

# Optimal Taxation with Incomplete Markets

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## Lucas and Stokey, 1983

*... the option to issue state-contingent debt is important: tax policies that are optimal under uncertainty have an essential 'insurance' aspect to them.*

# Commitment, representative agent, no capital

- ▶ **Incomplete markets**

- A single, possibly risky asset

- ▶ **Linear tax schedules**

- Proportional tax on labor earnings (maybe plus *nonnegative* transfers)

- ▶ **Aggregate shocks**

- To productivities, government expenditures

# Questions

1. Should a government accumulate or decumulate assets?
2. Why might different economic fundamentals lead governments to want different amounts of debt?
3. Existing answers hinge on polar assumptions:
  - + Lucas Stokey (1984), complete markets: non history dependent debt quantities inherited from initial debt
  - + AMSS (2002), a risk-free bond only, quasi-linear preferences: govt. accumulates *assets* sufficient to finance activities using interest revenues
4. Unknown after AMSS (2002): what if interest rates fluctuate?

# Environment

- ▶ **Uncertainty:** Markov aggregate shocks  $s_t \in \mathcal{S}$ ;  $S \times S$  stochastic matrix  $\Pi$ ;  $g_t = g(s_t)$ ;  $\theta_t = \theta(s_t)$
- ▶ **Demography:** Infinitely lived representative agent plus a benevolent planner
- ▶ **Preferences** (representative household)

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c(s^t), l(s^t))$$

- ▶ **Technology:** Output  $y_t = \theta_t l_t$

## Environment, II

- ▶ **Asset market:**

- ▶  $S \times S$  matrix  $\mathbb{P}$  with time  $t$  payoff being

$$p_t = \mathbb{P}(s_t | s_{t-1})$$

- ▶ **Linear Taxes:** Representative consumer's tax bill

$$-T_t + \tau_t \theta_t l_t, \quad T_t \geq 0$$

- ▶ **Budget constraints**  $q_t$  is price of asset

- ▶ Household:  $c_t + b_t = (1 - \tau_t) \theta_t l_t + \frac{p_t}{q_{t-1}} b_{t-1} + T_t$
  - ▶ Government:  $g_t + B_t + T_t = \tau_t \theta_t l_t + \frac{p_t}{q_{t-1}} B_{t-1}$

- ▶ **Feasibility:**  $c_t + g_t = \theta_t l_t$
  - ▶ **Market Clearing for Asset:**  $b_t + B_t = 0$

- ▶ **Initial conditions:** Assets  $b_{-1} = -B_{-1}$  and  $s_{-1}$

# Ramsey Problem

## Definition

**Allocation, price system, government policy**

## Definition

**Competitive equilibrium:** Given  $(b_{-1} = -B_{-1}, s_{-1})$  and  $\{\tau_t, T_t\}_{t=0}^{\infty}$ , the allocation is individually rational, markets clear <sup>1</sup>

## Definition

**Optimal competitive equilibrium:** A welfare-maximizing competitive equilibrium for a given  $(b_{-1}, B_{-1}, s_{-1})$

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<sup>1</sup>Usually, we impose only “natural” debt limits.

# Ramsey problem

1. **Primal approach:** To eliminate tax rates and prices, use household's first order conditions:

$$U_{c,t} q_t = \beta \mathbb{E}_t p_{t+1} U_{c,t+1}$$

$$(1 - \tau_t) \theta_t U_{c,t} = -U_{l,t}$$

2. **Implementability constraints:** Derive by iterating the household's budget equation forward at every history  
 $\Rightarrow$  for  $t \geq 1$ , these impose *measurability restrictions* on Ramsey allocations
3. The  $t \geq 1$  **measurability constraints** contribute the only difference from Lucas-Stokey's Ramsey problem.



# Ramsey problem

4. **Transfers:** We temporarily restrict transfers  $T_t = 0 \forall t$ . This is convenient for our analytical results. We eventually show that this assumption is not restrictive.

