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STRUCTURING LOGICAL SPACE

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It is natural to assume that mathematics is an attempt to discover and describe facts about mathematical phenomena—much like physics, geology and economics are attempts to discover and describe facts about physical, geological and economic phenomena. But it has proven difficult to say what the mathematical facts are, and to explain how our mathematical practice could reliably get at such facts.[1]

The challenge is particularly pressing if we assume that our mathematical theories are largely correct, and that our epistemic capacities are ultimately to be understood in broadly naturalistic terms. So it is not surprising that each of these assumptions has been denied.

Some think we have no good reason to take our mathematical beliefs to be true. Mathematical theorizing can only tell us what things *would be* like if our mathematical beliefs were broadly correct. If our mathematical practice is somehow helpful in inquiry, it is not because of its success at what it sets out to do. Others think we underestimate our cognitive abilities: perhaps we should conclude that we have a non-natural faculty of 'mathematical intuition' that gives us access to the relevant facts. Others yet think we should hold on to these assumptions and reject as unreasonable any demand for a 'philosophical' explanation of the success of our mathematical practice.[2]

But one assumption remains unchallenged: that we have mathematical beliefs (or at least belief-like attitudes of some kind—supposition, make-believe, or what have you.[3]) Few would deny, in other words, that mathematics is an

1 Cf. Benacerraf 1973; Field 1982.

2 Cf. Field 1980, Gödel 1947, and Burgess and Rosen 1997, pp. 45-49 ("the question or challenge is essentially just a demand for a philosophical 'foundation' for common sense and science [...] of the kind that Quine's naturalized epistemology rejects"), respectively. Some varieties of fictionalism (e.g. Yablo 2001) don't fall into either of these categories, but they maintain that mathematical thought should be analyzed in terms of belief-like attitudes.

3 One exception seems to be Bishop Berkeley—cf. Berkeley 1732, VII, §14, as well as the *Treatise*, §20—and perhaps David Hilbert, on some interpretations. For discussion, see Detlefsen 2005.

attempt to discover and describe facts of some kind. My goal in this paper is to provide a principled way of denying just that.

To accept a mathematical theory, on my view, is not to have a belief about some subject matter—at least not if we think of beliefs as essentially attempts to accurately represent some realm of facts. The point of mathematical practice is not to gather a distinctive kind of 'mathematical information'. It is rather to *structure logical space* in an epistemically useful way. When I accept a mathematical theory, I do not change my view on what the world is like—I do not, to use a familiar metaphor, rule out a way the world could be.[4] Instead, I adopt conceptual resources that allow me to make distinctions between ways things could be—to structure the space of possibilities in ways conducive to discovering and understanding what the world is like.

My suggestion is similar in spirit to some *non-cognitivist* views in metaethics. On these views, to think that cannibalism is wrong is not to take a stance on what the world is like. Morality is not about getting at the moral facts—rather, it is about how to live, what to do. Similarly, on my view, to make a mathematical judgment is not to take a stance on what the world is like. Mathematics is not about getting at the mathematical facts—rather, it is about how to structure the space of hypotheses with which we theorize about the world.

Non-cognitivists often offer their view as the best way of making sense of the motivating force of moral judgments. But non-cognitivism can also be seen as a way of dissolving the well-known difficulties of accounting for our moral practice. As with mathematics, these difficulties arise around what is perhaps the central question in moral epistemology: how do we come to know moral facts? The non-cognitivists deny a presupposition of the question, viz. that our moral practice should be understood as involving a relation between ourselves and a realm of moral facts.

I seek to reject a similar presupposition in the metaphysics and epistemology of mathematics. On my view, what needs to be explained is not how we can relate to some realm of mathematical facts, nor how our mathematical practice can reliably reflect what goes on in a far away realm. What we need is an account of the goals of our mathematical practice that does not make it a mystery how creatures with our interests and abilities could successfully engage in it.

My view is thus a form of *nonfactualism* about mathematical thought, in the following sense: our mathematical theorizing does not aim to discover a

4 Some think that we cannot fully characterize the contents of the attitudes involved in cognition with sets of possible worlds alone—e.g. Soames 1987. I agree. My proposal is an attempt to go beyond the possible-world model in order to give a better picture of our mathematical thought. Cf. fn. 14.

particular sort of fact.[5] A nonfactualist view about mathematical practice may seem like a non-starter: mathematics is perhaps the paradigm of a rationally constrained enterprise. If mathematics is not to be measured up against an independent domain of facts, how else can we explain the discipline of our mathematical theorizing?

To be sure, any reasonable account of mathematical thought must explain how our mathematical theorizing is rationally constrained. But these constraints need not arise out of some putative domain of facts that we are trying to track. On the nonfactualist account I will develop, those constraints arise instead out of our more general epistemic goals. On my view, we should seek mathematical theories that allow us to isolate information about the physical world that is conducive to our knowledge and understanding.

1 PREVIEW

There are two main aspects to our mathematical practice: deducing new claims and accepting new theories. Most everyday mathematics involves deducing new claims from previously accepted ones. But when we set forth the axioms of our theory of arithmetic, we did not deduce them from something else. Nor was deduction what led to the 'discovery' of real numbers, or of permutation groups. We simply took up some new mathematical structure as an object of study. We *accepted a new theory* about this structure.

I want to start by focusing on what we do when we accept a new mathematical theory. I will not discuss deduction until §5. Until then, you can take me to be focusing exclusively on logically omniscient agents. With a logically omniscient being there will be no room for the type of change involved in discovering a new theorem. But there *will* be room, I submit, for the type of change involved in accepting a new theory. It is perfectly coherent to imagine a logically omniscient being who does not know anything about topology. And when she comes to learn about topology—by reading about it in a book, say—there will be a distinctive change in her cognitive state. That is the type of change I will focus on until §5.

Of course, a full account of mathematical thought must explain the cognitive accomplishment involved in proving a particular theorem (e.g. that there

5 It is worth reiterating that this is primarily a claim about mathematical *thought*. Nonfactualism is often characterized as a semantic thesis (as in e.g. Boghossian 1989). But if one subscribes, as I do, to the view that mental content is prior to linguistic content—a claim I cannot defend here—the question of whether a particular fragment of language should be understood in nonfactualist terms is ultimately a question of how best to understand the relevant mental states. So I think the interesting question is whether or not factualism about mathematical *thought* is correct.

are infinitely many primes, or that every set is smaller than its power set). But it would be a mistake to try to do so in isolation. After all, talk of 'discovering that there are infinitely many primes' only makes sense against the background of a large body of arithmetical assumptions—a mathematical theory. We first need an account of what it is to accept a mathematical theory before we can say what it is to draw a logical consequence from that theory.

Here is what I will do. I will begin (§2) by isolating an important role that mathematical theorizing plays in our cognitive economy. I will use that in §3 to build an account of what it is to accept a mathematical theory. I will show how the account differs from one on which we have mathematical beliefs in the ordinary sense. This will lead to an account of the cognitive utility of mathematics, and of how rationality constrains our mathematical theorizing even if we are not aiming to track some putative domain of facts (§4). I will turn to the question of deductive reasoning in §5. Before concluding, I will list what I take to be the most pressing outstanding issues (§6).

My goal here is to sketch an alternative to factualist accounts of mathematics. I will not be arguing against factualist accounts directly. In part, this is because that would take us too far afield. But more importantly, this is because we can only make a choice between factualism and nonfactualism once the nonfactualist alternative is on the table. To my knowledge, no such alternative has been developed in any detail. I hope to change that here.

2 MATHEMATICS AS A SOURCE OF CONCEPTUAL RESOURCES

What effect does accepting a new mathematical theory have on our cognitive lives? How is this reflected in our overall mental state?

Here is one uncontroversial, if partial, answer: when we accept a new mathematical theory we gain conceptual resources. We gain the ability to articulate propositions about the concrete world that we would be unable to articulate otherwise.[6]

Consider Newtonian mechanics and the discovery of the calculus, or Quantum mechanics and the discovery of Hilbert spaces. In each case, nontrivial amounts of mathematics are necessary to formulate crucial aspects of the relevant physical theories—theories that make claims about what the world is like.[7]

6 To be sure, accepting a mathematical theory has broader repercussions in one's overall cognitive system. But my hypothesis is that this will all be a consequence of adopting new conceptual resources.

