Optimal Taxation with Incomplete Markets

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Abstract

This paper characterizes tax and debt dynamics in Ramsey plans for an incomplete markets economies that generalize a Aiyagari et al. (2002) economy by allowing the single asset traded by the government possibly to be risky rather than risk-free. Long run debt and tax dynamics can be attracted not only to the first-best continuation allocations discovered by Aiyagari et al. but instead often to an allocation that is associated with a particular level or range of levels of (marginal utility scaled) government debt prevailing in a Lucas-Stokey economy that starts from a particular initial level of government debt. The paper formulates, analyzes, and numerically solves two Bellman equations that encode a recursive formulation of a Ramsey plan.

KEYWORDS: Complete markets, incomplete markets, Ramsey plan, taxes, debt, competitive equilibrium, implementability constraints, dynamic programming squared

> "... the option to issue state-contingent debt is important: tax policies that are optimal under uncertainty have an essential 'insurance' aspect to them." Lucas and Stokey (1983, p. 88)

1 Introduction

Controversies sparked by Reinhart and Rogoff (2010) motivated us to reassess what we know and don't know about two elementary questions. What is an optimal government debt? And is government debt even a pertinent state variable? Lucas and Stokey (1983) and Aiyagari et al. (2002) offer different answers to these two questions in the context of economic environments that are identical in all respects but one: Lucas and Stokey (1983) allow the government to issue a complete set of Arrow securities, while Aiyagari et al. (2002) allow the government to issue only a one-period risk-free government bond. For Lucas and Stokey, under an optimal tax and debt policy, government debt is not an independent state variable but instead is an exact function of the Markov state variable that drives government expenditures. In Lucas and Stokey's model, the optimal state-by-state levels of government debt depend on the initial government debt. By way of contrast, for Aiyagari et al. (2002) government debt is an independent state variable with a limiting value or distribution of values that does not depend on the initial government debt. The quote by Lucas and Stokey pinpoints the source of these differences: the government's purchase of *insurance* from the private sector through explicit state-contingent securities underlies Lucas and Stokey's answers to our two questions; while a government's *self-insurance* achieved through its past accumulation of a risk-free asset underlies Aiyagari et al.'s answers.

This paper revisits our two questions in the context of a generalization of the Aiyagari et al. (2002) environment that continues to restrict the government to issue only a single security, but that allows that security to be risky. The government manages the single security it is allowed to issue as best it can. We study the implications for government debt dynamics of alternative exogenous designs of that single security. We use this generalization of Aiyagari et al.'s setup to attack questions left unanswered by Aiyagari et al. and also to say some new things about alternative ways that the government can achieve insurance in an equilibrium of the original Lucas and Stokey (1983) model. Our analysis exploits new, or at least previously unstated, connections between the Lucas and Stokey and Aiyagari et al. economies.

Aiyagari et al. obtained their sharpest results for an economy with a quasi-linear household one-period utility function. Linearity of utility in consumption tied down the risk-free one-period interest rate and enabled them to attain their result that in the long run the government accumulates a big enough stock of the risk-free asset to finance its expenditures entirely from interest earned from its claims on the private sector; so the tail of the Aiyagari et al. Ramsey plan features a zero distorting tax on labor and a first-best allocation. Aiyagari et al. were able to say much less about outcomes for preferences that exhibit risk-aversion in consumption because then the Lagrange multiplier on the key incomplete markets implementability constraint becomes a risk-adjusted martingale rather than the pure martingale that it is under quasi-linearity. Here we are able to say much more than Aiyagari et al.. We accomplish this by recognizing connections to limits of (our generalization of) their economy and the allocation associated with a Lucas and Stokey economy for a particular initial level of government debt. With preferences that exhibit risk aversion in consumption, an attractor for the limiting debt dynamics of our economy is not associated with the first-best allocation active for the quasi-linear economy of Aiyagari et al. but rather an allocation associated with a Lucas-Stokey economy or one close to it.

Our analysis sheds light on the risk-sharing theme in the quotation with which we begin this paper. We exploit insights about exactly *how* the Ramsey planner in a Lucas-Stokey economy delivers the insurance through state-contingent debt that Lucas and Stokey stress is part and parcel of an optimal tax plan: fluctuations in equilibrium interest rates do part of the job. We can construct examples in which the Lucas-Stokey Ramsey planner chooses to issue risk-free debt and to achieve the required state-contingencies entirely through equilibrium fluctuations in the risk-free interest rate.

Our basic strategy is first to find the tail of an incomplete markets Ramsey allocation, then to ask whether that long-run continuation allocation coincides with a Lucas-Stokey complete markets Ramsey allocation for *some* initial government debt. We describe conditions in which the answer is 'yes' or 'almost yes'.

The structure of our analysis is related to but differs from conceptually distinct inquiries of Angeletos (2002), Buera and Nicolini (2004), and Shin (2007). Like us, they want understand the link among possible limits on the state contingency government debt, interest rate fluctuations, and an optimal tax and debt management plan. They begin with a Lucas-Stokey complete market Ramsey allocation and construct conditions under which it can be supported by a limited collection of non-contingent debts of different maturities. Equilibrium interest rate fluctuations let state-contingent returns help.

In addition to the intrinsic interest that we attach to the two questions with which we began, this paper can be viewed as a prolegomenon to an an analysis of debt dynamics in a more complicated economic environment featured in Bhandari et al. (2013). There a Ramsey planner levies a distorting tax on labor partly to finance exogenous government and partly to redistribute goods among heterogeneously skilled households. Debt dynamics are driven by some forces similar to those present in the simpler environment of this paper, but those forces are obscured by the presence of additional ones. We find it enlightening to isolate the underlying forces by studying a simpler setting.

2 Environment

We analyze economies that share the following features. Government expenditures at time t, $g_t = g(s_t)$, and a productivity shock $\theta_t = \theta(s_t)$ are both functions of a Markov shock $s_t \in \mathcal{S}$ having $S \times S$ transition matrix Π and initial condition s_{-1}. We will denote time t histories with s^t and z_t will refer to a generic random variable measurable with respect to s^t. Sometimes we will denote $z_t(s^t)$ indicate a particular realization of z_t. An infinitely lived representative consumer has preferences over allocations $\{c_t, l_t\}_{t=0}^{\infty}$ of consumption and labor supply that are ordered by

$$\mathbb{E}_{-1} \sum_{t=0}^{\infty} \beta^t U(c_t, l_t), \tag{1}$$ eqn:obj

where U is the period utility function for consumption and labor. For most of the paper, we shall assume that U separable in consumption and labor. We describe additional assumptions later. Labor produces output via the linear technology

$$y_t = \theta_t l_t$$

The representative consumer's tax bill at time $t \geq 0$ is

$$-T_t + \tau_t \theta_t l_t, \;\; T_t \geq 0,$$

where $\tau_t(s^t,)$ is a flat rate tax on labor income and T_t is a nonnegative transfer. Often, we'll set $T_t = 0$. The government and consumer trade a single possibly risky asset whose time t payoff p_t is described by an $S \times S$ matrix $\mathbb{P}$:

$$p_t = \mathbb{P}(s_t, s_{t-1})$$

Let B_t denote the government's holdings of the asset and b_t be the consumer's holdings. Let $q_t = q_t(s^t)$ be the price of the single asset at time t. At $t \geq 0$, the household's time budget constraint is

$$c_t + b_t = (1 - \tau_t)\,\theta_t l_t + \frac{p_t}{q_{t-1}} b_{t-1} + T_t \tag{2}$$ eqn:HHbudget

and the government's is

$$g_t + B_t + T_t = \tau_t \theta_t l_t + \frac{p_t}{q_{t-1}} B_{t-1}. \tag{3}$$ eqn:Govbudget

Feasible allocations satisfy

$$c_t + g_t = \theta_t l_t, \ \forall t \geq 0 \tag{4}$$ eqn:ResFeas

Clearing in the time $t \geq 0$ market for the single asset requires

$$b_t + B_t = 0. \tag{5}$$ eqn:bondmarket

Initial assets satisfy $b_{-1} = -B_{-1}$[1] An initial value of the exogenous state s_{-1} is given. Equilibrium objects including $\{c_t, l_t, \tau_t\}_{t=0}^{\infty}$ will come in the form of sequences of functions of initial government debt b_{-1} and $s^t = [s_t, s_{t-1}, \ldots, s_0, s_{-1}]$.

