An LLL algorithm for module lattices

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and Alexandre Wallet[3]

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Lattices: Geometry, Algorithms and Hardness
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https://eprint.iacr.org/2019/1035

KU LEUVEN

ENS DE LYON

NTT

What is this talk about?

LLL-type algorithm for modules over a ring of integers

- all number fields
- approx-factor $\approx$ exponential in module rank
- quantum poly-time ... given a CVP oracle depending on K

Structured lattices

Motivation (crypto)

Improve efficiency of lattice-based schemes using structured lattices

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Example: NIST post-quantum standardization process

- 26 remaining candidates (2nd round)
- 12 lattice-based
- 11 using structured lattices

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	Frodo (unstructured lattices)	Kyber (structured lattices)
public key size (in Bytes)	9 616	800
ciphertexts size (in Bytes)	9 720	736

(CCA2 KEMs, round 2 version, security level 1)

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All 11 schemes use module lattices

Module

- $K = \mathbb{Q}[X]/P(X)$
- $R = \mathbb{Z}[X]/P(X)$ (or $R = O_K$)

with P monic and irreducible, degree d

Module

A (free) module M is a subset of K^k of the form $M = \{B\vec{x} \mid \vec{x} \in R^k\}$, with $B \in K^{k \times k}$ invertible. B is a basis of M; k is the rank of M.

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A module is a 'lattice' of rank k over R.

Module lattice

Reminder:

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- $M = \{B\vec{x} \,:\, \vec{x} \in R^d\}$, with $B = (\vec{b}_1, \cdots, \vec{b}_k)$ linearly independent

Module lattice

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Canonical embedding

$$\begin{aligned}\sigma : K &\to \mathbb{C}^d \\ x &\mapsto (x(\alpha_1), \cdots, x(\alpha_d))\end{aligned}$$

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$$\boxed{\mathcal{L}(M) = \{\sigma(\vec{x}) \,:\, \vec{x} \in M\} \subset \mathbb{C}^{kd}}$$

$\mathcal{L}(M)$ is a lattice of rank kd, spanned by $(\sigma(\vec{b}_1), \sigma(x\vec{b}_1), \cdots, \sigma(x^{d-1}\vec{b}_k))$

Motivation

Module lattices

- dimension $n = kd$ over $\mathbb{Z}$
- dimension k over R

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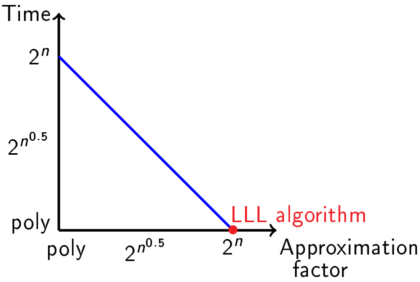
  typically $500 \leq kd \leq 1500$
- dimension k over R
  typically $k \leq 10$

Motivation



Lattice reduction over $\mathbb{Z}$
(in blue: BKZ trade-offs [Sch87, SE94])

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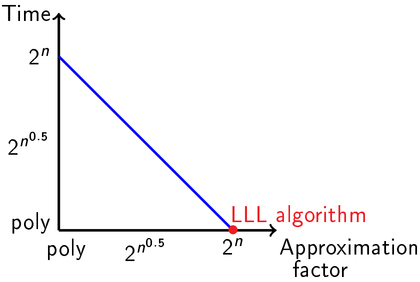
[LLL82] A. K. Lenstra, H. W. Lenstra, L. Lovász. Factoring polynomials with rational coefficients. Mathematische

Motivation



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Can we extend the LLL algorithm to lattices over R?

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What is small in K?

Remember

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Two generalization of $|\cdot|$:

- $\|x\| := \|\sigma(x)\|$ (Euclidean norm)
- $\mathcal{N}(x) := \prod_i |x(\alpha_i)|$ (algebraic norm)

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$$\|x\| \text{ small } \overset{\Rightarrow}{\underset{\not\Leftarrow}{}} \mathcal{N}(x) \text{ small}$$

Previous works and result

	Bound on quality	Bound on runtime
Napias [Nap96]: ▸ specific number fields	✗	✗

[Nap96] H. Napias. A generalization of the LLL-algorithm over Euclidean rings or orders. Journal de théorie des nombres de Bordeaux.

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▶ totally real fields	✗	✓

[FP96] C. Fieker, M. E. Pohst. Lattices over number fields. ANTS.