# Ramsey problem (Lucas-Stokey)

$$\max_{\{c_t, l_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t, l_t)$$

subject to

(a) **Feasibility**

$$c_t + g_t = \theta_t l_t$$

(b) **Implementability constraint**

$$b_{-1} = \frac{1}{U_{c,0}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (U_{c,t} c_t + U_{l,t} l_t)$$

# Ramsey problem (BEGS)

$$\max_{\{c_t, l_t, b_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t, l_t)$$

(a) **Feasibility**

$$c_t + g_t = \theta_t l_t$$

(b) **Lucas-Stokey implementability constraint**

$$b_{-1} = \frac{1}{U_{c,0}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (U_{c,t} c_t + U_{l,t} l_t)$$

(c) **Measurability constraints**

$$\frac{b_{t-1} U_{c,t-1}}{\beta} = \frac{\mathbb{E}_{t-1} p_t U_{c,t}}{p_t U_{c,t}} \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j (U_{c,t+j} c_{t+j} + U_{l,t+j} l_{t+j}) \text{ for } t \geq 1$$

# Roadmap, analytic strategy

- ▶ Ramsey allocation – especially asymptotic properties – varies with **asset returns** that reflect
  - ▶ Prices  $\{q_t(s^t|B_{-1}, s_{-1})\}_t$
  - ▶ Payoffs  $\mathbb{P}$
- ▶ To focus on the exogenous  $\mathbb{P}$  part of returns, we first study quasi-linear preferences that pin down  $q_t = \beta \mathbb{E}_t \mathbb{P}(s_{t+1}|s_t)$
- ▶ Activate risk aversion and fluctuating  $q_t$  later

# Battlefield

*What is government debt in long-run?*

	risk-free bond	risky bond
Quasi-Linear		
Risk Aversion		

# Analysis with quasi-linear preferences

Quasilinear preferences  $U(c, l) = c - \frac{l^{1+\gamma}}{1+\gamma}$

To characterize **long-run** debt and taxes, we construct and then invert mapping  $\mathbb{P}^*(b)$

- ▶ Given **arbitrary** initial govt. debt  $b_{-1}$ , what is an **optimal** asset payoff matrix  $\mathbb{P}^* = \mathbb{P}^*(b_{-1})$ ?
- ▶ Under a Ramsey plan for an **arbitrary** payoff matrix  $\mathbb{P}$ , when would  $b_t \rightarrow b^*$ , where

$$\mathbb{P} = \mathbb{P}^*(b^*) \text{ or } b^* = \mathbb{P}^{*-1}(\mathbb{P})?$$

## Roadmap, the answers

- ▶ We first reverse engineer an optimal  $\mathbb{P}^*(b_{-1})$  from a Lucas-Stokey Ramsey allocation
- ▶ In a binary IID world, we identify a big set of  $\mathbb{P}$ 's that imply that  $b_t$  under a Ramsey plan converges to  $b^*$  that solves

$$\mathbb{P} = \mathbb{P}^*(b^*)$$

- ▶ For more general shock structures, we numerically verify an ergodic set of  $b_t$ 's hovering around  $\tilde{b}$ . The optimal  $\mathbb{P}^*$  associated with  $\tilde{b}$  seems close to  $\mathbb{P}$ :

$$\mathbb{P} \approx \mathbb{P}^*(\tilde{b})$$

## Optimal asset payoff matrix $\mathbb{P}^*$

1. Given  $b_{-1}$ , compute a Lucas-Stokey Ramsey allocation
2. Notice that the measurability constraints are invariant to scaling of  $p_t$  by a constant  $k_{t-1}$  that can depend on  $s^{t-1}$ .
3. From this class we select a  $p_t$  that imposes the normalization  $\mathbb{E}_{t-1} U_{c,t} p_t = 1$

$$p_t = \frac{\beta}{U_{c,t-1} b_{t-1} U_{c,t}} \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j (U_{c,t+j} c_{t+j} + U_{l,t+j} l_{t+j})$$

4. By construction,  $p_t$  disarms the time  $t \geq 1$  measurability constraints.
5. Using the fact that the Lucas-Stokey allocation is stationary, we can construct the optimal payoff matrix

$$\mathbb{P}^*(s_t, s_{t-1} | b_{-1}) = p_t$$



## Quasilinear preferences $U(c, l) = c - \frac{l^{1+\gamma}}{1+\gamma}$

Given initial govt. debt  $b_{-1}$ , let  $\mu(b_{-1})$  be the Lagrange multiplier on the Lucas-Stokey implementability constraint

