

# **Fuzzy Reasoning and Fuzzy Optimization**

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# Chapter 1

## Introduction

”Always honor a threat.”

Richard Herman, Jr.

Many decision-making tasks are too complex to be understood quantitatively, however, humans succeed by using knowledge that is imprecise rather than precise.

*Fuzzy logic* resembles human reasoning in its use of imprecise information to generate decisions. Unlike classical logic which requires a deep understanding of a system, exact equations, and precise numeric values, fuzzy logic incorporates an *alternative way of thinking*, which allows modeling complex systems using a higher level of abstraction originating from our knowledge and experience.

*Fuzzy logic* allows expressing this knowledge with subjective concepts such as *very big* and *a long time* which are mapped into exact numeric ranges. Since knowledge can be expressed in a more natural way by using fuzzy sets, many decision (and engineering) problems can be greatly simplified.

*Fuzzy logic* provides an inference morphology that enables approximate human reasoning capabilities to be applied to knowledge-based systems. The theory of fuzzy logic provides a mathematical strength to capture the uncertainties associated with human cognitive processes, such as thinking and reasoning. The conventional approaches to knowledge representation lack the means for representing the meaning of fuzzy concepts. As a consequence, the approaches based on first order logic do not provide an appropriate conceptual framework for dealing with the representation of commonsense knowledge, since such knowledge is by its nature both lexically imprecise and noncategorical. The development of fuzzy logic was motivated in large measure by the need for a conceptual framework which can address the issue of lexical imprecision. Some of the essential characteristics of fuzzy logic relate to the following [445].

- In fuzzy logic, exact reasoning is viewed as a limiting case of approximate reasoning.
- In fuzzy logic, everything is a matter of degree.
- In fuzzy logic, knowledge is interpreted as a collection of elastic or, equivalently, fuzzy constraints on a collection of variables.
- Inference is viewed as a process of propagation of elastic constraints.

- Any logical system can be fuzzified.

There are two main characteristics of fuzzy systems that give them better performance for specific applications.

- Fuzzy systems are suitable for uncertain or approximate reasoning, especially for systems with mathematical models that are difficult to derive.
- Fuzzy logic allows decision making with estimated values under incomplete or uncertain information.

This monograph summarizes the author's works in the nineties on fuzzy optimization and fuzzy reasoning. Many of the theorems and principles presented in this book have been obtained jointly, in scientific collaboration, with Elio Canestrelli, Christer Carlsson, Patrik Eklund, Mario Fedrizzi, Silvio Giove, Tibor Keresztfalvi, Margit Kovács, Luisa Mich, Eberhard Triesch, Fodor Pavlovich Vasiljev, Brigitte Werners and Hans-Jürgen Zimmermann.

The book is organized as follows. It begins, in chapter 2 'Fuzzy Sets and Fuzzy Logic', with a short historical survey of development of fuzzy thinking and progresses through an analysis of the extension principle, in which we derive exact formulas for t-norm-based operations on fuzzy numbers of *LR*-type, show a generalization of Nguyen's theorem [342] on  $\alpha$ -level sets of sup-min-extended functions to sup-t-norm-extended ones and provide a fuzzy analogue of Chebyshev's theorem [86].

Fuzzy set theory provides a host of attractive aggregation connectives for integrating membership values representing uncertain information. These connectives can be categorized into the following three classes *union*, *intersection* and *compensation* connectives. Union produces a high output whenever any one of the input values representing degrees of satisfaction of different features or criteria is high. Intersection connectives produce a high output only when all of the inputs have high values. Compensative connectives have the property that a higher degree of satisfaction of one of the criteria can compensate for a lower degree of satisfaction of another criteria to a certain extent. In the sense, union connectives provide full compensation and intersection connectives provide no compensation. In a decision process the idea of *trade-offs* corresponds to viewing the global evaluation of an action as lying between the *worst* and the *best* local ratings. This occurs in the presence of conflicting goals, when a compensation between the corresponding compatibilities is allowed. Averaging operators realize trade-offs between objectives, by allowing a positive compensation between ratings.

In Chapter 3 'Fuzzy Multicriteria Decision Making', we illustrate the applicability of Ordered Weighted Averaging [418] operators to a doctoral student selection problem. In many applications of fuzzy sets such as multi-criteria decision making, pattern recognition, diagnosis and fuzzy logic control one faces the problem of weighted aggregation. In 1994 Yager [422] discussed the issue of weighted min and max aggregations and provided for a formalization of the process of importance weighted transformation. We introduce fuzzy implication operators for importance weighted transformation containing as a subset those ones introduced by Yager.

Then we discuss the issue of weighted aggregations and provide a possibilistic approach to the process of importance weighted transformation when both the importances (interpreted as *benchmarks*) and the ratings are given by symmetric triangular fuzzy numbers. Furthermore, we show that using the possibilistic approach (i) small changes in the membership function of the importances can cause only small variations in the weighted aggregate; (ii) the weighted

aggregate of fuzzy ratings remains stable under small changes in the *nonfuzzy* importances; (iii) the weighted aggregate of crisp ratings still remains stable under small changes in the crisp importances whenever we use a continuous implication operator for the importance weighted transformation.

In 1973 Zadeh [439] introduced the compositional rule of inference and six years later [443] the theory of approximate reasoning. This theory provides a powerful framework for reasoning in the face of imprecise and uncertain information. Central to this theory is the representation of propositions as statements assigning fuzzy sets as values to variables. In Chapter 4 'Fuzzy Reasoning', we show two very important features of the compositional rule of inference under triangular norms. Namely, we prove that (i) if the t-norm defining the composition and the membership function of the observation are continuous, then the conclusion depends continuously on the observation; (ii) if the t-norm and the membership function of the relation are continuous, then the observation has a continuous membership function. The stability property of the conclusion under small changes of the membership function of the observation and rules guarantees that small rounding errors of digital computation and small errors of measurement of the input data can cause only a small deviation in the conclusion, i.e. every successive approximation method can be applied to the computation of the linguistic approximation of the exact conclusion.

Possibilistic linear equality systems (PLES) are linear equality systems with fuzzy coefficients, defined by the Zadeh's extension principle. Kovács [291] showed that the fuzzy solution to PLES with symmetric triangular fuzzy numbers is stable with respect to small changes of centres of fuzzy parameters. First, in Chapter 5 'Fuzzy Optimization', we generalize Kovács's results to PLES with (Lipschitzian) fuzzy numbers and flexible linear programs, and illustrate the sensitivity of the fuzzy solution by several one- and two-dimensional PLES. Then we consider linear (and quadratic) possibilistic programs and show that the possibility distribution of their objective function remains stable under small changes in the membership functions of the fuzzy number coefficients. Furthermore, we present similar results for multiobjective possibilistic linear programs with noninteractive and weakly-noninteractive fuzzy numbers.

In Chapter 6 'Fuzzy Reasoning for Fuzzy Optimization', we interpret fuzzy linear programming (FLP) problems with fuzzy coefficients and fuzzy inequality relations as multiple fuzzy reasoning schemes (MFR), where the antecedents of the scheme correspond to the constraints of the FLP problem and the fact of the scheme is the objective of the FLP problem. Then the solution process consists of two steps: first, for every decision variable, we compute the (fuzzy) value of the objective function, via sup-min convolution of the antecedents/constraints and the fact/objective, then an (optimal) solution to FLP problem is any point which produces a maximal element of the set of fuzzy values of the objective function (in the sense of the given inequality relation). We show that this solution process for a classical (crisp) LP problem results in a solution in the classical sense, and (under well-chosen inequality relations and objective function) coincides with those suggested by Buckley [30], Delgado et al. [105, 106], Negoita [337], Ramik and Rimanek [351], Verdegay [401, 402] and Zimmermann [450].

Typically, in complex, real-life problems, there are some unidentified factors which effect the values of the objective functions. We do not know them or can not control them; i.e. they have an impact we can not control. The only thing we can observe is the values of the objective functions at certain points. And from this information and from our knowledge about the problem we may be able to formulate the impacts of unknown factors (through the observed values of the objectives). First we state the multiobjective decision problem with independent



objectives and then adjust our model to reality by introducing interdependences among the objectives. Interdependences among the objectives exist whenever the computed value of an objective function is not equal to its observed value. We claim that the real values of an objective function can be identified by the help of feed-backs from the values of other objective functions, and show the effect of various kinds (linear, nonlinear and compound) of additive feed-backs on the compromise solution.

Even if the objective functions of a multiobjective decision problem are exactly known, we can still measure the *complexity* of the problem, which is derived from the *grades of conflict* between the objectives. Then we introduce concave utility functions for those objectives that support the majority of the objectives, and convex utility functions for those ones that are in conflict with the majority of the objectives. Finally, to find a good compromise solution we employ the following heuristic: increase the value of those objectives that support the majority of the objectives, because the gains on their (concave) utility functions surpass the losses on the (convex) utility functions of those objectives that are in conflict with the majority of the objectives.

Finally, in Chapter 7 'Applications in Management' we present three management applications. In the first case, *Nordic Paper Inc.*, we outline an algorithm for strategic decisions for the planning period 1996-2000 based on the interdependencies between the criteria. In the second case, *Adaptive fuzzy cognitive maps for strategy formation process*, we show that the effectiveness and usefulness of hyperknowledge support systems for strategy formation can be further advanced using adaptive fuzzy cognitive maps. In the third case, *Soft computing techniques for portfolio evaluation*, we suppose that the value of our portfolio depends on the currency fluctuations on the global finance market. Furthermore, we suppose that our knowledge is given in the form of fuzzy if-then rules, where all of the linguistic values for the exchange rates and the portfolio values are represented by sigmoidal fuzzy numbers. It is relatively easy to create fuzzy if-then rules for portfolio evaluation, however it is time-consuming and difficult to fine-tune them. We compute the crisp portfolio values by Tsukamoto's inference mechanism and introducing some reasonable interdependences among the linguistic terms we show a simple method for tuning the membership functions in the rules.

Special thanks go to Christer Carlsson who led me from theoretical problems of an abstract world to real-world problems.

# Chapter 2

## Fuzzy Sets and Fuzzy Logic

### 2.1 Some historical notes

*Fuzzy set theory* was introduced by Zadeh (1965) as a means of representing and manipulating data that was not precise, but rather fuzzy. It was specifically designed to mathematically represent uncertainty and vagueness and to provide formalized tools for dealing with the imprecision intrinsic to many problems. However, the story of fuzzy logic started much more earlier (see James F. Brule's tutorial, [24] for details) ...

To devise a concise theory of logic, and later mathematics, *Aristotle* posited the so-called "Laws of Thought" (see [286]). One of these, the "Law of the Excluded Middle," states that every proposition must either be *True* (**T**) or *False* (**F**). Even when *Parmenides* proposed the first version of this law (around 400 Before Christ) there were strong and immediate objections: for example, *Heraclitus* proposed that things could be simultaneously *True* and *not True*. It was *Plato* who laid the foundation for what would become fuzzy logic, indicating that there was a third region (beyond **T** and **F**) where these opposites "tumbled about."

A systematic alternative to the bi-valued logic of Aristotle was first proposed by *Łukasiewicz* (see [310]) around 1920, when he described a three-valued logic, along with the mathematics to accompany it. The third value he proposed can best be translated as the term "possible," and he assigned it a numeric value between **T** and **F**. Eventually, he proposed an entire notation and axiomatic system from which he hoped to derive modern mathematics. Later, he explored four-valued logics, five-valued logics, and then declared that in principle there was nothing to prevent the derivation of an infinite-valued logic. *Łukasiewicz* felt that three- and infinite-valued logics were the most intriguing, but he ultimately settled on a four-valued logic because it seemed to be the most easily adaptable to Aristotelian logic. It should be noted that *Knuth* also proposed a three-valued logic similar to *Łukasiewicz's*, from which he speculated that mathematics would become even more elegant than in traditional bi-valued logic.

The notion of an infinite-valued logic was introduced in Zadeh's seminal work "Fuzzy Sets" [438] where he described the mathematics of fuzzy set theory, and by extension fuzzy logic. This theory proposed making the membership function (or the values **F** and **T**) operate over the range of real numbers  $[0, 1]$ . New operations for the calculus of logic were proposed, and showed to be in principle at least a generalization of classic logic.

## 2.2 Fuzzy sets

*Fuzzy sets* serve as a means of representing and manipulating data that was not precise, but rather fuzzy. There is a strong relationship between *Boolean* logic and the concept of a subset, there is a similar strong relationship between fuzzy logic and fuzzy subset theory. In classical set theory, a subset  $A$  of a set  $X$  can be defined by its characteristic function  $\chi_A$  as a mapping from the elements of  $X$  to the elements of the set  $\{0, 1\}$ ,

$$\chi_A: X \rightarrow \{0, 1\}.$$

This mapping may be represented as a set of ordered pairs, with exactly one ordered pair present for each element of  $X$ . The first element of the ordered pair is an element of the set  $X$ , and the second element is an element of the set  $\{0, 1\}$ . The value zero is used to represent non-membership, and the value one is used to represent membership. The truth or falsity of the statement " $x$  is in  $A$ " is determined by the ordered pair  $(x, \chi_A(x))$ . The statement is true if the second element of the ordered pair is 1, and the statement is false if it is 0.

Similarly, a *fuzzy subset*  $A$  of a set  $X$  can be defined as a set of ordered pairs, each with the first element from  $X$ , and the second element from the interval  $[0, 1]$ , with exactly one ordered pair present for each element of  $X$ . This defines a mapping,

$$\mu_A: X \rightarrow [0, 1],$$

between elements of the set  $X$  and values in the interval  $[0, 1]$ . The value zero is used to represent complete non-membership, the value one is used to represent complete membership, and values in between are used to represent intermediate degrees of membership. The set  $X$  is referred to as the universe of discourse for the fuzzy subset  $A$ . Frequently, the mapping  $\mu_A$  is described as a function, the membership function of  $A$ . The degree to which the statement " $x$  is in  $A$ " is true is determined by finding the ordered pair  $(x, \mu_A(x))$ . The degree of truth of the statement is the second element of the ordered pair.

**Definition 2.2.1.** [438] Let  $X$  be a nonempty set. A fuzzy set  $A$  in  $X$  is characterized by its membership function

$$\mu_A: X \rightarrow [0, 1]$$

and  $\mu_A(x)$  is interpreted as the degree of membership of element  $x$  in fuzzy set  $A$  for each  $x \in X$ .

It is clear that  $A$  is completely determined by the set of tuples

$$A = \{(x, \mu_A(x)) | x \in X\}.$$

It should be noted that the terms *membership function* and *fuzzy subset* get used interchangeably and frequently we will write simply  $A(x)$  instead of  $\mu_A(x)$ . The family of all fuzzy (sub)sets in  $X$  is denoted by  $\mathcal{F}(X)$ . Fuzzy subsets of the real line are called *fuzzy quantities*. If  $X = \{x_1, \dots, x_n\}$  is a finite set and  $A$  is a fuzzy set in  $X$  then we use the notation

$$A = \mu_1/x_1 + \dots + \mu_n/x_n$$

where the term  $\mu_i/x_i$ ,  $i = 1, \dots, n$  signifies that  $\mu_i$  is the grade of membership of  $x_i$  in  $A$  and the plus sign represents the union.

**Example.** Suppose we want to define the set of natural numbers "close to 1". This can be expressed by

$$A = 0.0/-2 + 0.3/-1 + 0.6/0 + 1.0/1 + 0.6/2 + 0.3/3 + 0.0/4.$$

where  $X = \{-2, -1, 0, 1, 2, 3, 4\}$ .

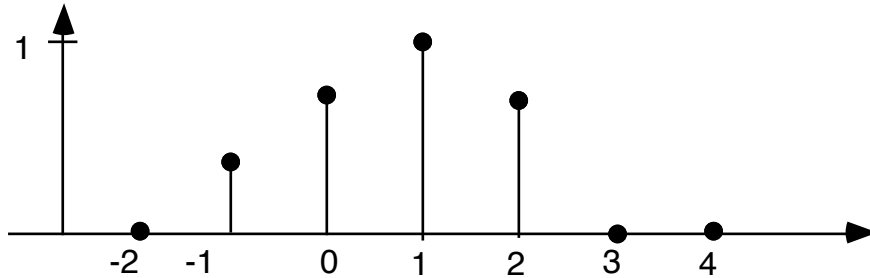


Figure 2.1: A discrete membership function for "x is close to 1".

**Example.** The membership function of the fuzzy set of real numbers "close to 1", is can be defined as

$$A(t) = \exp(-\beta(t-1)^2)$$

where  $\beta$  is a positive real number.

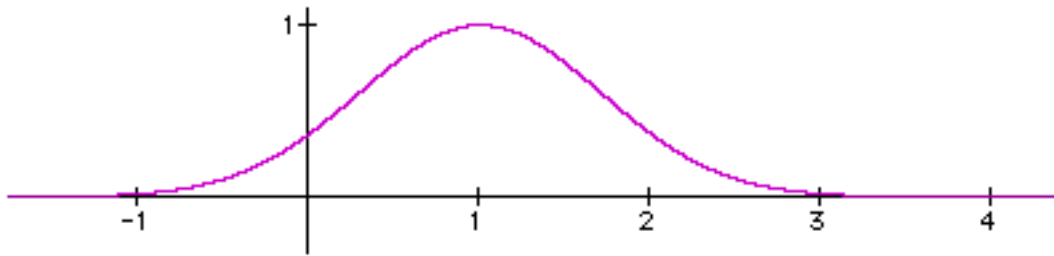


Figure 2.2: A membership function for "x is close to 1".

**Example.** Assume someone wants to buy a cheap car. Cheap can be represented as a fuzzy set on a universe of prices, and depends on his purse. For instance, from Figure 1 cheap is roughly interpreted as follows:

- Below 3000\$ cars are considered as cheap, and prices make no real difference to buyer's eyes.
- Between 3000\$ and 4500\$, a variation in the price induces a weak preference in favor of the cheapest car.
- Between 4500\$ and 6000\$, a small variation in the price induces a clear preference in favor of the cheapest car.
- Beyond 6000\$ the costs are too high (out of consideration).

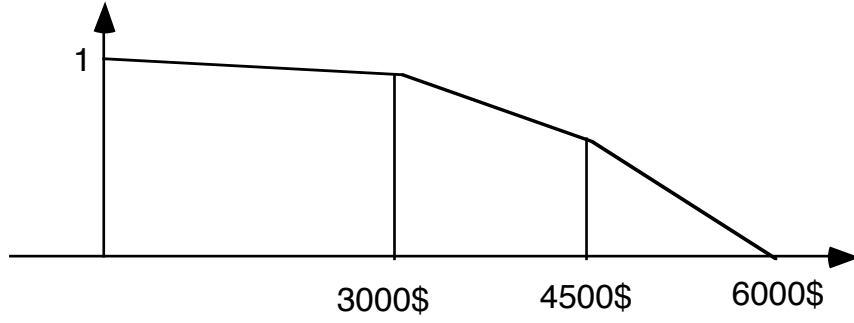


Figure 2.3: Membership function of "cheap".

**Definition 2.2.2.** Let  $A$  be a fuzzy subset of  $X$ ; the support of  $A$ , denoted  $\text{supp}(A)$ , is the crisp subset of  $X$  whose elements all have nonzero membership grades in  $A$ .

$$\text{supp}(A) = \{x \in X | A(x) > 0\}.$$

**Definition 2.2.3.** A fuzzy subset  $A$  of a classical set  $X$  is called normal if there exists an  $x \in X$  such that  $A(x) = 1$ . Otherwise  $A$  is subnormal.

**Definition 2.2.4.** An  $\alpha$ -level set (or  $\alpha$ -cut) of a fuzzy set  $A$  of  $X$  is a non-fuzzy set denoted by  $[A]^\alpha$  and defined by

$$[A]^\alpha = \begin{cases} \{t \in X | A(t) \geq \alpha\} & \text{if } \alpha > 0 \\ \text{cl}(\text{supp}A) & \text{if } \alpha = 0 \end{cases}$$

where  $\text{cl}(\text{supp}A)$  denotes the closure of the support of  $A$ .

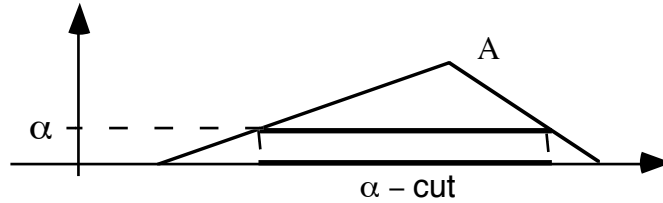


Figure 2.4: An  $\alpha$ -cut of a triangular fuzzy number.

**Example.** Assume  $X = \{-2, -1, 0, 1, 2, 3, 4\}$  and

$$A = 0.0/-2 + 0.3/-1 + 0.6/0 + 1.0/1 + 0.6/2 + 0.3/3 + 0.0/4.$$

Then,

$$[A]^\alpha = \begin{cases} \{-1, 0, 1, 2, 3\} & \text{if } 0 \leq \alpha \leq 0.3 \\ \{0, 1, 2\} & \text{if } 0.3 < \alpha \leq 0.6 \\ \{1\} & \text{if } 0.6 < \alpha \leq 1 \end{cases}$$

**Definition 2.2.5.** A fuzzy set  $A$  of  $X$  is called convex if  $[A]^\alpha$  is a convex subset of  $X$  for all  $\alpha \in [0, 1]$ .

In many situations people are only able to characterize numeric information imprecisely. For example, people use terms such as, about 5000, near zero, or essentially bigger than 5000. These are examples of what are called *fuzzy numbers*. Using the theory of fuzzy subsets we can represent these fuzzy numbers as fuzzy subsets of the set of real numbers. More exactly,

**Definition 2.2.6.** A fuzzy number  $A$  is a fuzzy set of the real line with a normal, (fuzzy) convex and continuous membership function of bounded support. The family of fuzzy numbers will be denoted by  $\mathcal{F}$ .

To distinguish a fuzzy number from a crisp (non-fuzzy) one, the former will sometimes be denoted with a tilde  $\sim$ .

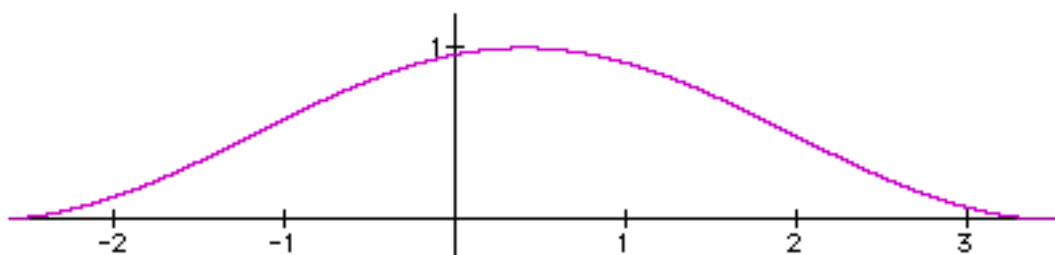


Figure 2.5: Fuzzy number.

**Definition 2.2.7.** A quasi fuzzy number  $A$  is a fuzzy set of the real line with a normal, fuzzy convex and continuous membership function satisfying the limit conditions

$$\lim_{t \rightarrow \infty} A(t) = 0, \quad \lim_{t \rightarrow -\infty} A(t) = 0.$$

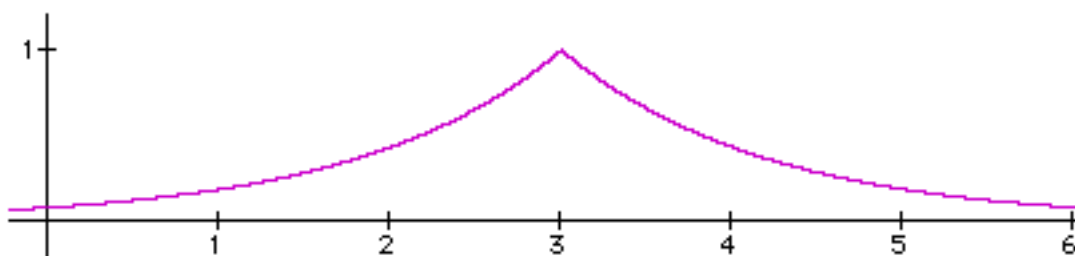


Figure 2.6: Quasi Fuzzy number.

In the literature the terms *fuzzy number* and *quasi fuzzy number* are often used interchangeably. It is easy to see that the membership function of a fuzzy number  $A$  has the following properties:

- $\mu_A(t) = 0$ , outside of some interval  $[c, d]$ ;
- there are real numbers  $a$  and  $b$ ,  $c \leq a \leq b \leq d$  such that  $\mu_A(t)$  is monotone increasing on the interval  $[c, a]$  and monotone decreasing on  $[b, d]$ ;
- $\mu_A(t) = 1$  for each  $x \in [a, b]$ .

Let  $A$  be a fuzzy number. Then  $[A]^\gamma$  is a closed convex (compact) subset of  $\mathbb{R}$  for all  $\gamma \in [0, 1]$ . Let us introduce the notations

$$a_1(\gamma) = \min[A]^\gamma, \quad a_2(\gamma) = \max[A]^\gamma$$

In other words,  $a_1(\gamma)$  denotes the left-hand side and  $a_2(\gamma)$  denotes the right-hand side of the  $\gamma$ -cut. It is easy to see that

$$\text{If } \alpha \leq \beta \text{ then } [A]^\alpha \supset [A]^\beta$$

Furthermore, the left-hand side function

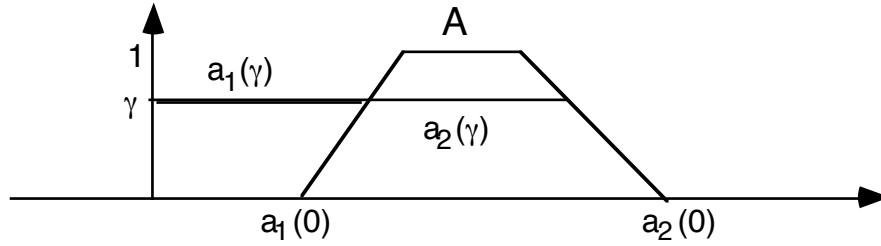


Figure 2.7: The support of  $A$  is  $(a_1(0), a_2(0))$ .

$$a_1: [0, 1] \rightarrow \mathbb{R}$$

is monoton increasing and lower semicontinuous, and the right-hand side function

$$a_2: [0, 1] \rightarrow \mathbb{R}$$

is monoton decreasing and upper semicontinuous. We shall use the notation

$$[A]^\gamma = [a_1(\gamma), a_2(\gamma)].$$

The support of  $A$  is the open interval  $(a_1(0), a_2(0))$ .

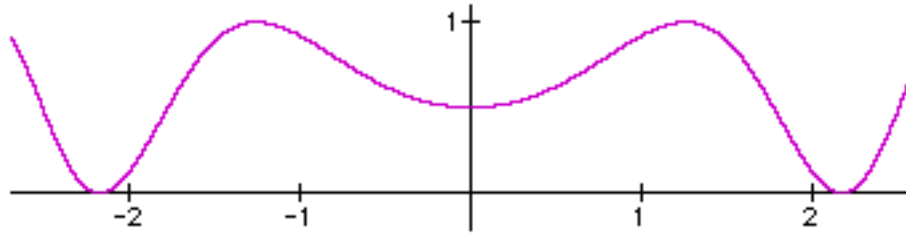


Figure 2.8: Not fuzzy number.

If  $A$  is not a fuzzy number then there exists an  $\gamma \in [0, 1]$  such that  $[A]^\gamma$  is not a convex subset of  $\mathbb{R}$ .

**Definition 2.2.8.** A fuzzy set  $A$  is called *triangular fuzzy number* with peak (or center)  $a$ , left width  $\alpha > 0$  and right width  $\beta > 0$  if its membership function has the following form

$$A(t) = \begin{cases} 1 - \frac{a-t}{\alpha} & \text{if } a - \alpha \leq t \leq a \\ 1 - \frac{t-a}{\beta} & \text{if } a \leq t \leq a + \beta \\ 0 & \text{otherwise} \end{cases}$$

and we use the notation  $A = (a, \alpha, \beta)$ . It can easily be verified that

$$[A]^\gamma = [a - (1 - \gamma)\alpha, a + (1 - \gamma)\beta], \quad \forall \gamma \in [0, 1].$$

The support of  $A$  is  $(a - \alpha, b + \beta)$ .

A triangular fuzzy number with center  $a$  may be seen as a fuzzy quantity

” $x$  is approximately equal to  $a$ ”.

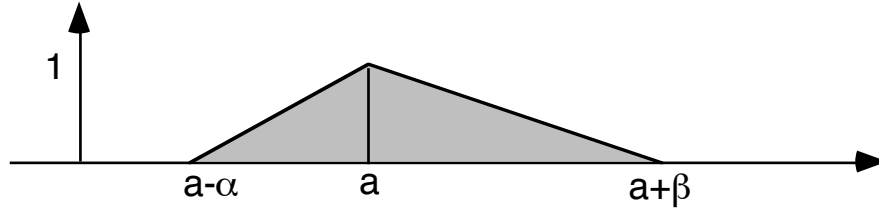


Figure 2.9: Triangular fuzzy number.

**Definition 2.2.9.** A fuzzy set  $A$  is called trapezoidal fuzzy number with tolerance interval  $[a, b]$ , left width  $\alpha$  and right width  $\beta$  if its membership function has the following form

$$A(t) = \begin{cases} 1 - \frac{a-t}{\alpha} & \text{if } a - \alpha \leq t \leq a \\ 1 & \text{if } a \leq t \leq b \\ 1 - \frac{t-b}{\beta} & \text{if } b \leq t \leq b + \beta \\ 0 & \text{otherwise} \end{cases}$$

and we use the notation

$$A = (a, b, \alpha, \beta). \quad (2.1)$$

It can easily be shown that

$$[A]^\gamma = [a - (1 - \gamma)\alpha, b + (1 - \gamma)\beta], \quad \forall \gamma \in [0, 1].$$

The support of  $A$  is  $(a - \alpha, b + \beta)$ .

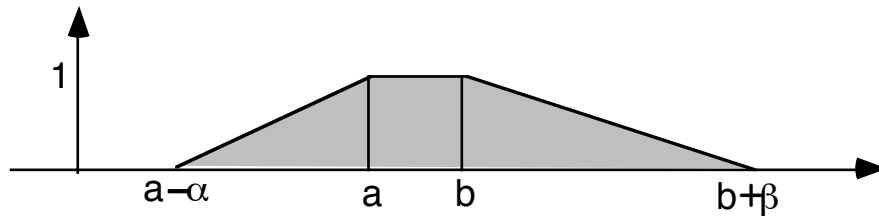


Figure 2.10: Trapezoidal fuzzy number.

A trapezoidal fuzzy number may be seen as a fuzzy quantity

” $x$  is approximately in the interval  $[a, b]$ ”.



**Definition 2.2.10.** Any fuzzy number  $A \in \mathcal{F}$  can be described as

$$A(t) = \begin{cases} L\left(\frac{a-t}{\alpha}\right) & \text{if } t \in [a-\alpha, a] \\ 1 & \text{if } t \in [a, b] \\ R\left(\frac{t-b}{\beta}\right) & \text{if } t \in [b, b+\beta] \\ 0 & \text{otherwise} \end{cases}$$

where  $[a, b]$  is the peak or core of  $A$ ,

$$L: [0, 1] \rightarrow [0, 1], \quad R: [0, 1] \rightarrow [0, 1]$$

are continuous and non-increasing shape functions with  $L(0) = R(0) = 1$  and  $L(1) = R(1) = 0$ . We call this fuzzy interval of LR-type and refer to it by

$$A = (a, b, \alpha, \beta)_{LR}$$

The support of  $A$  is  $(a - \alpha, b + \beta)$ .

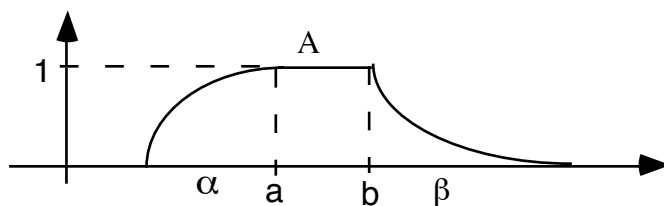


Figure 2.11: A fuzzy number of type LR.

**Definition 2.2.11.** (quasi fuzzy number of type LR) Any quasi fuzzy number  $A \in \mathcal{F}(\mathbb{R})$  can be described as

$$A(t) = \begin{cases} L\left(\frac{a-t}{\alpha}\right) & \text{if } t \leq a, \\ 1 & \text{if } t \in [a, b], \\ R\left(\frac{t-b}{\beta}\right) & \text{if } t \geq b, \end{cases}$$

where  $[a, b]$  is the peak or core of  $A$ ,

$$L: [0, \infty) \rightarrow [0, 1], \quad R: [0, \infty) \rightarrow [0, 1]$$

are continuous and non-increasing shape functions with  $L(0) = R(0) = 1$  and

$$\lim_{t \rightarrow \infty} L(t) = 0, \quad \lim_{t \rightarrow \infty} R(t) = 0.$$

**Definition 2.2.12.** Let  $A = (a, b, \alpha, \beta)_{LR}$  be a fuzzy number of type LR. If  $a = b$  then we use the notation

$$A = (a, \alpha, \beta)_{LR} \quad (2.2)$$

and say that  $A$  is a quasi-triangular fuzzy number. Furthermore if  $L(x) = R(x) = 1 - x$ , then instead of  $A = (a, b, \alpha, \beta)_{LR}$  we simply write

$$A = (a, b, \alpha, \beta).$$

**Definition 2.2.13.** Let  $A$  and  $B$  are fuzzy subsets of a classical set  $X$ . We say that  $A$  is a subset of  $B$  if  $A(t) \leq B(t)$  for all  $t \in X$ .

**Definition 2.2.14.** (equality of fuzzy sets) Let  $A$  and  $B$  are fuzzy subsets of a classical set  $X$ .  $A$  and  $B$  are said to be equal, denoted  $A = B$ , if  $A \subset B$  and  $B \subset A$ . We note that  $A = B$  if and only if  $A(x) = B(x)$  for all  $x \in X$ .

**Example.** Let  $A$  and  $B$  be fuzzy subsets of  $X = \{-2, -1, 0, 1, 2, 3, 4\}$ .

$$A = 0.0/-2 + 0.3/-1 + 0.6/0 + 1.0/1 + 0.6/2 + 0.3/3 + 0.0/4,$$

$$B = 0.1/-2 + 0.3/-1 + 0.9/0 + 1.0/1 + 1.0/2 + 0.3/3 + 0.2/4.$$

It is easy to check that  $A \subset B$  holds.

**Definition 2.2.15.** The empty fuzzy subset of  $X$  is defined as the fuzzy subset  $\emptyset$  of  $X$  such that  $\emptyset(x) = 0$  for each  $x \in X$ .

It is easy to see that  $\emptyset \subset A$  holds for any fuzzy subset  $A$  of  $X$ .

**Definition 2.2.16.** The largest fuzzy set in  $X$ , called universal fuzzy set in  $X$ , denoted by  $1_X$ , is defined by  $1_X(t) = 1, \forall t \in X$ .

It is easy to see that  $A \subset 1_X$  holds for any fuzzy subset  $A$  of  $X$ .

**Definition 2.2.17.** Let  $A$  be a fuzzy number. If  $\text{supp}(A) = \{x_0\}$  then  $A$  is called a fuzzy point and we use the notation  $A = \bar{x}_0$ .

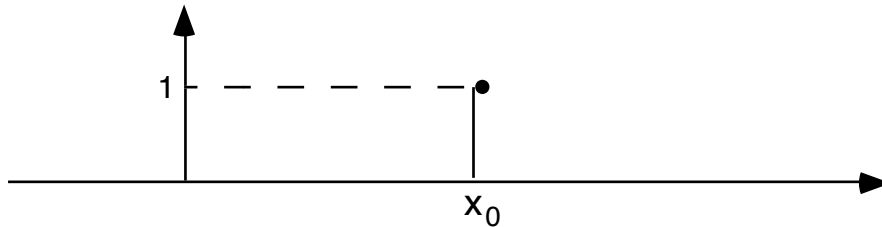


Figure 2.12: Fuzzy point.

Let  $A = \bar{x}_0$  be a fuzzy point. It is easy to see that  $[A]^\gamma = [x_0, x_0] = \{x_0\}, \forall \gamma \in [0, 1]$ .

## 2.3 Fuzzy relations

A classical relation can be considered as a set of tuples, where a tuple is an ordered pair. A binary tuple is denoted by  $(u, v)$ , an example of a ternary tuple is  $(u, v, w)$  and an example of  $n$ -ary tuple is  $(x_1, \dots, x_n)$ .

**Definition 2.3.1.** Let  $X_1, \dots, X_n$  be classical sets. The subsets of the Cartesian product  $X_1 \times \dots \times X_n$  are called  $n$ -ary relations. If  $X_1 = \dots = X_n$  and  $R \subset X^n$  then  $R$  is called an  $n$ -ary relation in  $X$ . Let  $R$  be a binary relation in  $\mathcal{R}$ . Then the characteristic function of  $R$  is defined as

$$\chi_R(u, v) = \begin{cases} 1 & \text{if } (u, v) \in R \\ 0 & \text{otherwise} \end{cases}$$

**Example.** Let  $X$  be the domain of men  $\{\text{John, Charles, James}\}$  and  $Y$  the domain of women  $\{\text{Diana, Rita, Eva}\}$ , then the relation "married to" on  $X \times Y$  is, for example

$$\{(\text{Charles, Diana}), (\text{John, Eva}), (\text{James, Rita})\}$$

**Example.** The following rectangle

$$\chi_R(u, v) = \begin{cases} 1 & \text{if } (u, v) \in [a, b] \times [0, c] \\ 0 & \text{otherwise,} \end{cases}$$

describes the relation  $R$  such that  $(u, v) \in R$  iff  $u \in [a, b]$  and  $v \in [0, c]$ .

**Definition 2.3.2.** Let  $X$  and  $Y$  be nonempty sets. A fuzzy relation  $R$  is a fuzzy subset of  $X \times Y$ . In other words,  $R \in \mathcal{F}(X \times Y)$ . If  $X = Y$  then we say that  $R$  is a binary fuzzy relation in  $X$ .

Let  $R$  be a binary fuzzy relation on  $\mathcal{R}$ . Then  $R(u, v)$  is interpreted as the degree of membership of  $(u, v)$  in  $R$ .

**Example.** A simple example of a binary fuzzy relation on  $U = \{1, 2, 3\}$ , called "approximately equal" can be defined as

$$\begin{aligned} R(1, 1) &= R(2, 2) = R(3, 3) = 1, \\ R(1, 2) &= R(2, 1) = R(2, 3) = R(3, 2) = 0.8, \\ R(1, 3) &= R(3, 1) = 0.3 \end{aligned}$$

The membership function of  $R$  is given by

$$R(u, v) = \begin{cases} 1 & \text{if } u = v \\ 0.8 & \text{if } |u - v| = 1 \\ 0.3 & \text{if } |u - v| = 2 \end{cases} \quad \text{or} \quad R = \begin{bmatrix} 1 & 0.8 & 0.3 \\ 0.8 & 1 & 0.8 \\ 0.3 & 0.8 & 1 \end{bmatrix}.$$

Fuzzy relations are very important because they can describe interactions between variables. Let  $R$  and  $S$  be two binary fuzzy relations on  $X \times Y$ .

**Definition 2.3.3.** (intersection) The intersection of  $R$  and  $G$  is defined by

$$(R \cap G)(u, v) = \min\{R(u, v), G(u, v)\} = R(u, v) \wedge G(u, v), \quad (u, v) \in X \times Y.$$

Note that  $R: X \times Y \rightarrow [0, 1]$ , i.e. the domain of  $R$  is the whole Cartesian product  $X \times Y$ .

**Definition 2.3.4.** (union) The union of  $R$  and  $S$  is defined by

$$(R \cup G)(u, v) = \max\{R(u, v), G(u, v)\} = R(u, v) \vee G(u, v), (u, v) \in X \times Y.$$

**Example.** Let us define two binary relations  $R = "x \text{ is considerable smaller than } y"$  and  $G = "x \text{ is very close to } y"$ :

$$R = \begin{pmatrix} & y_1 & y_2 & y_3 & y_4 \\ x_1 & 0.5 & 0.1 & 0.1 & 0.7 \\ x_2 & 0 & 0.8 & 0 & 0 \\ x_3 & 0.9 & 1 & 0.7 & 0.8 \end{pmatrix},$$

and

$$G = \begin{pmatrix} & y_1 & y_2 & y_3 & y_4 \\ x_1 & 0.4 & 0 & 0.9 & 0.6 \\ x_2 & 0.9 & 0.4 & 0.5 & 0.7 \\ x_3 & 0.3 & 0 & 0.8 & 0.5 \end{pmatrix}.$$

The intersection of  $R$  and  $G$  means that " $x$  is considerable smaller than  $y$ " **and** " $x$  is very close to  $y$ ".

$$(R \cap G)(x, y) = \begin{pmatrix} & y_1 & y_2 & y_3 & y_4 \\ x_1 & 0.4 & 0 & 0.1 & 0.6 \\ x_2 & 0 & 0.4 & 0 & 0 \\ x_3 & 0.3 & 0 & 0.7 & 0.5 \end{pmatrix}.$$

The union of  $R$  and  $G$  means that " $x$  is considerable smaller than  $y$ " **or** " $x$  is very close to  $y$ ".

$$(R \cup G)(x, y) = \begin{pmatrix} & y_1 & y_2 & y_3 & y_4 \\ x_1 & 0.5 & 0.1 & 0.9 & 0.7 \\ x_2 & 0.9 & 0.8 & 0.5 & 0.7 \\ x_3 & 0.9 & 1 & 0.8 & 0.8 \end{pmatrix}.$$

Consider a classical relation  $R$  on  $\mathbb{R}$ .

$$R(u, v) = \begin{cases} 1 & \text{if } (u, v) \in [a, b] \times [0, c] \\ 0 & \text{otherwise} \end{cases}$$

It is clear that the *projection* (or **shadow**) of  $R$  on the  $X$ -axis is the closed interval  $[a, b]$  and its projection on the  $Y$ -axis is  $[0, c]$ .

**Definition 2.3.5.** (projection of classical relations) Let  $R$  be a classical relation on  $X \times Y$ . The projection of  $R$  on  $X$ , denoted by  $\Pi_X(R)$ , is defined as

$$\Pi_X(R) = \{x \in X \mid \exists y \in Y \text{ such that } (x, y) \in R\}$$

similarly, the projection of  $R$  on  $Y$ , denoted by  $\Pi_Y(R)$ , is defined as

$$\Pi_Y(R) = \{y \in Y \mid \exists x \in X \text{ such that } (x, y) \in R\}$$



Figure 2.13: Graph of a crisp binary relation.

**Definition 2.3.6.** (*projection of binary fuzzy relations*) Let  $R$  be a binary fuzzy relation on  $X \times Y$ . The projection of  $R$  on  $X$  is a fuzzy subset of  $X$ , denoted by  $\Pi_X(R)$ , defined as

$$\Pi_X(R)(x) = \sup\{R(x, y) \mid y \in Y\}$$

and the projection of  $R$  on  $Y$  is a fuzzy subset of  $Y$ , denoted by  $\Pi_Y(R)$ , defined as

$$\Pi_Y(R)(y) = \sup\{R(x, y) \mid x \in X\}$$

If  $R$  is fixed then instead of  $\Pi_X(R)(x)$  we write simply  $\Pi_X(x)$ .

**Example.** Consider the fuzzy relation  $R = "x \text{ is considerable smaller than } y"$

$$R = \begin{pmatrix} & y_1 & y_2 & y_3 & y_4 \\ x_1 & 0.5 & 0.1 & 0.1 & 0.7 \\ x_2 & 0 & 0.8 & 0 & 0 \\ x_3 & 0.9 & 1 & 0.7 & 0.8 \end{pmatrix}$$

then the projection on  $X$  means that

- $x_1$  is assigned the highest membership degree from the tuples  $(x_1, y_1), (x_1, y_2), (x_1, y_3), (x_1, y_4)$ , i.e.  $\Pi_X(x_1) = 0.7$ , which is the maximum of the first row.
- $x_2$  is assigned the highest membership degree from the tuples  $(x_2, y_1), (x_2, y_2), (x_2, y_3), (x_2, y_4)$ , i.e.  $\Pi_X(x_2) = 0.8$ , which is the maximum of the second row.
- $x_3$  is assigned the highest membership degree from the tuples  $(x_3, y_1), (x_3, y_2), (x_3, y_3), (x_3, y_4)$ , i.e.  $\Pi_X(x_3) = 1$ , which is the maximum of the third row.

**Definition 2.3.7.** The Cartesian product of two fuzzy sets  $A \in \mathcal{F}(X)$  and  $B \in \mathcal{F}(Y)$  is defined by

$$(A \times B)(u, v) = \min\{A(u), B(v)\}, \quad (u, v) \in X \times Y.$$

It is clear that the Cartesian product of two fuzzy sets  $A \in \mathcal{F}(X)$  and  $B \in \mathcal{F}(Y)$  is a binary fuzzy relation in  $X \times Y$ , i.e.  $A \times B \in \mathcal{F}(X \times Y)$ .

Assume  $A$  and  $B$  are normal fuzzy sets. An interesting property of  $A \times B$  is that  $\Pi_Y(A \times B) = B$  and  $\Pi_X(A \times B) = A$ . Really,

$$\Pi_X(x) = \sup\{(A \times B)(x, y) \mid y \in Y\} = \sup\{\min\{A(x), B(y)\} \mid y \in Y\} =$$

$$\min\{A(x), \sup\{B(y) \mid y \in Y\}\} = A(x).$$

Similarly to the one-dimensional case, intersection and union operations on fuzzy relations can be defined via t-norms and t-conorms, respectively (see Section 2.4 for definitions).

**Definition 2.3.8.** (*t-norm-based intersection*) Let  $T$  be a t-norm and let  $R$  and  $G$  be binary fuzzy relations in  $X \times Y$ . Their  $T$ -intersection is defined by

$$(R \cap S)(u, v) = T(R(u, v), G(u, v)), \quad (u, v) \in X \times Y.$$

**Definition 2.3.9.** (*t-conorm-based union*) Let  $S$  be a t-conorm and let  $R$  and  $G$  be binary fuzzy relations in  $X \times Y$ . Their  $S$ -union is defined by

$$(R \cup S)(u, v) = S(R(u, v), G(u, v)), \quad (u, v) \in X \times Y.$$

**Definition 2.3.10.** (*sup-min composition*) Let  $R \in \mathcal{F}(X \times Y)$  and  $G \in \mathcal{F}(Y \times Z)$ . The sup-min composition of  $R$  and  $G$ , denoted by  $R \circ G$  is defined as

$$(R \circ S)(u, w) = \sup_{v \in Y} \min\{R(u, v), S(v, w)\}$$

It is clear that  $R \circ G$  is a binary fuzzy relation in  $X \times Z$ .

**Example.** Consider two fuzzy relations  $R = "x \text{ is considerable smaller than } y"$  and  $G = "y \text{ is very close to } z"$ :

$$R = \begin{pmatrix} & y_1 & y_2 & y_3 & y_4 \\ x_1 & 0.5 & 0.1 & 0.1 & 0.7 \\ x_2 & 0 & 0.8 & 0 & 0 \\ x_3 & 0.9 & 1 & 0.7 & 0.8 \end{pmatrix},$$

and

$$G = \begin{pmatrix} & z_1 & z_2 & z_3 \\ y_1 & 0.4 & 0.9 & 0.3 \\ y_2 & 0 & 0.4 & 0 \\ y_3 & 0.9 & 0.5 & 0.8 \\ y_4 & 0.6 & 0.7 & 0.5 \end{pmatrix}.$$

Then their sup – min composition is

$$R \circ G = \begin{pmatrix} & z_1 & z_2 & z_3 \\ x_1 & 0.6 & 0.8 & 0.5 \\ x_2 & 0 & 0.4 & 0 \\ x_3 & 0.7 & 0.9 & 0.7 \end{pmatrix}.$$

Formally,

$$\begin{pmatrix} & y_1 & y_2 & y_3 & y_4 \\ x_1 & 0.5 & 0.1 & 0.1 & 0.7 \\ x_2 & 0 & 0.8 & 0 & 0 \\ x_3 & 0.9 & 1 & 0.7 & 0.8 \end{pmatrix} \circ \begin{pmatrix} & z_1 & z_2 & z_3 \\ y_1 & 0.4 & 0.9 & 0.3 \\ y_2 & 0 & 0.4 & 0 \\ y_3 & 0.9 & 0.5 & 0.8 \\ y_4 & 0.6 & 0.7 & 0.5 \end{pmatrix} =$$

$$\begin{pmatrix} & z_1 & z_2 & z_3 \\ x_1 & 0.6 & 0.7 & 0.5 \\ x_2 & 0 & 0.4 & 0 \\ x_3 & 0.7 & 0.9 & 0.7 \end{pmatrix}$$

i.e., the composition of  $R$  and  $G$  is nothing else, but the classical product of the matrices  $R$  and  $G$  with the difference that instead of addition we use maximum and instead of multiplication we use minimum operator. For example,

$$(R \circ G)(x_1, z_1) = \max\{0.5 \wedge 0.4, 0.1 \wedge 0, 0.1 \wedge 0.9, 0.7 \wedge 0.6\} = 0.6$$

**Definition 2.3.11.** (*sup- $T$  composition*) Let  $T$  be a  $t$ -norm and let  $R \in \mathcal{F}(X \times Y)$  and  $G \in \mathcal{F}(Y \times Z)$ . The sup- $T$  composition of  $R$  and  $G$ , denoted by  $R \circ G$  is defined as

$$(R \circ S)(u, w) = \sup_{v \in Y} T(R(u, v), S(v, w)).$$

Following Zadeh [439] we can define the sup-min composition of a fuzzy set and fuzzy relation as follows

**Definition 2.3.12.** Let  $C \in \mathcal{F}(X)$  and  $R \in \mathcal{F}(X \times Y)$ . The membership function of the composition of a fuzzy set  $C$  and a fuzzy relation  $R$  is defined by

$$(C \circ R)(y) = \sup_{x \in X} \min\{C(x), R(x, y)\}, \quad \forall y \in Y.$$

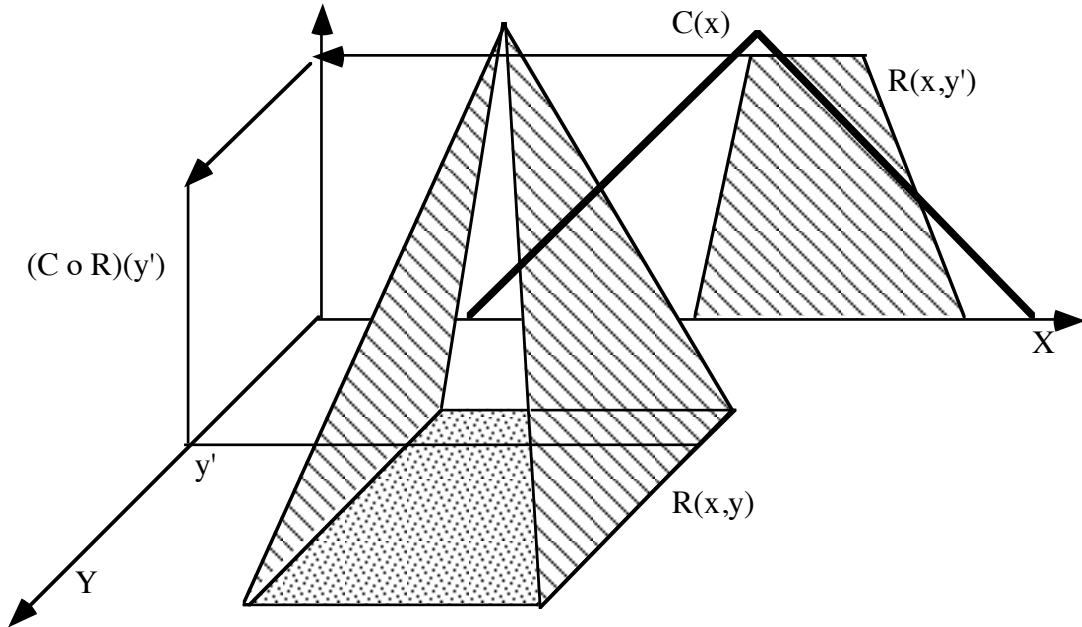


Figure 2.14: Composition of a fuzzy number and a fuzzy relation.

The composition of a fuzzy set  $C$  and a fuzzy relation  $R$  can be considered as the shadow of the relation  $R$  on the fuzzy set  $C$ . In the above definition we can use any  $t$ -norm for modeling the compositional operator.

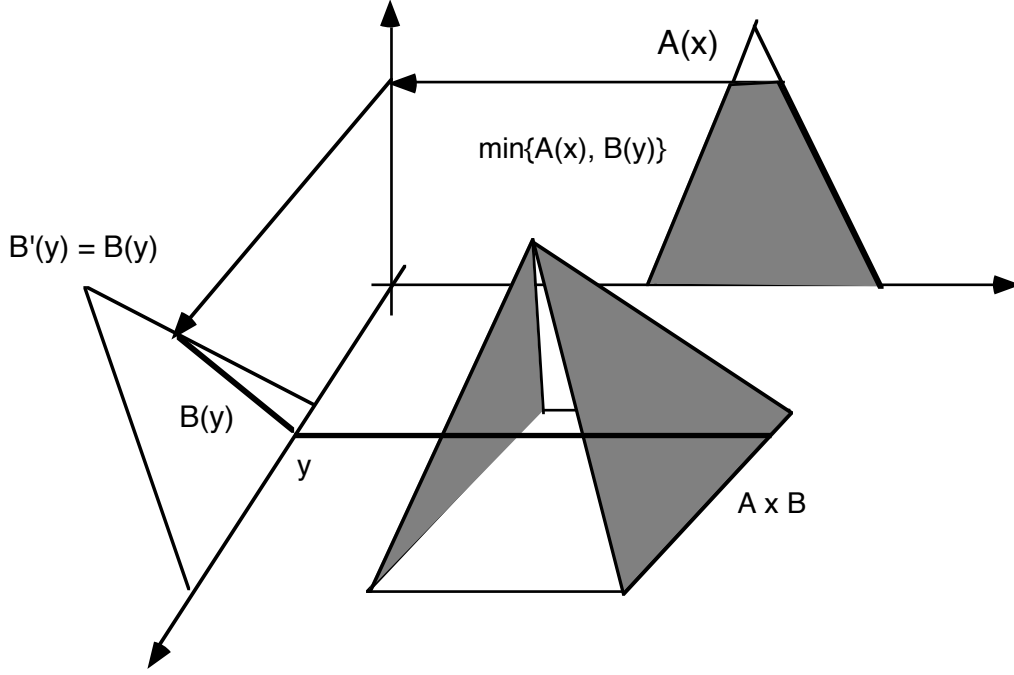


Figure 2.15:  $A \circ A \times B = B$ .

**Definition 2.3.13.** Let  $T$  be a  $t$ -norm  $C \in \mathcal{F}(X)$  and  $R \in \mathcal{F}(X \times Y)$ . The membership function of the composition of a fuzzy set  $C$  and a fuzzy relation  $R$  is defined by

$$(C \circ R)(y) = \sup_{x \in X} T(C(x), R(x, y)),$$

for all  $y \in Y$ .

For example, if  $T(x, y) = T_P(x, y) = xy$  is the product  $t$ -norm then the sup- $T$  composition of a fuzzy set  $C$  and a fuzzy relation  $R$  is defined by

$$(C \circ R)(y) = \sup_{x \in X} T_P(C(x), R(x, y)) = \sup_{x \in X} C(x)R(x, y)$$

and if  $T(x, y) = T_L(x, y) = \max\{0, x + y - 1\}$  is the Łukasiewicz  $t$ -norm then we get

$$(C \circ R)(y) = \sup_{x \in X} T_L(C(x), R(x, y)) = \sup_{x \in X} \max\{0, C(x) + R(x, y) - 1\}$$

for all  $y \in Y$ .

**Example.** Let  $A$  and  $B$  be fuzzy numbers and let  $R = A \times B$  a fuzzy relation. Observe the following property of composition

$$A \circ R = A \circ (A \times B) = B,$$

$$B \circ R = B \circ (A \times B) = A.$$

This fact can be interpreted as: if  $A$  and  $B$  have relation  $A \times B$  and then the composition of  $A$  and  $A \times B$  is exactly equal to  $B$ , and then the composition of  $B$  and  $A \times B$  is exactly equal to  $A$ .



**Example.** Let  $C$  be a fuzzy set in the universe of discourse  $\{1, 2, 3\}$  and let  $R$  be a binary fuzzy relation in  $\{1, 2, 3\}$ . Assume that

$$C = 0.2/1 + 1/2 + 0.2/3$$

and

$$R = \begin{pmatrix} 1 & 0.8 & 0.3 \\ 0.8 & 1 & 0.8 \\ 0.3 & 0.8 & 1 \end{pmatrix}$$

Using Definition 2.3.12 we get

$$C \circ R = (0.2/1 + 1/2 + 0.2/3) \circ \begin{pmatrix} 1 & 0.8 & 0.3 \\ 0.8 & 1 & 0.8 \\ 0.3 & 0.8 & 1 \end{pmatrix} = 0.8/1 + 1/2 + 0.8/3.$$

**Example.** Let  $C$  be a fuzzy set in the universe of discourse  $[0, 1]$  and let  $R$  be a binary fuzzy relation in  $[0, 1]$ . Assume that  $C(x) = x$  and  $R(x, y) = 1 - |x - y|$ . Using the definition of sup-min composition (2.3.12) we get

$$(C \circ R)(y) = \sup_{x \in [0, 1]} \min\{x, 1 - |x - y|\} = \frac{1 + y}{2},$$

for all  $y \in [0, 1]$ .

## 2.4 Operations on fuzzy sets

In this section we extend classical set theoretic operations from ordinary set theory to fuzzy sets. We note that all those operations which are extensions of crisp concepts reduce to their usual meaning when the fuzzy subsets have membership degrees that are drawn from  $\{0, 1\}$ . For this reason, when extending operations to fuzzy sets we use the same symbol as in set theory. Let  $A$  and  $B$  be fuzzy subsets of a crisp set  $X \neq \emptyset$ .

**Definition 2.4.1.** The intersection of  $A$  and  $B$  is defined as

$$(A \cap B)(t) = \min\{A(t), B(t)\} = A(t) \wedge B(t), \forall t \in X.$$

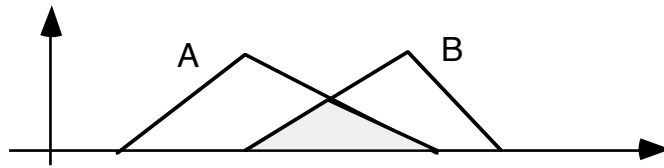


Figure 2.16: Intersection of two triangular fuzzy numbers.

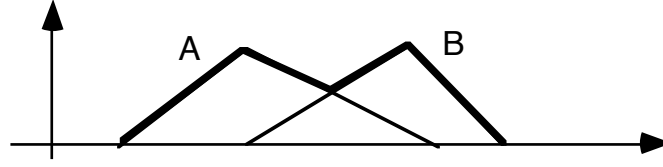


Figure 2.17: Union of two triangular fuzzy numbers.

**Definition 2.4.2.** *The union of  $A$  and  $B$  is defined as*

$$(A \cup B)(t) = \max\{A(t), B(t)\} = A(t) \vee B(t), \forall t \in X.$$

**Definition 2.4.3.** *The complement of a fuzzy set  $A$  is defined as*

$$(\neg A)(t) = 1 - A(t), \forall t \in X.$$

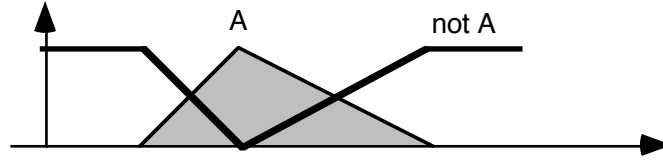


Figure 2.18:  $A$  and its complement.

Triangular norms were introduced by Schweizer and Sklar [387] to model distances in probabilistic metric spaces. In fuzzy sets theory triangular norms are extensively used to model logical connective *and*.

**Definition 2.4.4.** *(Triangular norm.) A mapping*

$$T: [0, 1] \times [0, 1] \rightarrow [0, 1]$$

*is a triangular norm (t-norm for short) iff it is symmetric, associative, non-decreasing in each argument and  $T(a, 1) = a$ , for all  $a \in [0, 1]$ . In other words, any t-norm  $T$  satisfies the properties:*

*Symmetry:*

$$T(x, y) = T(y, x), \forall x, y \in [0, 1].$$

*Associativity:*

$$T(x, T(y, z)) = T(T(x, y), z), \forall x, y, z \in [0, 1].$$

*Monotonicity:*

$$T(x, y) \leq T(x', y') \text{ if } x \leq x' \text{ and } y \leq y'.$$

*One identity:*

$$T(x, 1) = x, \forall x \in [0, 1].$$

These axioms attempt to capture the basic properties of set intersection. The basic t-norms are:

- minimum:  $\min(a, b) = \min\{a, b\}$ ,
- Łukasiewicz:  $T_L(a, b) = \max\{a + b - 1, 0\}$
- product:  $T_P(a, b) = ab$
- weak:

$$T_W(a, b) = \begin{cases} \min\{a, b\} & \text{if } \max\{a, b\} = 1 \\ 0 & \text{otherwise} \end{cases}$$

- Hamacher [212]:

$$H_\gamma(a, b) = \frac{ab}{\gamma + (1 - \gamma)(a + b - ab)}, \quad \gamma \geq 0 \quad (2.3)$$

- Dubois and Prade:

$$D_\alpha(a, b) = \frac{ab}{\max\{a, b, \alpha\}}, \quad \alpha \in (0, 1)$$

- Yager:

$$Y_p(a, b) = 1 - \min\{1, \sqrt[p]{(1 - a)^p + (1 - b)^p}\}, \quad p > 0$$

- Frank [161]:

$$F_\lambda(a, b) = \begin{cases} \min\{a, b\} & \text{if } \lambda = 0 \\ T_P(a, b) & \text{if } \lambda = 1 \\ T_L(a, b) & \text{if } \lambda = \infty \\ 1 - \log_\lambda \left[ 1 + \frac{(\lambda^a - 1)(\lambda^b - 1)}{\lambda - 1} \right] & \text{otherwise} \end{cases}$$

All t-norms may be extended, through associativity, to  $n > 2$  arguments. The minimum t-norm is automatically extended and

$$T_P(a_1, \dots, a_n) = a_1 \times a_2 \times \dots \times a_n,$$

$$T_L(a_1, \dots, a_n) = \max\left\{\sum_{i=1}^n a_i - n + 1, 0\right\}$$

A t-norm  $T$  is called strict if  $T$  is strictly increasing in each argument. A t-norm  $T$  is said to be Archimedean iff  $T$  is continuous and  $T(x, x) < x$  for all  $x \in (0, 1)$ . Every Archimedean t-norm  $T$  is representable by a continuous and decreasing function  $f: [0, 1] \rightarrow [0, \infty]$  with  $f(1) = 0$  and

$$T(x, y) = f^{-1}(\min\{f(x) + f(y), f(0)\}).$$

The function  $f$  is the additive generator of  $T$ . A t-norm  $T$  is said to be nilpotent if  $T(x, y) = 0$  holds for some  $x, y \in (0, 1)$ . Let  $T_1, T_2$  be t-norms. We say that  $T_1$  is weaker than  $T_2$  (and write  $T_1 \leq T_2$ ) if  $T_1(x, y) \leq T_2(x, y)$  for each  $x, y \in [0, 1]$ .

Triangular conorms are extensively used to model logical connective *or*.

**Definition 2.4.5.** (*Triangular conorm.*) A mapping

$$S: [0, 1] \times [0, 1] \rightarrow [0, 1],$$

is a triangular co-norm (*t-conorm*) if it is symmetric, associative, non-decreasing in each argument and  $S(a, 0) = a$ , for all  $a \in [0, 1]$ . In other words, any *t-conorm*  $S$  satisfies the properties:

$$S(x, y) = S(y, x) \quad (\text{symmetricity})$$

$$S(x, S(y, z)) = S(S(x, y), z) \quad (\text{associativity})$$

$$S(x, y) \leq S(x', y') \text{ if } x \leq x' \text{ and } y \leq y' \quad (\text{monotonicity})$$

$$S(x, 0) = x, \quad \forall x \in [0, 1] \quad (\text{zero identity})$$

If  $T$  is a t-norm then the equality

$$S(a, b) := 1 - T(1 - a, 1 - b),$$

defines a t-conorm and we say that  $S$  is derived from  $T$ . The basic t-conorms are:

- maximum:  $\max(a, b) = \max\{a, b\}$
- Łukasiewicz:  $S_L(a, b) = \min\{a + b, 1\}$
- probabilistic:  $S_P(a, b) = a + b - ab$
- strong:

$$STRONG(a, b) = \begin{cases} \max\{a, b\} & \text{if } \min\{a, b\} = 0 \\ 1 & \text{otherwise} \end{cases}$$

- Hamacher:

$$HOR_\gamma(a, b) = \frac{a + b - (2 - \gamma)ab}{1 - (1 - \gamma)ab}, \quad \gamma \geq 0$$

- Yager:

$$YOR_p(a, b) = \min\{1, \sqrt[p]{a^p + b^p}\}, \quad p > 0.$$

**Lemma 2.4.1.** Let  $T$  be a t-norm. Then the following statement holds

$$T_W(x, y) \leq T(x, y) \leq \min\{x, y\}, \quad \forall x, y \in [0, 1].$$

*Proof.* From monotonicity, symmetricity and the extremal condition we get

$$T(x, y) \leq T(x, 1) \leq x, \quad T(x, y) = T(y, x) \leq T(y, 1) \leq y.$$

This means that  $T(x, y) \leq \min\{x, y\}$ . □

**Lemma 2.4.2.** Let  $S$  be a t-conorm. Then the following statement holds

$$\max\{a, b\} \leq S(a, b) \leq STRONG(a, b), \quad \forall a, b \in [0, 1]$$

*Proof.* From monotonicity, symmetricity and the extremal condition we get

$$S(x, y) \geq S(x, 0) \geq x, S(x, y) = S(y, x) \geq S(y, 0) \geq y$$

This means that  $S(x, y) \geq \max\{x, y\}$ . □

**Lemma 2.4.3.**  $T(a, a) = a$  holds for any  $a \in [0, 1]$  if and only if  $T$  is the minimum norm.

*Proof.* If  $T(a, b) = \min(a, b)$  then  $T(a, a) = a$  holds obviously. Suppose  $T(a, a) = a$  for any  $a \in [0, 1]$ , and  $a \leq b \leq 1$ . We can obtain the following expression using monotonicity of  $T$

$$a = T(a, a) \leq T(a, b) \leq \min\{a, b\}.$$

From commutativity of  $T$  it follows that

$$a = T(a, a) \leq T(b, a) \leq \min\{b, a\}.$$

These equations show that  $T(a, b) = \min\{a, b\}$  for any  $a, b \in [0, 1]$ . □

**Lemma 2.4.4.** The distributive law of  $t$ -norm  $T$  on the max operator holds for any  $a, b, c \in [0, 1]$ .

$$T(\max\{a, b\}, c) = \max\{T(a, c), T(b, c)\}.$$

The operation *intersection* can be defined by the help of triangular norms.

**Definition 2.4.6.** ( $t$ -norm-based intersection) Let  $T$  be a  $t$ -norm. The  $T$ -intersection of  $A$  and  $B$  is defined as

$$(A \cap B)(t) = T(A(t), B(t)), \forall t \in X.$$

**Example.** Let  $T(x, y) = T_L(x, y) = \max\{x + y - 1, 0\}$  be the Łukasiewicz  $t$ -norm. Then we have

$$(A \cap B)(t) = \max\{A(t) + B(t) - 1, 0\} \forall t \in X.$$

Let  $A$  and  $B$  be fuzzy subsets of  $X = \{-2, -1, 0, 1, 2, 3, 4\}$ .

$$A = 0.0/-2 + 0.3/-1 + 0.6/0 + 1.0/1 + 0.6/2 + 0.3/3 + 0.0/4,$$

$$B = 0.1/-2 + 0.3/-1 + 0.9/0 + 1.0/1 + 1.0/2 + 0.3/3 + 0.2/4.$$

Then  $A \cap B$  has the following form

$$A \cap B = 0.0/-2 + 0.0/-1 + 0.5/0 + 1.0/1 + 0.6/2 + 0.0/3 + 0.2/4.$$

The operation *union* can be defined by the help of triangular conorms.

**Definition 2.4.7.** ( $t$ -conorm-based union) Let  $S$  be a  $t$ -conorm. The  $S$ -union of  $A$  and  $B$  is defined as

$$(A \cup B)(t) = S(A(t), B(t)), \forall t \in X.$$

**Example.** Let  $S(x, y) = \min\{x + y, 1\}$  be the Łukasiewicz t-conorm. Then we have

$$(A \cup B)(t) = \min\{A(t) + B(t), 1\}, \forall t \in X.$$

Let  $A$  and  $B$  be fuzzy subsets of  $X = \{-2, -1, 0, 1, 2, 3, 4\}$ .

$$A = 0.0/-2 + 0.3/-1 + 0.6/0 + 1.0/1 + 0.6/2 + 0.3/3 + 0.0/4,$$

$$B = 0.1/-2 + 0.3/-1 + 0.9/0 + 1.0/1 + 1.0/2 + 0.3/3 + 0.0/4.$$

Then  $A \cup B$  has the following form

$$A \cup B = 0.1/-2 + 0.6/-1 + 1.0/0 + 1.0/1 + 1.0/2 + 0.6/3 + 0.2/4.$$

In general, the law of the excluded middle and the noncontradiction principle properties are not satisfied by t-norms and t-conorms defining the intersection and union operations. However, the Łukasiewicz t-norm and t-conorm do satisfy these properties.

**Lemma 2.4.5.** If  $T(x, y) = T_L(x, y) = \max\{x + y - 1, 0\}$  then the law of noncontradiction is valid.

*Proof.* Let  $A$  be a fuzzy set in  $X$ . Then from the definition of t-norm-based intersection we get

$$(A \cap \neg A)(t) = T_L(A(t), 1 - A(t)) = (A(t) + 1 - A(t) - 1) \vee 0 = 0, \forall t \in X.$$

□

**Lemma 2.4.6.** If  $S(x, y) = S_L(x, y) = \min\{1, x + y\}$ , then the law of excluded middle is valid.

*Proof.* Let  $A$  be a fuzzy set in  $X$ . Then from the definition of t-conorm-based union we get

$$(A \cup \neg A)(t) = S_L(A(t), 1 - A(t)) = (A(t) + 1 - A(t)) \wedge 1 = 1,$$

for all  $t \in X$ .

□

## 2.5 The extension principle

In order to use fuzzy numbers and relations in any intelligent system we must be able to perform arithmetic operations with these fuzzy quantities. In particular, we must be able to to *add*, *subtract*, *multiply* and *divide* with fuzzy quantities. The process of doing these operations is called *fuzzy arithmetic*. We shall first introduce an important concept from fuzzy set theory called the *extension principle*. We then use it to provide for these arithmetic operations on fuzzy numbers. In general the extension principle plays a fundamental role in enabling us to extend any point operations to operations involving fuzzy sets. In the following we define this principle.

**Definition 2.5.1.** (Zadeh's extension principle) Assume  $X$  and  $Y$  are crisp sets and let  $f$  be a mapping from  $X$  to  $Y$ ,

$$f: X \rightarrow Y$$

such that for each  $x \in X$ ,  $f(x) = y \in Y$ . Assume  $A$  is a fuzzy subset of  $X$ , using the extension principle, we can define  $f(A)$  as a fuzzy subset of  $Y$  such that

$$f(A)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} A(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \quad (2.4)$$

where  $f^{-1}(y) = \{x \in X \mid f(x) = y\}$ .

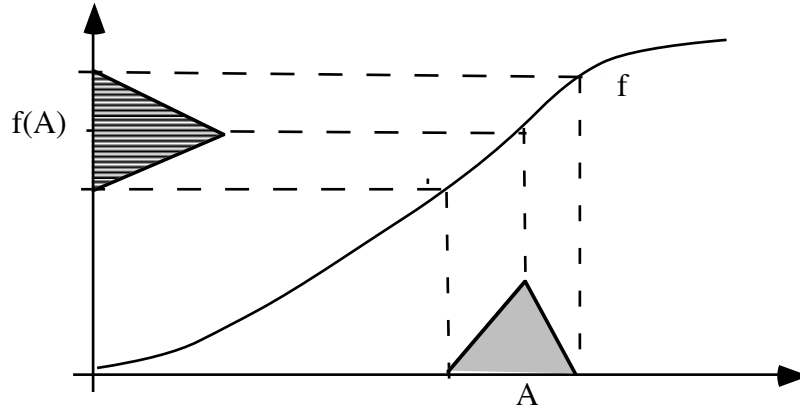


Figure 2.19: Extension of a monotone increasing function.

It should be noted that if  $f$  is strictly increasing (or strictly decreasing) then (2.4) turns into

$$f(A)(y) = \begin{cases} A(f^{-1}(y)) & \text{if } y \in \text{Range}(f) \\ 0 & \text{otherwise} \end{cases}$$

where  $\text{Range}(f) = \{y \in Y \mid \exists x \in X \text{ such that } f(x) = y\}$ .

**Example.** Let  $f(x) = x^2$  and let  $A \in \mathcal{F}$  be a symmetric triangular fuzzy number with membership function

$$A(x) = \begin{cases} 1 - \frac{|a - x|}{\alpha} & \text{if } |a - x| \leq \alpha \\ 0 & \text{otherwise} \end{cases}$$

Then using the extension principle we get

$$f(A)(y) = \begin{cases} A(\sqrt{y}) & \text{if } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

that is

$$f(A)(y) = \begin{cases} 1 - \frac{|a - \sqrt{y}|}{\alpha} & \text{if } |a - \sqrt{y}| \leq \alpha \text{ and } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

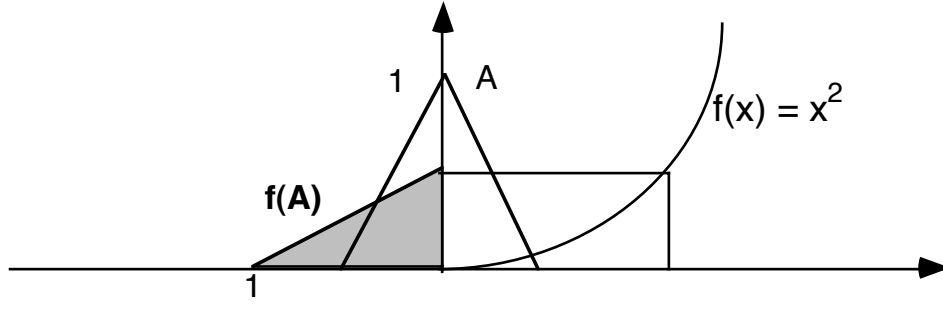


Figure 2.20: The quadratic image of a symmetric triangular fuzzy number.

**Example.** Let

$$f(x) = \frac{1}{1 + e^{-x}}$$

be a sigmoidal function and let  $A$  be a fuzzy number. Then from

$$f^{-1}(y) = \begin{cases} \ln\left(\frac{y}{1-y}\right) & \text{if } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

it follows that

$$f(A)(y) = \begin{cases} A\left(\ln\left(\frac{y}{1-y}\right)\right) & \text{if } 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let  $\lambda \neq 0$  be a real number and let  $f(x) = \lambda x$  be a linear function. Suppose  $A \in \mathcal{F}$  is a fuzzy number. Then using the extension principle we obtain

$$f(A)(y) = \sup\{A(x) \mid \lambda x = y\} = A(y/\lambda).$$

For  $\lambda = 0$  then we get

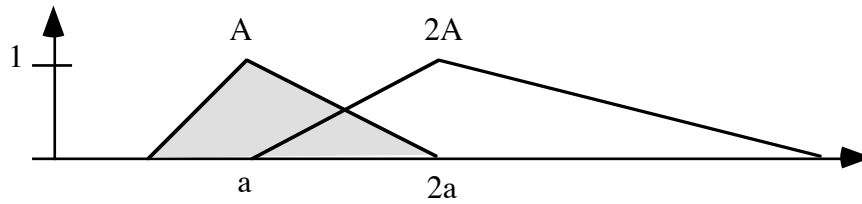


Figure 2.21: The fuzzy number  $\lambda A$  for  $\lambda = 2$ .

$$f(A)(y) = (0 \times A)(y) = \sup\{A(x) \mid 0x = y\} = \begin{cases} 0 & \text{if } y \neq 0 \\ 1 & \text{if } y = 0. \end{cases}$$

That is  $0 \times A = \bar{0}$  for all  $A \in \mathcal{F}$ . If  $f(x) = \lambda x$  and  $A \in \mathcal{F}$  then we will write  $f(A) = \lambda A$ .



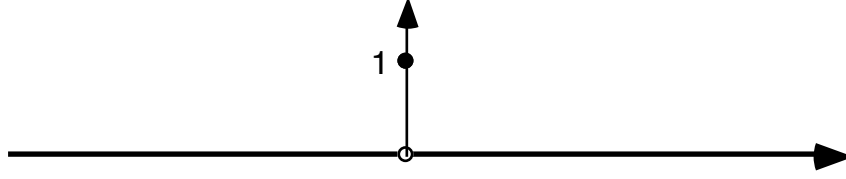


Figure 2.22:  $0 \times A$  is equal to  $\bar{0}$ .

Especially, if  $\lambda = -1$  then we have

$$(-1A)(x) = (-A)(x) = A(-x), \quad x \in \mathbb{R}.$$

It should be noted that Zadeh's extension principle is nothing else but a straightforward generalization of set-valued functions (see [315] for details). Namely, let  $f: X \rightarrow Y$  be a function. Then the image of a (crisp) subset  $A \subset X$  by  $f$  is defined by

$$f(A) = \{f(x) \mid x \in A\}$$

and the characteristic function of  $f(A)$  is

$$\chi_{f(A)}(y) = \sup\{\chi_A(x) \mid x \in f^{-1}(y)\}$$

Then replacing  $\chi_A$  by a fuzzy set  $\mu_A$  we get Zadeh's extension principle (2.4).

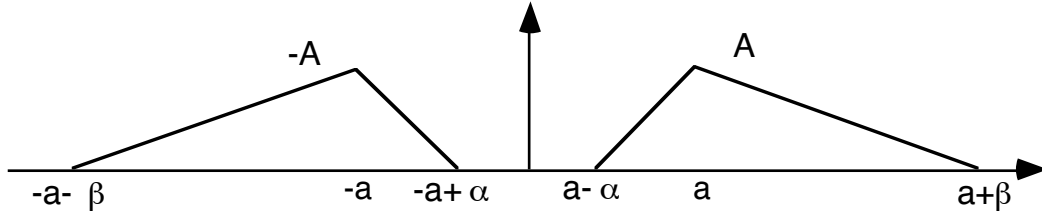


Figure 2.23: Fuzzy number  $A$  and  $-A$ .

### 2.5.1 The extension principle for $n$ -place functions

The extension principle can be generalized to  $n$ -place functions.

**Definition 2.5.2.** Let  $X_1, X_2, \dots, X_n$  and  $Y$  be a family of sets. Assume  $f$  is a mapping

$$f: X_1 \times X_2 \times \dots \times X_n \rightarrow Y,$$

that is, for each  $n$ -tuple  $(x_1, \dots, x_n)$  such that  $x_i \in X_i$ , we have  $f(x_1, x_2, \dots, x_n) = y \in Y$ . Let  $A_1, \dots, A_n$  be fuzzy subsets of  $X_1, \dots, X_n$ , respectively; then the extension principle allows for the evaluation of  $f(A_1, \dots, A_n)$ . In particular,  $f(A_1, \dots, A_n) = B$ , where  $B$  is a fuzzy subset of  $Y$  such that

$$f(A_1, \dots, A_n)(y) = \begin{cases} \sup\{\min\{A_1(x_1), \dots, A_n(x_n)\} \mid x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases} \quad (2.5)$$

For  $n = 2$  then the extension principle reads

$$f(A_1, A_2)(y) = \sup\{A_1(x_1) \wedge A_2(x_2) \mid f(x_1, x_2) = y\}.$$

**Example.** (extended addition) Let  $f: X \times X \rightarrow X$  be defined as

$$f(x_1, x_2) = x_1 + x_2,$$

i.e.  $f$  is the addition operator. Suppose  $A_1$  and  $A_2$  are fuzzy subsets of  $X$ . Then using the extension principle we get

$$f(A_1, A_2)(y) = \sup_{x_1+x_2=y} \min\{A_1(x_1), A_2(x_2)\}$$

and we use the notation  $f(A_1, A_2) = A_1 + A_2$ .

**Example.** (extended subtraction) Let  $f: X \times X \rightarrow X$  be defined as

$$f(x_1, x_2) = x_1 - x_2,$$

i.e.  $f$  is the subtraction operator. Suppose  $A_1$  and  $A_2$  are fuzzy subsets of  $X$ . Then using the extension principle we get

$$f(A_1, A_2)(y) = \sup_{x_1-x_2=y} \min\{A_1(x_1), A_2(x_2)\},$$

and we use the notation  $f(A_1, A_2) = A_1 - A_2$ .

We note that from the equality

$$\sup_{x_1-x_2=y} \min\{A_1(x_1), A_2(x_2)\} = \sup_{x_1+x_2=y} \min\{A_1(x_1), A_2(-x_2)\},$$

it follows that  $A_1 - A_2 = A_1 + (-A_2)$  holds. However, if  $A \in \mathcal{F}$  is a fuzzy number then

$$(A - A)(y) = \sup_{x_1-x_2=y} \min\{A(x_1), A(x_2)\}, \quad y \in \mathbb{R}$$

is not equal to the fuzzy number  $\bar{0}$ , where  $\bar{0}(t) = 1$  if  $t = 0$  and  $\bar{0}(t) = 0$  otherwise.

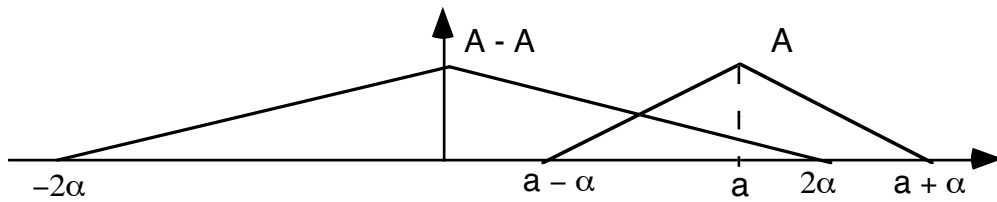


Figure 2.24: The membership function of  $A - A$ .