Borrowing from a standard boilerplate, we use the following:

Definition 2.1. *An* ***allocation*** *is a sequence $\{c_t, l_t\}_{t=0}^{\infty}$ for consumption and labor.An* ***asset profile*** *is a sequence $\{b_t, B_t\}_{t=0}^{\infty}$. A* ***price system*** *is a sequence of asset prices $\{q_t\}_{t=0}^{\infty}$. A* ***budget-feasible government policy*** *is a sequence of taxes and transfers $\{\tau_t, T_t\}_{t=0}^{\infty}$*

Definition 2.2. *Given $(b_{-1} = -B_{-1}, s_{-1})$ and a government policy, a* ***competitive equilibrium with distorting taxes*** *is a price system, an asset profile, a government policy, and an allocation such that (a) the allocation maximizes* (1) *subject to* (2)*, (b) given prices, $\{b_t\}_{t=0}^{\infty}$ is bounded; and (c) equations* (3), (4) *and* (5) *are satisfied.*

Definition 2.3. *Given (b_{-1}, B_{-1}, s_{-1}), a* ***Ramsey plan*** *is a welfare-maximizing competitive equilibrium with distorting taxes.*

3 Two Ramsey problems

Following Lucas and Stokey (1983) and Aiyagari et al. (2002), we use a "primal approach." To encode a government policy and price system as a restriction on an allocation, we first

[1] We assume that b_{-1} are obligations with accrued interest. This is equivalent to setting $q_{-1} = 1$.

⟨eqn:HHFOC⟩ obtain the representative household's first order conditions[2]

$$U_{c,t}q_t = \beta \mathbb{E}_t p_{t+1} U_{c,t+1} \tag{6a}$$ eqn:Euler

$$(1-\tau_t)\theta_t U_{c,t} = -U_{l,t} \tag{6b}$$?eqn:lcFOC?

We substitute these into the household's budget constraint to get a difference equation that we solve forward at every history for every $t \geq 0$. That yields *implementability constraints* on a Ramsey allocation that fall into two categories: (1) the time $t = 0$ version is identical with the *single* implementability constraint imposed by Lucas and Stokey (1983); (2) the time $t \geq 1$ implementability constraints are counterparts of the additional *measurability restrictions* that Aiyagari et al. (2002) impose on a Ramsey allocation.

We first state our Ramsey problem, then Lucas and Stokey's.

rob:RamseyBEGS⟩ **Problem 3.1.** *The Ramsey problem is to choose an allocation and an bounded government debt sequence* $\{b_t\}_{t=0}^{\infty}$ *that attain:*

$$\max_{\{c_t, l_t, b_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t, l_t) \tag{7}$$?eqn:Ramseyobj

subject to

$$c_t + g_t = \theta_t l_t, \;\; t \geq 0 \tag{8a}$$ eqn:feas

$$b_{-1} = \frac{1}{U_{c,0}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(U_{c,t}c_t + U_{l,t}l_t\right) \tag{8b}$$ eqn:LSimplemer

$$\frac{b_{t-1}U_{c,t-1}}{\beta} = \frac{\mathbb{E}_{t-1} p_t U_{c,t}}{p_t U_{c,t}} \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \left(U_{c,t+j}c_{t+j} + U_{l,t+j}l_{t+j}\right) \;\; \textit{for } t \geq 1 \tag{8c}$$ eqn:AMSSimplem

⟨prob:RamseyLS⟩ **Problem 3.2.** *Lucas and Stokey's Ramsey problem is to choose an allocation that attains*

$$\max_{\{c_t, l_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t, l_t) \tag{9}$$?eqn:Ramseyobj

subject to the single implementability constraint (8b) *and feasibility* (8a) *for all* t, s^t.

[2]We thus focus on interior equilibria. Arguments by Magill and Quinzii (1994) and Constantinides and Duffie (1996) can be used to show that $\{c_t,\, l_t,\, b_t\}_{t=0}^{\infty}$ with bounded $\{b_t\}$ that also satisfy equations (2) and (6) solve the consumers problem.

Remark 3.3. *Equation* (8a) *imposes feasibility, while equation* (8b) *is the single implementability constraint present in Lucas and Stokey (1983). Equations* (8c) *express additional implementability constraints at every node from time $t \geq 1$. These generalize the Aiyagari et al. (2002) measurability constraints on a Ramsey allocation to our more general payoff structure $\mathbb{P}$ for the single asset. The measurability constraints* (8c) *are cast in terms of the date, history $(t-1, s^{t-1})$ measurable state variable b_{t-1} that for $t \geq 1$ is absent from Lucas and Stokey's complete markets Ramsey problem. Evidently, Ramsey allocation for our incomplete markets economy automatically satisfies the single implementability constraint imposed by Lucas and Stokey.*

⟨rem:LSdebt⟩ **Remark 3.4.** *State-contingent, but not history-dependent, values of consumption, labor supply, and continuation government debt $\check{b}(s)$ solve the Lucas and Stokey (1983) Ramsey problem 3.2. As intermediated by the Lagrange multiplier on the implementability constraint* (8b)*, consumption, labor supply, and $\check{b}(s)$ are functions of initial government debt b_{-1} and the current state s_t, but not past history s^{t-1}.*

3.1 Motivation for quasi-linear U

sequasilinear⟩

Asymptotic properties of a Ramsey plan for our incomplete markets economy vary with asset returns $R_{t-1,t} \equiv \frac{\mathbb{P}(s_t|s_{t-1})}{q_{t-1}}$. We see that $\mathbb{P}$ affects these returns directly through the ex-post exogenous payoffs and indirectly through prices q_{t-1}. To focus exclusively on the exogenous $\mathbb{P}$ part of returns, we begin by studying an economy with quasi-linear utility function:

$$U(c,l) = c - \frac{l^{1+\gamma}}{1+\gamma}, \tag{10}$$?eqn:UQL?

which sets $U_{c,t} = 1$. Asymptotic properties of a Ramsey plan for our incomplete markets economy vary with asset returns that reflect properties of equilibrium prices $\{q_t(s^t|B_{-1}, s_{-1})\}_t$ and the exogenous asset payoff matrix $\mathbb{P}$. At an interior solution, quasi-linear preferences and the Euler equation (6a) pins down $q_t = \beta \mathbb{E}_t \mathbb{P}(s_{t+1}|s_t)$. After studying the consequences of quasi-linear utility, we shall solve for Ramsey plans for utility functions that express risk aversion with respect to consumption and so activate endogenous fluctuations in q_t.

4 Quasi-linear preferences

Throughout this section, we assume that U is quasi-linear and use an indirect three step approach to characterize the asymptotic behavior of government debt and the tax rate.

(1) **Construct an optimal payoff matrix:**

We pose the following problem:

b:PPoperator⟩?

Problem 4.1. *Given arbitrary initial government debt b_{-1}, what is an optimal asset payoff matrix?*

Let $\mathcal{P}$ be the set of all $\mathcal{S} \times \mathcal{S}$ real matrices. Define the indirect utility function $\mathcal{W}(\mathbb{P}; b_{-1})$ as the solution to problem 3.1 for $\mathbb{P} \in \mathcal{P}$ and initial debt b_{-1}. This induces an operator $\mathbb{P}^*$ that maps initial government debt into an optimal payoff matrix [3]

$$\mathbb{P}^*(b_{-1}) \in \arg\max_{\mathbb{P} \in \mathcal{P}} W(\mathbb{P}; b_{-1})$$

(2) **Apply the inverse of the operator $\mathbb{P}^*$.**

For an arbitrary payoff matrix $\mathbb{P}$, let

$$\mathbb{P}^{*-1}(\mathbb{P}) = \min_{b} \|\mathbb{P} - \mathbb{P}^*(b^*)\|, \tag{11}$$

?eqn:invPopera

where $\|\cdot\|$ is the Frobenius matrix norm. For initial government debt b_{-1} such that $\mathbb{P}^*(b_{-1}) = \mathbb{P}$, we shall show that a Ramsey plan for the incomplete markets economy has $b_t = b^*$ for all $t \geq 0$.

