

## Notes on Mathematics

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# 1 Calculus

## 1.1 Differentiation and Integration

**Lemma 1.1** (Simple Calculations).

1. For  $1 = xx^{-1}$  the product rule yields  $0 = x^{-1} + x(x^{-1})'$ . Hence

$$\frac{d}{dx}x^{-1} = -\frac{1}{x^2}$$

2. Similarly  $x = \sqrt{x}^2$  and  $1 = 2\sqrt{x}\sqrt{x}'$  and so

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

3. It is

$$\frac{d}{dx}x^n = nx^{n-1}$$

since via induction the product rule yields

$$\frac{d}{dx}x^n = \frac{d}{dx}xx^{n-1} = x^{n-1} + \frac{d}{dx}x^{n-1} = x^{n-1} + (n-1)x^{n-1} = nx^{n-1}$$

4. Again, applying the product rule gives

$$\left(\frac{1}{g}\right)' = \left(\frac{1}{x} \circ g\right)' = -\frac{g'}{g^2}$$

and the quotient rule

$$\left(\frac{f}{g}\right)' = \frac{f'}{g} + f\left(\frac{1}{g}\right)' = \frac{f'}{g} - \frac{fg'}{g^2} = \frac{gf' - fg'}{g^2}$$

5. Also  $x = f \circ f^{-1}$  and  $1 = (f^{-1})'f' \circ f^{-1}$ . Thus

$$(f^{-1})' = \frac{1}{f' \circ f^{-1}}$$

where defined. Especially for  $x \neq 0$

$$\log'(x) = \frac{1}{\exp'(\log(x))} = \frac{1}{x}$$

6.  $(1-q)(1+q+q^2+\dots+q^n) = 1-q+q-q^2+q^2-q^3+\dots+q^{n+1}$  gives

$$\sum_{k=0}^n q^k = \frac{1-q^{n+1}}{1-q} \text{ and } \sum_{k=m}^n q^k = \frac{q^m - q^{n+1}}{1-q}$$

**Lemma 1.2** (Exponential Function).

1. It is

$$\exp(x + y) = \exp(x) \exp(y)$$

Hence

$$\begin{aligned}\exp(0) &= 1 \\ \exp(-x) &= \exp(x)^{-1} \\ \exp(nx) &= \exp(x)^n\end{aligned}$$

2. For the derivative

$$\exp'(x) = \sum_{k=0}^{\infty} \frac{1}{k!} (x^k)' = \sum_{k=0}^{\infty} \frac{1}{k!} k x^{k-1} = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} x^{k-1} = \exp(x)$$

**Lemma 1.3** (Sinus and Cosinus).

1. Sinus and Cosinus power series

$$\begin{aligned}\cos(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k!} x^{2k} \\ \sin(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}\end{aligned}$$

2. Symmetry

$$\begin{aligned}\cos(-x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k!} (-x)^{2k} = \cos(x) \\ \sin(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (-x)^{2k+1} = -\sin(x)\end{aligned}$$

3. Derivatives

$$\begin{aligned}\cos'(x) &= \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)!} x^{2k-1} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!} x^{2k+1} = -\sin(x) \\ \sin'(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k!} x^{2k} = \cos(x)\end{aligned}$$

**Theorem 1.4** (Fermat Stationary Point). Let  $\Omega \subseteq \mathbb{R}$  be open and  $f \in C^1(\Omega)$ . If  $x^* \in \Omega$  is local extremum then  $f'(x^*) = 0$ .

*Proof.* Assume  $x^*$  is the minimum of  $f$  in  $\Omega$  and let  $f(x^*) > 0$ . Since  $f \in C^1(\Omega)$  there exist  $\varepsilon, \delta > 0$ , so that for  $|h| \leq \varepsilon$

$$\frac{f(x^* + h) - f(x^*)}{h} > \delta$$

Pick a negative  $h \in [-\varepsilon, 0)$ . Then

$$f(x^* + h) < f(x^*) + \delta h < f(x^*)$$

and  $x^*$  cannot be the minimum. Analog for maximum with a positive  $h$ , then apply to  $-f$ .  $\square$

**Theorem 1.5** (Rolle). *Let  $f \in C[a, b]$  with  $f(a) = f(b)$ . If  $f$  is differentiable in  $(a, b)$  then there exists a  $\xi \in (a, b)$  with  $f'(\xi) = 0$ .*

*Proof.* Assume  $f$  is not constant. Since  $[a, b]$  is compact there exists either a global minimum or maximum  $\xi \in (a, b)$  and Theorem 1.4 can be applied.  $\square$

**Theorem 1.6** (Mean Value). *Let  $f \in C[a, b]$  be differentiable in  $(a, b)$ . Then there exists a  $\xi \in (a, b)$  with*

