Optimal Taxation with Incomplete Markets

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Abstract

KEYWORDS:

> "... the option to issue state-contingent debt is important: tax policies that are optimal under uncertainty have an essential 'insurance' aspect to them." Lucas and Stokey (1983, p. 88)

1 Introduction

Controversial responses to Reinhart and Rogoff (2010) motivated us to reassess what we know and don't know about two elementary questions. What is an optimal government debt? And is government debt a pertinent state variable?

Lucas and Stokey (1983) and Aiyagari et al. (2002) offer different answers to these questions in the context of economic environments that are identical in all respects but one: Lucas and Stokey (1983) allow the government to issue a complete set of Arrow securities, while Aiyagari et al. (2002) allow the government to issue only a one-period risk-free government bond. For Lucas and Stokey, under an optimal policy government debt is not an independent state variable but instead is an exact function of a Markov state variable that drives government expenditures. The optimal state-by-state levels of government debt depend on the initial government debt. By way of contrast, for Aiyagari et al. (2002) government debt is an independent state variable with a limiting value or distribution of values that does not depend on the initial government debt. The quote by Lucas and Stokey pinpoints the source of these differences: the government's purchase of *insurance* from the private sector through explicit state-contingent securities underlies Lucas and Stokey's answers to our two questions; while a government's *self-insurance* by holding a risk-free asset underlies Aiyagari et al.'s answers.

This paper revisits our two questions in the context of a generalization of the Aiyagari et al. (2002) environment that continues to restrict the government to issue only a single security, but allows that security to be risky. The government must cope as best it can with the single security it is allowed to issue. We study the implications of alternative exogenous possibly risky securities for government debt dynamics. We use this generalization of Aiyagari et al.'s setup to attack questions left unresolved by Aiyagari et al. and also to say some new things about how the government achieves the insurance in an equilibrium in the original Lucas and Stokey (1983) model. Our analysis exploits new or at least previously unstated connections between the Lucas and Stokey and Aiyagari et al. economies.

Aiyagari et al. obtained their sharpest results for an economy with a quasi-linear household one-period utility function. That linearity of utility in consumption tied down the risk-free one-period interest rate enabled them to attain their result that in the long run the government accumulates a big enough stock of the risk-free asset entirely to finance its expenditures from interest earned from its claims on the private sector, so the tail of the Ramsey plan exhibits a zero distorting tax on labor and a first-best allocation. Aiyagari et al. were able to say much less for preferences that exhibit risk-aversion in consumption because then the Lagrange multiplier on the key incomplete markets implementability constraint becomes a risk-adjusted martingale rather than the martingale that it is under quasi-linearity. Here we are able to say much more than Aiyagari et al.. We accomplish this by recognizing connections to limits of (our generalization of) their economy and the allocation associated with a Lucas and Stokey economy for a particular initial level of government debt. With preferences that exhibit risk aversion in consumption, an attractor for the limiting debt dynamics of our economy is not associated with the first-best allocation active for the quasi-linear economy of Aiyagari et al. but rather an allocation associated with a Lucas-Stokey economy or one close to it.

Our analysis sheds light on the risk-sharing theme in the quotation with which we begin this paper. We exploit insights about exactly *how* the Ramsey planner in a Lucas-Stokey economy delivers the insurance through state-contingent debt that Lucas and Stokey stress is part and parcel of an optimal tax plan: fluctuations in equilibrium interest rates do part of the job. We can construct examples in which the Lucas-Stokey Ramsey planner chooses to issue risk-free debt and to achieve the required state-contingencies entirely through equilibrium fluctuations in the risk-free rate of interest rate.

The heart of our analysis is first to find the tail of an incomplete markets Ramsey allocation, then to ask whether that long-run allocation coincides with a Lucas-Stokey complete markets Ramsey allocation for *some* initial government debt. We describe conditions in which the answer is 'yes' or 'almost'.

The structure of our analysis is related to but differs from inquiries of Angeletos (2002), Buera and Nicolini (2004), and Shin (2007). Like us, they want understand the link between state-contingent government debt and an optimal tax plan. They begin with a Lucas-Stokey complete market Ramsey allocation and study how it can be supported by a limited collection of non-contingent debts of different maturities. Equilibrium interest rate fluctuations let state-contingent returns help.

In addition to the intrinsic interest that we attach to the two questions with which we began, this paper can be viewed as a prolegomenon to an an analysis of debt dynamics in a more complicated economic environment featured in Bhandari et al. (2013). There a Ramsey planner levies a distorting tax on labor partly to finance exogenous government and partly to redistribute among heterogeneously skilled households. Debt dynamics are driven by some forces similar to those present in the simpler environment of this paper, but they are obscured by additional features. We find it enlightening to study the underlying forces in a simpler setting.

2 Environment

We analyze economies that share the following features. Government expenditures at time t, $g_t = g(s_t)$, and a productivity shock $\theta_t = \theta(s_t)$ are both functions of a Markov shock $s_t \in \mathcal{S}$ having $S \times S$ transition matrix Π and initial condition s_{-1}. We will denote time t histories with s^t and z_t will refer to a generic random variable measurable with respect to s^t. Sometimes we will denote $z_t(s^t)$ indicate a particular realization of z_t. An infinitely lived representative consumer has preferences over allocations $\{c_t, l_t\}_{t=0}^{\infty}$ of consumption and labor supply that are ordered by

$$\mathbb{E}_{-1} \sum_{t=0}^{\infty} \beta^t U\left(c_t, l_t\right), \tag{1}$$

eqn:obj

where U is the period utility function for consumption and labor. For most of the paper, we shall assume that U separable in consumption and labor. We describe additional assumptions later. Labor produces output via the linear technology

$$y_t = \theta_t l_t$$

The representative consumer's tax bill at time $t \geq 0$ is

$$-T_t + \tau_t \theta_t l_t, \ T_t \geq 0,$$

where $\tau_t(s^t,)$ is a flat rate tax on labor income and T_t is a nonnegative transfer. Often, we'll set $T_t = 0$. The government and consumer trade a single possibly risky asset whose time t payoff p_t is described by an $S \times S$ matrix $\mathbb{P}$:

$$p_t = \mathbb{P}(s_t, s_{t-1})$$

Let B_t denote the government's holdings of the asset and b_t be the consumer's holdings. Let $q_t = q_t(s^t)$ be the price of the single asset at time t. At $t \geq 0$, the household's time budget constraint is

$$c_t + b_t = \left(1 - \tau_t\right)\theta_t l_t + \frac{p_t}{q_{t-1}} b_{t-1} + T_t \tag{2}$$

eqn:HHbudget

and the government's is

$$g_t + B_t + T_t = \tau_t \theta_t l_t + \frac{p_t}{q_{t-1}} B_{t-1}. \tag{3}$$

eqn:Govbudget

Feasible allocations satisfy

$$c_t + g_t = \theta_t l_t, \ \forall t \geq 0 \tag{4}$$

eqn:ResFeas

Clearing in the time $t \geq 0$ market for the single asset requires

$$b_t + B_t = 0. \tag{5}$$

eqn:bondmarket

Initial assets satisfy $b_{-1} = -B_{-1}$[1] An initial value of the exogenous state s_{-1} is given. Equilibrium objects including $\{c_t, l_t, \tau_t\}_{t=0}^{\infty}$ will come in the form of sequences of functions of initial government debt b_{-1} and $s^t = [s_t, s_{t-1}, \ldots, s_0, s_{-1}]$.

