Algebraic lattices in cryptography

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La Rochelle

Motivation: cryptography

Cryptographic primitives

public key encryption | signature | homomorphic encryption | $\cdots$

error correcting codes | lattices | isogenies

factoring | discrete logarithm | $\cdots$

(Supposedly intractable) algorithmic problems

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(Supposedly intractable) algorithmic problems
in a quantum world

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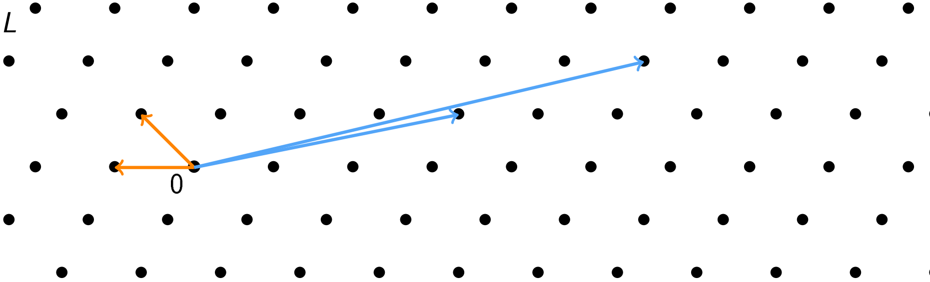
(Supposedly intractable) algorithmic problems in a quantum world

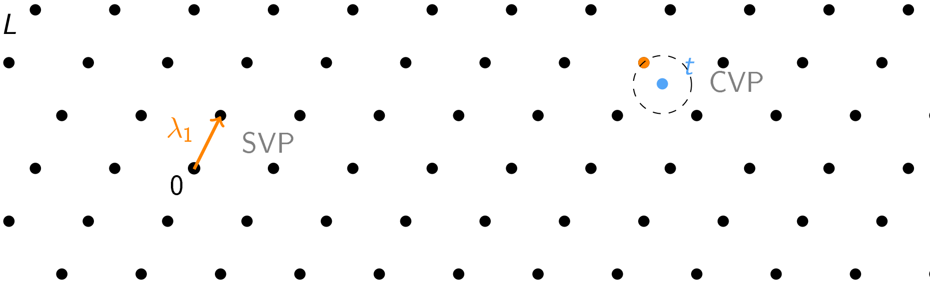
Lattices

Lattices



- $L = \{Bx \mid x \in \mathbb{Z}^n\}$ is a lattice
- $B \in \mathrm{GL}_n(\mathbb{R})$ is a basis
- n is the dimension of L