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

*Proof.* Apply Theorem 1.5 to

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a)$$

$\square$

**Remark 1.7.**

1. More generally choose any  $\varphi \in C^1[a, b]$  with  $\varphi(a) = 0$  and  $\varphi(b) = f(b) - f(a)$ . Set  $g(x) = f(x) - \varphi(x)$  to see there is a  $\xi \in (a, b)$  with  $f'(\xi) = \varphi'(\xi)$ .
2. Another useful generalization: let  $\Omega \subseteq \mathbb{R}^n$  be open and  $f \in C^1(\Omega)$ . For  $x, y \in \Omega$  define  $\varphi(t) = f(tx + (1 - t)y)$  and apply the chain rule for differentiation

$$\varphi'(\xi) = \nabla f(\xi x + (1 - \xi)y)^T(x - y) = f(x) - f(y)$$

3. The Cauchy Schwarz inequality then yields

$$\|f(x) - f(y)\| \leq \|\nabla f(\xi x + (1 - \xi)y)\| \|x - y\|$$

**Theorem 1.8** (Differentiation Theorem). *Let  $f \in C[a, b]$  and define*

$$F(x) = \int_a^x f(t) dt$$

*Then  $F \in C^1[a, b]$  with  $F'(x) = f(x)$  for  $x \in [a, b]$ .*

*Proof.* Applying the Mean Value Theorem of Integration gives

$$F(x + h) - F(x) = \int_x^{x+h} f(t) dt = f(\xi)h$$

for some  $\xi \in (x, x + h)$ .  $\square$

**Theorem 1.9** (Fundamental Theorem of Calculus). *Let  $F \in C^1[a, b]$  with  $F' = f$  Then*

$$F(b) - F(a) = \int_a^b f(t) dt$$

**Lemma 1.10** (Integration by Substitution). *Let  $I \subseteq \mathbb{R}$  be an interval and  $f \in C(I)$ . For  $\varphi \in C([a, b], I)$  it follows*

$$\int_{\varphi(a)}^{\varphi(b)} f(x) dx = \int_a^b f(\varphi(t))\varphi'(t) dt$$

*Proof.* Let  $F \in C^1(I)$  with  $F' = f$ . Then the chain rule for differentiation yields

$$\begin{aligned} \int_{\varphi(a)}^{\varphi(b)} f(x) dx &= F(\varphi(b)) - F(\varphi(a)) \\ &= F \circ \varphi(b) - F \circ \varphi(a) \\ &= \int_a^b (F \circ \varphi)'(t) dt \\ &= \int_a^b f(\varphi(t))\varphi'(t) dt \end{aligned}$$

□

**Examples 1.11.**

1. For  $\varphi(x) = x^2 + 1$  it is  $\varphi(0) = 1$  and  $\varphi(2) = 5$ . Thus

$$\int_0^2 x \cos(x^2 + 1) dx = \frac{1}{2} \int_0^2 2x \cos(x^2 + 1) dx = \frac{1}{2} \int_1^5 \cos(t) dt = \frac{1}{2}(\sin(5) - \sin(1))$$

2. Consider  $\varphi(x) = \sin(x)$  where  $\varphi(0) = 0$  and  $\varphi(\pi/2) = 1$ . Since  $\cos(t) = \sqrt{1 - \sin^2(t)}$  it follows

$$\int_0^1 \sqrt{1 - x^2} dx = \int_{\cos(0)}^{\cos(\pi/2)} \sqrt{1 - x^2} dx = \int_0^{\pi/2} \sqrt{1 - \sin^2(t)} \cos(t) dt = \int_0^{\pi/2} \cos^2(t) dt$$

3. Let  $f \in C[a, b]$  and  $\varphi(x) = a + t(b - a)$ . Then

$$\int_a^b f(x) dx = (b - a) \int_0^1 f(a + t(b - a)) dt$$

4. Let  $f(x) = x^n$  and  $\varphi(x) = t^m$ . As expected

$$\int_0^1 x^n dx = \int_0^1 t^{nm} m t^{m-1} dt = m \int_0^1 t^{m(n+1)-1} dt = \left[ \frac{m}{m(n+1)} t^{m(n+1)} \right]_0^1 = \frac{1}{n+1}$$

## 1.2 Directional Derivative and Gradients

**Lemma 1.12** (Directional Derivative). *Let  $\Omega \subseteq \mathbb{R}^n$  be open and  $f \in C^1(\Omega)$ . Then*

$$\frac{\partial f}{\partial d}(x) = \nabla f(x)^T d$$

for any  $d \in \mathbb{R}^n$ .