Borrowing from a standard boilerplate, we use the following:

Definition 2.1. *An* ***allocation*** *is a sequence* $\{c_t, l_t\}_{t=0}^{\infty}$ *for consumption and labor.An* ***asset profile*** *is a sequence* $\{b_t, B_t\}_{t=0}^{\infty}$*. A* ***price system*** *is a sequence of asset prices* $\{q_t\}_{t=0}^{\infty}$*. A* ***budget-feasible government policy*** *is a sequence of taxes and transfers* $\{\tau_t, T_t\}_{t=0}^{\infty}$

Definition 2.2. *Given* $(b_{-1} = -B_{-1}, s_{-1})$ *and a government policy, a* ***competitive equilibrium with distorting taxes*** *is a price system, an asset profile, a government policy, and an allocation such that (a) the allocation maximizes* (1) *subject to* (2)*, (b) given prices,* $\{b_t\}_{t=0}^{\infty}$ *is bounded; and (c) equations* (3), (4) *and* (5) *are satisfied.*

Definition 2.3. *Given* (b_{-1}, B_{-1}, s_{-1})*, a* ***Ramsey plan*** *is a welfare-maximizing competitive equilibrium with distorting taxes.*

3 Two Ramsey problems

Following Lucas and Stokey (1983) and Aiyagari et al. (2002), we use a "primal approach." To encode a government policy and price system as a restriction on an allocation, we first obtain the representative household's first order conditions[2]

⟨eqn:HHFOC⟩

$$U_{c,t} q_t = \beta \mathbb{E}_t p_{t+1} U_{c,t+1} \tag{6a}$$ eqn:Euler

$$(1 - \tau_t)\theta_t U_{c,t} = -U_{l,t} \tag{6b}$$?eqn:lcFOC?

We substitute these into the household's budget constraint to get a difference equation that we solve forward at every history for every $t \geq 0$. That yields *implementability constraints* on a Ramsey allocation that fall into two categories: (1) the time $t = 0$ version is identical with the *single* implementability constraint imposed by Lucas and Stokey (1983); (2) the time $t \geq 1$ implementability constraints are counterparts of the additional *measurability restrictions* that Aiyagari et al. (2002) impose on a Ramsey allocation.

[1] We assume that b_{-1} are obligations with accrued interest. This is equivalent to setting $q_{-1} = 1$.

[2] We thus focus on interior equilibria. Arguments by Magill and Quinzii (1994) and Constantinides and Duffie (1996) can be used to show that $\{c_t, l_t, b_t\}_{t=0}^{\infty}$ with bounded $\{b_t\}$ that also satisfy equations (2) and (6) solve the consumers problem.

We first state our Ramsey problem, then Lucas and Stokey's.

rob:RamseyBEGS⟩ **Problem 3.1.** *The Ramsey problem is to choose an allocation and an bounded government debt sequence* $\{b_t\}_{t=0}^{\infty}$ *that attain:*

$$\max_{\{c_t,l_t,b_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t, l_t) \tag{7}$$?eqn:Ramseyobj

subject to

$$c_t + g_t = \theta_t l_t, \;\; t \geq 0 \tag{8a}$$ eqn:feas

$$b_{-1} = \frac{1}{U_{c,0}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(U_{c,t} c_t + U_{l,t} l_t\right) \tag{8b}$$ eqn:LSimplemer

$$\frac{b_{t-1} U_{c,t-1}}{\beta} = \frac{\mathbb{E}_{t-1} p_t U_{c,t}}{p_t U_{c,t}} \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \left(U_{c,t+j} c_{t+j} + U_{l,t+j} l_{t+j}\right) \;\; \textit{for } t \geq 1 \tag{8c}$$ eqn:AMSSimplem

⟨prob:RamseyLS⟩ **Problem 3.2.** *Lucas and Stokey's Ramsey problem is to choose an allocation that attains*

$$\max_{\{c_t,l_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t, l_t) \tag{9}$$?eqn:Ramseyobj

subject to the single implementability constraint (8b) *and feasibility* (8a) *for all* t, s^t.

Remark 3.3. *Equation* (8a) *imposes feasibility, while equation* (8b) *is the single implementability constraint present in Lucas and Stokey (1983). Equations* (8c) *express additional implementability constraints at every node from time* $t \geq 1$. *These generalize the Aiyagari et al. (2002) measurability constraints on a Ramsey allocation to our more general payoff structure* $\mathbb{P}$ *for the single asset. The measurability constraints* (8c) *are cast in terms of the date, history* $(t-1, s^{t-1})$ *measurable state variable* b_{t-1} *that for* $t \geq 1$ *is absent from Lucas and Stokey's complete markets Ramsey problem. Evidently, Ramsey allocation for our incomplete markets economy automatically satisfies the single implementability constraint imposed by Lucas and Stokey.*

⟨rem:LSdebt⟩ **Remark 3.4.** *State-contingent, but not history-dependent, values of consumption, labor supply, and continuation government debt* $\check{b}(s)$ *solve the Lucas and Stokey (1983) Ramsey problem 3.2. As intermediated by the Lagrange multiplier on the implementability constraint* (8b), *consumption, labor supply, and* $\check{b}(s)$ *are functions of initial government debt* b_{-1} *and the current state* s_t, *but not past history* s^{t-1}.

3.1 Motivation for quasi-linear U

usequasilinear⟩

Asymptotic properties of a Ramsey plan for our incomplete markets economy vary with asset returns $R_{t-1,t} \equiv \frac{\mathbb{P}(s_t|s_{t-1})}{q_{t-1}}$. We see that $\mathbb{P}$ affects these returns directly through the ex-post exogenous payoffs and indirectly through prices q_{t-1}. To focus exclusively on the exogenous $\mathbb{P}$ part of returns, we begin by studying an economy with quasi-linear utility function:

$$U(c,l) = c - \frac{l^{1+\gamma}}{1+\gamma}, \tag{10}$$?eqn:UQL?

which sets $U_{c,t} = 1$. Asymptotic properties of a Ramsey plan for our incomplete markets economy vary with asset returns that reflect properties of equilibrium prices $\{q_t(s^t|B_{-1}, s_{-1})\}_t$ and the exogenous asset payoff matrix $\mathbb{P}$. At an interior solution, quasi-linear preferences and the Euler equation (6a) pins down $q_t = \beta \mathbb{E}_t \mathbb{P}(s_{t+1}|s_t)$. After studying the consequences of quasi-linear utility, we shall solve for Ramsey plans for utility functions that express risk aversion with respect to consumption and so activate endogenous fluctuations in q_t.

4 Quasi-linear preferences

Throughout this section, we assume that U is quasi-linear and use an indirect three step approach to characterize the asymptotic behavior of government debt and the tax rate.

(1) **Construct an optimal payoff matrix:**

We pose the following problem:

ob:PPoperator⟩?

Problem 4.1. *Given arbitrary initial government debt b_{-1}, what is an optimal asset payoff matrix?*

Let $\mathcal{P}$ be the set of all $\mathcal{S} \times \mathcal{S}$ real matrices. Define the indirect utility function $\mathcal{W}(\mathbb{P}; b_{-1})$ as the solution to problem 3.1 for $\mathbb{P} \in \mathcal{P}$ and initial debt b_{-1}. This induces an operator $\mathbb{P}^*$ that maps initial government debt into an optimal payoff matrix [3]

$$\mathbb{P}^*(b_{-1}) \in \arg\max_{\mathbb{P} \in \mathcal{P}} W(\mathbb{P}; b_{-1})$$

(2) **Apply the inverse of the operator $\mathbb{P}^*$.**

[3]We will demonstrate existence of a maximizer that is unique up to a constant factor along each row of the matrix.

For an arbitrary payoff matrix $\mathbb{P}$, let

$$\mathbb{P}^{*-1}(\mathbb{P}) = \min_{b} \|\mathbb{P} - \mathbb{P}^{*}(b^{*})\|, \tag{11}$$?eqn:invPopera

where $\|\cdot\|$ is the Frobenius matrix norm. For initial government debt b_{-1} such that $\mathbb{P}^{*}(b_{-1}) = \mathbb{P}$, we shall show that a Ramsey plan for the incomplete markets economy has $b_t = b^*$ for all $t \geq 0$.

(3) **Long run assets**

Starting from an arbitrary initial government b_{-1} and an arbitrary payoff matrix $\mathbb{P}$, establish conditions under which $b_t \to b^*$ under a Ramsey plan.

