Generalized Immodesty Principles in Epistemic Utility Theory

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According to one of the better known constraints on epistemic utility functions, each probabilistically coherent function should be *immodest* in a particular sense: for any probabilistically coherent credence function P and any alternative $Q \neq P$ to P, the expected epistemic utility of P relative to P should be greater than that of Q relative to P. This constraint, often known as *Strict Propriety*, is usually motivated by appealing to a combination of two independent claims. The first is a certain kind of admissibility principle: that any probabilistically coherent function can sometimes be epistemically rational.[1] The second is an abstract principle linking epistemic utility and rationality: that an epistemically rational credence function should always expect itself to be epistemically better than any of its alternatives.[2] If we assume, as most typically do, that the alternatives to any probabilistically coherent function are all and only those credence functions with the same domain, these two principles arguably entail Strict Propriety.

What happens if we enlarge the class of alternatives to include all a wider range of probability functions, including some with a different domain? This would strengthen the principle linking epistemic utility and rationality: it would no longer suffice, for a credence function to be deemed epistemically rational, that it expects itself to be doing better, epistemically, than credence functions with the same domain. And this stronger principle would arguably give us a more plausible theory of epistemic rationality, at least on some ways of widening the range of alternatives. Suppose an agent with a credence function defined over a collection of propositions takes herself to be doing better, epistemically, than she would be by having another credence function defined over the same collection of propositions. But suppose she thinks she would be doing better, epistemically, having a credence function defined over a smaller collection of propositions—perhaps she thinks she would be doing better, epistemically, not having certain defective concepts and thus that she would be doing better, epistemically, simply

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1 Cf. Joyce 2009, p. 279 on 'Minimal Coherence'.
2 Cf. the principle 'Immodest Dominance' in Pettigrew 2016, p. 24. See also Joyce 2009, p. 280 on what he calls 'Coherent Admissibility'.

not having propositions with those concepts as constituents in the domain of her credence function. Such an agent would seem to be irrational in much the same way as an agent who thinks she would be doing better, epistemically, by assigning different credences to the propositions she assigns credence to.[3]

Now, our interest here is not with the question what is the right principle linking epistemic utility and rationality. Rather, we are interested in understanding how strong a principle we can consistently endorse: we are interested in the kinds of constraints on epistemic utility functions that come from different views on how epistemic utility and epistemic rationality are related to one another. So we start by considering the strongest version of our principle linking epistemic utility and rationality, one that says that an epistemically rational credence function should take itself to be doing better than any other credence function, regardless of its domain. As we will see, the resulting immodesty constraint is far too strong, in that, perhaps surprisingly, it cannot be satisfied by any reasonable epistemic utility function—that this is so is a consequence of our main results (§3.1–§3.2).[4]

We then consider different possible ways of strengthening our principle and study the resulting constraints on epistemic utility functions, their relationship to one another, and establish a few characterization results for the class of epistemic utility functions satisfying these constraints (§3.3). Before concluding, we discuss (§4) how our results relate to recent work on the question whether epistemic utility theory is incompatible with imprecise, or 'mushy', credences.

1 Introduction

Fix a collection W of possible worlds and a finite partition π of W—a collection of pairwise disjoint, jointly exhaustive subsets of W, which we call *cells*.

We say that a real-valued function P defined over π is *coherent* iff for each $s \in \pi$, $P(s) \in [0,1]$, and $\sum_{s \in \pi} P(s) = 1$. A coherent function over π uniquely determines a probability function over the Boolean closure of π. Accordingly, and slightly abusing notation, we will refer to coherent functions over π as *probability functions over π*.[5]

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3 See REDACTED for more discussion.

4 Previous work on related issues include Carr 2015, Pettigrew 2018, Talbot 2019. Unlike those authors, we make very minimal assumptions about the nature of epistemic utility—we do not assume, for example, that epistemic utility functions are simply measures of accuracy (our results are thus independent of whether we endorse the program of 'accuracy first' epistemology), nor that the epistemic utility of a credence function at a world is determined by the epistemic utility of individual credence assignments to propositions (our results do not rely an atomistic (in the sense of Joyce 2009, § 5) conception of epistemic utility). For related discussion, see REDACTED.

5 Since we will be taking Probabilism for granted, we can work with these simplified definitions without loss of generality.

Let $\mathcal{P}_\pi$ denote the collection of probability functions over π. An *epistemic utility function* (*for π*) is a function $\mathfrak{u} : \mathcal{P}_\pi \times W \longrightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ such that for each $P \in \mathcal{P}_\pi$, $\mathfrak{u}(P, \cdot) : W \longrightarrow \overline{\mathbb{R}}$ is π-measurable—where $f : W \longrightarrow \overline{\mathbb{R}}$ is π-measurable iff for each $r \in \overline{\mathbb{R}}$, $\{w : f(w) = r\}$ is in the Boolean closure of π. footnoteThe standard definition of π-measurability of course requires that the preimage of any open set in $\overline{\mathbb{R}}$ be in the Boolean closure of π. But since π is finite, this simpler definition is equivalent to the standard one.

Throughout, we assume that epistemic utility functions are *bounded above* (for each $\mathfrak{u}$ there exists a finite M such that $\mathfrak{u} < M$) and *truth-directed* in the following sense: if for a given $w \in W$ and (i) for any proposition s in π, $P(s)$ is at least as close as $P'(s)$ is to the truth-value of s in w, while (ii) for some proposition s in π, $P(s)$ strictly closer than $P'(s)$ is to s's truth-value in w, then $\mathfrak{u}(P, w) > \mathfrak{u}(P', w)$.[6]

By definition, for fixed $\mathfrak{u}$ and $Q \in \mathcal{P}_\pi$, $\mathfrak{u}(Q, \cdot)$ is a discrete random variable. Accordingly, for $P \in \mathcal{P}_\pi$ we let $\mathbb{E}_P[\mathfrak{u}(Q)] := \mathbb{E}_{X \sim P}[\mathfrak{u}(Q, X)]$ denote the *expectation* of $\mathfrak{u}(Q)$ relative to P, so that

$$\mathbb{E}_P[\mathfrak{u}(Q)] = \sum_{s \in \pi} P(s)\mathfrak{u}(Q, w_s),$$

where $s \mapsto w_s$ is a *choice function*, in that for each w and s, $w_s \in s$. The π-measurability of $\mathfrak{u}(Q, \cdot)$ ensures that our definition does not depend on our choice function. Indeed, we will simply write $\mathfrak{u}(Q, s)$ to denote $\mathfrak{u}(Q, w_s)$, so that $\mathbb{E}_P[\mathfrak{u}(Q)] = \sum_{s \in \pi} P(s)\mathfrak{u}(Q, s)$.

We say that an epistemic utility function $\mathfrak{u}$ for π is *proper* iff for each $P, Q \in \mathcal{P}_\pi$,

$$\mathbb{E}_P[\mathfrak{u}(P)] \geq \mathbb{E}_P[\mathfrak{u}(Q)].$$

We say that $\mathfrak{u}$ is *strictly proper* iff for each $P \neq Q \in \mathcal{P}_\pi$, the above inequality is always strict. A variety of characterization results can be found in the literature—see especially (Gneiting & Raftery 2007).

Strictly proper epistemic utility functions have been the subject of considerable interest. In discussions of how to reward a forecaster's predictions, strictly proper functions are of interest because they reward honesty—someone whose forecasts will be rewarded using a strictly proper epistemic utility function cannot expect to do better than by reporting her true credences (Brier 1950, Savage 1971). In general discussions of epistemic value, strictly proper functions are of interest because they incorporate a certain kind of *immodesty*—if your epistemic values are represented by a strictly proper epistemic utility function and you are

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6 Arguably, epistemic utility functions would need to satisfy additional constraints to count as genuinely epistemic ways of comparing probability functions relative to a given state of the world. For a sense of the wide range of possible constraints, see Joyce 2009.

rational, you will never expect any other credence function to be doing better, epistemically, than your own (Gibbard 2008, Greaves & Wallace 2006, Horowitz 2014, Joyce 2009, *inter alia*). And in discussions of justifications of Probabilism—the requirement on degrees of belief functions that they satisfy the axioms of the probability calculus—strictly proper utility functions have played a starring role in a range of dominance results to the effect that probabilistic credences strictly dominate non-probabilistic credences and are never dominated by any other credence function (Joyce 1998, 2009, Leitgeb & Pettigrew 2010, Pettigrew 2016, Predd et al. 2009).

One natural question to ask is how to generalize the framework of epistemic utility theory to allow for comparisons of probability functions defined over distinct algebras of propositions. And given such a generalization, an equally natural question is how to generalize the notion of (strict) propriety. Let us take each of these questions in turn.

2   Generalizing the Framework

Let Π denote the collection of finite partitions of W. For $\pi, \pi' \in \Pi$, say that π is a *refinement* of π' iff for each $s \in \pi$ there is $s' \in \pi'$ such that $s \subseteq s'$. If π is a refinement of π', we say that π' is a *coarsening* of π. Of course, the refinement relation induces a partial ordering over Π, which we denote by $\sqsubseteq$, where $\pi' \sqsubseteq \pi$ iff π' is a refinement of π. In fact, the resulting partially ordered set constitutes a *lattice*, in that any subset $\mathcal{S}$ of Π admits of an *infimum* (a coarsest partition that is a refinement of all elements of $\mathcal{S}$) and an *supremum* (a coarsening of each element of $\mathcal{S}$ that refines any other partition that coarsens each element of $\mathcal{S}$).

Define now $\mathcal{P} := \bigcup_\Pi \mathcal{P}_\pi$, and, for a given $P \in \mathcal{P}$, let π_P denote the domain of P. If $\pi' \sqsubseteq \pi$, $P \in \mathcal{P}_\pi$, and $Q \in \mathcal{P}_{\pi'}$, we say that Q is an *extension of P to π'* (and P is a *restriction of P to π*) iff for each $s \in \pi$, $\sum Q(s') = P(s)$, where s' ranges over elements of π' that are subsets of s.

We say that P is a *restriction of Q* (and Q an *extension of P*) iff $\pi_P \sqsupseteq \pi_Q$ and P is a restriction of Q to π_P. We say that P and Q are *compatible* if they admit of a common extension.

A *generalized epistemic utility function* is a real-valued function $\mathfrak{u}$ defined over $\mathcal{P} \times W$ such that for each $\pi \in \Pi$, the restriction[7] of $\mathfrak{u}$ to $\mathcal{P}_\pi \times W$ is a truth-directed, epistemic utility function for π.