(3) **Long run assets**

Starting from an arbitrary initial government b_{-1} and an arbitrary payoff matrix $\mathbb{P}$, establish conditions under which $b_t \to b^*$ under a Ramsey plan.

In particular, where $S = 2$ and shocks s_t are IID, we describe a large set of $\mathbb{P}$'s for which government debt b_t under a Ramsey plan converges to b^*. For more general shock processes, we numerically find an ergodic set of b_t's hovering around the debt level b^*. We execute steps (1), (2) and (3) in sections 4.1, 4.2, and 4.3.

4.1 The Optimal Payoff Matrix

⟨sec:David41⟩

We construct an optimal payoff matrix by first solving problem 3.2 for a Lucas-Stokey Ramsey allocation associated with a given b_{-1}. Next we construct a sequence $\{p_t\}_t$ that satisfies the implementability constraints imposed in (8c). Note that these implementability constraints are invariant to scaling of p_t by a constant k_{t-1} that can depend on s^{t-1}. From

[3]We will demonstrate existence of a maximizer that is unique up to a constant factor along each row of the matrix.

this equivalence class of $\{p_t\}_t$'s we select a $\{p_t\}_t$ that satisfies a normalization $\mathbb{E}_{t-1} p_t = 1$ and also satisfies

$$p_t = \frac{\beta}{b_{t-1}} \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \left(c_{t+j} + U_{l,t+j} l_{t+j} \right), \tag{12}$$ eqn:pdisarm

where

$$b_{t-1} = \beta \mathbb{E}_{t-1} \sum_{j=0}^{\infty} \beta^j (c_{t+j} + U_{l,t+j} l_{t+j}). \tag{13}$$ eqn:bt-1

The term $c_{t+j} + U_{l,t+j} l_{t+j} = (1 - \tau_{t+j}) l_{t+j} - g_{t+j}$ is the net-of-interest government surplus at time $t + j$. From equations (12) and (13), note that $\frac{1}{p_t} - 1$ is the percentage innovation in the present value government surplus at time t.

Note that by construction, p_t disarms the time $t \geq 1$ measurability constraints[4]. Using the remark 3.4 fact that the Lucas-Stokey Ramsey allocation is not history-dependent, construct the optimal payoff matrix as

$$\mathbb{P}^*(s_t, s_{t-1} | b_{-1}) = p_t.$$

Thus, given initial government debt b_{-1}, let $\mu(b_{-1})$ be the Lagrange multiplier on the Lucas-Stokey implementability constraint (8b) at the Lucas-Stokey Ramsey allocation. The tax rate in the Ramsey allocation is $\tau(\mu) = \frac{\gamma\mu}{(1+\gamma)\mu - 1}$, which implies a net-of-interest government surplus $S(s, \tau)$ that satisfies

$$S(s, \tau) = \theta(s)^{\frac{\gamma}{1+\gamma}} (1 - \tau)^{\frac{1}{\gamma}} \tau - g(s)$$

If the aggregate state process s_t is i.i.d. then the 'disarm-the-measurability-constraints' equation (12) implies that the optimal payoff matrix is

$$\mathbb{P}^*(s, s_- | b_{-1}) = \beta \frac{S(s, \tau)}{b_{-1}} + \beta = (1 - \beta) \frac{S(s, \tau)}{\mathbb{E} S(s, \tau)} + \beta, \tag{14}$$ eqn:optPP

which is independent of s_-.

Equation (14) lets us depict an optimal payoff matrix as a function of initial government

[4] Although we assume quasi-linear preferences throughout this particular construction, please note that equation (12) can be generalized to preferences with curvature via

$$p_t = \frac{\beta}{U_{c,t-1} b_{t-1} U_{c,t}} \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \left(U_{c,t+j} c_{t+j} + U_{l,t+j} l_{t+j} \right)$$

with the normalization $\mathbb{E}_{t-1} U_{c,t} p_t = 1$

debt. Figure 4.1 plots the optimal payoff in both states of the world when either government expenditures or TFP follows a 2 state i.i.d. process. In both cases, we see that the ordering of the payoff flips on either side of zero government debt.

⟨fig:optP⟩

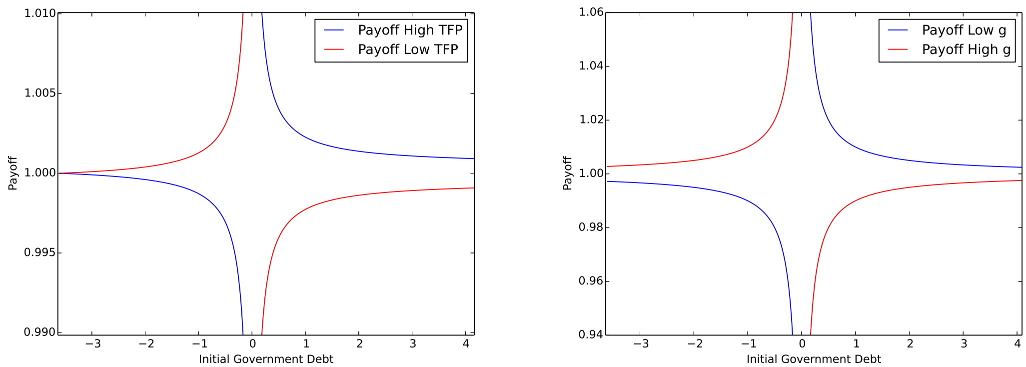

Figure 1: Optimal asset payoff structure as a function of initial government debt when TFP follows a 2 shock i.i.d. process (left) and when government expenditures follow a 2 shock i.i.d process (right).

To appreciate how the initial government debt level influences the optimal asset payoff structure via formula (14), call a state s "adverse" if it implies either "high" government expenditures or "low " TFP; formally, say that s is "adverse" if

$$g(s)\mathbb{E}\theta^{\frac{\gamma}{1+\gamma}} - \theta(s)^{\frac{\gamma}{1+\gamma}}\mathbb{E}g > 0$$

A "good" state is the opposite of an "adverse" state. "Adverse" states have the property that for wide range of initial government debts, the net-of-interest government surplus is lower than in "good" states. When initial government assets are positive, (14) implies that $\mathbb{P}^*$ pays *more* in "adverse" states, while when initial government assets are negative, $\mathbb{P}^*$ pays *less* in "adverse" states.

4.2 The Inverse of $\mathbb{P}^*$ Again

⟨sec:David42⟩ Temporarily assume that s_t is i.i.d and $S = 2$. In this case, note that (14) implies that the optimal payoff matrix $\mathbb{P}^*$ has identical rows. This lets us restrict our attention to $\mathbb{P}(s, s_{\_})$ that have payoffs that are independent of $s_{\_}$. This in turn lets us summarize $\mathbb{P}$ with a vector. Under the normalization $\mathbb{E}\mathbb{P}(s) = 1$, payoffs on the single asset are determined by a scalar $\boldsymbol{p}$, the payoff in state 1. A risk-free bond is then a security for which $\boldsymbol{p} = 1$. Without loss of generality, we shall assume that $g(1)\mathbb{E}\theta^{\frac{\gamma}{1+\gamma}} - \theta(1)^{\frac{\gamma}{1+\gamma}}\mathbb{E}g < 0$, and thus, $\boldsymbol{p}$ is the payoff in the "good" state of the world. Because the optimal payoff matrix can be summarized by a single scalar variable, we can recast the optimal matrix map $\mathbb{P}^*(b)$ as a single scalar function $\boldsymbol{p}^*(b)$. The steady state level of debt associated with an exogenous payoff $\boldsymbol{p}$ is then

$$b^* = \boldsymbol{p}^{*-1}(\boldsymbol{p}). \tag{15}$$

eq-ss

p:ssexistence⟩ **Proposition 4.2.** *There exists $0 \geq \alpha_2 \geq \alpha_1 \geq 1$ such that*

a. If $\boldsymbol{p} \leq \alpha_1$, then $b^* < 0$

b. If $\boldsymbol{p} \geq \alpha_2$, then $b^* > 0$

c. If $\alpha_1 > \boldsymbol{p} > \alpha_2$, then b^* solving (15) does not exist