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Kim-Lee [KL17]:		
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▸ biquadratic norm euclidean	✓	✓

[KL17] T. Kim, C. Lee. Lattice reductions over euclidean rings with applications to cryptanalysis. IMACC.

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This work [LPSW19]:		
▸ any number field	✓	$\approx$

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[LPSW19] C. Lee, A. Pellet-Mary, D. Stehlé, A. Wallet. An LLL algorithm for module lattices.

Outline of the talk

1. The LLL algorithm
2. The Lagrange-Gauss algorithm
3. Computing the relaxed Euclidean division

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High-level overview of LLL

LLL over $\mathbb{Z}$

- γ'-SVP in dim k
  $\leq$ 1-SVP in dim 2
  - $\gamma' = 2^{O(k)}$
  - poly time

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LLL over $R = \mathbb{Z}[X]/(X^d + 1)$

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- γ'-SVP in rank-k $\leq$ γ-SVP in rank-2
  - $\gamma' = (\gamma d)^{O(k)}$
  - poly time
- Algorithm for γ-SVP in rank-2
  - $\gamma = 2^{(\log d)^{O(1)}}$
  - heuristic, quantum
  - poly time if oracle solving CVP in a fixed lattice (depending only on R)

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High-level overview of LLL

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- Lagrange-Gauss algo for SVP in dim 2
  - poly time

LLL over $R = \mathbb{Z}[X]/(X^d + 1)$

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  - time

needs QR-factorisation

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next section

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Inner product over R

For $\vec{a} = (a_1, \cdots, a_k) \in K^k$ and $\vec{b} = (b_1, \cdots, b_k) \in K^k$,

$$\boxed{\langle \vec{a}, \vec{b} \rangle_K = \sum_i a_i \overline{b_i} \in K \quad (\text{or } K_{\mathbb{R}})}$$

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Properties

- $\| \, \|\vec{a}\|_K \, \| = \|\sigma(\vec{a})\|$

$\|x\| = \|\sigma(x)\|$

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Properties

- $\| \; \|\vec{a}\|_K \; \| = \|\sigma(\vec{a})\|$
- $\mathcal{N}(\|\vec{a}\|_K) = \Delta_K^{-1/2} \cdot \det(\mathcal{L}(\vec{a}))$

$\mathcal{N}(x) = \prod_{i=1}^d |\sigma(x)_i|$

QR factorization over R

Let $B = (\vec{b}_1, \cdots, \vec{b}_k) \in K^k$, define

$$\vec{b}_i^* = \vec{b}_i - \sum_{j<i} \mu_{ij} \vec{b}_j^*, \quad \text{with } \mu_{ij} = \frac{\langle \vec{b}_i, \vec{b}_j^* \rangle_K}{\langle \vec{b}_j^*, \vec{b}_j^* \rangle_K}$$

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QR-factorisation: $B = QR$, with

- $r_{ii} = \|\vec{b}_i^*\|_K$, $r_{ij} = \mu_{ji} r_{ii}$ for $i < j$ and $r_{ij} = 0$ otherwise
- columns of Q are $\vec{b}_i^* / \|\vec{b}_i^*\|_K$

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Properties

- $\forall \vec{x}, \vec{y}$, $\boxed{\langle B\vec{x}, B\vec{y} \rangle_K = \langle R\vec{x}, R\vec{y} \rangle_K}$

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Properties

- $\forall \vec{x}, \vec{y}$, $\boxed{\langle B\vec{x}, B\vec{y} \rangle_K = \langle R\vec{x}, R\vec{y} \rangle_K}$
- $\forall \vec{v} \in \mathcal{L}(B)$, $\mathcal{N}(\|\vec{v}\|_K) \geq \min \mathcal{N}(r_{ii})$

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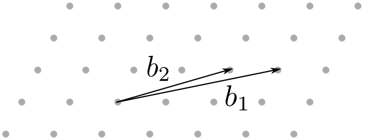
Lagrange-Gauss algorithm (over $\mathbb{Z}$)



$$M = \begin{pmatrix} 10 & 7 \\ 2 & 2 \end{pmatrix}$$

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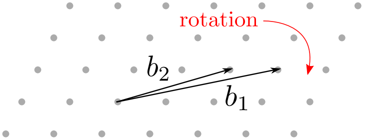

rotation

$$M = \begin{pmatrix} 10 & 7 \\ 2 & 2 \end{pmatrix}$$

Compute QR factorization

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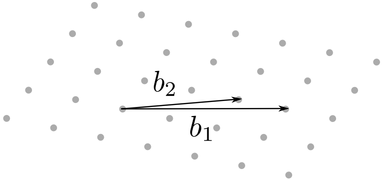

