Linearization of Policy Functions with Incomplete Markets

January 31, 2014

The Problem

Our problem can be written recursively as follows

$$V(b) = \max_{c(s), l(s), b'(s)} \sum_s \Pi(s) \left[c(s) - \frac{l(s)^{1+\gamma}}{1+\gamma} + \beta V(b'(s)) \right]$$

subject to the constraints

$$c(s) - l(s)^{1+\gamma} + b'(s) = \frac{p_s b}{\beta} \tag{1a}$$

$$c(s) + g_s \leq l(s) \tag{1b}$$

- We can show that this problem is convex and $V(b)$ is concave.
- Let $\mu = V'(b)$, then there exists a mapping $b(\mu)$ that maps the multiplier on the implementability constraint into government debt.

First Order Conditions

- The first order conditions can be written succinctly as finding a function $b(\mu)$ such that the following system of equations can be solved for all μ.

$$\frac{b(\mu)p_s}{\beta \mathbb{E}p} = I(\mu'(s)) - g_s + b(\mu'(s)) \tag{2}$$

$$\mu = \frac{\mathbb{E}\mu' p}{\mathbb{E}p} \tag{3}$$

- Where

$$I(\mu) = (1-\tau(\mu))^{\frac{1}{\gamma}}\tau(\mu) \text{ and } \tau(\mu) = \frac{\gamma\mu}{(1+\gamma)\mu - 1}$$

Payoffs with a Steay State

- We wish to find payoff $\overline{p}_s$ (normalized so that $\mathbb{E}\overline{p} = 1$) such that the policy functions $\mu'(\overline{\mu}, s) = \overline{\mu}$ is optimal.
- Plugging this into our implementability constraint we see

$$\frac{\overline{b}\overline{p}_s}{\beta} = I(\overline{\mu}) - g_s + \overline{b}$$

- Subtracting s' from s we have

$$\frac{\overline{b}(\overline{p}_s - \overline{p}_{s'})}{\beta} = g_{s'} - g_{s'}$$

or

$$\overline{p}_s - \overline{p}_{s'} = \frac{-\beta}{\overline{b}}(g_{s'} - g_s)$$

Payoffs with a Steady State

- The steady state level of debt associated with $\overline{\mu}$ is

$$\overline{b}(\overline{\mu}) = \frac{\beta}{1-\beta}\left(I(\overline{\mu}) - \mathbb{E}g\right)$$

- With both of these equations we can construct the payoff vector associated with a steady state multiplier $\overline{\mu}$

$$\overline{p}(\overline{\mu})_s = 1 - \frac{\beta}{\overline{b}(\overline{\mu})}(g_s - \mathbb{E}g)$$

- A key element to note here is that if $\overline{p}_s$ has a complete markets steady state then $\overline{p}$ is perfectly correlated with g.

Larger State Space

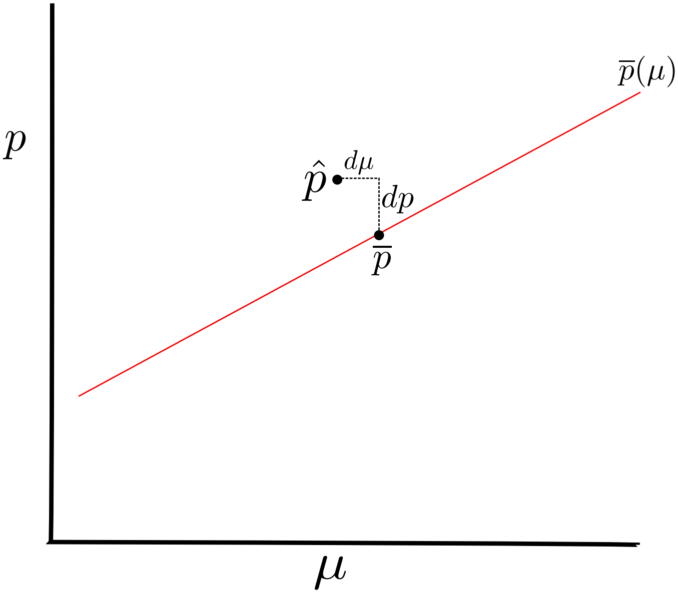
- Until this point we have considered our policy rules $\mu'(s,\mu)$ and $b(\mu)$ to be solely functions of μ.
- These functions must satisfy