### 1. Multiplier $\rightarrow$ Tax rate:

$$\tau(\mu) = \frac{\gamma\mu}{(1+\gamma)\mu - 1}$$

### 2. Tax rate $\rightarrow$ net of interest surplus:

$$S(s, \tau) = \theta(s)^{\frac{\gamma}{1+\gamma}} (1 - \tau)^{\frac{1}{\gamma}} \tau - g(s)$$

### 3. Surplus $\rightarrow$ optimal payoff structure:

$$\mathbb{P}^*(s, s_- | b_{-1}) = (1 - \beta) \frac{S(s, \tau)}{\mathbb{E}_{s_-} S(s, \tau)} + \beta$$

# Initial holdings influence optimal asset payoff structure

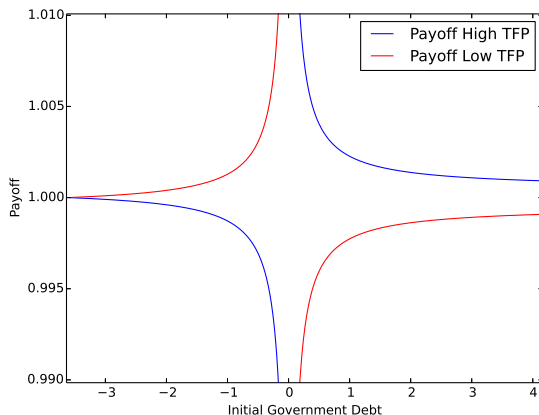
Denote state  $s$  as “adverse” if it has “high” govt. expenditures or “low ” TFP; formally,  $s$  is “adverse” if

$$g(s)\mathbb{E}_{s_-}\theta^{\frac{\gamma}{1+\gamma}} - \theta(s)^{\frac{\gamma}{1+\gamma}}\mathbb{E}_{s_-}g > 0$$

Properties of optimal payoff matrix  $\mathbb{P}$

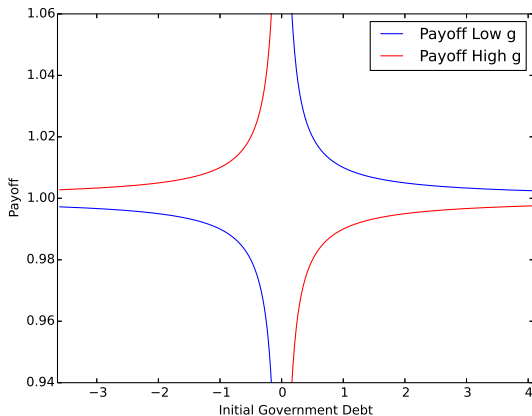
- ▶ With positive initial govt. assets: want an asset that pays *more* in “adverse” states
- ▶ With negative initial govt. assets: want an asset that pays *less* in “adverse” states

# Optimal Payoff Structure: TFP shocks



**Figure:** Optimal asset payoff structure as a function of initial government debt when TFP follows a 2 shock i.i.d process

# Optimal Payoff Structure: Expenditure shocks



**Figure:** Optimal asset payoff structure as a function of initial government debt when government expenditures follow 2 shock i.i.d process

# Inverting the $\mathbb{P}^*$ mapping

1. **Exogenous payoff structure:** Suppose  $\mathbb{P} \neq \mathbb{P}^*(b_{-1})$
2. **Steady States:** A steady state is a government debt  $b^*$  such that

$$b_t = b^* \text{ implies } b_{t+\tau} = b^* \quad \forall \tau > 0$$

3. **Characterization:** Given an asset payoff structure  $\mathbb{P}$ 
  - ▶ Does a steady state exist? Is it unique?
  - ▶ Value of  $b^*$ ?
  - ▶ For what *initial government debts*  $b_{-1}$  does  $b_t$  converge to  $b^*$ ?