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7 I'm using 'restriction' here in the standard way, where the restriction of a function f defined over X to some $Y \subseteq X$ just is a function $f \restriction_Y$ whose domain is Y and is such that for all $y \in Y$, $f \restriction_Y (y) = f(x)$. The terminological ambiguity here is merely apparent: if P is a credence function over π, $\pi' \sqsupseteq \pi$, and Q the restriction of P to π', we can identify P with a unique probability function defined over the smallest algebra $\mathbb{A}_\pi$ containing π and Q with the restriction of that probability function to the smallest algebra containing π', which of course is a subset of $\mathbb{A}_\pi$.

As before, for a fixed π and any $P, Q \in \mathcal{P}_\pi$, we can define the expectation of $\mathfrak{u}(Q)$ relative to P. Indeed, since π-measurability entails π'-measurability whenever π is a refinement of π', we can define the expectation of $\mathfrak{u}(Q)$ relative to any P defined over a refinement of π_Q, where if P_Q is the restriction of P to the domain of Q, we have

$$\mathbb{E}_P[\mathfrak{u}(Q)] = \mathbb{E}_{P_Q}[\mathfrak{u}(Q)]. \tag{1}$$

In general, though, for given $P, Q \in \mathcal{P}$, the expectation of $\mathfrak{u}(Q)$ relative to P is not well-defined. Nonetheless, we can introduce some ways of estimating the value of $\mathfrak{u}(Q)$ relative to P as follows.

Fix P and let π be some partition of W. We will denote by $[P]_\pi$ the collection of all probability functions defined over π that are compatible with P. Slightly abusing notation, for a given Q and P, we will use $[P]_Q$ as shorthand for $[P]_{\pi_Q}$. (Note that if π is a refinement of π_P, $[P]_\pi$ is just the set of extensions of P to π, and that if π is a coarsening of π_P, $[P]_\pi$ is just the singleton set of the restriction of P to π.) We now define the *lower expectation*[8] of $\mathfrak{u}(Q)$ relative to P, which we denote by $\underline{\mathbb{E}}_P[\mathfrak{u}(Q)]$, by

$$\underline{\mathbb{E}}_P[\mathfrak{u}(Q)] := \inf_{P' \in [P]_Q} \mathbb{E}_{P'}[\mathfrak{u}(Q)].$$

Similarly, we define the *upper expectation* of $\mathfrak{u}(Q)$ relative to P, which we denote by $\overline{\mathbb{E}}_P[\mathfrak{u}(Q)]$, by

$$\overline{\mathbb{E}}_P[\mathfrak{u}(Q)] := \sup_{P' \in [P]_Q} \mathbb{E}_{P'}[\mathfrak{u}(Q)].$$

Finally, for $\alpha \in [0, 1]$, we can define the *α-expectation* of $\mathfrak{u}(Q)$ relative to P, which we denote by $\mathbb{E}_P^\alpha[\mathfrak{u}(Q)]$, by

$$\mathbb{E}_P^\alpha[\mathfrak{u}(Q)] := \alpha \cdot \overline{\mathbb{E}}_P[\mathfrak{u}(Q)] + (1 - \alpha) \cdot \underline{\mathbb{E}}_P[\mathfrak{u}(Q)].$$

Intuitively, the lower expectation of $\mathfrak{u}(Q)$ relative to P can be thought of as P's worst-case estimate for the value of $\mathfrak{u}(Q)$; similarly, the upper expectation of $\mathfrak{u}(Q)$ relative to P can be thought of as P's best-case estimate for the value

8 Note that we can think of $[P]_\pi$ as an *imprecise* probability function, most naturally identified with a *representor* in the sense of van Fraassen 1990, viz. a set of probability functions. In this case, we can think of $[P]_\pi$ as an imprecise probability function that assigns precise values to each member of π_P and imprecise values to any other member of π. The definitions to follow can thus be seen as variants of the familiar definition of upper and lower expectation for imprecise probabilities (Gilboa 1987, Satia & Lave 1973), at least assuming that all representors are *convex*, in the sense that representors are closed under linear combinations (with non-negative weights adding up to one). See Troffaes 2007 for a recent overview of these and related works. For reasons that will emerge in §4, however, our concerns are somewhat orthogonal to questions those animating the debate over the rationality of imprecise probability functions.

of $\mathfrak{u}(Q)$. (For a given α, the α-expectation of $\mathfrak{u}(Q)$ relative to P is a weighted average of the two estimates.)

Clearly,

$$\underline{\mathbb{E}}_P[\mathfrak{u}(Q)] \le \overline{\mathbb{E}}_P[\mathfrak{u}(Q)],$$

with equality if $\pi_Q \sqsubseteq \pi_P$, in which case

$$\underline{\mathbb{E}}_P[\mathfrak{u}(Q)] = \mathbb{E}_P[\mathfrak{u}(Q)] = \overline{\mathbb{E}}_P[\mathfrak{u}(Q)].$$

Also note that for any $\alpha \in [0,1]$,

$$\underline{\mathbb{E}}_P[\mathfrak{u}(Q)] = \mathbb{E}_P^0[\mathfrak{u}(Q)] \le \mathbb{E}_P^\alpha[\mathfrak{u}(Q)] \le \mathbb{E}_P^1[\mathfrak{u}(Q)] = \overline{\mathbb{E}}_P[\mathfrak{u}(Q)]. \qquad (2)$$

3 Generalized Immodesty Principles

We say that a generalized epistemic utility function $\mathfrak{u}$ is *partition-wise proper* iff for each $\pi \in \Pi$, the restriction $\mathfrak{u}_\pi$ of $\mathfrak{u}$ to $\mathcal{P}_\pi \times W$ is proper. We say that $\mathfrak{u}$ is (partition-wise) *strictly proper* iff $\mathfrak{u}_\pi$ is strictly proper for all $\pi \in \Pi$.

It is straightforward to define generalized epistemic utility functions that are partition-wise proper. For example, take the generalized version of the *Brier score*, defined by

$$\mathfrak{b}(P, w) = - \sum_{s \in \pi_P} \big(P(s) - \mathbb{1}\{w \in s\} \big)^2,$$

where $\mathbb{1}\{w \in s\}$ equals 1 if $w \in s$ and 0 otherwise. It is easy to check that $\mathfrak{b}$ is a generalized epistemic utility function that is partition-wise strictly proper. Indeed, for any family $\{\mathfrak{u}_\pi : \pi \in \Pi\}$ of functions such that $\mathfrak{u}_\pi$ is a partition-wise strictly proper utility function for each π, the function $\mathfrak{u}(P, w) = \mathfrak{u}_{\pi_P}(P, w)$ is a generalized epistemic utility function that is partition-wise strictly proper.

If we are working with a fixed partition and only considering probability functions defined over that partition, a strictly proper epistemic utility function for that partition ensures the kind of immodesty that is allegedly a feature of epistemic rationality (Lewis 1971). And in the context of elicitation, strictly proper epistemic utility functions for a given partition can be used to devise systems of penalties and rewards that ensure the kind of honest reporting of forecasts over that partition that made epistemic utility functions, or *scoring rules*, play the starring role in a wide body of literature.[9]

Once we relax the assumption that we are working with a fixed partition, however, partition-wise strict propriety does not suffice to ensure immodesty, nor to encourage honest reporting. Indeed, if we consider functions defined over

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9 Again, see Gneiting & Raftery 2007 and references therein.

any partition and use the Brier score to measure epistemic utility, it turns out that any probability function that is not *maximally opinionated*—any probability function that assigns a value other than 0 or 1 to some proposition—will assign a greater expected epistemic utility to a probability function other than itself.[10] Similarly, if we do not fix a partition but allow a forecaster to choose which partition to report her forecasts on, using the Brier score will not encourage the forecaster to report her full credence function—as long as her credence function is not maximally opinionated, she will expect to do better, according to the Brier score, by reporting a strict restriction of her credence function.[11]

Example 1. Suppose P is not maximally opinionated. Let π^* be the coarsening of π_P such that the restriction P^* of P is maximally opinionated. Note now that for $s \in \pi^*$ with $P^*(s) \neq 0$, $\mathfrak{b}(P^*, s) = 0$, and hence that

$$\mathbb{E}_P[\mathfrak{b}(P^*)] = \mathbb{E}_{P^*}[\mathfrak{b}(P^*)] = 0.$$

Now let $s_0 \in \pi_P$ be such that $P(s_0) \in (0, 1)$, and let $\pi_P^- = \pi_P \smallsetminus \{s_0\}$. By definition,

$$\mathfrak{b}(P, s_0) = -(P(s_0) - 1)^2 - \sum_{s \in \pi_P^-} P(s)^2 < - \sum_{s \in \pi_P^-} P(s)^2 \leq 0,$$

and thus

$$\mathbb{E}_P[\mathfrak{b}(P)] < 0 = \mathbb{E}_P[\mathfrak{b}(P^*)].$$

$\square$

An interesting question, then, is whether there are epistemic utility functions that capture the relevant kind of immodesty once we consider probability functions defined over any partition. In other words, the question is whether there are epistemic utility functions such that, for any probability function P, P 'takes itself' to be doing better than any other $Q \neq P$ in terms of epistemic utility. But in order to answer this question, of course, we need to make clear what it is for some probability function to 'take itself' to be doing better than another in terms of epistemic utility. After all, we cannot just use the familiar notion of expectation here, since in general the expectation of $\mathfrak{u}(Q)$ relative to P is not well-defined unless $\pi_Q \sqsupseteq \pi_P$.

In light of our discussion in the previous section, we have at least three different options. First, we can use *lower expectation*: we could seek epistemic

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10 Cf. Carr 2015, Pettigrew 2018 and REDACTED.

11 Note that our example below suffices to show that the normalized version of the Brier score, defined by

$$\mathfrak{b}^*(P, w) = -\frac{1}{|\pi_P|} \sum_{s \in \pi_P} (P(s) - \mathbb{1}\{w \in s\})^2,$$

is also not downwards proper.

