Notes on Mathematics

Contents

1 Calculus

1.1 Differentation and Integration

Lemma 1.1 (Simple Calculations).

1. *For* $1 = xx^{-1}$ *the product rule yields* $0 = x^{-1} + x(x^{-1})'$. *Hence*
$$\frac{d}{dx}x^{-1} = -\frac{1}{x^2}$$

2. *Similarly* $x = \sqrt{x}^2$ *and* $1 = 2\sqrt{x}\sqrt{x}'$ *and so*
$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

3. *It is*
$$\frac{d}{dx}x^n = nx^{n-1}$$
since via induction the product rule yields
$$\frac{d}{dx}x^n = \frac{d}{dx}xx^{n-1} = x^{n-1} + \frac{d}{dx}x^{n-1} = x^{n-1} + (n-1)x^{n-1} = nx^{n-1}$$

4. *Again, applying the product rule gives*
$$\left(\frac{1}{g}\right)' = \left(\frac{1}{x} \circ g\right)' = -\frac{g'}{g^2}$$
and the quotient rule
$$\left(\frac{f}{g}\right)' = \frac{f'}{g} + f\left(\frac{1}{g}\right)' = \frac{f'}{g} - \frac{fg'}{g^2} = \frac{gf' - fg'}{g^2}$$

5. *Also* $x = f \circ f^{-1}$ *and* $1 = (f^{-1})'f' \circ f^{-1}$. *Thus*
$$(f^{-1})' = \frac{1}{f' \circ f^{-1}}$$
where defined. Especially for $x \neq 0$
$$\log'(x) = \frac{1}{\exp'(\log(x))} = \frac{1}{x}$$

6. $(1-q)(1+q+q^2+\cdots+q^n) = 1-q+q-q^2+q^2-q^3+\cdots+q^{n+1}$ *gives*
$$\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q} \quad \text{and} \quad \sum_{k=m}^{n} q^k = \frac{q^m - q^{n+1}}{1-q}$$

Lemma 1.2 (Exponential Function).

1. *It is*
$$\exp(x+y) = \exp(x)\exp(y)$$
Hence
$$\exp(0) = 1$$
$$\exp(-x) = \exp(x)^{-1}$$
$$\exp(nx) = \exp(x)^n$$

2. *For the derivative*
$$\exp'(x) = \sum_{k=0}^{\infty} \frac{1}{k!}(x^k)' = \sum_{k=0}^{\infty} \frac{1}{k!} k x^{k-1} = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} x^{k-1} = \exp(x)$$

Lemma 1.3 (Sinus and Cosinus)**.**

1. *Sinus and Cosinus power series*
$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k!} x^{2k}$$
$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

2. *Symmetry*
$$\cos(-x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k!} (-x)^{2k} = \cos(x)$$
$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (-x)^{2k+1} = -\sin(x)$$

3. *Derivatives*
$$\cos'(x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)!} x^{2k-1} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!} x^{2k+1} = -\sin(x)$$
$$\sin'(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k!} x^{2k} = \cos(x)$$

Theorem 1.4 (Fermat Stationary Point)**.** *Let $\Omega \subseteq \mathbb{R}$ be open and $f \in C^1(\Omega)$. If $x^* \in \Omega$ is local extremum then $f'(x^*) = 0$.*

Proof. Assume x^* is the minimum of f in Ω and let $f'(x^*) > 0$. Since $f \in C^1(\Omega)$ there exist $\varepsilon, \delta > 0$, so that for $|h| \le \varepsilon$
$$\frac{f(x^*+h) - f(x^*)}{h} > \delta$$
Pick a negative $h \in [-\varepsilon, 0)$. Then
$$f(x^*+h) < f(x^*) + \delta h < f(x^*)$$
and x^* cannot be the minimum. Analog for maximum with a positive h, then apply to $-f$. $\square$

Theorem 1.5 (Rolle)**.** *Let $f \in C[a,b]$ with $f(a) = f(b)$. If f is differentiable in (a,b) then there exists a $\xi \in (a,b)$ with $f'(\xi) = 0$.*

Proof. Assume f is not constant. Since $[a,b]$ is compact there exists either a global minimum or maximum $\xi \in (a,b)$ and Theorem 1.4 can be applied. □

Theorem 1.6 (Mean Value)**.** *Let $f \in C[a,b]$ be differentiable in (a,b). Then there exists a $\xi \in (a,b)$ with*

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

Proof. Apply Theorem 1.5 to

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a)$$

□

Remark 1.7.

