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Gene Freudenburg

Algebraic Theory of Locally Nilpotent Derivations

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To my wife Sheryl and our wonderful children, Jenna, Kathryn, and Ella Marie, whom I love very much.

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Evansville, IN, USA (May 2006) Gene Freudenburg
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Introduction

But, in the further development of a branch of mathematics, the human mind, encouraged by the success of its solutions, becomes conscious of its independence. It evolves from itself alone, often without appreciable influence from without, by means of logical combination, generalization, specialization, by separating and collecting ideas in fortunate new ways, new and fruitful problems, and appears then itself as the real questioner.

David Hilbert, *Mathematical Problems*

The study of locally nilpotent derivations and $\mathbb{G}_a$-actions has recently emerged from the long shadows of other branches of mathematics, branches whose provenance is older and more distinguished. The subject grew out of the rich environment of Lie theory, invariant theory, and differential equations and continues to draw inspiration from these and other fields.

At the heart of the present exposition lie 16 principles for locally nilpotent derivations, laid out in *Chap. 1*. These provide the foundation on which the subsequent theory is built. We would like to distinguish which properties of a locally nilpotent derivation are due to its being a derivation and which are special to the locally nilpotent condition. Thus, we first consider general properties of derivations. The 16 First Principles which follow can then be seen as belonging especially to the locally nilpotent derivations.

Of course, one must choose one's category. While $\mathbb{G}_a$-actions can be investigated in a characteristic-free environment, locally nilpotent derivations are, by nature, objects belonging to rings of characteristic zero. Most of the basic results about derivations in *Chaps. 1* and *2* are stated for a commutative k-domain B, where k is a field of characteristic zero.

In discussing geometric aspects of the subject, it is also generally assumed that B is affine and that the underlying field k is algebraically closed. The associated geometry falls under the rubric of affine algebraic geometry. Miyanishi writes: "There is no clear definition of affine algebraic geometry. It is one branch of algebraic geometry which deals with the affine spaces and the polynomial rings, hence affine algebraic varieties as subvarieties of the affine spaces and finitely generated algebras as the residue rings of the polynomial rings" [306]. Due to their obvious importance, special attention is given throughout the book to polynomial rings and affine spaces $\mathbb{A}^n_k$.

Chapter 3 explores the case of polynomial rings over k. Here, the Jacobian derivations are of central importance. *Chapter 4* considers polynomial rings in two variables, first over a field k and second over other base rings. Two proofs of Rentschler's Theorem are given, and this result is applied to give proofs for Jung's Theorem and the Structure Theorem for the planar automorphism group. This effectively classifies all locally nilpotent derivations of $k[x, y]$ and likewise all algebraic $\mathbb{G}_a$-actions on the plane $\mathbb{A}^2$. *Chapter 5* documents the progress in our understanding of the three-dimensional case which has been made over the past three decades, beginning with the Bass-Nagata example and Miyanishi's Theorem. We now have a large catalogue of interesting and instructive examples, in addition to the impressive Daigle-Russell classification in the homogeneous case and Kaliman's classification of the free $\mathbb{G}_a$-actions. These feats notwithstanding, a classification of the locally nilpotent derivations of $k[x, y, z]$ remains elusive.

Chapter 6 examines the case of linear actions of $\mathbb{G}_a$ on affine spaces, and it is here that the oldest literature on the subject of $\mathbb{G}_a$-actions can be found. One of the main results of the chapter is the Maurer-Weitzenböck Theorem, a classical result showing that a linear action of $\mathbb{G}_a$ on $\mathbb{A}^n$ has a finitely generated ring of invariants.[1]

Nagata's famous counterexamples to the Fourteenth Problem of Hilbert showed that the Maurer-Weitzenböck Theorem does not generalize to higher-dimensional groups, i.e., it can happen that a linear $\mathbb{G}_a^m$-action on affine space has a nonfinitely generated ring of invariants when $m > 1$. It can also happen that a nonlinear $\mathbb{G}_a$-action has nonfinitely generated invariant ring, and these actions form the main topic of *Chap. 7*.

Chapter 8 discusses various algorithms associated with locally nilpotent derivations, including the van den Essen algorithm for calculating kernels of finite type. *Chapter 9* focuses on the Makar-Limanov and Derksen invariants of a ring and illustrates how they can be applied. *Chapter 10* shows how locally nilpotent derivations can be found and used in a range of important problems, such as the cancellation problem and embedding problem. The concluding chapter, *Chap. 11*, is devoted to questions and open problems.

In addition to the articles found in the *References*, there are four larger works used in preparing this manuscript. These are the books of Nowicki (1994) and van den Essen (2000) and the extensive lecture notes written by Makar-Limanov (1998) and Daigle (2003). In addition, I found in the books of Kraft (1985), Popov (1992), Grosshans (1997), Borel (2001), Derksen and Kemper (2002), and Dolgachev (2003) a wealth of pertinent references and historical background regarding invariant theory.

[1]This result is commonly attributed only to R. Weitzenböck, but after reading Borel's *Essays in the History of Lie Groups and Algebraic Groups*, it becomes clear that L. Maurer should receive at least equal credit.

The reader will find that this book focuses on the algebraic aspects of locally nilpotent derivations, as the book's title indicates. The subject is simply too large and diverse to include a complete geometric treatment in a volume of this size. The outstanding survey articles of Kaliman [228] and Miyanishi [306] will serve to fill this void.

Historical Overview

The study of locally nilpotent derivations in its present form appears to have emerged in the 1960s, and was first made explicit in the work of several mathematicians working in France, including Dixmier, Gabriel and Nouazé, and Rentschler. Their motivation came from the areas of Lie algebras and Lie groups, where the connections between derivations, vector fields, and group actions were well-explored.

Linear $\mathbb{G}_a$-actions were one of the main objects of interest for invariant theory in the nineteenth century. Gordan (1868) gave an algorithm to calculate the invariants of the basic $\mathbb{G}_a$-actions and found their invariant rings up to dimension 7. Stroh (1888) gave a transcendence basis for the invariants of the basic $\mathbb{G}_a$-actions, and Hilbert calculated the invariants of the basic actions up to integral closure (see [188], §10, Note 1). In 1899, Maurer outlined his proof showing the finite generation of invariant rings for one-dimensional group actions. In 1932, Weitzenböck gave a more complete version of Maurer's proof, which used ideas of Gordan and M. Roberts dating to 1868 and 1871, respectively, in addition to the theory developed by Hilbert. In their paper dating to 1876, Gordan and M. Nöther studied certain systems of differential operators and were led to investigate special kinds of non-linear $\mathbb{G}_a$-actions on $\mathbb{C}^n$, though they did not use this language; see *Chap. 3*.

It seems that the appearance of Nagata's counterexamples to Hilbert's Fourteenth Problem in 1958 spurred a renewed interest in $\mathbb{G}_a$-actions and more general unipotent actions, since the theorem of Maurer and Weitzenböck could then be seen in sharp contrast to the case of higher-dimensional vector group actions. It was shortly thereafter, in 1962, that Seshadri published his well-known proof of the Maurer-Weitzenböck result. Nagata's 1962 paper [322] contains significant results about connected unipotent groups acting on affine varieties, and his classic Tata lecture notes [323] appeared in 1965. The case of algebraic $\mathbb{G}_a$-actions on affine varieties was considered by Białynicki-Birula in the mid-1960s [30–32]. In 1966, Hadziev published his famous theorem [198], which is a finiteness result for the maximal unipotent subgroups of reductive groups. In 1967, Shioda gave the first complete system of generators for the basic $\mathbb{G}_a$-action in dimension 9. The 1969 article of Horrocks [212] considered connectedness and fundamental groups for certain kinds of unipotent actions, and the 1973 paper of Hochschild and Mostow [208] remains a standard reference for unipotent actions. Grosshans began his work on unipotent actions in the early 1970s; his 1997 book [188] provides an excellent overview of the subject. Another notable body of research from the 1970s is due to Fauntleroy [148–150] and Fauntleroy and Magid [151, 152]. The papers of Pommerening also began to appear in the late 1970s (see [188, 340]), and Tan's algorithm for computing invariants of basic $\mathbb{G}_a$-actions appeared in 1989. In his 2002 thesis, Cröni gave a complete set of generators for the basic $\mathbb{G}_a$-action in dimension 8.

In a famous paper published in 1968, Rentschler classified the locally nilpotent derivations of the polynomial ring in two variables over a field of characteristic zero and pointed out how this gives the equivalent classification of all the algebraic $\mathbb{G}_a$-actions on the plane $\mathbb{A}^2$. This article is highly significant, in that it was the first publication devoted to the study of certain locally nilpotent derivations (even though its title mentions only $\mathbb{G}_a$-actions). Indeed, Rentschler's landmark paper crystallized the definitions and concepts for locally nilpotent derivations in their modern form, and further provided a compelling illustration of their importance, namely, a simple proof of Jung's Theorem using locally nilpotent derivations.

It should be noted that the classification of planar $\mathbb{G}_a$-actions in characteristic zero was first given by Ebey in 1962 [133]. Ebey's paper clearly deserves more recognition than it receives. Of the more than 400 works listed in the *References* of this book, only the papers of Shafarevich (1966) and Koshevoi (1967) cite it [252, 380]. Ebey's paper was an outgrowth of his thesis, written under the direction of Max Rosenlicht. Rather than using the standard theorems of Jung (1942) or van der Kulk (1953) on planar automorphisms, the author used an equivalent result of Engel, dating to 1958.

The crucial Slice Theorem appeared in the 1967 paper of Gabriel and Nouazé [178], which is cited in Rentschler's paper. Other proofs of the Slice Theorem were given by Dixmier in 1974 ([116], 4.7.5), Miyanishi in 1978 ([297], 1.4), and Wright in 1981 ([426], 2.1). In Dixmier's proof we find the implicit definition and use of what is herein referred to as the Dixmier map. Wright's proof also uses such a construction. The first explicit definition and use of this map is found in van den Essen [141], 1993, and in Deveney and Finston [101], 1994. Arguably, the Dixmier map is to unipotent actions what the Reynolds operator is to reductive group actions (see [142], 9.2).

Certainly, one impetus for the study of locally nilpotent derivations is the Jacobian Conjecture. This famous problem and its connection to derivations is briefly described in *Chap. 3* and is thoroughly investigated in the book of van den Essen [142]. It seems likely that the conjecture provided, at least partly, the motivation behind Vasconcelos's Theorem on locally nilpotent derivations, which appeared in 1969. In the paper of Wright mentioned above, locally nilpotent derivations play a central role in his discussion of the conjecture.

There are not too many papers about locally nilpotent derivations or $\mathbb{G}_a$-actions from the decade of the 1970s. A notable exception is found in the work of Miyanishi, who was perhaps the first researcher to systematically investigate $\mathbb{G}_a$-actions throughout his career. Already in 1968, his paper [293] dealt with locally finite higher iterative derivations. These objects were first defined by Hasse and Schmidt [204] in 1937 and serve to generalize the definition of locally nilpotent derivations in order to give a correspondence with $\mathbb{G}_a$-actions in arbitrary characteristic.

Miyanishi's 1971 paper [294] is about planar $\mathbb{G}_a$-actions in positive characteristic, giving the analogue of Rentschler's Theorem in this case. His 1973 paper [295] uses $\mathbb{G}_a$-actions to give a proof of the cancellation theorem of Abhyankar, Eakin, and Heinzer. In his 1978 book [297], Miyanishi entitled the first section "Locally Nilpotent Derivations" (Sect. 1.1). In these few pages, Miyanishi organized and proved many of the fundamental properties of locally nilpotent derivations: the correspondence of locally nilpotent derivations and exponential automorphisms (Lemma 1.2) the fact that the kernel is factorially closed the Slice Theorem and its local version (Lemma 1.5). While these results already existed elsewhere in the literature, this publication constituted an important new resource for the study of locally nilpotent derivations. A later section of the book, called "Locally Nilpotent Derivations in Connection with the Cancellation Problem" (Sect. 1.6), proved some new cases in which the cancellation problem has a positive solution, based on locally nilpotent derivations. Miyanishi's 1980 paper [298] and 1981 book [299] include some of the earliest results about $\mathbb{G}_a$-actions on $\mathbb{A}^3$. Ultimately, his 1985 paper [301] outlined the proof of his well-known theorem about invariant rings of $\mathbb{G}_a$-actions on $\mathbb{A}^3$. In many other papers, Miyanishi used $\mathbb{G}_a$-actions extensively in the classification of surfaces, characterization of affine spaces, and the like.

In 1984, Bass produced a non-triangularizable $\mathbb{G}_a$-action on $\mathbb{A}^3$, based on the automorphism published by Nagata in 1972. This example, together with the 1985 theorem of Miyanishi, marked the beginning of the next generation of research on $\mathbb{G}_a$-actions and locally nilpotent derivations. The subject gathered momentum in the late 1980s, with significant new results of Popov, Snow, M. Smith, Winkelmann, and Zurkowski [344, 386–388, 421, 431, 432].[2] This trend continued in the early 1990s, especially in several papers due to van den Essen, and Deveney and Finston, which began a more systematic approach to the study of locally nilpotent derivations. Paul Roberts' counterexample to the Fourteenth Problem of Hilbert appeared in 1990, and it was soon realized that his example was the invariant ring of a $\mathbb{G}_a$-action on $\mathbb{A}^7$. The 1994 book of Nowicki [333] includes a chapter about locally nilpotent derivations. The book of van den Essen, published in 2000, is about polynomial automorphisms and the Jacobian Conjecture and takes locally nilpotent derivations as one of its central themes.

By the mid-1990s, Daigle, Kaliman, Makar-Limanov, Russell, Bhatwadekar, and Dutta began making significant contributions to our understanding of the subject. The introduction by Makar-Limanov in 1996 of the ring of absolute constants (now called the Makar-Limanov invariant) brought widespread recognition to locally nilpotent derivations as a tool in understanding affine geometry and commutative ring theory. Extensive (unpublished) lecture notes on the subject of locally nilpotent derivations were written by Makar-Limanov (1998) and by Daigle (2003). Papers of Kaliman which appeared in 2004 contain important results about $\mathbb{C}^+$-actions on threefolds, bringing to bear a wide range of tools from topology and algebraic geometry.

[2] My own work in this area began in 1993, and I "went to school" on these papers.

The Makar-Limanov invariant is currently one of the central themes in the classification of algebraic surfaces. In particular, families of surfaces having a trivial Makar-Limanov invariant have been classified by Bandman and Makar-Limanov, Daigle and Russell, Dubouloz, and Gurjar and Miyanishi [9, 88, 126, 193]. Already in 1983, Bertin [24] had studied surfaces which admit a $\mathbb{C}^+$-action.

By the late 1990s, locally nilpotent derivations began to appear in some thesis work, especially from the Nijmegen School, i.e., students of van den Essen at the University of Nijmegen. It appears that Z. Wang's 1999 PhD thesis, written under the direction of Daigle at the University of Ottawa, holds the distinction of being the first thesis devoted to the subject of locally nilpotent derivations.

The foregoing overview is by no means a complete account of the subject's development. Significant work in this area from many other researchers can be found in the *References*, much of which is discussed in the following chapters. In a conversation with the author about locally nilpotent derivations and $\mathbb{G}_a$-actions, Białynicki-Birula remarked: "I believe that we are just at the beginning of our understanding of this wonderful subject."

Notes on the Second Edition

New material presented in the second edition includes an overview of results about linear $\mathbb{G}_a$-actions from the nineteenth century, with the disclaimer that, given the vast body of literature on classical invariant theory, this is done in only the most cursory fashion. In this volume, I have also endeavored to better represent the work of certain researchers, including that of Bhatwadekar and Dutta and of Deveney and Finston.

There remain 16 First Principles, but a new principle (the Generating Principle) is introduced, taking the place of the original Principle 14, which can be seen as a consequence of Principle 15. *Chapter 1* discusses degree functions, gradings, and associated graded rings in a more general setting and devotes a new section to degree modules for locally nilpotent derivations and the canonical factorization of the quotient morphism for the induced $\mathbb{G}_a$-action; *Chap. 8* then gives a new algorithm for calculating degree modules. *Chapter 2* has been expanded to become a gathering place for a large number of fundamental results used in later chapters. New topics found there include cable algebras, transvectants, *G*-critical elements, and the AB and ABC theorems. In particular, the cable algebra structure on a ring induced by a locally nilpotent derivation can be viewed as a generalization of Jordan block form for a nilpotent linear operator. Other new topics include the down operator in *Chap. 7* and the Pham-Brieskorn surfaces in *Chap. 9*.

This edition features a new proof for the Abhyankar-Eakin-Heinzer Theorem for algebraically closed fields of characteristic 0, based on a proof given by Makar-Limanov. It also includes a new proof of nonfinite generation for the triangular derivation in dimension 5 due to Daigle and the author, showing that this kernel is a cable algebra of an especially simple type. *Chapter 10* gives a new proof that the Danielewski surfaces are not cancelative. In addition, readers are introduced to the theory of affine modifications relative to $\mathbb{G}_a$-actions, which was developed by Kaliman and Zaidenberg in the late 1990s and which underlies the discussion of canonical factorizations found in *Chap. 1*.

Above all, this second edition is intended to be an improved reference for locally nilpotent derivations and $\mathbb{G}_a$-actions and their applications.

Contents

Chapter 1
First Principles

Why should we study derivations? Answer: they occur naturally all over the place.

Irving Kaplansky [242]

Throughout this chapter, assume that B is an integral domain containing a field k of characteristic zero. B is referred to as a k-domain. B^* denotes the group of units of B and $\mathrm{frac}(B)$ denotes the field of fractions of B. Further, $\mathrm{Aut}(B)$ denotes the group of ring automorphisms of B, and $\mathrm{Aut}_k(B)$ denotes the group of automorphisms of B as a k-algebra. If $A \subset B$ is a subring, then $\mathrm{tr.deg}_A B$ denotes the transcendence degree of $\mathrm{frac}(B)$ over $\mathrm{frac}(A)$. The ideal generated by $x_1, \dots, x_n \in B$ is denoted by either $(x_1, \dots, x_n)$ or $x_1 B + \cdots + x_n B$. The ring of $m \times n$ matrices with entries in B is indicated by $\mathcal{M}_{m\times n}(B)$ and the ring of $n \times n$ matrices with entries in B is indicated by $\mathcal{M}_n(B)$. The transpose of a matrix M is M^T.

The term **affine k-domain** will mean a commutative k-domain which is finitely generated as a k-algebra. An algebraic k-variety V is an **affine k-variety** if its coordinate ring $k[V]$ is affine. V is a **quasi-affine k-variety** if it is isomorphic to an open subset of an affine k-variety. A commutative k-domain R is a **quasi-affine k-domain** if $R = k[V]$ for some quasi-affine k-variety V.

The standard notations $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are used throughout to denote the fields of rational, real, and complex numbers, respectively. Likewise, $\mathbb{Z}$ denotes the ring of integers, $\mathbb{N}$ is the monoid of non-negative integers, and $\mathbb{Z}_+$ is the set of positive integers. S_n will denote the symmetric group on n letters. The term unique factorization domain is abbreviated by UFD, and principal ideal domain by PID.

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1.1 Preliminaries

A **derivation** of B is a function $D : B \to B$ which satisfies the following conditions: For all $a, b \in B$,

(C.1) $D(a + b) = Da + Db$
(C.2) $D(ab) = aDb + bDa$

Condition (C.2) is usually called the **Leibniz rule** or **product rule**. The set of all derivations of B is denoted by $\mathrm{Der}(B)$. If A is any subring of B, then $\mathrm{Der}_A(B)$ denotes the subset of all $D \in \mathrm{Der}(B)$ with $D(A) = 0$. The set $\ker D = \{b \in B \mid Db = 0\}$ is the **kernel** of D. Given $D \in \mathrm{Der}(B)$, the following properties hold.

(C.3) $\ker D$ is a subring of B for any $D \in \mathrm{Der}(B)$.
(C.4) The subfield $\mathbb{Q} \subset k$ has $\mathbb{Q} \subset \ker D$ for any $D \in \mathrm{Der}(B)$.
(C.5) $\mathrm{Aut}(B)$ acts on $\mathrm{Der}(B)$ by conjugation: $\alpha \cdot D = \alpha D \alpha^{-1}$.
(C.6) Given $b \in B$ and $D, E \in \mathrm{Der}(B)$, if $[D, E] = DE - ED$, then bD, $D + E$, and $[D, E]$ are again in $\mathrm{Der}(B)$.

Verification of properties (C.3)–(C.6) is an easy exercise.

We are especially interested in $\mathrm{Der}_k(B)$, called the **k-derivations** of B. For k-derivations, the conditions above imply that D is uniquely defined by its image on any set of generators of B as a k-algebra, and that $\mathrm{Der}_k(B)$ forms a Lie algebra over k. If A is a subring of B containing k, then $\mathrm{Der}_A(B)$ is a Lie subalgebra of $\mathrm{Der}_k(B)$.

Given $D \in \mathrm{Der}(B)$, let $A = \ker D$. We define several terms and notations for D.

1. Given $n \geq 0$, D^n denotes the n-fold composition of D with itself, where it is understood that D^0 is the identity map.
2. A commonly used alternate term for the kernel of D is the **ring of constants** of D, with alternate notation B^D.
3. The **image** of D is denoted DB.
4. The B-ideal generated by the image DB is denoted (DB).
5. The **image ideals** of D are the ideals of A defined by:

$$I_n = A \cap D^n B \quad (n \geq 0) \quad \text{and} \quad I_\infty = \cap_{n \geq 0} I_n$$

   Note that $I_0 = A$ and $I_{n+1} \subset I_n$ for $n \geq 0$. I_1 is called the **plinth ideal**[1] for D, denoted $\mathrm{pl}(D)$, and I_∞ is called the **core ideal** for D. (See *Proposition 1.9*.)
6. An ideal $J \subset B$ is an **integral ideal** for D if and only if $DJ \subset J$ [304]. (Some authors call such J a **differential ideal**, e.g., [333].)
7. An element $f \in B$ is an **integral element** for D if and only if fB is an integral ideal for D.
8. D is **reducible** if and only if there exists a non-unit $b \in B$ such that $DB \subset b \cdot B$. Otherwise, D is **irreducible**.

[1] The term *plinth* commonly refers to the base of a column or statue.

9. Any element $s \in B$ with $Ds = 1$ is a **slice** for D. Any $s \in B$ such that $Ds \neq 0$ and $D^2 s = 0$ is a **local slice** for D. (Some authors use the term **pre-slice** instead of local slice, e.g., [142].)
10. Given $b \in B$, we say D is **nilpotent at** b if and only if there exists $n \in \mathbb{N}$ with $D^n b = 0$. The set of all elements of B at which D is nilpotent is denoted $\mathrm{Nil}(D)$.
11. Let $S \subset B$ be a non-empty subset, and let $k \subset R \subset A$ be a subring. Define the subring:

$$R[S, D] = R[D^i s \mid s \in S, i \geq 0]$$

Then D restricts to $R[S, D]$ and $R[S, D]$ is the smallest subring of B containing R and S to which D restricts.

1.1.1 Rings and Modules

For a ring A, the polynomial ring in one variable t over A is defined in the usual way, and is denoted by $A[t]$. It is also common to write $A^{[1]}$ for this ring. More generally, polynomial rings over a coefficient ring are defined as follows: If A is any commutative ring, then $A^{[0]} := A$, and for $n \geq 0$, $A^{[n+1]} := A^{[n]}[t]$, where t is a variable over $A^{[n]}$. We say that R is a **polynomial ring in** n **variables over** A if and only if $A \subset R$ and R is A-isomorphic to $A^{[n]}$. In this case, we simply write $R = A^{[n]}$.

Given a subring $A \subset B$, an element $x \in B$ is **algebraic** over A if there exists nonzero $P \in A^{[1]}$ such that $P(x) = 0$. If P can be chosen to be monic over A, then x is an **integral** element over A. The **algebraic closure** of A in B is the subring $\bar{A}$ of B consisting of all $x \in B$ which are algebraic over A. A is said to be **algebraically closed** in B if $\bar{A} = A$. B is an **algebraic extension** of A if $\bar{A} = B$. The terms **integrally closed** subring, **integral closure**, and **integral extension** are defined analogously.