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utility functions u that are *strictly universally l-proper* (resp. *universally l-proper*), in that for each $P \neq Q$, the lower expectation of $u(P)$ relative to P is greater than (resp. greater than or equal to) that of $u(Q)$ relative to P. Alternatively, we can use *upper expectation*: we could seek epistemic utility functions u that are *strictly universally u-proper* (resp. *universally u-proper*), in that for each $P \neq Q$, the upper expectation of $u(P)$ relative to P is greater than (resp. greater than or equal to) that of $u(Q)$ relative to P. Finally, we can use some convex combination of lower and upper expectations: for $\alpha \in [0,1]$, we could seek epistemic utility functions u that are *strictly universally α-proper* (resp. *universally α-proper*), in that for each $P \neq Q$, the α-expectation of $u(P)$ relative to P is greater than (resp. greater than or equal to) that of $u(Q)$ relative to P.

It follows from (2) that universal u-propriety entails universal l-propriety, and thus that universal u-propriety entails universal α-propriety for all α. (Note too that universal 0-propriety is equivalent to universal l-propriety and universal 1-propriety is equivalent to universal u-propriety.) We thus get three general immodesty principles, of increasing strengths, according to whether we require all epistemic utility functions to be strictly universally l-proper, universally α-proper for some $\alpha \in (0,1)$, or universally u-proper. So the first question to ask is whether there are any universally u-proper epistemic utility functions. And the answer, perhaps unsurprisingly, is *no*.

3.1 There are no (strictly) universally u-proper epistemic utility functions

Say that an epistemic utility function u is *downwards proper* iff for each P and each Q defined over a coarsening of π_P,

$$\mathbb{E}_P[u(P)] \geq \mathbb{E}_P[u(Q)],$$

and *strictly downwards proper* iff the above inequality is always strict. Say that u is *upwards u-proper* iff for each P and each Q defined over a refinement of π_P,

$$\overline{\mathbb{E}}_P[u(P)] \geq \overline{\mathbb{E}}_P[u(Q)]$$

and *strictly upwards u-proper* iff the above inequality is always strict.[12]

Using these definitions we can make a few simple observations. First, and most clearly, (strict) downwards propriety and (strict) upwards u-propriety individually suffice for (strict) partition-wise propriety. Second, for partition-wise proper epistemic utility functions, (strict) downwards propriety (resp. (strict) upwards u-propriety) can be established by looking only at comparisons between credence functions and their restrictions (resp. extensions).

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12 A helpful mnemonic: for *downwards* propriety you compare by going *down* in size: you compare a credence function only with those defined over a smaller domain; for *upwards* propriety, you go *up* in size: you check only those with a *larger* domain.

Fact 3.1. *Suppose* u *is partition-wise proper. Then:*

(i) u *is downwards proper iff for each P and each restriction Q of P,* $\mathbb{E}_P[u(P)] \geq \mathbb{E}_P[u(Q)]$; u *is strictly downwards proper iff* $\mathbb{E}_P[u(P)] > \mathbb{E}_P[u(Q)]$ *whenever Q is a restriction of P.*

(ii) u *is upwards u-proper iff for each P and each extension Q of P,* $\mathbb{E}_P[u(P)] \geq \overline{\mathbb{E}}_P[u(Q)]$; u *is strictly upwards u-proper iff* $\mathbb{E}_P[u(P)] > \overline{\mathbb{E}}_P[u(Q)]$ *whenever Q is an extension of P and $Q \neq P$.*

Proof. Only the right to left direction of each biconditional is non-trivial, and that of (i) follows immediately from (1) and the fact that if Q is defined over a coarsening of P and P_Q is the restriction of P to π_Q, partition-wise propriety ensures that $\mathbb{E}_{P_Q}[u(P_Q)] \geq \mathbb{E}_{P_Q}[u(P)]$.

For the right to left direction of (ii), simply note that for P and Q with $\pi_Q \sqsubseteq \pi_P$, our assumptions ensure that for each extension P^+ of P to π_Q,

$$\mathbb{E}_P[u(P)] \geq \mathbb{E}_{P^+}[u(P^+)] \geq \mathbb{E}_{P^+}[u(Q)],$$

which ensures $\mathbb{E}_P[u(P)] \geq \overline{\mathbb{E}}_P[u(Q)]$. $\qquad\square$

Say that an extension Q of P is *opinionated* iff for each $s \in \pi_P$ there is $s_Q \in \pi_Q$ with $s_Q \subseteq s$ and $Q(s_Q) = P(s)$—in other words, an extension is opinionated if for each cell of π_P, Q assigns all of the probability P assigns to it to a single one of its subsets in π_Q. In order to determine the value of the upper or lower expectation of any extension Q of P, all we need to look at are the opinionated extensions of P defined over π_Q.

Fact 3.2. *Fix an epistemic utility function u, a probability function P and any Q defined over a refinement of π_P. There are opinionated extensions P_Q^+ and P_Q^- of P defined over π_Q such that*

$$\overline{\mathbb{E}}_P[u(Q)] = \mathbb{E}_{P_Q^+}[u(Q)] \quad and \quad \underline{\mathbb{E}}_P[u(Q)] = \mathbb{E}_{P_Q^-}[u(Q)].$$

Proof. For each $s \in \pi_P$, pick $s_Q^+, s_Q^- \in \pi_Q$, with $s_Q^+ \subseteq s$ and $s_Q^- \subseteq s$, such that for all $t \in \pi_Q$ with $t \subseteq S$

$$u(Q, s_Q^+) \geq u(Q, t) \quad and \quad u(Q, s_Q^-) \leq u(Q, t),$$

and let P_Q^+ and P_Q^-) be the unique opinionated extensions of P such that for all s, $P_Q^+(s_Q^+) = P(s)$ and $P_Q^-(s_Q^-) = P(s)$. $\qquad\square$

A consequence of the last two results is that for determining whether u is upwards u-proper, we don't really need to compute upper-expectations.

Corollary 3.3. *A partition-wise proper epistemic utility function $\mathfrak{u}$ is upwards u-proper (resp. strictly upwards u-proper) iff for each P and each opinionated extension Q of P, $\mathbb{E}_P[\mathfrak{u}(P)] \geq \mathbb{E}_Q[\mathfrak{u}(Q)]$ (resp. $\mathbb{E}_P[\mathfrak{u}(P)] > \mathbb{E}_Q[\mathfrak{u}(Q)]$ when $Q \neq P$.)*

Proof. The left to right direction follows immediately from (1) and the definition of upper expectation. For the right to left direction, take P and fix Q defined over a refinement of π_P. From Fact 3.2, we know that there is an opinionated extension P_Q^+ of P to π_Q such that $\overline{\mathbb{E}}_P[\mathfrak{u}(Q)] = \mathbb{E}_{P_Q^+}[\mathfrak{u}(Q)]$.

But by assumption,

$$\overline{\mathbb{E}}_P[\mathfrak{u}(P)] = \mathbb{E}_P[\mathfrak{u}(P)] \geq \mathbb{E}_{P_Q^+}[\mathfrak{u}(P_Q^+)]$$

(resp. the above inequality is strict when $Q \neq P$), and from partition-wise propriety we know that $\mathbb{E}_{P_Q^+}[\mathfrak{u}(P_Q^+)] \geq \mathbb{E}_{P_Q^+}[\mathfrak{u}(Q)]$. We can thus conclude that $\overline{\mathbb{E}}_P[\mathfrak{u}(P)] \geq \overline{\mathbb{E}}_P[\mathfrak{u}(Q)]$ (resp. $\overline{\mathbb{E}}_P[\mathfrak{u}(P)] > \overline{\mathbb{E}}_P[\mathfrak{u}(Q)]$ when $Q \neq P$.) From Fact 3.1, we conclude that $\mathfrak{u}$ is upwards u-proper (resp. strictly upwards u-proper). $\square$

Corollary 3.4. *A partition-wise proper epistemic utility function $\mathfrak{u}$ is upwards u-proper (resp. strictly upwards u-proper) iff for each P and each extension Q of P, $\mathbb{E}_P[\mathfrak{u}(P)] \geq \mathbb{E}_Q[\mathfrak{u}(Q)]$ (resp. $\mathbb{E}_P[\mathfrak{u}(P)] > \mathbb{E}_Q[\mathfrak{u}(Q)]$ when $Q \neq P$.)*

Proof. The right to left direction is an immediate consequence of Corollary 3.3. For the converse, simply note that if P_Q^+ is an opinionated extension of P such that

$$\overline{\mathbb{E}}_P[\mathfrak{u}(Q)] = \mathbb{E}_{P_Q^+}[\mathfrak{u}(Q)],$$

the definition of upper expectation entails that $\mathbb{E}_{P_Q^+}[\mathfrak{u}(Q)] \geq \mathbb{E}_Q[\mathfrak{u}(Q)]$, so the left to right direction of Corollary 3.3 yields the desired result. $\square$

Now, a natural question to ask is whether there are epistemic utility functions that are both (strictly) upwards u-proper and (strictly) downwards proper. But this just turns out to be the question whether there are universally u-proper epistemic utility functions.

Fact 3.5. *An epistemic utility function is universally u-proper (resp. strictly universally u-proper) iff it is downwards proper (resp. strictly downwards proper) and upwards u-proper (resp. strictly upwards u-proper).*

Proof. The left to right direction is immediate. For the right to left direction, suppose $\mathfrak{u}$ is downwards proper and upwards u-proper and fix $P \neq Q$. Let P_Q be an arbitrary probability function defined over π_Q that is compatible with P,

let π^* be the coarsest partition that refines both π_P and π_Q, and let P_Q^* be some common extension of P_Q and P to π^*. From the fact that $\mathfrak{u}$ is upwards u-proper and the definition of upper expectation, we know that

$$\overline{\mathbb{E}}_P[\mathfrak{u}(P)] \geq \overline{\mathbb{E}}_P[\mathfrak{u}(P_Q^*)] \geq \mathbb{E}_{P_Q^*}[\mathfrak{u}(P_Q^*)].$$

And since $\mathfrak{u}$ is downwards proper, we can infer using (1) that

$$\mathbb{E}_{P_Q^*}[\mathfrak{u}(P_Q^*)] \geq \mathbb{E}_{P_Q^*}[\mathfrak{u}(P_Q)] = \mathbb{E}_{P_Q}[\mathfrak{u}(P_Q)] \geq \mathbb{E}_{P_Q}[\mathfrak{u}(Q)],$$

where the last inequality follows from the fact that $\mathfrak{u}$ is partition-wise proper. We thus have that for any P_Q defined over π_Q that is compatible with P, $\overline{\mathbb{E}}_P[\mathfrak{u}(P)] \geq \mathbb{E}_{P_Q}[\mathfrak{u}(Q)]$, which entails $\overline{\mathbb{E}}_P[\mathfrak{u}(P)] \geq \overline{\mathbb{E}}_P[\mathfrak{u}(Q)]$, as desired. To show that strict universal u-propriety follows from strict downwards propriety and strict upwards u-propriety, simply note that if $P \neq Q$ but π_Q is neither a refinement nor a coarsening of π_P, it follows that for any P_Q defined over π_Q that is compatible with P, $P_Q^* \neq Q$ and thus strict upwards u-propriety ensures $\overline{\mathbb{E}}_P[\mathfrak{u}(P)] > \overline{\mathbb{E}}_P[\mathfrak{u}(P_Q^*)]$. Combined with the reasoning above, this ensures that $\mathfrak{u}$ is strictly universally proper, as desired. $\square$

And as announced above, there just are no strictly universally u-proper epistemic utility functions.