1. *More generally choose any $\varphi \in C^1[a,b]$ with $\varphi(a) = 0$ and $\varphi(b) = f(b) - f(a)$. Set $g(x) = f(x) - \varphi(x)$ to see there is a $\xi \in (a,b)$ with $f'(\xi) = \varphi'(\xi)$.*
2. *Another useful generalization: let $\Omega \subseteq \mathbb{R}^n$ be open and $f \in C^1(\Omega)$. For $x, y \in \Omega$ define $\varphi(t) = f(tx + (1-t)y)$ and apply the chain rule for differentiation*

$$\varphi'(\xi) = \nabla f(\xi x + (1-\xi)y)^T (x - y) = f(x) - f(y)$$

3. *The Cauchy Schwarz inequality then yields*

$$\|f(x) - f(y)\| \leq \|\nabla f(\xi x + (1-\xi)y)\| \|(x - y)\|$$

Theorem 1.8 (Differentiation Theorem)**.** *Let $f \in C[a,b]$ and define*

$$F(x) = \int_a^x f(t)\, dt$$

Then $F \in C^1[a,b]$ with $F'(x) = f(x)$ for $x \in [a,b]$.

Proof. Applying the Mean Value Theorem of Integration gives

$$F(x+h) - F(x) = \int_x^{x+h} f(t)\, dt = f(\xi)h$$

for some $\xi \in (x, x+h)$. □

Theorem 1.9 (Fundamental Theorem of Calculus)**.** *Let $f \in C[a,b]$. Then*

$$F(b) - F(a) = \int_a^b f(t)\, dt$$

1.2 Directional Derivative and Gradients

Lemma 1.10 (Directional Derivative)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be open and $f \in C^1(\Omega)$. Then*

$$\frac{\partial f}{\partial d}(x) = \nabla f(x)^T d$$

for any $d \in \mathbb{R}^n$.

Proof. Let $\varphi(t) = f(x + td)$. Then $\varphi \in C^1[-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$ and the chain rule yields

$$\varphi'(t) = \nabla f(x + td)^T d$$

Hence

$$\varphi'(0) = \lim_{t \to 0} \frac{\varphi(x + td) - \varphi(0)}{t} = \lim_{t \to 0} \frac{f(x + td) - f(x)}{t} = \nabla f(x)^T d$$

□

Remark 1.11.

1. *Note that by definition the directional derivative is invariant under multiplication with any $\lambda \neq 0$.*
2. *A similar proposition holds under the weaker assumption that d is a only feasable direction for f in x*
3. *For $d = \nabla f(x)/\|\nabla f(x)\|$ it follows that*

$$\frac{\partial f}{\partial d}(x) = \|\nabla f(x)\| > 0$$

and for any other $d \in \mathbb{R}^n$ with $\|d\| = 1$ the Cauchy Schwarz inequality yields

$$|\frac{\partial f}{\partial d}(x)| = |\nabla f(x)^T d| \leq \|\nabla f(x)\| \|d\| = \|\nabla f(x)\|$$

Hence $\nabla f(x)$ is the direction of the greatest ascent and respectively, $-\nabla f(x)$ is the direction of the greatest descent.

Theorem 1.12 (First Order Necessary Condition)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be open and $f \in C^1(\Omega)$. If $x^* \in \Omega$ is a local minimzer then $\nabla f(x^*) = 0$.*

Proof. Let $h \in \mathbb{R}^n$ and $\delta > 0$ so that $x^* + th \in \Omega$ for all $t \in (-\delta, \delta)$. Then 0 is local minimizer for $\varphi(t) = f(x^* + th)$ and

$$\varphi'(0) = \nabla f(x^*)^T h = 0$$

Now let $h = \nabla f(x^*)$. □

Theorem 1.13 (Banach Fixed-Point Theorem)**.** *Let X be a Banach space and $f \in C(X,X)$ a contraction*

$$\|f(x) - f(y)\| \leq q\|x - y\| \text{ for all } x, y \in X$$

for some $0 < q < 1$. Then there exists a unique fix point $x^* \in X$ with

$$f(x^*) = x^*$$

Furthermore for any $x_0 \in X$ the sequence defined by

$$x_{n+1} = f(x_n)$$

converges aganist x^*.