$$F(\mu', b, \mu) = \vec{0}$$

  where F was our first order conditions above.
- In truth, these policy functions are also functions of the payoff vector p so $\mu'(s,\mu,p)$ and $b'(\mu,p)$.
- Moreover, we know that at $\overline{\mu}, \overline{p}(\overline{\mu})$ we have

$$F(\mu', b, \overline{\mu}, \overline{p}(\overline{\mu})) = 0$$

  as well as $\mu'(s,\overline{\mu},\overline{p}(\overline{\mu})) = \overline{\mu}$ and $\overline{b}(\overline{\mu},\overline{p}(\overline{\mu})) = \overline{b}(\overline{\mu})$


p
$\hat{p}$
$d\mu$
dp
$\overline{p}$
$\overline{p}(\mu)$
μ

Linearize with Respect to μ

- Differentiating the first order conditions with respect to μ around $(\overline{\mu}, \overline{p})$ we obtain

$$\frac{\overline{p}_s}{\beta}\frac{\partial b}{\partial \mu} = \left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu}\right]\frac{\partial \mu'(s)}{\partial \mu},$$

and

$$1 = \sum_{s'} \Pi_{s'}\overline{p}_{s'}\frac{\partial \mu'(s')}{\partial \mu}$$

- Applying $\sum_{s'} \Pi_{s'}\overline{p}_{s'}\cdot$ to the first equation we get

$$\frac{\partial b}{\partial \mu} = \frac{I'(\overline{\mu})}{\frac{\mathbb{E}\overline{p}^2}{\beta} - 1} \text{ and } \frac{\partial \mu'(s)}{\partial \mu} = \frac{p_s}{\mathbb{E}\overline{p}^2}$$

Linearize with Respect to p

- Differentiating with respect to p_s around $(\overline{\mu}, \overline{p})$ we get

$$\frac{\overline{p}_{s'}}{\beta}\frac{\partial b}{\partial p_s} + 1_{s,s'}\frac{\overline{b}}{\beta} - \frac{\Pi_s \overline{b}\overline{p}_{s'}}{\beta} = \left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu} \right] \frac{\partial \mu'(s')}{\partial p_s} \tag{4}$$

and

$$0 = \sum_{s'} \Pi_{s'} \overline{p}_{s'} \frac{\partial \mu'(s')}{\partial p_s}$$

- The same trick as last slide applied to equation (4) gives

$$\frac{\partial b}{\partial p_s} = \Pi_s \overline{b} \frac{\mathbb{E}\overline{p}^2 - \overline{p}_s}{\mathbb{E}\overline{p}^2} \tag{5}$$

and

$$\frac{\partial \mu'(s')}{\partial p_s} = \frac{\overline{b}}{\beta \left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu} \right]} \left(1_{s,s'} - \frac{\Pi_s \overline{p}_s \overline{p}_{s'}}{\mathbb{E}\overline{p}^2} \right) \tag{6}$$

Linearized system

- For a given p_s near $\overline{p}$ we can construct a linearized system

$$\hat{\mu}_{t+1} = B\hat{\mu}_t + C$$

  where $\hat{\mu} = \mu_t - \overline{\mu}$ and B and C are random.
- B is just

$$B_{s'} = \frac{\partial \mu'(s')}{\partial \mu}$$

- While C is given by

$$C_{s'} = \sum_s \frac{\partial \mu'(s')}{\partial p_s}(p_s - \overline{p}_s)$$

Ergodic Distribution

- With a little algebra we can characterize the moments of the ergodic distribution of $\hat{\mu}$.
- Specifically we obtain that

$$\mathbb{E}\hat{\mu} = \frac{\overline{C}}{1-\overline{B}}$$

where $\overline{C} = \sum_{s'} C_{s'}\Pi_{s'}$ and $\overline{B} = \sum_{s'} \Pi_{s'} B_{s'}$

