

## Notes on Mathematics

### Contents

<b>1</b>	<b>Calculus</b>	<b>1</b>
1.1	Differentiation and Integration . . . . .	1
1.2	Directional Derivative and Gradients . . . . .	5
<b>2</b>	<b>Nonlinear Optimization</b>	<b>7</b>
2.1	Minimization without Constraints . . . . .	7
2.2	One Dimensional Minimization and Direct Search . . . . .	9
<b>3</b>	<b>Functional Analysis</b>	<b>11</b>
3.1	The Hahn-Banach Theorem . . . . .	11
3.2	The Riesz Representation Theorem . . . . .	11
<b>4</b>	<b>The Road to Reality</b>	<b>12</b>
4.1	Hyperbolic Geometry . . . . .	12
4.2	Complex Numbers . . . . .	13
4.3	Exponential Function and Logarithms . . . . .	14
4.4	Complex Analysis . . . . .	15
<b>5</b>	<b>Neural Networks</b>	<b>22</b>
5.1	The Perceptron . . . . .	22
5.2	Feedforward Neural Networks . . . . .	26
5.3	The Backtracking Algorithm . . . . .	27

## 1 Calculus

### 1.1 Differentiation and Integration

**Lemma 1.1** (Simple Calculations).

1. For  $1 = xx^{-1}$  the product rule yields  $0 = x^{-1} + x(x^{-1})'$ . Hence

$$\frac{d}{dx}x^{-1} = -\frac{1}{x^2}$$

2. Similarly  $x = \sqrt{x^2}$  and  $1 = 2\sqrt{x}\sqrt{x}'$  and so

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

3. It is

$$\frac{d}{dx}x^n = nx^{n-1}$$

since via induction the product rule yields

$$\frac{d}{dx}x^n = \frac{d}{dx}xx^{n-1} = x^{n-1} + \frac{d}{dx}x^{n-1} = x^{n-1} + (n-1)x^{n-1} = nx^{n-1}$$

4. Again, applying the product rule gives

$$\left(\frac{1}{g}\right)' = \left(\frac{1}{x} \circ g\right)' = -\frac{g'}{g^2}$$

and the quotient rule

$$\left(\frac{f}{g}\right)' = \frac{f'}{g} + f\left(\frac{1}{g}\right)' = \frac{f'}{g} - \frac{fg'}{g^2} = \frac{gf' - fg'}{g^2}$$

5. Also  $x = f \circ f^{-1}$  and  $1 = (f^{-1})'f' \circ f^{-1}$ . Thus

$$(f^{-1})' = \frac{1}{f' \circ f^{-1}}$$

where defined. Especially for  $x \neq 0$

$$\log'(x) = \frac{1}{\exp'(\log(x))} = \frac{1}{x}$$

6.  $(1-q)(1+q+q^2+\dots+q^n) = 1-q+q-q^2+q^2-q^3+\dots+q^{n+1}$  gives

$$\sum_{k=0}^n q^k = \frac{1-q^{n+1}}{1-q} \text{ and } \sum_{k=m}^n q^k = \frac{q^m - q^{n+1}}{1-q}$$

**Lemma 1.2** (Exponential Function).

1. It is

$$\exp(x + y) = \exp(x) \exp(y)$$

Hence

$$\begin{aligned}\exp(0) &= 1 \\ \exp(-x) &= \exp(x)^{-1} \\ \exp(nx) &= \exp(x)^n\end{aligned}$$

2. For the derivative

$$\exp'(x) = \sum_{k=0}^{\infty} \frac{1}{k!} (x^k)' = \sum_{k=0}^{\infty} \frac{1}{k!} k x^{k-1} = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} x^{k-1} = \exp(x)$$

**Lemma 1.3** (Sinus and Cosinus).

1. Sinus and Cosinus power series

$$\begin{aligned}\cos(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k!} x^{2k} \\ \sin(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}\end{aligned}$$

2. Symmetry

$$\begin{aligned}\cos(-x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k!} (-x)^{2k} = \cos(x) \\ \sin(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (-x)^{2k+1} = -\sin(x)\end{aligned}$$

3. Derivatives

$$\begin{aligned}\cos'(x) &= \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)!} x^{2k-1} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!} x^{2k+1} = -\sin(x) \\ \sin'(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k!} x^{2k} = \cos(x)\end{aligned}$$

**Theorem 1.4** (Fermat Stationary Point). Let  $\Omega \subseteq \mathbb{R}$  be open and  $f \in C^1(\Omega)$ . If  $x^* \in \Omega$  is local extremum then  $f'(x^*) = 0$ .

*Proof.* Assume  $x^*$  is the minimum of  $f$  in  $\Omega$  and let  $f(x^*) > 0$ . Since  $f \in C^1(\Omega)$  there exist  $\varepsilon, \delta > 0$ , so that for  $|h| \leq \varepsilon$

$$\frac{f(x^* + h) - f(x^*)}{h} > \delta$$

Pick a negative  $h \in [-\varepsilon, 0)$ . Then

$$f(x^* + h) < f(x^*) + \delta h < f(x^*)$$

and  $x^*$  cannot be the minimum. Analog for maximum with a positive  $h$ , then apply to  $-f$ .  $\square$

**Theorem 1.5** (Rolle). *Let  $f \in C[a, b]$  with  $f(a) = f(b)$ . If  $f$  is differentiable in  $(a, b)$  then there exists a  $\xi \in (a, b)$  with  $f'(\xi) = 0$ .*

*Proof.* Assume  $f$  is not constant. Since  $[a, b]$  is compact there exists either a global minimum or maximum  $\xi \in (a, b)$  and Theorem 1.4 can be applied.  $\square$

**Theorem 1.6** (Mean Value). *Let  $f \in C[a, b]$  be differentiable in  $(a, b)$ . Then there exists a  $\xi \in (a, b)$  with*

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

*Proof.* Apply Theorem 1.5 to

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a)$$

$\square$

**Remark 1.7.**

1. More generally choose any  $\varphi \in C^1[a, b]$  with  $\varphi(a) = 0$  and  $\varphi(b) = f(b) - f(a)$ . Set  $g(x) = f(x) - \varphi(x)$  to see there is a  $\xi \in (a, b)$  with  $f'(\xi) = \varphi'(\xi)$ .
2. Let  $f$  be differentiable in  $(a, b)$  with  $f' = 0$ . For  $x, y \in (a, b)$

$$0 = f'(\xi) = \frac{f(y) - f(x)}{y - x}$$

and  $f$  is a constant.