Proof. Since $\|x_{n+1} - x_n\| = \|f(x_n) - f(x_{n-1})\| \leq q\|x_n - x_{n-1}\|$ it follows, that

$$\|x_{n+1} - x_n\| \leq q^n\|x_1 - x_0\|$$

Furthermore

$$\|x_n - x_m\| \leq \sum_{k=m}^{n} q^k\|x_1 - x_0\| = \frac{q^m - q^{n+1}}{1-q}\|x_1 - x_0\|$$

and (x_n) is a Cauchy sequence. For its limit x^* we have

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(x^*)$$

For any other $y^* \in X$ with $f(y^*) = y^*$ it follows, that

$$\|x^* - y^*\| = \|f(x^*) - f(y^*)\| \leq q\|x^* - y^*\|$$

and therefore $x^* = y^*$.

□

2 Nonlinear Optimization

2.1 Minimization without Constraints

Lemma 2.1 (Gradient Inequality). *Let $M \subseteq \mathbb{R}^n$ be a convex set and $f \in C^1(M)$. Then f is convex if and only if*

$$f(x) \geq f(y) + \nabla f(y)^T (x - y)$$

for all $x, y \in M$.

Proof. Let f be convex and $x, y \in M$. For $0 \leq \lambda \leq 1$ we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) = \lambda f(x) - \lambda f(y) + f(y)$$

and

$$f(x) - f(y) \geq \frac{f(\lambda x + (1-\lambda)y) - f(y)}{\lambda} = \frac{f(y + \lambda(x-y)) - f(y)}{\lambda}$$

For $d = x - y$ and $\lambda \to 0$ the term on the right converges to the direction derivative of f in d

$$\frac{\partial f}{\partial d}(y) = \nabla f(y)^T d = \nabla f(y)^T (x - y)$$

Now let $x, y \in M$ and $0 \leq \lambda \leq 1$. For $z = \lambda x + (1-\lambda)y \in M$ it follows that

$$\begin{aligned} \lambda f(x) &\geq \lambda f(z) + \lambda \nabla f(z)^T (x - z) \\ (1-\lambda) f(y) &\geq (1-\lambda) f(z) + (1-\lambda) \nabla f(z)^T (y - z) \end{aligned}$$

Adding the two inequalities gives

$$\begin{aligned} \lambda f(x) + (1-\lambda) f(y) &\geq f(z) + \nabla f(z)^T (\lambda x - \lambda z + (1-\lambda) y - (1-\lambda) z) \\ &= f(z) + \nabla f(z)^T (\lambda x + (1-\lambda) y - z) \\ &= f(z) \end{aligned}$$

□

Exercise 2.2 (Facility Locations). *The facilities are located at:*

$$(3, 0), (0, -3), (1, 4)$$

Proof. Let

$$\begin{aligned} f(x) &= (x-3)^2 + y^2 + x^2 + (y+3)^2 + (x-1)^2 + (y-4)^2 \\ &= x^2 - 6x + 9 + y^2 + x^2 + y^2 + 6y + 9 + x^2 - 2x + 1 + y^2 - 8y + 16 \\ &= 3x^2 + 3y^2 - 8x - 2y + 35 \end{aligned}$$

Then

$$\nabla f(x, y) = (6x - 8, 6y - 2) \text{ and } \nabla^2 f(x, y) = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} > 0$$

Hence $(4/3, 1/3)$ is the gobal minimum. □

Exercise 2.3 (Convex Functions)**.** *The sum of convex functions is convex.*

Proof. Let $x, y \in M$. Since $\alpha_i > 0$ we have

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= \sum_{i=1}^{m} \alpha_i f_i(\lambda x + (1-\lambda)y)) \\ &\leq \sum_{i=1}^{m} \alpha_i \lambda f_i(x) + \sum_{i=1}^{m} \alpha_i (1-\lambda) f_i(y) = \lambda f(x) + (1-\lambda) f(y) \end{aligned}$$

Let $f(x) = x^2$. Then $-f$ is not convex, e.g. $x = 1, y = -1$ and $\lambda = 0.5$.