**Example.** Let  $f: X \times X \rightarrow X$  be defined as

$$f(x_1, x_2) = \lambda_1 x_1 + \lambda_2 x_2,$$

Suppose  $A_1$  and  $A_2$  are fuzzy subsets of  $X$ . Then using the extension principle we get

$$f(A_1, A_2)(y) = \sup_{\lambda_1 x_1 + \lambda_2 x_2 = y} \min\{A_1(x_1), A_2(x_2)\}$$

and we use the notation  $f(A_1, A_2) = \lambda_1 A_1 + \lambda_2 A_2$ .

**Example.** (extended multiplication) Let  $f: X \times X \rightarrow X$  be defined as

$$f(x_1, x_2) = x_1 x_2,$$

i.e.  $f$  is the multiplication operator. Suppose  $A_1$  and  $A_2$  are fuzzy subsets of  $X$ . Then using the extension principle we get

$$f(A_1, A_2)(y) = \sup_{x_1 x_2 = y} \min\{A_1(x_1), A_2(x_2)\}$$

and we use the notation  $f(A_1, A_2) = A_1 A_2$ .

**Example.** (extended division) Let  $f: X \times X \rightarrow X$  be defined as

$$f(x_1, x_2) = x_1 / x_2,$$

i.e.  $f$  is the division operator. Suppose  $A_1$  and  $A_2$  are fuzzy subsets of  $X$ . Then using the extension principle we get

$$f(A_1, A_2)(y) = \sup_{x_1/x_2=y, x_2 \neq 0} \min\{A_1(x_1), A_2(x_2)\}$$

and we use the notation  $f(A_1, A_2) = A_1 / A_2$ .

The extension principle for  $n$ -place functions is also a straightforward generalization of set-valued functions. Namely, let  $f: X_1 \times X_2 \rightarrow Y$  be a function. Then the image of a (crisp) subset  $(A_1, A_2) \subset X_1 \times X_2$  by  $f$  is defined by

$$f(A_1, A_2) = \{f(x_1, x_2) \mid x_1 \in A_1 \text{ and } x_2 \in A_2\}$$

and the characteristic function of  $f(A_1, A_2)$  is

$$\chi_{f(A_1, A_2)}(y) = \sup\{\min\{\chi_{A_1}(x), \chi_{A_2}(x)\} \mid x \in f^{-1}(y)\}.$$

Then replacing the characteristic functions by fuzzy sets we get Zadeh's extension principle for  $n$ -place functions (2.5).

Let  $A = (a_1, a_2, \alpha_1, \alpha_2)_{LR}$ , and  $B = (b_1, b_2, \beta_1, \beta_2)_{LR}$ , be fuzzy numbers of LR-type. Using the (sup-min) extension principle we can verify the following rules for addition and subtraction of fuzzy numbers of LR-type.

$$A + B = (a_1 + b_1, a_2 + b_2, \alpha_1 + \beta_1, \alpha_2 + \beta_2)_{LR}$$

$$A - B = (a_1 - b_2, a_2 - b_1, \alpha_1 + \beta_2, \alpha_2 + \beta_1)_{LR}$$

furthermore, if  $\lambda \in \mathbb{R}$  is a real number then  $\lambda A$  can be represented as

$$\lambda A = \begin{cases} (\lambda a_1, \lambda a_2, \alpha_1, \alpha_2)_{LR} & \text{if } \lambda \geq 0 \\ (\lambda a_2, \lambda a_1, |\lambda| \alpha_2, |\lambda| \alpha_1)_{LR} & \text{if } \lambda < 0 \end{cases}$$

In particular if  $A = (a_1, a_2, \alpha_1, \alpha_2)$  and  $B = (b_1, b_2, \beta_1, \beta_2)$  are fuzzy numbers of trapezoidal form then

$$A + B = (a_1 + b_1, a_2 + b_2, \alpha_1 + \beta_1, \alpha_2 + \beta_2) \quad (2.6)$$

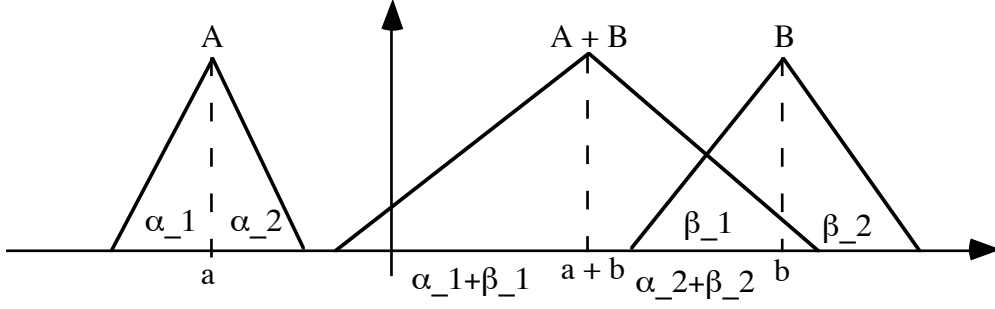


Figure 2.25: Addition of triangular fuzzy numbers.

$$A - B = (a_1 - b_2, a_2 - b_1, \alpha_1 + \beta_2, \alpha_2 + \beta_1). \quad (2.7)$$

If  $A = (a, \alpha_1, \alpha_2)$  and  $B = (b, \beta_1, \beta_2)$  are fuzzy numbers of triangular form then

$$A + B = (a + b, \alpha_1 + \beta_1, \alpha_2 + \beta_2),$$

$$A - B = (a - b, \alpha_1 + \beta_2, \alpha_2 + \beta_1)$$

and if  $A = (a, \alpha)$  and  $B = (b, \beta)$  are fuzzy numbers of symmetric triangular form then

$$A + B = (a + b, \alpha + \beta), \quad A - B = (a - b, \alpha + \beta), \quad \lambda A = (\lambda a, |\lambda| \alpha).$$

The above results can be generalized to linear combinations of fuzzy numbers.

**Lemma 2.5.1.** *Let  $A_i = (a_i, \alpha_i)$  be a fuzzy number of symmetric triangular form and let  $\lambda_i$  be a real number,  $i = 1, \dots, n$ . Then their linear combination*

$$A := \lambda_1 A_1 + \dots + \lambda_n A_n,$$

*can be represented as*

$$A = (\lambda_1 a_1 + \dots + \lambda_n a_n, |\lambda_1| \alpha_1 + \dots + |\lambda_n| \alpha_n).$$

Assume  $A_i = (a_i, \alpha)$ ,  $i = 1, \dots, n$  are fuzzy numbers of symmetric triangular form and  $\lambda_i \in [0, 1]$ , such that  $\lambda_1 + \dots + \lambda_n = 1$ . Then their *convex linear combination* can be represented as

$$A = (\lambda_1 a_1 + \dots + \lambda_n a_n, \lambda_1 \alpha + \dots + \lambda_n \alpha) = (\lambda_1 a_1 + \dots + \lambda_n a_n, \alpha).$$

## 2.5.2 Fuzzy functions

Let  $X \neq \emptyset$  and  $Y \neq \emptyset$  be crisp sets and let  $f$  be a function from  $\mathcal{F}(X)$  to  $\mathcal{F}(Y)$ . Then  $f$  is called a fuzzy function (or mapping) and we use the notation

$$f: \mathcal{F}(X) \rightarrow \mathcal{F}(Y).$$

It should be noted, however, that a fuzzy function is not necessarily defined by Zadeh's extension principle. It can be any function which maps a fuzzy set  $A \in \mathcal{F}(X)$  into a fuzzy set  $B := f(A) \in \mathcal{F}(Y)$ .

**Definition 2.5.3.** Let  $X \neq \emptyset$  and  $Y \neq \emptyset$  be crisp sets. A fuzzy mapping  $f: \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$  is said to be monoton increasing if from  $A, A' \in \mathcal{F}(X)$  and  $A \subset A'$  it follows that  $f(A) \subset f(A')$ .

**Theorem 2.5.1.** Let  $X \neq \emptyset$  and  $Y \neq \emptyset$  be crisp sets. Then every fuzzy mapping  $f: \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$  defined by the extension principle is monoton increasing.

*Proof.* Let  $A, A' \in \mathcal{F}(X)$  such that  $A \subset A'$ . Then using the definition of sup-min extension principle we get

$$f(A)(y) = \sup_{x \in f^{-1}(y)} A(x) \leq \sup_{x \in f^{-1}(y)} A'(x) = f(A')(y)$$

for all  $y \in Y$ . □

**Lemma 2.5.2.** Let  $A, B \in \mathcal{F}$  be fuzzy numbers and let  $f(A, B) = A + B$  be defined by sup-min extension principle. Then  $f$  is monoton increasing.

*Proof.* Let  $A, A', B, B' \in \mathcal{F}$  such that  $A \subset A'$  and  $B \subset B'$ . Then using the definition of sup-min extension principle we get

$$(A + B)(z) = \sup_{x+y=z} \min\{A(x), B(y)\} \leq \sup_{x+y=z} \min\{A'(x), B'(y)\} = (A' + B')(z)$$

Which ends the proof. □

The following lemma can be proved in a similar way.

**Lemma 2.5.3.** Let  $A, B \in \mathcal{F}$  be fuzzy numbers, let  $\lambda_1, \lambda_2$  be real numbers and let

$$f(A, B) = \lambda_1 A + \lambda_2 B$$

be defined by sup-min extension principle. Then  $f$  is a monoton increasing fuzzy function.

### 2.5.3 Nguyen's theorems

Let  $A$  and  $B$  be fuzzy numbers with  $[A]^\alpha = [a_1(\alpha), a_2(\alpha)]$  and  $[B]^\alpha = [b_1(\alpha), b_2(\alpha)]$ . Then it can easily be shown that

$$[A + B]^\alpha = [a_1(\alpha) + b_1(\alpha), a_2(\alpha) + b_2(\alpha)],$$

$$[A - B]^\alpha = [a_1(\alpha) - b_2(\alpha), a_2(\alpha) - b_1(\alpha)]$$

$$[\lambda A]^\alpha = \lambda[A]^\alpha$$

where  $[\lambda A]^\alpha = [\lambda a_1(\alpha), \lambda a_2(\alpha)]$  if  $\lambda \geq 0$  and  $[\lambda A]^\alpha = [\lambda a_2(\alpha), \lambda a_1(\alpha)]$  if  $\lambda < 0$  for all  $\alpha \in [0, 1]$ , i.e. any  $\alpha$ -level set of the extended sum of two fuzzy numbers is equal to the sum of their  $\alpha$ -level sets. The following two theorems (Nguyen, 1978) show that similar representatins are valid for any extended continuous function.

**Theorem 2.5.2.** [342] Let  $f: X \rightarrow X$  be a continuous function and let  $A$  be fuzzy numbers. Then

$$[f(A)]^\alpha = f([A]^\alpha)$$

where  $f(A)$  is defined by the extension principle (2.4) and

$$f([A]^\alpha) = \{f(x) \mid x \in [A]^\alpha\}.$$

If  $[A]^\alpha = [a_1(\alpha), a_2(\alpha)]$  and  $f$  is monoton increasing then from the above theorem we get

$$[f(A)]^\alpha = f([A]^\alpha) = f([a_1(\alpha), a_2(\alpha)]) = [f(a_1(\alpha)), f(a_2(\alpha))].$$

**Theorem 2.5.3.** [342] Let  $f: X \times X \rightarrow X$  be a continuous function and let  $A$  and  $B$  be fuzzy numbers. Then

$$[f(A, B)]^\alpha = f([A]^\alpha, [B]^\alpha)$$

where

$$f([A]^\alpha, [B]^\alpha) = \{f(x_1, x_2) \mid x_1 \in [A]^\alpha, x_2 \in [B]^\alpha\}.$$

Let  $f(x, y) = xy$  and let  $[A]^\alpha = [a_1(\alpha), a_2(\alpha)]$  and  $[B]^\alpha = [b_1(\alpha), b_2(\alpha)]$  be two fuzzy numbers. Applying Theorem 2.5.3 we get

$$[f(A, B)]^\alpha = f([A]^\alpha, [B]^\alpha) = [A]^\alpha [B]^\alpha.$$

However the equation

$$[AB]^\alpha = [A]^\alpha [B]^\alpha = [a_1(\alpha)b_1(\alpha), a_2(\alpha)b_2(\alpha)]$$

holds if and only if  $A$  and  $B$  are both nonnegative, i.e.  $A(x) = B(x) = 0$  for  $x \leq 0$ .

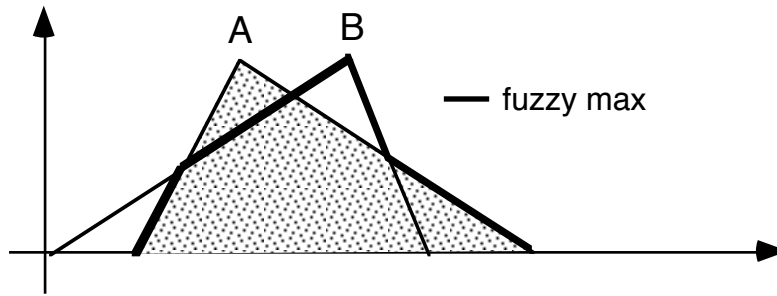


Figure 2.26: Fuzzy max of triangular fuzzy numbers.

If  $B$  is nonnegative then we have

$$[A]^\alpha [B]^\alpha = [\min\{a_1(\alpha)b_1(\alpha), a_1(\alpha)b_2(\alpha)\}, \max\{a_2(\alpha)b_1(\alpha), a_2(\alpha)b_2(\alpha)\}]$$

In general case we obtain a very complicated expression for the  $\alpha$  level sets of the product  $AB$

$$[A]^\alpha [B]^\alpha = [\min\{a_1(\alpha)b_1(\alpha), a_1(\alpha)b_2(\alpha), a_2(\alpha)b_1(\alpha), a_2(\alpha)b_2(\alpha)\}, \\ \max\{a_1(\alpha)b_1(\alpha), a_1(\alpha)b_2(\alpha), a_2(\alpha)b_1(\alpha), a_2(\alpha)b_2(\alpha)\}]$$

The above properties of extended operations *addition*, *subtraction* and *multiplication by scalar* of fuzzy fuzzy numbers of type LR are often used in *fuzzy neural networks*.

**Definition 2.5.4.** (fuzzy max) Let  $f(x, y) = \max\{x, y\}$  and let  $[A]^\alpha = [a_1(\alpha), a_2(\alpha)]$  and  $[B]^\alpha = [b_1(\alpha), b_2(\alpha)]$  be two fuzzy numbers. Applying Theorem 2.5.3 we get

$$[f(A, B)]^\alpha = f([A]^\alpha, [B]^\alpha) = \max\{[A]^\alpha, [B]^\alpha\} = [a_1(\alpha) \vee b_1(\alpha), a_2(\alpha) \vee b_2(\alpha)]$$

and we use the notation  $\max\{A, B\}$ .

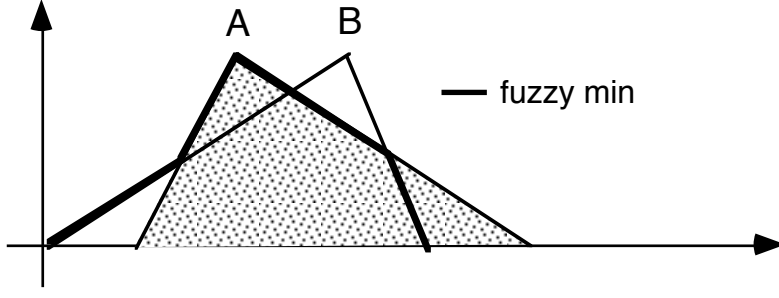


Figure 2.27: Fuzzy min of triangular fuzzy numbers.

**Definition 2.5.5.** (*fuzzy min*) Let  $f(x, y) = \min\{x, y\}$  and let  $[A]^\alpha = [a_1(\alpha), a_2(\alpha)]$  and  $[B]^\alpha = [b_1(\alpha), b_2(\alpha)]$  be two fuzzy numbers. Applying Theorem 2.5.3 we get

$$[f(A, B)]^\alpha = f([A]^\alpha, [B]^\alpha) = \min\{[A]^\alpha, [B]^\alpha\} = [a_1(\alpha) \wedge b_1(\alpha), a_2(\alpha) \wedge b_2(\alpha)]$$

and we use the notation  $\min\{A, B\}$ .

The fuzzy max and min are commutative and associative operations. Furthermore, if  $A, B$  and  $C$  are fuzzy numbers then

$$\max\{A, \min\{B, C\}\} = \min\{\max\{A, B\}, \max\{A, C\}\},$$

$$\min\{A, \max\{B, C\}\} = \max\{\min\{A, B\}, \min\{A, C\}\},$$

i.e. min and max are distributive.

## 2.6 t-norm-based operations on fuzzy numbers

In the definition of the extension principle one can use any t-norm for modeling the conjunction operator.

**Definition 2.6.1.** Let  $T$  be a t-norm and let  $f$  be a mapping from  $X_1 \times X_2 \times \cdots \times X_n$  to  $Y$ , Assume  $(A_1, \dots, A_n)$  is a fuzzy subset of  $X_1 \times X_2 \times \cdots \times X_n$ , using the extension principle, we can define  $f(A_1, A_2, \dots, A_n)$  as a fuzzy subset of  $Y$  such that

$$f(A_1, A_2, \dots, A_n)(y) = \begin{cases} \sup\{T(A_1(x), \dots, A_n(x)) \mid x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

This is called the *sup-T extension principle*.

Specially, if  $T$  is a t-norm and  $*$  is a binary operation on  $\mathbb{R}$  then  $*$  can be extended to fuzzy quantities in the sense of the sup-T extension principle as

$$(A_1 * A_2)(z) = \sup_{x_1 * x_2 = z} T(A_1(x_1), A_2(x_2)), \quad z \in \mathbb{R}.$$

For example, if  $A$  and  $B$  are fuzzy numbers,  $T_P(u, v) = uv$  is the product t-norm and  $f(x_1, x_2) = x_1 + x_2$  is the addition operation on the real line then the sup-product extended sum of  $A$  and  $B$ , called product-sum and denoted by  $A + B$ , is defined by

$$f(A, B)(y) = (A + B)(y) = \sup_{x_1 + x_2 = y} T(A_1(x_1), A_2(x_2)) = \sup_{x_1 + x_2 = y} A_1(x_1)A_2(x_2),$$

and if  $f(x_1, x_2) = x_1 x_2$  is the multiplication operation on the real line then the sup-Łukasiewicz extended product of  $A$  and  $B$ , denoted by  $A \times B$ , is defined by

$$(A \times B)(y) = \sup_{x_1 x_2 = y} T_L(A_1(x_1), A_2(x_2)) = \sup_{x_1 x_2 = y} \max\{A_1(x_1) + A_2(x_2) - 1, 0\}.$$

and if  $f(x_1, x_2) = x_1/x_2$  is the division operation on the real line then the sup- $H_\gamma$  extended division of  $A$  and  $B$ , denoted by  $A/B$ , is defined by

$$(A/B)(y) = \sup_{x_1/x_2 = y} H_\gamma(A_1(x_1), A_2(x_2)) = \sup_{x_1/x_2 = y} \frac{A_1(x_1)A_2(x_2)}{\gamma + (1 - \gamma)(A_1(x_1) + A_2(x_2) - A_1(x_1)A_2(x_2))},$$

where  $H_\gamma$  is the Hamacher t-norm (2.3) with parameter  $\gamma \geq 0$ .

The sup- $T$  extension principle is a very important in fuzzy arithmetic. For example, we have a sequence of symmetric triangular fuzzy numbers  $\tilde{a}_i$ ,  $i \in \mathbb{N}$  then their sup-min extended sum  $\tilde{a}_1 + \tilde{a}_2 + \cdots + \tilde{a}_n + \cdots$  is always the universal fuzzy set in  $\mathbb{R}$  independently of  $\alpha$ . This means that the minimum norm, because it is too big, might be inappropriate in situations where we have to manipulate with many fuzzy quantities (for example, fuzzy time series analysis, fuzzy linear programming problems, fuzzy control with a large number of rules, etc.).

## 2.7 Product-sum of triangular fuzzy numbers

Following Fullér [171] we will calculate the membership function of the product-sum  $\tilde{a}_1 + \tilde{a}_2 + \cdots + \tilde{a}_n + \cdots$  where  $\tilde{a}_i$ ,  $i \in \mathbb{N}$  are fuzzy numbers of triangular form. The next theorem can be interpreted as a central limit theorem for mutually product-related identically distributed fuzzy variables of symmetric triangular form (see [350]).

**Theorem 2.7.1.** [171] *Let  $\tilde{a}_i = (a_i, \alpha)$ ,  $i \in \mathbb{N}$ . If*

$$A := a_1 + a_2 + \cdots + a_n + \cdots = \sum_{i=1}^{\infty} a_i,$$

*exists and is finite, then with the notations*

$$\tilde{A}_n := \tilde{a}_1 + \cdots + \tilde{a}_n, \quad A_n := a_1 + \cdots + a_n, \quad n \in \mathbb{N},$$

*we have*

$$\left( \lim_{n \rightarrow \infty} \tilde{A}_n \right) (z) = \exp \left( -\frac{|A - z|}{\alpha} \right), \quad z \in \mathbb{R}.$$

*Proof.* It will be sufficient to show that

$$\tilde{A}_n(z) = \begin{cases} \left( 1 - \frac{|A_n - z|}{n\alpha} \right)^n & \text{if } |A_n - z| \leq n\alpha \\ 0 & \text{otherwise} \end{cases} \quad (2.8)$$



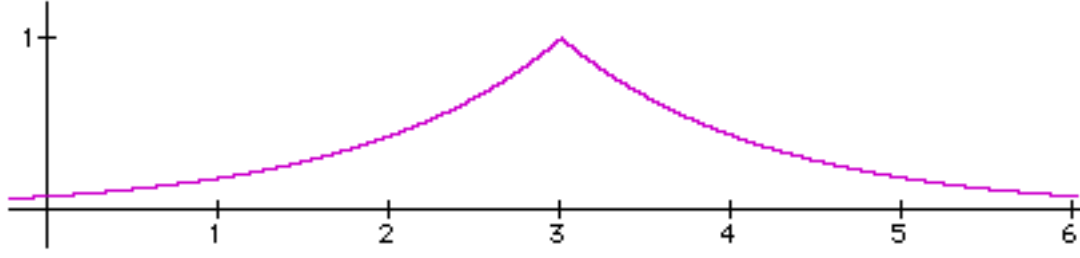


Figure 2.28: The limit distribution of the product-sum of  $\tilde{a}_i$ 's.

for each  $n \geq 2$ , because from (2.8) it follows that

$$\begin{aligned} \left( \lim_{n \rightarrow \infty} \tilde{A}_n \right) (z) &= \lim_{n \rightarrow \infty} \left( 1 - \frac{|A_n - z|}{n\alpha} \right)^n = \exp \left( - \frac{|\lim_{n \rightarrow \infty} A_n - z|}{\alpha} \right) = \\ &= \exp \left( - \frac{|A - z|}{\alpha} \right), \quad z \in \mathbb{R}. \end{aligned}$$

From the definition of product-sum of fuzzy numbers it follows that

$$\text{supp} \tilde{A}_n = \text{supp}(\tilde{a}_1 + \cdots + \tilde{a}_n) = \text{supp} \tilde{a}_1 + \cdots + \text{supp} \tilde{a}_n =$$

$$[a_1 - \alpha, a_1 + \alpha] + \cdots + [a_n - \alpha, a_n + \alpha] = [A_n - n\alpha, A_n + n\alpha], \quad n \in \mathbb{N}.$$

We prove (2.8) by making an induction argument on  $n$ . Let  $n = 2$ . In order to determine  $\tilde{A}_2(z)$ ,  $z \in [A_2 - 2\alpha, A_2 + 2\alpha]$  we need to solve the following mathematical programming problem:

$$\begin{aligned} \left( 1 - \frac{|a_1 - x|}{\alpha} \right) \left( 1 - \frac{|a_2 - y|}{\alpha} \right) &\rightarrow \max \\ \text{subject to } |a_1 - x| &\leq \alpha, \\ |a_2 - y| &\leq \alpha, \quad x + y = z. \end{aligned}$$

By using Lagrange's multipliers method and decomposition rule of fuzzy numbers into two separate parts (see [120]) it is easy to see that  $\tilde{A}_2(z)$ ,  $z \in [A_2 - 2\alpha, A_2 + 2\alpha]$  is equal to the optimal value of the following mathematical programming problem:

$$\begin{aligned} \left( 1 - \frac{a_1 - x}{\alpha} \right) \left( 1 - \frac{a_2 - z + x}{\alpha} \right) &\rightarrow \max \\ \text{subject to } a_1 - \alpha &\leq x \leq a_1, \\ a_2 - \alpha &\leq z - x \leq a_2, \quad x + y = z. \end{aligned} \tag{2.9}$$

Using Lagrange's multipliers method for the solution of (2.9) we get that its optimal value is

$$\left( 1 - \frac{|A_2 - z|}{2\alpha} \right)^2$$

and its unique solution is

$$x = \frac{a_1 - a_2 + z}{2},$$

where the derivative vanishes. Indeed, it can be easily checked that the inequality

$$\left(1 - \frac{|A_2 - z|}{2\alpha}\right)^2 \geq 1 - \frac{A_2 - z}{\alpha}$$

holds for each  $z \in [A_2 - 2\alpha, A_2]$ .

In order to determine  $\tilde{A}_2(z)$ ,  $z \in [A_2, A_2 + 2\alpha]$  we need to solve the following mathematical programming problem:

$$\begin{aligned} \left(1 + \frac{a_1 - x}{\alpha}\right) \left(1 + \frac{a_2 - z + x}{\alpha}\right) &\rightarrow \max \\ \text{subject to } a_1 &\leq x \leq a_1 + \alpha, \\ a_2 &\leq z - x \leq a_2 + \alpha. \end{aligned} \tag{2.10}$$

In a similar manner we get that the optimal value of (2.10) is

$$\left(1 - \frac{|z - A_2|}{2\alpha}\right)^2.$$

Let us assume that (2.8) holds for some  $n \in \mathbb{N}$ . By similar arguments we obtain

$$\begin{aligned} \tilde{A}_{n+1}(z) &= (\tilde{A}_n + \tilde{a}_{n+1})(z) = \\ \sup_{x+y=z} \tilde{A}_n(x) \cdot \tilde{a}_{n+1}(y) &= \sup_{x+y=z} \left(1 - \frac{|A_n - x|}{n\alpha}\right) \left(1 - \frac{|a_{n+1} - y|}{\alpha}\right) = \\ \left(1 - \frac{|A_{n+1} - z|}{(n+1)\alpha}\right)^{n+1}, & \quad z \in [A_{n+1} - (n+1)\alpha, A_{n+1} + (n+1)\alpha], \end{aligned}$$

and

$$\tilde{A}_{n+1}(z) = 0, \quad z \notin [A_{n+1} - (n+1)\alpha, A_{n+1} + (n+1)\alpha],$$

This ends the proof. □

**Theorem 2.7.2.** [169] Let  $T(x, y) = xy$  and  $\tilde{a}_i = (a_i, \alpha_i)$ ,  $i = 1, 2$ . Then

$$(\tilde{a}_1 + \tilde{a}_2)(z) = \begin{cases} c_1(z) & \text{if } |a_1 + a_2 - z| \leq |\alpha_1 - \alpha_2| \\ c_2(z) & \text{otherwise} \\ 0 & \text{if } |a_1 + a_2 - z| > \alpha_1 + \alpha_2 \end{cases}$$

where

$$c_1(z) = 1 - \frac{|a_1 + a_2 - z|}{\alpha_1 \vee \alpha_2}, \quad c_2(z) = \frac{(\alpha_1 + \alpha_2)^2}{4\alpha_1\alpha_2} \left(1 - \frac{|a_1 + a_2 - z|}{\alpha_1 + \alpha_2}\right)^2,$$

and  $\alpha_1 \vee \alpha_2 = \max\{\alpha_1, \alpha_2\}$ .

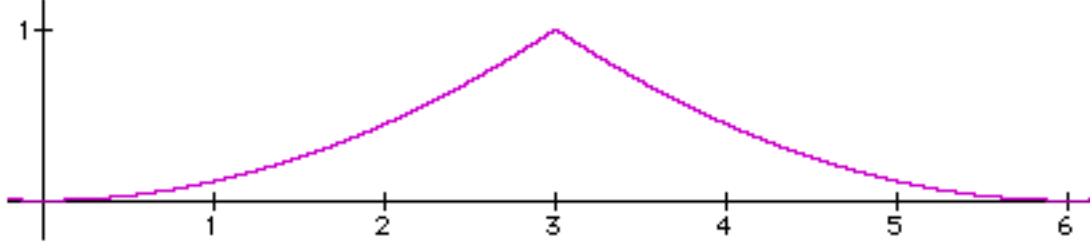


Figure 2.29: Product-sum of fuzzy numbers  $\tilde{a}_1 = (1, 3/2)$  and  $\tilde{a}_2 = (2, 3/2)$ .

Let  $\tilde{a}_i = (a_i, \alpha, \beta)_{LR}$ ,  $1 \leq i \leq n$  be fuzzy numbers of LR-type. In [171] Fullér asked the following question: On what condition will the membership function of the product-sum  $\tilde{A}_n$  have the following form

$$\tilde{A}_n(z) = \begin{cases} L^n \left( \frac{A_n - z}{n\alpha} \right) & \text{if } A_n - n\alpha \leq z \leq A_n, \\ R^n \left( \frac{z - A_n}{n\beta} \right) & \text{if } A_n \leq z \leq A_n + n\beta \end{cases} \quad (2.11)$$

Triesch [399] provided a partial answer to this question that  $\tilde{A}_n$  is given by (2.11) if  $\log R$  and  $\log L$  are concave functions. However, Hong [234] pointed out that the condition given by Triesch is not only sufficient but necessary, too.

## 2.8 Hamacher-sum of triangular fuzzy numbers

If  $\tilde{a}$  and  $\tilde{b}$  are fuzzy numbers and  $\gamma \geq 0$  a real number, then their Hamacher-sum ( $H_\gamma$ -sum for short) is defined as

$$(\tilde{a} + \tilde{b})(z) = \sup_{x+y=z} H_\gamma(\tilde{a}(x), \tilde{b}(y)) = \sup_{x+y=z} \frac{\tilde{a}(x)\tilde{b}(y)}{\gamma + (1 - \gamma)(\tilde{a}(x) + \tilde{b}(y) - \tilde{a}(x)\tilde{b}(y))},$$

for  $x, y, z \in \mathbb{R}$ , where  $H_\gamma$  the Hamacher t-norm (2.3) with parameter  $\gamma$ .

In the next two lemmas we shall calculate the exact membership function of  $H_\gamma$ -sum of two symmetric triangular fuzzy numbers having common width  $\alpha > 0$  for each permissible value of parameter  $\gamma$ .

**Lemma 2.8.1.** [173] Let  $0 \leq \gamma \leq 2$  and  $\tilde{a}_i = (a_i, \alpha)$ ,  $i = 1, 2$ . Then their  $H_\gamma$ -sum,  $\tilde{A}_2 = \tilde{a}_1 + \tilde{a}_2$ , has the following membership function:

$$\tilde{A}_2(z) = \frac{\left(1 - \frac{|A_2 - z|}{2\alpha}\right)^2}{1 + (\gamma - 1) \left(\frac{|A_2 - z|}{2\alpha}\right)^2}$$

if  $|A_2 - z| < 2\alpha$  and  $\tilde{A}_2(z) = 0$ , otherwise, where  $A_2 = a_1 + a_2$ .

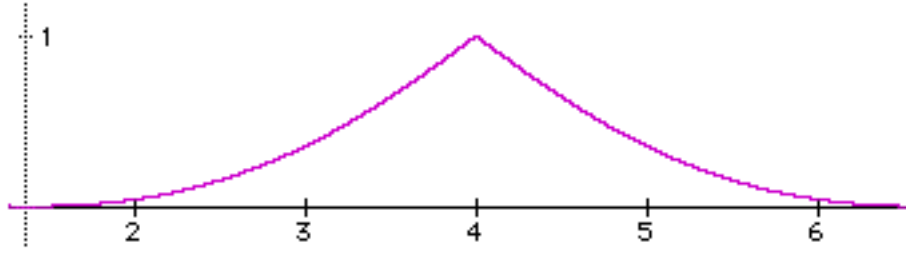


Figure 2.30:  $H_{1.5}$ -sum of fuzzy numbers  $\tilde{a}_1 = (1, 1.3)$  and  $\tilde{a}_2 = (3, 1.3)$ .

*Proof.* We need to determine the value of  $\tilde{a}_2(z)$  from the following relationships:

$$\tilde{A}_2(z) = (\tilde{a}_1 + \tilde{a}_2)(z) = \sup_{x+y=z} \frac{\tilde{a}_1(x)\tilde{a}_2(y)}{\gamma + (1-\gamma)(\tilde{a}_1(x) + \tilde{a}_2(y) - \tilde{a}_1(x)\tilde{a}_2(y))}, \text{ if } |A_2 - z| < 2\alpha,$$

and  $\tilde{A}_2(z) = 0$  otherwise.

According to the decomposition rule of fuzzy numbers into two separate parts,  $\tilde{A}_2(z)$ ,  $A_2 - 2\alpha < z \leq A_2$ , is equal to the optimal value of the following mathematical programming problem:

$$\begin{aligned} \phi(x) := & \frac{[1 - (a_1 - x)/\alpha][1 - (a_2 - z + x)/\alpha]}{\gamma + (1-\gamma)\{2 - (a_1 + a_2 - z)/\alpha - [1 - (a_1 - x)/\alpha][1 - (a_2 - z + x)/\alpha]\}} \rightarrow \max \\ & \text{subject to } a_1 - \alpha < x \leq a_1, \\ & a_2 - \alpha < z - x \leq a_2. \end{aligned}$$

Using Lagrange's multipliers method for the solution of the above problem we get that its optimal value is

$$\frac{\left(1 - \frac{A_2 - z}{2\alpha}\right)^2}{1 + (\gamma - 1) \left(\frac{A_2 - z}{2\alpha}\right)^2}$$

and its unique solution is

$$x = \frac{a_1 - a_2 + z}{2},$$

where the derivative vanishes. Indeed, from the inequality

$$\frac{\left(1 - \frac{A_2 - z}{2\alpha}\right)^2}{1 + (\gamma - 1) \left(\frac{A_2 - z}{2\alpha}\right)^2} \geq 1 - \frac{A_2 - z}{\alpha},$$

for  $A_2 - 2\alpha < z \leq A_2$ , and

$$\phi''\left(\frac{1}{2}(a_1 - a_2 + z)\right) < 0$$

follows that the function  $\phi$  attains its conditional maximum at the single stationary point

$$\frac{a_1 - a_2 + z}{2}.$$

If  $A_2 \leq z < A_2 + 2\alpha$ , then  $\tilde{A}_2(z)$  is equal to the optimal value of the the following mathematical programming problem

$$\begin{aligned} & \frac{[1 - (x - a_1)/\alpha][1 - (z - x - a_2)/\alpha]}{\gamma + (1 - \gamma)\{2 - (z - a_1 - a_2)/\alpha - [1 - (x - a_1)/\alpha][1 - (z - x - a_2)/\alpha]\}} \rightarrow \max \\ & \text{subject to } a_1 \leq x < a_1 + \alpha, \\ & a_2 \leq z - x < a_2 + \alpha. \end{aligned} \quad (2.12)$$

In a similar manner we get that the optimal value of (2.12) is

$$\frac{[1 - (z - A_2)/(2\alpha)]^2}{1 + (\gamma - 1)[(z - A_2)/(2\alpha)]^2}$$

and the unique solution of (2.12) is  $x = (a_1 - a_2 + z)/2$  (where the derivative vanishes). Which ends the proof.  $\square$

**Lemma 2.8.2.** [173] Let  $2 < \gamma < \infty$  and  $\tilde{a}_i = (a_i, \alpha)$ ,  $i = 1, 2$ . Then their  $H_\gamma$ -sum  $\tilde{A}_2 := \tilde{a}_1 + \tilde{a}_2$  has the following membership function

$$\tilde{A}_2(z) = \begin{cases} h_1(z) & \text{if } 1 - \frac{1}{\gamma - 1} \leq \frac{|A_2 - z|}{\alpha} < 2, \\ h_2(z) & \text{if } \frac{|A_2 - z|}{\alpha} < 1 - \frac{1}{\gamma - 1}, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$h_1(z) = \frac{[1 - (A_2 - z)/(2\alpha)]^2}{1 + (\gamma - 1)[(A_2 - z)/(2\alpha)]^2}, \quad h_2(z) = 1 - \frac{|A_2 - z|}{\alpha}$$

and  $A_2 = a_1 + a_2$ .

The following theorems can be interpreted as central limit theorems for mutually  $H_\gamma$ -related fuzzy variables of symmetric triangular form (see [350]).

**Theorem 2.8.1.** [173] Let  $\gamma = 0$  and  $\tilde{a}_i = (a_i, \alpha)$ ,  $i \in \mathbb{N}$ . Suppose that  $A := \sum_{i=1}^{\infty} a_i$  exists and is finite, then with the notation

$$\tilde{A}_n = \tilde{a}_1 + \cdots + \tilde{a}_n, \quad A_n = a_1 + \cdots + a_n$$

we have

$$\left( \lim_{n \rightarrow \infty} \tilde{A}_n \right) (z) = \frac{1}{1 + |A - z|/\alpha}, \quad z \in \mathbb{R}.$$

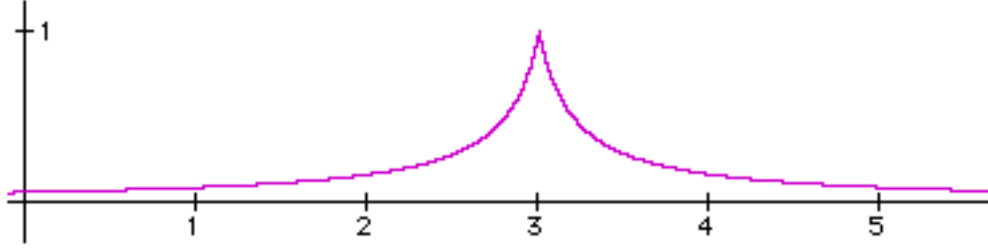


Figure 2.31: The limit distribution of the  $H_0$ -sum of  $\tilde{a}_i$ 's.

**Theorem 2.8.2.** [173] (Einstein-sum). Let  $\gamma = 2$  and  $\tilde{a}_i = (a_i, \alpha)$ ,  $i \in \mathbb{N}$ . If  $A := \sum_{i=1}^{\infty} a_i$  exists and is finite, then we have

$$\left( \lim_{n \rightarrow \infty} \tilde{A}_n \right) (z) = \frac{2}{1 + \exp \left[ \frac{-2|A - z|}{\alpha} \right]}, \quad z \in \mathbb{R}.$$

According to the decomposition rule of fuzzy numbers into two separate parts, the above theorems remain valid for sequences of non-symmetric fuzzy numbers of triangular form  $\tilde{a}_1 = (a_1, \alpha, \beta)$ ,  $\tilde{a}_2 = (a_2, \alpha, \beta)$ , ... with the difference that in the membership function of their  $H_\gamma$ -sum instead of  $\alpha$  we write  $\beta$  if  $z \geq A$ .

## 2.9 t-norm-based addition of fuzzy numbers

Recall that a t-norm  $T$  is Archimedean iff  $T$  is continuous and  $T(x, x) < x$  for all  $x \in (0, 1)$ . Every Archimedean t-norm  $T$  is representable by a continuous and decreasing function  $f: [0, 1] \rightarrow [0, \infty]$  with  $f(1) = 0$  and

$$T(x, y) = f^{[-1]}(f(x) + f(y)),$$

where  $f^{[-1]}$  is the pseudo-inverse of  $f$ , defined by

$$f^{[-1]}(y) = \begin{cases} f^{-1}(y) & \text{if } y \in [0, f(0)] \\ 0 & \text{otherwise.} \end{cases}$$

The function  $f$  is the additive generator of  $T$ .

If  $T$  is an Archimedean t-norm and  $\tilde{a}_1$  and  $\tilde{a}_2$  are fuzzy sets of the real line (i.e. fuzzy quantities) then their  $T$ -sum  $\tilde{A}_2 := \tilde{a}_1 + \tilde{a}_2$  is defined by

$$\tilde{A}_2(z) = \sup_{x_1 + x_2 = z} T(\tilde{a}_1(x_1), \tilde{a}_2(x_2)), \quad z \in \mathbb{R},$$

which expression can be written in the form

$$\tilde{A}_2(z) = f^{[-1]}(f(\tilde{a}_1(x_1)) + f(\tilde{a}_2(x_2))),$$

where  $f$  is the additive generator of  $T$ . By the associativity of  $T$ , the membership function of the  $T$ -sum  $\tilde{A}_n := \tilde{a}_1 + \dots + \tilde{a}_n$  of fuzzy quantities  $\tilde{a}_1, \dots, \tilde{a}_n$  can be written as

$$\tilde{A}_n(z) = \sup_{x_1 + \dots + x_n = z} f^{[-1]} \left( \sum_{i=1}^n f(\tilde{a}_i(x_i)) \right).$$

Since  $f$  is continuous and decreasing,  $f^{[-1]}$  is also continuous and non-increasing, we have

$$\tilde{A}_n(z) = f^{[-1]} \left( \inf_{x_1 + \dots + x_n = z} \sum_{i=1}^n f(\tilde{a}_i(x_i)) \right).$$

Following Fullér and Keresztfalvi [180] we shall determine a class of t-norms in which the addition of fuzzy numbers is very simple.

**Theorem 2.9.1.** [180] *Let  $T$  be an Archimedean t-norm with additive generator  $f$  and let  $\tilde{a}_i = (a_i, b_i, \alpha, \beta)_{LR}$ ,  $i = 1, \dots, n$ , be fuzzy numbers of LR-type. If  $L$  and  $R$  are twice differentiable, concave functions, and  $f$  is twice differentiable, strictly convex function then the membership function of the  $T$ -sum  $\tilde{A}_n = \tilde{a}_1 + \dots + \tilde{a}_n$  is*

$$\tilde{A}_n(z) = \begin{cases} 1 & \text{if } A_n \leq z \leq B_n \\ f^{[-1]} \left( n \times f \left( L \left( \frac{A_n - z}{n\alpha} \right) \right) \right) & \text{if } A_n - n\alpha \leq z \leq A_n \\ f^{[-1]} \left( n \times f \left( R \left( \frac{z - B_n}{n\beta} \right) \right) \right) & \text{if } B_n \leq z \leq B_n + n\beta \\ 0 & \text{otherwise} \end{cases}$$

where  $A_n = a_1 + \dots + a_n$  and  $B_n = b_1 + \dots + b_n$ .

*Proof.* As it was mentioned above, the investigated membership function is

$$\tilde{A}_n(z) = f^{[-1]} \left( \inf_{x_1 + \dots + x_n = z} f(\tilde{a}_1(x_1)) + \dots + f(\tilde{a}_n(x_n)) \right). \quad (2.13)$$

It is easy to see that the support of  $\tilde{A}_n$  is included in the interval  $[A_n - n\alpha, B_n + n\beta]$ . From the decomposition rule of fuzzy numbers into two separate parts it follows that the peak of  $\tilde{A}_n$  is  $[A_n, B_n]$ . Moreover, if we consider the right hand side of  $\tilde{A}_n$  (i.e.  $B_n \leq z \leq B_n + n\beta$ ) then only the right hand sides of terms  $\tilde{a}_n$  come into account in (2.13) (i.e.  $b_i \leq x_i \leq b_i + \beta$ ,  $i = 1, \dots, n$ ). The same thing holds for the left hand side of  $\tilde{A}_n$  (i.e. if  $A_n - n\alpha \leq z \leq A_n$  then  $a_i - \alpha \leq x_i \leq a_i$ ,  $i = 1, \dots, n$ ).

Let us now consider the right hand side of  $\tilde{A}_n$ , so let  $B_n \leq z \leq B_n + n\beta$ . (A similar method can be used if  $A_n - n\alpha \leq z \leq A_n$ .) The constraints

$$x_1 + \dots + x_n = z, \quad b_i \leq x_i \leq b_i + \beta, \quad i = 1, \dots, n$$

determine a compact and convex domain  $\mathcal{K} \subset \mathbb{R}^n$  which can be considered as the section of the brick

$$\mathcal{B} := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid b_i \leq x_i \leq b_i + \beta, \quad i = 1, \dots, n\}$$

by the hyperplane

$$\mathcal{P} := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 + \dots + x_n = z\}$$

In order to determine  $\tilde{A}_n(z)$  we need to find the conditional minimum value of the function  $\phi : \mathcal{B} \rightarrow \mathbb{R}$ ,

$$\phi(x_1, \dots, x_n) := f(\tilde{a}_1(x_1)) + \dots + f(\tilde{a}_n(x_n))$$

subject to condition  $(x_1, \dots, x_n) \in \mathcal{K}$ .  $\mathcal{K}$  is compact and  $\phi$  is continuous, this is why we could change the infimum with minimum.

Following the Lagrange's multipliers method we are searching the stationary points of the auxiliary function  $\Phi : \mathcal{B} \rightarrow \mathbb{R}$

$$\Phi(x_1, \dots, x_n) := \phi(x_1, \dots, x_n) + \lambda(z - (x_1 + x_2 + \dots + x_n))$$

i.e. the points  $(x_1, \dots, x_n, \lambda) \in \mathcal{B} \times \mathbb{R}$  where its derivative vanishes.

It is clear, that  $\Phi$  is twice differentiable, and its partial derivative with respect to  $x_i$  is

$$\partial_i \Phi(x_1, \dots, x_n, \lambda) = f'(R(\sigma_i(x_i)))R'(\sigma_i(x_i))\frac{1}{\beta} - \lambda,$$

where  $\sigma_i(x_i) := (x_i - b_i)/\beta$  and with respect to  $\lambda$  is

$$\partial_\lambda \Phi(x_1, \dots, x_n, \lambda) = z - (x_1 + \dots + x_n)$$

We get the system of equations:

$$f'(R(\sigma_i(x_i)))R'(\sigma_i(x_i))\frac{1}{\beta} - \lambda = 0, \quad i = 1, \dots, n,$$

$$z - (x_1 + \dots + x_n) = 0.$$

If we take

$$\hat{x}_i = b_i + \frac{z - B_n}{n}, \quad i = 1, \dots, n,$$

then

$$\sigma_1(x_1) = \dots = \sigma_n(x_n)$$

and we can define

$$\lambda = f'(R(\sigma_i(x_i)))R'(\sigma_i(x_i))\frac{1}{\beta}.$$

It is easy to verify that  $(\hat{x}_1, \dots, \hat{x}_n, \lambda)$  is a solution of the system of equations above.

This means that  $\phi$  attains its absolute conditional minimum at the point  $\hat{x}$ . Thus we have obtained a stationary point of the auxiliary function  $\Phi$ . It is left to show, that  $\Phi|_{\mathcal{K}}$  attains its absolute minimum at  $\hat{x} := (\hat{x}_1, \dots, \hat{x}_n) \in \mathcal{K}$ . For, then

$$\tilde{A}_n(z) = f^{[-1]}(f(\tilde{a}_1(\hat{x}_1)) + \dots + f(\tilde{a}_n(\hat{x}_n))) =$$

$$f^{[-1]}(f(R(\sigma_1(\hat{x}_1))) + \dots + f(R(\sigma_n(\hat{x}_n)))) =$$

$$f^{[-1]} \left( n \cdot f \left( R \left( \frac{z - B_n}{n\beta} \right) \right) \right).$$

It is easy to see that the partial derivatives of the second order of  $\phi$  at any point  $x = (x_1, \dots, x_n) \in \mathcal{B}$  have the following properties:

$$\partial_{ij}\phi(x) = 0, \quad \text{if } i \neq j$$



and for  $i = 1, \dots, n$ ,

$$\partial_{ii}\phi(x) = \frac{1}{\beta^2} [f''(R(\sigma_i(x_i)))(R'(\sigma_i(x_i)))^2 + f'(R(\sigma_i(x_i)))R''(\sigma_i(x_i))].$$

Therefore, we have  $\partial_{ii}(x) > 0$  for each  $i = 1, \dots, n$ . Indeed,

- $R'(\sigma_i(x_i)) \neq 0$  since  $R$  is non-increasing and concave, hence strictly decreasing in a neighbourhood of  $\sigma_i(x_i)$ ;
- $f' < 0$ ,  $f'' > 0$  and  $R'' < 0$  hold by monotonicity and strict convexity of  $f$  and concavity of  $R$ .

The matrix of the derivative of the second order of  $\phi$  at any point inside  $\mathcal{B}$  has only (nonzero) elements in its diagonal, which are positive. Therefore, it is positive definite in  $\mathcal{B}$ . Now we show that  $\phi(\hat{x})$  is the minimum of  $\phi$  in  $\mathcal{K}$ .

Consider an arbitrary point  $x = (x_1, \dots, x_n) \in \mathcal{K}$ . From convexity of  $\mathcal{K}$  it follows that the segment  $[\hat{x}, x]$  lies within  $\mathcal{K}$ . By virtue of Lagrange's mean value theorem, there exists  $\xi = (\xi_1, \dots, \xi_n) \in [\hat{x}, x]$  such that

$$\phi(x) = \phi(\hat{x}) + \sum_{i=1}^n \partial_i \phi(\hat{x})(x_i - \hat{x}_i) + \sum_{i,j=1}^n \partial_{ij} \phi(\xi)(x_i - \hat{x}_i)(x_j - \hat{x}_j)$$

and using the properties  $\partial_1 \phi(\hat{x}) = \dots = \partial_n \phi(\hat{x}) = 0$ ,  $\partial_{ij} \phi(\hat{x}) = 0$ , if  $i \neq j$  and  $\partial_{ii} \phi(\hat{x}) > 0$ , for each  $i$ , we obtain that  $\phi(x) > \phi(\hat{x})$ . This means that  $\phi$  attains its absolute conditional minimum at the point  $\hat{x}$ .  $\square$

It should be noted, that from the concavity of shape functions it follows that the fuzzy numbers in question can not have infinite support.

## 2.9.1 Extensions

Theorem 2.9.1 has been improved and generalized later by Kawaguchi and Da-Te ([276, 277]), Hong ([235, 242]), Hong and Kim ([240]), Hong and Hwang ([234, 236, 241, 243, 244]), Marková [326], Mesiar [329, 330, 331]), De Baets and Markova ([6, 7]).

Namely, in 1994 Hong and Hwang ([234]) proved that Theorem 2.9.1 remains valid for convex additive generator  $f$ , and concave shape functions  $L$  and  $R$ . In 1994 Hong and Hwang ([236]) provided an upper bound for the membership function of  $T$ -sum of  $LR$ -fuzzy numbers with different spreads:

**Theorem 2.9.2.** [236] *Let  $T$  be an Archimedean  $t$ -norm with additive generator  $f$  and let  $\tilde{a}_i = (a_i, \alpha_i, \beta_i)_{LR}$ ,  $i = 1, 2$ , be fuzzy numbers of  $LR$ -type. If  $L$  and  $R$  are concave functions, and  $f$  is a convex function then the membership function of the  $T$ -sum  $\tilde{A}_2 = \tilde{a}_1 + \tilde{a}_2$  is less than or equal to*

$$A_2^*(z) =$$

$$\left\{ \begin{array}{ll} f^{[-1]} \left( 2f \left( L \left( 1/2 + \frac{(A_2 - z) - \alpha^*}{2\alpha_*} \right) \right) \right) & \text{if } A_2 - \alpha_1 - \alpha_2 \leq z \leq A_2 - \alpha^* \\ f^{[-1]} \left( 2f \left( L \left( \frac{A_2 - z}{2\alpha^*} \right) \right) \right) & \text{if } A_2 - \alpha^* \leq z \leq A_2 \\ f^{[-1]} \left( 2f \left( R \left( \frac{z - A_2}{2\beta^*} \right) \right) \right) & \text{if } A_2 \leq z \leq A_2 + \beta^* \\ f^{[-1]} \left( 2f \left( R \left( 1/2 + \frac{(z - A_2) - \beta^*}{2\beta_*} \right) \right) \right) & \text{if } A_2 + \beta^* \leq z \leq A_2 + \beta_1 + \beta_2 \\ 0 & \text{otherwise} \end{array} \right.$$

where  $\beta^* = \max\{\beta_1, \beta_2\}$ ,  $\beta_* = \min\{\beta_1, \beta_2\}$ ,  $\alpha^* = \max\{\alpha_1, \alpha_2\}$ ,  $\alpha_* = \min\{\alpha_1, \alpha_2\}$  and  $A_2 = a_1 + a_2$ .

In 1996 Mesiar [329] showed that Theorem 2.9.1 remains valid if both  $L \circ f$  and  $R \circ f$  are convex functions.

In 1997 Hong and Hwang [244] gave upper and lower bounds of  $T$ -sums of  $LR$ -fuzzy numbers  $\tilde{a}_i = (a_i, \alpha_i, \beta_i)_{LR}$ ,  $i = 1, \dots, n$ , with different spreads where  $T$  is an Archimedean  $t$ -norm. The theorems in question are stated as follows.

**Theorem 2.9.3.** [244] *Let  $T$  be an Archimedean  $t$ -norm with additive generator  $f$  and let  $\tilde{a}_i = (a_i, \alpha_i, \beta_i)_{LR}$ ,  $i = 1, \dots, n$ , be fuzzy numbers of  $LR$ -type. If  $f \circ L$  and  $f \circ R$  are convex functions, then the membership function of their  $T$ -sum  $\tilde{A}_n = \tilde{a}_1 + \dots + \tilde{a}_n$  is less than or equal to*

$$A_n^*(z) = \left\{ \begin{array}{ll} f^{[-1]} \left( n f \left( L \left( \frac{1}{n} I_L(A_n - z) \right) \right) \right) & \text{if } A_n - (\alpha_1 + \dots + \alpha_n) \leq z \leq A_n \\ f^{[-1]} \left( n f \left( R \left( \frac{1}{n} I_R(z - A_n) \right) \right) \right) & \text{if } A_n \leq z \leq A_n + (\beta_1 + \dots + \beta_n) \\ 0 & \text{otherwise,} \end{array} \right.$$

where

$$I_L(z) = \inf \left\{ \frac{x_1}{\alpha_1} + \dots + \frac{x_n}{\alpha_n} \mid x_1 + \dots + x_n = z, 0 \leq x_i \leq \alpha_i, i = 1, \dots, n \right\},$$

and

$$I_R(z) = \inf \left\{ \frac{x_1}{\beta_1} + \dots + \frac{x_n}{\beta_n} \mid x_1 + \dots + x_n = z, 0 \leq x_i \leq \beta_i, i = 1, \dots, n \right\}.$$

**Theorem 2.9.4.** [244] *Let  $T$  be an Archimedean  $t$ -norm with additive generator  $f$  and let  $\tilde{a}_i = (a_i, \alpha_i, \beta_i)_{LR}$ ,  $i = 1, \dots, n$ , be fuzzy numbers of  $LR$ -type. Then*

$$\tilde{A}_n(z) \geq A_n^{**}(z) =$$

$$\begin{cases} f^{[-1]} \left( n f \left( L \left( \frac{A_n - z}{\alpha_1 + \dots + \alpha_n} \right) \right) \right) & \text{if } A_n - (\alpha_1 + \dots + \alpha_n) \leq z \leq A_n \\ f^{[-1]} \left( n f \left( R \left( \frac{A_n - z}{\beta_1 + \dots + \beta_n} \right) \right) \right) & \text{if } A_n \leq z \leq A_n + (\beta_1 + \dots + \beta_n) \\ 0 & \text{otherwise,} \end{cases}$$

## 2.9.2 Illustrations

We shall illustrate the Theorem 2.9.1 by Yager's, Dombi's and Hamacher's parametrized t-norms. For simplicity we shall restrict our consideration to the case of symmetric fuzzy numbers

$$\tilde{a}_i = (a_i, a_i, \alpha, \alpha)_{LL}, \quad i = 1, \dots, n.$$

Denoting

$$\sigma_n := \frac{|A_n - z|}{n\alpha},$$

we get the following formulas for the membership function of t-norm-based sum

$$\tilde{A}_n = \tilde{a}_1 + \dots + \tilde{a}_n$$

(i) Yager's t-norm with  $p > 1$ :

$$T(x, y) = 1 - \min \left\{ 1, \sqrt[p]{(1-x)^p + (1-y)^p} \right\} \quad \text{with } f(x) = (1-x)^p$$

and then

$$\tilde{A}_n(z) = \begin{cases} 1 - n^{1/p}(1 - L(\sigma_n)) & \text{if } \sigma_n < L^{-1}(1 - n^{-1/p}) \\ 0 & \text{otherwise.} \end{cases}$$

(ii) Hamacher's t-norm with  $p \leq 2$ :

$$T(x, y) = \frac{xy}{p + (1-p)(x+y-xy)} \quad \text{with } f(x) = \ln \frac{p + (1-p)x}{x}.$$

Then

$$\tilde{A}_n(z) = \begin{cases} v(p, \sigma) & \text{if } \sigma_n < 1 \\ 0 & \text{otherwise.} \end{cases}$$

where

$$v(p, \sigma) = \frac{p}{[(p + (1-p)L(\sigma_n))/L(\sigma_n)]^n - 1 + p}.$$

Specially, for the product t-norm (that is  $H_1$ ):

$$T(x, y) = xy \quad \text{with } f(x) = -\ln x$$

Then

$$\tilde{A}_n(z) = L^n(\sigma_n), \quad z \in \mathbb{R}.$$

(iii) Dombi's t-norm with  $p > 1$ :

$$T(x, y) = \frac{1}{1 + \sqrt[p]{(1/x - 1)^p + (1/y - 1)^p}} \text{ with } f(x) = \left(\frac{1}{x} - 1\right)^p.$$

Then

$$\tilde{A}_n(z) = \begin{cases} (1 + n^{1/p}(1/L(\sigma_n) - 1))^{-1} & \text{if } \sigma_n < 1 \\ 0 & \text{otherwise.} \end{cases}$$

### 2.9.3 A functional relationship between $T$ -extended addition and multiplication

A fuzzy quantity  $M$  of  $\mathbb{R}$  is said to be positive if  $\mu_M(x) = 0$  for all  $x < 0$ , and a fuzzy number is negative if  $\mu_M(x) = 0$  for all  $x > 0$ . The following theorem (Fullér and Keresztfalvi, [183]) gives a functional relationship between the membership functions of fuzzy numbers  $M_1 + \dots + M_n$  and  $M_1 \times \dots \times M_n$ , where  $M_i, i = 1, \dots, n$ , are positive  $LR$  fuzzy numbers of the same form  $M_i = M = (a, b, \alpha, \beta)_{LR}$ .

**Theorem 2.9.5.** [183] *Let  $T$  be an Archimedean  $t$ -norm with an additive generator  $f$  and let  $M_i = M = (a, b, \alpha, \beta)_{LR}$  be positive fuzzy numbers of  $LR$  type. If  $L$  and  $R$  are twice differentiable, concave functions, and  $f$  is twice differentiable, strictly convex function, then*

$$(M_1 + \dots + M_n)(n \cdot z) = (M_1 \times \dots \times M_n)(z^n) = f^{[-1]}(n \cdot f(M(z))). \quad (2.14)$$

*Proof.* Let  $z \geq 0$  be arbitrarily fixed. According to the decomposition rule of fuzzy numbers into two separate parts, we can assume without loss of generality that  $z < a$ . From Theorem 2.9.1 it follows that

$$\begin{aligned} (M_1 + \dots + M_n)(n \cdot z) &= f^{[-1]} \left( n \cdot f \left( L \left( \frac{na - nz}{n\alpha} \right) \right) \right) = \\ &= f^{[-1]} \left( n \cdot f \left( L \left( \frac{a - z}{\alpha} \right) \right) \right) = f^{[-1]}(n \cdot f(M(z))) \end{aligned}$$

The proof will be complete if we show that

$$\begin{aligned} (M \times \dots \times M)(z) &= \sup_{x_1, \dots, x_n = z} T(M(x_1), \dots, M(x_n)) = \\ &= T(M(\sqrt[n]{z}), \dots, M(\sqrt[n]{z})) = f^{[-1]}(n \cdot f(M(\sqrt[n]{z}))) \end{aligned} \quad (2.15)$$

We shall justify it by induction:

- (i) for  $n = 1$  (2.15) is obviously valid;
- (ii) Let us suppose that (2.15) holds for some  $n = k$ , i. e.

$$\begin{aligned} (M^k)(z) &= \sup_{x_1, \dots, x_k = z} T(M(x_1), \dots, M(x_k)) = \\ &= T(M(\sqrt[k]{z}), \dots, M(\sqrt[k]{z})) = \end{aligned}$$

$$f^{[-1]} \left( k \cdot f \left( M(\sqrt[k]{z}) \right) \right),$$

and verify the case  $n = k + 1$ . It is clear that

$$\begin{aligned} (M^{k+1})(z) &= \sup_{x \cdot y = z} T \left( M^k(x), M(y) \right) = \\ &= \sup_{x \cdot y = z} T \left( M(\sqrt[k]{x}), \dots, M(\sqrt[k]{x}), M(y) \right) = \\ &= g^{[-1]} \left( \inf_{x \cdot y = z} \left( k \cdot g \left( M(\sqrt[k]{x}) \right) + g \left( M(y) \right) \right) \right) = \\ &= g^{[-1]} \left( \inf_x \left( k \cdot g \left( M(\sqrt[k]{x}) \right) + g \left( M(z/x) \right) \right) \right) \end{aligned}$$

The support and the peak of  $M^{k+1}$  are

$$\begin{aligned} [M^{k+1}]^1 &= [M]^{1^{k+1}} = [a^{k+1}, b^{k+1}] \\ \text{Supp}(M^{k+1}) &\subset \left( \text{Supp}(M) \right)^{k+1} = [(a - \alpha)^{k+1}, (a + \beta)^{k+1}] \end{aligned}$$

According to the decomposition rule we can consider only the left hand side of  $M$ , that is let  $z \in [(a - \alpha)^{k+1}, a^{k+1}]$ . We need to find the minimum of the mapping

$$x \mapsto k \cdot g \left( M(\sqrt[k]{x}) \right) + g \left( M(z/x) \right)$$

in the interval  $[(a - \alpha)^k, a^k]$ . Let us introduce the auxiliary variable  $t = \sqrt[k]{x}$  and look for the minimum of the function

$$t \mapsto \varphi(t) := k \cdot f \left( M(t) \right) + f \left( M(z/t^k) \right)$$

in the interval  $[a - \alpha, a]$ . Dealing with the left hand side of  $M$  we have

$$M(t) = L \left( \frac{a - t}{\alpha} \right) \quad \text{and} \quad M(z/t^k) = L \left( \frac{a - z/t^k}{\alpha} \right)$$

The derivative of  $\varphi$  is equal to zero when

$$\begin{aligned} \varphi'(t) &= \\ k f'(M(t)) L' \left( \frac{a - t}{\alpha} \right) \cdot \frac{-1}{\alpha} + f'(M(z/t^k)) L' \left( \frac{a - z/t^k}{\alpha} \right) \cdot \frac{-1}{\alpha} \cdot \left( -k \cdot \frac{z}{t^{k+1}} \right) &= 0, \end{aligned}$$

i. e.

$$t \cdot f'(M(t)) \cdot L' \left( \frac{a - t}{\alpha} \right) = \frac{z}{t^k} \cdot f'(M(z/t^k)) \cdot L' \left( \frac{a - z/t^k}{\alpha} \right) \quad (2.16)$$

which obviously holds taking  $t = z/t^k$ . So  $t_0 = \sqrt[k+1]{z}$  is a solution of (2.16), furthermore, from the strict monotony of

$$t \mapsto t \cdot f'(M(t)) \cdot L' \left( \frac{a - t}{\alpha} \right)$$

follows that there are no other solutions.

It is easy to check, that  $\varphi''(t_0) > 0$ , which means that  $\varphi$  attains its absolute minimum at  $t_0$ . Finally, from the relations  $\sqrt[k]{x_0} = \sqrt[k+1]{z}$  and  $z/x_0 = \sqrt[k+1]{z}$ , we get

$$(M^{k+1})(z) = T\left(M(\sqrt[k+1]{z}), \dots, M(\sqrt[k+1]{z}), M(\sqrt[k+1]{z})\right) =$$

$$f^{[-1]}\left(k \cdot f(M(\sqrt[k+1]{z})) + f(M(\sqrt[k+1]{z}))\right) = f^{[-1]}\left((k+1) \cdot g(M(\sqrt[k+1]{z}))\right).$$

Which ends the proof.  $\square$

As an immediate consequence of Theorem 2.9.5 we can easily calculate the exact possibility distribution of expressions of the form

$$e_n^*(M) := \frac{M + \dots + M}{n}$$

and the limit distribution of  $e_n^*(M)$  as  $n \rightarrow \infty$ . Namely, from (2.14) we have

$$(e_n^*(M))(z) = \left(\frac{M + \dots + M}{n}\right)(z) =$$

$$(M + \dots + M)(n \cdot z) = f^{[-1]}\left(n \cdot f(M(z))\right)$$

therefore, from  $f(x) > 0$  for  $0 \leq x < 1$  and

$$\lim_{x \rightarrow \infty} f^{[-1]}(x) = 0$$

we get

$$\left[\lim_{n \rightarrow \infty} e_n^*(M)\right](z) = \lim_{n \rightarrow \infty} (e_n^*(M))(z) =$$

$$\lim_{n \rightarrow \infty} f^{[-1]}\left(n \cdot f(M(z))\right) = \begin{cases} 1 & \text{if } z \in [a, b] \\ 0 & \text{if } z \notin [a, b], \end{cases}$$

that is

$$\lim_{n \rightarrow \infty} e_n^*(M) = [a, b] \tag{2.17}$$

which is the peak of  $M$ . It can be shown [120] that (2.17) remains valid for the (non-Archimedean) weak t-norm. It is easy to see [171] that when  $T(x, y) = xy$ :

$$(M_1 + \dots + M_n)(n \cdot z) = (M_1 \times \dots \times M_n)(z^n) = (M(z))^n.$$

## 2.10 On generalization of Nguyen's theorems

In this Section we generalize Nguyen's theorem (Theorem 2.5.3 in Section 2.5) and following Fullér and Keresztfalvi [172], we give a necessary and sufficient condition for obtaining the equality

$$[f(A, B)]^\alpha = \bigcup_{T(\xi, \eta) \geq \alpha} f([A]^\xi, [B]^\eta), \quad \alpha \in (0, 1], \tag{2.18}$$

where  $f : X \times Y \rightarrow Z$ ,  $T$  is a t-norm,  $A$  and  $B$  are fuzzy subsets of  $X$  and  $Y$ , respectively,  $f(A, B)$  is defined via sup- $T$ -norm convolution,  $[A]^\alpha$  and  $[B]^\alpha$  are the  $\alpha$ -level sets of  $A$  and  $B$ , respectively, and  $[f(A, B)]^\alpha$  is the  $\alpha$ -level set of  $f(A, B)$ .

Furthermore, we shall define a class of fuzzy subsets in which this equality holds for all upper semicontinuous  $T$  and continuous  $f$ . It should be noted that in the special case  $T(x, y) = \min\{x, y\}$ , the equation (2.18) yields

$$[f(A, B)]^\alpha = f([A]^\alpha, [B]^\alpha), \alpha \in (0, 1],$$

which coincides with Nguyen's result. Additionally, since fuzzy logics are defined in terms of t-norms rather just min-max operators, the result of this Section can be useful for workers in the field in the implementation of algorithms.

The symbol  $\mathcal{F}(X)$  denotes the family of all fuzzy subsets of a set  $X$ . When  $X$  is a topological space, we denote by  $\mathcal{F}(X, \mathcal{K})$  the set of all fuzzy subsets of  $X$  having upper semicontinuous, compactly-supported membership function.

Recall that if  $T$  is a t-norm,  $f : X \times Y \rightarrow Z$ ,  $A \in \mathcal{F}(X)$  and  $B \in \mathcal{F}(Y)$  then the fuzzy set  $f(A, B) \in \mathcal{F}(Z)$  is defined via the extension principle by

$$f(A, B)(z) = \sup_{f(x,y)=z} T(A(x), B(y)), z \in Z.$$

The following theorem illustrates that, if instead of min-norm in Zadeh's extension principle, we use an arbitrary t-norm, then we obtain results similar to those of Nguyen.

**Theorem 2.10.1.** [172] *Let  $X \neq \emptyset$ ,  $Y \neq \emptyset$ ,  $Z \neq \emptyset$  be sets and let  $T$  be a t-norm. If  $f : X \times Y \rightarrow Z$  is a two-place function and  $A \in \mathcal{F}(X)$ ,  $B \in \mathcal{F}(Y)$  then a necessary and sufficient condition for the equality*

$$[f(A, B)]^\alpha = \bigcup_{T(\xi, \eta) \geq \alpha} f([A]^\xi, [B]^\eta), \alpha \in (0, 1], \quad (2.19)$$

*is, that for each  $z \in Z$ ,*

$$\sup_{f(x,y)=z} T(A(x), B(y))$$

*is attained.*

The next theorem shows that the equality (2.18) holds for all upper semicontinuous  $T$  and continuous  $f$  in the class of upper semicontinuous and compactly-supported fuzzy subsets. In the following,  $X, Y, Z$  are locally compact topological spaces.

**Theorem 2.10.2.** [172] *If  $f : X \times Y \rightarrow Z$  is continuous and the t-norm  $T$  is upper semicontinuous, then*

$$[f(A, B)]^\alpha = \bigcup_{T(\xi, \eta) \geq \alpha} f([A]^\xi, [B]^\eta), \alpha \in (0, 1],$$

*holds for each  $A \in \mathcal{F}(X, \mathcal{K})$  and  $B \in \mathcal{F}(Y, \mathcal{K})$ .*

The following examples illustrate that the  $\alpha$ -cuts of the fuzzy set  $f(A, B)$  can be generated in a simple way when the t-norm in question has a simple form.

**Example.** If  $T(x, y) = \min(x, y)$ , then using the fact that  $\xi \geq \alpha$  and  $\eta \geq \alpha$  implies

$$f([A]^\xi, [B]^\eta) \subset f([A]^\alpha, [B]^\alpha),$$

equation (2.18) is reduced to the well-known form of Nguyen:

$$[f(A, B)]^\alpha = f([A]^\alpha, [B]^\alpha) \quad \alpha \in (0, 1],$$

**Example.** If  $T(x, y) = T_W(x, y)$ , where

$$T_W(x, y) = \begin{cases} \min\{x, y\} & \text{if } \max\{x, y\} = 1 \\ 0 & \text{otherwise} \end{cases}$$

is the weak  $t$ -norm, then (2.18) turns into

$$[f(A, B)]^\alpha = f([A]^1, [B]^\alpha) \cup f([A]^\alpha, [B]^1) \quad \alpha \in (0, 1],$$

since  $T_W(\xi, \eta) \geq \alpha > 0$  holds only if  $\xi = 1$  or  $\eta = 1$ .

Thus if  $[A]^1 = \emptyset$  or  $[B]^1 = \emptyset$ , then  $[f(A, B)]^\alpha = \emptyset$ ,  $\forall \alpha \in (0, 1]$ . If there exist unique  $x_0$  and  $y_0$  such that  $A(x_0) = B(y_0) = 1$ , then we obtain

$$[f(A, B)]^\alpha = f(x_0, [B]^\alpha) \cup f([A]^\alpha, y_0) \quad \alpha \in (0, 1],$$

**Example.** If  $T(x, y) = xy$ , then the equation (2.18) yields

$$[f(A, B)]^\alpha = \bigcup_{\xi \in [\alpha, 1]} f([A]^\xi, [B]^{\alpha/\xi}), \quad \alpha \in (0, 1].$$

**Example.** If  $T(x, y) = \max\{0, x + y - 1\}$ , then

$$[f(A, B)]^\alpha = \bigcup_{\xi \in [\alpha, 1]} f([A]^\xi, [B]^{\alpha+1-\xi}), \quad \alpha \in (0, 1].$$

## 2.11 Measures of possibility and necessity

Fuzzy numbers can also be considered as possibility distributions [127]. If  $A \in \mathcal{F}$  is a fuzzy number and  $x \in \mathbb{R}$  a real number then  $A(x)$  can be interpreted as the degree of possibility of the statement " $x$  is  $A$ ".

Let  $A, B \in \mathcal{F}$  be fuzzy numbers. The degree of possibility that the proposition " $A$  is less than or equal to  $B$ " is true denoted by  $\text{Pos}[A \leq B]$  and defined by the extension principle as

$$\text{Pos}[A \leq B] = \sup_{x \leq y} \min\{A(x), B(y)\} = \sup_{z \leq 0} (A - B)(z), \quad (2.20)$$

In a similar way, the degree of possibility that the proposition " $A$  is greater than or equal to  $B$ " is true, denoted by  $\text{Pos}[A \geq B]$ , is defined by

$$\text{Pos}[A \geq B] = \sup_{x \geq y} \min\{A(x), B(y)\} = \sup_{z \geq 0} (A - B)(z). \quad (2.21)$$



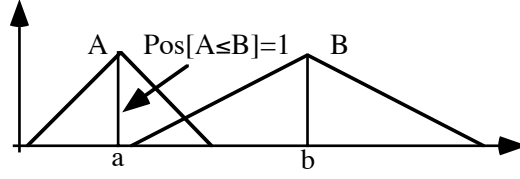


Figure 2.32:  $\text{Pos}[A \leq B] = 1$ , because  $a \leq b$ .

Finally, the degree of possibility that the proposition is true "A is equal to B" and denoted by  $\text{Pos}[A = B]$ , is defined by

$$\text{Pos}[A = B] = \sup_x \min\{A(x), B(x)\} = (A - B)(0), \quad (2.22)$$

Let  $A = (a, \alpha)$  and  $B = (b, \beta)$  fuzzy numbers of symmetric triangular form. It is easy to compute that,

$$\text{Pos}[A \leq B] = \begin{cases} 1 & \text{if } a \leq b \\ 1 - \frac{a - b}{\alpha + \beta} & \text{otherwise} \\ 0 & \text{if } a \geq b + \alpha + \beta \end{cases} \quad (2.23)$$

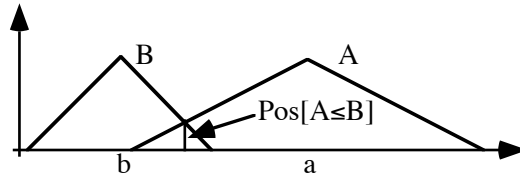


Figure 2.33:  $\text{Pos}[A \leq B] < 1$ , because  $a > b$ .

The degree of necessity that the proposition "A is less than or equal to B" is true, denoted by  $\text{Pos}[A \leq B]$ , is defined as

$$\text{Nes}[A \leq B] = 1 - \text{Pos}[A \geq B].$$

If  $A = (a, \alpha)$  and  $B = (b, \beta)$  are fuzzy numbers of symmetric triangular form then

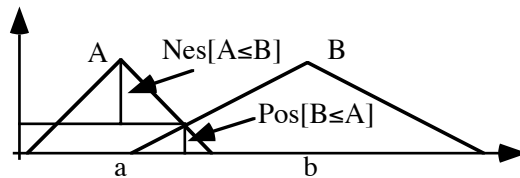


Figure 2.34:  $\text{Nes}[A \leq B] < 1$ , ( $a < b$ ,  $A \cap B \neq \emptyset$ ).

$$\text{Nes}[A \leq B] = \begin{cases} 1 & \text{if } a \leq b - \alpha - \beta \\ \frac{b - a}{\alpha + \beta} & \text{otherwise} \\ 0 & \text{if } a \geq b \end{cases} \quad (2.24)$$

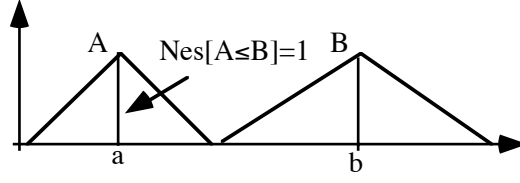


Figure 2.35:  $Nes[A \leq B] = 1$ , ( $a < b$  and  $A \cap B = \emptyset$ ).

Let  $\xi \in \mathcal{F}$  be a fuzzy number. Given a subset  $D \subset \mathbb{R}$ , the grade of possibility of the statement "D contains the value of  $\xi$ " is defined by

$$Pos(\xi | D) = \sup_{x \in D} \xi(x) \quad (2.25)$$

The quantity  $1 - Pos(\xi | \bar{D})$ , where  $\bar{D}$  is the complement of  $D$ , is denoted by  $Nes(\xi | D)$  and

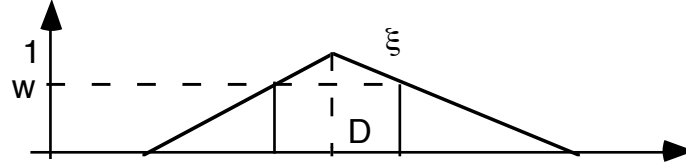


Figure 2.36:  $Pos(\xi | D) = 1$  és  $Nes(\xi | D) = 1 - w$ .

is interpreted as the grade of necessity of the statement "D contains the value of  $\xi$ ". It satisfies dual property with respect to (2.25):

$$Nes(\xi | D) = 1 - Pos(\xi | \bar{D}).$$

If  $D = [a, b] \subset \mathbb{R}$  then instead of  $Nes(\xi | [a, b])$  we shall write  $Nes(a \leq \xi \leq b)$  and if  $D = \{x\}$ ,  $x \in \mathbb{R}$  we write  $Nes(\xi = x)$ .

Let  $\xi_1, \xi_2, \dots$  be a sequence of fuzzy numbers. We say that  $\{\xi_n\}$  converges pointwise to a fuzzy set  $\xi$  (and write  $\lim_{n \rightarrow \infty} \xi_n = \xi$ ) if

$$\lim_{n \rightarrow \infty} \xi_n(x) = \xi(x),$$

for all  $x \in \mathbb{R}$ .

## 2.12 A law of large numbers for fuzzy numbers

Following Fullér [177, 182] we study the problem:

If  $\xi_1, \xi_2, \dots$  are fuzzy numbers with modal values  $M_1, M_2, \dots$ , then what is the strongest  $t$ -norm for which

$$\lim_{n \rightarrow \infty} Nes\left(m_n - \epsilon \leq \frac{\xi_1 + \dots + \xi_n}{n} \leq m_n + \epsilon\right) = 1,$$

for any  $\epsilon > 0$ , where

$$m_n = \frac{M_1 + \cdots + M_n}{n},$$

the arithmetic mean

$$\frac{\xi_1 + \cdots + \xi_n}{n}$$

is defined via sup- $t$ -norm convolution and Nes denotes necessity. Given two fuzzy numbers,  $\xi$  and  $\eta$ , their  $T$ -sum  $\xi + \eta$  is defined by

$$(\xi + \eta)(z) = \sup_{x+y=z} T(\xi(x), \eta(y)), \quad x, y, z \in \mathbb{R}$$

where  $T$  t-norm. The function  $H_0 : [0, 1] \times [0, 1] \rightarrow [0, 1]$ , defined by

$$H_0(u, v) = \frac{uv}{u + v - uv},$$

is called Hamacher-norm with parameter zero ( $H_0$ -norm for short) [212].

Let  $T_1, T_2$  be t-norms. We say that  $T_1$  is weaker than  $T_2$  (and write  $T_1 \leq T_2$ ) if  $T_1(x, y) \leq T_2(x, y)$  for each  $x, y \in [0, 1]$ . We shall provide a fuzzy analogue of Chebyshev's theorem [86].

**Theorem 2.12.1.** (Chebyshev's theorem.) *If  $\xi_1, \xi_2, \dots$  is a sequence of pairwise independent random variables having finite variances bounded by the same constant*

$$D\xi_1 \leq C, D\xi_2 \leq C, \dots, D\xi_n \leq C, \dots$$

and

$$M = \lim_{n \rightarrow \infty} \frac{M_1 + \cdots + M_n}{n}$$

exists, then for any positive constant  $\epsilon$

$$\lim_{n \rightarrow \infty} \text{Prob} \left( \left| \frac{\xi_1 + \cdots + \xi_n}{n} - \frac{M_1 + \cdots + M_n}{n} \right| < \epsilon \right) = 1$$

where  $M_n = M\xi_n$  and Prob denotes probability.

In this section we shall prove that if  $\xi_1 = (M_1, \alpha)$ ,  $\xi_2 = (M_2, \alpha) \dots$  is a sequence of symmetric triangular fuzzy numbers and  $T$  is a t-norm (by which the sequence of arithmetic means

$$\left\{ \frac{\xi_1 + \cdots + \xi_n}{n} \right\},$$

is defined) then the relation

$$\lim_{n \rightarrow \infty} \text{Nes} \left( m_n - \epsilon \leq \frac{\xi_1 + \cdots + \xi_n}{n} \leq m_n + \epsilon \right) = 1, \text{ for any } \epsilon > 0 \quad (2.26)$$

holds for any  $T \leq H_0$ ; and the relation (2.26) is not valid for the "min"-norm.

**Definition 2.12.1.** Let  $T$  be a  $t$ -norm and let  $\xi, \xi_2, \dots$  be a sequence of fuzzy numbers. We shall say that the sequence  $\{\xi_n\}$  obeys the law of large numbers if it satisfies the relation (2.26).

**Lemma 2.12.1.** Let  $\xi$  and  $\eta$  be fuzzy sets of  $\mathbb{R}$ . If  $\xi \subseteq \eta$  (i.e.  $\xi(x) \leq \eta(x)$ , for each  $x \in \mathbb{R}$ ) then

$$\text{Nes}(\xi = x) \geq \text{Nes}(\eta = x), \text{ for each } x \in \mathbb{R}.$$

*Proof.* From the definition of necessity we have

$$\text{Nes}(\xi = x) = 1 - \text{Pos}(\xi | \mathbb{R} \setminus \{x\}) = 1 - \sup_{t \neq x} \xi(t) \geq 1 - \sup_{t \neq x} \eta(t) = \text{Nes}(\eta = x).$$

Which ends the proof.  $\square$

The proof of the next two lemmas follows from the definition of  $t$ -sum of fuzzy numbers.

**Lemma 2.12.2.** Let  $T_1$  and  $T_2$  be  $t$ -norms and let  $\xi_1$  and  $\xi_2$  be fuzzy numbers. If  $T_1 \leq T_2$  then

$$(\xi_1 + \xi_2)_1 \subseteq (\xi_1 + \xi_2)_2$$

where  $(\xi_1 + \xi_2)_i$  denotes the  $T_i$ -sum of fuzzy numbers  $\xi_1$  and  $\xi_2$ ,  $i = 1, 2$ .