*Proof.* Let  $\varphi(t) = f(x + td)$ . Then  $\varphi \in C^1[-\varepsilon, \varepsilon]$  for some  $\varepsilon > 0$  and the chain rule yields

$$\varphi'(t) = \nabla f(x + td)^T d$$

Hence

$$\varphi'(0) = \lim_{t \rightarrow 0} \frac{\varphi(x + td) - \varphi(0)}{t} = \lim_{t \rightarrow 0} \frac{f(x + td) - f(x)}{t} = \nabla f(x)^T d$$

□

**Remark 1.13.**

1. Note that by definition the directional derivative is invariant under multiplication with any  $\lambda \neq 0$ .
2. A similar proposition holds under the weaker assumption that  $d$  is a only feasible direction for  $f$  in  $x$
3. For  $d = \nabla f(x) / \|\nabla f(x)\|$  it follows that

$$\frac{\partial f}{\partial d}(x) = \|\nabla f(x)\| > 0$$

and for any other  $d \in \mathbb{R}^n$  with  $\|d\| = 1$  the Cauchy Schwarz inequality yields

$$|\frac{\partial f}{\partial d}(x)| = |\nabla f(x)^T d| \leq \|\nabla f(x)\| \|d\| = \|\nabla f(x)\|$$

Hence  $\nabla f(x)$  is the direction of the greatest ascent and respectively,  $-\nabla f(x)$  is the direction of the greatest descent.

**Theorem 1.14** (First Order Necessary Condition). *Let  $\Omega \subseteq \mathbb{R}^n$  be open and  $f \in C^1(\Omega)$ . If  $x^* \in \Omega$  is a local minimizer then  $\nabla f(x^*) = 0$ .*

*Proof.* Let  $h \in \mathbb{R}^n$  and  $\delta > 0$  so that  $x^* + th \in \Omega$  for all  $t \in (-\delta, \delta)$ . Then 0 is local minimizer for  $\varphi(t) = f(x^* + th)$  and

$$\varphi'(0) = \nabla f(x^*)^T h = 0$$

Now let  $h = \nabla f(x^*)$ .

□

**Theorem 1.15** (Banach Fixed-Point Theorem). *Let  $X$  be a Banach space and  $f \in C(X, X)$  a contraction*

$$\|f(x) - f(y)\| \leq q \|x - y\| \text{ for all } x, y \in X$$

*for some  $0 < q < 1$ . Then there exists a unique fix point  $x^* \in X$  with*

$$f(x^*) = x^*$$

*Furthermore for any  $x_0 \in X$  the sequence defined by*

$$x_{n+1} = f(x_n)$$

*converges against  $x^*$ .*

*Proof.* Since  $\|x_{n+1} - x_n\| = \|f(x_n) - f(x_{n-1})\| \leq q\|x_n - x_{n-1}\|$  it follows, that

$$\|x_{n+1} - x_n\| \leq q^n \|x_1 - x_0\|$$

Furthermore

$$\|x_n - x_m\| \leq \sum_{k=m}^n q^k \|x_1 - x_0\| = \frac{q^m - q^{n+1}}{1 - q} \|x_1 - x_0\|$$

and  $(x_n)$  is a Cauchy sequence. For its limit  $x^*$  we have

$$x^* = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} f(x_n) = f(x^*)$$

For any other  $y^* \in X$  with  $f(y^*) = y^*$  it follows, that

$$\|x^* - y^*\| = \|f(x^*) - f(y^*)\| \leq q\|x^* - y^*\|$$

and therefore  $x^* = y^*$ .

□

## 2 Nonlinear Optimization

### 2.1 Minimization without Constraints

**Lemma 2.1** (Gradient Inequality). *Let  $M \subseteq \mathbb{R}^n$  be a convex set and  $f \in C^1(M)$ . Then  $f$  is convex if and only if*

$$f(x) \geq f(y) + \nabla f(y)^T(x - y)$$

for all  $x, y \in M$ .

*Proof.* Let  $f$  be convex and  $x, y \in M$ . For  $0 \leq \lambda \leq 1$  we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) = \lambda f(x) - \lambda f(y) + f(y)$$

and

$$f(x) - f(y) \geq \frac{f(\lambda x + (1 - \lambda)y) - f(y)}{\lambda} = \frac{f(y + \lambda(x - y)) - f(y)}{\lambda}$$

