Notes on Mathematics

Contents

1 Calculus

1.1 Differentation and Integration

Lemma 1.1 (Simple Calculations)**.**

1. *For* $1 = xx^{-1}$ *the product rule yields* $0 = x^{-1} + x(x^{-1})'$. *Hence*
$$\frac{d}{dx}x^{-1} = -\frac{1}{x^2}$$

2. *Similarly* $x = \sqrt{x}^2$ *and* $1 = 2\sqrt{x}\sqrt{x}'$ *and so*
$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

3. *It is*
$$\frac{d}{dx}x^n = nx^{n-1}$$
since via induction the product rule yields
$$\frac{d}{dx}x^n = \frac{d}{dx}xx^{n-1} = x^{n-1} + \frac{d}{dx}x^{n-1} = x^{n-1} + (n-1)x^{n-1} = nx^{n-1}$$

4. *Again, applying the product rule gives*
$$\left(\frac{1}{g}\right)' = \left(\frac{1}{x} \circ g\right)' = -\frac{g'}{g^2}$$
and the quotient rule
$$\left(\frac{f}{g}\right)' = \frac{f'}{g} + f\left(\frac{1}{g}\right)' = \frac{f'}{g} - \frac{fg'}{g^2} = \frac{gf' - fg'}{g^2}$$

5. *Also* $x = f \circ f^{-1}$ *and* $1 = (f^{-1})'f' \circ f^{-1}$. *Thus*
$$(f^{-1})' = \frac{1}{f' \circ f^{-1}}$$
where defined. Especially for $x \neq 0$
$$\log'(x) = \frac{1}{\exp'(\log(x))} = \frac{1}{x}$$

6. $(1-q)(1+q+q^2+\cdots+q^n) = 1-q+q-q^2+q^2-q^3+\cdots+q^{n+1}$ *gives*
$$\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q} \quad \text{and} \quad \sum_{k=m}^{n} q^k = \frac{q^m - q^{n+1}}{1-q}$$

Lemma 1.2 (Exponential Function)**.**

1. *It is*
$$\exp(x+y) = \exp(x)\exp(y)$$
Hence
$$\begin{aligned}\exp(0) &= 1\\ \exp(-x) &= \exp(x)^{-1}\\ \exp(nx) &= \exp(x)^n\end{aligned}$$
2. *For the derivative*
$$\exp'(x) = \sum_{k=0}^{\infty} \frac{1}{k!}(x^k)' = \sum_{k=0}^{\infty} \frac{1}{k!} k x^{k-1} = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} x^{k-1} = \exp(x)$$

Lemma 1.3 (Sinus and Cosinus)**.**

1. *Sinus and Cosinus power series*
$$\begin{aligned}\cos(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k!} x^{2k}\\ \sin(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}\end{aligned}$$
2. *Symmetry*
$$\begin{aligned}\cos(-x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k!} (-x)^{2k} = \cos(x)\\ \sin(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (-x)^{2k+1} = -\sin(x)\end{aligned}$$
3. *Derivatives*
$$\begin{aligned}\cos'(x) &= \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)!} x^{2k-1} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!} x^{2k+1} = -\sin(x)\\ \sin'(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k!} x^{2k} = \cos(x)\end{aligned}$$

Theorem 1.4 (Fermat Stationary Point)**.** *Let $\Omega \subseteq \mathbb{R}$ be open and $f \in C^1(\Omega)$. If $x^* \in \Omega$ is local extremum then $f'(x^*) = 0$.*

Proof. Assume x^* is the minimum of f in Ω and let $f(x^*) > 0$. Since $f \in C^1(\Omega)$ there exist $\varepsilon, \delta > 0$, so that for $|h| \le \varepsilon$
$$\frac{f(x^*+h) - f(x^*)}{h} > \delta$$
Pick a negative $h \in [-\varepsilon, 0)$. Then
$$f(x^*+h) < f(x^*) + \delta h < f(x^*)$$
and x^* cannot be the minimum. Analog for maximum with a positive h, then apply to $-f$. $\square$

Theorem 1.5 (Rolle)**.** *Let $f \in C[a,b]$ with $f(a) = f(b)$. If f is differentiable in (a,b) then there exists a $\xi \in (a,b)$ with $f'(\xi) = 0$.*

Proof. Assume f is not constant. Since $[a,b]$ is compact there exists either a global minimum or maximum $\xi \in (a,b)$ and Theorem 1.4 can be applied. □

Theorem 1.6 (Mean Value)**.** *Let $f \in C[a,b]$ be differentiable in (a,b). Then there exists a $\xi \in (a,b)$ with*

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

Proof. Apply Theorem 1.5 to

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a)$$

□

Remark 1.7.

1. *More generally choose any $\varphi \in C^1[a,b]$ with $\varphi(a) = 0$ and $\varphi(b) = f(b) - f(a)$. Set $g(x) = f(x) - \varphi(x)$ to see there is a $\xi \in (a,b)$ with $f'(\xi) = \varphi'(\xi)$.*
2. *Let f be differentiable in (a,b) with $f' = 0$. For $x, y \in (a,b)$*

$$0 = f'(\xi) = \frac{f(y) - f(x)}{y - x}$$

and f is a constant.