**Lemma 2.12.3.** Let  $T = H_0$  and  $\xi_i = (a_i, \alpha)$ ,  $i = 1, 2, \dots, n$ . Then with the notations

$$\eta_n = \xi_1 + \dots + \xi_n, \quad A_n = a_1 + \dots + a_n$$

we have

$$(i) \quad \eta_n(z) = \begin{cases} \frac{1 - |A_n - z|(n\alpha)^{-1}}{1 + (n-1)|A_n - z|(n\alpha)^{-1}} & \text{if } |A_n - z| \leq n\alpha, \\ 0 & \text{otherwise,} \end{cases}$$

$$(ii) \quad \left(\frac{\eta_n}{n}\right)(z) = \begin{cases} \frac{1 - |A_n/n - z|\alpha^{-1}}{1 + (n-1)|A_n/n - z|\alpha^{-1}} & \text{if } |A_n/n - z| \leq \alpha, \\ 0 & \text{otherwise,} \end{cases}$$

*Proof.* We prove (i) by making an induction argument on  $n$ . Let  $n = 2$ . Then we need to determine the value of  $\eta_2(z)$  from the following relationship:

$$\eta_2(z) = \sup_{x+y=z} \frac{\xi_1(x)\xi_2(y)}{\xi_1(x) + \xi_2(y) - \xi_1(x)\xi_2(y)} = \sup_{x+y=z} \frac{1}{\frac{1}{\xi_1(x)} + \frac{1}{\xi_2(y)} - 1},$$

if  $z \in (a_1 + a_2 - 2\alpha, a_1 + a_2 + 2\alpha)$  and  $\eta_2(z) = 0$  otherwise.

According to the decomposition rule of fuzzy numbers into two separate parts,  $\eta_2(z)$ ,  $z \in (a_1 + a_2 - 2\alpha, a_1 + a_2]$ , is equal to the value of the following mathematical programming problem

$$\frac{1}{\frac{1}{1 - \frac{a_1 - x}{\alpha}} + \frac{1}{1 - \frac{a_2 - z + x}{\alpha}} - 1} \rightarrow \max \quad (2.27)$$

subject to  $a_1 - \alpha < x \leq a_1$ ,

$a_2 - \alpha < z - x \leq a_2$ .

Using Lagrange's multipliers method for the solution of the problem (2.27) we get that its value is

$$\frac{1 - \frac{a_1 + a_2 - z}{2\alpha}}{1 + \frac{a_1 + a_2 - z}{2\alpha}} = \frac{1 - \frac{A_2 - z}{2\alpha}}{1 + \frac{A_2 - z}{2\alpha}}$$

and the solution of (2.27) is

$$x = \frac{a_1 - a_2 + z}{2}$$

(where the first derivative vanishes). If  $a_1 + a_2 \leq z < a_1 + a_2 + 2\alpha$  then we need to solve the following problem

$$\begin{aligned} & \frac{1}{\frac{1}{1 - \frac{x - a_1}{\alpha}} + \frac{1}{1 - \frac{z - x - a_2}{\alpha}} - 1} \rightarrow \max \\ & \text{subject to } a_1 < x < a_1 + \alpha, \\ & a_2 < z - x < a_2 + \alpha. \end{aligned} \tag{2.28}$$

In a similar manner we get that the value of (2.28) is

$$\frac{1 - \frac{z - A_2}{2\alpha}}{1 + \frac{z - A_2}{2\alpha}}$$

and the solution of (2.28) is

$$x = \frac{a_1 - a_2 + z}{2}$$

(where the first derivative vanishes). Let us assume that (i) holds for some  $n \in \mathbb{N}$ . Then,

$$\eta_{n+1}(z) = (\eta_n + \xi_{n+1})(z), \quad z \in \mathbb{R},$$

and by similar arguments it can be shown that (i) holds for  $\eta_{n+1}$ . The statement (ii) can be proved directly using the relationship  $(\eta_n/n)(z) = \eta_n(nz)$ ,  $z \in \mathbb{R}$ . This ends the proof.  $\square$

The following lemma shows that if instead of Necessity we used Possibility in (2.26), then every sequence of fuzzy numbers would obey the law of large numbers.

**Lemma 2.12.4.** *Let  $T$  be a  $t$ -norm and let  $\xi_1, \xi_2, \dots$  be a sequence of fuzzy numbers with modal values  $M_1, M_2, \dots$  then with the notations*

$$\eta_n = \xi_1 + \dots + \xi_n, \quad m_n = \frac{M_1 + \dots + M_n}{n}$$

*we have  $\text{Pos}(\eta_n/n = m_n) = 1$ ,  $n \in \mathbb{N}$ .*

*Proof.* From the Lemmas it follows that  $(\eta_n/n)(m_n) = 1, n \in \mathbb{N}$ . Which ends the proof.  $\square$

The theorem in question can be stated as follows:

**Theorem 2.12.2.** (Law of large numbers for fuzzy numbers, [182]) Let  $T \leq H_0$  and let  $\xi_i = (M_i, \alpha), i \in \mathbb{N}$  be fuzzy numbers. If

$$M = \lim_{n \rightarrow \infty} \frac{M_1 + \dots + M_n}{n}$$

exists, then for any  $\epsilon > 0$ ,

$$\lim_{n \rightarrow \infty} \text{Nes} \left( m_n - \epsilon \leq \frac{\xi_1 + \dots + \xi_n}{n} \leq m_n + \epsilon \right) = 1, \quad (2.29)$$

where

$$m_n = \frac{M_1 + \dots + M_n}{n}.$$

*Proof.* If  $\epsilon \geq \alpha$  then we get (2.29) trivially. Let  $\epsilon < \alpha$ , then from Lemma 2.12.1 and Lemma 2.12.2 it follows that we need to prove (2.29) only for  $T = H_0$ . Using Lemma 2.12.3 we get

$$\begin{aligned} & \text{Nes} \left( m_n - \epsilon \leq \frac{\eta_n}{n} \leq m_n + \epsilon \right) = \\ & 1 - \text{Pos} \left( \frac{\eta_n}{n} | (-\infty, m_n - \epsilon) \cup (m_n + \epsilon, \infty) | \right) = \\ & - \sup_{x \notin [m_n - \epsilon, m_n + \epsilon]} \left( \frac{\eta_n}{n} \right) (x) = \\ & 1 - \frac{1 - |m_n - (m_n + \epsilon)|/\alpha}{1 + (n-1)|m_n - (m_n \pm \epsilon)|/\alpha} = \\ & 1 - \frac{1 - \epsilon/\alpha}{1 + (n-1)\epsilon/\alpha} \end{aligned}$$

and, consequently,

$$\lim_{n \rightarrow \infty} \text{Nes} \left( m_n - \epsilon \leq \frac{\eta_n}{n} \leq m_n + \epsilon \right) = 1 - \lim_{n \rightarrow \infty} \frac{1 - \epsilon/\alpha}{1 + (n-1)\epsilon/\alpha} = 1.$$

Which ends the proof.  $\square$

Theorem 2.12.2 can be interpreted as a law of large numbers for mutually T-related fuzzy variables. Strong laws of large numbers for fuzzy random variables were proved in [303, 335]. Especially, if  $T(u, v) = H_1(u, v) = uv$  then we get [171]

$$\lim_{n \rightarrow \infty} \text{Nes} \left( m_n - \epsilon \leq \frac{\xi_1 + \dots + \xi_n}{n} \leq m_n + \epsilon \right) =$$

$$1 - \frac{\eta_n}{n}(m_n - \epsilon) = 1 - \lim_{n \rightarrow \infty} \left(1 - \frac{\epsilon}{\alpha}\right)^n = 1.$$

The following theorem shows that if  $T = \text{"min"}$  then the sequence  $\xi_1 = (M_1, \alpha)$ ,  $\xi_2 = (M_2, \alpha)$  ... does not obey the law of large numbers for fuzzy numbers.

**Theorem 2.12.3.** *Let  $T(u, v) = \min\{u, v\}$  and  $\xi_i = (M_i, \alpha)$ ,  $i \in \mathbb{N}$ . Then for any positive  $\epsilon$ , such that  $\epsilon < \alpha$  we have*

$$\lim_{n \rightarrow \infty} \text{Nes} \left( m_n - \epsilon \leq \frac{\xi_1 + \dots + \xi_n}{n} \leq m_n + \epsilon \right) = \frac{\epsilon}{\alpha}.$$

$$\text{Nes} \left( \lim_{n \rightarrow \infty} \frac{\eta_n}{n} = M \right) = 0$$

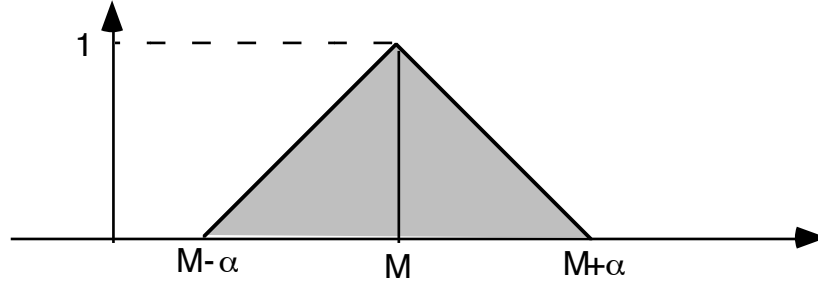


Figure 2.37: The limit distribution of  $\eta_n/n$  if  $T = \text{"min"}$

*Proof.* The proof of this theorem follows from the equalities  $\eta_n/n = (m_n, \alpha)$ ,  $n \in \mathbb{N}$  and

$$\lim_{n \rightarrow \infty} \frac{\eta_n}{n} = (M, \alpha).$$

□

From the addition rule of LR-type fuzzy numbers via sup-min convolution it follows that Theorem 2.12.2 remains valid for any sequence  $\xi_1 = (M_1, \alpha)_{LL}$ ,  $\xi_2 = (M_2, \alpha)_{LL}$ , ... of LL-type fuzzy numbers with continuous shape function  $L$ .

### 2.12.1 Extensions

The results presented in this Section have been extended and improved by Triesch [398], Hong [237], Hong and Kim [240], and Jang and Kwon [261].

Namely, in 1993 Triesch [398] showed that the class of Archimedean t-norms can be characterised by the validity of a very general law of large numbers for sequences of  $LR$  fuzzy numbers. The theorem in question is stated as follows:

**Theorem 2.12.4.** [398] Suppose that  $T$  is a continuous  $t$ -norm. Then  $T$  is Archimedean if and only if all sequences of fuzzy numbers  $\xi_1 = (M_1, \alpha_1, \beta_1)_{LR}$ ,  $\xi_2 = (M_2, \alpha_2, \beta_2)_{LR}$ ,  $\dots$ , such that  $\alpha_n \leq C$  and  $\beta_n \leq C$  for all  $n$  and some constant  $C$  obey the law of large numbers with respect to  $T$ .

In 1994 Hong [237] answered the following question stated in [182]

Let  $T$  be a  $t$ -norm such that  $H_0 < T < \min$  and let  $\xi_1 = (M_1, \alpha)$ ,  $\xi_2 = (M_2, \alpha)$   $\dots$  be a sequence of symmetric triangular fuzzy numbers. Does this sequence obey the law of large numbers?

and showed that the  $t$ -norm defined by

$$T(u, v) = \begin{cases} g^{-1}(g(u) + g(v)) & \text{if } u, v \in (1/2, 1] \\ \min\{u, v\} & \text{otherwise,} \end{cases}$$

where

$$g(x) = \frac{2 - 2x}{2x - 1}$$

satisfies the property  $H_0 < T < \min$ , but the sequence of identical fuzzy numbers  $\xi_i = (0, 1)$ ,  $i = 1, 2, \dots$  does not obey the law of large numbers for fuzzy numbers.

Generalizing the law of large numbers for fuzzy numbers to Banach spaces, Hong and Kim [240] proved the following extension of Theorem 2.12.2.

**Theorem 2.12.5.** [240] Suppose  $\{\xi_i = (M_i, \alpha_i, \beta_i)_{LR}\}$ ,  $i \in \mathbf{N}$  is a sequence of  $LR$ -fuzzy numbers with left and right spreads  $\alpha_i$  and  $\beta_i$ , respectively, such that  $\alpha_i \leq C$  and  $\beta_i \leq C$  for all  $i$  and some constant  $C$ . Suppose further that  $T$  is an Archimedean  $t$ -norm. Then for any  $\epsilon > 0$

$$\lim_{n \rightarrow \infty} \text{Nes} \left( m_n - \epsilon \leq \frac{\xi_1 + \dots + \xi_n}{n} \leq m_n + \epsilon \right) = 1.$$

Moreover, if  $M = \lim_{n \rightarrow \infty} m_n$  exists and is finite, then

$$\lim_{n \rightarrow \infty} \text{Nes} \left( M - \epsilon \leq \frac{\xi_1 + \dots + \xi_n}{n} \leq M + \epsilon \right) = 1.$$

## 2.13 Metrics for fuzzy numbers

Let  $A$  and  $B$  be fuzzy numbers with  $[A]^\alpha = [a_1(\alpha), a_2(\alpha)]$  and  $[B]^\alpha = [b_1(\alpha), b_2(\alpha)]$ . We metricize the set of fuzzy numbers by the following metrics

- **Hausdorff distance**

$$D(A, B) = \sup_{\alpha \in [0, 1]} \max\{|a_1(\alpha) - b_1(\alpha)|, |a_2(\alpha) - b_2(\alpha)|\}.$$

i.e.  $D(A, B)$  is the maximal distance between the  $\alpha$ -level sets of  $A$  and  $B$ . For example, if  $A = (a, \alpha)$  and  $B = (b, \alpha)$  are fuzzy numbers of symmetric triangular form with the same width  $\alpha > 0$  then

$$D(A, B) = |a - b|,$$



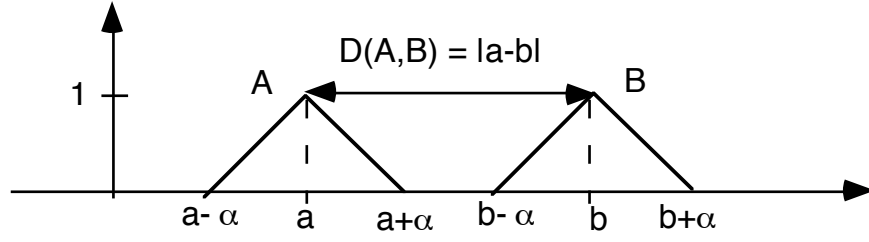


Figure 2.38: Hausdorff distance between  $A = (a, \alpha)$  and  $B = (b, \alpha)$ .

and if  $A = (a, \alpha)$  and  $B = (b, \beta)$  then

$$D(A, B) = |a - b| + |\alpha - \beta|.$$

- **$C_\infty$  distance**

$$C_\infty(A, B) = \sup\{|A(u) - B(u)| : u \in \mathbb{R}\}.$$

i.e.  $C_\infty(A, B)$  is the maximal distance between the membership grades of  $A$  and  $B$ . The following statement holds  $0 \leq C_\infty(A, B) \leq 1$ .

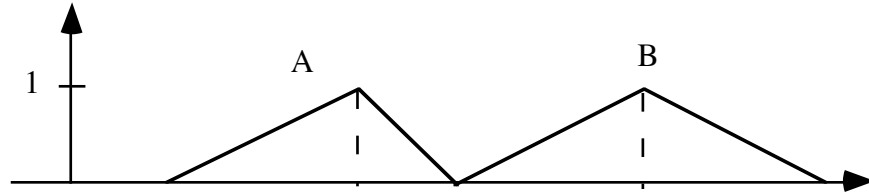


Figure 2.39:  $C_\infty(A, B) = 1$  whenever the supports of  $A$  and  $B$  are disjunctive.

- **Hamming distance** Suppose  $A$  and  $B$  are fuzzy sets in  $X$ . Then their Hamming distance, denoted by  $H(A, B)$ , is defined by

$$H(A, B) = \int_X |A(x) - B(x)| dx.$$

- **Discrete Hamming distance** Suppose  $A$  and  $B$  are discrete fuzzy sets in  $X = \{x_1, \dots, x_n\}$  defined by

$$A = \mu_1/x_1 + \dots + \mu_n/x_n,$$

$$B = \nu_1/x_1 + \dots + \nu_n/x_n.$$

Then their Hamming distance is defined by

$$H(A, B) = \sum_{j=1}^n |\mu_j - \nu_j|.$$

It should be noted that  $D(A, B)$  is a better measure of *similarity* than  $C_\infty(A, B)$ , because  $C_\infty(A, B) \leq 1$  holds even though the supports of  $A$  and  $B$  are very far from each other.

**Definition 2.13.1.** Let  $f$  be a fuzzy function from  $\mathcal{F}$  to  $\mathcal{F}$ . Then  $f$  is said to be continuous in metric  $D$  if  $\forall \epsilon > 0$  there exists  $\delta > 0$  such that if  $D(A, B) \leq \delta$  then

$$D(f(A), f(B)) \leq \epsilon$$

In a similar way we can define the continuity of fuzzy functions in metric  $C_\infty$ .

**Definition 2.13.2.** Let  $f$  be a fuzzy function from  $\mathcal{F}(\mathbb{R})$  to  $\mathcal{F}(\mathbb{R})$ . Then  $f$  is said to be continuous in metric  $C_\infty$  if  $\forall \epsilon > 0$  there exists  $\delta > 0$  such that if  $C_\infty(A, B) \leq \delta$  then

$$C_\infty(f(A), f(B)) \leq \epsilon.$$

We note that in the definition of continuity in metric  $C_\infty$  the domain and the range of  $f$  can be the family of all fuzzy subsets of the real line, while in the case of continuity in metric  $D$  the domain and the range of  $f$  is the set of fuzzy numbers.

We will use the following inequality relations between fuzzy numbers  $[A]^r = [a_1(r), a_2(r)]$  and  $[B]^r = [b_1(r), b_2(r)]$

$$A \leq B \iff \max\{A, B\} = B \quad (2.30)$$

that is,

$$A \leq B \iff a_1(r) \leq b_2(r) \text{ and } a_2(r) \leq b_2(r),$$

for all  $r \in [0, 1]$ , and [204],

$$A \leq B \iff \mathcal{W}(A) = \int_0^1 r(a_1(r) + a_2(r))dr \leq \mathcal{W}(B) = \int_0^1 r(b_1(r) + b_2(r))dr \quad (2.31)$$

Equation (2.30) is derived directly from Zadeh' extension principle, and (2.31) compares fuzzy numbers based on their weighted center of gravity, where the weights are the membership degrees.

## 2.14 Auxiliary lemmas

The following lemmas build up connections between  $C_\infty$  and  $D$  distances of fuzzy numbers.

**Lemma 2.14.1.** [272]. Let  $\tilde{a}, \tilde{b}, \tilde{c}$  and  $\tilde{d}$  be fuzzy numbers. Then

$$D(\tilde{a} + \tilde{c}, \tilde{b} + \tilde{d}) \leq D(\tilde{a}, \tilde{b}) + D(\tilde{c}, \tilde{d}),$$

$$D(\tilde{a} - \tilde{c}, \tilde{b} - \tilde{d}) \leq D(\tilde{a}, \tilde{b}) + D(\tilde{c}, \tilde{d})$$

and  $D(\lambda\tilde{a}, \lambda\tilde{b}) = |\lambda|D(\tilde{a}, \tilde{b})$  for  $\lambda \in \mathbb{R}$  and

Let  $\tilde{a} \in \mathcal{F}$  be a fuzzy number. Then for any  $\theta \geq 0$  we define  $\omega(\tilde{a}, \theta)$ , the modulus of continuity of  $\tilde{a}$  as

$$\omega(\tilde{a}, \theta) = \max_{|u-v| \leq \theta} |\tilde{a}(u) - \tilde{a}(v)|.$$

The following statements hold [216]:

$$\text{If } 0 \leq \theta \leq \theta' \text{ then } \omega(\tilde{a}, \theta) \leq \omega(\tilde{a}, \theta') \quad (2.32)$$

$$\text{If } \alpha > 0, \beta > 0, \text{ then } \omega(\tilde{a}, \alpha + \beta) \leq \omega(\tilde{a}, \alpha) + \omega(\tilde{a}, \beta). \quad (2.33)$$

$$\lim_{\theta \rightarrow 0} \omega(\tilde{a}, \theta) = 0 \quad (2.34)$$

Recall, if  $\tilde{a}$  and  $\tilde{b}$  are fuzzy numbers with  $[\tilde{a}]^\alpha = [a_1(\alpha), a_2(\alpha)]$  and  $[\tilde{b}]^\alpha = [b_1(\alpha), b_2(\alpha)]$  then

$$[\tilde{a} + \tilde{b}]^\alpha = [a_1(\alpha) + b_1(\alpha), a_2(\alpha) + b_2(\alpha)]. \quad (2.35)$$

**Lemma 2.14.2.** [151, 174] Let  $\lambda \neq 0, \mu \neq 0$  be real numbers and let  $\tilde{a}$  and  $\tilde{b}$  be fuzzy numbers. Then

$$\omega(\lambda\tilde{a}, \theta) = \omega\left(\tilde{a}, \frac{\theta}{|\lambda|}\right), \quad (2.36)$$

$$\omega(\lambda\tilde{a} + \lambda\tilde{b}, \theta) \leq \omega\left(\frac{\theta}{|\lambda| + |\mu|}\right), \quad (2.37)$$

where

$$\omega(\theta) := \max\{\omega(\tilde{a}, \theta), \omega(\tilde{b}, \theta)\}$$

for  $\theta \geq 0$ .

*Proof.* From the equation  $(\lambda\tilde{a})(t) = \tilde{a}(t/\lambda)$  for  $t \in \mathbb{R}$  we have

$$\begin{aligned} \omega(\lambda\tilde{a}, \theta) &= \max_{|u-v| \leq \theta} |(\lambda\tilde{a})(u) - (\lambda\tilde{a})(v)| = \max_{|u-v| \leq \theta} |\tilde{a}(u/\lambda) - \tilde{a}(v/\lambda)| = \\ &= \max_{|u/\lambda - v/\lambda| \leq \theta/|\lambda|} |\omega(u/\lambda) - \omega(v/\lambda)| = \omega(\tilde{a}, \theta/|\lambda|), \end{aligned}$$

which proves (2.36).

As to (2.37), let  $\theta > 0$  be arbitrary and  $u, t \in \mathbb{R}$  such that  $|u - t| \leq \theta$ . Then with the notations  $\tilde{c} := \lambda\tilde{a}, \tilde{d} := \mu\tilde{b}$  we need to show that

$$|(\tilde{c} + \tilde{d})(u) - (\tilde{c} + \tilde{d})(t)| \leq \omega\left(\frac{\theta}{|\lambda| + |\mu|}\right).$$

We assume without loss of generality that  $t < u$ . From (2.35) it follows that there are real numbers  $t_1, t_2, u_1, u_2$  with the properties

$$t = t_1 + t_2, \quad u = u_1 + u_2, \quad t_1 \leq u_1, \quad t_2 \leq u_2 \quad (2.38)$$

$$\tilde{c}(t_1) = \tilde{d}(t_2) = (\tilde{c} + \tilde{d})(t), \quad \tilde{c}(u_1) = \tilde{d}(u_2) = (\tilde{c} + \tilde{d})(u). \quad (2.39)$$

Since from (2.36) we have

$$|\tilde{c}(u_1) - \tilde{c}(t_1)| \leq \omega(\tilde{a}, |u_1 - t_1|/|\lambda|)$$

and

$$|\tilde{d}(u_2) - \tilde{d}(t_2)| \leq \omega(\tilde{b}, |u_2 - t_2|/|\mu|),$$

it follows by (2.32), (2.38) and (2.39) that

$$|(\tilde{c} + \tilde{d})(u) - (\tilde{c} + \tilde{d})(t)| \leq$$

$$\begin{aligned} \min\{\omega(\tilde{a}, |u_1 - t_1|/|\lambda|), \omega(\tilde{b}, |u_2 - t_2|/|\mu|)\} &\leq \\ \min\{\omega(|u_1 - t_1|/|\lambda|), \omega(|u_2 - t_2|/|\mu|)\} &\leq \end{aligned}$$

$$\omega\left(\frac{|u_1 - t_1| + |u_2 - t_2|}{|\lambda| + |\mu|}\right) = \omega\left(\frac{|u - t|}{|\lambda| + |\mu|}\right) \leq \omega\left(\frac{\theta}{|\lambda| + |\mu|}\right),$$

which proves the lemma.  $\square$

The following lemma can be proved directly by using the definition of  $\alpha$ -level sets.

**Lemma 2.14.3.** *Let  $\tilde{a} \in \mathcal{F}$  be a fuzzy number and. Then  $a_1: [0, 1] \rightarrow \mathbb{R}$  is strictly increasing and*

$$a_1(\tilde{a}(t)) \leq t,$$

*for  $t \in cl(\text{supp}\tilde{a})$ , furthermore  $\tilde{a}(a_1(\alpha)) = \alpha$ , for  $\alpha \in [0, 1]$  and*

$$a_1(\tilde{a}(t)) \leq t \leq a_1(\tilde{a}(t) + 0),$$

*for  $a_1(0) \leq t < a_1(1)$ , where*

$$a_1(\tilde{a}(t) + 0) = \lim_{\epsilon \rightarrow +0} a_1(\tilde{a}(t) + \epsilon). \quad (2.40)$$

**Lemma 2.14.4.** *Let  $\tilde{a}$  and  $\tilde{b}$  be fuzzy numbers. Then*

- (i)  $D(\tilde{a}, \tilde{b}) \geq |a_1(\alpha + 0) - b_1(\alpha + 0)|$ , for  $0 \leq \alpha < 1$ ,
- (ii)  $\tilde{a}(a_1(\alpha + 0)) = \alpha$ , for  $0 \leq \alpha < 1$ ,
- (iii)  $a_1(\alpha) \leq a_1(\alpha + 0) < a_1(\beta)$ , for  $0 \leq \alpha < \beta \leq 1$ .

*Proof.* (i) From the definition of the metric  $D$  we have

$$\begin{aligned} |a_1(\alpha + 0) - b_1(\alpha + 0)| &= \lim_{\epsilon \rightarrow +0} |a_1(\alpha + \epsilon) - b_1(\alpha + \epsilon)| = \\ \lim_{\epsilon \rightarrow +0} |a_1(\alpha + \epsilon) - b_1(\alpha + \epsilon)| &\leq \sup_{\gamma \in [0, 1]} |a_1(\gamma) - b_1(\gamma)| \leq D(\tilde{a}, \tilde{b}). \end{aligned}$$

(ii) Since  $\tilde{a}(a_1(\alpha + \epsilon)) = \alpha + \epsilon$ , for  $\epsilon \leq 1 - \alpha$ , we have

$$\tilde{a}(a_1(\alpha + 0)) = \lim_{\epsilon \rightarrow +0} \tilde{a}(a_1(\alpha + \epsilon)) = \lim_{\epsilon \rightarrow +0} (\alpha + \epsilon) = \alpha.$$

(iii) From strictly monotonicity of  $a_1$  it follows that  $a_1(\alpha + \epsilon) < a_1(\beta)$ , for  $\epsilon < \beta - \alpha$ . Therefore,

$$a_1(\alpha) \leq a_1(\alpha + 0) = \lim_{\epsilon \rightarrow +0} a_1(\alpha + \epsilon) < a_1(\beta),$$

which completes the proof.  $\square$

The following lemma shows that if all the  $\alpha$ -level sets of two (continuous) fuzzy numbers are close to each other, then there can be only a small deviation between their membership grades.

**Lemma 2.14.5.** Let  $\delta \geq 0$  and let  $\tilde{a}, \tilde{b}$  be fuzzy numbers. If  $D(\tilde{a}, \tilde{b}) \leq \delta$ , then

$$\sup_{t \in \mathbb{R}} |\tilde{a}(t) - \tilde{b}(t)| \leq \max\{\omega(\tilde{a}, \delta), \omega(\tilde{b}, \delta)\}. \quad (2.41)$$

*Proof.* Let  $t \in \mathbb{R}$  be arbitrarily fixed. It will be sufficient to show that

$$|\tilde{a}(t) - \tilde{b}(t)| \leq \max\{\omega(\tilde{a}, \delta), \omega(\tilde{b}, \delta)\}.$$

If  $t \notin \text{supp} \tilde{a} \cup \text{supp} \tilde{b}$  then we obtain (2.41) trivially. Suppose that  $t \in \text{supp} \tilde{a} \cup \text{supp} \tilde{b}$ . With no loss of generality we will assume  $0 \leq \tilde{b}(t) < \tilde{a}(t)$ . Then either of the following must occur:

- (a)  $t \in (b_1(0), b_1(1))$ ,
- (b)  $t \leq b_1(0)$ ,
- (c)  $t \in (b_2(1), b_2(0))$
- (d)  $t \geq b_2(0)$ .

In this case of (a) from Lemma 2.14.4 (with  $\alpha = \tilde{b}(t), \beta = \tilde{a}(t)$ ) and Lemma 2.14.3(iii) it follows that

$$\tilde{a}(a_1(\tilde{b}(t) + 0)) = \tilde{b}(t), \quad t \geq a_1(\tilde{a}(t)) \geq a_1(\tilde{b}(t) + 0)$$

and

$$D(\tilde{a}, \tilde{b}) \geq |a_1(\tilde{b}(t) + 0) - a_1(\tilde{b}(t) + 0)|.$$

Therefore from continuity of  $\tilde{a}$  we get

$$\begin{aligned} |\tilde{a}(t) - \tilde{b}(t)| &= |\tilde{a}(t) - \tilde{a}(a_1(\tilde{b}(t) + 0))| = \omega(\tilde{a}, |t - a_1(\tilde{b}(t) + 0)|) = \\ &= \omega(\tilde{a}, t - a_1(\tilde{b}(t) + 0)) \leq \omega(\tilde{a}, b_1(\tilde{b}(t) + 0) - a_1(\tilde{b}(t) + 0)) \leq \omega(\tilde{a}, \delta). \end{aligned}$$

In this case of (b) we have  $\tilde{b}(t) = 0$ ; therefore from Lemma 2.14.3(i) it follows that

$$\begin{aligned} |\tilde{a}(t) - \tilde{b}(t)| &= |\tilde{a}(t) - 0| = |\tilde{a}(t) - \tilde{a}(a_1(0))| \leq \omega(\tilde{a}, |t - a_1(0)|) \\ &\leq \omega(\tilde{a}, |b_1(0) - a_1(0)|) \leq \omega(\tilde{a}, \delta). \end{aligned}$$

A similar reasoning yields in the cases of (c) and (d); instead of properties  $a_1$  we use the properties of  $a_2$ .  $\square$

**Remark.** It should be noted that if  $\tilde{a}$  or  $\tilde{b}$  are discontinuous fuzzy quantities, then there can be a big deviation between their membership grades even if  $D(\tilde{a}, \tilde{b})$  is arbitrarily small.

Let  $L > 0$  be a real number. By  $\mathcal{F}(L)$  we denote the set of all fuzzy numbers  $\tilde{a} \in \mathcal{F}$  with membership function satisfying the Lipschitz condition with constant  $L$ , i.e.

$$|\tilde{a}(t) - \tilde{a}(t')| \leq L|t - t'|, \quad \forall t, t' \in \mathbb{R}.$$

In the following lemma (which is a direct consequence of Lemma 2.14.2 and Lemma 2.14.5) we see that (i) linear combinations of Lipschitzian fuzzy numbers are also Lipschitzian ones, and (ii) if all the  $\alpha$ -level sets of two Lipschitzian fuzzy numbers are closer to each other than  $\delta$ , then there can be maximum  $L\delta$  difference between their membership grades.

**Lemma 2.14.6.** Let  $L > 0$ ,  $\lambda \neq 0$ ,  $\mu \neq 0$  be real numbers and let  $\tilde{a}, \tilde{b} \in \mathcal{F}(L)$  be fuzzy numbers. Then

$$\lambda \tilde{a} \in \mathcal{F}\left(\frac{L}{|\lambda|}\right), \quad \lambda \tilde{a} + \mu \tilde{b} \in \mathcal{F}\left(\frac{L}{|\lambda| + |\mu|}\right).$$

Furthermore, if  $D(\tilde{a}, \tilde{b}) \leq \delta$ , then

$$\sup_t |\tilde{a}(t) - \tilde{b}(t)| \leq L\delta.$$

If the fuzzy  $\tilde{a}$  and  $\tilde{b}$  are of suymmetric triangular form then Lemma 2.14.6 reads

**Lemma 2.14.7.** Let  $\delta > 0$  be a real number and let  $\tilde{a} = (a, \alpha)$ ,  $\tilde{b} = (b, \beta)$  be symmetric triangular fuzzy numbers. Then

$$\lambda \tilde{a} \in \mathcal{F}\left[\frac{1}{\alpha|\lambda|}\right], \quad \lambda \tilde{a} + \mu \tilde{b} \in \mathcal{F}\left(\frac{\max\{1/\alpha, 1/\beta\}}{|\lambda| + |\mu|}\right).$$

Furthermore, from the inequality

$$D(\tilde{a}, \tilde{b}) \leq \delta,$$

it follows that

$$\sup_t |\tilde{a}(t) - \tilde{b}(t)| \leq \max\left\{\frac{\delta}{\alpha}, \frac{\delta}{\beta}\right\}.$$

## 2.15 Fuzzy implications

Let  $p = "x \text{ is in } A"$  and  $q = "y \text{ is in } B"$  be crisp propositions, where  $A$  and  $B$  are crisp sets for the moment. The implication  $p \rightarrow q$  is interpreted as  $\neg(p \wedge \neg q)$ . The full interpretation of the material implication  $p \rightarrow q$  is that the degree of truth of  $p \rightarrow q$  quantifies to what extend  $q$  is at least as true as  $p$ , i.e.

$$\tau(p \rightarrow q) = \begin{cases} 1 & \text{if } \tau(p) \leq \tau(q) \\ 0 & \text{otherwise} \end{cases}$$

where  $\tau(\cdot)$  denotes the truth value of a proposition.

**Example.** Let  $p = "x \text{ is bigger than } 10"$  and let  $q = "x \text{ is bigger than } 9"$ . It is easy to see that  $p \rightarrow q$  is true, because it can never happen that  $x$  is bigger than 10 and at the same time  $x$  is not bigger than 9.

Consider the implication statement: if "pressure is high" then "volume is small". The membership function of the fuzzy set  $A = "big \text{ pressure}"$  is defined by

$$A(u) = \begin{cases} 1 & \text{if } u \geq 5 \\ 1 - (5 - u)/4 & \text{if } 1 \leq u \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

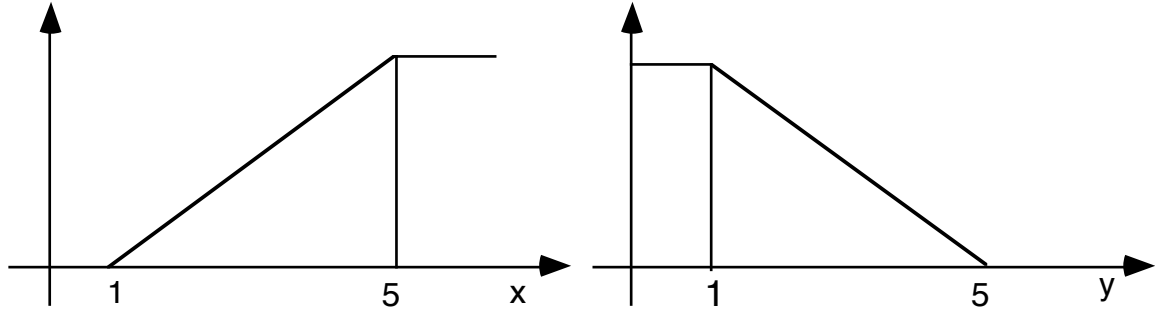


Figure 2.40: "x is big pressure" and "y is small volume".

The membership function of the fuzzy set  $B$ , *small volume* is given by

$$B(v) = \begin{cases} 1 & \text{if } v \leq 1 \\ 1 - (v - 1)/4 & \text{if } 1 \leq v \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

If  $p$  is a proposition of the form "x is  $A$ " where  $A$  is a fuzzy set, for example, "big pressure" and  $q$  is a proposition of the form "y is  $B$ " for example, "small volume" then one encounters the following problem: *How to define the membership function of the fuzzy implication  $A \rightarrow B$ ?* It is clear that  $(A \rightarrow B)(x, y)$  should be defined *pointwise* i.e.  $(A \rightarrow B)(x, y)$  should be a function of  $A(x)$  and  $B(y)$ . That is

$$(A \rightarrow B)(u, v) = I(A(u), B(v)).$$

We shall use the notation  $(A \rightarrow B)(u, v) = A(u) \rightarrow B(v)$ . In our interpretation  $A(u)$  is considered as the truth value of the proposition "u is big pressure", and  $B(v)$  is considered as the truth value of the proposition "v is small volume".

$$u \text{ is big pressure} \rightarrow v \text{ is small volume} \equiv A(u) \rightarrow B(v)$$

One possible extension of material implication to implications with intermediate truth values is

$$A(u) \rightarrow B(v) = \begin{cases} 1 & \text{if } A(u) \leq B(v) \\ 0 & \text{otherwise} \end{cases}$$

This implication operator is called *Standard Strict*.

$$\text{"4 is big pressure"} \rightarrow \text{"1 is small volume"} = A(4) \rightarrow B(1) = 0.75 \rightarrow 1 = 1$$

However, it is easy to see that this fuzzy implication operator is not appropriate for real-life applications. Namely, let  $A(u) = 0.8$  and  $B(v) = 0.8$ . Then we have

$$A(u) \rightarrow B(v) = 0.8 \rightarrow 0.8 = 1$$

Let us suppose that there is a small error of measurement or small rounding error of digital computation in the value of  $B(v)$ , and instead 0.8 we have to proceed with 0.7999. Then from the definition of Standard Strict implication operator it follows that

$$A(u) \rightarrow B(v) = 0.8 \rightarrow 0.7999 = 0$$

This example shows that small changes in the input can cause a big deviation in the output, i.e. our system is very sensitive to rounding errors of digital computation and small errors of measurement.

A smoother extension of material implication operator can be derived from the equation

$$X \rightarrow Y = \sup\{Z | X \cap Z \subset Y\}$$

where  $X, Y$  and  $Z$  are classical sets. Using the above principle we can define the following fuzzy implication operator

$$A(u) \rightarrow B(v) = \sup\{z | \min\{A(u), z\} \leq B(v)\}$$

that is,

$$A(u) \rightarrow B(v) = \begin{cases} 1 & \text{if } A(u) \leq B(v) \\ B(v) & \text{otherwise} \end{cases}$$

This operator is called *Gödel* implication. Using the definitions of negation and union of fuzzy subsets the material implication  $p \rightarrow q = \neg p \vee q$  can be extended by

$$A(u) \rightarrow B(v) = \max\{1 - A(u), B(v)\}$$

This operator is called *Kleene-Dienes* implication.

In many practical applications one uses Mamdani's implication operator to model causal relationship between fuzzy variables. This operator simply takes the minimum of truth values of fuzzy predicates

$$A(u) \rightarrow B(v) = \min\{A(u), B(v)\}$$

It is easy to see this is not a correct extension of material implications, because  $0 \rightarrow 0$  yields zero. However, in knowledge-based systems, we are usually not interested in rules, in which the antecedent part is false. There are three important classes of fuzzy implication operators:

- ***S-implications***: defined by

$$x \rightarrow y = S(n(x), y)$$

where  $S$  is a t-conorm and  $n$  is a negation on  $[0, 1]$ . These implications arise from the Boolean formalism  $p \rightarrow q = \neg p \vee q$ . Typical examples of *S-implications* are the Łukasiewicz and Kleene-Dienes implications.

- ***R-implications***: obtained by residuation of continuous t-norm  $T$ , i.e.

$$x \rightarrow y = \sup\{z \in [0, 1] | T(x, z) \leq y\}$$

These implications arise from the *Intuitionistic Logic* formalism. Typical examples of *R-implications* are the Gödel and Gaines implications.

- ***t-norm implications***: if  $T$  is a t-norm then

$$x \rightarrow y = T(x, y)$$

Although these implications do not verify the properties of material implication they are used as model of implication in many applications of fuzzy logic. Typical examples of t-norm implications are the Mamdani ( $x \rightarrow y = \min\{x, y\}$ ) and Larsen ( $x \rightarrow y = xy$ ) implications.



The most often used fuzzy implication operators are listed in the following table.

Name	Definition
Early Zadeh	$x \rightarrow y = \max\{1 - x, \min(x, y)\}$
Łukasiewicz	$x \rightarrow y = \min\{1, 1 - x + y\}$
Mamdani	$x \rightarrow y = \min\{x, y\}$
Larsen	$x \rightarrow y = xy$
Standard Strict	$x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$
Gödel	$x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$
Gaines	$x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise} \end{cases}$
Kleene-Dienes	$x \rightarrow y = \max\{1 - x, y\}$
Kleene-Dienes-Łukasiewicz	$x \rightarrow y = 1 - x + xy$
Yager	$x \rightarrow y = y^x$

**Table 1.** Fuzzy implication operators.

## 2.16 Linguistic variables

The use of fuzzy sets provides a basis for a systematic way for the manipulation of vague and imprecise concepts. In particular, we can employ fuzzy sets to represent linguistic variables. A linguistic variable can be regarded either as a variable whose value is a fuzzy number or as a variable whose values are defined in linguistic terms.

**Definition 2.16.1.** *A linguistic variable is characterized by a quintuple*

$$(x, T(x), U, G, M)$$

*in which  $x$  is the name of variable;  $T(x)$  is the term set of  $x$ , that is, the set of names of linguistic values of  $x$  with each value being a fuzzy number defined on  $U$ ;  $G$  is a syntactic rule for generating the names of values of  $x$ ; and  $M$  is a semantic rule for associating with each value its meaning.*

For example, if *speed* is interpreted as a linguistic variable, then its term set  $T$  (speed) could be

$$T = \{\text{slow, moderate, fast, very slow, more or less fast, slightly slow, } \dots\}$$

where each term in  $T$  (speed) is characterized by a fuzzy set in a universe of discourse  $U = [0, 100]$ . We might interpret

- *slow* as "a speed below about 40 mph"
- *moderate* as "a speed close to 55 mph"
- *fast* as "a speed above about 70 mph"

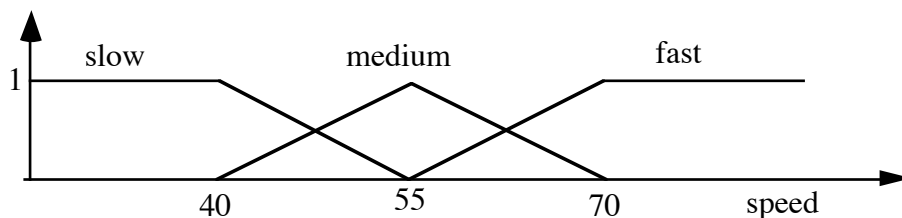


Figure 2.41: Values of linguistic variable *speed*.

These terms can be characterized as fuzzy sets whose membership functions are

$$\text{slow}(v) = \begin{cases} 1 & \text{if } v \leq 40 \\ 1 - (v - 40)/15 & \text{if } 40 \leq v \leq 55 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{moderate}(v) = \begin{cases} 1 - |v - 55|/30 & \text{if } 40 \leq v \leq 70 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{fast}(v) = \begin{cases} 1 & \text{if } v \geq 70 \\ 1 - \frac{70 - v}{15} & \text{if } 55 \leq v \leq 70 \\ 0 & \text{otherwise} \end{cases}$$

In many practical applications we normalize the domain of inputs and use the following type of fuzzy partition:

NVB	(Negative Very Big)
NB	(Negative Big),
NM	(Negative Medium),
NS	(Negative Small),
ZE	(Zero),
PS	(Positive Small),
PM	(Positive Medium),
PB	(Positive Big)
PVB	(Positive Very Big)

### 2.16.1 The linguistic variable *Truth*

Truth also can be interpreted as linguistic variable with a possible term set

$$T = \{\text{Absolutely false, Very false, False, Fairly true, True, Very true, Absolutely true}\}$$

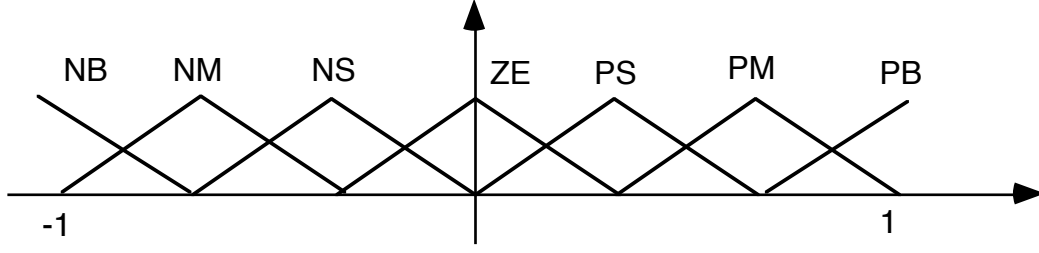


Figure 2.42: A usual fuzzy partition of  $[-1, 1]$ .

One may define the membership function of linguistic terms of truth as

$$\text{True}(u) = u, \quad \text{False}(u) = 1 - u$$

for each  $u \in [0, 1]$ , and

$$\text{Absolutely false}(u) = \begin{cases} 1 & \text{if } u = 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{Absolutely true}(u) = \begin{cases} 1 & \text{if } u = 1 \\ 0 & \text{otherwise} \end{cases}$$

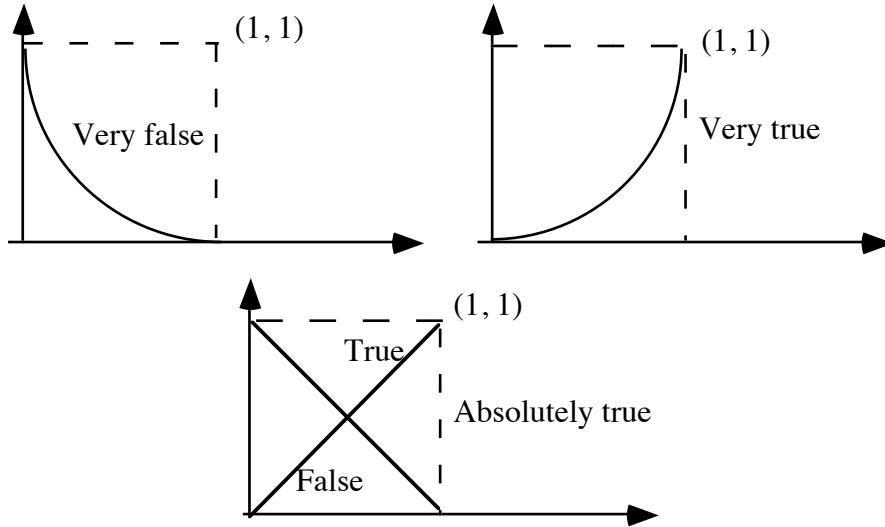


Figure 2.43: Some values of linguistic variable *Truth*.

The words "Fairly" and "Very" are interpreted as

$$\text{Fairly true}(u) = \sqrt{u}, \quad \text{Very true}(u) = u^2,$$

$$\text{Fairly false}(u) = \sqrt{1 - u}, \quad \text{Very false}(u) = (1 - u)^2$$

for each  $u \in [0, 1]$ . Suppose we have the fuzzy statement " $x$  is  $A$ ". Let  $\tau$  be a term of linguistic variable *Truth*. Then the statement " $x$  is  $A$  is  $\tau$ " is interpreted as " $x$  is  $\tau \circ A$ ". Where

$$(\tau \circ A)(u) = \tau(A(u))$$

for each  $u \in [0, 1]$ . For example, let  $\tau = \text{"true"}$ . Then " $x$  is  $A$  is true" is defined by " $x$  is  $\tau \circ A$ " = " $x$  is  $A$ " because

$$(\tau \circ A)(u) = \tau(A(u)) = A(u)$$

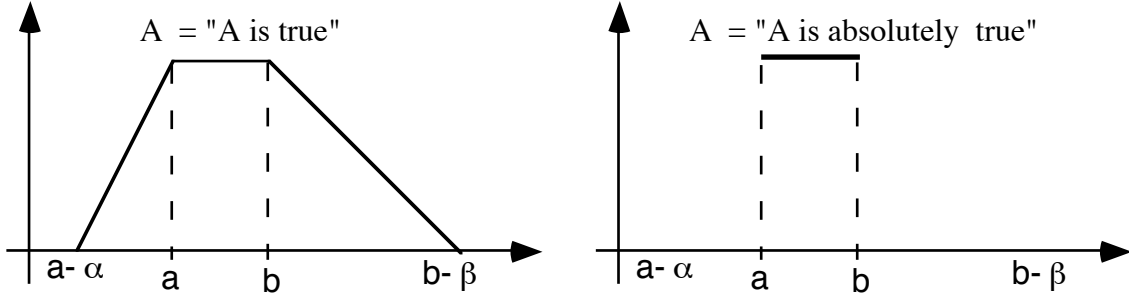


Figure 2.44: "A is true" and "A is absolutely true".

for each  $u \in [0, 1]$ . It is why "everything we write is considered to be true".

Let  $\tau$  = "absolutely true". Then the statement " $x$  is  $A$  is Absolutely true" is defined by " $x$  is  $\tau \circ A$ ", where

$$(\tau \circ A)(x) = \begin{cases} 1 & \text{if } A(x) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Let  $\tau$  = "fairly true". Then the statement " $x$  is  $A$  is Fairly true" is defined by " $x$  is  $\tau \circ A$ ",

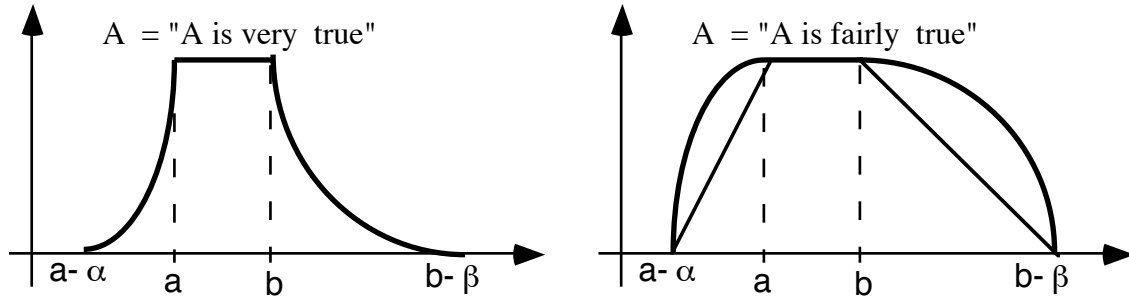


Figure 2.45: "A is very true" and "A is fairly true".

where

$$(\tau \circ A)(x) = \sqrt{A(x)}$$

Let  $\tau$  = "very true". Then the statement " $x$  is  $A$  is Fairly true" is defined by " $x$  is  $\tau \circ A$ ", where

$$(\tau \circ A)(x) = (A(x))^2$$

Let  $\tau$  = "false". Then the statement " $x$  is  $A$  is false" is defined by " $x$  is  $\tau \circ A$ ", where

$$(\tau \circ A)(x) = 1 - A(x)$$

Let  $\tau$  = "absolutely false". Then the statement " $x$  is  $A$  is Absolutely false" is defined by " $x$  is  $\tau \circ A$ ", where

$$(\tau \circ A)(x) = \begin{cases} 1 & \text{if } A(x) = 0 \\ 0 & \text{otherwise} \end{cases}$$

We will use the following parametrized standard fuzzy partition of the unit interval.

Suppose that  $U = [0, 1]$  and  $T(x)$  consists of  $K + 1$ ,  $K \geq 2$ , terms,

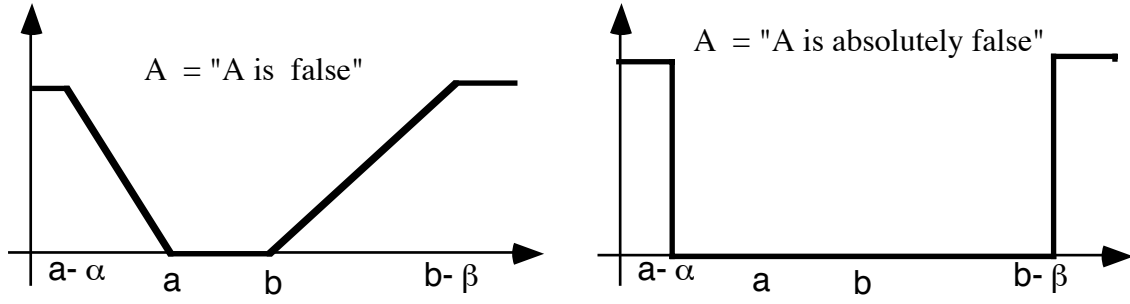


Figure 2.46: "A is false" and "A is absolutely false".

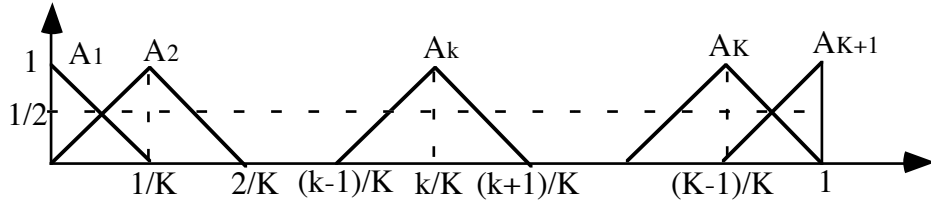


Figure 2.47: Standard fuzzy partition of  $[0, 1]$  with  $(K + 1)$  terms.

$$\mathcal{T} = \{\text{small}_1, \text{around } 1/K, \text{around } 2/K, \dots, \text{around } (K-1)/K, \text{big}_K\}$$

which are represented by triangular membership functions  $\{A_1, \dots, A_{K+1}\}$  of the form

$$A_1(u) = [\text{small}_1](u) = \begin{cases} 1 - Ku & \text{if } 0 \leq u \leq 1/K \\ 0 & \text{otherwise} \end{cases} \quad (2.42)$$

$$A_k(u) = [\text{around } k/K](u) = \begin{cases} Ku - k + 1 & \text{if } (k-1)/K \leq u \leq k/K \\ k + 1 - Ku & \text{if } k/K \leq u \leq (k+1)/K \\ 0 & \text{otherwise} \end{cases} \quad (2.43)$$

for  $1 \leq k \leq (K - 1)$ , and

$$A_{K+1}(u) = [\text{big}_K](u) = \begin{cases} Ku - K + 1 & \text{if } (K-1)/K \leq u \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (2.44)$$

If  $K = 1$  then the fuzzy partition for the  $[0, 1]$  interval consists of two linguistic terms  $\{\text{small}, \text{big}\}$  which are defined by

$$\text{small}(t) = 1 - t, \quad \text{big}(t) = t, \quad t \in [0, 1]. \quad (2.45)$$

Suppose that  $U = [0, 1]$  and  $\mathcal{T}(x)$  consists of  $2K + 1$ ,  $K \geq 2$ , terms,

$$\mathcal{T} = \{\text{small}_1, \dots, \text{small}_K = \text{small}, \text{big}_0 = \text{big}, \text{big}_1, \dots, \text{big}_K\}$$

which are represented by triangular membership functions as

$$\text{small}_k(u) = \begin{cases} 1 - \frac{K}{k}u & \text{if } 0 \leq u \leq k/K \\ 0 & \text{otherwise} \end{cases} \quad (2.46)$$

for  $k \leq k \leq K$ ,

$$\text{big}_k(u) = \begin{cases} \frac{u - k/K}{1 - k/K} & \text{if } k/K \leq u \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (2.47)$$

for  $0 \leq k \leq K - 1$ .

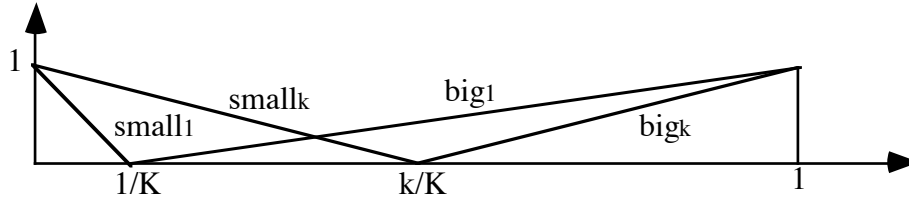


Figure 2.48: Fuzzy partition of  $[0,1]$  with monotone membership functions.

# Chapter 3

## Fuzzy Multicriteria Decision Making

### 3.1 Averaging operators

Fuzzy set theory provides a host of attractive aggregation connectives for integrating membership values representing uncertain information. These connectives can be categorized into the following three classes *union*, *intersection* and *compensation* connectives.

Union produces a high output whenever any one of the input values representing degrees of satisfaction of different features or criteria is high. Intersection connectives produce a high output only when all of the inputs have high values. Compensative connectives have the property that a higher degree of satisfaction of one of the criteria can compensate for a lower degree of satisfaction of another criteria to a certain extent. In the sense, union connectives provide full compensation and intersection connectives provide no compensation. In a decision process the idea of *trade-offs* corresponds to viewing the global evaluation of an action as lying between the *worst* and the *best* local ratings. This occurs in the presence of conflicting goals, when a compensation between the corresponding compatibilities is allowed. Averaging operators realize trade-offs between objectives, by allowing a positive compensation between ratings.

**Definition 3.1.1.** An averaging operator  $M$  is a function  $M: [0, 1] \times [0, 1] \rightarrow [0, 1]$ , satisfying the following properties

- *Idempotency*

$$M(x, x) = x, \quad \forall x \in [0, 1],$$

- *Commutativity*

$$M(x, y) = M(y, x), \quad \forall x, y \in [0, 1],$$

- *Extremal conditions*

$$M(0, 0) = 0, \quad M(1, 1) = 1$$

- *Monotonicity*

$$M(x, y) \leq M(x', y') \text{ if } x \leq x' \text{ and } y \leq y',$$

- *$M$  is continuous.*

Averaging operators represent a wide class of aggregation operators. We prove that whatever is the particular definition of an averaging operator,  $M$ , the global evaluation of an action will lie between the *worst* and the *best* local ratings:

**Lemma 3.1.1.** *If  $M$  is an averaging operator then*

$$\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}, \forall x, y \in [0, 1]$$

*Proof.* From idempotency and monotonicity of  $M$  it follows that

$$\min\{x, y\} = M(\min\{x, y\}, \min\{x, y\}) \leq M(x, y)$$

and  $M(x, y) \leq M(\max\{x, y\}, \max\{x, y\}) = \max\{x, y\}$ . Which ends the proof.  $\square$

Averaging operators have the following interesting properties [123]:

**Property.** *A strictly increasing averaging operator cannot be associative.*

**Property.** *The only associative averaging operators are defined by*

$$M(x, y, \alpha) = \text{med}(x, y, \alpha) = \begin{cases} y & \text{if } x \leq y \leq \alpha \\ \alpha & \text{if } x \leq \alpha \leq y \\ x & \text{if } \alpha \leq x \leq y \end{cases}$$

where  $\alpha \in (0, 1)$ .

An important family of averaging operators is formed by quasi-arithmetic means

$$M(a_1, \dots, a_n) = f^{-1}\left(\frac{1}{n} \sum_{i=1}^n f(a_i)\right)$$

This family has been characterized by Kolmogorov as being the class of all decomposable continuous averaging operators. For example, the quasi-arithmetic mean of  $a_1$  and  $a_2$  is defined by

$$M(a_1, a_2) = f^{-1}\left(\frac{f(a_1) + f(a_2)}{2}\right).$$

The next table shows the most often used mean operators.

Name	$M(x, y)$
harmonic mean	$2xy/(x + y)$
geometric mean	$\sqrt{xy}$
arithmetic mean	$(x + y)/2$
dual of geometric mean	$1 - \sqrt{(1 - x)(1 - y)}$
dual of harmonic mean	$(x + y - 2xy)/(2 - x - y)$
median	$\text{med}(x, y, \alpha), \alpha \in (0, 1)$
generalized $p$ -mean	$((x^p + y^p)/2)^{1/p}, p \geq 1$

**Table 2.** Mean operators.



The process of information aggregation appears in many applications related to the development of intelligent systems. One sees aggregation in neural networks, fuzzy logic controllers, vision systems, expert systems and multi-criteria decision aids. In [418] Yager introduced a new aggregation technique based on the ordered weighted averaging (OWA) operators.

**Definition 3.1.2.** An OWA operator of dimension  $n$  is a mapping  $F: \mathbb{R}^n \rightarrow \mathbb{R}$ , that has an associated weighting vector  $W = (w_1, w_2, \dots, w_n)^T$  such as  $w_i \in [0, 1]$ ,  $1 \leq i \leq n$ , and

$$w_1 + \dots + w_n = 1.$$

Furthermore

$$F(a_1, \dots, a_n) = w_1 b_1 + \dots + w_n b_n = \sum_{j=1}^n w_j b_j$$

where  $b_j$  is the  $j$ -th largest element of the bag  $\langle a_1, \dots, a_n \rangle$ .

**Example.** Assume  $W = (0.4, 0.3, 0.2, 0.1)^T$  then

$$F(0.7, 1, 0.2, 0.6) = 0.4 \times 1 + 0.3 \times 0.7 + 0.2 \times 0.6 + 0.1 \times 0.2 = 0.75.$$

A fundamental aspect of this operator is the re-ordering step, in particular an aggregate  $a_i$  is not associated with a particular weight  $w_i$  but rather a weight is associated with a particular ordered position of aggregate. When we view the OWA weights as a column vector we shall find it convenient to refer to the weights with the low indices as weights at the top and those with the higher indices with weights at the bottom.

It is noted that different OWA operators are distinguished by their weighting function. In [418] Yager pointed out three important special cases of OWA aggregations:

- $F^*$ : In this case  $W = W^* = (1, 0 \dots, 0)^T$  and

$$F^*(a_1, \dots, a_n) = \max\{a_1, \dots, a_n\},$$

- $F_*$ : In this case  $W = W_* = (0, 0 \dots, 1)^T$  and

$$F_*(a_1, \dots, a_n) = \min\{a_1, \dots, a_n\},$$

- $F_A$ : In this case  $W = W_A = (1/n, \dots, 1/n)^T$  and

$$F_A(a_1, \dots, a_n) = \frac{a_1 + \dots + a_n}{n}.$$

A number of important properties can be associated with the OWA operators. We shall now discuss some of these. For any OWA operator  $F$  holds

$$F_*(a_1, \dots, a_n) \leq F(a_1, \dots, a_n) \leq F^*(a_1, \dots, a_n).$$

Thus the upper and lower star OWA operator are its boundaries. From the above it becomes clear that for any  $F$

$$\min\{a_1, \dots, a_n\} \leq F(a_1, \dots, a_n) \leq \max\{a_1, \dots, a_n\}.$$

The OWA operator can be seen to be *commutative*. Let  $\langle a_1, \dots, a_n \rangle$  be a bag of aggregates and let  $\{d_1, \dots, d_n\}$  be any *permutation* of the  $a_i$ . Then for any OWA operator

$$F(a_1, \dots, a_n) = F(d_1, \dots, d_n).$$

A third characteristic associated with these operators is *monotonicity*. Assume  $a_i$  and  $c_i$  are a collection of aggregates,  $i = 1, \dots, n$  such that for each  $i$ ,  $a_i \geq c_i$ . Then

$$F(a_1, \dots, a_n) \geq F(c_1, c_2, \dots, c_n)$$

where  $F$  is some fixed weight OWA operator.

Another characteristic associated with these operators is *idempotency*. If  $a_i = a$  for all  $i$  then for any OWA operator

$$F(a_1, \dots, a_n) = a.$$

From the above we can see the OWA operators have the basic properties associated with an *averaging operator*.

**Example.** A window type OWA operator takes the average of the  $m$  arguments around the center. For this class of operators we have

$$w_i = \begin{cases} 0 & \text{if } i < k \\ \frac{1}{m} & \text{if } k \leq i < k+m \\ 0 & \text{if } i \geq k+m \end{cases} \quad (3.1)$$

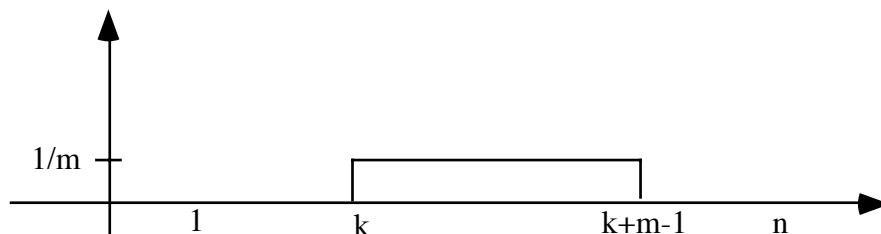


Figure 3.1: Window type OWA operator.

In order to classify OWA operators in regard to their location between *and* and *or*, a measure of *orness*, associated with any vector  $W$  is introduced by Yager [418] as follows

$$\text{orness}(W) = \frac{1}{n-1} \sum_{i=1}^n (n-i)w_i.$$

It is easy to see that for any  $W$  the  $\text{orness}(W)$  is always in the unit interval. Furthermore, note that the nearer  $W$  is to an *or*, the closer its measure is to one; while the nearer it is to an *and*, the closer is to zero.

**Lemma 3.1.2.** Let us consider the the vectors

$$W^* = (1, 0, \dots, 0)^T, \quad W_* = (0, 0, \dots, 1)^T, \quad W_A = (1/n, \dots, 1/n)^T.$$

Then it can easily be shown that  $\text{orness}(W^*) = 1$ ,  $\text{orness}(W_*) = 0$  and  $\text{orness}(W_A) = 0.5$ .

A measure of *andness* is defined as

$$\text{andness}(W) = 1 - \text{orness}(W).$$

Generally, an OWA operator with much of nonzero weights near the top will be an *orlike* operator, that is,

$$\text{orness}(W) \geq 0.5$$

and when much of the weights are nonzero near the bottom, the OWA operator will be *andlike*, that is,

$$\text{andness}(W) \geq 0.5.$$

**Example.** Let  $W = (0.8, 0.2, 0.0)^T$ . Then

$$\text{orness}(W) = \frac{1}{3}(2 \times 0.8 + 0.2) = 0.6,$$

and

$$\text{andness}(W) = 1 - \text{orness}(W) = 1 - 0.6 = 0.4.$$

This means that the OWA operator, defined by

$$F(a_1, a_2, a_3) = 0.8b_1 + 0.2b_2 + 0.0b_3 = 0.8b_1 + 0.2b_2,$$

where  $b_j$  is the  $j$ -th largest element of the bag  $\langle a_1, a_2, a_3 \rangle$ , is an *orlike* aggregation.

The following theorem shows that as we move weight up the vector we increase the orness, while moving weight down causes us to decrease *orness*( $W$ ).

**Theorem 3.1.1.** [419] Assume  $W$  and  $W'$  are two  $n$ -dimensional OWA vectors such that

$$W = (w_1, \dots, w_n)^T,$$

and

$$W' = (w_1, \dots, w_j + \epsilon, \dots, w_k - \epsilon, \dots, w_n)^T$$

where  $\epsilon > 0$ ,  $j < k$ . Then  $\text{orness}(W') > \text{orness}(W)$ .

*Proof.* From the definition of the measure of *orness* we get

$$\begin{aligned} \text{orness}(W') &= \frac{1}{n-1} \sum_i (n-i)w'_i = \\ &= \frac{1}{n-1} \sum_i (n-1)w_i + (n-j)\epsilon - (n-k)\epsilon, \\ \text{orness}(W') &= \text{orness}(W) + \frac{1}{n-1}\epsilon(k-j). \end{aligned}$$

Since  $k > j$ ,  $\text{orness}(W') > \text{orness}(W)$ . □

In [418] Yager defined the measure of dispersion (or entropy) of an OWA vector by

$$disp(W) = - \sum_i w_i \ln w_i.$$

We can see when using the OWA operator as an averaging operator  $Disp(W)$  measures the degree to which we use all the aggregates equally. If  $F$  is an OWA aggregation with weights  $w_i$  the dual of  $F$  denoted  $\hat{F}$ , is an OWA aggregation of the same dimension where with weights  $\hat{w}_i$

$$\hat{w}_i = w_{n-i+1}.$$

We can easily see that if  $F$  and  $\hat{F}$  are duals then

$$disp(\hat{F}) = disp(F), \quad orness(\hat{F}) = 1 - orness(F) = andness(F)$$

Thus if  $F$  is orlike its dual is andlike.

**Example.** Let  $W = (0.3, 0.2, 0.1, 0.4)^T$ . Then  $\hat{W} = (0.4, 0.1, 0.2, 0.3)^T$  and

$$orness(F) = 1/3(3 \times 0.3 + 2 \times 0.2 + 0.1) \approx 0.466,$$

$$orness(\hat{F}) = 1/3(3 \times 0.4 + 2 \times 0.1 + 0.2) \approx 0.533.$$

An important application of the OWA operators is in the area of quantifier guided aggregations [418]. Assume

$$\{A_1, \dots, A_n\},$$

is a collection of criteria. Let  $x$  be an object such that for any criterion  $A_i$ ,  $A_i(x) \in [0, 1]$  indicates the degree to which this criterion is satisfied by  $x$ . If we want to find out the degree to which  $x$  satisfies "all the criteria" denoting this by  $D(x)$ , we get following Bellman and Zadeh [10]:

$$D(x) = \min\{A_1(x), \dots, A_n(x)\}. \quad (3.2)$$

In this case we are essentially requiring  $x$  to satisfy " $A_1$  and  $A_2$  and  $\dots$  and  $A_n$ ".

If we desire to find out the degree to which  $x$  satisfies "at least one of the criteria", denoting this  $E(x)$ , we get

$$E(x) = \max\{A_1(x), \dots, A_n(x)\}.$$

In this case we are requiring  $x$  to satisfy " $A_1$  or  $A_2$  or  $\dots$  or  $A_n$ ".

In many applications rather than desiring that a solution satisfies one of these extreme situations, "all" or "at least one", we may require that  $x$  satisfies *most* or *at least half* of the criteria. Drawing upon Zadeh's concept [444] of linguistic quantifiers we can accomplish these kinds of quantifier guided aggregations.

**Definition 3.1.3.** A quantifier  $Q$  is called

- regular monotonically non-decreasing if

$$Q(0) = 0, \quad Q(1) = 1, \quad \text{if } r_1 > r_2 \text{ then } Q(r_1) \geq Q(r_2).$$

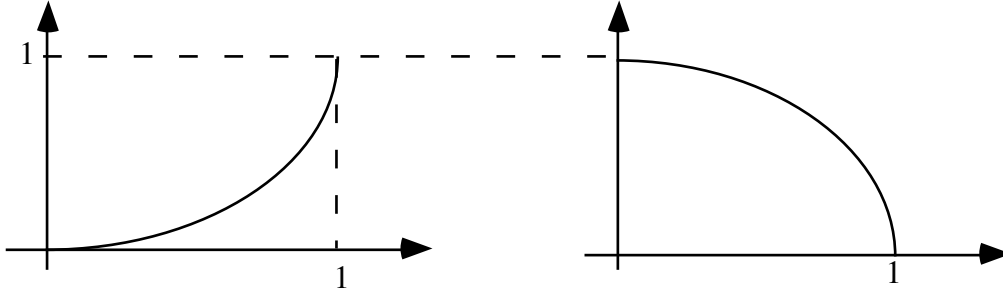


Figure 3.2: Monoton linguistic quantifiers.

- regular monotonically non-increasing *if*

$$Q(0) = 1, \quad Q(1) = 0, \quad \text{if } r_1 < r_2 \text{ then } Q(r_1) \geq Q(r_2).$$

- regular unimodal *if*

$$Q(r) = \begin{cases} 0 & \text{if } r = 0 \\ \text{monotone increasing} & \text{if } 0 \leq r \leq a \\ 1 & \text{if } a \leq r \leq b, 0 < a < b < 1 \\ \text{monotone decreasing} & \text{if } b \leq r \leq 1 \\ 0 & \text{if } r = 1 \end{cases}$$

With  $a_i = A_i(x)$  the overall valuation of  $x$  is  $F_Q(a_1, \dots, a_n)$  where  $F_Q$  is an OWA operator. The weights associated with this quantified guided aggregation are obtained as follows

$$w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right), \quad i = 1, \dots, n. \quad (3.3)$$

The next figure graphically shows the operation involved in determining the OWA weights directly from the quantifier guiding the aggregation.

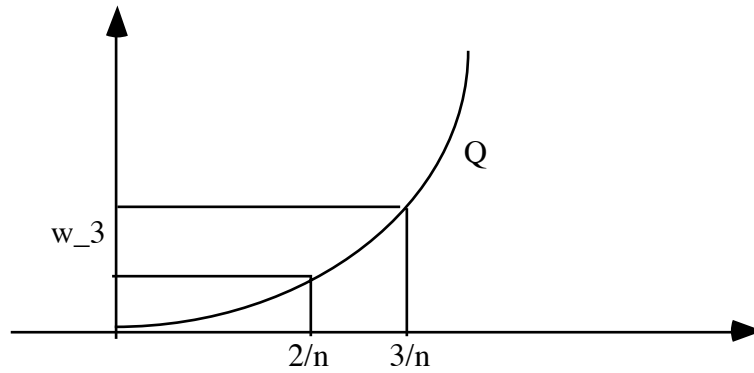


Figure 3.3: Determining weights from a quantifier.

**Theorem 3.1.2.** [419] If we construct  $w_i$  via the method (3.3) we always get  $\sum w_i = 1$  and  $w_i \in [0, 1]$  for any function

$$Q: [0, 1] \rightarrow [0, 1],$$

satisfying the conditions of a regular nondecreasing quantifier.

*Proof.* We first see that from the non-decreasing property  $Q(i/n) \geq Q((i-1)/n)$  hence  $w_i \geq 0$  and since  $Q(r) \leq 1$  then  $w_i \leq 1$ . Furthermore we see

$$\sum_{i=1}^n w_i = \sum_{i=1}^n \left[ Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right) \right] = Q\left[\frac{n}{n}\right] - Q\left[\frac{0}{n}\right] = 1 - 0 = 1.$$

Which proves the theorem. □

We call any function satisfying the conditions of a regular non-decreasing quantifier an *acceptable OWA weight generating function*. Let us look at the weights generated from some basic types of quantifiers. The quantifier, for all  $Q_*$ , is defined such that

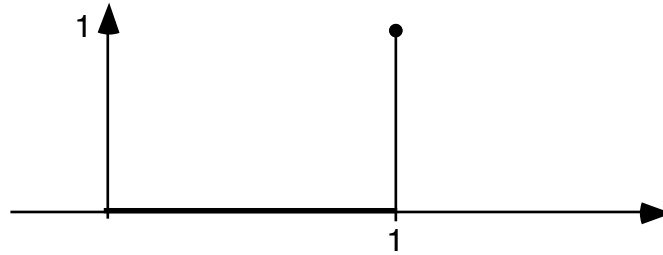


Figure 3.4: The quantifier *all*.

$$Q_*(r) = \begin{cases} 0 & \text{for } r < 1, \\ 1 & \text{for } r = 1. \end{cases}$$

Using our method for generating weights  $w_i = Q_*(i/n) - Q_*((i-1)/n)$  we get

$$w_i = \begin{cases} 0 & \text{for } i < n, \\ 1 & \text{for } i = n. \end{cases}$$

This is exactly what we previously denoted as  $W_*$ . For the quantifier *there exists* we have

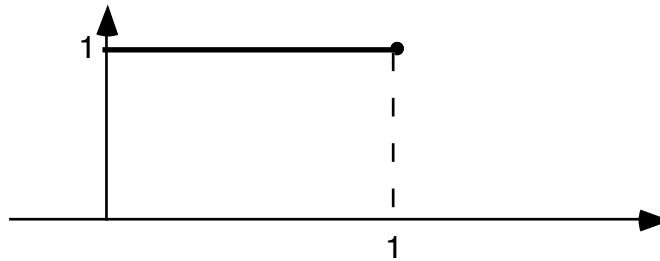


Figure 3.5: The quantifier *there exists*.

$$Q^*(r) = \begin{cases} 0 & \text{for } r = 0, \\ 1 & \text{for } r > 0. \end{cases}$$

In this case we get

$$w_1 = 1, \quad w_i = 0, \text{ for } i \neq 1.$$

This is exactly what we denoted as  $W^*$ . Consider next the quantifier defined by  $Q(r) = r$ .

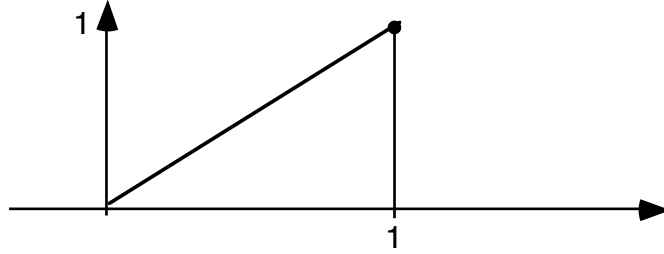


Figure 3.6: The identity quantifier.

This is an *identity* or *linear type* quantifier. In this case we get

$$w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right) = \frac{i}{n} - \frac{i-1}{n} = \frac{1}{n}.$$

This gives us the pure averaging OWA aggregation operator. Recapitulating using the approach suggested by Yager if we desire to calculate

$$F_Q(a_1, \dots, a_n)$$

for  $Q$  being a regular non-decreasing quantifier we proceed as follows:

- Calculate

$$w_i = Q\left[\frac{i}{n}\right] - Q\left[\frac{i-1}{n}\right],$$

- Calculate

$$F_Q(a_i, \dots, a_n) = w_1 b_1 + \dots + w_n b_n.$$

where  $b_i$  is the  $i$ -th largest of the  $a_j$ . For example, the weights of the window-type OWA

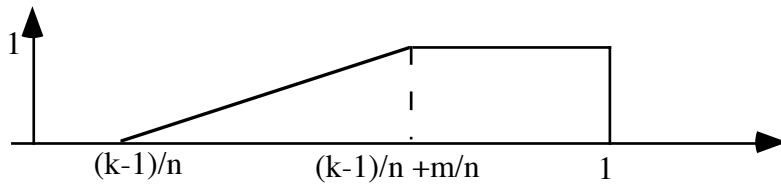


Figure 3.7: Quantifier for a window-type OWA operator.

operator given by equation (3.1) can be derived from the quantifier

$$Q(r) = \begin{cases} 0 & \text{if } r \leq (k-1)/n \\ 1 - \frac{(k-1+m) - nr}{m} & \text{if } (k-1)/n \leq r \leq (k-1+m)/n \\ 1 & \text{if } (k-1+m)/n \leq r \leq 1 \end{cases}$$

## 3.2 OWA Operators for Ph.D. student selection

Following Carlsson and Fullér [70, 72] we illustrate the applicability of OWA operators to a doctoral student selection problem at the Graduate School of Turku Centre for Computer Science (TUCS).

TUCS offers a programme for gaining the Doctoral (PhD) degree in Computer Science and Information Systems. It is open for students from everywhere. The teaching language of the school is English. Prerequisites are either a Master's or a Bachelor's degree in Computer Science or in a closely related field. Study time is expected to be 4 years when starting from Master's level and 6 years from Bachelor's level.

The Graduate School offers advanced level courses in Computer Science and supervision of students within existing research projects. The main areas of research are *Algorithmics*, *Discrete Mathematics*, *Embedded Systems*, *Information Systems*, *Software Engineering*. Students are expected to take courses from at least two of these areas. Each student is assigned a supervisor from one of the fields indicated above.

The Graduate School is open for applications. There are no specific application forms. Applicants to TUCS graduate school should write a letter to the Director of TUCS. The letter should contain a formal application to the school, together with the following enclosures:

- Curriculum vitae
- Financing plan for studies
- Application for financial support, if requested
- Two letters of recommendation with referees' full contact addresses
- Official copy of examinations earned with official English translation
- Certificate of knowledge of English
- Short description of research interests

As certificate of knowledge of English, TOEFL test (minimum 550 points) or corresponding knowledge in English is required for applicants outside Finland.

Since the number of applicants (usually between 20 and 40) is much greater than the number of available scholarships (around 6) we have to rank the candidates based on their performances. It can also happen that only a part of available scholarships will be awarded, because the number of *good* candidates is smaller than the number of available places.

The problem of selecting *young promising doctoral researchers* can be seen to consist of three components. The first component is a collection

$$X = \{x_1, \dots, x_p\}$$

of applicants for the Ph.D. program.

The second component is a collection of 6 criteria (see Table 3) which are considered relevant in the ranking process.



<b>Research interests</b>	(excellent)	(average)	(weak)
- Fit in research groups	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
- On the frontier of research	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
- Contributions	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
<b>Academic background</b>			
- University	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
- Grade average	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
- Time for acquiring degree	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
Letters of recommendation	<input type="checkbox"/> Y	<input type="checkbox"/> N	
Knowledge of English	<input type="checkbox"/> Y	<input type="checkbox"/> N	

**Table 3** Evaluation sheet.

For simplicity we suppose that all applicants are *young* and have Master's degree acquired more than one year before. In this case all the criteria are meaningful, and are of approximately the same importance.

For applicants with Bachelor's degree the first three criteria *Fit in research groups*, *Contributions* and *On the frontier of research* are meaningless, because we have an undergraduate student without any research record. An applicant with Bachelor's degree or *just acquired* Master's degree should have excellent university record from a good university to be competitive.

For *old* applicants we encounter the problem of trade-offs between the age and the research record, and in this case their ratings on the last three criteria *University*, *Grade average* and *Time for acquiring degree* do not really matter. An *old* applicant should have a very good research record and a history of scientific cooperation with a TUCS research group to be competitive.

The third component is a group of 11 experts whose opinions are solicited in ranking the alternatives. The experts are selected from the following 9 research groups

- Algorithmics Group
- Coding Theory Group
- Computational Intelligence for Business

- Information Systems Research group
- Institute for Advanced Management Systems Research
- Probabilistic Algorithms and Software Quality
- Programming Methodology Group
- Strategic Information Systems Planning
- Theory Group: Mathematical Structures in Computer Science

So we have a **Multi Expert-Multi Criteria Decision Making** (ME-MCDM) problem. The ranking system described in the following is a two stage process. In the first stage, individual experts are asked to provide an evaluation of the alternatives. This evaluation consists of a rating for each alternative on each of the criteria, where the ratings are chosen from the scale  $\{1, 2, 3\}$ , where 3 stands for *excellent*, 2 stands for *average* and 1 means *weak* performance. Each expert provides a 6-tuple

$$(a_1, \dots, a_6)$$

for each applicant, where  $a_i \in \{1, 2, 3\}$ ,  $i = 1, \dots, 6$ . The next step in the process is to find the overall evaluation for an alternative by a given expert.

In the second stage we aggregate the individual experts evaluations to obtain an overall value for each applicant.