Theorem 3.6. *There are no strictly universally u-proper epistemic utility functions.*

Proof. Suppose $\mathfrak{u}$ is strictly downwards proper. Fix P and $\pi^* \sqsubseteq \pi_P$ with $\pi \neq \pi_P$, and let P^* be some extension of P to π^*. Strict downwards propriety entails $\mathbb{E}_{P*}[\mathfrak{u}(P^*)] > \mathbb{E}_{P*}[\mathfrak{u}(P)]$. And combined with the definition of upper-expectation and (1), this entails

$$\overline{\mathbb{E}}_P[\mathfrak{u}(P^*)] \geq \mathbb{E}_{P*}[\mathfrak{u}(P^*)] > \mathbb{E}_{P*}[\mathfrak{u}(P)] = \mathbb{E}_P[\mathfrak{u}(P)],$$

which shows that $\mathfrak{u}$ is not strictly upwards u-proper. $\square$

We can strengthen Theorem 3.6 if we restrict ourselves to the class of partition-wise strictly proper epistemic utility functions.

Theorem 3.7. *There are no universally u-proper epistemic utility functions that are strictly partition-wise proper.*

Proof. Let $\mathfrak{u}$ be strictly partition-wise proper, and suppose $\mathfrak{u}$ is downwards proper. Pick P and let π^* be a refinement of π_P such that such that for all $S \in \pi_P \smallsetminus \pi^*$, $P(s) \neq 0$. Let Q be an extension of P to π^* that is not opinionated and pick $s_0 \in \pi_P$ and $s_0^* \in \pi^*$ such that $s_0^* \in s_0$ and $Q(s_0^*) \neq P(s_0)$. From Fact 3.2 and the

fact that $\mathfrak{u}$ is downwards proper we know, again using (1) and the definition of upper-expectation, that there is an opinionated extension P_Q^+ of P defined over π_Q such that

$$\mathbb{E}_{P_Q^+}[\mathfrak{u}(Q)] = \overline{\mathbb{E}}_P[\mathfrak{u}(Q)] \geq \mathbb{E}_Q[\mathfrak{u}(Q)] \geq \mathbb{E}_Q[\mathfrak{u}(P)] = \mathbb{E}_P[\mathfrak{u}(P)].$$

Since by construction $P_Q^+ \neq Q$, strict partition-wise propriety ensures that $\mathbb{E}_{P_Q^+}[\mathfrak{u}(P_Q^+)] > \mathbb{E}_{P_Q^+}[\mathfrak{u}(Q)]$. Putting all of this together and using the definition of upper-expectation, we have that there is an extension P_Q^+ of P such that

$$\overline{\mathbb{E}}_P[\mathfrak{u}(P_Q^+)] \geq \mathbb{E}_{P_Q^+}[\mathfrak{u}(P_Q^+)] > \mathbb{E}_{P_Q^+}[\mathfrak{u}(Q)] \geq \mathbb{E}_P[\mathfrak{u}(P)],$$

which shows that $\mathfrak{u}$ is not upwards u-proper.

$\square$

3.2 There are no (strictly) universally l-proper epistemic utility functions

The next questions to ask are whether there are any strictly universally α-proper epistemic utility functions and, if not, whether there are any strictly universally l-proper epistemic utility functions. If we make one additional assumption about epistemic utility functions, the answer to each of these questions also turns out to be *no*.

Start by noting that for a given $\mathfrak{u}$ and π, we can think of the restriction of $\mathfrak{u}$ to $\mathcal{P}_\pi \times W$ as a function $f_\mathfrak{u}^\pi : \Delta^{N-1} \longrightarrow \overline{\mathbb{R}}$, where $N = |\pi|$ and for each n, Δ^n is the standard n-simplex, that is,

$$\Delta^n := \{\mathbf{x} = \langle x_1, \ldots, x_{n+1} \rangle \in \mathbb{R}^{n+1} : \sum x_i = 1, \text{ and for all } i \leq n + 1, x_i \geq 0\}.$$

(Simply fix an enumeration of π and identify each $P \in \mathcal{P}_\pi$ with the vector $\langle P(s_1), \ldots, P(s_N) \rangle$.) We say that an epistemic utility function $\mathfrak{u}$ is *continuous* iff for each π and w, the function $\mathbf{x} \mapsto f_\mathfrak{u}^\pi(\mathbf{x}, w)$ is continuous.

Our main result is that no continuous, strictly downwards proper epistemic utility function can be upwards l-proper.[13]

Before stating our result, let us define an analogue of upwards u-propriety that relies on the lower expectation, rather than on upper expectation, in the obvious way: $\mathfrak{u}$ is *upwards l-proper* iff for each P and each refinement Q of P,

$$\underline{\mathbb{E}}_P[\mathfrak{u}(P)] \geq \underline{\mathbb{E}}_P[\mathfrak{u}(Q)].$$

Before asking whether there are strictly universally l-proper epistemic utility functions, we could ask whether there are any epistemic utility functions that

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13 In fact, something slightly weaker than the full continuity assumption may be all that is really needed— see Grünwald & Dawid 2004 for a more general result, especially their Theorem 6.2 together with their Corollary 4.2.

are both strictly downwards proper and strictly upwards l-proper. If we restrict ourselves to the class of continuous epistemic utility functions, we can answer this question in the negative.

Theorem 3.8. *If $\mathfrak{u}$ is continuous and strictly downwards-proper, then it is not upwards l-proper.*

So we can conclude that if $\mathfrak{u}$ is continuous, it is not strictly universally l-proper.

Corollary 3.9. *There are no continuous, strictly universally l-proper epistemic utility functions.* $\qquad\qquad\square$

And of course, using Theorem 3.6 we can conclude that for any $0 \leq \alpha \leq 1$, there are no continuous, strictly universally α-proper epistemic utility functions.

Corollary 3.10. *If $\mathfrak{u}$ is a continuous epistemic utility function, and $\alpha \in [0,1]$, $\mathfrak{u}$ is not strictly universally α-proper.* $\qquad\qquad\square$

Proof of Theorem 3.8. This result follows immediately from the following lemma (essentially due to Grünwald & Dawid 2004), a proof of which is in the appendix.

Lemma 3.11. *Suppose $\mathfrak{u}$ is continuous and partition-wise strictly proper. For any $P \in \mathcal{P}$ and any $\pi \sqsubseteq \pi_P$, there is some $\hat{P} \in [P]_\pi$ such that, for all $Q \in \mathcal{P}_\pi$, and all $P^* \in [P]_\pi$*

$$\underline{\mathbb{E}}_P[\mathfrak{u}(Q)] \leq \underline{\mathbb{E}}_P[\mathfrak{u}(\hat{P})] = \mathbb{E}_{\hat{P}}[\mathfrak{u}(\hat{P})] \leq \mathbb{E}_{P^*}[\mathfrak{u}(P^*)].$$

$\qquad\qquad\square$

Before concluding this subsection, let us note two consequences of Lemma 3.11, which serve as counterparts to Corollary 3.4 and Fact 3.5.

Fact 3.12. *Suppose $\mathfrak{u}$ is partition-wise proper and continuous. Then $\mathfrak{u}$ is upwards l-proper (resp. strictly upwards l-proper) iff for each P and each $\pi \sqsubseteq \pi_P$ there is $P^* \in [P]_\pi$ such that $\mathbb{E}_P[\mathfrak{u}(P)] \geq \mathbb{E}_{P^*}[\mathfrak{u}(P^*)]$ (resp. $\mathbb{E}_P[\mathfrak{u}(P)] > \mathbb{E}_{P^*}[\mathfrak{u}(P^*)]$ if $\pi \neq \pi_P$).*

Proof. From Lemma 3.11, we know that for each P and each $\pi \sqsupseteq \pi_P$ there is $\hat{P} \in [P]_\pi$ such that

$$\max_{Q \in \mathcal{P}_P} \underline{\mathbb{E}}_P[\mathfrak{u}(Q)] = \underline{\mathbb{E}}_P[\mathfrak{u}(\hat{P})] = \mathbb{E}_{\hat{P}}[\mathfrak{u}(\hat{P})] = \min_{Q \in [P]_\pi} \mathbb{E}_Q[\mathfrak{u}(Q)].$$

The left to right direction now follows immediately (simply let $\hat{P} = P^*$). For the right to left direction, simply note that for each P and $\pi \sqsupseteq \pi_P$ we have $P^* \in [P]_\pi$ with $\mathbb{E}_P[\mathfrak{u}(P)] \geq \mathbb{E}_{P^*}[\mathfrak{u}(P^*)]$ (resp. $\mathbb{E}_P[\mathfrak{u}(P)] > \mathbb{E}_{P^*}[\mathfrak{u}(P^*)]$). But of course,

$$\mathbb{E}_{P^*}[\mathfrak{u}(P^*)] \geq \min_{Q \in [P]_\pi} \mathbb{E}_Q[\mathfrak{u}(Q)] = \max_{Q \in \mathcal{P}_P} \underline{\mathbb{E}}_P[\mathfrak{u}(Q)],$$

where the last equality follows from Lemma 3.11. We can thus conclude that $\mathfrak{u}$ is upwards l-proper (resp. strictly upwards l-proper). $\square$

Fact 3.13. *A continuous epistemic utility function* $\mathfrak{u}$ *is (strictly) universally l-proper iff it is (strictly) upwards l-proper and (strictly) downwards proper.*

Proof. Again, only the right to left direction is non-trivial. So fix $P \neq Q$, and let π^* be a refinement of π_P and π_Q. From Fact 3.12 and the fact that $\mathfrak{u}$ is continuous and upwards l-proper, we know that there is $P^* \in [P]_{\pi^*}$ such that $\mathbb{E}_P[\mathfrak{u}(P)] \geq \mathbb{E}_{P^*}[\mathfrak{u}(P^*)]$. Letting P_Q^* be the restriction of P^* to π_Q, we know from the fact that $\mathfrak{u}$ is downwards proper that

$$\mathbb{E}_{P^*}[\mathfrak{u}(P^*)] \geq \mathbb{E}_{P^*}[\mathfrak{u}(P_Q^*)] = \mathbb{E}_{P_Q^*}[\mathfrak{u}(Q)].$$

But by definition, P_Q^* is compatible with P, and so

$$\mathbb{E}_P[\mathfrak{u}(P)] \geq \mathbb{E}_{P^*}[\mathfrak{u}(P_Q^*)] \geq \mathbb{E}_P[\mathfrak{u}(Q)],$$

as required.