7 A particularly clear example is the second law of motion. Without the calculus, the law couldn't have been formulated. Cf. Friedman 2001, p.35ff for discussion and other related examples in the context of what he calls the 'relativized *a priori*'.

Here is a simpler example:

(1) The number of houses on Elm St is odd.

Whatever your views on number talk, you should agree that (1) tells us something about the concrete world. It is something that would be true if any of the following were true:

> There is exactly one house on Elm St.
>
> There are exactly three houses on Elm St.
>
> There are exactly five houses on Elm St.
>
> ...

What (1) entails about the concrete world might be expressed by the infinite disjunction of all of such claims. Of course, we do not (and could not) have an infinitary language. We are finite beings after all. But with a little bit of mathematics we are able to learn that there is an odd number of houses on Elm St.[8]

Here is a more interesting example.[9] One of Leonhard Euler's most well-known achievements was the solution to the *Königsberg Bridges problem*:

(KB) Is it possible to tour the city of Königsberg (see Fig. 1) crossing each of its seven bridges exactly once, and ending at the starting place?

We can reconstruct his solution in two steps. First, he isolated a proposition about the city—call it *Euler's proposition*. Once understood, this proposition can easily be seen to be true. Second, he proved that Euler's proposition entailed that the answer to (KB) is *no*.

I want to focus on the first step. (I will turn to the second step in §5.) It is a nice illustration of the way in which new mathematical theories improve our conceptual resources.

Let me first introduce a bit of terminology. Think of a *graph* as a collection of points, or *vertices*, connected to each other by one or more *edges*—see Figure 2 for an example.[10] A *path* in a graph is a sequence of vertices and edges,

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8 Some nominalists will object to this—see, e.g. Field 1980. They will insist that our mathematical talk is merely shorthand: we could, if we worked hard enough, express everything we need to express about the concrete world in a finitary non-mathematical language. See Burgess and Rosen 1997 for discussion of the limitations of these reconstructive programs.

9 Cf. Pincock 2007.

10 Formally, we can identify a graph with an ordered triple $\langle V, E, f\rangle$, where V and E are the sets of vertices and edges (respectively), and f is a function assigning to each $e \in E$ a two-membered subset of V, so that $f(e)$ is the set of e's vertices.

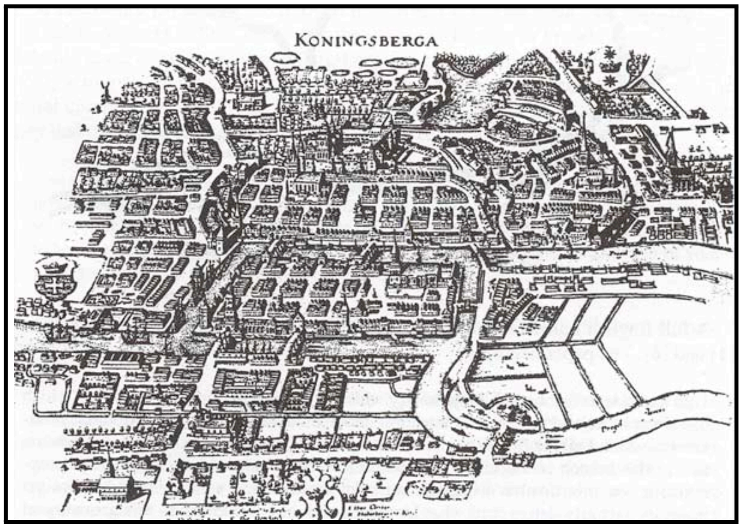

Figure 1: Königsberg ca. 1652.

where each edge is between its two vertices. Call a path containing every edge in the graph exactly once an *Euler path*. An *Euler tour* is an Euler path that starts and ends with the same vertex.

Euler's first insight was that the solution to (KB) depends essentially on whether there is an Euler tour in the graph in Fig. 2 (where the edges represent the bridges, and the vertices the landmasses). He then proceeded to give a

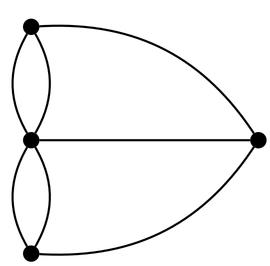

Figure 2: A graph representing the structure of Königsberg.

proof of *Euler's theorem*: that a graph contains an Euler tour if and only if each of its vertices is of even *valence*, where a vertex is of even (resp. odd) *valence* iff it is reached by an even (resp. odd) number of edges.

Introducing this small amount of graph theory allowed Euler to isolate a true proposition about the bridges of Königsberg, what I called 'Euler's proposition', viz.[11]:

11 I am assuming that (EP) is *entirely* about the bridges of Königsberg: it simply claims that they

(EP) The structure of the bridges of Königsberg is a graph at least one of whose vertices is of odd valence.

As you can see by looking at Fig. 2, every vertex in the graph representing Königsberg and its bridges is of *odd* valence. In short, (EP) is true. But this proposition is not just one more truth about the bridges of Königsberg. It is one whose connections to other propositions about the bridges are made apparent because of how it is embedded in the theory of graphs. In particular, given Euler's theorem, it follows from (EP) that the answer to (KB) is *no*. So we have a solution to the Königsberg Bridges problem.

3 ACCEPTING A MATHEMATICAL THEORY

Accepting a mathematical theory can provide us with new conceptual resources. But how? In particular, how can it provide us with *fruitful* conceptual resources?[12]

I suppose the simplest answer is this: because to accept a mathematical theory is to adopt certain conceptual resources. I will now elaborate on this simple answer to give an account of what it is to accept a mathematical theory. But before doing that, we need to answer a preliminary question: what is it to adopt some conceptual resources?

On my view, to adopt new conceptual resources is to make new distinctions among possibilities. Let me explain.

Following Lewis, we can think of the collection of all possibilities as a 'logical' space. A believer, on the Lewisian metaphor, is a traveler trying to locate herself in logical space.[13] So we can think of an agent's belief state as a particular type of map: possibilities compatible with what she believes are spread out all over it. Her goal is to find the point on the map where she is located. When our agent finds out that p, she rules out all those possibilities in which it is not true that p. She thus comes closer to isolating the point on the map corresponding to the way things are.[14]

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are arranged in a particular way (so that there is an isomorphism from the graph in Fig. 2 to the city of Königsberg). But nothing hinges on this. If you think (EP) is not entirely about the concrete world, let 'Euler's proposition' refer to the strongest proposition about the concrete world that (EP) entails.

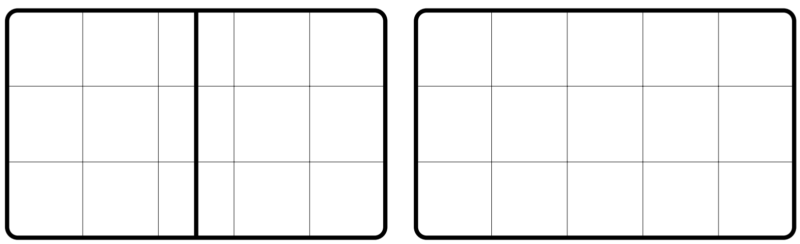
12 This is a version of the problem of accounting for the applicability of mathematics. See Steiner 1998, 2005 for discussion.

13 See, e.g. Lewis 1979a. This metaphor can probably be traced at least back to Wittgenstein, but it is most clearly associated with F. P. Ramsey—see Ramsey 1931.

14 I am assuming that beliefs should be analyzed in terms of epistemic possibilities. I think of these as metaphysically possible worlds, but nothing in what I will say hinges on this.

Some maps are more fine-grained than others. Consider a map that leaves out small streets, such as Carlisle St. Using that map alone, a traveler cannot locate herself to the North of Carlisle St, or to the South of Carlisle St. In other words, the agent cannot use the map to demarcate the region that is North of Carlisle St but South of Cambridge St (say), from the one that is South of Carlisle St but North of Hampshire St.

Likewise with beliefs. We can imagine an agent that cannot locate herself exactly in the region of logical space in which quarks are tiny, perhaps because she has never even heard of quarks before. She is thus unable to wonder whether quarks are tiny: she lacks the conceptual resources to distinguish worlds in which quarks are tiny from those in which they're not. It is only when she acquires the ability to make this distinction—the ability to entertain the proposition that quarks are tiny—that her map of logical space can go from the one in 3(b) to the one in 3(a).[15]



(a) A map that makes the distinction (b) A map that does not

Figure 3: Logical space divided by the proposition that quarks are tiny. Think of each point inside the two rectangles as a possible world. The lines correspond to distinctions between those worlds that are made by the map. The worlds in the right half of each rectangle are those in which quarks are tiny.

