Optimal Taxation with Incomplete Markets

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Abstract

KEYWORDS:

1 Introduction

2 Environment

We analyze economies that share the following features. Government expenditures at time t, $g_t = g(s_t)$, and a productivity shock $\theta_t = \theta(s_t)$ are both functions of a Markov shock $s_t \in \mathcal{S}$ having $S \times S$ transition matrix Π and initial condition s_{-1}. An infinitely lived representative consumer has preferences over allocations $\{c_t(s^t), l_t(s^t)\}_{t=0}^{\infty}$ of consumption and labor supply that are ordered by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U\left(c(s^t), l(s^t)\right)$$

where U satisfies XXXXXX. Labor produces output via the linear technology

$$y_t = \theta_t l_t$$

The representative consumer's tax bill at time $t \geq 0$ is

$$-T_t + \tau_t \theta_t l_t, \; T_t \geq 0,$$

where $\tau_t(s^t, b_{-1})$ is a flat rate tax on labor income and T_t is a nonnegative transfer. Often, we'll set $T_t = 0$. The government and consumer trade a single possibly risky asset whose time t payoff p_t is described by an $S \times S$ matrix $\mathbb{P}$:

$$p_t = \mathbb{P}(s_t | s_{t-1}).$$

Let B_t denote the government's holdings of the asset and b_t be the consumer's holdings. Let $q_t = q_t(s^t; b_{-1})$ be the price of the single asset at time t. At $t \geq 0$, the household's time budget constraint is

$$c_t + b_t = (1 - \tau_t)\, \theta_t l_t + \frac{p_t}{q_{t-1}} b_{t-1} + T_t$$

and the government's is

$$g_t + B_t + T_t = \tau_t \theta_t l_t + \frac{p_t}{q_{t-1}} B_{t-1}.$$

Feasible allocations satisfy

$$c_t + g_t = \theta_t l_t, \; \forall t \geq 0$$

Clearing in the time $t \geq 0$ market for the single asset requires

$$b_t + B_t = 0.$$

Initial assets satisfy $b_{-1} = -B_{-1}$ An initial value of the exogenous state s_{-1} is given. Equibrium objects including $\{c_t, l_t, \tau_t\}_{t=0}^{\infty}$ will come in the form of sequences of functions of initial government debt b_{-1} and $s^t = [s_t, s_{t-1}, \ldots, s_0, s_{-1}]$.

Borrowing from a standard boilerplate, we use the following:

Definition 2.1. *An allocation is XXXXX. A price system is XXXXX. A budget-feasible government policy is* $\{\tau_t, T_t\}_{t=0}^{\infty}$ *XXXXX*

Definition 2.2. *Given* $(b_{-1} = -B_{-1}, s_{-1})$ *and a government policy, a* ***competitive equilibrium with distorting taxes*** *is a price system, a budget-feasible government policy, and an allocation such that the allocation is individually rational and the bond market clears.*

Definition 2.3. *Given* (b_{-1}, B_{-1}, s_{-1})*, a* ***Ramsey plan*** *is a welfare-maximizing competitive equilibrium with distorting taxes.*

3 Two Ramsey problems

Following Lucas and Stokey (1983) and Aiyagari et al. (2002), we use use a "primal approach." To encode a government policy and and price system as a restriction on an allocation, we first obtain the representative household's first order conditions:

$$U_{c,t}q_t = \beta \mathbb{E}_t p_{t+1} U_{c,t+1}$$

$$(1-\tau_t)\theta_t U_{c,t} = -U_{l,t}$$

We substitute these into the household's budget constraint to get a difference equation that we solve forward at every history for every $t \geq 0$. We divide the consequent *implementability constraints* on a Ramsey allocation in two: (1) the time $t = 0$ version is identical with the *single* implementability constraint imposed by Lucas and Stokey (1983); (2) the time $t \geq 1$ implementability constraints constitute versions of the Aiyagari et al. (2002) *measurability restrictions* on Ramsey allocations. These augment and contribute the only difference vis a vis Lucas-Stokey's Ramsey problem.