Recall that a subring $A \subset B$ is **factorially closed** in B if and only if, given nonzero $f, g \in B$, the condition $fg \in A$ implies $f \in A$ and $g \in A$. Similarly, if $M \subset B$ is an A-module, then M is **factorially closed** in B if and only if $fg \in M$ implies $f \in M$ and $g \in M$. As we will see, factorially closed rings and modules play an important role in the subject at hand.

Note that the condition "in B" is important in this definition. For example, if B is an integral domain and $f \in B \setminus A$, then $ff^{-1} = 1 \in A$, but $f \notin A$. Nonetheless, when the ambient ring B is understood, we will often say simply that A is factorially closed.

Other terms used for this property are **saturated** and **inert**. When A is factorially closed in B, then $A^* = B^*$, A is algebraically closed in B, and every irreducible element of A is irreducible in B. In addition, the intersection of factorially closed subrings of B is a factorially closed subring of B.

1.1.2 Fields

If K is a field and $n \in \mathbb{N}$, then $K^{(n)}$ denotes the field of rational functions in n variables, i.e., $K^{(n)} = \text{frac}(K^{[n]})$. If L is a subfield of K and $x_1, \ldots, x_m \in K$, then $L(x_1, \ldots, x_m)$ denotes the subfield of K generated by $x_1, \ldots, x_m$ over L. Note that x_i is transcendental over L if and only if $L(x_i) = L^{(1)}$.

K is **ruled** if there exists a subfield L and element $t \in K$ with $K = L(t) = L^{(1)}$. If R is a commutative K-domain, then R is **rational** over K if and only if $\text{frac}(R) = K^{(n)}$ for some $n \geq 0$.

The famous Lüroth's Theorem asserts that, if $L \subset K \subset M$ are fields with $L \neq K$ and $M = L(x)$ for some $x \in M$ transcendental over L, then there exists $y \in M$ with $K = L(y) = L^{(1)}$. Lüroth's Theorem was proved by Lüroth for the field $L = \mathbb{C}$ in 1876, and for all fields by Steinitz in 1910 [274, 393]. One generalization states that, if $L \subset K \subset L(x_1, \ldots, x_n)$ and K is of transcendence degree one over L, then $K = L(y)$. This was proved by Gordan for $L = \mathbb{C}$ in 1887, and for all fields by Igusa in 1951 [183, 217]; other proofs appear in [324, 369]. In 1894, Castelnuovo showed that, if $\mathbb{C} \subset K \subset \mathbb{C}(x_1, \ldots, x_n)$ and K is of transcendence degree two over $\mathbb{C}$, then $K = \mathbb{C}(y_1, y_2)$ [45]. Castelnuovo's result does not extend to non-algebraically closed ground fields, or to fields K of higher transcendence degree. An excellent account of ruled fields and their variants can be found in [335].

Zariski asked whether the condition $K^{(n)} \cong_L M^{(n)}$ for fields K, M containing L implies that $K \cong_L M$. This was shown to be false, due to a famous example of Beauville, Colliot-Thélèn, Sansuc and Swinnerton-Dyer [15]. However, when $\text{tr.deg}_L K \leq 2$, then Zariski's question has a positive answer; see [241]. This positive result is known as the Cancellation Theorem for Fields.

1.1.3 Localizations

Let $S \subset B \setminus \{0\}$ be any multiplicatively closed subset. Then $S^{-1}B \subset \text{frac}(B)$ denotes the localization of B at S:

$$S^{-1}B = \{ab^{-1} \in \text{frac}(B) \mid a \in B \ , \ b \in S\}$$

In case $S = \{f^i\}_{i \geq 0}$ for some nonzero $f \in B$, then B_f denotes $S^{-1}B$. Likewise, if $S = B \setminus \mathfrak{p}$ for some prime ideal $\mathfrak{p}$ of B, then $B_{\mathfrak{p}}$ denotes $S^{-1}B$.

1.1.4 Degree Functions

An additive abelian group G is **totally ordered** if G has a total order $\leq$ which is translation invariant: For all $x, y, z \in G$, $x + z \leq y + z$ whenever $x \leq y$. Recall that an abelian group G admits a total order if and only if G is torsion free.

If G is a totally ordered abelian group, then G^+ denotes the submonoid $\{g \in G \mid 0 \leq g\}$, and $G \cup \{-\infty\}$ denotes the totally ordered set which extends the order of G by setting $-\infty < g$ for all $g \in G$.

A **degree function** on B is any map $\deg : B \to G \cup \{-\infty\}$ for a totally ordered abelian group G such that, for all $a, b \in B$, the following three conditions are satisfied.

(1) $\deg(a) = -\infty \Leftrightarrow a = 0$
(2) $\deg(ab) = \deg(a) + \deg(b)$
(3) $\deg(a + b) \leq \max\{\deg(a), \deg(b)\}$

Here, it is understood that $(-\infty) + (-\infty) = -\infty$, and $(-\infty) + g = -\infty$ for all $g \in G$. It is an easy exercise to show that equality holds in condition (3) if $\deg(a) \neq \deg(b)$. We will say that deg is *a degree function with values in G*. Unless specified otherwise, it will be assumed that $\deg c = 0$ for every nonzero c in the ground field k.

A degree function deg on B gives rise to the filtration $B = \bigcup_{g \in G} F_g$, where $F_g = \{b \in B \mid \deg(b) \leq g\}$. Note that F_0 is a subring of B. In addition, the restriction of deg to any subring of B is again a degree function. If $K = \mathrm{frac}(B)$, then deg extends to K by setting $\deg(u/v) = \deg u - \deg v$ for $u, v \in B$, $v \neq 0$.

If $\deg(B) \subset G^+ \cup \{-\infty\}$, then $B^* \subset F_0$, F_0 is a factorially closed subring of B, and F_g is an F_0-module for each $g \in G$. In this case, $\{F_g\}_{g \in G}$ is the set of **degree modules** induced by deg.

Given $D \in \mathrm{Der}_k(B)$, if the set $\{\deg(Dx) - \deg(x) \mid x \in B, x \neq 0\}$ has a maximum in $G \cup \{-\infty\}$, define the **degree** of D by:

$$\deg D = \max\{\deg(Dx) - \deg(x) \mid x \in B, x \neq 0\}$$

In this case, we say that $\deg D$ *is defined*. Note that, if $D = 0$, then $\deg D$ is defined and $\deg D = -\infty$; conversely, if $\deg D$ is defined and $\deg D = -\infty$, then $D = 0$. (See [77].)

We are primarily interested in degree functions with values in $\mathbb{N}$ or $\mathbb{Z}$.

1.1.5 Graded Rings and Homogeneous Derivations

Let G be an additive abelian group. A G**-grading** of B is a family $\{B_g\}_{g \in G}$ of subgroups of $(B, +)$ such that:

(1) $B = \bigoplus_{g \in G} B_g$
(2) $B_g B_h \subset B_{g+h}$ for all $g, h \in G$

Note that B_0 is a subring of B, $1 \in B_0$, and each B_g is a B_0-module. A G-grading of B **over** k is a G-grading where $k \subset B_0$. We will always assume that G-gradings are over k.

If $N \subset G$ is a submonoid, then an N**-grading** of B is a G-grading such that $B_g = \{0\}$ for all $g \in G \setminus N$.

If G is totally ordered, then $\bigoplus_{g \geq 0} B_g$ and $\bigoplus_{g \leq 0} B_g$ are subrings. If the grading is a G^+-grading, then given $\gamma \in G$, the set $\bigoplus_{g > \gamma} B_g$ is an ideal. In particular, $I = \bigoplus_{g>0} B_g$ is an ideal of B with $B/I = B_0$. A G-grading of B over k is **positive** if and only if it is a G^+-grading where I is a maximal ideal and $B_0 = B/I = k$.

If B is G-graded and M is a B-module, then a G**-grading** of M is a family of B-submodules $\{M_g\}_{g \in G}$ such that:

(3) $M = \bigoplus_{g \in G} M_g$
(4) $B_g M_h \subset M_{g+h}$ for all $g, h \in G$

For the G-grading $B = \bigoplus_{g \in G} B_g$, given $f \in B$, f is G**-homogeneous** if $f \in B_g$ for some $g \in G$, and if $f \neq 0$, then g is uniquely determined. More generally, any nonzero $f \in B$ determines a unique family $\{f_g\}_{g \in G}$ of elements $f_g \in B_g$ such that $f = \sum_{g \in G} f_g$. The G**-support** of f is the set $\mathrm{Supp}_G(f) \subset G$ defined by $\mathrm{Supp}_G(f) = \{g \in G \,|\, f_g \neq 0\}$. Note that $\mathrm{Supp}_G(f)$ is a finite set.

Given a subring $A \subset B$, define sets $A_g = A \cap B_g$ $(g \in G)$. Then A is a G**-graded subring** if and only if $\{A_g\}_{g \in G}$ defines a G-grading of A if and only if A is generated by G-homogeneous elements over A_0. Similarly, given an ideal $I \subset B$, define sets $I_g = I \cap B_g$ $(g \in G)$. Then I is a G**-homogeneous ideal** or G**-graded ideal** if and only if $\{I_g\}_{g \in G}$ defines a G-grading of I as a B-module if and only if I is generated by G-homogeneous elements. See [134], § 1.5 and [43], § 1.5.

Let $S \subset B \setminus \{0\}$ be a multiplicatively closed set of G-homogeneous elements and $K = \mathrm{frac}(B)$. Then $S^{-1}B$ is a G-graded ring, where:

$$(S^{-1}B)_g = \{a/s \in K \,|\, a \in B_u, s \in S_v, u - v = g\}$$

$S^{-1}B$ is a G**-homogeneous localization** of B, which is also denoted by B_S. The subring $(S^{-1}B)_0$ of B_S is denoted by $B_{(S)}$. The set $S_0 = S \cap B_0$ is multiplicatively closed in B_0, and $S_0^{-1}B_0$ is a subring of $B_{(S)}$. When S is the set of all nonzero G-homogeneous elements of B, then $S_0 = B_0 \setminus \{0\}$, and $S_0^{-1}B_0$ and $B_{(S)}$ are fields with $S_0^{-1}B_0 \subset B_{(S)}$

$D \in \mathrm{Der}(B)$ is a G**-homogeneous derivation** if and only if there exists $d \in G$ such that $DB_g \subset B_{g+d}$ for all $g \in G$. If $D \neq 0$, then d is unique, and is the **degree** of D. Note that the kernel of a G-homogeneous derivation of B is a G-graded subring. In addition, if D is G-homogeneous and $f \in B$ has $f = \sum_{g \in G} f_g$, then $Df = 0$ if and only if $Df_g = 0$ for every g. This is because the decomposition of Df into G-homogeneous summands is $\sum_{g \in G} Df_g$.

If the group G is totally ordered, then the function $\deg_G : B \to G \cup \{-\infty\}$ defined by

$$\deg_G(f) = \max \mathrm{Supp}_G(f) \ \ (f \neq 0) \ \text{and} \ \deg_G(0) = -\infty$$

is a degree function on B, and the degree $\deg_G D$ is defined for any G-homogeneous derivation D of B.

For rings graded by a totally ordered abelian group, we have the following result; see [169], Lemma 2.1.

Proposition 1.1 *Let G be a totally ordered abelian group, and let B be a G-graded integral domain.*

(a) *B_0 is algebraically closed in B.*
(b) *If $A \subset B$ is a G-graded subalgebra and H is the algebraic closure of A in B, then H is a G-graded subalgebra of B.*
(c) *If $S \subset B$ is the set of all nonzero G-homogeneous elements of B, then the field $B_{(S)}$ is algebraically closed in $K = \mathrm{frac}(B)$.*

Proof Given nonzero $b \in B$, let $\bar{b} = b_g$ for $g = \max \mathrm{Supp}_G(b)$.

For part (a), let $b \in B \setminus B_0$ be given. Then $v := b - b_0$ is not in B_0 and $\deg_G v \neq 0$. Since G is totally ordered, it is torsion free and the values $\deg_G(v^n) = n \deg_G v$, $n \geq 0$, are distinct. Therefore, given nonzero $P \in B_0[T] = B_0^{[1]}$, we have $\deg_G P(v) = (\deg_T P)(\deg_G v) \in G$, which implies that $P(v) \neq 0$. So v is transcendental over B_0, which implies b is transcendental over B_0. This proves part (a).

For part (b), given an integer $n \geq 0$, let $H(n)$ denote the ring obtained by adjoining to A all elements $h \in H$ such that $\#\mathrm{Supp}_G(h) \leq n$. In particular, $H(0) = A$. We show by induction on n that, for each $n \geq 1$:

$$H(n) \subset H(1) \tag{1.1}$$

This property implies $H = H(1)$, which is a G-graded subring of B, being generated by G-homogeneous elements over A.

Assume that, for some $n \geq 2$, $H(n-1) \subset H(1)$. Let $h \in H$ be given such that $\#\mathrm{Supp}_G(h) = n$, and let $\sum_{i \geq 0} a_i h^i = 0$ be a non-trivial dependence relation for h over A, where $a_i \in A$ for each i. Define:

$$d = \max_{i \geq 0} \{\deg_G a_i h^i\} \quad \text{and} \quad I = \{\, i \in \mathbb{Z} \mid i \geq 0 \ , \ \deg_G a_i h^i = d\}$$

Then I is non-empty and $\sum_{i \in I} \bar{a}_i \bar{h}^i = 0$. Since A is a G-graded subalgebra, $\bar{a}_i \in A$ for each i. Therefore, $\bar{h}$ is algebraic over A. Since $h = (h - \bar{h}) + \bar{h}$, it follows that $h \in H(n-1) + H(1)$. Since $H(n-1) \subset H(1)$, we see that $h \in H(1)$, thus proving by induction the inclusion claimed in (1.1). This proves part (b).

For part (c), let nonzero $w \in K$ and nonzero $p \in B_{(S)}[T] = B_{(S)}^{[1]}$ be given and extend $\deg_G$ to K. If $\deg_G w \neq 0$, then since G is totally ordered, it is torsion free and the values $\deg_G w^n$, $n \geq 0$, are distinct. So $p(w) \neq 0$ when $\deg_G w \neq 0$, and w is transcendental over $B_{(S)}$ in this case.

Suppose that $p(w) = 0$. Then $\deg_G w = 0$. If $t = \deg_T p$, then we have:

$$P(X, Y) := Y^t p(X/Y) \in B_{(S)}[X, Y] = B_{(S)}^{[2]}$$

Note that P is homogeneous for the $\mathbb{Z}$-grading of $B_{(S)}[X,Y]$ over $B_{(S)}$ for which X,Y are homogeneous of degree one. If $w = u/v$ for $u,v \in B$, then $P(u,v) = v^t p(w) = 0$. Consequently, $P(\bar{u},\bar{v}) = 0$, and if $\hat{w} = \bar{u}/\bar{v}$, then $\hat{w} \in B_{(S)}$ (since $\deg_G \bar{u} = \deg_G \bar{v}$) and:

$$p(\hat{w}) = p(\bar{u}/\bar{v}) = \bar{v}^{-t} P(\bar{u},\bar{v}) = 0$$

Therefore, $T - \hat{w}$ divides $p(T)$, which implies that the minimal polynomial of w over $B_{(S)}$ is linear, i.e., $w \in B_{(S)}$. This proves part (c). $\square$

We are primarily interested in G-gradings of B where G is a finitely generated free abelian group $G = \mathbb{Z}^n$, $n \geq 1$, though gradings by other groups—even those not admitting a total order, such as finite groups—are also important.

1.1.6 Associated Graded Rings

Let G be a totally ordered abelian group, and let R be a commutative k-domain. If R admits a G-filtration, then it is possible to construct an associated G-graded ring $\mathrm{Gr}(R)$. In general, $\mathrm{Gr}(R)$ can be difficult to work with, and we first consider a frequently encountered special case.

Suppose that B is a commutative k-domain and that $B = \bigoplus_{g \in G} B_g$ is a G-grading over k for the totally ordered abelian group G. Let $\deg_G$ denote the induced degree function on B. Given nonzero $f \in B$, write $f = \sum_{g \in G} f_g$, where $f_g \in B_g$, and let $\gamma = \deg_G f$. The **highest-degree homogeneous summand** of f is $\bar{f} = f_\gamma$. In case $f = 0$, define $\bar{0} = 0$.

For any k-subalgebra $R \subset B$, the **associated graded ring** $\mathrm{Gr}(R)$ is the subalgebra of B generated by the set $\{\bar{f} \mid f \in R\}$. Observe that $\mathrm{Gr}(R)$ is a G-graded subalgebra of B, and that $\mathrm{Gr}(R) = R$ if and only if R is a G-graded subalgebra. In particular, $\mathrm{Gr}(B) = B$.

Since G is totally ordered and B is an integral domain, it follows that $\overline{ab} = \bar{a} \cdot \bar{b}$ for any pair $a,b \in B$. Therefore, the function $\mathrm{gr} : R \to \mathrm{Gr}(R)$ defined by $\mathrm{gr}(f) = \bar{f}$ is a homomorphism of the multiplicative monoids $R \setminus \{0\}$ and $\mathrm{Gr}(R) \setminus \{0\}$. Note that this function does not generally respect addition.

Let $D \in \mathrm{Der}_k(R)$ be such that $\deg_G D$ is defined for the degree function $\deg_G$ restricted to R. Define the **associated G-homogeneous derivation** $\mathrm{gr}\, D : \mathrm{Gr}(R) \to \mathrm{Gr}(R)$ as follows: $(\mathrm{gr}\, D)(0) = 0$, and for nonzero $f \in R$:

$$(\mathrm{gr}\, D)(\bar{f}) = \begin{cases} \overline{Df} & \text{if } \deg_G(Df) - \deg_G(f) = \deg_G D \\ 0 & \text{if } \deg_G(Df) - \deg_G(f) < \deg_G D \end{cases}$$

(See [83].) Since the elements $\bar{f}$ ($f \in R$) generate $\mathrm{Gr}(R)$ as a k-algebra, this suffices to define $\mathrm{gr}\, D$. The reader can verify that $\mathrm{gr}\, D$ is a G-homogeneous element of $\mathrm{Der}_k(\mathrm{Gr}(R))$, $\deg_G(\mathrm{gr}\, D) = \deg_G(D)$, and $\mathrm{gr}(\ker D) \subset \ker(\mathrm{gr}\, D)$.

The following result specifies $\mathrm{Gr}(R)$ in certain cases.

Lemma 1.2 ([83], Lemma 3.7) *Let G be a totally ordered abelian group and $B = \bigoplus_{g\in G} B_g$ a G-graded integral domain. Let $A = \bigoplus_{g\le 0} B_g$, $x \in B$, and $R = A[x]$. Then* $\mathrm{Gr}(R) = A[\bar{x}]$.

Proof If $x \in A$ this is clear, since A is a G-graded subring of B. So assume $x \notin A$. Given nonzero $r \in R$, suppose that $r = \sum_{1\le i\le n} a_i x^{e_i}$ for nonzero $a_1, \dots, a_n \in A$ and distinct $e_1, \dots, e_n \in \mathbb{N}$ ($n \ge 2$). If $\deg_G(a_i x^{e_i}) = \deg_G(a_j x^{e_j})$ for $e_i > e_j$, then $\deg_G(a_i x^{e_i - e_j}) = \deg_G(a_j) \le 0$, which implies that $a_i x^{e_i - e_j} \in A$. We may therefore write r as a sum of $n-1$ terms of the form ax^e ($a \in A$, $e \in \mathbb{N}$). Since this kind of reduction can be carried out only a finite number of times, we can assume that:

$$r = \sum_{1\le i\le n} a_i x^{e_i} \quad \text{and} \quad \deg_G(a_i x^{e_i}) \ne \deg_G(a_j x^{e_j}) \quad \text{when} \quad i \ne j$$

Let m ($1 \le m \le n$) be such that $\deg_G(a_m x^{e_m}) = \deg_G r$. Then $\bar{r} = \bar{a}_m \bar{x}^{e_m}$. Since $\bar{a}_m \in A$, we see that $\bar{r} \in A[\bar{x}]$. Since $\mathrm{Gr}(R)$ is generated by elements $\bar{r}$ for $r \in R$, it follows that $\mathrm{Gr}(R) \subset A[\bar{x}]$, and the reverse inclusion is clear. □

We now turn our attention to the more general case, following the ideas of Makar-Limanov in [276] for $\mathbb{Z}$-filtrations. Let G be a totally ordered abelian group. By a ***G*-filtration** of B we mean a collection $\{B_i\}_{i\in G}$ of subsets of B with the following properties.

1. Each B_i is a vector space over k.
2. $B_j \subset B_i$ whenever $j \le i$.
3. $B = \cup_{i\in G} B_i$
4. $B_i B_j \subset B_{i+j}$ for all $i, j \in G$.

The filtration will be called a **proper *G*-filtration** if the following two properties also hold (where S^C denotes the complement of the set S in B).

5. $\cap_{i\in G} B_i = \{0\}$
6. If $a \in B_i \cap B_{i-1}^C$ and $b \in B_j \cap B_{j-1}^C$, then $ab \in B_{i+j} \cap B_{i+j-1}^C$.

Note that any degree function on B will give a proper G-filtration.

For k-vector spaces $W \subset V$, the notation V/W will denote the k-vector space V modulo W in the usual sense. Suppose $B = \cup B_i$ is a proper G-filtration, and define the **associated graded ring** $\mathrm{Gr}(B)$ as follows. The k-additive structure on $\mathrm{Gr}(B)$ is given by:

$$\mathrm{Gr}(B) = \oplus_{n\in G} B_n / B_{n-1}$$

Consider elements $a + B_{i-1}$ belonging to B_i/B_{i-1}, and $b + B_{j-1}$ belonging to B_j/B_{j-1}, where $a \in B_i$ and $b \in B_j$. Their product is the element of B_{i+j}/B_{i+j-1} defined by:

$$(a + B_i/B_{i-1})(b + B_j/B_{j-1}) = ab + B_{i+j-1}$$

Now extend this multiplication to all of $\mathrm{Gr}(B)$ by the distributive law.

Note that, because of axiom 6, $\mathrm{Gr}(B)$ is a commutative k-domain.

Because of axiom 5, for each nonzero $a \in B$, the set $\{ i \in G \,|\, a \in B_i \}$ has a minimum, which will be denoted $\iota(a)$. The natural map $\rho : B \to \mathrm{Gr}(B)$ is the one which sends each nonzero $a \in B$ to its class in B_i/B_{i-1}, where $i = \iota(a)$. We also define $\rho(0) = 0$.

Given $a \in B$, observe that $\rho(a) = 0$ if and only if $a = 0$. Note further that ρ is a multiplicative map, but is not an algebra homomorphism, since it does not generally respect addition.

In case B is already a G-graded ring, then B admits a filtration relative to which B and $\mathrm{Gr}(B)$ are canonically isomorphic via ρ. In particular, if $B = \oplus_{i \in G} A_i$, then a proper G-filtration is defined by $B_i = \oplus_{j \le i} A_j$.

Example 1.3 Let $B = k[x]$, a univariate polynomial ring over k, and let B_i consist of polynomials of degree at most i ($i \ge 0$). Then $k[x] = \cup B_i$ is a $\mathbb{Z}$-filtration (with $B_i = \{0\}$ for $i < 0$), and $\mathrm{Gr}(k[x]) = \oplus_{i \ge 0} kx^i \cong k[x]$.

Example 1.4 Let $B = k(x)$, a univariate rational function field over k. Given nonzero $p(x), q(x) \in k[x]$, define the degree of $p(x)/q(x)$ to be $\deg p(x) - \deg q(x)$. Let B_i consist of functions of degree at most i. Then $\mathrm{Gr}(k(x)) = k[x, x^{-1}]$, the ring of Laurent polynomials.

Now suppose $B = \cup B_i$ is a proper G-filtration. Given $D \in \mathrm{Der}_k(B)$, we say that D **respects the filtration** if there exists an integer t such that, for all $i \in G$, $D(B_i) \subset B_{i+t}$. Define a function $\mathrm{gr}(D) : \mathrm{Gr}(B) \to \mathrm{Gr}(B)$ as follows.

If $D = 0$, then $\mathrm{gr}(D)$ is the zero map.

