)}{1/c_2(\rho, s_l)} \tag{73}$$

[34]This can be seen from the fact $\lim_{\rho\to\underline{\rho}^+} 1 + \rho[l_1(\rho, s_l)^2 - 1] > 0$ and $\lim_{\rho\to\infty} 1 + \rho[l_1(\rho, s_l)^2 - 1] > -\infty$, thus $\overline{\rho}$ exists in $(\underline{\rho}, \infty)$

In order for a steady state to exist with constant interest rates there must be a root of the following function

$$
\begin{aligned}
f(\rho) &= \frac{1+\rho[l_1(\rho,s_h)^2-1]}{1+\rho[l_1(\rho,s_l)^2-1]} - \frac{\frac{1/c_2(\rho,s_h)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta(s_h)}{\frac{1/c_2(\rho,s_l)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta(s_l)} \\
&= \frac{1+\rho[l_1(\rho,s_h)^2-1]}{1+\rho[l_1(\rho,s_l)^2-1]} - \frac{\frac{1/c_2(\rho,s_h)}{\beta(s_h)\mathbb{E}[\frac{1}{c_2}](\rho)} - 1}{\frac{1/c_2(\rho,s_l)}{\beta(s_l)\mathbb{E}[\frac{1}{c_2}](\rho)} - 1}\frac{\beta(s_h)}{\beta(s_l)} \\
&= \frac{1+\rho[l_1(\rho,s_h)^2-1]}{1+\rho[l_1(\rho,s_l)^2-1]} - \frac{1/c_2(\rho,s_h)}{1/c_2(\rho,s_l)}
\end{aligned}
$$

Taking limits of $f(\rho)$ as ρ approaches $\underline{\rho}$ from the positive side we already demonstrated

$$
\lim_{\rho\to\underline{\rho}^+} \frac{1+\rho[l_1(\rho,s_h)^2-1]}{1+\rho[l_1(\rho,s_l)^2-1]} = 1
$$

From equation (69) and Lemma 9 it is straightforward to see that

$$
\lim_{\rho\to\underline{\rho}^+} \frac{1/c_2(\rho,s_h)}{1/c_2(\rho,s_l)} < 1
$$

which allows us to conclude that

$$
\lim_{\rho\to\underline{\rho}^+} f(\rho) > 0
$$

Taking limits as ρ approaches $\overline{\rho}$ from the negative direction we know that

$$
\lim_{\rho\to\overline{\rho}^-} \frac{1+\rho[l_1(\rho,s_h)^2-1]}{1+\rho[l_1(\rho,s_l)^2-1]} = -\infty
$$

As $\frac{1/c_2(\rho,s_h)}{1/c_2(\rho,s_h)} > 0$ for all ρ it is straightforward to conclude that

$$
\lim_{\rho\to\overline{\rho}^-} f(\rho) = -\infty
$$

Continuity then implies the existence of a ρ^{SS} such that $f(\rho^{SS}) = 0$, and thus there exists a $\beta(s_l)$ and $\beta(s_h)$ such that $R(s_l) = R(s_h)$ in steady state. From Lemma 9

$$
l(\rho,s_l) > l(\rho,s_h).
$$

In order for

$$
\frac{1+\rho^{SS}[l_1(\rho^{SS},s_h)^2-1]}{1+\rho^{SS}[l_1(\rho^{SS},s_l)^2-1]} = \frac{1/c_2(\rho^{SS},s_h)}{1/c_2(\rho^{SS},s_l)} < 1
$$

it is necessary that

$$
1+\rho^{SS}[l_1(\rho^{SS},s_l)^2-1] > 1+\rho^{SS}[l_1(\rho^{SS},s_h)^2-1] > 0
$$

implying that the steady state asset level

$$
x_{SS} = \frac{1+\rho_{SS}[l_1(\rho_{SS},s_l)-1]}{\frac{1/c_2(\rho,s_l)}{\mathbb{E}[\frac{1}{c_2}](\rho)} - \beta(s_l)} > 0
$$

Part 3 As noted before, since $g/\theta(s) < 1$ for all s we have

$$\lim_{\rho\to\infty} 1 + \rho[l_1(\rho, s)^2 - 1] = -\infty$$

Thus, there exists ρ_{SS} such that

$$0 > 1 + \rho_{SS}[l_1(\rho_{SS}, s_l)^2 - 1] > 1\rho_{SS}[l_1(\rho_{SS}, s_h)^2 - 1]$$

It is then possible to choose $\beta(s) < \frac{1/c_2(\rho_{SS},s)}{\mathbb{E}[\frac{1}{c_2}](\rho_{SS})}$ such that

$$1 > \frac{1 + \rho_{SS}[l_1(\rho_{SS}, s_l)^2 - 1]}{1 + \rho_{SS}[l_1(\rho_{SS}, s_h)^2 - 1]} = \frac{\frac{1/c_2(\rho_{SS},s_l)}{\mathbb{E}[\frac{1}{c_2}](\rho_{SS})} - \beta(s_l)}{\frac{1/c_2(\rho_{SS},s_h)}{\mathbb{E}[\frac{1}{c_2}](\rho_{SS})} - \beta(s_h)} \tag{74}$$

