An LLL algorithm for module lattices

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Institut Fourier,
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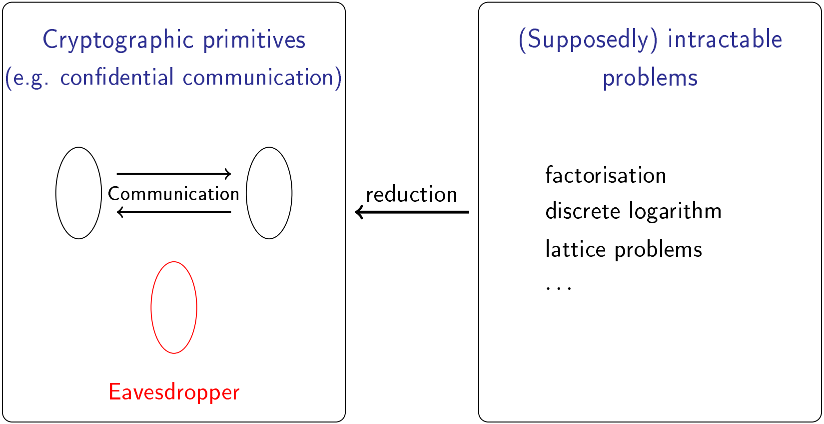
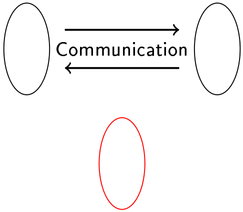
Cryptography and hard problems



Cryptography and hard problems

Cryptographic primitives
(e.g. confidential communication)



Eavesdropper

← reduction

(Supposedly) intractable problems

- factorisation
- discrete logarithm
- $\boxed{\text{lattice problems}}$
- ...

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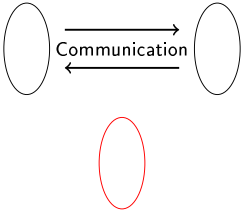


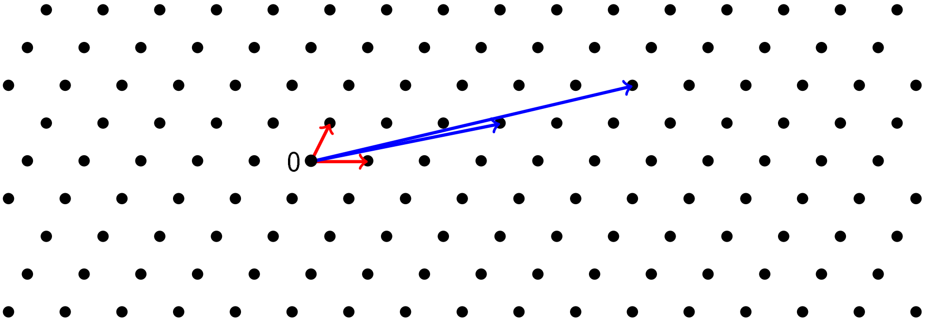
Eavesdropper

reduction ←

(Supposedly) intractable problems
in a quantum world

~~factorisation~~
~~discrete logarithm~~
lattice problems
...

Lattices

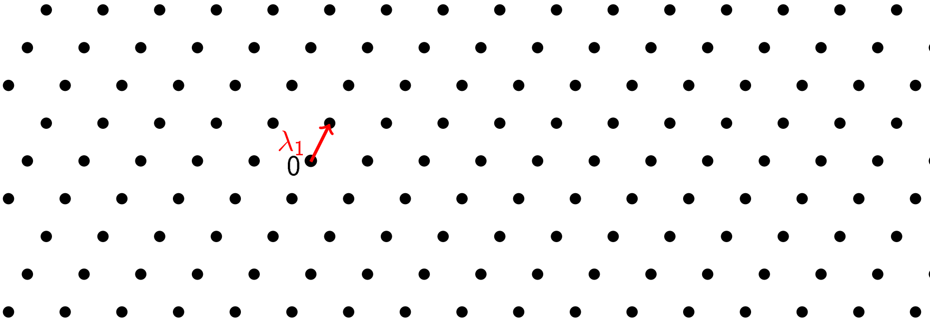


Lattice

A (full-rank) lattice L is a subset of $\mathbb{R}^n$ of the form $L = \{Bx \mid x \in \mathbb{Z}^n\}$, with $B \in \mathbb{R}^{n \times n}$ invertible. B is a basis of L.