On this way of thinking, acquiring new conceptual resources can be identified with being able to make new distinctions among possibilities. And we can think of the distinctions an agent is able to make as the propositions she is able to entertain.[16]

15 Cf. Leuenberger 2004 for this way of thinking about entertainability. For related discussion, see Swanson 2006; Yalcin 2008 and the discussion of 'digital' and 'analog' representation in Dretske 1981.

16 It might seem odd to identify conceptual resources with the ability to entertain some propositions. Although I cannot argue for this here, one can capture a lot of our talk of concepts in an apparatus that starts with propositions rather than concepts as its basic component. Very roughly, the idea is to think of concept possession as a closure condition that mimics Evans' (1982) *generality constraint*: to have the concept F just is a matter of being such that if one is able to entertain the thought that x *is* F, one is therefore able to entertain the thought that y *is* F for any y one is acquainted with.

Note that one can acquire new conceptual resources without *adopting* those resources. Consider an example: if Alice has not heard of quarks, her map of logical space will not make distinctions that depend on how things stand with quarks. But even if she comes to acquire the relevant concepts—even after she acquires the ability to make the distinctions—she may not actually make them: she may not include those propositions among those she takes to be worth gathering evidence for. How things stand with quarks may have no bearing on any question she cares about. So we can represent an agent's conceptual resources by the degree of granularity she is *able* to give to her map of logical space. Which level of granularity she will give to her *working picture of logical space*—those propositions she takes to be worth gathering evidence for, the space of hypotheses that she appeals to for theorizing about the world—will depend on whether she thinks those distinctions are worth making.

I can now give a more fleshed-out formulation of my proposal. Understanding a mathematical theory can increase an agent's conceptual resources. In coming to understand a mathematical theory, one acquires the ability to entertain some propositions. In coming to *accept* a particular mathematical theory, one comes to *adopt* the distinctions given by those propositions for the purposes of theorizing. To accept a new mathematical theory is thus to increase the granularity of one's working picture of logical space. Unlike coming to have a belief, accepting a mathematical theory does not involve eliminating any possibilities. Rather, it involves making new distinctions among possibilities.

To be sure, not all ways of increasing the granularity of one's picture of logical space correspond to the adoption of a new mathematical theory. When I first heard about possums, I acquired the ability to make new distinctions between possibilities—e.g. to distinguish between possibilities in which possums are pests from those in which they are not. Clearly, such a change in my picture of logical space does not involve accepting a new mathematical theory. So we need a principled way of distinguishing the addition of propositions about possums from those additions that do correspond to adopting a new mathematical theory.

Here is a natural suggestion. Mathematics allows us to isolate *structural* features of physical systems. Mathematics gives us ways of carving up logical space where worlds sharing a given structural feature are treated as equivalent. Let me call such propositions *structural propositions*.

It is a difficult question—for reasons that are independent of my view—what a structural feature is. It is thus equally difficult to give an account of structural propositions—structural propositions are those whose truth supervenes on structural features of the world. I cannot provide such an account here. But *very* roughly, a proposition is a structural proposition if its truth depends on the way in which the relevant objects and their parts relate to one another, and not on the identity of the objects themselves.

Some examples might help. An object's shape is a structural feature. A proposition about the spatial arrangement of some objects is arguably a structural proposition. In contrast, the proposition that there are possums in New Zealand is not a structural proposition. For it may be false in a possum-less world in which creatures that are functionally indistinguishable from possums are rampant in New Zealand.

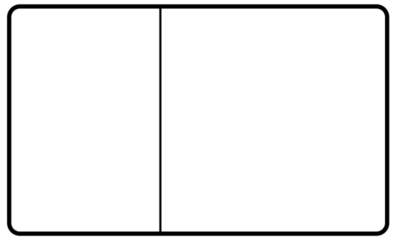
Perhaps more controversially, one might think that propositions whose truth is sensitive to which categorical properties an object has do not count as a structural propositions. See Chalmers 2003; Lewis 2009. Proponents of this view often go on to claim that science only tells us about structural properties of objects. If this turned out to be true, it would follow from my view that the relevant propositions require accepting a substantial amount of mathematics. But that may well be right, given the pervasive role that mathematics plays in the natural sciences. Note however that adopting a physical theory goes beyond adopting the resources provided by that theory: it also involves ruling out those possibilities that are incompatible with what the theory *says* about the world.

Recall the Königsberg example from §2. Euler's proposition is a structural proposition. Its truth does not depend on features of the city of Königsberg other than the relationships between the bridges and the landmasses they connect. In particular, it does not depend on which materials the bridges are made of, nor on which individuals inhabit the city.

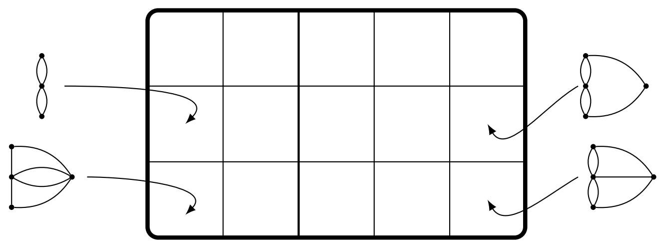
I can now contrast my proposal with other, factualist ones. All parties would agree that there was a change in Euler's cognitive state when he discovered graph-theoretic structures. He was now able to see the bridges of Königsberg as instantiating a particular graph-theoretic structure. (Figure 4 is a model of this change.[17])

According to the realist, there are some (epistemic) possibilities that Euler ruled out when he discovered graph theory. (If the realist thinks that graphs exist necessarily, she will say those possibilities are 'metaphysically impossi-

17 Let me flag something. Figure 4 suggests that the change in Euler's cognitive state involves making finer distinctions among possibilities. But it seems intuitively clear that one thing gained by the introduction of graph theory was the ability to see possible configurations of the city as having something in common—to abstract away from details of the city. Thus, it might be natural to think of Euler's change as involving some sort of *coarsening* of logical space. Strictly speaking, as a matter of algebraic fact, any addition of a new proposition to one's picture of logical space will be the result of a refinement, even if the epistemic benefit comes from the induced coarsening. See §5 for further discussion.



(a) Possible configurations of the city, divided by the proposition that the answer to (KB) is 'no'.



(b) Possibilities further classified according to the graph-theoretic properties of the bridges. Those in the same cell agree on what the bridges' structure is.

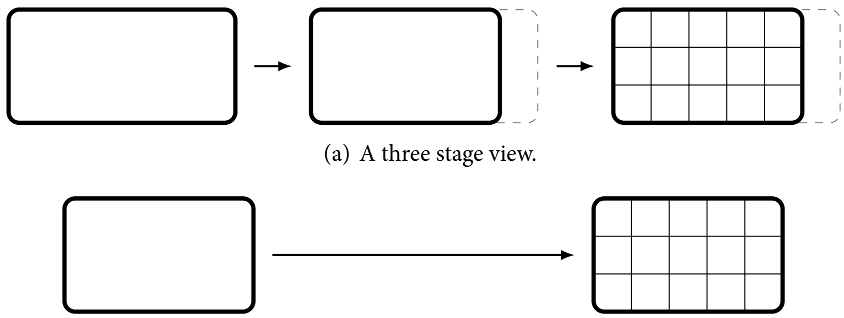
Figure 4: Two stages in the overall change of Euler's cognitive state. After discovering graphs, Euler acquired the ability to classify possibilities as in (b). Only after being able to classify possibilities this way did Euler gain the ability to entertain Euler's proposition.

ble'.) On her view, the change depicted in Fig. 4 was not immediate: it involved first eliminating those possibilities and *then* using the newly acquired beliefs to see the city of Königsberg as instantiating the graph in Fig. 2. (A realist need not think that these changes take place 'one at a time'. The point is that, on her picture, we can conceptually pull them apart.)

The fictionalist, on the other hand, would perhaps insist that Euler first learned something about some non-actualized possibilities—in which, contrary to fact, our mathematical theories are true. Still, possibilities are being ruled out according to whether they make true counterfactuals of the form 'if the fiction were fact, then *p*'. Euler then used these newly acquired beliefs in order to see the city of Königsberg as having the structure of the graph in Fig. 2.