The primal approach version of our Ramsey problem is:

$$\max_{\{c_t, l_t, b_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t, l_t) \tag{2}$$

subject to

$$c_t + g_t = \theta_t l_t, \; t \geq 0 \tag{3a}$$

$$b_{-1} = \frac{1}{U_{c,0}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(U_{c,t} c_t + U_{l,t} l_t \right) \tag{3b}$$

$$\frac{b_{t-1} U_{c,t-1}}{\beta} = \frac{\mathbb{E}_{t-1} p_t U_{c,t}}{p_t U_{c,t}} \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \left(U_{c,t+j} c_{t+j} + U_{l,t+j} l_{t+j} \right) \text{ for } t \geq 1 \tag{3c}$$

Remark 3.1. *Equation* (3a) *is feasibility, while equation* (3b) *the single implementability constraint present in Lucas and Stokey (1983). Equation* (3c) *expresses implementability constraints at every node from time* $t \geq 0$ *in terms of the date, history* $(t-1, s^{t-1})$ *measurable state variable* b_{t-1} *that for* $t \geq 1$ *is absent from Lucas and Stokey's complete markets Ramsey problem. These extend the Aiyagari et al. (2002) measurability constraints to our situation with its more general payoff structure* $\mathbb{P}$ *for the single asset in our incom-*

plete markets economy. Thus, a Ramsey allocation for our incomplete markets economy automatically satisfies the single implementability constraint imposed by Lucas and Stokey.

3.1 Roadmap, analytic strategy

Asymptotic properties of a Ramsey allocation vary with asset returns that reflect properties of equilibrium prices $\{q_t(s^t|B_{-1}, s_{-1})\}_t$ and the exogenous asset payoff matrix $\mathbb{P}$. It is enlightening first to focus on the exogenous $\mathbb{P}$ part of returns. We do that by studying an economy with quasi-linear preferences that pin down $q_t = \beta \mathbb{E}_t \mathbb{P}(s_{t+1}|s_t)$. After that, we shall study preferences that express risk aversion with respect to consumption and so activate fluctuating q_t.

3.2 Analysis with quasi-linear preferences

Quasilinear preferences $U(c, l) = c - \frac{l^{1+\gamma}}{1+\gamma}$

To characterize **long-run** debt and taxes, we construct and then invert mapping $\mathbb{P}^*(b)$

- Given **arbitrary** initial govt. assets b_{-1}, what is an **optimal** asset payoff matrix $\mathbb{P}^* = \mathbb{P}^*(b_{-1})$?

- Under a Ramsey plan for an **arbitrary** payoff matrix $\mathbb{P}$, when would $b_t \to b^*$, where

$$\mathbb{P} = \mathbb{P}^*(b^*) \text{ or } b^* = \mathbb{P}^{*-1}(\mathbb{P})?$$

- We first reverse engineer an optimal $\mathbb{P}^*(b_{-1})$ from a Lucas-Stokey Ramsey allocation

- In a binary IID world, we identify a big set of $\mathbb{P}$'s that imply that b_t under a Ramsey plan converges to b^* that solves

$$\mathbb{P} = \mathbb{P}^*(b^*)$$

- For more general shock structures, we numerically verify an ergodic set of b_t's hovering around $\tilde{b}$. The optimal $\mathbb{P}^*$ associated with $\tilde{b}$ seems close to $\mathbb{P}$:

$$\mathbb{P} \approx \mathbb{P}^*(\tilde{b})$$

3.3 Optimal asset payoff matrix $\mathbb{P}^*$

1. Given b_{-1}, compute a Lucas-Stokey Ramsey allocation

2. Notice that the measurability constraints are invariant to scaling of p_t by a constant k_{t-1} that can depend on s^{t-1}.