3. Another useful generalization: let  $\Omega \subseteq \mathbb{R}^n$  be open and  $f \in C^1(\Omega)$ . For  $x, y \in \Omega$  define  $\varphi(t) = f(tx + (1 - t)y)$  and apply the chain rule for differentiation

$$\varphi'(\xi) = \nabla f(\xi x + (1 - \xi)y)^T(x - y) = f(x) - f(y)$$

4. The Cauchy Schwarz inequality then yields

$$\|f(x) - f(y)\| \leq \|\nabla f(\xi x + (1 - \xi)y)\| \|x - y\|$$

**Theorem 1.8** (Differentiation Theorem). *Let  $f \in C[a, b]$  and define*

$$F(x) = \int_a^x f(t) dt$$

*Then  $F \in C^1[a, b]$  with  $F'(x) = f(x)$  for  $x \in [a, b]$ .*

*Proof.* Applying the Mean Value Theorem of Integration gives

$$F(x + h) - F(x) = \int_x^{x+h} f(t) dt = f(\xi)h$$

for some  $\xi \in (x, x + h)$ .  $\square$

**Theorem 1.9** (Fundamental Theorem of Calculus). *Let  $F \in C^1[a, b]$  with  $F' = f$ . Then*

$$F(b) - F(a) = \int_a^b f(t) dt$$

**Lemma 1.10** (Integration by Substitution). *Let  $I \subseteq \mathbb{R}$  be an interval and  $f \in C(I)$ . For  $\varphi \in C([a, b], I)$  it follows*

$$\int_{\varphi(a)}^{\varphi(b)} f(x) dx = \int_a^b f(\varphi(t))\varphi'(t) dt$$

*Proof.* Let  $F \in C^1(I)$  with  $F' = f$ . Then the chain rule for differentiation yields

$$\begin{aligned} \int_{\varphi(a)}^{\varphi(b)} f(x) dx &= F(\varphi(b)) - F(\varphi(a)) \\ &= F \circ \varphi(b) - F \circ \varphi(a) \\ &= \int_a^b (F \circ \varphi)'(t) dt \\ &= \int_a^b f(\varphi(t))\varphi'(t) dt \end{aligned}$$

□

**Examples 1.11.**

1. For  $\varphi(x) = x^2 + 1$  it is  $\varphi(0) = 1$  and  $\varphi(2) = 5$ . Thus

$$\int_0^2 x \cos(x^2 + 1) dx = \frac{1}{2} \int_0^2 2x \cos(x^2 + 1) dx = \frac{1}{2} \int_1^5 \cos(t) dt = \frac{1}{2}(\sin(5) - \sin(1))$$

2. Consider  $\varphi(x) = \sin(x)$  where  $\varphi(0) = 0$  and  $\varphi(\pi/2) = 1$ . Since  $\cos(t) = \sqrt{1 - \sin^2(t)}$  it follows

$$\int_0^1 \sqrt{1 - x^2} dx = \int_{\cos(0)}^{\cos(\pi/2)} \sqrt{1 - x^2} dx = \int_0^{\pi/2} \sqrt{1 - \sin^2(t)} \cos(t) dt = \int_0^{\pi/2} \cos^2(t) dt$$

3. Let  $f \in C[a, b]$  and  $\varphi(x) = a + t(b - a)$ . Then

$$\int_a^b f(x) dx = (b - a) \int_0^1 f(a + t(b - a)) dt$$

4. Let  $f(x) = x^n$  and  $\varphi(x) = t^m$ . As expected

$$\int_0^1 x^n dx = \int_0^1 t^{nm} m t^{m-1} dt = m \int_0^1 t^{m(n+1)-1} dt = \left[ \frac{m}{m(n+1)} t^{m(n+1)} \right]_0^1 = \frac{1}{n+1}$$

## 1.2 Directional Derivative and Gradients

**Lemma 1.12** (Directional Derivative). *Let  $\Omega \subseteq \mathbb{R}^n$  be open and  $f \in C^1(\Omega)$ . Then*

$$\frac{\partial f}{\partial d}(x) = \nabla f(x)^T d$$

for any  $d \in \mathbb{R}^n$ .

*Proof.* Let  $\varphi(t) = f(x + td)$ . Then  $\varphi \in C^1[-\varepsilon, \varepsilon]$  for some  $\varepsilon > 0$  and the chain rule yields

$$\varphi'(t) = \nabla f(x + td)^T d$$

Hence

$$\varphi'(0) = \lim_{t \rightarrow 0} \frac{\varphi(x + td) - \varphi(0)}{t} = \lim_{t \rightarrow 0} \frac{f(x + td) - f(x)}{t} = \nabla f(x)^T d$$

□

**Remarks 1.13.**

1. Note that by definition the directional derivative is invariant under multiplication with any  $\lambda \neq 0$ .
2. A similar proposition holds under the weaker assumption that  $d$  is a only feasible direction for  $f$  in  $x$
3. For  $d = \nabla f(x) / \|\nabla f(x)\|$  it follows that

$$\frac{\partial f}{\partial d}(x) = \|\nabla f(x)\| > 0$$

and for any other  $d \in \mathbb{R}^n$  with  $\|d\| = 1$  the Cauchy Schwarz inequality yields

$$\left| \frac{\partial f}{\partial d}(x) \right| = |\nabla f(x)^T d| \leq \|\nabla f(x)\| \|d\| = \|\nabla f(x)\|$$

Hence  $\nabla f(x)$  is the direction of the greatest ascent and respectively,  $-\nabla f(x)$  is the direction of the greatest descent.

**Theorem 1.14** (First Order Necessary Condition). *Let  $\Omega \subseteq \mathbb{R}^n$  be open and  $f \in C^1(\Omega)$ . If  $x^* \in \Omega$  is a local minimizer then  $\nabla f(x^*) = 0$ .*

*Proof.* Let  $h \in \mathbb{R}^n$  and  $\delta > 0$  so that  $x^* + th \in \Omega$  for all  $t \in (-\delta, \delta)$ . Then 0 is local minimizer for  $\varphi(t) = f(x^* + th)$  and

$$\varphi'(0) = \nabla f(x^*)^T h = 0$$

Now let  $h = \nabla f(x^*)$ .

□

**Theorem 1.15** (Banach Fixed-Point Theorem). *Let  $X$  be a Banach space and  $f \in C(X, X)$  a contraction*

$$\|f(x) - f(y)\| \leq q \|x - y\| \text{ for all } x, y \in X$$

for some  $0 < q < 1$ . Then there exists a unique fix point  $x^* \in X$  with

$$f(x^*) = x^*$$

Furthermore for any  $x_0 \in X$  the sequence defined by

$$x_{n+1} = f(x_n)$$

converges against  $x^*$ .

*Proof.* Since  $\|x_{n+1} - x_n\| = \|f(x_n) - f(x_{n-1})\| \leq q\|x_n - x_{n-1}\|$  it follows, that

$$\|x_{n+1} - x_n\| \leq q^n \|x_1 - x_0\|$$

Furthermore

$$\|x_n - x_m\| \leq \sum_{k=m}^n q^k \|x_1 - x_0\| = \frac{q^m - q^{n+1}}{1 - q} \|x_1 - x_0\|$$

and  $(x_n)$  is a Cauchy sequence. For its limit  $x^*$  we have

$$x^* = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} f(x_n) = f(x^*)$$

For any other  $y^* \in X$  with  $f(y^*) = y^*$  it follows, that

$$\|x^* - y^*\| = \|f(x^*) - f(y^*)\| \leq q\|x^* - y^*\|$$

and therefore  $x^* = y^*$ .

□

## 2 Nonlinear Optimization

### 2.1 Minimization without Constraints

**Lemma 2.1** (Gradient Inequality). *Let  $M \subseteq \mathbb{R}^n$  be a convex set and  $f \in C^1(M)$ . Then  $f$  is convex if and only if*

$$f(x) \geq f(y) + \nabla f(y)^T(x - y)$$

for all  $x, y \in M$ .