$\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 17 & 10 \\ 4 & 2 \end{pmatrix}$ are two bases of the above lattice.

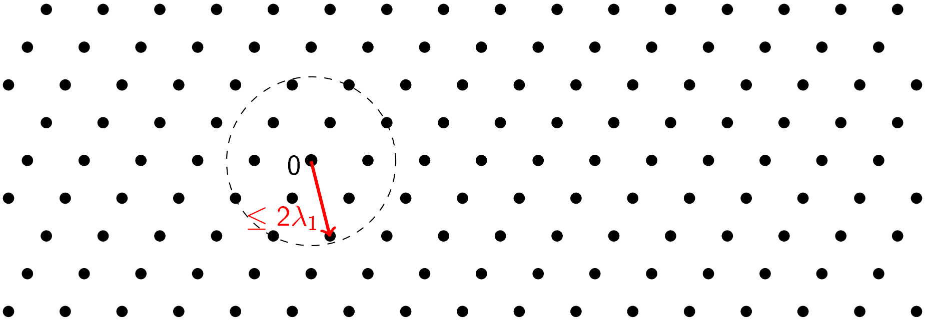
Lattice problems



Shortest Vector Problem (SVP)

Find a shortest (in Euclidean norm) non-zero vector.
Its Euclidean norm is denoted λ_1.

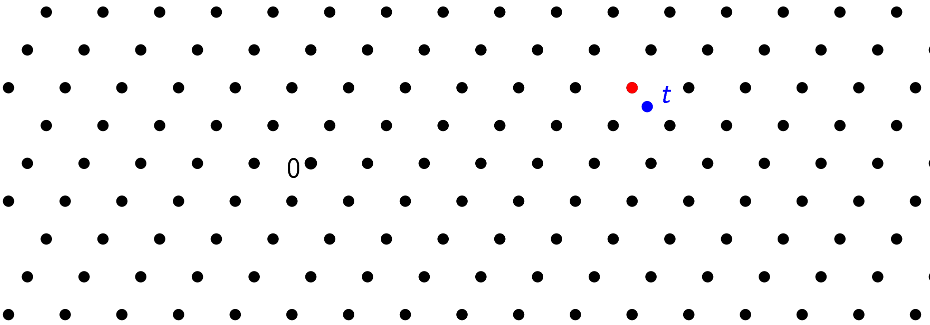
Lattice problems



Approximate Shortest Vector Problem (approx-SVP)

Find a short (in Euclidean norm) non-zero vector.
(e.g. of norm $\leq 2\lambda_1$).

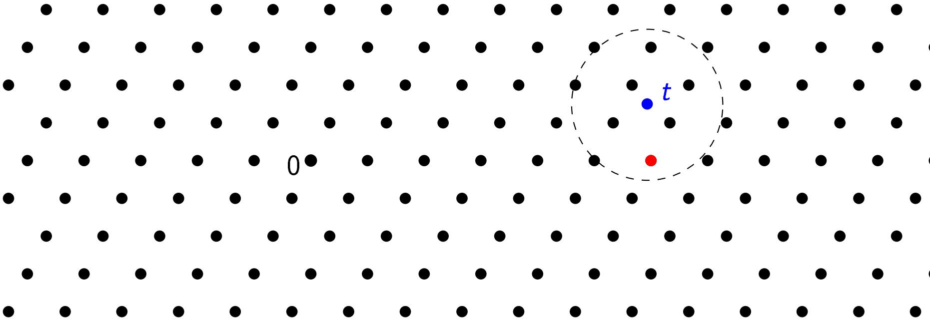
Lattice problems



Closest Vector Problem (CVP)

Given a target point t, find a point of the lattice closest to t.

Lattice problems

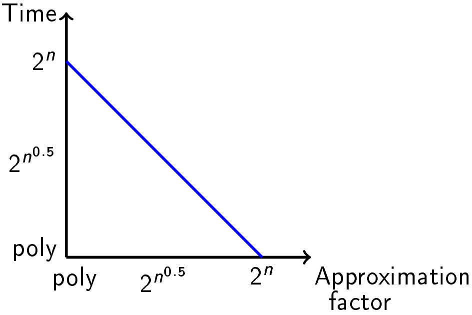


Approximate Closest Vector Problem (approx-CVP)

Given a target point t, find a point of the lattice close to t.

Hardness of lattice problems

Best Time/Approximation trade-off for SVP and CVP (even quantumly):
BKZ algorithm [Sch87,SE94]



[Sch87] C.-P. Schnorr. A hierarchy of polynomial time lattice basis reduction algorithms. TCS.