If $D \neq 0$, choose $t \in G$ to be minimal such that $D(B_i) \subset B_{i+t}$ for all $i \in G$. Then given $i \in G$, define

$$\mathrm{gr}(D) : B_i/B_{i-1} \to B_{i+t}/B_{i+t-1}$$

by the rule $\mathrm{gr}(D)(a + B_{i-1}) = Da + B_{i+t-1}$. Now extend $\mathrm{gr}(D)$ to all of $\mathrm{Gr}(B)$ by linearity. It is an easy exercise to check that $\mathrm{gr}(D)$ satisfies the product rule, and is therefore a homogeneous k-derivation of $\mathrm{Gr}(B)$. The reader should note that $\mathrm{gr}(D) = 0$ if and only if $D = 0$. In addition, observe that, by definition:

$$\rho(\ker D) \subset \ker(\mathrm{gr}(D))$$

Given $a \in B$, the notation $\mathrm{gr}(a)$ is commonly used to denote the image $\rho(a)$. In doing so, one must be careful to distinguish $\mathrm{gr}(D)(a)$ from $\mathrm{gr}(Da)$.

The case in which a filtration is given by ideals is treated in many sources, for example, Matsumura [285]. A categorical treatment of filtrations, degree functions and associated graded rings can be found in Škoda [413].

1.1.7 Locally Finite and Locally Nilpotent Derivations

A derivation $D \in \mathrm{Der}(B)$ is **locally finite** if and only if for each $f \in B$, the $\mathbb{Q}$-vector space spanned by the images $\{D^n f \mid n \geq 0\}$ is finite dimensional, or equivalently, if and only if there exists a monic polynomial $p(t) \in \mathbb{Q}[t]$ (depending on f) such that $p(D)(f) = 0$.

A derivation $D \in \mathrm{Der}(B)$ is **locally nilpotent** if and only if to each $f \in B$, there exists $n \in \mathbb{N}$ (depending on f) such that $D^n f = 0$, i.e., if and only if $\mathrm{Nil}(D) = B$. The term locally nilpotent derivation is sometimes abbreviated LND. Thus, the locally nilpotent derivations are special kinds of locally finite derivations. Let $\mathrm{LND}(B)$ denote the set of all $D \in \mathrm{Der}(B)$ which are locally nilpotent. Important examples of locally nilpotent derivations are the familiar partial derivative operators on a polynomial ring. If A is a subring of B, define $\mathrm{LND}_A(B) = \mathrm{Der}_A(B) \cap \mathrm{LND}(B)$.

Lemma 1.5 *Suppose that G is an abelian group, $B = \bigoplus_{g \in G} B_g$ is a G-grading over k, and $D \in \mathrm{Der}_k(B)$ is G-homogeneous. If $\dim_k B_g < \infty$ for each $g \in G$ and $\deg_G D = 0$, then D is locally finite.*

Proof Given $f \in B$, let $S = \mathrm{Supp}_G(f)$ and set $V = \bigoplus_{g \in S} B_g$. By hypothesis, V is a finite-dimensional vector space over k, and D restricts to a linear map $L : V \to V$. If $p \in k^{[1]}$ is the minimal polynomial of L, then $p(D)(V) = 0$. □

As is evident from its title, the derivations investigated in this book are the locally nilpotent derivations. Apart from being an interesting and important topic in its own right, the study of locally nilpotent derivations is motivated by their connection to algebraic group actions. Specifically, the condition "locally nilpotent" imposed on a derivation corresponds precisely to the condition "algebraic" imposed on the corresponding group action. This is explained in *Sect. 1.5* below.

For a discussion of derivations in a more general setting, the reader is referred to the books of Northcott [331] and Nowicki [333]. The topic of locally finite derivations is explored in Chap. 9 of Nowicki's book; in Chap. 1.3 of van den Essen's book [142]; and in papers of Zurkowski [431, 432].

1.1.8 Degree Function Induced by a Derivation

The degree function $\deg_D$ induced by a derivation D is a simple yet indispensable tool in working with D, especially in the locally nilpotent case. Given $D \in \mathrm{Der}(B)$ and $f \in \mathrm{Nil}(D)$, we know that $D^n f = 0$ for $n \gg 0$. If $f \neq 0$, define

$$\deg_D(f) = \min\{n \in \mathbb{N} \mid D^{n+1} f = 0\} \ .$$

In addition, define $\deg_D(0) = -\infty$. It is shown in *Proposition 1.10* that $\mathrm{Nil}(D)$ is a subalgebra of B and $\deg_D$ is a degree function on $\mathrm{Nil}(D)$.

If D is locally nilpotent, then $\deg_D$ induces the $\mathbb{N}$-filtration

$$B = \bigcup_{i \in \mathbb{N}} \mathcal{F}_i$$

where $\mathcal{F}_i = \{f \in B \mid \deg_D f \le i\}$, and $\deg_D D = -1$. This filtration is discussed in *Sect.* 1.6 below.

1.1.9 Exponential and Dixmier Maps

Given $D \in \mathrm{LND}(B)$, the **exponential function** determined by D is $\exp D : B \to B$, where:

$$\exp D(f) = \sum_{i \ge 0} \frac{1}{i!} D^i f$$

Likewise, for any local slice $r \in B$ of D, the **Dixmier map** induced by r is $\pi_r : B \to B_{Dr}$, where:

$$\pi_r(f) = \sum_{i \ge 0} \frac{(-1)^i}{i!} D^i f \frac{r^i}{(Dr)^i}$$

Here, B_{Dr} denotes localization at Dr. Note that, since D is locally nilpotent, both $\exp D$ and π_r are well-defined. These definitions rely on the fact that B contains $\mathbb{Q}$.

1.1.10 Derivative of a Polynomial

If A is a subring of B, and $B = A[t] \cong A^{[1]}$ for some $t \in B$, the **derivative** of B relative to the pair (A, t) is the derivation $(\frac{d}{dt})_A \in \mathrm{Der}_A(B)$ uniquely defined by $(\frac{d}{dt})_A(t) = 1$. (As mentioned, a derivation is uniquely determined by its image on a generating set.) Usually, if the subring A is understood, we denote this derivation more simply by $\frac{d}{dt}$; in this case, given $P(t) \in A[t]$, we also define

$$P'(t) := \frac{d}{dt}(P(t)) \ .$$

Likewise, given $n \ge 0$, the notations

$$P^{(n)}(t) \quad \text{and} \quad \frac{d^n P}{dt^n}$$

each denotes the n-fold composition:

$$\left(\frac{d}{dt}\right)^n (P(t))$$

Note that it is possible that $B = \tilde{A}[t] = A[t]$ for subrings $A \neq \tilde{A}$, in which case $(\frac{d}{dt})_A \neq (\frac{d}{dt})_{\tilde{A}}$. It can also happen that $B = A[t] = A[s]$ for elements $s \neq t$, in which case $(\frac{d}{dt})_A \neq (\frac{d}{ds})_A$. So one must be careful. See [2].

1.2 Basic Facts About Derivations

At the beginning of this chapter, two defining conditions (C.1) and (C.2) for a k-derivation D of B are given, which imply further conditions (C.3)–(C.6). We now examine the next layer of consequences implied by these conditions.

1.2.1 Algebraic Operations

Proposition 1.6 *Let $D \in \mathrm{Der}(B)$ be given, and let $A = \ker D$.*

(a) *$D(ab) = aDb$ for all $a \in A, b \in B$. Therefore, D is an A-module endomorphism of B.*
(b) ***power rule:*** *For any $t \in B$ and $n \geq 1$, $D(t^n) = nt^{n-1}Dt$.*
(c) ***quotient rule:*** *If $g \in B^*$ and $f \in B$, then $D(fg^{-1}) = g^{-2}(gDf - fDg)$.*
(d) ***higher product rule:*** *For any $a, b \in B$ and any integer $m \geq 0$:*

$$D^m(ab) = \sum_{i+j=m} \binom{m}{i} D^i a D^j b$$

Proof Property (a) is immediately implied by (C.1) and (C.2).

To prove (b), proceed by induction on n, the case $n = 1$ being clear. Given $n \geq 2$, assume by induction that $D(t^{n-1}) = (n-1)t^{n-2}Dt$. By the product rule (C.2):

$$D(t^n) = tD(t^{n-1}) + t^{n-1}Dt = t \cdot (n-1)t^{n-2}Dt + t^{n-1}Dt = nt^{n-1}Dt$$

So (b) is proved. Part (c) follows from the equation:

$$Df = D(g \cdot fg^{-1}) = gD(fg^{-1}) + fg^{-1}Dg$$

Finally, (d) is easily proved by inductive application of the product rule (C.2), together with (C.1) and (C.4). $\square$

Proposition 1.7 *Suppose A is a subring of B and $t \in B$ is transcendental over A. If $P(t) \in A[t]$ is given by $P(t) = \sum_{0 \le i \le m} a_i t^i$ for $a_i \in A$, then*

$$P'(t) = \sum_{1 \le i \le m} i a_i t^{i-1}$$

where $P'(t) = (\frac{d}{dt})_A(P(t))$.

Proof By parts (a) and (b) above, we have, for $1 \le i \le m$:

$$\frac{d}{dt}(a_i t^i) = a_i \frac{d}{dt}(t^i) = a_i(it^{i-1})$$

By now applying the additive property (C.1), the desired result follows. □

The proof of the following corollary is an easy exercise.

Corollary 1.8 (Taylor's Formula) *Let A be a subring of B. Given $s, t \in B$, and $P \in A^{[1]}$ of degree $n \ge 0$:*

$$P(s+t) = \sum_{i=0}^{n} \frac{P^{(i)}(s)}{i!} t^i$$

Proposition 1.9 *Let $D \in \mathrm{Der}(B)$ and let $A = \ker D$.*

- **(a)** *$A \cap D^n B$ is an ideal of A for each $n \ge 0$.*
- **(b)** *Any ideal of B generated by elements of A is an integral ideal for D.*
- **(c)** **chain rule:** *If $P \in A^{[1]}$ and $t \in B$, then $D(P(t)) = P'(t)Dt$.*
- **(d)** *A is an algebraically closed subring of B.*

Proof For (a), since $D : B \to B$ is an A-module homomorphism, both A and $D^n B$ are A-submodules of B. Thus, $A \cap D^n B$ is an A-submodule of A, i.e., an ideal of A. Part (b) is immediately implied by *Proposition 1.6 (a)*. Likewise, part (c) is easily implied by *Proposition 1.6 (a–c)*.

For (d), suppose $t \in B$ is an algebraic element over A, and let $P \in A^{[1]}$ be a nonzero polynomial of minimal degree such that $P(t) = 0$. Then part (b) implies $0 = D(P(t)) = P'(t)Dt$. If $Dt \ne 0$, then $P'(t) \ne 0$ as well, by minimality of P. Since B is a domain, this is impossible. Therefore, $Dt = 0$. □

Note that $P'(t)$ in part (b) above means evaluation of P' as defined on $A^{[1]}$.

1.2.2 Subalgebra Nil(D)

Proposition 1.10 (See Also [332]) *Let $D \in \mathrm{Der}(B)$ be given.*

- **(a)** $\deg_D(Df) = \deg_D(f) - 1$ *whenever $f \in \mathrm{Nil}(D) \setminus \ker(D)$.*
- **(b)** $\mathrm{Nil}(D)$ *is a $\mathbb{Q}$-subalgebra of B.*
- **(c)** $\deg_D$ *is a degree function on* $\mathrm{Nil}(D)$.

Proof For the given elements f and g, set $m = \deg_D(f)$ and $n = \deg_D(g)$. Assume $fg \neq 0$, so that $m \geq 0$ and $n \geq 0$. Since $0 = D^{m+1}f = D^m(Df)$, it follows that $Df \in \mathrm{Nil}(D)$. Assertion (a) now follows by definition of $\deg_D$.

In addition, if $\mu = \max\{m, n\}$, then $D^{\mu+1}(f+g) = D^{\mu+1}f + D^{\mu+1}g = 0$. So $\mathrm{Nil}(D)$ is closed under addition. This equation also implies that, for all $f, g \in \mathrm{Nil}(D)$, $\deg_D(f+g) \leq \max\{\deg_D(f), \deg_D(g)\}$.

By the higher product rule, we also see that:

$$D^{m+n+1}(fg) = \sum_{i+j=m+n+1} \binom{m+n+1}{i} D^i f D^j g$$

If $i + j = m + n + 1$ for non-negative i and j, then either $i > m$ or $j > n$. Thus, $D^i f D^j g = 0$, implying $D^{m+n+1}(fg) = 0$. Therefore, $\mathrm{Nil}(D)$ is closed under multiplication, and forms a subalgebra of B, and (b) is proved.

The reasoning above shows that $\deg_D(fg) \leq m + n$, and further shows that $D^{m+n}(fg) = \frac{(m+n)!}{m!n!} D^m f D^n g \neq 0$. Therefore, $\deg_D(fg) = m + n$, and (c) is proved. □

Corollary 1.11 *Suppose that G is an abelian group and $B = \bigoplus_{g \in G} B_g$ is a G-grading such that B is a finitely generated G-homogeneous B_0-algebra, i.e., B_1 is finitely generated as a B_0-module and $B = B_0[B_1]$. Then any B_0-module endomorphism $L : B_1 \to B_1$ defines G-homogeneous $D_L \in \mathrm{Der}_{B_0}(B)$ with $\deg_G(D_L) = 0$, and D_L is locally nilpotent if and only if L is nilpotent.*

Proof Since $\mathrm{Nil}(D_L)$ is a B_0-subalgebra of B (*Proposition 1.10*), and since B is generated by B_1 as a B_0-algebra, we see that $B = \mathrm{Nil}(D_L)$ if and only if $B_1 \subset \mathrm{Nil}(D_L)$. Therefore, D_L is locally nilpotent if and only if L is nilpotent. □

1.2.3 Kernels

Proposition 1.12 (See Nowicki [333], 3.3.2) *Let $K \subset L$ be fields of characteristic zero. The following are equivalent.*

(a) *There exists $d \in \mathrm{Der}(L)$ such that $K = \ker d$.*
(b) *K is algebraically closed in L.*

From this Nowicki further characterizes all k-subalgebras of B which are kernels of k-derivations.

Proposition 1.13 ([333], 4.1.4) *Let B be an affine k-domain, and $A \subset B$ a k-subalgebra. The following are equivalent.*

(a) *There exists $D \in \mathrm{Der}_k(B)$ such that $A = \ker D$.*
(b) *A is integrally closed in B and $\mathrm{frac}(A) \cap B = A$.*

We also have:

Proposition 1.14 *Suppose B is an algebraic extension of the subring B'. If $D, E \in \mathrm{Der}(B)$ and $Df = Ef$ for every $f \in B'$, then $D = E$.*

Proof We have that $D-E \in \mathrm{Der}(B)$, and that $\ker(D-E)$ contains B'. Since $\ker(D-E)$ is algebraically closed in B, it contains the algebraic closure of B' in B, i.e., $B \subset \ker(D-E)$. This means $D-E = 0$, and thus $D = E$. □

Remark 1.15 Observe that, while $\ker D$ is algebraically closed in B for $D \in \mathrm{Der}(B)$, $\mathrm{Nil}(D)$ may fail to be. For example, if δ is the extension of the derivative $\frac{d}{dx}$ on $k[x]$ to the field $k(x)$, then $\mathrm{Nil}(\delta) = k[x]$ in $k(x)$. As another example (due to Daigle), consider the power series ring $B = \mathbb{Q}[[x]]$ and its natural derivation $D = \frac{d}{dx}$: we have $\ker D = \mathbb{Q}$ and $\mathrm{Nil}(D) = \mathbb{Q}[x]$. Thus, $\mathrm{Nil}(D)$ is not even integrally closed in B, since $\sqrt{1+x} \notin \mathrm{Nil}(D)$, whereas $1 + x \in \mathrm{Nil}(D)$.

Remark 1.16 If $\mathcal{O}$ is a commutative k-algebra which is not an integral domain, then the kernel of a k-derivation, even a locally nilpotent derivation, can fail to be algebraically closed in $\mathcal{O}$. For example, let $R = k[t]$ where $t \neq 0$ and $t^2 = 0$. Define $\mathcal{O} = R[v] = R^{[1]}$ and define $d \in \mathrm{LND}_R(\mathcal{O})$ by $dv = tv$. Then $\ker d = R[tv, tv^2, tv^3, \ldots] = k[t, tv, tv^2, tv^3, \ldots]$. Similarly, suppose that $\mathcal{O}' = k[T]$ where $T^2 \neq 0$ and $T^3 = 0$. Then $\mathcal{O}' = k \oplus kT \oplus kT^2$. Define $e \in \mathrm{LND}_k(\mathcal{O}')$ by the nilpotent operator

$$e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

in the k-basis $\{1, T, T^2\}$. Then $\ker e = k + kT^2$.

1.2.4 Localization

Let S be any multiplicatively closed subset of B not containing 0, and let $D \in \mathrm{Der}_k(B)$. Then D extends uniquely to a **localized derivation** $S^{-1}D$ on the localization $S^{-1}B$ via the quotient rule above. Note that $(S \cap \ker D)^{-1}(\ker D) \subset \ker(S^{-1}D)$, with equality when $S \subset \ker D$.

As a matter of notation, if $S = \{f^i\}_{i \geq 0}$ for some nonzero $f \in B$, then D_f denotes the induced derivation $S^{-1}D$ on B_f. Likewise, if $S = B \setminus \mathfrak{p}$ for some prime ideal $\mathfrak{p}$ of B, then $D_{\mathfrak{p}}$ denotes the induced derivation $S^{-1}D$ on $B_{\mathfrak{p}}$.

1.2.5 Integral Ideals

Suppose $D \in \mathrm{Der}_k(B)$ and $I \subset B$ is an integral ideal of D, meaning $DI \subset I$. Then the pair (D, I) induces a well-defined **quotient derivation** D/I on the quotient B/I in an obvious way: $D/I([b]) = [Db]$, where $[b]$ denotes the congruence class of $b \in B$, modulo I. Conversely, any ideal $I \subset B$ for which D/I is well-defined is an integral ideal of D. Miyanishi gives the following basic properties of integral ideals for derivations.

Proposition 1.17 ([304], Lemma 1.1) *Suppose that B is a commutative noetherian k-domain, and let $\delta \in \mathrm{Der}_k(B)$. If $I, J \subset B$ are integral ideals for δ, then:*

(a) *$I + J$, IJ, and $I \cap J$ are integral ideals for δ.*
(b) *Every prime divisor $\mathfrak{p}$ of I is an integral ideal for δ.*
(c) *The radical $\sqrt{I}$ is an integral ideal for δ.*

1.2.6 Extension of Scalars

Nowicki gives parts (a) and (b) of the following result, and part (c) is an easy consequence of the definitions.

Proposition 1.18 ([333], 5.1.1 and 5.1.3) *Let $D \in \mathrm{Der}_k(B)$ be given, and suppose $k \subset k'$ is a field extension. Let $B' = k' \otimes_k B$ and $D' = 1 \otimes D \in \mathrm{Der}_{k'}(B')$.*

(a) *$k' \otimes_k B^D$ and $(B')^{D'}$ are isomorphic as k'-algebras.*
(b) *B^D is a finitely generated k-algebra if and only if $(B')^{D'}$ is a finitely generated k'-algebra.*
(c) *$D \in \mathrm{LND}(B)$ if and only if $D' \in \mathrm{LND}(B')$*

A similar result for group actions is given by Nagata in [326], 8.9 and 8.10. See also [142], 1.2.7.

The reader is reminded that, although B is an integral domain, it does not follow that B' is an integral domain. Nonetheless, in certain cases B' is a domain, for example, if $B = k^{[n]}$, then $B' = (k')^{[n]}$.

1.2.7 Integral Extensions and Conductor Ideals

The following fundamental result is due to Seidenberg. The reader is referred to [376] or [142], 1.2.15, for its proof.

Proposition 1.19 (Seidenberg's Theorem) *Let R be a noetherian integral domain containing $\mathbb{Q}$, and let $\mathcal{O}$ be the integral closure of R in $\mathrm{frac}(R)$. Then every $D \in \mathrm{Der}(R)$ extends to $\mathcal{O}$.*

Definition 1.20 Let $R \subset S$ be integral domains. The **conductor** of S in R is:

$$\mathcal{C}_R(S) = \{r \in R \mid rS \subset R\}$$

If $\mathcal{O}_R$ is the integral closure of R in frac(R), then the **conductor ideal** of R is $\mathcal{C}_R(\mathcal{O}_R)$. Note that $\mathcal{C}_R(S)$ is an ideal of both R and S, and is the largest ideal of S contained in R. The following two properties of the conductor are easily verified.

(1) $\mathcal{C}_{R^{[n]}}(S^{[n]}) = \mathcal{C}_R(S) \cdot S^{[n]}$ for every $n \geq 0$
(2) $D\mathcal{C}_R(S) \subset \mathcal{C}_R(S)$ for every $D \in \text{Der}(S)$ restricting to R

Lemma 1.21 ([170], Lemma 4.1) *Let K be a field, let R be an integral domain containing K, and let $\mathfrak{C} \subset R$ be the conductor ideal of R. If R is affine over K, then $\mathfrak{C} \neq (0)$.*

Proof Since R is affine over K, its normalization $\mathcal{O}$ is also affine over K, and is finitely generated as an R-module (see [203], Ch. I, Thm. 3.9A). Let $\{\omega_1, \ldots, \omega_n\}$ be a generating set for $\mathcal{O}$ as an R-module, and let nonzero $r \in R$ be such that $r\omega_1, \ldots, r\omega_n \in R$. Then $r \in \mathfrak{C}$. □

1.3 Varieties and Group Actions

In the purely algebraic situation, it is advantageous to consider the general category of commutative k-domains. In the geometric setting, however, our primary interest relates to irreducible affine k-varieties X over an algebraically closed field k. The classical case is when $k = \mathbb{C}$.

So assume in this section that k is an algebraically closed field (of characteristic zero). We will consider irreducible affine varieties X over k, endowed with the Zariski topology. The coordinate ring of X, or ring of regular functions, is indicated by either $k[X]$ or $\mathcal{O}(X)$. X is **normal** if $k[X]$ is integrally closed in its field of fractions, and X is **factorial** if $k[X]$ is a UFD. The **cylinder** over X is $X \times \mathbb{A}^1$. The zero set of $f \in k[X]$ in X is denoted by $\mathcal{V}(f)$. If B is an affine k-domain, then $X = \text{Spec}(B)$ is the corresponding affine variety. Affine n-space over k will be denoted by $\mathbb{A}^n_k$, or simply $\mathbb{A}^n$ when the ground field k is understood. We also wish to consider algebraic groups G over k. The reader can find these standard definitions in many sources, some of which are given at the end of this section.

Suppose that G is an algebraic k-group, and that G acts algebraically on the irreducible affine k-variety X.[2] In this case, X is called a ***G*-variety**. G acts by automorphisms on the coordinate ring $B = k[X]$, and the **ring of invariants** for

[2]Note that the term **regular action** is also used in the literature to indicate an algebraic action. A regular action is thus distinguished from a **rational action**, which refers to the action of a group by birational automorphisms.

the action is:

$$B^G = \{f \in B \mid g \cdot f = f \text{ for all } g \in G\}$$

Some authors refer to B^G as the **fixed ring** of the action. An element $f \in B^G$ is called an **invariant function** for the action. Likewise, $f \in B$ is called a **semi-invariant** for the action if there exists a character $\chi : G \to k^*$ such that $g \cdot f = \chi(g)f$ for all $g \in G$. In this case, χ is the **weight** of the semi-invariant f. Certain important groups, like the special linear group $SL_2(k)$ and the additive group k^+ of the field k, have no nontrivial characters.

The set of **fixed points** for the action is:

$$X^G = \{x \in X \mid g \cdot x = x \text{ for all } g \in G\}$$

The action is **fixed point free**, or simply **free**, if X^G is empty.

The **orbit** of $x \in X$ is $Gx = \{g \cdot x \mid g \in G\}$, and the **stabilizer** of x is $G_x = \{g \in G \mid g \cdot x = x\}$. The consideration of orbits leads naturally to the important (and subtle) question of forming quotients. Questions about orbits are at the heart of Geometric Invariant Theory.

Since we wish to navigate within the category of affine varieties, we define the **categorical quotient** for the action (if it exists) to be an affine variety Z, together with a morphism $\pi : X \to Z$, satisfying: (a) π is constant on the orbits, and (b) for any other affine variety Z' with morphism $\phi : X \to Z'$ which is constant on the orbits, ϕ factors uniquely through π. The categorical quotient is commonly denoted by $X/\!\!/G$.