## Existence and $\mathbb{P}^{*-1}$

When shocks are i.i.d and take two values

1.  $\mathbb{P}(s_-, s)$  is independent of  $s_-$  (so  $\mathbb{P}$  can be a vector)
2. Under the normalization  $q_t = \beta$ ,  $\mathbb{E}\mathbb{P}(s) = 1$ . Payoffs are then determined by a scalar  $\mathbf{p}$ .
  - ▶  $\mathbf{p}$  is the asset's payoff in the “good” state  $s$
  - ▶ A risk-free bond is a security for which  $\mathbf{p} = 1$
3. A steady state is obtained by inverting the optimal payoff mapping  $p^*$  **Anmol and David XXXXX: we should fix this slide – we have used new notation to define the operator**

$$b^* \text{ satisfies } \mathbf{p} = \mathbf{p}^*(b^*) \text{ or } p^{*-1}(p) = b^*$$

One equation in one unknown  $b^*$

## Existence regions in $\mathbf{p}$ space

The payoff  $\mathbf{p}$  in good state  $\in (0, \infty)$ .

We categorize a set of economies with different asset payoffs into 3 regions via thresholds  $\alpha_2 \geq \alpha_1 \geq 1$

- ▶ Low enough  $\mathbf{p}(\leq \alpha_1)$ : government holds assets in steady state
- ▶ High enough  $\mathbf{p}(\geq \alpha_2)$ : government issues debt in steady state
- ▶ Intermediate  $\mathbf{p}(\alpha_1 > \mathbf{p} > \alpha_2)$ : steady state does not exist

## Thresholds: $\alpha_1 < \alpha_2$

- ▶ With only government expenditure shocks

$$\alpha_1 = 1 \text{ and } \alpha_2 = (1 - \beta) \frac{\theta^{\frac{\gamma}{1+\gamma}} \left( \frac{1}{1+\gamma} \right)^{\frac{1}{\gamma}} \frac{\gamma}{1+\gamma} - g(s_1)}{\theta^{\frac{\gamma}{1+\gamma}} \left( \frac{1}{1+\gamma} \right)^{\frac{1}{\gamma}} \frac{\gamma}{1+\gamma} - \mathbb{E}g} + \beta > 1$$

- ▶ With only TFP shocks

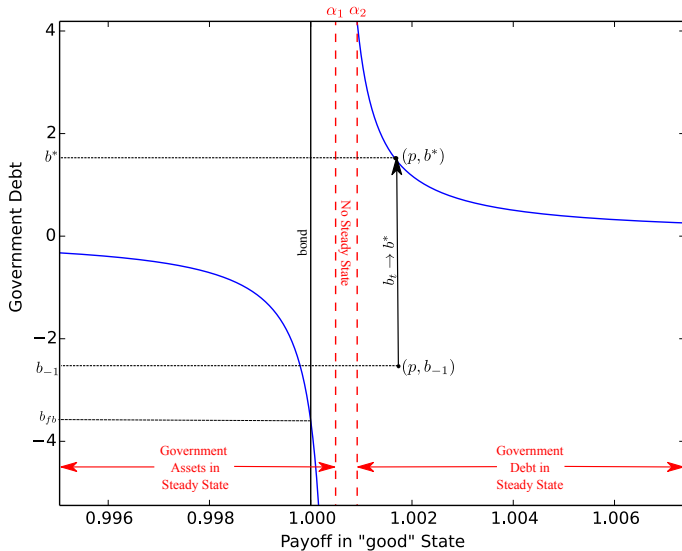
$$\alpha_1 = (1 - \beta) \frac{\theta(s_1)^{\frac{\gamma}{1+\gamma}}}{\mathbb{E}\theta^{\frac{\gamma}{1+\gamma}}} + \beta > 1$$

and

$$\alpha_2 = (1 - \beta) \frac{\theta(s_1)^{\frac{\gamma}{1+\gamma}} \left( \frac{1}{1+\gamma} \right)^{\frac{1}{\gamma}} \frac{\gamma}{1+\gamma} - g}{\mathbb{E}\theta^{\frac{\gamma}{1+\gamma}} \left( \frac{1}{1+\gamma} \right)^{\frac{1}{\gamma}} \frac{\gamma}{1+\gamma} - g} + \beta > \alpha_1$$



