Lab 4

Inverted Pendulum

Purpose

This experiment is designed for students to apply the controller design techniques learnt in the class, such as pole-placement, to a rod balancing and cart tracking problem. Linearization technique is used to form a model which is suitable for linear control theory. A controller which would stabilize the rod and keep the cart in a desired position is then designed. This controller is then implemented using SIMULINK for hardware-in-the-loop simulation.

System Description

The whole system is shown in Figure 4.1. The system consists of a cart and a rod. The cart, with a mass M_c, slides on a stainless steel shaft and is equipped with a motor. A rod, attached with a ball, is mounted on the cart whose axis of rotation is perpendicular to the direction of the motion of the cart. The rod has an evenly distributed mass M_p and a length L. The ball, with a mass M_b, can be regarded as a mass point. The card position $x(t)$ and the pendulum angle $\theta(t)$ can be measured. The input is the force $f(t)$ applied on the cart.

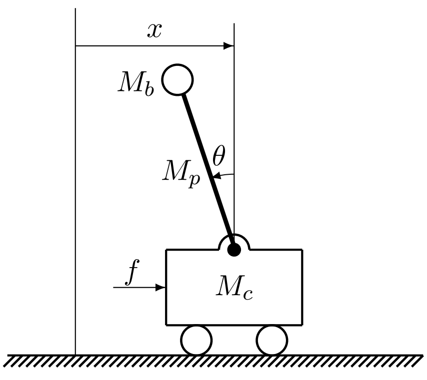

Figure 4.1: Inverted pendulum system.

System Modeling

Applying Newton's law to the horizontal direction of Figure 4.1, we get

$$f(t) = M_c\frac{d^2x(t)}{dt^2} + M_p\frac{d^2[x(t) - L/2\sin\theta(t)]}{dt^2} + M_b\frac{d^2[x(t) - L\sin\theta(t)]}{dt^2}.$$

This gives

$$(M_p+M_c+M_b)\ddot{x}(t)-(\frac{M_p}{2}+M_b)L\ddot{\theta}(t)\cos\theta(t)+(\frac{M_p}{2}+M_b)L\dot{\theta}^2(t)\sin\theta(t) = f(t). \tag{4.1}$$

Note that the moment of inertia of the rod with respect to the pivot point can be computed as

$$J_p = \int_0^L \ell^2(\frac{M_p}{L}d\ell) = \frac{1}{3}M_pL^2.$$

Similarly, the moment of inertia of the ball is given by

$$J_b = M_bL^2.$$

Writing the rotational version of Newton's second law for the pendulum and the ball around the pivotal point, we obtain

$$J_p\ddot{\theta}(t)+J_b\ddot{\theta}(t) = M_p\ddot{x}(t)\cos\theta(t)\frac{L}{2}+M_pg\sin\theta(t)\frac{L}{2}+M_bL\ddot{x}(t)\cos\theta(t)+M_bgL\sin\theta(t).$$

This gives

$$(\frac{1}{3}M_p + M_b)L\ddot{\theta}(t) - (\frac{M_p}{2} + M_b)\ddot{x}(t)\cos\theta(t) - (\frac{M_p}{2} + M_b)g\sin\theta(t) = 0. \tag{4.2}$$

Therefore the differential equation model of the system is given by the equation (4.1) and (4.2). This is a highly nonlinear system.

In the present experimental setups, the parameters for the above equations are given in Table 4.1.

Mass of rod (M_p)	0.07 Kg
Mass of the cart (M_c)	1.42 Kg
Mass of the ball (M_b)	0.05 Kg
Gravity (g)	9.8 m/s^2
Length of the rod (L)	0.335 m

Table 4.1: System Parameters

Linearization

We need a linearized model of the system for controller design. To linearize, simply replace $\sin\theta(t)$ by $\theta(t)$ and $\cos\theta(t)$ by 1 and neglect all terms with second order effect. Then we get

$$(M_p + M_c + M_b)\ddot{x}(t) - (\frac{M_p}{2} + M_b)L\ddot{\theta}(t) = f(t), \tag{4.3}$$

and

$$(\frac{1}{3}M_p + M_b)L\ddot{\theta}(t) - (\frac{M_p}{2} + M_b)\ddot{x}(t) - (\frac{M_p}{2} + M_b)g\theta(t) = 0. \tag{4.4}$$