3. From this class we select a p_t that imposes the normalization $\mathbb{E}_{t-1} U_{c,t} p_t = 1$

$$p_t = \frac{\beta}{U_{c,t-1} b_{t-1} U_{c,t}} \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \left(U_{c,t+j} c_{t+j} + U_{l,t+j} l_{t+j} \right)$$

4. By construction, p_t disarms the time $t \geq 1$ measurability constraints.

5. Using the fact that the Lucas-Stokey allocation is stationary, we can construct the optimal payoff matrix

$$\mathbb{P}^*(s_t, s_{t-1} | b_{-1}) = p_t$$

3.4 Quasilinear preferences $U(c,l) = c - \frac{l^{1+\gamma}}{1+\gamma}$

Given initial assets b_{-1}, let $\mu(b_{-1})$ be the Lagrange multiplier on the Lucas-Stokey implementability constraint

1. **Multiplier → Tax rate:**

$$\tau(\mu) = \frac{\gamma \mu}{(1+\gamma)\mu - 1}$$

2. **Tax rate → net of interest surplus:**

$$S(s, \tau) = \theta(s)^{\frac{\gamma}{1+\gamma}} (1-\tau)^{\frac{1}{\gamma}} \tau - g(s)$$

3. **Surplus → optimal payoff structure:**

$$\mathbb{P}^*(s, s_{\_} | b_{-1}) = (1-\beta) \frac{S(s,\tau)}{\mathbb{E}_{s_{\_}} S(s,\tau)} + \beta$$

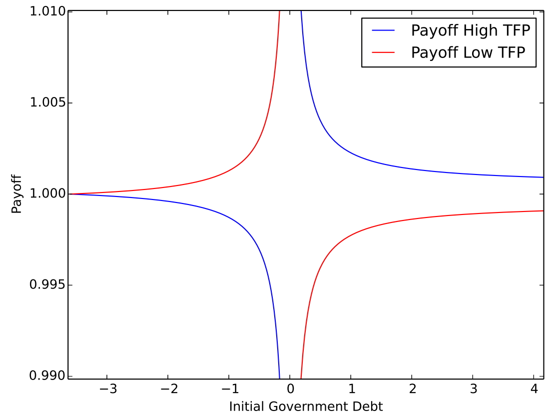

Figure 1: Optimal asset payoff structure as a function of initial government debt when TFP follows a 2 shock i.i.d process

3.5 Initial holdings influence optimal asset payoff structure

Denote state s as "adverse" if it has "high" govt. expenditures or "low " TFP; formally, s is "adverse" if

$$g(s)\mathbb{E}_{s\_}\theta^{\frac{\gamma}{1+\gamma}} - \theta(s)^{\frac{\gamma}{1+\gamma}}\mathbb{E}_{s\_}g > 0$$

Properties of optimal payoff matrix $\mathbb{P}$

- With positive initial govt. assets: want an asset that pays *more* in "adverse" states
- With negative initial govt. assets: want an asset that pays *less* in "adverse" states

3.6 Inverting the $\mathbb{P}^*$ mapping

1. **Exogenous payoff structure:** Suppose $\mathbb{P} \neq \mathbb{P}^*(b_{-1})$
2. **Steady States:** A steady state is a government debt b^* such that

$$b_t = b^* \text{ implies } b_{t+\tau} = b^* \quad \forall \tau > 0$$

3. **Characterization:** Given an asset payoff structure $\mathbb{P}$
    - Does a steady state exist? Is it unique?

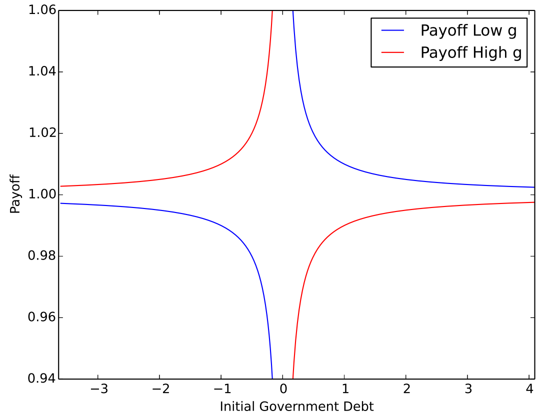

Figure 2: Optimal asset payoff structure as a function of initial government debt when government expenditures follow 2 shock i.i.d process

- Value of b^*?
- For what *initial government debts* b_{-1} does b_t converge to b^*?