In [418] Yager suggested an approach to the aggregation of criteria satisfactions guided by a regular non-decreasing quantifier  $Q$ . If  $Q$  is Regular Increasing Monotone (RIM) quantifier then we measure the overall success of the alternative  $x = (a_1, \dots, a_n)$  by

$$F_Q(a_1, \dots, a_n)$$

where  $F_Q$  is an OWA operator derived from  $Q$ , i.e. the weights associated with this quantified guided aggregation are obtained as follows

$$w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right)$$

for  $i = 1, \dots, n$ . The standard degree of orness associated with a (RIM) linguistic quantifier  $Q$

$$\text{orness}(Q) = \int_0^1 Q(r) dr$$

is equal to the area under the quantifier [424]. This definition for the measure of orness of quantifier provides a simple useful method for obtaining this measure.

Consider the family of RIM quantifiers

$$Q_\alpha(r) = r^\alpha, \alpha \geq 0. \quad (3.4)$$

It is clear that

$$\text{orness}(Q_\alpha) = \int_0^1 r^\alpha dr = \frac{1}{\alpha + 1}$$

and  $\text{orness}(Q_\alpha) < 0.5$  for  $\alpha > 1$ ,  $\text{orness}(Q_\alpha) = 0.5$  for  $\alpha = 1$  and  $\text{orness}(Q_\alpha) > 0.5$  for  $\alpha < 1$ .

Taking into consideration that we have 6 criteria (see Table 3) the weights derived from  $Q_\alpha$  are determined as

$$\begin{aligned} w_1 &= \left[\frac{1}{6}\right]^\alpha - 0, \\ w_2 &= \left[\frac{2}{6}\right]^\alpha - \left[\frac{1}{6}\right]^\alpha, \\ w_3 &= \left[\frac{3}{6}\right]^\alpha - \left[\frac{2}{6}\right]^\alpha, \\ w_4 &= \left[\frac{4}{6}\right]^\alpha - \left[\frac{3}{6}\right]^\alpha, \\ w_5 &= \left[\frac{5}{6}\right]^\alpha - \left[\frac{4}{6}\right]^\alpha, \\ w_6 &= 1 - \left[\frac{5}{6}\right]^\alpha \end{aligned}$$

Furthermore, whatever is the linguistic quantifier,  $Q_\alpha$ , representing the statement *most criteria are satisfied by x*, we see that

$$1 \leq F_\alpha(a_1, \dots, a_6) \leq 3$$

holds for each alternative  $x = (a_1, \dots, a_6)$  since  $a_i \in \{1, 2, 3\}$ ,  $i = 1, \dots, 6$ .

We search for an index  $\alpha \geq 0$  such that the associated linguistic quantifier  $Q_\alpha$  from the family (3.4) approximates the experts' preferences as much as possible. After interviewing the experts we found that all of them agreed on the following principles

- (i) if an applicant has more than two weak performances then his overall performance should be less than two,
- (ii) if an applicant has maximum two weak performances then his overall performance should be more than two,
- (iii) if an applicant has all but one excellent performances then his overall performance should be about 2.75,
- (iv) if an applicant has three weak performances and one of them is on the criterion *on the frontier of research* then his overall performance should not be above 1.5,

From (i) we get

$$F_\alpha(3, 3, 3, 1, 1, 1) = 3 \times (w_1 + w_2 + w_3) + w_4 + w_5 + w_6 < 2,$$

that is,

$$3 \times \left[\frac{3}{6}\right]^\alpha + 1 - \left[\frac{3}{6}\right]^\alpha < 2 \iff \left[\frac{1}{2}\right]^\alpha < \left[\frac{1}{2}\right] \iff \alpha > 1,$$

and from (ii) we obtain

$$F_\alpha(3, 3, 3, 2, 1, 1) = 3 \times (w_1 + w_2 + w_3) + 2 \times w_4 + w_5 + w_6 > 2$$

that is,

$$3 \times \left[\frac{3}{6}\right]^\alpha + 2 \times \left(\left[\frac{4}{6}\right]^\alpha - \left[\frac{3}{6}\right]^\alpha\right) + 1 - \left[\frac{4}{6}\right]^\alpha > 2 \iff \left[\frac{1}{2}\right]^\alpha + \left[\frac{2}{3}\right]^\alpha > 1$$

which holds if  $\alpha < 1.293$ . So from (i) and (ii) we get

$$1 < \alpha \leq 1.293,$$

which means that  $Q_\alpha$  should be *andlike* (or risk averse) quantifier with a degree of compensation just below the arithmetic average.

It is easy to verify that (iii) and (iv) can not be satisfied by any quantifier  $Q_\alpha$ ,  $1 < \alpha \leq 1.293$ , from the family (3.4). In fact, (iii) requires that  $\alpha \approx 0.732$  which is smaller than 1 and (iv) can be satisfied if  $\alpha \geq 2$  which is bigger than 1.293. Rules (iii) and (iv) have priority whenever they are applicable.

In the second stage the technique for combining the expert's evaluation to obtain an overall evaluation for each alternative is based upon the OWA operators. Each applicant is represented by an 11-tuple

$$(b_1, \dots, b_{11})$$

where  $b_i \in [1, 3]$  is the unit score derived from the  $i$ -th expert's ratings. We suppose that the  $b_i$ 's are organized in descending order, i.e.  $b_i$  can be seen as the worst of the  $i$ -th top scores.

Taking into consideration that the experts are selected from 9 different research groups there exists no applicant that scores overall well on the first criterion "Fit in research group". After a series of negotiations all experts agreed that the support of at least four experts is needed for qualification of the applicant.

Since we have 11 experts, applicants are evaluated based on their top four scores

$$(b_1, \dots, b_4)$$

and if at least three experts agree that the applicant is excellent then his final score should be 2.75 which is a cut-off value for the best student. That is

$$F_\alpha(3, 3, 3, 1) = 3 \times (w_1 + w_2 + w_3) + w_4 = 2.75,$$

that is,

$$3 \times \left[\frac{3}{4}\right]^\alpha + 1 - \left[\frac{3}{4}\right]^\alpha = 2.75 \iff \left[\frac{3}{4}\right]^\alpha = 0.875 \iff \alpha \approx 0.464$$

So in the second stage we should choose an *orlike* OWA operator with  $\alpha \approx 0.464$  for aggregating the top six scores of the applicant to find the final score.

If the final score is less than 2 then the applicant is disqualified and if the final score is at least 2.5 then the scholarship should be awarded to him. If the final score is between 2 and 2.5 then the scholarship can be awarded to the applicant pending on the total number of scholarships available.

We have presented a two stage process for doctoral student selection problem. In the first stage we have used an *andlike* OWA operator to implement some basic rules derived from certain (extremal) situations. In the second stage we have applied an *orlike* OWA operator, because the final score of applicants should be high if at least three experts find his record attractive (we do not require support from *all experts*).

It can happen (and it really happened) that some experts (a minority) forms a coalition and deliberately *overrate* some candidates in order to qualify them even though the majority of experts finds these candidates overall weak. We can resolve this problem by adding an extra criterion to the set of criteria measuring the competency of individual experts, or we issue an alarm message about the attempted cheating.

To determine the most appropriate linguistic quantifier in the first stage we can also try to *identify* interdependences between criteria [57, 58, 63].

### 3.2.1 Example

Let us choose  $\alpha = 1.2$  for the aggregation of the ratings in the first stage. Consider some applicant with the following scores (after re-ordering the scores in descending order):

	<i>Unit score</i>						
Expert 1	3	3	3	2	2	1	2.239
Expert 2	3	3	3	2	2	2	2.435
Expert 3	3	2	2	2	2	1	1.920
Expert 4	3	3	3	3	2	2	2.615
Expert 5	3	3	2	2	2	1	2.071
Expert 6	3	3	3	2	2	1	2.239
Expert 7	3	3	2	2	2	1	2.071
Expert 8	3	3	2	2	1	1	1.882
Expert 9	3	2	2	2	2	1	1.920
Expert 10	3	3	2	2	1	1	1.882
Expert 11	2	2	2	2	1	1	1.615

The weights associated with this linguistic quantifier are

$$(0.116, 0.151, 0.168, 0.180, 0.189, 0.196)$$

In the second stage we choose  $\alpha = 0.464$  and obtain the following weights

$$(0.526, 0.199, 0.150, 0.125).$$

The best four scores of the applicant are

$$(2.615, 2.435, 2.239, 2.239).$$

The final score is computed as

$$F_{\alpha}(2.615, 2.435, 2.239, 2.239) = 2.475.$$

So the applicant has good chances to get the scholarship.

### 3.3 Possibility and necessity in weighted aggregation

Yager [422] discussed the issue of weighted min and max aggregations and provided for a formalization of the process of importance weighted transformation.

Following Carlsson and Fullér [60, 69] we introduce fuzzy implication operators for importance weighted transformation. It should be noted that this new class of transfer functions contains as a subset those ones introduced by Yager in 1994. First we provide the definitions of terms needed in the process of weighted aggregation. Recall the three important classes of fuzzy implication operators:

- ***S*-implications:** defined by

$$x \rightarrow y = S(n(x), y) \quad (3.5)$$

where  $S$  is a t-conorm and  $n$  is a negation on  $[0, 1]$ . We shall use the following  $S$ -implications:  $x \rightarrow y = \min\{1 - x + y, 1\}$  (Łukasiewicz) and  $x \rightarrow y = \max\{1 - x, y\}$  (Kleene-Dienes).

- ***R*-implications:** obtained by residuation of continuous t-norm  $T$ , i.e.

$$x \rightarrow y = \sup\{z \in [0, 1] \mid T(x, z) \leq y\}$$

We shall use the following  $R$ -implication:  $x \rightarrow y = 1$  if  $x \leq y$  and  $x \rightarrow y = y$  if  $x > y$  (Gödel),  $x \rightarrow y = \min\{1 - x + y, 1\}$  (Łukasiewicz)

- **t-norm implications:** if  $T$  is a t-norm then

$$x \rightarrow y = T(x, y)$$

We shall use the minimum-norm as t-norm implication (Mamdani).

Let  $A$  and  $B$  be two fuzzy predicates defined on the real line  $\mathbb{R}$ . Knowing that ' $X$  is  $B$ ' is true, the degree of possibility that the proposition ' $X$  is  $A$ ' is true, denoted by  $\text{Pos}[A|B]$  or  $\text{Pos}[A = B]$ , is given by

$$\text{Pos}[A|B] = \sup_{t \in \mathbb{R}} A(t) \wedge B(t), \quad (3.6)$$

the degree of necessity that the proposition ' $X$  is  $A$ ' is true,  $N[A|B]$ , is given by

$$\text{Nes}[A|B] = 1 - \text{Pos}[\neg A|B],$$

where  $A$  and  $B$  are the possibility distributions defined by the predicates  $A$  and  $B$ , respectively, and

$$(\neg A)(t) = 1 - A(t)$$

for any  $t$ . We can use any t-norm  $T$  in (3.6) to model the logical connective *and*:

$$\text{Pos}[A|B] = \sup\{T(A(t), B(t)) \mid t \in \mathbb{R}\}, \quad (3.7)$$

Then for the measure of necessity of  $A$ , given  $B$  we get

$$\text{Nes}[A|B] = 1 - \text{Pos}[\neg A|B] = 1 - \sup_t T(1 - A(t), B(t)).$$

Let  $S$  be a t-conorm derived from t-norm  $T$ , then

$$1 - \sup_t T(1 - A(t), B(t)) = \inf_t \{1 - T(1 - A(t), B(t))\} =$$

$$\inf_t \{S(1 - B(t), A(t))\} = \inf_t \{B(t) \rightarrow A(t)\},$$

where the implication operator is defined in the sense of (3.5). That is,

$$\text{Nes}[A|B] = \inf_t \{B(t) \rightarrow A(t)\}.$$

Let  $A$  and  $W$  be discrete fuzzy sets in the unit interval, such that

$$A = a_1/(1/n) + a_2/(2/n) + \cdots + a_n/1,$$

and

$$W = w_1/(1/n) + w_2/(2/n) + \cdots + w_n/1,$$

where  $n > 1$ , and the terms  $a_j/(j/n)$  and  $w_j/(j/n)$  signify that  $a_j$  and  $w_j$  are the grades of membership of  $j/n$  in  $A$  and  $W$ , respectively, i.e.

$$A(j/n) = a_j, \quad W(j/n) = w_j$$

for  $j = 1, \dots, n$ , and the plus sign represents the union. Then we get the following simple formula for the measure of necessity of  $A$ , given  $W$

$$\text{Nes}[A|W] = \min_{j=1, \dots, n} \{W(j/n) \rightarrow A(j/n)\} = \min_{j=1, \dots, n} \{w_j \rightarrow a_j\} \quad (3.8)$$

and we use the notation

$$\text{Nes}[A|W] = N[(a_1, a_2, \dots, a_n)|(w_1, w_2, \dots, w_n)]$$

A classical MADM problem can be expressed in a matrix format. A decision matrix is an  $m \times n$  matrix whose element  $x_{ij}$  indicates the performance rating of the  $i$ -th alternative,  $x_i$ , with respect to the  $j$ -th attribute,  $c_j$ :

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix}$$

In fuzzy case the values of the decision matrix are given as degrees of "how an alternative satisfies a certain attribute". Let  $x$  be an alternative such that for any criterion  $C_j(x) \in [0, 1]$  indicates the degree to which this criterion is satisfied by  $x$ . So, in fuzzy case we have the following decision matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

where  $a_{ij} = C_j(x_{ij})$ , for  $i = 1, \dots, m$  and  $j = 1, \dots, n$ . Let  $x$  be an alternative and let

$$(a_1, a_2, \dots, a_n)$$

denote the degrees to which  $x$  satisfies the criteria, i.e.

$$a_j = C_j(x), \quad i = 1, \dots, n.$$

In many applications of fuzzy sets as multi-criteria decision making, pattern recognition, diagnosis and fuzzy logic control one faces the problem of weighted aggregation. The issue of weighted aggregation has been studied by Carlsson and Fullér [60], Dubois and Prade [123, 124, 126], Fodor and Roubens [160] and Yager [414, 418, 419, 420, 421, 422, 423, 424].

Assume associated with each fuzzy set  $C_j$  is a weight  $w_j \in [0, 1]$  indicating its importance in the aggregation procedure,  $j = 1, \dots, n$ . The general process for the inclusion of importance in the aggregation involves the transformation of the fuzzy sets under the importance. Let **Agg** indicate an aggregation operator, max or min, to find the weighted aggregation. Yager [422] first transforms each of the membership grades using the weights

$$g(w_i, a_i) = \hat{a}_i,$$

for  $i = 1, \dots, n$ , and then obtain the weighted aggregate

$$\mathbf{Agg}(\hat{a}_1, \dots, \hat{a}_n).$$

The form of  $g$  depends upon the type of aggregation being performed, the operation **Agg**.

As discussed by Yager in incorporating the effect of the importances in the min operation we are interested in reducing the effect of the elements which have low importance. In performing the min aggregation it is the elements with low values that play the most significant role in this type of aggregation, one way to reduce the effect of elements with low importance is to transform them into big values, values closer to one. Yager introduced a class of functions which can be used for the inclusion of importances in the min aggregation

$$g(w_i, a_i) = S(1 - w_i, a_i)$$

where  $S$  is a t-conorm, and then obtain the weighted aggregate

$$\min\{\hat{a}_1, \dots, \hat{a}_n\} = \min\{S(1 - w_1, a_1), \dots, S(1 - w_n, a_n)\} \quad (3.9)$$

We first note that if  $w_i = 0$  then from the basic property of t-conorms it follows that

$$S(1 - w_i, a_i) = S(1, w_i) = 1$$

Thus, zero importance gives us one. Yager notes that the formula can be seen as a measure of the degree to which an alternative satisfies the following proposition:

*All important criteria are satisfied*

**Example.** Let

$$(0.3, 0.2, 0.7, 0.6)$$



be the vector of weights and let

$$(0.4, 0.6, 0.6, 0.4)$$

be the vector of aggregates. If  $g(w_i, a_i) = \max\{1 - w_i, a_i\}$  then we get

$$g(w_1, a_1) = (1 - 0.3) \vee 0.4 = 0.7,$$

$$g(w_2, a_2) = (1 - 0.2) \vee 0.6 = 0.8$$

$$g(w_3, a_3) = (1 - 0.7) \vee 0.6 = 0.6,$$

$$g(w_4, a_4) = (1 - 0.6) \vee 0.4 = 0.4$$

That is

$$\min\{g(w_1, a_1), \dots, g(w_4, a_4)\} = \min\{0.7, 0.8, 0.6, 0.4\} = 0.4$$

As for the max aggregation operator: Since it is the large values that play the most important role in the aggregation we desire to transform the low importance elements into small values and thus have them not play a significant role in the max aggregation. Yager suggested a class of functions which can be used for importance transformation in max aggregation

$$g(w_i, a_i) = T(w_i, a_i)$$

where  $T$  is a t-norm. We see that if  $w_i = 0$  then  $T(w_i, a_i) = 0$  and the element plays no role in the max.

Let **Agg** indicate any aggregation operator and let

$$(a_1, a_2, \dots, a_n)$$

denote the vector of aggregates. We define the weighted aggregation as

$$\mathbf{Agg} \langle g(w_1, a_1), \dots, g(w_n, a_n) \rangle.$$

where the function  $g$  satisfies the following properties

- if  $a > b$  then  $g(w, a) \geq g(w, b)$
- $g(w, a)$  is monotone in  $w$
- $g(0, a) = id$ ,  $g(1, a) = a$

where the identity element,  $id$ , is such that if we add it to our aggregates it doesn't change the aggregated value. Let us recall formula (3.8)

$$\text{Nes}[(a_1, a_2, \dots, a_n)|(w_1, w_2, \dots, w_n)] = \min\{w_1 \rightarrow a_1, \dots, w_n \rightarrow a_n\} \quad (3.10)$$

where

$$A = a_1/(1/n) + a_2/(2/n) + \dots + a_n/1$$

is the fuzzy set of performances and

$$W = w_1/(1/n) + w_2/(2/n) + \dots + w_n/1$$

is the fuzzy set of weights; and the formula for weighted aggregation by the minimum operator

$$\min\{\hat{a}_1, \dots, \hat{a}_n\}$$

where

$$\hat{a}_i = g(w_i, a_i) = S(1 - w_i, a_i)$$

and  $S$  is a t-conorm.

It is easy to see that if the implication operator in (3.10) is an  $S$ -implication then from the equality

$$w_j \rightarrow a_j = S(1 - w_j, a_j)$$

it follows that the weighted aggregation of the  $a_i$ 's is nothing else, but

$$\text{Nes}[(a_1, a_2, \dots, a_n)|(w_1, w_2, \dots, w_n)],$$

the necessity of performances, given weights.

This observation leads us to a new class of transfer functions introduced by Carlsson and Fullér [60, 69] (which contains Yager's functions as a subset):

$$\hat{a}_i = g(w_i, a_i) = w_i \rightarrow a_i \quad (3.11)$$

where  $\rightarrow$  is an arbitrary implication operator. Then we combine the  $\hat{a}_i$ 's with an appropriate aggregation operator **Agg**.

However, we first select the implication operator, and then the aggregation operator **Agg** to combine the  $\hat{a}_i$ 's. If we choose a t-norm implication in (3.11) then we will select the max operator, and if we choose an  $R$ - or  $S$ -implication then we will select the min operator to aggregate the  $\hat{a}_i$ 's.

It should be noted that if we choose an  $R$ -implication in (3.11) then the equation

$$\min\{w_1 \rightarrow a_1, \dots, w_n \rightarrow a_n\} = 1$$

holds iff  $w_i \leq a_i$  for all  $i$ , i.e. when each performance rating is at least as big as its associated weight. In other words, if a performance rating with respect to an attribute exceeds the value of the weight of this attribute then this rating does not matter in the overall rating. However, ratings which are well below of the corresponding weights play a significant role in the overall rating. Thus the formula (3.10) with an  $R$ -implication can be seen as a measure of the degree to which an alternative satisfies the following proposition:

*All scores are bigger than or equal to the importances*

It should be noted that the min aggregation operator does not allow any compensation, i.e. a higher degree of satisfaction of one of the criteria can not compensate for a lower degree of satisfaction of another criteria. Averaging operators realize *trade-offs* between objectives, by allowing a positive compensation between ratings.

Another possibility is to use an *andlike* or an *orlike* OWA-operator to aggregate the elements of the bag

$$\langle w_1 \rightarrow a_1, \dots, w_n \rightarrow a_n \rangle.$$

Let  $A$  and  $W$  be discrete fuzzy sets in  $[0, 1]$ , where  $A(t)$  denotes the performance rating and  $W(t)$  denotes the weight of a criterion labeled by  $t$ . Then the weighted aggregation of  $A$  can be defined by,

- a t-norm-based measure of necessity of  $A$ , given  $W$ :

$$\text{Nes}[A|W] = \min_t \{W(t) \rightarrow A(t)\}$$

For example, the Kleene-Dienes implication operator,

$$w_i \rightarrow a_i = \max\{1 - w_i, a_i\},$$

implements Yager's approach to fuzzy screening [419].

- a t-norm-based measure of possibility of  $A$ , given  $W$ :

$$\text{Pos}[A|W] = \max_t \{T(A(t), W(t))\}$$

- an OWA-operator defined on the bag

$$\langle W(t) \rightarrow A(t) \mid t \rangle$$

Other possibility is to take the value

$$\frac{\int_0^1 A(t) \wedge W(t) dt}{\int_0^1 W(t) dt}$$

for the overall score of  $A$ . If  $A(t) \geq W(t)$  for all  $t \in [0, 1]$  then the overall score of  $A$  is equal to one. However, the bigger the set

$$\{t \in [0, 1] \mid A(t) \leq W(t)\}$$

the smaller the overall rating of  $A$ .

### 3.3.1 Example

Let  $(0.3, 0.2, 0.7, 0.6)$  be the vector of weights and let  $(0.4, 0.6, 0.6, 0.4)$  be the vector of aggregates. If

$$g(w_i, a_i) = \min\{1, 1 - w_i + a_i\}$$

is the Łukasiewicz implication then we compute

$$\begin{aligned} g(w_1, a_1) &= 0.3 \rightarrow 0.4 = 1, \\ g(w_2, a_2) &= 0.2 \rightarrow 0.6 = 1, \\ g(w_3, a_3) &= 0.7 \rightarrow 0.6 = 0.9, \\ g(w_4, a_4) &= 0.6 \rightarrow 0.4 = 0.8. \end{aligned}$$

That is

$$\min\{g(w_1, a_1), \dots, g(w_4, a_4)\} = \min\{1, 1, 0.9, 0.8\} = 0.8$$

If  $g(w_i, a_i)$  is implemented by the Gödel implication then we get

$$\begin{aligned} g(w_1, a_1) &= 0.3 \rightarrow 0.4 = 1, \\ g(w_2, a_2) &= 0.2 \rightarrow 0.6 = 1, \\ g(w_3, a_3) &= 0.7 \rightarrow 0.6 = 0.6, \\ g(w_4, a_4) &= 0.6 \rightarrow 0.4 = 0.4. \end{aligned}$$

That is

$$\min\{g(w_1, a_1), \dots, g(w_4, a_4)\} = \min\{1, 1, 0.6, 0.4\} = 0.4$$

If  $g(w_i, a_i) = w_i a_i$  is the Larsen implication then we have

$$g(w_1, a_1) = 0.3 \times 0.4 = 0.12,$$

$$g(w_2, a_2) = 0.2 \times 0.6 = 0.12,$$

$$g(w_3, a_3) = 0.7 \times 0.6 = 0.42,$$

$$g(w_4, a_4) = 0.6 \times 0.4 = 0.24.$$

That is

$$\max\{g(w_1, a_1), \dots, g(w_4, a_4)\} = \max\{0.12, 0.12, 0.42, 0.24\} = 0.42$$

Generalizing Yager's principles for weighted min and max aggregations we introduced fuzzy implication operators as a means for importance weighted transformation. Weighted aggregations are important in decision problems where we have multiple attributes to consider and where the outcome is to be judged in terms of attributes which are not equally important for the decision maker. The importance is underscored if there is a group of decision makers with varying value judgments on the attributes and/or if this group has factions promoting some subset of attributes.

### 3.4 Benchmarking in linguistic importance weighted aggregations

In this Section we concentrate on the issue of weighted aggregations and provide a possibilistic approach to the process of importance weighted transformation when both the importances (interpreted as *benchmarks*) and the ratings are given by symmetric triangular fuzzy numbers. Following Carlsson and Fullér [75, 76] we will show that using the possibilistic approach

- (i) small changes in the membership function of the importances can cause only small variations in the weighted aggregate;
- (ii) the weighted aggregate of fuzzy ratings remains stable under small changes in the *nonfuzzy* importances;
- (iii) the weighted aggregate of crisp ratings still remains stable under small changes in the crisp importances whenever we use a continuous implication operator for the importance weighted transformation.

In many applications of fuzzy sets such as multi-criteria decision making, pattern recognition, diagnosis and fuzzy logic control one faces the problem of weighted aggregation. The issue of weighted aggregation has been studied extensively by Carlsson and Fullér [57, 69, 70], Delgado et al [110], Dubois and Prade [123, 124, 126], Fodor and Roubens [160], Herrera et al [222, 223, 226, 227, 228] and Yager [414, 418, 419, 420, 421, 422, 423, 424, 427, 429].

Unlike Herrera and Herrera-Viedma [227] who perform direct computation on a finite and totally ordered term set, we use the membership functions to aggregate the values of the linguistic variables *rate* and *importance*. The main problem with finite term sets is that the impact of small changes in the weighting vector can be disproportionately large on the weighted aggregate (because the set of possible output values is finite, but the set of possible weight vectors is a subset of  $\mathbb{R}^n$ ). For example, the *rounding* operator in the *convex combination of linguistic labels*, defined by Delgado et al. [110], is very sensitive to the values around 0.5 ( $\text{round}(0.499) = 0$  and  $\text{round}(0.501) = 1$ ).

Following Carlsson and Fullér [76] we consider the process of importance weighted aggregation when both the aggregates and the importances are given by an infinite term set, namely by the values of the linguistic variables "rate" and "importance". In this approach the importances are considered as benchmark levels for the performances, i.e. an alternative performs well on all criteria if the degree of satisfaction to each of the criteria is at least as big as the associated benchmark.

The proposed "stable" method in [76] ranks the alternatives by measuring the degree to which they satisfy the proposition:

"All ratings are larger than or equal to their importance".

We will also use OWA operators to measure the degree to which an alternative satisfies the proposition:

"Most ratings are larger than or equal to their importance",

where the OWA weights are derived from a well-chosen linguistic quantifier.

Recall that a fuzzy set  $A$  is called a symmetric triangular fuzzy number with center  $a$  and width  $\alpha > 0$  if its membership function has the following form

$$A(t) = \begin{cases} 1 - \frac{|a - t|}{\alpha} & \text{if } |a - t| \leq \alpha \\ 0 & \text{otherwise} \end{cases}$$

and we use the notation  $A = (a, \alpha)$ . If  $\alpha = 0$  then  $A$  collapses to the characteristic function of  $\{a\} \subset \mathbb{R}$  and we will use the notation  $A = \bar{a}$ .

We will use symmetric triangular fuzzy numbers to represent the values of linguistic variables [440] *rate* and *importance* in the universe of discourse  $I = [0, 1]$ . The set of all symmetric triangular fuzzy numbers in the unit interval will be denoted by  $\mathcal{F}(I)$ .

Let  $A = (a, \alpha)$  and  $B = (b, \beta)$ . The degree of possibility that the proposition "A is less than or equal to B" is true, denoted by  $\text{Pos}[A \leq B]$ , is defined by (2.20) and computed by

$$\text{Pos}[A \leq B] = \begin{cases} 1 & \text{if } a \leq b \\ 1 - \frac{a - b}{\alpha + \beta} & \text{if } 0 < a - b < \alpha + \beta \\ 0 & \text{otherwise} \end{cases} \quad (3.12)$$

Let  $A$  be an alternative with ratings  $(A_1, A_2, \dots, A_n)$ , where  $A_i = (a_i, \alpha_i) \in \mathcal{F}(I)$ ,  $i = 1, \dots, n$ . For example, the symmetric triangular fuzzy number  $A_j = (0.8, \alpha)$  when  $0 < \alpha \leq 0.2$  can represent the property

”the rating on the  $j$ -th criterion is around 0.8”

and if  $\alpha = 0$  then  $A_j = (0.8, \alpha)$  is interpreted as

”the rating on the  $j$ -th criterion is equal to 0.8”

and finally, the value of  $\alpha$  can not be bigger than 0.2 because the domain of  $A_j$  is the unit interval.

Assume that associated with each criterion is a weight  $W_i = (w_i, \gamma_i)$  indicating its importance in the aggregation procedure,  $i = 1, \dots, n$ . For example, the symmetric triangular fuzzy number  $W_j = (0.5, \gamma) \in \mathcal{F}(I)$  when  $0 < \gamma \leq 0.5$  can represent the property

”the importance of the  $j$ -th criterion is approximately 0.5”

and if  $\gamma = 0$  then  $W_j = (0.5, \gamma)$  is interpreted as

”the importance of the  $j$ -th criterion is equal to 0.5”

and finally, the value of  $\gamma$  can not be bigger than 0.5 because the domain of  $W_j$  is the unit interval.

The general process for the inclusion of importance in the aggregation involves the transformation of the ratings under the importance. Following Carlsson and Fullér [76] we suggest the use of the transformation function

$$g: \mathcal{F}(I) \times \mathcal{F}(I) \rightarrow [0, 1],$$

where,

$$g(W_i, A_i) = \text{Pos}[W_i \leq A_i],$$

for  $i = 1, \dots, n$ , and then obtain the weighted aggregate,

$$\phi(A, W) = \mathbf{Agg}(\text{Pos}[W_1 \leq A_1], \dots, \text{Pos}[W_n \leq A_n]). \quad (3.13)$$

where **Agg** denotes an aggregation operator.

For example if we use the min function for the aggregation in (3.13), that is,

$$\phi(A, W) = \min\{\text{Pos}[W_1 \leq A_1], \dots, \text{Pos}[W_n \leq A_n]\} \quad (3.14)$$

then the equality

$$\phi(A, W) = 1$$

holds iff  $w_i \leq a_i$  for all  $i$ , i.e. when the mean value of each performance rating is at least as large as the mean value of its associated weight. In other words, if a performance rating with respect to a criterion exceeds the importance of this criterion with possibility one, then this rating does not matter in the overall rating. However, ratings which are well below the corresponding importances (in possibilistic sense) play a significant role in the overall rating. In this sense the importance can be considered as *benchmark* or *reference level* for the performance. Thus, formula (3.13) with the min operator can be seen as a measure of the degree to which an alternative satisfies the following proposition:

”All ratings are larger than or equal to their importance”.

It should be noted that the min aggregation operator does not allow any compensation, i.e. a higher degree of satisfaction of one of the criteria can not compensate for a lower degree of satisfaction of another criterion.

Averaging operators realize *trade-offs* between criteria, by allowing a positive compensation between ratings. We can use an *andlike* or an *orlike* OWA-operator [420] to aggregate the elements of the bag

$$\langle \text{Pos}[W_1 \leq A_1], \dots, \text{Pos}[W_n \leq A_n] \rangle.$$

In this case (3.13) becomes,

$$\phi(A, W) = \mathbf{OWA} \langle \text{Pos}[W_1 \leq A_1], \dots, \text{Pos}[W_n \leq A_n] \rangle,$$

where **OWA** denotes an Ordered Weighted Averaging Operator. Formula (3.13) does not make any difference among alternatives whose performance ratings exceed the value of their importance with respect to all criteria with possibility one: the overall rating will always be equal to one. Penalizing ratings that are ”larger than the associated importance, but not large enough” (that is, their intersection is not empty) we can modify formula (3.13) to measure the degree to which an alternative satisfies the following proposition:

”All ratings are essentially larger than their importance”.

In this case the transformation function can be defined as

$$g(W_i, A_i) = \text{Nes}[W_i \leq A_i] = 1 - \text{Pos}[W_i > A_i],$$

for  $i = 1, \dots, n$ , and then obtain the weighted aggregate,

$$\phi(A, W) = \min\{\text{Nes}[W_1 \leq A_1], \dots, \text{Nes}[W_n \leq A_n]\}. \quad (3.15)$$

If we do allow a positive compensation between ratings then we can use OWA-operators in (3.15). That is,

$$\phi(A, W) = \mathbf{OWA} \langle \text{Nes}[W_1 \leq A_1], \dots, \text{Nes}[W_n \leq A_n] \rangle.$$

The following theorem shows that if we choose the min operator for **Agg** in (3.13) then small changes in the membership functions of the weights can cause only a small change in the weighted aggregate, i.e. the weighted aggregate depends continuously on the weights.

**Theorem 3.4.1.** [76] Let  $A_i = (a_i, \alpha) \in \mathcal{F}(I)$ ,  $\alpha_i > 0$ ,  $i = 1, \dots, n$  and let  $\delta > 0$  such that

$$\delta < \alpha := \min\{\alpha_1, \dots, \alpha_n\}$$

If  $W_i = (w_i, \gamma_i)$  and  $W_i^\delta = (w_i^\delta, \gamma^\delta) \in \mathcal{F}(I)$ ,  $i = 1, \dots, n$ , satisfy the relationship

$$\max_i D(W_i, W_i^\delta) \leq \delta \quad (3.16)$$

then the following inequality holds,

$$|\phi(A, W) - \phi(A, W^\delta)| \leq \frac{\delta}{\alpha} \quad (3.17)$$

where  $\phi(A, W)$  is defined by (3.14) and

$$\phi(A, W^\delta) = \min\{\text{Pos}[W_1^\delta \leq A_1], \dots, \text{Pos}[W_n^\delta \leq A_n]\}.$$

*Proof.* It is sufficient to show that

$$|\text{Pos}[W_i \leq A_i] - \text{Pos}[W_i^\delta \leq A_i]| \leq \frac{\delta}{\alpha} \quad (3.18)$$

for  $1 \leq i \leq n$ , because (3.17) follows from (3.18). Using the representation (2.20) we need to show that

$$|\sup_{t \leq 0} (W_i - A_i)(t) - \sup_{t \leq 0} (W_i^\delta - A_i)(t)| \leq \frac{\delta}{\alpha}.$$

Using (3.16) and applying Lemma 2.14.7 to

$$W_i - A_i = (w_i - a_i, \alpha_i + \gamma_i) \text{ and } W_i^\delta - A_i = (w_i - a_i, \alpha_i + \gamma_i^\delta),$$

we find

$$D(W_i - A_i, W_i^\delta - A_i) = D(W_i, W_i^\delta) \leq \delta,$$

and

$$\begin{aligned} & \left| \sup_{t \leq 0} (W_i - A_i)(t) - \sup_{t \leq 0} (W_i^\delta - A_i)(t) \right| \leq \\ & \sup_{t \leq 0} \left| (W_i - A_i)(t) - (W_i^\delta - A_i)(t) \right| \leq \\ & \sup_{t \in \mathbf{R}} \left| (W_i - A_i)(t) - (W_i^\delta - A_i)(t) \right| \leq \\ & \max \left\{ \frac{1}{\alpha_i + \gamma_i}, \frac{1}{\alpha_i + \gamma_i^\delta} \right\} \times \delta \leq \frac{\delta}{\alpha}. \end{aligned}$$

Which ends the proof. □

From (3.16) and (3.17) it follows that

$$\lim_{\delta \rightarrow 0} \phi(A, W^\delta) = \phi(A, W)$$

for any  $A$ , which means that if  $\delta$  is small enough then  $\phi(A, W^\delta)$  can be made arbitrarily close to  $\phi(A, W)$ .

As an immediate consequence of (3.17) we can see that Theorem 3.4.1 remains valid for the case of crisp weighting vectors, i.e. when  $\gamma_i = 0$ ,  $i = 1, \dots, n$ . In this case

$$\text{Pos}[\bar{w}_i \leq A_i] = \begin{cases} 1 & \text{if } w_i \leq a_i \\ A(w_i) & \text{if } 0 < w_i - a_i < \alpha_i \\ 0 & \text{otherwise} \end{cases}$$

where  $\bar{w}_i$  denotes the characteristic function of  $w_i \in [0, 1]$ ; and the weighted aggregate, denoted by  $\phi(A, w)$ , is computed as

$$\phi(A, w) = \mathbf{Agg}\{\text{Pos}[\bar{w}_1 \leq A_1], \dots, \text{Pos}[\bar{w}_n \leq A_n]\}$$

If  $\mathbf{Agg}$  is the minimum operator then we get

$$\phi(A, w) = \min\{\text{Pos}[\bar{w}_1 \leq A_1], \dots, \text{Pos}[\bar{w}_n \leq A_n]\} \quad (3.19)$$



If both the ratings and the importances are given by crisp numbers (i.e. when  $\gamma_i = \alpha_i = 0$ ,  $i = 1, \dots, n$ ) then  $\text{Pos}[\bar{w}_i \leq \bar{a}_i]$  implements the *standard strict* implication operator, i.e.,

$$\text{Pos}[\bar{w}_i \leq \bar{a}_i] = w_i \rightarrow a_i = \begin{cases} 1 & \text{if } w_i \leq a_i \\ 0 & \text{otherwise} \end{cases}$$

It is clear that whatever is the aggregation operator in

$$\phi(a, w) = \mathbf{Agg}\{\text{Pos}[\bar{w}_1 \leq \bar{a}_1], \dots, \text{Pos}[\bar{w}_n \leq \bar{a}_n]\},$$

the weighted aggregate,  $\phi(a, w)$ , can be very sensitive to small changes in the weighting vector  $w$ .

However, we can still sustain the *benchmarking character* of the weighted aggregation if we use an  $R$ -implication operator to transform the ratings under importance [57, 69]. For example, for the operator

$$\phi(a, w) = \min\{w_1 \rightarrow a_1, \dots, w_n \rightarrow a_n\}. \quad (3.20)$$

where  $\rightarrow$  is an  $R$ -implication operator, the equation

$$\phi(a, w) = 1,$$

holds iff  $w_i \leq a_i$  for all  $i$ , i.e. when the value of each performance rating is at least as big as the value of its associated weight. However, the crucial question here is: Does the

$$\lim_{w^\delta \rightarrow w} \phi(a, w^\delta) = \phi(a, w), \quad \forall a \in I,$$

relationship still remain valid for any  $R$ -implication?

The answer is negative.  $\phi$  will be continuous in  $w$  if and only if the implication operator is continuous. For example, if we choose the Gödel implication in (3.20) then  $\phi$  will not be continuous in  $w$ , because the Gödel implication is not continuous.

To illustrate the sensitivity of  $\phi$  defined by the Gödel implication (6.2) consider (3.20) with  $n = 1$ ,  $a_1 = w_1 = 0.6$  and  $w_1^\delta = w_1 + \delta$ . In this case

$$\phi(a_1, w_1) = \phi(w_1, w_1) = \phi(0.6, 0.6) = 1,$$

but

$$\phi(a, w_1^\delta) = \phi(w_1, w_1 + \delta) = \phi(0.6, 0.6 + \delta) = (0.6 + \delta) \rightarrow 0.6 = 0.6,$$

that is,

$$\lim_{\delta \rightarrow 0} \phi(a_1, w_1^\delta) = 0.6 \neq \phi(a_1, w_1) = 1.$$

But if we choose the (continuous) Łukasiewicz implication in (3.20) then  $\phi$  will be continuous in  $w$ , and therefore, small changes in the importance can cause only small changes in the weighted aggregate. Thus, the following formula

$$\phi(a, w) = \min\{(1 - w_1 + a_1) \wedge 1, \dots, (1 - w_n + a_n) \wedge 1\}. \quad (3.21)$$

not only keeps up the benchmarking character of  $\phi$ , but also implements a stable approach to importance weighted aggregation in the nonfuzzy case.

If we do allow a positive compensation between ratings then we can use an OWA-operator for aggregation in (3.21). That is,

$$\phi(a, w) = \mathbf{OWA} \langle (1 - w_1 + a_1) \wedge 1, \dots, (1 - w_n + a_n) \wedge 1 \rangle. \quad (3.22)$$

Taking into consideration that OWA-operators are usually continuous, equation (3.22) also implements a stable approach to importance weighted aggregation in the nonfuzzy case.

### 3.4.1 Examples

We illustrate our approach by several examples.

- *Crisp importance and crisp ratings*

Consider the aggregation problem with

$$a = \begin{pmatrix} 0.7 \\ 0.5 \\ 0.8 \\ 0.9 \end{pmatrix}$$

and

$$w = \begin{pmatrix} 0.8 \\ 0.7 \\ 0.9 \\ 0.6 \end{pmatrix}.$$

Using formula (3.21) for the weighted aggregate we find

$$\begin{aligned} \phi(a, w) &= \min\{0.8 \rightarrow 0.7, 0.7 \rightarrow 0.5, 0.9 \rightarrow 0.8, 0.6 \rightarrow 0.9\} = \\ &= \min\{0.9, 0.8, 0.9, 1\} = 0.8 \end{aligned}$$

- *Crisp importance and fuzzy ratings*

Consider the aggregation problem with

$$a = \begin{pmatrix} (0.7, 0.2) \\ (0.5, 0.3) \\ (0.8, 0.2) \\ (0.9, 0.1) \end{pmatrix}$$

and

$$w = \begin{pmatrix} 0.8 \\ 0.7 \\ 0.9 \\ 0.6 \end{pmatrix}.$$

Using formula (3.19) for the weighted aggregate we find

$$\phi(A, w) = \min\{1/2, 1/3, 1/2, 1\} = 1/3.$$

The essential reason for the low performance of this object is that it performed low on the second criterion which has a high importance. If we allow positive compensations and use an OWA operator with weights, for example,  $(1/6, 1/3, 1/6, 1/3)$  then we find

$$\begin{aligned} \phi(A, w) &= \mathbf{OWA}\langle 1/2, 1/3, 1/2, 1 \rangle = \\ &= 1/6 + 1/2 \times (1/3 + 1/6) + 1/3 \times 1/3 = 19/36 \approx 0.5278 \end{aligned}$$

- *Fuzzy importance and fuzzy ratings*

Consider the aggregation problem with

$$A = \begin{pmatrix} (0.7, 0.2) \\ (0.5, 0.3) \\ (0.8, 0.2) \\ (0.9, 0.1) \end{pmatrix}$$

and

$$W = \begin{pmatrix} (0.8, 0.2) \\ (0.7, 0.3) \\ (0.9, 0.1) \\ (0.6, 0.2) \end{pmatrix}.$$

Using formula (3.14) for the weighted aggregate we find

$$\phi(A, W) = \min\{3/4, 2/3, 2/3, 1\} = 2/3.$$

The reason for the relatively high performance of this object is that, even though it performed low on the second criterion which has a high importance, the second importance has a relatively large tolerance level, 0.3.

In this Section we have introduced a possibilistic approach to the process of importance weighted transformation when both the importances and the aggregates are given by triangular fuzzy numbers. In this approach the importances have been considered as benchmark levels for the performances, i.e. an alternative performs well on all criteria if the degree of satisfaction to each of the criteria is at least as big as the associated benchmark. We have suggested the use of measure of necessity to be able to distinguish alternatives with overall rating one (whose performance ratings exceed the value of their importance with respect to all criteria with possibility one).

We have shown that using the possibilistic approach (i) small changes in the membership function of the importances can cause only small variations in the weighted aggregate; (ii) the weighted aggregate of fuzzy ratings remains stable under small changes in the *nonfuzzy* importances; (iii) the weighted aggregate of crisp ratings still remains stable under small changes in the crisp importances whenever we use a continuous implication operator for the importance weighted transformation.

These results have further implications in several classes of multiple criteria decision making problems, in which the aggregation procedures are rough enough to make the finely tuned formal selection of an optimal alternative meaningless.

# Chapter 4

## Fuzzy Reasoning

### 4.1 The theory of approximate reasoning

In 1979 *Zadeh* introduced the theory of approximate reasoning [443]. This theory provides a powerful framework for reasoning in the face of imprecise and uncertain information. Central to this theory is the representation of propositions as statements assigning fuzzy sets as values to variables. Suppose we have two interactive variables  $x \in X$  and  $y \in Y$  and the causal relationship between  $x$  and  $y$  is completely known. Namely, we know that  $y$  is a function of  $x$ , that is  $y = f(x)$ . Then we can make inferences easily

$$"y = f(x)" \ \& \ "x = x_1" \longrightarrow "y = f(x_1)".$$

This inference rule says that if we have  $y = f(x)$ , for all  $x \in X$  and we observe that  $x = x_1$  then  $y$  takes the value  $f(x_1)$ . More often than not we do not know the complete causal link  $f$

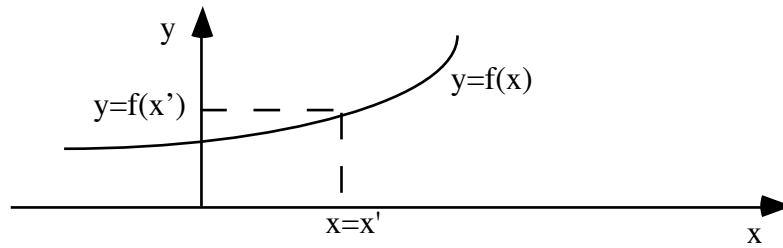


Figure 4.1: Simple crisp inference.

between  $x$  and  $y$ , only we now the values of  $f(x)$  for some particular values of  $x$ , that is

$$\begin{aligned} \mathfrak{R}_1 : & \text{ If } x = x_1 \text{ then } y = y_1 \\ \mathfrak{R}_2 : & \text{ If } x = x_2 \text{ then } y = y_2 \\ & \dots \\ \mathfrak{R}_n : & \text{ If } x = x_n \text{ then } y = y_n \end{aligned}$$

If we are given an  $x' \in X$  and want to find an  $y' \in Y$  which corresponds to  $x'$  under the rule-base  $\mathfrak{R} = \{\mathfrak{R}_1, \dots, \mathfrak{R}_m\}$  then we have an interpolation problem.

Let  $x$  and  $y$  be linguistic variables, e.g. " $x$  is high" and " $y$  is small". The basic problem of approximate reasoning is to find the membership function of the consequence  $C$  from the rule-base  $\{\mathfrak{R}_1, \dots, \mathfrak{R}_n\}$  and the fact  $A$ .

$\mathfrak{R}_1 :$	if $x$ is $A_1$ then $y$ is $C_1$ ,
$\mathfrak{R}_2 :$	if $x$ is $A_2$ then $y$ is $C_2$ ,
	.....
$\mathfrak{R}_n :$	if $x$ is $A_n$ then $y$ is $C_n$
fact:	$x$ is $A$
<hr/>	
consequence:	$y$ is $C$

In [443] Zadeh introduced a number of translation rules which allow us to represent some common linguistic statements in terms of propositions in our language. In the following we describe some of these translation rules.

**Definition 4.1.1.** *Entailment rule:*

$x$ is $A$	$Mary$ is very young
$A \subset B$	$very\ young \subset young$
<hr/>	<hr/>
$x$ is $B$	$Mary$ is young

**Definition 4.1.2.** *Conjunction rule:*

$x$ is $A$	$pressure$ is not very high
$x$ is $B$	$pressure$ is not very low
<hr/>	<hr/>
$x$ is $A \cap B$	$pressure$ is not very high and not very low

**Definition 4.1.3.** *Disjunction rule:*

$x$ is $A$	$pressure$ is not very high
or $x$ is $B$	or $pressure$ is not very low
<hr/>	<hr/>
$x$ is $A \cup B$	$pressure$ is not very high or not very low

**Definition 4.1.4.** *Projection rule:*

$(x, y)$ have relation $R$	$(x, y)$ have relation $R$
<hr/>	<hr/>
$x$ is $\text{Pos}_X(R)$	$y$ is $\text{Pos}_Y(R)$
$(x, y)$ is close to $(3, 2)$	$(x, y)$ is close to $(3, 2)$
<hr/>	<hr/>
$x$ is close to 3	$y$ is close to 2

**Definition 4.1.5.** *Negation rule:*

$$\frac{\text{not } (x \text{ is } A)}{x \text{ is } \neg A} \qquad \frac{\text{not } (x \text{ is high})}{x \text{ is not high}}$$

In fuzzy logic and approximate reasoning, the most important fuzzy inference rule is the *Generalized Modus Ponens* (GMP).

The classical *Modus Ponens* inference rule says:

$$\frac{\begin{array}{l} \text{premise} \\ \text{fact} \end{array} \qquad \begin{array}{l} \text{if } p \text{ then } q \\ p \end{array}}{\text{consequence} \qquad q}$$

This inference rule can be interpreted as: If  $p$  is true and  $p \rightarrow q$  is true then  $q$  is true.

If we have fuzzy sets,  $A \in \mathcal{F}(U)$  and  $B \in \mathcal{F}(V)$ , and a fuzzy implication operator in the premise, and the fact is also a fuzzy set,  $A' \in \mathcal{F}(U)$ , (usually  $A \neq A'$ ) then the consequence,  $B' \in \mathcal{F}(V)$ , can be derived from the premise and the fact using the compositional rule of inference suggested by Zadeh [439]. The *Generalized Modus Ponens* inference rule says

$$\frac{\begin{array}{l} \text{premise} \\ \text{fact} \end{array} \qquad \begin{array}{l} \text{if } x \text{ is } A \text{ then } y \text{ is } B \\ x \text{ is } A' \end{array}}{\text{consequence:} \qquad y \text{ is } B'}$$

where the consequence  $B'$  is determined as a composition of the fact and the fuzzy implication operator

$$B' = A' \circ (A \rightarrow B)$$

that is,

$$B'(v) = \sup_{u \in U} \min\{A'(u), (A \rightarrow B)(u, v)\}, \quad v \in V.$$

The consequence  $B'$  is nothing else but the shadow of  $A \rightarrow B$  on  $A'$ . The *Generalized Modus Ponens*, which reduces to classical modus ponens when  $A' = A$  and  $B' = B$ , is closely related to the forward data-driven inference which is particularly useful in the *Fuzzy Logic Control*.

In many practical cases instead of sup-min composition we use sup- $T$  composition, where  $T$  is a t-norm.

**Definition 4.1.6.** (*sup- $T$  compositional rule of inference*)

$$\frac{\begin{array}{l} \text{premise} \\ \text{fact} \end{array} \qquad \begin{array}{l} \text{if } x \text{ is } A \text{ then } y \text{ is } B \\ x \text{ is } A' \end{array}}{\text{consequence:} \qquad y \text{ is } B'}$$

where the consequence  $B'$  is determined as a composition of the fact and the fuzzy implication operator

$$B' = A' \circ (A \rightarrow B)$$

that is,

$$B'(v) = \sup\{T(A'(u), (A \rightarrow B)(u, v)) \mid u \in U\}, \quad v \in V.$$

It is clear that  $T$  can not be chosen independently of the implication operator.

**Example.** The GMP with Larsen's product implication, where the membership function of the consequence  $B'$  is defined by

$$B'(y) = \sup \min\{A'(x), A(x)B(y)\},$$

for all  $y \in \mathbb{R}$ .

The classical *Modus Tollens* inference rule says: If  $p \rightarrow q$  is true and  $q$  is false then  $p$  is false. The *Generalized Modus Tollens*, inference rule says,

$$\frac{\begin{array}{l} \text{premise} \quad \text{if } x \text{ is } A \text{ then } y \text{ is } B \\ \text{fact} \quad y \text{ is } B' \end{array}}{\text{consequence: } x \text{ is } A'}$$

which reduces to "Modus Tollens" when  $B = \neg B$  and  $A' = \neg A$ , is closely related to the backward goal-driven inference which is commonly used in expert systems, especially in the realm of *medical diagnosis*.

Suppose that  $A$ ,  $B$  and  $A'$  are fuzzy numbers. The Generalized Modus Ponens should satisfy some rational properties

**Property. Basic property:**

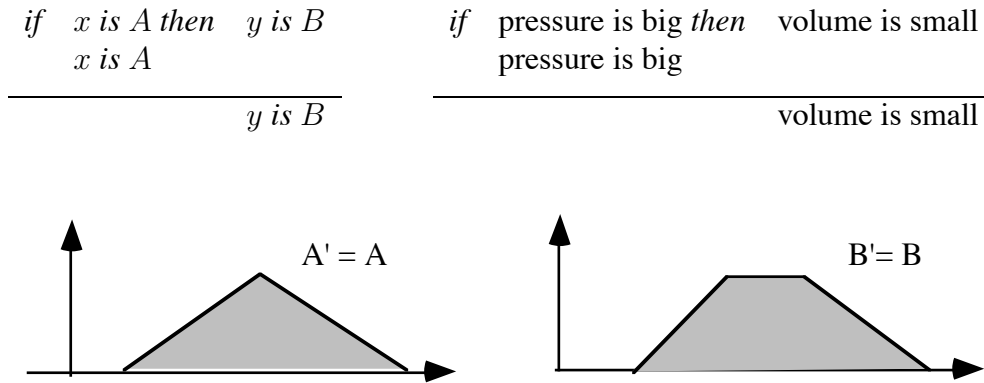


Figure 4.2: Basic property.

**Property. Total indeterminance:**

$\frac{\begin{array}{l} \text{if } x \text{ is } A \text{ then } y \text{ is } B \\ x \text{ is } \neg A \end{array}}{y \text{ is unknown}}$	$\frac{\begin{array}{l} \text{if pres. is big then volume is small} \\ \text{pres. is not big} \end{array}}{\text{volume is unknown}}$
--	--

**Property. Subset:**

$\frac{\begin{array}{l} \text{if } x \text{ is } A \text{ then } y \text{ is } B \\ x \text{ is } A' \subset A \end{array}}{y \text{ is } B}$	$\frac{\begin{array}{l} \text{if pres. is big then volume is small} \\ \text{pres. is very big} \end{array}}{\text{volume is small}}$
---	---

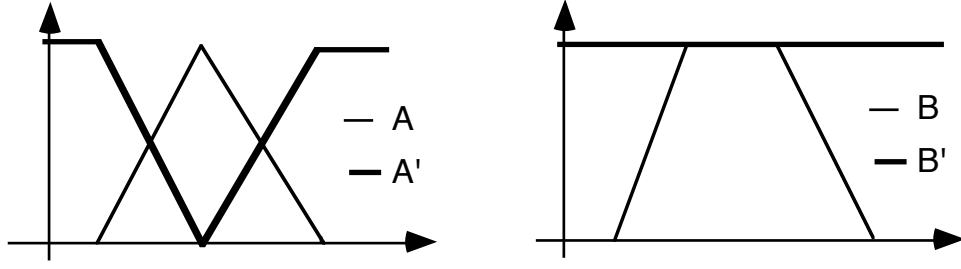


Figure 4.3: Total indeterminance.

**Property. Superset:**

$$\frac{\begin{array}{l} \text{if } x \text{ is } A \text{ then } y \text{ is } B \\ x \text{ is } A' \end{array}}{y \text{ is } B' \supset B}$$

#### 4.1.1 The GMP with Mamdani implication

Suppose that  $A$ ,  $B$  and  $A'$  are fuzzy numbers. We show that the GMP with Mamdani implication operator does not satisfy all the four properties listed above.

The GMP with Mamdani implication inference rule says

$$\frac{\begin{array}{l} \text{if } x \text{ is } A \text{ then } y \text{ is } B \\ x \text{ is } A' \end{array}}{y \text{ is } B'}$$

where the membership function of the consequence  $B'$  is defined by

$$B'(y) = \sup\{A'(x) \wedge A(x) \wedge B(y) | x \in \mathbb{R}\}, \quad y \in \mathbb{R}.$$

- *Basic property.* Let  $A' = A$  and let  $y \in \mathbb{R}$  be arbitrarily fixed. Then we have

$$\begin{aligned} B'(y) &= \sup_x \min\{A(x), \min\{A(x), B(y)\}\} = \\ &= \sup_x \min\{A(x), B(y)\} = \min\{B(y), \sup_x A(x)\} = \\ &= \min\{B(y), 1\} = B(y). \end{aligned}$$

So the basic property is satisfied.

- *Total indeterminance.* Let  $A' = \neg A = 1 - A$  and let  $y \in \mathbb{R}$  be arbitrarily fixed. Then we have

$$\begin{aligned} B'(y) &= \sup_x \min\{1 - A(x), \min\{A(x), B(y)\}\} = \\ &= \sup_x \min\{A(x), 1 - A(x), B(y)\} = \\ &= \min\{B(y), \sup_x \min\{A(x), 1 - A(x)\}\} = \end{aligned}$$



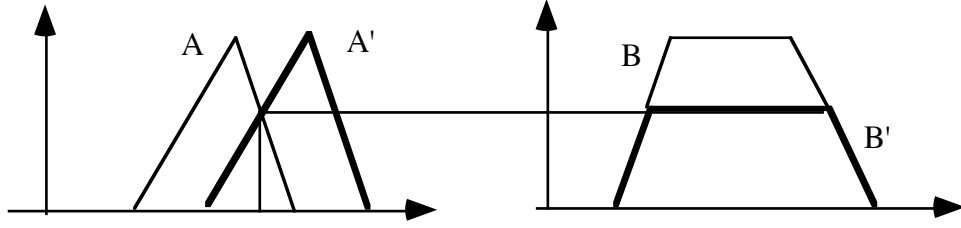


Figure 4.4: The GMP with Mamdani's implication operator.

$$\min\{B(y), 1/2\} = 1/2B(y) < 1$$

this means that the total indeterminance property is not satisfied.

- *Subset.* Let  $A' \subset A$  and let  $y \in \mathbb{R}$  be arbitrarily fixed. Then we have

$$\begin{aligned} B'(y) &= \sup_x \min\{A'(x), \min\{A(x), B(y)\}\} = \\ &= \sup_x \min\{A(x), A'(x), B(y)\} = \\ &= \min\{B(y), \sup_x A'(x)\} = \min\{B(y), 1\} = B(y) \end{aligned}$$

So the subset property is satisfied.

- *Superset.* Let  $y \in \mathbb{R}$  be arbitrarily fixed. Then we have

$$\begin{aligned} B'(y) &= \sup_x \min\{A'(x), \min\{A(x), B(y)\}\} = \\ &= \sup_x \min\{A(x), A'(x), B(y)\} \leq B(y). \end{aligned}$$

So the superset property is also not satisfied by Mamdani's implication operator.

### 4.1.2 The GMP with Gödel implication

We show now that the GMP with Gödel implication does satisfy all the four properties listed above. In this case the membership function of the consequence  $B'$  is defined by

$$B'(y) = \sup_x \min\{A'(x), A(x) \rightarrow B(y)\},$$

for all  $y \in \mathbb{R}$ , where

$$A(x) \rightarrow B(y) = \begin{cases} 1 & \text{if } A(x) \leq B(y) \\ B(y) & \text{otherwise.} \end{cases}$$

- *Basic property.*

Let  $A' = A$  and let  $x, y \in \mathbb{R}$  be arbitrarily fixed. On the one hand from the definition of Gödel implication operator we obtain

$$\min\{A(x), A(x) \rightarrow B(y)\} = \begin{cases} A(x) & \text{if } A(x) \leq B(y) \\ B(y) & \text{if } A(x) > B(y) \end{cases}$$

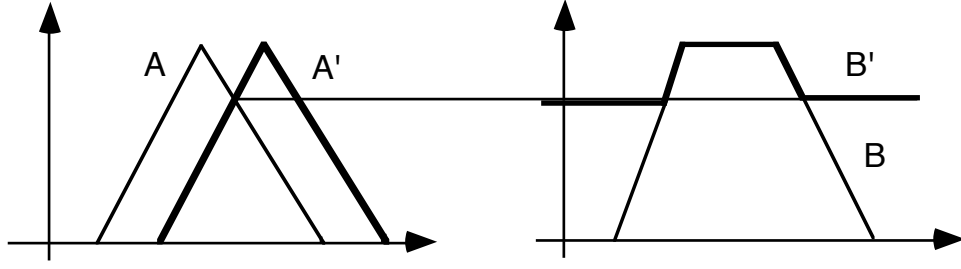


Figure 4.5: Inference with Gödel implication operator.

That is,

$$B'(y) = \sup_x \min\{A(x), A(x) \rightarrow B(y)\} \leq B(y)$$

On the other hand from continuity and normality of  $A$  it follows that there exists an  $x' \in \mathbb{R}$  such that  $A(x') = B(y)$ . So

$$B'(y) = \sup_x \min\{A(x), A(x) \rightarrow B(y)\} \geq \min\{A(x'), A(x') \rightarrow B(y)\} = B(y)$$

- *Total indeterminance.* Let  $x' \notin \text{supp}(A)$  be arbitrarily chosen. Then from  $A(x') = 0$  it follows that

$$B'(y) = \sup_x \min\{1 - A(x), A(x) \rightarrow B(y)\} \geq \min\{1 - A(x'), A(x') \rightarrow B(y)\} = 1,$$

for any  $y \in \mathbb{R}$ .

- *Subset.* Let  $A'(x) \leq A(x), \forall x \in \mathbb{R}$ . Then

$$B'(y) = \sup_x \min\{A'(x), A(x) \rightarrow B(y)\} \leq \sup_x \min\{A(x), A(x) \rightarrow B(y)\} = B(y).$$

- *Superset.* From  $A' \in \mathcal{F}$  it follows that there exists an  $x' \in \mathbb{R}$  such that  $A'(x') = 1$ . Then

$$B'(y) = \sup_x \min\{A'(x), A(x) \rightarrow B(y)\} \geq \min\{A'(x'), A(x') \rightarrow B(y)\} = A(x') \rightarrow B(y) \geq B(y).$$

## 4.2 Aggregation in fuzzy system modeling

Many applications of fuzzy set theory involve the use of a fuzzy rule base to model complex and perhaps ill-defined systems. These applications include fuzzy logic control, fuzzy expert systems and fuzzy systems modeling. Typical of these situations are set of  $n$  rules of the form

$$\begin{aligned} \mathfrak{R}_1 : & \text{ if } x \text{ is } A_1 \text{ then } y \text{ is } C_1 \\ \mathfrak{R}_2 : & \text{ if } x \text{ is } A_2 \text{ then } y \text{ is } C_2 \\ & \dots\dots\dots \\ \mathfrak{R}_n : & \text{ if } x \text{ is } A_n \text{ then } y \text{ is } C_n \end{aligned}$$

The fuzzy inference process consists of the following four step algorithm [421]:

- Determination of the relevance or matching of each rule to the current input value.
- Determination of the output of each rule as fuzzy subset of the output space. We shall denote these individual rule outputs as  $R_j$ .
- Aggregation of the individual rule outputs to obtain the overall fuzzy system output as fuzzy subset of the output space. We shall denote this overall output as  $R$ .
- Selection of some action based upon the output set.

Our purpose here is to investigate the requirements for the operations that can be used to implement this reasoning process. We are particularly concerned with the third step, the rule output aggregation.

Let us look at the process for combining the individual rule outputs. A basic assumption we shall make is that the operation is *pointwise* and *likewise*. By pointwise we mean that for every  $y$ ,  $R(y)$  just depends upon  $R_j(y)$ ,  $j = 1, \dots, n$ . By likewise we mean that the process used to combine the  $R_j$  is the same for all of the  $y$ .

Let us denote the pointwise process we use to combine the individual rule outputs as

$$F(y) = \mathbf{Agg}(R_1(y), \dots, R_n(y))$$

In the above **Agg** is called the aggregation operator and the  $R_j(y)$  are the arguments. More generally, we can consider this as an operator

$$a = \mathbf{Agg}(a_1, \dots, a_n)$$

where the  $a_i$  and  $a$  are values from the membership grade space, normally the unit interval.

Let us look at the minimal requirements associated with **Agg**. We first note that the combination of the individual rule outputs should be independent of the choice of indexing of the rules. This implies that a required property that we must associate with the **Agg** operator is that of commutativity, the indexing of the arguments does not matter. We note that the commutativity property allows to represent the arguments of the **Agg** operator, as an unordered collection of possible duplicate values; such an object is a **bag**.

For an individual rule output,  $R_j$ , the membership grade  $R_j(y)$  indicates the degree or strength to which this rule suggests that  $y$  is the appropriate solution. In particular if for a pair of elements  $y'$  and  $y''$  it is the case that  $R_j(y') \geq R_j(y'')$ , then we are saying that rule  $j$  is preferring  $y'$  as the system output over  $y''$ . From this we can reasonably conclude that if all rules prefer  $y'$  over  $y''$  as output then the overall system output should prefer  $y'$  over  $y''$ . This observation requires us to impose a monotonicity condition on the **Agg** operation. In particular if

$$R_j(y') \geq R_j(y''),$$

for all  $j$ , then  $R(y') \geq R(y'')$ .

There appears one other condition we need to impose upon the aggregation operator. Assume that there exists some rule whose firing level is zero. The implication of this is that the

rule provides no information regarding what should be the output of the system. It should not affect the final  $R$ . The first observation we can make is that whatever output this rule provides should not make any distinction between the potential outputs. Thus, we see that the aggregation operator needs an identity element.

In summary, we see that the aggregation operator, **Agg** must satisfy three conditions: *commutativity*, *monotonicity*, *must contain a fixed identity*. These conditions are based on the three requirements: that the indexing of the rules be unimportant, a positive association between individual rule output and total system output, and non-firing rules play no role in the decision process.

These operators are called MICA (Monotonic Identity Commutative Aggregation) operators introduced by Yager [421]. MICA operators are the most general class for aggregation in fuzzy modeling. They include t-norms, t-conorms, averaging and compensatory operators.

Assume  $X$  is a set of elements. A bag drawn from  $X$  is any collection of elements which is contained in  $X$ . A bag is different from a subset in that it allows multiple copies of the same element. A bag is similar to a set in that the ordering of the elements in the bag does not matter. If  $A$  is a bag consisting of  $a, b, c, d$  we denote this as  $A = \langle a, b, c, d \rangle$ . Assume  $A$  and  $B$  are two bags. We denote the sum of the bags  $C = A \oplus B$  where  $C$  is the bag consisting of the members of both  $A$  and  $B$ .

**Example.** Let  $A = \langle a, b, c, d \rangle$  and  $B = \langle b, c, c \rangle$  then

$$A \oplus B = \langle a, b, c, d, b, c, c \rangle$$

In the following we let  $\text{Bag}(X)$  indicate the set of all bags of the set  $X$ .

**Definition 4.2.1.** A function

$$F: \text{Bag}(X) \rightarrow X$$

is called a bag mapping from  $\text{Bag}(X)$  into the set  $X$ .

An important property of bag mappings are that they are commutative in the sense that the ordering of the elements does not matter.

**Definition 4.2.2.** Assume  $A = \langle a_1, \dots, a_n \rangle$  and  $B = \langle b_1, \dots, b_n \rangle$  are two bags of the same cardinality  $n$ . If the elements in  $A$  and  $B$  can be indexed in such way that  $a_i \geq b_i$  for all  $i$  then we shall denote this  $A \geq B$ .

**Definition 4.2.3.** (MICA operator) [421] A bag mapping  $M: \text{Bag}([0, 1]) \rightarrow [0, 1]$  is called MICA operator if it has the following two properties

- If  $A \geq B$  then  $M(A) \geq M(B)$  (monotonicity)
- For every bag  $A$  there exists an element,  $u \in [0, 1]$ , called the identity of  $A$  such that if  $C = A \oplus \langle u \rangle$  then  $M(C) = M(A)$  (identity)

Thus the MICA operator is endowed with two properties in addition to the inherent commutativity of the bag operator, monotonicity and identity: (i) the requirement of *monotonicity* appears natural for an aggregation operator in that it provides some connection between the arguments and the aggregated value; (ii) the property of identity allows us to have the facility for aggregating data which does not affect the overall result. This becomes useful for enabling us to include importances among other characteristics.

### 4.3 Multiple fuzzy reasoning schemes

Suppose we are given one block of fuzzy rules of the form

$$\begin{array}{ll}
 \mathfrak{R}_1 : & \text{if } x \text{ is } A_1 \text{ then } z \text{ is } C_1, \\
 \mathfrak{R}_2 : & \text{if } x \text{ is } A_2 \text{ then } z \text{ is } C_2, \\
 & \dots\dots\dots \\
 \mathfrak{R}_n : & \text{if } x \text{ is } A_n \text{ then } z \text{ is } C_n \\
 \text{fact:} & x \text{ is } A \\
 \hline
 \text{consequence:} & z \text{ is } C
 \end{array}$$

where the rules are connected with the (hidden) sentence connective *also*. The  $i$ -th fuzzy rule  $\mathfrak{R}_i$ , from this rule-base,  $\mathfrak{R} = \{\mathfrak{R}_1, \dots, \mathfrak{R}_n\}$ , is implemented by a *fuzzy implication*  $R_i$  and is defined as

$$R_i(u, w) = A_i(u) \rightarrow C_i(w)$$

There are two main approaches to determine the consequence  $C$ :

1. *Combine the rules first.* In this approach, we first combine all the rules by an aggregation operator **Agg** into one rule which used to obtain  $C$  from  $A$ .

$$R = \mathbf{Agg}(\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_n)$$

If the implicate sentence connective *also* is interpreted as *and* then we get

$$R(u, w) = \bigcap_{i=1}^n R_i(u, w) = \min(A_i(u) \rightarrow C_i(w))$$

or by using a t-norm  $T$  for modeling the connective *and*

$$R(u, w) = T(R_1(u, w), \dots, R_n(u, w))$$

If the implicate sentence connective *also* is interpreted as *or* then we get

$$R(u, w) = \bigcup_{i=1}^n R_i(u, v, w) = \max(A_i(u) \rightarrow C_i(w))$$

or by using a t-conorm  $S$  for modeling the connective *or*

$$R(u, w) = S(R_1(u, w), \dots, R_n(u, w))$$

Then we compute  $C$  from  $A$  by the compositional rule of inference as

$$C = A \circ R = A \circ \mathbf{Agg}(R_1, R_2, \dots, R_n).$$

2. *Fire the rules first.* Fire the rules individually, given  $A$ , and then combine their results into  $C$ . We first compose  $A$  with each  $R_i$  producing intermediate result

$$C'_i = A \circ R_i$$

for  $i = 1, \dots, n$  and then combine the  $C'_i$  component wise into  $C'$  by some aggregation operator **Agg**

$$C' = \mathbf{Agg}(C'_1, \dots, C'_n) = \mathbf{Agg}(A \circ R_1, \dots, A \circ R_n).$$

The next Lemma shows that the sup-min compositional operator and the sentence connective *also* interpreted as the union operator are commutative. Thus the consequence,  $C$ , inferred from the complete set of rules is equivalent to the aggregated result,  $C'$ , derived from individual rules.

**Lemma 4.3.1.** *Let*

$$C = A \circ \bigcup_{i=1}^n R_i$$

*be defined by standard sup-min composition as*

$$C(w) = \sup_u \min\{A(u), \max\{R_1(u, w), \dots, R_n(u, w)\}\}$$

*and let*

$$C' = \bigcup_{i=1}^n A \circ R_i$$

*defined by the sup-min composition as*

$$C'(w) = \max_{i=1, \dots, n} \left\{ \sup_u A(u) \wedge R_i(u, w) \right\}.$$

*Then  $C(w) = C'(w)$  for all  $w$  from the universe of discourse  $W$ .*

It should be noted that the sup-product compositional operator and the sentence connective *also* interpreted as the union operator are also commutative. However, the sup-min compositional operator and the sentence connective *also* interpreted as the intersection operator are not usually commutative. In this case, the consequence,  $C$ , inferred from the complete set of rules is included in the aggregated result,  $C'$ , derived from individual rules.

**Lemma 4.3.2.** *Let*

$$C = A \circ \bigcap_{i=1}^n R_i$$

*be defined by standard sup-min composition as*

$$C(w) = \sup_u \min\{A(u), \min\{R_1(u, w), \dots, R_n(u, w)\}\}$$

*and let*

$$C' = \bigcap_{i=1}^n A \circ R_i$$

*defined by the sup-min composition as*

$$C'(w) = \min\left\{ \sup_u \{A(u) \wedge R_i(u, w)\}, \dots, \sup_u \{A(u) \wedge R_n(u, w)\} \right\}.$$

*Then  $C \subset C'$ , i.e  $C(w) \leq C'(w)$  holds for all  $w$  from the universe of discourse  $W$ .*

*Proof.* From the relationship

$$A \circ \bigcap_{i=1}^n R_i \subset A \circ R_i$$

for each  $i = 1, \dots, n$ , we get

$$A \circ \bigcap_{i=1}^n R_i \subset \bigcap_{i=1}^n A \circ R_i.$$

Which ends the proof.  $\square$

Similar statement holds for the sup-t-norm compositional rule of inference, i.e the sup-product compositional operator and the connective *also* interpreted as the intersection operator are not commutative. In this case, the consequence,  $C$ , inferred from the complete set of rules is included in the aggregated result,  $C'$ , derived from individual rules.

**Lemma 4.3.3.** *Let*

$$C = A \circ \bigcap_{i=1}^n R_i$$

*be defined by sup-T composition as*

$$C(w) = \sup_u T(A(u), \min\{R_1(u, w), \dots, R_n(u, w)\})$$

*and let*

$$C' = \bigcap_{i=1}^n A \circ R_i$$

*defined by the sup-T composition. Then  $C \subset C'$ , i.e  $C(w) \leq C'(w)$  holds for all  $w$  from the universe of discourse  $W$ .*

**Example.** *We illustrate Lemma 4.3.2 by a simple example. Assume we have two fuzzy rules of the form*

$\mathfrak{R}_1$  : *if  $x$  is  $A_1$  then  $z$  is  $C_1$*

$\mathfrak{R}_2$  : *if  $x$  is  $A_2$  then  $z$  is  $C_2$*

*where  $A_1, A_2$  and  $C_1, C_2$  are discrete fuzzy numbers of the universe of discourses  $\{x_1, x_2\}$  and  $\{z_1, z_2\}$ , respectively. Suppose that we input a fuzzy set  $A = a_1/x_1 + a_2/x_2$  to the system and let*

$$R_1 = \begin{pmatrix} & z_1 & z_2 \\ x_1 & 0 & 1 \\ x_2 & 1 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} & z_1 & z_2 \\ x_1 & 1 & 0 \\ x_2 & 0 & 1 \end{pmatrix}$$

*represent the fuzzy rules. We first compute the consequence  $C$  by*

$$C = A \circ (R_1 \cap R_2).$$

*Using the definition of intersection of fuzzy relations we get*

$$C = (a_1/x_1 + a_2/x_2) \circ \left[ \begin{pmatrix} & z_1 & z_2 \\ x_1 & 0 & 1 \\ x_2 & 1 & 0 \end{pmatrix} \cap \begin{pmatrix} & z_1 & z_2 \\ x_1 & 1 & 0 \\ x_2 & 0 & 1 \end{pmatrix} \right] =$$

$$(a_1/x_1 + a_2/x_2) \circ \begin{pmatrix} & z_1 & z_2 \\ x_1 & 0 & 0 \\ x_2 & 0 & 0 \end{pmatrix} = \emptyset$$

Let us compute now the membership function of the consequence  $C'$  by

$$C' = (A \circ R_1) \cap (A \circ R_2)$$

Using the definition of sup-min composition we get

$$A \circ R_1 = (a_1/x_1 + a_2/x_2) \circ \begin{pmatrix} & z_1 & z_2 \\ x_1 & 0 & 1 \\ x_2 & 1 & 0 \end{pmatrix}.$$

Plugging into numerical values

$$(A \circ R_1)(z_1) = \max\{a_1 \wedge 0, a_2 \wedge 1\} = a_2, \quad (A \circ R_1)(z_2) = \max\{a_1 \wedge 1, a_2 \wedge 0\} = a_1,$$

So,

$$A \circ R_1 = a_2/z_1 + a_1/z_2$$

and from

$$A \circ R_2 = (a_1/x_1 + a_2/x_2) \circ \begin{pmatrix} & z_1 & z_2 \\ x_1 & 1 & 0 \\ x_2 & 0 & 1 \end{pmatrix} =$$

we get

$$A \circ R_2 = a_1/z_1 + a_2/z_2.$$

Finally,

$$C' = a_2/z_1 + a_1/z_2 \cap a_1/z_1 + a_2/z_2 = a_1 \wedge a_2/z_1 + a_1 \wedge a_2/z_2.$$

Which means that  $C$  is a proper subset of  $C'$  whenever  $\min\{a_1, a_2\} \neq 0$ .

**Lemma 4.3.4.** Consider one block of fuzzy rules of the form

$$\mathcal{R}_i: \text{if } x \text{ is } A_i \text{ then } z \text{ is } C_i, \quad 1 \leq i \leq n$$

and suppose that the input to the system is a fuzzy singleton. Then the consequence,  $C$ , inferred from the complete set of rules is equal to the aggregated result,  $C'$ , derived from individual rules. This statements holds for any kind of aggregation operators used to combine the rules.

*Proof.* Suppose that the input of the system  $A = \bar{x}_0$  is a fuzzy singleton. On the one hand we have

$$C(w) = (A \circ \text{Agg} \langle R_1, \dots, R_n \rangle)(w) = \text{Agg} \langle R_1(x_0, w), \dots, R_n(x_0, w) \rangle.$$

On the other hand

$$C'(w) = \text{Agg} \langle A \circ R_1, \dots, A \circ R_n \rangle(w) = \text{Agg} \langle R_1(x_0, w), \dots, R_n(x_0, w) \rangle = C(w).$$

Which ends the proof. □



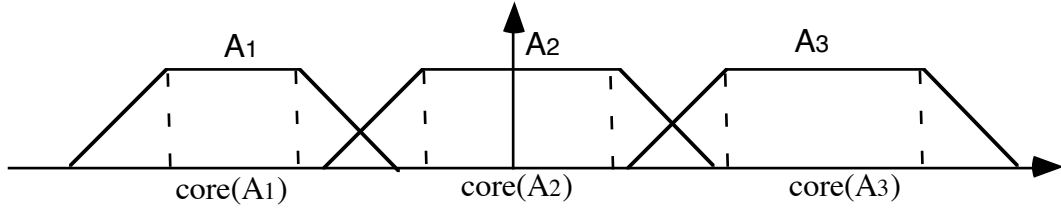


Figure 4.6: Separated rule-base.

Consider one block of fuzzy rules of the form

$$\mathfrak{R} = \{A_i \rightarrow C_i, 1 \leq i \leq n\}$$

where  $A_i$  and  $C_i$  are fuzzy numbers.

**Lemma 4.3.5.** *Suppose that in  $\mathfrak{R}$  the supports of  $A_i$  are pairwise disjunctive:*

$$\text{supp}A_i \cap \text{supp}A_j = \emptyset, \text{ for } i \neq j.$$

*If the Gödel implication operator is used in  $\mathfrak{R}$  then we get*

$$\bigcap_{i=1}^n A_i \circ (A_i \rightarrow C_i) = C_i$$

*holds for  $i = 1, \dots, n$*

*Proof.* Since the GMP with Gödel implication satisfies the basic property we get

$$A_i \circ (A_i \rightarrow C_i) = A_i.$$

From  $\text{supp}(A_i) \cap \text{supp}(A_j) = \emptyset$ , for  $i \neq j$  it follows that

$$A_i \circ (A_j \rightarrow C_j) = \mathbf{1}, i \neq j$$

where  $\mathbf{1}$  is the universal fuzzy set. So,

$$\bigcap_{i=1}^n A_i \circ (A_i \rightarrow C_i) = C_i \cap \mathbf{1} = C_i.$$

□

This property means that deleting any of the rules from  $\mathfrak{R}$  leaves a point  $\hat{x}$  to which no rule applies. It means that every rule is useful.

**Definition 4.3.1.** *The rule-base  $\mathfrak{R}$  is said to be separated if the core of  $A_i$ , defined by*

$$\text{core}(A_i) = \{x \mid A_i(x) = 1\},$$

*is not contained in*

$$\bigcap_{j \neq i} \text{supp}A_j$$

*for  $i = 1, \dots, n$ .*

The following theorem shows that Lemma 4.3.5 remains valid for separated rule-bases.

**Theorem 4.3.1.** [127] *Let  $\mathfrak{R}$  be separated. If the implication is modelled by the Gödel implication operator then*

$$\bigcap_{i=1}^n A_i \circ (A_i \rightarrow C_i) = C_i$$

*holds for  $i = 1, \dots, n$*

## 4.4 MISO fuzzy reasoning schemes

If several linguistic variables are involved in the antecedents and the conclusions of the rules then the system will be referred to as a multi-input-multi-output fuzzy system. For example, the case of two-input-single-output (MISO) fuzzy systems is of the form

$$\mathfrak{R}_i : \text{if } x \text{ is } A_i \text{ and } y \text{ is } B_i \text{ then } z \text{ is } C_i$$

where  $x$  and  $y$  are the process state variables,  $z$  is the control variable,  $A_i$ ,  $B_i$ , and  $C_i$  are linguistic values of the linguistic variables  $x$ ,  $y$  and  $z$  in the universes of discourse  $U$ ,  $V$ , and  $W$ , respectively, and an implicit sentence connective *also* links the rules into a rule set or, equivalently, a rule-base. The procedure for obtaining the fuzzy output of such a knowledge base consists from the following three steps:

- Find the firing level of each of the rules.
- Find the output of each of the rules.
- Aggregate the individual rule outputs to obtain the overall system output.

To infer the output  $z$  from the given process states  $x$ ,  $y$  and fuzzy relations  $R_i$ , we apply the compositional rule of inference:

$$\begin{array}{ll} \mathfrak{R}_1 : & \text{if } x \text{ is } A_1 \text{ and } y \text{ is } B_1 \text{ then } z \text{ is } C_1 \\ \mathfrak{R}_2 : & \text{if } x \text{ is } A_2 \text{ and } y \text{ is } B_2 \text{ then } z \text{ is } C_2 \\ & \dots\dots\dots \\ \mathfrak{R}_n : & \text{if } x \text{ is } A_n \text{ and } y \text{ is } B_n \text{ then } z \text{ is } C_n \\ \text{fact :} & x \text{ is } \bar{x}_0 \text{ and } y \text{ is } \bar{y}_0 \\ \hline \text{consequence :} & z \text{ is } C \end{array}$$

where the consequence is computed by

$$\text{consequence} = \text{Agg} \langle \text{fact} \circ \mathfrak{R}_1, \dots, \text{fact} \circ \mathfrak{R}_n \rangle.$$

That is,

$$C = \text{Agg}(\bar{x}_0 \times \bar{y}_0 \circ R_1, \dots, \bar{x}_0 \times \bar{y}_0 \circ R_n)$$

taking into consideration that  $\bar{x}_0(u) = 0, u \neq x_0$  and  $\bar{y}_0(v) = 0, v \neq y_0$ , the computation of the membership function of  $C$  is very simple:

$$C(w) = \text{Agg}\{A_1(x_0) \times B_1(y_0) \rightarrow C_1(w), \dots, A_n(x_0) \times B_n(y_0) \rightarrow C_n(w)\}$$

for all  $w \in W$ . The procedure for obtaining the fuzzy output of such a knowledge base can be formulated as

- The firing level of the  $i$ -th rule is determined by

$$A_i(x_0) \times B_i(y_0).$$

- The output of the  $i$ -th rule is calculated by

$$C'_i(w) := A_i(x_0) \times B_i(y_0) \rightarrow C_i(w)$$

for all  $w \in W$ .