If $\mathfrak{u}$ is strictly upwards l-proper and strictly downwards proper, then repeat the above reasoning after first assuming π_Q is neither a refinement nor a coarsening of π_P, so that π^* is a refinement of π_P with $\pi^* \neq \pi_P$. Using Fact 3.12 we conclude that there is $P^* \in [P]_{\pi^*}$ such that $\mathbb{E}_P[\mathfrak{u}(P)] > \mathbb{E}_{P^*}[\mathfrak{u}(P^*)]$, and the above reasoning will then show, *mutatis mutandis*, what we need, viz.

$$\mathbb{E}_P[\mathfrak{u}(P)] > \mathbb{E}_{P^*}[\mathfrak{u}(P_Q^*)] \geq \mathbb{E}_P[\mathfrak{u}(Q)],$$

as required. $\square$

3.3 Downwards propriety and upwards propriety

We have seen that there are no strictly universally u-proper or l-proper epistemic utility functions. But we can easily find examples of downwards proper and upwards u-proper (and hence upwards l-proper) epistemic utility functions.

Say that an epistemic utility function $\mathfrak{u}$ is an *additive accuracy measure*[14] iff there is a function $u : [0,1] \times \{0,1\} \longrightarrow \overline{\mathbb{R}}$ such that

$$\mathfrak{u}(P, w) = \sum_{s \in \pi_P} u(P(s), \mathbb{1}\{w \in s\}).$$

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14 I am thus implicitly assuming that accuracy measures satisfy what Joyce 2009, p. 273 calls 'Extensionality'. Nothing hinges on this assumption, of course—you can simply read 'additive accuracy measure' as shorthand for 'extensional, additive accuracy measure'.

Say that a function $u : [0,1] \times \{0,1\} \longrightarrow \overline{\mathbb{R}}$ is *proper* iff for all $x \neq y \in [0,1]$,

$$x \cdot u(x,1) + (1-x) \cdot u(x,0) \geq x \cdot u(y,1) + (1-x) \cdot u(y,0),$$

and say that u is *strictly proper* iff the above inequality is always strict.

If u is an additive accuracy measure, we call u its *local accuracy measure*. It is easy to see that an additive accuracy measure is partition-wise proper (resp. strictly proper) iff its local accuracy measure is proper (resp. strictly proper).

For a given local accuracy measure u we call $\phi_u : [0,1] \longrightarrow \overline{\mathbb{R}}$ its *self-expectation function*, where

$$\phi_u(x) := x \cdot u(x,1) + (1-x) \cdot u(x,0).$$

The linearity of expectation ensures that if u is an additive accuracy measure with local accuracy measure u, $\mathbb{E}_P[u(P)] = \sum \phi_u(P(s))$ (with s ranging over π_P). From this we can easily derive the following characterization result.[15]

Theorem 3.14. *An additive accuracy measure u is downwards proper (resp. strictly downwards proper) iff its local accuracy measure u is proper (resp. strictly proper) and the function $\phi_u : [0,1] \longrightarrow \overline{\mathbb{R}}$ defined by*

$$\phi_u(x) := x \cdot u(x,1) + (1-x) \cdot u(x,0)$$

is subadditive, in that for $x, y \in [0,1]$, with $x + y \in [0,1]$

$$\phi_u(x+y) \leq \phi_u(x) + \phi_u(y),$$

(resp. strictly subadditive, in that the above inequality is always strict). □

Proof. For the left to right direction, start by taking a three celled partition $\pi = \{s_0, s_1, s_2\}$ and let $\pi^* = \{s_0, s^*\}$ be a coarsening of π (and hence, $s^* = s_0 \cup s_1$). Fix $x, y \in [0,1]$ with $x + y \in [0,1]$ and let P be the unique probability function in $\mathcal{P}_\pi$ assigning x to s_1 and y to s_2. Let P^* be the restriction of P to π^* and note that

$$\mathbb{E}_P[u(P)] = \sum_{s \in \pi} \phi_u(P(s)) = \phi_u(1 - (x+y)) + \phi_u(x) + \phi_u(y)$$

and

$$\mathbb{E}_{P^*}[u(P^*)] = \sum_{s \in \pi^*} \phi_u(P^*(s)) = \phi_u(1 - (x+y)) + \phi_u(x+y).$$

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15 See REDACTED for a slightly more general result.

Since $\mathfrak{u}$ is downwards proper, we know that $\mathbb{E}_{P*}[\mathfrak{u}(P^*)] = \mathbb{E}_P[\mathfrak{u}(P^*)] \leq \mathbb{E}_P[\mathfrak{u}(P)]$, and hence that $\phi_{\mathfrak{u}}(x + y) \leq \phi_{\mathfrak{u}}(x) + \phi_{\mathfrak{u}}(y)$, as required.

For the converse, fix P and $\pi \sqsupseteq \pi_P$ with $|\pi_P| - |\pi| = 1$. In other words, π is a coarsening of π_P such that there are $s_0, s_1 \in \pi_P$ with $s_0 \cup s_1 = s^* \in \pi$ such that $\pi = \{s^*\} \cup (\pi_P \smallsetminus \{s_0, s_1\})$. Let P^* be the restriction of P to π, set $x_0 = P(s_0)$, $x_1 = P(s_1)$, and note that, letting t range over elements of π,

$$\mathbb{E}_P[\mathfrak{u}(P)] = \sum_{s \in \pi_P} \phi_{\mathfrak{u}}(P(s)) = \phi_{\mathfrak{u}}(x_0) + \phi_{\mathfrak{u}}(x_1) + \sum_{t \neq s^*} \phi_{\mathfrak{u}}(P(t))$$

and

$$\mathbb{E}_{P*}[\mathfrak{u}(P^*)] = \sum_{s \in \pi^*} \phi_{\mathfrak{u}}(P^*(s)) = \phi_{\mathfrak{u}}(x_0 + x_1) + \sum_{t \neq s^*} \phi_{\mathfrak{u}}(P^*(t))$$
$$= \phi_{\mathfrak{u}}(x_0 + x_1) + \sum_{t \neq s^*} \phi_{\mathfrak{u}}(P(t))$$

Since $\phi_{\mathfrak{u}}$ is subadditive, we conclude that $\mathbb{E}_{P*}[\mathfrak{u}(P^*)] = \mathbb{E}_P[\mathfrak{u}(P^*)] \leq \mathbb{E}_P[\mathfrak{u}(P)]$. A simple inductive argument on the size of $|\pi_P| - |\pi_Q|$ shows that for any P and any restriction Q of P, $\mathbb{E}_P[\mathfrak{u}(P)] \geq \mathbb{E}_P[\mathfrak{u}(Q)]$, as required. $\square$

As we saw in Example 1, the (generalized) Brier score is not downwards proper, but the (generalized version of the) well-known *spherical score* (which, like the Brier score, is an additive accuracy measure) is strictly downwards proper.

Example 2. Define $\mathsf{s} : [0,1] \times \{0,1\} \longrightarrow \overline{\mathbb{R}}$ by

$$\mathsf{s}(x, i) := \frac{|1 - (i + x)|}{\sqrt{x^2 + (1 - x)^2}}$$

and let

$$\mathfrak{s}(P, w) := \sum_{s \in \pi_P} \mathsf{s}(P(s), \mathbb{1}\{w \in s\}).$$

Clearly, the restriction of $\mathfrak{s}$ to any partition is just the familiar spherical score, which is strictly proper, so that $\mathfrak{s}$ is strictly partition-wise proper. But $\mathfrak{s}$ is also strictly downwards proper.

To see why, note that for any $x \in [0,1]$,

$$\phi_{\mathsf{s}}(x) = x \cdot \left(\frac{x}{\sqrt{x^2 + (1-x)^2}}\right) + (1-x) \cdot \left(\frac{1-x}{\sqrt{x^2 + (1-x)^2}}\right) = \sqrt{x^2 + (1-x)^2}$$
$$= \|\langle x, (1-x)\rangle\|,$$

where $\|\cdot\|$ is the Euclidean norm.

Since the Euclidean norm is a norm, it satisfies the triangle inequality, and thus for any $x, y \in [0,1]$ with $x + y \in [0,1]$,

$$\phi_s(x) + \phi_s(y) = \|\langle x, 1-x \rangle\| + \|\langle y, 1-y \rangle\|$$
$$\geq \|\langle x+y, 1+1-(x+y) \rangle\| > \|\langle x+y, 1-(x+y) \rangle\| = \phi_s(x+y),$$

which means ϕ_s is strictly subadditive and thus that $\mathfrak{s}$ is strictly downwards proper.

$\square$

We also need not look far to find an example of an upwards proper additive accuracy measure.

Example 3. Let

$$\mathfrak{l}(P, w) = \log(P([w]_P)),$$

where $[w]_P$ is the unique $s \in \pi_P$ with $w \in s$. As is well known, $\mathfrak{l}$ is partition-wise proper. But it is also upwards u-proper. To see that, first note that for any P,

$$\mathbb{E}_P[\mathfrak{l}(P)] = \sum_{s \in \pi_P} P(s) \cdot \mathfrak{l}(P, s) = \sum_{s \in \pi_P} P(s) \log(P(s)).$$

Fix now P, let P^* be an opinionated extension of P, and for each $s \in \pi_P$ let s^* denote the unique $t \in \pi_{P^*}$ with $t \subseteq s$ and $P^*(t) \neq 0$. Note now that

$$\mathbb{E}_P[\mathfrak{l}(P)] = \sum_{s \in \pi_P} P(s) \cdot \log(P(s)) = \sum_{s \in \pi_P} P^*(s^*) \cdot \log(P^*(s^*)).$$

And of course,

$$\sum_{s \in \pi_P} P^*(s^*) \cdot \log(P^*(s^*)) = \sum_{t \in \pi_{P*}} P^*(t) \cdot \mathfrak{l}(P^*, t) = \mathbb{E}_{P*}[\mathfrak{l}(P^*)].$$

From Corollary 3.3 we conclude that $\mathfrak{l}$ is upwards u-proper, and thus upwards l-proper.