3.7 Existence and $\mathbb{P}^{*-1}$

When shocks are i.i.d and take two values

1. $\mathbb{P}(s_-, s)$ is independent of s_- (so $\mathbb{P}$ can be a vector)
2. Under the normalization $q_t = \beta$, $\mathbb{E}\mathbb{P}(s) = 1$. Payoffs are then determined by a scalar $\boldsymbol{p}$.
   - $\boldsymbol{p}$ is the asset's payoff in the "good" state s
   - A risk-free bond is a security for which $\boldsymbol{p} = 1$
3. A steady state is obtained by inverting the optimal payoff mapping p^*

$$b^* \text{ satisfies } \boldsymbol{p} = \boldsymbol{p}^*(b^*) \text{ or } p^{*-1}(p) = b^*$$

   One equation in one unknown b^*

3.8 Existence regions in $\boldsymbol{p}$ space

The payoff $\boldsymbol{p}$ in good state $\in (0, \infty)$.

We categorize a set of economies with different asset payoffs into 3 regions via thresholds $\alpha_2 \geq \alpha_1 \geq 1$

- Low enough $\boldsymbol{p}(\leq \alpha_1)$: government holds assets in steady state
- High enough $\boldsymbol{p}(\geq \alpha_2)$: government issues debt in steady state
- Intermediate $\boldsymbol{p}(\alpha_1 > \boldsymbol{p} > \alpha_2)$: steady state does not exist

3.9 Thresholds: $\alpha_1 < \alpha_2$

- With only government expenditure shocks

$$\alpha_1 = 1 \text{ and } \alpha_2 = (1-\beta)\frac{\theta^{\frac{\gamma}{1+\gamma}}\left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}}\frac{\gamma}{1+\gamma} - g(s_1)}{\theta^{\frac{\gamma}{1+\gamma}}\left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}}\frac{\gamma}{1+\gamma} - \mathbb{E}g} + \beta > 1$$

- With only TFP shocks

$$\alpha_1 = (1-\beta)\frac{\theta(s_1)^{\frac{\gamma}{1+\gamma}}}{\mathbb{E}\theta^{\frac{\gamma}{1+\gamma}}} + \beta > 1$$

and

$$\alpha_2 = (1-\beta)\frac{\theta(s_1)^{\frac{\gamma}{1+\gamma}}\left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}}\frac{\gamma}{1+\gamma} - g}{\mathbb{E}\theta^{\frac{\gamma}{1+\gamma}}\left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}}\frac{\gamma}{1+\gamma} - g} + \beta > \alpha_1$$

3.10 Convergence

- Our analysis verifies existence of a steady state in a 2-state i.i.d. economy.
- To study long-run properties of a Ramsey allocation, we want to know whether steady state is stable
- **Risk-adjusted martingale:**