Proof. Let g_1 and θ_1 denote government expenditures and TFP, respectively in the "good" state of the world. In state s, the government surplus is

$$S(s, \tau) = \theta(s)^{\frac{\gamma}{1+\gamma}}(1-\tau)^{\frac{1}{\gamma}}\tau - g(s),$$

which is maximized at $\tau = \frac{\gamma}{1+\gamma}$ when $(1-\tau)^{\frac{1}{\gamma}}\tau$ is also maximized. Furthermore, in the region $(-\infty, \frac{\gamma}{1+\gamma}]$, $S(\cdot, \tau)$ is an increasing function of τ. In an i.i.d. world with complete markets, government debt at a constant tax rate τ would be

$$\frac{\beta}{1-\beta}\sum_s \Pi(s)S(s,\tau), \tag{16}$$

eqn:tax_to_deb

which is an increasing function of τ. The maximal initial government debt sustainable with *incomplete* markets is then

$$\bar{b} = \frac{1}{1-\beta}\sum_s \Pi(s)\theta(s)^{\frac{\gamma}{1+\gamma}}\left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}}\frac{\gamma}{1+\gamma} - g(s).$$

Inverting the equation (16) mapping from the tax rate into government debt gives us a function $\tau(b)$ that maps initial government debt into an optimal tax rate. The function $\tau(b)$ is an increasing function of b on the domain of possible complete markets initial debts $(-\infty, \bar{b}]$, with $\tau((-\infty, \bar{b}]) = \left(-\infty, \frac{\gamma}{1+\gamma}\right]$.

Substituting the formula for $S(s,\tau)$ into equation (14), we obtain

$$\boldsymbol{p}^*(\tau) = (1-\beta)\frac{\theta_1^{\frac{\gamma}{1+\gamma}}(1-\tau)^{\frac{1}{\gamma}}\tau - g_1}{\mathbb{E}\theta^{\frac{\gamma}{1+\gamma}}(1-\tau)^{\frac{1}{\gamma}}\tau - \mathbb{E}g} + \beta.$$

Solving for $(1-\tau)^{\frac{1}{\gamma}}\tau$ gives

$$(1-\tau)^{\frac{1}{\gamma}}\tau = \frac{(\boldsymbol{p}^* - \beta)\mathbb{E}g - (1-\beta)g_1}{(\boldsymbol{p}^* - \beta)\mathbb{E}\theta^{\frac{\gamma}{1+\gamma}} - (1-\beta)\theta_1^{\frac{\gamma}{1+\gamma}}}.$$

The set of complete market optimal tax rates is $(-\infty, \frac{\gamma}{1+\gamma}]$. Since the mapping $(1-\tau)^{\frac{1}{\gamma}}\tau$ is one to one and $b(\tau)$ is increasing on this domain, we conclude that $\boldsymbol{p}^*(b)$ is one to one. Differentiating $\boldsymbol{p}^*(\tau)$ with respect to τ yields

$$\frac{d}{d\tau}\boldsymbol{p}^*(\tau) = (1-\beta)(1-\tau)^{\frac{1}{\gamma}-1}\left[\gamma - (1+\gamma)\tau\right]\frac{g_1\mathbb{E}\theta^{\frac{\gamma}{1+\gamma}} - \theta_1^{\frac{\gamma}{1+\gamma}}\mathbb{E}g}{(\mathbb{E}\theta^{\frac{\gamma}{1+\gamma}}(1-\tau)^{\frac{1}{\gamma}}\tau - \mathbb{E}g)^2} < 0,$$

implying that $\boldsymbol{p}^*(b)$ is decreasing in b. Since $b = 0$ implies that $\mathbb{E}S(\tau(b)) = 0$, the function function $\boldsymbol{p}^*(b)$ has a pole at $b = 0$. That $\boldsymbol{p}^*(b)$ decreasing in b must therefore imply that $\lim_{b\to 0^-}\boldsymbol{p}^*(b) = -\infty$ and $\lim_{b\to 0^+}\boldsymbol{p}^*(b) = \infty$. We conclude that

$$\boldsymbol{p}^*((-\infty,\bar{b}]) = \boldsymbol{p}^*((-\infty,0)) \cup \boldsymbol{p}^*((0,\bar{b}]) = (-\infty,\alpha_1) \cup [\alpha_2,\infty).$$

We compute the bounds α_1 and α_2 by taking the limits of $\boldsymbol{p}^*$ as b approaches $-\infty$ and the upper bound for government debt under complete markets $\bar{b}$, or equivalently as τ approaches $-\infty$ and $\frac{\gamma}{1+\gamma}$, respectively. □

With only government expenditure shocks, we compute

$$\alpha_1 = 1 \text{ and } \alpha_2 = (1-\beta)\frac{\theta^{\frac{\gamma}{1+\gamma}}\left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}}\frac{\gamma}{1+\gamma} - g(s_1)}{\theta^{\frac{\gamma}{1+\gamma}}\left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}}\frac{\gamma}{1+\gamma} - \mathbb{E}g} + \beta > 1$$

With only TFP shocks, we compute

$$\alpha_1 = (1-\beta)\frac{\theta(s_1)^{\frac{\gamma}{1+\gamma}}}{\mathbb{E}\theta^{\frac{\gamma}{1+\gamma}}} + \beta > 1$$

and

$$\alpha_2 = (1-\beta)\frac{\theta(s_1)^{\frac{\gamma}{1+\gamma}}\left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}}\frac{\gamma}{1+\gamma} - g}{\mathbb{E}\theta^{\frac{\gamma}{1+\gamma}}\left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}}\frac{\gamma}{1+\gamma} - g} + \beta > \alpha_1$$

Remark 4.3. *With only TFP shocks, the bond payoff has the special property that it is associated with a steady state asset level that supports the first-best allocation,* $\boldsymbol{p}^{*-1}(1) = b_{fb}$. *At the first-best taxes are zero, so the net-of-interest government surplus is constant across states.*

We illustrate Proposition 4.2 in figure 4.2. The blue curve is the inverse map $\boldsymbol{p}^{*-1}$. Two constants α_1 and α_2 divide possible payoff structures into three regions: one in which a steady state exists with the government holding assets, another in which a steady state exists with the government owing debt, and yet another in which where a steady state does not exist.

⟨fig:noStable⟩

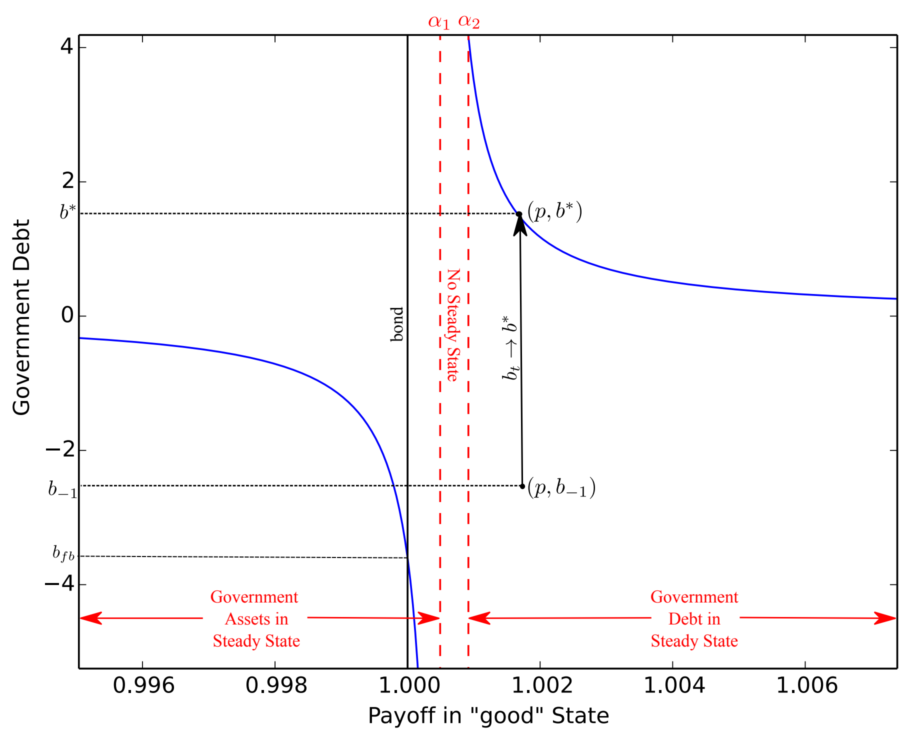

Figure 2: Three regions in $\boldsymbol{p}$ space.