In particular, where $S = 2$ and shocks s_t are IID, we describe a large set of $\mathbb{P}$'s for which government debt b_t under a Ramsey plan converges to b^*. For more general shock processes, we numerically find an ergodic set of b_t's hovering around the debt level b^*. We execute steps (1), (2) and (3) in sections 4.1, 4.2, and 4.3.

4.1 The Optimal Payoff Matrix

⟨sec:David41⟩

We construct an optimal payoff matrix by first solving problem 3.2 for a Lucas-Stokey Ramsey allocation associated with a given b_{-1}. Next we construct a sequence $\{p_t\}_t$ that satisfies the implementability constraints imposed in (8c). Note that these implementability constraints are invariant to scaling of p_t by a constant k_{t-1} that can depend on s^{t-1}. From this equivalence class of $\{p_t\}_t$'s we select a $\{p_t\}_t$ that satisfies a normalization $\mathbb{E}_{t-1} p_t = 1$ and also satisfies

$$p_t = \frac{\beta}{b_{t-1}} \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \left(c_{t+j} + U_{l,t+j} l_{t+j}\right), \tag{12}$$ eqn:pdisarm

where

$$b_{t-1} = \beta \mathbb{E}_{t-1} \sum_{j=0}^{\infty} \beta^j (c_{t+j} + U_{l,t+j} l_{t+j}). \tag{13}$$ eqn:bt-1

The term $c_{t+j} + U_{l,t+j} l_{t+j} = (1 - \tau_{t+j}) l_{t+j} - g_{t+j}$ is the net-of-interest government surplus at time $t + j$. From equations (12) and (13), note that $\frac{1}{p_t} - 1$ is the percentage innovation in the present value government surplus at time t.

Note that by construction, p_t disarms the time $t \geq 1$ measurability constraints[4]. Using the remark 3.4 fact that the Lucas-Stokey Ramsey allocation is not history-dependent,

[4]Although we assume quasi-linear preferences throughout this particular construction, please note that

construct the optimal payoff matrix as

$$\mathbb{P}^*(s_t, s_{t-1}|b_{-1}) = p_t.$$

Thus, given initial government debt b_{-1}, let $\mu(b_{-1})$ be the Lagrange multiplier on the Lucas-Stokey implementability constraint (8b) at the Lucas-Stokey Ramsey allocation. The tax rate in the Ramsey allocation is $\tau(\mu) = \frac{\gamma\mu}{(1+\gamma)\mu-1}$, which implies a net-of-interest government surplus $S(s,\tau)$ that satisfies

$$S(s,\tau) = \theta(s)^{\frac{\gamma}{1+\gamma}}(1-\tau)^{\frac{1}{\gamma}}\tau - g(s)$$

If the aggregate state process s_t is i.i.d. then the 'disarm-the-measurability-constraints' equation (12) implies that the optimal payoff matrix is

$$\mathbb{P}^*(s, s_-|b_{-1}) = \beta\frac{S(s,\tau)}{b_{-1}} + \beta = (1-\beta)\frac{S(s,\tau)}{\mathbb{E}S(s,\tau)} + \beta, \tag{14}$$

eqn:optPP

which is independent of s_-.

To appreciate how the initial government debt level influences the optimal asset payoff structure via formula (14), call a state s "adverse" if it implies either "high" government expenditures or "low " TFP; formally, say that s is "adverse" if

$$g(s)\mathbb{E}\theta^{\frac{\gamma}{1+\gamma}} - \theta(s)^{\frac{\gamma}{1+\gamma}}\mathbb{E}g > 0$$

A "good" state is the opposite of an "adverse" state. "Adverse" states have the property that for wide range of initial government debts, the net-of-interest government surplus is lower than in "good" states. When initial government assets are positive, (14) implies that $\mathbb{P}^*$ pays *more* in "adverse" states, while when initial government assets are negative, $\mathbb{P}^*$ pays *less* in "adverse" states.

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equation (12) can be generalized to preferences with curvature via

$$p_t = \frac{\beta}{U_{c,t-1}b_{t-1}U_{c,t}}\mathbb{E}_t\sum_{j=0}^{\infty}\beta^j\left(U_{c,t+j}c_{t+j} + U_{l,t+j}l_{t+j}\right)$$

with the normalization $\mathbb{E}_{t-1}U_{c,t}p_t = 1$

4.2 The Inverse of $\mathbb{P}^*$ Again

⟨sec:David42⟩ Temporarily assume that s_t is i.i.d and $S = 2$. In this case, note that (14) implies that the optimal payoff matrix $\mathbb{P}^*$ has identical rows. This lets us restrict our attention to $\mathbb{P}(s, s_{\_})$ that have payoffs that are independent of s_{-}. This in turn lets us summarize $\mathbb{P}$ with a vector. Under the normalization $\mathbb{E}\mathbb{P}(s) = 1$, payoffs on the single asset are determined by a scalar $\boldsymbol{p}$, the payoff in state 1. A risk-free bond is then a security for which $\boldsymbol{p} = 1$. Without loss of generality, we shall assume that $g(1)\mathbb{E}\theta^{\frac{\gamma}{1+\gamma}} - \theta(1)^{\frac{\gamma}{1+\gamma}}\mathbb{E}g < 0$, and thus, $\boldsymbol{p}$ is the payoff in the "good" state of the world. Because the optimal payoff matrix can be summarized by a single scalar variable, we can recast the optimal matrix map $\mathbb{P}^*(b)$ as a single scalar function $\boldsymbol{p}^*(b)$. The steady state level of debt associated with an exogenous payoff $\boldsymbol{p}$ is then

$$b^* = \boldsymbol{p}^{*-1}(\boldsymbol{p}). \tag{15}$$ eq-ss

p:ssexistence⟩? **Proposition 4.2.** *There exists $0 \geq \alpha_2 \geq \alpha_1 \geq 1$ such that*

a. If $\boldsymbol{p} \leq \alpha_1$, then $b^* < 0$

b. If $\boldsymbol{p} \geq \alpha_2$, then $b^* > 0$

c. If $\alpha_1 > \boldsymbol{p} > \alpha_2$, then b^* solving (15) does not exist

Proof. Let g_1 and θ_1 denote government expenditures and TFP, respectively in the "good" state of the world. In state s, the government surplus is

$$S(s,\tau) = \theta(s)^{\frac{\gamma}{1+\gamma}}(1-\tau)^{\frac{1}{\gamma}}\tau - g(s),$$

which is maximized at $\tau = \frac{\gamma}{1+\gamma}$ when $(1-\tau)^{\frac{1}{\gamma}}\tau$ is also maximized. Furthermore, in the region $(-\infty, \frac{\gamma}{1+\gamma}]$, $S(\cdot,\tau)$ is an increasing function of τ. In an i.i.d. world with complete markets, government debt at a constant tax rate τ would be

$$\frac{\beta}{1-\beta}\sum_s \Pi(s)S(s,\tau),$$

which is an increasing function of τ. The maximal initial government debt sustainable with *incomplete* markets is then

$$\bar{b} = \frac{1}{1-\beta}\sum_s \Pi(s)\theta(s)^{\frac{\gamma}{1+\gamma}}\left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}}\frac{\gamma}{1+\gamma} - g(s).$$

Inverting the mapping [tjs]David - let's number and label the mapping, then refer to it here. from the tax rate into government debt would give us a function $\tau(b)$ that maps initial government debt into an optimal tax rate. The function $\tau(b)$ is an increasing function of b on the domain of possible complete markets initial debts $(-\infty, \bar{b}]$, with $\tau((-\infty, \bar{b}]) = \left(-\infty, \frac{\gamma}{1+\gamma}\right]$.