*Proof.* Let  $f$  be convex and  $x, y \in M$ . For  $0 \leq \lambda \leq 1$  we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) = \lambda f(x) - \lambda f(y) + f(y)$$

and

$$f(x) - f(y) \geq \frac{f(\lambda x + (1 - \lambda)y) - f(y)}{\lambda} = \frac{f(y + \lambda(x - y)) - f(y)}{\lambda}$$

For  $d = x - y$  and  $\lambda \rightarrow 0$  the term on the right converges to the direction derivative of  $f$  in  $d$

$$\frac{\partial f}{\partial d}(y) = \nabla f(y)^T d = \nabla f(y)^T(x - y)$$

Now let  $x, y \in M$  and  $0 \leq \lambda \leq 1$ . For  $z = \lambda x + (1 - \lambda)y \in M$  it follows that

$$\begin{aligned} \lambda f(x) &\geq \lambda f(z) + \lambda \nabla f(z)^T(x - z) \\ (1 - \lambda)f(y) &\geq (1 - \lambda)f(z) + (1 - \lambda)\nabla f(z)^T(y - z) \end{aligned}$$

Adding the two inequalities gives

$$\begin{aligned} \lambda f(x) + (1 - \lambda)f(y) &\geq f(z) + \nabla f(z)^T(\lambda x - \lambda z + (1 - \lambda)y - (1 - \lambda)z) \\ &= f(z) + \nabla f(z)^T(\lambda x + (1 - \lambda)y - z) \\ &= f(z) \end{aligned}$$

□

**Exercise 2.2** (Facility Locations). *The facilities are located at:*

$$(3, 0), (0, -3), (1, 4)$$

*Proof.* Let

$$\begin{aligned} f(x) &= (x - 3)^2 + y^2 + x^2 + (y + 3)^2 + (x - 1)^2 + (y - 4)^2 \\ &= x^2 - 6x + 9 + y^2 + x^2 + y^2 + 6y + 9 + x^2 - 2x + 1 + y^2 - 8y + 16 \\ &= 3x^2 + 3y^2 - 8x - 2y + 35 \end{aligned}$$

Then

$$\nabla f(x, y) = (6x - 8, 6y - 2) \text{ and } \nabla^2 f(x, y) = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} > 0$$

Hence  $(4/3, 1/3)$  is the global minimum.

□

**Exercise 2.3** (Convex Functions). *The sum of convex functions is convex.*



*Proof.* Let  $x, y \in M$ . Since  $\alpha_i > 0$  we have

$$\begin{aligned} f(\lambda x + (1 - \lambda)y) &= \sum_{i=1}^m \alpha_i f_i(\lambda x + (1 - \lambda)y) \\ &\leq \sum_{i=1}^m \alpha_i \lambda f_i(x) + \sum_{i=1}^m \alpha_i (1 - \lambda) f_i(y) = \lambda f(x) + (1 - \lambda) f(y) \end{aligned}$$

Let  $f(x) = x^2$ . Then  $-f$  is not convex, e.g.  $x = 1, y = -1$  and  $\lambda = 0.5$ .

**Exercise 2.4** (Solution of Quadratic Inequality). *Let*

$$f(x) = x^T A x + b^T x + c$$

*Proof.* The product rule gives

$$\nabla f(x) = x^T A + A x + b = (A^T + A)x + b = 2A x + b$$

Thus  $\nabla^2 f(x) = 2A > 0$  and  $f$  is convex. Hence the level set  $\Gamma_{-c}$  is convex. Since the intersection of convex sets is convex  $\Gamma_{-c} \cap \{x \in \mathbb{R}^n : g^T x + h = 0\}$  is convex, too.

**Exercise 2.5** (Line Search on Compact Convex Sets). *Let  $S \subset \mathbb{R}^n$  be compact and convex. Furthermore let  $f \in C^1(S)$  be convex,  $x \in S$  and  $d \in \mathbb{R}^n$  a descent direction of  $f$  in  $x$  with  $\nabla f(x)^T d < 0$ .*

*Proof.* If  $x + \lambda^* d$  is an optimal solution then  $\nabla f(x + \lambda^* d)^T d = 0$  according to Theorem 1.14. Let  $\nabla f(x + \lambda^* d)^T d = 0$ . Then Lemma 2.1 gives

$$f(x + \lambda d) \geq f(x + \lambda^* d) + (\lambda - \lambda^*) \nabla f(x + \lambda^* d)^T d = f(x + \lambda^* d)$$

and  $x + \lambda^* d$  is an optimal solution.

**Exercise 2.6** (Steepest Descent). *Let*

$$f(x) = \frac{1}{2} x^T A x + b^T x + c$$

where  $A$  is symmetrical and positive definite.

*Proof.* Since  $\nabla f(x) = A x + b$  and  $\nabla^2 f(x) = A > 0$  it follows  $x^* = -A^{-1}b$ . Let  $v$  be eigenvector with  $Av = \mu v$ . For  $x_0 = x^* + \theta v$  we have

$$\nabla f(x_0) = A x^* + \mu \theta v + b = \mu \theta v$$

and for  $\lambda \geq 0$

$$\arg \min \{f(x_0 - \lambda \nabla f(x_0))\} = \arg \min \{f(x^* + \theta v - \lambda \mu \theta v)\} = \mu^{-1}$$

Thus

$$x_1 = x_0 - \mu^{-1} \nabla f(x_0) = x^* + \theta v - \mu^{-1} \mu \theta v = x^*$$

and  $\nabla f(x_1) = 0$ . Hence the algorithm stops after the first iteration. Now let

$$x_0 = x^* + \sum_{i=0}^m \theta_i v_i$$

for orthogonal eigenvectors with  $Av_i = \mu_i$  and  $m \leq n$ . Then

$$\nabla f(x_0) = Ax^* + \sum_{i=0}^m \mu_i \theta_i v_i + b = \sum_{i=0}^m \mu_i \theta_i v_i$$

and

$$x_1 = x_0 - \lambda \sum_{i=0}^m \mu_i \theta_i v_i = x^* + \sum_{i=0}^m \theta_i v_i - \lambda \sum_{i=0}^m \mu_i \theta_i v_i = x^* + \sum_{i=0}^m (1 - \lambda \mu_i) \theta_i v_i$$

Since  $x^*$  is the minimum we have  $\nabla f(x_1) = 0$  iff  $\lambda = \mu_i^{-1}$  for all  $0 \leq i \leq m$ .  $\square$

## 2.2 One Dimensional Minimization and Direct Search

**Definition 2.7** (Unimodal Function). *A function  $f : [a, b] \rightarrow \mathbb{R}$  is called unimodal if there exists a  $\xi \in [a, b]$ , so that  $f$  is strictly decreasing in  $[a, \xi]$  and strictly increasing in  $[\xi, b]$ .*

In fact  $\xi$  is the unique minimum of  $f$  in  $[a, b]$ . According to the definition, for  $a \leq x < y \leq b$  we have

$$f(x) > f(y) \text{ for } x, y \in [a, \xi] \text{ and } f(x) < f(y) \text{ for } x, y \in [\xi, b]$$

Thus

$$\xi \in [a, y] \text{ if } f(x) < f(y) \text{ and } \xi \in [x, b] \text{ if } f(x) \geq f(y)$$