- The overall system output,  $C$ , is obtained from the individual rule outputs  $C'_i$  by

$$C(w) = \text{Agg}\{C'_1, \dots, C'_n\}$$

for all  $w \in W$ .

**Example.** If the sentence connective also is interpreted as anding the rules by using minimum-norm then the membership function of the consequence is computed as

$$C = (\bar{x}_0 \times \bar{y}_0 \circ R_1) \cap \dots \cap (\bar{x}_0 \times \bar{y}_0 \circ R_n).$$

That is,

$$C(w) = \min\{A_1(x_0) \times B_1(y_0) \rightarrow C_1(w), \dots, A_n(x_0) \times B_n(y_0) \rightarrow C_n(w)\}$$

for all  $w \in W$ .

**Example.** If the sentence connective also is interpreted as oring the rules by using minimum-norm then the membership function of the consequence is computed as

$$C = (\bar{x}_0 \times \bar{y}_0 \circ R_1) \cup \dots \cup (\bar{x}_0 \times \bar{y}_0 \circ R_n).$$

That is,

$$C(w) = \max\{A_1(x_0) \times B_1(y_0) \rightarrow C_1(w), \dots, A_n(x_0) \times B_n(y_0) \rightarrow C_n(w)\}$$

for all  $w \in W$ .

**Example.** Suppose that the Cartesian product and the implication operator are implemented by the  $t$ -norm  $T(u, v) = uv$ . If the sentence connective also is interpreted as oring the rules by using minimum-norm then the membership function of the consequence is computed as

$$C = (\bar{x}_0 \times \bar{y}_0 \circ R_1) \cup \dots \cup (\bar{x}_0 \times \bar{y}_0 \circ R_n).$$

That is,

$$C(w) = \max\{A_1(x_0)B_1(y_0)C_1(w), \dots, A_n(x_0)B_n(y_0)C_n(w)\}$$

for all  $w \in W$ .

We present three well-known inference mechanisms in MISO fuzzy systems. For simplicity we assume that we have two fuzzy rules of the form

$$\begin{array}{ll}
\mathfrak{R}_1 : & \text{if } x \text{ is } A_1 \text{ and } y \text{ is } B_1 \text{ then } z \text{ is } C_1 \\
\mathfrak{R}_2 : & \text{if } x \text{ is } A_2 \text{ and } y \text{ is } B_2 \text{ then } z \text{ is } C_2 \\
\text{fact :} & x \text{ is } \bar{x}_0 \text{ and } y \text{ is } \bar{y}_0 \\
\hline
\text{consequence :} & z \text{ is } C
\end{array}$$

#### 4.4.1 Tsukamoto

All linguistic terms are supposed to have monotonic membership functions. The firing levels of the rules are computed by

$$\begin{aligned}
\alpha_1 &= A_1(x_0) \wedge B_1(y_0), \\
\alpha_2 &= A_2(x_0) \wedge B_2(y_0).
\end{aligned}$$

In this mode of reasoning the individual crisp control actions  $z_1$  and  $z_2$  are computed from the equations

$$\alpha_1 = C_1(z_1), \quad \alpha_2 = C_2(z_2)$$

and the overall crisp control action is expressed as

$$z_0 = \frac{\alpha_1 z_1 + \alpha_2 z_2}{\alpha_1 + \alpha_2} = \frac{\alpha_1 C_1^{-1}(\alpha_1) + \alpha_2 C_2^{-1}(\alpha_2)}{\alpha_1 + \alpha_2}$$

i.e.  $z_0$  is computed by the discrete Center-of-Gravity method. If we have  $m$  rules in our rule-base then the crisp control action is computed as

$$z_0 = \frac{\alpha_1 z_1 + \cdots + \alpha_m z_m}{\alpha_1 + \cdots + \alpha_m},$$

where  $\alpha_i$  is the firing level and  $z_i$  is the (crisp) output of the  $i$ -th rule,  $i = 1, \dots, m$ .

#### 4.4.2 Sugeno and Takagi

Sugeno and Takagi use the following architecture [393]

$$\begin{array}{ll}
\mathfrak{R}_1 : & \text{if } x \text{ is } A_1 \text{ and } y \text{ is } B_1 \text{ then } z_1 = a_1 x + b_1 y \\
\mathfrak{R}_2 : & \text{if } x \text{ is } A_2 \text{ and } y \text{ is } B_2 \text{ then } z_2 = a_2 x + b_2 y \\
\text{fact :} & x \text{ is } \bar{x}_0 \text{ and } y \text{ is } \bar{y}_0 \\
\hline
\text{consequence :} & z_0
\end{array}$$

The firing levels of the rules are computed by

$$\begin{aligned}
\alpha_1 &= A_1(x_0) \wedge B_1(y_0), \\
\alpha_2 &= A_2(x_0) \wedge B_2(y_0),
\end{aligned}$$

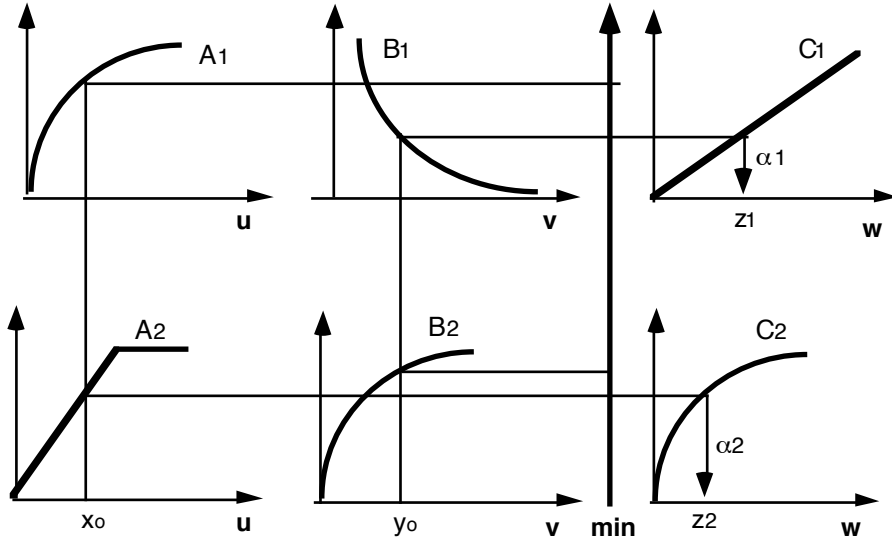


Figure 4.7: Tsukamoto's inference mechanism [400].

then the individual rule outputs are derived from the relationships

$$z_1^* = a_1 x_0 + b_1 y_0,$$

$$z_2^* = a_2 x_0 + b_2 y_0,$$

and the crisp control action is expressed as

$$z_0 = \frac{\alpha_1 z_1^* + \alpha_2 z_2^*}{\alpha_1 + \alpha_2}$$

If we have  $m$  rules in our rule-base then the crisp control action is computed as

$$z_0 = \frac{\alpha_1 z_1^* + \cdots + \alpha_m z_m^*}{\alpha_1 + \cdots + \alpha_m},$$

where  $\alpha_i$  denotes the firing level of the  $i$ -th rule,  $i = 1, \dots, m$ .

**Example.** We illustrate Sugeno's reasoning method by the following simple example

$$\begin{array}{llll} \text{if } x \text{ is SMALL} & \text{and } y \text{ is BIG} & \text{then } z = x - y \\ \text{if } x \text{ is BIG} & \text{and } y \text{ is SMALL} & \text{then } z = x + y \\ \text{if } x \text{ is BIG} & \text{and } y \text{ is BIG} & \text{then } z = x + 2y \end{array}$$

where the membership functions *SMALL* and *BIG* are defined by

$$SMALL(v) = \begin{cases} 1 & \text{if } v \leq 1 \\ 1 - \frac{v-1}{4} & \text{if } 1 \leq v \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

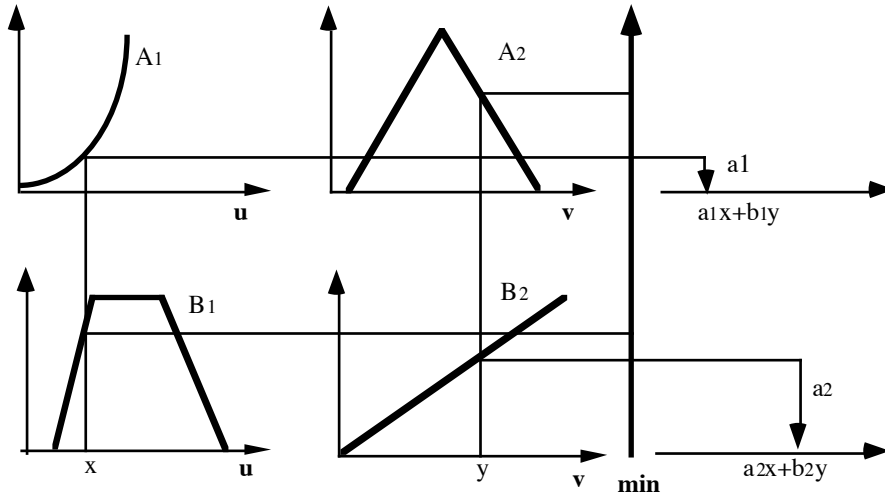


Figure 4.8: Sugeno's inference mechanism.

$$BIG(u) = \begin{cases} 1 & \text{if } u \geq 5 \\ 1 - \frac{5-u}{4} & \text{if } 1 \leq u \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

Suppose we have the inputs  $x_0 = 3$  and  $y_0 = 3$ . What is the output of the system?

The firing level of the first rule is

$$\alpha_1 = \min\{SMALL(3), BIG(3)\} = \min\{0.5, 0.5\} = 0.5$$

the individual output of the first rule is  $z_1 = x_0 - y_0 = 3 - 3 = 0$ . The firing level of the second rule is

$$\alpha_1 = \min\{BIG(3), SMALL(3)\} = \min\{0.5, 0.5\} = 0.5$$

the individual output of the second rule is  $z_2 = x_0 + y_0 = 3 + 3 = 6$ . The firing level of the third rule is

$$\alpha_1 = \min\{BIG(3), BIG(3)\} = \min\{0.5, 0.5\} = 0.5$$

the individual output of the third rule is  $z_3 = x_0 + 2y_0 = 3 + 6 = 9$ . and the system output,  $z_0$ , is computed from the equation

$$z_0 = \frac{0 \times 0.5 + 6 \times 0.5 + 9 \times 0.5}{1.5} = 5.0.$$

### 4.4.3 Simplified fuzzy reasoning

In this context, the word *simplified* means that the individual rule outputs are given by crisp numbers, and therefore, we can use their weighted sum (where the weights are the firing strengths of the corresponding rules) to obtain the overall system output:

$\mathfrak{R}_1$ :	if $x_1$ is $A_{11}$ and ... and $x_n$ is $A_{1n}$	then $y = z_1$
	.....	
$\mathfrak{R}_m$ :	if $x_1$ is $A_{m1}$ and ... and $x_n$ is $A_{mn}$	then $y = z_m$
fact:	$x_1$ is $u_1$ and ... and $x_n$ is $u_n$	
consequence:		$y$ is $z_0$

where  $A_{ij}$  are values of the linguistic variables  $x_1, \dots, x_n$ . We derive  $z_0$  from the initial content of the data base,  $\{u_1, \dots, u_n\}$ , and from the fuzzy rule base  $\mathfrak{R} = \{\mathfrak{R}_1, \dots, \mathfrak{R}_m\}$ . by the simplified fuzzy reasoning scheme as

$$z_0 = \frac{z_1\alpha_1 + \dots + z_m\alpha_m}{\alpha_1 + \dots + \alpha_m}$$

where  $\alpha_i = (A_{i1} \times \dots \times A_{in})(u_1, \dots, u_n)$ ,  $i = 1, \dots, m$ .

## 4.5 Some properties of the compositional rule of inference

Following Fullér and Zimmermann [176, 184], and Fullér and Werners [181] we show two very important features of the compositional rule of inference under triangular norms. Namely, we prove that (i) if the t-norm defining the composition and the membership function of the observation are continuous, then the conclusion depends continuously on the observation; (ii) if the t-norm and the membership function of the relation are continuous, then the observation has a continuous membership function. We consider the compositional rule of inference with different observations  $P$  and  $P'$ :

Observation:	$X$ has property $P$
Relation:	$X$ and $Y$ are in relation $R$
Conclusion:	$Y$ has property $Q$
<hr/>	
Observation:	$X$ has property $P'$
Relation m:	$X$ and $Y$ are in relation $R$
Conclusion:	$Y$ has property $Q'$

According to Zadeh's compositional rule of inference,  $Q$  and  $Q'$  are computed as

$$Q = P \circ R, \quad Q' = P' \circ R$$

i.e.,

$$\mu_Q(y) = \sup_{x \in R} T(\mu_P(x), \mu_R(x, y)), \quad \mu_{Q'}(y) = \sup_{x \in R} T(\mu_{P'}(x), \mu_R(x, y)).$$

The following theorem shows that when the observations are close to each other in the metric  $D$ , then there can be only a small deviation in the membership functions of the conclusions.

**Theorem 4.5.1.** [184] Let  $\delta \geq 0$  and  $T$  be a continuous triangular norm, and let  $P, P'$  be fuzzy intervals. If  $D(P, P') \leq \delta$  then

$$\sup_{y \in \mathbb{R}} |\mu_Q(y) - \mu_{Q'}(y)| \leq \omega_T(\max\{\omega_P(\delta), \omega_{P'}(\delta)\}).$$

where  $\omega_P(\delta)$  and  $\omega_{P'}(\delta)$  denotes the modulus of continuity of  $P$  and  $P'$  at  $\delta$ .

*Proof.* Let  $y \in \mathbb{R}$  be arbitrarily fixed. From Lemma 2.14.5 it follows that

$$\begin{aligned} |\mu_Q(y) - \mu_{Q'}(y)| &= \\ |\sup_{x \in \mathbb{R}} T(\mu_P(x), \mu_R(x, y)) - \sup_{x \in \mathbb{R}} T(\mu_{P'}(x), \mu_R(x, y))| &\leq \\ \sup_{x \in \mathbb{R}} |T(\mu_P(x), \mu_R(x, y)) - T(\mu_{P'}(x), \mu_R(x, y))| &\leq \\ \sup_{x \in \mathbb{R}} \omega_T(|\mu_P(x) - \mu_{P'}(x)|) &\leq \\ \sup_{x \in \mathbb{R}} \omega_T(\max\{\omega_P(\delta), \omega_{P'}(\delta)\}) &= \\ \omega_T(\max\{\omega_P(\delta), \omega_{P'}(\delta)\}). \end{aligned}$$

Which proves the theorem.  $\square$

It should be noted that the stability property of the conclusion  $Q$  with respect to small changes in the membership function of the observation  $P$  in the compositional rule of inference scheme is independent from the relation  $R$  (it's membership function can be discontinuous). Since the membership function of the conclusion in the compositional rule of inference can have unbounded support, it is possible that the maximal distance between the  $\alpha$ -level sets of  $Q$  and  $Q'$  is infinite, but their membership grades are arbitrarily close to each other.

The following theorem establishes the continuity property of the conclusion in the compositional rule of inference scheme.

**Theorem 4.5.2.** [184] Let  $R$  be continuous fuzzy relation, and let  $T$  be a continuous  $t$ -norm. Then  $Q$  is continuous and

$$\omega_Q(\delta) \leq \omega_T(\omega_R(\delta)),$$

for each  $\delta \geq 0$ .

*Proof.* Let  $\delta \geq 0$  be a real number and let  $u, v \in \mathbb{R}$  such that  $|u - v| \leq \delta$ . Then

$$\begin{aligned} |\mu_Q(u) - \mu_Q(v)| &= \\ |\sup_{x \in \mathbb{R}} T(\mu_P(x), \mu_R(x, u)) - \sup_{x \in \mathbb{R}} T(\mu_P(x), \mu_R(x, v))| &\leq \\ \sup_{x \in \mathbb{R}} |T(\mu_P(x), \mu_R(x, u)) - T(\mu_P(x), \mu_R(x, v))| &\leq \\ \sup_{x \in \mathbb{R}} \omega_T(|\mu_R(x, u) - \mu_R(x, v)|) &\leq \\ \sup_{x \in \mathbb{R}} \omega_T(\omega_R(|u - v|)) = \omega_T(\omega_R(|u - v|)) &\leq \omega_T(\omega_R(\delta)). \end{aligned}$$

Which ends the proof.  $\square$



From Theorem 4.5.2 it follows that the continuity property of the membership function of the conclusion  $Q$  in the compositional rule of inference scheme is independent from the observation  $P$  (it's membership function can be discontinuous). The next theorem shows that the stability property of the conclusion under small changes in the membership function of the observation holds in the discrete case, too.

**Theorem 4.5.3.** [184] *Let  $T$  be a continuous  $t$ -norm. If the observation  $P$  and the relation matrix  $R$  are finite, then*

$$H(Q, Q') \leq \omega_T(H(P, P')) \quad (4.1)$$

where  $H$  denotes the Hamming distance and the conclusions  $Q$  and  $Q'$  are computed as

$$\mu_Q(y_j) = \max_{i=1, \dots, m} T(\mu_P(x_i), \mu_R(x_i, y_j)), \quad \mu_{Q'}(y_j) = \max_{i=1, \dots, m} T(\mu_{P'}(x_i), \mu_R(x_i, y_j)),$$

for  $j = 1, \dots, n$ ,  $\text{supp}(\mu_Q) = \text{supp}(\mu_{Q'}) = \{y_1, \dots, y_n\}$  and  $\text{supp}(\mu_P) = \text{supp}(\mu_{P'}) = \{x_1, \dots, x_m\}$ .

The proof of this theorem is carried out analogously to the proof of Theorem 4.5.1. It should be noted that in the case of  $T(u, v) = \min\{u, v\}$  (4.1) yields

$$H(Q, Q') \leq H(P, P').$$

Theorems 4.5.1 and 4.5.2 can be easily extended to the compositional rule of inference with several relations:

$$\begin{array}{ll} \text{Observation:} & X \text{ has property } P \\ \text{Relation 1:} & X \text{ and } Y \text{ are in relation } W_1 \\ \dots & \\ \text{Relation } m: & X \text{ and } Y \text{ are in relation } W_m \\ \hline \text{Conclusion:} & Y \text{ has property } Q \end{array}$$
  

$$\begin{array}{ll} \text{Observation:} & X \text{ has property } P' \\ \text{Relation 1:} & X \text{ and } Y \text{ are in relation } W_1 \\ \dots & \\ \text{Relation } m: & X \text{ and } Y \text{ are in relation } W_m \\ \hline \text{Conclusion:} & Y \text{ has property } Q'. \end{array}$$

According to Zadeh's compositional rule of inference,  $Q$  and  $Q'$  are computed by sup- $T$  composition as follows

$$Q = \bigcap_{i=1}^m P \circ W_i \quad Q' = \bigcap_{i=1}^m P' \circ W_i \quad (4.2)$$

Generalizing Theorems 4.5.1 and 4.5.2 about the case of single relation, we show that when the observations are close to each other in the metric  $D$ , then there can be only a small deviation in the membership function of the conclusions even if we have several relations.

**Theorem 4.5.4.** [181] Let  $\delta \geq 0$  and  $T$  be a continuous triangular norm, and let  $P, P'$  be continuous fuzzy intervals. If

$$D(P, P') \leq \delta$$

then

$$\sup_{y \in \mathbf{R}} |\mu_Q(y) - \mu_{Q'}(y)| \leq \omega_T(\max\{\omega_P(\delta), \omega_{P'}(\delta)\})$$

where  $Q$  and  $Q'$  are computed by (4.2).

In the following theorem we establish the continuity property of the conclusion under continuous fuzzy relations  $W_i$  and continuous t-norm  $T$ .

**Theorem 4.5.5.** [181] Let  $W_i$  be continuous fuzzy relation,  $i=1, \dots, m$  and let  $T$  be a continuous t-norm. Then  $Q$  is continuous and

$$\omega_Q(\delta) \leq \omega_T(\omega(\delta)) \quad \text{for each } \delta \geq 0$$

where  $\omega(\delta) = \max\{\omega_{W_1}(\delta), \dots, \omega_{W_m}(\delta)\}$ .

The above theorems are also valid for Multiple Fuzzy Reasoning (MFR) schemes:

$$\begin{array}{lcl} \text{Observation:} & P & P' \\ \text{Implication 1:} & P_1 \rightarrow Q_1 & P'_1 \rightarrow Q'_1 \\ & \dots & \dots \\ \text{Implication } m: & P_m \rightarrow Q_m & P'_m \rightarrow Q'_m \\ \hline \text{Conclusion:} & Q & Q' \end{array}$$

where  $Q$  and  $Q'$  are computed by sup- $T$  composition as follows

$$Q = P \circ \bigcap_{i=1}^m P_i \rightarrow Q_i, \quad Q' = P' \circ \bigcap_{i=1}^m P'_i \rightarrow Q'_i,$$

i.e.,

$$\begin{aligned} \mu_Q(y) &= \sup_{x \in \mathbf{R}} T(\mu_P(x), \min_{i=1, \dots, m} \mu_{P_i}(x) \rightarrow \mu_{Q_i}(y)), \\ \mu_{Q'}(y) &= \sup_{x \in \mathbf{R}} T(\mu_{P'}(x), \min_{i=1, \dots, m} \mu_{P'_i}(x) \rightarrow \mu_{Q'_i}(y)). \end{aligned}$$

Then the following theorems hold.

**Theorem 4.5.6.** [181] Let  $\delta \geq 0$ , let  $T$  be a continuous triangular norm, let  $P, P', P_i, P'_i, Q_i, Q'_i, i = 1, \dots, m$ , be fuzzy intervals and let  $\rightarrow$  be a continuous fuzzy implication operator. If

$$\max\{D(P, P'), \max_{i=1, \dots, m} D(P_i, P'_i), \max_{i=1, \dots, m} D(Q_i, Q'_i)\} \leq \delta,$$

then

$$\sup_{y \in \mathbf{R}} |\mu_Q(y) - \mu_{Q'}(y)| \leq \omega_T(\max\{\omega(\delta), \omega_{\rightarrow}(\omega(\delta))\}),$$

where

$$\omega(\delta) = \max\{\omega_{P_i}(\delta), \omega_{P'_i}(\delta), \omega_{Q_i}(\delta), \omega_{Q'_i}(\delta)\},$$

and  $\omega_{\rightarrow}$  denotes the modulus of continuity of the fuzzy implication operator.

**Theorem 4.5.7.** [181] Let  $\rightarrow$  be a continuous fuzzy implication operator, let  $P, P', P_i, P'_i, Q_i, Q'_i, i = 1, \dots, m$ , be fuzzy intervals and let  $T$  be a continuous t-norm. Then  $Q$  is continuous and

$$\omega_Q(\delta) \leq \omega_T(\omega_{\rightarrow}(\omega(\delta))) \quad \text{for each } \delta \geq 0,$$

where

$$\omega(\delta) = \max\{\omega_{P_i}(\delta), \omega_{P'_i}(\delta), \omega_{Q_i}(\delta), \omega_{Q'_i}(\delta)\},$$

and  $\omega_{\rightarrow}$  denotes the modulus of continuity of the fuzzy implication operator.

From  $\lim_{\delta \rightarrow 0} \omega(\delta) = 0$  and Theorem 4.5.6 it follows that

$$\|\mu_Q - \mu_{Q'}\|_{\infty} = \sup_y |\mu_Q(y) - \mu_{Q'}(y)| \rightarrow 0$$

whenever  $D(P, P') \rightarrow 0$ ,  $D(P_i, P'_i) \rightarrow 0$  and  $D(Q_i, Q'_i) \rightarrow 0$ ,  $i = 1, \dots, m$ , which means the stability of the conclusion under small changes of the observation and rules.

The stability property of the conclusion under small changes of the membership function of the observation and rules guarantees that small rounding errors of digital computation and small errors of measurement of the input data can cause only a small deviation in the conclusion, i.e. every successive approximation method can be applied to the computation of the linguistic approximation of the exact conclusion.

## 4.6 Computation of the compositional rule of inference under t-norms

In approximate reasoning there are several kinds of inference rules, which deal with the problem of deduction of conclusions in an imprecise setting. An important problem is the (approximate) computation of the membership function of the conclusion in these schemes. Throughout this Section shall use  $\phi$ -functions [215] for the representation of linguistic terms in the compositional rule of inference.

**Definition 4.6.1.** A  $\phi$ -function is defined by

$$\phi(x; a, b, c, d) = \begin{cases} 1 & \text{if } b \leq x \leq c \\ \phi_1\left(\frac{x-a}{b-a}\right) & \text{if } a \leq x \leq b, a < b, \\ \phi_2\left(\frac{x-c}{d-c}\right) & \text{if } c \leq x \leq d, c < d, \\ 0 & \text{otherwise} \end{cases} \quad (4.3)$$

where  $\phi_1: [0, 1] \rightarrow [0, 1]$  is continuous, monoton increasing function and  $\phi_1(0) = 0$ ,  $\phi_1(1) = 1$ ;  $\phi_2: [0, 1] \rightarrow [0, 1]$  is continuous, monoton decreasing function and  $\phi_2(0) = 1$ ,  $\phi_2(1) = 0$ . So  $\phi$  is a function which is 0 left of  $a$ , increases to 1 in  $(a, b)$ , is 1 in  $[b, c]$ , decreases to 0 in  $(c, d)$  and is 0 right of  $d$  (for the sake of simplicity, we do not consider the cases  $a = b$  or  $c = d$ ).

It should be noted that  $\phi$  can be considered as the membership function of the fuzzy interval  $\tilde{a} = (b, c, b - a, d - c)_{LR}$ , with  $R(x) = \phi_2(x)$  and  $L(x) = \phi_1(1 - x)$ .

In [215] Hellendoorn showed the closure property of the compositional rule of inference under sup-min composition and presented exact calculation formulas for the membership function of the conclusion when both the observation and relation parts are given by  $S^-$ ,  $\pi^-$ , or  $\phi$ -function. Namely, he proved the following theorem.

**Theorem 4.6.1.** [215] *In the compositional rule of inference under minimum norm,*

$$\begin{array}{lcl} \text{Observation:} & X \text{ has property } P & \\ \text{Relation:} & X \text{ and } Y \text{ are in relation } W & \\ \hline \text{Conclusion:} & Y \text{ has property } Q & \end{array}$$

*is true that, when  $\mu_P(x) = \phi(x; a_1, a_2, a_3, a_4)$  and  $\mu_W(x, y) = \phi(y - x; b_1, b_2, b_3, b_4)$  then*

$$\mu_Q(y) = \phi(y; a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4),$$

*where the function  $\phi$  is defined by (4.3) .*

In this Section, following Fullér and Werners [178], and Fullér and Zimmermann [179], generalizing Hellendoorn's results, we derive exact calculation formulas for the compositional rule of inference under triangular norms when both the observation and the part of the relation (rule) are given by concave  $\phi$ -function [215]; and the t-norm is Archimedean with a strictly convex additive generator function. The efficiency of this method stems from the fact that the distributions, involved in the relation and observation, are represented by a parametrized  $\phi$ -function. The deduction process then consists of some simple computations performed on the parameters.

We consider the compositional rule of inference, where, the membership functions of  $P$  and  $W$  are defined by means of a particular  $\phi$ -function, and the membership function of the conclusion  $Q$  is defined by sup- $T$  composition of  $P$  and  $W$

$$Q(y) = (P \circ W)(y) = \sup_x T(P(x), W(x, y)), y \in \mathbb{R}.$$

The following theorem presents an efficient method for the exact computation of the membership function of the conclusion.

**Theorem 4.6.2.** [179] *Let  $T$  be an Archimedean t-norm with additive generator  $f$  and let  $P(x) = \phi(x; a, b, c, d)$  and  $W(x, y) = \phi(y - x; a + u, b + u, c + v, d + v)$ . If  $\phi_1$  and  $\phi_2$  are twice differentiable, concave functions, and  $f$  is a twice differentiable, strictly convex function, then*

$$Q(y) = \begin{cases} 1 & \text{if } 2b + u \leq y \leq 2c + v \\ f^{[-1]} \left( 2f \left( \phi_1 \left[ \frac{y - 2a - u}{2(b - a)} \right] \right) \right) & \text{if } 2a + u \leq y \leq 2b + u \\ f^{[-1]} \left( 2f \left( \phi_2 \left[ \frac{y - 2c - v}{2(d - c)} \right] \right) \right) & \text{if } 2c + v \leq y \leq 2d + v \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* Using Theorem 2.9.1 with

$$\tilde{a}_1 = (b, c, b - a, d - c)_{LR} \text{ and } \tilde{a}_2 = (b + u, c + v, b - a, d - c)_{LR},$$

where  $R(x) = \phi_2(x)$  and  $L(x) = \phi_1(1 - x)$ , we have

$$\begin{aligned} \mu_Q(y) &= \sup_{x \in R} T(\mu_P(x), \mu_W(x, y)) = \\ &= \sup_{x \in R} T((\phi(x; a, b, c, d), \phi(y - x; a + u, b + u, c + v, d + v)) = (\tilde{a}_1 + \tilde{a}_2)(y). \end{aligned}$$

Which proves the theorem.  $\square$

It should be noted that we have calculated the membership function of  $Q$  under the assumption that the left and right spreads of  $P$  do not differ from the left and right spreads of  $W$  (the lengths of their tops can be different). To determine the exact membership function of  $Q$  in the general case:  $P(x) = \phi(x; a_1, a_2, a_3, a_4)$  and  $W(x, y) = \phi(y - x; b_1, b_2, b_3, b_4)$  can be very tricky (see [236]).

Using Theorem 4.6.2 we shall compute the exact membership function of the conclusion  $Q$  in the case of Yager's, Dombi's and Hamacher's parametrized t-norm. Let us consider the following scheme

$$\begin{array}{rcl} P(x) & = & \phi(x; a, b, c, d) \\ W(y, x) & = & \phi(y - x; a + u, b + u, c + v, d + v) \\ \hline Q(y) & = & (P \circ W)(y) \end{array}$$

Denoting

$$\sigma := \frac{(y - 2a - u)}{2(b - a)}, \quad \theta := \frac{y - 2c - v}{2(d - c)},$$

we get the following formulas for the membership function of the conclusion  $Q$ .

- Yager's t-norm with  $p > 1$ . Here

$$T(x, y) = 1 - \min \left\{ 1, \sqrt[p]{(1 - x)^p + (1 - y)^p} \right\}.$$

with generator  $f(t) = (1 - t)^p$ , and

$$Q(y) = \begin{cases} 1 - 2^{1/p}(1 - \phi_1(\sigma)) & \text{if } 0 < \sigma < \phi_1^{-1}(2^{-1/p}), \\ 1 & \text{if } 2b + u \leq y \leq 2c + v, \\ 1 - 2^{1/p}(1 - \phi_2(\theta)) & \text{if } 0 < \theta < \phi_2^{-1}(1 - 2^{1/p}), \end{cases} \quad (4.4)$$

- Hamacher's t-norm with  $p \leq 2$ . Here

$$T(x, y) = \frac{xy}{p + (1 - p)(x + y - xy)}$$

with generator

$$f(t) = \ln \frac{p + (1 - p)t}{t},$$

and

$$Q(y) = \begin{cases} \frac{p}{\tau_1^2 - 1 + p} & \text{if } 0 < \sigma < 1, \\ 1 & \text{if } 2b + u \leq y \leq 2c + v, \\ \frac{p}{\tau_2^2 - 1 + p} & \text{if } 0 < \theta < 1, \end{cases}$$

where

$$\tau_1 = \frac{p + (1-p)\phi_1(\sigma)}{\phi_1(\sigma)}, \quad \tau_2 = \frac{p + (1-p)\phi_2(\sigma)}{\phi_2(\sigma)}$$

- Dombi's t-norm with  $p > 1$ . Here

$$T(x, y) = \frac{1}{1 + \sqrt[p]{(1/x - 1)^p + (1/y - 1)^p}}$$

with additive generator

$$f(t) = \left(\frac{1}{t} - 1\right)^p,$$

and

$$Q(y) = \begin{cases} \frac{1}{1 + 2^{1/p}\tau_3} & \text{if } 0 < \sigma < 1, \\ 1 & \text{if } 2b + u \leq y \leq 2c + v, \\ \frac{1}{1 + 2^{1/p}\tau_4} & \text{if } 0 < \theta < 1, \end{cases}$$

where

$$\tau_3 = \frac{1}{\phi_1(\sigma)} - 1, \quad \tau_4 = \frac{1}{\phi_2(\sigma)} - 1$$

**Example.** We illustrate Theorem 4.6.2 by the following example (see Fig. 4.9-11):

$x$ is close to 3	$\phi(x; 1, 3, 4, 7)$
$x$ and $y$ are approximately equal	$\phi(y - x; -2, 0, 0, 3)$
$y$ is more or less close to $[3, 4]$	$Q(y)$

where  $Q(y)$  is computed by Yager's t-norm (4.4).

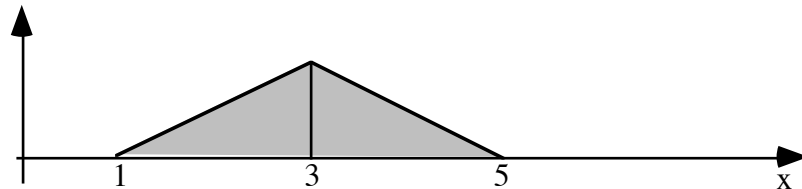


Figure 4.9: "x is close to  $[3, 4]$ "

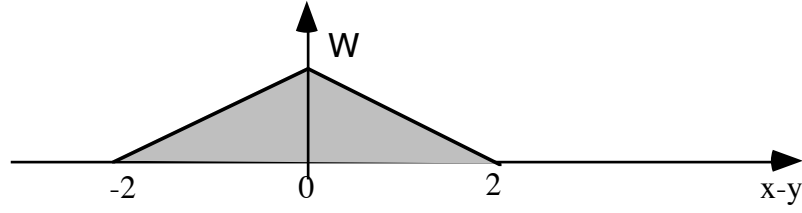


Figure 4.10: "x and y are approximately equal"

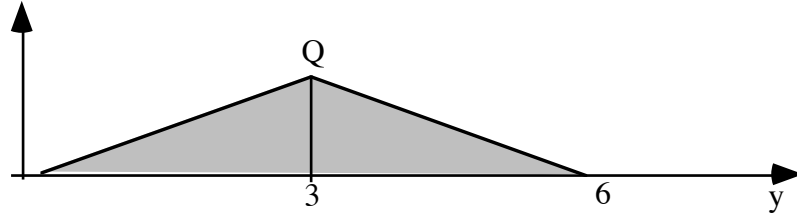


Figure 4.11: "y is more or less close to [3, 4]", Yager's t-norm.

We have used the membership function  $\phi(y - x; -2, 0, 0, 3)$  to describe "x and y are approximately equal". This means that the membership degree is one, iff x and y are equal in the classical sense. If  $y - x > 2$  or  $x - y > 3$ , then the degree of membership is 0. The conclusion Q has been called "y is more or less close to [3, 4]", because  $P(t) = Q(T) = 1$ , when  $t \in [3, 4]$  and  $P(t) < Q(t)$  otherwise.

## 4.7 On the generalized method-of-case inference rule

In this Section we will deal with the generalized method-of-case (GMC) inference scheme with fuzzy antecedents, which has been introduced by Da in [102]. We show that when the fuzzy numbers involved in the observation part of the scheme have continuous membership functions; and the t-norm, t-conorm used in the definition of the membership function of the conclusion are continuous, then the conclusion defined by the compositional rule of inference depends continuously on the observation.

When the predicates are crisp then the method of cases reads

Observation:	$A$	or	$B$
Antecedent 1:	if $A$	then	$C$
Antecedent 2:	if $B$	then	$C$
Conclusion:	$C$		

This equivalent to saying that the formula is a tautology in binary logic where  $A$ ,  $B$  and  $C$  are propositional variables.

The proof of many theorems in conventional mathematics is based on this scheme, e.g. theorems involving the absolute value of a real variable are usually proved by considering separately positive and nonpositive values of the variable, and the conclusion is derived in each of these cases.

We will investigate the effect of small changes of the observation to the conclusion of similar deduction schemes when the antecedents involve fuzzy concepts.

Let  $X, Y$  and  $Z$  be variables taking values in universes  $U, V$  and  $W$ , respectively and let  $A, A' \in \mathcal{F}(U)$ ,  $B, B' \in \mathcal{F}(U)$ , and  $C \in \mathcal{F}(W)$ , then the generalized method of cases reads:

$$\begin{array}{ll} \text{Observation:} & X \text{ is } A' \text{ OR } Y \text{ is } B' \\ \text{Antecedent 1:} & \text{IF } X \text{ is } A \text{ THEN } Z \text{ is } C \\ \text{Antecedent 2:} & \text{IF } Y \text{ is } B \text{ THEN } Z \text{ is } C \\ \hline \text{Conclusion:} & Z \text{ is } C' \end{array}$$

The conclusion  $C'$  is given by applying the general compositional rule of inference

$$C'(w) = \sup_{(u,v) \in U \times V} T(S(A'(u), B'(v)), I(A(u), C(w)), I(B(v), C(w))) \quad (4.5)$$

where  $T$  is an arbitrary triangular norm,  $S$  is an arbitrary conorm and  $I$  represents an arbitrary fuzzy implication operator. For instance,

$$\begin{array}{ll} \text{Observation:} & \text{This bunch of grapes is fairly sweet OR this bunch} \\ & \text{of grapes is more or less yellow} \\ \text{Antecedent 1:} & \text{IF a bunch of grapes is yellow THEN the bunch of} \\ & \text{grapes is ripe} \\ \text{Antecedent 2:} & \text{IF a bunch of grapes is sweet THEN the bunch of} \\ & \text{grapes is ripe} \\ \hline \text{Conclusion:} & \text{This bunch of grapes is more or less ripe} \end{array}$$

Consider now the generalized method-of-case scheme with different fuzzy observations  $A', A'', B', B''$ :

$$\begin{array}{ll} X \text{ is } A' \text{ OR } Y \text{ is } B' & X \text{ is } A'' \text{ OR } Y \text{ is } B'' \\ \text{IF } X \text{ is } A \text{ THEN } Z \text{ is } C & \text{IF } X \text{ is } A \text{ THEN } Z \text{ is } C \\ \text{IF } Y \text{ is } B \text{ THEN } Z \text{ is } C & \text{IF } Y \text{ is } B \text{ THEN } Z \text{ is } C \\ \hline Z \text{ is } C' & Z \text{ is } C'' \end{array}$$

where  $C'$  and  $C''$  are defined by the compositional rule of inference, in the sense of (4.5), i.e.

$$C'(w) = \sup_{(u,v) \in U \times V} T(S(A'(u), B'(v)), I(A(u), C(w)), I(B(v), C(w))) \quad (4.6)$$

$$C''(w) = \sup_{(u,v) \in U \times V} T(S(A''(u), B''(v)), I(A(u), C(w)), I(B(v), C(w))) \quad (4.7)$$

The following theorem gives an upper estimation for the distance between the conclusions  $C'$  and  $C''$  obtained from GMC schemes above.

**Theorem 4.7.1.** [175] *Let  $T$  and  $S$  be continuous functions and let  $A', A'', B'$  and  $B''$  be continuous fuzzy numbers. Then with the notation*

$$\Delta = \max\{\omega_{A'}(D(A', A'')), \omega_{A''}(D(A', A'')), \omega_{B'}(D(B', B'')), \omega_{B''}(D(B', B'')), \}$$



we have

$$\sup_{w \in W} |C'(w) - C''(w)| \leq \omega_T(\omega_S(\Delta)), \quad (4.8)$$

where

$$\Delta = \max\{\omega_{A''}(\Delta), \omega_{A''}(\Delta), \omega_{B'}(\Delta), \omega_{B''}(\Delta), \},$$

and the conclusions  $C'$ ,  $C''$  are defined by (4.6) and (4.7), respectively.

It should be noted that: (i) from (4.8) it follows that  $C' \rightarrow C''$  uniformly as  $\Delta \rightarrow 0$ , which means the stability (in the classical sense) of the conclusion under small changes of the fuzzy terms; (ii) the stability or instability of the conclusion does not depend on the implication operator  $I$ .

**Remark.** In 1992 Fedrizzi and Fullér [150] considered a Group Decision Support System (GDSS) logic architecture in which linguistic variables and fuzzy production rules were used for reaching consensus, and showed that the degrees of consensus (defined by a certain similarity measure) relative to each alternative are stable under small changes in the experts' opinions.

### 4.7.1 Illustration

For illustration of this theorem consider the following schemes with arbitrary continuous fuzzy numbers  $A$ ,  $B$  and  $C$ :

$X \text{ is } A \text{ OR } Y \text{ is } B$	$X \text{ is } A \text{ OR } Y \text{ is more or less } B$
$\text{IF } X \text{ is } A \text{ THEN } Z \text{ is } C$	$\text{IF } X \text{ is } A \text{ THEN } Z \text{ is } C$
$\text{IF } Y \text{ is } B \text{ THEN } Z \text{ is } C$	$\text{IF } Y \text{ is } B \text{ THEN } Z \text{ is } C$
$Z \text{ is } C'$	$Z \text{ is } C''$

where

$$(\text{more or less } B)(y) := \sqrt{B(y)},$$

for  $y \in \mathbb{R}$ ,

$$T(u, v) = \min\{u, v\},$$

(minimum norm);

$$S(x, y) = \max\{u, v\},$$

(maximum conorm);

$$I(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

(Gödel's implication operator).

Following Da ([102], p.125), we get  $C' = C$  and  $C'' = \text{more or less } C'$ , i.e.

$$C''(w) = \sqrt{C(w)}, \quad w \in \mathbb{R}.$$

So,

$$\sup_{w \in \mathbb{R}} |C'(w) - C''(w)| = \sup_{w \in \mathbb{R}} |C(w) - \sqrt{C(w)}| = 1/4$$

On the other hand, using the relationships,

$$D(A, A) = 0, \quad D(B, \text{more or less } B) \leq 1/4;$$

$$\omega_S(\Delta) = \Delta, \omega_T(\Delta) = \Delta, \quad \Delta > 0;$$

Theorem 4.7.1 gives

$$\sup_{w \in R} |C'(w) - C''(w)| \leq \max\{\omega_B(1/4), \omega_{\{\text{more or less } B\}}(1/4)\} \leq 1/4$$

which means, that our estimation (4.8) is sharp, i.e. there exist  $C'$  and  $C''$ , such that

$$\sup_{w \in R} |C'(w) - C''(w)| = \omega_T(\omega_S(\Delta)).$$



# Chapter 5

## Fuzzy Optimization

### 5.1 Possibilistic linear equality systems

Modelling real world problems mathematically we often have to find a solution to a linear equality system

$$a_{i1}x_1 + \cdots + a_{in}x_n = b_i, \quad i = 1, \dots, m, \quad (5.1)$$

or shortly,

$$Ax = b,$$

where  $a_{ij}$ ,  $b_i$  and  $x_j$  are real numbers. It is known that system (5.1) generally belongs to the class of ill-posed problems, so a small perturbation of the parameters  $a_{ij}$  and  $b_i$  may cause a large deviation in the solution.

A possibilistic linear equality system is

$$\tilde{a}_{i1}x_1 + \cdots + \tilde{a}_{in}x_n = \tilde{b}_i, \quad i = 1, \dots, m, \quad (5.2)$$

or shortly,

$$\tilde{A}x = \tilde{b},$$

where  $\tilde{a}_{ij}$ ,  $\tilde{b}_i \in \mathcal{F}(\mathbb{R})$  are fuzzy quantities,  $x \in \mathbb{R}^n$ , the operations addition and multiplication by a real number of fuzzy quantities are defined by Zadeh's extension principle and the equation is understood in possibilistic sense. Recall the truth value of the assertion " $\tilde{a}$  is equal to  $\tilde{b}$ ", written as  $\tilde{a} = \tilde{b}$ , denoted by  $\text{Pos}(\tilde{a} = \tilde{b})$ , is defined as

$$\text{Pos}(\tilde{a} = \tilde{b}) = \sup_t \{\tilde{a}(t) \wedge \tilde{b}(t)\} = (\tilde{a} - \tilde{b})(0). \quad (5.3)$$

We denote by  $\mu_i(x)$  the degree of satisfaction of the  $i$ -th equation in (5.2) at the point  $x \in \mathbb{R}^n$ , i.e.

$$\mu_i(x) = \text{Pos}(\tilde{a}_{i1}x_1 + \cdots + \tilde{a}_{in}x_n = \tilde{b}_i).$$

Following Bellman and Zadeh [10] the fuzzy solution (or the fuzzy set of feasible solutions) of system (5.2) can be viewed as the intersection of the  $\mu_i$ 's such that

$$\mu(x) = \min\{\mu_1(x), \dots, \mu_m(x)\}. \quad (5.4)$$

A measure of consistency for the possibilistic equality system (5.2) is defined as

$$\mu^* = \sup\{\mu(x) \mid x \in \mathbb{R}^n\}. \quad (5.5)$$

Let  $X^*$  be the set of points  $x \in \mathbb{R}^n$  for which  $\mu(x)$  attains its maximum, if it exists. That is

$$X^* = \{x^* \in \mathbb{R}^n \mid \mu(x^*) = \mu^*\}$$

If  $X^* \neq \emptyset$  and  $x^* \in X^*$ , then  $x^*$  is called a maximizing (or best) solution of (5.2).

If  $\tilde{a}$  and  $\tilde{b}$  are fuzzy numbers with  $[a]^\alpha = [a_1(\alpha), a_2(\alpha)]$  and  $[b]^\alpha = [b_1(\alpha), b_2(\alpha)]$  then their Hausdorff distance is defined as

$$D(\tilde{a}, \tilde{b}) = \sup_{\alpha \in [0,1]} \max\{|a_1(\alpha) - b_1(\alpha)|, |a_2(\alpha) - b_2(\alpha)|\}.$$

i.e.  $D(\tilde{a}, \tilde{b})$  is the maximal distance between the  $\alpha$ -level sets of  $\tilde{a}$  and  $\tilde{b}$ .

Let  $L > 0$  be a real number. By  $\mathcal{F}(L)$  we denote the set of all fuzzy numbers  $\tilde{a} \in \mathcal{F}$  with membership function satisfying the Lipschitz condition with constant  $L$ , i.e.

$$|\tilde{a}(t) - \tilde{a}(t')| \leq L|t - t'|, \quad \forall t, t' \in \mathbb{R}.$$

In many important cases the fuzzy parameters  $\tilde{a}_{ij}, \tilde{b}_i$  of the system (5.2) are not known exactly and we have to work with their approximations  $\tilde{a}_{ij}^\delta, \tilde{b}_i^\delta$  such that

$$\max_{i,j} D(\tilde{a}_{ij}, \tilde{a}_{ij}^\delta) \leq \delta, \quad \max_i D(\tilde{b}_i, \tilde{b}_i^\delta) \leq \delta, \quad (5.6)$$

where  $\delta \geq 0$  is a real number. Then we get the following system with perturbed fuzzy parameters

$$\tilde{a}_{i1}^\delta x_1 + \cdots + \tilde{a}_{in}^\delta x_n = \tilde{b}_i^\delta, \quad i = 1, \dots, m \quad (5.7)$$

or shortly,

$$\tilde{A}^\delta x = \tilde{b}^\delta.$$

In a similar manner we define the solution

$$\mu^\delta(x) = \min\{\mu_1^\delta(x), \dots, \mu_m^\delta(x)\},$$

and the measure of consistency

$$\mu^*(\delta) = \sup\{\mu^\delta(x) \mid x \in \mathbb{R}^n\},$$

of perturbed system (5.7), where

$$\mu_i^\delta(x) = \text{Pos}(\tilde{a}_{i1}^\delta x_1 + \cdots + \tilde{a}_{in}^\delta x_n = \tilde{b}_i^\delta)$$

denotes the degree of satisfaction of the  $i$ -th equation at  $x \in \mathbb{R}^n$ . Let  $X^*(\delta)$  denote the set of maximizing solutions of the perturbed system (5.7).

Kovács [291] showed that the fuzzy solution to system (5.2) with symmetric triangular fuzzy numbers is a stable with respect to small changes of centres of fuzzy parameters. Following Fullér [170] in the next theorem we establish a stability property (with respect to perturbations (5.6)) of the solution of system (5.2).

**Theorem 5.1.1.** [170] Let  $L > 0$  and  $\tilde{a}_{ij}, \tilde{a}_{ij}^\delta, \tilde{b}_i, \tilde{b}_i^\delta \in \mathcal{F}(L)$ . If (5.6) holds, then

$$\|\mu - \mu^\delta\|_\infty = \sup_{x \in \mathbb{R}^n} |\mu(x) - \mu^\delta(x)| \leq L\delta, \quad (5.8)$$

where  $\mu(x)$  and  $\mu^\delta(x)$  are the (fuzzy) solutions to systems (5.2) and (5.7), respectively.

*Proof.* It is sufficient to show that

$$|\mu_i(x) - \mu_i^\delta(x)| \leq L\delta$$

for each  $x \in \mathbb{R}^n$  and  $i = 1, \dots, m$ . Let  $x \in \mathbb{R}^n$  and  $i \in \{1, \dots, m\}$  be arbitrarily fixed. From (5.3) it follows that

$$\mu_i(x) = \left( \sum_{j=1}^n \tilde{a}_{ij} x_j - \tilde{b}_i \right) (0), \quad \mu_i^\delta(x) = \left( \sum_{j=1}^n \tilde{a}_{ij}^\delta x_j - \tilde{b}_i^\delta \right) (0).$$

Applying Lemma 2.14.1 we have

$$D \left( \sum_{j=1}^n \tilde{a}_{ij} x_j - \tilde{b}_i, \sum_{j=1}^n \tilde{a}_{ij}^\delta x_j - \tilde{b}_i^\delta \right) \leq \sum_{j=1}^n |x_j| D(\tilde{a}_{ij}, \tilde{a}_{ij}^\delta) + D(\tilde{b}_i, \tilde{b}_i^\delta) \leq \delta(|x|_1 + 1),$$

where  $|x|_1 = |x_1| + \dots + |x_n|$ . Finally, by Lemma 2.14.6 we have

$$\sum_{j=1}^n \tilde{a}_{ij} x_j - \tilde{b}_i \in \mathcal{F} \left( \frac{L}{|x|_1 + 1} \right), \quad \sum_{j=1}^n \tilde{a}_{ij}^\delta x_j - \tilde{b}_i^\delta \in \mathcal{F} \left( \frac{L}{|x|_1 + 1} \right)$$

and, therefore

$$\begin{aligned} |\mu_i(x) - \mu_i^\delta(x)| &= \left| \left( \sum_{j=1}^n \tilde{a}_{ij} x_j - \tilde{b}_i \right) (0) - \left( \sum_{j=1}^n \tilde{a}_{ij}^\delta x_j - \tilde{b}_i^\delta \right) (0) \right| \leq \\ &\sup_{t \in \mathbb{R}} \left| \left( \sum_{j=1}^n \tilde{a}_{ij} x_j - \tilde{b}_i \right) (t) - \left( \sum_{j=1}^n \tilde{a}_{ij}^\delta x_j - \tilde{b}_i^\delta \right) (t) \right| \leq \\ &\frac{L}{|x|_1 + 1} \times \delta(|x|_1 + 1) = L\delta. \end{aligned}$$

Which proves the theorem. □

From (5.8) it follows that

$$|\mu^* - \mu^*(\delta)| \leq L\delta,$$

where  $\mu^*, \mu^*(\delta)$  are the measures of consistency for the systems (5.2) and (5.7), respectively. It is easily checked that in the general case  $\tilde{a}_{ij}, \tilde{b}_i \in \mathcal{F}(\mathbb{R})$  the solution to possibilistic linear

equality system (5.2) may be unstable (in metric  $C_\infty$ ) under small variations in the membership function of fuzzy parameters (in metric D).

When the problem is to find a maximizing solution to a possibilistic linear equality system (5.2), then according to Negoita [338], we are led to solve the following optimization problem

$$\begin{aligned} & \text{maximize} \quad \lambda \\ & \mu_1(x_1, \dots, x_n) \geq \lambda, \\ & \quad \dots \\ & \mu_m(x_1, \dots, x_n) \geq \lambda, \\ & x \in \mathbb{R}^n, \quad 0 \leq \lambda \leq 1. \end{aligned} \tag{5.9}$$

Finding the solutions of problem (5.9) generally requires the use of nonlinear programming techniques, and could be tricky. However, if the fuzzy numbers in (5.2) are of trapezoidal form, then the problem (5.9) turns into a quadratically constrained programming problem.

Even though the fuzzy solution and the measure of consistency of system (5.2) have a stability property with respect to changes of the fuzzy parameters, the behavior of the maximizing solution towards small perturbations of the fuzzy parameters can be very fortuitous, i.e. supposing that,  $X^*$ , the set of maximizing solutions to system (5.2) is not empty, the distance between  $x^*(\delta)$  and  $X^*$  can be very big, where  $x^*(\delta)$  is a maximizing solution of the perturbed possibilistic equality system (5.7).

Consider now the possibilistic equality system (5.2) with fuzzy numbers of symmetric triangular form

$$(a_{i1}, \alpha)x_1 + \dots + (a_{in}, \alpha)x_n = (b_i, \alpha), \quad i = 1, \dots, m,$$

or shortly,

$$(A, \alpha)x = (b, \alpha) \tag{5.10}$$

Then following Kovács and Fullér [293] the fuzzy solution of (5.10) can be written in a compact form

$$\mu(x) = \begin{cases} 1 & \text{if } Ax = b \\ 1 - \frac{\|Ax - b\|_\infty}{\alpha(|x|_1 + 1)} & \text{if } 0 < \|Ax - b\|_\infty \leq \alpha(|x|_1 + 1) \\ 0 & \text{if } \|Ax - b\|_\infty > \alpha(|x|_1 + 1) \end{cases}$$

where

$$\|Ax - b\|_\infty = \max\{|\langle a_1, x \rangle - b_1|, \dots, |\langle a_m, x \rangle - b_m|\}.$$

If

$$[\mu]^1 = \{x \in \mathbb{R}^n \mid \mu(x) = 1\} \neq \emptyset$$

then the set of maximizing solutions,  $X^* = [\mu]^1$ , of (5.10) coincides with the solution set, denoted by  $X^{**}$ , of the crisp system  $Ax = b$ . The stability theorem for system (5.10) reads

**Theorem 5.1.2.** [291] *If*

$$D(\tilde{A}, \tilde{A}^\delta) = \max_{i,j} |a_{ij} - a_{ij}^\delta| \leq \delta, \quad D(\tilde{b}, \tilde{b}^\delta) = \max_i |b_i - b_i^\delta| \leq \delta$$

hold, then

$$\|\mu - \mu^\delta\|_\infty = \sup_x |\mu(x) - \mu^\delta(x)| \leq \frac{\delta}{\alpha},$$

where  $\mu(x)$  and  $\mu^\delta(x)$  are the fuzzy solutions to possibilistic equality systems

$$(A, \alpha)x = (b, \alpha),$$

and

$$(A^\delta, \alpha)x = (b^\delta, \alpha),$$

respectively.

Theorem 5.1.1 can be extended to possibilistic linear equality systems with (continuous) fuzzy numbers.

**Theorem 5.1.3.** [174] Let  $\tilde{a}_{ij}, \tilde{a}_{ij}^\delta, \tilde{b}_i, \tilde{b}_i^\delta \in \mathcal{F}$  be fuzzy numbers. If (5.6) holds, then

$$\|\mu - \mu^\delta\|_\infty \leq \omega(\delta),$$

where  $\omega(\delta)$  denotes the maximum of modulus of continuity of all fuzzy coefficients at  $\delta$  in (5.2) and (5.7).

In 1992 Kovács [299] showed a wide class of fuzzified systems that are well-posed extensions of ill-posed linear equality and inequality systems.

### 5.1.1 Examples

Consider the following two-dimensional possibilistic equality system

$$(1, \alpha)x_1 + (1, \alpha)x_2 = (0, \alpha) \tag{5.11}$$

$$(1, \alpha)x_1 - (1, \alpha)x_2 = (0, \alpha)$$

Then its fuzzy solution is

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0 \\ \tau_2(x) & \text{if } 0 < \max\{|x_1 - x_2|, |x_1 + x_2|\} \leq \alpha(|x_1| + |x_2| + 1) \\ 0 & \text{if } \max\{|x_1 - x_2|, |x_1 + x_2|\} > \alpha(|x_1| + |x_2| + 1) \end{cases}$$

where

$$\tau_2(x) = 1 - \frac{\max\{|x_1 - x_2|, |x_1 + x_2|\}}{\alpha(|x_1| + |x_2| + 1)},$$

and the only maximizing solution of system (5.11) is  $x^* = (0, 0)$ . There is no problem with stability of the solution even for the crisp system

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

because  $\det(A) \neq 0$ .



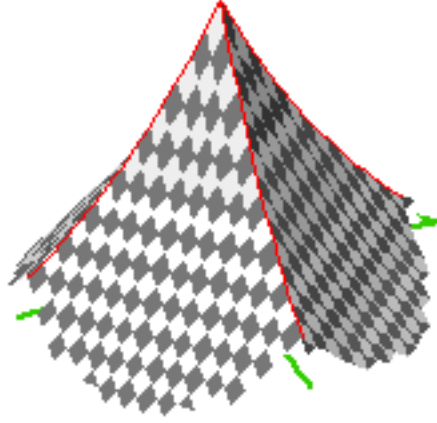


Figure 5.1: The graph of fuzzy solution of system (5.11) with  $\alpha = 0.4$ .

The fuzzy solution of possibilistic equality system

$$(1, \alpha)x_1 + (1, \alpha)x_2 = (0, \alpha) \quad (5.12)$$

$$(1, \alpha)x_1 + (1, \alpha)x_2 = (0, \alpha)$$

is

$$\mu(x) = \begin{cases} 1 & \text{if } |x_1 + x_2| = 0 \\ 1 - \frac{|x_1 + x_2|}{\alpha(|x_1| + |x_2| + 1)} & \text{if } 0 < |x_1 + x_2| \leq \alpha(|x_1| + |x_2| + 1) \\ 0 & \text{if } |x_1 + x_2| > \alpha(|x_1| + |x_2| + 1) \end{cases}$$

and the set of its maximizing solutions is

$$X^* = \{x \in \mathbb{R}^2 \mid x_1 + x_2 = 0\}.$$

In this case we have

$$X^* = X^{**} = \{x \in \mathbb{R}^2 \mid Ax = b\}.$$

We might experience problems with the stability of the solution of the crisp system

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

because  $\det(A) = 0$ .

Really, the fuzzy solution of possibilistic equality system

$$(1, \alpha)x_1 + (1, \alpha)x_2 = (\delta_1, \alpha) \quad (5.13)$$

$$(1, \alpha)x_1 + (1, \alpha)x_2 = (\delta_2, \alpha)$$

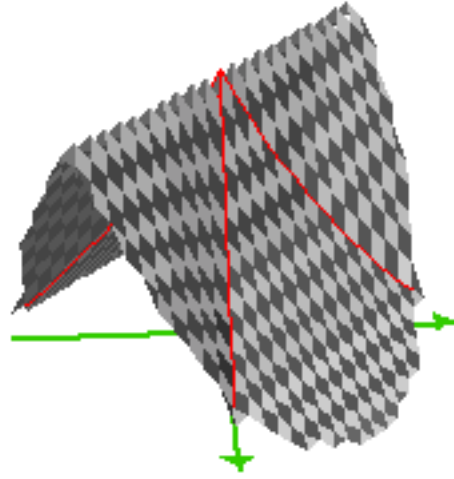


Figure 5.2: The graph of fuzzy solution of system (5.12) with  $\alpha = 0.4$ .

where  $\delta_1 = 0.3$  and  $\delta_2 = -0.3$ , is

$$\mu(x) = \begin{cases} \tau_1(x) & \text{if } 0 < \max\{|x_1 + x_2 - 0.3|, |x_1 + x_2 + 0.3|\} \leq \alpha(|x_1| + |x_2| + 1) \\ 0 & \text{if } \max\{|x_1 + x_2 - 0.3|, |x_1 + x_2 + 0.3|\} > \alpha(|x_1| + |x_2| + 1) \end{cases}$$

where

$$\tau_1(x) = 1 - \frac{\max\{|x_1 + x_2 - 0.3|, |x_1 + x_2 + 0.3|\}}{\alpha(|x_1| + |x_2| + 1)}$$

and the set of the maximizing solutions of (5.13) is empty, and  $X^{**}$  is also empty. Even though

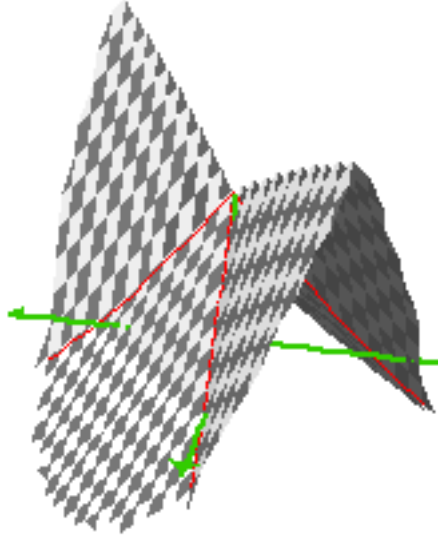


Figure 5.3: The graph of fuzzy solution of system (5.13) with  $\alpha = 0.4$ .

the set of maximizing solution of systems (5.12) and (5.13) varies a lot under small changes of the centers of fuzzy numbers of the right-hand side,  $\delta_1$  and  $\delta_2$ , their fuzzy solutions can be

made arbitrary close to each other by letting

$$\frac{\max\{\delta_1, \delta_2\}}{\alpha}$$

to tend to zero.

## 5.2 Sensitivity analysis of $\tilde{a}x = \tilde{b}$ and $\tilde{a}^\delta x = \tilde{b}^\delta$ .

We illustrate Theorem 5.1.1 by a very simple possibilistic equality system

$$\tilde{a}x = \tilde{b}, \quad (5.14)$$

where  $\tilde{a} = (a, \alpha) \in \mathcal{F}(1/\alpha)$  and  $\tilde{b} = (b, \alpha) \in \mathcal{F}(1/\alpha)$  are (Lipschitzian) fuzzy numbers of symmetric triangular form with the same width  $\alpha > 0$ .

It is easy to check that the fuzzy solution to system (5.14) is

$$\mu(x) = \text{Pos}(\tilde{a}x = \tilde{b}) = \text{Pos}[(ax, \alpha|x|) = (b, \alpha)] = 1 - \frac{|ax - b|}{\alpha(|x| + 1)},$$

if  $|ax - b| \leq \alpha(|x| + 1)$  and  $\mu(x) = 0$  otherwise. If  $a \neq 0$  then the only maximizing solution is

$$x^* = \frac{b}{a}$$

which is also the unique solution, denoted by  $x^{**}$ , of the crisp equation  $ax = b$ . Suppose we are given the following perturbed possibilistic equality system

$$\tilde{a}^\delta x = \tilde{b}^\delta, \quad (5.15)$$

where  $\tilde{a}^\delta = (a^\delta, \alpha) \in \mathcal{F}(1/\alpha)$  and  $\tilde{b}^\delta = (b^\delta, \alpha) \in \mathcal{F}(1/\alpha)$  are (Lipschitzian) fuzzy numbers of symmetric triangular form with the original (exact) width  $\alpha > 0$ .

The fuzzy solution to system (5.15) is

$$\mu^\delta(x) = \text{Pos}(\tilde{a}^\delta x = \tilde{b}^\delta) = \text{Pos}[(a^\delta x, \alpha|x|) = (b^\delta, \alpha)] = 1 - \frac{|a^\delta x - b^\delta|}{\alpha(|x| + 1)},$$

if  $|a^\delta x - b^\delta| \leq \alpha(|x| + 1)$  and  $\mu^\delta(x) = 0$  otherwise. If  $a^\delta \neq 0$  then the only maximizing solution is

$$x^*(\delta) = \frac{b^\delta}{a^\delta}$$

which is also the unique solution, denoted by  $x^{**}(\delta)$ , of the crisp equation  $a^\delta x = b^\delta$ .

Suppose, furthermore, that  $\tilde{a}$ ,  $\tilde{a}^\delta$ ,  $\tilde{b}$  and  $\tilde{b}^\delta$  satisfy the following inequalities

$$D(\tilde{a}, \tilde{a}^\delta) = |a - a^\delta| \leq \delta, \quad D(\tilde{b}, \tilde{b}^\delta) = |b - b^\delta| \leq \delta,$$

where  $\delta > 0$  is a small positive number. Then we can easily give an upper bound for the  $C_\infty$  distance between the fuzzy solutions  $\mu$  and  $\mu^\delta$  by

$$|\mu(x) - \mu^\delta(x)| = \left| 1 - \frac{|ax - b|}{\alpha(|x| + 1)} - \left( 1 - \frac{|a^\delta x - b^\delta|}{\alpha(|x| + 1)} \right) \right| =$$

$$\left| \frac{|ax - b| - |a^\delta x - b^\delta|}{\alpha(|x| + 1)} \right| \leq$$

$$\left| \frac{|x||a - a^\delta| + |b - b^\delta|}{\alpha(|x| + 1)} \right| \leq$$

$$\left| \frac{|x|\delta + \delta}{\alpha(|x| + 1)} \right| \leq \frac{\delta}{\alpha}.$$

for any  $x \in \mathbb{R}$ , which coincides with the upper bound derived from Theorem 5.1.1 with  $L = 1/\alpha$ .

What if  $a = 0$  and  $b = 0$ ? In this case the crisp system becomes

$$0x = 0,$$

and its solution set is the whole real line. However, depending on the values of  $a^\delta$  and  $b^\delta$ , the perturbed crisp system

$$a^\delta x = b^\delta,$$

either has no solution (if  $a^\delta = 0$  and  $b^\delta \neq 0$ ), has the unique solution  $b^\delta/a^\delta$  (if  $a^\delta \neq 0$ ) or its solution set is the whole real line (if  $a^\delta = 0$  and  $b^\delta = 0$ ). So, even a very small change in the crisp coefficients can cause a very large deviation in the solution.

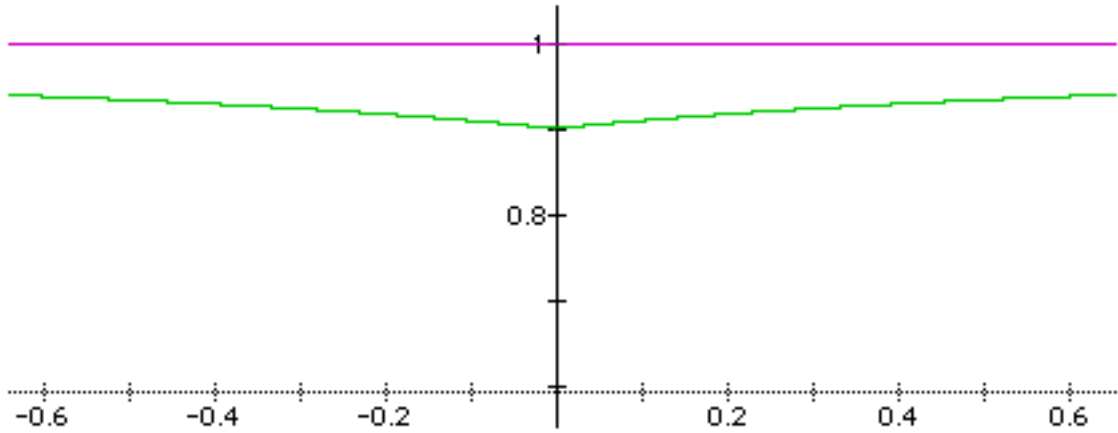


Figure 5.4: Fuzzy solution of  $(0, \alpha)x = (b^\delta, \alpha)$  with  $\alpha = 0.2$  and  $\delta = 0.02$ .

The fuzzified systems, however, behave totally differently. Consider the possibilistic systems

$$(0, \alpha)x = (0, \alpha), \tag{5.16}$$

and

$$(a^\delta, \alpha)x = (b^\delta, \alpha), \tag{5.17}$$

where  $|a^\delta| \leq \delta$  and  $|b^\delta| \leq \delta$  are small numbers. Then the fuzzy solution of (5.16) is

$$\mu(x) = 1 - \frac{|0x - 0|}{\alpha(|x| + 1)} = 1,$$

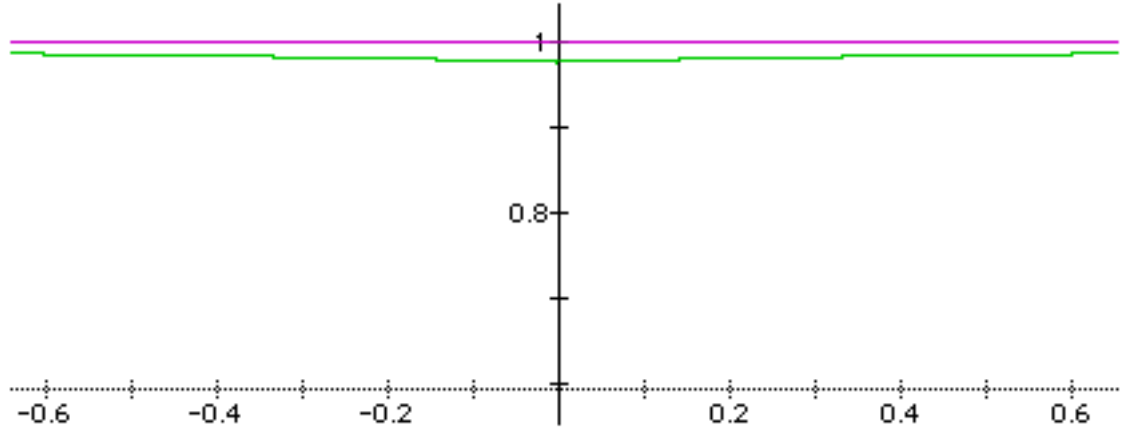


Figure 5.5: Fuzzy solution of  $(0, \alpha)x = (b^\delta, \alpha)$  with  $\alpha = 0.2$  and  $\delta = 0.005$ .

for all  $x \in \mathbb{R}$ , so  $\mu$  is the universal fuzzy set in  $\mathbb{R}$ , and the fuzzy solution of (5.17) is

$$\mu^\delta(x) = 1 - \frac{|a^\delta x - b^\delta|}{\alpha(|x| + 1)},$$

and in the sense of Theorem 5.1.1 we get that

$$\|\mu - \mu^\delta\| = \sup_x |\mu(x) - \mu^\delta(x)| \leq \frac{\delta}{\alpha},$$

which means that the fuzzy solutions of the original and the perturbed systems can be made arbitrarily close to each other if  $\delta$  is sufficiently small. If  $b^\delta = 0$  but  $a^\delta \neq 0$  then the fuzzy solution of

$$(a^\delta, \alpha)x = (0, \alpha)$$

is computed as

$$\mu^\delta(x) = 1 - \frac{|a^\delta x|}{\alpha(|x| + 1)},$$

and its unique maximizing solution is zero.

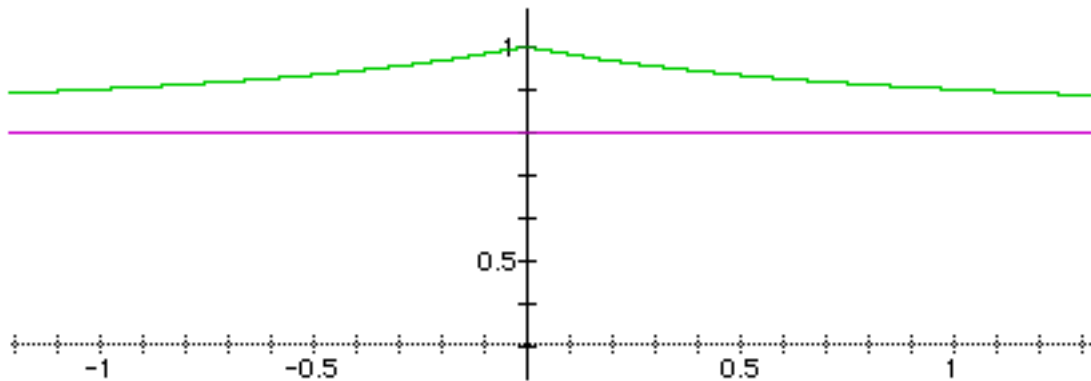


Figure 5.6: Fuzzy solution of  $(a^\delta, \alpha)x = (0, \alpha)$  with  $\alpha = 0.05$  and  $\delta = 0.01$ .

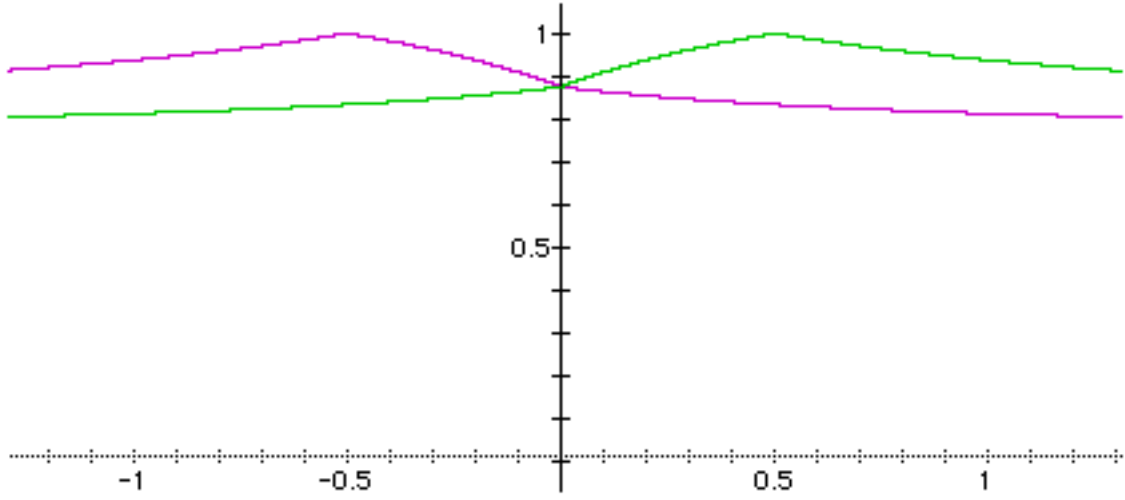


Figure 5.7: Fuzzy solutions of  $(a^\delta, \alpha)x = (b^\delta, \alpha)$  and  $(a, \alpha)x = (b, \alpha)$  with  $a^\delta = -0.01$ ,  $a = 0.01$ ,  $b^\delta = b = 0.005$ ,  $\alpha = 0.04$  and  $\delta = 0.02$ . The maximizing solutions are  $x^*(\delta) = -0.5$  and  $x^* = 0.5$ .

Finally, if  $a \neq 0$ ,  $b \neq 0$ ,  $a^\delta \neq 0$  and  $b^\delta \neq 0$  then the solutions

$$x^* = \frac{b}{a} \text{ and } x^*(\delta) = \frac{b^\delta}{a^\delta},$$

of the crisp systems  $ax = b$  and  $a^\delta x = b^\delta$  can be very far from each other even for very small  $\delta$ . However, the fuzzy solutions of  $(a^\delta, \alpha)x = (b^\delta, \alpha)$  and  $(a, \alpha)x = (b, \alpha)$  can be made arbitrarily close to each other (depending on the relationship between  $\delta$  and  $\alpha$ ).

Even though the fuzzy solutions are very close to each other, the distance between the maximizing solutions can be very big, because the maximizing solutions of the fuzzy system with triangular fuzzy numbers coincide with the solutions of the crisp systems.

### 5.3 Possibilistic systems with trapezoid fuzzy numbers

Consider now a possibilistic linear equality system

$$\tilde{a}_{i1}x_1 + \cdots + \tilde{a}_{in}x_n = \tilde{b}_i, \quad i = 1, \dots, m, \quad (5.18)$$

where  $\tilde{a}_{ij} \in \mathcal{F}$  and  $\tilde{b}_i \in \mathcal{F}$  are symmetric trapezoid fuzzy numbers with the same width  $\alpha > 0$  and tolerance intervals  $[a_{ij} - \theta, a_{ij} + \theta]$  and  $[b_i - \theta, b_i + \theta]$ , respectively, and represented by (2.1) as

$$\tilde{a}_{ij} = (a_{ij} - \theta, a_{ij} + \theta, \alpha, \alpha), \quad \tilde{b}_i = (b_i - \theta, b_i + \theta, \alpha, \alpha).$$

Suppose that we are given the following perturbed possibilistic linear equality system

$$\tilde{a}_{i1}^\delta x_1 + \cdots + \tilde{a}_{in}^\delta x_n = \tilde{b}_i^\delta, \quad i = 1, \dots, m, \quad (5.19)$$

where

$$\tilde{a}_{ij}^\delta = (a_{ij}^\delta - \theta, a_{ij}^\delta + \theta, \alpha, \alpha), \quad \tilde{b}_i^\delta = (b_i^\delta - \theta, b_i^\delta + \theta, \alpha, \alpha).$$

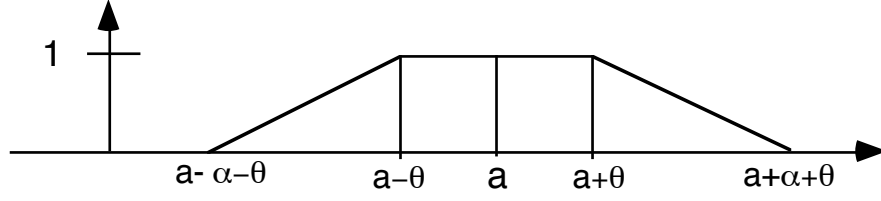


Figure 5.8: A symmetric trapezoid fuzzy number with center  $a$ .

Following Kovács, Vasiljev and Fullér [169] the fuzzy solutions to system (5.18) and (5.19) can be written as

$$\mu(x) = \begin{cases} 1 & \text{if } \|Ax - b\|_\infty \leq \theta(|x|_1 + 1) \\ 1 + \frac{\theta}{\alpha} - \frac{\|Ax - b\|_\infty}{\alpha(|x|_1 + 1)} & \text{if } \theta(|x|_1 + 1) < \|Ax - b\|_\infty \leq (\theta + \alpha)(|x|_1 + 1) \\ 0 & \text{if } \|Ax - b\|_\infty > (\theta + \alpha)(|x|_1 + 1) \end{cases}$$

and

$$\mu^\delta(x) = \begin{cases} 1 & \text{if } \|A^\delta x - b^\delta\|_\infty \leq \theta(|x|_1 + 1) \\ 1 + \frac{\theta}{\alpha} - \frac{\|A^\delta x - b^\delta\|_\infty}{\alpha(|x|_1 + 1)} & \text{if } \theta(|x|_1 + 1) < \|A^\delta x - b^\delta\|_\infty \leq (\theta + \alpha)(|x|_1 + 1) \\ 0 & \text{if } \|A^\delta x - b^\delta\|_\infty > (\theta + \alpha)(|x|_1 + 1) \end{cases}$$

The following theorem [169] shows that the stability property of fuzzy solutions of systems (5.18) and (5.19) does not depend on  $\theta$ .

**Theorem 5.3.1.** [169] Let  $\delta > 0$  and let  $\mu$  and  $\mu^\delta$  be the solutions of possibilistic equality systems (5.18) and (5.19), respectively. If  $a_{ij}$ ,  $a_{ij}^\delta$ ,  $b_i$  and  $b_i^\delta$  satisfy the inequalities

$$D(\tilde{a}_{ij}, \tilde{a}_{ij}^\delta) = |a_{ij} - a_{ij}^\delta| \leq \delta, \quad D(\tilde{b}_i, \tilde{b}_i^\delta) = |b_i - b_i^\delta| \leq \delta, \quad (5.20)$$

then

$$\|\mu - \mu^\delta\|_\infty = \sup_{x \in \mathbb{R}^n} |\mu(x) - \mu^\delta(x)| \leq \frac{\delta}{\alpha}. \quad (5.21)$$

To find a maximizing solution to (5.18) we have to solve the following nonlinear programming problem

$$\begin{aligned} \gamma &\rightarrow \max; \quad (x, \gamma) \in Z, \\ Z &= \left\{ (x, \gamma) \mid 1 + \frac{\theta}{\alpha} - \frac{\|Ax - b\|_\infty}{\alpha(|x|_1 + 1)} \geq \gamma, \quad 0 \leq \gamma \leq 1 \right\} \end{aligned}$$

The next theorem shows that if the crisp equality system  $Ax = b$  has a solution and  $\theta > 0$  then the sets of maximizing solutions of systems (5.18) and (5.19) can be made close to each other.

**Theorem 5.3.2.** [169] Suppose the set  $X^{**} = \{x \in \mathbb{R}^n \mid Ax = b\}$  is not empty. If  $a_{ij}^\delta$  and  $b_i^\delta$  satisfy the relationships (5.20) and  $0 \leq \delta \leq \theta$  then

$$\rho(x, X^*) = \inf_{y \in X^*} |x - y| \leq C_0(\delta + \theta)(|x|_1 + 1), \quad x \in X^*(\delta)$$

where  $X^*$  is the set of maximizing solutions of (5.18) and

$$X^*(\delta) = \{x \in \mathbb{R}^n \mid \mu^\delta(x) = 1\}$$

is the set of maximizing solutions of (5.19) and  $C_0$  is a positive constant depending only on the  $a_{ij}$ 's.

Theorem 5.3.2 states nothing else but if the maximizing solutions of possibilistic equality systems (5.18) and (5.19) can be made close to each other supposing the sets  $X^*(\delta)$ ,  $\forall \delta > 0$  are uniformly bounded.

**Example.** Specially, the fuzzy solution of a possibilistic equation

$$(a - \theta, a + \theta, \alpha, \alpha)x = (b - \theta, b + \theta, \alpha, \alpha), \quad (5.22)$$

can be written as

$$\mu(x) = \begin{cases} 1 & \text{if } |ax - b| \leq \theta(|x| + 1) \\ 1 + \frac{\theta}{\alpha} - \frac{|ax - b|}{\alpha(|x| + 1)} & \text{if } \theta(|x| + 1) < |ax - b| \leq (\theta + \alpha)(|x| + 1) \\ 0 & \text{if } |ax - b| > (\theta + \alpha)(|x| + 1) \end{cases}.$$

It is clear that the set of maximizing solutions of (5.22)

$$X^* = \{x \in \mathbb{R} : |ax - b| \leq \theta(|x| + 1)\}$$

always contains the solution set,  $X^{**}$ , of the equality  $ax = b$ .