3. *Another useful generalization: let $\Omega \subseteq \mathbb{R}^n$ be open and $f \in C^1(\Omega)$. For $x, y \in \Omega$ define $\varphi(t) = f(tx + (1-t)y)$ and apply the chain rule for differentiation*

$$\varphi'(\xi) = \nabla f(\xi x + (1-\xi)y)^T(x - y) = f(x) - f(y)$$

4. *The Cauchy Schwarz inequality then yields*

$$\|f(x) - f(y)\| \le \|\nabla f(\xi x + (1-\xi)y)\| \|(x - y)\|$$

Theorem 1.8 (Differentiation Theorem)**.** *Let $f \in C[a,b]$ and define*

$$F(x) = \int_a^x f(t)\,dt$$

Then $F \in C^1[a,b]$ with $F'(x) = f(x)$ for $x \in [a,b]$.

Proof. Applying the Mean Value Theorem of Integration gives

$$F(x+h) - F(x) = \int_x^{x+h} f(t)\,dt = f(\xi)h$$

for some $\xi \in (x, x+h)$. □

Theorem 1.9 (Fundamental Theorem of Calculus)**.** *Let $F \in C^1[a,b]$ with $F' = f$ Then*

$$F(b) - F(a) = \int_a^b f(t)\,dt$$

Lemma 1.10 (Integration by Substitution)**.** *Let $I \subseteq \mathbb{R}$ be an interval and $f \in C(I)$. For $\varphi \in C([a,b], I)$ it follows*

$$\int_{\varphi(a)}^{\varphi(b)} f(x)\,dx = \int_a^b f(\varphi(t))\varphi'(t)\,dt$$

Proof. Let $F \in C^1(I)$ with $F' = f$. Then the chain rule for differentiation yields

$$\begin{aligned}\int_{\varphi(a)}^{\varphi(b)} f(x)\,dx &= F(\varphi(b)) - F(\varphi(a)) \\ &= F \circ \varphi(b) - F \circ \varphi(a) \\ &= \int_a^b (F \circ \varphi)'(t)\,dt \\ &= \int_a^b f(\varphi(t))\varphi'(t)\,dt\end{aligned}$$

□

Examples 1.11.

1. *For $\varphi(x) = x^2 + 1$ it is $\varphi(0) = 1$ and $\varphi(2) = 5$. Thus*

$$\int_0^2 x\cos(x^2+1)\,dx = \frac{1}{2}\int_0^2 2x\cos(x^2+1)\,dx = \frac{1}{2}\int_1^5 \cos(t)\,dt = \frac{1}{2}(\sin(5) - \sin(1))$$

2. *Consider $\varphi(x) = \sin(x)$ where $\varphi(0) = 0$ and $\varphi(\pi/2) = 1$. Since $\cos(t) = \sqrt{1-\sin^2(t)}$ it follows*

$$\int_0^1 \sqrt{1-x^2}\,dx = \int_{\cos(0)}^{\cos(\pi/2)} \sqrt{1-x^2}\,dx = \int_0^{\pi/2} \sqrt{1-\sin^2(t)}\cos(t)\,dt = \int_0^{\pi/2} \cos^2(t)\,dt$$

3. *Let $f \in C[a,b]$ and $\varphi(x) = a + t(b-a)$. Then*

$$\int_a^b f(x)\,dx = (b-a)\int_0^1 f(a + t(b-a))\,dt$$

4. *Let $f(x) = x^n$ and $\varphi(x) = t^m$. As expected*

$$\int_0^1 x^n\,dx = \int_0^1 t^{nm} m t^{m-1}\,dt = m\int_0^1 t^{m(n+1)-1}\,dt = \left[\frac{m}{m(n+1)} t^{m(n+1)}\right]_0^1 = \frac{1}{n+1}$$

1.2 Directional Derivative and Gradients

Lemma 1.12 (Directional Derivative). *Let $\Omega \subseteq \mathbb{R}^n$ be open and $f \in C^1(\Omega)$. Then*

$$\frac{\partial f}{\partial d}(x) = \nabla f(x)^T d$$

for any $d \in \mathbb{R}^n$.

Proof. Let $\varphi(t) = f(x + td)$. Then $\varphi \in C^1[-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$ and the chain rule yields

$$\varphi'(t) = \nabla f(x + td)^T d$$

Hence

$$\varphi'(0) = \lim_{t \to 0} \frac{\varphi(x + td) - \varphi(0)}{t} = \lim_{t \to 0} \frac{f(x + td) - f(x)}{t} = \nabla f(x)^T d$$

□

Remark 1.13.