Consider now a symmetrical partitioning of the interval  $[0, 1]$  where two consecutive partitionings hold the same ratio respectively:

$$\sigma = 1 - \tau \text{ and } \frac{1}{\tau} = \frac{\tau}{\sigma}$$

Then  $1 - \tau = \tau^2$  and solving the quadratic equation  $\tau^2 + \tau = 1$  yields

$$\tau = \frac{\sqrt{5} - 1}{2} \approx 0.61803$$

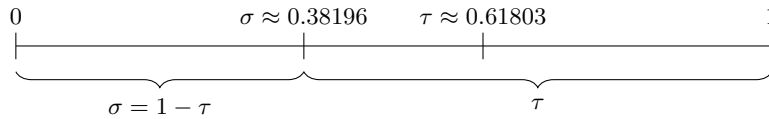


Figure 1: Golden Section

Let now  $[a_0, b_0] = [a, b]$  and define

$$[a_{k+1}, b_{k+1}] = \begin{cases} [a_k, y_k] & \text{if } f(x_k) < f(y_k) \\ [x_k, b_k] & \text{if } f(x_k) \geq f(y_k) \end{cases}$$

where

$$\begin{aligned}x_k &= b_k - \tau(b_k - a_k) \\ y_k &= a_k + \tau(b_k - a_k)\end{aligned}$$

It follows that  $[a_k, b_k] \supset [a_{k+1}, b_{k+1}]$  is a decreasing series of intervals with

$$(b_{k+1} - a_{k+1}) = \tau(b_k - a_k)$$

where the interval converges to  $\xi$ . This leads to the following algorithm:

**Algorithm 2.8** (Golden Section Search).

---

```
"""Basic implementation of the golden section search, this easily can be
improved by storing and resuing the results of the previous iteration
"""
```

```
import math
```

```
def golden_section_search(f, I, eps=0.00001):
    t = 0.5 * (math.sqrt(5) - 1)
    a, b = I
    while abs(b - a) > eps:
        x, y = b - t * (b - a), a + t * (b - a)
        if f(x) > f(y):
            a = x
        else:
            b = y
    return (a + b) / 2
```

---

**Algorithm 2.9** (Steepest Descent).

Let  $f \in C^1(\mathbb{R}^n)$  and  $x_0 \in \mathbb{R}^n$ . For  $0 < \alpha \leq \beta < 1$  and  $\gamma < 1$  let

**Exercise 2.10** (Surprising Convergence). *Example for  $f \in C^2(\mathbb{R})$  with a sequence of strict local minima converging to a strict local maximum.*

*Proof.* Let  $f \in C[a, b]$  and  $\xi \in (a, b)$  so that  $f$  is strictly increasing in  $(a, \xi]$  and strictly decreasing in  $[\xi, b)$ . Define

$$g(x) = \int_{\xi-x}^{\xi+x} f(t) dt$$

□

### 3 Functional Analysis

#### 3.1 The Hahn-Banach Theorem

**Remarks 3.1.**

1. Let  $f = u + iv$  be a complex valued linear functional. Then  $f(ix) = if(x)$  gives

$$u(ix) = -v(x) \text{ and } v(ix) = u(x)$$

Thus

$$f(x) = u(x) - iu(ix)$$

2. Assume  $|u(x)| \leq C\|x\|$  and choose  $t \in \mathbb{R}$  so that  $|f(x)| = e^{it}f(x)$ . Then

$$|f(x)| = e^{it}f(x) = f(e^{it}x) = u(e^{it}x) \leq C\|e^{it}x\| = C\|x\|$$

3. See also the [Cauchy-Rieman Equations 4.10](#).

**Theorem 3.2** (Hahn-Banach Theorem). *Let  $E$  be a normed space and  $F$  a linear subspace. For any linear functional  $f \in F^*$  there exists a  $\tilde{f} \in E^*$  so that*

$$\tilde{f}|_F = f \text{ and } \|\tilde{f}\| = \|f\|$$

*Proof.* Assume that  $f : E \rightarrow \mathbb{R}$  is a real valued linear functional and  $x' \in E \setminus F$ . Consider the subspace  $F' = \{x + \lambda x' | x \in F\}$  and define

$$f'(x + \lambda x') = f(x) + \lambda r$$

for some  $r \in \mathbb{R}$ . As sum of two linear functionals  $f'$  again is a linear functional with  $\|f'\| \geq \|f\|$ .

Using Zorn's Lemma allows the extension to  $E$  while the remark above can be used to extend to complex valued linear functionals.  $\square$

#### 3.2 The Riesz Representation Theorem

**Theorem 3.3** (The Riesz Representation Theorem). *Let  $X$  be a locally compact Hausdorff space and  $L \in C_c(X)$  a positive linear functional. Then there exists a  $\sigma$ -algebra  $M$  in  $X$  and a unique positive measure  $\mu$  on  $M$  so that*

$$L(f) = \int_X f d\mu$$

for every  $f \in C_c(X)$ .

## 4 The Road to Reality

### 4.1 Hyperbolic Geometry

The ratio between the area  $A$  and  $A'$  of two similar shapes is given by

$$A' = k^2 A$$

**Theorem 4.1** (Pythagoras).

$$a^2 + b^2 = c^2$$

*Proof.* Let  $A, B$  and  $C$  be the areas of the three triangles respectively. All triangles are similar, hence

$$B = \frac{b^2}{a^2} A \text{ and } C = \frac{c^2}{b^2} B$$

Since  $A + B = C$  it follows that

$$a^2 + b^2 = \frac{b^2 A}{B} + b^2 = \frac{b^2(A + B)}{B} = \frac{b^2 C}{B} = c^2$$

□

**Lemma 4.2** (Conformal and Projective Representation). *The mapping from conformal and projective representation of any point is given by the radial expansion of the following factor*

$$\frac{2R}{R^2 + r^2}$$

*Proof.* For any point the distance from the origin with regard to the two representations is given by

$$\log \frac{R+r}{R-r} = \frac{1}{2} \log \frac{R+r'}{R-r'} = \log \frac{(R+r')^2}{(R-r')^2}$$

This gives

$$(R-r)^2(R+r') = (R+r)^2(R-r') \text{ and } -4R^2r + 2R^2r' + 2r^2r' = 0$$

Hence

$$r' = \frac{2R^2}{R^2 + r^2} r$$

□

## 4.2 Complex Numbers

**Lemma 4.3** (Basic Formulas).

1. *It is*

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

2. *Thus*

$$(a + ib)^2 = (a^2 - b^2) + i2ab$$

and

$$(a + ib)(a - ib) = a^2 + iab - iab - i^2b^2 = a^2 + b^2$$

3. *Hence*

$$\frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + i \frac{bc - ad}{c^2 + d^2}$$

4. *For*

$$z = \sqrt{\frac{1}{2}(a + \sqrt{a^2 + b^2})} + i\sqrt{\frac{1}{2}(-a + \sqrt{a^2 + b^2})}$$

*it follows*

$$z^2 = \frac{1}{2}(a + \sqrt{a^2 + b^2}) - \frac{1}{2}(-a + \sqrt{a^2 + b^2}) + i2\sqrt{\frac{1}{4}(\sqrt{a^2 + b^2})^2} - a^2 = a + ib$$