[SE94] C.-P. Schnorr and M. Euchner. Lattice basis reduction: improved practical algorithms and solving subset sum problems. Mathematical programming.

Hardness of lattice problems

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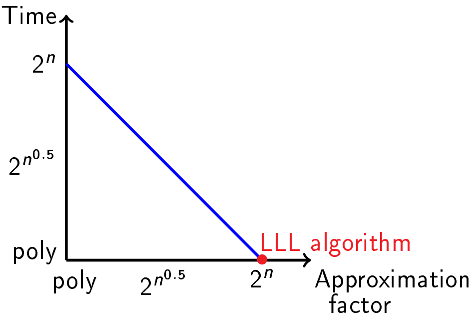

[LLL82] A. K. Lenstra, H. W. Lenstra, L. Lovász. Factoring polynomials with rational coefficients. Mathematische Annalen.

Structured lattices

Motivation

Schemes using lattices are usually not very efficient
(storage: n^2, matrix-vector mult: n^2)
$\Rightarrow$ improve efficiency using structured lattices

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- 26 candidates (2nd round)
- 12 lattice-based
- 11 using structured lattices

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- 26 candidates (2nd round)
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- 11 using structured lattices

	Frodo (lvl 1) (unstructured lattices)	Kyber (lvl 1) (structured lattices)
secret key size (in Bytes)	19 888	1 632
public key size (in Bytes)	9 616	800

Ideal lattices

$$M_{\mathbf{a}} = \begin{pmatrix} a_0 & a_{n-1} & \cdots & a_1 \\ a_1 & a_0 & \cdots & a_2 \\ \vdots & \ddots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \dots & a_0 \end{pmatrix}$$

Ideal lattices

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multiplication by $a_0 + a_1 X + \cdots + a_{n-1} X^{n-1}$ mod $X^n - 1$

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$R = \mathbb{Z}[X]/(X^n - 1)$

$$\sigma : \quad R \to \mathbb{R}^n$$
$$\sum_{i=0}^{n-1} a_i X^i \mapsto (a_0, \cdots, a_{n-1})$$

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$$M_{\mathbf{a}} = \begin{pmatrix} | & | & & | \\ \sigma(\mathbf{a}) & \sigma(X\mathbf{a}) & \cdots & \sigma(X^{n-1}\mathbf{a}) \\ | & | & & | \end{pmatrix}$$

$$\boxed{\mathcal{L}(M_{\mathbf{a}}) = \sigma(\langle \mathbf{a} \rangle)}$$

(principal) ideal lattice

Ideal lattices (2)

Notations

- P irreducible monic polynomial of degree d
- $K = \mathbb{Q}[X]/P$
- R ring of integers of K

An ideal lattice is $\sigma(I)$ for I an ideal of R

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An ideal lattice is $\sigma(I)$ for I an ideal of R

For simplicity in this talk

- only principal ideals $I = \langle a \rangle$
- $P = X^d + 1$ $(d = 2^k)$
  or $P = X^d - X - 1$ (d prime)

Module lattices

Definition

A (full rank) free module $\mathcal{M} \subset K^k$ is

$$\mathcal{M} = \{\sum_i x_i \vec{b}_i \ : \ x_i \in R\},$$

where the $\vec{b}_i \in K^k$ are linearly independent vectors and $1 \leq i \leq k$.
k is the rank of $\mathcal{M}$.

A (free) module lattice is $\sigma(\mathcal{M}) \subset \mathbb{R}^{kd}$

where $\sigma((x_1, \cdots, x_k)) = \sigma(x_1) \| \cdots \| \sigma(x_k)$

Duality of module lattices

Recall

- M_a = matrix of multiplication by a
- $\mathcal{M} = \{\sum_i x_i \vec{b}_i \ : \ x_i \in R\}$, with $\vec{b}_i = (a_{1i}, \cdots, a_{ki})$.

$$d \updownarrow \begin{array}{|c|c|c|c|} \hline M_{a_{11}} & M_{a_{12}} & \cdots & M_{a_{1k}} \\ \hline M_{a_{21}} & M_{a_{22}} & \cdots & M_{a_{2k}} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline M_{a_{k1}} & M_{a_{k2}} & \cdots & M_{a_{kk}} \\ \hline \end{array} \updownarrow dk$$

Basis of $\mathcal{L}(\mathcal{M})$ over $\mathbb{Z}$

(dim $n = dk$)