Consider now a possibilistic linear equality and inequality system

$$\tilde{a}_{i1}x_1 + \cdots + \tilde{a}_{in}x_n * \tilde{b}_i, \quad i = 1, \dots, m, \quad (5.23)$$

where  $\tilde{a}_{ij} \in \mathcal{F}$  and  $\tilde{b}_i \in \mathcal{F}$  are symmetric trapezoid fuzzy numbers represented by

$$\tilde{a}_{ij} = (a_{ij} - \theta_{ij}, a_{ij} + \theta_{ij}, \alpha_{ij}, \alpha_{ij}), \quad \tilde{b}_i = (b_i - \beta_i, b_i + \beta_i, \eta_i, \eta_i).$$

and  $*$  stands for extended  $\leq, =$  or  $\geq$ .

Then the fuzzy solution of system (5.23) is defined

$$\mu(x) = \min\{\mu_1(x), \dots, \mu_m(x)\},$$

where

$$\mu_i(x) = \text{Pos}(\tilde{a}_{i1}x_1 + \cdots + \tilde{a}_{in}x_n * \tilde{b}_i),$$

denotes the degree of satisfaction of the  $i$ -th restriction at  $x \in \mathbb{R}^n$ , and  $\mu_i(x)$  can be computed as follows [167, 293]

(i) If  $*$  denotes  $=$  then

$$\mu_i(x) = \begin{cases} 1 & \text{if } x \in \text{Le} \\ 1 + \frac{\langle |x|, \theta_i \rangle + \eta_i}{\langle |x|, \alpha_i \rangle + \beta_i} - \frac{|\langle a_i, x \rangle - b_i|}{\langle |x|, \alpha_i \rangle + \beta_i} & \text{otherwise} \\ 0 & \text{if } x \in \text{Ge} \end{cases}$$



where

$$\begin{aligned} \text{Ge} &= \{x \in \mathbb{R}^n : |\langle a_i, x \rangle - b_i| > \langle |x|, \theta_i \rangle + \eta_i + \langle |x|, \alpha_i \rangle + \beta_i\} \\ \text{Le} &= \{x \in \mathbb{R}^n : |\langle a_i, x \rangle - b_i| \leq \langle |x|, \theta_i \rangle + \eta_i\}, \end{aligned}$$

and  $|x| = (|x_1|, \dots, |x_n|)$ ,  $\alpha_i = (\alpha_{i1}, \dots, \alpha_{in})$ , and  $\theta_i = (\theta_{i1}, \dots, \theta_{in})$ .

(ii) If  $*$  denotes  $\leq$  then

$$\mu_i(x) = \begin{cases} 1 & \text{if } x \in \text{Ll} \\ 1 + \frac{\langle |x|, \theta_i \rangle + \eta_i}{\langle |x|, \alpha_i \rangle + \beta_i} - \frac{\langle a_i, x \rangle - b_i}{\langle |x|, \alpha_i \rangle + \beta_i} & \text{otherwise} \\ 0 & \text{if } x \in \text{Gl} \end{cases}$$

where

$$\begin{aligned} \text{Gl} &= \{x \in \mathbb{R}^n : \langle a_i, x \rangle - b_i > \langle |x|, \theta_i \rangle + \eta_i + \langle |x|, \alpha_i \rangle + \beta_i\} \\ \text{Ll} &= \{x \in \mathbb{R}^n : \langle a_i, x \rangle - b_i \leq \langle |x|, \theta_i \rangle + \eta_i\}, \end{aligned}$$

(iii) If  $*$  denotes  $\geq$  then

$$\mu_i(x) = \begin{cases} 1 & \text{if } x \in \text{Lg} \\ 1 + \frac{\langle |x|, \theta_i \rangle + \eta_i}{\langle |x|, \alpha_i \rangle + \beta_i} - \frac{b_i - \langle a_i, x \rangle}{\langle |x|, \alpha_i \rangle + \beta_i} & \text{otherwise} \\ 0 & \text{if } x \in \text{Gg} \end{cases}$$

where

$$\begin{aligned} \text{Gg} &= \{x \in \mathbb{R}^n : \langle a_i, x \rangle - b_i < -(\langle |x|, \theta_i \rangle + \eta_i + \langle |x|, \alpha_i \rangle + \beta_i)\} \\ \text{Lg} &= \{x \in \mathbb{R}^n : \langle a_i, x \rangle - b_i \geq -\langle |x|, \theta_i \rangle - \eta_i\}. \end{aligned}$$

## 5.4 Flexible linear programming

The conventional model of linear programming (LP) can be stated as

$$\langle a_0, x \rangle \rightarrow \min$$

$$\text{subject to } Ax \leq b.$$

In many real-world problems instead of minimization of the objective function  $\langle a_0, x \rangle$  it may be sufficient to determine an  $x$  such that

$$a_{01}x_1 + \dots + a_{0n}x_n \leq b_0, \tag{5.24}$$

$$\text{subject to } Ax \leq b.$$

where  $b_0$  is a predetermined aspiration level.

Assume that all parameters in (5.24) are fuzzy quantities and are described by symmetric triangular fuzzy numbers. Then the following flexible (or fuzzy) linear programming (FLP) problem can be obtained by replacing crisp parameters  $a_{ij}$ ,  $b_i$  with symmetric triangular fuzzy numbers  $\tilde{a}_{ij} = (a_{ij}, \alpha)$  and  $\tilde{b}_i = (b_i, d_i)$  respectively,

$$(a_{i1}, \alpha)x_1 + \cdots + (a_{in}, \alpha)x_n \leq (b_i, d_i), \quad i = 0, \dots, m. \quad (5.25)$$

Here  $d_0$  and  $d_i$  are interpreted as the tolerance levels for the objective function and the  $i$ -th constraint, respectively. The parameter  $\alpha > 0$  will guarantee the stability property of the solution of (5.25) under small changes in the coefficients  $a_{ij}$  and  $b_i$ .

We denote by  $\mu_i(x)$  the degree of satisfaction of the  $i$ -th restriction at the point  $x \in R^n$  in (5.25), i.e.

$$\mu_i(x) = \text{Pos}(\tilde{a}_{i1}x_1 + \cdots + \tilde{a}_{in}x_n \leq \tilde{b}_i).$$

Then the (fuzzy) solution of the FLP problem (5.25) is defined as a fuzzy set on  $R^n$  whose membership function is given by

$$\mu(x) = \min\{\mu_0(x), \mu_1(x), \dots, \mu_m(x)\},$$

and the maximizing solution  $x^*$  of the FLP problem (5.25) satisfies the equation

$$\mu(x^*) = \mu^* = \max_x \mu(x).$$

From (2.23) it follows that the degree of satisfaction of the  $i$ -th restriction at  $x$  in (5.25) is the following:

$$\mu_i(x) = \begin{cases} 1 & \text{if } \langle a_i, x \rangle \leq b_i, \\ 1 - \frac{\langle a_i, x \rangle - b_i}{\alpha|x|_1 + d_i} & \text{otherwise,} \\ 0 & \text{if } \langle a_i, x \rangle > b_i + \alpha|x|_1 + d_i, \end{cases} \quad (5.26)$$

where  $|x|_1 = |x_1| + \cdots + |x_n|$  and  $\langle a_i, x \rangle = a_{i1}x_1 + \cdots + a_{in}x_n$ ,  $i = 0, 1, \dots, m$ .

In the extremal case  $\alpha = 0$  but  $d_i > 0$  in (5.26), we get a linear membership function for  $\mu_i$ , i.e. Zimmermann's principle [451]. Really, for  $\alpha = 0$  we get

$$(a_{i1}, 0)x_1 + \cdots + (a_{in}, 0)x_n \leq (b_i, d_i), \quad (5.27)$$

and the  $\mu_i$ 's have a very simple form

$$\mu_i(x) = \begin{cases} 1 & \text{if } \langle a_i, x \rangle \leq b_i, \\ 1 - \frac{\langle a_i, x \rangle - b_i}{d_i} & \text{if } b_i < \langle a_i, x \rangle \leq b_i + d_i, \\ 0 & \text{if } \langle a_i, x \rangle > b_i + d_i, \end{cases}$$

for  $i = 0, 1, \dots, m$ .

If  $\alpha = 0$  then  $\mu_i$  has an easy interpretation: If for an  $x \in \mathbb{R}^n$  the value of  $\langle a_i, x \rangle$  is less or equal than  $b_i$  then  $x$  satisfies the  $i$ -th constraint with the maximal conceivable degree one; if  $b_i < \langle a_i, x \rangle < b_i + d_i$  then  $x$  is not feasible in classical sense, but the decision maker can still tolerate the violation of the crisp constraint, and accept  $x$  as a solution with a positive degree,

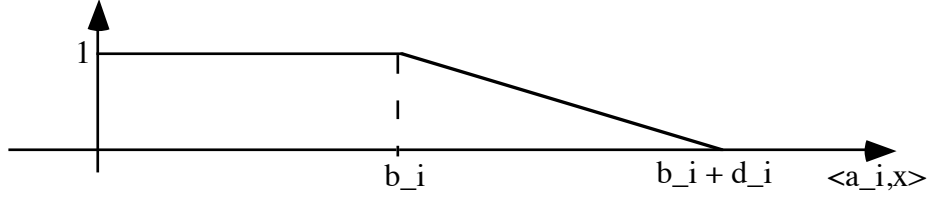


Figure 5.9: The graph of  $\mu_i$  if  $\alpha = 0$ .

however, the bigger the violation the less is the degree of acceptance; and if  $\langle a_i, x \rangle > b_i + d_i$  then the violation of the  $i$ -th constraint is intolerable by the decision maker, that is,  $\mu_i(x) = 0$ .

Sensitivity analysis in FLP problems (with crisp parameters and soft constraints) was first considered in [211], where a functional relationship between changes of parameters of the right-hand side and those of the optimal value of the primal objective function was derived for almost all conceivable cases. In [395] a FLP problem (with symmetric triangular fuzzy numbers) was formulated and the value of information was discussed via sensitivity analysis.

Following Fullér [168] we investigate the stability of the solution in FLP problems (with symmetric triangular fuzzy numbers and extended operations and inequalities) with respect to changes of fuzzy parameters and show that the solution to these problems is stable (in metric  $C_\infty$ ) under small variations in the membership functions of the fuzzy coefficients.

Consider now the perturbed FLP problem

$$(a_{i1}^\delta, \alpha)x_1 + \dots + (a_{in}^\delta, \alpha)x_n \leq (b_i^\delta, d_i), \quad i = 0, \dots, m. \quad (5.28)$$

where  $a_{ij}^\delta$  and  $b_i^\delta$  satisfy the inequalities

$$\max_{i,j} |a_{ij} - a_{ij}^\delta| \leq \delta, \quad \max_i |b_i - b_i^\delta| \leq \delta. \quad (5.29)$$

In a similar manner we can define the solution of FLP problem (5.28) by

$$\mu^\delta(x) = \min\{\mu_0^\delta(x), \mu_1^\delta(x), \dots, \mu_m^\delta(x)\}, \quad x \in \mathbb{R}^n,$$

where  $\mu_i^\delta(x)$  denotes the degree of satisfaction of the  $i$ -th restriction at  $x \in \mathbb{R}^n$  and the maximizing solution  $x^*(\delta)$  of FLP problem (5.28) satisfies the equation

$$\mu^\delta(x^*(\delta)) = \mu^*(\delta) = \sup_x \mu^\delta(x). \quad (5.30)$$

In the following theorem we establish a stability property of the fuzzy solution of FLP problem (5.25).

**Theorem 5.4.1.** [168] *Let  $\mu(x)$  and  $\mu^\delta(x)$  be solution of FLP problems (5.25) and (5.28) respectively. Then*

$$\|\mu - \mu^\delta\|_\infty = \sup_{x \in \mathbb{R}^n} |\mu(x) - \mu^\delta(x)| \leq \delta \left[ \frac{1}{\alpha} + \frac{1}{d} \right] \quad (5.31)$$

where  $d = \min\{d_0, d_1, \dots, d_m\}$ .

*Proof.* First let  $\delta \geq \min\{\alpha, d\}$ . Then from  $|\mu(x) - \mu^\delta(x)| \leq 1, \forall x \in \mathbb{R}^n$  and

$$\frac{\delta}{\alpha + d} \geq 1,$$

we obtain (5.31). Suppose that

$$0 < \delta < \min\{\alpha, d\}.$$

It will be sufficient to show that

$$|\mu_i(x) - \mu_i^\delta(x)| \leq \delta \left[ \frac{1}{\alpha} + \frac{1}{d} \right], \quad \forall x \in \mathbb{R}^n, \quad i = 0, \dots, m, \quad (5.32)$$

because from (5.32) follows (5.31). Let  $x \in \mathbb{R}^n$  and  $i \in \{0, \dots, m\}$  be arbitrarily fixed. Consider the following cases:

- (1)  $\mu_i(x) = \mu_i^\delta(x)$ . In this case (5.32) is trivially obtained.
- (2)  $0 < \mu_i(x) < 1$  and  $0 < \mu_i^\delta(x) < 1$ . In this case from (5.26), (5.29) we have

$$\begin{aligned} |\mu_i(x) - \mu_i^\delta(x)| &= \left| 1 - \frac{\langle a_i, x \rangle - b_i}{\alpha|x|_1 + d_i} - \left( 1 - \frac{\langle a_i^\delta, x \rangle - b_i^\delta}{\alpha|x|_1 + d_i} \right) \right| = \\ &= \frac{|\langle a_i, x \rangle - b_i - \langle a_i^\delta, x \rangle + b_i^\delta|}{\alpha|x|_1 + d_i} \leq \\ &= \frac{|b_i - b_i^\delta| + |\langle a_i^\delta - a_i, x \rangle|}{\alpha|x|_1 + d_i} \leq \\ &\leq \frac{\delta + |a_i^\delta - a_i|_\infty |x|_1}{\alpha|x|_1 + d_i} \leq \frac{\delta + \delta|x|_1}{\alpha|x|_1 + d_i} \leq \\ &\leq \delta \left[ \frac{1}{\alpha} + \frac{1}{d_i} \right] \leq \delta \left[ \frac{1}{\alpha} + \frac{1}{d} \right], \end{aligned}$$

where  $a_i^\delta = (a_{i1}^\delta, \dots, a_{in}^\delta)$  and  $|a_i^\delta - a_i|_\infty = \max_j |a_{ij}^\delta - a_{ij}|$ .

- (3)  $\mu_i(x) = 1$  and  $0 < \mu_i^\delta(x) < 1$ . In this case we have  $\langle a_i, x \rangle \leq b_i$ . Hence

$$\begin{aligned} |\mu_i(x) - \mu_i^\delta(x)| &= \left| 1 - \left[ 1 - \frac{\langle a_i^\delta, x \rangle - b_i^\delta}{\alpha|x|_1 + d_i} \right] \right| = \\ &= \frac{\langle a_i^\delta, x \rangle - b_i^\delta}{\alpha|x|_1 + d_i} \leq \frac{(\langle a_i^\delta, x \rangle - b_i^\delta) - (\langle a_i, x \rangle - b_i)}{\alpha|x|_1 + d_i} \leq \delta \left[ \frac{1}{\alpha} + \frac{1}{d} \right]. \end{aligned}$$

- (4)  $0 < \mu_i(x) < 1$  and  $\mu_i^\delta(x) = 1$ . In this case the proof is carried out analogously to the proof of the preceding case.

(5)  $0 < \mu_i(x) < 1$  and  $\mu_i^\delta(x) = 0$ . In this case from

$$\langle a_i^\delta, x \rangle - b_i^\delta > \alpha|x|_1 + d_i$$

it follows that

$$\begin{aligned} |\mu_i(x) - \mu_i^\delta(x)| &= \left| 1 - \frac{\langle a_i, x \rangle - b_i}{\alpha|x|_1 + d_i} \right| = \\ &= \frac{1}{\alpha|x|_1 + d_i} \times \left| \alpha|x|_1 + d_i - (\langle a_i, x \rangle - b_i) \right| \leq \\ &= \frac{|\langle a_i(\delta), x \rangle - b_i(\delta) - (\langle a_i, x \rangle - b_i)|}{\alpha|x|_1 + d_i} \leq \delta \left[ \frac{1}{\alpha} + \frac{1}{d} \right]. \end{aligned}$$

(6)  $\mu_i(x) = 0$  and  $0 < \mu_i^\delta(x) < 1$ . In this case the proof is carried out analogously to the proof of the preceding case.

(7)  $\mu_i(x) = 1$ ,  $\mu_i^\delta(x) = 0$ , or  $\mu_i(x) = 0$ ,  $\mu_i^\delta(x) = 1$ . These cases are not reasonable. For instance suppose that case  $\mu_i(x) = 1$ ,  $\mu_i^\delta(x) = 0$  is conceivable. Then from (5.29) it follows that

$$\begin{aligned} |\langle a_i, x \rangle - b_i - (\langle a_i(\delta), x \rangle - b_i(\delta))| &\leq \\ |b_i - b_i^\delta| + |a_i^\delta - a_i|_\infty |x|_1 &\leq \\ \leq \delta(|x|_1 + 1). \end{aligned}$$

On the other hand we have

$$\begin{aligned} |\langle a_i, x \rangle - b_i - (\langle a_i^\delta, x \rangle - b_i^\delta)| &\geq \\ |\langle a_i^\delta, x \rangle - b_i^\delta| &\geq \alpha|x|_1 + d_i > \\ \delta|x|_1 + \delta &= \delta(|x|_1 + 1). \end{aligned}$$

So we arrived at a contradiction, which ends the proof.

□

From (5.31) it follows that

$$|\mu^* - \mu^*(\delta)| \leq \delta \left[ \frac{1}{\alpha} + \frac{1}{d} \right]$$

and  $\|\mu - \mu^\delta\|_C \rightarrow 0$  if  $\delta/\alpha \rightarrow 0$  and  $\delta/d \rightarrow 0$ , which means stability with respect to perturbations (5.29) of the solution and the measure of consistency in FLP problem (5.25). To find a maximizing solution to FLP problem (5.25) we have to solve the following nonlinear programming problem

$$\begin{aligned} \max \lambda \\ \lambda(\alpha|x|_1 + d_0) - \alpha|x|_1 + \langle a_0, x \rangle &\leq b_0 + d_0, \\ \lambda(\alpha|x|_1 + d_1) - \alpha|x|_1 + \langle a_1, x \rangle &\leq b_1 + d_1, \end{aligned}$$

.....

$$\begin{aligned} \lambda(\alpha|x|_1 + d_m) - \alpha|x|_1 + \langle a_m, x \rangle &\leq b_m + d_m, \\ 0 \leq \lambda \leq 1, \quad x &\in \mathbb{R}^n. \end{aligned}$$

It is easily checked that in the extremal case  $\alpha = 0$  but  $d_i > 0$ , the solution of FLP problem (5.25) may be unstable with respect to changes of the crisp parameters  $a_{ij}, b_i$ .

### 5.4.1 Example

As an example consider the following simple FLP

$$(1, \alpha)x \rightarrow \min \tag{5.33}$$

$$\text{subject to } (-1, \alpha)x \leq (-1, d_1), \quad x \in \mathbb{R},$$

with  $b_0 = 0.5, \alpha = 0.4, d_0 = 0.6$  and  $d_1 = 0.5$ .

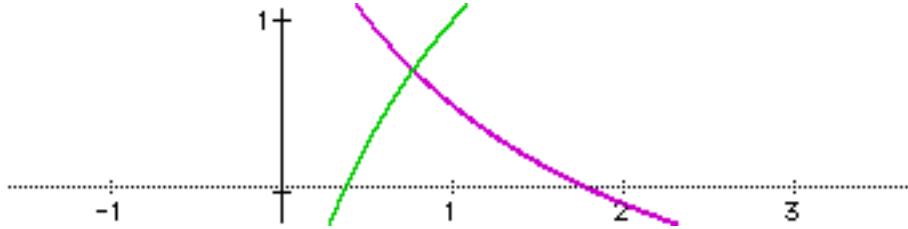


Figure 5.10:  $\mu_0$  and  $\mu_1$ .

That is,

$$(1, 0.4)x \leq (0.5, 0.6)$$

$$(-1, 0.4)x \leq (-1, 0.5), \quad x \in \mathbb{R},$$

and

$$\mu_0(x) = \begin{cases} 1 & \text{if } x \leq 0.5 \\ 1 - \frac{x - 0.5}{0.4|x| + 0.6} & \text{if } 0.5 \leq x \leq 1.1 \\ 0 & \text{if } x > 1.1 \end{cases}$$

$$\mu_1(x) = \begin{cases} 1 & \text{if } -x \leq -1 \\ 1 - \frac{-x + 1}{0.4|x| + 0.5} & \text{if } -1 < -x \leq -0.5 \\ 0 & \text{if } -x > -0.5 \end{cases}$$

The unique maximizing solution of (5.33) is  $x^* = 0.764$  and the degree of consistency is  $\mu^* = 0.707$ . The degree of consistency is smaller than one, because the aspiration level,  $b_0 = 0.5$ , is set below one, the minimum of the crisp goal function,  $1 \times x$  under the crisp constraint  $x \geq 1$ .

## 5.5 Fuzzy linear programming with crisp relations

Following Fullér [165, 166] we consider LP problems, in which all of the coefficients are fuzzy numbers

$$\begin{aligned} \tilde{c}_1 x_1 + \cdots + \tilde{c}_n x_n &\rightarrow \max \\ \tilde{a}_{i1} x_1 + \cdots + \tilde{a}_{in} x_n &\leq \tilde{b}_i, \quad i = 1, \dots, m, \quad x \in \mathbb{R}^n. \end{aligned} \quad (5.34)$$

Suppose that the crisp inequality relation between fuzzy numbers is defined by (2.31), i.e. if  $[\tilde{a}]^\gamma = [a_1(\gamma), a_2(\gamma)]$  and  $[\tilde{b}]^\gamma = [b_1(\gamma), b_2(\gamma)]$  then

$$\begin{aligned} \tilde{a} \leq \tilde{b} &\iff \mathcal{W}(\tilde{a}) = \int_0^1 \gamma(a_1(\gamma) + a_2(\gamma))d\gamma \leq \\ &\mathcal{W}(\tilde{b}) = \int_0^1 \gamma(b_1(\gamma) + b_2(\gamma))d\gamma. \end{aligned}$$

In this way (5.34) can be stated as follows

$$\begin{aligned} \mathcal{W}(\tilde{c}_1 x_1 + \cdots + \tilde{c}_n x_n) &\rightarrow \max \\ \mathcal{W}(\tilde{a}_{i1} x_1 + \cdots + \tilde{a}_{in} x_n) &\leq \mathcal{W}(\tilde{b}_i), \quad i = 1, \dots, m, \quad x \in \mathbb{R}^n. \end{aligned} \quad (5.35)$$

First we observe that

$$\mathcal{W}: \mathcal{F} \rightarrow \mathbb{R},$$

is a linear mapping, in the sense that

$$\mathcal{W}(\tilde{a} + \tilde{b}) = \mathcal{W}(\tilde{a}) + \mathcal{W}(\tilde{b}),$$

and

$$\mathcal{W}(\lambda \tilde{a}) = \lambda \mathcal{W}(\tilde{a}). \quad (5.36)$$

for any  $\lambda \in \mathbb{R}$ . Really, from the equation

$$[\tilde{a} + \tilde{b}]^\gamma = [a_1(\gamma) + b_1(\gamma), a_2(\gamma) + b_2(\gamma)],$$

we have

$$\begin{aligned} \mathcal{W}(\tilde{a} + \tilde{b}) &= \int_0^1 \gamma(a_1(\gamma) + b_1(\gamma) + a_2(\gamma) + b_2(\gamma))d\gamma = \\ &\int_0^1 \gamma(a_1(\gamma) + a_2(\gamma))d\gamma + \int_0^1 \gamma(b_1(\gamma) + b_2(\gamma))d\gamma = \mathcal{W}(\tilde{a}) + \mathcal{W}(\tilde{b}), \end{aligned}$$

and (5.36) follows from the relationship

$$[\lambda \tilde{a}]^\gamma = \lambda [\tilde{a}]^\gamma.$$

by

$$\mathcal{W}(\lambda \tilde{a}) = \int_0^1 \gamma(\lambda a_1(\gamma) + \lambda a_2(\gamma))d\gamma = \int_0^1 \gamma(\lambda a_2(\gamma) + \lambda a_1(\gamma))d\gamma = \lambda \mathcal{W}(\tilde{a}).$$

Using the linearity of  $\mathcal{W}$  the LP problem (5.35) with fuzzy number coefficients turns into the following crisp LP problem

$$\mathcal{W}(\tilde{c}_1)x_1 + \cdots + \mathcal{W}(\tilde{c}_n)x_n \rightarrow \max \quad (5.37)$$

$$\mathcal{W}(\tilde{a}_{i1})x_1 + \cdots + \mathcal{W}(\tilde{a}_{in})x_n \leq \mathcal{W}(\tilde{b}_i), \quad i = 1, \dots, m, \quad x \in \mathbb{R}^n.$$

or shortly,

$$\begin{aligned} \langle \mathcal{W}(\tilde{c}), x \rangle &\rightarrow \max, \\ \text{s.t. } \mathcal{W}(\tilde{A})x &\leq \mathcal{W}(\tilde{b}), \quad x \in \mathbb{R}^n. \end{aligned} \quad (5.38)$$

Consider (5.35) with symmetric fuzzy quasi-triangular fuzzy number coefficients (2.2) of the form

$$\tilde{a}_{ij} = (a_{ij}, \alpha_{ij})_{LL}, \quad \tilde{b}_i = (b_i, \beta_i)_{LL}, \quad \tilde{c}_j = (c_j, \theta_j)_{LL}.$$

Then from the representations

$$\begin{aligned} [\tilde{a}_{ij}]^\gamma &= [a_{ij} - \alpha_{ij}L^{-1}(1 - \gamma), a_{ij} + \alpha_{ij}L^{-1}(1 - \gamma)], \\ [\tilde{b}_i]^\gamma &= [b_i - \beta_iL^{-1}(1 - \gamma), b_i + \beta_iL^{-1}(1 - \gamma)], \\ [\tilde{c}_j]^\gamma &= [c_j - \theta_jL^{-1}(1 - \gamma), c_j + \theta_jL^{-1}(1 - \gamma)], \end{aligned}$$

we get

$$\begin{aligned} \mathcal{W}(\tilde{a}_{ij}) &= \int_0^1 \gamma(a_{ij} - \alpha_{ij}L^{-1}(1 - \gamma) + a_{ij} + \alpha_{ij}L^{-1}(1 - \gamma))d\gamma = a_{ij}, \\ \mathcal{W}(\tilde{b}_i) &= \int_0^1 \gamma(b_i - \beta_iL^{-1}(1 - \gamma) + b_i + \beta_iL^{-1}(1 - \gamma))d\gamma = b_i, \\ \mathcal{W}(\tilde{c}_j) &= \int_0^1 \gamma(c_j - \theta_jL^{-1}(1 - \gamma) + c_j + \theta_jL^{-1}(1 - \gamma))d\gamma = c_j, \end{aligned}$$

in this way FLP problem (5.38) turns into the crisp LP

$$\begin{aligned} \langle c, x \rangle &\rightarrow \max \\ \text{subject to } Ax &\leq b, \quad x \in \mathbb{R}^n. \end{aligned}$$

where the coefficients are the centres of the corresponding fuzzy coefficients.

## 5.6 Possibilistic linear programming

We consider certain possibilistic linear programming problems, which have been introduced by Buckley in [29]. In contrast to classical linear programming (where a small error of measurement may produce a large variation in the objective function), we show that the possibility distribution of the objective function of a possibilistic linear program with continuous fuzzy number parameters is stable under small perturbations of the parameters. First, we will briefly review possibilistic linear programming and set up notations. A possibilistic linear program is

$$\max/\min Z = x_1\tilde{c}_1 + \cdots + x_n\tilde{c}_n, \quad (5.39)$$



subject to  $x_1 \tilde{a}_{i1} + \dots + x_n \tilde{a}_{in} * \tilde{b}_i, 1 \leq i \leq m, x \geq 0.$

where  $\tilde{a}_{ij}, \tilde{b}_i, \tilde{c}_j$  are fuzzy numbers,  $x = (x_1, \dots, x_n)$  is a vector of (nonfuzzy) decision variables, and  $*$  denotes  $<, \leq, =, \geq$  or  $>$  for each  $i$ .

We will assume that all fuzzy numbers  $\tilde{a}_{ij}, \tilde{b}_i$  and  $\tilde{c}_j$  are non-interactive. Non-interactivity means that we can find the joint possibility distribution of all the fuzzy variables by calculating the min-intersection of their possibility distributions.

Following Buckley [29], we define  $\text{Pos}[Z = z]$ , the possibility distribution of the objective function  $Z$ . We first specify the possibility that  $x$  satisfies the  $i$ -th constraints. Let

$$\Pi(a_i, b_i) = \min\{\tilde{a}_{i1}(a_{i1}), \dots, \tilde{a}_{in}(a_{in}), \tilde{b}_i(b_i)\},$$

where  $a_i = (a_{i1}, \dots, a_{in})$ , which is the joint distribution of  $\tilde{a}_{ij}, j = 1, \dots, n$ , and  $\tilde{b}_i$ . Then

$$\text{Pos}[x \in \mathcal{F}_i] = \sup_{a_i, b_i} \{\Pi(a_i, b_i) \mid a_{i1}x_1 + \dots + a_{in}x_n * b_i\},$$

which is the possibility that  $x$  is feasible with respect to the  $i$ -th constraint. Therefore, for  $x \geq 0$ ,

$$\text{Pos}[x \in \mathcal{F}] = \min_{1 \leq i \leq m} \text{Pos}[x \in \mathcal{F}_i],$$

which is the possibility that  $x$  is feasible. We next construct  $\text{Pos}[Z = z|x]$  which is the conditional possibility that  $Z$  equals  $z$  given  $x$ . The joint distribution of the  $\tilde{c}_j$  is

$$\Pi(c) = \min\{\tilde{c}_1(c_1), \dots, \tilde{c}_n(c_n)\}$$

where  $c = (c_1, \dots, c_n)$ . Therefore,

$$\text{Pos}[Z = z|x] = \sup_c \{\Pi(c) \mid c_1x_1 + \dots + c_nx_n = z\}.$$

Finally, applying Bellman and Zadeh's method for fuzzy decision making [10], the possibility distribution of the objective function is defined as

$$\text{Pos}[Z = z] = \sup_{x \geq 0} \min\{\text{Pos}[Z = z|x], \text{Pos}[x \in \mathcal{F}]\}.$$

It should be noted that Buckley [30] showed that the solution to an appropriate linear program gives the correct  $z$  values in  $\text{Pos}[Z = z] = \alpha$  for each  $\alpha \in [0, 1]$ .

An important question [128, 211, 453] is the influence of the perturbations of the fuzzy parameters to the possibility distribution of the objective function. We will assume that there is a collection of fuzzy parameters  $\tilde{a}_{ij}^\delta, \tilde{b}_i^\delta$  and  $\tilde{c}_j^\delta$  available with the property

$$D(\tilde{A}, \tilde{A}^\delta) \leq \delta, D(\tilde{b}, \tilde{b}^\delta) \leq \delta, D(\tilde{c}, \tilde{c}^\delta) \leq \delta, \quad (5.40)$$

where

$$D(\tilde{A}, \tilde{A}^\delta) := \max_{i,j} D(\tilde{a}_{ij}, \tilde{a}_{ij}^\delta), D(\tilde{b}, \tilde{b}^\delta) := \max_i D(\tilde{b}_i, \tilde{b}_i^\delta), D(\tilde{c}, \tilde{c}^\delta) := \max_j D(\tilde{c}_j, \tilde{c}_j^\delta).$$

Then we have to solve the following perturbed problem:

$$\max/\min Z^\delta = x_1 \tilde{c}_1^\delta + \dots + x_n \tilde{c}_n^\delta \quad (5.41)$$

$$\text{subject to } x_1 \tilde{a}_{i1}^\delta + \cdots + x_n \tilde{a}_{in}^\delta * \tilde{b}_i^\delta, \quad 1 \leq i \leq m, \quad x \geq 0.$$

Let us denote by  $\text{Pos}[x \in \mathcal{F}_i^\delta]$  the possibility that  $x$  is feasible with respect to the  $i$ -th constraint in (5.41). Then the possibility distribution of the objective function  $Z^\delta$  is defined as follows:

$$\text{Pos}[Z^\delta = z] = \sup_{x \geq 0} (\min\{\text{Pos}[Z^\delta = z \mid x], \text{Pos}[x \in \mathcal{F}^\delta]\}).$$

The next theorem shows a stability property (with respect to perturbations (5.40) of the possibility distribution of the objective function of the possibilistic linear programming problems (5.39) and (5.41).

**Theorem 5.6.1.** *Let  $\delta \geq 0$  be a real number and let  $\tilde{a}_{ij}$ ,  $\tilde{b}_i$ ,  $\tilde{a}_{ij}^\delta$ ,  $\tilde{c}_j$ ,  $\tilde{c}_j^\delta$  be (continuous) fuzzy numbers. If (5.40) hold, then*

$$\sup_{z \in \mathbb{R}} |\text{Pos}[Z^\delta = z] - \text{Pos}[Z = z]| \leq \omega(\delta) \quad (5.42)$$

where

$$\omega(\delta) = \max_{i,j} \{\omega(\tilde{a}_{ij}, \delta), \omega(\tilde{a}_{ij}^\delta, \delta), \omega(\tilde{b}_i, \delta), \omega(b_i^\delta, \delta), \omega(\tilde{c}_j, \delta), \omega(\tilde{c}_j^\delta, \delta)\}.$$

*Proof.* It is sufficient to show that

$$|\text{Pos}[Z = z \mid x] - \text{Pos}[Z^\delta = z \mid x]| \leq \omega(\delta), \quad z \in \mathbb{R}, \quad (5.43)$$

$$|\text{Pos}[x \in \mathcal{F}_i] - \text{Pos}[x \in \mathcal{F}_i^\delta]| \leq \omega(\delta), \quad (5.44)$$

for each  $x \in \mathbb{R}$  and  $1 \leq i \leq m$ . We shall prove only (5.44), because the proof of (5.43) is carried out analogously. Let  $x \in \mathbb{R}$  and  $i \in \{1, \dots, m\}$  arbitrarily fixed. From the definition of possibility it follows that

$$\begin{aligned} \text{Pos}[x \in \mathcal{F}_i] &= \sup_{t \neq 0} \left( \sum_{j=1}^n \tilde{a}_{ij} x_j - B_i \right) (t), \\ \text{Pos}[x \in \mathcal{F}_i^\delta] &= \sup_{t \neq 0} \left( \sum_{j=1}^n \tilde{a}_{ij}^\delta x_j - B_i^\delta \right) (t), \end{aligned}$$

Applying Lemma 2.14.1 we have

$$\begin{aligned} D \left( \sum_{j=1}^n \tilde{a}_{ij} x_j - \tilde{b}_i, \sum_{j=1}^n \tilde{a}_{ij}^\delta x_j - \tilde{b}_i^\delta \right) &\leq \\ \sum_{j=1}^n |x_j| D(\tilde{a}_{ij}, \tilde{a}_{ij}^\delta) + D(\tilde{b}_i, \tilde{b}_i^\delta) &\leq \delta(|x|_1 + 1), \end{aligned}$$

By Lemma 2.14.2 we get

$$\max \left\{ \omega \left( \sum_{j=1}^n \tilde{a}_{ij} x_j - \tilde{b}_i, \delta \right), \omega \left( \sum_{j=1}^n \tilde{a}_{ij}^\delta x_j - \tilde{b}_i^\delta, \delta \right) \right\} \leq \omega \left( \frac{\delta}{|x|_1 + 1} \right).$$

Finally, applying Lemma 2.14.5 we have

$$\begin{aligned}
& | \text{Pos}[x \in \mathcal{F}_i] - \text{Pos}[x \in \mathcal{F}_i^\delta] | = \\
& \left| \sup_{t \neq 0} \left( \sum_{j=1}^n \tilde{a}_{ij} x_j - \tilde{b}_i \right) (t) - \sup_{t \neq 0} \left( \sum_{j=1}^n \tilde{a}_{ij}^\delta x_j - \tilde{b}_i^\delta \right) (t) \right| \leq \\
& \sup_{t \neq 0} \left| \left( \sum_{j=1}^n \tilde{a}_{ij} x_j - \tilde{b}_i \right) (t) - \left( \sum_{j=1}^n \tilde{a}_{ij}^\delta x_j - \tilde{b}_i^\delta \right) (t) \right| \leq \\
& \sup_{t \in \mathbb{R}} \left| \left( \sum_{j=1}^n \tilde{a}_{ij} x_j - \tilde{b}_i \right) (t) - \left( \sum_{j=1}^n \tilde{a}_{ij}^\delta x_j - \tilde{b}_i^\delta \right) (t) \right| \leq \\
& \omega \left[ \frac{\delta(|x|_1 + 1)}{|x|_1 + 1} \right] = \omega(\delta),
\end{aligned}$$

which proves the theorem.  $\square$

From (5.42) follows that  $\sup_z |\text{Pos}[Z^\delta = z] - \text{Pos}[Z = z]| \rightarrow 0$  as  $\delta \rightarrow 0$ , which means the stability of the possibility distribution of the objective function with respect to perturbations (5.40). As an immediate consequence of this theorem we obtain the following result: If the fuzzy numbers in (5.39) and (5.41) satisfy the Lipschitz condition with constant  $L > 0$ , then

$$\sup_{z \in \mathbb{R}} | \text{Pos}[Z^\delta = z] - \text{Pos}[Z = z] | \leq L\delta$$

Furthermore, similar estimations can be obtained in the case of symmetric trapezoidal fuzzy number parameters [292] and in the case of symmetric triangular fuzzy number parameters [168, 291]. It is easy to see that in the case of non-continuous fuzzy parameters the possibility distribution of the objective function may be unstable under small changes of the parameters.

### 5.6.1 Example

As an example, consider the following possibilistic linear program

$$\max/\min \tilde{c}x \tag{5.45}$$

$$\text{subject to } \tilde{a}x \leq \tilde{b}, x \geq 0.$$

where  $\tilde{a} = (1, 1)$ ,  $\tilde{b} = (2, 1)$  and  $\tilde{c} = (3, 1)$  are fuzzy numbers of symmetric triangular form. Here  $x$  is one-dimensional ( $n = 1$ ) and there is only one constraint ( $m = 1$ ).

We find

$$\text{Pos}[x \in \mathcal{F}] = \begin{cases} 1 & \text{if } x \leq 2, \\ \frac{3}{x+1} & \text{if } x > 2. \end{cases}$$

and



Figure 5.11: The graph of  $\text{Pos}[Z = z|x]$  for  $z = 8$ .

$$\text{Pos}[Z = z|x] = \text{Pos}[\tilde{c}x = z] = \begin{cases} 4 - z/x & \text{if } z/x \in [3, 4], \\ z/x - 2 & \text{if } z/x \in [2, 3], \\ 0 & \text{otherwise,} \end{cases}$$

for  $x \neq 0$ , and

$$\text{Pos}[Z = z|0] = \text{Pos}[0 \times \tilde{c} = z] = \begin{cases} 1 & \text{if } z = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\text{Pos}[Z = z] = \sup_{x \geq 0} \min \left\{ \frac{3}{x+1}, 1 - \left| \frac{z}{x} - 3 \right| \right\}$$

if  $z > 6$  and  $\text{Pos}[Z = z] = 1$  if  $0 \leq z \leq 6$ . That is,

$$\text{Pos}[Z = z] = \begin{cases} 1 & \text{if } 0 \leq z \leq 6, \\ v(z) & \text{otherwise.} \end{cases}$$

where

$$v(z) = \frac{24}{z + 7 + \sqrt{z^2 + 14z + 1}}.$$

This result can be understood if we consider the crisp LP problem with the centers of the fuzzy numbers

$$\begin{aligned} & \max / \min 3x \\ & \text{subject to } x \leq 2, x \geq 0. \end{aligned}$$

All negative values as possible solutions to the crisp problem are excluded by the constraint  $x \geq 0$ , and the possible values of the objective function are in the interval  $[0, 6]$ . However, due to the fuzziness in (5.45), the objective function can take bigger values than six with a non-zero degrees of possibility. Therefore to find an optimal value of the problem

$$(3, 1)x \rightarrow \max \tag{5.46}$$

$$\text{subject to } (1, 1)x \leq (2, 1) \quad x \geq 0.$$

requires a determination a trade-off between the increasing value of  $z$  and the decreasing value of  $\text{Pos}[Z = z]$ . If we take the product operator for modeling the trade-offs then we see that the

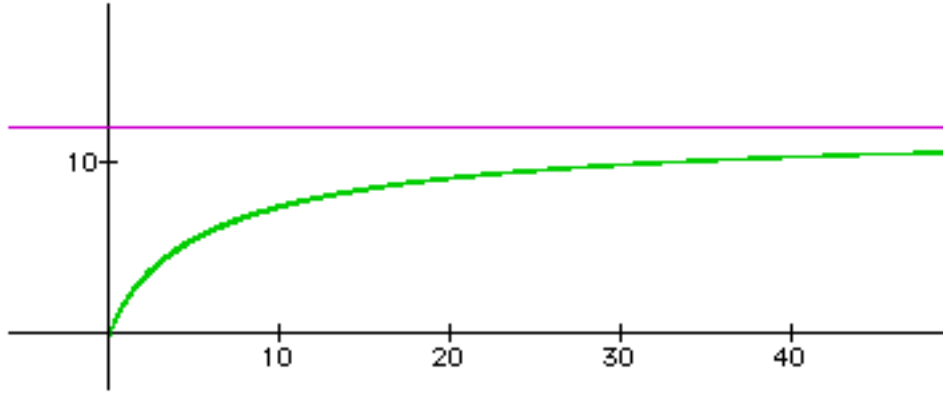


Figure 5.12:  $z \times \text{Pos}[Z = z]$  tends to 12 as  $z \rightarrow \infty$ .

resulting problem

$$z \times \text{Pos}[Z = z] = \frac{24z}{z + 7 + \sqrt{z^2 + 14z + 1}} \rightarrow \max$$

subject to  $z \geq 0$ .

does not have a finite solution, because the function  $z \times \text{Pos}[Z = z]$  is strictly increasing if  $z \geq 0$ .

## 5.7 Possibilistic quadratic programming

A possibilistic quadratic program is

$$\begin{aligned} & \text{maximize} && Z := x^T \tilde{C} x + \langle \tilde{d}, x \rangle \\ & \text{subject to} && \langle \tilde{a}_i, x \rangle \leq \tilde{b}_i, \quad 1 \leq i \leq m, \quad x \geq 0 \end{aligned} \quad (5.47)$$

where  $\tilde{C} = (\tilde{c}_{kj})$  is a matrix of fuzzy numbers,  $\tilde{a}_i = (\tilde{a}_{ij})$  and  $\tilde{d} = (\tilde{d}_j)$  are vectors of fuzzy numbers,  $\tilde{b}_i$  is a fuzzy number and

$$\langle \tilde{d}, x \rangle = \tilde{d}_1 x_1 + \cdots + \tilde{d}_n x_n.$$

We will assume that all fuzzy numbers are non-interactive. We define,  $\text{Pos}[Z = z]$ , the possibility distribution of the objective function  $Z$ . We first specify the possibility that  $x$  satisfies the  $i$ -th constraint. Let

$$\Pi(a_i, b_i) = \min\{\tilde{a}_{i1}(a_{i1}), \dots, \tilde{a}_{in}(a_{in}), \tilde{b}_i(b_i)\}$$

where  $a_i = (a_{i1}, \dots, a_{in})$ , which is the joint possibility distribution of  $\tilde{a}_i$ ,  $1 \leq j \leq n$  and  $\tilde{b}_i$ . Then

$$\text{Pos}[x \in \mathcal{F}_i] = \sup_{a_i, b_i} \{\Pi(a_i, b_i) \mid a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i\}$$

which is the possibility that  $x$  is feasible with respect to the  $i$ -th constraint. Therefore, for  $x \geq 0$ ,

$$\text{Pos}[x \in \mathcal{F}] = \min\{\text{Pos}[x \in \mathcal{F}_1], \dots, \text{Pos}[x \in \mathcal{F}_m]\}.$$

We next construct  $\text{Pos}[Z = z|x]$  which is the conditional possibility that  $Z$  equals  $z$  given  $x$ . The joint possibility distribution of  $\tilde{C}$  and  $\tilde{d}$  is

$$\Pi(C, d) = \min_{k,j} \{\tilde{C}_{kj}(c_{kj}), \tilde{d}_j(d_j)\}$$

where  $C = (c_{kj})$  is a crisp matrix and  $d = (d_j)$  a crisp vector. Therefore,

$$\text{Pos}[Z = z|x] = \sup_{C,d} \{\Pi(C, d) \mid x^T C x + \langle d, x \rangle = z\}.$$

Finally, the possibility distribution of the objective function is defined as

$$\text{Pos}[Z = z] = \sup_{x \geq 0} \min \{\text{Pos}[Z = z|x], \text{Pos}[x \in \mathcal{F}]\}.$$

We show that possibilistic quadratic programs with crisp decision variables and continuous fuzzy number coefficients are well-posed, i.e. small changes in the membership function of the coefficients may cause only a small deviation in the possibility distribution of the objective function. We will assume that there is a collection of fuzzy parameters  $\tilde{A}^\delta, \tilde{b}^\delta, \tilde{C}^\delta$  and  $\tilde{d}^\delta$  are available with the property

$$D(\tilde{A}, \tilde{A}^\delta) \leq \delta, D(\tilde{C}, \tilde{C}^\delta) \leq \delta, D(\tilde{b}, \tilde{b}^\delta) \leq \delta, D(\tilde{d}, \tilde{d}^\delta) \leq \delta, \quad (5.48)$$

Then we have to solve the following perturbed problem:

$$\begin{aligned} & \text{maximize} && x^T \tilde{C}^\delta x + \langle \tilde{d}^\delta, x \rangle \\ & \text{subject to} && \tilde{A}^\delta x \leq \tilde{b}^\delta, x \geq 0 \end{aligned} \quad (5.49)$$

Let us denote by  $\text{Pos}[x \in \mathcal{F}_i^\delta]$  that  $x$  is feasible with respect to the  $i$ -th constraint in (5.49). Then the possibility distribution of the objective function  $Z^\delta$  is defined as follows

$$\text{Pos}[Z^\delta = z] = \sup_{x \geq 0} \min \{\text{Pos}[Z^\delta = z|x], \text{Pos}[x \in \mathcal{F}^\delta]\}.$$

The next theorem shows a stability property of the possibility distribution of the objective function of the possibilistic quadratic programs (5.47) and (5.49).

**Theorem 5.7.1.** [39] *Let  $\delta > 0$  be a real number and let  $\tilde{c}_{kj}, \tilde{a}_{ij}, \tilde{d}_j, \tilde{b}_i, \tilde{c}_{kj}^\delta, \tilde{a}_{ij}^\delta, \tilde{d}_j^\delta, \tilde{b}_i^\delta \in \mathcal{F}$  be fuzzy numbers. If (5.48) hold then*

$$\sup_{z \in \mathbf{R}} |\text{Pos}[Z^\delta = z] - \text{Pos}[Z = z]| \leq \omega(\delta)$$

where  $\omega(\delta)$  denotes the maximum of modulus of continuity of all fuzzy number coefficients at  $\delta$  in (5.47) and (5.49).

From Theorem 5.7.1 it follows that  $\sup_z |\text{Pos}[Z^\delta = z] - \text{Pos}[Z = z]| \rightarrow 0$  as  $\delta \rightarrow 0$  which means the stability of the possibility distribution of the objective function with respect to perturbations (5.48).

## 5.8 Multiobjective possibilistic linear programming

Stability and sensitivity analysis becomes more and more attractive also in the area of multiple objective mathematical programming (for excellent surveys see e.g. Gal [199] and Rios Insua [358]). Publications on this topic usually investigate the impact of parameter changes (in the righthand side or/and the objective functions or/and the 'A-matrix' or/and the domination structure) on the solution in various models of vector maximization problems, e.g. linear or nonlinear, deterministic or stochastic, static or dynamic [112, 359].

Following Fullér and Fedrizzi [187], in this Section we show that the possibility distribution of the objectives of an multiobjective possibilistic linear program (MPLP) with (continuous) fuzzy number coefficients is stable under small changes in the membership function of the fuzzy parameters.

A multiobjective possibilistic linear program (MPLP) is

$$\max/\min Z = (\tilde{c}_{11}x_1 + \cdots + \tilde{c}_{1n}x_n, \dots, \tilde{c}_{k1}x_1 + \cdots + \tilde{c}_{kn}x_n) \quad (5.50)$$

$$\text{subject to } \tilde{a}_{i1}x_1 + \cdots + \tilde{a}_{in}x_n * \tilde{b}_i, \quad i = 1, \dots, m, \quad x \geq 0,$$

where  $\tilde{a}_{ij}$ ,  $\tilde{b}_i$ , and  $\tilde{c}_{lj}$  are fuzzy quantities,  $x = (x_1, \dots, x_n)$  is a vector of (non-fuzzy) decision variables and  $ast$  denotes  $<$ ,  $\leq$ ,  $=$ ,  $\geq$  or  $>$  for each  $i$ ,  $i = 1, \dots, m$ .

Even though  $*$  may vary from row to row in the constraints, we will rewrite the MPLP (5.50) as

$$\max/\min Z = (\tilde{c}_1x, \dots, \tilde{c}_kx)$$

$$\text{subject to } \tilde{A}x * \tilde{b}, \quad x \geq 0,$$

where  $\tilde{a} = \{\tilde{a}_{ij}\}$  is an  $m \times n$  matrix of fuzzy numbers and  $\tilde{b} = (\tilde{b}_1, \dots, \tilde{b}_m)$  is a vector of fuzzy numbers. The fuzzy numbers are the possibility distributions associated with the fuzzy variables and hence place a restriction on the possible values the variable may assume [440, 441]. For example,  $\text{Pos}[\tilde{a}_{ij} = t] = \tilde{a}_{ij}(t)$ . We will assume that all fuzzy numbers  $\tilde{a}_{ij}$ ,  $\tilde{b}_i$ ,  $\tilde{c}_l$  are non-interactive.

Following Buckley [32], we define  $\text{Pos}[Z = z]$ , the possibility distribution of the objective function  $Z$ . We first specify the possibility that  $x$  satisfies the  $i$ -th constraints. Let

$$\Pi(a_i, b_i) = \min\{\tilde{a}_{i1}(a_{i1}), \dots, \tilde{a}_{in}(a_{in}), \tilde{b}_i(b_i)\},$$

where  $a_i = (a_{i1}, \dots, a_{in})$ , which is the joint distribution of  $\tilde{a}_{ij}$ ,  $j = 1, \dots, n$ , and  $\tilde{b}_i$ . Then

$$\text{Pos}[x \in \mathcal{F}_i] = \sup_{a_i, b_i} \{\Pi(a_i, b_i) \mid a_{i1}x_1 + \cdots + a_{in}x_n * b_i\},$$

which is the possibility that  $x$  is feasible with respect to the  $i$ -th constraint. Therefore, for  $x \geq 0$ ,

$$\text{Pos}[x \in \mathcal{F}] = \min\{\text{Pos}[x \in \mathcal{F}_1], \dots, \text{Pos}[x \in \mathcal{F}_m]\}.$$

which is the possibility that  $x$  is feasible. We next construct  $\text{Pos}[Z = z|x]$  which is the conditional possibility that  $Z$  equals  $z$  given  $x$ . The joint distribution of the  $\tilde{c}_{lj}$ ,  $j = 1, \dots, n$ , is

$$\Pi(c_l) = \min\{\tilde{c}_{l1}(c_{l1}), \dots, \tilde{c}_{ln}(c_{ln})\}$$

where  $c_l = (c_{l1}, \dots, c_{ln})$ ,  $l = 1, \dots, k$ . Therefore,

$$\text{Pos}[Z = z|x] = \text{Pos}[\tilde{c}_1 x = z_1, \dots, \tilde{c}_k x = z_k] = \min_{1 \leq l \leq k} \text{Pos}[\tilde{c}_l x = z_l] =$$

$$\min_{1 \leq l \leq k} \sup_{c_{l1}, \dots, c_{lk}} \{\Pi(c_l) \mid c_{l1}x_1 + \dots + c_{ln}x_n = z_l\}.$$

Finally, the possibility distribution of the objective function is defined as

$$\text{Pos}[Z = z] = \sup_{x \geq 0} \min\{\text{Pos}[Z = z|x], \text{Pos}[x \in \mathcal{F}]\}$$

We will assume that there is a collection of fuzzy parameters  $\tilde{a}_{ij}^\delta$ ,  $\tilde{b}_i^\delta$ ,  $\tilde{c}_{lj}^\delta$  available with the property

$$\max_{i,j} D(\tilde{a}_{ij}, \tilde{a}_{ij}^\delta) \leq \delta, \quad \max_i D(\tilde{b}_i, \tilde{b}_i^\delta) \leq \delta, \quad \max_{l,j} D(\tilde{c}_{lj}, \tilde{c}_{lj}^\delta) \leq \delta. \quad (5.51)$$

Then we have to solve the following problem:

$$\max/\min Z^\delta = (\tilde{c}_1^\delta x, \dots, \tilde{c}_k^\delta x) \quad (5.52)$$

$$\text{subject to } \tilde{A}^\delta x * \tilde{b}^\delta, x \geq 0.$$

Let us denote by  $\text{Pos}[x \in \mathcal{F}_i^\delta]$  the possibility that  $x$  is feasible with respect to the  $i$ -th constraint in (5.52). Then the possibility distribution of the objective function  $Z^\delta$  in (5.52) is defined as:

$$\text{Pos}[Z^\delta = z] = \sup_{x \geq 0} (\min\{\text{Pos}[Z^\delta = z \mid x], \text{Pos}[x \in \mathcal{F}^\delta]\}).$$

The next theorem shows a stability property (with respect to perturbations (5.51) of the possibility distribution of the objective function,  $Z$ , of multiobjective possibilistic linear programming problems (5.50) and (5.52).

**Theorem 5.8.1.** [187] *Let  $\delta \geq 0$  be a real number and let  $\tilde{a}_{ij}$ ,  $\tilde{b}_i$ ,  $\tilde{a}_{ij}^\delta$ ,  $\tilde{c}_{lj}$ ,  $\tilde{c}_{lj}^\delta$  be (continuous) fuzzy numbers. If (5.51) hold, then*

$$\sup_{z \in \mathbf{R}^k} |\text{Pos}[Z^\delta = z] - \text{Pos}[Z = z]| \leq \omega(\delta)$$

where  $\omega(\delta)$  is the maximum of moduli of continuity of all fuzzy numbers at  $\delta$ .

From Theorem 5.8.1 it follows that

$$\sup_{z \in \mathbf{R}^k} |\text{Pos}[Z^\delta = z] - \text{Pos}[Z = z]| \rightarrow 0 \text{ as } \delta \rightarrow 0$$

which means the stability of the possibility distribution of the objective function with respect to perturbations (5.51). It is easy to see that in the case of non-continuous fuzzy parameters the possibility distribution of the objective function may be unstable under small changes of the parameters.



### 5.8.1 Example

As an example, consider the following biobjective possibilistic linear program

$$\max/\min (\tilde{c}x, \tilde{c}x) \quad (5.53)$$

$$\text{subject to } \tilde{a}x \leq \tilde{b}, x \geq 0.$$

where  $\tilde{a} = (1, 1)$ ,  $\tilde{b} = (2, 1)$  and  $\tilde{c} = (3, 1)$  are fuzzy numbers of symmetric triangular form. Here  $x$  is one-dimensional ( $n = 1$ ) and there is only one constraint ( $m = 1$ ). We find

$$\text{Pos}[x \in \mathcal{F}] = \begin{cases} 1 & \text{if } x \leq 2, \\ \frac{3}{x+1} & \text{if } x > 2. \end{cases}$$

and

$$\text{Pos}[Z = (z_1, z_2)|x] = \min\{\text{Pos}[\tilde{c}x = z_1], \text{Pos}[\tilde{c}x = z_2]\}$$

where

$$\text{Pos}[\tilde{c}x = z_i] = \begin{cases} 4 - \frac{z_i}{x} & \text{if } z_i/x \in [3, 4], \\ \frac{z_i}{x} - 2 & \text{if } z_i/x \in [2, 3], \\ 0 & \text{otherwise,} \end{cases}$$

for  $i = 1, 2$  and  $x \neq 0$ , and

$$\text{Pos}[Z = (z_1, z_2)|0] = \text{Pos}[0 \times \tilde{c} = z] = \begin{cases} 1 & \text{if } z = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Both possibilities are nonlinear functions of  $x$ , however the calculation of  $\text{Pos}[Z = (z_1, z_2)]$  is easily performed and we obtain

$$\text{Pos}[Z = (z_1, z_2)] = \begin{cases} \theta_1 & \text{if } z \in M_1, \\ \min\{\theta_1, \theta_2, \theta_3\} & \text{if } z \in M_2, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$M_1 = \{z \in \mathbb{R}^2 \mid |z_1 - z_2| \leq \min\{z_1, z_2\}, z_1 + z_2 \leq 12\},$$

$$M_2 = \{z \in \mathbb{R}^2 \mid |z_1 - z_2| \leq \min\{z_1, z_2\}, z_1 + z_2 > 12\},$$

and

$$\theta_i = \frac{24}{z_i + 7 + \sqrt{z_i^2 + 14z_i + 1}}.$$

for  $i = 1, 2$  and

$$\theta_3 = \frac{4 \min\{z_1, z_2\} - 2 \max\{z_1, z_2\}}{z_1 + z_2}.$$

Consider now a perturbed biobjective problem with two different objectives (derived from (5.53) by a simple  $\delta$ -shifting of the centres of  $\tilde{a}$  and  $\tilde{c}$ ):

$$\max/\min (\tilde{c}x, \tilde{c}^\delta x) \quad (5.54)$$

$$\text{subject to } \tilde{a}^\delta x \leq \tilde{b}, x \geq 0.$$

where  $\tilde{a} = (1 + \delta, 1)$ ,  $\tilde{b} = (2, 1)$ ,  $\tilde{c} = (3, 1)$ ,  $\tilde{c}^\delta = (3 - \delta, 1)$  and  $\delta \geq 0$  is the error of measurement. Then

$$\text{Pos}[x \in \mathcal{F}^\delta] = \begin{cases} 1 & \text{if } x \leq \frac{2}{1 + \delta}, \\ \frac{3 - \delta x}{x + 1} & \text{if } x > \frac{2}{1 + \delta}. \end{cases}$$

and

$$\text{Pos}[Z^\delta = (z_1, z_2)|x] = \min\{\text{Pos}[\tilde{c}x = z_1], \text{Pos}[\tilde{c}^\delta x = z_2]\}$$

where

$$\text{Pos}[\tilde{c}x = z_1] = \begin{cases} 4 - \frac{z_1}{x} & \text{if } z_1/x \in [3, 4], \\ \frac{z_1}{x} - 2 & \text{if } z_1/x \in [2, 3], \\ 0 & \text{otherwise,} \end{cases}$$

$$\text{Pos}[\tilde{c}^\delta x = z_2] = \begin{cases} 4 - \delta - \frac{z_2}{x} & \text{if } z_2/x \in [3 - \delta, 4 - \delta], \\ \frac{z_2}{x} - 2 + \delta & \text{if } z_2/x \in [2 - \delta, 3 - \delta], \\ 0 & \text{otherwise,} \end{cases}$$

$x \neq 0$ , and

$$\text{Pos}[Z^\delta = (z_1, z_2)|0] = \text{Pos}[0 \times \tilde{c} = z] = \begin{cases} 1 & \text{if } z = 0, \\ 0 & \text{otherwise.} \end{cases}$$

So,

$$\text{Pos}[Z^\delta = (z_1, z_2)] = \begin{cases} \theta_1(\delta) & \text{if } z \in M_1(\delta), \\ \min\{\theta_1(\delta), \theta_2(\delta), \theta_3(\delta)\} & \text{if } z \in M_2(\delta), \\ 0 & \text{otherwise,} \end{cases}$$

where

$$M_1(\delta) = \left\{ z \in \mathbb{R}^2 \mid |z_1 - z_2| \leq (1 - 0.5\delta) \min\{z_1, z_2\}, z_1 + z_2 \leq \frac{2(6 - \delta)}{1 + \delta} \right\},$$

$$M_2(\delta) = \left\{ z \in \mathbb{R}^2 \mid |z_1 - z_2| \leq (1 - 0.5\delta) \min\{z_1, z_2\}, z_1 + z_2 > \frac{2(6 - \delta)}{1 + \delta} \right\},$$

$$\theta_1(\delta) = \frac{24 + \delta \left( 7 - z_1 - \sqrt{z_1^2 + 14z_1 + 1 + 4z_1\delta} \right)}{z_1 + 7 + \sqrt{z_1^2 + 14z_1 + 1 + 4z_1\delta} + 2\delta}$$

$$\theta_2(\delta) = \frac{24 - \delta \left( \delta + z_2 - 1 + \sqrt{(1 - \delta - z_2)^2 + 16z_2} \right)}{z_2 + 7 + \sqrt{(1 - \delta - z_2)^2 + 16z_2} + \delta}$$

and

$$\theta_3(\delta) = \frac{(4 - \delta) \min\{z_1, z_2\} - 2 \max\{z_1, z_2\}}{z_1 + z_2}.$$

It is easy to check that

$$\begin{aligned} \sup_{x \geq 0} |\text{Pos}[x \in \mathcal{F}] - \text{Pos}[x \in \mathcal{F}^\delta]| &\leq \delta, \\ \sup_z |\text{Pos}[Z = z|x] - \text{Pos}[Z^\delta = z|x]| &\leq \delta, \quad \forall x \geq 0, \\ \sup_z |\text{Pos}[Z = z] - \text{Pos}[Z^\delta = z]| &\leq \delta. \end{aligned}$$

On the other hand, from the definition of metric  $D$  the modulus of continuity and Theorem 5.8.1 it follows that

$$\begin{aligned} D(\tilde{a}, \tilde{a}^\delta) &= \delta, \\ D(\tilde{c}, \tilde{c}^\delta) &= \delta, \\ D(\tilde{c}, \tilde{c}) &= 0, \\ D(\tilde{b}, \tilde{b}) &= 0, \\ \omega(\delta) &= \delta, \end{aligned}$$

and, therefore,

$$\sup_z |\text{Pos}[Z = z] - \text{Pos}[Z^\delta = z]| \leq \delta.$$

## 5.9 MPLP with weakly non-interactive fuzzy coefficients

Assume now that all fuzzy numbers  $\tilde{a}_{ij}$ ,  $\tilde{b}_i$ ,  $\tilde{a}_{ij}^\delta$ ,  $\tilde{b}_i^\delta$ ,  $\tilde{c}_{lj}$  and  $\tilde{c}_{lj}^\delta$  in (5.50) and (5.52) are weakly non-interactive [440]. Weakly-noninteractivity means that there exists a triangular norm  $T$ , such that we can find the joint possibility distribution of all the fuzzy variables by calculating the  $T$ -intersection of their possibility distributions.

The next theorem shows a stability property of the possibility distribution of the objective functions of multiobjective possibilistic linear programs with  $T$ -weakly non-interactive fuzzy number coefficients (5.50) and (5.52).

**Theorem 5.9.1.** [142] *Let  $\delta \geq 0$  be a real number and let  $\tilde{a}_{ij}$ ,  $\tilde{b}_i$ ,  $\tilde{a}_{ij}^\delta$ ,  $\tilde{b}_i^\delta$ ,  $\tilde{c}_{lj}$  and  $\tilde{c}_{lj}^\delta$  be  $T$ -weakly-noninteractivity fuzzy numbers. If (5.51) hold, and  $T$  is a continuous  $t$ -norm then*

$$\sup_{z \in \mathbb{R}^k} |\text{Pos}[Z^\delta = z] - \text{Pos}[Z = z]| \leq \omega(T, \Omega(\delta))$$

where  $\Omega(\delta)$  is the maximum of modulus of continuity of all fuzzy number coefficients at  $\delta$  in (5.50) and (5.52), and  $\omega(T, \cdot)$  denotes the modulus of continuity of  $T$ .

*Proof.* It is sufficient to show that

$$|\text{Pos}[Z^\delta = z] - \text{Pos}[Z = z]| \leq \omega(T, \Omega(\delta))$$

for any  $z = (z_1, \dots, z_k) \in \mathbb{R}^k$ . Applying Lemma 2.14.5 we have

$$\begin{aligned} & |\text{Pos}[Z^\delta = z] - \text{Pos}[Z = z]| = \\ & \left| \sup_{x \geq 0} \min\{\text{Pos}[Z = z|x], \text{Pos}[x \in \mathcal{F}]\} - \sup_{x \geq 0} \min\{\text{Pos}[Z^\delta = z|x], \text{Pos}[x \in \mathcal{F}^\delta]\} \right| \leq \end{aligned}$$

$$\sup_{x \geq 0} \max \left\{ |\text{Pos}[Z = z|x] - \text{Pos}[Z^\delta = z|x]|, |\text{Pos}[x \in \mathcal{F}] - \text{Pos}[x \in \mathcal{F}^\delta]| \right\} \leq$$

$$\sup_{x \geq 0} \max \left\{ \max_{1 \leq l \leq k} \{ |\text{Pos}[\tilde{c}_l x = z_l] - \text{Pos}[\tilde{c}_l^\delta x = z_l]| \}, \right.$$

$$\left. \max_{1 \leq i \leq m} \{ |\text{Pos}[x \in \mathcal{F}_i] - \text{Pos}[x \in \mathcal{F}_i^\delta]| \} \right\} =$$

$$\sup_{x \geq 0} \max \left\{ \max_{1 \leq l \leq k} \left| \sup_{c_{l1}, \dots, c_{ln}} \{ T(\tilde{c}_{l1}(c_{l1}), \dots, \tilde{c}_{ln}(c_{ln})) \mid c_{l1}x + \dots + c_{ln}x_n = z_l \} - \right. \right.$$

$$\left. \sup_{c_{l1}, \dots, c_{ln}} \{ T(\tilde{c}_{l1}^\delta(c_{l1}), \dots, \tilde{c}_{ln}^\delta(c_{ln})) \mid c_{l1}x_1 + \dots + c_{ln}x_n = z_l \} \right|, \right.$$

$$\left. \max_{1 \leq i \leq m} \left| \sup_{a_{i1}, \dots, a_{in}} \{ T(\tilde{a}_{i1}(a_{i1}), \dots, \tilde{a}_{in}(a_{in}), \tilde{b}_i(b_i)) \mid a_{i1}x_1 + \dots + a_{in}x_n * b_i \} - \right. \right.$$

$$\left. \sup_{a_{i1}, \dots, a_{in}} \{ T(\tilde{a}_{i1}^\delta(a_{i1}), \dots, \tilde{a}_{in}^\delta(a_{in}), \tilde{b}_i^\delta(b_i)) \mid a_{i1}x_1 + \dots + a_{in}x_n * b_i \} \right| \right\} \leq$$

$$\sup_{x \geq 0} \max \left\{ \max_{1 \leq l \leq k} \sup_{c_{l1}, \dots, c_{ln}} \max \{ \omega(T, |\tilde{c}_{lj}(c_{lj}) - \tilde{c}_{lj}^\delta(c_{lj})|) \mid c_{l1}x_1 + \dots + c_{ln}x_n = z_l \}, \right.$$

$$\left. \max_{1 \leq i \leq m} \sup_{a_{i1}, \dots, a_{in}} \max_{1 \leq j \leq n} \left\{ \max \{ \omega(T, |\tilde{a}_{ij}(a_{ij}) - \tilde{a}_{ij}^\delta(a_{ij})|), \right. \right.$$

$$\left. \omega(T, |\tilde{b}_i(b_i) - \tilde{b}_i^\delta(b_i)|) \} \mid a_{i1}x_1 + \dots + a_{in}x_n * b_i \right\} \right\} \leq$$

$$\sup_{a_{ij}, b_i, c_{lj}} \max \left\{ \omega(T, |\tilde{a}_{ij}^\delta(a_{ij}) - \tilde{a}_{ij}(a_{ij})|), \omega(T, |\tilde{b}_i^\delta(b_i) - \tilde{b}_i(b_i)|), \right.$$

$$\begin{aligned}
& \omega(T, |\tilde{c}_{lj}^\delta(c_{lj}) - \tilde{c}_{lj}(c_{lj})|) \Big\} \leq \\
& \max_{i,j,l} \max \left\{ \omega(T, \omega(\tilde{a}_{ij}, \delta)), \omega(T, \omega(\tilde{a}_{ij}^\delta, \delta)), \omega(T, \omega(\tilde{b}_i, \delta)), \omega(T, \omega(\tilde{b}_i^\delta, \delta)), \right. \\
& \left. \omega(T, \omega(\tilde{c}_{lj}, \delta)), \omega(\tilde{c}_{lj}^\delta, \delta) \right\} \leq \omega(T, \Omega(\delta)).
\end{aligned}$$

Which ends the proof. □

# Chapter 6

## Fuzzy Reasoning for Fuzzy Optimization

### 6.1 Fuzzy reasoning for FMP

Following Fullér and Zimmermann [185], we interpret fuzzy linear programming (FLP) problems with fuzzy coefficients and fuzzy inequality relations as multiple fuzzy reasoning schemes (MFR), where the antecedents of the scheme correspond to the constraints of the FLP problem and the fact of the scheme is the objective of the FLP problem.

Then the solution process consists of two steps: first, for every decision variable  $x \in \mathbb{R}^n$ , we compute the (fuzzy) value of the objective function,  $\text{MAX}(x)$ , via sup-min convolution of the antecedents/constraints and the fact/objective, then an (optimal) solution to FLP problem is any point which produces a maximal element of the set

$$\{\text{MAX}(x) \mid x \in \mathbb{R}^n\}$$

(in the sense of the given inequality relation). We show that this solution process for a classical (crisp) LP problem results in a solution in the classical sense, and (under well-chosen inequality relations and objective function) coincides with those suggested by Buckley [30], Delgado et al. [105, 106], Negoita [337], Ramik and Rimanek [351], Verdegay [401, 402] and Zimmermann [450].

We consider FLP problems of the form

$$\tilde{c}_1 x_1 + \cdots + \tilde{c}_n x_n \rightarrow \max \quad (6.1)$$

$$\tilde{a}_{i1} x_1 + \cdots + \tilde{a}_{in} x_n \lesssim \tilde{b}_i, \quad i = 1, \dots, m,$$

or, shortly,

$$\begin{aligned} \langle \tilde{c}, x \rangle &\rightarrow \max \\ \text{subject to } \tilde{A}x &\lesssim \tilde{b}, \end{aligned}$$

where  $x \in \mathbb{R}^n$  is the vector of decision variables,  $\tilde{a}_{ij}$ ,  $\tilde{b}_i$  and  $\tilde{c}_j$  are fuzzy quantities, the operations addition and multiplication by a real number of fuzzy quantities are defined by Zadeh's extension principle, the inequality relation,  $\lesssim$ , for the constraints is given by a certain fuzzy relation and the objective function is to be maximized in the sense of a given crisp inequality relation,  $\leq$ , between fuzzy quantities.

The FLP problem (6.1) can be stated as follows: Find  $x^* \in \mathbb{R}^n$  such that

$$\tilde{c}_1 x_1 + \cdots + \tilde{c}_n x_n \leq \tilde{c}_1 x_1^* + \cdots + \tilde{c}_n x_n^*$$

$$\tilde{a}_{i1} x_1 + \cdots + \tilde{a}_{in} x_n \lesssim \tilde{b}_i, \quad i = 1, \dots, m,$$

i.e. we search for an alternative,  $x^*$ , which maximizes the objective function subject to constraints. Now we set up the notations and recall some fuzzy inference rules needed for the proposed solution principle.

In the following  $\bar{a}$  denotes the characteristic function of the singleton  $a \in \mathbb{R}$ , i.e

$$\bar{a}(t) = \begin{cases} 1 & \text{if } t = a \\ 0 & \text{otherwise.} \end{cases}$$

Let  $X$  be a non-empty set. The empty fuzzy set in  $X$ , denoted by  $\emptyset_X$ , is defined as  $\emptyset_X(x) = 0, \forall x \in X$ . A binary fuzzy relation  $W$  in  $X$  is a fuzzy subset of the Cartesian product  $X \times X$  and defined by its membership function  $\mu_W$  (or simply  $W$  if not confusing). If  $\mu_W(u, v) \in \{0, 1\}, \forall u, v \in X$  then  $W$  is called a crisp relation in  $X$ . Throughout this Section we shall use the terms relation and inequality relation interchangeably, i.e. we do not require any additional property for the later. However, we can get unexpected solutions if we use unjustifiable inequality relations to compare fuzzy quantities.

Let  $\leq$  be a crisp inequality relation in  $\mathcal{F}$ . Then for all pairs  $\tilde{a}, \tilde{b} \in \mathcal{F}$  it induces a crisp binary relation in  $\mathbb{R}$  defined by

$$(\tilde{a} \leq \tilde{b})(u, v) = \begin{cases} 1 & \text{if } u = v, \text{ and } \tilde{a} \text{ and } \tilde{b} \text{ are in relation } \leq, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that  $(\tilde{a} \leq \tilde{b}) = \emptyset$  iff and  $\tilde{a}$  and  $\tilde{b}$  are not in relation  $\leq$ . If the inequality relation  $\leq$  is modeled by a fuzzy implication operator  $I$  then for all pairs  $\tilde{a}, \tilde{b} \in \mathcal{F}$  it induces a fuzzy binary relation in  $\mathbb{R}$  defined by

$$(\tilde{a} \leq \tilde{b})(u, v) = I(\tilde{a}(u), \tilde{b}(v)),$$

e.g. if  $\leq$  is given by the Gödel implication operator then we have

$$(\tilde{a} \leq \tilde{b})(u, v) = \begin{cases} 1 & \text{if } \tilde{a}(u) \leq \tilde{b}(v), \\ \tilde{b}(v) & \text{otherwise.} \end{cases} \quad (6.2)$$

If an inequality relation  $\leq$  in  $\mathcal{F}$  is not crisp then we will usually write  $\lesssim$  instead. We will use the following crisp inequality relations in  $\mathcal{F}$ :

$$\tilde{a} \leq \tilde{b} \iff \text{m}\tilde{\text{ax}} \{ \tilde{a}, \tilde{b} \} = \tilde{b} \quad (6.3)$$

where  $\text{m}\tilde{\text{ax}}$  is the ordinary extension of the two-placed max function defined as

$$\tilde{a} \leq \tilde{b} \iff \tilde{a} \subseteq \tilde{b}, \quad (6.4)$$

where  $\tilde{a} \subseteq \tilde{b}$  if  $\tilde{a}(u) \leq \tilde{b}(u)$ , for all  $u \in \mathbb{R}$ ,

$$\tilde{a} \leq \tilde{b} \iff \text{peak}(\tilde{a}) \leq \text{peak}(\tilde{b}) \quad (6.5)$$

where  $\tilde{a}$  and  $\tilde{b}$  are fuzzy numbers, and  $\text{peak}(\tilde{a})$  and  $\text{peak}(\tilde{b})$  denote their peaks,

$$\bar{a} \leq \bar{b} \iff a \leq b, \quad (6.6)$$

where  $\bar{a}$  and  $\bar{b}$  are fuzzy singletons. Let  $\Gamma$  be an index set,  $\tilde{a}_\gamma \in \mathcal{F}$ ,  $\gamma \in \Gamma$ , and let  $\leq$  a crisp inequality relation in  $\mathcal{F}$ . We say that  $\tilde{a}$  is a maximal element of the set

$$\mathcal{G} := \{\tilde{a}_\gamma \mid \gamma \in \Gamma\} \quad (6.7)$$

if  $\tilde{a}_\gamma \leq \tilde{a}$  for all  $\gamma \in \Gamma$  and  $\tilde{a} \in \mathcal{G}$ . A fuzzy quantity  $\tilde{A}$  is called an upper bound of  $\mathcal{G}$  if  $\tilde{a}_\gamma \leq \tilde{A}$  for all  $\gamma \in \Gamma$ . A fuzzy quantity  $\tilde{A}$  is called a least upper bound (supremum) of  $\mathcal{G}$  if it is an upper bound and if there exists an upper bound  $\tilde{B}$ , such that  $\tilde{B} \leq \tilde{A}$ , then  $\tilde{A} \leq \tilde{B}$ . If  $\tilde{A}$  is a least upper bound of  $\mathcal{G}$ , then we write

$$\tilde{A} = \sup\{\tilde{a}_\gamma \mid \gamma \in \Gamma\}$$

It is easy to see that, depending on the definition of the inequality relation, the set (6.7) may have many maximal elements (suprema) or the set of maximal elements (suprema) may be empty. For example, (i) if  $\{\text{peak}(\tilde{a}) \mid \gamma \in \Gamma\}$  is a bounded and closed subset of the real line then  $\mathcal{G}$  has at least one maximal element in the sense of relation (6.5); (ii)  $\mathcal{G}$  always has a unique supremum in relation (6.4), but usually does not have maximal elements; (iii) if there exists  $u \in \mathbb{R}$ , such that  $\tilde{a}_\gamma(v) = 0$ , for all  $v \geq u$  and  $\gamma \in \Gamma$  then  $\mathcal{G}$  has infinitely many suprema in relation (6.3).

The degree of possibility of the statement " $\tilde{a}$  is smaller or equal to  $\tilde{b}$ ", which we write  $\text{Pos}[\tilde{a} \leq \tilde{b}]$ , induces the following relation in  $\mathbb{R}$

$$(\tilde{a} \lesssim \tilde{b})(u, v) = \begin{cases} \text{Pos}[\tilde{a} \leq \tilde{b}] & \text{if } u = v \\ 0 & \text{otherwise} \end{cases} \quad (6.8)$$

We shall use the compositional rule of inference scheme with several relations (called Multiple Fuzzy Reasoning Scheme) [439] which has the general form

Fact	$X$ has property $P$
Relation 1:	$X$ and $Y$ are in relation $W_1$
$\dots$	$\dots$
Relation m:	$X$ and $Y$ are in relation $W_m$
Consequence:	$Y$ has property $Q$

where  $X$  and  $Y$  are linguistic variables taking their values from fuzzy sets in classical sets  $U$  and  $V$ , respectively,  $P$  and  $Q$  are unary fuzzy predicates in  $U$  and  $V$ , respectively, and  $W_i$  is a binary fuzzy relation in  $U \times V$ ,  $i=1, \dots, m$ . The consequence  $Q$  is determined by

$$Q = P \circ \bigcap_{i=1}^m W_i$$

or in detail,

$$\mu_Q(y) = \sup_{x \in U} \min \{\mu_P(x), \mu_{W_1}(x, y), \dots, \mu_{W_m}(x, y)\}.$$

We consider FLP problems as MFR schemes, where the antecedents of the scheme correspond to the constraints of the FLP problem and the fact of the scheme is interpreted as the objective



function of the FLP problem. Then the solution process consists of two steps: first, for every decision variable  $x \in \mathbb{R}^n$ , we compute the value of the objective function,  $MAX(x)$ , via sup-min convolution of the antecedents/constraints and the fact/objective, then an (optimal) solution to the FLP problem is any point which produces a maximal element of the set

$$\{MAX(x)|x \in \mathbb{R}^n\} \quad (6.9)$$

in the sense of the given inequality relation. We interpret the FLP problem (6.1) as MFR schemes of the form

Antecedent 1	$Constraint_1(x) := \tilde{a}_{11}x_1 + \cdots + \tilde{a}_{1n}x_n \lesssim \tilde{b}_1$
$\cdots$	$\cdots$
Antecedent m	$Constraint_m(x) := \tilde{a}_{m1}x_1 + \cdots + \tilde{a}_{mn}x_n \lesssim \tilde{b}_m$
Fact	$Goal(x) := \tilde{c}_1x_1 + \cdots + \tilde{c}_nx_n$
<hr style="border: 0.5px solid black;"/>	
Consequence	$MAX(x)$

where  $x \in \mathbb{R}^n$  and the consequence (i.e. the value of the objective function subject to constraints at  $x$ )  $MAX(x)$  is computed as follows

$$MAX(x) = Goal(x) \circ \bigcap_{i=1}^m Constraint_i(x),$$

i.e.

$$\mu_{MAX(x)}(v) = \sup_u \min \left\{ \mu_{Goal(x)}(u), \mu_{Constraint_1(x)}(u, v), \dots, \mu_{Constraint_m(x)}(u, v) \right\}. \quad (6.10)$$

Then an optimal value of the objective function of problem (6.1), denoted by  $M$ , is defined as

$$M := \sup \{MAX(x)|x \in \mathbb{R}^n\}, \quad (6.11)$$

where sup is understood in the sense of the given crisp inequality relation for the objective function. Finally, a solution  $x^* \in \mathbb{R}^n$  to problem (6.1) is obtained from the equation

$$MAX(x^*) = M.$$

The set of solutions of problem (6.1) is non-empty iff the set of maximizing elements of (6.9) is non-empty. Apart from the deterministic LP,

$$\langle c, x \rangle \rightarrow \max$$

$$\text{subject to } Ax \leq b,$$

where we simply compute the value of the objective function as

$$c_1y_1 + \cdots + c_ny_n$$

at any feasible point  $y \in \mathbb{R}^n$  and do not care about non-feasible points, in FLP problem (6.1) we have to consider the whole decision space, because each  $y$  from  $\mathbb{R}^n$  has a (fuzzy) degree of feasibility (given by the fuzzy relations  $Constraint_i(y)$ ,  $i = 1, \dots, m$ ). We have a right to compute the value of the objective function of (6.1) at  $y \in \mathbb{R}^n$  as  $\tilde{c}_1y_1 + \cdots + \tilde{c}_ny_n$  if there are

no constraints at all (if there are no rules in a fuzzy reasoning scheme then the consequence takes the value of the observation automatically).

To determine a maximal element of the set (6.9) even for a *crisp* inequality relation is usually a very complicated process. However, this problem can lead to a crisp LP problem (see Zimmermann [451], Buckley [30]), crisp multiple criteria parametric linear programming problem (see Delgado et al. [105, 106], Verdegay [401, 402]) or nonlinear mathematical programming problem (see Zimmermann [455]). If the inequality relation for the objective function is not crisp but fuzzy, then we somehow have to find an element from the set (6.9) which can be considered as a best choice in the sense of the given fuzzy inequality relation (see Ovchinnikov [348], Orlovski [345], Ramik and Rimanek [351], Rommelfanger [360], Roubens and Vincke [366], Tanaka and Asai [394]).