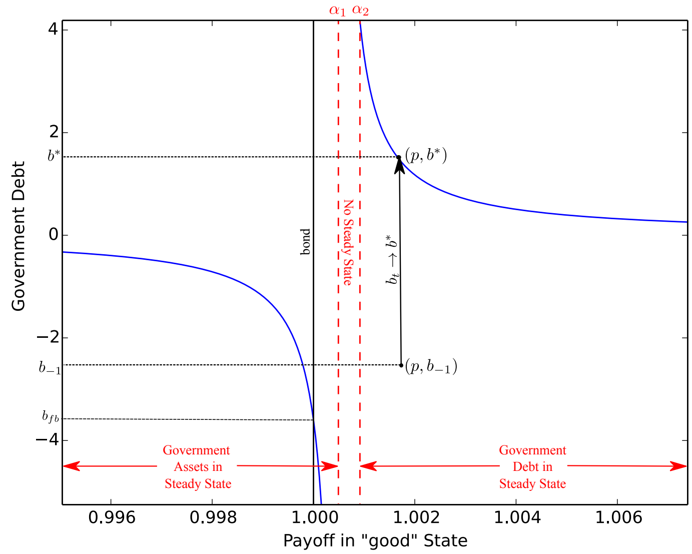

Figure 3: Existence regions in $\boldsymbol{p}$ space

The Lagrange multiplier μ_t on the implementability constraint satisfies

$$\mu_t = \mathbb{E}_t p_{t+1} \mu_{t+1}$$

or

$$\mathbb{E}_t \mu_{t+1} = \mu_t - Cov_t(p_{t+1}, \mu_{t+1})$$

- **Stability criterion:** Away from a steady state, is the drift of μ_t big enough?

3.11 Characterizing convergence under quasi-linearity, iid, and $S = 2$

- Reminder: $\boldsymbol{p}$ is the payoff in the "good" state.
- We partition the "$\boldsymbol{p}$ space" into stable and unstable regions

Theorem 3.2. *Let b^* denote steady state govt. debt and b_{fb} be govt. debt that supports the first-best allocation with complete markets. Then*

1. ***Low p**: If $\boldsymbol{p} \leq \min(\alpha_1, 1)$ then $b_{fb} < b^* < 0$ and $b_t \to b^*$ with probability 1.*

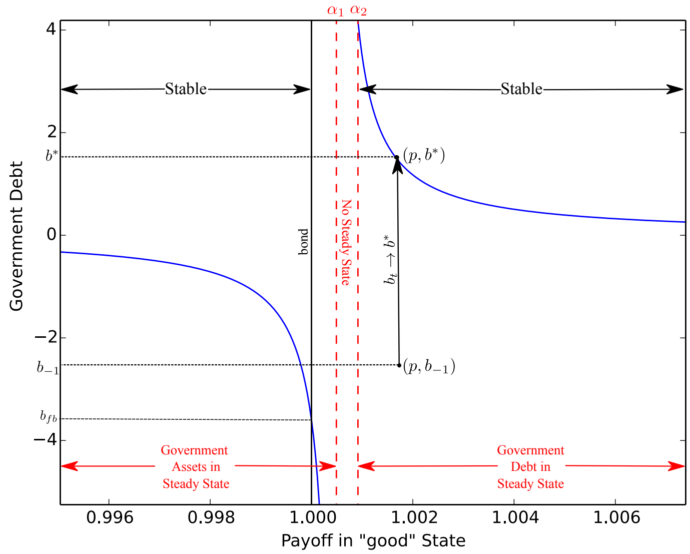

Figure 4: Stability regions

2. ***High p**: If $\mathbf{p} \geq \alpha_2$ then $0 < b^*$ and $b_t \to b^*$ with probability 1.*

For the intermediate region where b^* either does not exist or is unstable, there is a tendency to run up debt

Stability regions

3.12 Intuition for Convergence

- The Ramsey policy with incomplete markets smooths the welfare costs of distorting labor taxes by manipulating debt positions
- With a risk-free bond, the marginal cost of raising funds μ_t is a martingale. Changes in debt levels help smooth tax distortions across time.
- If the payoff matrix of the asset differs across states, then by generating state contingent revenues, the level of government debt smooths tax distortions across states.
- The steady state b^* is a unique debt level that provides enough "state contingency" completely to overcome missing assets markets
- When issuing debt, the government takes takes this benefit into account by distorting the martingale and either accumulating or decumulating debt.