Exercise 2.4 (Solution of Quadratic Inequality)**.** *Let*

$$f(x) = x^T Ax + b^T x + c$$

Proof. The product rule gives

$$\nabla f(x) = x^T A + Ax + b = (A^T + A)x + b = 2Ax + b$$

Thus $\nabla^2 f(x) = 2A > 0$ and f is convex. Hence the level set Γ_{-c} is convex. Since the intersection of convex sets is convex $\Gamma_{-c} \cap \{x \in \mathbb{R}^n : g^T x + h = 0\}$ is convex, too.

Exercise 2.5 (Line Search on Compact Convex Sets)**.** *Let $S \subset \mathbb{R}^n$ be compact and convex. Furthermore let $f \in C^1(S)$ be convex, $x \in S$ and $d \in \mathbb{R}^n$ a descent direction of f in x with $\nabla f(x)^T d < 0$.*

Proof. If $x + \lambda^* d$ is an optimal solution then $\nabla f(x + \lambda^* d)^T d = 0$ according to Theorem 1.12. Let $\nabla f(x + \lambda^* d)^T d = 0$. Then Lemma 2.1 gives

$$f(x + \lambda d) \geq f(x + \lambda^* d) + (\lambda - \lambda^*) \nabla f(x + \lambda^* d)^T d = f(x + \lambda^* d)$$

and $x + \lambda^* d$ is an optimal solution.

Exercise 2.6 (Steepest Descent)**.** *Let*

$$f(x) = \frac{1}{2} x^T Ax + b^T x + c$$

where A is symmetrical and positive definite.

Proof. Since $\nabla f(x) = Ax + b$ and $\nabla^2 f(x) = A > 0$ it follows $x^* = -A^{-1} b$. Let v be eigenvector with $Av = \mu v$. For $x_0 = x^* + \theta v$ we have

$$\nabla f(x_0) = Ax^* + \mu \theta v + b = \mu \theta v$$

and for $\lambda \geq 0$

$$\arg\min\{f(x_0 - \lambda \nabla f(x_0)\} = \arg\min\{f(x^* + \theta v - \lambda \mu \theta v)\} = \mu^{-1}$$

Thus

$$x_1 = x_0 - \mu^{-1}\nabla f(x_0) = x^* + \theta v - \mu^{-1}\mu\theta v = x^*$$

and $\nabla f(x_1) = 0$. Hence the algorithm stops after the first iteration. Now let

$$x_0 = x^* + \sum_{i=0}^{m} \theta_i v_i$$

for orthogonal eigenvectors with $Av_i = \mu_i$ and $m \leq n$. Then

$$\nabla f(x_0) = Ax^* + \sum_{i=0}^{m} \mu_i\theta_i v_i + b = \sum_{i=0}^{m} \mu_i\theta_i v_i$$

and

$$x_1 = x_0 - \lambda\sum_{i=0}^{m} \mu_i\theta_i v_i = x^* + \sum_{i=0}^{m} \theta_i v_i - \lambda\sum_{i=0}^{m} \mu_i\theta_i v_i = x^* + \sum_{i=0}^{m}(1-\lambda\mu_i)\theta_i v_i$$

Since x^* is the minimum we have $\nabla f(x_1) = 0$ iff $\lambda = \mu^{-1}$ for all $0 \leq i \leq m$. $\square$

2.2 One Dimensional Minimization and Direct Search

Definition 2.7 (Unimodal Function)**.** *A function $f : [a, b] \to \mathbb{R}$ is called unimodal if there exists a $\xi \in [a, b]$, so that f is strictly decreasing in $[a, \xi]$ and strictly increasing in $[\xi, b]$.*

In fact ξ is the unique minimum of f in $[a, b]$. According to the definition, for $a \leq x < y \leq b$ we have

$$f(x) > f(y) \text{ for } x, y \in [a, \xi) \text{ and } f(x) < f(y) \text{ for } x, y \in (\xi, b]$$

Thus

$$\xi \in [a, y] \text{ if } f(x) < f(y) \text{ and } \xi \in [x, b] \text{ if } f(x) \geq f(y)$$