Implying that for discount factor shocks $\beta(s)$, ρ_{SS} is a steady state level for the ratio of marginal utilities, with steady state marginal utility weighted government debt

$$x_{SS} = \frac{1 + \rho_{SS}[l_1(\rho_{SS}, s_l)^2 - 1]}{\frac{1/c_2(\rho_{SS},s_l)}{\mathbb{E}[\frac{1}{c_2}](\rho_{SS})} - \beta(s_l)} < 0$$

Thus, in the steady state, the government is holding debt, under the normalization that the unproductive worker holds no assets. As $\frac{1/c_2(\rho,s_l)}{\mathbb{E}[\frac{1}{c_2}](\rho)} > \frac{1/c_2(\rho,s_h)}{\mathbb{E}[\frac{1}{c_2}](\rho)}$, in order for equation (74) to hold we need $\beta_l > \beta_h$. We can then rewrite equation (74) as

$$1 > \frac{\beta(s_h)}{\beta(s_l)} > \frac{\frac{1/c_2(\rho_{SS},s_l)}{\beta_l\mathbb{E}[\frac{1}{c_2}](\rho_{SS})} - 1}{\frac{1/c_2(\rho_{SS},s_h)}{\beta(s_h)\mathbb{E}[\frac{1}{c_2}](\rho_{SS})} - 1}$$

Thus

$$R(s_l)\frac{1/c_2(\rho_{SS}, s_l)}{\beta(s_l)\mathbb{E}[\frac{1}{c_2}](\rho_{SS})} < \frac{1/c_2(\rho_{SS}, s_h)}{\beta(s_h)\mathbb{E}[\frac{1}{c_2}](\rho_{SS})} = R(s_h) \tag{75}$$

in the steady state interest rates are positively correlated with TFP.

■

9.5 Constrained optimum with unobservable assets

We contrast allocations in the economy described in the previous section with allocations in an economy in which the government does not observe agents' assets. To define optimal allocations, we need to take a stand on what assets agents can trade. In line with our previous analysis, we assume that they can trade only a one-period risk-free bond. Later we will explain how our conclusions would change with more general asset structures.

We borrow a general formulation of the constrained optimal problem from Golosov and Tsyvinski (2007). The mechanism designer collects agents' reports about their types and chooses allocations of labor services and consumption. The agents can re-trade consumption as they choose. Asset prices are determined by market clearing conditions. Let $\{e_{i,t}, y_{i,t}\}_{i,t}$ be the allocation of consumption and labor services that the planner assigns to agents. The utility of agent i who chooses basket $\{e_{j,t}, y_{j,t}\}_t$ and faces interest rates $\{R_t\}_t$ is

$$\mathcal{V}_i\left(\{e_{j,t}, y_{j,t}, R_t\}_t\right) = \max_{\left\{c^j_{i,t}, b^j_{i,t}\right\}_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U_i\left(c^j_{i,t}, \frac{y_{j,t}}{\theta_i}\right)$$

subject to

$$c^j_{i,t} + b^j_{i,t} = e_{j,t} + R_{t-1} b^j_{i,t-1}.$$

The mechanism designer chooses an allocation such that no agent wants to re-trade along the equilibrium path. The optimal allocation solves[35]

$$\max_{\{c_{i,t}, y_{i,t}, R_t\}_{i,t}} \mathbb{E}_0 \sum_{i=1}^{I} \alpha_i \pi_i \sum_{t=0}^{\infty} \beta^t U^i\left(c_{i,t}, \frac{y_{i,t}}{\theta_i}\right)$$

subject to (2), (13) and

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U^i\left(c_{i,t}, \frac{y_{i,t}}{\theta_i}\right) \geq \mathcal{V}_i\left(\{c_{j,t}, y_{j,t}, R_t\}_t\right) \text{ for all } i, j.$$

An optimal allocation can be decentralized by a non-linear tax $\{T_t(y_t, ..., y_0)\}_t$. We can define a competitive equilibrium in a fashion that parallels Definition 3. To maintain consistency with the informational requirement of the mechanism design problem, we assume that the government does not know who holds government debt (if government debt were to be issued in an equilibrium). As discussed in Section 3, Theorem 1 continues to hold in this economy. Thus, again the distribution of net assets $\left\{\tilde{b}_{i,t}\right\}$ is uniquely determined, but not agents' gross assets. These results show that dynamic properties of optimal taxes and allocations depend crucially on whether the government can observe and tax individuals' asset holdings.

Below we construct an example that shows that optimal allocations are generally history dependent, in contrast to the history independent optimal allocations that emerge for an economy where the government can observe all agents' asset holdings.