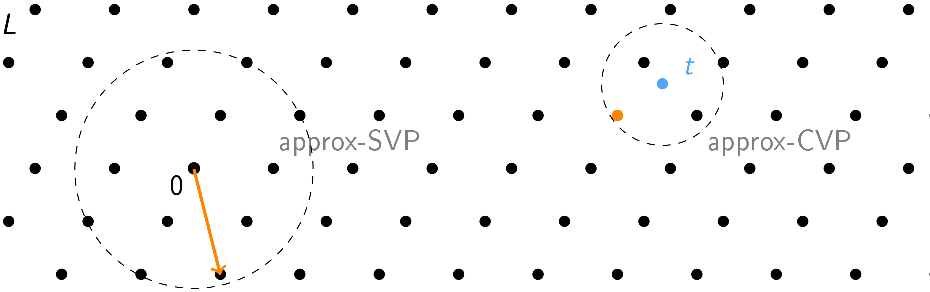
Algorithmic problems



SVP : Shortest Vector Problem

CVP : Closest Vector Problem

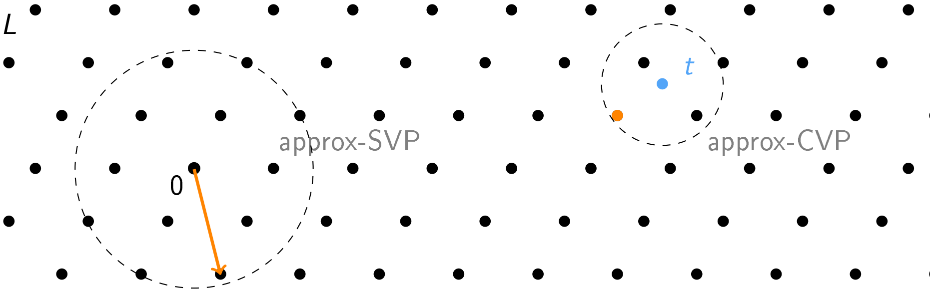
Algorithmic problems



approx-SVP : Shortest Vector Problem

approx-CVP : Closest Vector Problem

Algorithmic problems



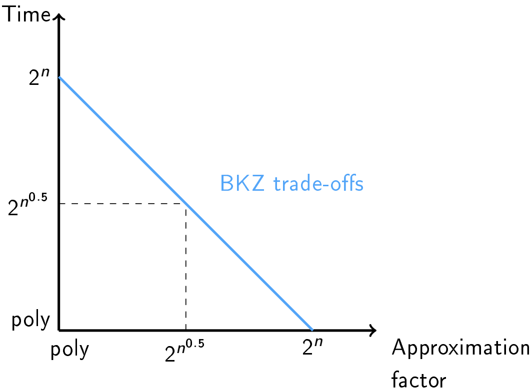
approx-SVP : Shortest Vector Problem   approx-CVP : Closest Vector Problem

Supposedly hard to solve when n is large (input: a bad basis of L)

- even with a quantum computer
- even with a small approximation factor ($\mathrm{poly}(n)$)

Hardness of SVP and CVP

Best Time/Approximation trade-off for SVP, CVP (even quantumly):
BKZ algorithm [Sch87,SE94]



[Sch87] C.-P. Schnorr. A hierarchy of polynomial time lattice basis reduction algorithms. TCS.

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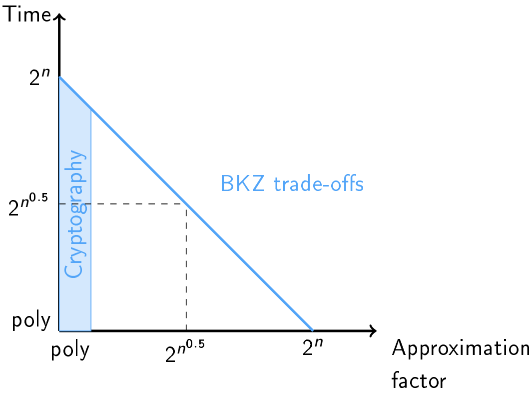

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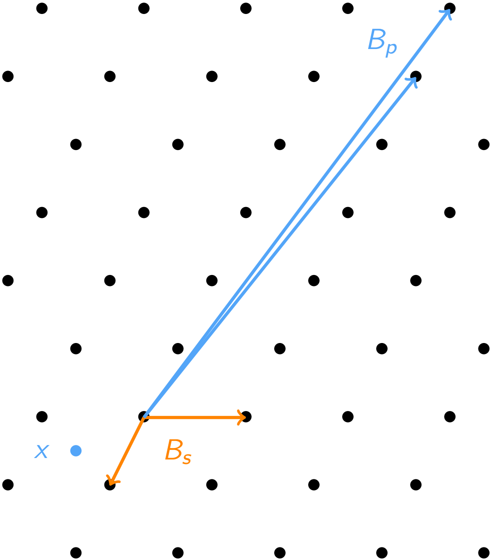
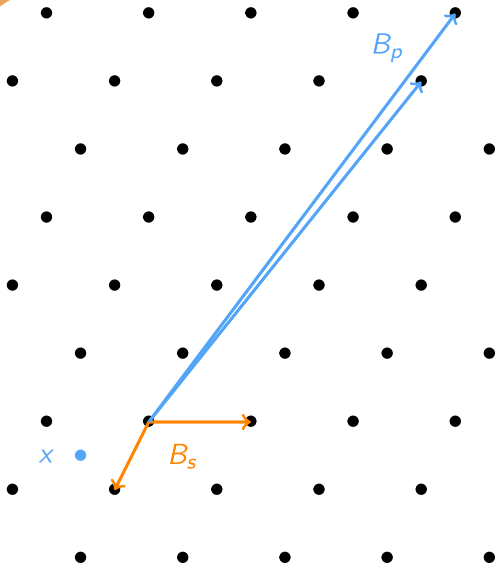
Public key encryption from lattices