**Lemma 4.4** (Binomial Theorem).

1. *For the binomial coefficient Pascal's identity holds*

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

2. *The following equation states the binomial identity*

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k} = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

3. *For  $a = 1$  follows*

$$(1 + x)^n = \sum_{k=0}^n \binom{n}{k} x^k$$

*Proof.* It is

$$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} = \frac{n!(n+1-k) + n!k!}{k!(n+1-k)!} = \binom{n+1}{k}$$

Furthermore by using induction

$$\begin{aligned}
(a+b)^{n+1} &= \sum_{k=0}^n \binom{n}{k} a^{k+1} b^{n-k} + \sum_{k=0}^n \binom{n}{k} a^k b^{n+1-k} \\
&= \sum_{k=1}^{n+1} \binom{n}{k-1} a^k b^{n+1-k} + \sum_{k=0}^n \binom{n}{k} a^k b^{n+1-k} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k}
\end{aligned}$$

□

### 4.3 Exponential Function and Logarithms

**Exercise 4.5** (Exponential Function). *The Cauchy product yields*

$$\sum_{n=0}^{\infty} a_n \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \sum_{k=0}^n a_k b_{n-k}$$

*if at least one of the series is absolutely convergent. Hence*

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{n!} z^n \sum_{n=0}^{\infty} \frac{1}{n!} w^n &= \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{1}{k!} z^k \frac{1}{(n-k)!} w^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} z^k w^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} (z+w)^n
\end{aligned}$$

Let  $t \in \mathbb{R}$ . Then

$$\begin{aligned}
e^{it} &= \sum_{k=0}^{\infty} \frac{1}{k!} (it)^k \\
&= \sum_{k=0}^{\infty} \frac{1}{2k!} (it)^{2k} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (it)^{2k+1} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k!} t^{2k} + i \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k+1} \\
&= \cos t + i \sin t
\end{aligned}$$

More generally for  $z = \log r + it$

$$e^z = e^{\log r + it} = r e^{it} = r(\cos t + i \sin t)$$

For  $r = 1$  and  $t = 2\pi$  this yields

$$e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1$$

and for  $t = 2\pi$  we get

**Lemma 4.6** (Euler Equation).

$$e^{\pi i} + 1 = 0$$

**Exercise 4.7.**

1. If  $e^z = w$  then  $z + \pi i$  is a logarithm to  $-w$ :  $e^{z+\pi i} = e^z e^{\pi i} = -e^z = -w$ .

2. Since  $e^{i(s+t)} = e^{is} e^{it}$  it follows

$$\begin{aligned}\cos(s+t) + i \sin(s+t) &= (\cos s + i \sin s)(\cos t + i \sin t) \\ &= \cos s \cos t - \sin s \sin t + i(\cos s \sin t + \sin s \cos t)\end{aligned}$$

Hence

$$\begin{aligned}\cos(s+t) &= \cos s \cos t - \sin s \sin t \\ \sin(s+t) &= \cos s \sin t + \sin s \cos t\end{aligned}$$

3. It is  $e^{3it} = (e^{it})^3$  and thus

$$\cos 3t + i \sin 3t = (\cos t + i \sin t)^3 = \cos^3 t - 3 \cos t \sin^2 t + i(\cos^2 t \sin t - \sin^3 t)$$

4. Fun facts

$$e^{1-4\pi^2} = e^{1+(2i\pi)^2} = e e^{2\pi i} e^{2\pi i} = e$$

and  $i = e^{i\pi/2}$  gives

$$i^i = e^{i \log i} = e^{i i \pi/2} = e^{-\pi/2} \in \mathbb{R}$$

## 4.4 Complex Analysis

**Definition 4.8** (holomorphic Function). Let  $\Omega \subseteq \mathbb{C}$  be open. A function  $f : \Omega \rightarrow \mathbb{C}$  is called differentiable at  $z \in \Omega$  if the limit

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}$$

exists.  $f$  is called holomorphic on  $\Omega$  if  $f$  is complex differentiable at all points of  $\Omega$  and  $f' : \Omega \rightarrow \mathbb{C}$  is called the derivative of  $f$ .

**Remarks 4.9.**

1.  $f$  is differentiable at  $z_0 \in \Omega$  iff the limit

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists

2. If  $f$  is differentiable at  $z_0 \in \Omega$  and  $\varepsilon > 0$  then there exists a small enough environment of  $z_0$ , so that

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| < \varepsilon |z - z_0|$$



**Theorem 4.10** (Cauchy Riemann Equations). *Let  $f = u + iv$  be holomorphic. Then  $f$  satisfies the Cauchy Riemann equations*

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}\end{aligned}$$

*Proof.* For  $h \in \mathbb{R}$  follows

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \frac{\partial u}{\partial x}(z) + i \frac{\partial v}{\partial x}(z)$$

and

$$\lim_{h \rightarrow 0} \frac{f(z+ih) - f(z)}{ih} = \frac{\partial u}{i\partial y}(z) + \frac{\partial v}{\partial y}(z) = \frac{\partial v}{\partial y}(z) - i \frac{\partial u}{\partial y}(z)$$

□

**Examples 4.11.**

1. Let  $f(z) = z^3$ . Then  $u(x, y) + iv(x, y) = x^3 - 3xy^2 + i(3x^2y - y^3)$  and as expected

$$\begin{aligned}\frac{\partial u}{\partial x}(x, y) &= x^3 - 3y^2 & \text{and} & & \frac{\partial u}{\partial y}(x, y) &= -6xy \\ \frac{\partial v}{\partial x}(x, y) &= 6xy & \text{and} & & \frac{\partial v}{\partial y}(x, y) &= x^3 - 3y^2\end{aligned}$$

**Lemma 4.12.** *Let  $D \subseteq \mathbb{C}$  be connected. For arbitrary  $z, w \in D$  there exists a polygonal path from  $z$  to  $w$ .*

*Proof.* For any path from  $z$  to  $w$  the image is compact, which can be used to define a finite subcover of disks. Use the center points to define the polygonal path. □

**Lemma 4.13.** *Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  a smooth path,  $\psi : [c, d] \rightarrow [a, b]$  a smooth and increasing bijection and  $f$  continuous.*

$$\int_{\gamma} f(z) dz = \int_{\gamma \circ \psi} f(z) dz$$

*Proof.* It is

$$\begin{aligned}\int_{\gamma \circ \psi} f(z) dz &= \int_c^d f(\gamma \circ \psi(t))(\gamma \circ \psi)'(t) dt \\ &= \int_{\psi(a)}^{\psi(b)} f(\gamma(\psi(t)))\gamma'(\psi(t))\psi'(t) dt \\ &= \int_a^b f(\gamma(s))\gamma'(s) ds = \int_{\gamma} f(z) dz\end{aligned}$$

□

**Lemma 4.14.** For a smooth path  $\gamma : [a, b] \rightarrow \mathbb{C}$  define  $-\gamma(t) = a + b - t$ . Then

$$\int_{-\gamma} f(z) dz = - \int_{\gamma} f(z) dz$$

*Proof.* Using integration by substitution

$$\int_{-\gamma} f(z) dz = - \int_a^b f(\gamma(a + b - t)) \gamma'(a + b - t) dt = \int_b^a f(\gamma(s)) \gamma'(s) ds = - \int_{\gamma} f(z) dz$$

□

In order to use the results from real calculus recall the fact, that for every  $z \in \mathbb{C}$  there exists a  $t \in [0, 2\pi]$ , so that  $z = |z|e^{it}$  and hence  $|z| = ze^{-it}$ .