4.3 Long Run Assets

⟨sec:David43⟩ In subsection 4.2, we provided conditions under which there exists b^* such $\boldsymbol{p}^*(b^*) = \boldsymbol{p}$. By construction, if $b_{-1} = b^*$ then the allocation that solves complete markets Ramsey problem 3.2 for initial condition b^* automatically satisfies the measurability constraints (8c). That allocation therefore solves the incomplete markets Ramsey problem 3.1. This implies that if $b_{-1} = b^*$, then $b_t = b^*$ for all t. Thus, $p^{*-1}(\boldsymbol{p})$ corresponds to a "steady state". It remains to be determined whether the incomplete markets Ramsey b_t converges to b^* for arbitrary b_{-1}. Theorem 4.4 provides sufficient conditions for convergence.

m:convergence⟩ **Theorem 4.4.** *Let b_{fb} denote the level of government debt associated with the first-best allocation with complete markets. Then*

a. *If $\boldsymbol{p} \leq \min(\alpha_1, 1)$, then $b_{fb} < b^* < 0$ and $b_t \to b^*$ with probability 1.*

b. *If $\boldsymbol{p} \geq \alpha_2$, then $0 < b^*$ and $b_t \to b^*$ with probability 1.*

c. *If $\min(\alpha_1, 1) < \boldsymbol{p} < \alpha_2$, b^* either does not exist or is unstable.*

For $\boldsymbol{p}$ in region (c), the government run up debt over time.

Proof. The optimal allocation can be represented recursively in terms of functions $c_t(\mu_t), l_t(\mu_t), b_t(\mu_t)$ together with a law of motion for $\mu' = \mu'(\mu, s)$ for μ. We shall establish show global stability under the assumption that $\mu'(\mu, s)$ an increasing function of μ The heart of the proof revolves around the twisted-martingale equation for μ:

$$\mu_t = \sum_s \Pi(s) p_s \mu'(\mu_t, s) = \mathbb{E}_t p_{t+1} \mu_{t+1}.$$

We have shown that there is at most one μ^* such that $\mu'(\mu^*, s) = \mu^*$ for all s. Here we focus on showing global stability for $\mu < \mu^*$. The twisted-martingale equation can be decomposed as follows

$$\mu_t = \mathbb{E}_t \mu_{t+1} + Cov_t(p_{t+1}, \mu_{t+1}).$$

By signing $Cov_t(p_{t+1}, \mu_{t+1})$, we can determine whether μ_t follows a sub or super-martingale. Given that μ_t is bounded from above,[5] we can verify global convergence to the steady state if μ_t is a supermartingale. As in the statement of the theorem, we will split the proof up into three cases. Recall that $\boldsymbol{p}$ is the payoff in the "good" state 1.

[5]Since $\mu'(\mu^*, s) = \mu^*$ and $\mu'(\mu, s)$ is increasing in μ, we know that if $\mu_t < \mu^*$, then $\mu_{t+j} < \mu^*$ for all histories s^{t+j}.

1. $\boldsymbol{p} < \min\{1, \alpha_1\}$**:** Let $\overline{b}_s$ be maximal debt with which the government could enter a period and be able to pay off, assuming that it receives shock s from this period onward. Then

$$\overline{b}_s = \left(\frac{p_s}{\beta} - 1\right)^{-1} \left(\theta_s^{\frac{\gamma}{1+\gamma}} \left(\frac{1}{1+\gamma}\right)^{\frac{1}{\gamma}} \frac{\gamma}{1+\gamma} - g_s\right)$$

because the government maximizes tax revenue by setting τ to $\frac{\gamma}{1+\gamma}$. For $\boldsymbol{p} < \alpha_2$, it is possible to show that $\overline{b}_1 > \overline{b}_2$, and thus the natural debt limit is attained under repetition of the "adverse" state. This implies that $\lim_{\mu \to -\infty} b(\mu) = \overline{b}_2$ and $\lim_{\mu \to -\infty} \mu'(\mu, 2) = -\infty$. In order for the period-by-period budget constraint

$$\frac{p_s}{\beta} b(\mu) = S(\mu'(\mu, s)) + b(\mu'(\mu, s))$$

to be satisfied for all s, it must be true that $\lim_{\mu \to -\infty} \mu'(\mu, 1) > -\infty$ (as $\overline{b}_1 > \overline{b}_2$). Continuity of μ together with the uniqueness of the steady state μ^* then implies that $\mu'(\mu, 1) > \mu'(\mu, 2)$ for all $\mu < \mu^*$. $\boldsymbol{p} < 1$ implies that $p_1 < p_2$, allowing us to conclude that $Cov_t(p_{t+1}, \mu_{t+1}) < 0$. We then have that

$$\mu_t < \mathbb{E}_t \mu_{t+1}$$

for $\mu_t < \mu^*$. Since $\mu'(\mu, s)$ is increasing and continuous, and since $\mu'(\mu^*, s) = \mu^*$, we can iterate on the policy functions to show that if $\mu_t < \mu^*$, then for all $j > 0$, we must have $\mu_{t+j} < \mu^*$. Thus, if $\mu_t < \mu^*$, then μ_t is a supermartingale bounded from below. That implies that $\mu_t \to \tilde{\mu}$ for some constant $\tilde{\mu}$ with probability 1. Then we can use the continuity of $\mu'(\mu, s)$ to show that

$$\mu'(\tilde{u}, s) = \tilde{\mu},$$

implying that $\tilde{\mu} = \mu^*$, as μ^* is the unique steady state. The steady state is then globally stable since $\mu_t \to \mu^*$ with probability 1.

2. $\boldsymbol{p} \geq \alpha_2$**:** Following the same approach used in for case 1, we know for $\boldsymbol{p} > \alpha_2$ that $\overline{b}_1 < \overline{b}_2$, implying that the natural debt limit is attained under repetition of the "good" state. As in case 1, by taking limits we obtain $\lim_{\mu \to -\infty} \mu'(\mu, 1) = -\infty$ and $\lim_{\mu \to -\infty} \mu'(\mu, 2) > -\infty$. This implies that $\mu'(\mu, 1) < \mu'(\mu, 2)$, which along with $p_1 > p_2$ implies $Cov_t(p_{t+1}, \mu_{t+1}) < 0$. As in case 1, we then have global stability of

the steady state for $\mu_t < \mu^*$.

3. $\min(\alpha_1, 1) < \boldsymbol{p} < \alpha_2$**:** In this case, either there exists a steady state if $1 < \boldsymbol{p} \leq \alpha_1$ or there does not exist a steady state. In either case the analysis for case 1 implies that $\mu'(\mu, 1) > \mu'(\mu, 2)$ for $\mu < \mu^*$.[6] Since $\boldsymbol{p} > 1$ implies that $p_1 > p_2$, we can conclude that $Cov_t(p_{t+1}, \mu_{t+1}) > 0$, implying that

$$\mu_t > \mathbb{E}_t \mu_{t+1}.$$

We thus cannot apply the martingale convergence theorem, leaving open the possibility that the steady state is not stable.

□

Remark 4.5. *Figure 4.3 illustrates Theorem 4.4. In addition to depicting values of* $\boldsymbol{p}$ *for which a steady state exists, it also highlights regions where a steady state is stable. The theorem asserts that there exist* $\boldsymbol{p}$ *for which a steady state exists but is unstable.*

An important aspect of this model is that small changes in primitives (specifically $\boldsymbol{p}$) can lead to major differences in long run allocations. To illustrate this, in Figure 4.3 we plot two sample paths where the only difference is the asset restriction $\boldsymbol{p}$.

4.4 Economic forces driving convergence

In summary, when the aggregate state follows a 2-state i.i.d. process, government debt b_t often converges to b^*, while the tail of the allocation equals Ramsey allocation for an economy with complete markets and initial government debt b^*. The level and sign of b^* depend on the asset payoff structure, which we have expressed in terms of a scalar $\boldsymbol{p}$ that concisely captures what in more general settings we represented with the asset payoff matrix $\mathbb{P}$.