$\text{pk} = (B_p, x)$
$\text{sk} = B_s$

Public key encryption from lattices



$\textcolor{blue}{\text{pk} = (B_p, x)}$
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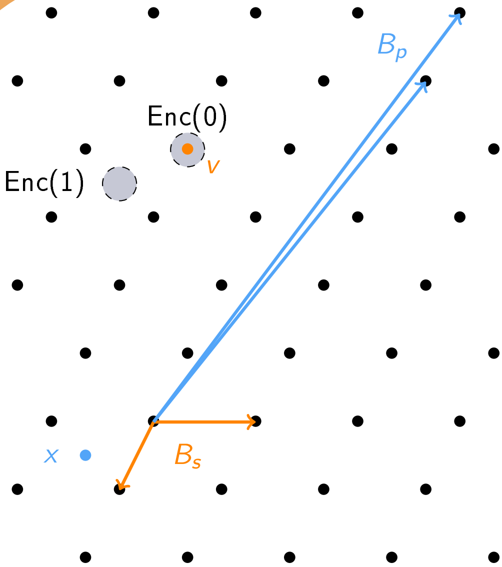
message: $m \in \{0, 1\}$

Public key encryption from lattices



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Encryption(m, pk):

- sample random $v \in L$
- sample small $e \in \mathbb{R}^n$
- return $c = v + e + m \cdot x$

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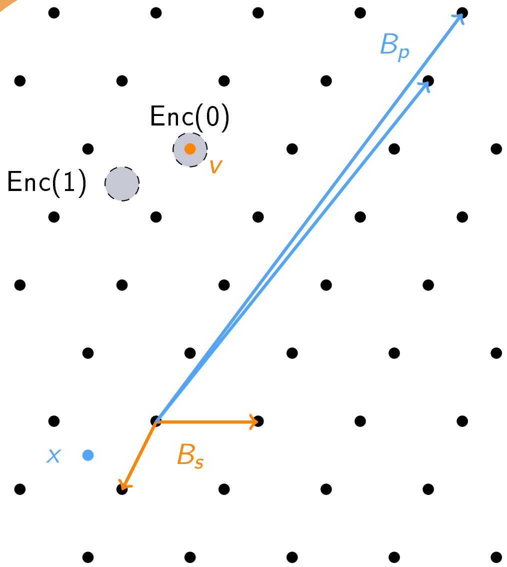

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Decryption(c, sk):

- find $w \in L$ closest to c
- if c is very close to w, return $m = 0$
- otherwise return $m = 1$

Structured lattices

Why?

Motivation

Schemes using lattices are usually not efficient

(storage: n^2, matrix-vector mult: n^2)

$\Rightarrow$ improve efficiency using structured lattices

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$\Rightarrow$ improve efficiency using structured lattices

Two examples: (submitted to the NIST post-quantum standardization process)

	Frodo (unstructured lattices)	Kyber (structured lattices)
secret key size (in Bytes)	19 888	1 632
public key size (in Bytes)	9 616	800

Structured lattices: example

$$M_a = \begin{pmatrix} a_1 & -a_n & \cdots & -a_2 \\ a_2 & a_1 & \cdots & -a_3 \\ \vdots & \ddots & \ddots & \vdots \\ a_n & a_{n-1} & \dots & a_1 \end{pmatrix}$$

basis of a special case of
ideal lattice

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basis of a special case of
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$$\begin{array}{c|cccc|c} n & M_{a_{11}} & M_{a_{12}} & \cdots & M_{a_{1m}} & \\ & M_{a_{21}} & M_{a_{22}} & \cdots & M_{a_{2m}} & nm \\ & \vdots & \vdots & \ddots & \vdots & \\ & M_{a_{m1}} & M_{a_{m2}} & \cdots & M_{a_{mm}} & \end{array}$$