We consider a version of the economy where $I = 2$ with Agent 1 having quasilinear preferences and Agent 2's utility is given by $u(c) - h_2\left(\frac{y}{\theta_2}\right)$. We assume that the planner assigns a sufficiently high Pareto weight on Agent 1 so that it is agent 2 incentive constraint which binds.

The equilibrium can either be interior, in which case $c_{1,t}$ is always above $\underline{c}$, or the non-negativity constraint eventually binds. The latter case automatically implies history-dependence of allocations, so we consider the former case. The optimal tax problem is

$$\max_{\left\{c^j_{i,t}, b^j_{i,t}, y_{i,t}, R_t\right\}_{t,i,j}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\alpha_1\left(c_{1,t} - h_1\left(\frac{y_{1,t}}{\theta_1}\right)\right) + \alpha_2\left(u(c_{2,t}) - h_2\left(\frac{y_{2,t}}{\theta_2}\right)\right)\right]$$

subject to

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(u(c_{2,t}) - h_2\left(\frac{y_{2,t}}{\theta_2}\right)\right) \geq \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(u(c^1_{2,t}) - h_2\left(\frac{y_{1,t}}{\theta_2}\right)\right)$$

[35] For details see **?**.

$$c_{2,t} + b_{2,t} = e_{2,t} + \frac{1}{\beta} b_{2,t-1}$$

$$u'(c_{2,t+1}) = \mathbb{E}_t u'(c_{2,t+1})$$

$$c^1_{2,t} + b^1_{2,t} = e_{1,t} + \frac{1}{\beta} b^1_{2,t-1}$$

$$u'\left(c^1_{2,t+1}\right) = \mathbb{E}_t u'\left(c^1_{2,t+1}\right)$$

$$c_{1,t} + b_{1,t} = e_{1,t} + \frac{1}{\beta} b_{1,t-1}$$

$$c_{1,t} + c_{2,t} + g_t = y_{1,t} + y_{2,t}$$

$$b^j_{i,t} \geq \underline{B}.$$

There are a few redundant equations here. Without loss of generality we can set $b_{1,t} = 0$ for all t in which case $c_{1,t} = e_{1,t}$. Moreover, it is clear that the Lagrange multiplier on the second constraint must be zero. Therefore, we have

$$\max_{\left\{c^j_{i,t}, b^j_{i,t}, y_{i,t}\right\}_{t,i,j}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\alpha_1 \left(c_{1,t} - h_1\left(\frac{y_{1,t}}{\theta_1}\right)\right) + \alpha_2 \left(u(c_{2,t}) - h_2\left(\frac{y_{2,t}}{\theta_2}\right)\right)\right]$$

s.t.

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(u(c_{2,t}) - h_2\left(\frac{y_{2,t}}{\theta_2}\right)\right) \geq \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(u(c^1_{2,t}) - h_2\left(\frac{y_{1,t}}{\theta_2}\right)\right)$$

$$c^1_{2,t} + b^1_{2,t} = c_{1,t} + \frac{1}{\beta} b^1_{2,t-1}$$

$$u'(c_{2,t+1}) = \mathbb{E}_t u'(c_{2,t+1})$$

$$u'\left(c^1_{2,t+1}\right) = \mathbb{E}_t u'\left(c^1_{2,t+1}\right)$$

$$c_{1,t} + c_{2,t} + g_t = y_{1,t} + y_{2,t}$$

Guess that neither Euler equation constraint binds. Take the first-order conditions

$$\alpha_1 - \eta^1_{2,t} = \lambda_t \tag{76a}$$

$$u'(c_{2,t})(\alpha_2 + \mu) = \lambda_t \tag{76b}$$

$$\mu u'\left(c^1_{2,t}\right) = \eta^1_{2,t} \tag{76c}$$

$$\eta^1_{2,t} = \mathbb{E}_t \eta^1_{2,t+1} \tag{76d}$$

From (76d) $\eta^1_{2,t}$ is a martingale, therefore from (76a) , λ_t is a martingale, and therefore $u'(c_{2,t})$ and $u'\left(c^1_{2,t}\right)$ are martingales, which confirms our guess. Moreover, $u'(c_{2,t})$, $u'\left(c^1_{2,t}\right)$, λ_t and $\eta^1_{2,t}$ all must converge. Next we discuss what they must converge to.

Note that since λ_t converges to a constant, the first-order conditions for $y_{2,t}$ and $y_{1,t}$ imply that they also converge to constants. Since $c_{2,t}$ converges to a constant, $c_{1,t}$ must fluctuate to offset fluctuations in g_t. If $c_{1,t}$ fluctuates, then $u'\left(c^1_{2,t}\right) \to 0$ and therefore $\eta^1_{2,t} \to 0$. This implies that in the long run this economy converges to the constrained optimal allocations discussed in Section 3.2. Intuitively, what happens is that as $c_{1,t}$ fluctuates, the agent 2, if he deviates, accumulates infinitely large amount of assets. When he does that, smoothing fluctuations in $c_{1,t}$ has no effect on his welfare, and we get back to the fully constrained optimum allocations.