1. *Note that by definition the directional derivative is invariant under multiplication with any $\lambda \neq 0$.*
2. *A similar proposition holds under the weaker assumption that d is a only feasable direction for f in x*
3. *For $d = \nabla f(x)/\|\nabla f(x)\|$ it follows that*

   $$\frac{\partial f}{\partial d}(x) = \|\nabla f(x)\| > 0$$

   and for any other $d \in \mathbb{R}^n$ with $\|d\| = 1$ the Cauchy Schwarz inequality yields

   $$|\frac{\partial f}{\partial d}(x)| = |\nabla f(x)^T d| \leq \|\nabla f(x)\| \|d\| = \|\nabla f(x)\|$$

   Hence $\nabla f(x)$ is the direction of the greatest ascent and respectively, $-\nabla f(x)$ is the direction of the greatest descent.

Theorem 1.14 (First Order Necessary Condition). *Let $\Omega \subseteq \mathbb{R}^n$ be open and $f \in C^1(\Omega)$. If $x^* \in \Omega$ is a local minimzer then $\nabla f(x^*) = 0$.*

Proof. Let $h \in \mathbb{R}^n$ and $\delta > 0$ so that $x^* + th \in \Omega$ for all $t \in (-\delta, \delta)$. Then 0 is local minimizer for $\varphi(t) = f(x^* + th)$ and

$$\varphi'(0) = \nabla f(x^*)^T h = 0$$

Now let $h = \nabla f(x^*)$. □

Theorem 1.15 (Banach Fixed-Point Theorem). *Let X be a Banach space and $f \in C(X, X)$ a contraction*

$$\|f(x) - f(y)\| \leq q\|x - y\| \text{ for all } x, y \in X$$

for some $0 < q < 1$. Then there exists a unique fix point $x^* \in X$ with

$$f(x^*) = x^*$$

Furthermore for any $x_0 \in X$ the sequence defined by

$$x_{n+1} = f(x_n)$$

converges aganist x^*.

Proof. Since $\|x_{n+1} - x_n\| = \|f(x_n) - f(x_{n-1})\| \leq q\|x_n - x_{n-1}\|$ it follows, that

$$\|x_{n+1} - x_n\| \leq q^n \|x_1 - x_0\|$$

Furthermore

$$\|x_n - x_m\| \leq \sum_{k=m}^{n} q^k \|x_1 - x_0\| = \frac{q^m - q^{n+1}}{1-q} \|x_1 - x_0\|$$

and (x_n) is a Cauchy sequence. For its limit x^* we have

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(x^*)$$

For any other $y^* \in X$ with $f(y^*) = y^*$ it follows, that

$$\|x^* - y^*\| = \|f(x^*) - f(y^*)\| \leq q\|x^* - y^*\|$$

and therefore $x^* = y^*$.

$\square$

2 Nonlinear Optimization

2.1 Minimization without Constraints

Lemma 2.1 (Gradient Inequality)**.** *Let $M \subseteq \mathbb{R}^n$ be a convex set and $f \in C^1(M)$. Then f is convex if and only if*

$$f(x) \geq f(y) + \nabla f(y)^T (x - y)$$

for all $x, y \in M$.

Proof. Let f be convex and $x, y \in M$. For $0 \leq \lambda \leq 1$ we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) = \lambda f(x) - \lambda f(y) + f(y)$$

and

$$f(x) - f(y) \geq \frac{f(\lambda x + (1-\lambda)y) - f(y)}{\lambda} = \frac{f(y + \lambda(x-y)) - f(y)}{\lambda}$$

For $d = x - y$ and $\lambda \to 0$ the term on the right converges to the direction derivative of f in d

$$\frac{\partial f}{\partial d}(y) = \nabla f(y)^T d = \nabla f(y)^T (x - y)$$

Now let $x, y \in M$ and $0 \leq \lambda \leq 1$. For $z = \lambda x + (1-\lambda)y \in M$ it follows that

$$\begin{aligned} \lambda f(x) &\geq \lambda f(z) + \lambda \nabla f(z)^T (x - z) \\ (1-\lambda) f(y) &\geq (1-\lambda) f(z) + (1-\lambda) \nabla f(z)^T (y - z) \end{aligned}$$

Adding the two inequalities gives

$$\begin{aligned} \lambda f(x) + (1-\lambda) f(y) &\geq f(z) + \nabla f(z)^T (\lambda x - \lambda z + (1-\lambda) y - (1-\lambda) z) \\ &= f(z) + \nabla f(z)^T (\lambda x + (1-\lambda) y - z) \\ &= f(z) \end{aligned}$$

□

Exercise 2.2 (Facility Locations)**.** *The facilities are located at:*

$$(3, 0), (0, -3), (1, 4)$$

Proof. Let

$$\begin{aligned} f(x) &= (x-3)^2 + y^2 + x^2 + (y+3)^2 + (x-1)^2 + (y-4)^2 \\ &= x^2 - 6x + 9 + y^2 + x^2 + y^2 + 6y + 9 + x^2 - 2x + 1 + y^2 - 8y + 16 \\ &= 3x^2 + 3y^2 - 8x - 2y + 35 \end{aligned}$$