**Lemma 4.15.** Let  $f \in C[a, b]$ . Then

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx$$

*Proof.* Using the estimation for integrals from real calculus

$$\left| \int_a^b f(x) dx \right| = e^{-it} \int_a^b f(x) dx \leq \int_a^b |e^{-it} f(x)| dx = \int_a^b |f(x)| dx$$

□

Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  be a smooth path and  $a = t_0 < t_1 < \dots < t_n = b$  a partitioning of  $[a, b]$ . Then

$$\sum_{k=1}^n |\gamma(t_k) - \gamma(t_{k-1})| = \sum_{k=1}^n \left| \frac{\gamma(t_k) - \gamma(t_{k-1})}{t_k - t_{k-1}} \right| (t_k - t_{k-1}) = \sum_{k=1}^n |\gamma'(\xi_k)| (t_k - t_{k-1})$$

yields a reasonable approximation of the length of the path. Hence

**Definition 4.16.** For a smooth path  $\gamma : [a, b] \rightarrow \mathbb{C}$

$$L(\gamma) = \int_a^b |\gamma'(t)| dt$$

is called the length of  $\gamma$ .

**Lemma 4.17** (Estimation Lemma). Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  be a smooth path. Then

$$\left| \int_{\gamma} f(z) dz \right| \leq L(\gamma) \max_{\gamma[a, b]} f$$

*Proof.* Using the definition above

$$\left| \int_{\gamma} f(z) dz \right| = \left| \int_a^b f(\gamma(t)) \gamma'(t) dt \right| \leq \int_a^b |f(\gamma(t)) \gamma'(t)| dt \leq \max_{\gamma[a,b]} |f| \int_a^b |\gamma'(t)| dt$$

□

**Examples 4.18.**

1. Let  $\gamma(t) = t + it$ . Then

$$\int_{\gamma} z^2 dz = \int_0^1 (t + it)^2 (1 + i) dt = (1 + i) \int_0^1 2it^2 dt = [(-2 + 2i)t^3]_0^1 = -\frac{2}{3} + i\frac{2}{3}$$

2. For  $\gamma(t) = t^2 + it$

$$\begin{aligned} \int_{\gamma} z^2 dz &= \int_0^1 (t^2 + it)^2 (2t + i) dt = \int_0^1 (2t^5 - 4t^3) + i(5t^4 - t^2) dt \\ &= \left[ \frac{1}{3}t^6 - t^4 \right]_0^1 + i \left[ t^5 - \frac{1}{3}t^3 \right]_0^1 = -\frac{2}{3} + i\frac{2}{3} \end{aligned}$$

3. And  $\gamma(t) = i + e^{it}$

$$\begin{aligned} \int_{\gamma} z^2 dz &= \int_{3/2\pi}^{2\pi} (i + e^{it})^2 ie^{it} dt = \int_{3/2\pi}^{2\pi} (-1 + 2ie^{it} + e^{2it})ie^{it} dt \\ &= \int_{3/2\pi}^{2\pi} -ie^{it} - 2e^{2it} + ie^{3it} dt = \left[ -e^{it} + ie^{2it} + \frac{1}{3}e^{3it} \right]_{3/2\pi}^{2\pi} \\ &= \left( -1 + i + \frac{1}{3} \right) - \left( i - i + \frac{1}{3}i \right) = -\frac{2}{3} + i\frac{2}{3} \end{aligned}$$

4. Let  $\gamma(t) = e^{it}$  and  $k \neq -1$ . Then

$$\int_{\gamma} z^k dz = \int_0^{2\pi} e^{ikt} ie^{it} dt = \int_0^{2\pi} ie^{i(k+1)t} dt = 0$$

**Theorem 4.19.** Let  $D \subseteq \mathbb{C}$  be a connected domain and  $f \in C(D)$ . Then the following assertions are equivalent

1.  $f$  has an antiderivative
2. For every closed path  $\gamma$

$$\int_{\gamma} f(z) dz = 0$$

*Proof.* Let  $F' = f$ . Since  $\gamma$  is closed

$$\int_{\gamma} f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt = \int_a^b (F \circ \gamma)'(t) dt = F(\gamma(b)) - F(\gamma(a)) = 0$$

Now fix some arbitrary  $a \in D$ . For  $z \in D$  let  $\gamma_z$  be a path from  $a$  to  $z$  and define

$$F(z) = \int_{\gamma_z} f(\zeta) d\zeta$$

This is well defined since the integral of  $f$  vanishes over each closed path. Moreover, since  $\gamma_{z+h} + [z+h, z] - \gamma_z$  defines a closed path

$$F(z+h) - F(z) = \int_{\gamma_{z+h}} f(z) dz - \int_{\gamma_z} f(z) dz = \int_{[z, z+h]} f(z) dz = h \int_0^1 f(z+th) dt$$

Here the latter integral is continuous at 0 with respect to  $h$

$$\left| \int_0^1 f(z+th) - f(z) dt \right| \leq \int_0^1 |f(z+th) - f(z)| dt \leq \max_{t \in [0,1]} |f(z+th) - f(z)|$$

□

**Corollary 4.20.** *The second assertion can be weakened to*

$$\int_{\partial \Delta} f(z) dz = 0$$

for every triangle  $\Delta \subset D$ , where e.g.  $D$  is convex or star shaped. Here the antiderivative can directly be defined as

$$F(z) = \int_{[a,z]} f(\zeta) d\zeta$$

similar to the real calculus approach. Note, that under this conditions  $f$  always has a local antiderivative.

**Examples 4.21.**

1. Let  $z_0 \in \mathbb{C}$  and  $\gamma(t) = z_0 + e^{it}$  for  $t \in [0, 2\pi]$ . Then

$$\int_{\gamma} \frac{1}{z - z_0} dz = \int_0^{2\pi} \frac{ie^{it}}{z_0 + e^{it} - z_0} dt = \int_0^{2\pi} i dt = 2\pi i$$

and thus  $1/(z - z_0)$  has no antiderivative on  $\mathbb{C} \setminus \{z_0\}$

2. Let  $z_0 \in \mathbb{C}$  and  $z \in D = D_r(z_0)$ . Applying [Theorem 4.19](#). to  $\partial D$  and a small enough circle around  $z$  gives

$$\int_{\partial D} \frac{1}{\zeta - z} d\zeta = \int_{\partial D} \frac{1}{\zeta - z_0} d\zeta = 2\pi i$$

**Theorem 4.22** (Goursat). *Let  $\Omega \subseteq \mathbb{C}$  be open and  $f$  holomorphic on  $\Omega$ . Then*

$$\int_{\partial\Delta} f(z) dz = 0$$

*for every triangle  $\Delta \subset \Omega$ .*

*Proof.* Choose a sequence of triangles  $\Delta \supset \Delta_0 \supset \Delta_1 \cdots \supset \Delta_k$  as depicted. Since all the triangles are compact with a vanishing diameter there exists a unique  $z_0 \in \Omega$  with  $\bigcap \Delta_k = \{z_0\}$ . Thus

$$\left| \int_{\partial\Delta} f(z) dz \right| \leq 4^k \left| \int_{\partial\Delta_k} f(z) dz \right| = 4^k \left| \int_{\partial\Delta_k} f(z) - f(z_0) - f'(z_0)(z - z_0) dz \right|$$