Facing incomplete markets, the Ramsey planner recognizes that the government's debt *level* combines with the payoff structure on its debt instrument to affect the welfare costs associated with varying the distorting labor tax rate across states. When the instrument is a risk-free bond, the government's marginal cost of raising funds μ_t is a martingale. In this situation, *changes* in debt levels help smooth tax distortions across time. However, if the payoff on the debt instrument varies across states, then by affecting its state-contingent

[6] When a steady state does not exist, take μ^* to be ∞.

⟨fig:stable⟩

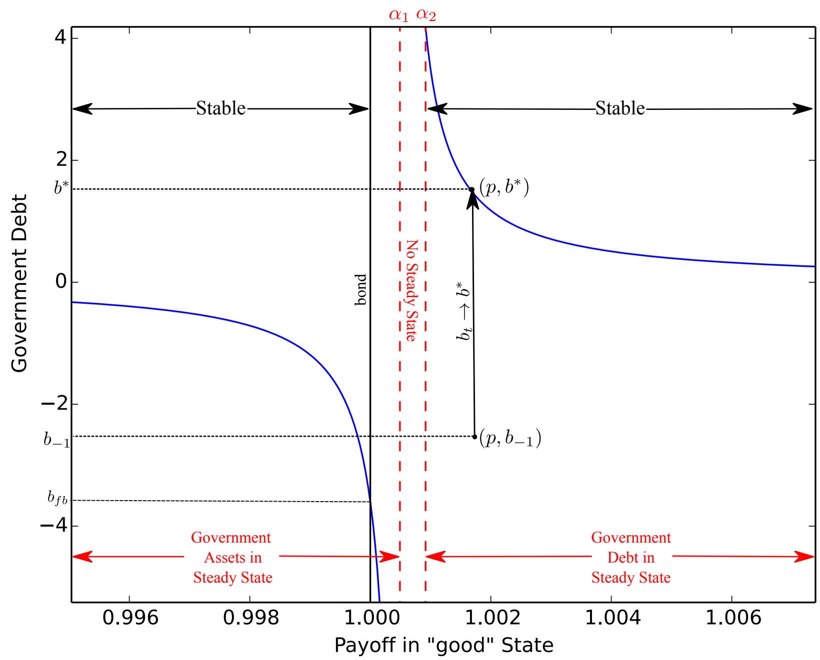

Figure 3: Stability regions in $\boldsymbol{p}$ space.

revenues, the *level* of government debt can help smooth tax distortions across states. For our two state, iid shock process, the steady state debt level b^*, when it exists, is the unique amount of government debt that provides just enough "state contingency" completely to fill the void left by missing assets markets. The Ramsey planner takes into account the additional benefits from tax smoothing as the government debt approaches b^*; that puts a risk-adjustment into the martingale governing μ and leads the government either to accumulate or decumulate debt. Although accumulating government assets requires raising distorting taxes, locally the welfare costs of higher taxes are second-order and so are dominated by the welfare gains from approaching b^*, which are first-order.

5 Turning on risk-aversion

We now depart from quasi-linearity of $U(c, l)$ and thus activate an additional source of return fluctuations coming from endogenous fluctuations in prices of the asset q_t. To obtain fig:2ps_figure⟩

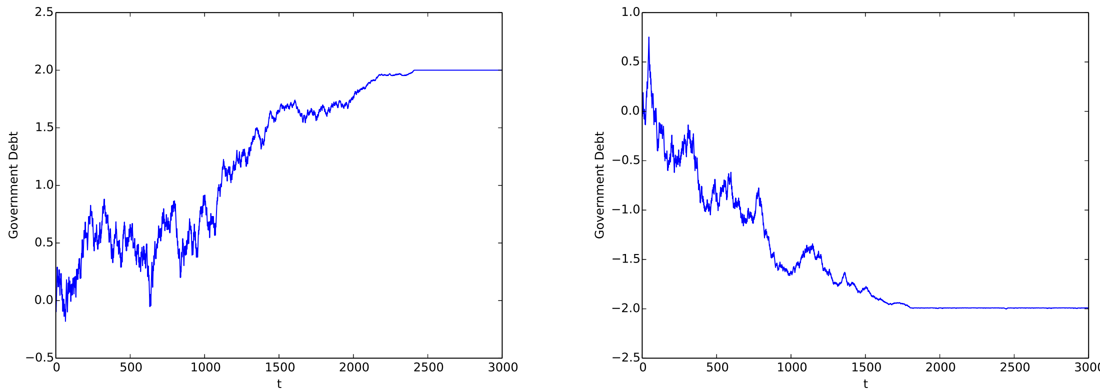

Figure 4: A sample path with $\boldsymbol{p} > 1$ (left) $\boldsymbol{p} < 1$ (right)

a recursive representation of a Ramsey plan, we employ the endogenous state variable

$$x_t = u_{c,t} b_t,$$

and study how long-run properties of x_t depend on equilibrium returns $R_{t,t+1} = \frac{\mathbb{P}(s_t, s_{t+1})}{q_t(s^t)}$. Activating risk aversion in consumption makes q_t vary in interesting ways.

Commitment to a Ramsey plan implies that government actions at $t \geq 1$ are constrained by the household's anticipations about them at $s < t$ Following Kydland and Prescott (1980), we use the marginal utility of consumption that the Ramsey planner promises to the household to account for that 'forward looking' restriction on the Ramsey planner. That comes from the fact that the Euler equation restricts allocations such that expected marginal utility in time t is constrained by consumption choices in time $t-1$. It is convenient for us that scaling the household's budget constraint by the marginal utility of consumption makes Ramsey problem recursive in $x = U_c b$. In particular, implementability constraints (8c) can be represented as

$$\frac{x_{t-1}\mathbb{P}(s_t, s_{t-1}) U_{c,t}}{\beta \mathbb{E}_{t-1} \mathbb{P} U_{c,t}} = U_{c,t} c_t + U_{l,t} l_t + x_t, \ t \geq 1 \tag{17}$$ {?}

RamseyBellman⟩ **Problem 5.1.** *Let $V(x, s_{-1})$ be the* expected *continuation value of the Ramsey plan at $t \geq 1$ given promised marginal utility government debt inherited from the past $x = U_{c,t} b_t$ and time $t-1$ Markov state s_{-1}. After the realization of time 0 Markov shock s_0, let $W(b_{-1}, s_0)$ be the value of the Ramsey plan when initial government debt is b_{-1}. The (ex* *ante) Bellman equation for $t \geq 1$ is*

$$V(x, s_{\_}) = \max_{c(s),l(s),x'(s)} \sum_s \Pi(s, s_{\_}) \Big(U(c(s), l(s)) + \beta V(x'(s), s) \Big) \tag{18}$$ eqn:Bellman1

subject to $x'(s) \in [\underline{x}, \overline{x}]$ and

$$\frac{x\mathbb{P}(s, s_{\_})U_c(s)}{\beta \mathbb{E}_{s_{\_}} \mathbb{P} U_c} = U_c(s)c(s) + U_l(s)l(s) + x'(s) \tag{19}$$ timetBellimple

$$c(s) + g(s) = \theta(s)l(s) \tag{20}$$ timetfeas

Equation (19) is the implementability constraint and (20) is feasibility. Given an initial debt b_{-1},time 0 Markov state s_0, and continuation value function $V(x, s_{\_})$, the (ex post) time 0 Bellman equation is

$$W(b_{-1}, s_0) = \max_{c_0,l_0,x_0} U(c, l) + \beta V(x_0, s0) \tag{21}$$ eqn:Bellman0

subject to time zero implementability constraint

$$U_c(c_0, l_0)c + U_l(c_0, l_0)l_0 + x_0 = U_c(c_0, l_0)b_{-1}$$

and the resource constraint

$$c_0 + g(s_0) = \theta(s_0)l_0$$

and

$$x_0 \in [\underline{x}, \overline{x}]$$

Lemma 5.2. *Let V, W be the optimal value functions for problem 5.1. The allocation given by the corresponding optimal policy function solves problem 3.1.*

5.1 Computational and analytic strategy

[tjs]David and Anmol XXXXXX: please edit and add to this section. A brief paragraph describing the computational strategy and a reference to an appendix, perhaps separate from the paper, is all that are needed. The analysis in this section is based on two pillars: (1) a suite of `python` computer programs that solves Bellman equations (18) and (21); and (2) some mathematical analysis of first-order conditions satisfied by the optimal policy function that attain the right sides of these Bellman equations. As for pillar 1, we approximate the

value function U, V by XXXXX and the policy rules by XXXXX. [tjs]David and Anmol XXXXX: please complete the above sentence and paragraph. Please add a remark or two foretelling some of the analysis to come. You can be brief.