# Existence regions in $p$ space



# Convergence

- ▶ Our analysis verifies existence of a steady state in a 2-state i.i.d. economy.
- ▶ To study long-run properties of a Ramsey allocation, we want to know whether steady state is stable
- ▶ **Risk-adjusted martingale:**  
The Lagrange multiplier  $\mu_t$  on the implementability constraint satisfies

$$\mu_t = \mathbb{E}_t p_{t+1} \mu_{t+1}$$

or

$$\mathbb{E}_t \mu_{t+1} = \mu_t - \text{Cov}_t(p_{t+1}, \mu_{t+1})$$

- ▶ **Stability criterion:** Away from a steady state, is the drift of  $\mu_t$  big enough?

# Characterizing convergence under quasi-linearity, iid, and $S = 2$

- ▶ Reminder:  $\mathbf{p}$  is the payoff in the “good” state.
- ▶ We partition the “ $\mathbf{p}$  space” into stable and unstable regions

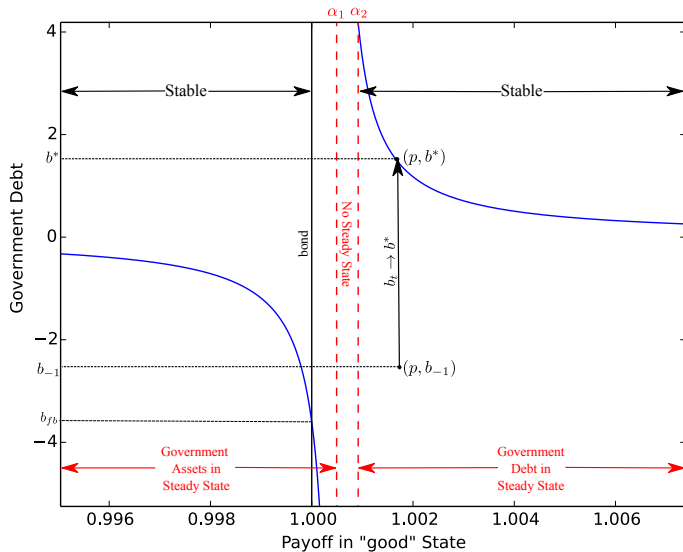
## Theorem

*Let  $b^*$  denote steady state govt. debt and  $b_{fb}$  be govt. debt that supports the first-best allocation with complete markets. Then*

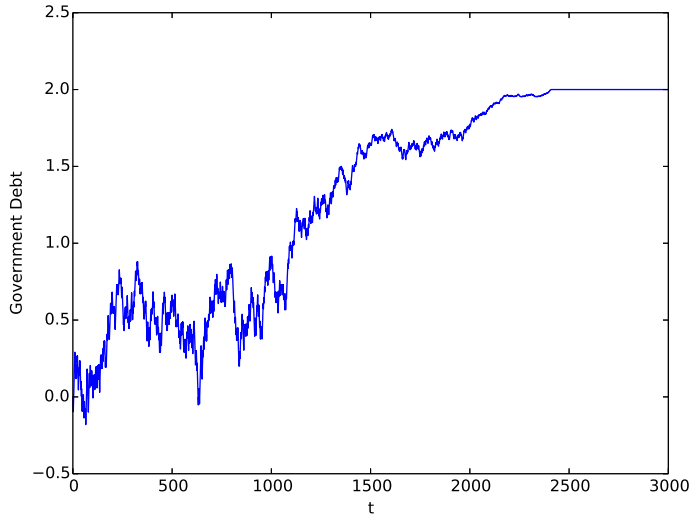
1. **Low  $\mathbf{p}$ :** *If  $\mathbf{p} \leq \min(\alpha_1, 1)$  then  $b_{fb} < b^* < 0$  and  $b_t \rightarrow b^*$  with probability 1.*
2. **High  $\mathbf{p}$ :** *If  $\mathbf{p} \geq \alpha_2$  then  $0 < b^*$  and  $b_t \rightarrow b^*$  with probability 1.*

For the intermediate region where  $b^*$  either does not exist or is unstable, there is a tendency to run up debt