Then

$$\nabla f(x, y) = (6x - 8, 6y - 2) \text{ and } \nabla^2 f(x, y) = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} > 0$$

Hence $(4/3, 1/3)$ is the gobal minimum. □

Exercise 2.3 (Convex Functions)**.** *The sum of convex functions is convex.*

Proof. Let $x, y \in M$. Since $\alpha_i > 0$ we have

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= \sum_{i=1}^{m} \alpha_i f_i(\lambda x + (1-\lambda)y)) \\ &\leq \sum_{i=1}^{m} \alpha_i \lambda f_i(x) + \sum_{i=1}^{m} \alpha_i (1-\lambda) f_i(y) = \lambda f(x) + (1-\lambda) f(y) \end{aligned}$$

Let $f(x) = x^2$. Then $-f$ is not convex, e.g. $x = 1, y = -1$ and $\lambda = 0.5$.

Exercise 2.4 (Solution of Quadratic Inequality)**.** *Let*

$$f(x) = x^T A x + b^T x + c$$

Proof. The product rule gives

$$\nabla f(x) = x^T A + Ax + b = (A^T + A)x + b = 2Ax + b$$

Thus $\nabla^2 f(x) = 2A > 0$ and f is convex. Hence the level set Γ_{-c} is convex. Since the intersection of convex sets is convex $\Gamma_{-c} \cap \{x \in \mathbb{R}^n : g^T x + h = 0\}$ is convex, too.

Exercise 2.5 (Line Search on Compact Convex Sets)**.** *Let $S \subset \mathbb{R}^n$ be compact and convex. Furthermore let $f \in C^1(S)$ be convex, $x \in S$ and $d \in \mathbb{R}^n$ a descent direction of f in x with $\nabla f(x)^T d < 0$.*

Proof. If $x + \lambda^* d$ is an optimal solution then $\nabla f(x + \lambda^* d)^T d = 0$ according to Theorem 1.14. Let $\nabla f(x + \lambda^* d)^T d = 0$. Then Lemma 2.1 gives

$$f(x + \lambda d) \geq f(x + \lambda^* d) + (\lambda - \lambda^*) \nabla f(x + \lambda^* d)^T d = f(x + \lambda^* d)$$

and $x + \lambda^* d$ is an optimal solution.

Exercise 2.6 (Steepest Descent)**.** *Let*

$$f(x) = \frac{1}{2} x^T A x + b^T x + c$$

where A is symmetrical and positive definite.

Proof. Since $\nabla f(x) = Ax + b$ and $\nabla^2 f(x) = A > 0$ it follows $x^* = -A^{-1} b$. Let v be eigenvector with $Av = \mu v$. For $x_0 = x^* + \theta v$ we have

$$\nabla f(x_0) = Ax^* + \mu \theta v + b = \mu \theta v$$

and for $\lambda \geq 0$

$$\arg\min\{f(x_0 - \lambda \nabla f(x_0)\} = \arg\min\{f(x^* + \theta v - \lambda \mu \theta v)\} = \mu^{-1}$$

Thus

$$x_1 = x_0 - \mu^{-1} \nabla f(x_0) = x^* + \theta v - \mu^{-1} \mu \theta v = x^*$$

and $\nabla f(x_1) = 0$. Hence the algorithm stops after the first iteration. Now let

$$x_0 = x^* + \sum_{i=0}^{m} \theta_i v_i$$

for orthogonal eigenvectors with $Av_i = \mu_i$ and $m \leq n$. Then

$$\nabla f(x_0) = Ax^* + \sum_{i=0}^{m} \mu_i \theta_i v_i + b = \sum_{i=0}^{m} \mu_i \theta_i v_i$$

and

$$x_1 = x_0 - \lambda \sum_{i=0}^{m} \mu_i \theta_i v_i = x^* + \sum_{i=0}^{m} \theta_i v_i - \lambda \sum_{i=0}^{m} \mu_i \theta_i v_i = x^* + \sum_{i=0}^{m} (1 - \lambda \mu_i) \theta_i v_i$$

Since x^* is the minimum we have $\nabla f(x_1) = 0$ iff $\lambda = \mu^{-1}$ for all $0 \leq i \leq m$. □

2.2 One Dimensional Minimization and Direct Search

Definition 2.7 (Unimodal Function)**.** *A function $f : [a, b] \to \mathbb{R}$ is called unimodal if there exists a $\xi \in [a, b]$, so that f is strictly decreasing in $[a, \xi]$ and strictly increasing in $[\xi, b]$.*

In fact ξ is the unique minimum of f in $[a, b]$. According to the definition, for $a \leq x < y \leq b$ we have

$$f(x) > f(y) \text{ for } x, y \in [a, \xi) \text{ and } f(x) < f(y) \text{ for } x, y \in (\xi, b]$$

Thus

$$\xi \in [a, y] \text{ if } f(x) < f(y) \text{ and } \xi \in [x, b] \text{ if } f(x) \geq f(y)$$