### 6.1.1 Extension to nonlinear FMP

We show how the proposed approach can be extended to nonlinear FMP problems with fuzzy coefficients. Generalizing the classical mathematical programming (MP) problem

$$\begin{array}{ll} \text{maximize} & g(c, x) \\ \text{subject to} & f_i(a_i, x) \leq b_i, \quad i = 1, \dots, m, \end{array}$$

where  $x \in \mathbb{R}^n$ ,  $c = (c_1, \dots, c_k)$  and  $a_i = (a_{i1}, \dots, a_{il})$  are vectors of crisp coefficients, we consider the following FMP problem

$$\begin{array}{ll} \text{maximize} & g(\tilde{c}_1, \dots, \tilde{c}_k, x) \\ \text{subject to} & f_i(\tilde{a}_{i1}, \dots, \tilde{a}_{il}, x) \lesssim \tilde{b}_i, \quad i = 1, \dots, m, \end{array}$$

where  $x \in \mathbb{R}^n$ ,  $\tilde{c}_h$ ,  $h = 1, \dots, k$ ,  $\tilde{a}_{is}$ ,  $s = 1, \dots, l$ , and  $\tilde{b}_i$  are fuzzy quantities, the functions  $g(\tilde{c}, x)$  and  $f_i(\tilde{a}_i, x)$  are defined by Zadeh's extension principle, and the inequality relation  $\lesssim$  is defined by a certain fuzzy relation. We interpret the above FMP problem as MFR schemes of the form

Antecedent 1:	$\text{Constraint}_1(x) := f_i(\tilde{a}_{i1}, \dots, \tilde{a}_{il}, x) \lesssim \tilde{b}_i$
...	...
Antecedent m:	$\text{Constraint}_m(x) := f_m(\tilde{a}_{m1}, \dots, \tilde{a}_{ml}, x) \lesssim \tilde{b}_m$
Fact:	$\text{Goal}(x) := g(\tilde{c}_1, \dots, \tilde{c}_k, x)$
Consequence	$\text{MAX}(x)$

Then the solution process is carried out analogously to the linear case, i.e an optimal value of the objective function,  $M$ , is defined by (6.11), and a solution  $x^* \in \mathbb{R}^n$  is obtained by solving the equation  $\text{MAX}(x^*) = M$ .

### 6.1.2 Relation to classical LP problems

We show that our solution process for classical LP problems results in a solution in the classical sense. A classical LP problem can be stated as follows

$$\max \langle c, x \rangle; \text{ subject to } Ax \leq b, \quad x \in \mathbb{R}^n. \quad (6.12)$$

Let  $X^*$  be the set of solutions and if  $X^* \neq \emptyset$  then let  $v^* = \langle c, x^* \rangle$  denote the optimal value of the objective function of (6.12). An element  $x$  from  $\mathbb{R}^n$  is said to be feasible if it satisfies the inequality  $Ax \leq b$ . Generalizing the crisp LP problem (6.12) we consider the FLP problem (6.1) with fuzzy singletons and crisp inequality relations (6.6)

$$\begin{aligned} & \text{maximize} && \text{Goal}(x) := \bar{c}_1 x_1 + \cdots + \bar{c}_n x_n \\ & \text{subject to} && \text{Constraint}_1(x) := \bar{a}_{11} x_1 + \cdots + \bar{a}_{1n} x_n \leq \bar{b}_1 \\ & && \dots \\ & && \text{Constraint}_m(x) := \bar{a}_{m1} x_1 + \cdots + \bar{a}_{mn} x_n \leq \bar{b}_m \end{aligned} \quad (6.13)$$

where  $\bar{a}_{ij}$ ,  $\bar{b}_i$  and  $\bar{c}_j$  denote the characteristic function of the crisp coefficients  $a_{ij}$ ,  $b_i$  and  $c_j$ , respectively, and the inequality relation  $\leq$  is defined by

$$\bar{a}_{i1} x_1 + \cdots + \bar{a}_{in} x_n \leq \bar{b}_i \iff a_{i1} x_1 + \cdots + a_{in} x_n \leq b_i,$$

i.e.

$$(\bar{a}_{i1} x_1 + \cdots + \bar{a}_{in} x_n \leq \bar{b}_i)(u, v) = \begin{cases} 1 & \text{if } u = v \text{ and } \langle a_i, x \rangle \leq b_i \\ 0 & \text{otherwise} \end{cases} \quad (6.14)$$

Then from (6.10) we get

$$\mu_{MAX(x)}(v) = \begin{cases} 1 & \text{if } v = \langle c, x \rangle \text{ and } Ax \leq b \\ 0 & \text{otherwise,} \end{cases}$$

which can be written in the form

$$MAX(x) = \begin{cases} \overline{\langle c, x \rangle} & \text{if } x \text{ is feasible} \\ 0 & \text{otherwise,} \end{cases}$$

consequently, if  $x$  and  $x'$  are feasible then

$$MAX(x) \leq MAX(x') \iff \langle c, x \rangle \leq \langle c, x' \rangle,$$

and if  $x'$  is feasible, but  $x''$  is not feasible then

$$MAX(x'') \leq MAX(x'),$$

since  $MAX(x'')$  is empty. Therefore from (6.11) we get  $M = \overline{v^*}$ , and  $x^*$  satisfies the equality  $MAX(x^*) = M$  if and only if  $v^* = \langle c, x^* \rangle$ , i.e.  $x^* \in X^*$ . This means that LP problem (6.12) and FLP problem (6.13) have the same solution-set, and the optimal value of the FLP problem is the characteristic function of the optimal value of the LP problem.

### 6.1.3 Crisp objective and fuzzy coefficients in constraints

FLP problems with crisp inequality relations in fuzzy constraints and crisp objective function can be formulated as follows (see Negoita's robust programming [337], Ramik and Rimanek [351], and Werners [411])

$$\begin{aligned} & \max && \langle c, x \rangle \\ & \text{subject to} && \tilde{a}_{i1} x_1 + \cdots + \tilde{a}_{in} x_n \leq \tilde{b}_i, \quad i = 1, \dots, m. \end{aligned} \quad (6.15)$$

It is easy to see that problem (6.15) is equivalent to the crisp MP problem

$$\max \langle c, x \rangle; \text{ subject to } x \in X, \quad (6.16)$$

where,

$$X = \bigcap_{i=1}^m X_i = \bigcap_{i=1}^m \left\{ x \in \mathbb{R}^n \mid \tilde{a}_{i1}x_1 + \cdots + \tilde{a}_{in}x_n \leq \tilde{b}_i \right\}.$$

Now we show that our approach leads to the same crisp MP problem (6.15). Consider problem (6.1) with fuzzy singletons in the objective function

$$\begin{aligned} \max \quad & \langle \bar{c}, x \rangle \\ \text{subject to} \quad & \tilde{a}_{i1}x_1 + \cdots + \tilde{a}_{in}x_n \leq \tilde{b}_i, \quad i = 1, \dots, m. \end{aligned}$$

where the inequality relation  $\leq$  is defined by

$$(\tilde{a}_{i1}x_1 + \cdots + \tilde{a}_{in}x_n \leq \tilde{b}_i)(u, v) = \begin{cases} 1 & \text{if } u = v \text{ and } x \in X_i \\ 0 & \text{otherwise} \end{cases}$$

Then we have

$$\mu_{MAX(x)}(v) = \begin{cases} 1 & \text{if } v = \langle c, x \rangle \text{ and } x \in X \\ 0 & \text{otherwise} \end{cases}$$

Thus, to find a maximizing element of the set  $\{MAX(x) \mid x \in \mathbb{R}^n\}$  in the sense of the given inequality relation we have to solve the crisp problem (6.16).

#### 6.1.4 Fuzzy objective function and crisp constraints

Consider the FLP problem (6.1) with fuzzy coefficients in the objective function and fuzzy singletons in the constraints

$$\begin{aligned} \text{maximize} \quad & \tilde{c}_1x_1 + \cdots + \tilde{c}_nx_n \\ \text{subject to} \quad & \bar{a}_{i1}x_1 + \cdots + \bar{a}_{in}x_n \leq \bar{b}_i, \quad i = 1, \dots, m, \end{aligned} \quad (6.17)$$

where the inequality relation for constraints is defined by (6.14) and the objective function is to be maximized in relation (6.3), i.e.

$$MAX(x') \leq MAX(x'') \iff \tilde{\max}\{MAX(x'), MAX(x'')\} = MAX(x'').$$

Then  $\mu_{MAX(x)}(v)$ ,  $\forall v \in \mathbb{R}$ , is the optimal value of the following crisp MP problem

$$\begin{aligned} \text{maximize} \quad & (\tilde{c}_1x_1 + \cdots + \tilde{c}_nx_n)(v) \\ \text{subject to} \quad & Ax \leq b, \quad x \in \mathbb{R}^n \end{aligned}$$

and the problem of computing a solution to FLP problem (6.17) leads to the same crisp multiple objective parametric linear programming problem obtained by Delgado et al. [105, 106] and Verdegay [401, 402].

### 6.1.5 Relation to Zimmermann's soft constraints

Consider Zimmermann's LP with crisp coefficients and soft constraints: Find  $x$  such that

$$\langle c, x \rangle \lesssim z; \langle a_i, x \rangle \lesssim b_i, \quad i = 1, \dots, m, \quad (6.18)$$

where the inequality relation  $\lesssim$  is defined by

$$\langle a_i, x \rangle \lesssim b_i = \begin{cases} 1 & \text{if } \langle a_i, x \rangle \leq b_i \\ 1 - \frac{b_i - \langle a_i, x \rangle}{d_i} & \text{if } b_i \leq \langle a_i, x \rangle \leq b_i + d_i \\ 0 & \text{otherwise} \end{cases} \quad (6.19)$$

for  $i = 1, \dots, m$ , and

$$\langle c, x \rangle \lesssim z = \begin{cases} 1 & \text{if } \langle c, x \rangle \leq z \\ 1 - \frac{z - \langle c, x \rangle}{d_0} & \text{if } z \leq \langle c, x \rangle \leq z + d_0 \\ 0 & \text{otherwise} \end{cases} \quad (6.20)$$

An optimal solution  $x^*$  to (6.18) is determined from the crisp LP

$$\begin{aligned} & \lambda \rightarrow \max \\ & 1 - \frac{z - \langle c, x \rangle}{d_0} \geq \lambda, \quad 1 - \frac{z - \langle a_i, x \rangle}{d_i} \geq \lambda, \quad i = 1, \dots, m, \\ & x \in \mathbb{R}^n, \quad 0 \leq \lambda \leq 1. \end{aligned} \quad (6.21)$$

The following theorem can be proved directly by using the definitions (6.4) and (6.11).

**Theorem 6.1.1.** *The FLP problem*

$$\begin{aligned} & \text{maximize} \quad 1_R(x) \\ & \text{subject to} \quad \bar{c}_1 x_1 + \dots + \bar{c}_n x_n \lesssim \bar{z} \\ & \quad \quad \quad \bar{a}_{i1} x_1 + \dots + \bar{a}_{in} x_n \lesssim \bar{b}_i, \quad i = 1, \dots, m. \end{aligned} \quad (6.22)$$

where  $1_R(u) = 1, \forall u \in \mathbb{R}$ , the objective function is to be maximized in relation (6.4) and  $\lesssim$  is defined by (6.19) and (6.20), i.e.

$$(\langle a_i, x \rangle \lesssim b_i)(u, v) = \begin{cases} 1 & \text{if } \langle a_i, x \rangle \leq b_i \\ 1 - \frac{b_i - \langle a_i, x \rangle}{d_i} & \text{if } b_i \leq \langle a_i, x \rangle \leq b_i + d_i \\ 0 & \text{otherwise} \end{cases}$$

for  $i = 1, \dots, m$ , and

$$(\langle c, x \rangle \lesssim z)(u, v) = \begin{cases} 1 & \text{if } \langle c, x \rangle \leq z \\ 1 - \frac{z - \langle c, x \rangle}{d_0} & \text{if } z \leq \langle c, x \rangle \leq z + d_0 \\ 0 & \text{otherwise} \end{cases}$$

has the same solution-set as problem (6.21).

### 6.1.6 Relation to Buckley's possibilistic LP

We show that when the inequality relations in an FLP problem are defined in a possibilistic sense then the optimal value of the objective function is equal to the possibility distribution of the objective function defined by Buckley [29]. Consider a possibilistic LP

$$\begin{aligned} & \text{maximize} && Z := \tilde{c}_1 x_1 + \cdots + \tilde{c}_n x_n \\ & \text{subject to} && \tilde{a}_{i1} x_1 + \cdots + \tilde{a}_{in} x_n \leq \tilde{b}_i, \quad i = 1, \dots, m. \end{aligned} \quad (6.23)$$

The possibility distribution of the objective function  $Z$ , denoted by  $\text{Pos}[Z = z]$ , is defined by [29]

$$\text{Pos}[Z = z] = \sup_x \min \left\{ \text{Pos}[Z = z \mid x], \text{Pos}[\langle \tilde{a}_1, x \rangle \leq \tilde{b}_1], \dots, \text{Pos}[\langle \tilde{a}_m, x \rangle \leq \tilde{b}_m] \right\},$$

where  $\text{Pos}[Z = z \mid x]$ , the conditional possibility that  $Z = z$  given  $x$ , is defined by

$$\text{Pos}[Z = z \mid x] = (\tilde{c}_1 x_1 + \cdots + \tilde{c}_n x_n)(z).$$

The following theorem can be proved directly by using the definitions of  $\text{Pos}[Z = z]$  and  $\mu_M(v)$ .

**Theorem 6.1.2.** *For the FLP problem*

$$\begin{aligned} & \text{maximize} && \tilde{c}_1 x_1 + \cdots + \tilde{c}_n x_n \\ & \text{subject to} && \tilde{a}_{i1} x_1 + \cdots + \tilde{a}_{in} x_n \lesssim \tilde{b}_i, \quad i = 1, \dots, m. \end{aligned} \quad (6.24)$$

where the inequality relation  $\lesssim$  is defined by (6.8) and the objective function is to be maximized in relation (6.4), i.e.

$$\text{MAX}(x') \leq \text{MAX}(x'') \iff \text{MAX}(x') \subseteq \text{MAX}(x''),$$

the following equality holds

$$\mu_M(v) = \text{Pos}[Z = v],$$

for all  $v \in \mathbb{R}$ , where  $M$  is defined by (6.11).

So, if the inequality relations for constraints are defined in a possibilistic sense and the objective function is to be maximized in relation (6.4) then the optimal value of the objective function of FLP problem (6.24) is equal to the possibility distribution of the objective function of possibilistic LP (6.23).

### 6.1.7 Examples

We illustrate our approach by two simple FMP problems. Consider first the FLP problem

$$\begin{aligned} & \text{maximize} && \tilde{c}x \\ & \text{subject to} && \tilde{a} \lesssim \tilde{a}, \quad 0 \leq x \leq 4, \end{aligned} \quad (6.25)$$

where  $\tilde{c} = (1, 1)$  is a fuzzy number of symmetric triangular form,  $\tilde{a}$  is a fuzzy number with membership function

$$\tilde{a}(u) = \begin{cases} 1 - u/4 & \text{if } 0 \leq u \leq 4 \\ 0 & \text{otherwise,} \end{cases}$$

the inequality relation for the constraint is defined by

$$(\tilde{a} \lesssim \tilde{a})(u, v) = \begin{cases} 1 & \text{if } \tilde{a}(u) \leq \tilde{a}(v) \\ \tilde{a}(v) & \text{otherwise} \end{cases}$$

and the inequality relation for the objective function is given by (6.4). Then the corresponding fuzzy reasoning scheme is

Antecedent	$\tilde{a} \lesssim \tilde{a}$
Fact	$\tilde{c}x$
<hr/>	
Consequence	$\text{MAX}(x)$

It is easy to compute that for  $0 \leq x \leq 2$

$$\mu_{\text{MAX}(x)}(v) = \begin{cases} 1 & \text{if } 0 \leq v \leq x \\ \frac{4-v}{4-x} & \text{if } x < v \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

and for  $2 \leq x \leq 4$

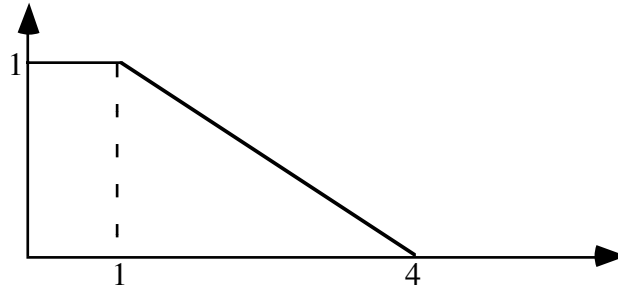


Figure 6.1: The membership function of  $\text{MAX}(1)$ .

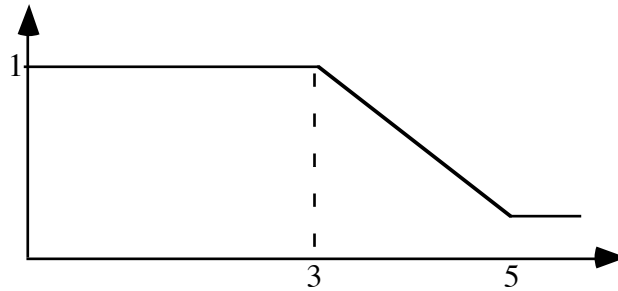


Figure 6.2: The membership function of  $\text{MAX}(2.5)$ .

$$\mu_{\text{MAX}(x)}(v) = \begin{cases} 1 & \text{if } 0 \leq v \leq x \\ (1/x - 4/x^2)v + 4/x & \text{if } x < v \leq 2x \\ 2 - 4/x & \text{otherwise} \end{cases}$$

So, if  $0 \leq x' \leq x'' \leq 4$  then from (6.4) we get

$$\text{MAX}(x') \leq \text{MAX}(x'') \leq \text{MAX}(4) = 1_R.$$

This means that  $x^* = 4$  is the unique solution and  $1_R$  is the optimal value of (6.25).

It differs from the *defuzzified* case

$$\begin{array}{ll} \text{maximize} & x \\ \text{subject to} & 0 \leq 0, 0 \leq x \leq 4 \end{array}$$

where the coefficients are the peaks of the fuzzy coefficients of FLP problem (6.25), because the solution  $x^* = 4$  of the crisp problem is equal to the solution of (6.25), but the optimal value of the FLP problem is too large  $\mu_M(v) = 1$  for all  $v \in \mathbb{R}$  (because the Gödel implication enlarges  $\text{MAX}(x)$  for all  $x \in \mathbb{R}$  by taking into account all membership values  $\tilde{a}(u)$  and  $\tilde{a}(v)$  separately).

Consider next the FMP problem

$$\begin{array}{ll} \text{maximize} & \tilde{c}x \\ \text{subject to} & (\tilde{a}x)^2 \lesssim \tilde{b}, x \geq 0, \end{array} \quad (6.26)$$

where  $\tilde{a} = (2, 1)$ ,  $\tilde{b} = (1, 1)$  and  $\tilde{c} = (3, 1)$  are fuzzy numbers of symmetric triangular form, the inequality relation  $\lesssim$  is defined in a possibilistic sense, i.e.

$$(\tilde{a}x \lesssim \tilde{b})(u, v) = \begin{cases} \text{Pos}[\tilde{a}x \leq \tilde{b}] & \text{if } u = v, \\ 0 & \text{otherwise} \end{cases}$$

and the inequality relation for the values of the objective function is defined by (6.5) (with the difference that subnormal values of the objective function are considered smaller than normal ones), i.e.

$$\text{MAX}(x') \leq \text{MAX}(x'') \iff \text{peak}(\text{MAX}(x')) \leq \text{peak}(\text{MAX}(x'')),$$

where  $\text{MAX}(x')$  and  $\text{MAX}(x'')$  are fuzzy numbers, and  $\text{MAX}(x') \leq \text{MAX}(x'')$  if  $\text{MAX}(x')$  is subnormal fuzzy quantity and  $\text{MAX}(x'')$  is a fuzzy number. It is easy to compute that

$$\text{Pos}[(\tilde{a}x)^2 \lesssim \tilde{b}] = \begin{cases} 1 & \text{if } x \leq 1/2, \\ \frac{-1 - 2x^2 + \sqrt{1 + 12x^2}}{2x^2} & \text{if } 1/2 \leq x \leq \sqrt{2} \\ 0 & \text{if } x \geq \sqrt{2} \end{cases}$$

and  $\text{MAX}(x)$  is a fuzzy number if  $0 \leq x \leq 1/2$ . Therefore, the unique solution to FMP problem (6.26) is  $x^* = 1/2$  and the optimal value of the objective function is

$$\mu_{\text{MAX}(x)}(v) = \mu_{\text{MAX}(1/2)}(v) = \begin{cases} 4 - 2v & \text{if } 3/2 \leq v \leq 2, \\ 2v - 2 & \text{if } 1 \leq v \leq 3/2 \\ 0 & \text{otherwise} \end{cases}$$

So, the optimal solution to FMP problem (6.26) is equal to the optimal solution of crisp problem

$$\begin{array}{ll} \text{maximize} & 3x \\ \text{subject to} & (2x)^2 \leq 1, x \geq 0, \end{array}$$



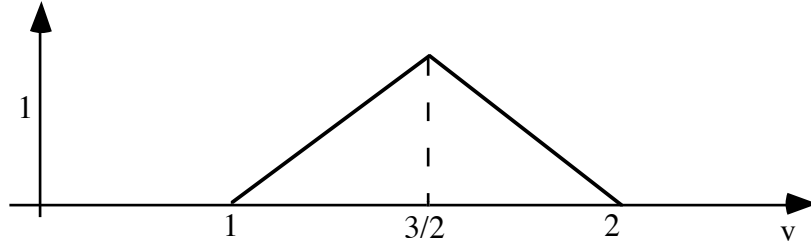


Figure 6.3: "v is approximately equal to 3/2".

where the coefficients are the peaks of the fuzzy coefficients of problem (6.26), and the optimal value of problem (6.26), which can be called "v is approximately equal to 3/2", can be considered as an approximation of the optimal value of the crisp problem  $v^* = 3/2$ .

We have interpreted FLP problems with fuzzy coefficients and fuzzy inequality relations as MFR schemes and shown a method for finding an optimal value of the objective function and an optimal solution. In the general case the computerized implementation of the proposed solution principle is not easy. To compute  $\text{MAX}(x)$  we have to solve a generally non-convex and non-differentiable mathematical programming problem. However, the stability property of the consequence in MFR schemes under small changes of the membership function of the antecedents [171] guarantees that small rounding errors of digital computation and small errors of measurement in membership functions of the coefficients of the FLP problem can cause only a small deviation in the membership function of the consequence,  $\text{MAX}(x)$ , i.e. every successive approximation method can be applied to the computation of the linguistic approximation of the exact  $\text{MAX}(x)$ .

However, to find an optimal value of the objective function,  $M$ , from the equation  $\text{MAX}(x) = M$  can be a very complicated process (for related works see [42, 47, 337, 345, 401, 402, 453, 455]) and very often we have to put up with a compromise solution [360]. An efficient fuzzy-reasoning-based method is needed for the exact computation of  $M$ . The solution principle described above can be applied to multiple criteria mathematical programming problems with fuzzy coefficients [56].

## 6.2 Optimization with linguistic variables

Following Carlsson and Fullér [77, 79] we introduce a novel statement of fuzzy mathematical programming problems and to provide a method for finding a fair solution to these problems. Suppose we are given a mathematical programming problem in which the functional relationship between the decision variables and the objective function is not completely known. Our knowledge-base consists of a block of fuzzy if-then rules, where the antecedent part of the rules contains some linguistic values of the decision variables, and the consequence part consists of a linguistic value of the objective function. We suggest the use of Tsukamoto's fuzzy reasoning method to determine the crisp functional relationship between the objective function and the decision variables, and solve the resulting (usually nonlinear) programming problem to find a fair optimal solution to the original fuzzy problem.

When Bellman and Zadeh [10], and a few years later Zimmermann [449], introduced fuzzy

sets into optimization problems, they cleared the way for a new family of methods to deal with problems which had been inaccessible to and unsolvable with standard mathematical programming techniques.

Fuzzy optimization problems can be stated and solved in many different ways (for good surveys see [256, 460]). Usually the authors consider optimization problems of the form

$$\begin{aligned} & \max/\min f(x), \\ & \text{subject to } x \in X, \end{aligned}$$

where  $f$  or/and  $X$  are defined by fuzzy terms. Then they are searching for a crisp  $x^*$  which (in certain) sense maximizes  $f$  under the (fuzzy) constraints  $X$ . For example, fuzzy linear programming (FLP) problems are stated as [365]

$$\begin{aligned} & \max/\min \quad f(x) := \tilde{c}_1 x_1 + \cdots + \tilde{c}_n x_n \\ & \text{subject to} \quad \tilde{a}_{i1} x_1 + \cdots + \tilde{a}_{in} x_n \lesssim \tilde{b}_i, \quad i = 1, \dots, m, \end{aligned} \quad (6.27)$$

where  $x \in \mathbb{R}^n$  is the vector of crisp decision variables,  $\tilde{a}_{ij}$ ,  $\tilde{b}_i$  and  $\tilde{c}_j$  are fuzzy quantities, the operations addition and multiplication by a real number of fuzzy quantities are defined by Zadeh's extension principle [440], the inequality relation,  $\lesssim$ , is given by a certain fuzzy relation,  $f$  is to be maximized in the sense of a given crisp inequality relation between fuzzy quantities, and the (implicite)  $X$  is a fuzzy set describing the concept "x satisfies all the constraints".

Unlike in (6.27) the fuzzy value of the objective function  $f(x)$  may not be known for any  $x \in \mathbb{R}^n$ . In many cases we are able to describe the causal link between  $x$  and  $f(x)$  linguistically using fuzzy if-then rules. Following Carlsson and Fullér [79] we consider a new statement of constrained fuzzy optimization problems, namely

$$\begin{aligned} & \max/\min f(x) \\ & \text{subject to } \{\mathfrak{R}(x) \mid x \in X\} \end{aligned} \quad (6.28)$$

where  $x_1, \dots, x_n$  are linguistic variables,  $X \subset \mathbb{R}^n$  is a (crisp or fuzzy) set of constraints on the domains of  $x_1, \dots, x_n$ , and  $\mathfrak{R}(x) = \{\mathfrak{R}_1(x), \dots, \mathfrak{R}_m(x)\}$  is a fuzzy rule base, and

$$\mathfrak{R}_i(x) : \text{if } x_1 \text{ is } A_{i1} \text{ and } \dots \text{ and } x_n \text{ is } A_{in} \text{ then } f(x) \text{ is } C_i,$$

constitutes the only knowledge available about the (linguistic) values of  $f(x)$ , and  $A_{ij}$  and  $C_i$  are fuzzy numbers.

Generalizing the fuzzy reasoning approach introduced by Carlsson and Fullér [56] we determine the crisp value of  $f$  at  $y \in X$  by Tsukamoto's fuzzy reasoning method, and obtain an optimal solution to (6.28) by solving the resulting (usually nonlinear) optimization problem  $\max/\min f(y)$ , subject to  $y \in X$ .

The use of fuzzy sets provides a basis for a systematic way for the manipulation of vague and imprecise concepts. In particular, we can employ fuzzy sets to represent linguistic variables. A linguistic variable [440] can be regarded either as a variable whose value is a fuzzy number or as a variable whose values are defined in linguistic terms. Fuzzy points are used to represent crisp values of linguistic variables. If  $x$  is a linguistic variable in the universe of discourse  $X$  and  $y \in X$  then we simply write " $x = y$ " or " $x$  is  $\bar{y}$ " to indicate that  $y$  is a crisp value of the linguistic variable  $x$ .

Recall the three basic t-norms: (i) minimum:  $T(a, b) = \min\{a, b\}$ , (ii) Łukasiewicz:  $T(a, b) = \max\{a + b - 1, 0\}$ , and (iii) product (or probabilistic):  $T(a, b) = ab$ .

We briefly describe Tsukamoto's fuzzy reasoning method [400]. Consider the following fuzzy inference system,

$\mathfrak{R}_1$ :	if $x_1$ is $A_{11}$ and ... and $x_n$ is $A_{1n}$ then	$z$ is $C_1$
...		
$\mathfrak{R}_m$ :	if $x_1$ is $A_{m1}$ and ... and $x_n$ is $A_{mn}$ then	$z$ is $C_m$
Input:	$x_1$ is $\bar{y}_1$ and ... and $x_n$ is $\bar{y}_n$	
Output:		$z_0$

where  $A_{ij} \in \mathcal{F}(U_j)$  is a value of linguistic variable  $x_j$  defined in the universe of discourse  $U_j \subset \mathbb{R}$ , and  $C_i \in \mathcal{F}(W)$  is a value of linguistic variable  $z$  defined in the universe  $W \subset \mathbb{R}$  for  $i = 1, \dots, m$  and  $j = 1, \dots, n$ . We also suppose that  $W$  is bounded and each  $C_i$  has strictly monotone (increasing or decreasing) membership function on  $W$ . The procedure for obtaining the crisp output,  $z_0$ , from the crisp input vector  $y = \{y_1, \dots, y_n\}$  and fuzzy rule-base  $\mathfrak{R} = \{\mathfrak{R}_1, \dots, \mathfrak{R}_m\}$  consists of the following three steps:

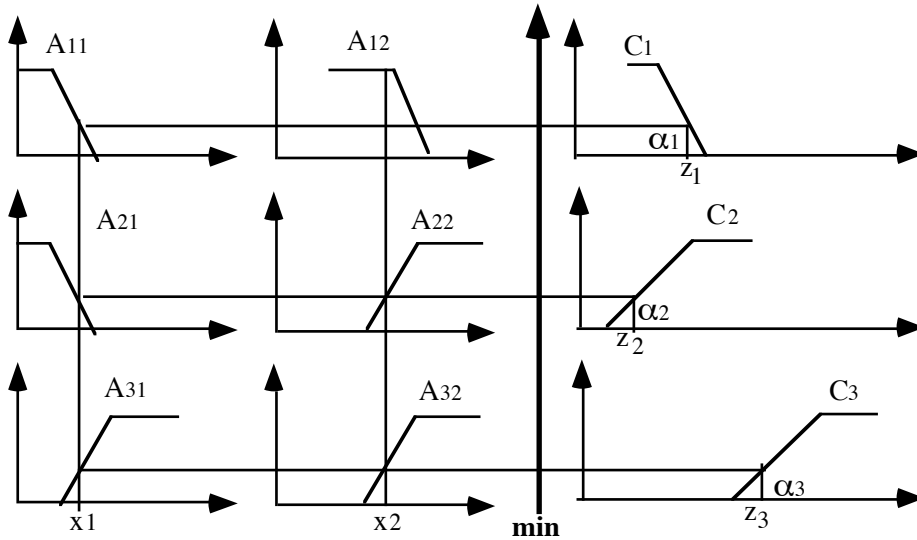


Figure 6.4: Illustration of Tsukamoto's inference mechanism.

- We find the firing level of the  $i$ -th rule as

$$\alpha_i = T(A_{i1}(y_1), \dots, A_{in}(y_n)), \quad i = 1, \dots, m, \quad (6.29)$$

where  $T$  usually is the minimum or the product t-norm.

- We determine the (crisp) output of the  $i$ -th rule, denoted by  $z_i$ , from the equation  $\alpha_i = C_i(z_i)$ , that is,

$$z_i = C_i^{-1}(\alpha_i), \quad i = 1, \dots, m,$$

where the inverse of  $C_i$  is well-defined because of its strict monotonicity.

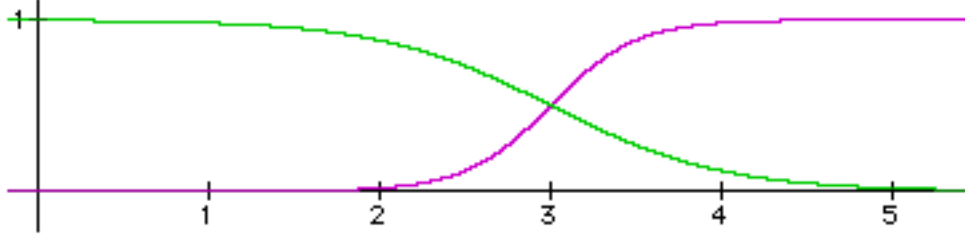


Figure 6.5: Sigmoid membership functions for "z is small" and "z is big".

- The overall system output is defined as the weighted average of the individual outputs, where associated weights are the firing levels. That is,

$$z_0 = \frac{\alpha_1 z_1 + \cdots + \alpha_m z_m}{\alpha_1 + \cdots + \alpha_m} = \frac{\alpha_1 C_1^{-1}(\alpha_1) + \cdots + \alpha_m C_m^{-1}(\alpha_m)}{\alpha_1 + \cdots + \alpha_m}$$

i.e.  $z_0$  is computed by the discrete Center-of-Gravity method.

If  $W = \mathbb{R}$  then all linguistic values of  $x_1, \dots, x_n$  also should have strictly monotone membership functions on  $\mathbb{R}$  (that is,  $0 < A_{ij}(x) < 1$  for all  $x \in \mathbb{R}$ ), because  $C_i^{-1}(1)$  and  $C_i^{-1}(0)$  do not exist. In this case  $A_{ij}$  and  $C_i$  usually have sigmoid membership functions of the form

$$\text{big}(t) = \frac{1}{1 + \exp(-b(t - c))}, \quad \text{small}(t) = \frac{1}{1 + \exp(b'(t - c'))}$$

where  $b, b' > 0$  and  $c, c' > 0$ .

Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be a function and let  $X \subset \mathbb{R}^n$ . A constrained optimization problem can be stated as

$$\min f(x); \text{ subject to } x \in X.$$

In many practical cases the function  $f$  is not known exactly. In this Section we consider the following fuzzy optimization problem

$$\min f(x); \text{ subject to } \{\mathfrak{R}_1(x), \dots, \mathfrak{R}_m(x) \mid x \in X\}, \quad (6.30)$$

where  $x_1, \dots, x_n$  are linguistic variables,  $X \subset \mathbb{R}^n$  is a (crisp or fuzzy) set of constraints on the domain of  $x_1, \dots, x_n$ , and the only available knowledge about the values of  $f$  is given as a block fuzzy if-then rules of the form

$$\mathfrak{R}_i(x) : \text{if } x_1 \text{ is } A_{i1} \text{ and } \dots \text{ and } x_n \text{ is } A_{in} \text{ then } f(x) \text{ is } C_i,$$

here  $A_{ij}$  are fuzzy numbers (with continuous membership function) representing the linguistic values of  $x_i$  defined in the universe of discourse  $U_j \subset \mathbb{R}$ ; and  $C_i, i = 1, \dots, m$ , are linguistic values (with strictly monotone and continuous membership functions) of the objective function  $f$  defined in the universe  $W \subset \mathbb{R}$ . To find a fair solution to the fuzzy optimization problem (6.30) we first determine the crisp value of the objective function  $f$  at  $y \in X$  from the fuzzy rule-base  $\mathfrak{R}$  using Tsukamoto's fuzzy reasoning method as

$$f(y) := \frac{\alpha_1 C_1^{-1}(\alpha_1) + \cdots + \alpha_m C_m^{-1}(\alpha_m)}{\alpha_1 + \cdots + \alpha_m}$$

where the firing levels,

$$\alpha_i = T(A_{i1}(y_1), \dots, A_{in}(y_n)),$$

for  $i = 1, \dots, m$ , are computed according to (6.29). To determine the firing level of the rules, we suggest the use of the product t-norm (to have a smooth output function).

In this manner our constrained optimization problem (6.30) turns into the following crisp (usually nonlinear) mathematical programming problem

$$\min f(y); \text{ subject to } y \in X.$$

The same principle is applied to constrained maximization problems

$$\max f(x); \text{ subject to } \{\mathfrak{R}_1(x), \dots, \mathfrak{R}_m(x) \mid x \in X\}. \quad (6.31)$$

**Remark.** If  $X$  is a fuzzy set in  $U_1 \times \dots \times U_n \subset \mathbb{R}^n$  with membership function  $\mu_X$  (e.g. given by soft constraints as in [449]) and  $W = [0, 1]$  then following Bellman and Zadeh [10] we define the fuzzy solution to problem (6.31) as

$$D(y) = \min\{\mu_X(y), f(y)\},$$

for  $y \in U_1 \times \dots \times U_n$ , and an optimal (or maximizing) solution,  $y^*$ , is determined from the relationship

$$D(y^*) = \sup_{y \in U_1 \times \dots \times U_n} D(y) \quad (6.32)$$

### 6.2.1 Examples

Consider the optimization problem

$$\min f(x); \{x_1 + x_2 = 1/2, 0 \leq x_1, x_2 \leq 1\}, \quad (6.33)$$

and  $f(x)$  is given linguistically as

$$\begin{aligned} \mathfrak{R}_1 : & \text{ if } x_1 \text{ is small and } x_2 \text{ is small then } f(x) \text{ is small,} \\ \mathfrak{R}_2 : & \text{ if } x_1 \text{ is small and } x_2 \text{ is big then } f(x) \text{ is big,} \end{aligned}$$

and the universe of discourse for the linguistic values of  $f$  is also the unit interval  $[0, 1]$ .

We will compute the firing levels of the rules by the product t-norm. Let the membership functions in the rule-base  $\mathfrak{R}$  be defined by (2.45) and let  $[y_1, y_2] \in [0, 1] \times [0, 1]$  be an input vector to the fuzzy system. Then the firing levels of the rules are

$$\begin{aligned} \alpha_1 &= (1 - y_1)(1 - y_2), \\ \alpha_2 &= (1 - y_1)y_2, \end{aligned}$$

It is clear that if  $y_1 = 1$  then no rule applies because  $\alpha_1 = \alpha_2 = 0$ . So we can exclude the value  $y_1 = 1$  from the set of feasible solutions. The individual rule outputs are

$$\begin{aligned} z_1 &= 1 - (1 - y_1)(1 - y_2), \\ z_2 &= (1 - y_1)y_2, \end{aligned}$$

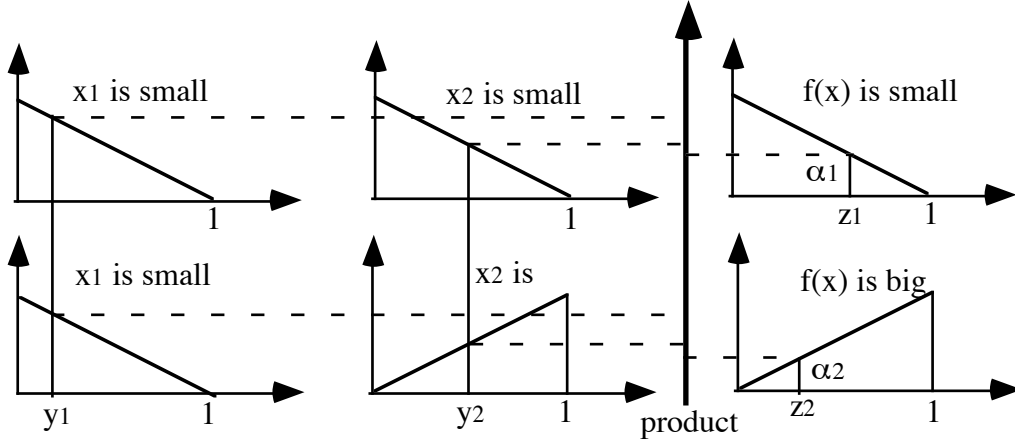


Figure 6.6: Illustration of Example (6.33).

and, therefore, the overall system output, interpreted as the crisp value of  $f$  at  $y$ , is

$$f(y) := \frac{(1 - y_1)(1 - y_2)(1 - (1 - y_1)(1 - y_2)) + (1 - y_1)y_2(1 - y_1)y_2}{(1 - y_1)(1 - y_2) + (1 - y_1)y_2} =$$

$$y_1 + y_2 - 2y_1y_2$$

Thus our original fuzzy problem

$$\min f(x); \text{ subject to } \{\mathfrak{R}_1(x), \mathfrak{R}_2(x) \mid x \in X\},$$

turns into the following crisp nonlinear mathematical programming problem

$$(y_1 + y_2 - 2y_1y_2) \rightarrow \min$$

$$y_1 + y_2 = 1/2,$$

$$0 \leq y_1 < 1, 0 \leq y_2 \leq 1.$$

which has the optimal solution

$$y_1^* = y_2^* = 1/4$$

and its optimal value is

$$f(y^*) = 3/8.$$

It is clear that if there were no other constraints on the crisp values of  $x_1$  and  $x_2$  then the optimal solution to (6.33) would be  $y_1^* = y_2^* = 0$  with  $f(y^*) = 0$ .

This example clearly shows that we can not just choose the rule with the smallest consequence part (the first first rule) and fire it with the maximal firing level ( $\alpha_1 = 1$ ) at  $y^* \in [0, 1]$ , and take  $y^* = (0, 0)$  as an optimal solution to (6.30).

The rules represent our knowledge-base for the fuzzy optimization problem. The fuzzy partitions for linguistic variables will not ususally satisfy  $\varepsilon$ -completeness, normality and convexity. In many cases we have only a few (and contradictory) rules. Therefore, we can not make any preselection procedure to remove the rules which *do not play any role* in the optimization problem. All rules should be considered when we derive the crisp values of the

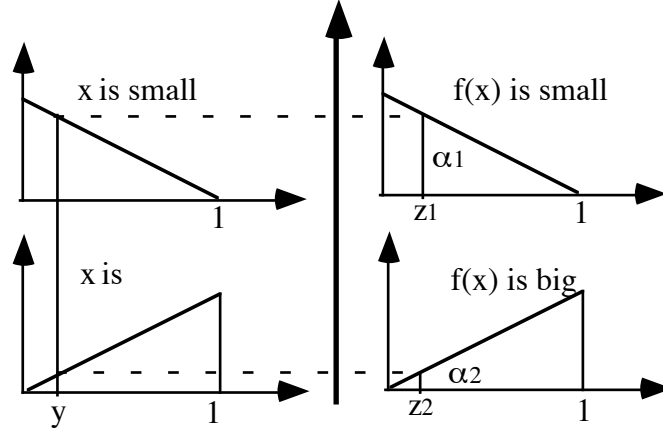


Figure 6.7: Illustration of Example (6.34).

objective function. We have chosen Tsukamoto's fuzzy reasoning scheme, because the individual rule outputs are crisp numbers, and therefore, the functional relationship between the input vector  $y$  and the system output  $f(y)$  can be relatively easily identified (the only thing we have to do is to perform inversion operations).

Consider the problem

$$\max_X f(x) \quad (6.34)$$

where  $X$  is a fuzzy subset of the unit interval with membership function

$$\mu_X(y) = \frac{1}{1+y}, \quad y \in [0, 1],$$

and the fuzzy rules are

$$\mathfrak{R}_1 : \text{ if } x \text{ is small then } f(x) \text{ is small,}$$

$$\mathfrak{R}_2 : \text{ if } x \text{ is big then } f(x) \text{ is big,}$$

Let  $y \in [0, 1]$  be an input to the fuzzy system  $\{\mathfrak{R}_1, \mathfrak{R}_2\}$ . Then the firing levels of the rules are

$$\alpha_1 = 1 - y,$$

$$\alpha_2 = y.$$

the individual rule outputs are computed by

$$z_1 = (1 - y)y,$$

$$z_2 = y^2,$$

and, therefore, the overall system output is

$$f(y) = (1 - y)y + y^2 = y.$$

Then according to (6.32) our original fuzzy problem (6.34) turns into the following crisp biobjective mathematical programming problem

$$\max \min \left\{ y, \frac{1}{1+y} \right\}; \text{ subject to } y \in [0, 1],$$

which has the optimal solution

$$y^* = \frac{\sqrt{5} - 1}{2}$$

and its optimal value is  $f(y^*) = y^*$ .

Consider the following one-dimensional problem

$$\max f(x); \text{ subject to } \{\mathfrak{R}_1(x), \dots, \mathfrak{R}_{K+1}(x) \mid x \in X\}, \quad (6.35)$$

where  $U = W = [0, 1]$ ,

$$\mathfrak{R}_i(x) : \text{ if } x \text{ is } A_i \text{ then } f(x) \text{ is } C_i.$$

and  $A_i$  is defined by equations (2.42, 2.43, 2.44), the linguistic values of  $f$  are selected from (2.46, 2.47),  $i = 1, \dots, K + 1$ . It is clear that exactly two rules fire with nonzero degree for any input  $y \in [0, 1]$ . Namely, if

$$y \in I_k := \left[ \frac{k-1}{K}, \frac{k}{K} \right],$$

then  $\mathfrak{R}_k$  and  $\mathfrak{R}_{k+1}$  are applicable, and therefore we get

$$f(y) = (k - Ky)C_k^{-1}(k - Ky) + (Ky - k + 1)C_{k+1}^{-1}(Ky - k + 1)$$

for any  $k \in \{1, \dots, K\}$ . In this way the fuzzy maximization problem (6.35) turns into  $K$  independent maximization problem

$$\max_{k=1, \dots, K} \left\{ \max_{X \cap I_k} (k - Ky)C_k^{-1}(k - Ky) + (Ky - k + 1)C_{k+1}^{-1}(Ky - k + 1) \right\}$$

If  $x \in \mathbb{R}^n$ , with  $n \geq 2$  then a similar reasoning holds, with the difference that we use the same fuzzy partition for all the linguistic variables,  $x_1, \dots, x_n$ , and the number of applicable rules grows to  $2^n$ .

It should be noted that we can refine the fuzzy rule-base by introducing new linguistic variables modeling the linguistic dependencies between the variables and the objectives [53, 57, 155].

## 6.3 Multiobjective optimization with linguistic variables

The principles presented above can be extended to multiple objective optimization problems under fuzzy if-then rules [78]. Namely, consider the following statement of multiple objective optimization problem

$$\max/\min \{f_1(x), \dots, f_K(x)\}; \text{ subject to } \{\mathfrak{R}_1(x), \dots, \mathfrak{R}_m(x) \mid x \in X\}, \quad (6.36)$$

where  $x_1, \dots, x_n$  are linguistic variables, and

$$\mathfrak{R}_i(x) : \text{ if } x_1 \text{ is } A_{i1} \text{ and } \dots \text{ and } x_n \text{ is } A_{in} \text{ then } f_1(x) \text{ is } C_{i1} \text{ and } \dots \text{ and } f_K(x) \text{ is } C_{iK},$$

constitutes the only knowledge available about the values of  $f_1, \dots, f_K$ , and  $A_{ij}$  and  $C_{ik}$  are fuzzy numbers.



To find a fair solution to the fuzzy optimization problem (6.36) with continuous  $A_{ij}$  and with strictly monotone and continuous  $C_{ik}$ , representing the linguistic values of  $f_k$ , we first determine the crisp value of the  $k$ -th objective function  $f_k$  at  $y \in \mathbb{R}^n$  from the fuzzy rule-base  $\mathfrak{R}$  using Tsukamoto's fuzzy reasoning method as

$$f_k(y) := \frac{\alpha_1 C_{1k}^{-1}(\alpha_1) + \cdots + \alpha_m C_{mk}^{-1}(\alpha_m)}{\alpha_1 + \cdots + \alpha_m}$$

where

$$\alpha_i = T(A_{i1}(y_1), \dots, A_{in}(y_n))$$

denotes the firing level of the  $i$ -th rule,  $\mathfrak{R}_i$  and  $T$  is a t-norm. To determine the firing level of the rules, we suggest the use of the product t-norm (to have a smooth output function). In this manner the constrained optimization problem (6.36) turns into the crisp (usually nonlinear) multiobjective mathematical programming problem

$$\max/\min \{f_1(y), \dots, f_K(y)\}; \text{ subject to } y \in X. \quad (6.37)$$

### 6.3.1 Example

Consider the optimization problem

$$\max \{f_1(x), f_2(x)\}; \{x_1 + x_2 = 3/4, 0 \leq x_1, x_2 \leq 1\}, \quad (6.38)$$

where  $f_1(x)$  and  $f_2(x)$  are given linguistically by

$\mathfrak{R}_1(x)$  : if  $x_1$  is *small* and  $x_2$  is *small* then  $f_1(x)$  is *small* and  $f_2(x)$  is *big*,

$\mathfrak{R}_2(x)$  : if  $x_1$  is *small* and  $x_2$  is *big* then  $f_1(x)$  is *big* and  $f_2(x)$  is *small*,

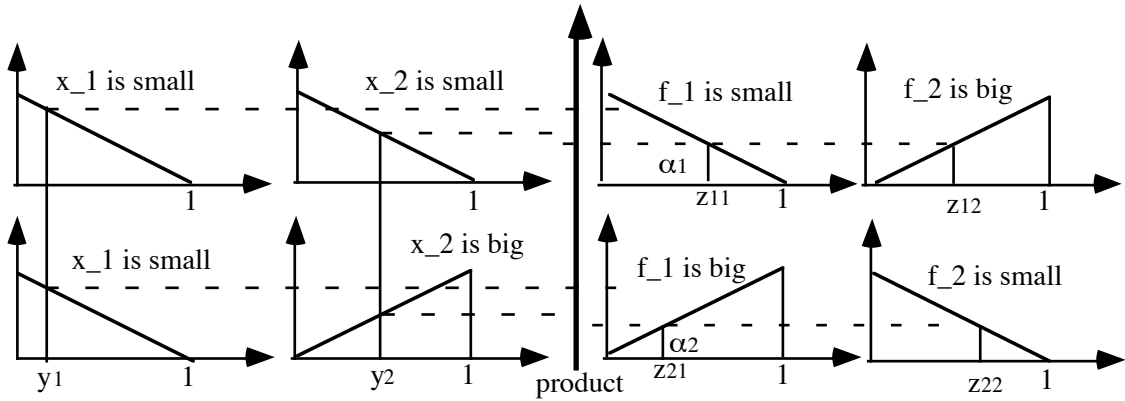


Figure 6.8: The fuzzy rule-base for (6.38).

and the universe of discourse for the linguistic values of  $f_1$  and  $f_2$  is also the unit interval  $[0, 1]$ . We will compute the firing levels of the rules by the product t-norm. Let the membership functions in the rule-base  $\mathfrak{R} = \{\mathfrak{R}_1, \mathfrak{R}_2\}$  be defined by

$$\text{small}(t) = 1 - t, \quad \text{big}(t) = t.$$

Let  $0 \leq y_1, y_2 \leq 1$  be an input to the fuzzy system. Then the firing leveles of the rules are

$$\begin{aligned}\alpha_1 &= (1 - y_1)(1 - y_2), \\ \alpha_2 &= (1 - y_1)y_2.\end{aligned}$$

It is clear that if  $y_1 = 1$  then no rule applies because  $\alpha_1 = \alpha_2 = 0$ . So we can exclude the value  $y_1 = 1$  from the set of feasible solutions. The individual rule outputs are

$$\begin{aligned}z_{11} &= 1 - (1 - y_1)(1 - y_2), \\ z_{21} &= (1 - y_1)y_2, \\ z_{12} &= (1 - y_1)(1 - y_2), \\ z_{22} &= 1 - (1 - y_1)y_2,\end{aligned}$$

and, therefore, the overall system outputs are

$$f_1(y) = \frac{(1 - y_1)(1 - y_2)(1 - (1 - y_1)(1 - y_2)) + (1 - y_1)y_2(1 - y_1)y_2}{(1 - y_1)(1 - y_2) + (1 - y_1)y_2} = y_1 + y_2 - 2y_1y_2,$$

and

$$f_2(y) = \frac{(1 - y_1)(1 - y_2)(1 - y_1)(1 - y_2) + (1 - y_1)y_2(1 - (1 - y_1)y_2)}{(1 - y_1)(1 - y_2) + (1 - y_1)y_2} = 1 - (y_1 + y_2 - 2y_1y_2).$$

Modeling the anding of the objective functions by the minimum t-norm our original fuzzy problem (6.38) turns into the following crisp nonlinear mathematical programming problem

$$\begin{aligned}\max \min \{ &y_1 + y_2 - 2y_1y_2, 1 - (y_1 + y_2 - 2y_1y_2) \} \\ \text{subject to } &\{y_1 + y_2 = 3/4, 0 \leq y_1 < 1, 0 \leq y_2 \leq 1\}.\end{aligned}$$

which has the following optimal solutions

$$y^* = \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/4 \end{pmatrix},$$

and

$$\begin{pmatrix} 1/4 \\ 1/2 \end{pmatrix},$$

from symmetricity, and its optimal value is

$$(f_1(y^*), f_2(y^*)) = (1/2, 1/2).$$

We can introduce trade-offs among the objectives function by using an OWA-operator in (6.37). However, as Yager has pointed out in [427], constrained OWA-aggregations are not easy to solve, because the usually lead to a mixed integer mathematical programming problem of very big dimension.

## 6.4 Interdependent multiple criteria decision making

Decision making with interdependent multiple criteria is normal in standard business decision making; in mcdm theory the standard assumption is to assume that the criteria are independent, which makes optimal mcdm solutions less useful than they could be. Following Carlsson and Fullér [57] in this Section we describe a method for both dealing with and making use of the interdependence of multiple criteria.

Interdependence is a fairly obvious concept: consider a decision problem, in which we have to find a  $x^* \in X$  such that three different criteria  $c_1$ ,  $c_2$  and  $c_3$  all are satisfied, when  $c_1$  and  $c_2$  are *supportive* of each others,  $c_2$  and  $c_3$  are *conflicting*, and  $c_1$  and  $c_3$  again are *supportive* of each others (with respect to some directions). Unless it is obvious, the choice of an optimal decision alternative will become a very complex process with an increasing number of criteria.

There has been a growing interest and activity in the area of multiple criteria decision making (MCDM), especially in the last 20 years. Modeling and optimization methods have been developed in both crisp and fuzzy environments. The overwhelming majority of approaches for finding best compromise solutions to MCDM problems do not make use of the interdependences among the objectives. However, as has been pointed out by Carlsson [51], in modeling real world problems (especially in management sciences) we often encounter MCDM problems with interdependent objectives.

In this Section we introduce measures of interdependences between the objectives, in order to provide for a better understanding of the decision problem, and to find effective and more correct solutions to MCDM problems.

P.L. Yu explains that we have habitual ways of thinking, acting, judging and responding, which when taken together form our habitual domain (HD) [436]. This domain is very nicely illustrated with the following example ([436] p 560):

A retiring chairman wanted to select a successor from two finalists ( $A$  and  $B$ ). The chairman invited  $A$  and  $B$  to his farm, and gave each finalist an equally good horse. He pointed out the course of the race and the rules saying, "From this point whoever's horse is slower reaching the final point will be the new chairman". This rule of horse racing was outside the habitual ways of thinking of  $A$  and  $B$ . Both of them were puzzled and did not know what to do. After a few minutes,  $A$  all of a sudden got a great idea. he jumped out of the constraint of his HD. He quickly mounted  $B$ 's horse and rode as fast as possible, leaving his own horse behind. When  $B$  realized what was going on, it was too late.  $A$  became the new chairman.

Part of the HD of multiple criteria decision making is the intuitive assumption that all criteria are independent; this was initially introduced as a safeguard to get a feasible solution to a multiple criteria problem, as there were no means available to deal with interdependence. Then, gradually, conflicts were introduced as we came to realize that multiple goals or objectives almost by necessity represent conflicting interests [448, 418]. Here we will "jump out of the constraints" of the HD of MCDM and leave out the assumption of independent criteria.

The existence of the HD is a partial explanation of why MCDM is not an explicit part of managerial decision making, although it is always claimed that it is implicitly pursued

by all economic agents under most circumstances [448]. By not allowing interdependence multiple criteria problems are simplified beyond recognition and the solutions reached by the traditional algorithms have only marginal interest. Zeleny also points to other circumstances [448] which have reduced the visibility and usefulness of MCDM: (i) time pressure reduces the number of criteria to be considered; (ii) the more complete and precise the problem definition, the less criteria are needed; (iii) autonomous decision makers are bound to use more criteria than those being controlled by a strict hierarchical decision system; (iv) isolation from the perturbations of changing environment reduces the need for multiple criteria; (v) the more complete, comprehensive and integrated knowledge of the problem the more criteria will be used - but partial, limited and non-integrated knowledge will significantly reduce the number of criteria; and (vi) cultures and organisations focused on central planning and collective decision making rely on aggregation and the reduction of criteria in order to reach consensus. When we combine these circumstances with the HD we get a most challenging field of research: to make MCDM both more realistic and more relevant to the decision makers of the business world. We believe that this can be done both by introducing interdependence and by developing MCDM-based decision support systems; here we will pursue the first task.

A typical approach to solving multiple criteria decision problems is the SIMOLP procedure introduced by Reeves and Franz [354]; we have the following multiobjective linear programming formulation,

$$\begin{aligned} & \max \{ \langle c^1, x \rangle = z_1 \} \\ & \max \{ \langle c^2, x \rangle = z_2 \} \\ & \quad \dots \\ & \max \{ \langle c^k, x \rangle = z_k \} \\ & \text{subject to } x \in X = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0, b \in \mathbb{R}^m\} \end{aligned}$$

for which the optimal solution is found in the following sequence [354]:

- (i) optimize each objective function individually over the feasible region; solve the  $k$  single objective linear programming problems and obtain  $k$  efficient points  $x^i$ ,  $i = 1, \dots, k$  and the  $k$  nondominated criterion vectors  $z^i$ ; define these vectors in both the decision and the criteria space as  $E^* = \{x^i, i = 1, \dots, k\}$ ;  $N^* = \{z^i, i = 1, \dots, k\}$ ;
- (ii) have the decision maker (DM) review the  $k$  elements of  $N^*$ ; if the DM finds a preferred element, the procedure terminates; if none of the elements is preferred, set  $i = k$  and continue with the next step;
- (iii) set  $i = i + 1$ ; form the hyperplane  $z_i$  which passes through the  $k$  elements of  $N^*$ ; solve the following LP problem,

$$\max_{x \in X} z_i$$

to obtain the efficient point  $x^i$  and the nondominated vector  $z^i$ .

- (iv) if  $z^i$  is not a member of  $N^*$  and if  $z^i$  is preferred to at least one element in  $N^*$ , then replace this element with  $z^i$  and return to step iii.
- (v) if  $z^i$  is a member of  $N^*$  or if the DM does not prefer  $z^i$  to any element of  $N^*$ , have the DM select the most preferred element of  $N^*$  and stop;

As can be seen from this procedure the DM should have a *dilemma*: he is expected to be able to formulate his preferences regarding the objectives, but he cannot have much more than an intuitive grasp of the trade-offs he is probably doing among the objectives. This is taken care of with a convenient assumption: the DM is taken to be a fully informed, rational decision maker who relies on some underlying utility function as a basis for his preferences.

It is well known that there does not exist any concept of *optimal solution* universally accepted and valid for any multiobjective problem [446]. Delgado et al [108] provided a unified framework to use fuzzy sets and possibility theory in multicriteria decision and multiobjective programming. Felix [154] presented a novel theory for multiple attribute decision making based on fuzzy relations between objectives, in which the interactive structure of objectives is inferred and represented explicitly. Carlsson [51] used the fuzzy Pareto optimal set of nondominated alternatives as a basis for an OWA-type operator [418] for finding a best compromise solution to MCDM problems with interdependent criteria.

We provide a new method for finding a compromise solution to MCDM problems by using explicitly the interdependences among the objectives and combining the results of [49, 51, 108, 154, 451]. First we define interdependences between the objectives of a decision problem defined in terms of multiple objectives. Consider the following problem

$$\max_{x \in X} \{f_1(x), \dots, f_k(x)\} \quad (6.39)$$

where  $f_i: \mathbb{R}^n \rightarrow \mathbb{R}$  are objective functions,  $x \in \mathbb{R}^n$  is the decision variable, and  $X$  is a subset of  $\mathbb{R}^n$  without any additional conditions for the moment.

**Definition 6.4.1.** An  $x^* \in X$  is said to be *efficient* for (6.39) iff there exists no  $x^{**} \in X$  such that  $f_i(x^{**}) \geq f_i(x^*)$  for all  $i$  with strict inequality for at least one  $i$ .

**Definition 6.4.2.** Let  $f_i$  and  $f_j$  be two objective functions of (6.39). We say that

- (i)  $f_i$  supports  $f_j$  on  $X$  (denoted by  $f_i \uparrow f_j$ ) if  $f_i(x') \geq f_i(x)$  entails  $f_j(x') \geq f_j(x)$ , for all  $x', x \in X$ ;
- (ii)  $f_i$  is in conflict with  $f_j$  on  $X$  (denoted by  $f_i \downarrow f_j$ ) if  $f_i(x') \geq f_i(x)$  entails  $f_j(x') \leq f_j(x)$ , for all  $x', x \in X$ ;
- (iii)  $f_i$  and  $f_j$  are independent on  $X$ , otherwise.

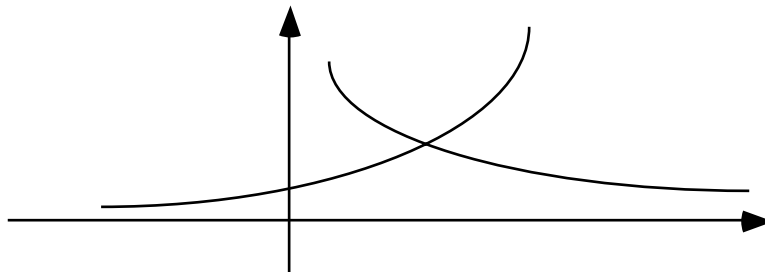


Figure 6.9: A typical example of conflict on  $\mathbb{R}$ .

If  $X = \mathbb{R}^n$  then we say that  $f_i$  supports (or is in conflict with)  $f_j$  *globally*.

If the objective functions are differentiable on  $X$  then we have

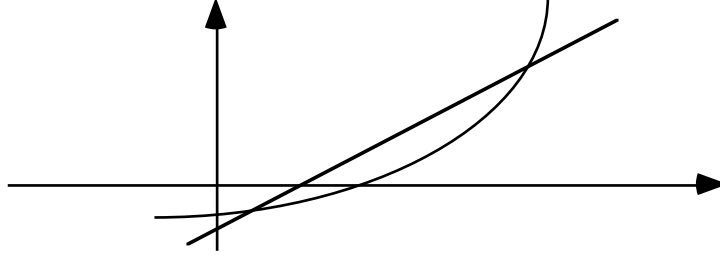


Figure 6.10: Supportive functions on  $\mathbb{R}$ .

$$f_i \uparrow f_j \text{ on } X \iff \partial_e f_i(x) \partial_e f_j(x) \geq 0 \text{ for all } e \in \mathbb{R}^n \text{ and } x \in X,$$

$$f_i \downarrow f_j \text{ on } X \iff \partial_e f_i(x) \partial_e f_j(x) \leq 0 \text{ for all } e \in \mathbb{R}^n \text{ and } x \in X,$$

where  $\partial_e f_i(x)$  denotes the derivative of  $f_i$  with respect to the direction  $e \in \mathbb{R}^n$  at  $x \in \mathbb{R}^n$ . If for a given direction  $e \in \mathbb{R}^n$ ,

$$\partial_e f_i(x) \partial_e f_j(x) \geq 0 \quad [\partial_e f_i(x) \partial_e f_j(x) \leq 0]$$

holds for all  $x \in X$  then we say that  $f_i$  supports  $f_j$  [ $f_i$  is in conflict with  $f_j$ ] with respect to the direction  $e$  on  $X$ . Let  $f_i$  be an objective function of (6.39). Then we define the grade of interdependency, denoted by  $\Delta(f_i)$ , of  $f_i$  as

$$\Delta(f_i) = \sum_{f_i \uparrow f_j, i \neq j} 1 - \sum_{f_i \downarrow f_j} 1, \quad i = 1, \dots, k. \quad (6.40)$$

If  $\Delta(f_i)$  is positive and large then  $f_i$  supports a majority of the objectives, if  $\Delta(f_i)$  is negative and large then  $f_i$  is in conflict with a majority of the objectives, if  $\Delta(f_i)$  is positive and small then  $f_i$  supports more objectives than it hinders, and if  $\Delta(f_i)$  is negative and small then  $f_i$  hinders more objectives than it supports. Finally, if  $\Delta(f_i) = 0$  then  $f_i$  is independent from the others or supports the same number of objectives as it hinders.

### 6.4.1 The linear case

If the objective functions are linear then their derivatives are constant. So if two objectives are parallel and growing in the same direction then they support each others, otherwise we can globally measure only the conflict between them. Consider the following problem with multiple objectives

$$\max_{x \in X} \{f_1(x), \dots, f_k(x)\} \quad (6.41)$$

where  $f_i(x) = \langle c_i, x \rangle = c_{i1}x_1 + \dots + c_{in}x_n$  and  $\|c_i\| = 1$ ,  $i = 1, \dots, k$ .

**Definition 6.4.3.** Let  $f_i(x) = \langle c_i, x \rangle$  and  $f_j(x) = \langle c_j, x \rangle$  be two objective functions of (6.41). Then the measure of conflict between  $f_i$  and  $f_j$ , denoted by  $\kappa(f_i, f_j)$ , is defined by

$$\kappa(f_i, f_j) = \frac{1 - \langle c_i, c_j \rangle}{2}.$$

We illustrate the meaning of the measure of conflict by a biobjective two-dimensional decision problem

$$\max_{x \in X} \{\alpha(x), \beta(x)\},$$

where  $\alpha(x) = \langle \mathbf{n}, x \rangle$  and  $\beta(x) = \langle \mathbf{m}, x \rangle$ .

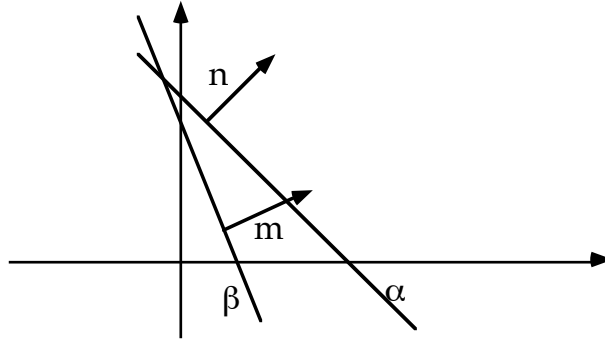


Figure 6.11: The measure of conflict between  $\alpha$  and  $\beta$  is  $|\mathbf{n}||\mathbf{m}| \cos(\mathbf{n}, \mathbf{m})$ .

The bigger the angle between the lines  $\alpha$  and  $\beta$  the bigger the degree of conflict between them.

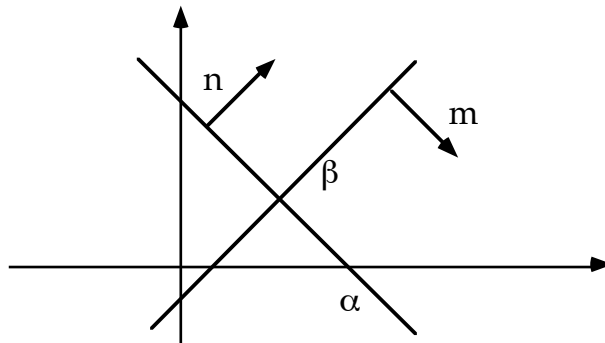


Figure 6.12:  $\kappa(\alpha, \beta) = 1/2$  - the case of perpendicular objectives.

If  $\kappa(\alpha, \beta) = 1/2$  and the set of feasible solutions is a convex polyhedron in  $\mathbb{R}^n$  then  $\alpha$  and  $\beta$  attain their independent maximum at neighbour vertexes of  $X$ .

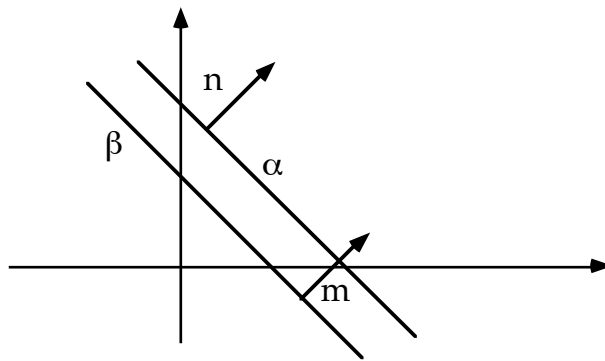


Figure 6.13:  $\kappa(\alpha, \beta) = 0$  - the case of parallel objectives.

If  $\kappa(\alpha, \beta) = 0$  and the set of feasible solutions is a convex polyhedron subset of  $\mathbb{R}^n$  then  $\alpha$  and  $\beta$  attain their independent maximum at the same vertex of  $X$ .

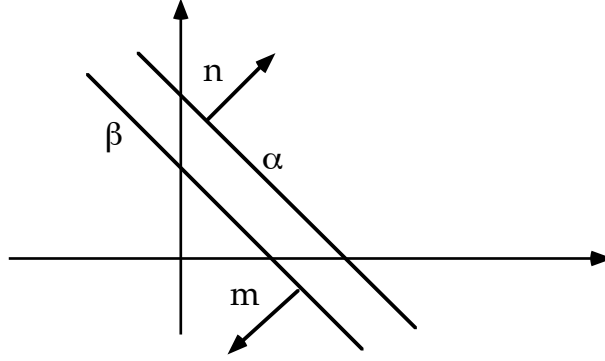


Figure 6.14:  $\kappa(\alpha, \beta) = 1$  - the case of opposite objectives.

**Definition 6.4.4.** The complexity of the problem (6.41) is defined as

$$\Omega = \frac{\sum_{i,j}^k \kappa(f_i, f_j)}{2}.$$

It is clear that  $\Omega = 0$  iff all the objectives are parallel, i.e. we have a single objective problem. Let  $f_i(x) = \langle c_i, x \rangle$  and  $f_j(x) = \langle c_j, x \rangle$  with  $c_i \neq c_j$ . If

$$\text{sign } c_{ir} = \text{sign } \partial_r f_i(x) = \text{sign } \partial_r f_j(x) = \text{sign } c_{jr},$$

for some basic direction  $r$ , then  $f_i \uparrow f_j$  with respect to direction  $r$ . This information can be useful in the construction of a scalarizing function, when we search for a nondominated solution being closest to an ideal point in a given metric.

## 6.4.2 Application functions

Following [451, 108] we introduce an application function

$$h_i: \mathbb{R} \rightarrow [0, 1]$$

such that  $h_i(t)$  measures the degree of fulfillment of the decision maker's requirements about the  $i$ -th objective by the value  $t$ . In other words, with the notation of

$$H_i(x) = h_i(f(x)),$$

$H_i(x)$  may be considered as the degree of membership of  $x$  in the fuzzy set "good solutions" for the  $i$ -th objective. Then a "good compromise solution" to (6.39) may be defined as an  $x \in X$  being "as good as possible" for the whole set of objectives. Taking into consideration the nature of  $H_i, i = 1, \dots, k$ , it is quite reasonable to look for such a kind of solution by means of the following auxiliary problem

$$\max_{x \in X} \{H_1(x), \dots, H_k(x)\} \quad (6.42)$$



As  $\max \{H_1(x), \dots, H_k(x)\}$  may be interpreted as a synthetical notation of a conjunction statement (maximize jointly all objectives) and  $H_i(x) \in [0, 1]$ , it is reasonable to use a t-norm  $T$  to represent the connective AND. In this way (6.42) turns into the single-objective problem

$$\max_{x \in X} T(H_1(x), \dots, H_k(x)).$$

There exist several ways to introduce application functions [268]. Usually, the authors consider increasing membership functions (the bigger is better) of the form

$$h_i(t) = \begin{cases} 1 & \text{if } t \geq M_i \\ v_i(t) & \text{if } m_i \leq t \leq M_i \\ 0 & \text{if } t \leq m_i \end{cases} \quad (6.43)$$

where

$$m_i := \min\{f_i(x) \mid x \in X\},$$

is the independent minimum and

$$M_i := \max\{f_i(x) \mid x \in X\},$$

is the independent maximum of the  $i$ -th criterion. As it has been stated before, our idea is to use explicitly the interdependences in the solution method. To do so, first we define  $H_i$  by

$$H_i(x) = \begin{cases} 1 & \text{if } f_i(x) \geq M_i \\ 1 - \frac{M_i - f_i(x)}{M_i - m_i} & \text{if } m_i \leq f_i(x) \leq M_i \\ 0 & \text{if } f_i(x) \leq m_i \end{cases}$$

i.e. all membership functions are defined to be linear. Then from (6.40) we compute  $\Delta(f_i)$  for

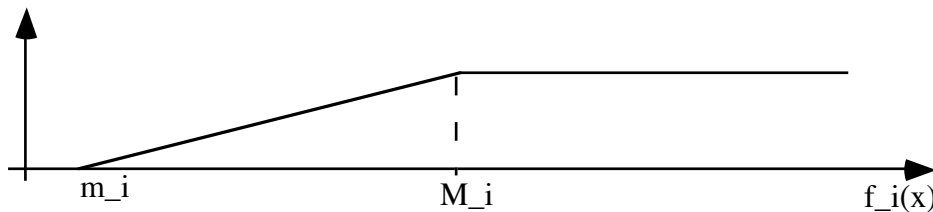


Figure 6.15: The case of linear membership function.

$i = 1, \dots, k$ , and we change the shapes of  $H_i$  according to the value of  $\Delta(f_i)$  as follows

- If  $\Delta(f_i) = 0$  then we do not change the shape.
- If  $\Delta(f_i) > 0$  then we use a concave membership function defined as

$$H_i(x, \Delta(f_i)) = \begin{cases} 1 & \text{if } f_i(x) \geq M_i \\ \tau_2(x) & \text{if } m_i \leq f_i(x) \leq M_i \\ 0 & \text{if } f_i(x) \leq m_i \end{cases}$$

where

$$\tau_2(x) = \left(1 - \frac{M_i - f_i(x)}{M_i - m_i}\right)^{\frac{1}{\Delta(f_i) + 1}}$$

- If  $\Delta(f_i) < 0$  then we use a convex membership function defined as

$$H_i(x, \Delta(f_i)) = \begin{cases} 1 & \text{if } f_i(x) \geq M_i \\ \tau_3(x) & \text{if } m_i \leq f_i(x) \leq M_i \\ 0 & \text{if } f_i(x) \leq m_i \end{cases}$$

where

$$\tau_3(x) = \left(1 - \frac{M_i - f_i(x)}{M_i - m_i}\right)^{|\Delta(f_i)| + 1}$$

Then we solve the following auxiliary problem

$$\max_{x \in X} T(H_1(x, \Delta(f_1)), \dots, H_k(x, \Delta(f_k))) \quad (6.44)$$

Let us suppose that we have a decision problem with many ( $k \geq 7$ ) objective functions. It is clear (due to the interdependences between the objectives), that we find optimal compromise solutions rather closer to the values of independent minima than maxima.

The basic idea of introducing this type of shape functions can be explained then as follows: if we manage to increase the value of the  $i$ -th objective having a large positive  $\Delta(f_i)$  then it entails the growth of the majority of criteria (because it supports the majority of the objectives), so we are getting essentially closer to the optimal value of the scalarizing function (because the losses on the other objectives are not so big, due to their definition).

One of the most important questions is the efficiency of the obtained compromise solutions. Delgado et al obtained the following result [108]:

**Theorem 6.4.1.** [108] *Let  $x^*$  be an optimal solution to*

$$\max_{x \in X} T(H_1(x), \dots, H_k(x)) \quad (6.45)$$

*where  $T$  is a  $t$ -norm,  $H_i(x) = h_i(f_i(x))$  and  $h_i$  is an application function of the form (6.43),  $i = 1, \dots, k$ . If  $h_i$  is strictly increasing on  $[m_i, M_i]$ ,  $i = 1, \dots, k$  then  $x^*$  is efficient for the problem*

$$\max_{x \in X} \{f_1(x), \dots, f_k(x)\} \quad (6.46)$$

*if either (i)  $x^*$  is unique; (ii)  $T$  is strict and  $0 < H_i(x^*) < 1$ ,  $i = 1, \dots, k$ .*

It is easy to see that our application functions are strictly increasing on  $[m_i, M_i]$ , and, therefore any optimal solution  $x^*$  to the auxiliary problem (6.44) is an efficient solution to (6.46) if either (i)  $x^*$  is unique; (ii)  $T$  is strict and  $0 < H_i(x^*) < 1$ ,  $i = 1, \dots, k$ .

The choice of a particular  $t$ -norm depends upon several factors such as the nature of the problem, the environment or decision maker's knowledge representation model. Minimum and

product t-norms are primarily used in literature to solve (6.46) through (6.45). The associated problems are, respectively

$$\begin{aligned} \min \{H_1(x, \Delta(f_1)), \dots, H_k(x, \Delta(f_k))\} &\rightarrow \max; \text{ subject to } x \in X, \\ H_1(x, \Delta(f_1)) \times \dots \times H_k(x, \Delta(f_k)) &\rightarrow \max; \text{ subject to } x \in X. \end{aligned}$$

We prefer to use the Łukasiewicz t-norm,  $T_L$ , in (6.44), because it contains the sum of the particular application functions, which is increasing rapidly if we manage to improve the value of an objective function supporting the majority of the objectives.

Then we get the following auxiliary problem

$$\max_{x \in X} \max \left\{ \sum_{i=1}^k H_i(x, \Delta(f_i)) - k + 1, 0 \right\} \quad (6.47)$$

The Łukasiewicz t-norm is not strict, so an optimal solution  $x^*$  to (6.47) is efficient for (6.46) iff  $x^*$  is the only optimal solution to (6.47).

### 6.4.3 Example

We illustrate the proposed method by an 5-objective one dimensional decision problem. Consider the problem

$$\max_{x \in X} \{f_1(x), \dots, f_5(x)\} \quad (6.48)$$

with objective functions

$$\begin{aligned} f_1(x) &= x, \\ f_2(x) &= (x + 1)^2 - 1, \\ f_3(x) &= 2x + 1, \\ f_4(x) &= x^4 - 1, \\ f_5(x) &= -3x + 1 \end{aligned}$$

and  $X = [0, 2]$ . It is easy to check that we have the following interdependences

$$f_1 \uparrow f_2, f_2 \uparrow f_3, f_3 \uparrow f_4, f_4 \downarrow f_5$$

Then the grades of interdependences are

$$\Delta(f_1) = \Delta(f_2) = \Delta(f_3) = \Delta(f_4) = 3, \Delta(f_5) = -4,$$

and we get

$$\begin{aligned} H_1(x, \Delta(f_1)) &= H_3(x, \Delta(f_3)) = \left(\frac{x}{2}\right)^{1/4}, \\ H_2(x, \Delta(f_2)) &= \left(\frac{x(x+2)}{8}\right)^{1/4}, \\ H_4(x, \Delta(f_4)) &= \frac{x}{2}, \\ H_5(x, \Delta(f_5)) &= \left(1 - \frac{x}{2}\right)^5. \end{aligned}$$

And if the product t-norm is chosen to represent the decision maker's preferences, we get the following single objective MP

$$\left(\frac{1}{2}\right)^{1/4} \left(\frac{x}{2}\right)^{7/4} \left(\frac{x}{2} + 1\right)^{1/4} \left(1 - \frac{x}{2}\right)^5 \rightarrow \max$$

subject to  $0 \leq x \leq 2$ .

This problem has a unique solution

$$x^* = \frac{-19 + \sqrt{1145}}{28} \approx 0.53$$

which is a nondominated solution to the problem (6.48) with the values of the objective functions

$$(0.53, 1.34, 2.06, -0.92, -0.59).$$

We think that the traditional assumption used in MCDM-modelling - that the criteria should be independent - is rather an unfortunate one. In some of the MOLP-methods there are pairwise trade offs among conflicting objectives, but further interdependences among the objectives are not recognized. It makes the model unrealistic and its recommendations rather abstract: a decision maker who accepts an optimal solution from the model cannot be sure that he has made the correct trade offs among the objectives. There is another type of interdependence which should be recognized: some of the objectives might *support* each others, which should be exploited in a problem solving method. Zeleny recognized these possibilities [448] when he pointed out the fallacy with using weights independent from criterion performance, but he did not take this insight further. In this Section we have presented a method for explicitly using interdependence among the criteria of a multiple criteria problem. We have shown that it will give a well-defined solution and we have illustrated the technique with a simple numerical example.

## 6.5 Multiple objective programming with interdependent objectives

We consider multiple objective programming (MOP) problems with additive interdependences, i.e. when the states of some chosen objective are attained through *supportive* or *inhibitory* feed-backs from several other objectives. MOP problems with independent objectives (i.e. when the cause-effect relations between the decision variables and the objectives are completely known) will be treated as special cases of the MOP in which we have interdependent objectives. We illustrate our ideas by a simple three-objective and a real-life seven-objective problem.

In their classical text *Theory of Games and Economic Behavior* John von Neumann and Oskar Morgenstern (1947) described the problem with *interdependence*; in their outline of a social exchange economy they discussed the case of two or more persons exchanging goods with each others ([341], page 11):

... then the results for each one will depend in general not merely upon his own actions but on those of others as well. Thus each participant attempts to maximize a function ... of which he does not control all variables. This is certainly no maximum problem, but a peculiar and disconcerting mixture of several conflicting maximum problems. Every participant is guided by another principle and neither determines all variables which affects his interest.

This kind of problem is nowhere dealt with in classical mathematics. We emphasize at the risk of being pedantic that this is no conditional maximum problem, no problem of the calculus of variations, of functional analysis, etc. It arises in full clarity, even in the most "elementary" situations, e.g., when all variables can assume only a finite number of values.

The interdependence is part of the economic theory and all market economies, but in most modelling approaches in multiple criteria decision making there seems to be an implicit assumption that objectives should be independent. This appears to be the case, if not earlier than at least at the moment when we have to select some optimal compromise among the set of nondominated decision alternatives. Milan Zeleny [446] - and many others - recognizes one part of the interdependence (page 1),

Multiple and conflicting objectives, for example, "minimize cost" and "maximize the quality of service" are the real stuff of the decision maker's or manager's daily concerns. Such problems are more complicated than the convenient assumptions of economics indicate. Improving achievement with respect to one objective can be accomplished only at the expense of another.

but not the other part: objectives could support each others.

Situations with multiple interdependent objectives are not only derived from some axiomatic framework as logical conclusions, or built as illustrations of complex decision problems in classical text books, there are real life situations which, if we ponder them systematically, reveal themselves to have interdependent objectives.

A well-known negotiation problem is the *Buyer/Seller* dilemma [349], in which it is unclear for both parties at which price they are going to settle when they start the negotiation process:



Figure 6.16: Buyer/Seller negotiation problem.

Their objectives are clearly conflicting: the Buyer wants the price to be as low as possible; the Seller tries to keep the price as high as possible. There are two points, the reservation prices, beyond which the negotiations break down. The Buyer will not discuss a price higher than the (Buyer)-point; the Seller will find a price lower than the (Seller)-point insulting. If both parties compromise they can eventually settle on a price somewhere at the mid-point of the interval. The problem becomes interesting and challenging when none of the parties can be sure of the other party's reservation price, but the setup is simple in the sense that the objectives are conflicting, and the problem can be solved with standard methods.

Let us now assume that there is a third party, the *Government*, involved and that the Government reacts to the negotiation process by working out consequences of both the reservation prices and the offers made during the negotiations. Let us also assume that the Government wants to intervene in the process in order to promote its own objectives.

In this way the negotiation problem becomes much more complex as there are a number of new objectives involved over which the primary parties have no control.