Consider now a symmetrical partioning of the interval $[0, 1]$ where two consecutive partionings hold the same ratio respectively:

$$\sigma = 1 - \tau \text{ and } \frac{1}{\tau} = \frac{\tau}{\sigma}$$

Then $1 - \tau = \tau^2$ and solving the quadratic equation $\tau^2 + \tau = 1$ yields

$$\tau = \frac{\sqrt{5} - 1}{2} \approx 0.61803$$

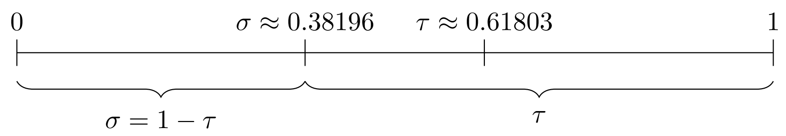

Figure 1: Golden Section

Let now $[a_0, b_0] = [a, b]$ and define

$$[a_{k+1}, b_{k+1}] = \begin{cases} [a_k, y_k] & \text{if } f(x_k) < f(y_k) \\ [x_k, b_k] & \text{if } f(x_k) \geq f(y_k) \end{cases}$$

where

$$\begin{aligned} x_k &= b_k - \tau(b_k - a_k) \\ y_k &= a_k + \tau(b_k - a_k) \end{aligned}$$

It follows that $[a_k, b_k] \supset [a_{k+1}, b_{k+1}]$ is a decreasing series of intervals with

$$(b_{k+1} - a_{k+1}) = \tau(b_k - a_k)$$

where the interval converges to ξ. This leads to the following algorithm:

Algorithm 2.8 (Golden Section Search)**.**

```python
"""Basic implementation of the golden section search, this easily can be
improved by storing and resuing the results of the previous iteration
"""

import math


def golden_section_search(f, I, eps=0.00001):
    t = 0.5 * (math.sqrt(5) - 1)
    a, b = I
    while abs(b - a) > eps:
        x, y = b - t * (b - a), a + t * (b - a)
        if f(x) > f(y):
            a = x
        else:
            b = y
    return (a + b) / 2


if __name__ == '__main__':
    p, q, I = 0, 0, (-10, 10)
    p, q, I = -4, 1, (-10, 10)
    f = lambda x: (x + p) ** 2 + q
    x0 = golden_section_search(f, I)
    print(f'arg min f on {I}: {x0}')
```

Algorithm 2.9 (Steepest Descent)**.**

Let $f \in C^1(\mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$. For $0 < \alpha \leq \beta < 1$ and $\gamma < 1$ let

Exercise 2.10 (Surprising Convergence)**.** *Example for $f \in C^2(\mathbb{R})$ with a sequence of strict local minima converging to a strict local maximum.*

Proof. Let $f \in C[a,b]$ and $\xi \in (a,b)$ so that f is strictly increasing in $(a,\xi]$ and strictly decreasing in $[\xi,b)$. Define

$$g(x) = \int_{\xi-x}^{\xi+x} f(t)\,dt$$

$\square$

3 The Road to Reality

3.1 Hyperbolic Geometry

The ratio between the area A and A' of two similar shapes is given by

$$A' = k^2 A$$

Theorem 3.1 (Pythagoras)**.**

$$a^2 + b^2 = c^2$$

Proof. Let A, B and C be the areas of the three triangles respectively. All triangles are similar, hence

$$B = \frac{b^2}{a^2} A \text{ and } C = \frac{c^2}{b^2} B$$

Since $A + B = C$ it follows that

$$a^2 + b^2 = \frac{b^2 A}{B} + b^2 = \frac{b^2(A+B)}{B} = \frac{b^2 C}{B} = c^2$$