5.2 Motivation to focus on risk-free bond economy

riskfreeonly⟩?

As mentioned in section 3.1, properties of a Ramsey plan for our incomplete markets economy vary sensitively with asset returns that reflect properties of equilibrium prices $\{q_t(s^t|B_{-1}, s_{-1})\}_t$ and the exogenous asset payoff matrix $\mathbb{P}$. By studying quasi-linear preferences, we eliminated fluctuations in returns coming from prices. Here we turn the table and by studying an economy with a risk-free bond, we eliminate fluctuations in returns coming from the exogenous asset payoff matrix $\mathbb{P}$. Thus, we set $\mathbb{P}(s|s_{\_}) = 1 \;\forall\; (s, s_{\_})$.

Let $x'(s; x, s_{\_})$ be the decision rule for x' that attains the right side of the $t \geq 1$ Bellman equation (18). A steady state x^* satisfies $x^* = x'(s; x^*, s_{\_})$ for all $s, s_{\_}$. A steady state is a node at which the continuation allocation and tax rate have no further history dependence.

op:existenceU⟩

Proposition 5.3. *Assume that U is separable and iso-elastic, $U(c,l) = \frac{c^{1-\sigma}}{1-\sigma} - \frac{l^{1+\gamma}}{1+\gamma}$. Assume that*

[apb]XXXX We are changing notation to represent good states with $s = s_g$ instead of $s = 1$?

the Markov state s take two values is i.i.d with s_b being the "adverse" state (either low TFP or high govt. expenditures) and s_g begin the good state. Let x_{fb} be the discounted present value of marginal utility weighted government surpluses associated with the first best allocation. Let $q_{fb}(s)$ be the shadow price of government debt in state s at the first best allocation. If

$$\frac{1 - q_{fb}(s_b)}{1 - q_{fb}(s_g)} > \frac{g(s_b)}{g(s_g)} \geq 1, \tag{22}$$

eqn:prop52suff

then there exists a steady state with $x_{fb} > x^* > 0$

Proof. As in the quasi-linear case, a steady state is associated with a continuation allocation of a complete markets allocation starting from some initial debt level. We can index such continuation allocations by their associated multiplier μ on the implementability constraint. Letting $S(\mu, s)$ be the government surplus at state s and multiplier μ, a steady state has a multiplier μ^* at which the budget constraint in both states of the world is satisfied:

$$\frac{S(\mu^*, s_g)}{\frac{c(\mu^*, s_g)^{-\sigma}}{\beta \mathbb{E} c(\mu^*)^{-\sigma}} - 1} = \frac{S(\mu^*, s_b)}{\frac{c(\mu^*, s_b)^{-\sigma}}{\beta \mathbb{E} c(\mu^*)^{-\sigma}} - 1}.$$

By choosing μ_1 so that $S(\mu_1, s_g) = 0$, we conclude that

$$0 = \frac{S(\mu_1, s_g)}{\frac{c(\mu_1,s_g)^{-\sigma}}{\beta\mathbb{E}c(\mu_1)^{-\sigma}} - 1} > \frac{S(\mu_1, s_b)}{\frac{c(\mu_1,s_b)^{-\sigma}}{\beta\mathbb{E}c(\mu_1)^{-\sigma}} - 1}.$$

We derived this equation directly from $S(\mu, s_g) < S(\mu, s_b)$ for all μ and $c(\mu, s_g) > c(\mu, s_b)$ for all μ.

Eliminating q_{fb}, equation (22) can expressed as

$$\frac{g(s_g)}{1 - \frac{\beta\mathbb{E}c_{fb}^{-\sigma}}{c_{fb}(s_g)^{-\sigma}}} > \frac{g(s_b)}{1 - \frac{\beta\mathbb{E}c_{fb}^{-\sigma}}{c_{fb}(s_b)^{-\sigma}}}.$$

Multiplying both sides by -1 and factoring out $\beta\mathbb{E}c_{fb}^{-\sigma}$, this equation simplifies to

$$\frac{-c_{fb}(s_g)^{-\sigma}g(s_g)}{\frac{c_{fb}(s_g)^{-\sigma}}{\beta\mathbb{E}c_{fb}^{-\sigma}} - 1} < \frac{-c_{fb}(s_b)^{-\sigma}g(s_b)}{\frac{c_{fb}(s_b)^{-\sigma}}{\beta\mathbb{E}c_{fb}^{-\sigma}} - 1}$$

or

$$\frac{S(0, s_g)}{\frac{c(0,s_g)^{-\sigma}}{\beta\mathbb{E}c(0)^{-\sigma}} - 1} < \frac{S(0, s_b)}{\frac{c(0,s_b)^{-\sigma}}{\beta\mathbb{E}c(0)^{-\sigma}} - 1}.$$

Existence of μ^* follows directly from the Intermediate Value Theorem. □

o:convergenceU⟩ **Proposition 5.4.** *There exist $\underline{x} < x^*$ and $\overline{x} > 0$ such that if $\{c_t(s^t), l_t(s^t), x_t(s^{t-1})\}$ solves the incomplete markets Ramsey problem 5.1 with bounds $\underline{x}$ and $\overline{x}$, then $x_t(s^t) \to x^*$ as $t \to \infty$ with probability 1.*

Proof. The proof relies on the concavity of the value function V and two lemmas that describe the structure of the policy functions. Proofs of the lemmas appear in the appendix.

⟨lem:c_order⟩ **Lemma 5.5.** *Consumption is ordered by the state of the world. In particular, there exist $\underline{x}$ and $\overline{x}$ such that for all $x \in [\underline{x}, \overline{x}]$, the policy function for consumption satisfies $c(x, s_g) > c(x, s_b)$.*

This lemma assures that for the same level of marginal utility weighted government debt, consumption is larger in "good" states of the world than in "adverse" states of the world.

⟨lem:x_order⟩ **Lemma 5.6.** *There exist $\underline{x}$ and $\overline{x}$ such that the optimal government savings policy $x'(x, s)$ satisfies*

1. *For* $x \in (x^*, \overline{x}]$*,* $x'(x, s_g) < x'(x, s_b)$

2. *For* $x \in [\underline{x}, x^*)$*,* $x'(x, s_g) > x'(x, s_b)$

Furthermore, $x'(x, \cdot)$ *is increasing in* x*.*

Property 1 states that if government debt exceeds its steady state value, then the government issues more debt in bad states of the world than in good states of the world. Property 2 states that if government debt is smaller than its steady state amount, then the government has accumulated enough assets[7] such that the lower interest rate in the "adverse" state of the world allow it to purchase more assets (issue less debt) than in the "good" states of the world. The last part of the lemma guarantees that if the government enters with more debt, it will pass on more debt to future periods. We can now prove global convergence. We will focus on the case where $x_t \geq x^*$, since the analysis of the other case is symmetric. Since $x'(x, \cdot)$ is increasing in x, we can iterate the policy functions forward to conclude that $x_{t+j} > x^*$ for all j as long as $x_t > x^*$. Letting $\mu_t = V'(x_t)$ be the multiplier on the implementability constraint and $\overline{\lambda}_t$ be the multiplier on the constraint $x_t \leq \overline{x}$, we have

$$\mu_t = \frac{1}{\mathbb{E}_t[c_{t+1}^{-\sigma}]}\mathbb{E}_t[\mu_{t+1}c_{t+1}^{-\sigma}] - \overline{\lambda}_t$$

Lemma 5.6. along with concavity of V allows us to conclude that $\mu_{t+1}(s_g) > \mu_{t+1}(s_b)$. From Lemma 5.5, we know that $c_{t+1}(s_g) > c_{t+1}(s_b)$, which implies that $Cov_t(\mu_{t+1}, c_{t+1}^{-\sigma}) < 0$, so

$$\frac{1}{\mathbb{E}_t[c_{t+1}^{-\sigma}]}\mathbb{E}_t[\mu_{t+1}c_{t+1}^{-\sigma}] < \mathbb{E}_t[\mu_{t+1}].$$

Since $\overline{\lambda}_t \geq 0$, we conclude that

$$\mu_t < \mathbb{E}_t[\mu_{t+1}].$$

Moreover $\mu_t < V'(x^*) = \mu^*$, so μ_t is a submartingale that is bounded from above. Applying the martingale convergence theorem, we conclude that $\mu_t \to \mu^*$ with probability 1. Continuity of the policy functions and uniqueness of the steady state in the region $[x^*, \overline{x}]$ implies that $x_t \to x^*$ with probability 1. □

Remark 5.7. *In this economy, fluctuations in the risk-free interest rate come from fluctuations in marginal utility of consumption. The interest rate is low in "good" states (i.e.,*

[7] Remember that in the steady state, the government owns a positive amount of the risk-free asset.

when TFP is high or government expenditures are low). In a steady state, the government holds claims against the private sector, an outcome that resembles those in economies with quasi-linear utility and low $\boldsymbol{p}$. For all admissible initial levels of government debt, an incomplete markets Ramsey allocation converges to a particular Lucas-Stokey Ramsey allocation.