$$M = \begin{pmatrix} 10.2 & 7.3 \\ 0 & 0.6 \end{pmatrix}$$

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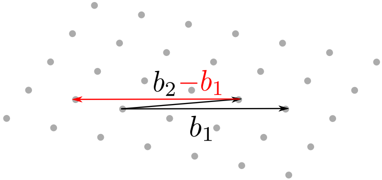
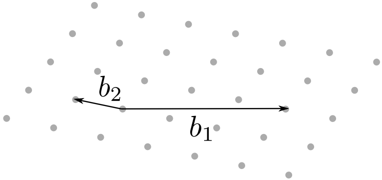


reduce b_2 with b_1

$$M = \begin{pmatrix} 10.2 & 7.3 \\ 0 & 0.6 \end{pmatrix}$$

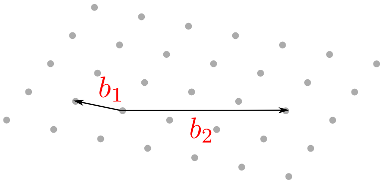
"Euclidean division" (over $\mathbb{R}$) of 7.3 by 10.2

Lagrange-Gauss algorithm (over $\mathbb{Z}$)



$$M = \begin{pmatrix} 10.2 & -2.9 \\ 0 & 0.6 \end{pmatrix}$$

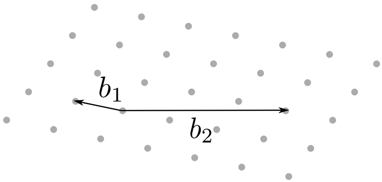
Lagrange-Gauss algorithm (over $\mathbb{Z}$)



swap

$$M = \begin{pmatrix} -2.9 & 10.2 \\ 0.6 & 0 \end{pmatrix}$$

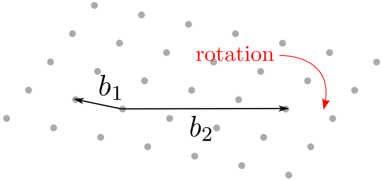
Lagrange-Gauss algorithm (over $\mathbb{Z}$)



start again

$$M = \begin{pmatrix} -2.9 & 10.2 \\ 0.6 & 0 \end{pmatrix}$$

Lagrange-Gauss algorithm (over $\mathbb{Z}$)



rotation

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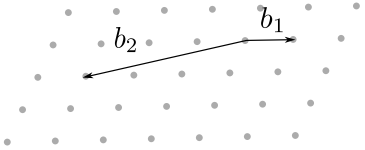
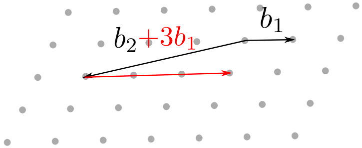
Lagrange-Gauss algorithm (over $\mathbb{Z}$)



rotation

$$M = \begin{pmatrix} 3 & -10 \\ 0 & -2 \end{pmatrix}$$

Lagrange-Gauss algorithm (over $\mathbb{Z}$)



reduce b_2 with b_1

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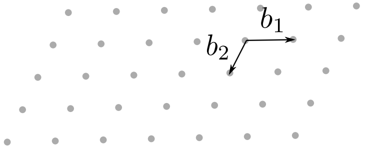
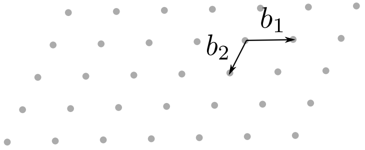
"Euclidean division" (over $\mathbb{R}$) of -10 by 3

Lagrange-Gauss algorithm (over $\mathbb{Z}$)



$$M = \begin{pmatrix} 3 & -1 \\ 0 & -2 \end{pmatrix}$$

Lagrange-Gauss algorithm (over $\mathbb{Z}$)



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For the Lagrange-Gauss algorithm over R, we need

- Rotation
- Euclidean division

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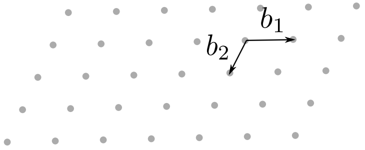

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For the Lagrange-Gauss algorithm over R, we need

- Rotation $\Rightarrow$ ok
- Euclidean division $\Rightarrow$?