Both the realist's and the fictionalist's accounts are thus versions of a *three-stage view*. In contrast, on my proposal the change corresponding to the newly acquired beliefs about graphs does not involve ruling out possibilities of any kind. Rather, it just involves undergoing the change depicted in Fig. 4. (See

Fig. 5 for a contrast between simplified versions of the story according to each of these views.)



(a) A three stage view.

(b) A two stage view

Figure 5: Two contrasting accounts of the change in Euler's cognitive state. Here, the grayed out areas correspond to possibilities that have been ruled out. On factualist models, Euler's change occurs in three stages. He first proceeds to eliminate some possibilities (those in which there are no graphs, perhaps), and then puts those new beliefs to use in classifying the different configurations of the bridges. On a nonfactualist model, the change does not involve eliminating any possibilities.

The distinctions corresponding to differences in the graph-theoretic structure of the bridges of Königsberg are not the only ones that can be made with the introduction of graph theory. We can ascribe a particular graph-theoretic structure to the bridges of *any* city we are familiar with. To understand the theory of graphs is to be able to put a variety of related distinctions to use in one's epistemic endeavors, and to be able to draw connections between those distinctions.

Say that a physical system lends itself to graph-theoretic interpretation if it has the following features:

- There is an assignment pairing each non-logical expression in the language of graph-theory to a property or relation of the right type that figures in the system; and

- when the language is interpreted so that each non-logical expression is assigned the corresponding property in the system, the axioms of the theory of graphs are true.

In other words, a physical system lends itself to graph-theoretic interpretation if it can be seen as an interpretation—in the standard, model-theoretic sense—of the axioms of graph-theory.[18] In discovering the theory of graphs, Euler acquired the ability to ascribe graph-theoretic structure to any physical system that lends itself to a graph-theoretic interpretation. Less precisely, though more vividly: Euler acquired the ability to *see* physical systems as graph-theoretic structures.

This observation can be generalized to many mathematical theories. Consider the gain in conceptual resources when an agent manages to see physical systems as interpretations of our language of arithmetic:[19] to take a simple example, she acquired the ability to see any group of three pebbles as having something in common with all and only all groups of $2n + 1$ pebbles for any n (viz. as being a *odd* number of pebbles).[20] Similarly, consider the abilities an agent gains by understanding the calculus: e.g., the ability to see different sets of data points as being instances of the same function, or as being all generated by polynomials; the ability to place the claim that a system evolves in a continuous manner within a larger network of relevant claims about theoretically interesting properties of the system.

But we need not assume that mathematical theories were developed with applications in mind. For acquiring the ability to *see* the bridges of Königsberg as a graph-theoretic structure can be done without having this (or any other) application in mind. Very roughly, accepting a mathematical theory is tantamount to acquiring the disposition to apply that theory in suitable circumstances, whether or not one goes on to do so.

The natural question to ask is whether this proposal can be generalized to all mathematical theories. This is no doubt a difficult question. A full answer is beyond the scope of this paper. But in §6 I will briefly sketch what I take to be a promising path.

4 CONSTRAINTS ON MATHEMATICAL THEORIZING

Our theorizing about the concrete world is constrained by the facts—by what the world is like. I have proposed that our mathematical theorizing does not involve a relation between ourselves and a realm of mathematical facts. What then constraints our mathematical theorizing? How can we evaluate mathematical theories from an *epistemic* point of view?

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18 Note that it is straightforward to make room for partial interpretations—not every bit of the mathematical vocabulary needs to be given a physical interpretation.

19 Note that for this to happen they need not have had anything like *our* language of arithmetic.

20 More generally, for any mathematical structure and any physical system, we can ask whether there is a (partial) isomorphism from the mathematical structure to the physical system. The propositions generated by M will be the closure under Boolean operations of all propositions of the form 'S is a physical system isomorphic to M'.

On my account, to accept a mathematical theory is to modify one's working hypothesis space—by making new distinctions, or by abstracting away from others. In other words, it is to take on a particular way of carving up logical space for theorizing about the world. Given that good inquiry is partly a matter of formulating the right hypotheses, one's *epistemic* goals must constrain which way of carving up possibilities one should adopt—and thus, which mathematical theories one should adopt.[21]

We can say more. Which propositions make up one's picture of logical space will constrain which beliefs one will come to have. While different yet compatible sets of beliefs may not differ in how many truths they contain, they may differ in other epistemically significant ways. In particular, which propositions are included in one's system of beliefs will constrain what kind of explanations one can provide. This is not a matter of how many truths a system of beliefs contains: two systems of beliefs that are equally accurate may differ in the type of explanations they can provide.

Take Putnam's famous example.[22] Alice has a small board in front of her with two holes on it, *A* and *B*. *A* is a circle one inch in diameter; *B* is a square one inch in height. She tries to get a cubical peg slightly less than one inch high to go through each hole. Alice believes that the peg can pass through hole *B*, but not through hole *A*.

Compare two different systems of beliefs that could be Alice's. One contains a true description of the microphysical structure of the system consisting of the board and the peg. It also contains a list of the laws of particle mechanics. These, we can suppose, entail that given the microphysical structure of the system, the peg cannot pass through hole *A*, but can pass through hole *B*.

The second system of beliefs takes no stance on what the microphysical structure of the system is. However, it contains the proposition that the peg is cube-shaped, the proposition that hole *A* is round, and the proposition that hole *B* is square. These three (true) propositions in turn entail that the peg cannot pass through hole *A*, but can pass through hole *B*.

We can assume that the two systems of beliefs do not differ in any other significant respect. In particular, they do not differ in how many true propositions they contain. Yet we know enough to see that the second system of beliefs is superior to the first in at least one respect: it allows for a better explanation of why the peg can pass through hole *B*, but not through hole *A*.

21 Cf. Bromberger 1966, 1988. Frege makes vivid the importance of drawing new boundaries in inquiry: "The more fruitful type of definition is a matter of *drawing boundary lines that were not previously given at all*. What we shall be able to infer from it, cannot be inspected in advance; here we are not simply taking out of the box again what we have just put in. The conclusions we draw from it *extend our knowledge* [...]." (Frege 1884, §88, my emphasis)

22 Putnam 1975.

This is not to say that the second system of beliefs is better than the first *all things considered*. But it should be uncontroversial that there is one *pro tanto* reason for preferring it, epistemically, to the first. When evaluating systems of beliefs, we need to look not only at the accuracy of each system, but also at *which* propositions the system includes.

We can return to Euler's example to illustrate this point as well: the addition of graph-theoretic resources is a non-trivial cognitive achievement, one that led to an epistemic improvement in Euler's system of beliefs. Euler used Euler's proposition to explain why one cannot tour the city of Königsberg by crossing each of its bridges exactly once. Now, he could have in principle explained this in terms of the microphysical structure of the city of Königsberg. But even for someone with the computational resources to understand that explanation, the one in terms of Euler's proposition is better for two closely related reasons.[23]

First, the explanation in terms of Euler's proposition is more general: it can apply to a wider variety of cases. General explanations tend to be more satisfactory and thus can be expected to have a high explanatory value.[24] This is why explaining my opening the door by appealing to the claim that someone knocked on it seems more satisfactory than the explanation that appeals to the fact that Tom knocked on it.

Second, the explanation in terms of Euler's proposition manages to abstract away from prima facie *irrelevant* features of the city of Königsberg. *Ex ante*, we are inclined to think that small changes in the microphysical structure of the city of Königsberg would not affect the answer to (KB). Explanations that rely on what we take to be inessential details are worse than those that do not. (This is why appealing to beliefs and desires to explain my behavior can be more satisfying than giving a full account of my brain state.[25])

The explanation in terms of Euler's proposition has these virtues because Euler's proposition is a structural proposition. We can expect explanations in terms of structural propositions to have these explanatory virtues. So if what we are after is an increase in valuable explanatory resources, it is worth taking on the expansions that correspond to accepting a mathematical theory.

Now, this does not give us a story about why we have accepted the math-

23 More carefully: it is a better explanation for the particular explanatory task at hand. But the two features of the explanation in terms of Euler's propositions that I will go on to discuss tend to make for good explanations more generally, at least given the kinds of things we want to explain.

24 Highly disjunctive explanations may be the exception—so not *any* way of weakening the explanans leads to good explanations. It is not clear why.