A categorical quotient is a **geometric quotient** if the points of the underlying space correspond to the orbits of G on X. The geometric quotient is commonly denoted by X/G. Dolgachev writes:

> The main problem here is that the quotient space X/G may not exist in the category of algebraic varieties. The reason is rather simple. Since one expects that the canonical projection $f : X \to X/G$ is a regular map of algebraic varieties and so has closed fibres, all orbits must be closed subsets in the Zariski topology of X. This rarely happens when G is not a finite group. (Introduction to [118])

A third kind of quotient is the **algebraic quotient** $Y = \mathrm{Spec}(B^G)$. While B^G is not necessarily an affine ring, Winkelmann has shown that if it is normal, then it is quasi-affine, i.e., the ring of regular functions on an open subset of an affine variety [422]. The function $X \to Y$ induced by the inclusion $B^G \hookrightarrow B$ is the **algebraic quotient map**.

In many situations, B^G is indeed an affine ring, and the algebraic quotient map is a morphism of affine k-varieties. In this case, the categorical quotient $X/\!\!/G$ exists and equals the algebraic quotient $Y = \mathrm{Spec}(B^G)$. (This is easily verified using the universal mapping property of $X/\!\!/G$, and the fact that the inclusion map $B^G \to B$ is injective.) So hereafter in this book, the terms **quotient** and **quotient map** will mean the algebraic/categorical quotient and its associated morphism, with the underlying assumption that B^G is finitely generated.

It may happen that the invariant functions B^G do not separate orbits, so that even when $\mathcal{O}(X/G)$ exists, it may not equal B^G, since the geometric and algebraic quotients may not admit a bijective correspondence. Dolgachev points out the simple example of $G = GL_n(k)$ acting on $\mathbb{A}^n$ in the natural way: This action has two orbits, whereas $B^G = k$.

If X is an affine G-variety with an affine quotient, then the quotient map $\pi : X \to Y$ is uniquely defined by the inclusion $k[X]^G \hookrightarrow k[X]$. Therefore, π does not distinguish between two different G-actions with the same quotient. For example, if G is the additive group of k, then the G-actions on the plane $\mathbb{A}^2$ defined by

$$t \cdot (x, y) = (x, y + t) \quad \text{and} \quad t \cdot (x, y) = (x, y + tx)$$

have the same ring of invariants $k[x]$ but are clearly not the same action: The first action is fixed point free and the second fixes the line $x = 0$.

If G acts on two varieties X and X', then a morphism $\phi : X \to X'$ is called **equivariant** relative to these two actions if and only if $\phi(g \cdot x) = g \cdot \phi(x)$ for all $g \in G$ and $x \in X$.

The G-action $\rho : G \times X \to X$ is called **proper** if and only if the morphism $G \times X \to X \times X$, $(g, x) \mapsto (x, \rho(g, x))$, is proper as a map of algebraic k-varieties (see [118], 9.2). If $k = \mathbb{C}$, then properness has its usual topological meaning, i.e., the inverse image of a compact set is compact.

The action ρ is called **locally finite** if and only if the linear span of the orbit of every $f \in k[X]$ is a finite-dimensional vector space over k.

In case the underlying space X is a k-vector space $X = V$ and G acts by vector space automorphisms $G \to GL(V)$, ρ is said to be a **linear** action, and V is called a ***G*-module**.

Given $n \geq 1$, let U_n denote the subgroup of $GL_n(k)$ consisting of upper triangular matrices with each diagonal entry equal to 1. A linear algebraic group $G \subset GL_n(k)$ is called **unipotent** if and only if it is conjugate to a subgroup of U_n; equivalently, the only eigenvalue of G is 1. G is **reductive** if and only if G is connected and contains no nontrivial connected normal unipotent subgroup. It is well-known that, when G is a reductive group acting on X, then $k[X]^G$ is affine (so the categorical and algebraic quotients are the same), and the quotient map is always surjective and separates closed orbits (see Kraft [253]).

Regarding group actions, our primary interest is in algebraic $\mathbb{G}_a$-actions, where $\mathbb{G}_a$ denotes the **additive group** of the field k. Also important are the algebraic $\mathbb{G}_m$-actions, where $\mathbb{G}_m$ denotes the **multiplicative group of units** of k. Other common notations are $\mathbb{G}_a(k)$ or k^+ for $\mathbb{G}_a$, and $\mathbb{G}_m(k)$,$k^\times$, or k^* for $\mathbb{G}_m$. Under the assumption k is algebraically closed, $\mathbb{G}_a$ and $\mathbb{G}_m$ are the only irreducible algebraic k-groups of dimension 1 (20.5 of [213]). For non-algebraically closed fields, there may be other such groups, for example, the circle group S^1 over the field $\mathbb{R}$ of real numbers (which is reductive).

Any group $\mathbb{G}_m^n$ $(n \geq 1)$ is called an **algebraic torus** of dimension n. Likewise, the group $\mathbb{G}_a^n$ $(n \geq 1)$ is called the **vector group** of dimension n. The algebraic tori $\mathbb{G}_m^n$ are reductive, and the vector groups $\mathbb{G}_a^n$ are unipotent.

Suppose $\rho : \mathbb{G}_a \times X \to X$ is an algebraic $\mathbb{G}_a$-action on the algebraic variety X, noting that $\mathbb{G}_a \cong \mathbb{A}^1$. Then $\mathbb{G}_a$ also acts on the coordinate ring $k[X]$.

- ρ is **equivariantly trivial** or **globally trivial** if and only if there exists an affine variety Y and an equivariant isomorphism $\phi : X \to Y \times \mathbb{A}^1$, where $\mathbb{G}_a$ acts trivially on Y, and $\mathbb{G}_a$ acts on $\mathbb{A}^1$ by translation: $t \cdot x = x + t$.
- ρ is **locally trivial** if and only if there exists a covering $X = \cup_{i=1}^n X_i$ by Zariski open sets such that the action restricts to an equivariantly trivial action on each X_i.

Observe that these definitions can be extended to other categories. For example, we are assuming $\mathbb{G}_a$ acts by algebraic automorphisms, with ϕ being an algebraic isomorphism; but when $k = \mathbb{C}$, $\mathbb{G}_a$ may act by holomorphic automorphisms, with ϕ being a holomorphic isomorphism.

On the subject of affine k-varieties, there are a number of good references, including Harris [202], Hartshorne [203], Kunz [259], or Miyanishi [303]. For references to actions of algebraic groups and classical invariant theory, the reader should see Bass [13], Derksen and Kemper [98], Dieudonné and Carrell [113], Dolgachev [118], Humphreys [213], Kraft [253], Kraft and Procesi [256], Popov [345], together with the nice review of Popov's book written by Schwarz [373], or van den Essen [142], Chap. 9. The article of Greuel and Pfister [186] and the book of Grosshans [188] focus on the invariant theory of unipotent groups. The latter includes a wealth of historical references.

If X is an affine variety, elements of the Lie algebra $\mathrm{Der}_k(k[X])$ can be viewed as **vector fields** on X with polynomial coefficients, and can be used to study X; see the article of Siebert [385]. For a discussion of locally finite group actions relative to derivations and vector fields, see Drensky and Yu [123], Draisma [120], or Cohen and Draisma [54].

1.4 First Principles for Locally Nilpotent Derivations

We next turn our attention to the locally nilpotent case, with the ongoing assumption that k is a field of characteristic zero, and B is a commutative k-domain.

Principle 1 *Suppose* $D \in \mathrm{LND}(B)$.

(a) $\ker D$ *is factorially closed.*
(b) $B^* \subset \ker D$. *In particular,* $\mathrm{LND}(B) = \mathrm{LND}_k(B)$.
(c) *If* $D \neq 0$, *then* D *admits a local slice* $r \in B$.
(d) $\mathrm{Aut}_k(B)$ *acts on* $\mathrm{LND}(B)$ *by conjugation.*

Proof By *Proposition 1.10*, $\deg_D$ is a degree function with values in $\mathbb{N}$, and it was observed earlier that for any such degree function, the set of degree-zero elements forms a factorially closed subring containing B^*. This is the content of (a) and (b).

For (c), choose $b \in B$ such that $Db \neq 0$, and set $n = \deg_D(b) \geq 1$. Then $D^n b \neq 0$ and $D^{n+1}b = 0$, so we may take $r = D^{n-1}b$.

Part (d) is due to the observation that $(\alpha D \alpha^{-1})^n = \alpha D^n \alpha^{-1}$ for any $\alpha \in \mathrm{Aut}_k(B)$ and $n \geq 0$. □

Note that, while derivations of fields are of interest and importance, the foregoing result shows that the only locally nilpotent derivation of a field is the zero derivation.

Corollary 1.22 *If K is a field of characteristic 0, then* $\mathrm{LND}(K) = \{0\}$.

Principle 2 *Let S be any subset of B which generates B as k-algebra, and let $D \in$* $\mathrm{Der}_k(B)$. *Then $D \in \mathrm{LND}(B)$ if and only if $S \subset \mathrm{Nil}(D)$.*

Proof This follows immediately from the fact that $\mathrm{Nil}(D)$ is a subalgebra (*Proposition 1.10(b)*). □

Suppose B is finitely generated over $A = \ker D$, namely, $B = A[x_1, \ldots, x_n]$. *Principle 2* implies that $D \in \mathrm{LND}(B)$ if and only if there exists $N \in \mathbb{Z}$ such that $D^N x_i = 0$ for each i.

Principle 3 *Suppose $D \in \mathrm{LND}(B)$ is nonzero, and set $A = \ker D$. If $P \in A^{[1]}$ and $t \in B$ are such that neither t nor $P(t)$ is zero, then:*

$$\deg_D(P(t)) = (\deg_t P) \cdot \deg_D(t)$$

Proof The case $\deg_D(t) = 0$ is clear, so assume $\deg_D(t) > 0$. For each $a \in A$ ($a \neq 0$) and $i \in \mathbb{N}$ we have $\deg_D(at^i) = i \deg_D(t)$. Thus, the nonzero terms in $P(t)$ have distinct $\deg_D$-values, and the desired conclusion follows. □

Principle 4 *Given an ideal I of B, and given $D \in \mathrm{LND}(B)$, $D \neq 0$, D induces a well-defined quotient derivation $D/I \in \mathrm{LND}(B/I)$ if and only if I is an integral ideal of D. In addition, if I is an integral ideal of D which is a maximal ideal of B, then $DB \subset I$.*

Proof Only the second assertion requires demonstration. If I is a maximal ideal, then B/I is a field, and thus $\mathrm{LND}(B/I) = \{0\}$. In particular, $D/I = 0$, so $DB \subset I$. □

Principle 5 *Let $D \in \mathrm{LND}(B)$ and $f_1, \ldots, f_n \in B$ ($n \geq 1$) be given. Suppose that there exist positive integers $m_1, \ldots, m_n$ and a permutation $\sigma \in S_n$ such that $D^{m_i} f_i \in f_{\sigma(i)}B$ for each i. Then in each orbit of σ there is an i with $D^{m_i} f_i = 0$.*

Proof Suppose $D^{m_i} f_i \neq 0$ for each i, and choose $a_1, \ldots, a_n \in B$ such that $D^{m_i} f_i = a_i f_{\sigma(i)}$. Then $\deg_D(f_i) \geq 1$ and $\deg_D(a_i) \geq 0$ for each i. It follows that, for each i:

$$\begin{aligned} \deg_D(f_i) - m_i = \deg_D(D^{m_i} f_i) &= \deg_D(a_i f_{\sigma(i)}) \\ &= \deg_D(a_i) + \deg_D(f_{\sigma(i)}) \geq \deg_D(f_{\sigma(i)}) \end{aligned}$$

Therefore,

$$\sum_{1\le i\le n} (\deg_D(f_i) - m_i) \ge \sum_{1\le i\le n} \deg_D(f_{\sigma(i)})$$

which is absurd, since $\sum_i \deg_D f_i = \sum_i \deg_D f_{\sigma(i)}$. Therefore, $D^{m_i} f_i = 0$ for at least one i. Now apply this result to the decomposition of σ into disjoint cycles, and the desired result follows. □

The case $n = 1$ above is especially important.

Corollary 1.23 *If $D^m f \in fB$ for $D \in \mathrm{LND}(B)$, $m \ge 1$ and $f \in B$, then $D^m f = 0$.*

A similar result holds for the radical of a principal ideal.

Lemma 1.24 ([174], Lemma 2.1) *Let B be a commutative k-domain and $D \in \mathrm{LND}(B)$. If $t \in B$ and $\sqrt{tB}$ is an integral ideal of D, then $Dt = 0$.*

Proof Assume that $D \in \mathrm{LND}(B)$ is nonzero and that $\sqrt{tB}$ is an integral ideal of D for nonzero $t \in B$. If $n = \deg_D(t)$, then $D^n t$ is a nonzero element of $\sqrt{tB} \cap \ker D$. If $m \ge 0$ is such that $(D^n t)^m \in tB$, then $(D^n t)^m \in tB \cap \ker D$, which means that $tB \cap \ker D \ne \{0\}$. Since $\ker D$ is factorially closed in B (see *Principle 1(a)*), it follows that $t \in \ker D$. □

Principle 6 (See [159], Prop. 1) *Let B be a commutative k-domain and $B[t] = B^{[1]}$. Assume that $D \in \mathrm{Der}_k(B[t])$, $A = \ker D$, D restricts to B, and $D|_B$ is locally nilpotent. The following are equivalent.*

1. $D \in \mathrm{LND}(B[t])$
2. $Dt \in B$
3. $[D, \frac{d}{dt}] = 0$

In this case, $\frac{d}{dt}$ restricts to A. If $D|_B \ne 0$, then $\frac{d}{dt}|_A \ne 0$ and $\mathrm{LND}(A) \ne \{0\}$.

Proof (2) ⇒ (1): If $Dt \in B$, then since $B \subset \mathrm{Nil}(D)$, it follows that $t \in \mathrm{Nil}(D)$. So in this case, D is locally nilpotent by *Principle 2* above.

(1) ⇒ (2): Assume D is locally nilpotent, but that $Dt \notin B$. Choose $N \ge 1$ such that $D^N t \notin B$, but $D^{N+1} t \in B$, which is possible since D is locally nilpotent. Then $P(t) := D^N t$ is of positive t-degree. Suppose $\deg_t P(t) = m \ge 1$ and $\deg_t Dt = n \ge 1$, and write $P(t) = \sum_{0\le i\le m} b_i t^i$ for $b_i \in B$. Then

$$D(P(t)) = P'(t)Dt + \sum_{0\le i\le m} (Db_i)t^i$$

which belongs to B, and thus has t-degree 0. It follows that $(m-1)+n \le m$, so $n = 1$, i.e., Dt is linear in t. This implies that $\deg_t D^i t \le 1$ for all $i \ge 0$, and in particular we must have $m = 1$. Write $P(t) = at + b$ and $Dt = ct + d$ for $a, b, c, d \in B$ and $a, c \ne 0$. Then $D(at + b) = (ac + Da)t + ad + Db$ belongs to B, meaning that $ac + Da = 0$. But then $Da \in aB$, so by *Corollary 1.23*, $Da = 0$. But then $ac = 0$, a contradiction. Therefore, $Dt \in B$.

(3) ⇒ (2): We have $0 = [D, \frac{d}{dt}](t) = -\frac{d}{dt}(Dt)$, meaning that $Dt \in B$.

(2) ⇒ (3): If $Dt \in B$, then $[D, \frac{d}{dt}](t) = -\frac{d}{dt}(Dt) = 0$. In addition, for $b \in B$ we have $[D, \frac{d}{dt}](b) = -\frac{d}{dt}(Db) = 0$. Therefore, $[D, \frac{d}{dt}] = 0$. □

Principle 7 *Given $D \in \mathrm{Der}_k(B)$, and given nonzero $f \in B$:*

$$fD \in \mathrm{LND}(B) \quad \Leftrightarrow \quad D \in \mathrm{LND}(B) \text{ and } f \in \ker D$$

Proof Suppose $fD \in \mathrm{LND}(B)$ but $\mathrm{Nil}(D) \neq B$. Then $D \neq 0$. Set $N = \deg_{fD}(f) \geq 0$, and choose $g \in B - \mathrm{Nil}(D)$. It follows that $g \neq 0$, $\deg_{fD}(g) \geq 0$, and $\deg_{fD}(D^n g) \geq 0$ for all $n \geq 1$. On the one hand, we have:

$$\deg_{fD}(f \cdot D^n g) = \deg_{fD}((fD)(D^{n-1}g)) = \deg_{fD}(D^{n-1}g) - 1$$

On the other hand, we see that:

$$\deg_{fD}(f \cdot D^n g) = \deg_{fD}(f) + \deg_{fD}(D^n g) = N + \deg_{fD}(D^n g)$$

Therefore:

$$\deg_{fD}(D^n g) = \deg_{fD}(D^{n-1}g) - (N+1) \quad \text{for all } n \geq 1$$

This implies

$$\deg_{fD}(D^n g) = \deg_{fD}(g) - n(N+1)$$

which is absurd since it means $\deg_{fD}$ has values in the negative integers. Therefore, $D \in \mathrm{LND}(B)$. To see that $f \in \ker D$, note that $(fD)(f) \in fB$. By *Corollary 1.23*, it follows that $f \in \ker(fD)$; and since B is a domain, $\ker(fD) = \ker D$.

The converse is immediate. □

Remark 1.25 Nowicki gives an example to show that the result above may fail for a non-reduced ring. Let $R = \mathbb{Q}[x]/(x^3) = \mathbb{Q}[\bar{x}]$, and let $d \in \mathrm{Der}_{\mathbb{Q}}(R)$ be defined by $d\bar{x} = \bar{x}^2$. Then $d, \bar{x}d \in \mathrm{LND}(R)$, but $\bar{x} \notin \ker d$ ([333], 8.1.3).

Principle 8 *Suppose $B = A[t]$, where A is a subring of B and $t \in B$ is transcendental over A.*

(a) $\frac{d}{dt} \in \mathrm{LND}_A(A[t])$
(b) $\ker(\frac{d}{dt}) = A$
(c) $\mathrm{LND}_A(A[t]) = A \cdot \frac{d}{dt}$

Proof Part (a) is an immediate consequence of *Proposition 1.7*, since this shows that each application of $\frac{d}{dt}$ reduces degree in $A[t]$ by one.

By definition, $A \subset \ker(\frac{d}{dt})$. Conversely, suppose $P(t) \in \ker(\frac{d}{dt})$. If $\deg P \geq 1$, then since this kernel is algebraically closed, it would follow that $t \in \ker(\frac{d}{dt})$, a contradiction. Therefore, $\ker(\frac{d}{dt}) = A$.

For (c), let $D \in \mathrm{LND}_A(B)$ be given, $D \neq 0$. By *Proposition 1.9(c)*, for any $p(t) \in A[t]$, $D(p(t)) = p'(t)Dt$. Consequently, $D = Dt\frac{d}{dt}$. Since both D and $\frac{d}{dt}$ are locally nilpotent, *Principle 7* implies that $Dt \in A$. Therefore, $\mathrm{LND}_A(A[t]) \subseteq A \cdot \frac{d}{dt}$. The reverse inclusion is implied by *Principle 7*. □

Principle 9 *Let $S \subset B \setminus \{0\}$ be a multiplicatively closed set, and let $D \in \mathrm{Der}_k(B)$ be given. Then:*

$$S^{-1}D \in \mathrm{LND}(S^{-1}B) \quad \Leftrightarrow \quad D \in \mathrm{LND}(B) \text{ and } S \subset \ker D$$

In this case, $\ker(S^{-1}D) = S^{-1}(\ker D)$.

Proof Suppose $S^{-1}D$ is locally nilpotent, noting that D is clearly locally nilpotent in this case. Since $S \subset (S^{-1}B)^* \subset \ker(S^{-1}D)$, it follows that $S \subset \ker D$.

Conversely, suppose D is locally nilpotent and $S \subset \ker D$. Let $f/g \in S^{-1}B$ be given. Since $g \in \ker D$, it follows immediately from the quotient rule that $g^{-1} \in \ker(S^{-1}D)$. Thus:

$$(S^{-1}D)^n(f/g) = g^{-1}(S^{-1}D)^n(f) = g^{-1}D^n f = 0 \quad \text{for} \quad n \gg 0$$

Therefore, $S^{-1}D$ is locally nilpotent.

Assuming $S \subset \ker D$, it follows that:

$$S^{-1}D(f/g) = 0 \Leftrightarrow (1/g)Df = 0 \Leftrightarrow Df = 0 \Leftrightarrow f/g \in S^{-1}(\ker D)$$

□

Principle 10 *Suppose $D \in \mathrm{LND}(B)$.*

(a) $\exp D \in \mathrm{Aut}_k(B)$
(b) *If $[D, E] = 0$ for $E \in \mathrm{LND}(B)$, then $D + E \in \mathrm{LND}(B)$ and:*

$$\exp(D + E) = \exp D \circ \exp E$$

(c) *The subgroup of $\mathrm{Aut}_k(B)$ generated by $\{\exp D \mid D \in \mathrm{LND}(B)\}$ is normal.*

Proof Since every function D^i is additive, $\exp D(f) = \sum_{i \geq 0} \frac{1}{i!} D^i f$ is an additive function. To see that $\exp D$ respects multiplication, suppose $f, g \in B$ are nonzero, with $\deg_D(f) = m$ and $\deg_D(g) = n$. Then $D^i f = D^j g = 0$ for $i > m$ and $j > n$, and:

$$\begin{aligned}
(\exp D)(f) \cdot (\exp D)(g) &= \left(\sum_{0 \le i \le m} \frac{1}{i!} D^i f \right) \cdot \left(\sum_{0 \le j \le n} \frac{1}{j!} D^j g \right) \\
&= \sum_{0 \le i+j \le m+n} \frac{1}{i!j!} D^i f D^j g \\
&= \sum_{0 \le i+j \le m+n} \frac{1}{(i+j)!} \binom{i+j}{j} D^i f D^j g \\
&= \sum_{0 \le t \le m+n} \frac{1}{t!} \left(\sum_{i+j=t} \binom{i+j}{j} D^i f D^j g \right) \\
&= \sum_{0 \le t \le m+n} \frac{1}{t!} D^t (fg) \\
&= (\exp D)(fg)
\end{aligned}$$

The penultimate line follows from the preceding line by the higher product rule. Thus, $\exp D$ is an algebra homomorphism.

Next, let $f \in B$ be given, and choose $m \ge 0$ so that $D^m f = E^m f = 0$. Set $n = 2m$. Since D and E commute:

$$(D + E)^n (f) = \sum_{i+j=n} \binom{n}{i} D^i E^j (f)$$

For each term of this sum, either $i \ge m$ or $j \ge m$, and it follows that $D^i E^j (f) = E^j D^i (f) = 0$ for each pair i, j. Therefore, $D + E \in \mathrm{LND}(B)$. Further, by using this same expansion for $(D + E)^n$, the proof that $\exp(D + E) = \exp D \circ \exp E$ now follows exactly as above. Thus, (b) is proved.

In addition, note that by *Principle 7*, $-D \in \mathrm{LND}(B)$. Thus, by part (b), it follows that:

$$\exp D \circ \exp(-D) = \exp(-D) \circ \exp D = \exp 0 = I$$

Therefore, $\exp D$ is an automorphism, and (a) is proved.

Finally, part (c) follows from the observation that

$$\alpha (\exp D) \alpha^{-1} = \exp(\alpha D \alpha^{-1})$$

for any $\alpha \in \mathrm{Aut}_k(B)$, and that $\alpha D \alpha^{-1}$ is again locally nilpotent. □

Principle 11 *Let $D \in \mathrm{LND}(B)$ be given, $D \neq 0$, and set $A = \ker D$. Choose a local slice $r \in B$ of D, and let $\pi_r : B \to B_{Dr}$ denote the Dixmier map defined by r.*

(a) $\pi_r(B) \subset A_{Dr}$
(b) *π_r is a k-algebra homomorphism.*
(c) $\ker \pi_r = rB_{Dr} \cap B$
(d) $B_{Dr} = A_{Dr}[r]$
(e) *The transcendence degree of B over A is 1.*

Proof Consider first the case $Ds = 1$ for some $s \in B$.

For (a), recall the definition:

$$\pi_s(h) = \sum_{i \geq 0} \frac{(-1)^i}{i!}(D^i h)s^i$$

From this, one verifies immediately that $D(\pi_s(h)) = 0$ for all $h \in B$. Therefore, $\pi_s(B) \subset A = A_{Ds}$.