Euclidean division

Over $\mathbb{Z}$

Input: $a, b \in \mathbb{Z}$, $a \neq 0$
Output: $r \in \mathbb{Z}$
such that $|b + ra| \leq |a|/2$

Euclidean division

Over $\mathbb{Z}$

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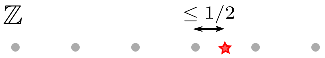
CVP in $\mathbb{Z}$ with target $-b/a$.



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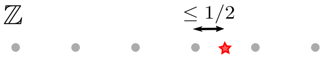

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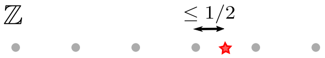

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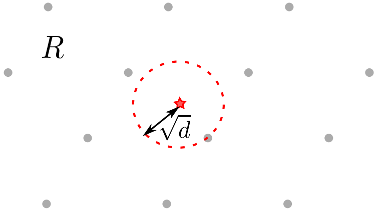

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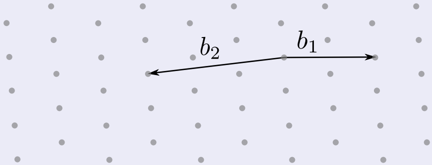

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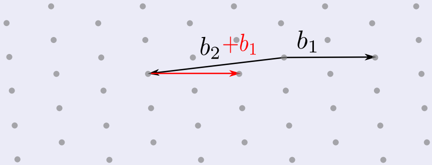

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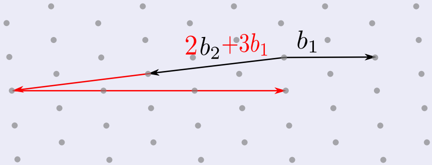

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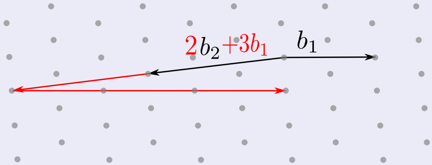

Outline of the talk

1. The LLL algorithm
2. The Lagrange-Gauss algorithm
3. Computing the relaxed Euclidean division

The Log space

Reminder: $\sigma(r) = (r(\alpha_1), \cdots, r(\alpha_d))^T$

(multiplication is coefficient-wise)

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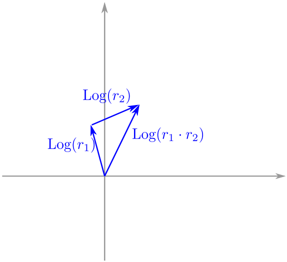

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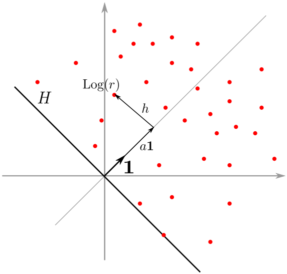

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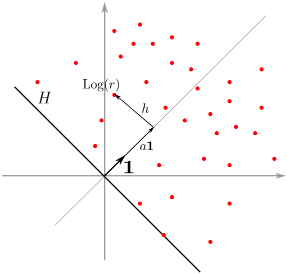

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Let's say $d = 1$

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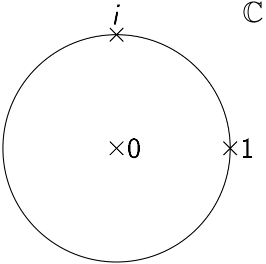

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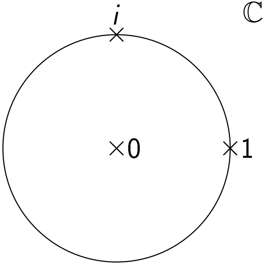

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$$\boxed{\begin{array}{l} |\mathrm{Log}(u) - \mathrm{Log}(v)| \leq \varepsilon \\ \not\Rightarrow |u - v| \leq \varepsilon' \end{array}}$$

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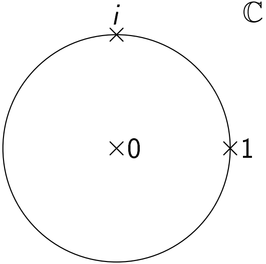

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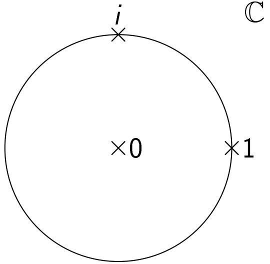

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New objective

Find $x, y \in R$ such that

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Idea of the algorithm

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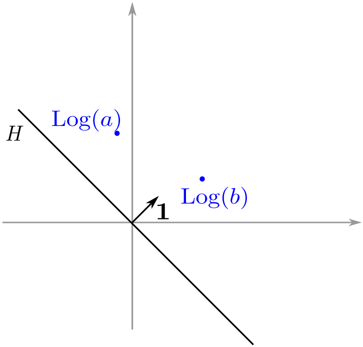