$\square$

Note that the log score is also an additive accuracy measure with local accuracy measure $\mathfrak{l}$, where

$$\mathfrak{l}(x, i) = i \cdot \log(x).$$

Note too that $\mathfrak{l}(0,0) = 0$. Interestingly, any additive accuracy measure $\mathfrak{u}$ whose local accuracy measure u satisfies $u(0,0) = 0$ will be upwards u-proper, as the following makes clear.

Theorem 3.15. *An additive accuracy measure $\mathfrak{u}$ is upwards u-proper (resp. strictly upwards proper) iff $u(0,0) \leq 0$ (resp. $u(0,0) < 0$), where u is $\mathfrak{u}$'s local accuracy measure.*

Proof. Suppose $\mathfrak{u}$ is an upwards u-proper additive accuracy measure with local accuracy measure u. Take a two cell partition $\pi = s_0, s_1$ of W. Let P_0 be the unique probability function in $\mathcal{P}_\pi$ that assigns probability 1 to s_0 and let $P_\top$ be the unique probability function defined over the trivial partition $\{W\}$. Of course P_0 is an opinionated extension of $P_\top$, so that the upwards u-propriety of $\mathfrak{u}$ and Fact 3.2 entails $\mathbb{E}_{P_\top}[\mathfrak{u}(P_\top)] \geq \mathbb{E}_{P_0}[\mathfrak{u}(P_0)]$. But $\mathbb{E}_{P_\top}[\mathfrak{u}(P_\top)] = u(1,1)$, and $\mathbb{E}_{P_0}[\mathfrak{u}(P_0)] = u(1,1) + u(0,0)$, and thus $u(0,0) \leq 0$. A similar argument shows that $\mathfrak{u}$ is strictly upwards u-proper, then $u(0,0) < 0$.

To establish the other direction, fix P and let P^* be an opinionated extension of P. For each $s \in \pi_P$, let s^* be the unique $t \in \pi_{P*}$ such that $t \subseteq s$ and $P^*(t) \neq 0$, and let $n_s = |\{t \in \pi_{P*} : t \subseteq s\}|$.

Note now that

$$\mathbb{E}_{P*}[\mathfrak{u}(P^*)] = \sum_{t \in \pi_{p*}} \phi_\mathfrak{u}(P^*(t)) = \sum_{s \in \pi_P} \phi_\mathfrak{u}(P^*(s^*)) + \sum_{s \in \pi_P} (n_s - 1) \cdot u(0,0).$$

And since clearly

$$\sum_{s \in \pi_P} \phi_\mathfrak{u}(P^*(s^*)) = \mathbb{E}_P[\mathfrak{u}(P)],$$

$u(0,0) \leq 0$ (resp. $u(0,0) < 0$) entails that $\mathbb{E}_{P*}[\mathfrak{u}(P^*)] \geq \mathbb{E}_P[\mathfrak{u}(P)]$ (resp. $\mathbb{E}_{P*}[\mathfrak{u}(P^*)] > \mathbb{E}_P[\mathfrak{u}(P)]$), and hence, using Fact 3.2, we can conclude that $\mathfrak{u}$ is upwards u-proper (resp. strictly upwards u-proper). $\square$

To conclude this section, let us state one final characterization result, this time for the class of upwards l-proper additive accuracy measures.

Theorem 3.16. *A continuous, additive accuracy measure $\mathfrak{u}$ with a strictly proper local accuracy measure u is upwards l-proper (resp. strictly upwards l-proper) iff for each $x \in [0,1]$ there is $y \in [0,1]$ with $y < x$ and $\phi_\mathfrak{u}(x) \geq \phi(y) + \phi_\mathfrak{u}(x - y)$ (resp. $\phi_\mathfrak{u}(x) > \phi(y) + \phi_\mathfrak{u}(x - y)$).*

Proof. We know from Fact 3.12 that $\mathfrak{u}$ is upwards l-proper iff for each P and each $\pi \sqsupseteq \pi_P$ there is $P^* \in [P]_\pi$ such that

$$\mathbb{E}_P[\mathfrak{u}(P)] \geq \mathbb{E}_{P*}[\mathfrak{u}(P^*)],$$

and that $\mathfrak{u}$ is strictly upwards l-proper iff the above inequality is always strict whenever $\pi \neq \pi_P$. For the left to right direction, assume $\mathfrak{u}$ is upwards l-proper (resp. strictly upwards l-proper). Given $x \in [0,1]$, take P defined over a two-celled partition $\{s_0, s_1\}$ with $P(s_0) = x$ and take a three-celled refinement $\pi = \{s_0^0, s_0^1, s_1\}$ of π. From Fact 3.12 we know that there is some $P^* \in [P]^*$ such that $\mathbb{E}_P[\mathfrak{u}(P)] \geq \mathbb{E}_{P*}[\mathfrak{u}(P^*)]$ (resp. $\mathbb{E}_P[\mathfrak{u}(P)] \geq \mathbb{E}_{P*}[\mathfrak{u}(P^*)]$). Let $y = P^*(s_0^0)$, and note that the above inequality entails that $\phi_\mathfrak{u}(x) \geq \phi_\mathfrak{u}(y) + \phi_\mathfrak{u}(x - y)$, since $x - y = P^*(s_0^1)$.

For the right to left direction, start by fixing P, $\pi \sqsupseteq \pi_P$ with $|\pi| - |\pi_P| = 1$, and let $s^* \in \pi_P$ and $s_0^*, s_1^* \in \pi$ be such that $s^* = s_0^* \cup s_1^*$. Let $x = P(s^*)$ and fix $y < x$ such that $\phi_u(x) \geq \phi_u(y) + \phi_u(x - y)$. Let P^* be the unique extension of P to π that assigns probability y to s_0^*, and note that, letting s range over π_P,

$$\mathbb{E}_P[u(P)] = \phi_u(x) + \sum_{s \neq s^*} \phi_u(x)$$

and

$$\mathbb{E}_{P^*}[u(P^*)] = \phi_u(y) + \phi u(x - y) + \sum_{s \neq s^*} \phi_u(x),$$

whence $\mathbb{E}_P[u(P)] \geq \mathbb{E}_{P^*}[u(P^*)]$. A simple induction argument on the size of $|\pi| - |\pi_P|$ allows us to conclude that for each $\pi \sqsupseteq \pi_P$ there is P^* with $\mathbb{E}_P[u(P)] \geq \mathbb{E}_{P^*}[u(P^*)]$, and thus that u is upwards l-proper. Parallel reasoning shows that if for each $x \in [0,1]$ there is $y < x$ such that $\phi_u(x) > \phi_u(y) + \phi_u(x - y)$, u is strictly upwards l-proper.

$\square$

4 Imprecise probabilities and partial credence functions

According to the standard, Bayesian picture we have been taking for granted, an agent's epistemic state can be adequately represented with a single probability function. But many think this is a mistake: on their view, an agent's epistemic state is best represented not with a single probability function but with a *set* thereof. This view can model any agent the more standard Bayesian picture can equally well—identify each probability function with its singleton set. But it is, at least on the face of it, more flexible. It can, for example, represent the kind of epistemic state most of us are arguably in with respect to the proposition that the last person to arrive in Australia in the year 2000 was wearing a white shirt: a state that seems hard to represent by assigning any one number to that proposition.

Grant that proponents of this dissenting view are right—grant, in other words, that one can be in the kind of epistemic state that is better modeled with a set of probability functions than with a single probability function.[16] An interesting question is whether it is ever epistemically *rational* to be in the kind of state that cannot be aptly represented with a unique probability function.

16 Cf. Joyce 2010, p. 283: ''It is rare, outside casinos, to find opinions that are anywhere near definite or univocal enough to admit of quantification. An agent with a precisecredence for, say, the proposition that it will rain in Detroit next July 4th should be able to assign an exact 'fair price' to a wager that pays $100 if the proposition is true and costs $50 if it is false. The best most people can do, however, is to specify some vague range.''

There has been much debate around this question and it is not my purpose here to take a stance either way.[17] But a family of related and interesting results that emerged from this debate bear some resemblance to the results established in this paper and it is worth clarifying exactly how they differ from our results.[18]

In the literature on epistemic utility theory, it is by and large taken for granted that something like the following principle captures an important relationship between epistemic utility and epistemic rationality:[19]

DOMINANCE: If for any world w, the epistemic utility of P at w is strictly lower than that of P' at w, and if for some world w, the epistemic utility of P at w is strictly less than that of P' at w, then P is epistemically irrational.

So much attention has been paid to the question what kinds of reasonable epistemic utility functions can be defined that allow us to compare the epistemic utility of a 'precise' credence function at a world with that of an 'imprecise' one—here we think of sets of probability functions as 'imprecise' or 'indeterminate' credence functions since for many propositions they do not determine a unique degree of credence.

For example, generalizing some results in (Schoenfield 2017), Berger & Das (2020) have argued that for any imprecise credence function there is a precise credence function that is at least as accurate relative to any world—at least given some assumptions about what a measure of accuracy must be like. And this, at least if we think that epistemic utility functions should be measures of accuracy, arguably shows that no epistemic utility function can be strictly upwards l-proper, and *a fortiori* that no epistemic utility function can be strictly universally l-proper or strictly universally u-proper.

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17 See e.g. Joyce 2005, Levi 1974, White 2010. For a helpful overview of this vast body of literature, see Bradley 2019.

18 See e.g. Berger & Das 2020, Konek forthcoming, Mayo-Wilson & Wheeler 2016, Schoenfield 2017, Seidenfeld, Schervish & Kadane 2012.