For (b), let t be transcendental over B, and extend D to $B[t]$ via $Dt = 0$. Let $\iota : B \hookrightarrow B[t]$ be inclusion, and let $\epsilon : B[t] \to B$ be the evaluation map $\epsilon(t) = s$. By Principles 7 and 10, $\exp(-tD)$ is an automorphism of $B[t]$. In addition, $\pi_s = \epsilon \circ \exp(-tD) \circ \iota$. Therefore, π_s is a homomorphism.

For (c), note that $\pi_s(s) = s - (Ds)s = 0$. Therefore, $\pi(sB) = 0$. Conversely, if $\pi_s(f) = 0$, then since $\pi_s(f) = f + sb$ for some $b \in B$, we conclude that $f \in sB$. Therefore, $\ker(\pi_s) = sB$ when $Ds = 1$.

Next, since the kernel of D on $B[t]$ equals $A[t]$, π_s extends to a homomorphism $\pi_s : B[t] \to A[t]$. Define the homomorphism $\phi : B \to A[s]$ by $\phi = \epsilon \circ \pi_s \circ \exp(tD) \circ \iota$. Specifically, ϕ is defined by:

$$\phi(g) = \sum_{n \geq 0} \frac{1}{n!}\pi_s(D^n g)s^n$$

Then ϕ is a surjection, since $\phi(a) = a$ for $a \in A$, and $\phi(s) = s$. Also, if $\phi(g) = 0$, then since s is transcendental over A by *Proposition 1.9*, it follows that each coefficient of $\phi(g) \in A[s]$ is zero. If $g \neq 0$, then the highest-degree coefficient of $\phi(g)$ equals $(1/n!)\pi_s(D^n g)$, where $n = \deg_D(g) \geq 0$. Thus, $D^n g \in \ker \pi_s = sB$, and since also $D^n g \in A \setminus \{0\}$, we conclude that $s \in A$ (since A is factorially closed). But $s \notin A$, so it must be that $g = 0$. Therefore, ϕ is an isomorphism, and (d) is proved.

We have now proved (a)–(d) in the special case $Ds = 1$.

For the general case, suppose that, for the local slice r, $Dr = f \in A$. Let D_f denote the extension of D to B_f. Then $s := r/f$ is a slice of D_f. Since π_r is the restriction to B of the homomorphism $\pi_s : B_f \to B_f$, it follows that π_r is a homomorphism. The kernel is $sB_f \cap B = rB_f \cap B$, and $B_f = A_f[s] = A_f[r]$. Therefore, results (a)–(d) hold in the general case.

Finally, (e) follows immediately from (d). □

Several important corollaries are implied by this result.

Corollary 1.26 (Slice Theorem) *Suppose $D \in \mathrm{LND}(B)$ admits a slice $s \in B$, and let $A = \ker D$. Then:*

(a) $B = A[s]$ *and* $D = \frac{d}{ds}$
(b) $A = \pi_s(B)$ *and* $\ker \pi_s = sB$
(c) *If B is affine, then A is affine.*

Proof Immediate. □

Example 1.27 Let S^1 be the unit circle over the field $\mathbb{R}$ of real numbers, and let $\mathcal{T}(S^1)$ be the algebraic tangent bundle of S^1. The coordinate rings of these varieties are given by:

$$U = \mathbb{R}[S^1] = \mathbb{R}[x_1, x_2]/(x_1^2 + x_2^2 - 1)$$

and

$$B = \mathbb{R}[\mathcal{T}(S^1)] = U[y_1, y_2]/(x_1y_1 + x_2y_2)$$

Define $D \in \mathrm{LND}_U(B)$ by $Dy_1 = x_2$ and $Dy_2 = -x_1$, and set $A = \ker D$. If $s = x_2y_1 - x_1y_2$, then $Ds = 1$. By *Corollary 1.26(a)*, $B = A[s]$, and by *Corollary 1.26(b)*, $A = U[\pi_s(y_1), \pi_s(y_2)]$. Since $\pi_s(y_i) = 0$ for $i = 1, 2$, we see that $A = U$ and $B = U[s]$. Geometrically, this implies:

$$\mathcal{T}(S^1) = S^1 \times \mathbb{R}^1$$

Corollary 1.28 *If S is a commutative k-domain such that $\mathrm{tr.deg.}_k S = 1$ and* $\mathrm{LND}(S) \neq \{0\}$*, then $S = K^{[1]}$, where K is a field algebraic over k. If, in addition, k is algebraically closed in S, then $S = k^{[1]}$.*

Proof Suppose $\delta \in \mathrm{LND}(S)$ is nonzero, and set $K = \ker \delta$. We have $k \subset K \subset S$, where $\mathrm{tr.\,deg.}_K S = 1$. Therefore, $\mathrm{tr.deg.}_k K = 0$, i.e., K is an algebraic field extension of k. If $r \in S$ is a local slice of δ, then $\delta r \in K^*$, so $s = (\delta r)^{-1} r$ is a slice of δ. It follows that $S = K[s]$. □

Corollary 1.29 *Let B be any commutative k-domain, and let $D \in$* $\mathrm{LND}(B)$. *Let δ denote the extension of D to a derivation of the field* $\mathrm{frac}(B)$ *(so δ is not locally nilpotent if $D \neq 0$). Then* $\ker \delta = \mathrm{frac}(\ker D)$.

Proof Let $A = \ker D$. If r is a local slice of D and $f = Dr$, then $B \subset B_f = A_f[r]$. We therefore have $\mathrm{frac}(B) \subset \mathrm{frac}(A)(r) \subset \mathrm{frac}(B)$, which implies $\mathrm{frac}(B) = \mathrm{frac}(A)(r)$. Now suppose $\delta g = 0$ for $g \in \mathrm{frac}(B)$, and write $g = P(r)$ for the rational function P having coefficients in $\mathrm{frac}(A)$. Then $0 = P'(r)\delta r$, and since $\delta r \neq 0$, $P'(r) = 0$. It follows that $g = P(r) \in \mathrm{frac}(A)$, which shows $\ker \delta \subset \mathrm{frac}(\ker D)$. The reverse containment is obvious. □

Another immediate implication of *Principle 11*, combined with *Proposition 1.10*, is the following degree formula. (See also Cor. 2.2 of [101] and 1.3.32 of [142].)

Corollary 1.30 *Given $D \in \mathrm{LND}(B)$, set $L = \mathrm{frac}(B)$ and $K = \mathrm{frac}(B^D)$. If $b \in B$ and $b \notin B^D$, then $\deg_D(b) = [L : K(b)]$.*

The next result examines the case in which two kernels coincide.

Principle 12 ([70], Lemma 1.1) *Suppose $D, E \in \mathrm{LND}(B)$ have the same kernel, and define $A = \ker D = \ker E$.*

(a) *There exist nonzero $a, b \in A$ such that $aD = bE$.*
(b) $\deg_D = \deg_E$

Proof We may assume $D, E \neq 0$. By *Principle 11(d)*, there exists $c \in A$ and $t \in B$ such that $B_c = A_c[t]$. By *Principle 9*, $\ker(D_c) = \ker(E_c) = A_c$. Therefore, by *Principle 8*, $D_c = \beta \cdot (d/dt)$ and $E_c = \alpha \cdot (d/dt)$ for some $\alpha, \beta \in A_c$. Consequently, $\alpha D_c = \beta E_c$. Choose $n \in \mathbb{Z}_+$ so that $a := c^n\alpha$ and $b := c^n\beta$ belong to A. Then $aD_c = bE_c$. Restriction to B gives the desired result, and part (a) is proved. Part (b) follows from part (a), since:

$$\deg_D = \deg_{aD} = \deg_{bE} = \deg_E$$

□

In case B is an affine ring, the following result characterizes those subrings of B which occur as the kernel of some locally nilpotent derivation.

Principle 13 ([70], Prop. 1.4) *Let A be a subalgebra of B other than B itself, $S = A \setminus \{0\}$, and $K = S^{-1}A$, the field of fractions of A. Consider the following statements.*

1. *$A = \ker D$ for some $D \in \mathrm{LND}(B)$.*
2. *$S^{-1}B = K^{[1]}$ and $A = K \cap B$.*

Then (1) implies (2). If B is finitely generated over A, then (2) implies (1).

Proof That (1) implies (2) follows immediately from part (d) of *Principle 11*, together with *Principle 9*. Conversely, assume (2) holds and $B = A[b_1, \dots, b_n]$ for some $b_i \in B$. Since $\frac{d}{dt}(b_i) \in K[t]$ for each i, there exists $s \in S$ so that $s\frac{d}{dt}(B) \subset B$. Since $s \in K$, $s\frac{d}{dt}$ is locally nilpotent. If D denotes the restriction of $s\frac{d}{dt}$ to B, it follows that D is also locally nilpotent, and:

$$\ker D = B \cap \ker\left(s\,\frac{d}{dt}\right) = B \cap K = A$$

□

Principle 14 (Generating Principle) *Let $D \in \mathrm{LND}(B)$ be nonzero, $A = \ker D$, $r \in B$ a local slice and $f = Dr$. Suppose that $R \subset B$ is a subalgebra which satisfies:*

1. *$A[r] \subset R \subset B$*
2. *$fB \cap R = fR$*

Then $R = B$.

Proof Let $b \in B$ be given. Since r is a local slice of D, there exists an integer $n \geq 0$ with $f^n b \in A[r] \subset R$. Repeated application of property (2) yields:

$$f^n B \cap R = f^n R$$

Since $f^n b \in R$ and B is a domain, it follows that $b \in R$. $\square$

Principle 15 ([278], Lemma 4, [276], Lemma 3) *Let G be a totally ordered abelian group, let* deg *be a degree function on B with values in G, and let* $\mathrm{Gr}(B)$ *be the graded ring associated to the filtration of B induced by* deg*. Given $D \in \mathrm{LND}(B)$, if* $\deg D$ *is defined, then* $\mathrm{gr}\, D \in \mathrm{LND}(\mathrm{Gr}(R))$.

Proof It is easy to see from the definitions in *Sect. 1.1.6* that, for each $n \geq 0$ and $f \in B$, either $(\mathrm{gr}\, D)^n(\mathrm{gr}(f)) = 0$ or $(\mathrm{gr}\, D)^n(\mathrm{gr}(f)) = \mathrm{gr}(D^n f)$. Therefore:

$$\mathrm{gr}(\mathrm{Nil}(D)) \subset \mathrm{Nil}(\mathrm{gr}\, D)$$

Since $\mathrm{Nil}(D) = B$, and since $\mathrm{gr}(B)$ generates $\mathrm{Gr}(B)$ as a k-algebra, it follows that $\mathrm{Nil}(\mathrm{gr}\, D) = \mathrm{Gr}(B)$. $\square$

Corollary 1.31 (Compare to [140], Princ. II) *Let G be a totally ordered abelian group and $B = \bigoplus_{g \in G} B_g$ a G-grading. Given $D \in \mathrm{LND}(B)$, suppose that there exists a finite set $I \subset G$ and $D_i \in \mathrm{Der}_k(B)$, $i \in I$, such that D_i is G-homogeneous, $D_i \neq 0$, $\deg_G D_i = i$ for each i, and $D = \sum_{i \in I} D_i$.*

(a) *$D_m, D_n \in \mathrm{LND}(B)$, where $m = \min I$ and $n = \max I$.*
(b) *Given $f \in \ker D$, $f_u \in \ker D_m$ and $f_v \in \ker D_n$, where $u = \min \mathrm{Supp}_G(f)$ and $v = \max \mathrm{Supp}_G(f)$.*

Proof For part (a), $D_n \in \mathrm{LND}(B)$ is implied by *Principle 15*. To get $D_m \in \mathrm{LND}(B)$, we may reverse the ordering on G.

For part (b), assume that $f = \sum_{g \in G} f_g$ for $f_g \in B_g$. We have:

$$0 = Df = \sum_{\substack{m \leq i \leq n \\ u \leq j \leq v}} D_i f_j$$

Each term $D_i f_j$ is homogeneous. If $D_n f_v \neq 0$, then the degree of $D_n f_v$ exceeds that of any other term, which would imply $Df \neq 0$. Therefore, $D_n f_v = 0$. A similar argument shows that $D_u f_m = 0$. This proves part (b). $\square$

The final basic principle in our list is due to Vasconcelos; the reader is referred to [410] for its proof.[3] Other proofs appear in van den Essen [142], 1.3.37; and in Wright [426], Prop. 2.5.

[3] Vasconcelos's definition of "locally finite" is the same as the present definition of locally nilpotent. Apparently, the terminology at the time (1969) was not yet settled.

Principle 16 (Vasconcelos's Theorem) *Suppose $R \subset B$ is a subring over which B is integral. If $D \in \mathrm{Der}_k(B)$ restricts to a locally nilpotent derivation of R, then $D \in \mathrm{LND}(B)$.*

By combining the theorems of Seidenberg and Vasconcelos, we obtain the following.

Corollary 1.32 *Let R be a noetherian integral domain containing $\mathbb{Q}$ and $\mathcal{O}$ the integral closure of R in $\mathrm{frac}(R)$. Given $D \in \mathrm{LND}(R)$, let $r \in R$ be a local slice of D, $L = \mathrm{frac}(\ker D)$ and D_L the extension of D to $L[r]$. Then $R \subset \mathcal{O} \subset L[r]$ and D_L restricts to $\mathcal{O}$.*

Proof Let $K = \mathrm{frac}(R)$ and let $D_K \in \mathrm{Der}_k(K)$ be the extension of D to K. By Seidenberg's Theorem, D_K restricts to $\mathcal{O}$, and by Vasconcelos's Theorem, $\mathcal{O} \subset \mathrm{Nil}(D_K)$. Since $R_{Dr} = A_{Dr}[r] = A_{Dr}^{[1]}$, we have $K = L(r)$ and $D_K = \lambda \frac{d}{dr}$ for some $\lambda \in L$. Therefore, $\mathrm{Nil}(D_K) = L[r]$. □

Remark 1.33 The kernel of a locally finite derivation may fail to be factorially closed. For example, on the polynomial ring $k[x,y]$ define a k-derivation D by $Dx = y$ and $Dy = x$. Then $D(x^2 - y^2) = 0$, whereas neither $D(x-y)$ nor $D(x+y)$ is 0. Likewise, there are factorially closed subrings $A \subset B$ with $\mathrm{tr.deg.}_A B = 1$ which are not the kernel of any locally nilpotent derivation. For example, we may take $A = k[x^2 - y^3]$ in $B = k[x,y]$ (see [70] p. 226, first remark).

Remark 1.34 For any derivation D, an exponential map $\exp(tD) : B[[t]] \to B[[t]]$ can be defined by $\exp(tD)(f) = \sum_{i \geq 0}(1/i!)(D^i f)t^i$, where t is transcendental over B. Again, this is a ring automorphism, and the proof is identical to the one above. This map can be useful in proving that a given derivation is locally finite or locally nilpotent, as for example in [410].

Remark 1.35 One difficulty in working with locally nilpotent derivations is that $\mathrm{LND}(B)$ admits no obvious algebraic structure. For example, for the standard derivative $\frac{d}{dt}$ on the polynomial ring $k[t]$, we have seen that $\frac{d}{dt}$ is locally nilpotent, whereas $t\frac{d}{dt}$ is not. Thus, $\mathrm{LND}(B)$ is not closed under multiplication by elements of B, and does not form a B-module.

Likewise, if $k[x,y]$ is a polynomial ring in two variables over k, the derivations $D_1 = y(\partial/\partial x)$ and $D_2 = x(\partial/\partial y)$ are locally nilpotent, where $\partial/\partial x$ and $\partial/\partial y$ denote the usual partial derivatives. However, neither $D_1 + D_2$ nor $[D_1, D_2]$ is locally nilpotent. So $\mathrm{LND}(B)$ is also not closed under addition or bracket multiplication.

1.5 $\mathbb{G}_a$-Actions

In this section, assume that k is algebraically closed (of characteristic zero). Let B be an affine k-domain, and let $X = \mathrm{Spec}(B)$ be the corresponding affine variety.

1.5.1 Correspondence with LNDs

Given $D \in \text{LND}(B)$, by combining *Principle 7* and *Principle 10*, we obtain a group homomorphism:

$$\eta : (\ker D, +) \to \text{Aut}_k(B) \quad , \ \eta(a) = \exp(aD)$$

In addition, if $D \neq 0$, then η is injective.

If $H \subset \ker D$ is a subgroup, then $H \cong \mathbb{G}_a = (k, +)$ if and only if $H = kf$ for $f \in \ker D$, $f \neq 0$. In this case, restricting η to $H = kf$ gives the algebraic representation $\eta : \mathbb{G}_a \hookrightarrow \text{Aut}_k(B)$. Geometrically, this means that D induces the faithful algebraic $\mathbb{G}_a$-action $\exp(tfD)$ on X ($t \in k$).

Conversely, let $\rho : \mathbb{G}_a \times X \to X$ be an algebraic $\mathbb{G}_a$-action over k. Then ρ induces a derivation $\rho'(0)$, where differentiation takes places relative to $t \in \mathbb{G}_a$.

To be more precise, at the level of coordinate rings, $\rho^* : B \to B[t]$ is a k-algebra homomorphism (since ρ is a morphism of algebraic k-varieties). Given $t \in k$ and $f \in B$, denote the action of t on f by $t \cdot f$. Define $\delta : B \to B$ by the composition

$$B \xrightarrow{\rho^*} B[t] \xrightarrow{d/dt} B[t] \xrightarrow{t=0} B$$

i.e., $\delta = \epsilon \frac{d}{dt} \rho^*$, where ϵ denotes evaluation at $t = 0$.

Proposition 1.36 $\delta \in \text{LND}(B)$.

Proof To see this, we first verify conditions (C.1) and (C.2) from *Sect. 1.1*. Condition (C.1) holds, since δ is composed of k-module homomorphisms. For (C.2), observe that, given $a \in B$, if $\rho^*(a) = P(t) \in B[t]$, then for each $t_0 \in k$, $t_0 \cdot a = P(t_0)$. In particular, $a = 0 \cdot a = P(0) = \epsilon\rho^*(a)$. Therefore, given $a, b \in B$:

$$\delta(ab) = \epsilon \frac{d}{dt}(\rho^*(a)\rho^*(b)) = \epsilon \left(\rho^*(a) \frac{d}{dt} \rho^*(b) + \rho^*(b) \frac{d}{dt} \rho^*(a) \right) = a\delta b + b\delta a$$

So condition (C.2) holds, and δ is a derivation.

To see that δ is locally nilpotent, let $f \in B$ be given, and suppose $\rho^*(f) = P(t) = \sum_{0 \le i \le n} f_i t^i$ for $f_i \in B$. For general $s, t \in k$, we have:

$$(s + t) \cdot f = s \cdot (t \cdot f) = \sum_{0 \le i \le n} (s \cdot f_i) t^i$$

On the other hand, it follows from Taylor's formula that:

$$(s + t) \cdot f = P(s + t) = \sum_{0 \le i \le n} \frac{P^{(i)}(s)}{i!} t^i$$

Equating coefficients yields: $s \cdot f_i = (1/i!)P^{(i)}(s)$ for all $s \in k$.

We now proceed by induction on the t-degree of $\rho^*(f)$. If the degree is zero, then $\delta(f) = P'(0) = 0$, and thus $f \in \mathrm{Nil}(\delta)$. Assume $g \in \mathrm{Nil}(\delta)$ whenever the degree of $\rho^*(g)$ is less than n. Then $\delta(f) = P'(0) = f_1$ and $\deg \rho^*(f_1) = \deg P'(s) = n - 1$. Thus, $\delta(f) \in \mathrm{Nil}(\delta)$, which implies $f \in \mathrm{Nil}(\delta)$ as well. □

The reader can check that $D = (\exp(tD))'(0)$, and conversely $\rho = \exp(t\rho'(0))$. For other proofs, see [188], §8; [142], 9.5.2; and [68], §4.

In summary, there is a bijective correspondence between $\mathrm{LND}(B)$ and the set of all algebraic $\mathbb{G}_a$-actions on $X = \mathrm{Spec}(B)$, where $D \in \mathrm{LND}(B)$ induces the action $\exp(tD)$, and where the action ρ induces the derivation $\delta = \rho'(0)$, as described above. In addition, the kernel of the derivation coincides with the invariant ring of the corresponding action: $\ker D = B^{\mathbb{G}_a}$, since $Df = 0$ if and only if $\exp(tD)(f) = f$ for all $t \in k$.

Example 1.37 Let (x, y, z) be a system of coordinates on $\mathbb{A}^3$ and let $X \subset \mathbb{A}^3$ be the cone defined by $xz + y^2 = 0$. Define the $\mathbb{G}_a$-action $\rho : \mathbb{G}_a \times X \to X$ by:

$$\rho(t, (x, y, z)) = (x, y + tx, z - 2ty - t^2x)$$

The orbit of the point $p = (x_0, y_0, z_0) \in X$ is given by

$$\rho(t, (x_0, y_0, z_0)) = (x_0, y_0 + tx_0, z_0 - 2ty_0 - t^2x_0)$$

which is a line if $x_0 \neq 0$ or $y_0 \neq 0$, and a fixed point if $p = (0, 0, z_0)$. Differentiation gives $\rho'(0, (x, y, z)) = (0, x, -2y)$, thus defining the derivation $Dx = 0$, $Dy = x$ and $Dz = -2y$ on the coordinate ring $k[x, y, z]$ of X.

Many of the algebraic results we have established can be translated into geometric language. For example, *Corollary 1.28* becomes:

Corollary 1.38 *Let k be an algebraically closed field of characteristic zero and C an affine curve over k. If C admits a nontrivial algebraic $\mathbb{G}_a$-action, then $C \cong_k \mathbb{A}^1$.*

T.A. Springer (1981) gives the following fundamental result, where F is any field and X is an irreducible affine variety over F.

Theorem 1.39 ([389], Prop. 14.2.2) *Suppose that $\mathbb{G}_a$ acts non-trivially on X. There exists an affine F-variety Y with the following properties.*

1. *There is an F-isomorphism ϕ of $\mathbb{G}_a \times Y$ onto an open F-subvariety of X.*
2. *There is an F-morphism $\psi : \mathbb{G}_a \times Y \to \mathbb{G}_a$ such that:*

$$\psi(a + b, y) = \psi(a, y) + \psi(b, y) \quad \text{and} \quad a \cdot \phi(b, y) = \phi(\psi(a, y) + b, y)$$

1.5.2 Orbits, Vector Fields and Fixed Points

A classical theorem of Rosenlicht asserts that every orbit of a unipotent group acting algebraically on a quasiaffine algebraic variety is closed [361]. For a short proof of

Rosenlicht's theorem, see [38], Prop. 4.10. Note that Rosenlicht's theorem implies *Corollary 1.38*.

Consider the algebraic $\mathbb{G}_a$-action $\rho : \mathbb{G}_a \times X \to X$ on the irreducible affine k-variety $X = \mathrm{Spec}(B)$. Note that, since $\mathbb{G}_a \cong_k \mathbb{A}^1$ as k-varieties, every orbit of this action is either a line $\mathbb{A}^1$ or a single point. By Rosenlicht's theorem, these orbits are closed in X, and when the action is nontrivial, the union of the one-dimensional orbits forms a Zariski-dense open subset of X.

Given $p \in X$, the orbit $\mathbb{G}_a \cdot p$ is parametrized by $\rho(t, p)$, where $\rho(0, p) = p$. This defines the tangent vector $\rho'(0, p)$ to X at p. In this way, the $\mathbb{G}_a$-action defines a polynomial (tangent) vector field on X. The fixed points $X^{\mathbb{G}_a}$ are precisely those points p at which this vector field vanishes, i.e., $\rho'(0, p) = 0$.

Observe that the set of fixed points $X^{\mathbb{G}_a}$ includes all isolated singular points of X: If $p \in X$ is an isolated singularity, then every point of the orbit $\mathbb{G}_a \cdot p$ is also an isolated singular point of X, which implies $\mathbb{G}_a \cdot p = \{p\}$.

A fundamental result on the fixed points of unipotent group actions is due to Białynicki-Birula.

Theorem 1.40 ([32]) *If X is irreducible and affine, and* $\dim X \geq 1$*, then the algebraic action of any connected unipotent group G on X has no isolated fixed points.*

See also [30, 31, 176, 212] for related results.