Note that $c_{2,t}$ cannot be constant in all t. If it were, then $c^1_{2,t}$ would also have to be constant in all t, which is impossible since $c_{1,t}$ must fluctuate.

9.6 Numerical Methods

This section will outline our numerical methods used to solve for and linearize around a complete markets steady state in the case of a 2 state iid process for the aggregate state. We will specialize to the two agent case and for the remainder of this section it will be assumed that the Bellman equation is written in the form

$$V(x,\rho) = \max_{c_i(s),l_i(s),x'(s),\rho'(s)} \sum_s P(s)\left(\left[\sum_i \pi_i\alpha_i U(c_i(s),l_i(s))\right] + \beta(s)V(x'(s),\rho'(s))\right) \tag{77}$$

$$U_{c,2}(s)c_2(s) + U_{l,2}(s)l_2(s) - \rho'(s)\left[U_{c,1}(s)c_1(s) + U_{l,1}(s)l_1(s)\right] + \beta(s)x'(s) = \frac{xU_{c,2}(s)}{\mathbb{E}U_{c,2}} \tag{78a}$$

$$\sum_s P(s)U_{c,1}(s)(\rho(s)-\rho) = 0 \tag{78b}$$

$$\frac{\rho'(s)}{\theta_1(s)}U_{l,1}(s) = \frac{1}{\theta_2(s)}U_{l,2}(s) \tag{78c}$$

$$\pi_1 c_1(s) + \pi_2 c_2(s) + g(s) = \pi_1\theta_1(s)l_1(s) + \pi_2\theta_2(s)l_2(s) \tag{78d}$$

$$U_{c,2}(s) = \rho'(s)U_{c,1}(s) \tag{78e}$$

Note that some of the constraints have been modified a little for ease of differentiation. Associated with these constraints we have the Lagrange multipliers $P(s)\mu(s)$, λ,$P(s)\phi(s)$,$P(s)\xi(s)$, and $P(s)\zeta(s)$.

Steady State

By taking first order conditions it is possible to numerically determine the location of a stationary steady state. The first order conditions with respect to the choice variables are as follows (note we will be using the notation $\mathbb{E}z$ to represent $\sum_s P(s)z(s)$ for some variable z)

$c_1(s)$**:**

$$\pi_1\alpha_1 U_{c,1}(s) + \mu(s)\rho'(s)\left[U_{cc,1}(s)c_1(s) + U_{c,1}(s)\right] + \lambda U_{cc,1}(s)(\rho'(s)-\rho) - \pi_1\xi(s) + \zeta(s)\rho'(s)U_{cc,1}(s) = 0 \tag{79a}$$

$c_2(s)$**:**

$$\pi_2\alpha_2 U_{c,2}(s) - \mu(s)\left[U_{cc,2}(s)c_2(s) + U_{c,2}(s)\right] + \frac{xU_{cc,2}(s)}{\mathbb{E}U_{c,2}}\left(\mu(s) - \frac{\mathbb{E}\mu U_{c,2}}{\mathbb{E}U_{c,2}}\right) - \pi_2\xi(s) - \zeta(s)U_{cc,2}(s) = 0 \tag{79b}$$

$l_1(s)$**:**

$$\pi_1\alpha_1 U_{l,1}(s) + \mu(s)\rho(s)\left[U_{ll,1}(s)l_1(s) + U_{l,1}(s)\right] - \frac{\rho'(s)}{\theta_1(s)}\phi(s)U_{ll,1}(s) + \pi_1\theta_1(s)\xi(s) = 0 \tag{79c}$$

$l_2(s)$**:**

$$\pi_2\alpha_2 U_{l,2}(s) - \mu(s)\left[U_{ll,2}(s)l_2(s) + U_{l,2}(s)\right] + \frac{\phi(s)}{\theta_2(s)}U_{ll,2}(s) + \pi_2\theta_2(s)\xi(s) = 0 \tag{79d}$$

$x'(s)$:

$$V_x(x'(s), \rho'(s)) - \mu(s) = 0 \tag{79e}$$

$\rho'(s)$:

$$\beta(s)V\rho(x'(s), \rho'(s)) + \mu(s)\left[U_{c,1}(s)c_1(s) + U_{l,1}(s)l_1(s)\right] + \lambda U_{c,1}(s) - \phi(s)\frac{U_{l,1}(s)}{\theta_1(s)} + U_{c,1}(s)\zeta(s) = 0 \tag{79f}$$