19 We say 'something like' because the principle as stated is in need of clarification and arguably subject to a number of powerful objections. For one thing, we need to clarify whether the principle holds for any admissible measure of epistemic utility, or whether it needs to be understood as quantifying over all admissible ways of measuring epistemic utility—Schoenfield 2017 opts for the latter, in formulating a principle she calls 'Permission', but Pettigrew 2016 opts for the former (see especially his discussion of the well-known 'Bronfman objection' in ch. 5). (As stated, the principle is perhaps closest to what Joyce 2009 calls 'Admissibility'.) For another, the principle might be subject to counterexamples in cases where *every* credence function suffers from a similar defect—if any credence function is dominated by another, say (if for any P there is some P' that has always at least as much and sometimes more epistemic utility than P), we may think some dominated credence functions are rationally permissible, even if not rationally required (see again the formulation of 'Permission' in Schoenfield 2017). Other, weaker, alternatives to Dominance include the principles Pettigrew 2016 calls 'Undominated Dominance' and 'Immodest Dominance', as well as the principle that Joyce 1998 relies on in his argument for Probabilism—the principle Pettigrew 2016 calls 'Dominance', which is weaker than what we are calling 'Dominance'.

To see why, note that for a fixed π and a refinement π' of π, we can identify any probability function P defined over π with an imprecise probability function defined over π'—essentially, we identify P with $[P]_\pi$ (see fn. 8). Berger & Das's results can then be used to show that on any reasonable measure of accuracy, for any P defined over π there is some P' defined over π' that is as accurate as P relative to any state of the world. Thus, if we identify epistemic utility functions with measures of accuracy satisfying their constraints, their result can be used to show that for any π, any refinement π' of π, and any P defined over π, there is P' defined over π' such that for any w, the epistemic utility of P at w equals that of P' at w. And this in turn would suffice to show that there are no strictly upwards l-proper epistemic utility function.

Now, we can first observe that in a sense our results are more general, in that they do not make any substantive assumptions about epistemic utility functions—at most, we assume that epistemic utility functions are continuous and truth-directed.[20] We do not, for instance, assume that epistemic utility is atomistic (we do not assume that the epistemic utility of a credence function at a world is determined by the utility of the individual credence assignments that make up that credence function at that world), nor that it is extensional (we do not assume that the epistemic utility of a credence function at a world is independent of the content of the propositions it assigns credence to).[21]

But there is a more significant difference between our results and those from the literature on imprecise probability functions. The question at the center of impossibility results for imprecise probability functions takes as given a fixed partition and asks whether there are reasonable ways of measuring accuracy or epistemic utility *for that partition* that will sometimes have imprecise probability functions doing better than precise probability functions. And one assumption all in the literature seem to take for granted—an assumption which seems perfectly natural given the presuppositions of the question—is that any reasonable measure of accuracy for a given partition π should satisfy the following constraint:[22]

PERFECTION: For any w, there is a credence function P_w that has maximal epistemic utility with respect to π: the epistemic utility of any credence

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20 An additional assumption, worth pointing out since it may go unnoticed, is that epistemic utility functions are real-valued. For a way of thinking about epistemic utility for imprecise probabilities that does without this assumption—a view on epistemic utility on which imprecise probabilities are only partially ranked in terms of epistemic utility relative to any world—see Seidenfeld, Schervish & Kadane 2012.

21 The results in Seidenfeld, Schervish & Kadane 2012 and Schoenfield 2017 rely on similar assumptions.

22 Cf. the principle Schoenfield 2017 calls 'Boundedness' (p. 672), and what Berger & Das 2020 call 'Local Boundedness' (p. 13), a principle which is implicitly assumed in Seidenfeld, Schervish & Kadane 2012 (see e.g. the proof of their Proposition 5 in (p. 1256)).

function, precise or not, defined over π and different from P_w, is strictly less than that of P_w.

To get a handle on what Perfection says, it helps to focus on a simple case with a two-cell partition $\pi^* = \{s_1, s_2\}$. Take some world $w \in s_1$ and consider the credence function P_w that assigns 1 to s_1 and 0 to s_2. It seems natural to say that, with respect to π^*, any credence function different from P_w has lower epistemic utility, at w, than P_w has. In particular, relative to w, any (non-trivially) imprecise credence function—any set of probability functions defined over π with more than one element—is worse, epistemically and with respect to π^*, than P_w. After all, relative to w, P_w has a legitimate claim to being as good as it gets, epistemically *with respect to π^**.

Now, in our current framework we have not trafficked in anything quite like the notion of epistemic utility relative to a partition. So it is not completely straightforward to translate Perfection into a constraint on the kind of epistemic utility functions we have been interested in. But there is a somewhat natural way to recast Perfection into a constraint on generalized epistemic utility functions in our sense. And once we see what that constraint amounts to, we will see both that it is not quite so plausible (as a constraint on generalized epistemic utility functions) and that our results do not depend on it.

Recall that in comparing the discussion of imprecise probability functions over a partition with our discussion of credence functions whose domain does not include elements of that partition, we identified a (precise) credence function defined over a coarsening π' of π with an imprecise credence function defined over π. Hence, saying that P is better, epistemically and relative to w, than any imprecise credence function defined over the same domain as P, entails that that for any coarsening of π, P is better, relative to w, than any other credence function defined over that coarsening. So in our framework, Perfection amounts to the claim that relative to any π, any non-trivial coarsening π' of π, and any w, there is some credence function P_w defined over π that is better, epistemically relative to w, than any credence function defined over π'. Equivalently, in our framework Perfection amounts to the claim that for any partition, any world w, and any refinement of π, there is a credence function defined over that refinement that is better relative to w than any credence function defined over π:

REFINEMENT: For any π, any refinement π' of π, and any w, there is $P_w^{\pi'}$ defined over π' such that for any P defined over π, $\mathfrak{u}(P_w^{\pi}, w) > \mathfrak{u}(P, w)$.

Now, it should be clear that our results do not depend on anything like Refinement. After all, Refinement rules out as admissible any upwards u-proper epistemic utility function, whereas our assumptions are compatible with the

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admissible of such epistemic utility functions. So, strictly speaking, this is another sense in which our results are more general. But it is worth highlighting that, whereas in the discussion of imprecise probability functions something like Refinement may well be uncontroversial, in the present context it is far from it.

The constraint imposed by Refinement is incompatible with thinking of some refinements as an unalloyed epistemic bad: if epistemic utility satisfies Refinement, there can be no proposition such that that you are epistemically worse off no matter what when you come to form an opinion on that proposition. Whether it be a proposition about phlogiston, or about miasma, Refinement entails that it is always in principle possible to do better, epistemically, by forming a view on that proposition.

Of course, it may be that this is the right way to think about epistemic utility, but it is certainly not *obviously* the right way to think about it. One might, for example, think that there is an ideal language for theorizing about the world, and that the ideal epistemic state is the one that is maximally accurate with respect to propositions expressible in that ideal language and simply fails to even entertain hypotheses that cannot be formulated in that language. If that's how we think about epistemic utility, we will want to reject Refinement.[23]

At any rate, it is not our goal here to suggest that the right way to think about epistemic utility is incompatible with Refinement. But we do want to point that is yet another substantive assumption about epistemic utility that is required for the impossibility results mentioned above to go through. In contrast, our results make no substantive assumptions about epistemic utility. Rather, they establish that no matter *how* we think of epistemic utility, there are hard limits on the degree of immodesty we can expect to come from epistemic rationality.[24]

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23 Note that this view is incompatible with Extensionality—the thesis that the epistemic utility of a credence function at a world is independent of the content of the propositions it assigns credence to. Indeed, it may be that a commitment to Extensionality all but requires a commitment to Refinement.

24 We should add that whereas our results rely on much weaker assumption than those from the literature on imprecise credence functions, they are not stronger than them, since the conclusions they derive from their stronger assumptions are stronger than those we derive from our weaker assumptions. For instance, as mentioned above, Berger & Das show that, given their assumptions on epistemic utility functions, for any imprecise credence function there will be precise credence function with the same domain that is as good, epistemically, than the imprecise credence function relative to any world. The analogous conclusion, in our framework, would be that for any credence function P defined over some partition π, and any refinement π' of π, there is a credence function defined over π that is as good, epistemically, than P relative to any world. Without any additional assumptions on what epistemic utility functions are like, this cannot be guaranteed. (For one thing, without additional assumptions, we could have epistemic utility functions that make any credence function defined over π dominate any credence function defined over π'.) It is an interesting question, beyond the scope of this paper, what additional constraints on generalized epistemic utility functions are needed to establish this analogous result.

5 Conclusion

In contexts where probability functions are stipulated to all be defined over a fixed domain, strictly proper epistemic utility functions arguably capture a certain kind of immodesty. Once we move on to contexts where probability functions are allowed to be defined over different domains, strictly proper epistemic utility functions do not capture the relevant sense of immodesty. Our question was whether there was a way of characterizing immodesty in this general setting. We considered a variety of strong, generalized immodesty principles and showed that, under minimal assumptions, no epistemic utility function satisfies any of these stronger immodesty principles.

We also considered some very weak generalizations of strict propriety and showed that some of the familiar epistemic utility functions satisfy one or another of these weak immodesty principles. One interesting question left outstanding is how strong an immodesty principle can be imposed without ruling out every reasonable epistemic utility function. In particular, one interesting question is whether there are immodesty principles that distinguish among partitions—say, immodesty principles that say that for any partition of a certain kind, all credence functions defined over that partition take themselves to be doing better, in terms of epistemic utility, than any of their restrictions or extensions.

We have not, of course, argued that epistemic utility functions ought to satisfy any of these stronger immodesty principles. But it is at the very least not obvious that strict partition-wise propriety suffices to capture the sense in which epistemic rationality is said to be immodest. What else, if anything, suffices to capture that kind of immodesty is a question for some other time.

Appendix

Our proof of Lemma 3.11 will rely on a fundamental result in game theory, which we will simply state without proof.[25] Before stating the result, we need to introduce some minimal background.

A *two-person, zero-sum game* (henceforth, a *game*) is a tripe $\mathcal{G} = (A, B, f)$, where A is the set of *pure strategies* for player I, B is the set of pure strategies for player II, and $f : A \times B \longrightarrow \overline{\mathbb{R}}$ is a *payoff function*. When player I chooses to play $a \in A$ and player II chooses to play $b \in B$, player I gets $f(a, b)$ from player II if $f(a, b) > 0$ and gives player II $-f(a, b)$ if $f(a, b) < 0$ (nothing is exchanged if $f(a, b) = 0$, and let's not bother to think of an 'intuitive' interpretation of a situation in which $f(a, b)$ is non-finite).

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25 Our proof strategy follows some of the reasoning in the first five sections of Grünwald & Dawid 2004.