Furthermore  $L(\partial\Delta) = 2^{-k} L(\partial\Delta_k)$  and

$$|z - z_0| < L(\partial\Delta_k) = 2^{-k} L(\partial\Delta)$$

for any  $z \in \Delta_k$ . Since  $f$  is holomorphic at  $z_0$  for any given  $\varepsilon > 0$  there exists a sufficiently large enough  $k$ , so that

$$\begin{aligned} \left| \int_{\partial\Delta} f(z) dz \right| &\leq 4^k L(\partial\Delta_k) \max_{z \in \Delta_k} |f(z) - f(z_0) - f'(z_0)(z - z_0)| \\ &\leq 4^k L(\partial\Delta_k) \varepsilon \max_{z \in \Delta_k} |z - z_0| \\ &\leq L(\partial\Delta)^2 \varepsilon \end{aligned}$$

□

**Corollary 4.23.**

1. *A holomorphic function always has a local antiderivative*
2. *A holomorphic function on a star shaped domain has a global antiderivative and*

$$\int_{\gamma} f(z) dz = 0$$

*for any closed path*

3. *The prerequisites of Goursat theorem can be weakened to continuous and holomorphic with the exception of a finite number of points: adequate partitioning of the original triangle*

**Theorem 4.24** (Cauchy's Integral Formula). *Let  $\Omega \subseteq \mathbb{C}$  be open and  $f$  holomorphic on  $\Omega$ . Further let  $D \subset \Omega$  be a disc. Then*

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} dz$$

*for  $z \in D$ .*

*Proof.* For  $z \in D$  define

$$h(\zeta) = \frac{f(\zeta) - f(z)}{\zeta - z}$$

for  $\zeta \neq z$  and  $f'(z)$  for  $\zeta = z$ . Then  $h$  is holomorphic on  $D \setminus \{z\}$  and continuous at  $z$

$$0 = \int_{\partial D} h(\zeta) d\zeta = \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta - f(z) \int_{\partial D} \frac{1}{\zeta - z} d\zeta = \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta - 2\pi i f(z)$$

□

## 5 Neural Networks

### 5.1 The Perceptron

**Definition 5.1** (Binary Classifiers). *Let  $X \subset \mathbb{R}^n$  be the union of two finite disjoint sets  $X = M \cup N$ .*

1. *A binary classification problem is the task to find a mapping  $f : X \rightarrow \{0, 1\}$  with*

$$f(x) = \begin{cases} 1 & \text{for } x \in M \\ 0 & \text{for } x \in N \end{cases}$$

*$f$  then is called a binary classifier for  $X$*

2.  *$X$  is called separable if there exists a weight vector  $w \in \mathbb{R}^n$  and a bias  $b \in \mathbb{R}$  so that*

$$\begin{aligned} wx + b &> 0 & \text{for } x \in M \\ wx + b &< 0 & \text{for } x \in N \end{aligned}$$

3. *The weight  $w$  and the bias  $b$  are called solution to the classification problem. They implicitly define a binary classifier via*

$$f(x) = \begin{cases} 1 & \text{if } wx + b > 0 \\ 0 & \text{if } wx + b < 0 \end{cases}$$

#### Examples 5.2.

1. *Let  $X = \{0, 1\} \times \{0, 1\}$  and consider the and operator  $f(1, 1) = 1$  and  $f(x, y) = 0$  elsewhere. Then  $w = (3, 3)$  and  $b = -5$  yield a solution to the classification problem  $M = f^{-1}(1)$  and  $N = f^{-1}(0)$*
2. *Again let  $X = \{0, 1\} \times \{0, 1\}$  and  $f(1, 0) = f(0, 1) = 1$  and  $f(0, 0) = f(1, 1) = 0$ , the xor operator. Thus for any weight  $(w_1, w_2)$  and any bias  $b$*

$$\begin{aligned} w_1 + b &> 0 \\ w_2 + b &> 0 \end{aligned}$$

$$\begin{aligned} w_1 + w_2 + b &\leq 0 \\ b &\leq 0 \end{aligned}$$

*Adding two equations respectively shows that there cannot be a solution*

3. *The bias can be integrated into the weight vector via  $w' = (w, b) \in \mathbb{R}^{n+1}$  and  $x' = (x, 1) \in \mathbb{R}^{n+1}$ . Separability then reduces to*

$$w'x' > 0$$

### Geometrical Interpretation

The idea for the perceptron most likely has its origin in a simple geometrical observation. Recall that for  $x, y \in \mathbb{R}^n$  the dot product can be expressed as

$$xy = \|x\|\|y\| \cos \alpha$$

where  $\alpha$  is the angle between the two vectors. Hence the product is positive if the angle is less than  $90^\circ$  degrees and negative if the angle is between  $90^\circ$  and  $180^\circ$  degrees

$$xy > 0 \quad \text{for } 0 \leq \alpha < \pi/2$$

$$xy < 0 \quad \text{for } \pi/2 < \alpha \leq \pi$$

Note, that the sign does not depend on the vector lengths, but solely on the angle.

For any two vectors it is easy enough to find a weight that satisfies  $wx > 0$  and  $wy > 0$ . Generally  $w = x + y$  is a good guess, but not always correct as shown below in [Figure 2](#).

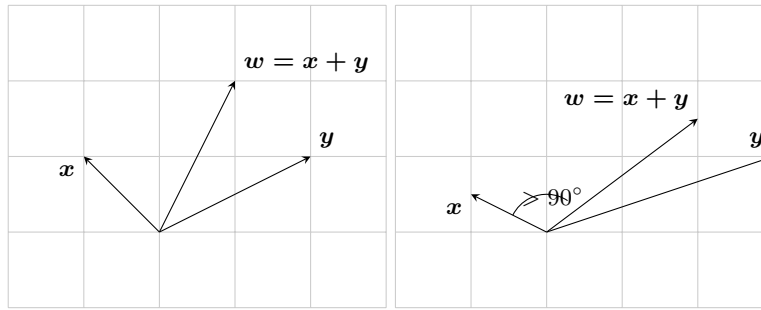


Figure 2: Dot Product and Angle

But, the more similar the lengths of the two vectors are the more likely  $x + y$  works. The actual threshold is given by the following

**Lemma 5.3.** *Let  $x, y \in \mathbb{R}^n$  with  $\|x\| < \|y\|$  and  $xy = \|x\|\|y\| \cos \alpha$ . If*

$$\|x\| > -\|y\| \cos \alpha$$

*then  $x(x + y) > 0$ .*

*Proof.* Let  $x(x + y) = \|x\|\|x + y\| \cos \beta$ . Since

$$x(x + y) = xx + xy = \|x\|^2 + \|x\|\|y\| \cos \alpha$$

it follows

$$\|x + y\| \cos \beta = \|x\| + \|y\| \cos \alpha$$

Hence  $\cos \beta > 0$  if the inequality above holds. □



An iterative approach is to repeatedly increase  $w = x + y$  in the direction of the shorter vector aka the one with the angle greater than  $90^\circ$  degrees.