Consider now a symmetrical partioning of the interval $[0, 1]$ where two consecutive partionings hold the same ratio respectively:

$$\sigma = 1 - \tau \text{ and } \frac{1}{\tau} = \frac{\tau}{\sigma}$$

Then $1 - \tau = \tau^2$ and solving the quadratic equation $\tau^2 + \tau = 1$ yields

$$\tau = \frac{\sqrt{5} - 1}{2} \approx 0.61803$$

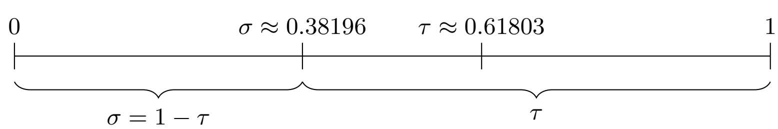

Figure 1: Golden Section

Let now $[a_0, b_0] = [a, b]$ and define

$$[a_{k+1}, b_{k+1}] = \begin{cases} [a_k, y_k] & \text{if } f(x_k) < f(y_k) \\ [x_k, b_k] & \text{if } f(x_k) \geq f(y_k) \end{cases}$$

where

$$x_k = b_k - \tau(b_k - a_k)$$
$$y_k = a_k + \tau(b_k - a_k)$$

It follows that $[a_k, b_k] \supset [a_{k+1}, b_{k+1}]$ is a decreasing series of intervals with

$$(b_{k+1} - a_{k+1}) = \tau(b_k - a_k)$$

where the interval converges to ξ. This leads to the following algorithm:

Algorithm 2.8 (Golden Section Search)**.**

```
"""Basic implementation of the golden section search, this easily can be
improved by storing and resuing the results of the previous iteration
"""

import math


def golden_section_search(f, I, eps=0.00001):
    t = 0.5 * (math.sqrt(5) - 1)
    a, b = I
    while abs(b - a) > eps:
        x, y = b - t * (b - a), a + t * (b - a)
        if f(x) > f(y):
            a = x
        else:
            b = y
    return (a + b) / 2


if __name__ == '__main__':
    p, q, I = 0, 0, (-10, 10)
    p, q, I = -4, 1, (-10, 10)
    f = lambda x: (x + p) ** 2 + q
    x0 = golden_section_search(f, I)
    print(f'arg min f on {I}: {x0}')
```

Algorithm 2.9 (Steepest Descent)**.**

Let $f \in C^1(\mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$. For $0 < \alpha \leq \beta < 1$ and $\gamma < 1$ let

Exercise 2.10 (Surprising Convergence)**.** *Example for $f \in C^2(\mathbb{R})$ with a sequence of strict local minima converging to a strict local maximum.*

Proof. Let $f \in C[a, b]$ and $\xi \in (a, b)$ so that f is strictly increasing in $(a, \xi]$ and strictly decreasing in $[\xi, b)$. Define

$$g(x) = \int_{\xi - x}^{\xi + x} f(t)\, dt$$

□

3 The Road to Reality

3.1 Hyperbolic Geometry

The ratio between the area A and A' of two similar shapes is given by

$$A' = k^2 A$$

Theorem 3.1 (Pythagoras)**.**

$$a^2 + b^2 = c^2$$

Proof. Let A, B and C be the areas of the three triangles respectively. All triangles are similar, hence

$$B = \frac{b^2}{a^2} A \text{ and } C = \frac{c^2}{b^2} B$$

Since $A + B = C$ it follows that

$$a^2 + b^2 = \frac{b^2 A}{B} + b^2 = \frac{b^2(A+B)}{B} = \frac{b^2 C}{B} = c^2$$

□

Lemma 3.2 (Conformal and Projective Representation)**.** *The mapping from conformal and projective representation of any point is given by the radial expansion of the following factor*

$$\frac{2R}{R^2 + r^2}$$

Proof. For any point the distance from the origin with regard to the two representations is given by

$$\log \frac{R+r}{R-r} = \frac{1}{2} \log \frac{R+r'}{R-r'} = \log \frac{(R+r')^2}{(R-r')^2}$$

This gives

$$(R-r)^2(R+r') = (R+r)^2(R-r') \text{ and } -4R^2 r + 2R^2 r' + 2r^2 r' = 0$$

Hence

$$r' = \frac{2R^2}{R^2 + r^2} r$$

□

3.2 Complex Numbers

Lemma 3.3 (Basic Formulas)**.**

1. *It is*
$$(a+\mathrm{i}b)(c+\mathrm{i}d) = (ac-bd) + \mathrm{i}(ad+bc)$$

2. *Thus*
$$(a+\mathrm{i}b)^2 = (a^2-b^2) + \mathrm{i}2ab$$
and
$$(a+\mathrm{i}b)(a-\mathrm{i}b) = a^2 + \mathrm{i}ab - \mathrm{i}ab - \mathrm{i}^2b^2 = a^2+b^2$$