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$$

Basis of $\mathcal{L}(\mathcal{M})$ over $\mathbb{Z}$
(dim $n = dk$)

Basis of $\mathcal{M}$ over R
(dim k)

Duality of module lattices

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Basis of $\mathcal{L}(\mathcal{M})$ over $\mathbb{Z}$

(dim $n = dk$)

Typically $500 \leq dk \leq 1000$

$$1 \updownarrow \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix} \updownarrow k$$

Basis of $\mathcal{M}$ over R

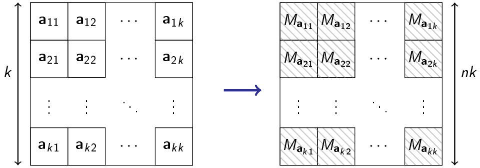
(dim k)

Typically $k \leq 10$

Canonical embedding

Reminder

$K = \mathbb{Q}[X]/P(X)$



dimension k over R — dimension nk over $\mathbb{Z}$

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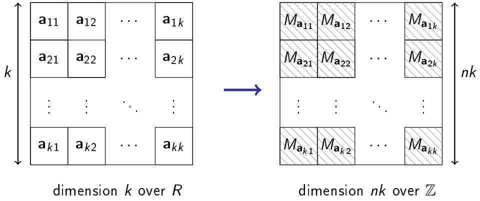

Coefficient embedding

$$\begin{array}{rccl} \sigma : & K & \to & \mathbb{R}^n \\ & \mathbf{a} = a_0 + a_1 X + \cdots + a_{n-1} X^{n-1} & \mapsto & (a_0, a_1, \cdots, a_{n-1})^T \end{array}$$

$$\mathbf{a} \longrightarrow M_{\mathbf{a}} = \begin{pmatrix} | & | & & | \\ \sigma(\mathbf{a}) & \sigma(X\mathbf{a}) & \cdots & \sigma(X^{n-1}\mathbf{a}) \\ | & | & & | \end{pmatrix}$$

Canonical embedding

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$\alpha_1, \cdots, \alpha_n$ roots of P

$$k \left\updownarrow \begin{array}{|c|c|c|c|} \hline \mathbf{a}_{11} & \mathbf{a}_{12} & \cdots & \mathbf{a}_{1k} \\ \hline \mathbf{a}_{21} & \mathbf{a}_{22} & \cdots & \mathbf{a}_{2k} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathbf{a}_{k1} & \mathbf{a}_{k2} & \cdots & \mathbf{a}_{kk} \\ \hline \end{array} \right. \longrightarrow \left. \begin{array}{|c|c|c|c|} \hline M_{\mathbf{a}_{11}} & M_{\mathbf{a}_{12}} & \cdots & M_{\mathbf{a}_{1k}} \\ \hline M_{\mathbf{a}_{21}} & M_{\mathbf{a}_{22}} & \cdots & M_{\mathbf{a}_{2k}} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline M_{\mathbf{a}_{k1}} & M_{\mathbf{a}_{k2}} & \cdots & M_{\mathbf{a}_{kk}} \\ \hline \end{array} \right\updownarrow nk$$

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Is SVP still hard when restricted to module lattices?

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Module lattices

- large dimension over $\mathbb{Z}$
- small dimension over R

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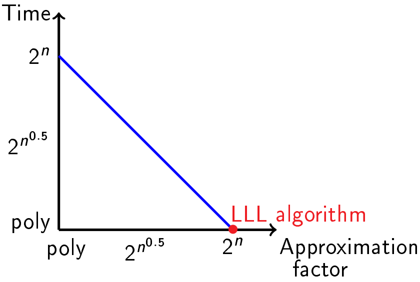

Lattice reduction over $\mathbb{Z}$

Module lattices

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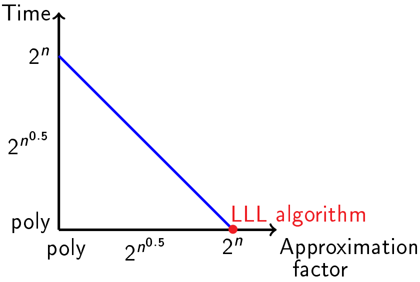

Lattice reduction over $\mathbb{Z}$

Module lattices

- large dimension over $\mathbb{Z}$
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Can we extend the LLL algorithm to lattices over R?

Previous works and result

[Nap96] LLL for some specific number fields
no bound on quality / run-time

[Nap96] H. Napias. A generalization of the LLL-algorithm over Euclidean rings or orders. Journal de théorie des nombres de Bordeaux.