basis of a special case of
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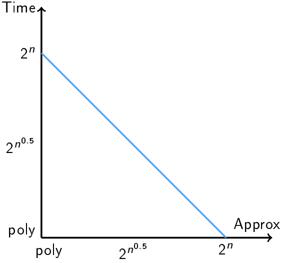
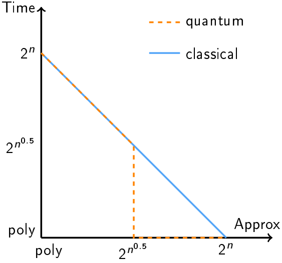
Is SVP still hard when restricted to ideal/module lattices?

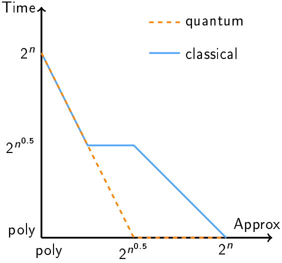
SVP in modules and ideals



Modules
(rank ≥ 2)

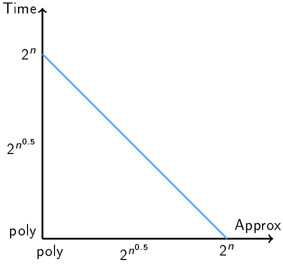


ideals
(in cyclotomic fields)



ideals
(with $2^{O(n)}$ pre-processing)

SVP in modules and ideals



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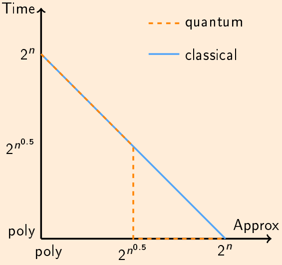

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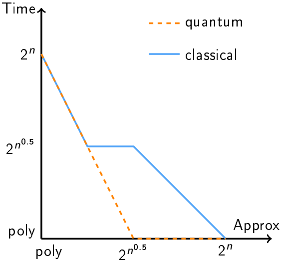

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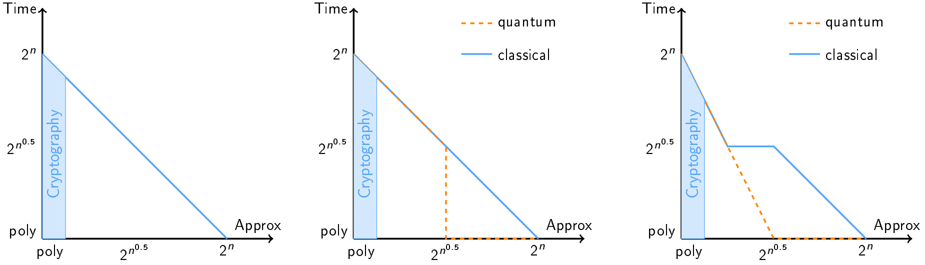
Impact on cryptography



Modules
(rank ≥ 2)

ideals
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Impact on cryptography

$$
\begin{tikzcd}
\text{Many protocols} \arrow[d, leftrightarrow] & & \text{Few protocols} \arrow[d, leftrightarrow] \\
\text{SVP in Modules of rank} \geq 2 & & \text{SVP in Ideals} \arrow[ll, "\geq"']
\end{tikzcd}
$$

Algorithms for ideal lattices

History: algorithms for ideal-SVP

[RBV04]: principal ideals in small dimension

[RBV04] G. Rekaya, J.-C. Belfiore, E. Viterbo. A very efficient lattice reduction tool on fast fading channels. ISITA.

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[CDPR16] R. Cramer, L. Ducas, C. Peikert and O. Regev. Recovering short generators of principal ideals in cyclotomic rings. Eurocrypt.

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[PHS19] A. Pellet-Mary, G. Hanrot, D. Stehlé. Approx-SVP in ideal lattices with pre-processing. Eurocrypt.