For  $d = x - y$  and  $\lambda \rightarrow 0$  the term on the right converges to the direction derivative of  $f$  in  $d$

$$\frac{\partial f}{\partial d}(y) = \nabla f(y)^T d = \nabla f(y)^T(x - y)$$

Now let  $x, y \in M$  and  $0 \leq \lambda \leq 1$ . For  $z = \lambda x + (1 - \lambda)y \in M$  it follows that

$$\begin{aligned} \lambda f(x) &\geq \lambda f(z) + \lambda \nabla f(z)^T(x - z) \\ (1 - \lambda)f(y) &\geq (1 - \lambda)f(z) + (1 - \lambda)\nabla f(z)^T(y - z) \end{aligned}$$

Adding the two inequalities gives

$$\begin{aligned} \lambda f(x) + (1 - \lambda)f(y) &\geq f(z) + \nabla f(z)^T(\lambda x - \lambda z + (1 - \lambda)y - (1 - \lambda)z) \\ &= f(z) + \nabla f(z)^T(\lambda x + (1 - \lambda)y - z) \\ &= f(z) \end{aligned}$$

□

**Exercise 2.2** (Facility Locations). *The facilities are located at:*

$$(3, 0), (0, -3), (1, 4)$$

*Proof.* Let

$$\begin{aligned} f(x) &= (x - 3)^2 + y^2 + x^2 + (y + 3)^2 + (x - 1)^2 + (y - 4)^2 \\ &= x^2 - 6x + 9 + y^2 + x^2 + y^2 + 6y + 9 + x^2 - 2x + 1 + y^2 - 8y + 16 \\ &= 3x^2 + 3y^2 - 8x - 2y + 35 \end{aligned}$$

Then

$$\nabla f(x, y) = (6x - 8, 6y - 2) \text{ and } \nabla^2 f(x, y) = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} > 0$$

Hence  $(4/3, 1/3)$  is the global minimum.

□

**Exercise 2.3** (Convex Functions). *The sum of convex functions is convex.*



*Proof.* Let  $x, y \in M$ . Since  $\alpha_i > 0$  we have

$$\begin{aligned} f(\lambda x + (1 - \lambda)y) &= \sum_{i=1}^m \alpha_i f_i(\lambda x + (1 - \lambda)y) \\ &\leq \sum_{i=1}^m \alpha_i \lambda f_i(x) + \sum_{i=1}^m \alpha_i (1 - \lambda) f_i(y) = \lambda f(x) + (1 - \lambda) f(y) \end{aligned}$$

Let  $f(x) = x^2$ . Then  $-f$  is not convex, e.g.  $x = 1, y = -1$  and  $\lambda = 0.5$ .

**Exercise 2.4** (Solution of Quadratic Inequality). *Let*

$$f(x) = x^T A x + b^T x + c$$

*Proof.* The product rule gives

$$\nabla f(x) = x^T A + A x + b = (A^T + A)x + b = 2Ax + b$$

Thus  $\nabla^2 f(x) = 2A > 0$  and  $f$  is convex. Hence the level set  $\Gamma_{-c}$  is convex. Since the intersection of convex sets is convex  $\Gamma_{-c} \cap \{x \in \mathbb{R}^n : g^T x + h = 0\}$  is convex, too.

**Exercise 2.5** (Line Search on Compact Convex Sets). *Let  $S \subset \mathbb{R}^n$  be compact and convex. Furthermore let  $f \in C^1(S)$  be convex,  $x \in S$  and  $d \in \mathbb{R}^n$  a descent direction of  $f$  in  $x$  with  $\nabla f(x)^T d < 0$ .*

*Proof.* If  $x + \lambda^* d$  is an optimal solution then  $\nabla f(x + \lambda^* d)^T d = 0$  according to Theorem 1.14. Let  $\nabla f(x + \lambda^* d)^T d = 0$ . Then Lemma 2.1 gives

$$f(x + \lambda d) \geq f(x + \lambda^* d) + (\lambda - \lambda^*) \nabla f(x + \lambda^* d)^T d = f(x + \lambda^* d)$$

and  $x + \lambda^* d$  is an optimal solution.

**Exercise 2.6** (Steepest Descent). *Let*

$$f(x) = \frac{1}{2} x^T A x + b^T x + c$$

where  $A$  is symmetrical and positive definite.

*Proof.* Since  $\nabla f(x) = Ax + b$  and  $\nabla^2 f(x) = A > 0$  it follows  $x^* = -A^{-1}b$ . Let  $v$  be eigenvector with  $Av = \mu v$ . For  $x_0 = x^* + \theta v$  we have

$$\nabla f(x_0) = Ax^* + \mu \theta v + b = \mu \theta v$$

and for  $\lambda \geq 0$

$$\arg \min \{f(x_0 - \lambda \nabla f(x_0))\} = \arg \min \{f(x^* + \theta v - \lambda \mu \theta v)\} = \mu^{-1}$$