Equations (78a)-(78e) and (79a)-(79f) then define the necessary conditions for an interior maximization of the planners problem for the state (x, ρ). In addition to these we have the two envelop conditions

$$V_x(x, \rho) = \frac{\sum_s P(s)\mu(s)U_{c,2}(s)}{\mathbb{E}U_{c,2}(s)} = \frac{\mathbb{E}\mu U_{c,2}}{\mathbb{E}U_{c,2}} \tag{80a}$$

and

$$V\rho(x, \rho) = -\lambda \mathbb{E}U_{c,1} \tag{80b}$$

A steady state will then be a set of allocations and Lagrange multipliers $\{c_1(s), c_2(s), l_1(s), l_2(s), x, \rho, \mu(s), \lambda, \phi(s), \xi(s), \zeta(s)\}$ that solve equations (78a)-(78e), (79a)-(79f), and (80a)-(80b). It should be noted that taking $\mu(s) = \mu = V_x(x, \rho)$ will solve both equations (79e) and (80a). Thus a steady state will be a set of allocations and Lagrange multipliers $\{c_1(s), c_2(s), l_1(s), l_2(s), x, \rho, \mu, \lambda, \phi(s), \xi(s), \zeta(s)\}$ that solves the following system of equations

$$U_{c,2}(s)c_2(s) + U_{l,2}(s)l_2(s) - \rho\left[U_{c,1}(s)c_1(s) + U_{l,1}(s)l_1(s)\right] + \beta(s)x - \frac{xU_{c,2}(s)}{\mathbb{E}U_{c,2}} \tag{81a}$$

$$\frac{U_{l,2}(s)}{\theta_2(s)} - \frac{\rho U_{l,1}(s)}{\theta_1(s)} = 0 \tag{81b}$$

$$\pi_1 c_1(s) + \pi_2 c_2(s) + g(s) = \pi_1\theta_1(s)l_1(s) + \pi_2\theta_2(s)l_2(s) = 0 \tag{81c}$$

$$\rho U_{c,1}(s) - U_{c,2}(s) = 0\pi_1\alpha_1 U_{c,1}(s) + \mu\rho\left[U_{cc,1}(s)c_1(s) + U_{c,1}(s)\right] - \pi_1\xi(s) + \rho\eta(s)U_{cc,1}(s) = 0 \tag{81d}$$

$$\pi_2\alpha_2 U_{cc,2}(s) - \mu\left[U_{cc,2}(s)c_2(s) + U_{c,2}(s)\right] - \pi_2\xi(s) - \eta(s)U_{cc,2}(s) = 0 \tag{81e}$$

$$\pi_1\alpha_1 U_{l,1}(s) + \mu\rho\left[U_{ll,1}(s)l_1(s) + U_{l,1}(s)\right] - \frac{\rho}{\theta_1(s)}\phi(s)U_{ll,1}(s) + \pi_1\theta_1(s)\xi(s) = 0 \tag{81f}$$

$$\pi_2\alpha_2 U_{l,2}(s) - \mu\left[U_{ll,2}(s)l_2(s) + U_{l,2}(s)\right] + \frac{\phi(s)}{\theta_2(s)}U_{ll,2}(s) + \pi_2\theta_2(s)\xi(s) = 0 \tag{81g}$$

$$\lambda\left[U_{c,1}(s) - \beta(s)\mathbb{E}U_{c,1}\right] + \mu\left[U_{c,1}(s)c_1(s) + U_{l,1}(s)l_1(s)\right] - \phi(s)\frac{U_{l,1}(s)}{\theta_1(s)} + U_{c,1}(s)\xi(s) = 0 \tag{81h}$$

Taking z to be the stacked vector $\{c_1(s), c_2(s), l_1(s), l_2(s), x, \rho, \mu, \lambda, \phi(s), \xi(s), \zeta(s)\}$, the equations (81a)-(81h) then define a vector function $F_{SS}(X)$ for which the steady state must satisfy $F_{SS}(X_{SS}) = 0$. An observant reader will have noted that the system $F_{SS}(X) = 0$ is only square when then number of aggregates states is 2. In the case of where there are 2 aggregate states we have verified numerically (using a non-linear equation solver) that a steady state does exist for nearly all of the parameter space. When the number of states is greater than two, there generically does not exist a steady state, however we find numerically that there exists an $\tilde{X}$ such that $F_{SS}(\tilde{X}) \approx 0$. Our solutions to Bellman equation find that the economy will converge in the long run to a region a round this point.