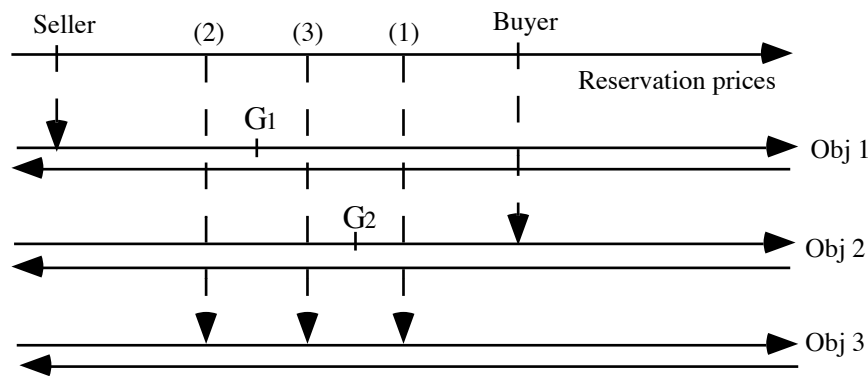


Figure 6.17: A modified Buyer/seller negotiation problem [71].

The Buyer and the Seller influence two different objectives of the Government respectively (Obj 1 and Obj 2):

- (i) a low Seller reservation price will support Obj 1, but a high Seller reservation price will be in conflict with Obj 1 after some point G1 ;
- (ii) a high Buyer reservation price will support Obj 2, but a low Buyer reservation price will be in conflict with Obj 2 after some point G2 ,
- (iii) Obj 1 and Obj 2 are conflicting (as often is the case with political objectives).

The negotiation process ((1), (2), (3)) influences Obj 3 of the Government:

- (iv) short steps and a small interval both support the attainment of Obj 3, but
- (v) large, oscillating steps and a large interval are hindrances for the attainment.

There are a number of cases of labor market negotiations in which these types of interdependences are present. Because Obj 1-3 are influenced by the Buyer-Seller negotiations the Government can not remain passive, but will influence the objectives of the primary negotiators. Then we get a set of decision problems for all parties involved, in which we have multiple interdependent objectives; these problems are not easy to cope with and to resolve. In some labor market negotiations in the spring 1996 the Finnish government managed to both define such levels of its own objectives and to push the reservation prices of the primary negotiators to levels which were unattainable. When the government finally agreed to face-saving compromises the country was 12 hours from a general strike.

We will in the following explore the consequences of allowing objectives to be interdependent.

## 6.6 Additive linear interdependences

Objective functions of a multiple objective programming problem are usually considered to be independent from each other, i.e. they depend only on the decision variable  $x$ . A typical statement of an MOP with independent objective functions is

$$\max_{x \in X} \{f_1(x), \dots, f_k(x)\} \quad (6.49)$$

where  $f_i$  is the  $i$ -th objective function,  $x$  is the decision variable, and  $X$  is a subset, usually defined by functional inequalities. Throughout this Section we will assume that the objective functions are normalized, i.e.  $f_i(x) \in [0, 1]$  for each  $x \in X$ .

However, as has been shown in some earlier work by Carlsson and Fullér [53, 57, 58], and Felix [155], there are management issues and negotiation problems, in which one often encounters the necessity to formulate MOP models with interdependent objective functions, in such a way that the objective functions are determined not only by the decision variables but also by one or more other objective functions.

*Typically, in complex, real-life problems, there are some unidentified factors which effect the values of the objective functions. We do not know them or can not control them; i.e. they have an impact we can not control. The only thing we can observe is the values of the objective functions at certain points. And from this information and from our knowledge about the problem we may be able to formulate the impacts of unknown factors (through the observed values of the objectives).*

First we state the multiobjective decision problem with independent objectives and then adjust our model to reality by introducing interdependences among the objectives. Interdependences among the objectives exist whenever the computed value of an objective function is not equal to its observed value. We claim that the real values of an objective function can be identified by the help of feed-backs from the values of other objective functions.

Suppose now that the objectives of (6.49) are interdependent, and the value of an objective function is determined by a *linear combination* of the values of other objectives functions. That is

$$f'_i(x) = f_i(x) + \sum_{j=1, j \neq i}^k \alpha_{ij} f_j(x), \quad 1 \leq i \leq k \quad (6.50)$$

or, in matrix format

$$\begin{pmatrix} f'_1(x) \\ f'_2(x) \\ \vdots \\ f'_k(x) \end{pmatrix} = \begin{pmatrix} 1 & \alpha_{12} & \dots & \alpha_{1k} \\ \alpha_{21} & 1 & \dots & \alpha_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{k1} & \alpha_{k2} & \dots & 1 \end{pmatrix} \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_k(x) \end{pmatrix},$$

where  $\alpha_{ij}$  is a real numbers denoting the grade of interdependency between  $f_i$  and  $f_j$ .

If  $\alpha_{ij} > 0$  then we say that  $f_i$  is supported by  $f_j$ ; if  $\alpha_{ij} < 0$  then we say that  $f_i$  is hindered by  $f_j$ ; if  $\alpha_{ij} = 0$  then we say that  $f_i$  is independent from  $f_j$  (or the states of  $f_j$  are irrelevant to the states of  $f_i$ ). The matrix of interdependences,  $(\alpha_{ij})$ , denoted by  $I(f_1, \dots, f_k)$ , and defined

by

$$I(f_1, \dots, f_k) = \begin{pmatrix} 1 & \alpha_{12} & \dots & \alpha_{1k} \\ \alpha_{21} & 1 & \dots & \alpha_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{k1} & \alpha_{k2} & \dots & 1 \end{pmatrix}$$

is called the interdependency matrix of (6.49).

In such cases, i.e. when the feed-backs from the objectives are directly proportional to their independent values, then we say that the objectives are *linearly* interdependent. It is clear that if  $\alpha_{ij} = 0, \forall i \neq j$ , i.e.

$$I(f_1, \dots, f_k) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

then we have an MOP problem with independent objective functions.

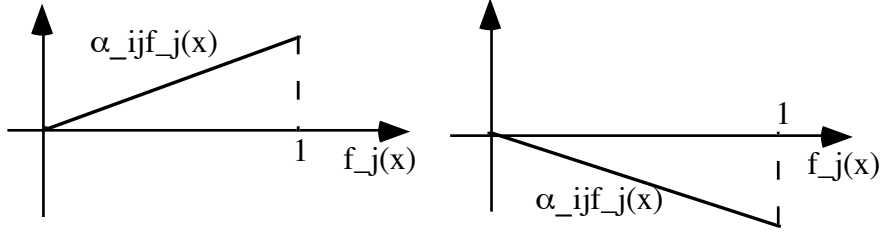


Figure 6.18: Linear feed-back with  $\alpha_{ij} > 0$  and  $\alpha_{ij} < 0$

The grade of interdependency, denoted by  $\Delta(f_i)$ , of an objective function  $f_i$  is defined by (6.40)

$$\Delta(f_i) = \sum_{i \neq j} \text{sign}(\alpha_{ji}) = \sum_{\substack{\alpha_{ji} > 0, \\ i \neq j}} 1 - \sum_{\alpha_{ji} < 0} 1$$

i.e.  $\Delta(f_i)$  is nothing else that the number of objectives supported by  $f_i$  minus the number of objectives hindered by  $f_i$ ,  $i = 1, \dots, k$ .

Taking into consideration the linear interdependences among the objective functions (6.50), (6.49) turns into the following problem (which is treated as an independent MOP)

$$\max_{x \in X} \{f'_1(x), \dots, f'_k(x)\} \quad (6.51)$$

It is clear that the solution-sets of (6.49) and (6.51) are usually not identical.

A typical case of interdependence is the following (almost) real world situation. We want to buy a house for which we have defined the following three objectives

- $f_1$ : the house should be non-expensive
- $f_2$ : as we do not have the necessary skills, the house should not require much maintenance or repair work



- $f_3$ : the house should be more than 10 year old so that the garden is fully grown and we need not look at struggling bushes and flowers

We have the following interdependences:

- $f_1$  is supported by both  $f_2$  and  $f_3$  as in certain regions it is possible to find 10 year old houses which (for the moment) do not require much repair and maintenance work, and which are non-expensive.
- $f_2$  can be conflicting with  $f_3$  for some houses as the need for maintenance and repair work increases with the age of the house; thus  $f_3$  is also conflicting with  $f_2$ .
- $f_3$  is supporting  $f_1$  for some houses; if the garden is well planned it could increase the price, in which case  $f_3$  would be in partial conflict with  $f_1$ ; if the neighbourhood is completed and no newbuilding takes place, prices could rise and  $f_3$  be in conflict with  $f_1$ .

To explain the issue more exactly, consider a three-objective problem with linearly inter-dependent objective functions

$$\max_{x \in X} \{f_1(x), f_2(x), f_3(x)\} \quad (6.52)$$

Taking into consideration that the objectives are linearly interdependent, the interdependent values of the objectives can be expressed by

$$\begin{aligned} f'_1(x) &= f_1(x) + \alpha_{12}f_2(x) + \alpha_{13}f_3(x), \\ f'_2(x) &= f_2(x) + \alpha_{21}f_1(x) + \alpha_{23}f_3(x), \\ f'_3(x) &= f_3(x) + \alpha_{31}f_1(x) + \alpha_{32}f_2(x). \end{aligned}$$

That is

$$\begin{pmatrix} f'_1(x) \\ f'_2(x) \\ f'_3(x) \end{pmatrix} = \begin{pmatrix} 1 & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & 1 & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & 1 \end{pmatrix} \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix}.$$

For example, depending on the values of  $\alpha_{ij}$  we can have the following simple linear interdependences among the objectives of (6.52)

- if  $\alpha_{12} = 0$  then we say that  $f_1$  is independent from  $f_2$ ;
- if  $\alpha_{12} > 0$  then we say that  $f_2$  unilaterally supports  $f_1$ ;
- if  $\alpha_{12} < 0$  then we say that  $f_2$  hinders  $f_1$ ;
- if  $\alpha_{12} > 0$  and  $\alpha_{21} > 0$  then we say that  $f_1$  and  $f_2$  mutually support each others;
- if  $\alpha_{12} < 0$  and  $\alpha_{21} < 0$  then we say that  $f_1$  and  $f_2$  are conflicting;
- if  $\alpha_{12} + \alpha_{21} = 0$  then we say that  $f_1$  and  $f_2$  are in a trade-off relation;

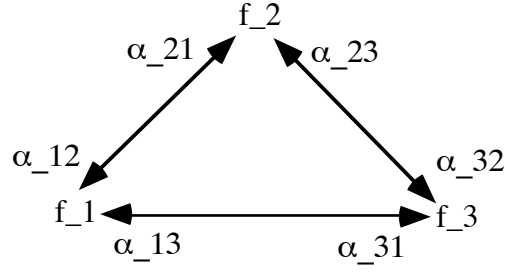


Figure 6.19: A three-objective interdependent problem with linear feed-backs.

It is clear, for example, that if  $f_2$  unilaterally supports  $f_1$  then the larger the improvement  $f_2$  (supporting objective function) the more significant is its contribution to  $f_1$  (supported objective function).

To illustrate our ideas consider the following simple decision problem.

$$\begin{aligned} \max \{x, 1 - x\} \\ x \in [0, 1] \end{aligned} \quad (6.53)$$

Choosing the minimum-norm to aggregate the values of objective functions this problem has a unique solution  $x^* = 1/2$  and the optimal values of the objective functions are  $(0.500, 0.500)$ .

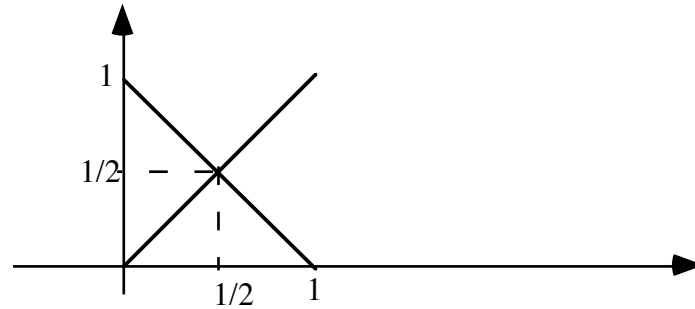


Figure 6.20: Independent problem.

Suppose that for example  $f_1$  is unilaterally supported by  $f_2$  on the whole decision space  $[0, 1]$  and the degree of support is given by

$$f'_1(x) = f_1(x) + 1/2 f_2(x) = x + 1/2(1 - x) = 1/2 + x/2$$

Then (6.53) turns into the following problem

$$\begin{aligned} \max \{1/2 + x/2, 1 - x\} \\ x \in [0, 1] \end{aligned}$$

Choosing the minimum-norm to aggregate the values of objective functions this problem has a unique solution  $x^* = 1/3$  and the optimal values of the objective functions are  $(0.667, 0.667)$ .

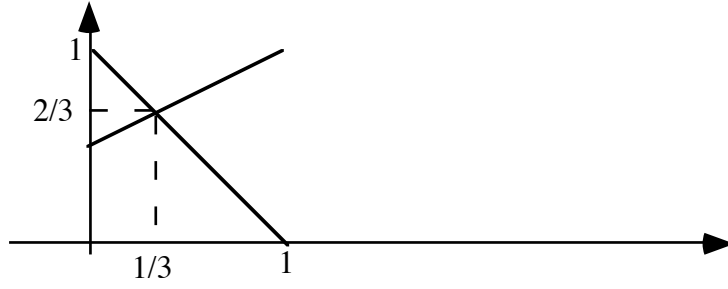


Figure 6.21: Unilateral support.

Suppose now that  $f_1$  and  $f_2$  support each other mutually, i.e. the better the value of  $f_1$  the more significant is its support to  $f_2$  and *vica versa*. The degrees of supports are given by

$$f'_1(x) = f_1(x) + 1/2f_2(x) = x + 1/2(1 - x) = 1/2(1 + x),$$

$$f'_2(x) = f_2(x) + 1/2f_1(x) = (1 - x) + 1/2x = 1 - x/2.$$

In this case our interdependent problem turns into

$$\begin{aligned} \max \{ & 1/2(1 + x), 1 - x/2 \} \\ & x \in [0, 1] \end{aligned}$$

Choosing the minimum-norm to aggregate the values of objective functions this problem has a unique solution  $x^* = 1/2$  and the optimal values of the objective functions are (0.750, 0.750). Suppose now that  $f_2$  hinders  $f_1$ , i.e. the better the value of  $f_2$  the more signifi-

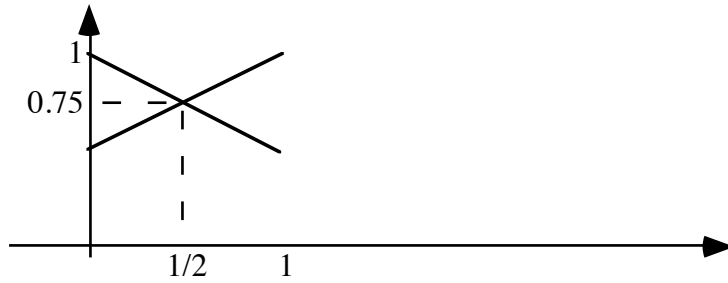


Figure 6.22: Mutual support.

cant is its negative feed-back to  $f_1$ . The degree of hindering is

$$f'(x) = f_1(x) - 1/2(1 - x) = x - 1/2 + 1/2x = 3/2x - 1/2.$$

So our interdependent problem turns into

$$\begin{aligned} \max \{ & 3/2x - 1/2, 1 - x \} \\ & x \in [0, 1] \end{aligned}$$

Choosing the minimum-norm to aggregate the values of objective functions this problem has a unique solution  $x^* = 3/5$  and the optimal values of the objective functions are (0.400, 0.400).

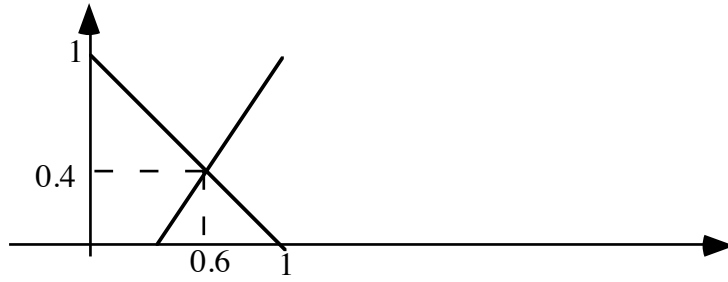


Figure 6.23: Hindering.

Suppose now that  $f_2$  hinders  $f_1$ , but  $f_1$  supports  $f_2$

$$f'_1(x) = f_1(x) - 1/2 f_2(x) = x - 1/2(1 - x) = 3/2x - 1/2,$$

$$f'_2(x) = f_2(x) + 1/2 f_1(x) = (1 - x) + 1/2x = 1 - x/2.$$

So our interdependent problem turns into

$$\begin{aligned} \max \{ & 3/2x - 1/2, 1 - x/2 \} \\ & x \in [0, 1] \end{aligned}$$

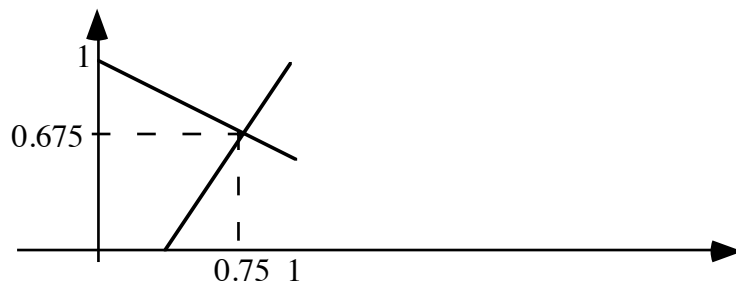


Figure 6.24: Hindering and support.

Choosing the minimum-norm to aggregate the values of objective functions this problem has a unique solution  $x^* = 3/4$  and the optimal values of the objective functions are  $(0.625, 0.625)$ . These findings can be summarized as follows:

case	solution	optimal values
independent objectives	0.5	(0.500, 0.500)
$f_1$ is supported by $f_2$	0.333	(0.667, 0.667)
mutual support	0.5	(0.750, 0.750)
$f_2$ hinders $f_1$	0.6	(0.400, 0.400)
$f_2$ hinders $f_1$ and $f_1$ supports $f_2$	0.75	(0.625, 0.625)

**Table 4** Cases and solutions.

## 6.7 Additive nonlinear interdependences

Suppose now that the objectives of (6.49) are interdependent, and the value of an objective function is determined by an *additive combination* of the feed-backs of other objectives functions

$$f'_i(x) = f_i(x) + \sum_{j=1, j \neq i}^k \alpha_{ij}[f_j(x)], \quad 1 \leq i \leq k \quad (6.54)$$

or, in matrix format

$$\begin{pmatrix} f'_1(x) \\ f'_2(x) \\ \vdots \\ f'_k(x) \end{pmatrix} = \begin{pmatrix} id & \alpha_{12} & \dots & \alpha_{1k} \\ \alpha_{21} & id & \dots & \alpha_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{k1} & \alpha_{k2} & \dots & id \end{pmatrix} \circ \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_k(x) \end{pmatrix}$$

where  $\alpha_{ij}: [0, 1] \rightarrow [0, 1]$  is a - usually nonlinear - function defining the value of feed-back from  $f_j$  to  $f_i$ ,  $id(z) = z$  denotes the identity function on  $[0, 1]$  and  $\circ$  denotes the composition operator.

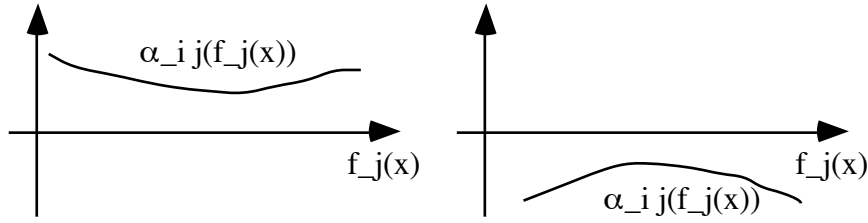


Figure 6.25: Nonlinear unilateral support and hindering.

If  $\alpha_{ij}(z) > 0, \forall z$  we say that  $f_i$  is supported by  $f_j$ ; if  $\alpha_{ij}(z) < 0, \forall z$  then we say that  $f_i$  is hindered by  $f_j$ ; if  $\alpha_{ij}(z) = 0, \forall z$  then we say that  $f_i$  is independent from  $f_j$ . If  $\alpha_{ij}(z_1) > 0$  and  $\alpha_{ij}(z_2) < 0$  for some  $z_1$  and  $z_2$ , then  $f_i$  is supported by  $f_j$  if the value of  $f_j$  is equal to  $z_1$  and  $f_i$  is hindered by  $f_j$  if the value of  $f_j$  is equal to  $z_2$ .

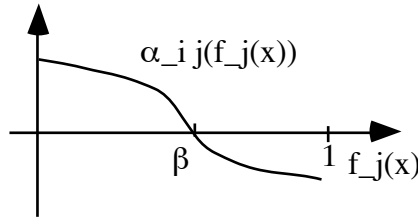


Figure 6.26:  $f_j$  supports  $f_i$  if  $f_j(x) \leq \beta$  and  $f_j$  hinders  $f_i$  if  $f_j(x) \geq \beta$ .

Consider again a three-objective problem

$$\max_{x \in X} \{f_1(x), f_2(x), f_3(x)\}$$

with nonlinear interdependences. Taking into consideration that the objectives are interdependent, the interdependent values of the objectives can be expressed by

$$f'_1(x) = f_1(x) + \alpha_{12}[f_2(x)] + \alpha_{13}[f_3(x)],$$

$$\begin{aligned} f'_2(x) &= f_2(x) + \alpha_{21}[f_1(x)] + \alpha_{23}[f_3(x)], \\ f'_3(x) &= f_3(x) + \alpha_{31}[f_1(x)] + \alpha_{32}[f_2(x)]. \end{aligned}$$

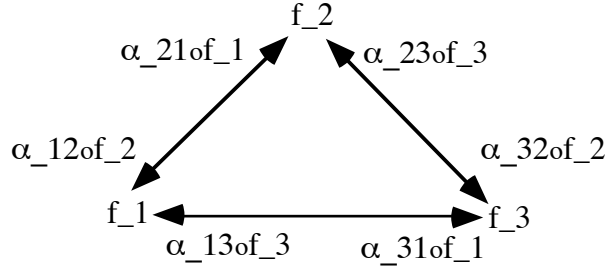


Figure 6.27: A three-objective interdependent problem with nonlinear feed-backs.

That is

$$\begin{pmatrix} f'_1(x) \\ f'_2(x) \\ f'_3(x) \end{pmatrix} = \begin{pmatrix} id & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & id & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & id \end{pmatrix} \circ \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix}$$

For example, depending on the values of the correlation functions  $\alpha_{12}$  and  $\alpha_{21}$  we can have the following simple interdependences among the objectives of (6.52)

- if  $\alpha_{12}(z) = 0, \forall z$  then we say that  $f_1$  is independent from  $f_2$ ;
- if  $\alpha_{12}(z) > 0, \forall z$  then we say that  $f_2$  unilaterally supports  $f_1$ ;
- if  $\alpha_{12}(z) < 0, \forall z$  then we say that  $f_2$  hinders  $f_1$ ;
- if  $\alpha_{12}(z) > 0$  and  $\alpha_{21}(z) > 0, \forall z$  then we say that  $f_1$  and  $f_2$  mutually support each others;
- if  $\alpha_{12}(z) < 0$  and  $\alpha_{21}(z) < 0$  for each  $z$  then we say that  $f_1$  and  $f_2$  are conflicting;
- if  $\alpha_{12}(z) + \alpha_{21}(z) = 0$  for each  $z$  then we say that  $f_1$  and  $f_2$  are in a trade-off relation;

However, despite of the linear case, we can have here more complex relationships between two objective functions, e.g.

- if for some  $\beta \in [0, 1]$

$$\alpha_{12}(z) = \begin{cases} \text{positive} & \text{if } 0 \leq z \leq \beta \\ \text{negative} & \text{if } \beta \leq z \leq 1 \end{cases}$$

then  $f_2$  unilaterally supports  $f_1$  if  $f_2(x) \leq \beta$  and  $f_2$  hinders  $f_1$  if  $f_2(x) \geq \beta$ .

- if for some  $\beta, \gamma \in [0, 1]$

$$\alpha_{12}(z) = \begin{cases} \text{positive} & \text{if } 0 \leq z \leq \beta \\ 0 & \text{if } \beta \leq z \leq \gamma \\ \text{negative} & \text{if } \gamma \leq z \leq 1 \end{cases}$$

then  $f_2$  unilaterally supports  $f_1$  if  $f_2(x) \leq \beta$ ,  $f_2$  does not affect  $f_1$  if  $\beta \leq f_2(x) \leq \gamma$  and then  $f_2$  hinders  $f_1$  if  $f_2(x) \geq \gamma$ .

## 6.8 Compound interdependences

Let us now more consider the case with compound interdependences in multiple objective programming, which is - so far - the most general case.

Assume again that the objectives of (6.49) are interdependent, and the value of an objective function is determined by an *additive combination* of the feed-backs from other objectives functions

$$f'_i(x) = \sum_{j=1}^k \alpha_{ij}[f_1(x), \dots, f_k(x)], \quad 1 \leq i \leq k \quad (6.55)$$

where  $\alpha_{ij}: [0, 1]^k \rightarrow [0, 1]$  is a - usually nonlinear - function defining the value of feed-back from  $f_j$  to  $f_i$ . We note that  $\alpha_{ij}$  depends not only on the value of  $f_j$ , but on the values of other objectives as well (this is why we call it compound interdependence [64]). Let us again consider the three-objective problem with nonlinear interdependences

$$\max_{x \in X} \{f_1(x), f_2(x), f_3(x)\}$$

With the assumptions of (6.55) the interdependent values of the objectives can be expressed by

$$f'_1(x) = \alpha_{11}[f_1(x), f_2(x), f_3(x)] + \alpha_{12}[f_1(x), f_2(x), f_3(x)] + \alpha_{13}[f_1(x), f_2(x), f_3(x)],$$

$$f'_2(x) = \alpha_{22}[f_1(x), f_2(x), f_3(x)] + \alpha_{21}[f_1(x), f_2(x), f_3(x)] + \alpha_{23}[f_1(x), f_2(x), f_3(x)],$$

$$f'_3(x) = \alpha_{33}[f_1(x), f_2(x), f_3(x)] + \alpha_{31}[f_1(x), f_2(x), f_3(x)] + \alpha_{32}[f_1(x), f_2(x), f_3(x)].$$

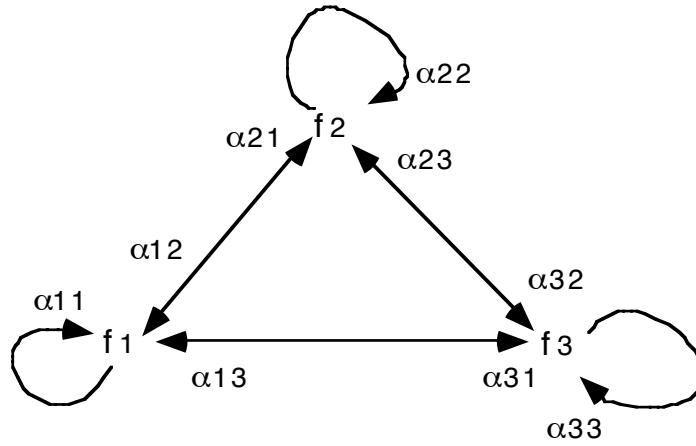


Figure 6.28: A 3-objective interdependent problem with compound interdependences.

Here we can have more complicated interrelations between  $f_1$  and  $f_2$ , because the feedback from  $f_2$  to  $f_1$  can depend not only on the value of  $f_2$ , but also on the values of  $f_1$  (self feedback) and  $f_3$ . Unfortunately, in real life cases we usually have compound interdependences [58].

We have considered only additive interdependences and time independent feed-backs. It should be noted, however, that in negotiation processes the feed-backs from other objectives are always time-dependent.

Time-dependent additive linear interdependences in MOP (6.49) can be defined as follows

$$f'_i(x) = f_i(x) + \sum_{\substack{j=1, \\ j \neq i}}^k \alpha_{ij}(t) f_j(x), \quad 1 \leq i \leq k$$

where  $\alpha_{ij}(t)$  denotes the dynamical grade of interdependency between functions  $f_i$  and  $f_j$  at time  $t$ .

Interdependence among criteria used in decision making is part of the classical economic theory even if most of the modelling efforts in the theory for multiple criteria decision making has been aimed at (the simplified effort of) finding optimal solutions for cases where the criteria are multiple but independent.

Decision making with interdependent objectives is not an easy task. However, with the methods proposed in this Section we are able to at least start dealing with interdependence. If the exact values of the objective functions can be measured (at least partially, or in some points), then from this information and some (incomplete or preliminary) model we may be able to approximate the effects of other objective functions, and of the set of decision variables we have found to be appropriate for the problem. In this way we will be able to deal with more complex decision problems in a more appropriate way.

In this Section we have tried to tackle interdependence head-on, i.e. we have deliberately formulated decision problems with interdependent criteria and found ways to deal with the "anomalies" thus created.

In the next Chapter we will demonstrate, with a fairly extensive case, called Nordic Paper Inc, that the situations we first described as just principles do have justifications in real world decision problems. It turned out that the introduction of interdependences creates complications for solving the decision problem, and there are no handy tools available for dealing with more complex patterns of interdependence. We can have the case, in fact, that problem solving strategies deciding the attainment of some subset of objectives will effectively cancel out all possibilities of attaining some other subset of objectives.

Allowing for additive, interdependent criteria appears to open up a new category of decision problems.



# Chapter 7

## Applications in Management

### 7.1 Nordic Paper Inc.

We show an example to illustrate the interdependencies by a real-life problem.

*Nordic Paper Inc.* (NPI) is one of the more successful paper producers in Europe and has gained a reputation among its competitors as a leader in quality, timely delivery to its customers, innovations in production technology and customer relationships of long duration. Still it does not have a dominating position in any of its customer segments, which is not even advisable in the European Common market, as there are always 2-5 competitors with sizeable market shares. NPI would, nevertheless, like to have a position which would be dominant against any chosen competitor when defined for all the markets in which NPI operates.

We will consider strategic decisions for the planning period 1996-2000.

Decisions will be how many tons of 6-9 paper qualities to produce for 3-4 customer segments in Germany, France, UK, Benelux, Italy and Spain. NPI is operating 9 paper mills which together cover all the qualities to be produced. Price/ton of paper qualities in different market segments are known and forecasts for the planning period are available. Capacities of the paper mills for different qualities are known and production costs/ton are also known and can be forecasted for the planning period. The operating result includes distribution costs from paper mills to the markets, and the distribution costs/ton are also known and can be forecasted for the planning period.

Decisions will also have to be made on how much more added capacity should be created through investments, when to carry out these investments and how to finance them. Investment decisions should consider target levels on productivity and competitive advantages to be gained through improvements in technology, as well as improvements in prices/ton and product qualities.

There are about 6 significant competitors in each market segment, with about the same (or poorer) production technology as the one operated by NPI. Competition is mainly on paper qualities, just-in-time deliveries, long-term customer relationships and production technology; all the competitors try to avoid competing with prices. Competition is therefore complex: if NPI manages to gain some customers for some specific paper quality in Germany by taking

these customers away from some competitor, the competitive game will not be fought in Germany, but the competitor will try to retaliate in (for instance) France by offering some superior paper quality at better prices to our customers; this offer will perhaps not happen immediately but over time, so that the game is played out over the strategic planning interval. NPI is looking for a long-term strategy to gain an overall dominance over its competitors in the European arena.

Decisions will have to be made on how to attain the best possible operating results over the planning period, how to avoid both surplus and negative cash flow, how to keep productivity at a high and stable level, and how to keep up with market share objectives introduced by shareholders, who believe that attaining dominating positions will increase share prices over time.

### 7.1.1 Objectives

There are several objectives which can be defined for the 1996-2000 strategic planning period.

- Operating result [ $C_1$ ]  
should either be as high as possible for the period or as close as possible to some acceptable level.
- Productivity [ $C_2$ ],  
defined as output (in ton) / input factors, should either be as high as possible or as close as possible to yearly defined target levels.
- Available capacity [ $C_3$ ]  
defined for all the available paper mills, should be used as much as possible, preferably to their operational limits.
- Market share [ $C_4$ ]  
objectives for the various market segments should be attained as closely as possible.
- Competitive position [ $C_5$ ]  
assessed as a relative strength to competitors in selected market segments, should be built up and consolidated over the planning period.
- Return on investments [ $C_6$ ]  
should be as high as possible when new production technology is allocated to market segments with high and stable prices and growing demand.
- Financing [ $C_7$ ]  
target levels should be attained as closely as possible when investment programs are decided and implemented; both surplus financial assets and needs for loans should be avoided.

## 7.1.2 Interdependence among objectives

There seems to be the following forms of interdependence among these objectives:

- $C_1$  and  $C_4$  are in conflict, as increased market share is gained at the expense of operating result; if  $C_5$  reaches a dominant level in a chosen market segment, then  $C_4$  will support  $C_1$ ; if  $C_5$  reaches dominant levels in a sufficient number of market segments then  $C_4$  will support  $C_1$  overall.
- $C_4$  supports  $C_5$ , as a high market share will form the basis for a strong competitive position;  $C_5$  supports  $C_4$  as a strong competitive position will form the basis for increasing market shares; there is a time lag between these objectives.
- $C_3$  supports  $C_2$ , as using most of the available capacity will increase productivity.
- $C_2$  supports  $C_1$  as increasing productivity will improve operating results.
- $C_3$  is in conflict, partially, with  $C_1$ , as using all capacity will reduce prices and have a negative effect on operating result.
- $C_6$  is supporting  $C_1$ ,  $C_4$  and  $C_5$ , as increasing return on investment will improve operating result, market share and competitive position;  $C_4$  and  $C_5$  support  $C_6$  as both objectives will improve return on investment;  $C_6$  is in conflict with  $C_3$  as increasing return on investment will increase capacity.
- $C_7$  supports  $C_1$  as a good financial stability will improve the operating result.
- $C_5$  supports  $C_2$  as a strong competitive position will improve productivity, because prices will be higher and demand will increase, which is using more of the production capacity.
- $C_4$  and  $C_6$  are in conflict as increasing market share is counterproductive to improving return on investment, which should focus on gaining positions only in market segments with high prices and stable growths.

### Preliminary outline of an algorithm

Let  $X$  be a set of possible strategic activities of relevance for the context in the sense that they are instrumental for attaining the objectives  $C_1$  -  $C_7$ . Strategic activities are decisions and action programs identified as appropriate and undertaken in order to establish positions of sustainable competitive advantages over the strategic planning period. As the objectives are interdependent the strategic activities need to be chosen or designed in such a way that the interdependences can be exploited, i.e. we can make the attainment of the various objectives more and more effective.

Let  $X$  be composed of several context-specific strategic activities:

$$X \subset \{X_{MP}, X_{CP}, X_{PROD}, X_{INV}, X_{FIN}, X_{PROF}\},$$

where the context-specific activities are defined as follows:

- $X_{MP}$ , market-oriented activities for demand, selling prices and market shares
- $X_{CP}$ , activities used for building competitive positions
- $X_{PROD}$ , production technology and productivity-improving activities
- $X_{INV}$ , investment decisions
- $X_{FIN}$ , financing of investments and operations
- $X_{PROF}$ , activities aimed at enhancing and consolidating profitability

It is clear that these activities have some temporal interdependences; it is, for instance, normally the case that a market position will influence the corresponding competitive position with some delay - in some markets this can be 2-3 months, in other markets 6-12 months. In the interest of simplicity we will disregard these interdependences.

### 7.1.3 An algorithm

- 1.1 check through database on markets, customers for intuitive view on potential changes in demand, prices, sales;
- 1.2 work out  $X_{MP}$  and list expected consequences on demand, selling prices and market shares;
- 1.3 work out consequences for  $C_4$  and check if the objective will be attained during the planning period; if not got to 1.1, otherwise proceed;
- 1.4.1 work out the impact of  $C_4$  on  $C_1$ ; if  $C_1$  is untenable, go to 1.2, otherwise proceed;
- 1.4.2 work out the impact of  $C_4$  on  $C_5$ , and the impact of  $C_5$  on  $C_4$ ; if  $C_5$  is tenable, proceed, otherwise go to 1.2;
- 1.4.3 work out the impact of  $C_4$  on  $C_6$ ; if  $C_6$  is tenable, proceed, otherwise go to 1.2;
- 1.4.4 work out the impact of  $C_6$  on  $C_4$ ; if  $C_4$  is tenable, proceed, otherwise go to 1.2;
- 2.1 check through database on markets, customers for intuitive view on the positions of key competitors;
- 2.2 work out  $X_{CP}$  and list expected consequences on overall status on critical success factors and competitive positions;
- 2.3 work out consequences for  $C_5$  and check if the objective will be attained during the planning period; if not got to 2.1, otherwise proceed;
- 2.4.1 work out the impact of  $C_5$  on  $C_4$  and  $C_1$ ; if  $C_1$ ,  $C_4$  are untenable, go to 2.2, otherwise proceed;
- 2.4.2 work out the impact of  $C_4$  on  $C_5$ , and the impact of  $C_5$  on  $C_4$ ; if  $C_4$  is tenable, proceed, otherwise go to 2.2;

- 2.4.3 work out the impact of  $C_5$  on  $C_6$ ; if  $C_6$  is tenable, proceed, otherwise go to 2.2;
- 2.4.4 work out the impact of  $C_5$  on  $C_2$ ; if  $C_2$  is tenable, proceed, otherwise go to 2.2;
- 3.1 check through database on markets, customers for intuitive view on potential changes in product demand, quality constraints, requirements on technology;
- 3.2 work out  $X_{PROD}$  and list expected consequences on production program, required selling prices and market shares;
- 3.3 work out consequences for  $C_2$  and check if the objective will be attained during the planning period; if not got to 3.1, otherwise proceed;
- 3.4.1 work out the impact of  $C_3$  on  $C_2$ ; if  $C_2$  is tenable, proceed, otherwise go to 3.1;
- 3.4.2 work out the impact of  $C_2$  on  $C_1$ ; if  $C_1$  is tenable, proceed, otherwise go to 3.2;
- 3.4.3 work out the impact of  $C_5$  on  $C_2$ ; if  $C_2$  is tenable, proceed, otherwise go to 3.2;
- 4.1 check through  $X_{MP}$ ,  $X_{CP}$ ,  $X_{PROD}$ ;
- 4.2 work out  $X_{INV}$  and list expected consequences on productivity, competitive position and market position;
- 4.3 work out consequences for  $C_6$  and check if the objective will be attained during the planning period; if not got to 4.1, otherwise proceed;
- 4.4.1 work out the impact of  $C_6$  on  $C_1$ ,  $C_4$  and  $C_5$ ; if all of them are tenable, proceed; otherwise go to 4.2;
- 4.4.2 work out the impact of  $C_4$  and  $C_5$  on  $C_6$ ; if  $C_6$  is tenable, proceed, otherwise go to 4.2;
- 4.4.3 work out the impact of  $C_6$  on  $C_3$ ; if  $C_3$  is tenable, proceed, otherwise go to 4.2;
- 5.1 check through  $X_{MP}$ ,  $X_{CP}$ ,  $X_{PROD}$ ,  $X_{INV}$ ;
- 5.2 work out  $X_{FIN}$  and list expected consequences on profitability and cash flow;
- 5.3 work out consequences for  $C_7$  and check if the objective will be attained during the planning period; if not got to 5.1, otherwise proceed;
- 5.4.1 work out the impact of  $C_7$  on  $C_1$ ; if  $C_1$  is tenable, proceed, otherwise go to 5.2;
- 6.1 check through  $X_{MP}$ ,  $X_{CP}$ ,  $X_{PROD}$ ,  $X_{INV}$ ;
- 6.2 work out  $X_{PROF}$  and list expected consequences on profitability, capital structure, cash flow and key ratios;
- 6.3 work out consequences for  $C_1$  and check if the objective will be attained during the planning period; if not got to 6.1 (or possibly 1.1), otherwise proceed;
- 6.4.1 work out the impact of  $C_1$  on  $C_4$ ; if  $C_4$  is untenable, go to 6.2, otherwise proceed;

6.4.2 work out the impact of  $C_5$  on  $C_4$ , and the impact of  $C_4$  on  $C_1$ ; if  $C_4$  is tenable, proceed, otherwise go to 6.2;

6.4.3 work out the impact of  $C_2$  on  $C_1$ ; if  $C_1$  is tenable, proceed, otherwise go to 6.2;

6.4.4 work out the impact of  $C_3$  on  $C_1$ ; if  $C_4$  is untenable, go to 6.2, otherwise proceed;

6.4.5 work out the impact of  $C_6$  on  $C_1$ ; if  $C_1$  is tenable, proceed, otherwise go to 6.2;

6.4.6 work out the impact of  $C_7$  on  $C_1$ ; if  $C_1$  is tenable, proceed, otherwise go to 6.2;

There are second and third degree interdependences between the objectives, and there are degrees to the interdependences; all with an impact on the the design of the set of strategic activities:

$$X \subset \{X_{MP}, X_{CP}, X_{PROD}, X_{INV}, X_{FIN}, X_{PROF}\}.$$

These will not be worked out here, as this illustration is sufficient to show the inherent complexity.

## 7.2 Adaptive fuzzy cognitive maps for hyperknowledge representation

In [61] Carlsson demonstrated that all the pertinent conceptual constructs of strategic management theory can be represented with a knowledge based support (KBS)-system with hyperknowledge properties. Furthermore, he showed that cognitive maps can be used to trace the impact of the support and to generalize the experiences of the users. Following Carlsson and Fullér [67] in this Section we will show that the effectiveness and usefulness of this hyperknowledge support system can be further advanced using adaptive fuzzy cognitive maps.

*Strategic Management* is defined as a system of action programs which form sustainable competitive advantages for a corporation, its divisions and its business units in a strategic planning period. A research team of the IAMSR institute developed a support system for strategic management, called the *Woodstrat*, in two major Finnish forest industry corporations in 1992-96. The system is modular and is built around the actual business logic of strategic management in the two corporations, i.e. the main modules cover the

<i>market position</i>	(MP),
<i>competitive position</i>	(CP),
<i>productivity position</i>	(PROD),
<i>profitability</i>	(PROF),
<i>investments</i>	(INV)
<i>financing of investments</i>	(FIN).

The innovation in *Woodstrat* is that these modules are linked together in a hyperknowledge fashion, i.e. when a strong market position is built in some market segment it will have an immediate impact on profitability through links running from key assumptions on expected

developments to the projected income statement. There are similar links making the competitive position interact with the market position, and the productivity position interact with both the market and the competitive positions, and with the profitability and financing positions.

The *Woodstrat* offers an intuitive and effective strategic planning support with object-oriented expert systems elements and a hyperknowledge user interface. In this Section we will show that the effectiveness and usefulness of a hyperknowledge support system can be further advanced using adaptive fuzzy cognitive maps.

### 7.2.1 Hyperknowledge and cognitive maps

*Hyperknowledge* is formed as a system of sets of interlinked concepts [83], much in the same way as hypertext is built with interlinked text strings [343]; then hyperknowledge-functions would be constructs which link concepts/systems of concepts in some predetermined or wanted way.

There are some useful characteristics of a hyperknowledge environment [83]: (i) the user can navigate through and work with diverse concepts; (ii) concepts can be different epistemologically, (iii) concepts can be organized in cognitive maps, (iv) the concepts can be made interrelated and interdependent, (v) relations can be structured or dynamic, and (vi) relations can change with or adapt to the context.

*Cognitive maps* were introduced by Axelrod [5] to represent crisp cause-effect relationships which are perceived to exist among the elements of a given environment. *Fuzzy cognitive maps* (FCM) are fuzzy signed directed graphs with feedbacks, and they model the world as a collection of concepts and causal relations between concepts [287].

When addressing strategic issues cognitive maps are used as action-oriented representations of the context the managers are discussing. They are built to show and simulate the interaction and interdependences of multiple belief systems as these are described by the participants - by necessity, these belief systems are qualitative and will change with the context and the organizations in which they are developed. They represent a way to make sure, that the intuitive belief that strategic issues should have consequences and implications, that every strategy is either constrained or enhanced by a network of other strategies, can be adequately described and supported. For simplicity, in this Section we illustrate the strategy building process by the following fuzzy cognitive map with six states (see Fig. 7.1)

The causal connections between the states MP (Market position), CP (Competitive position), PROF (Profitability), FIN (Financing position), PROD (Productivity position) and INV (Investments) are derived from the opinions of managers' of different Strategic Business Units.

It should be noted that the cause-effect relationships among the elements of the strategy building process may be defined otherwise (you may want to add other elements or delete some of these, or you may draw other arrows or rules or swap their signs or weight them in some new way).

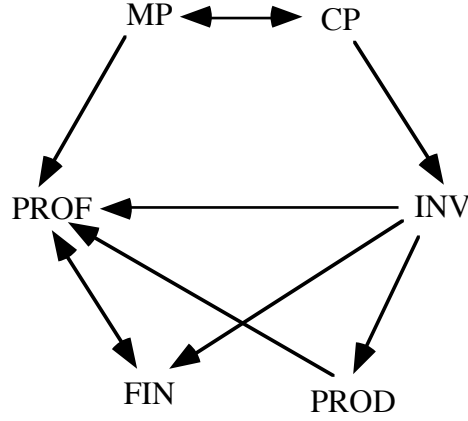


Figure 7.1: Essential elements of the strategy building process.

### 7.2.2 Adaptive FCM for strategy formation

It is relatively easy to create cause-effect relationships among the elements of the strategy building process, however it is time-consuming and difficult to fine-tune them. Neural nets give a shortcut to tuning fuzzy cognitive maps. The trick is to let the fuzzy causal edges change as if they were synapses (weights) in a neural net.

Each arrow in Fig. 7.1 defines a fuzzy rule. We weight these rules or arrows with a number from the interval  $[-1, 1]$ , or alternatively we could use *word weights* like *little*, or *somewhat*, or *more or less*. The states or nodes are fuzzy too. Each state can fire to some degree from 0% to 100%. In the crisp case the nodes of the network are *on* or *off*. In a real FCM the nodes are fuzzy and fire more as more causal juice flows into them.

Adaptive fuzzy cognitive maps can learn the weights from historical data. Once the FCM is trained it lets us play what-if games (e.g. *What if demand goes up and prices remain stable? - i.e. we improve our MP*) and can predict the future.

In the following we describe a learning mechanism for the FCM of the strategy building process, and illustrate the effectiveness of the map by a simple training set. Fig. 7.2 shows the structure of the FCM of the strategy building process.

Inputs of states are computed as the weighted sum of the outputs of its causing states

$$\text{net} = W o$$

where  $W$  denotes the matrix of weights,  $o$  is the vector of computed outputs, and  $\text{net}$  is the vector of inputs to the states. In our case the weight matrix is given by

$$W = \begin{pmatrix} 0 & w_{12} & 0 & 0 & 0 & 0 \\ w_{21} & 0 & 0 & 0 & 0 & 0 \\ w_{31} & 0 & 0 & w_{34} & w_{35} & w_{36} \\ 0 & w_{42} & 0 & 0 & 0 & 0 \\ 0 & 0 & w_{53} & w_{54} & 0 & 0 \\ 0 & 0 & 0 & w_{64} & 0 & 0 \end{pmatrix}$$

where the zero elements denote no causal link between the states, and



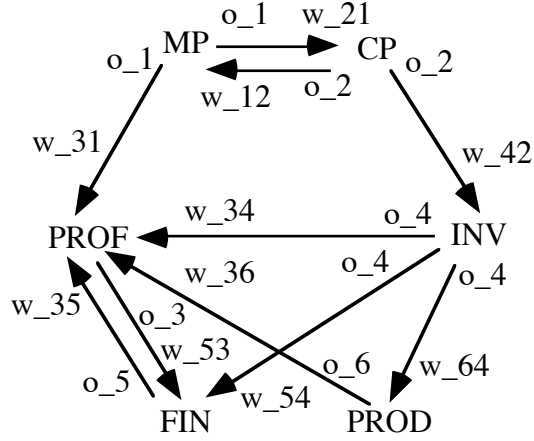


Figure 7.2: Adaptive fuzzy cognitive map for the strategy formation process.

$$\text{net} = \begin{bmatrix} \text{net}_1 \\ \text{net}_2 \\ \text{net}_3 \\ \text{net}_4 \\ \text{net}_5 \\ \text{net}_6 \end{bmatrix} = \begin{bmatrix} \text{net}(MP) \\ \text{net}(CP) \\ \text{net}(PROF) \\ \text{net}(INV) \\ \text{net}(FIN) \\ \text{net}(PROD) \end{bmatrix}$$

and

$$o = \begin{bmatrix} o_1 \\ o_2 \\ o_3 \\ o_4 \\ o_5 \\ o_6 \end{bmatrix} = \begin{bmatrix} o(MP) \\ o(CP) \\ o(PROF) \\ o(INV) \\ o(FIN) \\ o(PROD) \end{bmatrix}$$

That is,

$$\begin{aligned} \text{net}_1 &= \text{net}(MP) = w_{12}o_2, \\ \text{net}_2 &= \text{net}(CP) = w_{21}o_1, \\ \text{net}_3 &= \text{net}(PROF) = w_{31}o_1 + w_{34}o_4 + w_{35}o_5 + w_{36}o_6, \\ \text{net}_4 &= \text{net}(INV) = w_{42}o_2, \\ \text{net}_5 &= \text{net}(FIN) = w_{54}o_4 + w_{53}o_3, \\ \text{net}_6 &= \text{net}(PROD) = w_{64}o_4 \end{aligned}$$

The output of state  $i$  is computed by a squashing function

$$o_i = \frac{1}{1 + \exp(-\text{net}_i)}$$

Suppose we are given a set of historical training data

$$(MP(t), CP(t), PROF(t), INV(t), FIN(t), PROD(t))$$



Figure 7.3: Unipolar transfer function.

where  $t = 1, \dots, K$ . Here  $MP(t)$  is the observed value of the market position,  $CP(t)$  is the value of the competitive position at time  $t$ , and so on. Using an error correction learning procedure we find the weights by minimizing the overall error

$$E(W) = \frac{1}{2} \sum_{t=1}^K \left\{ (MP(t) - o_1(t))^2 + (CP(t) - o_2(t))^2 + \right. \\ (PROF(t) - o_3(t))^2 + (INV(t) - o_4(t))^2 + \\ \left. (FIN(t) - o_5(t))^2 + (PROD(t) - o_6(t))^2 \right\}$$

where  $o_i(t)$ , the computed value of the  $i$ -th state at time  $t$ , is determined as

$$o_i(t) = \frac{1}{1 + \exp[-\text{net}_i(t-1)]} = \frac{1}{1 + \exp[-\sum_j w_{ij} o_j(t-1)]}$$

where  $j$  is a causing state for state  $i$ . The weights are initialized at small random values. The rule for changing the weights of the states is derived from gradient descent method.

### 7.2.3 Example

Consider a simple training set of historical data shown in Table 5.

The observed values of the states are measured from the interval  $[1, 5]$ , where 1 stands for *weak*, 2 stands for *rather weak*, 3 stands for *medium*, 4 stands for *rather strong* and 5 stands for *strong*, intermediate values are denoted by  $\{1.5, 2.5, 3.5, 4.5\}$ .

For example, at reference time 7 we have a medium market position, weak-rather weak competitive position, rather strong profitability, strong investments, weak financing and rather strong productivity position. After the training we get the following weight matrix

$$W = \begin{pmatrix} 0 & 0.65 & 0 & 0 & 0 & 0 \\ 0.46 & 0 & 0 & 0 & 0 & 0 \\ 0.54 & 0 & 0 & 0.33 & 0.14 & -0.05 \\ 0 & 0.23 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.18 & 0.31 & 0 & 0 \\ 0 & 0 & 0 & 0.27 & 0 & 0 \end{pmatrix}.$$

Our findings can be interpreted as the market and competitive positions are the driving forces for the overall profitability position.

	MP	CP	PROF	INV	FIN	PROD
1.	3	3	3	3	3	3
2.	4	3.5	3.5	3	4	3
3.	4	4	3.5	4	5	3.5
4.	3	4	3.5	4	4	3.5
5.	3	3.5	4	4	3	4
6.	2	3	4	4	2	4
7.	3	2.5	4	5	1	4
8.	3	3	4	5	2	3.5
9.	4	3	4	5	3	3.5
10.	3	3.5	4	5	4	3

**Table 5.** A training set.

The extensions of *Woodstrat* to a fuzzy hyperknowledge support system shown here will have the effect to (i) support approximate reasoning schemes in linking the MP, CP, PROD, PROF, INV and FIN elements of strategic management; (ii) approximate reasoning gives us conclusions from imprecise premises; (iii) fuzzy sets and fuzzy logic support a synthesis of quantitative and qualitative concepts, which develops strategic planning to a strategy formation process.

### 7.3 Soft computing techniques for portfolio evaluation

Suppose that the value of our portfolio depends on the currency fluctuations on the global finance market. Our knowledge is given in the form of fuzzy if-then rules, where all of the linguistic values for the exchange rates and the portfolio values are represented by sigmoidal fuzzy numbers. It is relatively easy to create fuzzy if-then rules for portfolio evaluation, however it is time-consuming and difficult to fine-tune them.

Following Carlsson and Fullér [68, 73] in this Section we compute the crisp portfolio values by Tsukamoto's inference mechanism and introducing some reasonable interdependences among the linguistic terms we show a simple method for tuning the membership functions in the rules. For modeling the partially known causal link between the exchange rates and the portfolio value we employ fuzzy if-then rules of the following type

$$\mathfrak{R}_i: \text{if } x_1 \text{ is } A_{i1} \text{ and } \dots \text{and } x_n \text{ is } A_{in} \text{ then } PV \text{ is } C_i$$

where  $PV$  is the linguistic variable for the portfolio value,  $x_1, \dots, x_n$  are the linguistic variables for exchange rates having effects on the portfolio value. Each  $x_i$  has two linguistic terms *low* and *high*, denoted by  $L_i$  and  $H_i$ , which satisfy the equality  $L_i(t) + H_i(t) = 1$  for each

$t$ . The portfolio value can have four terms: *very big* (VB), *big* (B), *small* (S) and *very small* (VS). It is clear that the value of the portfolio can not be negative (in the worst case we lose everything). The membership functions for the portfolio are supposed to satisfy the properties  $B(t) + S(t) = 1$ ,  $VS(t) = S(t + c)$  and  $VB(t) = B(t - c)$  for some constant  $c$  and for each  $t$ .

We believe that two linguistic terms  $\{low, high\}$  are sufficient for exchange rates, because the term "exchange rate is medium" can be derived from the terms "exchange rate is low" and "exchange rate is high". In a similar manner we consider our portfolio value as *small* if its value is smaller or exceeds a little bit the value of our investment, and *big* if its value is definitively bigger than our investment. The term "portfolio value is medium" is rapidly changing and can be derived from other terms.

Under these assumptions it seems to be reasonable to derive the daily portfolio values from the actual exchange rates and from the rule-base  $\mathcal{R} = \{\mathcal{R}_1, \dots, \mathcal{R}_m\}$  by using Tsukamoto's reasoning mechanism, which requires monoton membership functions for all linguistic terms. Consider a simple case with the following three fuzzy if-then rules in our knowledge-base:

- $\mathcal{R}_1$  : **if**  $x_1$  is  $L_1$  **and**  $x_2$  is  $L_2$  **and**  $x_3$  is  $L_3$  **then** PV is  $VB$   
 $\mathcal{R}_2$  : **if**  $x_1$  is  $H_1$  **and**  $x_2$  is  $H_2$  **and**  $x_3$  is  $L_3$  **then** PV is  $B$   
 $\mathcal{R}_3$  : **if**  $x_1$  is  $H_1$  **and**  $x_2$  is  $H_2$  **and**  $x_3$  is  $H_3$  **then** PV is  $S$

where  $x_1$ ,  $x_2$  and  $x_3$  denote the exchange rates between USD and DEM, USD and SEK, and USD and FIM, respectively. The rules are interpreted as:

- $\mathcal{R}_1$  : If the US dollar is weak against the German mark Swedish crown and the Finnish mark then our portfolio value is very big.  
 $\mathcal{R}_2$  : If the US dollar is strong against the German mark and the Swedish crown and the US dollar is weak against the Finnish mark then our portfolio value is big.  
 $\mathcal{R}_3$  : If the US dollar is strong against the German mark the Swedish crown and the Finnish mark then our portfolio value is small.

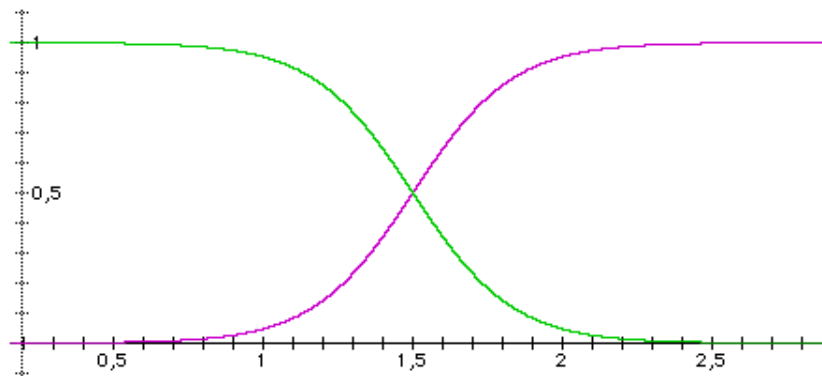


Figure 7.4: Initial fuzzy sets of "x<sub>1</sub> is low" and "x<sub>1</sub> is high",  $b_1 = 6$  and  $c_1 = 1.5$ .

The fuzzy sets  $L_1 = \text{"USD/DEM is low"}$  and  $H_1 = \text{"USD/DEM is high"}$  are given by the following membership functions

$$L_1(t) = \frac{1}{1 + \exp(b_1(t - c_1))}, \quad H_1(t) = \frac{1}{1 + \exp(-b_1(t - c_1))}$$

It is easy to check that the equality  $L_1(t) + H_1(t) = 1$  holds for all  $t$ .

The fuzzy sets  $L_2 = \text{"USD/SEK is low"}$  and  $H_2 = \text{"USD/SEK is high"}$  are given by the following membership functions

$$H_2(t) = \frac{1}{1 + \exp(b_2(t - c_2))}, \quad L_2(t) = \frac{1}{1 + \exp(-b_2(t - c_2))}$$

It is easy to check that the equality  $L_2(t) + H_2(t) = 1$  holds for all  $t$ . The fuzzy sets  $L_3$

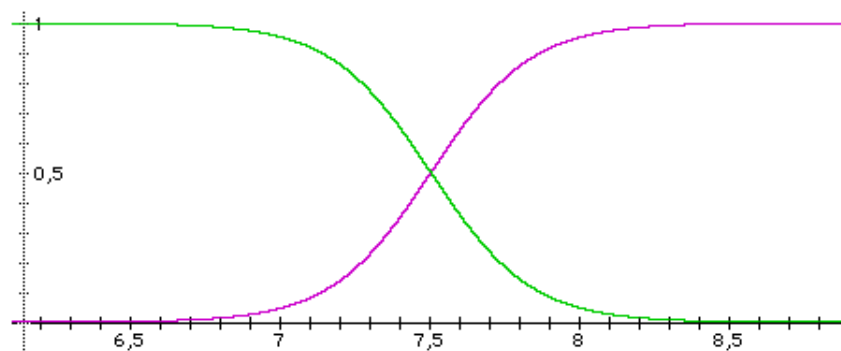


Figure 7.5: Initial fuzzy sets for " $x_2$  is low" and " $x_2$  is high",  $b_2 = 6$  and  $c_2 = 7.5$ .

$= \text{"USD/FIM is low"}$  and  $H_3 = \text{"USD/FIM is high"}$  are given by the following membership function

$$L_3(t) = \frac{1}{1 + \exp(b_3(t - c_3))}, \quad H_3(t) = \frac{1}{1 + \exp(-b_3(t - c_3))}$$

It is easy to check that the equality  $L_3(t) + H_3(t) = 1$  holds for all  $t$ . The fuzzy sets  $VB = \text{"portfolio value is very big"}$  and  $VB = \text{"portfolio value is very small"}$  are given by the following membership functions

$$VS(t) = \frac{1}{1 + \exp(b_4(t - c_4 - c_5))}, \quad VB(t) = \frac{1}{1 + \exp(-b_4(t - c_4 + c_5))},$$

The fuzzy sets  $B = \text{"portfolio value is big"}$  and  $S = \text{"portfolio value is small"}$  are given by the following membership function

$$B(t) = \frac{1}{1 + \exp(-b_4(t - c_4))}, \quad S(t) = \frac{1}{1 + \exp(b_4(t - c_4))}$$

It is easy to check that the equality  $B(t) + S(t) = 1$  holds for all  $t$ .

We evaluate the daily portfolio value by Tsukamoto's reasoning mechanism, i.e.

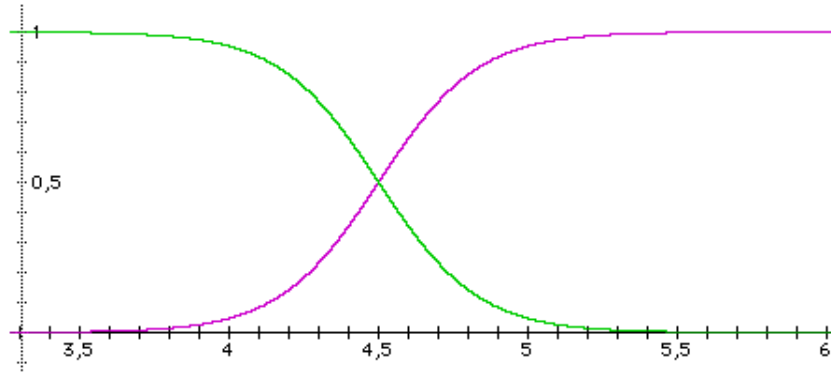


Figure 7.6: Initial fuzzy sets for "x<sub>3</sub> is low" and "x<sub>3</sub> is high",  $b_1 = 6$  and  $c_1 = 4.5$ .

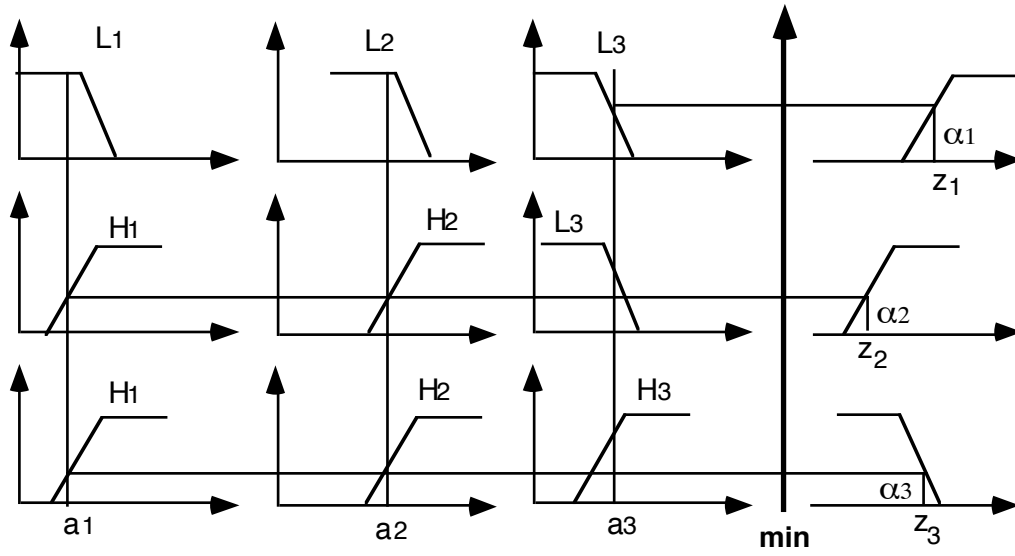


Figure 7.7: Tsukamoto's reasoning mechanism with three inference rules.

- The firing levels of the rules are computed by

$$\alpha_1 = L_1(a_1) \wedge L_2(a_2) \wedge L_3(a_3),$$

$$\alpha_2 = H_1(a_1) \wedge H_2(a_2) \wedge L_3(a_3),$$

$$\alpha_3 = H_1(a_1) \wedge H_2(a_2) \wedge H_3(a_3),$$

- The individual rule outputs are derived from the relationships

$$z_1 = VB^{-1}(\alpha_1) = c_4 + c_5 + \frac{1}{b_4} \ln \frac{1 - \alpha_1}{\alpha_1}, \quad (7.1)$$

$$z_2 = B^{-1}(\alpha_2) = c_4 + \frac{1}{b_4} \ln \frac{1 - \alpha_2}{\alpha_2} \quad (7.2)$$

$$z_3 = S^{-1}(\alpha_3) = c_4 - \frac{1}{b_4} \ln \frac{1 - \alpha_3}{\alpha_3} \quad (7.3)$$

- The overall system output is expressed as

$$z_0 = \frac{\alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3}{\alpha_1 + \alpha_2 + \alpha_3}$$

where  $a_1$ ,  $a_2$  and  $a_3$  are the inputs to the system.

### 7.3.1 Tuning the membership functions

We describe a simple method for learning of membership functions of the antecedent and consequent parts of fuzzy if-then rules. A hybrid neural net [33] computationally identical to our fuzzy system is shown in Figure 7.9.

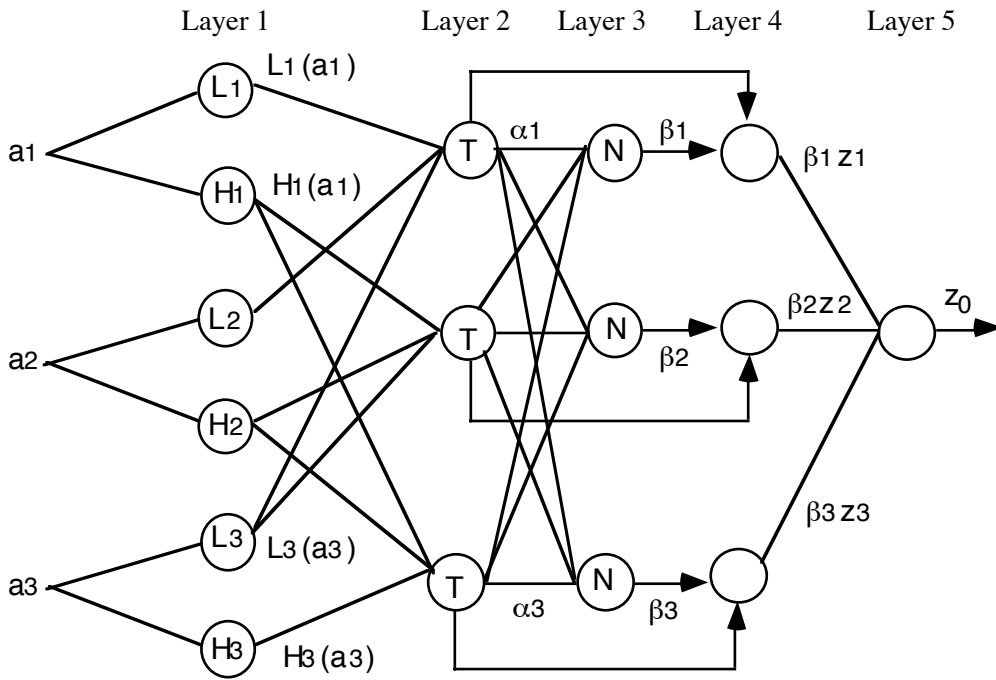


Figure 7.8: A hybrid neural net (ANFIS architecture [260]) which is computationally equivalent to Tsukamoto's reasoning method.

- **Layer 1** The output of the node is the degree to which the given input satisfies the linguistic label associated to this node.
- **Layer 2** Each node computes the firing strength of the associated rule.

The output of top neuron is

$$\alpha_1 = L_1(a_1) \wedge L_2(a_2) \wedge L_3(a_3),$$

the output of the middle neuron is

$$\alpha_2 = H_1(a_1) \wedge H_2(a_2) \wedge L_3(a_3),$$

and the output of the bottom neuron is

$$\alpha_3 = H_1(a_1) \wedge H_2(a_2) \wedge H_3(a_3).$$

All nodes in this layer is labeled by  $T$ , because we can choose other t-norms for modeling the logical *and* operator. The nodes of this layer are called *rule nodes*.

- **Layer 3** Every node in this layer is labeled by  $N$  to indicate the normalization of the firing levels.

The output of the top, middle and bottom neuron is the normalized firing level of the corresponding rule

$$\beta_1 = \frac{\alpha_1}{\alpha_1 + \alpha_2 + \alpha_3}, \quad \beta_2 = \frac{\alpha_2}{\alpha_1 + \alpha_2 + \alpha_3}, \quad \beta_3 = \frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3},$$

- **Layer 4** The output of the top, middle and bottom neuron is the product of the normalized firing level and the individual rule output of the corresponding rule

$$\beta_1 z_1 = \beta_1 V B^{-1}(\alpha_1), \quad \beta_2 z_2 = \beta_2 B^{-1}(\alpha_2), \quad \beta_3 z_3 = \beta_3 S^{-1}(\alpha_3),$$

- **Layer 5** The single node in this layer computes the overall system output as the sum of all incoming signals, i.e.

$$z_0 = \beta_1 z_1 + \beta_2 z_2 + \beta_3 z_3.$$

Suppose we have the following crisp training set

$$\{(x_1, y_1), \dots, (x_K, y_K)\}$$

where  $x_k$  is the vector of the actual exchange rates and  $y_k$  is the real value of our portfolio at time  $k$ . We define the measure of error for the  $k$ -th training pattern as usually

$$E_k = \frac{1}{2}(y_k - o_k)^2$$

where  $o_k$  is the computed output from the fuzzy system  $\mathfrak{R}$  corresponding to the input pattern  $x_k$ , and  $y_k$  is the real output,  $k = 1, \dots, K$ .

The steepest descent method is used to learn the parameters of the conditional and the consequence parts of the fuzzy rules. We show now how to tune the shape parameters  $b_4$ ,  $c_4$  and  $c_5$  of the portfolio value. From (7.1), (7.2) and (7.3) we get the following learning rule for the slope,  $b_4$ , of the portfolio values

$$b_4(t+1) = b_4(t) - \eta \frac{\partial E_k}{\partial b_4} = b_4(t) - \frac{\eta}{b_4^2} \delta_k \frac{\alpha_1 + \alpha_2 - \alpha_3}{\alpha_1 + \alpha_2 + \alpha_3},$$

In a similar manner we can derive the learning rules for the center  $c_4$

$$c_4(t+1) = c_4(t) - \eta \frac{\partial E_k}{\partial c_4} = c_4(t) + \eta \delta_k \frac{\alpha_1 + \alpha_2 + \alpha_3}{\alpha_1 + \alpha_2 + \alpha_3} = c_4(t) + \eta \delta_k,$$



and for the *shifting value*  $c_5$

$$c_5(t+1) = c_5(t) - \eta \frac{\partial E_k}{\partial c_5} = c_5(t) + \eta \delta_k \frac{\alpha_1}{\alpha_1 + \alpha_2 + \alpha_3}$$

where  $\delta_k = (y_k - o_k)$  denotes the error,  $\eta > 0$  is the learning rate and  $t$  indexes the number of the adjustments. The learning rules for the shape parameters of the antecedent part of the rules can be derived in a similar way.

Table 6 shows some mean exchange rates, the computed portfolio values (CPV) with the initial membership functions and real portfolio values (RPV) from 1995.

Date	USD/DEM	USD/SEK	USD/FIM	CPV	RPV
January 11, 1995	1.534	7.530	4.779	14.88	19
May 19, 1995	1.445	7.393	4.398	17.55	19.4
August 11, 1995	1.429	7.146	4.229	19.25	22.6
August 28, 1995	1.471	7.325	4.369	17.71	20

Table 6. Performance of the fuzzy system before the training.

Table 7 shows some mean exchange rates, the computed portfolio values with the final membership functions and real portfolio values from 1995.

Date	USD/DEM	USD/SEK	USD/FIM	CPV	RPV
January 11, 1995	1.534	7.530	4.779	18.92	19
May 19, 1995	1.445	7.393	4.398	19.37	19.4
August 11, 1995	1.429	7.146	4.229	22.64	22.6
August 28, 1995	1.471	7.325	4.369	19.9	20

Table 7. Performance of the fuzzy system after the training.

### 7.3.2 Illustration

We illustrate the tuning process by a very simple example. Consider two fuzzy rules of the form

$$\mathfrak{R}_1 : \text{ if } x \text{ is } A_1 \text{ then } y = z_1$$

$$\mathfrak{R}_2 : \text{ if } x \text{ is } A_2 \text{ then } y = z_2$$

where the fuzzy terms  $A_1$  "small" and  $A_2$  "big" have sigmoid membership functions defined by

$$A_1(x) = \frac{1}{1 + \exp(b_1(x - a_1))}, \quad A_2(x) = \frac{1}{1 + \exp(b_2(x - a_2))}$$

where  $a_1, a_2, b_1$  and  $b_2$  are the parameter set for the premises. Let  $x$  be the input to the fuzzy system. The firing levels of the rules are computed by

$$\alpha_1 = A_1(x) = \frac{1}{1 + \exp(b_1(x - a_1))} \quad \alpha_2 = A_2(x) = \frac{1}{1 + \exp(b_2(x - a_2))}$$

and the output of the system is computed by the discrete center-of-gravity defuzzification method as

$$o = \frac{\alpha_1 z_1 + \alpha_2 z_2}{\alpha_1 + \alpha_2} = \frac{A_1(x)z_1 + A_2(x)z_2}{A_1(x) + A_2(x)}.$$

Suppose further that we are given a training set

$$\{(x^1, y^1), \dots, (x^K, y^K)\}$$

obtained from the unknown nonlinear function  $f$ .

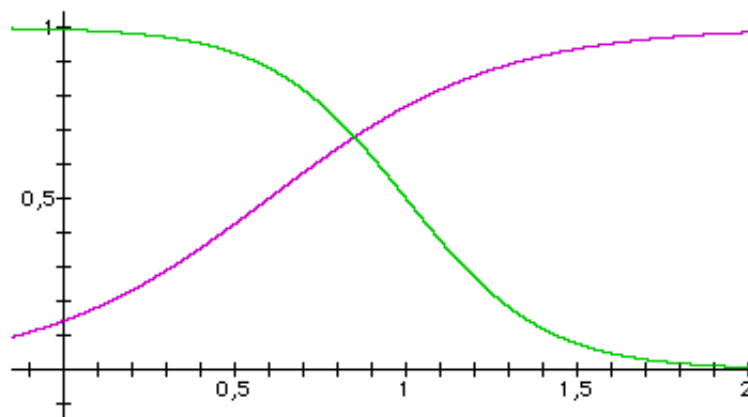


Figure 7.9: Initial sigmoid membership functions.

Our task is construct the two fuzzy rules with appropriate membership functions and consequent parts to generate the given input-output pairs. That is, we have to learn the following parameters

- $a_1, b_1, a_2$  and  $b_2$ , the parameters of the fuzzy numbers representing the linguistic terms "small" and "big",
- $z_1$  and  $z_2$ , the values of consequent parts.

We define the measure of error for the  $k$ -th training pattern as usually

$$E_k = E_k(a_1, b_1, a_2, b_2, z_1, z_2) = \frac{1}{2}(o^k(a_1, b_1, a_2, b_2, z_1, z_2) - y^k)^2$$

where  $o^k$  is the computed output from the fuzzy system corresponding to the input pattern  $x^k$  and  $y^k$  is the desired output,  $k = 1, \dots, K$ .

The steepest descent method is used to learn  $z_i$  in the consequent part of the  $i$ -th fuzzy rule. That is,

$$\begin{aligned} z_1(t+1) &= z_1(t) - \eta \frac{\partial E_k}{\partial z_1} = z_1(t) - \eta \frac{\partial}{\partial z_1} E_k(a_1, b_1, a_2, b_2, z_1, z_2) = \\ &= z_1(t) - \eta(o^k - y^k) \frac{\alpha_1}{\alpha_1 + \alpha_2} = z_1(t) - \eta(o^k - y^k) \frac{A_1(x^k)}{A_1(x^k) + A_2(x^k)} \\ z_2(t+1) &= z_2(t) - \eta \frac{\partial E_k}{\partial z_2} = z_2(t) - \eta \frac{\partial}{\partial z_2} E_k(a_1, b_1, a_2, b_2, z_1, z_2) = \\ &= z_2(t) - \eta(o^k - y^k) \frac{\alpha_2}{\alpha_1 + \alpha_2} = z_2(t) - \eta(o^k - y^k) \frac{A_2(x^k)}{A_1(x^k) + A_2(x^k)} \end{aligned}$$

where  $\eta > 0$  is the learning constant and  $t$  indexes the number of the adjustments of  $z_i$ .

In a similar manner we can find the shape parameters (center and slope) of the membership functions  $A_1$  and  $A_2$ .

$$\begin{aligned} a_1(t+1) &= a_1(t) - \eta \frac{\partial E_k}{\partial a_1}, \\ b_1(t+1) &= b_1(t) - \eta \frac{\partial E_k}{\partial b_1}, \\ a_2(t+1) &= a_2(t) - \eta \frac{\partial E_k}{\partial a_2}, \\ b_2(t+1) &= b_2(t) - \eta \frac{\partial E_k}{\partial b_2}, \end{aligned}$$

where  $\eta > 0$  is the learning constant and  $t$  indexes the number of the adjustments of the parameters. We show now how to compute analytically the partial derivative of the error function  $E_k$  with respect to  $a_1$ , the center of the fuzzy number  $A_1$ .

$$\frac{\partial E_k}{\partial a_1} = \frac{\partial}{\partial a_1} E_k(a_1, b_1, a_2, b_2, z_1, z_2) =$$

$$\frac{1}{2} \frac{\partial}{\partial a_1} (o^k(a_1, b_1, a_2, b_2, z_1, z_2) - y^k)^2 = (o^k - y^k) \frac{\partial o^k}{\partial a_1},$$

where

$$\begin{aligned} \frac{\partial o^k}{\partial a_1} &= \frac{\partial}{\partial a_1} \left[ \frac{A_1(x^k)z_1 + A_2(x^k)z_2}{A_1(x^k) + A_2(x^k)} \right] = \\ &= \frac{\frac{\partial}{\partial a_1} \left[ \frac{\frac{z_1}{1 + \exp(b_1(x^k - a_1))} + \frac{z_2}{1 + \exp(b_2(x^k - a_2))}}{\frac{1}{1 + \exp(b_1(x^k - a_1))} + \frac{1}{1 + \exp(b_2(x^k - a_2))}} \right]}{\frac{1}{1 + \exp(b_1(x^k - a_1))} + \frac{1}{1 + \exp(b_2(x^k - a_2))}} = \\ &= \frac{\frac{\partial}{\partial a_1} \left[ \frac{z_1[1 + \exp(b_2(x^k - a_2))] + z_2[1 + \exp(b_1(x^k - a_1))]}{2 + \exp(b_1(x^k - a_1)) + \exp(b_2(x^k - a_2))} \right]}{2 + \exp(b_1(x^k - a_1)) + \exp(b_2(x^k - a_2))} = \end{aligned}$$

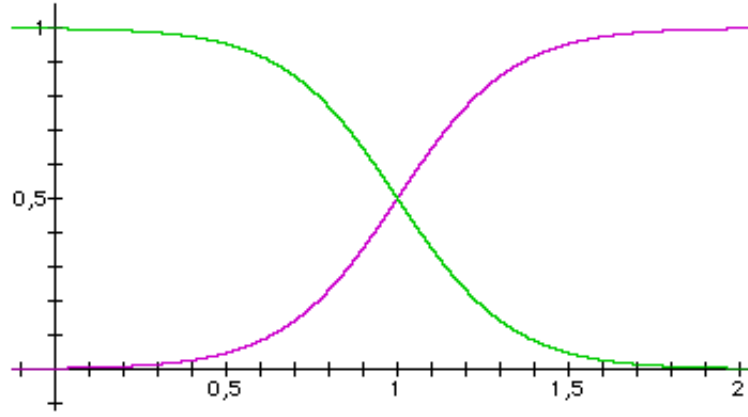


Figure 7.10: Symmetric membership functions.

$$\frac{-b_1 z_2 \epsilon_2 (2 + \epsilon_1 + \epsilon_2) + b_1 \epsilon_1 (z_1 (1 + \epsilon_2) + z_2 (1 + \epsilon_1))}{(2 + \epsilon_1 + \epsilon_2)^2},$$

where we used the notations  $\epsilon_1 = \exp(b_1(x^k - a_1))$  and  $\epsilon_2 = \exp(b_2(x^k - a_2))$ .

The learning rules are simplified if we use the following fuzzy partition

$$A_1(x) = \frac{1}{1 + \exp(-b(x - a))}, \quad A_2(x) = \frac{1}{1 + \exp(b(x - a))},$$

where  $a$  and  $b$  are the *shared* parameters of  $A_1$  and  $A_2$ . In this case the equation

$$A_1(x) + A_2(x) = 1,$$

holds for all  $x$  from the domain of  $A_1$  and  $A_2$ . The weight adjustments are defined as follows

$$z_1(t+1) = z_1(t) - \eta \frac{\partial E_k}{\partial z_1} = z_1(t) - \eta(o^k - y^k)A_1(x^k),$$

$$z_2(t+1) = z_2(t) - \eta \frac{\partial E_k}{\partial z_2} = z_2(t) - \eta(o^k - y^k)A_2(x^k),$$

$$a(t+1) = a(t) - \eta \frac{\partial E_k(a, b)}{\partial a},$$

$$b(t+1) = b(t) - \eta \frac{\partial E_k(a, b)}{\partial b},$$

where

$$\frac{\partial E_k(a, b)}{\partial a} = (o^k - y^k) \frac{\partial o^k}{\partial a} =$$

$$(o^k - y^k) \frac{\partial}{\partial a} [z_1 A_1(x^k) + z_2 A_2(x^k)] =$$

$$(o^k - y^k) \frac{\partial}{\partial a} [z_1 A_1(x^k) + z_2 (1 - A_1(x^k))] =$$

$$(o^k - y^k)(z_1 - z_2) \frac{\partial A_1(x^k)}{\partial a} =$$

$$(o^k - y^k)(z_1 - z_2) b A_1(x^k) (1 - A_1(x^k)) =$$

$$(o^k - y^k)(z_1 - z_2) b A_1(x^k) A_2(x^k).$$

and

$$\frac{\partial E_k(a, b)}{\partial b} = (o^k - y^k)(z_1 - z_2) \frac{\partial A_1(x^k)}{\partial b} =$$

$$-(o^k - y^k)(z_1 - z_2)(x^k - a) A_1(x^k) A_2(x^k).$$

Jang [260] showed that fuzzy inference systems with simplified fuzzy if-then rules are universal approximators, i.e. they can approximate any continuous function on a compact set to arbitrary accuracy. This means that the more fuzzy terms (and consequently more rules) are used in the rule base, the closer is the output of the fuzzy system to the desired values of the function to be approximated.



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