25 Cf. Jackson and Pettit 1988. See also Garfinkel 1981; Strevens 2004, and the discussion of *stability* in White 2005.

ematical theories we actually have. But it shows how one can have principled ways of evaluating distinct expansions of one's picture of logical space, and how these can be seen as arising out of our epistemic goals.[26] More importantly, it gives us the beginnings of an explanation of how creatures with goals and interests like our own could have developed something like our mathematical practice. To have something like our mathematical practice is essentially to have a picture of logical space rich in structural propositions.[27] If our picture of logical space evolved partly by trying to acquire propositions with high explanatory value, it is not surprising that we came to have something like our mathematical practice.

Increasing explanatory resources is not the only goal that our mathematical theories can help us meet. They also allow us to systematize data and make predictions that would be obscured by irrelevant details. More generally, mathematical theorizing can sometimes provide us with helpful computational resources. The example of the bridges of Königsberg shows that much. The discovery of graph theory was crucial for providing an explanation for why the answer to (KB) is what it is. But it was also crucial for proving *that* the answer to (KB) was *no*. To understand how accepting a mathematical theory can play such a role, we need to say something about the role of deductive reasoning in mathematics.

5 DEDUCTION

Suppose I am right that to accept a mathematical theory is to add structural propositions to one's working picture of logical space. How does deductive reasoning work on this picture? In particular, what is it to accept a logical consequence of a theory one accepts?

To answer these questions, I will first introduce an abstract framework for thinking about deductive reasoning for factual beliefs.[28] I will build on this framework to sketch an account of deductive reasoning for mathematical thought.

26 In Pérez Carballo 2014 I examine this question in more detail. By placing the discussion within a general framework of rational dynamics—on which rational epistemic change involves maximizing expected *epistemic* utility—I argue that one can make sense of expansions that are epistemically rational. The key claim is that expansions can lead to epistemic states that are more stable, and that epistemic utility maximizers seek to increase the stability of their epistemic states.

27 Note that, on my view, mathematics is not a tool we could in principle dispense with if only we could access those structural propositions 'directly', as it were. Thanks here to an anonymous reviewer.

28 Cf. Lewis 1982; Powers 1978; Stalnaker 1991, *inter alia*.

More often than not, our beliefs are not deductively closed. David Lewis tells the story of how he used to think that "Nassau Street ran roughly east-west; that the railroad nearby ran roughly north-south; and that the two were roughly parallel."[29] While Lewis did believe all these things, it would be a stretch to say that he believed their conjunction—an obvious inconsistency in light of his background beliefs.

Lewis' proposal was to think of his belief corpus as compartmentalized: rather than thinking of his actions as governed by one inconsistent body of beliefs, we should think of them as governed by distinct bodies of beliefs in different contexts. In some contexts, his actions were guided by the belief that Nassau St (and the railroad) ran roughly north-south. Perhaps when asked where north was, while on Nassau St, he would point in a direction parallel to it. In other contexts, his actions were guided by the belief that the railroad ran roughly east-west. Perhaps when asked where north was, while on the train, he would point in a direction perpendicular to the tracks.

The moral is that we should think of all agents who appear to have inconsistent beliefs as having distinct consistent fragments that are incompatible with each other.[30]

On this view, failures of logical omniscience are due to fragmentation. Lewis believes that Nassau St runs roughly north-south, that Nassau St and the railroad run roughly parallel to each other, but he fails to believe that the railroad runs roughly north-south. He has two fragmented bodies of belief: one of them includes the proposition that the railroads run roughly north-south, the other one doesn't. When railroads are under discussion, the fragment that does not have the railroad tracks running north-south is active. This fragment gives an answer to the question that is incompatible with the one the other fragment gives. Thus, we have a case of intuitively inconsistent beliefs. But we can imagine a small variant of the case, where the fragment activated for the purposes of discussing railroads in New Jersey is simply undecided as to whether the railroad runs roughly north-south. The moral is that when an agent believes p but fails to believe q, even though p implies q, we should model her belief state by two fragments. According to one of the fragments, p is true; according to the other, neither p nor its negation is. The point of deductive inquiry is (partly) to aggregate one's belief fragments.[31]

29 Lewis 1982, p. 436.

30 See Stalnaker 1984, ch. 5 for another version of this suggestion.

31 In light of general results in social choice theory and the theory of judgment aggregation, it is safe to conclude that there will be no easy answer to how exactly such aggregation should proceed. (See e.g. List 2008.) But this should not come as a surprise. As Gilbert Harman has often pointed out, after realizing that p follows from q an agent who believed p can either come to believe q or instead abandon her belief in p. It is an open question which option she will take.

This way of thinking about deduction raises a host of difficult questions that are beyond the scope of this paper.[32] But it is a promising strategy for thinking about deductive inquiry that is motivated by a natural way of understanding the phenomena. In what follows, I will assume that it is on the right track. I want to build on this model to give an account of deductive reasoning in mathematics that is compatible with my proposal.

On the view I favor, fragments can differ not only in what worlds they take to be possible, but also in how those worlds are carved up—in other words, in what propositions make up the space of hypotheses of each of the fragments.[33] So fragments can differ not only in what factual beliefs they include, but also in what mathematical theories they accept—for accepting a mathematical theory is a matter of carving up logical space in a particular way.

Now, one reason fragmentation is attractive in the case of straightforward beliefs is that, while each fragment is perfectly consistent, they are in conflict with each other. An agent who has contradictory beliefs is to some extent defective. Modeling her cognitive state by a fragmented belief system captures a sense in which her beliefs are somehow defective: the fragments are inconsistent with each other. There is a lack of unity in her picture of the world.

But consider an agent who accepts two inconsistent mathematical theories. I am suggesting we model her belief state by two fragments: each one would be partitioned by the conceptual resources generated by one of the theories. Again, an agent who accepts inconsistent mathematical theories is in some way defective. Yet it is hard to see what conflict there could be between two fragments that divide the same set of possibilities in different ways. Why not think of the agent as having one belief system consisting of the given set of possibilities and containing each of the propositions available at either fragment?[34]

Perhaps our agent makes use of distinct hypothesis spaces in different contexts. But while this might serve as a motivation for thinking of different partitions of logical space as belonging to different fragments, it is not enough to do justice to the phenomena. When an agent's descriptive beliefs are inconsistent, we are often *forced* to treat her system of beliefs as fragmented. The different fragments are in genuine conflict with each other. Assume we posit fragmentation because our agent uses different hypotheses spaces in different

Cf. Harman 1986 on the distinction between inference and implication.

32 In particular, we need to get clear on how fragments are to be individuated.

33 See Yalcin 2008 on how partition-sensitivity can be used to motivate this model of deductive reasoning.

34 This worry is related, although subtly distinct from, the so-called *negation problem* for metaethical expressivism. See Dreier 2006; Schroeder 2008 for discussion and references.

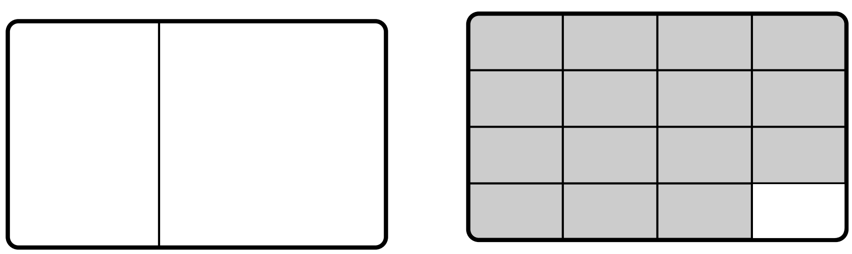
contexts, but suppose both fragments agree on which worlds are possible. Can we nevertheless claim that the two fragments are in conflict with each other?

One hypothesis is that the conflict arises out of limitations in an agent's cognitive resources. Our agent may be unable to incorporate the two partitions into an all-purpose one. But there is a deeper reason why different fragments may be in conflict even when they agree on what worlds are possible.

To see that, let me go back to an aspect of the Königsberg bridges example that I set aside in §2. Recall that Euler's solution could be split in two steps: first, the introduction of graph-theoretic resources; second, the realization that the answer to (KB) is *no*. The second step is where deduction comes in.

Plausibly, Euler already knew that each landmass in the city of Königsberg was reached by an odd number of bridges. And this, we now know, entails that the answer to (KB) is no. So we have a simple case of failure of deductive closure, one to which we can apply fragmentation.