Linearization

In order to check local stability we developed a way of linearizing locally around the steady state. Given the discussion above we will therefore be restricting ourselves to a 2 state iid process for the aggregate state. The optimal policy function, which we will denote as $z(x,\rho)$, must satisfy $F(z,y,g(z))=0$ where F represents the system of equations (78a)-(79f), y is the state vector (x,ρ), and g is the mapping of next periods policies into the derivatives of the value function. In other words

$$g(z)=\begin{pmatrix}V_x(x'(1),\rho'(1))\\ V\rho(x'(1),\rho'(1)\\ V_x(x'(2),\rho'(2))\\ V\rho(x'(2),\rho'(2))\end{pmatrix}$$

The optimal policy function is then a function $z(x,R)$ that satisfies the relationship $F(z(y),y,g(z(y)))=0$. Taking total derivatives around the steady stat $\overline{y}$ and $\overline{z}=z(\overline{y})$

$$D_zF(\overline{z},\overline{y},g(\overline{z}))D_yz(\overline{y})+D_yF(\overline{z},\overline{y},g(\overline{z}))+D_vF(\overline{z},\overline{y},g(\overline{z}))Dg(\overline{z})D_yz(\overline{z})=0$$

In order to linearize $z(y)$ around the steady state $\overline{y}$ we need to compute $D_yz(\overline{y})$. The problem is that g depends on the planners value function which is difficult to compute. The envelope conditions (80a)-(80b) tell us that V_x and $V\rho$ depend on the policies, i.e.

$$\begin{pmatrix}V_x(x,\rho)\\ V\rho(x,\rho)\end{pmatrix}=v(z(x,\rho))=\begin{pmatrix}-\sum_{s'}\frac{\mu(s')P(s')U_{c,2}(s)}{\mathbb{E}[U_{c,2}]}\\ -\lambda\mathbb{E}\left[U_{c,1}\right]\end{pmatrix}$$

If we let Φ_s be the matrix that maps $z(x,R)$ into $\begin{pmatrix}x'(s)\\ R'(s)\end{pmatrix}$ then we can write $g(x,R)$ as a function of z and v as follows

$$g(z)=\begin{pmatrix}v(z(\Phi_1 z))\\ v(z(\Phi_2 z))\end{pmatrix}$$

taking derivatives we quickly obtain that

$$\begin{aligned}D_zg(\overline{z})&=\begin{pmatrix}Dv(z(\Phi_1\overline{z})) & 0\\ 0 & Dv(z(\Phi_2\overline{z}))\end{pmatrix}\begin{pmatrix}D_yz(\Phi_1\overline{z}) & 0\\ 0 & D_yz(\Phi_1\overline{z})\end{pmatrix}\underbrace{\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}}_{\Phi}\\ &=\begin{pmatrix}Dv(\overline{z}) & 0\\ 0 & Dv(\overline{z})\end{pmatrix}\begin{pmatrix}D_yz(\overline{y}) & 0\\ 0 & D_yz(\overline{y})\end{pmatrix}\Phi\end{aligned}$$

We can then go back to our original matrix equation to obtain

$$\begin{aligned}D_yz(\overline{y})=&-D_zF(\overline{z},\overline{y},g(\overline{z}))^{-1}D_yF(\overline{z},\overline{y},g(\overline{z}))\\ &-D_zF(\overline{z},\overline{y},g(\overline{z}))^{-1}D_vF(\overline{z},\overline{y},g(\overline{z}))\begin{pmatrix}Dv(\overline{z}) & 0\\ 0 & Dv(\overline{z})\end{pmatrix}\begin{pmatrix}D_yz(\overline{y}) & 0\\ 0 & D_yz(\overline{y})\end{pmatrix}\Phi D_yz(\overline{y})\end{aligned}\tag{82}$$

This is now a non-linear matrix equation for $D_yz(\overline{y})$, where all the other terms can be computed using the steady state values $\overline{z}$ and $\overline{y}$ (note $g(\overline{z})$ is known from the envelope conditions at the steady state). Furthermore, $D_yz(\overline{y})$ gives us the linearization of the policy rules since to first order

$$z\approx\overline{z}+D_yz(\overline{y})(y-\overline{y})$$

Our procedure for computing the linearization proceeds as follows

1. Find the steady state by solving the system of equations (81a - 81h). Numerically, we have found that this is very robust to the parameters of the model.

2. Compute $D_z F(\overline{z}, \overline{y}, g(\overline{z}))$, $D_z F(\overline{z}, \overline{y}, g(\overline{z}))$ and $D_v F(\overline{z}, \overline{y}, g(\overline{z}))$ by numerically differentiating F (note, if greater precision is needed formulas for these derivative could in principle be derived by hand)

3. Compute DV around the steady state values of c_1,c_2, μ and λ.

4. Solve the matrix equation (82). This is done using the NAG non-linear equation solver. The method is somewhat problematic as there exists a near zero that the non-linear solver often finds. Thus the solution method involves guessing 100 random starting points and choosing the best solution.

Given the linearized policy rules it is then possible to evaluate the local stability of the steady state. We find that in the absence of discount factor shocks the steady state is stable generically across the parameter space.

References

Aiyagari, S. Rao, Albert Marcet, Thomas J. Sargent, and Juha Seppälä. 2002. "Optimal Taxation without State Contingent Debt." *Journal of Political Economy*, 110(6): 1220–1254, URL: http://www.jstor.org/stable/10.1086/343744.