3. *Hence*
$$\frac{a+\mathrm{i}b}{c+\mathrm{i}d} = \frac{(a+\mathrm{i}b)(c-\mathrm{i}d)}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \mathrm{i}\frac{bc-ad}{c^2+d^2}$$

4. *For*
$$z = \sqrt{\frac{1}{2}(a+\sqrt{a^2+b^2})} + \mathrm{i}\sqrt{\frac{1}{2}(-a+\sqrt{a^2+b^2})}$$
it follows
$$z^2 = \frac{1}{2}(a+\sqrt{a^2+b^2}) - \frac{1}{2}(-a+\sqrt{a^2+b^2}) + \mathrm{i}2\sqrt{\frac{1}{4}(\sqrt{a^2+b^2}^2) - a^2} = a+\mathrm{i}b$$

Lemma 3.4 (Binomial Theorem)**.**

1. *For the binomial coefficient Pascal's identity holds*
$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

2. *The following equation states the binomial identity*
$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k} = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

3. *For $a=1$ follows*
$$(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$$

Proof. It is
$$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} = \frac{n!(n+1-k)+n!k!}{k!(n+1-k)!} = \binom{n+1}{k}$$

Furthermore by using induction

$$\begin{aligned}(a+b)^{n+1} &= \sum_{k=0}^{n}\binom{n}{k}a^{k+1}b^{n-k} + \sum_{k=0}^{n}\binom{n}{k}a^k b^{n+1-k}\\ &= \sum_{k=1}^{n+1}\binom{n}{k-1}a^k b^{n+1-k} + \sum_{k=0}^{n}\binom{n}{k}a^k b^{n+1-k}\\ &= \sum_{k=0}^{n+1}\binom{n+1}{k}a^k b^{n+1-k}\end{aligned}$$

□

3.3 Exponential Function and Logarithms

Exercise 3.5 (Exponential Function)**.** *The Cauchy product yields*

$$\sum_{n=0}^{\infty} a_n \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k}$$

if at least one of the series is absolutely convergent. Hence

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{n!}z^n \sum_{n=0}^{\infty}\frac{1}{n!}w^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{k!}z^k\frac{1}{(n-k)!}w^{n-k}\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}z^k w^{n-k}\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}(z+w)^n\end{aligned}$$

Let $t \in \mathbb{R}$. Then

$$\begin{aligned}\mathrm{e}^{\mathrm{i}t} &= \sum_{k=0}^{\infty}\frac{1}{k!}(\mathrm{i}t)^k\\ &= \sum_{k=0}^{\infty}\frac{1}{2k!}(\mathrm{i}t)^{2k} + \sum_{k=0}^{\infty}\frac{1}{(2k+1)!}(\mathrm{i}t)^{2k+1}\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{2k!}t^{2k} + \mathrm{i}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}t^{2k+1}\\ &= \cos t + \mathrm{i}\sin t\end{aligned}$$

More generally for $z = \log r + \mathrm{i}t$

$$\mathrm{e}^z = \mathrm{e}^{\log r + \mathrm{i}t} = r\mathrm{e}^{\mathrm{i}t} = r(\cos t + \mathrm{i}\sin t)$$

For $r = 1$ and $t = 2\pi$ this yields

$$\mathrm{e}^{2\pi\mathrm{i}} = \cos 2\pi + \mathrm{i}\sin 2\pi = 1$$

and for $t = 2\pi$ we get

Lemma 3.6 (Euler Equation)**.**

$$e^{\pi i} + 1 = 0$$

Exercise 3.7.

1. *If* $e^z = w$ *then* $z + \pi i$ *is a logarithm to* $-w$*:* $e^{z+\pi i} = e^z e^{\pi i} = -e^z = -w$*.*
2. *Since* $e^{i(s+t)} = e^{is}e^{it}$ *it follows*

$$\begin{aligned}\cos(s+t) + i\sin(s+t) &= (\cos s + i\sin s)(\cos t + i\sin t)\\ &= \cos s\cos t - \sin s\sin t + i(\cos s\sin t + \sin s\cos t)\end{aligned}$$

Hence

$$\begin{aligned}\cos(s+t) &= \cos s\cos t - \sin s\sin t\\ \sin(s+t) &= \cos s\sin t + \sin s\cos t\end{aligned}$$

3. *It is* $e^{3it} = (e^{it})^3$ *and thus*

$$\cos 3t + i\sin 3t = (\cos t + i\sin t)^3 = \cos^3 t - 3\cos t\sin^2(t) + i(\cos^2 t\sin t - \sin^3 t)$$

4. *Fun facts*

$$e^{1-4\pi^2} = e^{1+(2i\pi)^2} = ee^{2\pi i}e^{2\pi i} = e$$

and $i = e^{i\pi/2}$ *gives*

$$i^i = e^{i\log i} = e^{ii\pi/2} = e^{-\pi/2} \in \mathbb{R}$$

3.4 Complex Analysis

Theorem 3.8 (Cauchy Riemann Equations)**.** *Let* $f = u + iv$ *be holomorphic. Then* f *satisfies the Cauchy Riemann equations*

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}\end{aligned}$$

Proof. For $h \in \mathbb{R}$ follows

$$\lim_{h\to 0}\frac{f(z+h) - f(z)}{h} = \frac{\partial u}{\partial x}(z) + i\frac{\partial v}{\partial x}(z)$$

and

$$\lim_{h\to 0}\frac{f(z+ih) - f(z)}{ih} = \frac{\partial u}{i\partial y}(z) + \frac{\partial v}{\partial y}(z) = \frac{\partial v}{\partial y}(z) - i\frac{\partial u}{\partial y}(z)$$

□

Examples 3.9.