We can think of the two fragments as in Figure 6. The first is carved up by the two answers to (KB), and contains possibilities corresponding to each answer. The second is carved up by the answers to the question: which landmass is reached by an even number of bridges? Here, we can assume that the only answer compatible with Euler's beliefs is *none*.



(a) Possible configurations of the city, divided by the proposition that the answer to (KB) is 'no'.

(b) Possibilities classified according to which landmasses are reached by an even number of bridges.

Figure 6: Two fragments of Euler's cognitive state. The only answer to (KB) compatible with the fragment in (b) is *no*, since according to it the proposition that no landmass is reached by an even number of bridges (the one not grayed out in the figure) is true. The fragment in (a), in contrast, does not settle (KB).

At first, Euler was unable to use his knowledge about how many landmasses are reached by an even number of bridges to answer (KB). This can be represented by a fragmented belief state. In this case, different possibilities are compatible with each fragment. Possibilities in which all landmasses are reached by an even number of bridges are compatible with one of the fragments (the one that is carved up by the answers to (KB)), but incompatible with the other.

But go back to a point *before* Euler realized that each landmass is reached by an odd number of bridges. We can assume, to make things simpler, that his non-mathematical beliefs were deductively closed. He did not know what the answer to (KB) was, but he also did not know how many bridges reached each landmass in the city. Nevertheless, I submit, we should model his cognitive system as fragmented. For he was disposed to have a fragmented belief state: he was disposed to form the belief that there was an odd number of bridges reaching each of the landmasses *without* forming the belief that the answer to (KB) was *no*. And this is because he was unable to use evidence that could settle the questions carving up one fragment—is each landmass reached by an odd number of bridges?—in order to answer the questions carving up the other—viz., (KB). He was unable to see how hypotheses from the two fragments relate to one another.

Deductive reasoning can eliminate inconsistencies in one's descriptive beliefs. But it can also improve one's *information transfer abilities*. An agent whose descriptive beliefs are in conflict with each other hasn't transferred information from one fragment to another. An agent whose mathematical views are inconsistent is *disposed* to be in that situation. This is because she hasn't acquired the ability to use evidence settling one question to answer another, logically related one.

Note that once we model things this way, we can see how the discovery of graph-theory could have helped with determining that the answer to (KB) is *no*. Suppose you could partition logical space in such a way that it was easy to tell (i) which cell of that partition the actual world belongs to and (ii) what the answer to (KB) was, given the answer to (i). The coarse partition given by (KB) itself makes (ii) trivial, but is of no help with (i). The fine partition given by detailed descriptions of the city might make (i) easy, but not (ii). Euler's accomplishment in introducing the theory of graphs was to provide a partition meeting both (i) and (ii).

Indeed, we can think of this partition as mediating the transition from the one fragment in Figure 6 to the other. This is the cognitive accomplishment involved in the proof of Euler's theorem: connecting the proposition that an odd number of bridges reaches each landmass in Königsberg, through Euler's proposition, to the proposition that the answer to (KB) is *no*. The transition from the former to Euler's proposition, and that from Euler's proposition to the latter can each be seen as simpler, more immediate ones (cf. Figure 7). Euler's lack of logical omniscience was manifested in his inability, before proving the theorem, to transfer information from the fragment triggered by the question 'how many landmasses are reached by an odd number of bridges?' to the fragment triggered by (KB).

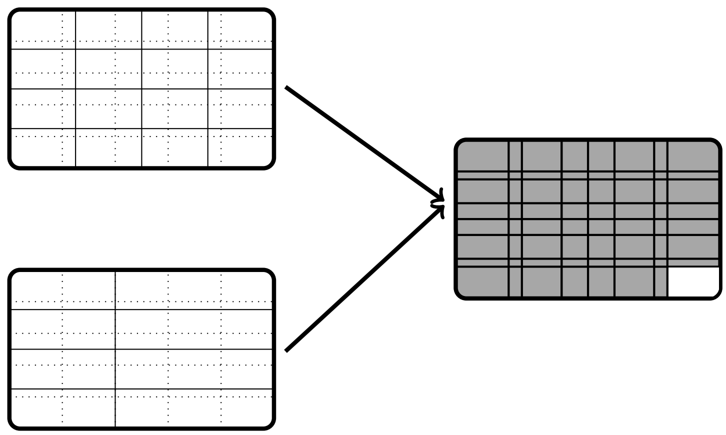


Figure 7: Refinements can be thought of as helping to calibrate two fragments. By arriving at a common refinement of two fragments, it is easier to see how cells in each fragment relate to cells in the other.

The interaction between fragmentation and my account of mathematical thought can thus be used to illuminate the way in which mathematical theorizing can increase our computational resources. But it can also be used to sketch an account of the role of deductive reasoning in mathematical thought.

6 OUTSTANDING ISSUES

Many issues remain outstanding. Here are two of the most pressing ones.

First, is my proposal compatible with a plausible semantics for mathematical language? To give a full account of mathematical practice we certainly need to give a compositional semantics for the relevant fragments of our language. On my view this needn't be the first step: a different starting point can give us a more illuminating theory. Formal semantics is often seen as a neutral ground on which disputes about the nature of mathematical practice should take place. But I see no reason why semantics should be the royal road to understanding our mathematical practice. The complexity of this practice goes beyond anything that can be explained by giving a semantics for a fragment of natural language. And focusing on the details of a compositional semantics might make us lose perspective.

If I am right, the goal of mathematical practice is to arrive at new ways of carving up logical space. But we engage in inquiry as a community, and we need to agree on how to carve up logical space in order to communicate with each other. It would be surprising if we did not have some way of fostering such coordination. A language is well-suited to do this: it is a device for expressing our mental states and trying to arrive at some common state. The difficulty is to give a detailed story of how *our* mathematical language could be understood as playing that role. I suspect that recent work done on nonfactualism—in particular (Gibbard 1990, 2003)—will be helpful for sketching a semantics for a fragment of our mathematical language. However, this is a task for some other time.[35]

That said, I should note that there are general reasons for thinking that giving a semantics compatible with my view will prove much less difficult than it might seem. On my view, an assignment of semantic values to sentences places fewer constraints on our theory of the relevant mental states than is usually supposed. All it tells us is that the algebra one uses to provide the semantics for a language is isomorphic to the algebra one uses to characterize the inferential relations among the relevant mental states, and the way these states evolve during a conversation. And the crucial point is that there is more than one way of understanding the role of the algebra that we use to characterize our mental states.[36]

For example, on a conventional interpretation, for a state s to be weaker (in the sense of the algebra) than state s' is for the *truth-conditional content* of s' to entail the *truth-conditional content* of s. On my interpretation, in contrast, for s to be weaker than s' is for s' to deploy richer conceptual resources than s. The upshot is that we can give a rather non-revisionary semantics, along the lines I gesture at in the appendix, for the language of mathematics. And we can do so without assuming that to accept a mathematical theory is to take a stance on what the world is like.

A second outstanding issue is whether my proposal can be generalized to deal with mathematical theories that are more abstract—theories involving large cardinal axioms, say, or theories that seem non-applicable to the natural sciences. I don't have an answer to this question, yet. But here is one line of thought worth exploring. In the same way that accepting 'lower-level' mathematical theories can be seen as adopting ways of making new distinctions among ordinary, descriptive propositions, accepting 'higher-level' mathematical theories can be seen as making new distinctions among lower-level theories. We can motivate this strategy by noting that more abstract branches of mathematics often arise out of reflection on more 'concrete' mathematical

35 It is not hard to sketch such a semantics based on the notion of *scorekeeping*, much along the lines of Lewis 1979b and Stalnaker 1973. The main observation—which I develop in the appendix—is that we can easily represent the evolution of the part of the score corresponding to the partition presupposed by conversational participants as proceeding by 'elimination' of alternatives.

36 Cf. Pérez Carballo forthcoming.

theories. This gives us a hierarchical picture, on which the kind of distinctions we make at one level will be constrained by the theories we accept at higher levels.[37]

7 CONCLUSION

Most attempts at making sense of our discursive practices proceed in full generality. They ask what asserting a declarative sentence is, or what judging that so-and-so amounts to. A less ambitious strategy—favored by metaethical expressivists—is to leave open the possibility of treating different domains of discourse differently. The strategy is to ask what it is to make a *moral* judgment, or a *mathematical* judgment, not simply what it is to judge that so-and-so. This is the strategy I have adopted. It would be nice if we could say something general about all the different aspects of inquiry. Reflection on our discursive practices suggests that they have much in common. But this is not a nonnegotiable constraint on the project.