Thus

$$x_1 = x_0 - \mu^{-1} \nabla f(x_0) = x^* + \theta v - \mu^{-1} \mu \theta v = x^*$$

and  $\nabla f(x_1) = 0$ . Hence the algorithm stops after the first iteration. Now let

$$x_0 = x^* + \sum_{i=0}^m \theta_i v_i$$

for orthogonal eigenvectors with  $Av_i = \mu_i$  and  $m \leq n$ . Then

$$\nabla f(x_0) = Ax^* + \sum_{i=0}^m \mu_i \theta_i v_i + b = \sum_{i=0}^m \mu_i \theta_i v_i$$

and

$$x_1 = x_0 - \lambda \sum_{i=0}^m \mu_i \theta_i v_i = x^* + \sum_{i=0}^m \theta_i v_i - \lambda \sum_{i=0}^m \mu_i \theta_i v_i = x^* + \sum_{i=0}^m (1 - \lambda \mu_i) \theta_i v_i$$

Since  $x^*$  is the minimum we have  $\nabla f(x_1) = 0$  iff  $\lambda = \mu_i^{-1}$  for all  $0 \leq i \leq m$ .  $\square$

## 2.2 One Dimensional Minimization and Direct Search

**Definition 2.7** (Unimodal Function). *A function  $f : [a, b] \rightarrow \mathbb{R}$  is called unimodal if there exists a  $\xi \in [a, b]$ , so that  $f$  is strictly decreasing in  $[a, \xi]$  and strictly increasing in  $[\xi, b]$ .*

In fact  $\xi$  is the unique minimum of  $f$  in  $[a, b]$ . According to the definition, for  $a \leq x < y \leq b$  we have

$$f(x) > f(y) \text{ for } x, y \in [a, \xi] \text{ and } f(x) < f(y) \text{ for } x, y \in (\xi, b]$$

Thus

$$\xi \in [a, y] \text{ if } f(x) < f(y) \text{ and } \xi \in [x, b] \text{ if } f(x) \geq f(y)$$

Consider now a symmetrical partitioning of the interval  $[0, 1]$  where two consecutive partitionings hold the same ratio respectively:

$$\sigma = 1 - \tau \text{ and } \frac{1}{\tau} = \frac{\tau}{\sigma}$$

Then  $1 - \tau = \tau^2$  and solving the quadratic equation  $\tau^2 + \tau = 1$  yields

$$\tau = \frac{\sqrt{5} - 1}{2} \approx 0.61803$$



Figure 1: Golden Section

Let now  $[a_0, b_0] = [a, b]$  and define

$$[a_{k+1}, b_{k+1}] = \begin{cases} [a_k, y_k] & \text{if } f(x_k) < f(y_k) \\ [x_k, b_k] & \text{if } f(x_k) \geq f(y_k) \end{cases}$$

where

$$\begin{aligned}x_k &= b_k - \tau(b_k - a_k) \\ y_k &= a_k + \tau(b_k - a_k)\end{aligned}$$

It follows that  $[a_k, b_k] \supset [a_{k+1}, b_{k+1}]$  is a decreasing series of intervals with

$$(b_{k+1} - a_{k+1}) = \tau(b_k - a_k)$$

where the interval converges to  $\xi$ . This leads to the following algorithm:

**Algorithm 2.8** (Golden Section Search).

---

```

"""Basic implementation of the golden section search, this easily can be
improved by storing and resuing the results of the previous iteration
"""

import math

def golden_section_search(f, I, eps=0.00001):
    t = 0.5 * (math.sqrt(5) - 1)
    a, b = I
    while abs(b - a) > eps:
        x, y = b - t * (b - a), a + t * (b - a)
        if f(x) > f(y):
            a = x
        else:
            b = y
    return (a + b) / 2

if __name__ == '__main__':
    p, q, I = 0, 0, (-10, 10)
    p, q, I = -4, 1, (-10, 10)
    f = lambda x: (x + p) ** 2 + q
    x0 = golden_section_search(f, I)
    print(f'arg min f on {I}: {x0}')

```

---

**Algorithm 2.9** (Steepest Descent).

Let  $f \in C^1(\mathbb{R}^n)$  and  $x_0 \in \mathbb{R}^n$ . For  $0 < \alpha \leq \beta < 1$  and  $\gamma < 1$  let

**Exercise 2.10** (Surprising Convergence). *Example for  $f \in C^2(\mathbb{R})$  with a sequence of strict local minima converging to a strict local maximum.*