1. *Let $f(z) = z^3$. Then $u(x,y) + \mathrm{i}v(x,y) = x^3 - 3xy^2 + \mathrm{i}(3x^2y - y^3)$ and as expected*

$$\frac{\partial u}{\partial x}(x,y) = x^3 - 3y^2 \quad \text{and} \quad \frac{\partial u}{\partial y}(x,y) = -6xy$$
$$\frac{\partial v}{\partial x}(x,y) = 6xy \quad \text{and} \quad \frac{\partial v}{\partial y}(x,y) = x^3 - 3y^2$$

Lemma 3.10. *Let $D \subseteq \mathbb{C}$ be connected. For arbitrary $z, w \in D$ there exists a polygonal path from z to w.*

Proof. For any path from z to w the image is compact, which can be used to define a finite subcover of balls. Use the center points to define the polygonal path. □

Lemma 3.11. *Let $\gamma : [a,b] \to \mathbb{C}$ a smooth path, $\psi : [c,d] \to [a,b]$ a smooth and increasing bijection and f continous.*

$$\int_\gamma f(z)\,dz = \int_{\gamma\circ\psi} f(z)\,dz$$

Proof. It is

$$\begin{aligned}\int_{\gamma\circ\psi} f(z)\,dz &= \int_c^d f(\gamma\circ\psi(t))(\gamma\circ\psi)'(t))\,dt \\ &= \int_{\psi(a)}^{\psi(b)} f(\gamma(\psi(t))\gamma'(\psi(t))\psi'(t)\,dt \\ &= \int_a^b f(\gamma(s))\gamma'(s)\,ds = \int_\gamma f(z)\,dz\end{aligned}$$

□

Lemma 3.12. *For a smooth path $\gamma : [a,b] \to \mathbb{C}$ define $-\gamma(t) = a + b - t$. Then*

$$\int_{-\gamma} f(z)\,dz = -\int_\gamma f(z)\,dz$$

Proof. Using integration by substitution

$$\int_{-\gamma} f(z)\,dz = -\int_a^b f(\gamma(a+b-t))\gamma'(a+b-t))\,dt = \int_b^a f(\gamma(s))\gamma'(s))\,ds = -\int_\gamma f(z)\,dz$$

□

In order to use the results from real calculus remember the fact, that for every $z \in \mathbb{C}$ there exists a $t \in [0, 2\pi]$, so that $z = |z|e^{\mathrm{i}t}$ and hence $|z| = ze^{-\mathrm{i}t}$.

Lemma 3.13. *Let $f \in C[a,b]$. Then*

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

Proof. Using the estimation for integrals from real calculus

$$\left|\int_a^b f(x)\,dx\right| = e^{-\mathrm{i}t}\int_a^b f(x)\,dx \le \int_a^b |e^{-\mathrm{i}t} f(x)|\,dx = \int_a^b |f(x)|\,dx$$

□

Lemma 3.14 (Estimation Lemma)**.** *Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth path. Then*

$$\left|\int_\gamma f(z)\,dz\right| \le L(\gamma) \max_{\gamma[a.b]} f$$

Proof. Using the estimation above

$$\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(\gamma(t))\gamma'(t))\,dt\right| \le \int_a^b |f(\gamma(t))\gamma'(t))|\,dt \le \max_{\gamma[a.b]} f \int_a^b |\gamma'(t)|\,dt$$

□

Examples 3.15.

1. *Let $\gamma(t) = t + \mathrm{i}t$. Then*

$$\int_\gamma z^2\,dz = \int_0^1 (t+\mathrm{i}t)^2(1+\mathrm{i})\,dt = (1+\mathrm{i})\int_0^1 2\mathrm{i}t^2\,dt = \left[(-2+2\mathrm{i})t^2\right]_0^1 = -\frac{2}{3} + \mathrm{i}\frac{2}{3}$$

2. *For $\gamma(t) = t^2 + \mathrm{i}t$*

$$\begin{aligned}\int_\gamma z^2\,dz &= \int_0^1 (t^2+\mathrm{i}t)^2(2+\mathrm{i}t)\,dt = \int_0^1 (2t^5-4t^3) + \mathrm{i}(5t^4 - t^2t)\,dt\\ &= \left[\frac{1}{3}t^6 - t^4\right]_0^1 + \mathrm{i}\left[t^5 - \frac{1}{3}t^3\right]_0^1 = -\frac{2}{3} + \mathrm{i}\frac{2}{3}\end{aligned}$$