Aiyagari, S. Rao, and Ellen R. McGrattan. 1998. "The optimum quantity of debt." *Journal of Monetary Economics*, 42(May 1997): 447–469, URL: http://www.sciencedirect.com/science/article/pii/S0304393298000312.

Albanesi, S., and R. Armenter. 2012. "Intertemporal Distortions in the Second Best." *The Review of Economic Studies*, 79(4): 1271–1307, URL: http://restud.oxfordjournals.org/lookup/doi/10.1093/restud/rds014, DOI: http://dx.doi.org/10.1093/restud/rds014.

Autor, DH, LF Katz, and MS Kearney. 2008. "Trends in US wage inequality: Revising the revisionists." *The Review of Economics and Statistics*, 90(2): 300–323, URL: http://www.mitpressjournals.org/doi/abs/10.1162/rest.90.2.300, DOI: http://dx.doi.org/10.1162/rest.90.2.300.

Azzimonti, Marina, Eva de Francisco, and Per Krusell. 2008a. "Aggregation and Aggregation." *Journal of the European Economic Association*, 6(2-3): 381–394, URL: http://doi.wiley.com/10.1162/JEEA.2008.6.2-3.381, DOI: http://dx.doi.org/10.1162/JEEA.2008.6.2-3.381.

Azzimonti, Marina, Eva de Francisco, and Per Krusell. 2008b. "Production subsidies and redistribution." *Journal of Economic Theory*, 142(1): 73–99, URL: http://linkinghub.elsevier.com/retrieve/pii/S0022053107001020, DOI: http://dx.doi.org/10.1016/j.jet.2007.03.009.

Barro, Robert J. 1979. "On the determination of the public debt." *The Journal of Political Economy*, 87(5): 940–971, URL: http://www.jstor.org/stable/10.2307/1833077.

Bassetto, Marco, and Narayana Kocherlakota. 2004. "On the irrelevance of government debt when taxes are distortionary." *Journal of Monetary Economics*, 51(2): 299–304, URL: http://linkinghub.elsevier.com/retrieve/pii/S0304393203001430, DOI: http://dx.doi.org/10.1016/j.jmoneco.2002.12.001.

Battaglini, Marco, and Stephen Coate. 2007. "Inefficiency in Legislative Policymaking: A Dynamic Analysis." *American Economic Review*, 97(1): 118–149, URL: http://www.aeaweb.org/articles.php?doi=10.1257/aer.97.1.118, DOI: http://dx.doi.org/10.1257/aer.97.1.118.

Battaglini, Marco, and Stephen Coate. 2008. "A Dynamic Theory of Public Spending, Taxation, and Debt." *American Economic Review*, 98(1): 201–236, URL: http://www.aeaweb.org/articles.php?doi=10.1257/aer.98.1.201, DOI: http://dx.doi.org/10.1257/aer.98.1.201.

Boldrin, Michele, Lawrence J. Christiano, and Jonas DM Fisher. 2001. "Habit Persistence, Asset Returns, and the Business Cycle." *American Economic Review*, 91(1): 149–166, URL: http://ideas.repec.org/a/aea/aecrev/v91y2001i1p149-166.html.

Bryant, John, and Neil Wallace. 1984. "A Price Discrimination Analysis of Monetary Policy." *The Review of Economic Studies*, 51(2): , p. 279, URL: http://restud.oxfordjournals.org/lookup/doi/10.2307/2297692, DOI: http://dx.doi.org/10.2307/2297692.

Chari, V V, Lawrence J Christiano, and Patrick J Kehoe. 1994. "Optimal Fiscal Policy in a Business Cycle Model." *Journal of Political Economy*, 102(4): 617–652, URL: http://www.nber.org/papers/w4490http://www.jstor.org/stable/2138759, DOI: http://dx.doi.org/10.2307/2138759.

Constantinides, George, and Darrell Duffie. 1996. "Asset pricing with heterogeneous consumers." *Journal of Political economy*, 104(2): 219–240, URL: http://www.jstor.org/stable/10.2307/2138925.

Correia, Isabel. 2010. "Consumption taxes and redistribution." *American Economic Review*, 100(September): 1673–1694, URL: http://www.ingentaconnect.com/content/aea/aer/2010/00000100/00000004/art00014.

Faraglia, Elisa, Albert Marcet, and Andrew Scott. 2012. "Dealing with Maturity : Optimal Fiscal Policy with Long Bonds."

Farhi, Emmanuel. 2010. "Capital Taxation and Ownership When Markets Are Incomplete." *Journal of Political Economy*, 118(5): 908–948, URL: http://www.jstor.org/stable/10.1086/657996.

Golosov, Mikhail, and Aleh Tsyvinski. 2007. "Optimal Taxation with Endogenous Insurance Markets." *The Quarterly Journal of Economics*, 122(2): 487–534, URL: http://qje.oxfordjournals.org/content/122/2/487.short, DOI: http://dx.doi.org/10.1162/qjec.122.2.487.