*Proof.* Let  $f \in C[a, b]$  and  $\xi \in (a, b)$  so that  $f$  is strictly increasing in  $(a, \xi]$  and strictly decreasing in  $[\xi, b)$ . Define

$$g(x) = \int_{\xi-x}^{\xi+x} f(t) dt$$

□

### 3 The Road to Reality

#### 3.1 Hyperbolic Geometry

The ratio between the area  $A$  and  $A'$  of two similar shapes is given by

$$A' = k^2 A$$

**Theorem 3.1** (Pythagoras).

$$a^2 + b^2 = c^2$$

*Proof.* Let  $A, B$  and  $C$  be the areas of the three triangles respectively. All triangles are similar, hence

$$B = \frac{b^2}{a^2} A \text{ and } C = \frac{c^2}{b^2} B$$

Since  $A + B = C$  it follows that

$$a^2 + b^2 = \frac{b^2 A}{B} + b^2 = \frac{b^2(A + B)}{B} = \frac{b^2 C}{B} = c^2$$

□

**Lemma 3.2** (Conformal and Projective Representation). *The mapping from conformal and projective representation of any point is given by the radial expansion of the following factor*

$$\frac{2R}{R^2 + r^2}$$

*Proof.* For any point the distance from the origin with regard to the two representations is given by

$$\log \frac{R+r}{R-r} = \frac{1}{2} \log \frac{R+r'}{R-r'} = \log \frac{(R+r')^2}{(R-r')^2}$$

This gives

$$(R-r)^2(R+r') = (R+r)^2(R-r') \text{ and } -4R^2r + 2R^2r' + 2r^2r' = 0$$

Hence

$$r' = \frac{2R^2}{R^2 + r^2} r$$

□

### 3.2 Complex Numbers

**Lemma 3.3** (Basic Formulas).

1. It is

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

2. Thus

$$(a + ib)^2 = (a^2 - b^2) + i2ab$$

and

$$(a + ib)(a - ib) = a^2 + iab - iab - i^2b^2 = a^2 + b^2$$

3. Hence

$$\frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + i \frac{bc - ad}{c^2 + d^2}$$

4. For

$$z = \sqrt{\frac{1}{2}(a + \sqrt{a^2 + b^2})} + i\sqrt{\frac{1}{2}(-a + \sqrt{a^2 + b^2})}$$

it follows

$$z^2 = \frac{1}{2}(a + \sqrt{a^2 + b^2}) - \frac{1}{2}(-a + \sqrt{a^2 + b^2}) + i2\sqrt{\frac{1}{4}(\sqrt{a^2 + b^2})^2} - a^2 = a + ib$$

**Lemma 3.4** (Binomial Theorem).

1. For the binomial coefficient Pascal's identity holds

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

2. The following equation states the binomial identity

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k} = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

3. For  $a = 1$  follows

$$(1 + x)^n = \sum_{k=0}^n \binom{n}{k} x^k$$

*Proof.* It is

$$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} = \frac{n!(n+1-k) + n!k!}{k!(n+1-k)!} = \binom{n+1}{k}$$

Furthermore by using induction

$$\begin{aligned}
(a+b)^{n+1} &= \sum_{k=0}^n \binom{n}{k} a^{k+1} b^{n-k} + \sum_{k=0}^n \binom{n}{k} a^k b^{n+1-k} \\
&= \sum_{k=1}^{n+1} \binom{n}{k-1} a^k b^{n+1-k} + \sum_{k=0}^n \binom{n}{k} a^k b^{n+1-k} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k}
\end{aligned}$$

□

### 3.3 Exponential Function and Logarithms

**Exercise 3.5** (Exponential Function). *The Cauchy product yields*

$$\sum_{n=0}^{\infty} a_n \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \sum_{k=0}^n a_k b_{n-k}$$

*if at least one of the series is absolutely convergent. Hence*

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{n!} z^n \sum_{n=0}^{\infty} \frac{1}{n!} w^n &= \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{1}{k!} z^k \frac{1}{(n-k)!} w^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} z^k w^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} (z+w)^n
\end{aligned}$$

Let  $t \in \mathbb{R}$ . Then

$$\begin{aligned}
e^{it} &= \sum_{k=0}^{\infty} \frac{1}{k!} (it)^k \\
&= \sum_{k=0}^{\infty} \frac{1}{2k!} (it)^{2k} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (it)^{2k+1} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k!} t^{2k} + i \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k+1} \\
&= \cos t + i \sin t
\end{aligned}$$

More generally for  $z = \log r + it$

$$e^z = e^{\log r + it} = r e^{it} = r(\cos t + i \sin t)$$

For  $r = 1$  and  $t = 2\pi$  this yields

$$e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1$$

and for  $t = 2\pi$  we get

**Lemma 3.6** (Euler Equation).

$$e^{\pi i} + 1 = 0$$

**Exercise 3.7.**

1. If  $e^z = w$  then  $z + \pi i$  is a logarithm to  $-w$ :  $e^{z+\pi i} = e^z e^{\pi i} = -e^z = -w$ .
2. Since  $e^{i(s+t)} = e^{is} e^{it}$  it follows

$$\begin{aligned}\cos(s+t) + i \sin(s+t) &= (\cos s + i \sin s)(\cos t + i \sin t) \\ &= \cos s \cos t - \sin s \sin t + i(\cos s \sin t + \sin s \cos t)\end{aligned}$$