Algebraically, suppose that ρ is given by $\exp(tD)$ ($t \in \mathbb{G}_a$). It is easy to see that the fixed points are defined by the ideal (DB) generated by the image of D; we sometimes write $\mathrm{Fix}\, D$ for $X^{\mathbb{G}_a}$. The $\mathbb{G}_a$-action is fixed-point free if and only if $1 \in (DB)$. At the opposite extreme, suppose D is reducible, meaning $DB \subset fB$ for some non-unit $f \in B$. Then $\mathcal{V}(f) \subset X^{\mathbb{G}_a} \subset X$, which implies that D has the form fD' for some $D' \in \mathrm{LND}(B)$ and $f \in \ker D$ (*Principle 7*).

Consider the geometric implications of *Principle 11*. If $D \in \mathrm{LND}(B)$ has local slice $r \in B$, set $A = \ker D$ and $f = Dr$. Then $B_f = A_f[r]$ and the extension of D to B_f equals $\frac{d}{dr}$. Thus, the induced $\mathbb{G}_a$-action $\exp(tD)$ on $X = \mathrm{Spec}(B)$ restricts to an equivariantly trivial action on the principal open set U_f defined by f. Likewise, if $f_1, \dots, f_n \in \mathrm{pl}(D) = A \cap DB$ satisfy $f_1B + \cdots + f_nB = B$, then the principal open sets U_{f_i} cover X, and the $\mathbb{G}_a$-action on X is locally trivial (hence fixed-point free) relative to these open sets. And finally, if D admits a slice, then $X = Y \times \mathbb{A}^1$ for $Y = \mathrm{Spec}(A)$, and the action of $\mathbb{G}_a$ on X is equivariantly trivial relative to this decomposition: $t \cdot (y, z) = (y, z + t)$.

To summarize these algebro-geometric connections:

- free $\mathbb{G}_a$-action $\Leftrightarrow 1 \in (DB)$
- locally trivial $\mathbb{G}_a$-action $\Leftrightarrow 1 \in (\mathrm{pl}(D))$
- equivariantly trivial $\mathbb{G}_a$-action $\Leftrightarrow 1 \in DB$

Here, $(\mathrm{pl}(D))$ denotes the B-ideal generated by the plinth ideal $\mathrm{pl}(D)$. See also Thm. 2.5 of [111].

Remark 1.41 The bijection between locally nilpotent derivations and $\mathbb{G}_a$-actions described above remains valid over any field k of characteristic zero. The proofs require a more general geometric setting.

Remark 1.42 In certain situations, it may be advantageous to consider $\mathbb{G}_a$-actions on a reducible (reduced) algebraic variety Y. For example, an irreducible $\mathbb{G}_a$-variety X may have a $\mathbb{G}_a$-subvariety Y with multiple irreducible components $Y_1, \dots, Y_n$. It is shown in [64], Prop. 2.2, that the action restricts to each component Y_i.

Remark 1.43 Let M be an orientable topological manifold, and let $\chi(M)$ denote its Euler characteristic. A well-known theorem from topology asserts that, if M admits a nonvanishing (tangent) vector field, then $\chi(M) = 0$. See, for example, [215]. Suppose that X is an algebraic variety over $\mathbb{R}$. As remarked above, a $\mathbb{G}_a$-action on X defines a polynomial vector field on X, and the fixed points $X^{\mathbb{G}_a}$ correspond to points at which the vector field vanishes. We thus conclude that, if X is orientable and admits a free $\mathbb{G}_a$-action, then $\chi(X) = 0$.

Remark 1.44 Nowicki [334] showed that, if G is a connected algebraic group which acts algebraically on the polynomial ring B, then there exists $D \in \mathrm{Der}_k(B)$ with $\ker D = B^G$. In particular, this means B^G is an algebraically closed subring of B. Derksen [94] constructed a derivation whose kernel coincides with the fixed ring of the group action in Nagata's famous counterexample to Hilbert's Fourteenth Problem (see *Chap. 6*).

Remark 1.45 An early result of Nagata (Thm. 4.1 of [322], 1962) is that the invariant ring of a $\mathbb{G}_a$-action on a factorial affine variety V over a field K has the following property: If every unit of $K[V]$ belongs to $K[V]^{\mathbb{G}_a}$, and if $f \in K[V]^{\mathbb{G}_a}$, then each prime factor of f belongs to $K[V]^{\mathbb{G}_a}$. In particular, $K[V]^{\mathbb{G}_a}$ is a UFD. See also Lemma 1 of [294].

Remark 1.46 The 1967 paper of Koshevoi [252] studies rational actions of $\mathbb{G}_a$ or $\mathbb{G}_m$ on varieties. In particular, Thm. 2 shows the existence of a rational slice for $\mathbb{G}_a$-actions. More recent work on rational $\mathbb{G}_a$-actions can be found (for example) in [341].

1.6 Degree Resolution and Canonical Factorization

Assume that B is a commutative k-domain and that $A \subset B$ is a subalgebra such that $A = \ker D$ for some non-zero $D \in \mathrm{LND}(B)$. This section considers the induced degree resolution and canonical factorization, following [157].

Our primary interest is in the following three related objects.

1. The ascending $\mathbb{N}$-filtration of B by A-modules given by:

$$B = \bigcup_{n \geq 0} \mathcal{F}_n \quad \text{where} \quad \mathcal{F}_n = \ker D^{n+1} = \{f \in B \mid \deg_D f \leq n\}$$

   The modules $\mathcal{F}_n$ are the **degree modules** associated to D.
2. The descending $\mathbb{N}$-filtration of A by ideals given by:

$$A = I_0 \supset I_1 \supset I_2 \supset \cdots \quad \text{where} \quad I_n = A \cap D^n B = D^n \mathcal{F}_n$$

   The ideals I_n are the **image ideals** associated to D. Note that $\mathrm{pl}(D) = I_1$.
3. The ring

$$\mathrm{Gr}_D(B) = \bigoplus_{n \geq 0} I_n \cdot t^n \subset A[t] \cong A^{[1]}$$

   is the **associated graded ring** defined by D.

1.6.1 Degree Modules

Observe the following.

1. Each $\mathcal{F}_n$ is a factorially closed A-submodule of B.
2. The definition of $\mathcal{F}_n$ depends only on A, not on the particular derivation D.
3. Given integers n, i with $1 \leq i \leq n$, the following sequence of A-modules is exact.

$$0 \to \mathcal{F}_{i-1} \hookrightarrow \mathcal{F}_n \xrightarrow{D^i} D^i \mathcal{F}_n \to 0$$

   In particular, $I_n \cong \mathcal{F}_n / \mathcal{F}_{n-1}$.
4. Let $r \in B$ be a local slice for D. For each $n \geq 0$, define the submodule $\mathcal{G}_n(r) \subset \mathcal{F}_n$ by:

$$\mathcal{G}_n(r) = A[r] \cap \mathcal{F}_n = A \oplus Ar \oplus \cdots \oplus Ar^n$$

   If r is a slice for D, then $\mathcal{G}_n(r) = \mathcal{F}_n$ for each $n \geq 0$.

Lemma 1.47 *If A is a noetherian ring, then $\mathcal{F}_n$ is a noetherian A-module for each $n \geq 0$.*

Proof $\mathcal{F}_0 = A$ is noetherian by hypothesis. Given $n \geq 1$, assume by induction that $\mathcal{F}_m$ is noetherian for $0 \leq m \leq n-1$. By the inductive hypothesis, $\mathcal{F}_{i-1}$ and $\mathcal{F}_{n-i}$ are noetherian for $1 \leq i \leq n$. Therefore, the submodule $D^i \mathcal{F}_n$ of $\mathcal{F}_{n-i}$ is also noetherian.

Since the sequence in observation (3) above is exact, and since $\mathcal{F}_{i-1}$ and $D^i\mathcal{F}_n$ are noetherian, it follows that $\mathcal{F}_n$ is noetherian. □

Example 1.48 Assume that A is a noetherian ring. Then there exist an integer $m \geq 1$ and $a_1, \dots, a_m \in A$ such that $\mathrm{pl}(D) = a_1A + \cdots + a_mA$. Let $r_1, \dots, r_m \in \mathcal{F}_1$ be such that $Dr_i = a_i$ for $1 \leq i \leq m$. Given $s \in \mathcal{F}_1$, write $Ds = c_1a_1 + \cdots + c_ma_m$ for $c_i \in A$. Then $s - (c_1r_1 + \cdots c_mr_m) \in A$. It follows that:

$$\mathcal{F}_1 = A + Ar_1 + \cdots + Ar_m$$

See [5], Lemma 2.2.

1.6.2 Degree Resolutions

We continue the notation and assumptions of the preceding section.

Define subrings $B_i = k[\mathcal{F}_i]$. Then $B_0 = A$ and $B_i \subset B_{i+1}$ for $i \geq 0$. If B is G-graded by an abelian group G and A is a G-graded subalgebra, then each $\mathcal{F}_i$ is a G-graded submodule and each B_i is a G-graded subalgebra.

Lemma 1.49 *Let i be an integer with $i \geq 0$.*

(a) *D restricts to $D : B_i \to B_i$.*
(b) *If $i \geq 1$, then* $\mathrm{frac}(B_i) = \mathrm{frac}(B)$

Proof Given $i \geq 1$, the definition of $\mathcal{F}_i$ implies that $D(\mathcal{F}_i) \subset \mathcal{F}_{i-1} \subset \mathcal{F}_i$. Since D restricts to a generating set for B_i, it follows that D restricts to B_i, and part (a) is confirmed.

Part (b) follows from the observation that $S^{-1}B_i = S^{-1}B = S^{-1}A[r]$ for $S = A \setminus \{0\}$ and some $r \in \mathcal{F}_1$. □

Assume that B is an affine k-domain. In this case, $B_N = B$ for some $N \geq 0$. It is possible that $B_i = B_{i+1}$ for some i. Let n_i, $0 \leq i \leq r$, be the unique subsequence of $0, 1, \dots, N$ such that:

$$\{B_0, \dots, B_N\} = \{B_{n_0}, \dots, B_{n_r}\} \quad \text{and} \quad B_{n_i-1} \subsetneq B_{n_i} \subsetneq B_{n_{i+1}} \text{ for } 1 \leq i < r$$

Note that, when $D \neq 0$, $n_0 = 0$, $n_1 = 1$ and $B_{n_r} = B$. Let $\mathcal{N}_B(A) = \{0, 1, n_2, \dots, n_r\}$. Both the integer r and the sequence of subrings

$$A = B_0 \subset B_1 \subset B_{n_2} \subset \cdots \subset B_{n_r} = B \tag{1.2}$$

are uniquely determined by A.

Definition 1.50 The sequence of inclusions (1.2) is called the **degree resolution** of B over A. The integer r is the **index** of A in B, denoted $\mathrm{index}_B(A)$; we also say that r is the **index** of D.

We make the following observations.

1. $\text{index}_B(A) + 1 = |\mathcal{N}_B(A)|$
2. $\text{index}_{B_{n_i}}(A) = i + 1$
3. $\text{index}_B(A) = 0$ if and only if $A = B$ if and only if $D = 0$
4. If D has a slice, then $\text{index}_B(A) = 1$

Let $R \subset B$ be an affine subring such that D restricts to R. The induced filtration of R is $R = \bigcup_{i \geq 0} R \cap \mathcal{F}_i$ and if $R_{n_i} = R \cap B_{n_i}$ for $n_i \in \mathcal{N}_B(A)$, then the degree resolution of R over $R \cap A$ is a refinement of the sequence:

$$R \cap A = R_0 \subset R_1 \subset R_{n_2} \subset \cdots \subset R_{n_r} = R$$

Therefore, $\mathcal{N}_R(R \cap A) \subset \mathcal{N}_B(A)$ and $\text{index}_R(R \cap A) \leq \text{index}_B(A)$.

1.6.3 Equivariant Affine Modifications

In this section, we follow the notation and terminology of [235]

An **affine triple** over k is of the form (B, I, f), where B is an affine k-domain, $I \subset B$ is a nonzero ideal and $f \in I, f \neq 0$. For such a triple, define:

$$f^{-1}I = \{g \in B_f \,|\, fg \in I\}$$

Definition 1.51 Given the affine triple (B, I, f), the subring $B[f^{-1}I] \subset B_f$ is the **affine modification** of B **along** f with **center** I.

Let $b_1, \ldots, b_s \in B$ be such that $I = (b_1, \ldots, b_s)$. Then

$$B[f^{-1}I] = B[b_1/f, \ldots, b_s/f]$$

and $B[f^{-1}I]$ is an affine domain.

Let $X = \text{Spec}(B)$ and $X_{(I,f)} = \text{Spec}(B[f^{-1}I])$.

Definition 1.52 $X_{(I,f)}$ is the **affine modification** of X **along** f with **center** I. The morphism $p : X_{(I,f)} \to X$ induced by the inclusion $B \subset B[f^{-1}I]$ is the **associated morphism** for the affine modification.
Since $B_f = B[f^{-1}I]_f$, we see that the associated morphism $p : X_{(I,f)} \to X$ is birational, and that the restriction of p to the set $\{f \neq 0\}$ is an isomorphism. The **exceptional divisor** E of $X_{(I,f)}$ is defined by the ideal $IB[f^{-1}I]$.

Our main interest is in the following two theorems, which are due to Kaliman and Zaidenberg.

Theorem 1.53 ([235], Thm. 1.1) *Any birational morphism of affine varieties is the associated morphism of an affine modification.*

Assume that X is endowed with a $\mathbb{G}_a$-action.

Definition 1.54 The affine modification $X_{(I,f)}$ of X is $\mathbb{G}_a$-**equivariant** if the $\mathbb{G}_a$-action on X lifts to $X_{(I,f)}$, i.e., $X_{(I,f)}$ admits a $\mathbb{G}_a$-action for which the associated morphism is $\mathbb{G}_a$-equivariant.

Note that, if $X_{(I,f)}$ admits a $\mathbb{G}_a$-action for which the associated morphism is $\mathbb{G}_a$-equivariant, then the action is uniquely determined.

Suppose that $D \in \mathrm{LND}(B)$, $f \in \ker D$ is non-zero, and $I \subset B$ is a non-zero integral ideal for D (i.e., $DI \subset I$). Let D' be the extension of D to B_f. Then $D'(f^{-1}I) = f^{-1}D'I \subset f^{-1}I$. It follows that D' restricts (and D extends) to $B[f^{-1}I]$. We have thus shown:

Theorem 1.55 ([235], Cor. 2.3) *Let $\rho : \mathbb{G}_a \times X \to X$ be a $\mathbb{G}_a$-action and let $X_{(I,f)}$ be an affine modification of X with locus (I,f). If $f \in k[X]^{\mathbb{G}_a}$ and ρ restricts to an action on $\mathcal{V}(I)$, then $X_{(I,f)}$ is a $\mathbb{G}_a$-equivariant affine modification.*

1.6.4 Canonical Factorizations

We continue the notation and assumptions of the preceding sections, with the added assumption that both A and B are k-affine. In this case, the geometric content of *Lemma 1.49* is as follows.

Let $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$, and let $\pi : X \to Y$ be the quotient map for the $\mathbb{G}_a$-action on X determined by D. By *Theorem 1.47*, B_{n_i} is affine for each $n_i \in \mathcal{N}_B(A)$. Define $X_i = \mathrm{Spec}(B_{n_i})$, $0 \le i \le r$.

For $0 \le i \le r-1$, the inclusion $B_{n_i} \to B_{n_{i+1}}$ induces a dominant $\mathbb{G}_a$-equivariant morphism $\pi_i : X_{i+1} \to X_i$ which is birational if $i \neq 0$. Therefore, π factors into the uniquely determined sequence of dominant $\mathbb{G}_a$-equivariant morphisms

$$X = X_r \xrightarrow{\pi_{r-1}} X_{r-1} \to \cdots \to X_2 \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0 = Y \tag{1.3}$$

where each morphism $\pi_{r-1}, \ldots, \pi_1$ is birational.

Definition 1.56 The sequence of mappings (1.3) is the **canonical factorization** of the quotient morphism π for the $\mathbb{G}_a$-action determined by D. The integer r is the **index** of the $\mathbb{G}_a$-action.

From *Theorem 1.53* and *Theorem 1.55*, we conclude that the maps $\pi_1, \ldots, \pi_{r-1}$ in the canonical factorization (1.3) form a sequence of $\mathbb{G}_a$-equivariant affine modifications. Regarding fixed points, note that $\pi_{i-1}(X_i^{\mathbb{G}_a}) \subset X_{i-1}^{\mathbb{G}_a}$ for each i with $1 \le i \le r$. Note also that, for $1 \le i \le r$, the $\mathbb{G}_a$-action on X_i has index i, and its canonical factorization is given by $\pi_0\pi_1 \cdots \pi_{i-1}$.

We calculate several examples of degree resolutions and canonical factorizations in *Chap. 8*.

Chapter 2
Further Properties of LNDs

To understand a ring, study its LNDs.
To understand an LND, study its kernel.
To understand a kernel, study its LNDs.

The first three sections of this chapter investigate derivations in the case B has one or more nice divisorial properties, in addition to the ongoing assumption that B is a commutative k-domain, where k is a field of characteristic zero. Subsequent sections discuss quasi-extensions, G-critical elements, the degree of a derivation, trees and cables, exponential automorphisms, construction of kernel elements by transvectants and Wronskians, and recognition of polynomial rings.

2.1 Irreducible Derivations

B is said to satisfy the **ascending chain condition (ACC) on principal ideals** if and only if every infinite chain $(b_1) \subset (b_2) \subset (b_3) \subset \cdots$ of principal ideals of B stabilizes. Note that every UFD and every commutative noetherian ring satisfies this condition.

Lemma 2.1 *If B satisfies the ACC on principal ideals, so does $B^{[n]}$ for every $n \geq 0$.*

Proof By induction, it suffices to show that $B^{[1]}$ has the ACC on principal ideals. Suppose

$$(p_1(t)) \subset (p_2(t)) \subset (p_3(t)) \subset \cdots$$

is an infinite chain of principal ideals, where $p_i(t) \in B[t]$ (t an indeterminate over B). Since B is a domain, the degrees of the $p_i(t)$ must stabilize, so we may assume (truncating the chain if necessary) that for some positive integer d, $\deg_t p_i(t) = d$ for all i. (If $d = 0$ this is already a chain in B.) Thus, given i, there exist $e_i \in B$ with $p_i(t) = e_i p_{i+1}(t)$. For each integer m with $0 \leq m \leq d$, let $c_m(p_i)$ denote

G. Freudenburg, *Algebraic Theory of Locally Nilpotent Derivations*,
Encyclopaedia of Mathematical Sciences 136,
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the coefficient of t^m in $p_i(t)$. Equating coefficients, we have $c_m(p_i) = e_i c_m(p_{i+1})$, which yields

$$(c_m(p_1)) \subset (c_m(p_2)) \subset (c_m(p_3)) \subset \cdots$$

and this is an ascending chain of principal ideals in B. By the ACC, each such chain stabilizes, and since there are only $d + 1$ such chains, we conclude that the given chain in $B[t]$ also stabilizes. □

Next, we say that B is a **highest common factor ring**, or **HCF-ring**, if and only if the intersection of any two principal ideals of B is again principal. Examples of HCF-rings are: a UFD, a valuation ring, or a polynomial ring over a valuation ring.

Note that a UFD is an HCF-ring which also satisfies the ACC on principal ideals.

Recall that $D \in \mathrm{Der}_k(B)$ is irreducible if and only if DB is contained in no proper principal ideal. We will show that for commutative k-domains satisfying the ACC on principal ideals, a derivation is always a multiple of an irreducible derivation.

Proposition 2.2 (See Also [68], Lemma 2.18) *Let* $\delta \in \mathrm{Der}_k(B)$ *and* $\delta \neq 0$.

(a) *If B satisfies the ACC for principal ideals, then there exists an irreducible $D \in \mathrm{Der}_k(B)$ and $a \in B$ such that $\delta = aD$.*
(b) *If B is an HCF-ring, and if $aD = bE$ for $a, b \in B$ and irreducible k-derivations D and E, then $aB = bB$.*
(c) *If B is a UFD and $\delta = aD$ for irreducible D and $a \in B$, then D is unique up to multiplication by a unit.*

Proof Note first that, for any commutative k-domain B, if $D \in \mathrm{Der}_k(B)$ has $DB \subset aB$ for $a \in B$ and $a \neq 0$, then there exists $D' \in \mathrm{Der}_k(B)$ such that $D = aD'$. To see this, let $\Delta \in \mathrm{Der}_k(\mathrm{frac}B)$ be given by $\Delta = \frac{1}{a}D$. Then Δ is well-defined, and restricts to B, so we may take D' to be the restriction of Δ to B.

To prove part (a), suppose δ is not irreducible. Then $\delta B \subset a_1 B$ for some non-unit $a_1 \in B$. So there exists $D_1 \in \mathrm{Der}_k(B)$ with $\delta = a_1 D_1$, and since B is a domain, $\ker \delta = \ker D_1$. If D_1 is irreducible, we are done. Otherwise, continue in this way to obtain a sequence of derivations D_i, and non-units $a_i \in B$, such that $\ker D_i = \ker \delta$ for each i, and:

$$\delta = a_1 D_1 = a_1 a_2 D_2 = a_1 a_2 a_3 D_3 = \cdots$$

The process terminates after n steps if any D_n is irreducible, and part (a) will follow.

Otherwise this chain is infinite, with the property that every a_i is a non-unit of B. In this case, choose $f \in B$ not in $\ker \delta$. By the ACC on principal ideals, the chain

$$(\delta f) \subset (D_1 f) \subset (D_2 f) \subset (D_3 f) \subset \cdots$$

stabilizes: $(D_n f) = (D_{n+1} f)$ for all $n \gg 0$. Therefore, there exists a sequence of units $u_i \in B$ with $u_n D_{n+1} f = D_n f = a_{n+1} D_{n+1} f$. Since $D_{n+1} f \neq 0$, this implies

$u_n = a_{n+1}$, i.e., every a_n is a unit for $n \gg 0$. We arrive at a contradiction, meaning this case cannot occur. So part (a) is proved.

For part (b), set $T = aD = bE$. Since B is an HCF-ring, there exists $c \in B$ with $aB \cap bB = cB$. Therefore, $TB \subset cB$, and there exists a k-derivation F of B such that $T = cF$. Write $c = as = bt$ for $s, t \in B$. Then $cF = asF = aD$ implies $D = sF$ (since B is a domain), and likewise $cF = btF = bE$ implies $E = tF$. By irreducibility, s and t are units of B, and thus $aB = bB$. So part (b) is proved.

Finally, part (c) follows immediately from parts (a) and (b), and the unique factorization hypothesis, since a UFD is both a ring satisfying the ACC on principal ideals and an HCF-ring. ☐

Corollary 2.3 *If B is a UFD, $D \in \mathrm{LND}(B)$ is irreducible, and $A = \ker D$, then* $\mathrm{LND}_A(B) = \{aD \mid a \in A\}$.

See Ex. 2.16 and 2.17 in [68] for examples in which the conclusions of these results fail for other rings.

2.2 Minimal Local Slices

Minimal local slices for a locally nilpotent derivation D are defined, and their basic properties discussed. The number of equivalence classes of minimal local slices can be a useful invariant of D. This number is closely related to the plinth ideal $DB \cap \ker D$ of D.

We assume throughout this section that B is a commutative k-domain which satisfies the ACC on principal ideals.

Fix $D \in \mathrm{LND}(B)$, $D \neq 0$, and set $A = \ker D$. Let $B = \cup_{i \geq 0} \mathcal{F}_i$ be the filtration of B induced by D, i.e., elements $f \in \mathcal{F}_i$ have $\deg_D(f) \leq i$. Note that $\mathcal{F}_1 \setminus A$ is the set of local slices for D. An equivalence relation is defined on $\mathcal{F}_1$ via:

$$r \sim s \quad \Leftrightarrow \quad A[r] = A[s]$$

In particular, all kernel elements are equivalent.