Note that actually we have a SIDO system since both $x(t)$ and $\theta(t)$ are variables to be controlled. So the transfer function of the linearized system has the following form

$$P(s) = \frac{1}{a(s)} \begin{bmatrix} b_x(s) \\ b_\theta(s) \end{bmatrix}$$

where $a(s)$, $b_x(s)$, $b_\theta(s)$ are polynomials. Notice that in this particular system, polynomial $a(s)$ is of 4th order, i.e., the system is of 4th order. The plant has a block diagram as in Figure 4.2.

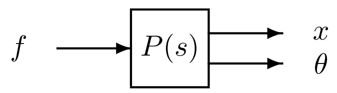

Figure 4.2: The linearized inverted pendulum system

Controller Design

First notice that we are trying to stabilize a SIMO plant. So we need to modify the controller design methods for SISO plants presented in the course notes. The feedback system structure in this case is shown in Figure 4.3, where w_f, w_x, w_θ are various noises.

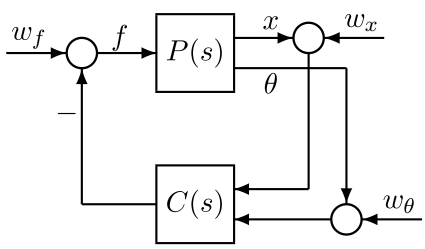

Figure 4.3: The inverted pendulum control system structure

We will use a PD controller to stabilize the inverted pendulum system. Since the plant has one input and two outputs, the controller has two inputs and one output. The PD controller in this case takes the following form

$$C(s) = [K_{x,P} + K_{x,D}s \quad K_{\theta,P} + K_{\theta,D}s],$$

i.e., the transfer function of each controller channel consists of the sum of two terms, one proportional term and one derivative term. After it is applied to the linearized model of the inverted pendulum, the closed-loop characteristic polynomial is

$$a(s) + (K_{x,P} + K_{x,D}s)b_x(s) + (K_{\theta,P} + K_{\theta,D}s)b_\theta(s).$$

The four design parameters $K_{x,P}, K_{x,D}, K_{\theta,P}, K_{\theta,D}$ can be chosen to place the four closed-loop poles arbitrarily.

Prelab

1. Obtain the numerical linearized transfer function model of the inverted pendulum system.
2. Write a MATLAB program to do the pole placement. This program should have the desired poles as its input and the parameters for the controller as the output.
3. Build the nonlinear system in SIMULINK, and carry out a closed-loop simulation with the controller designed in 2. In the simulation, set the external disturbances to zero but set the initial conditions of the plant states $[x(0),\ \dot{x}(0),\ \theta(0),\ \dot{\theta}(0)]^T$ to the value $[0, 0, 3, 0.5]^T$. Check whether the controller can indeed stabilize the rod. If not, redesign the controller and repeat the simulation.

Experiment procedure

1. Design a PD controller so that the closed-loop poles are $-10.27 \pm 11.15j$ and $-3.09 \pm 0.93j$. In SIMULINK, connect this controller to the nonlinear model built in the previous step and carry out the software simulation and observe that the closed-loop system is stabilized. Implement this PD controller and carry out the hardware-in-the-loop simulation. Check if the inverted pendulum is balanced.
2. Adjust the closed-loop poles in the last step and design another PD controller. Carry out the SIMULINK software simulation to see if you can improve the performance. Then carry out the hardware-in-the-loop simulation with the "best" controller you obtain.

Reports

Give detailed accounts on the following items:

1. Give answers to the Prelab questions.
2. Steps to derive a linearized mathematical models for the plant.
3. A printout of the MALTAB program for pole placement.
4. A printout of the SIMULINK block diagram of the nonlinear model.
5. Design steps for the controller.
6. Simulation results and a sketch of the actual response on the position of the cart and rod.

7. Comment on any difference between the simulation and experimental results.