In this paper, I sketched a novel account of mathematical practice. On this account, to discover a new mathematical theory (or structure) is not to acquire a new belief. Rather, it is to change the granularity of one's working picture of logical space—in other words, to change one's working hypothesis space. Discovering a new mathematical theory involves acquiring the ability to *see* any possible physical configuration as a potential instance of the theory.

The picture that emerges from my proposal is a form of nonfactualism. But it is one that can account for the ways in which our concern for the truth imposes substantial constraints on our mathematical theorizing. For how best to structure our inquiry into the physical world will depend on what our epistemic goals are. This opens the door to an account of the rationality of our mathematical practice that is compatible with a plausible picture of our cognitive lives.[38]

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37 E.g., the axiom of constructibility will rule out certain isomorphism-types from the mathematical universe that are available in models of ZFC in which $0^\sharp$ exists. See Maddy 1997, ch. 6 for discussion.

38 Thanks to Alex Byrne, Fabrizio Cariani, Tom Dougherty, Paul Égré, Paolo Santorio, Katia Vavova, and Kenny Walden for many helpful comments and advice. Thanks also to Phil Bricker, Rachael Briggs, Alexi Burgess, Sylvain Bromberger, Nina Emery, David Hills, Chris Meacham, Eliot Michaelson, Vann McGee, Dilip Ninan, Eliot Sober, Seth Yalcin, Roger White, and audiences at the University of Buenos Aires and the Ecole Normale Supérieure for comments on earlier versions of this material. Special thanks to Agustín Rayo, Bob Stalnaker, and Steve Yablo.

A APPENDIX: CONTENT AND SEMANTICS

Distinguish two questions:

(i) What is the functional role of a particular type of mental state?

(ii) What is the most perspicuous way of representing those mental states in a theory of the mind?

I have said something in answer to the first question. But what I have said does very little to constrain how we should answer the second one. Let me explain.

When I want a break, I am in a particular mental state. We might find it convenient to explain what that state *is* by appealing to the functional role it plays. We might say that it is a state that tends to bring it about that I look for an excuse to stop working, or that I get up and walk to the water cooler. This is no doubt a very partial characterization of what state I am in, but we can give a theory of *wants* along these lines.

Yet when deciding how to represent *wants* in our theory of the mind, we might want to use *propositions*. We might want to represent my wanting a break as my being in a particular relation to the proposition that I take a break. But this is not *forced* upon us by our account of what *wants* are. Rather, it is something we do in order to streamline our theory of the mind. For in assigning propositions as objects of my wants, as we do with beliefs, we can make useful generalizations about the way in which our wants and beliefs interact with one another, by looking at the logical relations between the propositions we assign to them.[39]

Sometimes we have additional pressure to represent attitudes as propositions. Consider attitudes like *expectations*. We not only expect rain, we also expect *that* it will rain. It seems natural to represent the latter state as involving a relation to a proposition (the proposition that it will rain), and it would be odd not to do the same with the former.

On my view, mathematical beliefs do not involve eliminating possible worlds. Rather, they involve taking on the conceptual resources that come with the relevant mathematical theory. But our mathematical beliefs relate to one another in much the same way that our ordinary, descriptive beliefs relate to one another. And our best way of thinking about the way in which our ordinary, descriptive beliefs relate to one another involves representing them as relations to propositions. It would be nice, therefore, if we could also

39 This point is nicely made by David Lewis (1979a, §1): "Our attitudes fit into a causal network. In combination, they cause much of our behavior (…). In attempting to systematize what we know about the causal roles of attitudes, we find it necessary to refer to the logical relations among the objects of the attitudes. Those relations will be hard to describe if the assigned objects are miscellaneous."

represent mathematical beliefs using proposition-like objects. For then we would be able to transfer all we know about our theorizing about ordinary, descriptive beliefs to our account of mathematical thought.

Fortunately, a simple trick will let us do so.

A.1 *Content*

Assume for a moment that we can associate with each mathematical structure a partition of the space of possibilities—i.e., a collection of propositions that are mutually exclusive and jointly exhaustive. This will correspond to the smallest set X of propositions such that any proposition that can be entertained given the conceptual resources the theory provides is a Boolean combination of elements of X. For example, the proposition that describes the graph-theoretic structure of the city of Königsberg is one that can be isolated using the graph in Figure 2—we can, as it were, point to that graph and say that the structure of the bridges is like *that*. I will speak of a proposition as being 'generated by a structure' whenever it is one of those built out of the partition associated with that structure.

Now, we can characterize the state of accepting a mathematical theory with a set of mathematical structures: those structures such that they each generate the (structural) propositions that make up our agent's picture of logical space. As we will see, we can think of those structures as being precisely what we normally think of as the *models* of the theory (in the model-theoretic sense).

If our agent comes to accept a claim that is independent of the theory she accepts, she will have narrowed down the set of structures that characterize her mental state—much as in forming a new descriptive belief an agent narrows down the set of worlds she takes to be possible. Which structures will be ruled out? Those that don't allow for the conceptual resources that the new theory generates.

A picture might help. Take a look at Figure 8, and think of i, j and k as standing for different mathematical structures. Each layer in the picture corresponds to a different way of partitioning logical space—$Q(i)$ corresponds to the partition generated by i, and so on. An agent that has not settled on which of $Q(i)$, $Q(j)$ or $Q(k)$ to take on as her working hypothesis space can be modeled by the set containing i, j and k. In particular, such a set would model our agent having adopted p, as part of her hypothesis space, but not yet q nor r. If she went on to include r as part of her picture of logical space, she could be modeled as having ruled out i, since it does not generate r (in other words, r is not a member of $Q(i)$).

Such a change could be represented as the transition from a partition that includes all and only those propositions in each of $Q(i)$, $Q(j)$ and $Q(k)$—

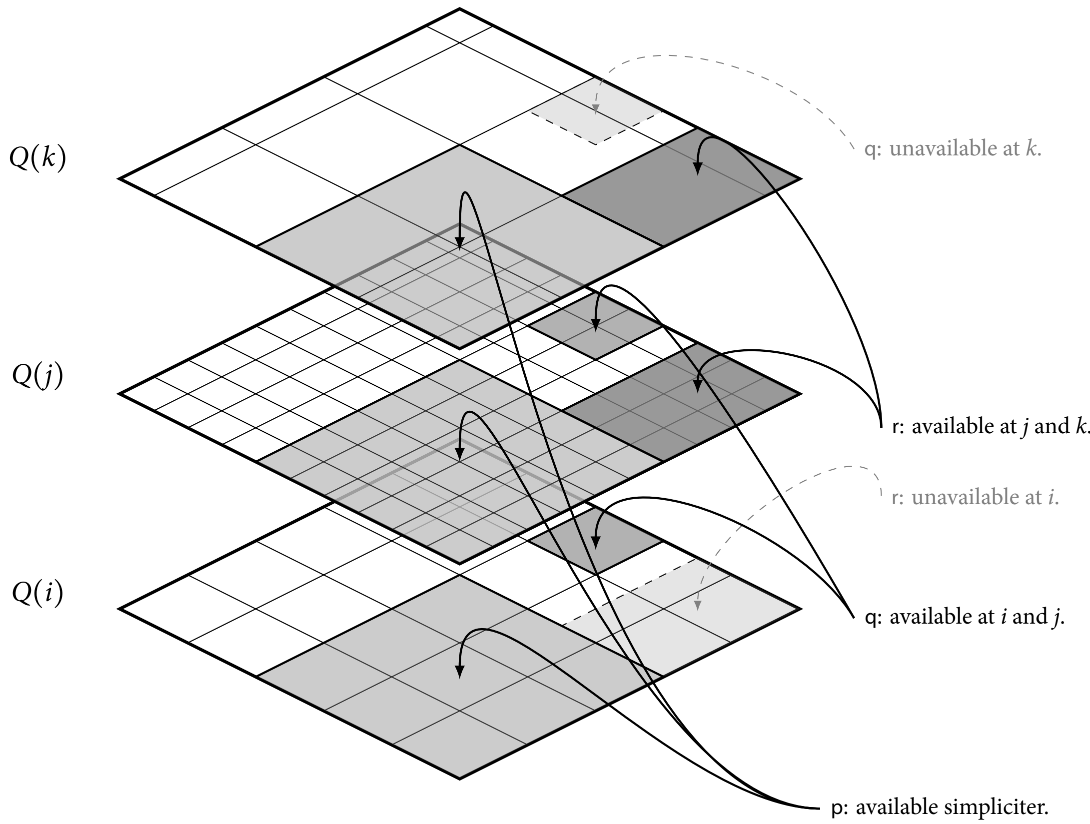
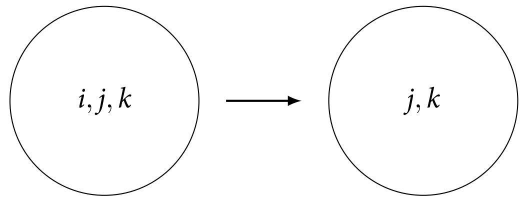


Figure 8: Alternative conceptual resources: This represents an agent that has p among her working hypothesis space, but has not yet taken in either q or r. We can model her coming to take on q as the result of her 'ruling out' *k*.