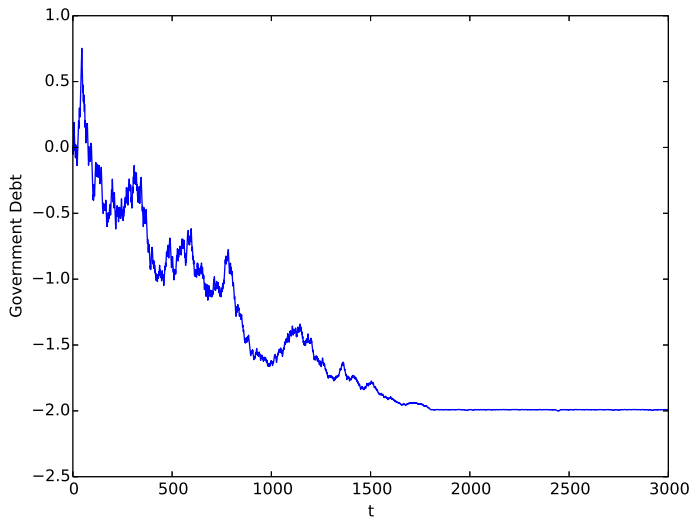
# Stability regions



## A sample path with $\mathbf{p} > 1$



## A sample path with $p < 1$



# Intuition for Convergence

- ▶ The Ramsey policy with incomplete markets smooths the welfare costs of distorting labor taxes by manipulating debt positions
- ▶ With a risk-free bond, the marginal cost of raising funds  $\mu_t$  is a martingale. Changes in debt levels help smooth tax distortions across time.
- ▶ If the payoff matrix of the asset differs across states, then by generating state contingent revenues, the level of government debt smooths tax distortions across states.
- ▶ The steady state  $b^*$  is a unique debt level that provides enough “state contingency” completely to overcome missing assets markets
- ▶ When issuing debt, the government takes this benefit into account by distorting the martingale and either accumulating or decumulating debt.
- ▶ Although this is achieved by raising taxes, locally the welfare costs of taxes are second order and dominated by the gains from coming closer to  $b^*$ , which are first order in terms of welfare.

# Outcomes with quasi-linear preferences

## Outcomes:

1. Often  $b_t \rightarrow b^*$  when the aggregate state follows a 2-state i.i.d. process
2. The level and sign of  $b^*$  depend on the **exogenous payoff structure**  $\mathbb{P}$
3. The limiting allocation corresponds to a complete market Ramsey allocation for initial govt. debt  $b^*$



# Turning on risk-aversion

## Modifications:

- ▶ Another source of return fluctuations – the risk-free interest rate
- ▶ Marginal utility adjusted debt encodes history dependence
- ▶ With binary i.i.d shock process,  $x_t = u_{c,t}b_t$  converges
- ▶ Long-run properties of  $x_t$  depend on equilibrium returns  $R_{t,t+1} = \frac{\mathbb{P}(s_t, s_{t+1})}{q_t(s^t)}$ . Now  $q_t$  varies in interesting ways

# Roadmap, II

Two subproblems

1.  $t = 0$  Bellman equation in value function  $W(b_{-1}, s_0)$
2.  $t \geq 1$  Bellman equation in value function  $V(x, s_-)$

Seek steady states  $x^*$  such that  $x_t \rightarrow x^*$

# A Recursive Formulation

1. Commitment implies that government actions at  $t \geq 1$  are constrained by the household's anticipations about them at  $s < t$
2. This contributes additional state variables like marginal utility of consumption
3. Scaling the budget constraint by marginal utility makes Ramsey problem recursive in  $x = U_c b$

$$\frac{x_{t-1} p_t U_{c,t}}{\beta \mathbb{E}_{t-1} p_t U_{c,t}} = U_{c,t} c_t + U_{l,t} l_t + x_t$$