Hence

$$\begin{aligned}\cos(s+t) &= \cos s \cos t - \sin s \sin t \\ \sin(s+t) &= \cos s \sin t + \sin s \cos t\end{aligned}$$

3. It is  $e^{3it} = (e^{it})^3$  and thus

$$\cos 3t + i \sin 3t = (\cos t + i \sin t)^3 = \cos^3 t - 3 \cos t \sin^2 t + i(\cos^2 t \sin t - \sin^3 t)$$

4. Fun facts

$$e^{1-4\pi^2} = e^{1+(2i\pi)^2} = e e^{2\pi i} e^{2\pi i} = e$$

and  $i = e^{i\pi/2}$  gives

$$i^i = e^{i \log i} = e^{i i \pi/2} = e^{-\pi/2} \in \mathbb{R}$$

### 3.4 Complex Analysis

**Theorem 3.8** (Cauchy Riemann Equations). *Let  $f = u + iv$  be holomorphic. Then  $f$  satisfies the Cauchy Riemann equations*

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}\end{aligned}$$

*Proof.* For  $h \in \mathbb{R}$  follows

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \frac{\partial u}{\partial x}(z) + i \frac{\partial v}{\partial x}(z)$$

and

$$\lim_{h \rightarrow 0} \frac{f(z+ih) - f(z)}{ih} = \frac{\partial u}{i \partial y}(z) + \frac{\partial v}{\partial y}(z) = \frac{\partial v}{\partial y}(z) - i \frac{\partial u}{\partial y}(z)$$

□

**Examples 3.9.**

1. Let  $f(z) = z^3$ . Then  $u(x, y) + iv(x, y) = x^3 - 3xy^2 + i(3x^2y - y^3)$  and as expected

$$\begin{aligned}\frac{\partial u}{\partial x}(x, y) &= x^3 - 3y^2 & \text{and} & & \frac{\partial u}{\partial y}(x, y) &= -6xy \\ \frac{\partial v}{\partial x}(x, y) &= 6xy & \text{and} & & \frac{\partial v}{\partial y}(x, y) &= x^3 - 3y^2\end{aligned}$$

**Lemma 3.10.** Let  $D \subseteq \mathbb{C}$  be connected. For arbitrary  $z, w \in D$  there exists a polygonal path from  $z$  to  $w$ .

*Proof.* For any path from  $z$  to  $w$  the image is compact, which can be used to define a finite subcover of balls. Use the center points to define the polygonal path.  $\square$

**Lemma 3.11.** Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  a smooth path,  $\psi : [c, d] \rightarrow [a, b]$  a smooth and increasing bijection and  $f$  continuous.

$$\int_{\gamma} f(z) dz = \int_{\gamma \circ \psi} f(z) dz$$

*Proof.* It is

$$\begin{aligned}\int_{\gamma \circ \psi} f(z) dz &= \int_c^d f(\gamma \circ \psi(t))(\gamma \circ \psi)'(t) dt \\ &= \int_{\psi(a)}^{\psi(b)} f(\gamma(\psi(t)))\gamma'(\psi(t))\psi'(t) dt \\ &= \int_a^b f(\gamma(s))\gamma'(s) ds = \int_{\gamma} f(z) dz\end{aligned}$$

$\square$

**Lemma 3.12.** For a smooth path  $\gamma : [a, b] \rightarrow \mathbb{C}$  define  $-\gamma(t) = a + b - t$ . Then

$$\int_{-\gamma} f(z) dz = - \int_{\gamma} f(z) dz$$

*Proof.* Using integration by substitution

$$\int_{-\gamma} f(z) dz = - \int_a^b f(\gamma(a + b - t))\gamma'(a + b - t) dt = \int_b^a f(\gamma(s))\gamma'(s) ds = - \int_{\gamma} f(z) dz$$

$\square$

**Lemma 3.13.** Let  $f \in C([a, b])$ . Then

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx$$



*Proof.* Using the results from real calculus

$$\left| \int_a^b f(x) dx \right| = e^{-it} \int_a^b f(x) dx \leq \int_a^b |e^{-it} f(x)| dx = \int_a^b |f(x)| dx$$