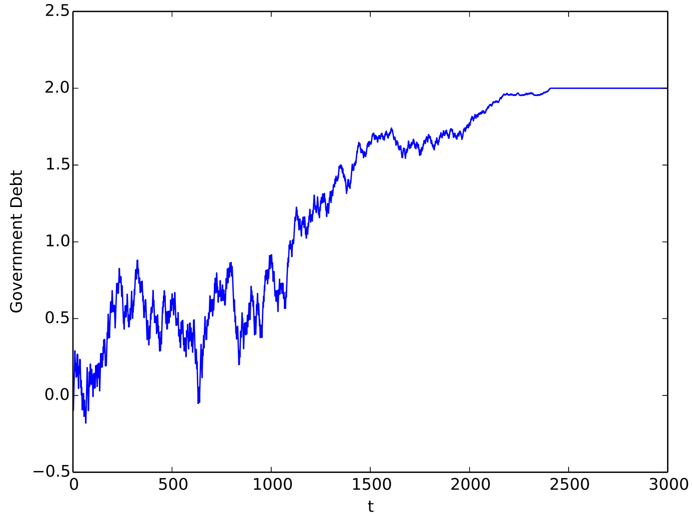

Figure 5: A sample path with $\boldsymbol{p} > 1$

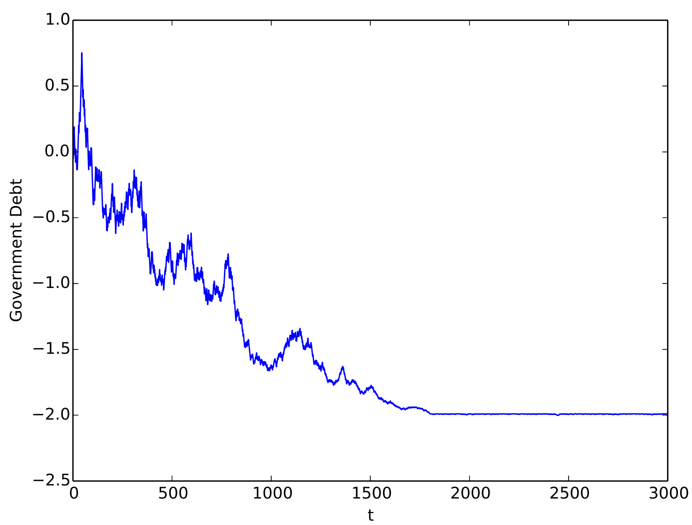

Figure 6: A sample path with $\boldsymbol{p} < 1$

- Although this is achieved by raising taxes, locally the welfare costs of taxes are second order and dominated by the gains from coming closer to b^*, which are first order in terms of welfare.

3.13 Outcomes with quasi-linear preferences

Outcomes:

1. Often $b_t \to b^*$ when the aggregate state follows a 2-state i.i.d. process
2. The level and sign of b^* depend on the **exogenous payoff structure** $\mathbb{P}$
3. The limiting allocation corresponds to a complete market Ramsey allocation for initial govt. debt b^*

3.14 Turning on risk-aversion

Modifications:

- Another source of return fluctuations – the risk-free interest rate
- Marginal utility adjusted debt encodes history dependence
- With binary i.i.d shock shock process, $x_t = u_{c,t} b_t$ converges
- Long-run properties of x_t depend on equilibrium returns $R_{t,t+1} = \frac{\mathbb{P}(s_t, s_{t+1})}{q_t(s^t)}$. Now q_t varies in interesting ways

3.15 Roadmap, II

Two subproblems

1. $t = 0$ Bellman equation in value function $W(b_{-1}, s_0)$
2. $t \geq 1$ Bellman equation in value function $V(x, s_{\text{-}})$

Seek steady states x^* such that $x_t \to x^*$

3.16 A Recursive Formulation

1. Commitment implies that government actions at $t \geq 1$ are constrained by the public's anticipations about them at $s < t$

2. This contributes additional state variables like marginal utility of consumption

3. Scaling the budget constraint by marginal utility makes Ramsey problem recursive in $x = U_c b$

$$\frac{x_{t-1} p_t U_{c,t}}{\beta \mathbb{E}_{t-1} p_t U_{c,t}} = U_{c,t} c_t + U_{l,t} l_t + x_t$$