Bellman equation for  $t \geq 1$  (*ex ante*)

$$V(x, s_-) = \max_{c(s), l(s), x'(s)} \sum_s \Pi(s, s_-) \left( U(c(s), l(s)) + \beta V(x'(s), s) \right)$$

subject to  $x'(s) \in [\underline{x}, \bar{x}]$

$$\frac{x \mathbb{P}(s) U_c(s)}{\beta \mathbb{E}_{s_-} \mathbb{P} U_c} = U_c(s) c(s) + U_l(s) l(s) + x'(s)$$

$$c(s) + g(s) = \theta(s) l(s)$$

## Time 0 Bellman equation (*ex post*)

Given an initial debt  $b_{-1}$ , state  $s_0$ , and continuation value function  $V(x, s_-)$

$$W(b_{-1}, s_0) = \max_{c_0, l_0, x_0} U(c, l) + \beta V(x_0, s_0)$$

subject to time zero implementability constraint

$$U_c(c_0, l_0)c + U_l(c_0, l_0)l_0 + x_0 = U_c(c_0, l_0)b_{-1}$$

and resource constraint

$$c_0 + g(s_0) = \theta(s_0)l_0$$

and

$$x_0 \in [\underline{x}, \bar{x}]$$

# Progress report

1. Existence proved only under special case of a risk-free bond  
 $\mathbb{P}(s|s_-) = 1 \quad \forall (s, s_-)$   
This focuses attention on *endogenous* component of returns coming from  $q_t(s^t)$
2.  $x^*$  is an initial condition for which the optimal asset payoff in a LS economy is a risk-free bond

## Revisiting steady states with risk aversion

Let  $x'(s; x, s_-)$  be an optimal law of motion for the state variable for the  $t \geq 1$  Bellman equation.

### Definition

A steady state  $x^*$  satisfies  $x^* = x'(s; x^*, s_-)$  for all  $s, s_-$

*Thus, a steady state is a node at which the continuation allocation and tax rate have no further history dependence.*

# Existence

1. For a class of economies with separable iso-elastic preferences
$$U(c, l) = \frac{c^{1-\sigma}}{1-\sigma} - \frac{l^{1+\gamma}}{1+\gamma}$$
2. Shocks that take two values and are i.i.d with  $s_b$  being the “adverse” state (either low TFP or high govt. expenditures)

Let  $x_{fb}$  be a value of the state  $x$  from which a government can implement first=best with complete markets

## Proposition

*Let  $q_{fb}(s)$  be the shadow price of government debt in state  $s$  using the first best allocation. If*

$$\frac{1 - q_{fb}(s_b)}{1 - q_{fb}(s_g)} > \frac{g(s_b)}{g(s_g)} > 1$$

*then there exists a steady state with  $x_{fb} > x^* > 0$*



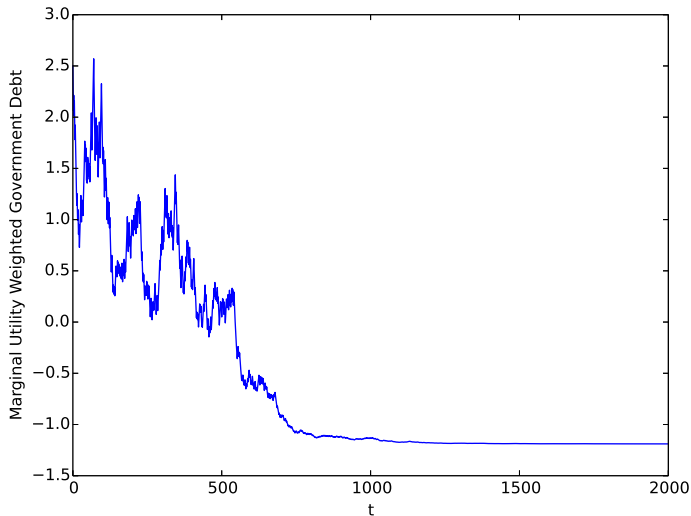
# Stability

1. Here interest rates are aligned with marginal utility of consumption; they are low in “good” states (high TFP or low expenditure)
2. In a steady state, the government holds claims against the private sector. Resembles the quasilinear case with low  $\mathbf{p}$
3. The incomplete markets Ramsey allocation converges to a Lucas-Stokey Ramsey allocation for all initial conditions