Remark 5.8. *Propositions 5.3 and 5.4 should be interpreted approximately as supplying a converse to Lemma 3 from section 5 of Aiyagari et al. (2002), which provided sufficient conditions for their incomplete markets Ramsey plan economy to* fail *to converge to a complete markets continuation allocation. Our propositions 5.3 and 5.4 provide sufficient conditions for a complete markets steady state continuation allocation to exist, and for the incomplete market Ramsey allocation to converge to that steady state continuation allocation. Note that propositions 5.3 and 5.4 assume a very special stochastic process for s. For more general stochastic processes, a steady state does not exist. But in simulations, we have found that the outcomes described in Propositions 5.3 and 5.4 do a good job of approximating long run dynamics of incomplete markets Ramsey plans for richer shock stochastic processes, in the sense that they converge to regions of low volatility.*

Figure 5.2 plots a simulation of the Ramsey plan. The path of marginal utility weighted government debt resembles the path of government debt for the quasilinear economy with low $\boldsymbol{p}$ plotted earlier in Figure 4.3.

5.3 Allowing nonnegative transfers

[tjs]Team XXXXX: Beware – please wear hard hat in this construction area. Add BS about AMSS and what transfers did for them. Write front end of this section – good low-skill job for Tom. Access to nonnegative transfers makes first-best level of assets trivially a "steady state." With nonnegative lump sum transfers, in cases where a steady state exists and is stable, if the initial debt of the government exceeds its steady state level, the economy converges with probability 1 to the steady state. Thus, counterpart to previous results continue to hold when initial government debt exceeds its steady state value. When initial government debt is less than a steady-state value, then [tjs]say something that is known or what is unknown.

⟨fig:conv_RA⟩

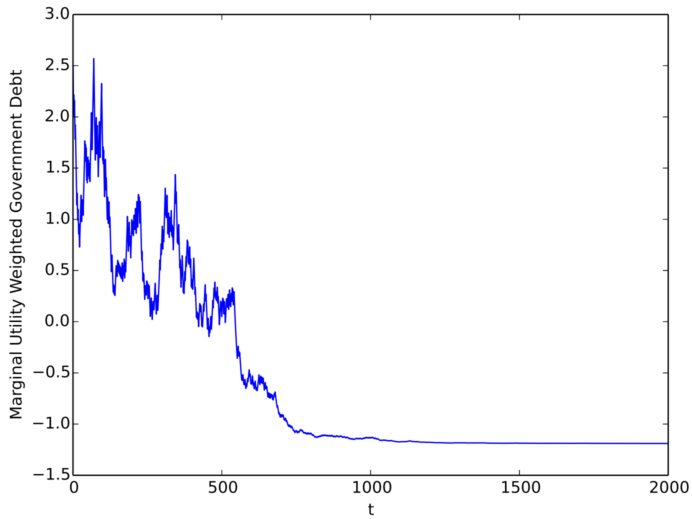

Figure 5: Sample path of x_t for an economy with risk aversion and a 2 state i.i.d. process for tjsDavid: please say more about this economy – is g random or TFP?

6 Concluding remarks

In the quotation with which we began this paper, Lucas and Stokey (1983) emphasize that the government's optimal administration of a flat rate tax on labor depends on its ability to trade a complete set of securities securities with the public whose payoffs are contingent on possible realizations of random variables that drive government expenditures. That he apparently prohibited the government from trading such securities helps account for quite different assertions about debt dynamics made by Barro (1979): government debt is a key state variable for Barro, one that that should be governed by a random walk; while it is not even an independent state variable for Lucas and Stokey, instead being an exact function of the Markov state driving government expenditures that is influenced by the initial level of government debt. By showing that government debt will have a unit-root-like component in a version of Lucas and Stokey's economy modified to allow the government to issue only risk-free debt, Aiyagari et al. (2002) went part way, but only part way, toward explaining the striking differences between the debt dynamics in Lucas and Stokey and Barro. Aiyagari et al. obtain analytical results that are both most complete and most consistent with Barro's assertions the special assumption of quasi-linear preferences that lets a fixed discount factor pin down a time-invariant risk-free interest rate. But even

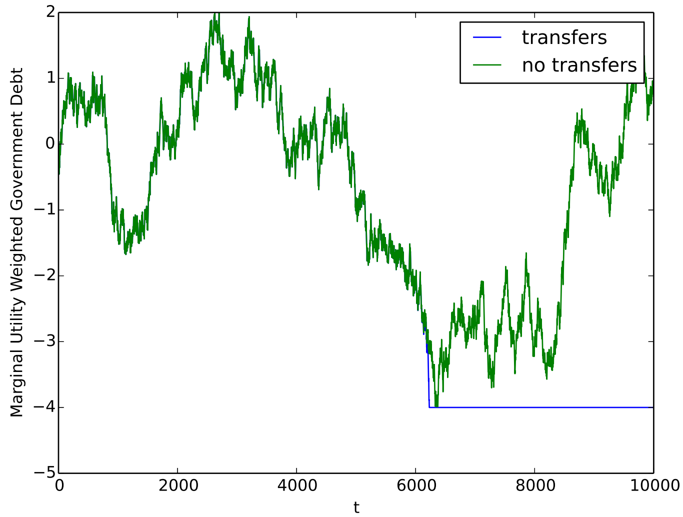

Figure 6: Quasilinear preferences and risk-free bond with and without nonnegative transfers

in that case, outcomes diverge from what Barro had asserted: Aiyagari et al. showed that if the government has access to nonnegative *transfers*, then eventually the government acquires large enough claims on the private sector to set the flat rate tax on labor to zero and to finance all expenditures from earnings on its assets. Aiyagari et al. were able to say much less about debt dynamics when preferences were not quasi-linear.

In this paper, we have gotten much further and evidently drawn at least the *limiting* aspects of optimal debt dynamics in an incomplete markets markets economy closer to outcomes prevailing in a Lucas-Stokey complete markets economy. There exist levels of government debt that let fluctuating returns on government debt – delivered either through fluctuating interest rates (when preferences show risk-aversion in consumption) and also possibly through fluctuations generated by random payoffs in the single risky security that we allow the government to trade – that give the government sufficient access to most of the risk-sharing that Lucas and Stokey stress as an important aspect of optimal taxation. For a wide range of economies, government debt is drawn toward that level of government debt (or assets), albeit at a rate that can be very slow. That slow rate of convergence that is possibly inherited from Barro's unit-root intuition. [tjs]Anmol and David XXXXX: this needs qualification and refinement

That interest rate fluctuations are a mechanism allowing a fiscal authority to hedge risks is a theme that plays an important role in contributions by Faraglia et al. (2012), Berndt et al. (2012). That avenue also is active in Bhandari et al. (2013), though what matters there are not government debt dynamics themselves but rather the dynamics of the relative debt positions of private agents one to another.

tjs David and Anmol XXXXX: some housekeeping things to do.

- The automatic figure numbering isn't working. Someone please look into it.
- The proofs of the lemmas should be included in the promised appendix.
- I have to write a concluding section and also polish and finish the introduction.
- Add a short appendix on how Bellman equations were solved numerically. Pat selves on back for doing so and display some policy functions – think of one or two things to do with those policy functions.
- Think of a couple of experiments that show off the policy function calculations.
- Edit section about turning on nonnegative transfers. x

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