Two formulations of the Brier score

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Fix a finite set $X = \{x_1, x_2, \ldots, x_N\}$. For any $A \subseteq X$, let $\mathbb{1}_A(x)$ denote the characteristic function of A. To reduce clutter, we will write $\mathbb{1}_x$ instead of $\mathbb{1}_{\{x\}}$ whenever $x \in X$. Similarly, if P is a probability function defined over $\wp(X)$, we will write $P(x)$ instead of $P(\{x\})$ whenever $x \in X$.

For any probability function P defined over $\wp(X)$ and any $x \in X$, let

$$\beta(P, x) = \sum_{y \in X} (P(y) - \mathbb{1}_x(y))^2$$

$$\beta_F(P, x) = \sum_{A \subseteq W} (P(A) - \mathbb{1}_A(x))^2$$

The purpose of this note is to give a proof of the following observation, due to Jim Joyce:

Theorem 1. *For any P and $x \in X$,*

$$\beta_F(P, x) = 2^{N-2} \cdot \beta(P, x).$$

Preliminaries

First, for the sake of reference, let us list a couple of simple facts that will prove helpful as we proceed.

Fact 2. *For any sequence $\{r_i : 1 \leq i \leq N\}$ of reals,*

$$\left(\sum_i r_i \right)^2 = \sum r_i^2 + \sum_{i \neq j} r_i r_j.$$

Corollary 3. *For any probability function P defined over $\wp(X)$ and any $x \in X$*

$$\beta(P, x) = -\sum_{i \neq j} (P(x_i) - \mathbb{1}_x(x_i))(P(x_j) - \mathbb{1}_x(x_j)).$$

Proof. It suffices to note that for any $x \in X$,

$$\sum_i (P(x_i) - \mathbb{1}_x(x_i)) = (P(x) - 1) + \sum_{x_i \neq x} P(x_i) = -1 + \sum_i P(x_i) = 0.$$

□

Fact 4.

$$\sum_{A\subseteq X}\sum_{\substack{i\neq j\\ x_i,x_j\in A}} x_i x_j = 2^{N-2}\sum_{\substack{i\neq j\\ i,j\in X}} x_i x_j.$$

Proof. Note that

$$\sum_{A\subseteq X}\sum_{\substack{i\neq j\\ x_i,x_j\in A}} x_i x_j = \sum_{i\neq j} M_{i,j}\cdot x_i x_j,$$

where

$$M_{i,j} = |\{A\subseteq X : x_i\in A, x_j\in A\}| = 2^{N-2}.$$

□

Proof of the main result

Our main result follows from Corollary 3 and the following lemma:

Lemma 5. *For any $x\in X$ and any probability function P defined over $\wp(X)$*

$$\beta_F(P,x) = -2^{N-2}\sum_{i\neq j}(P(x_i)-\mathbb{1}_x(x_i))(P(x_j)-\mathbb{1}_x(x_j)).$$

Proof. Start by noting that

$$\beta_F(P,x) = \sum_{A\subseteq X}\left(\sum_{x_i\in A} P(x_i)-\mathbb{1}_x(x_i)\right)^2,$$

which, together with Fact 2 entails that

$$\beta_F(P,x) = \sum_{A\subseteq X}\left(\sum_{x_i\in A}(P(x_i)-\mathbb{1}_x(x_i))^2 \right.$$
$$\left. + \sum_{\substack{i\neq j\\ x_i,x_j\in A}}(P(x_i)-\mathbb{1}_x(x_i))(P(x_j)-\mathbb{1}_x(x_j))\right).$$

Now,

$$\begin{aligned}\sum_{A\subseteq X}\left(\sum_{x_i\in A}(P(x_i)-\mathbb{1}_x(x_i))^2\right) &= \frac{1}{2}\sum_{A\subseteq X}\left(\sum_{i}(P(x_i)-\mathbb{1}_x(x_i))^2\right)\\ &= 2^{N-1}\sum_{i}(P(x_i)-\mathbb{1}_x(x_i)^2).\end{aligned}$$

And thus,

$$\beta_F(P,x) = 2^{N-1}\beta(P,x) + \sum_{A\subseteq X} \sum_{\substack{i\neq j \\ x_i,x_j\in A}} (P(x_i) - \mathbb{1}_x(x_i))(P(x_j) - \mathbb{1}_x(x_j)),$$

which together with Corollary 3 and Fact 4 entails

$$\begin{aligned}\beta_F(P,x) = -2^{N-1}\sum_{i\neq j}(P(x_i) - \mathbb{1}_x(x_i))(P(x_j) - \mathbb{1}_x(x_j)) \\ + 2^{N-2}\sum_{i\neq j}(P(x_i) - \mathbb{1}_x(x_i))(P(x_j) - \mathbb{1}_x(x_j)),\end{aligned}$$

as desired. □