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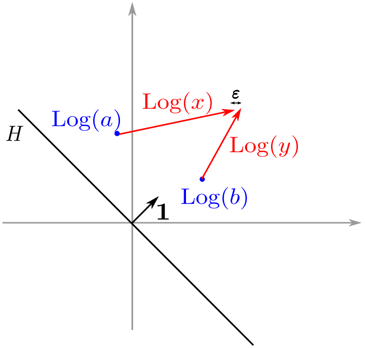

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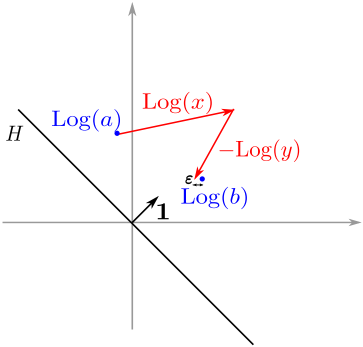

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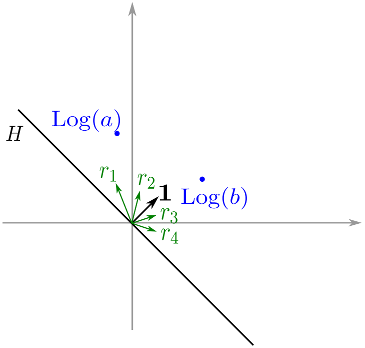

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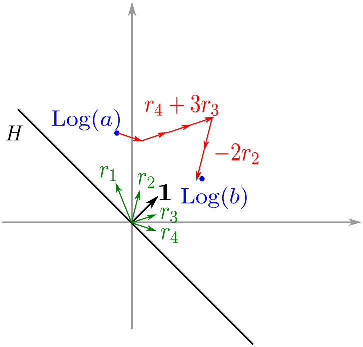

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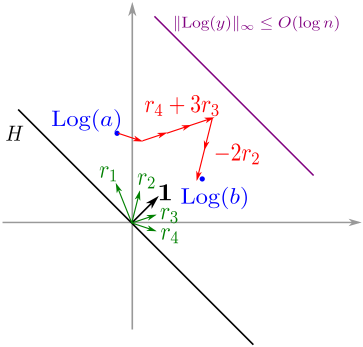

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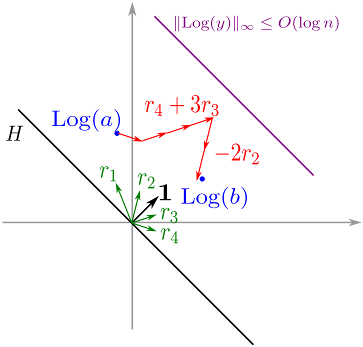

Solve **exact** CVP in L with target t

$$L = \begin{pmatrix} r_1 & \cdots & r_{d^2} \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}, \quad t = \begin{pmatrix} \mathrm{Log}(b/a) \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

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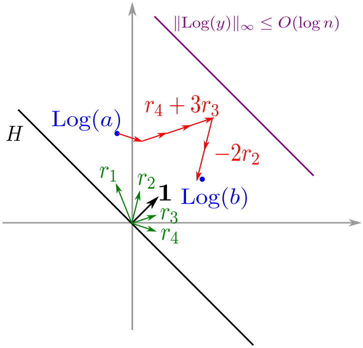

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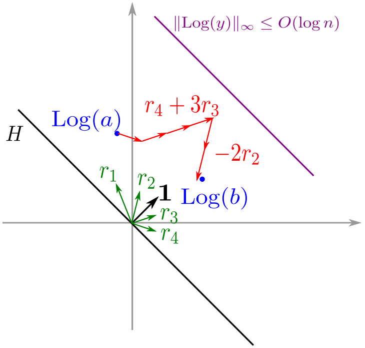

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Quantum poly time (with the oracle)

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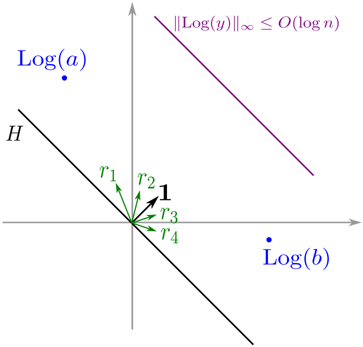