$$w' = \begin{cases} w + x & \text{if } wx \leq 0 \\ w + y & \text{if } wy \leq 0 \end{cases}$$

While this seems reasonable it is unclear whether the algorithm always yields a result after a finite number of iterations. The answer to this question will be given later as a special case of the [Perceptron Convergence Theorem 5.6](#)

**Algorithm 5.4** (Weight).

---

```
def add(v, w):
    return tuple(x + y for x, y in zip(v, w))

def dotprod(v, w):
    return sum(x * y for x, y in zip(v, w))

def weight(x, y):
    w = add(x, y)
    while True:
        if dotprod(w, x) <= 0:
            w = add(w, x)
        elif dotprod(w, y) <= 0:
            w = add(w, y)
        else:
            return w
```

---

**Examples 5.5.**

1. Let  $x = (-1, 1)$  and  $y = (6, 1)$ . Then

$$\begin{array}{lll} w_0 = (5, 2) & w_0x = -3 & w_0y = 32 \\ w_1 = (4, 3) & w_1x = -1 & w_1y = 27 \\ w_2 = (3, 4) & w_2x = 1 & w_2y = 22 \end{array}$$

2. Let  $x = (4, -6)$  and  $y = (-10, 5)$ . Then

$$\begin{array}{lll} w_0 = (-6, -1) & w_0x = -18 & w_0y = 55 \\ w_1 = (-2, -7) & w_1x = 34 & w_1y = -15 \\ w_2 = (-12, -2) & w_2x = -36 & w_2y = 110 \\ w_3 = (-8, -8) & w_3x = 16 & w_3y = 40 \end{array}$$

3. Let  $w = x/\|x\| + y/\|y\|$ . Then

$$wx = \frac{\|x\|^2}{\|x\|} + \frac{yx}{\|y\|} = \|x\| + \|x\| \cos \alpha = (1 + \cos \alpha)\|x\| > 0$$

and on the other hand  $wx = \|w\|\|x\| \cos \beta$ . Similarly  $wy = (1 + \cos \alpha)\|y\| = \|w\|\|y\| \cos \gamma$ . Hence

$$1 + \cos \alpha = \|w\| \cos \beta = \|w\| \cos \gamma$$

and thus  $\beta = \gamma$ . Also, as expected,  $wx > 0$  and  $wy > 0$

4. Task: for any given integer  $k$  find two vectors so that more than  $k$  steps are needed

The following theorem is the generalization of the approach above for binary classifiers. Roughly speaking the algorithm changes the weight only for misclassified vectors. The proof measures the change of angle against the change of length of the weight vector and provides an estimation for the maximum number of required steps.

**Theorem 5.6** (Perceptron Convergence Theorem). *Let  $X = M \cup N$  be separable by  $w^* \in \mathbb{R}^n$ . Define  $w_0 = 0$  and repeat to iterate over all  $x \in X$  via*

$$w_{k+1} = \begin{cases} w_k + x & \text{if } x \in M \text{ and } w_k x \leq 0 \\ w_k - x & \text{if } x \in N \text{ and } w_k x \geq 0 \\ w_k & \text{else} \end{cases}$$

*until no further changes occur. Suppose  $\|x\| \leq r$  and  $w^* x \geq \delta > 0$  for  $x \in X$ . Then the number of iterations before the algorithm stops is limited by*

$$k \leq \frac{r^2}{\delta^2}$$

*Proof.* Let  $x \in M$  with  $w_k x \leq 0$ . Then

$$w^* w_{k+1} = w^* w_k + w^* x \geq w^* w_k + \delta$$

Furthermore

$$\|w_{k+1}\|^2 = \|w_k + x\|^2 = \|w_k\|^2 + 2w_k x + \|x\|^2 \leq \|w_k\|^2 + r^2$$

The same estimations hold for  $x \in N$  with  $w_k x \geq 0$  and using induction yields

$$w^* w_k \geq k\delta \text{ and } \|w_k\|^2 \leq kr^2$$

Assuming  $\|w^*\| = 1$  now gives

$$k^2 \delta^2 \leq \|w_k\|^2 \leq kr^2$$

which proves the initial inequality. □

**Algorithm 5.7** (Perceptron).

---

```
def dotprod(v, w):
    return sum(x * y for x, y in zip(v, w))

def perceptron(M, t=0.1, max_iterations=50):
    n = len(M[0])
    w, b = n * (0.0,), 0.0
    for _ in range(max_iterations):
```

```

done = True
for x, y in M:
    if dotprod(w, x) + b >= 0:
        z = 1
    else:
        z = 0
    w = tuple(w[i] - t * (z - y) * x[i] for i in range(n))
    b = b - t * (z - y)
    if not y == z:
        done = False
if done:
    break
return w, b

```

---

**Remark 5.8.** Check [Banach Fixed-Point Theorem 1.15](#) for an alternative proof?!

## 5.2 Feedforward Neural Networks

**Definition 5.9** (Activation Functions).

1. The Heaviside function  $H : \mathbb{R} \rightarrow \{0, 1\}$  is defined as

$$H(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

2. The sigmoid function  $\sigma \in C^\infty(\mathbb{R})$  is defined as

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$

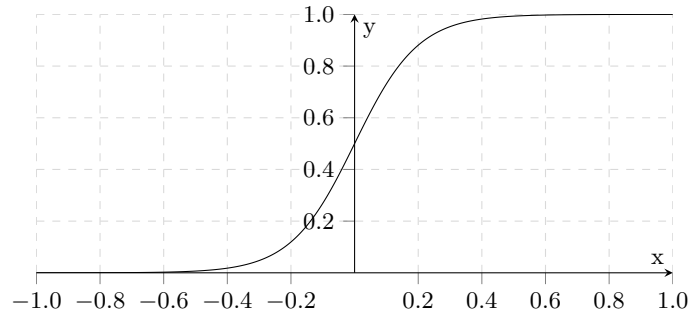


Figure 3: The sigmoid function  $\sigma(x) = \frac{1}{1+e^{-x}}$

**Remarks 5.10.**

1. Since the heaviside function is not continuous and therefore not differentiable at 0 the sigmoid function is often considered its smooth counterpart
2. The definition of the sigmoid function yields  $0 < \sigma(x) < 1$  as well as  $\sigma(x) \rightarrow 0$  for  $x \rightarrow -\infty$  and  $\sigma(x) \rightarrow 1$  for  $x \rightarrow \infty$
3. The quotient rule yields

$$\sigma'(x) = -\frac{-e^{-x}}{(1+e^{-x})^2} = \sigma(x) \frac{1+e^{-x}-1}{1+e^{-x}} = \sigma(x)(1-\sigma(x))$$

and  $\sigma$  is monotonically increasing over its domain

**Definition 5.11** (Single Layer Neural Network). Let  $w_k \in \mathbb{R}^n$  and  $a_k, b_k \in \mathbb{R}$ . Define  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  as

$$f(x) = \sum_{k=0}^m a_k \sigma(w_k x + b_k)$$

The  $f$  is called a feedforward neural network of  $m$  units.

**Theorem 5.12** (The Universal Approximation Theorem). The set of all single layer feedforward neural networks is dense in  $C(I^n)$ .

### 5.3 The Backtracking Algorithm