Previous works and result

[Nap96]	LLL for some specific number fields no bound on quality / run-time
[FP96]	LLL for any number fields no bound on quality / run-time bound on run-time for totally real number fields

[FP96] C. Fieker, M. E. Pohst. Lattices over number fields. ANTS.

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[Cam16]	LLL for Euclidean (for $\|\cdot\|_2$ norm) imaginary quadratic fields bound on run-time and on quality

[Cam17] T. Camus. Méthodes algorithmiques pour les réseaux algébriques. Thèse de doctorat.

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[KL17] T. Kim, C. Lee. Lattice reductions over euclidean rings with applications to cryptanalysis. IMACC.

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[KL17]	LLL for norm-Euclidean fields bound on run-time but not on quality bound on quality for biquadratic fields
[LPSW19]	LLL for any number field bound on quality and run-time if oracle solving CVP in a fixed lattice (depending on R)

[LPSW19] C. Lee, A. Pellet-Mary, D. Stehlé, A. Wallet. An LLL algorithm for module lattices. To appear at Asiacrypt 2019.

Outline of the talk

1. LLL over R^2
   - QR factorization
   - Euclidean division

2. LLL over R^k

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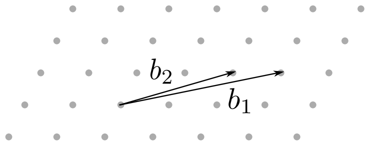
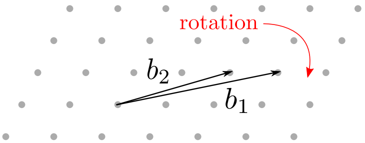
Lagrange-Gauss algorithm (over $\mathbb{Z}$)



$$M = \begin{pmatrix} 10 & 7 \\ 2 & 2 \end{pmatrix}$$

Lagrange-Gauss algorithm (over $\mathbb{Z}$)

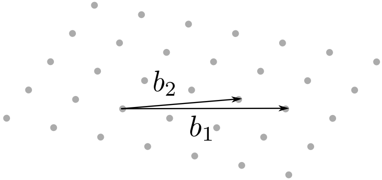


rotation

$$M = \begin{pmatrix} 10 & 7 \\ 2 & 2 \end{pmatrix}$$

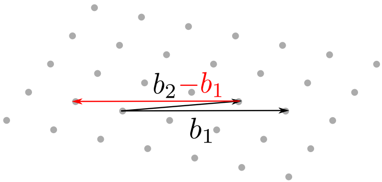
Compute QR factorization

Lagrange-Gauss algorithm (over $\mathbb{Z}$)



$$M = \begin{pmatrix} 10.2 & 7.3 \\ 0 & 0.6 \end{pmatrix}$$

Lagrange-Gauss algorithm (over $\mathbb{Z}$)

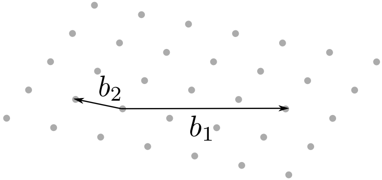


reduce b_2 with b_1

$$M = \begin{pmatrix} 10.2 & 7.3 \\ 0 & 0.6 \end{pmatrix}$$

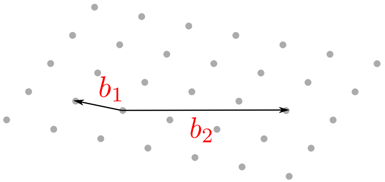
"Euclidean division" (over $\mathbb{R}$) of 7.3 by 10.2

Lagrange-Gauss algorithm (over $\mathbb{Z}$)



$$M = \begin{pmatrix} 10.2 & -2.9 \\ 0 & 0.6 \end{pmatrix}$$

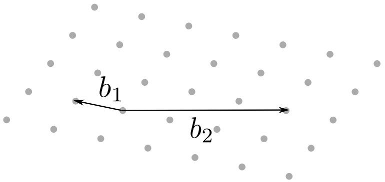
Lagrange-Gauss algorithm (over $\mathbb{Z}$)



swap

$$M = \begin{pmatrix} -2.9 & 10.2 \\ 0.6 & 0 \end{pmatrix}$$

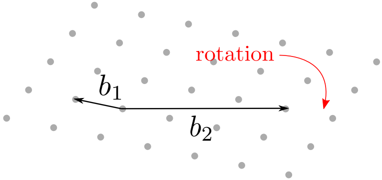
Lagrange-Gauss algorithm (over $\mathbb{Z}$)