3.17 Bellman equation for $t \geq 1$ (*ex ante*)

$$V(x, s_{\_}) = \max_{c(s), l(s), x'(s)} \sum_s \Pi(s, s_{\_}) \Big(U(c(s), l(s)) + \beta V(x'(s), s) \Big)$$

subject to $x'(s) \in [\underline{x}, \overline{x}]$

$$\frac{x \mathbb{P}(s) U_c(s)}{\beta \mathbb{E}_{s_{\_}} \mathbb{P} U c} = U_c(s) c(s) + U_l(s) l(s) + x'(s)$$

$$c(s) + g(s) = \theta(s) l(s)$$

3.18 Time 0 Bellman equation (*ex post*)

Given an initial debt b_{-1}, state s_0, and continuation value function $V(x, s_{\_})$

$$W(b_{-1}, s_0) = \max_{c_0, l_0, x_0} U(c, l) + \beta V(x_0, s0)$$

subject to time zero implementability constraint

$$U_c(c_0, l_0) c + U_l(c_0, l_0) l_0 + x_0 = U_c(c_0, l_0) b_{-1}$$

and resource constraint

$$c_0 + g(s_0) = \theta(s_0) l_0$$

and

$$x_0 \in [\underline{x}, \overline{x}]$$

3.19 Progress report

1. Existence proved only under special case of a risk-free bond $\mathbb{P}(s|s_{\_}) = 1 \ \forall \ (s, s_{\_})$

   This focuses attention on *endogenous* component of returns coming from $q_t(s^t)$

2. x^* is an initial condition for which the optimal asset payoff in a LS economy is a risk-free bond

3.20 Revisiting steady states with risk aversion

Let $x'(s; x, s_{-})$ be an optimal law of motion for the state variable for the $t \geq 1$ Bellman equation.

Definition 3.3. *A steady state x^* satisfies $x^* = x'(s; x^*, s_-)$ for all s, s_-*

Thus, a steady state is a node at which the continuation allocation and tax rate have no further history dependence.

3.21 Existence

1. For a class of economies with separable iso-elastic preferences $U(c, l) = \frac{c^{1-\sigma}}{1-\sigma} - \frac{l^{1+\gamma}}{1+\gamma}$

2. Shocks that take two values and are i.i.d with s_b being the "adverse" state (either low TFP or high govt. expenditures)

Let x_{fb} be a value of the state x from which a government can implement first=best with complete markets

Proposition 3.4. *Let $q_{fb}(s)$ be the shadow price of government debt in state s using the first best allocation. If*

$$\frac{1 - q_{fb}(s_b)}{1 - q_{fb}(s_g)} > \frac{g(s_b)}{g(s_g)} > 1$$

then there exists a steady state with $x_{fb} > x^* > 0$

3.22 Stability

1. Here interest rates are aligned with marginal utility of consumption; they are low in "good" states (high TFP or low expenditure)

2. In a steady state, the government holds claims against the private sector. Resembles the quasilinear case with low $\boldsymbol{p}$

3. The incomplete markets Ramsey allocation converges to a Lucas-Stokey Ramsey allocation for all initial conditions

Proposition 3.5. *Let $\{c_t(s^t), l_t(s^t), x_t(s^{t-1})\}$ solve the incomplete markets Ramsey problem with $x_0 > x^*$. Then $x_t(s^{t-1}) \to x^*$ as $t \to \infty$ with probability 1 for all initial conditions*