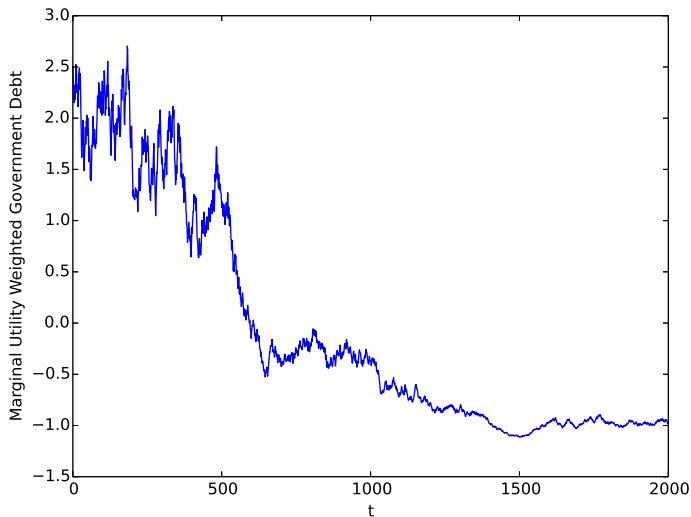
## Proposition

*Let  $\{c_t(s^t), l_t(s^t), x_t(s^{t-1})\}$  solve the incomplete markets Ramsey problem with  $x_0 > x^*$ . Then  $x_t(s^{t-1}) \rightarrow x^*$  as  $t \rightarrow \infty$  with probability 1 for all initial conditions*

## A sample path for 2 state i.i.d. process with risk aversion



## A sample path for economy with $S > 2$ states



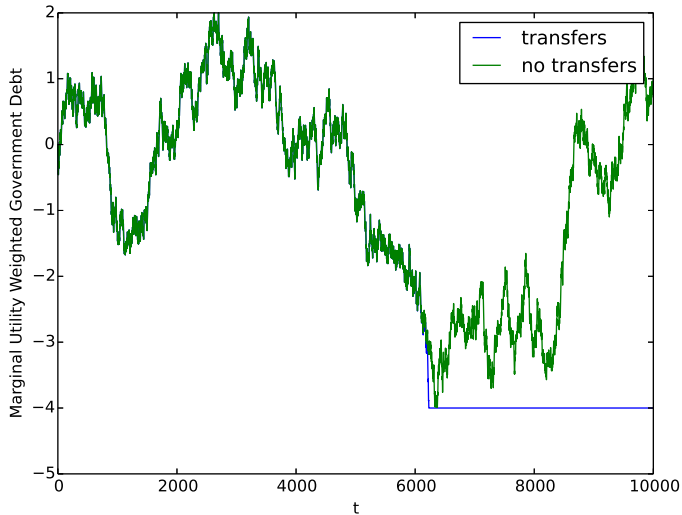
# Transfers

- ▶ Access to nonnegative transfers makes first-best level of assets trivially a “steady state”
- ▶ All results hold *on one side* of steady state

## Theorem

*With nonnegative lump sum transfers, in cases where a steady state exists and is stable, if the initial debt of the government exceeds its steady state level, the economy converges with probability 1 to the steady state.*

# Quasilinear preferences and risk-free bond with and without nonnegative transfers



# Battlefield

*What is government debt in long-run?*

With shocks that are IID and take two values

	risk-free bond	risky bond
Quasi-Linear	(AMSS) With $T_t \geq 0$ , govt. accumulates enough assets for first best.	Partition payoff space so that govt. either a) issues or b) runs up debt eventually
Risk Aversion	Conditions under which limiting govt. assets < first best	<i>Conjecture:</i> Similar to quasi-linear out- comes

We plan to study more general shocks processes

# Comparison to literature

1. Angeletos (2002), Buera and Nicolini (2004)
  - ▶ Begin with a complete market Ramsey allocation
  - ▶ Ask if this can be attained with a limited collection of non-contingent debts of different maturities
2. This paper
  - ▶ Begins with an incomplete markets Ramsey allocation
  - ▶ Asks whether the long-run allocation coincides with a complete markets Ramsey allocation for some initial govt debt
3. BEGS1 studies a related problem with heterogeneous agents