start again

$$M = \begin{pmatrix} -2.9 & 10.2 \\ 0.6 & 0 \end{pmatrix}$$

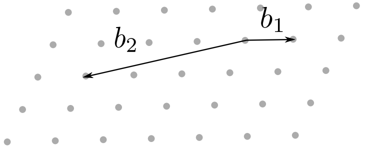
Lagrange-Gauss algorithm (over $\mathbb{Z}$)



rotation

$$M = \begin{pmatrix} -2.9 & 10.2 \\ 0.6 & 0 \end{pmatrix}$$

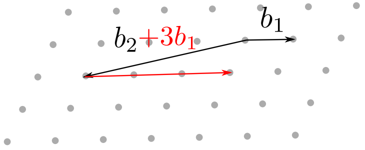
Lagrange-Gauss algorithm (over $\mathbb{Z}$)



rotation

$$M = \begin{pmatrix} 3 & -10 \\ 0 & -2 \end{pmatrix}$$

Lagrange-Gauss algorithm (over $\mathbb{Z}$)

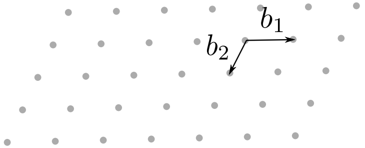


reduce b_2 with b_1

$$M = \begin{pmatrix} 3 & -10 \\ 0 & -2 \end{pmatrix}$$

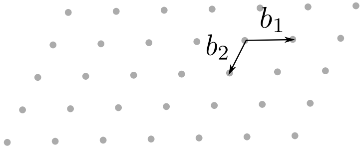
"Euclidean division" (over $\mathbb{R}$) of -10 by 3

Lagrange-Gauss algorithm (over $\mathbb{Z}$)



$$M = \begin{pmatrix} 3 & -1 \\ 0 & -2 \end{pmatrix}$$

Lagrange-Gauss algorithm (over $\mathbb{Z}$)



$$M = \begin{pmatrix} 3 & -1 \\ 0 & -2 \end{pmatrix}$$

For the Lagrange-Gauss algorithm over R, we need

- Rotation (i.e., QR factorization)
- Euclidean division

QR factorization over R

Inner product over R

For $\vec{a} = (a_1, \cdots, a_k) \in K^k$ and $\vec{b} = (b_1, \cdots, b_k) \in K^k$,

$$\boxed{\langle \vec{a}, \vec{b} \rangle_K = \sum_i a_i \overline{b_i} \in K} \quad (\text{or } K_{\mathbb{R}} = K \otimes \mathbb{R})$$

Inner product over R

For $\vec{a} = (a_1, \cdots, a_k) \in K^k$ and $\vec{b} = (b_1, \cdots, b_k) \in K^k$,

$$\boxed{\langle \vec{a}, \vec{b} \rangle_K = \sum_i a_i \overline{b_i} \in K} \quad (\text{or } K_{\mathbb{R}} = K \otimes \mathbb{R})$$

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- $\mathrm{Tr}(\|\vec{a}\|_K^2) = \|\sigma(\vec{a})\|_2^2$

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QR factorization over R

Let $B = (b_1, \cdots, b_k) \in K^k$, define

$$b_i^* = b_i - \sum_{j<i} \mu_{ij} b_j^*, \quad \text{with } \mu_{ij} = \frac{\langle b_i, b_j^* \rangle_K}{\langle b_j^*, b_j^* \rangle_K}$$

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Properties

- R is triangular
- $\overline{Q}^T Q = I_k$
- $\langle R\vec{u}, R\vec{v} \rangle = \langle B\vec{u}, B\vec{v} \rangle$

Euclidean division over R

Non-Euclidean rings

Objective

Input: $a, b \in K$, $a \neq 0$
Output: $r \in R$ such that $\|b + ra\| \leq \|a\|/2$

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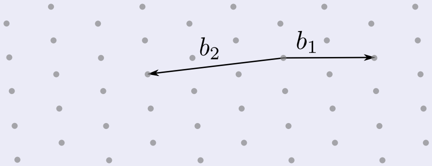

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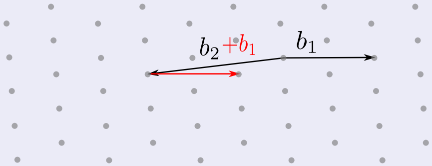