Golosov, Mikhail, Aleh Tsyvinski, and Ivan Werning. 2007. "New dynamic public finance: a user's guide." In *NBER Macroeconomics Annual 2006, Volume 21*. 21: MIT Press, 317–388, URL: http://www.nber.org/chapters/c11181.pdf.

Guvenen, Fatih, Serdar Ozkan, and Jae Song. 2012. "The Nature of Countercyclical Income Risk." Working Paper 18035, National Bureau of Economic Research.

Heathcote, Jonathan. 2005. "Fiscal Policy with Heterogeneous Agents and Incomplete Markets." *Review of Economic Studies*, 72(1): 161–188, URL: http://restud.oxfordjournals.org/lookup/doi/10.1111/0034-6527.00328, DOI: http://dx.doi.org/10.1111/0034-6527.00328.

Judd, Kenneth L., Lilia Maliar, and Serguei Maliar. 2011. "Numerically stable and accurate stochastic simulation approaches for solving dynamic economic models." *Quantitative Economics*, 2(2): 173–210, URL: http://doi.wiley.com/10.3982/QE14, DOI: http://dx.doi.org/10.3982/QE14.

Kocherlakota, Narayana R. 2010. *The new dynamic public finance.*: Princeton University Press.

Kydland, Finn E, and Edward C Prescott. 1980. "Dynamic optimal taxation, rational expectations and optimal control." *Journal of Economic Dynamics and Control*, 2(0): 79–91, URL: http://www.sciencedirect.com/science/article/pii/0165188980900524, DOI: http://dx.doi.org/http://dx.doi.org/10.1016/0165-1889(80)90052-4.

Lucas Jr., Robert E, and Nancy L Stokey. 1983. "Optimal fiscal and monetary policy in an economy without capital." *Journal of Monetary Economics*, 12(1): 55–93, URL: http://www.sciencedirect.com/science/article/pii/0304393283900491, DOI: http://dx.doi.org/http://dx.doi.org/10.1016/0304-3932(83)90049-1.

Magill, Michael, and Martine Quinzii. 1994. "Infinite Horizon Incomplete Markets." *Econometrica*, 62(4): 853–880, URL: http://www.jstor.org/stable/2951735, DOI: http://dx.doi.org/10.2307/2951735.

Newcomb, Simon. 1865. *A critical examination of our financial policy during the Southern rebellion.*

Ray, Debraj. 2002. "The Time Structure of Self-Enforcing Agreements." *Econometrica*, 70(2): 547–582, URL: http://onlinelibrary.wiley.com/doi/10.1111/1468-0262.00295/full.

Reinhart, Carmen M, and Kenneth S Rogoff. 2010. "Growth in a Time of Debt." *American Economic Review*, 100(2): 573–578, URL: http://pubs.aeaweb.org/doi/abs/10.1257/aer.100.2.573, DOI: http://dx.doi.org/10.1257/aer.100.2.573.

Sargent, Thomas J. 1987. *Dynamic macroeconomic theory.*: Harvard University Press.

Sargent, Thomas J., and Bruce D. Smith. 1987. "Irrelevance of open market operations in some economies with government currency being dominated in rate of return." *American Economic Review*, 77(1): 78–92, URL: http://ideas.repec.org/a/aea/aecrev/v77y1987i1p78-92.html.

Shin, Yongseok. 2006. "Ramsey meets Bewley: Optimal government financing with incomplete markets." *Unpublished manuscript, Washington University in St. Louis*(August): .

Wallace, Neil. 1981. "A Modigliani-Miller Theorem for Open-Market Operations." *The American Economic Review*, 71(3): 267–274, URL: http://www.jstor.org/stable/1802777, DOI: http://dx.doi.org/10.2307/1802777.

Werning, Iván. 2007. "Optimal Fiscal Policy with Redistribution,." *Quarterly Journal of Economics*, 122(August): 925–967, URL: http://qje.oxfordjournals.org/content/122/3/925.abstract, DOI: http://dx.doi.org/10.1162/qjec.122.3.925.

Werning, Iván. 2012. "Notes on Tax Smoothing with Heterogeneous Agents."

Yared, Pierre. 2010. "Politicians, Taxes and Debt." *Review of Economic Studies*, 77(2): 806–840, URL: `http://restud.oxfordjournals.org/lookup/doi/10.1111/j.1467-937X.2009.00584.x`, DOI: `http://dx.doi.org/10.1111/j.1467-937X.2009.00584.x`.

Yared, Pierre. 2013. "Public Debt Under Limited Private Credit." *Journal of the European Economic Association*, 11(2): 229–245, URL: `http://doi.wiley.com/10.1111/jeea.12010`, DOI: `http://dx.doi.org/10.1111/jeea.12010`.