Substituting the formula for $S(s,\tau)$ into equation (14), we obtain

$$\boldsymbol{p}^*(\tau) = (1-\beta)\frac{\theta_1^{\frac{\gamma}{1+\gamma}}(1-\tau)^{\frac{1}{\gamma}}\tau - g_1}{\mathbb{E}\theta^{\frac{\gamma}{1+\gamma}}(1-\tau)^{\frac{1}{\gamma}}\tau - \mathbb{E}g} + \beta.$$

Solving for $(1-\tau)^{\frac{1}{\gamma}}\tau$ gives

$$(1-\tau)^{\frac{1}{\gamma}}\tau = \frac{(\boldsymbol{p}^* - \beta)\mathbb{E}g - (1-\beta)g_1}{(\boldsymbol{p}^* - \beta)\mathbb{E}\theta^{\frac{\gamma}{1+\gamma}} - (1-\beta)\theta_1^{\frac{\gamma}{1+\gamma}}}.$$

The set of complete market optimal tax rates is $(-\infty, \frac{\gamma}{1+\gamma}]$. Since the mapping $(1-\tau)^{\frac{1}{\gamma}}\tau$ is one to one and $b(\tau)$ is increasing on this domain, we conclude that $\boldsymbol{p}^*(b)$ is one to one. Differentiating $\boldsymbol{p}^*(\tau)$ with respect to τ yields

$$\frac{d}{d\tau}\boldsymbol{p}^*(\tau) = (1-\beta)(1-\tau)^{\frac{1}{\gamma}-1}\left[\gamma - (1+\gamma)\tau\right]\frac{g_1\mathbb{E}\theta^{\frac{\gamma}{1+\gamma}} - \theta_1^{\frac{\gamma}{1+\gamma}}\mathbb{E}g}{(\mathbb{E}\theta^{\frac{\gamma}{1+\gamma}}(1-\tau)^{\frac{1}{\gamma}}\tau - \mathbb{E}g)^2} < 0,$$

implying that $\boldsymbol{p}^*(b)$ is decreasing in b. Since $b = 0$ implies that $\mathbb{E}S(\tau(b)) = 0$, the function function $\boldsymbol{p}^*(b)$ has a pole at $b = 0$. That $\boldsymbol{p}^*(b)$ decreasing in b must therefore imply that $\lim_{b\to 0^-}\boldsymbol{p}^*(b) = -\infty$ and $\lim_{b\to 0^+}\boldsymbol{p}^*(b) = \infty$. We conclude that

$$\boldsymbol{p}^*((-\infty, \bar{b}]) = \boldsymbol{p}^*((-\infty, 0)) \cup \boldsymbol{p}^*((0, \bar{b}]) = (-\infty, \alpha_1) \cup [\alpha_2, \infty).$$

We compute the bounds α_1 and α_2 by taking the limits of $\boldsymbol{p}^*$ as b approaches $-\infty$ and the upper bound for government debt under complete markets $\bar{b}$, or equivalently as τ approaches $-\infty$ and $\frac{\gamma}{1+\gamma}$, respectively. $\square$

With only government expenditure shocks, we compute

$$\alpha_1 = 1 \text{ and } \alpha_2 = (1-\beta)\frac{\theta^{\frac{\gamma}{1+\gamma}}\left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}}\frac{\gamma}{1+\gamma} - g(s_1)}{\theta^{\frac{\gamma}{1+\gamma}}\left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}}\frac{\gamma}{1+\gamma} - \mathbb{E}g} + \beta > 1$$

With only TFP shocks, we compute

$$\alpha_1 = (1-\beta)\frac{\theta(s_1)^{\frac{\gamma}{1+\gamma}}}{\mathbb{E}\theta^{\frac{\gamma}{1+\gamma}}} + \beta > 1$$

and

$$\alpha_2 = (1-\beta)\frac{\theta(s_1)^{\frac{\gamma}{1+\gamma}}\left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}}\frac{\gamma}{1+\gamma} - g}{\mathbb{E}\theta^{\frac{\gamma}{1+\gamma}}\left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}}\frac{\gamma}{1+\gamma} - g} + \beta > \alpha_1$$

Remark 4.3. *With only TFP shocks, the bond payoff has the special property that it is associated with a steady state asset level that supports the first-best allocation,* $\boldsymbol{p}^{*-1}(1) = b_{fb}$*. At a first-best taxes are zero, so the net-of-interest government surplus is constant across states.*

4.3 Long Run Assets

⟨sec:David43⟩ In subsection 4.2, we provided conditions under which there exists b^* such $\boldsymbol{p}^*(b^*) = \boldsymbol{p}$. By construction, if $b_{-1} = b^*$ then the allocation that solves complete markets Ramsey problem 3.2 for initial condition b^* automatically satisfies the measurability constraints (8c). That allocation therefore solve the incomplete markets Ramsey problem 3.1. This implies that if $b_{-1} = b^*$, then $b_t = b^*$ for all t. Thus, $p^{*-1}(\boldsymbol{p})$ corresponds to a "steady state". It remains to be determined whether the incomplete markets Ramsey b_t converges to b^* for arbitrary b_{-1}. Theorem 4.4 provides sufficient conditions for convergence. [tjs]David: please read the preceding carefully – I edited it.

[tjs]Tom stopped here at 11:30 pm Milan time on November 13

m:convergence⟩ **Theorem 4.4.** *Let b_{fb} denote the level of government debt associated with the first-best allocation with complete markets. Then*

a. *If $\boldsymbol{p} \leq \min(\alpha_1, 1)$, then $b_{fb} < b^* < 0$ and $b_t \to b^*$ with probability 1.*

b. *If $\boldsymbol{p} \geq \alpha_2$, then $0 < b^*$ and $b_t \to b^*$ with probability 1.*

c. *If $\min(\alpha_1, 1) < \boldsymbol{p} < \alpha_2$, b^* either does not exist or is unstable.*

For $\boldsymbol{p}$ in region (c.), the government tends to run up debt over time.

Proof. [dge]~~The first order conditions governing the optimal allocation allow us to treat μ_t the multiplier on the measurability constraints as the state variable.~~ [apb]The optimal allocation permits recursive representation as functions $c_t(\mu_t), l_t(\mu_t)$ and $b_t(\mu_t)$ with a law of motion for μ_t. $^{\text{dge}}$ We will show global stability under the assumption that $\mu'(\mu, s)$ an increasing function of μ[5] The heart of the proof revolves around the twisted-martingale equation for μ:

$$\mu_t = \sum_s \Pi(s) p_s \mu'(\mu_t, s) = \mathbb{E}_t p_{t+1} \mu_{t+1}$$

We have shown that there is at most an unique μ^* such that $\mu'(\mu^*, s) = \mu^*$ for all s. For this sketch we will focus on showing global stability for $\mu < \mu^*$. The twisted-martingale equation can be decomposed as follows

$$\mu_t = \mathbb{E}_t \mu_{t+1} + Cov_t(p_{t+1}, \mu_{t+1})$$

by signing $Cov_t(p_{t+1}, \mu_{t+1})$ we are able to determine if μ_t follows a sub or super-martingale. Given that μ_t is bounded from above[6], we can conclude global convergence to the steady state if μ_t is a supermartingale. As in the statement of the theorem we will split the proof up into three cases, recall $\boldsymbol{p}$ is the payoff in state 1 (the "good" state)

1. $\boldsymbol{p} < \min\{1, \alpha_1\}$: Let $\bar{b}_s$ be maximal debt the government ~~good~~ could enter with and be able to pay off assuming it received shock s from this period onward, then

$$\bar{b}_s = \left(\frac{p_s}{\beta} - 1\right)^{-1} \left(\theta_s^{\frac{\gamma}{1+\gamma}} \left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}} \frac{\gamma}{1+\gamma} - g_s\right)$$

as the government tax revenue is maximized by setting the proportional tax to $\frac{\gamma}{1+\gamma}$. For $\boldsymbol{p} < \alpha_2$ it is possible to show that $\bar{b}_1 > \bar{b}_2$ and thus the natural debt limit is obtained by the "adverse" state. This implies that $\lim_{\mu \to -\infty} b(\mu) = \bar{b}_2$ and thus $\lim_{\mu \to -\infty} \mu'(\mu, 2) = -\infty$. In order for the period by period budget constraint

$$\frac{p_s}{\beta} b(\mu) = S(\mu'(\mu, s)) + b(\mu'(\mu, s))$$

to be satisfied for all s it must be $\lim_{\mu \to -\infty} \mu'(\mu, 1) > -\infty$ (as $\bar{b}_1 > \bar{b}_2$). Continuity of μ with the uniqueness of the steady state μ^* then implies that $\mu'(\mu, 1) > \mu'(\mu, 2)$ for all $\mu < \mu^*$. $\boldsymbol{p} < 1$ implies that $p_1 < p_2$ allowing us to conclude that $Cov_t(p_{t+1}, \mu_{t+1}) < 0$.