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Math background

Notation

$K = \mathbb{Q}[X]/(X^n + 1)$, with $n = 2^k$ (or any cyclotomic field)

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- Units: $O_K^\times = \{a \in O_K \mid \exists b \in O_K, ab = 1\}$

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$O_K = \mathbb{Z}[X]/(X^n+1)$

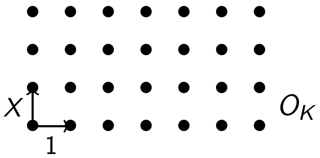
- Units: $O_K^\times = \{a \in O_K \mid \exists b \in O_K, ab = 1\}$
- Principal ideals: $\langle g \rangle = \{gr \mid r \in O_K\}$
  - g is a generator of $\langle g \rangle$
  - $\{$ generators of $\langle g \rangle$ $\} = \{gu \,|\, u \in O_K^\times\}$

Why is $\langle g \rangle$ a lattice?

O_K is a lattice

$$\begin{aligned} O_K = \mathbb{Z}[X]/(X^n+1) &\to \mathbb{C}^n \\ r(X) &\mapsto (r(\alpha_1), r(\alpha_2), \dots, r(\alpha_n)), \end{aligned}$$

where $\alpha_1, \dots, \alpha_n$ are the roots of $X^n + 1$ in $\mathbb{C}$



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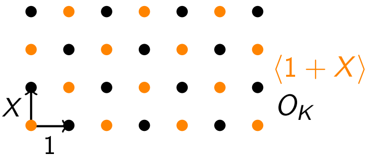
$$\begin{cases} \langle g \rangle \subseteq O_K \simeq \mathbb{Z}^n \\ \text{stable by '+' and '}-\text{'} \end{cases} \Rightarrow \text{ideal lattice}$$



Objective and first idea [CDPR,CGS14]

Objective: Given a basis of $\langle g \rangle$, find a (somehow) small element $gr \in \langle g \rangle$

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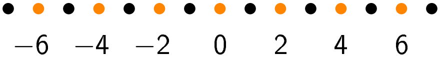
- If n $= 1$: e.g. $\langle 2 \rangle \Rightarrow 2$ and -2 are the smallest elements.



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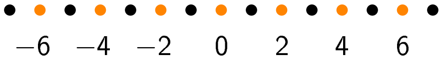
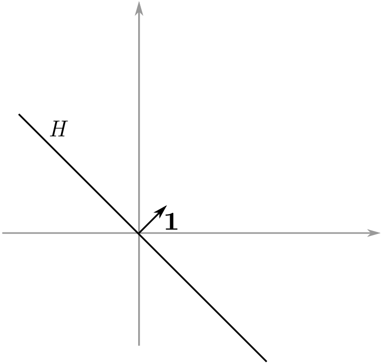


- For larger n: one of the generators is somehow small

The Log space

$\mathrm{Log} : O_K \to \mathbb{R}^n$ (take the log of every coordinate)

Let $\mathbf{1} = (1, \cdots, 1)$ and $H = \mathbf{1}^{\perp}$.



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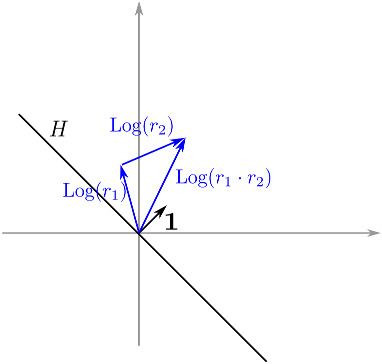
Properties ($r \in O_K$)

$\operatorname{Log} r = h + a \cdot 1$, with $h \in H$

- $\operatorname{Log}(r_1 \cdot r_2) = \operatorname{Log}(r_1) + \operatorname{Log}(r_2)$



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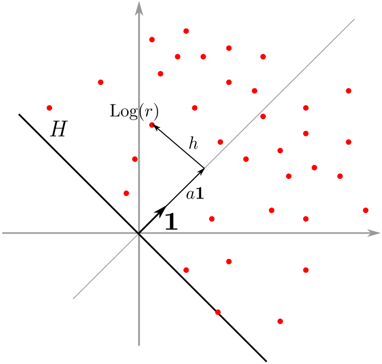