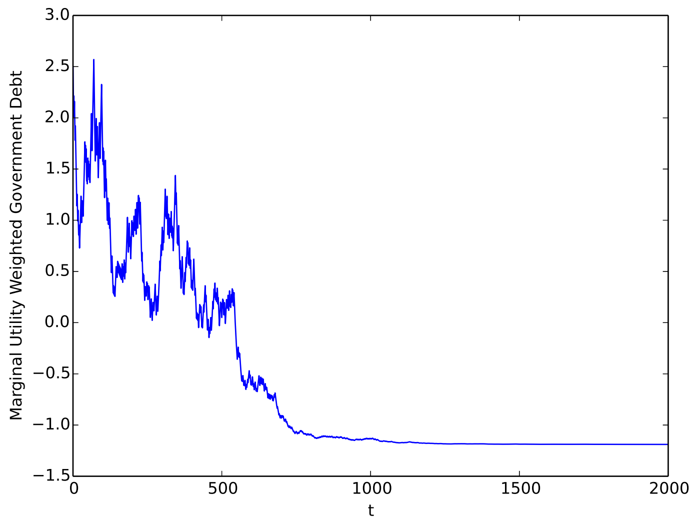

Figure 7: A sample path for 2 state i.i.d. process with risk aversion

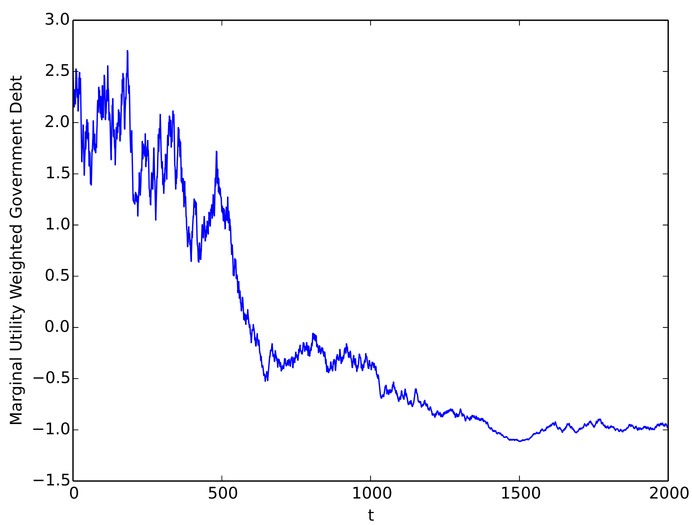

Figure 8: A sample path for economy with $S > 2$ states

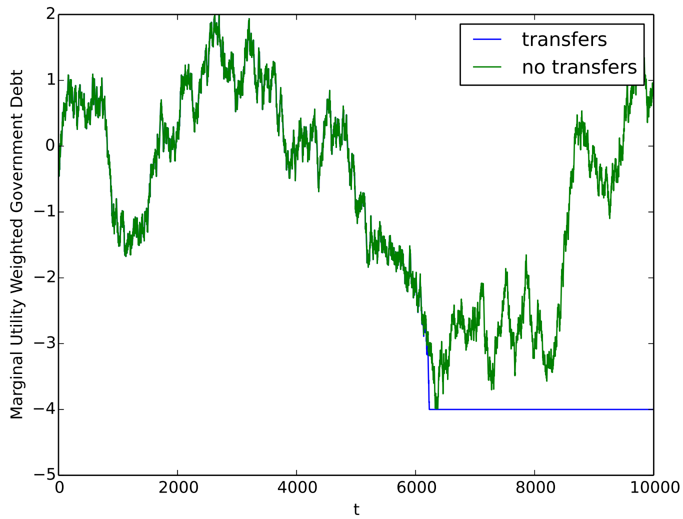

Figure 9: Quasilinear preferences and risk-free bond with and without nonnegative transfers

3.23 Transfers

- Access to nonnegative transfers makes first-best level of assets trivially a "steady state"
- All results hold *on one side* of steady state

Theorem 3.6. *With nonnegative lump sum transfers, in cases where a steady state exists and is stable, if the initial debt of the government exceeds its steady state level, the economy converges with probability 1 to the steady state.*

3.24 Comparison to literature

1. Angeletos (2002), Buera and Nicolini (2004)
    - Begin with a complete market Ramsey allocation
    - Ask if this can be attained with a limited collection of non-contingent debts of different maturities
2. This paper
    - Begins with an incomplete markets Ramsey allocation

- Asks whether the long-run allocation coincides with a complete markets Ramsey allocation for some initial govt debt

3. BEGS1 studies a related problem with heterogeneous agents

References

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