[5] $^{\text{dge}}$ I'm still working on this. I'm not sure if we'll need to include it as an assumption or not

[6] $^{\text{dge}}$Given that $\mu'(\mu^*, s) = \mu^*$ and $\mu'(\mu, s)$ is increasing in μ we know that if $\mu_t < \mu^*$ then $\mu_{t+j} < \mu^*$ for all histories s^{t+j}

We then have that

$$\mu_t < \mathbb{E}_t \mu_{t+1}$$

for $\mu_t < \mu^*$. As $\mu'(\mu, s)$ is increasing, continuous and $\mu'(\mu^*, s) = \mu^*$, we can iterate on the policy rules to show that if $\mu_t < \mu^*$ the for all $j > 0$ we must have $\mu_{t+j} < \mu^*$. Thus, if $\mu_t < \mu^*$ then μ_t is a supermartingale bounded from below implying that $\mu_t \to \tilde{\mu}$ for some constant $\tilde{\mu}$ with probability 1. It is then just a matter of using the continuity of $\mu'(\mu, s)$ to show that

$$\mu'(\tilde{u}, s) = \tilde{\mu}$$

implying that $\tilde{\mu} = \mu^*$, as μ^* is the unique steady state. The steady state is then globally stable as $\mu_t \to \mu^*$ with probability 1.

2. $\boldsymbol{p} \geq \alpha_2$**:** Following a similar method as in the previous case we know for $\boldsymbol{p} > \alpha_2$ that $\bar{b}_1 < \bar{b}_2$ implying that the natural debt limit is obtained using the "good" state. As in the previous case by taking limits we obtain $\lim_{\mu \to -\infty} \mu'(\mu, 1) = -\infty$ and $\lim_{\mu \to -\infty} \mu'(\mu, 2) > -\infty$. This implies that $\mu'(\mu, 1) < \mu'(\mu, 2)$ which, along with $p_1 > p_2$, implies $Cov_t(p_{t+1}, \mu_{t+1}) < 0$. As before, we then have global stability of the steady state for $\mu_t < \mu^*$.

3. $\min(\alpha_1, 1) < \boldsymbol{p} < \alpha_2$**:** In this case either there exists a steady state if $1 < \boldsymbol{p} \leq \alpha_1$ or there does not exist a steady state. In either case the analysis for the first case implies that $\mu'(\mu, 1) > \mu'(\mu, 2)$ for $\mu < \mu^*$.[7] As $\boldsymbol{p} > 1$ implies that $p_1 > p_2$, we can therefore conclude that $Cov_t(p_{t+1}, \mu_{t+1}) > 0$ implying that

$$\mu_t > \mathbb{E}_t \mu_{t+1}$$

We thus cannot apply the martingale convergence theorem and the steady state will locally be unstable. [apb]??? We havent defined local stability?

□

[7] [dge]In the case where there does not exists a steady state take μ^* to be ∞

4.4 Economic forces driving convergence

In summary, when the aggregate state follows a 2-state i.i.d. process, government debt b_t often converges to b^*, while the tail of the allocation equals Ramsey allocation for an economy with complete markets and initial government debt b^*. The level and sign of b^* depend on the asset payoff structure, which we have expressed in terms of a scalar $\boldsymbol{p}$ that concisely captures what in more general settings we represented with the asset payoff matrix $\mathbb{P}$.

Facing incomplete markets, the Ramsey planner recognizes that the government's debt *level* combines with the payoff structure on its debt instrument to affect the welfare costs associated with varying the distorting labor tax rate across states. When the instrument is a risk-free bond, the government's marginal cost of raising funds μ_t is a martingale. In this situation, *changes* in debt levels help smooth tax distortions across time. However, if the payoff on the debt instrument varies across states, then by affecting its state-contingent revenues, the *level* of government debt can help smooth tax distortions across states. For our two state, iid shock process, the steady state debt level b^*, when it exists, is the unique amount of government debt that provides just enough "state contingency" completely to fill the void left by missing assets markets. The Ramsey planner takes into account the additional benefits from tax smoothing as the government debt approaches b^*; that puts a risk-adjustment into the martingale governing μ and leads the government either to accumulate or decumulate debt. Although accumulating government assets requires raising distorting taxes, locally the welfare costs of higher taxes are second-order and so are dominated by the welfare gains from approaching b^*, which are first-order.

5 Turning on risk-aversion

We now depart from quasi-linearity of $U(c, l)$ and thus activate an additional source of return fluctuations coming from endogenous fluctuations in prices of the asset q_t. To obtain a recursive representation of a Ramsey plan, we employ the endogenous state variable

$$x_t = u_{c,t} b_t,$$

and study how long-run properties of x_t depend on equilibrium returns $R_{t,t+1} = \frac{\mathbb{P}(s_t, s_{t+1})}{q_t(s^t)}$. Activating risk aversion in consumption makes q_t vary in interesting ways.

Commitment to a Ramsey plan implies that government actions at $t \geq 1$ are constrained by the household's anticipations about them at $s < t$ Following Kydland and Prescott (1980), we use the marginal utility of consumption that the Ramsey planner promises to the household to account for that 'forward looking' restriction on the Ramsey planner. For eg., that comes from the fact that the Euler equation restricts allocations such that expected marginal utility in time t is constrained by consumption choices in time $t-1$. It is convenient for us that scaling the household's budget constraint by the marginal utility of consumption makes Ramsey problem recursive in $x = U_c b$. In particular, implementability constraints (8c) can be represented as

$$\frac{x_{t-1}\mathbb{P}(s_t, s_{t-1})U_{c,t}}{\beta \mathbb{E}_{t-1}\mathbb{P}U_{c,t}} = U_{c,t}c_t + U_{l,t}l_t + x_t, \ t \geq 1 \tag{16}$$ {?}

RamseyBellman⟩ **Problem 5.1.** *Before the realization of the time t Markov shock s_t, let $V(x, s_{-1})$ be the* expected *continuation value of the Ramsey plan at $t \geq 1$ given promised marginal utility government debt inherited from the past $x = U_{c,t}b_t$ and time $t-1$ Markov state s_{-1}. After the realization of time 0 Markov shock s_0, let $W(b_{-1}, s_0)$ be the value of the Ramsey plan when initial government debt is b_{-1}. The (ex ante) Bellman equation for $t \geq 1$ is*

$$V(x, s_{\_}) = \max_{c(s), l(s), x'(s)} \sum_s \Pi(s, s_{\_})\Big(U(c(s), l(s)) + \beta V(x'(s), s)\Big) \tag{17}$$ eqn:Bellman1

subject to $x'(s) \in [\underline{x}, \overline{x}]$ and

$$\frac{x\mathbb{P}(s, s_{\_})U_c(s)}{\beta \mathbb{E}_{s_{\_}}\mathbb{P}U_c} = U_c(s)c(s) + U_l(s)l(s) + x'(s) \tag{18}$$ timetBellimple

$$c(s) + g(s) = \theta(s)l(s) \tag{19}$$ timetfeas

Equation (18) is the implementability constraint and (19) is feasibility. Given an initial debt b_{-1},time 0 Markov state s_0, and continuation value function $V(x, s_{\_})$, the (ex post) time 0 Bellman equation is

$$W(b_{-1}, s_0) = \max_{c_0, l_0, x_0} U(c, l) + \beta V(x_0, s0) \tag{20}$$?eqn:Bellman0?

subject to time zero implementability constraint

$$U_c(c_0, l_0)c + U_l(c_0, l_0)l_0 + x_0 = U_c(c_0, l_0)b_{-1}$$

and the resource constraint

$$c_0 + g(s_0) = \theta(s_0) l_0$$

and

$$x_0 \in [\underline{x}, \overline{x}]$$

dge

Lemma 5.2. *Let V,W be the solution problem 5.1, then the allocation corresponding to the optimal policies solves problem 3.1.*

5.1 What we've done

[tjs]Tom XXXXXX: write a subsection describing what Anmol and David have done so far with this setup, computationally and analytically

5.2 Motivation to focus on risk-free bond economy

riskfreeonly⟩?