Proposition 2.4

(a) *B satisfies the ACC on subalgebras of the form $A[r]$, $r \in \mathcal{F}_1 \setminus A$.*

(b) *Given $r_0 \in \mathcal{F}_1 \setminus A$, the set*

$$\{A[r] \,:\, r \in \mathcal{F}_1 \setminus A,\, A[r_0] \subseteq A[r]\}$$

partially ordered by set inclusion contains at least one maximal element. Moreover, if $A[r]$ is maximal for this set, then $A[r]$ is also a maximal element of the superset $\{A[s] \,:\, s \in \mathcal{F}_1 \setminus A\}$, partially ordered by set inclusion.

Proof Suppose $A[r_1] \subseteq A[r_2] \subseteq A[r_3] \subseteq \cdots$ for $r_i \in \mathcal{F}_1 \setminus A$. Given $i \geq 1$, since $r_i \in A[r_{i+1}] \cong A^{[1]}$, the degree of r_i as a polynomial in r_{i+1} (over A) equals 1: otherwise $Dr_i \notin A$ (*Principle 3*). For each $i \geq 1$, write $r_i = a_i r_{i+1} + b_i$ for some $a_i, b_i \in A$. Then, for each $i \geq 1$, $Dr_i = a_i \cdot Dr_{i+1}$. We thus obtain an ascending chain of principal ideals:

$$(Dr_1) \subseteq (Dr_2) \subseteq (Dr_3) \subseteq \cdots$$

Since this chain must eventually stabilize, we conclude that all but finitely many of the a_i are units of B. It follows that $A[r_n] = A[r_{n+1}]$ for $n \gg 0$. This proves (a).

To prove (b), just use (a), and apply Zorn's Lemma. $\square$

We say that $r \in \mathcal{F}_1$ is a **minimal local slice** for D if and only if $A[r]$ is a maximal element of the set $\{A[s] \,:\, s \in \mathcal{F}_1\}$. The set of minimal local slices of D is denoted $\min(D)$, which by the proposition is nonempty if $D \neq 0$.

Proposition 2.5 *Let $\sigma \in \mathcal{F}_1 \setminus A$ be given. Then $\sigma \in \min(D)$ if and only if every $s \sim \sigma$ is irreducible.*

Proof If $\sigma \in \min(D)$ factors as $\sigma = ab$, then $1 = \deg_D(ab) = \deg_D(a) + \deg_D(b)$, which implies either $\deg_D(a) = 0$ and $\deg_D(b) = 1$, or $\deg_D(a) = 1$ and $\deg_D(b) = 0$. Thus, either $a \in A$ and $b \in \mathcal{F}_1$, or $a \in \mathcal{F}_1$ and $b \in A$. Assuming $a \in A$ and $b \in \mathcal{F}_1$, if $a \notin B^*$, then $A[\sigma]$ is properly contained in $A[b]$, which is impossible. Therefore $a \in B^*$, and σ is irreducible. Since every $s \sim \sigma$ is also in $\min(D)$, every such s is also irreducible.

Suppose $\sigma \notin \min(D)$. Then there exists $r \in \mathcal{F}_1$ such that $A[\sigma]$ is properly contained in $A[r]$. Since both σ and r are local slices, σ has degree 1 as a polynomial in r, i.e., $\sigma = ar + b$ for $a, b \in A$, $a \neq 0$, and $a \notin B^*$. Thus, $\sigma \sim ar$, which is reducible. $\square$

Proposition 2.6 *Let $r \in \mathcal{F}_1 \setminus A$ be given. If Dr is irreducible in B, then either $r \in \min(D)$ or D admits a slice.*

Proof If $r \notin \min(D)$, then $r = as + b$ for some $s \in \min(D)$ and some non-unit $a \in A$. Thus, $Dr = a \cdot Ds$, and since Dr is irreducible, $Ds \in B^*$. $\square$

Proposition 2.7 *Given $D \in \mathrm{LND}(B)$ with $A = \ker D$, the following are equivalent.*

1. *D has a unique minimal local slice (up to equivalence).*
2. *$\mathrm{pl}(D)$ is a principal ideal of A.*

Proof Suppose that D has only one minimal local slice $r \in B$, up to equivalence, and let $a \in \mathrm{pl}(D) = A \cap DB$ be given. Then there exists a local slice p of D such that $Dp = a$. By hypothesis, there exist $b, c \in A$ with $p = br + c$. Thus, $a = Dp = bDr \in Dr \cdot A$. Therefore, $\mathrm{pl}(D) = Dr \cdot A$.

Conversely, suppose $\mathrm{pl}(D) = fA$ for some $f \in A$, and let a minimal local slice r be given. Let $s \in B$ be such that $Ds = f$. Then s is also a local slice. If $Dr = fg$ for some $g \in A$, then $r - sg \in A$, which implies $A[r] \subset A[s]$. Since r is minimal, $A[r] = A[s]$, i.e., r and s are equivalent. $\square$

2.3 Four Lemmas About UFDs

Lemma 2.8 *If B is a UFD and $A \subset B$ is a factorially closed subring, then A is a UFD.*

Proof The property that A is factorially closed clearly implies:

$$A^* = A \cap B^* \quad \text{and} \quad aB \cap A = aA \quad \forall a \in A$$

Suppose that $a \in A$ is prime in B. If $fg \in aA$ for $f, g \in A$, then either $f \in aB \cap A = aA$ or $g \in aB \cap A = aA$. So a is prime in A.

Given nonzero $h \in A \setminus A^*$, write $h = b_1 \cdots b_n$, where $b_i \in B$ is prime, $1 \le i \le n$. Since A is factorially closed, $b_i \in A$, which implies that b_i is prime in A, $1 \le i \le n$. Since this factorization is unique in B up to order, it is also unique in A up to order. Therefore, A is a UFD. □

Lemma 2.9 *Suppose K is an algebraically closed field. If B is an affine UFD over K with tr.deg$_K B = 1$, then $B \cong_K K[t]_{f(t)}$ for some $f(t) \in K[t] = K^{[1]}$. If, in addition, $B^* = K^*$, then $B = K^{[1]}$.*

Proof Let X denote the curve Spec(B). Since B is a UFD, X is normal (hence smooth), and its class group Cl(X) is trivial (see, for example, [203], II.6.2). Embed X in a complete nonsingular algebraic curve Y as an open subset. Then there are points $P_i \in Y$, $1 \le i \le n$, such that $Y - X = \{P_1, \dots, P_n\}$. Since X is affine, $n \ge 1$.

Let F denote the subgroup of Cl(Y) generated by the divisor classes $[P_1], \dots, [P_n]$. Then $\{0\} = \text{Cl}(X) = \text{Cl}(Y)/F$, meaning Cl$(Y)$ is finitely generated. It is known that if C is any complete nonsingular curve over K which is not rational, then Cl(C) is not finitely generated. (This follows from the fact that the Jacobian variety of C is a divisible group; see Mumford [317], p. 62.) Therefore, Y is rational, which implies $Y = \mathbb{P}^1_K$ (the projective line over K).

It follows that X is the complement of n points in $\mathbb{P}^1_K$, which is isomorphic to the complement of $n-1$ points of $\mathbb{A}^1_K$. Therefore, $B = \mathcal{O}(X)$ has the form $K[t]_{f(t)}$ for some $f \in K[t] \cong K^{[1]}$. If $B^* = K^*$, then $B = K[t]$. □

Lemma 2.10 *Suppose k is an algebraically closed field of characteristic zero. If B is a UFD over k with tr. deg$_k B = 2$, then every irreducible element of* LND(B) *has a slice.*

Proof Suppose $D \in \text{LND}(B)$ is irreducible, and set $A = \ker D$. By *Proposition 2.4*, D has a minimal local slice y.

Suppose $Dy \notin B^*$. Then there exists irreducible $x \in B$ dividing Dy. Since A is factorially closed, $x \in A$. Since B is a UFD, x is prime and B/xB is a domain.

Let $\bar{D} = D \pmod{x}$ on $\bar{B} = B/xB$. Since D is irreducible, $\bar{D} \neq 0$. In addition, tr. deg.$_k \bar{B} = 1$. By *Corollary 1.28*, it follows that $\bar{B} = k^{[1]}$ and $\ker \bar{D} = k$. Since $\bar{D}\bar{y} = 0$, we see that $\bar{y} \in k$, and $y \in xB + k$. Write $y = xz + \lambda$ for some $z \in B$ and $\lambda \in k$. Then $y - \lambda = xz$ is irreducible, by *Proposition 2.5*. But this implies $z \in B^* \subset A$, and thus $y = xz + \lambda \in A$, a contradiction.

Therefore $Dy \in B^*$, and D has a slice. □

In case B is affine, we get the following geometric characterization.

Corollary 2.11 *Let k be an algebraically closed field of characteristic zero and let X be a factorial affine surface over k. X admits a non-trivial $\mathbb{G}_a$-action if and only if $X \cong_k Z \times \mathbb{A}^1$ for a rational plane curve Z defined by $yf(x) = 1$ for some $f(x) \in k[x, y] = k^{[2]}$.*

Proof Assume that X admits a non-trivial $\mathbb{G}_a$-action. By *Lemma 2.10*, there exists a subalgebra $A \subset k[X]$ such that $k[X] = A^{[1]}$. Therefore, A is an affine UFD and $\dim_k A = 1$. By *Lemma 2.9*, we conclude that $A = k[t]_{f(t)}$ for some $t \in A$ and $f \in k^{[1]}$. □

Lemma 2.12 *Let K be an algebraically closed field and B a UFD over K. Suppose that B is endowed with a degree function* deg *with values in $\mathbb{N}$ and degree modules $\{F_n\}_{n \geq 0}$ such that $F_0 = K$.*

(a) *Given $n \geq 0$ and $f \in B \setminus K[F_n]$ of minimal degree, $K[f]$ is factorially closed in B.*
(b) *If $tr.deg_K B = 1$, then $B = K^{[1]}$.*

Proof Given $\lambda \in K$, suppose that $uv = f - \lambda$ for $u, v \in B$. Since $\deg f > 0$, $\deg u + \deg v = \deg f$. If $\deg f \leq \deg u$, then:

$$0 < \deg f + \deg v \leq \deg u + \deg v = \deg f \;\Rightarrow\; \deg v \leq 0 \;\Rightarrow\; v \in K^*$$

Similarly, if $\deg f \leq \deg v$, then $u \in K^*$. If $\deg u < \deg f$ and $\deg v < \deg f$, then $u, v \in K[F_n]$. But this implies $f \in K[F_n]$, a contradiction. Therefore, $f - \lambda$ is irreducible for every $\lambda \in K$.

Suppose that $rs \in K[f]$ for $r, s \in B$. Then every irreducible factor of r (respectively, s) is of the form $f - \lambda$ for $\lambda \in K$. Therefore, $r, s \in K[f]$ and $K[f]$ is factorially closed in B.

If $\text{tr.deg}_K B = 1$, choose $f \in B \setminus K$ of minimal degree. By part (a), $K[f]$ is factorially closed in B. Therefore, $K[f]$ is algebraically closed in B, hence equal to B. □

Lemma 2.10 and *Lemma 2.12* are elementary. Combining them gives the following characterization. See *Sect.* 2.5 for the definition of $ML(B)$.

Corollary 2.13 *Let k be algebraically closed, and let B be a UFD over k such that $tr.deg_k B = 2$ and $ML(B) = k$. Given $D \in \mathrm{LND}(B)$, there exist $x, y \in B$ such that $B = k[x, y] = k^{[2]}$, $Dy \in k[x]$ and $Dx = 0$.*

Proof Let $A = \ker D$, and let $D = a\Delta$, where $\Delta \in \mathrm{LND}(B)$ is irreducible, $\ker \Delta = A$ and $a \in A$. By *Lemma 2.10*, Δ has a slice $y \in B$, so $B = A[y]$. By hypothesis, there exists $E \in \mathrm{LND}(B)$ such that $A \cap \ker E = k$. Therefore, $\deg_E$ defines a degree function on A for which the only elements of degree zero are those in k. By *Lemma 2.12(b)*, it follows that $A = k[x]$ for some $x \in A$. □

Remark 2.14 Notice that *Lemma 2.9* and *Lemma 2.12* are valid in any characteristic. *Lemma 2.9* and its proof seem to be well-known, although I could not find a reference. In addition, notice that we do not need to assume that B is an affine ring for the other three lemmas, or for *Corollary 2.13*. Fauntleroy and Magid proved *Lemma 2.10* in the case B is affine and the corresponding $\mathbb{G}_a$-action is free ([152], Thm. 3). Makar-Limanov proved *Corollary 2.13* in the case B is affine over k ([276], Lemma 19). Crachiola generalized *Lemma 2.10* to any algebraically closed field K, specifically: Let B be a UFD over K with $\text{tr.deg}_K B = 2$. If B admits a non-trivial $\mathbb{G}_a$-action, then $B = A^{[1]}$, where $A = B^{\mathbb{G}_a}$ ([60], Thm. 3.1).

2.4 Degree of a Derivation

This section features two results of Daigle giving cases in which the degree of a derivation is defined.

Theorem 2.15 ([77], Thm. 1.7(a) and L.1.8) *Let G be a totally ordered abelian group, and B a G-graded affine k-domain with induced degree function* $\deg_G$*. Suppose that R is a k-subalgebra of B such that B is a localization of R. If* deg *denotes the restriction of* $\deg_G$ *to R, then* $\deg D$ *is defined for every* $D \in \text{Der}_k(R)$.
It should be noted that Wang proved a special case of this result in his thesis, assuming that $G = \mathbb{Z}$, $R = B$ and D is locally nilpotent. See [414], Cor. 2.2.7.

Theorem 2.16 *Suppose that B is a commutative k-domain, of finite transcendence degree over k. Then for any pair* $D \in \text{Der}_k(B)$ *and* $E \in \text{LND}(B)$*,* $\deg_E D$ *is defined.*

The second of these two results is unpublished, and the remainder of this section provides a proof.

Suppose B is a commutative k-domain equipped with a degree function $\deg : B \to \mathbb{N} \cup \{-\infty\}$, and let B_0 denote the set of degree-zero elements, together with 0. Recall that B_0 is a factorially closed k-subalgebra of B, with $B^* \subset B_0$.

In addition to the degree function, let $D \in \text{Der}(B)$ be given. Together, these define an associated **defect function** $\text{def} : B \to \mathbb{Z} \cup \{-\infty\}$, namely:

$$\text{def}(b) = \deg(Db) - \deg(b) \quad \text{for} \quad b \neq 0 \ , \quad \text{and} \quad \text{def}(0) = -\infty$$

Likewise, for any non-empty subset $S \subset B$, $\text{def}(S)$ is defined by $\sup_{b \in S} \text{def}(b)$. Note that $\text{def}(S)$ takes its values in $\mathbb{Z} \cup \{\pm\infty\}$. The **defect** of D relative to deg is then defined to be $\text{def}(B)$, and is denoted by $\text{def}(D)$.

The reason for defining the defect of a derivation is that, when $D \neq 0$, $\deg D$ is defined if and only if $\text{def}(D)$ is finite. The defect has the following basic properties.

Lemma 2.17 *Let $a, b \in B$, and let S be a non-empty subset of B.*

- **(a)** $\text{def}(S) = -\infty$ *if and only if* $S \subset \ker D$.
- **(b)** $\text{def}(D) = -\infty$ *if and only if* $D = 0$.
- **(c)** *D is homogeneous relative to* deg *if and only if def is constant on* $B \setminus \{0\}$.
- **(d)** $\text{def}(ab) \le \max\{\text{def}(a), \text{def}(b)\}$*, with equality when* $\text{def}(a) \neq \text{def}(b)$.
- **(e)** $\text{def}(a^n) = \text{def}(a)$ *for all positive integers n.*

(f) *If* $a \in \ker D$, *then* $\operatorname{def}(ab) = \operatorname{def}(b)$.
(g) *If* $\deg(a) < \deg(b)$, *then* $\operatorname{def}(a+b) \le \max\{\operatorname{def}(a), \operatorname{def}(b)\}$.
(h) *If* $a, b \in B_0$, *then* $\operatorname{def}(a+b) \le \max\{\operatorname{def}(a), \operatorname{def}(b)\}$.

Proof Following is a proof of item (g); verification of the others is left to the reader.

$$\begin{aligned}
\operatorname{def}(a+b) &= \deg(D(a+b)) - \deg(a+b) \\
&= \deg(Da + Db) - \deg(b) \\
&\le \max\{\deg(Da), \deg(Db)\} - \deg(b) \\
&= \max\{\deg(Da) - \deg(b), \deg(Db) - \deg(b)\} \\
&\le \max\{\deg(Da) - \deg(a), \deg(Db) - \deg(b)\} \\
&= \max\{\operatorname{def}(a), \operatorname{def}(b)\}
\end{aligned}$$

□

The defect was used by Makar-Limanov in [275] to study locally nilpotent derivations, and independently by Wang in his thesis, which contains the following result.

Proposition 2.18 ([414], Lemma 2.2.5, (4)) *For any transcendence basis S of B_0 over k:*

$$\operatorname{def}(B_0) = \operatorname{def}(S)$$

In particular, if B_0 is finitely generated over k, then $\operatorname{def}(B_0) < \infty$.

Proof Note first that, since $S \subset k[S]$, we have $\operatorname{def}(S) \le \operatorname{def}(k[S])$. Conversely, let $f \in k[S]$ be given. Then there exist $t_1, \dots, t_n \in S$ such that f is a finite sum of monomials of the form $at_1^{e_1} \cdots t_n^{e_n}$, where $a \in k^*$ and $e_i \in \mathbb{N}$. From the properties in the lemma above, it follows that:

$$\operatorname{def}(f) \le \max_{1 \le i \le n} \operatorname{def}(t_i) \le \operatorname{def}(S)$$

Therefore, $\operatorname{def}(k[S]) \le \operatorname{def}(S)$, meaning $\operatorname{def}(k[S]) = \operatorname{def}(S)$. So if $B_0 = k[S]$, we are done.

Otherwise, choose $x \in B_0$ not in $k[S]$. By hypothesis, there exist $a_0, \dots, a_n \in k[S]$ such that, if T is indeterminate over $k[S]$ and $P(T) = \sum_i a_i T^i$, then $P(x) = 0$. Choose P of minimal positive T-degree with this property, so that $P'(x) \neq 0$, and set $Q(T) = \sum_i (Da_i) T^i$. Then by the product rule, $0 = D(P(x)) = Q(x) + P'(x)Dx$, which implies:

$$\deg(Dx) = \deg(P'(x)Dx) = \deg Q(x) \le \max\{\deg(Da_0), \dots, \deg(Da_n)\}$$

Since $\text{def}(b) = \deg(Db)$ for elements b of B_0, it follows that:

$$\text{def}(x) \leq \max\{\text{def}(a_0), \ldots, \text{def}(a_n)\} \leq \text{def}(k[S]) = \text{def}(S)$$

Therefore, $\text{def}(B_0) \leq \text{def}(S)$, meaning $\text{def}(B_0) = \text{def}(S)$. $\square$

Corollary 2.19 *Suppose $B = A[T] = A^{[1]}$ for some subring A of B which is of finite transcendence degree over k. Then, relative to T-degrees,* $\deg D$ *is defined for every nonzero $D \in \text{Der}_k(B)$.*

Proof Let $M = \max\{\text{def}(A), \text{def}(T)\}$, which is finite by the result above. Suppose $f(T) \in A[T]$ has degree $n \geq 1$, and write $f = \sum_i a_i T^i$ for $a_i \in A$. By property (g) from the lemma above, we have $\text{def}(f) \leq \max_{0 \leq i \leq n} \text{def}(a_i T^i)$. Using the other properties in the lemma, it follows that:

$$\text{def}(f) \leq \max\{\text{def}(a_0), \ldots, \text{def}(a_n), \text{def}(T)\} \leq M$$

Therefore, $\text{def}(D) \leq M < \infty$. $\square$

We can now prove the result of Daigle given at the beginning of this section.

Proof of Theorem 2.16 We need to prove that, relative to the degree function $\deg_E$ on B defined by E, the defect $\text{def}(D) < \infty$ for every $D \in \text{Der}_k(B)$. In case $D = 0$, this is clear, so assume $D \neq 0$. If $A = \ker E$, and if $r \in B$ is a local slice of D, then $B_f = A_f[r]$, where $f = Dr$. Let D_f denote the extension of D to B_f, and let Def denote the defect on B_f defined by degrees in r and the derivation D_f. Then by *Corollary 2.19* above, $\text{Def}(D_f)$ is finite.

Note that for any $b \in B$, $\deg_E(b)$ equals the r-degree of b as an element of $A_f[r]$. It follows that, for every nonzero $b \in B$:

$$\text{def}(b) = \deg_E(Db) - \deg_E(b) = \deg_r(D_f b) - \deg_r(b) = \text{Def}(b) \leq \text{Def}(D_f)$$

Therefore, $\text{def}(D) \leq \text{Def}(D_f) < \infty$. $\square$

Remark 2.20 In order to illustrate the necessity of assuming that the ring is of finite transcendence degree over k, consider $\mathcal{B} = k[x_1, x_2, \ldots]$, the ring of polynomials in a countably infinite number of variables x_i. Define $D \in \text{Der}_k(\mathcal{B})$ and $E \in \text{LND}(\mathcal{B})$ by $Dx_n = x_{2n}$ for all $n \geq 1$; and by $Ex_1 = 0$, and $Ex_n = x_{n-1}$ for $n \geq 2$. Then using degrees determined by E, we see that for all $n \geq 1$:

$$\text{def}(x_n) = \deg_E(Dx_n) - \deg_E(x_n) = 2n - n = n$$

See also Remark 5 (p. 21) of [414].

2.5 Makar-Limanov Invariant

The **Makar-Limanov Invariant** of the commutative k-domain B is the subalgebra of B defined by:

$$ML(B) = \bigcap_{D \in \mathrm{LND}(B)} \ker D$$

In the geometric setting, if X is an algebraic k-variety, then $ML(X)$ is the ring of regular functions on X which are invariant for all algebraic $\mathbb{G}_a$-actions on X.

This invariant was introduced by Makar-Limanov in [278]. He called it the **ring of absolute constants** or **absolute kernel** for B, and used the notation $AK(B)$ to denote this ring. We make the following observations.

1. $ML(B)$ is factorially closed in B and $B^* \subset ML(B)$.
2. $ML(B)$ is a characteristic subring of B, i.e., every element of $\mathrm{Aut}_k(B)$ restricts to an automorphism of $ML(B)$.

Example 2.21 If $B = k[x_1, \ldots, x_n] = k^{[n]}$, then $ML(B) = k$, since:

$$\bigcap_{1 \le i \le n} \ker \frac{\partial}{\partial x_i} = k$$

Lemma 2.22 *Let $S \subset ML(B) \setminus \{0\}$ be a multiplicatively closed set.*

(a) $ML(S^{-1}B) \subset S^{-1}ML(B)$
(b) *If B is affine, then $ML(S^{-1}B) = S^{-1}ML(B)$*

Proof Let $f \in ML(S^{-1}B)$ and $D \in \mathrm{LND}(B)$ be given. Write $f = b/s$ for $b \in B, s \in S$. Since $S \subset \ker D$, D extends to $S^{-1}D \in \mathrm{LND}(S^{-1}B)$. Therefore:

$$S^{-1}Df = 0 \quad \Rightarrow \quad sS^{-1}Db = 0 \quad \Rightarrow \quad S^{-1}Db = 0 \quad \Rightarrow \quad Db = 0$$

It follows that $b \in ML(B)$ and $f \in S^{-1}ML(B)$. This proves part (a).

Assume that B is affine. Let $g \in S^{-1}ML(B)$ and $\delta \in \mathrm{LND}(S^{-1}B)$ be given. Since B is affine, there exists $u \in S$ such that $u\delta$ restricts to B. Let E denote the restriction of $u\delta$ to B, and let $g = h/v$ for $h \in ML(B), v \in S$. Then:

$$Eh = 0 \quad \Rightarrow \quad u(\delta h) = (u\delta)h = 0 \quad \Rightarrow \quad \delta h = 0 \quad \Rightarrow \quad \delta g = \delta(v^{-1}h) = 0$$

It follows that $g \in ML(S^{-1}B)$. This proves part (b). □

A commutative k-domain B is **rigid** if and only if $ML(B) = B$. Similarly, B is **semi-rigid** if and only if there exists $D \in \mathrm{LND}(B)$ such that $ML(B) = \ker D$. Note that a rigid ring is semi-rigid, since we can take $D = 0$. A semi-rigid ring which is not rigid admits only one non-trivial locally nilpotent derivation up to equivalence, where two locally nilpotent derivations are equivalent if and only if their kernels coincide. Note that every field is rigid (*Corollary 1.22*).