- The variance of the ergodic distribution is given by

$$\sigma^2_{\hat{\mu}} = \frac{\sigma^2_B(\mathbb{E}\hat{\mu})^2 + \sigma_{BC}\mathbb{E}\hat{\mu} + \sigma^2_C}{1-\overline{B}^2-\sigma^2_B}$$

where σ^2_B, σ^2_C and σ_{BC} are the variance and covariance of B and C respectively

A Guess

- Given that we can approximate around any $(\overline{\mu}, \overline{p})$ a natural question is if we want to approximate the solution for some p, with $\mathbb{E}p = 1$, what is the best point to linearize around?
- A natural answer is that we want to minimize the distance between p and $\overline{p}$ given by

$$\|p - \overline{p}\|^2 = \sum_s \Pi_s (p_s - \overline{p}_s)^2$$

- Thus we wish to choose $\overline{\mu}, \overline{p}(\overline{\mu})$ to minimize

$$\|p - \overline{p}(\overline{\mu})\|^2$$

- We can show that choosing the point gives us additional benefits.

A Property of the Minimizer

- Taking the first order condition of the minimization problem we get
$$2\sum_{s'} \Pi'_s(p_{s'} - \overline{p}(\overline{\mu})_{s'})\overline{p}'(\overline{\mu})_{s'} = 0$$
- We noted before that $\overline{p}$ is a straight line in $\mathbb{R}^S$, thus $\overline{p}'(\overline{\mu}) \propto \overline{p} - 1$.
- Thus
$$\begin{aligned} 0 &= \sum_{s'} \Pi_{s'}(p_{s'} - \overline{p}(\overline{\mu})_{s'})(\overline{p}(\overline{\mu})_{s'} - 1) \\ &= -\sum_{s'} \Pi_{s'}(p_{s'} - \overline{p}(\overline{\mu})_{s'}) + \sum_{s'} \Pi_{s'}(p_{s'} - \overline{p}(\overline{\mu})_{s'})\overline{p}(\overline{\mu})_{s'} \\ &= \sum_{s'} \Pi_{s'}(p_{s'} - \overline{p}(\overline{\mu})_{s'})\overline{p}(\overline{\mu})_{s'} \\ &= \mathbb{E}\left[(p - \overline{p}(\overline{\mu}))\overline{p}(\overline{\mu})\right] \end{aligned} \tag{7}$$