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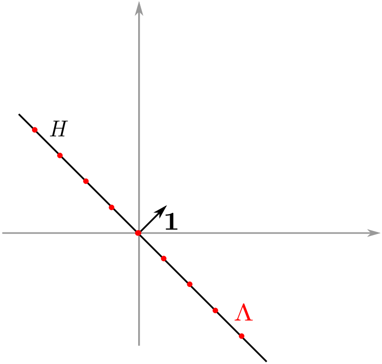

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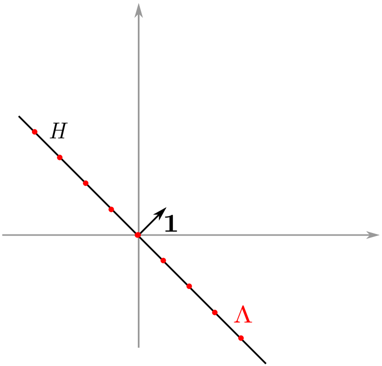

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The Log unit lattice

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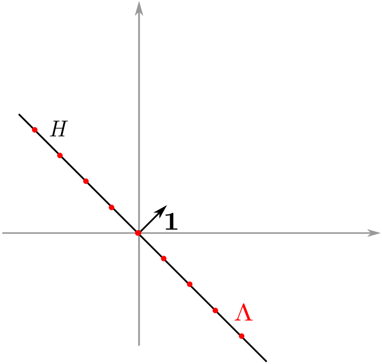

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- $\|r\| \simeq \exp(\|\mathrm{Log}\, r\|_\infty)$

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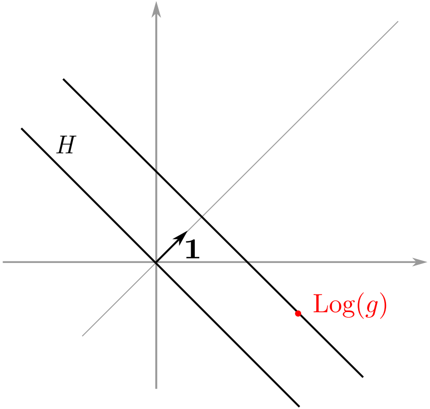
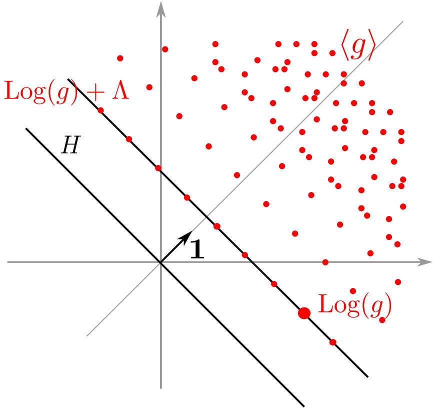
What does $\mathrm{Log}\langle g\rangle$ look like?



What does $\mathrm{Log}\langle g \rangle$ look like?



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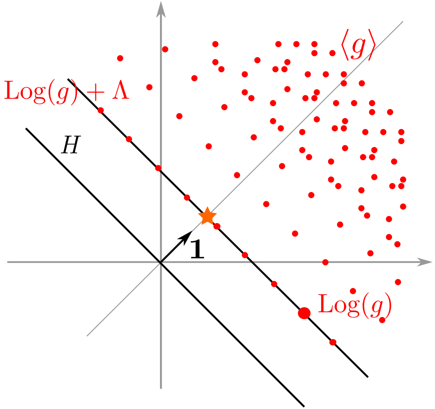
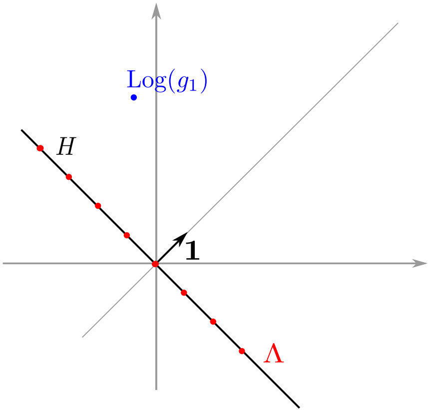