$\square$

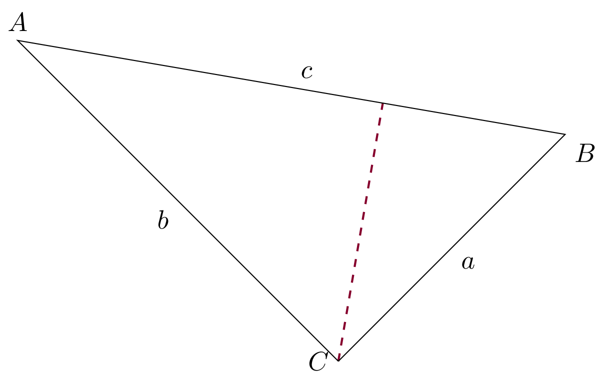

Figure 2: Pythagoras

Lemma 3.2 (Conformal and Projective Representation)**.** *The mapping from conformal and projective representation of any point is given by the radial expansion of the following factor*

$$\frac{2R}{R^2 + r^2}$$

Proof. For any point the distance from the origin with regard to the two representations is given by

$$\log\frac{R+r}{R-r}=\frac{1}{2}\log\frac{R+r'}{R-r'}=\log\frac{(R+r')^2}{(R-r')^2}$$

This gives

$$(R-r)^2(R+r')=(R+r)^2(R-r') \text{ and } -4R^2r+2R^2r'+2r^2r'=0$$

Hence

$$r'=\frac{2R^2}{R^2+r^2}r$$

□

3.2 Complex Numbers

Lemma 3.3 (Basic Formulas)**.**

1. *It is*

$$(a+\mathrm{i}b)(c+\mathrm{i}d)=(ac-bd)+\mathrm{i}(ad+bc)$$

2. *Thus*

$$(a+\mathrm{i}b)^2=(a^2-b^2)+\mathrm{i}2ab$$

   and

$$(a+\mathrm{i}b)(a-\mathrm{i}b)=a^2+\mathrm{i}ab-\mathrm{i}ab-\mathrm{i}^2b^2=a^2+b^2$$

3. *Hence*

$$\frac{a+\mathrm{i}b}{c+\mathrm{i}d}=\frac{(a+\mathrm{i}b)(c-\mathrm{i}d)}{c^2+d^2}=\frac{ac+bd}{c^2+d^2}+\mathrm{i}\frac{bc-ad}{c^2+d^2}$$

4. *For*

$$z=\sqrt{\frac{1}{2}(a+\sqrt{a^2+b^2})}+\mathrm{i}\sqrt{\frac{1}{2}(-a+\sqrt{a^2+b^2})}$$

   it follows

$$z^2=\frac{1}{2}(a+\sqrt{a^2+b^2})-\frac{1}{2}(-a+\sqrt{a^2+b^2})+\mathrm{i}2\sqrt{\frac{1}{4}(\sqrt{a^2+b^2}^2)-a^2}=a+\mathrm{i}b$$

Lemma 3.4 (Binomial Theorem)**.**

1. *For the binomial coefficient Pascal's identity holds*

$$\binom{n}{k-1}+\binom{n}{k}=\binom{n+1}{k}$$

2. *The following equation states the binomial identity*

$$(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k}=\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

3. For $a = 1$ follows

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Proof. It is

$$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} = \frac{n!(n+1-k)+n!k!}{k!(n+1-k)!} = \binom{n+1}{k}$$

Furthermore by using induction

$$\begin{aligned}
(a+b)^{n+1} &= \sum_{k=0}^{n} \binom{n}{k} a^{k+1} b^{n-k} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{n+1-k} \\
&= \sum_{k=1}^{n+1} \binom{n}{k-1} a^k b^{n+1-k} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{n+1-k} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k}
\end{aligned}$$

□

3.3 Exponential Function and Logarithms

Exercise 3.5 (Exponential Function)**.** *The Cauchy product yields*

$$\sum_{n=0}^{\infty} a_n \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k}$$

if at least one of the series is absolutely convergent. Hence

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{n!} z^n \sum_{n=0}^{\infty} \frac{1}{n!} w^n &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!} z^k \frac{1}{(n-k)!} w^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} (z+w)^n
\end{aligned}$$

Let $t \in \mathbb{R}$. Then

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}t} &= \sum_{k=0}^{\infty} \frac{1}{k!}(\mathrm{i}t)^k \\
&= \sum_{k=0}^{\infty} \frac{1}{2k!}(\mathrm{i}t)^{2k} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(\mathrm{i}t)^{2k+1} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k!}t^{2k} + \mathrm{i}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}t^{2k+1} \\
&= \cos t + \mathrm{i}\sin t
\end{aligned}$$