As mentioned in section 3.1, properties of a Ramsey plan for our incomplete markets economy vary sensitively with asset returns that reflect properties of equilibrium prices $\{q_t(s^t|B_{-1}, s_{-1})\}_t$ and the exogenous asset payoff matrix $\mathbb{P}$. By studying quasi-linear preferences, we eliminated fluctuations in returns coming from prices. Here we turn the table and by studying an economy with a risk-free bond, we eliminate fluctuations in returns coming from the exogenous asset payoff matrix $\mathbb{P}$. Thus, we set $\mathbb{P}(s|s_{\_}) = 1 \; \forall \; (s, s_{\_})$.

Let $x'(s; x, s_{\_})$ be the decision rule for x' that attains the right side of the $t \geq 1$ Bellman equation (17). A steady state x^* satisfies $x^* = x'(s; x^*, s_{-})$ for all $s, s_{\_}$. A steady state is a node at which the continuation allocation and tax rate have no further history dependence.

rop:existenceU⟩ **Proposition 5.3.** *Assume that U is separable and iso-elastic: $U(c,l) = \frac{c^{1-\sigma}}{1-\sigma} - \frac{l^{1+\gamma}}{1+\gamma}$. Assume that*

[apb]*XXXX We are changing notation to represent good states with $s = s_g$ instead of $s = 1$?*

the Markov state s take two values is i.i.d with s_b being the "adverse" state (either low TFP or high govt. expenditures) and s_g begin the good state. Let x_{fb} be a value of ~~the state x from which a government can implement first=best with complete markets~~ [dge]marginal *utility weighted debt associated with the first best allocation with complete markets. Let $q_{fb}(s)$ be the shadow price of government debt in state s at the first best allocation. If*

$$\frac{1-q_{fb}(s_b)}{1-q_{fb}(s_g)} > \frac{g(s_b)}{g(s_g)} \geq 1 \tag{21}$$

eqn:prop52suff

then there exists a steady state with $x_{fb} > x^* > 0$

Proof. [dge]As in the quasi-linear case a steady state is associated with the continuation allocation of a complete markets allocation with some initial debt level. Equivalently, we can index these allocations with their associated multiplier on the implementability constraint: μ. Letting $S(\mu, s)$ be the marginal utility government surplus in state s with multiplier μ, a steady state will be a multiplier μ^* where the budget constraint in both states of the world is satisfied

$$\frac{S(\mu^*, s_g)}{\frac{c(\mu^*,s_g)^{-\sigma}}{\beta\mathbb{E}c(\mu^*)^{-\sigma}}-1} = \frac{S(\mu^*, s_b)}{\frac{c(\mu^*,s_b)^{-\sigma}}{\beta\mathbb{E}c(\mu^*)^{-\sigma}}-1}$$

By choosing μ_1 such that $S(\mu_1, s_g) = 0$ we conclude that

$$0 = \frac{S(\mu_1, s_g)}{\frac{c(\mu_1,s_g)^{-\sigma}}{\beta\mathbb{E}c(\mu_1)^{-\sigma}}-1} > \frac{S(\mu_1, s_b)}{\frac{c(\mu_1,s_b)^{-\sigma}}{\beta\mathbb{E}c(\mu_1)^{-\sigma}}-1}$$

This is derived directly from $S(\mu, s_g) < S(\mu, s_b)$ for all μ and $c(\mu, s_g) > c(\mu, s_b)$ for all μ.

[dge]Substituting out for q_{fb}, equation (21) can equivalently be written as

$$\frac{g(s_g)}{1-\frac{\beta\mathbb{E}c_{fb}^{-\sigma}}{c_{fb}(s_g)^{-\sigma}}} > \frac{g(s_b)}{1-\frac{\beta\mathbb{E}c_{fb}^{-\sigma}}{c_{fb}(s_b)^{-\sigma}}}$$

Multiplying both sides by -1 and factoring out a $\beta\mathbb{E}c_{fb}^{-\sigma}$ this equation simplifies to

$$\frac{-c_{fb}(s_g)^{-\sigma}g(s_g)}{\frac{c_{fb}(s_g)^{-\sigma}}{\beta\mathbb{E}c_{fb}^{-\sigma}}-1} < \frac{-c_{fb}(s_b)^{-\sigma}g(s_b)}{\frac{c_{fb}(s_b)^{-\sigma}}{\beta\mathbb{E}c_{fb}^{-\sigma}}-1}$$

or

$$\frac{S(0, s_g)}{\frac{c(0,s_g)^{-\sigma}}{\beta\mathbb{E}c(0)^{-\sigma}}-1} < \frac{S(0, s_b)}{\frac{c(0,s_b)^{-\sigma}}{\beta\mathbb{E}c(0)^{-\sigma}}-1}$$

The existence of μ^* then comes directly from the Intermediate Value Theorem. □

p:convergenceU⟩

Proposition 5.4. ~~*Let* $\{c_t(s^t), l_t(s^t), x_t(s^{t-1})\}$ *solve the incomplete markets Ramsey problem with* $x_0 > x^*$. *Then* $x_t(s^{t-1}) \to x^*$ *as* $t \to \infty$ *with probability 1 for all initial conditions.*~~ dge*There exists* $\underline{x} < x^*$ *and* $\overline{x} > 0$ *such that if* $\{c_t(s^t), l_t(s^t), x_t(s^{t-1})\}$ *solves the incomplete markets Ramsey problem 5.1 with bounds* $\underline{x}$ *and* $\overline{x}$ *then* $x_t(s^t) \to x^*$ *as* $t \to \infty$ *with probability 1.*

Proof. dgeWe can break the proof of this proposition down into two lemmas describing the structure of the policy functions. These properties will be intuitive and can be proved in the appendix. Finally the proof will rely on the assumption of concavity of the value function,

⟨lem:c_order⟩ **Lemma 5.5.** *Consumption is ordered by the state of the world. That is there exists* $\underline{x}$ *and* $\overline{x}$ *such that for all* $x \in [\underline{x}, \overline{x}]$ *the policy functions for consumption satisfy* $c(x, s_g) > c(x, s_b)$.

This lemma guarantees that for the same level of marginal utility weighted government debt, consumption will be larger in "good" states of the world than "adverse" states of the world.

⟨lem:x_order⟩ **Lemma 5.6.** *There exits* $\underline{x}$ *and* $\overline{x}$ *such that the optimal government savings policy,* $x'(x, s)$ *satisfies*

1. *For* $x \in (x^*, \overline{x}]$ *we have* $x'(x, s_g) < x'(x, s_b)$
2. *For* $x \in [\underline{x}, x^*)$ *we have* $x'(x, s_g) > x'(x, s_b)$

Furthermore, $x'(x, \cdot)$ *is increasing in* x.