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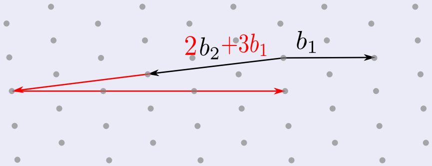

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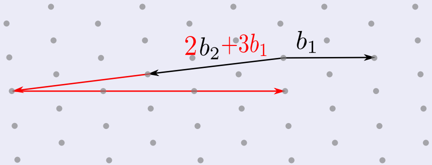
$\Rightarrow$ sufficient for Gauss' algo



The Log space

Reminder: $\sigma(r) = (r(\alpha_1), \cdots, r(\alpha_d))^T$

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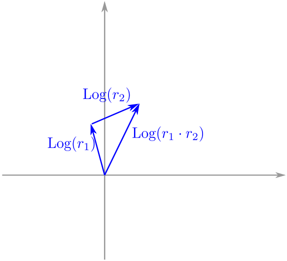

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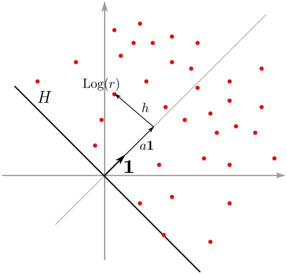

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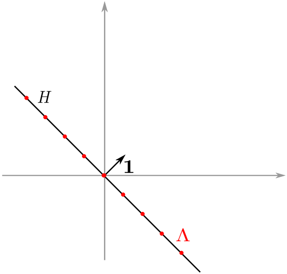

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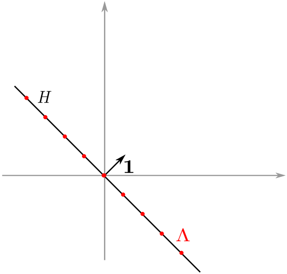

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- $\|r\| \simeq 2^{\|\mathrm{Log}\, r\|_\infty}$

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New objective

Find $x, y \in R$ such that

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- $\|\operatorname{Log}(y)\|_\infty \leq O(\log d)$

Idea

Objective: find $x, y \in R$ s.t.

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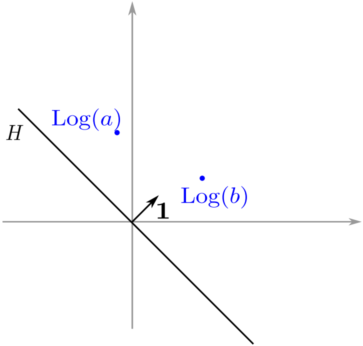

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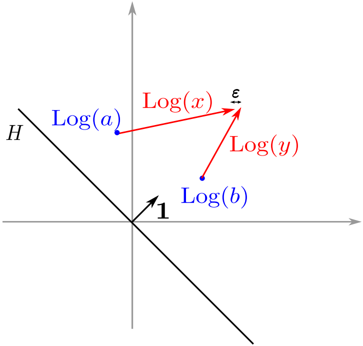

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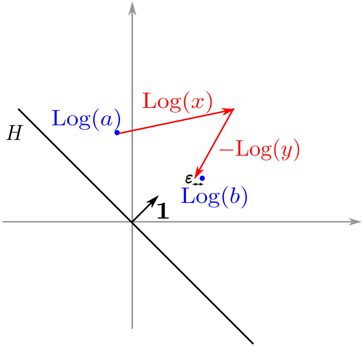

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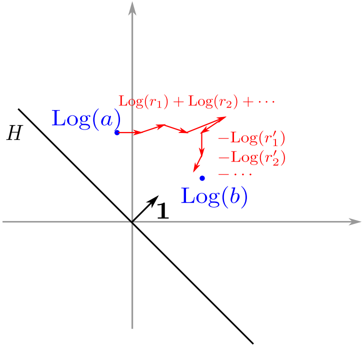

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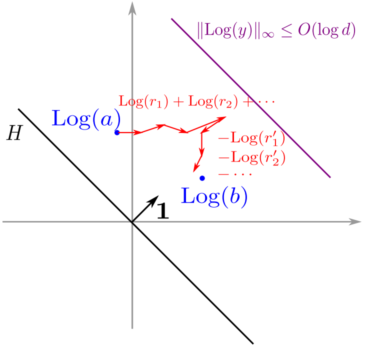