The algorithm [CGS14,CDPR16]

- Find a generator g_1 of $\langle g \rangle$.
  - [BS16]: quantum poly time



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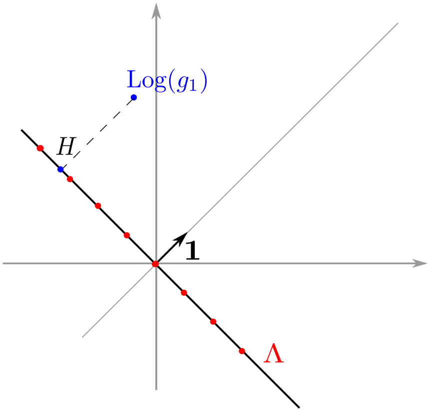

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- Find a generator g_1 of $\langle g \rangle$.
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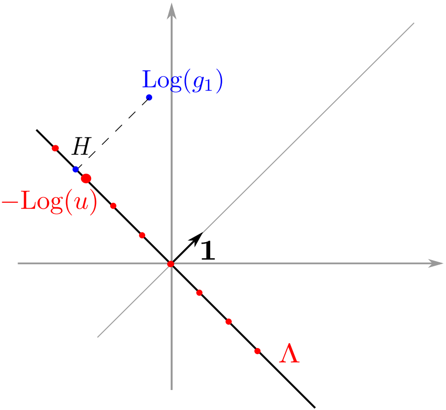
- Solve CVP in Λ



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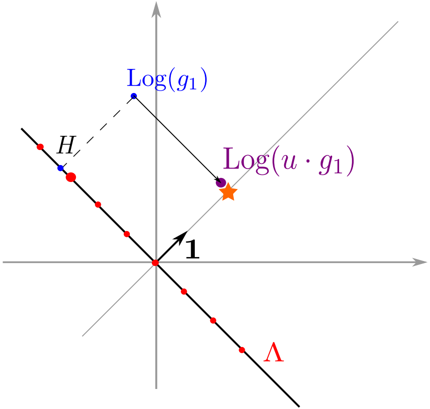

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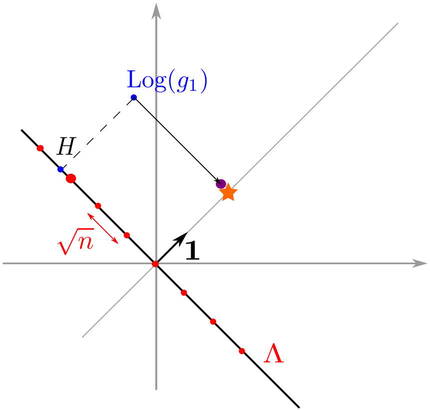

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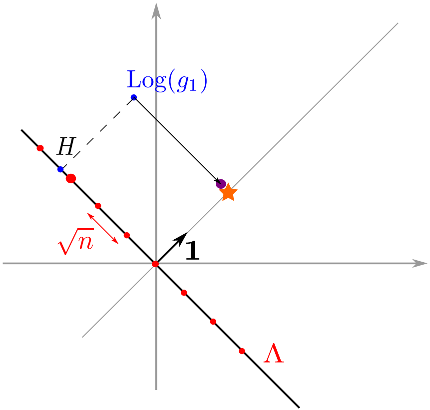

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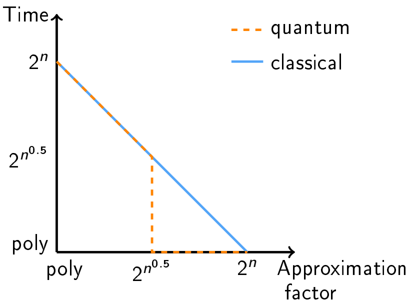

- Heuristic
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Conclusion

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Thank you