Property 1. states that if government debt is larger than the steady state, then the government will issue more debt in bad states of the world than good states of the world. Property 2. states that if government debt is smaller that the steady state debt then the government has accumulated enough assets[8] such that the lower interest rates in the "adverse" state of the world allow it to purchase more assets (issue less debt) than in the "good" states of the world. The last part of the lemma gaurantees that if the government enters with more debt it will pass on more debt to future periods. We can now prove global convergence, we will focus on the case where $x_t \geq x^*$ as the other case is symmetric. As $x'(x, \cdot)$ is increasing in x, we can iterate the policy functions forward to conclude that

[8]Remember in the steady state the government will hold assets

$x_{t+j} > x^*$ for all j as long as $x_t > x^*$. Letting $\mu_t = V'(x_t)$ be the multiplier on the implementability constraint and $\overline{\lambda}_t$ be the multiplier on the constraint $x_t \leq \overline{x}$ we have

$$\mu_t = \frac{1}{\mathbb{E}_t[c_{t+1}^{-\sigma}]}\mathbb{E}_t[\mu_{t+1}c_{t+1}^{-\sigma}] - \overline{\lambda}_t$$

Lemma 5.6. along with concavity of V allows us to conclude that $\mu_{t+1}(s_g) > \mu_{t+1}(s_b)$. From Lemma 5.5. we know that $c_{t+1}(s_g) > c_{t+1}(s_b)$ which implies that $Cov_t(\mu_{t+1}, c_{t+1}^{-\sigma} < 0$ and thus

$$\frac{1}{\mathbb{E}_t[c_{t+1}^{-\sigma}]}\mathbb{E}_t[\mu_{t+1}c_{t+1}^{-\sigma}] < \mathbb{E}_t[\mu_{t+1}]$$

As $\overline{\lambda}_t \geq 0$, we can conclude that

$$\mu_t < \mathbb{E}_t[\mu_{t+1}]$$

Moreover $\mu_t < V'(x^*) = \mu^*$, so μ_t is a submartingale bounded from above. Applying the martingale convergence theorem we conclude that $\mu_t \to \mu^*$ with probability 1. Continuity of the policy functions and uniqueness of the steady state in the region $[x^*, \overline{x}]$ implies that $x_t \to x^*$ with probability 1. $\square$

Remark 5.7. *In this economy, fluctuations in the risk-free interest rate come from fluctuations in marginal utility of consumption. The interest rate is low in "good" states (i.e., when TFP is high or government expenditures are low). In a steady state, the government holds claims against the private sector, an outcome that resembles those in economies with quasi-linear utility and low* $\boldsymbol{p}$*. For all admissible initial levels of government debt, an incomplete markets Ramsey allocation converges to a particular Lucas-Stokey Ramsey allocation.* [tjs] *Team xXXXXX: say a little more about the particular LS allocation and its initial debt level*

dge

Remark 5.8. *Propositions 5.3 and 5.4 can be thought of as a near converse to Lemma 3. of section 5 in Aiyagari et al. (2002). There they provide sufficient conditions for the non-convergence of the economy to a complete markets allocations. Our propositions provide sufficient conditions for the existence of the complete markets steady state and the long rung convergence to that steady state. While the results of Aiyagari et al. (2002) do hold, and for more general stochastic processes a steady state will not exist, we find numerically that the results Propositions 5.3 and 5.4 are informative as to the long run dynamics of* *incomplete markets economys. There will exist regions of low volatility and the economy will converge to these regions in the long run.*

[tjs]Team XXXXXX: let's add some modest self-promotion here telling just how much the results immediately above add in terms of filling in loose ends from AMSS – things they just weren't able to answer.

6 To do

1. Add above at appropriate place that until now $T_t \equiv 0$.
2. Edit section about turning on nonnegative transfers.
3. Tom to write introduction and concluding sections.
4. Fill in some notation about what objects are indexed by, e.g., s^t.
5. Fill in proofs.
6. Check flow and order.
7. Add a short appendix on how Bellman equations were solved numerically. Pat selves on back for doing so and display some policy functions – think of one or two things to do with those policy functions.
8. Think of a couple of experiments that show off the policy function calculations.

6.1 Allowing nonnegative transfers

[tjs]Team XXXXX: Beware – please wear hard hat in this construction area. Add BS about AMSS and what transfers did for them. Write front end of this section – good low-skill job for Tom. Access to nonnegative transfers makes first-best level of assets trivially a "steady state." With nonnegative lump sum transfers, in cases where a steady state exists and is stable, if the initial debt of the government exceeds its steady state level, the economy converges with probability 1 to the steady state. Thus, counterpart to previous results continue to hold when initial government debt exceeds its steady state value. When initial government debt is less than a steady-state value, then [tjs]say something that is known or what is unknown.

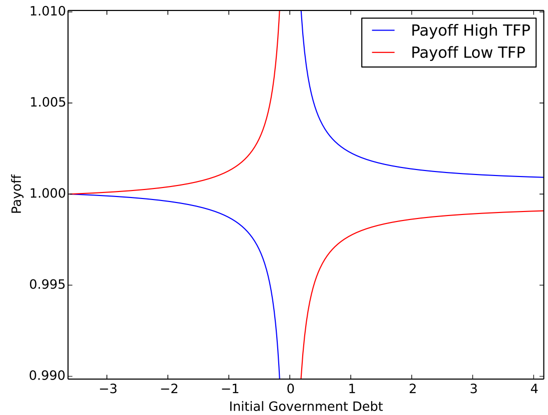

Figure 1: Optimal asset payoff structure as a function of initial government debt when TFP follows a 2 shock i.i.d process

7 Comparison to literature

tjs Team – please keep your hands off this section. Tom to use parts of it in the "womanly" sections (the introduction and concluding section).

1. Angeletos (2002), Buera and Nicolini (2004)
    - Begin with a complete market Ramsey allocation
    - Ask if this can be attained with a limited collection of non-contingent debts of different maturities
2. This paper
    - Begins with an incomplete markets Ramsey allocation
    - Asks whether the long-run allocation coincides with a complete markets Ramsey allocation for some initial govt debt
3. BEGS1 studies a related problem with heterogeneous agents

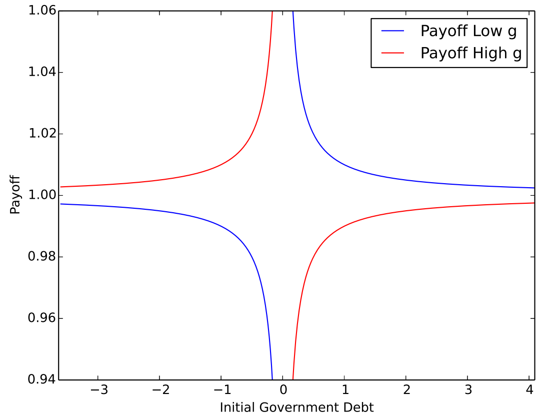

Figure 2: Optimal asset payoff structure as a function of initial government debt when government expenditures follow 2 shock i.i.d process