3. *And $\gamma(t) = \mathrm{i} + \mathrm{e}^{\mathrm{i}t}$*

$$\begin{aligned}\int_\gamma z^2\,dz &= \int_{3/2\pi}^{2\pi} (\mathrm{i}+\mathrm{e}^{\mathrm{i}t})^2\mathrm{i}\mathrm{e}^{\mathrm{i}t}\,dt = \int_{3/2\pi}^{2\pi} (-1+2\mathrm{i}\mathrm{e}^{\mathrm{i}t}+\mathrm{e}^{2\mathrm{i}t})\mathrm{i}\mathrm{e}^{\mathrm{i}t}\,dt\\ &= \int_{3/2\pi}^{2\pi} -\mathrm{i}\mathrm{e}^{\mathrm{i}t} - 2\mathrm{e}^{2\mathrm{i}t} + \mathrm{i}\mathrm{e}^{3\mathrm{i}t}\,dt = \left[(-\mathrm{e}^{\mathrm{i}t} + \mathrm{i}\mathrm{e}^{2\mathrm{i}t} + \frac{1}{3}\mathrm{e}^{3\mathrm{i}t}\right]_{3/2\pi}^{2\pi}\\ &= \left(-1+\mathrm{i}+\frac{1}{3}\right) - \left(\mathrm{i}-\mathrm{i}+\frac{1}{3}\mathrm{i}\right) = -\frac{2}{3} + \mathrm{i}\frac{2}{3}\end{aligned}$$

Theorem 3.16. *Let $D \subseteq \mathbb{C}$ be a connected domain and $f \in C(D)$. Then the following assertions are equivalent*

1. *f has an antiderivative*

2. *For every closd path γ*

$$\int_\gamma f(z)\,dz = 0$$

Proof. Let $F' = f$. Since γ is closed

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_a^b (F\circ\gamma)'(t)\,dt = F(\gamma(b)) - F(\gamma(a)) = 0$$

Now fix some arbitrary $a \in D$. For $z \in D$ let γ_z be a path from a to z and define

$$F(z) = \int_{\gamma_z} f(\zeta)\,d\zeta$$

This is well defined since the integral of f vanishes over each closed path. Moreover, since $\gamma_{z+h}+[z+h,z]-\gamma_z$ defines a closed path

$$F(z+h) - F(z) = \int_{\gamma_{z+h}} f(z)\,dz - \int_{\gamma_z} f(z)\,dz = \int_{[z,z+h]} f(z)\,dz = h\int_0^1 f(z+th)\,dt$$

Here the latter integral is continous at 0 with respect to h

$$\left|\int_0^1 f(z+th) - f(z)\,dt\right| \le \int_0^1 |f(z+th)-f(z)|\,dt \le \max_{t\in[0,1]} |f(z+th)-f(z)|$$

□

Corollary 3.17. *The second assertion can be weakend to*

$$\int_{\partial\Delta} f(z)\,dz = 0$$

for every triangle $\Delta \subset D$, where e.g. D is convex or star shaped. Here the anti derivative can directly be defined as

$$F(z) = \int_{[a,z]} f(\zeta)\,d\zeta$$

similarly to the real calculus approach. Note, that under this conditions f always has a local anit-derivative.

Examples 3.18.

1. *Let $z_0 \in \mathbb{C}$ and $\gamma(t) = z_0 + \mathrm{e}^{\mathrm{i}t}$ for $t \in [0, 2\pi]$. Then*

$$\int_\gamma \frac{1}{z-z_0}\,dz = \int_0^{2\pi} \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}t}}{z_0+\mathrm{e}^{\mathrm{i}t}-z_0}\,dt = \int_0^{2\pi} \mathrm{i}\,dt = 2\pi\mathrm{i}$$

   and thus $1/(z-z_0)$ has no antiderivative on $\mathbb{C}\setminus\{z_0\}$

Theorem 3.19 (Cauchy's Intergral Formula). *Let $\Omega \subseteq \mathbb{C}$ be open and f holomorphic on Ω. Further let $D \subset \Omega$ be a disc. Then*

$$f(z) = \frac{1}{2\pi\mathrm{i}}\int_{\partial D} \frac{f(\zeta)}{\zeta - z}\,dz$$

for $z \in D$.

Proof. For $z \in D$ define

$$h(\zeta) = \frac{f(\zeta) - f(z)}{\zeta - z}$$

for $\zeta \neq z$ and $f'(z)$ for $\zeta = z$. Then h is holomorphic on D and

$$0 = \int_{\partial D} h(\zeta)\, d\zeta = \int_{\partial D} \frac{f(\zeta)}{\zeta - z}\, d\zeta - f(z) \int_{\partial D} \frac{1}{\zeta - z}\, d\zeta = \int_{\partial D} \frac{f(\zeta)}{\zeta - z}\, d\zeta - 2\pi \mathrm{i} f(z)$$

□