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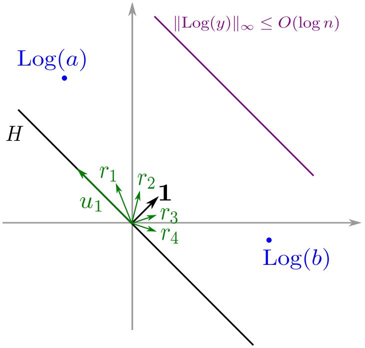

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For any target $t = (\beta \cdot x, 0, \cdots, 0)^T$, we have

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- Numerical experiments (up to dimension $d = 8$)

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- Numerical experiments (up to dimension $d = 8$)
- Hand waving counting argument
  - assuming $(\beta \cdot x \| \vec{v})$ with $v \xleftarrow{\$} \{\vec{w} \in \{0,1\}^{d^2} \mid \|\vec{w}\|_1 = d\}$ implies x (almost) uniform mod U.

Heuristic

$$L = \begin{pmatrix} \beta U & \beta r_1 & \cdots & \beta r_{d^2} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \qquad t = \begin{pmatrix} \beta \operatorname{Log}(b/a) \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Heuristic assumption

For any target $t = (\beta \cdot x, 0, \cdots, 0)^T$, we have

$$\operatorname{dist}(L, t) \leq \sqrt{d}.$$

- Numerical experiments (up to dimension $d = 8$)
- Hand waving counting argument
  - assuming $(\beta \cdot x \| \vec{v})$ with $v \xleftarrow{\$} \{\vec{w} \in \{0,1\}^{d^2} \mid \|\vec{w}\|_1 = d\}$ implies x (almost) uniform mod U.
  - related to Arakelov random walk $\Rightarrow$ see Léo and Benjamin's talk

Under the carpet

- Any module
  - use pseudo-basis
  - add class group to L (cf [Buc88])
- Proving correctness
  - Lovász' swap condition
  - switch between $\mathcal{N}(\cdot)$ and $\|\cdot\|$
  - handling bit sizes

[Buc88] J. Buchmann. A subexponential algorithm for the determination of class groups and regulators of algebraic number fields. Séminaire de théorie des nombres.

Summary and impact

LLL algorithm for cyclotomic fields

- Approx: quasi-poly(d)$^{O(k)} = 2^{(\log d)^{O(1)} \cdot k}$
- Time: quantum polynomial time
  if oracle solving CVP in L (of dim $O(d^{2+\varepsilon})$)

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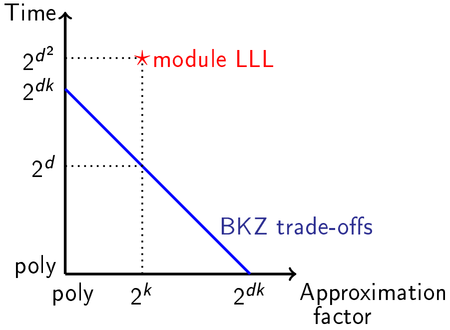

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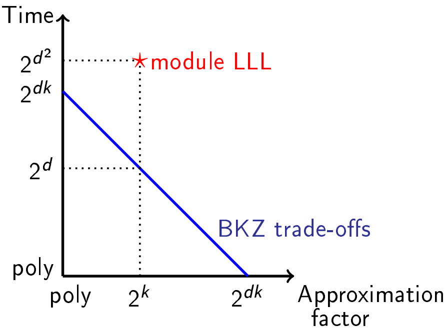

$\Rightarrow$ theoretical result
(not practical)

Open problems

(1) Improving CVP in L

- reduce its dimension to $\widetilde{O}(d)$?
- use the structure of L to improve CVP solvers?

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- CVP with pre-processing is NP-hard [Mic01]

"CVPP in NP-hard": $\forall n$, $\exists L$, $\forall \phi$ (SAT instance with n variables), $\exists \vec{t}$ and α $s.t.$

$$\phi \text{ is satisfiable } \Leftrightarrow \mathrm{dist}(L, \vec{t}) \leq \alpha.$$

[Mic01] D. Micciancio. The hardness of the closest vector problem with preprocessing. Transaction on Information Theory.

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(2) Generalizing LLL to all the BKZ trade-offs?

- reduction rank k to rank β (cf [MS19])
- sieving/enumeration in modules?

[MS19] T. Mukherjee, N. Stephens-Davidowitz. Lattice Reduction for Modules, or How to Reduce ModuleSVP to ModuleSVP, ePrint.

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Thank you