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The *lower value* of the game, $\underline{V}$ is defined as

$$\underline{V} := \sup_{a \in A} \inf_{b \in B} f(a, b).$$

This is the maximum payoff that player *I* can guarantee, since for each $a \in A$,

$$\inf_{b \in B} f(a, b)$$

is the best player I can do. The *upper value* of the game, $\overline{V}$ is analogously defined as

$$\overline{V} := \inf_{b \in B} \sup_{a \in A} f(a, b).$$

In general,

$$\underline{V} \leq \overline{V}.$$

We say that $\mathcal{G}$ *has a value* iff

$$\underline{V} = \overline{V}.$$

If a game has a value, we say that player I has an *optimal strategy* iff there is $a^* \in A$ that achieves

$$\sup_{a \in A} \inf_{b \in B} f(a, b).$$

Similarly, we say that player II has an optimal strategy iff there is $b^* \in B$ achieving

$$\inf_{b \in B} \sup_{a \in A} f(a, b).$$

If the game has a value and both players have an optimal strategy, the pair of optimal strategies corresponds in an intuitive way to an *equilibrium* in the game—a pair of strategies such that neither player prefers unilaterally deviating from it. Such a pair of strategy is called a *saddle-point*—thus, a saddle point in a game $\mathcal{G}$ is a pair of strategies (a^*, b^*) such that for all $a \in A$, $b \in B$, $f(a, b^*) \leq f(a^*, b^*) \leq f(a^*, b)$.

Not all games have a value. Some of the foundational results in game theory allow us to characterize classes of games that have a value. We will be relying on one such result for our proof of Lemma 3.11.

Recall that a function f on a vector space that takes values in $\overline{\mathbb{R}}$ is *convex* iff for each $\lambda \in (0, 1)$, $\lambda f(x) + (1 - \lambda) f(y) > f(\lambda x + (1 - \lambda) y)$ whenever the term on the left hand side is well-defined. We say that f and is *concave* iff $-f$ is convex, and that f is *affine* iff it is both convex and concave.

If X is a topological space, we say that a function $f : X \longrightarrow \overline{\mathbb{R}}$ is *upper-semi-continuous* (or *u.s.c.*) iff $f < \infty$, and for each $r \in X$, the set $\{x \in X : f(x) \geq r\}$ is closed in X. We say that f is *lower-semi-continuous* (or *l.s.c.*) iff $f > -\infty$ and for

each $r \in \mathbb{R}$, the set $\{x \in X : f(x) \le r\}$ is closed in X. (Here we follow Mertens, Sorin & Zamir 2015.) Of course, f is u.s.c. iff $-f$ is l.s.c. The result below is essentially Sion's minimax theorem (Sion 1958).[26]

Theorem A.1. *Let A and B be two convex topological spaces and suppose $f :$ $A \times B \longrightarrow \overline{\mathbb{R}}$ is concave and u.s.c. on the first argument, and convex and l.s.c. on the second—that is, for any $b \in B$ and $a \in A$, $f(x, b)$ and $-f(a, y)$ are concave, u.s.c. functions of x and y (respectively). Then the game $\mathcal{G} = (A, B, f)$ has a value. If A and B are compact, then the game has a saddle-point.* □

We can apply Theorem A.1 to show that, whereas many games of interest do not contain a saddle-point, if we allow players to *randomize* their choice of strategy, the resulting game does have a saddle-point. Let me explain.

For any compact $X \subseteq \mathbb{R}^n$, let $\Delta(X)$ denote the space of all Borel probability functions on X—the space of all countably additive probability functions on X whose domain is the smallest σ-algebra that contains all the open subsets of X. Of course, $\Delta(X)$ is a convex set, and from the fact that X is a compact subset of Euclidean space, we know that $\Delta(X)$ is compact.[27]

Now, fix $\mathcal{G} = (A, B, f)$, with $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ compact. We say that $\mathcal{G}^* = (A^*, B^*, f^*)$ is a *mixed extension* of $\mathcal{G}$ iff A^* (resp. B^*) is a closed and convex subset of $\Delta(A)$ (resp. $\Delta(B)$) and, for any $\alpha \in A^*$, $\beta \in B^*$,

$$f^*(\alpha, \beta) = \mathbb{E}_{X \sim \alpha} \mathbb{E}_{Y \sim \beta} f(X, Y).$$

Since each $a \in A$ (resp. $b \in B$) can be identified with the unique probability function $\alpha_a \in \Delta(A)$ (resp. $\beta_b \in B$) that assigns probability one to $\{a\}$ (resp. $\{b\}$), we will abuse notation and think of f^* as also defined over elements of $A \times B$.

If f is continuous and $f < \infty$, f^* is concave (since linear) and u.s.c. on the first argument and convex (since linear) and l.s.c. on the second. And since any closed subset of a compact topological space is compact, we know that A^* and B^* are compact and convex topological spaces. So we can apply Theorem A.1 to show that any mixed extension $\mathcal{G}^*$ has a saddle-point.

Corollary A.2. *Suppose $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ are compact and $f : A \times B \longrightarrow \mathbb{R}$ is separately continuous. If $f < \infty$, then any mixed extension of $\mathcal{G} = (A, B, f)$ has a saddle-point.* □

We can finally present our proof of Lemma 3.11.

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26 See e.g. Mertens, Sorin & Zamir 2015, Theorem 1.1.1, for a proof.
27 See e.g. Kechris 1995, Theorem 17.22.

Proof of Lemma 3.11. Suppose $\mathfrak{u}$ is a continuous epistemic utility function that is partition-wise strictly proper. Fix a probability function P and a refinement π of π_P. We define a game $\mathcal{G}_P = (A, B, f)$ as follows. First, let $N = |\pi|$, fix an enumeration $\{S_i\}$ of π, and let A be those elements of $\mathbb{R}^N$ of the form $\langle Q(s_1), \ldots Q(s_N) \rangle$ for $q \in \mathcal{P}_\pi$. Abusing notation, we will use Q, Q', etc. to denote elements of A, even though we will think of them as members of $\mathbb{R}^N$. Next let

$$B := \{Q \in A : Q(s_i) = 1 \text{ for some } i\}.$$

Again abusing notation, we will use $s_1, s_2, \ldots, s_N$ to denote the elements of B in the obvious way (with s_i corresponding to that $Q \in A$ assigning probability 1 to s_i). Finally, let

$$f(Q, s_i) = \mathfrak{u}(Q, s_i).$$

Note that A and B are compact subsets of $\mathbb{R}^N$, and since $\mathfrak{u} < \infty$, we know that any mixed extension of $\mathcal{G}_P$ has a saddle point.

   Our next step is to define a particular mixed extension of $\mathcal{G}_P$ and apply Corollary A.2. Before doing so, however, let us make a couple of observations. First, any element of $\Delta(A)$ corresponds to a probability function over $\mathcal{P}_\pi$. We will use μ, μ', etc. to denote elements of $\Delta(A)$, and will continue to abuse notation and use Q, Q', etc. to denote the element of $\Delta(A)$ that assigns probability 1 to the corresponding element of $\mathcal{P}_\pi$. Second, any element of $\Delta(B)$ corresponds to a probability function over π. We will thus abuse notation and use Q, Q', etc. to denote elements of $\Delta(B)$.

   Let now $A^* = \Delta(A)$ and $B^* = [P]_\pi$, and note that B^* is indeed a subset of $\Delta(B)$. Moreover, both A^* and B^* are closed and convex subsets of $\Delta(A)$ and $\Delta(B)$ (respectively), so that $\mathcal{G}_P^* = (A^*, B^*, f^*)$ is indeed a mixed extension of $\mathcal{G}_P$, with

$$f^*(\mu, Q) = \mathbb{E}_{X \sim \mu} \mathbb{E}_Q[\mathfrak{u}(X)],$$

and accordingly $f^*(Q', Q) = \mathbb{E}_Q[\mathfrak{u}(Q')]$.

   From Corollary A.2, we know that our game $\mathcal{G}_P^*$ has a saddle point $(\mu^*, \hat{P})$. We claim that this saddle point is in fact of the form $(\hat{P}, \hat{P})$. To see why, note that since μ^* is an optimal mixed strategy for player I, it follows that for any $Q \in \mathcal{P}_\pi$,

$$f^*(Q, \hat{P}) \leq f^*(\mu^*, \hat{P}),$$

and thus that

$$\mu^* \left(\arg\max_{Q \in \mathcal{P}_\pi} f^*(Q, \hat{P}) \right) = 1.$$

But since $\mathfrak{u}$ is strictly partition-wise proper,

$$\arg\max_{Q \in \mathcal{P}_\pi} f^*(Q, \hat{P}) = \{\hat{P}\}.$$

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Summing up, we have a saddle point of the form $(\hat{P}, \hat{P})$, and thus we know that for any $P^* \in [P]_\pi$ and any $Q \in \mathcal{P}_\pi$,

$$f^*(Q, \hat{P}) \leq f^*(\hat{P}, \hat{P}) \leq f^*(\hat{P}, P^*).$$

In other words,

$$\mathbb{E}_{\hat{P}}[\mathfrak{u}(Q)] \leq \mathbb{E}_{\hat{P}}[\mathfrak{u}(\hat{P})] \quad \text{for all } Q \in \mathcal{P}_\pi, \tag{3}$$

and

$$\mathbb{E}_{\hat{P}}[\mathfrak{u}(\hat{P})] \leq \mathbb{E}_{P^*}[\mathfrak{u}(\hat{P})] \quad \text{for all } P^* \in [P]_\pi. \tag{4}$$

But note that (3) entails both

$$\mathbb{E}_{\hat{P}}[\mathfrak{u}(\hat{P})] = \underline{\mathbb{E}}_P[\mathfrak{u}(\hat{P})], \tag{5}$$

by definition, and

$$\mathbb{E}_{\hat{P}}[\mathfrak{u}(\hat{P})] \leq \mathbb{E}_{P^*}[\mathfrak{u}(P^*)] \quad \text{for all } P^* \in [P]_\pi, \tag{6}$$

since $\mathfrak{u}$ is partition-wise proper.

Hence, from (4), (5), and (6), we have that for any $P^* \in [P]_\pi$ and any $Q \in \mathcal{P}_\pi$,

$$\mathbb{E}_{\hat{P}}[\mathfrak{u}(Q)] \leq \mathbb{E}_{\hat{P}}[\mathfrak{u}(\hat{P})] = \underline{\mathbb{E}}_P[\mathfrak{u}(\hat{P})] \leq \mathbb{E}_{P^*}[\mathfrak{u}(P^*)],$$

as desired.

$\square$

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