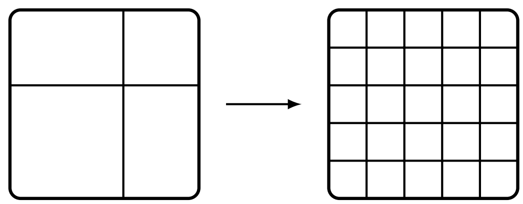
which we can denote by $Q(\{i, j, k\})$—to one that includes all and only those propositions in each of $Q(j)$ and $Q(k)$—which we denote by $Q(\{j, k\})$. Thus, such a change could be seen as an expansion in our agent's conceptual resources that can be represented by the elimination of one mathematical structure. You can look at the contrast in Figure 9 for illustration.

Note the analogy with our representation of changes in an agent's ordinary, descriptive beliefs. When we model an agent as having ruled out a possibility, it is because she comes to believe *more* propositions. Similarly, we can model our agent as ruling out a mathematical structure when she comes to take on *more* propositions as being part of her hypothesis space.

If we want to be able to assign proposition-like objects not only to 'pure'



(a) Ruling out points.



(b) Refining logical space. The transition from $\{i, j, k\}$ to $\{j, k\}$ is used to stand for a transition from $Q(\{i, j, k\})$ to $Q(\{j, k\})$.

Figure 9: Two ways of modeling one change in our agent's cognitive state

mathematical beliefs, but to 'mixed' ones as well, we can simply model an agent's cognitive state as a set of *pairs* $\langle w, i \rangle$,[40] where w is a possibility and i is a mathematical structure used to encode the conceptual resources we ascribe to an agent. Now suppose we represent an agent's cognitive state by a set of pairs H. We can read off from the set H which worlds are compatible with our agent's beliefs *and* which possible world propositions make up her hypothesis space. A possible world proposition A will be in our agent's hypothesis space iff it is in $Q(i)$ whenever i figures in a pair that is in H. A world w will be compatible with her beliefs just in case w figures in some pair that is in H.

To complete this sketch, we need to discharge the initial assumption that there is a good assignment of sets of propositions to each mathematical structure. Here is how. For any mathematical structure and any physical system, we can ask whether there is an isomorphism from the mathematical structure and the physical system.[41] Any proposition (or the negation of a proposition) of the form 'S is a physical system isomorphic to M' can be thought of as generated by M, as will any finite Boolean combination of such propositions. In other words, the set of propositions assigned to a mathematical structure M will be the largest collection of propositions A such that for any two worlds w

40 Cf. Gibbard 1990, 2003.

41 We can also ask whether there is a partial isomorphism, i.e. an isomorphism from a substructure of the given structure to the given physical system.

and w' in A, if a physical system in w is isomorphic to M, so is its counterpart in w'. To take an example, recall our graph in Figure 2: it can be used to generate the (true) proposition that the bridges of the city of Königsberg are isomorphic to it, the (false) proposition that the bridges of the city are not isomorphic to it, the (false) proposition that the bridges of Paris are isomorphic to it, the proposition that the faucets and pipes of a given house are isomorphic to it, and so on.[42]

We can thus use sets of subsets of I to represent what mathematical theories our agent accepts. By assigning to an agent a set $H \subset I$ as the 'content' of her mathematical beliefs, we attribute to them the conceptual resources that correspond to the intersection of those X_i such that $i \in H$. This means that we can use proposition-like objects to represent our agent's mathematical beliefs *even if* to have a belief represented by a set of points H is not a matter of taking a stance on what the world is like, but rather a matter of deploying certain conceptual resources in theorizing.

A.2 *Semantics*

A further advantage of this representation is that it allows us to give an adequate semantics for the language of mathematics.

To give a compositional semantics for a given language is to assigning in a recursive fashion an abstract object to any well-formed sentence in the language. On my view, the main constraint on this project is that these abstract objects can be used to model the effect of an utterance of that sentence in a conversation.[43] So in order to give a semantics for the language of mathematics, we need to answer three questions.

(Q1) What is the effect of an utterance of any such sentence in a conversation?

(Q2) What objects can be used to characterize the effects of utterancess in a conversation?

(Q3) How can we assign the relevant objects to well-formed sentences of the language in a recursive fashion?

I will answer each question in turn. Before I do that, I want to sketch a well-known abstract model of conversation. This will set the stage for what follows.[44]

42 Alternatively: it can be used to generate the proposition consisting of those worlds in which the bridges of Königsberg are isomorphic to the graph in Figure 2, etc.

43 For extended discussion, see Pérez Carballo forthcoming.

44 Cf. Lewis 1979b; Stalnaker 1973.

Think of a conversation as an activity whose purpose is to induce changes in the mental states of the participants. Each stage in the conversation can be characterized by the *conversational score*, which represents the states of mind that speakers take the participants to be in. The effect of an utterance can be captured by a transition rule: a rule that tells us how the conversational score should change if that utterance is accepted. The simplest example of this model is one in which the conversational score is just a set of possible worlds. This set contains those worlds that are taken to be live possibilities for the purpose of the conversation: they are the worlds that represent what speakers take each other to presuppose. To any sentence, we can assign a set of worlds. Uttering any sentence will eliminate, from the conversational score, those worlds not in the set of worlds assigned to that sentence.

The score can be more complex if there are different mental states we want to keep track of. Suppose we have a language that can induce changes in what speakers take each other to believe, and independently induce changes in what speakers take each other to accept as a standard of precision (i.e. whatever determines whether an utterance of a sentence like 'France is hexagonal' is appropriate). We will then want to have a score containing two elements: a set of worlds, and some abstract object that can encode information about what standards of precision are relevant for the conversation.

We can now go back to our three questions. First: how does uttering a sentence with mathematical vocabulary affect a conversation? On my view, to accept a mathematical theory is to adopt a particular way of structuring logical space. The point of uttering a sentence with mathematical vocabulary is in part to induce changes in what way of structuring logical space to accept for the purposes of the conversation—in what the hypothesis space of the conversation is.

What objects can be used to characterize the effect of these utterances? For any mathematical language $\mathcal{L}$, we can use subsets of the collection of $\mathcal{L}$-structures to characterize the hypothesis space of the conversation, using the construction in §A.1. Since the dynamics of refining logical space can be represented by 'narrowing down' that set, we can simply assign a set of mathematical structures to each sentence. Uttering that sentence will rule out the mathematical structures that are not in its associated set.

Finally, how can we assign these objects to our sentences in a systematic way? We can do so just as a descriptivist would. We can proceed by assigning to each sentence in a mathematical language $\mathcal{L}$ the set of $\mathcal{L}$-structures that are models of the sentence. This assignment can be done much in the same way that we do semantics for any first-order language.

It is slightly trickier to give a semantics for *mixed* sentences. But here is one possible way to do so. Start by having our conversational score contain a set of *pairs* of the form $\langle w, i\rangle$ where w is a possible world and i is a mathematical structure. Plausibly we can assign a set of such pairs to each such sentence: the set of $\langle w, i\rangle$ such that there is a (partial) isomorphism between the physical system—in w—that the sentence is about, and the structure i. Again, we can piggy-back on a relatively straightforward assignment of a set of pairs of a world and a mathematical structure to each mixed sentence. We can use these sets in order to represent the effect that any mixed sentence will have on the conversational score, roughly along the lines suggested by the construction in §A.1.

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