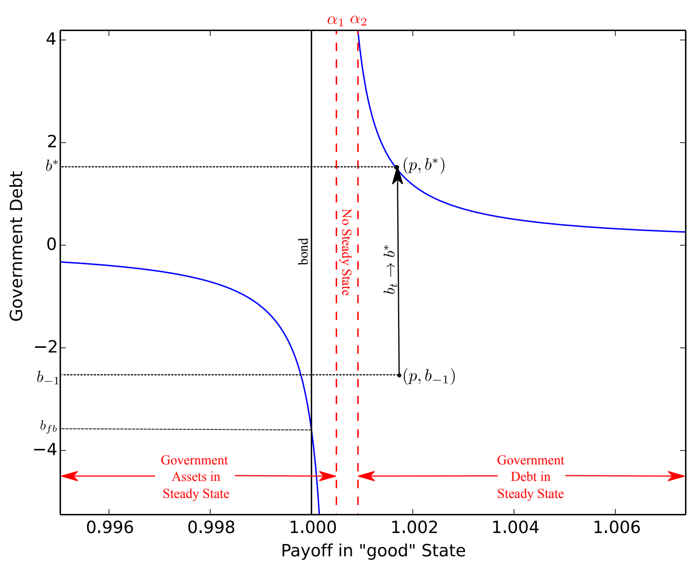

Figure 3: Existence regions in $\boldsymbol{p}$ space

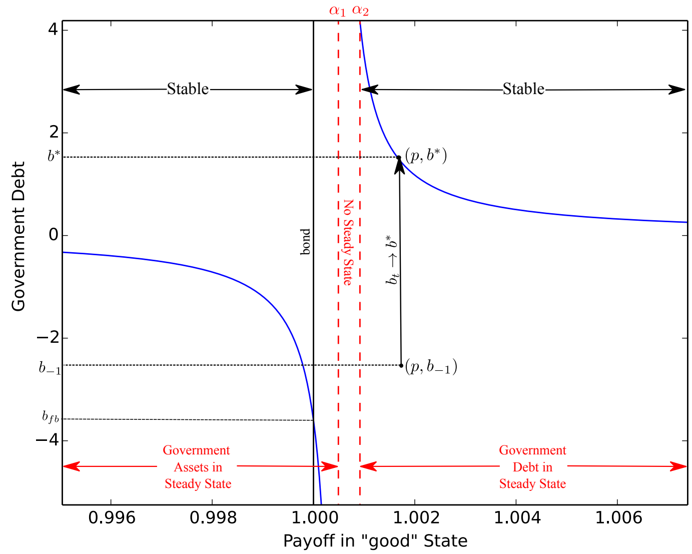

Figure 4: Stability regions

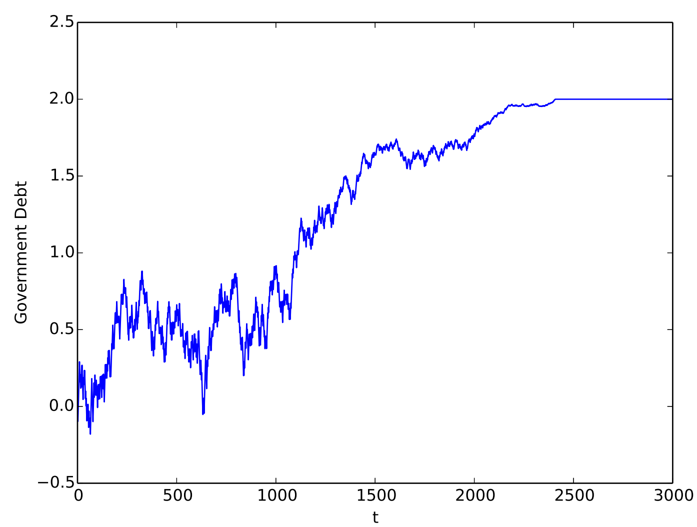

Figure 5: A sample path with $\boldsymbol{p} > 1$

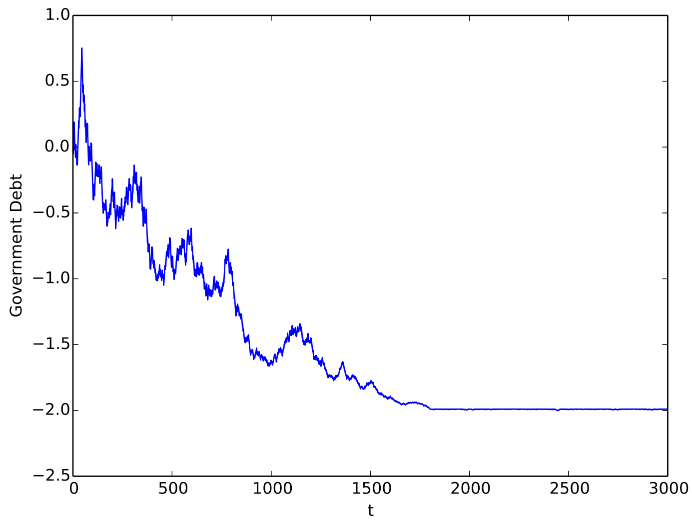

Figure 6: A sample path with $\boldsymbol{p} < 1$

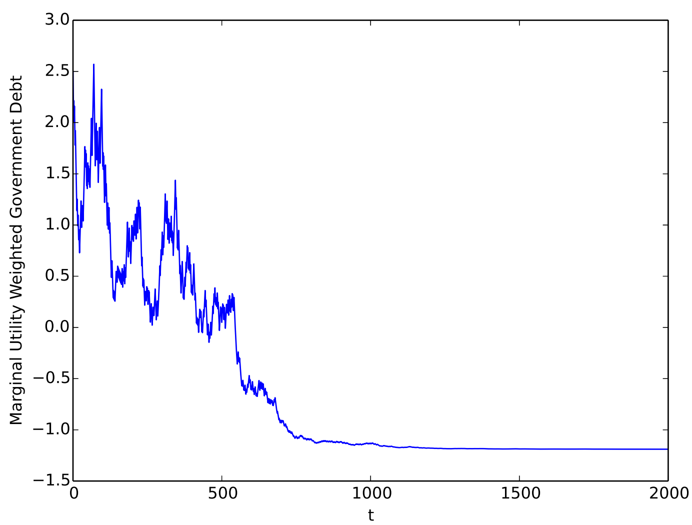

Figure 7: A sample path for 2 state i.i.d. process with risk aversion

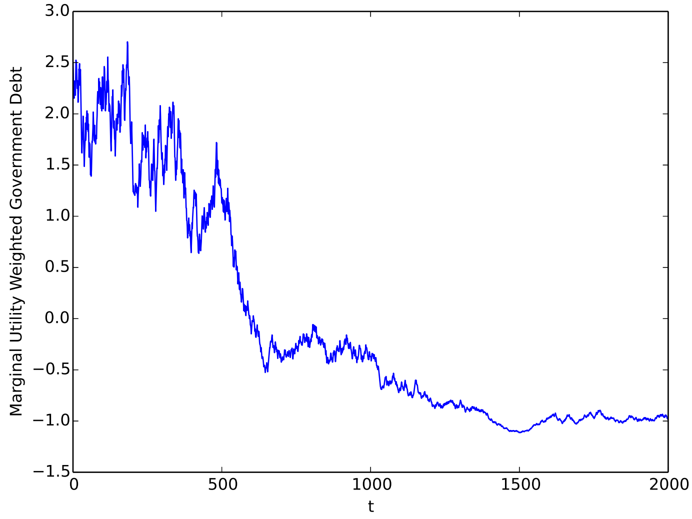

Figure 8: A sample path for economy with $S > 2$ states

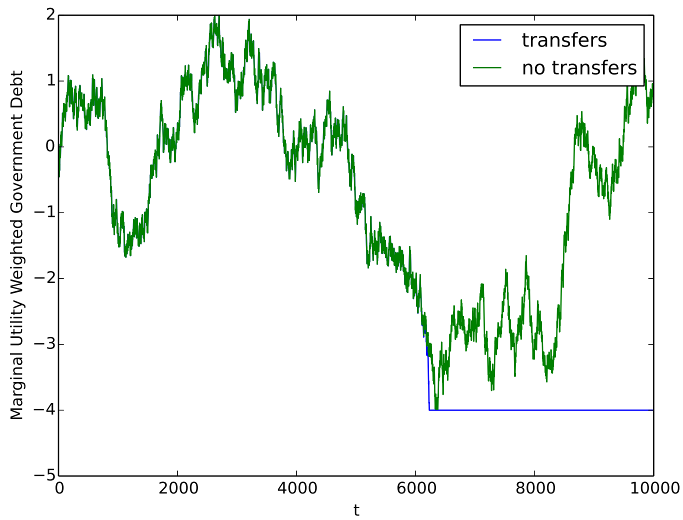

Figure 9: Quasilinear preferences and risk-free bond with and without nonnegative transfers

8 Figures

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