Here  $t$  is chosen, so that

$$\left| \int_a^b f(x) dx \right| e^{it} = \int_a^b f(x) dx$$

□

**Lemma 3.14.** *Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  be a smooth path. Then*

$$\left| \int_{\gamma} f(z) dz \right| \leq L(\gamma) \max_{\gamma[a, b]} f$$

*Proof.* Using the estimation above

$$\left| \int_{\gamma} f(z) dz \right| = \left| \int_a^b f(\gamma(t)) \gamma'(t) dt \right| \leq \int_a^b |f(\gamma(t)) \gamma'(t)| dt \leq \max_{\gamma[a, b]} f \int_a^b |\gamma'(t)| dt$$

□

**Examples 3.15.**

1. Let  $\gamma(t) = t + it$ . Then

$$\int_{\gamma} z^2 dz = \int_0^1 (t + it)^2 (1 + i) dt = (1 + i) \int_0^1 2it^2 dt = [(-2 + 2i)t^3]_0^1 = -\frac{2}{3} + i\frac{2}{3}$$

2. For  $\gamma(t) = t^2 + it$

$$\begin{aligned} \int_{\gamma} z^2 dz &= \int_0^1 (t^2 + it)^2 (2 + i) dt = \int_0^1 (2t^5 - 4t^3) + i(5t^4 - t^2) dt \\ &= \left[ \frac{1}{3}t^6 - t^4 \right]_0^1 + i \left[ t^5 - \frac{1}{3}t^3 \right]_0^1 = -\frac{2}{3} + i\frac{2}{3} \end{aligned}$$

3. And  $\gamma(t) = i + e^{it}$

$$\begin{aligned} \int_{\gamma} z^2 dz &= \int_{3/2\pi}^{2\pi} (i + e^{it})^2 i e^{it} dt = \int_{3/2\pi}^{2\pi} (-1 + 2ie^{it} + e^{2it}) i e^{it} dt \\ &= \int_{3/2\pi}^{2\pi} -ie^{it} - 2e^{2it} + ie^{3it} dt = \left[ -e^{it} + ie^{2it} + \frac{1}{3}e^{3it} \right]_{3/2\pi}^{2\pi} \\ &= \left( -1 + i + \frac{1}{3} \right) - \left( i - i + \frac{1}{3}i \right) = -\frac{2}{3} + i\frac{2}{3} \end{aligned}$$

**Theorem 3.16.** *Let  $D \subseteq \mathbb{C}$  a connected domain and  $f \in C(D)$ . Then the following assertions are equivalent*

1.  $f$  has an antiderivative

2. For every closed path  $\gamma$

$$\int_{\gamma} f(z) dz = 0$$

*Proof.* Let  $F' = f$ . Since  $\gamma$  is closed

$$\int_{\gamma} f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt = \int_a^b (F \circ \gamma)'(t) dt = F(\gamma(b)) - F(\gamma(a)) = 0$$

Now fix some arbitrary  $a \in D$ . For  $z \in D$  let  $\gamma_z$  be a path from  $a$  to  $z$  and define

$$F(z) = \int_{\gamma_z} f(\zeta) d\zeta$$

This is well defined since the integral of  $f$  vanishes over each closed path. Moreover, since  $\gamma_{z+h} + [z+h, z] - \gamma_z$  defines a closed path

$$F(z+h) - F(z) = \int_{\gamma_{z+h}} f(z) dz - \int_{\gamma_z} f(z) dz = \int_{[z, z+h]} f(z) dz = h \int_0^1 f(z+th) dt$$

Here the latter integral is continuous at 0 with respect to  $h$

$$\left| \int_0^1 f(z+th) - f(z) dt \right| \leq \int_0^1 |f(z+th) - f(z)| dt \leq \max_{t \in [0,1]} |f(z+th) - f(z)|$$

□

**Corollary 3.17.** *The second assertion can be weakened to*

$$\int_{\partial \Delta} f(z) dz = 0$$

for every triangle  $\Delta \subset D$ , where e.g.  $D$  is convex or star shaped. Here the anti derivative can directly be defined as

$$F(z) = \int_{[a,z]} f(\zeta) d\zeta$$

similarly to the real calculus approach. Note, that under this conditions  $f$  always has a local anti-derivative.

**Examples 3.18.**

1. Let  $z_0 \in \mathbb{C}$  and  $\gamma(t) = z_0 + e^{it}$  for  $t \in [0, 2\pi]$ . Then

$$\int_{\gamma} \frac{1}{z - z_0} dz = \int_0^{2\pi} \frac{ie^{it}}{z_0 + e^{it} - z_0} dt = \int_0^{2\pi} i dt = 2\pi i$$

and thus  $1/(z - z_0)$  has no antiderivative on  $\mathbb{C} \setminus \{z_0\}$