More generally for $z = \log r + \mathrm{i}t$

$$\mathrm{e}^z = \mathrm{e}^{\log r + \mathrm{i}t} = r\mathrm{e}^{\mathrm{i}t} = r(\cos t + \mathrm{i}\sin t)$$

For $r = 1$ and $t = 2\pi$ this yields

$$\mathrm{e}^{2\pi\mathrm{i}} = \cos 2\pi + \mathrm{i}\sin 2\pi = 1$$

and for $t = 2\pi$ we get

Lemma 3.6 (Euler Equation)**.**

$$\mathrm{e}^{\pi\mathrm{i}} + 1 = 0$$

Exercise 3.7.

1. *If $\mathrm{e}^z = w$ then $z + \pi\mathrm{i}$ is a logarithm to $-w$: $\mathrm{e}^{z+\pi\mathrm{i}} = \mathrm{e}^z\mathrm{e}^{\pi\mathrm{i}} = -\mathrm{e}^z = -w$.*
2. *Since $\mathrm{e}^{i(s+t)} = \mathrm{e}^{is}\mathrm{e}^{it}$ it follows*

$$\begin{aligned}
\cos(s+t) + \mathrm{i}\sin(s+t) &= (\cos s + \mathrm{i}\sin s)(\cos t + \mathrm{i}\sin t) \\
&= \cos s\cos t - \sin s\sin t + \mathrm{i}(\cos s\sin t + \sin s\cos t)
\end{aligned}$$

   Hence

$$\begin{aligned}
\cos(s+t) &= \cos s\cos t - \sin s\sin t \\
\sin(s+t) &= \cos s\sin t + \sin s\cos t
\end{aligned}$$

3. *It is $\mathrm{e}^{3it} = (\mathrm{e}^{it})^3$ and thus*

$$\cos 3t + \mathrm{i}\sin 3t = (\cos t + \mathrm{i}\sin t)^3 = \cos^3 t - 3\cos t\sin^2(t) + \mathrm{i}(\cos^2 t\sin t - \sin^3 t)$$

4. *Fun facts*

$$\mathrm{e}^{1-4\pi^2} = \mathrm{e}^{1+(2\mathrm{i}\pi)^2} = \mathrm{e}\mathrm{e}^{2\pi\mathrm{i}}\mathrm{e}^{2\pi\mathrm{i}} = \mathrm{e}$$

   and $\mathrm{i} = \mathrm{e}^{\mathrm{i}\pi/2}$ gives

$$\mathrm{i}^{\mathrm{i}} = \mathrm{e}^{\mathrm{i}\log\mathrm{i}} = \mathrm{e}^{\mathrm{i}\mathrm{i}\pi/2} = \mathrm{e}^{-\pi/2} \in \mathbb{R}$$

3.4 Complex Number Calculus

Theorem 3.8 (Cauchy Riemann Equations)**.** *Let $f = u + \mathrm{i}v$ be holomorphic. Then f satisfies the Cauchy Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Proof. For $h \in \mathbb{R}$ follows

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \frac{\partial u}{\partial x}(z) + \mathrm{i}\frac{\partial v}{\partial x}(z)$$

and

$$\lim_{h \to 0} \frac{f(z+\mathrm{i}h) - f(z)}{\mathrm{i}h} = \frac{\partial u}{\mathrm{i}\partial y}(z) + \frac{\partial v}{\partial y}(z) = \frac{\partial v}{\partial y}(z) - \mathrm{i}\frac{\partial u}{\partial y}(z)$$

□

Exercise 3.9.

1. *Let $z_0 \in \mathbb{C}$ and $\gamma(t) = z_0 + \mathrm{e}^{\mathrm{i}t}$ for $t \in [0, 2\pi]$. Then*

$$\int_\gamma \frac{1}{z - z_0}\, dz = \int_0^{2\pi} \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}t}}{z_0 + \mathrm{e}^{\mathrm{i}t} - z_0}\, dt = \int_0^{2\pi} \mathrm{i}\, dt = 2\pi\mathrm{i}$$

and thus $1/(z - z_0)$ is not holomorphic on $\mathbb{C} \setminus \{z_0\}$