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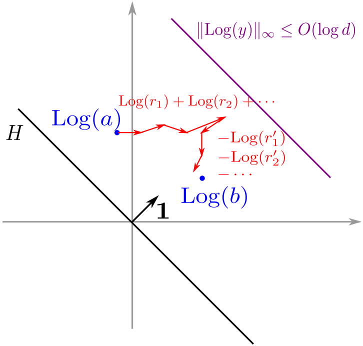

Solve **exact** CVP in L with target t

$$L = \begin{pmatrix} \Lambda & \mathrm{Log}\, r_1 & \cdots & \mathrm{Log}\, r_{n^2} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad t = \begin{pmatrix} \mathrm{Log}(b/a) \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

(L is fixed and independent of a and b)

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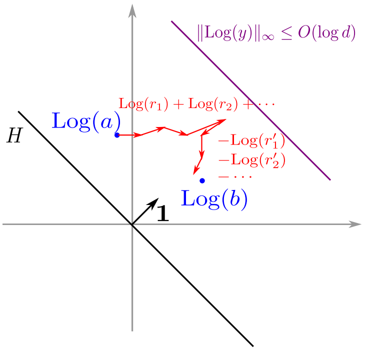

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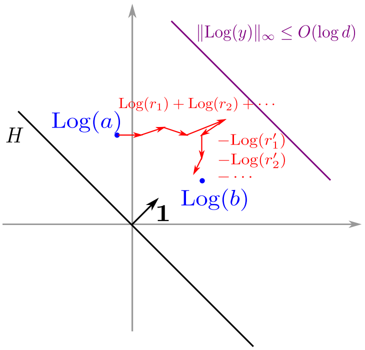

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Complexity

Poly time
(with the oracle)

Under the carpet

- Heuristics
  - maths justification
  - numerical experiments (in very small dimension)
- Any module / ideal
  - use pseudo-basis
  - add class group to L (cf [Buc88])
  - decompose ideals in class group $\Rightarrow$ quantum algorithm
- Full LLL algo over R^2
  - Lovász' swap condition
  - switch between $\mathcal{N}(\cdot)$ and $\|\cdot\|$
  - handling bit sizes

Outline of the talk

1. LLL over R^2
   - QR factorization
   - Euclidean division

2. LLL over R^k

Going to dimension k

LLL over $\mathbb{Z}$

- γ'-SVP in dim k
  $\leq$ 1-SVP in dim 2
  - $\gamma' = 2^{O(k)}$
  - poly time

[LLL82] A. K. Lenstra, H. W. Lenstra, L. Lovász. Factoring polynomials with rational coefficients. Mathematische Annalen.

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LLL over $R = \mathbb{Z}[X]/(X^d + 1)$

- γ'-SVP in rank-k
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- Algorithm for γ-SVP in rank-2
  - $\gamma = 2^{(\log d)^{O(1)}}$
  - heuristic, quantum
  - poly time if oracle solving CVP in a fixed lattice (depending only on R)

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needs QR-factorisation

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Summary and impact

LLL algorithm for power-of-two cyclotomic fields (or NTRUPrime)

- Approx: quasi-poly(d)$^{O(k)} = 2^{(\log d)^{O(1)} \cdot k}$
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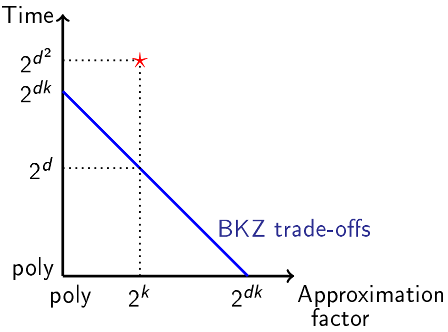

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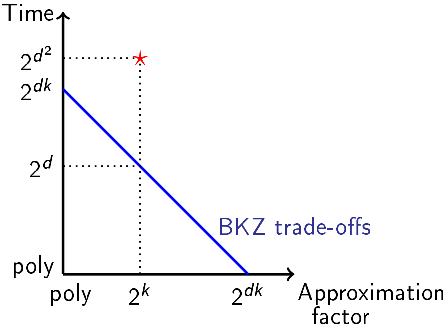

$\Rightarrow$ theoretical result (not practical)

Conclusion

Open problems:

- Better understanding of the lattice L
  - reduce its dimension to $\widetilde{O}(d)$?
  - prove the heuristics?
  - better CVP solver for L?

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  - sieving/enumeration in modules?

Thank you

[MS19] T. Mukherjee, N. Stephens-Davidowitz. Lattice Reduction for Modules, or How to Reduce ModuleSVP to ModuleSVP, ePrint.