Mean of the Ergodic Distribution

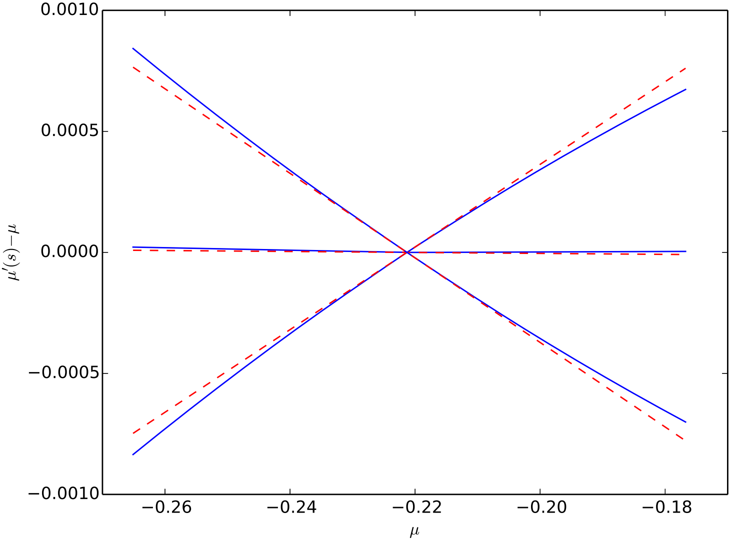
- Using our formula for C we have

$$\begin{aligned}\overline{C} &= \sum_s \left\{ \frac{\overline{b}}{\beta\left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu}\right]} \left(\Pi_s - \frac{\Pi_s \overline{p}_s}{\mathbb{E}\overline{p}^2}\right)(\hat{p}_s - \overline{p}_s) \right\} \\ &= \frac{\overline{b}}{\beta\left[I'(\overline{\mu}) + \frac{\partial b}{\partial \mu}\right]} \left(\mathbb{E}(\hat{p} - \overline{p}) - \frac{\mathbb{E}\left[(\hat{p} - \overline{p})\overline{p}\right]}{\mathbb{E}\overline{p}^2}\right) \\ &= 0\end{aligned} \tag{8}$$

- Thus we have that

$$\mathbb{E}\hat{\mu} = \frac{\overline{C}}{1 - \overline{B}} = 0$$

- Thus the linearized system will have the same mean for μ, $\overline{\mu}$, as the closest approximating steady state payoff structure.


0.0010
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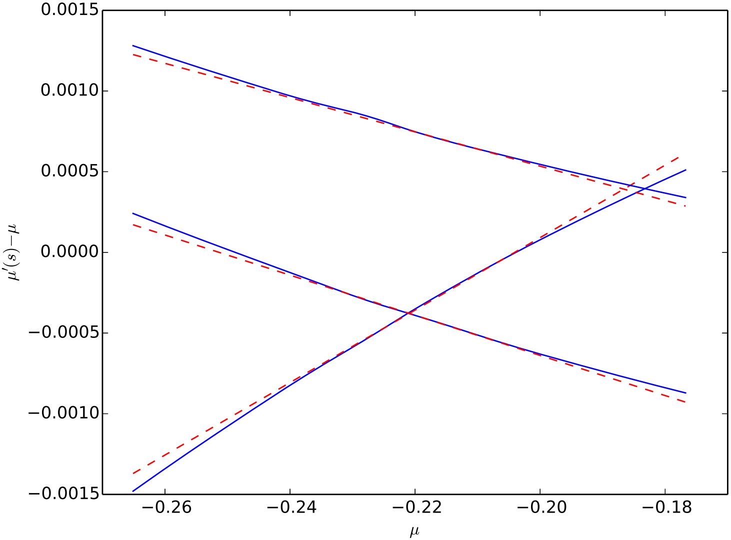
$\mu'(s) - \mu$
μ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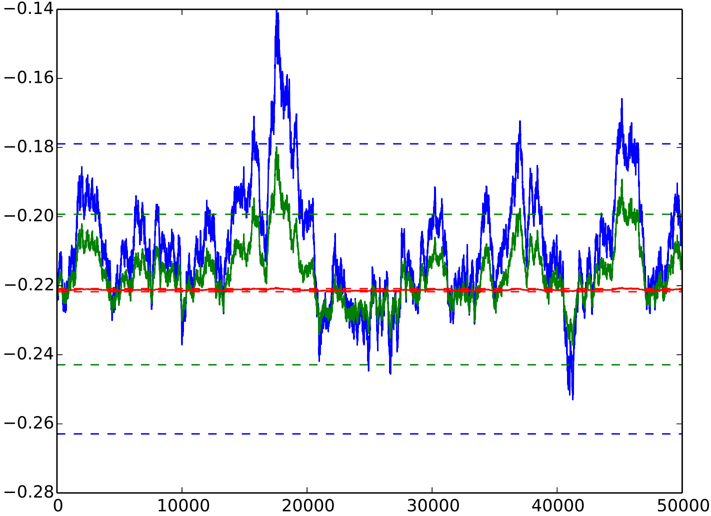
$\mu'(s) - \mu$
μ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