Rigid rings were first studied by Miyanishi [297] in the case B is affine, with an interest in cylinders over Spec(B). Rigid rings were later studied extensively by Makar-Limanov, who introduced the term rigid and proved *Theorem 2.24* below. In [125], Dubouloz introduced the term semi-rigid. The thesis of Alhajjar [5] studies certain families of semi-rigid rings. A semi-rigid ring which is not rigid has a unique non-trivial degree resolution. Alhajjar's thesis introduces degree resolutions in this special case, and they are an important tool in his study of semi-rigid rings.

Lemma 2.23 *Let B be an affine k-domain.*

(a) *If B is rigid, then every localization of B is rigid.*
(b) *If B is semi-rigid, then every localization of B is semi-rigid.*

Proof Part (a) follows directly from *Lemma 2.22*.

Assume that B is semi-rigid but not rigid and let $S \subset B \setminus \{0\}$ be a multiplicatively closed set. If $S^{-1}B$ is rigid, then part (b) holds. So assume that $S^{-1}B$ is not rigid and let nonzero $\delta \in \mathrm{LND}(S^{-1}B)$ be given. Note that $S \subset \ker \delta$. Since B is affine, there exists $s \in S$ such that $s\delta$ restricts to B. If $D = s\delta|_B$, then $D \in \mathrm{LND}(B)$ and $D \neq 0$. Therefore, $\ker D = ML(B)$ and $S \subset ML(B)$. By *Lemma 2.22*, it follows that:

$$ML(S^{-1}B) = S^{-1}ML(B) = \ker \delta$$

This proves part (b). □

The following fundamental result is due to Makar-Limanov [276].

Theorem 2.24 (Semi-Rigidity Theorem) *If A is a rigid commutative k-domain of finite transcendence degree over k, then $A^{[1]}$ is semi-rigid.*

Proof Let $A[x] = A^{[1]}$. Define the set $\mathcal{H} \subset \mathrm{LND}(A[x])$ by: $D \in \mathcal{H}$ if and only if $Dx = 0$. Any $D \in \mathcal{H}$ induces a quotient derivation $D_c \in \mathrm{LND}(A[x]/(x-c))$ for every $c \in k$, and we have:

$$A[x]/(x-c) \cong_k A \quad \Rightarrow \quad D_c = 0 \quad \Rightarrow \quad D(A[x]) \subset (x-c)A[x] \quad \forall c \in k$$

Since A is a domain, it follows that $D = 0$. Therefore, $\mathcal{H} = \{0\}$.

Let nonzero $D \in \mathrm{LND}(A[x])$ be given, and consider the $\mathbb{Z}$-grading of $A[x]$ over A for which x is $\mathbb{Z}$-homogeneous and $\deg_{\mathbb{Z}} x = 1$. Since the transcendence degree of $A[x]$ over k is finite, *Theorem 2.16* implies that $\deg_{\mathbb{Z}} D$ is defined. Since $A[x]$ is $\mathbb{Z}$-graded, the associated graded ring is $A[x]$. Therefore, D induces a nonzero $\mathbb{Z}$-homogeneous derivation $\bar{D}$ of $A[x]$, and by *Principle 14*, $\bar{D} \in \mathrm{LND}(A[x])$.

Given $f \in \ker D$, let $\bar{f}$ be the highest-degree $\mathbb{Z}$-homogeneous summand of f. Then $\bar{D}\bar{f} = 0$ (recall that $\mathrm{gr}(\ker D) \subset \ker(\mathrm{gr} D)$). Write $\bar{f} = ax^n$ for $a \in A$ and $n \geq 0$, noting that $a \neq 0$. If $n > 0$, then $\bar{D}x = 0$, which implies $\bar{D} \in \mathcal{H} = \{0\}$, a contradiction. Therefore, $n = 0$ and $f \in A$. Consequently, $\ker D = A$. □

Corollary 2.25 *Let A be a rigid affine k-domain, and let R be a localization of $A[x] = A^{[1]}$.*

(a) $\mathrm{LND}(R) = \mathrm{LND}_A(R)$

(b) *If R is algebraic over $A[R^*]$, then R is rigid*

Proof By *Theorem 2.24*, $A[x]$ is semi-rigid, and by *Lemma 2.23(b)*, R is semi-rigid. Since $A[x]$ is affine, there exists nonzero $D \in \mathrm{LND}(R)$ such that D restricts to $A[x]$. Therefore, $\ker(D|_{A[x]}) = A$, so $A \subset \ker D$. This proves part (a).

For part (b), let nonzero $\delta \in \mathrm{LND}(R)$ be given. Then $R^* \subset \ker\delta$. By part (a), we also have $A \subset \ker\delta$. So $A[R^*] \subset \ker\delta$. If R is algebraic over $A[R^*]$, then R is also algebraic over $\ker D$, meaning that $D = 0$. This proves part (b). □

Corollary 2.26 *Let A be a rigid affine k-domain and $A[x,y] = A^{[2]}$. If $D \in \mathrm{LND}(A[x,y])$ is nonzero and $A[x] \cap \ker D \not\subset A$, then $\ker D = A[x]$.*

Proof Suppose that $f \in A[x] \cap \ker D$ and $f \notin A$. Let $S = k[f] \setminus \{0\}$ and $R = S^{-1}A[x]$. By *Corollary 2.25(b)*, R is rigid. D extends to a locally nilpotent derivation $S^{-1}D$ of $S^{-1}A[x,y] = R[y] = R^{[1]}$. Therefore, by *Theorem 2.24*, $S^{-1}D(R) = 0$, which implies that $D(A[x]) = 0$. Consequently, $\ker D = A[x]$, since $A[x]$ is algebraically closed in $A[x,y]$. □

Note that the Semi-Rigidity Theorem does not generalize to arbitrary rings B of transcendence degree one over a rigid ring A. That is to say, there exist rings B which are not semi-rigid, but which have rigid kernels, as seen in the following example.

Example 2.27 Let $k[x,y] = k^{[2]}$ and write $k[x,y] = \bigoplus_{n\geq 0} V_n$, where V_n is the vector space of binary forms of degree n (see *Example 3.2*). Define $D_1, D_2 \in \mathrm{LND}(k[x,y])$ by $D_1 = x\frac{\partial}{\partial y}$ and $D_2 = y\frac{\partial}{\partial x}$. Then each D_i is linear, meaning $D_i(V_n) \subset V_n$ for every $n \geq 0$. Therefore, if $B = k[V_2, V_3]$, then D_1 and D_2 restrict to B. Let d_1, d_2 denote their respective restrictions to B. Then $\ker d_1 = k[x^2, x^3]$, $\ker d_2 = k[y^2, y^3]$ and $ML(B) = k$, meaning that B is not semi-rigid. However, $R = \ker d_2$ is rigid by *Corollary 1.28*, and $d_1 R \neq 0$.

A natural question is whether the Semi-Rigidity Theorem generalizes to $A^{[n]}$ for $n \geq 2$:

$$\text{If } A \text{ is a rigid affine } k\text{-domain, does } ML(A^{[n]}) = A\,?$$

It turns out that the answer is negative for $n = 2$. In [125], Dubouloz defines a family of smooth affine cubic surfaces $X \subset \mathbb{C}^3$ such that X has no non-trivial $\mathbb{G}_a$-action, but $X \times \mathbb{C}^2$ has $\mathbb{G}_a$-actions which do not stabilize X. Equivalently, $A = k[X]$ is rigid but the containment $\mathrm{LND}_A(A^{[2]}) \subset \mathrm{LND}(A^{[2]})$ is strict.

However, for rigid rings of transcendence degree one over k, this question has a positive answer.

Theorem 2.28 ([276], Lemma 28; [61], Lemma 2.3; [170]) *If A is a commutative k-domain with $\mathrm{tr.deg}_k A = 1$, then $ML(A^{[n]}) = ML(A)$ for every $n \geq 0$.*

We give two proofs of *Theorem 2.28*. The first assumes that k is algebraically closed and uses the theory of locally nilpotent derivations, following the ideas of

Makar-Limanov. This allows us to give an independent proof of the Abhyankar-Eakin-Heinzer Theorem for such fields; see *Theorem 10.3*. The second proof is for any field k of characteristic zero, and uses the Abhyankar-Eakin-Heinzer Theorem.

Proof (k Algebraically Closed of Characteristic 0) Since

$$ML(A^{[n]}) \subset ML(A) \subset A$$

we see that $ML(A^{[n]})$ is an algebraically closed subalgebra of A. Since $\text{tr.deg}_k A = 1$, either $ML(A^{[n]}) = A$ or $ML(A^{[n]}) = k$. Note that, if A is not rigid, then $ML(A^{[n]}) = k$. We show that the converse also holds, which gives the desired result.

Assume that $ML(A^{[n]}) = k$. There exists $D \in \text{LND}(A^{[n]})$ with $DA \neq 0$. If $\mathcal{O}$ is the integral closure of A in frac(A), then $\mathcal{O}^{[n]}$ is the integral closure of $A^{[n]}$ in frac$(A^{[n]})$. By property (1) in *Sect. 1.2.7*, if $\mathfrak{C}$ is the conductor ideal of A, then $\mathfrak{C} \cdot \mathcal{O}^{[n]}$ is the conductor ideal of $A^{[n]}$.

If s is a local slice of D and $K = \text{frac}(\ker D)$, then $A \subset K[s]$ and $A \not\subset K$. By *Lemma 2.29* below, A is k-affine and there exists $t \in \text{frac}(A)$ such that $\mathcal{O} = k[t]$. By the theorems of Seidenberg and Vasconcelos (*Proposition 1.19*, *Principle 16*), D extends to $D' \in \text{LND}(\mathcal{O}^{[n]})$; and by property (2) in *Sect. 1.2.7*, $D'(\mathfrak{C} \cdot \mathcal{O}^{[n]}) \subset \mathfrak{C} \cdot \mathcal{O}^{[n]}$.

By *Lemma 1.21*, $\mathfrak{C} \neq 0$. Since $\mathfrak{C}$ is an ideal of $k[t]$, there exists nonzero $h \in A$ with $\mathfrak{C} = h \cdot k[t]$. Thus, $\mathfrak{C} \cdot \mathcal{O}^{[n]} = h \cdot \mathcal{O}^{[n]}$ and $D'(h \cdot \mathcal{O}^{[n]}) \subset h \cdot \mathcal{O}^{[n]}$. Therefore, $D'h = 0$. If $h \notin k$, then $k[h] = k^{[1]}$. But then $k[h] \subset \ker D'$ implies $A \subset \ker D'$, which is not the case. Therefore, $h \in k^*$ and $A = k[t]$. $\square$

Proof (k of Characteristic 0) If A is not rigid, then $ML(A^{[n]}) = k = ML(A)$. If A is rigid, then A is not of the form $K^{[1]}$ for an algebraic extension field K of k (*Corollary 1.28*). In this case, the Abhyankar-Eakin-Heinzer Theorem ([2], 3.3) implies that $\phi(A) = A$ for any k-algebra automorphism ϕ of $A^{[n]}$.[1] In particular, $\exp D(A) = A$ for every $D \in \text{LND}(A^{[n]})$, meaning that $DA = 0$. Therefore $A \subset ML(A^{[n]})$, which implies $ML(A^{[n]}) = A = ML(A)$. $\square$

Lemma 2.29 ([276], Lemma 14; [61], Lemma 4.2; [170], Thm. 3.1) *Suppose that k is algebraically closed and R is a commutative k-domain with tr. $\deg_k R = 1$. If there exists a field K with $R \subset K^{[1]}$ and $R \not\subset K$, then the integral closure of R in* frac(R) *is a polynomial ring.*

Note that the versions of *Lemma 2.29* in the three cited papers are somewhat weaker than what appears here. However, the proof given in [170] for Thm. 3.1 yields the stated result.

Makar-Limanov proved:

Corollary 2.30 ([276], Lemma 19) *Suppose that k is algebraically closed, B is an affine UFD over k and tr.$\deg_k B = 2$. If $ML(B^{[1]}) = k$, then $B = k^{[2]}$.*

[1] This property is what the authors term **strong invariance**.

Proof By *Theorem* 2.24, B is not rigid. By *Lemma* 2.10, every irreducible $D \in \mathrm{LND}(B)$ has a slice. Therefore, there exists a subring $R \subset B$ such that $B = R^{[1]}$. We see that R is a UFD, $\mathrm{tr.deg_k R} = 1$ and $ML(R^{[2]}) = ML(B^{[1]}) = k$. By *Theorem* 2.28, $ML(R) = k$, so $R = k^{[1]}$, and therefore $B = k^{[2]}$. $\square$

Note that, if B is a commutative k-domain and $B^{[1]} = k^{[3]}$ for k algebraically closed, then *Corollary* 2.30 implies $B = k^{[2]}$. This is a case of the cancellation problem, discussed in *Chap.* 10. The reader should also compare *Corollary* 2.30 to *Corollary* 2.13.

Relaxing the UFD condition, Makar-Limanov also proves:

Proposition 2.31 ([276], Lemma 16) *If B is a commutative $\mathbb{C}$-domain with tr. deg.$_{\mathbb{C}}B = 2$ and $ML(B) = \mathbb{C}$, then B is isomorphic to a subring of $\mathbb{C}(x)[y]$.*
See *Example* 2.27 above. In addition, Crachiola showed the following.

Proposition 2.32 ([57], Cor. 5.15, Prop. 5.18) *Let B be an affine UFD over k with tr.deg$_k = 2$. If $ML(B) = B$ and $ML(B^{[n]}) \neq B$ for some $n \geq 2$, then $ML(B^{[n]}) = k$ and B is isomorphic to a subring of $k^{[2]}$.*

Note that, in the situation of *Theorem* 2.28, we have $A^{[n]} = A \otimes_k k^{[n]}$. For algebraically closed fields, Crachiola and Makar-Limanov give the following generalization of *Theorem* 2.28.

Theorem 2.33 ([61]) *Let k be algebraically closed and let A be a commutative k-domain with tr.deg$_kA = 1$ and $A \ncong k^{[1]}$. Then for any commutative k-domain B:*

$$ML(A \otimes_k B) = A \otimes_k ML(B)$$

The following lemma is used in subsequent chapters.

Lemma 2.34 *Let B be a commutative k-domain such that tr.deg$_kB = 2$, $ML(B) = k$ and B is integrally closed in its field of fractions. Suppose that at least one of the following conditions holds.*

1. *B is k-affine*
2. *k is algebraically closed*

Then $\ker D = k^{[1]}$ *for every nonzero* $D \in \mathrm{LND}(B)$.

Proof The case where condition (1) is assumed was proved by Kolhatkar [249], 2.14. So assume that condition (2) holds. Let $A = \ker D$, noting that $\mathrm{tr.deg}_k A = 1$. Since B is integrally closed in $\mathrm{frac}(B)$ and A is integrally closed in B, it follows that A is integrally closed in $\mathrm{frac}(A)$.

Since $ML(B) = k$, there exists $E \in \mathrm{LND}(B)$ such that $EA \neq 0$. If $s \in B$ is a local slice of E and K is the field of fractions of $\ker E$, then:

$$A \subset B \subset K[s] = K^{[1]}$$

By *Lemma* 2.29, A is a polynomial ring over k. $\square$

For this lemma, the case where both conditions (1) and (2) hold is well-known; see [88], 2.3.

2.6 Quasi-Extensions and $\mathbb{Z}_n$-Gradings

Let $D \in \mathrm{Der}_k(B)$, let $S \subset B$ be a subalgebra, and let $d \in \mathrm{Der}_k(S)$. Then D is a **quasi-extension** of d if there exists a nonzero $t \in B$ such that $Ds = t \cdot ds$ for all $s \in S$. Observe that, if D is a quasi-extension of d, then $S \cap \ker D = \ker d$.

Lemma 2.35 *Let $S \subset B$ be a subalgebra, and suppose that $D \in \mathrm{Der}_k(B)$ is a quasi-extension of $d \in \mathrm{Der}_k(S)$. If D is locally nilpotent on B, then d is locally nilpotent on S.*

Proof Suppose to the contrary that d is not locally nilpotent, and choose $s \in S$ for which $d^n s \neq 0$ for all $n \geq 1$. Then $\deg_D(d^n s) > 0$ for all $n \geq 0$. By hypothesis, there exists $t \in B$ such that $D|_S = td$. Set $\tau = \deg_D(t)$, noting that $\tau \geq 0$ (since $t \neq 0$). For every $n \geq 1$ we have:

$$D(d^{n-1}s) = td(d^{n-1}s) = t \cdot d^n s$$

Applying $\deg_D(\cdot)$ to each side of this equation yields:

$$\deg_D(d^{n-1}s) - 1 = \tau + \deg_D(d^n s) \;\Rightarrow\; \deg_D(d^n s) = \deg_D(d^{n-1}s) - (\tau + 1)$$

By induction, we obtain: $\deg_D(d^n s) = \deg_D(s) - n(\tau + 1)$. But this implies $\deg_D(d^n s) < 0$ for $n \gg 0$, a contradiction. Therefore, d is locally nilpotent. $\square$

As an application, suppose that the commutative k-domain B is given by

$$B = R \oplus Rx \oplus \cdots \oplus Rx^{n-1}$$

where R is a subalgebra, $x \in B$ and $x^n \in R$ for some $n \geq 2$. Let $f = x^n \in R$. Given $\delta \in \mathrm{Der}_k(R)$, define $D \in \mathrm{Der}_k(B)$ by:

$$Dr = x^{n-1}\delta r \quad (r \in R) \quad \text{and} \quad Dx = \tfrac{1}{n}\delta f$$

Then D is a quasi-extension of δ to B, called the **canonical quasi-extension** of δ.

Lemma 2.36 ([174], Lemma 3.2) *Assume that the conditions of the preceding paragraph hold. Given $\delta \in \mathrm{Der}_k(R)$, let $D \in \mathrm{Der}_k(B)$ be the canonical quasi-extension of δ to B. If $\delta \in \mathrm{LND}(R)$ and $\delta^2 f = 0$, then $D \in \mathrm{LND}(B)$.*

Proof Note first that, given $r \in R$, $\delta r = 0$ implies $Dr = 0$. Therefore, $\ker\delta \subset \mathrm{Nil}(D)$. Since $\delta f \in \ker\delta$, it follows that $Dx \in \ker D$, which implies $x \in \mathrm{Nil}(D)$. Since $B = R[x]$, it will therefore suffice to show that $R \subset \mathrm{Nil}(D)$. This is shown by induction on $N = \deg_\delta(r)$ for $r \in R$. The basis for induction has been established, since $\ker \delta \subset \mathrm{Nil}(D)$.

Given $N \geq 1$, assume $s \in \mathrm{Nil}(D)$ whenever $\deg_\delta(s) \leq N - 1$. In particular, if $r \in R$ is such that $\deg_\delta(r) = N$, then $\delta r \in \mathrm{Nil}(D)$, since $\deg_\delta(\delta r) = N-1$. It follows that:

$$Dr = x^{n-1}\delta r \in \mathrm{Nil}(D) \quad \Rightarrow \quad r \in \mathrm{Nil}(D)$$

Therefore, $\mathrm{Nil}(D) = B$, i.e., D is locally nilpotent. □

Let $\pi : \mathbb{Z} \to \mathbb{Z}_n$ be a surjection, and define a $\mathbb{Z}_n$-grading $B = \bigoplus_{\gamma \in \mathbb{Z}_n} B_\gamma$ by setting $B_{\pi(i)} = Rx^i$, $0 \leq i \leq n-1$, where x is $\mathbb{Z}_n$-homogeneous and $\deg_{\mathbb{Z}_n}(x)$ is a generator of $\mathbb{Z}_n$.

Theorem 2.37 ([174], Thm. 3.1) *If $D \in \mathrm{LND}(B)$ is $\mathbb{Z}_n$-homogeneous, then there exist $\delta \in \mathrm{LND}(R)$ and $m \in \mathbb{Z}$, $0 \leq m \leq n-1$, such that the following conditions hold.*

(a) *D is a quasi-extension of δ with $D|_R = x^m\delta$.*
(b) *$Dx, \delta(x^n) \in \ker \delta = R \cap \ker D$*
(c) *If $Dx \neq 0$, then $m = n-1$ and $\ker D = \ker \delta$.*

Proof Let $\gamma = \deg_{\mathbb{Z}_n} D$, and let $m \in \mathbb{Z}$ be such that $0 \leq m \leq n-1$ and $\pi(m) = \gamma$. Then $DR \subset R \cdot x^m$. Define $\delta \in \mathrm{Der}_k(R)$ by $\delta(r) = x^{-m}Dr$. Then D is a quasi-extension of δ and $\ker \delta = R \cap \ker D$. By *Lemma 2.35*, δ is locally nilpotent. This proves part (a).

If $Dx = 0$, then $\delta(x^n) = 0$, so part (b) holds in this case.

Assume $Dx \neq 0$. Since Dx is $\mathbb{Z}_n$-homogeneous, $Dx \in R \cdot x^l$ for some l ($0 \leq l \leq n-1$). If $l \neq 0$, then $Dx \in xB$ means $Dx = 0$, a contradiction (*Corollary 1.23*). Therefore, $l = 0$ and $Dx \in R$. Since $x \notin \ker D$, we also have $x^n \notin \ker D$, since $\ker D$ is factorially closed in B. Therefore:

$$x^n \in R \quad \Rightarrow \quad D(x^n) = nx^{n-1}Dx \in DR \subset R \cdot x^m \quad \Rightarrow \quad m = n-1$$

Since $Dx \in R$ and $DR \subset R \cdot x^{n-1}$, it follows that $D^2x \in R \cdot x^{n-1}$. Therefore, $D^2x \in xB$ implies $D^2x = 0$ (*Corollary 1.23*). In addition:

$$0 = D(x^n - x^n) = x^{n-1}\delta(x^n) - nx^{n-1}Dx \quad \Rightarrow \quad \delta(x^n) = nDx \in \ker D$$

This proves part (b) and the first statement of part (c).

Suppose that $b \in \ker D$ is $\mathbb{Z}_n$-homogeneous and write $b = rx^i$ for $r \in R$ and $0 \leq i \leq n-1$. If $i \neq 0$, then $x \in \ker D$, since $\ker D$ is factorially closed, a contradiction. Therefore, $i = 0$ and $b = r$. Since $\ker D$ is a $\mathbb{Z}_n$-graded subring, it is generated by $\mathbb{Z}_n$-homogeneous elements over $R \cap \ker D$. It follows that $\ker D \subset R$, and therefore $\ker D = R \cap \ker D = \ker \delta$. This proves part (c). □

Note that, in part (c) of *Theorem 2.37*, the condition $m = n-1$ is equivalent to the condition that D is the canonical quasi-extension of δ.

Corollary 2.38 ([174], Cor. 3.2) *Suppose that R is a $\mathbb{Z}$-graded affine k-domain, $f \in R$ is homogeneous,* $\deg f \neq 0$, *and $n \geq 2$ is an integer relatively prime to* $\deg f$. *Let x be an indeterminate over R and define the domain:*

$$B = R[x]/(f - x^n)$$

The following conditions are equivalent.

1. $\delta^2 f \neq 0$ *for every nonzero* $\delta \in \mathrm{LND}(R)$.
2. *B is rigid.*

Proof The given $\mathbb{Z}$-grading $R = \bigoplus_{i\in\mathbb{Z}} R_i$ induces an $n\mathbb{Z}$-grading $R = \bigoplus_{j\in n\mathbb{Z}} \tilde{R}_j$ by $R_i = \tilde{R}_{ni}$. This $n\mathbb{Z}$-grading of R extends to a $\mathbb{Z}$-grading $B = \bigoplus_{i\in\mathbb{Z}} B_i$ by letting x be homogeneous and $x \in B_{\deg f}$, where deg is the degree function for the given $\mathbb{Z}$-grading of R.

Assume that $\delta^2 f \neq 0$ for every nonzero $\delta \in \mathrm{LND}(R)$. Since B is a $\mathbb{Z}$-graded affine k-domain, a necessary and sufficient condition for B to be rigid is that $D = 0$ whenever $D \in \mathrm{LND}(B)$ is $\mathbb{Z}$-homogeneous.

Let $D \in \mathrm{LND}(B)$ be $\mathbb{Z}$-homogeneous. Then D is also homogeneous for the $\mathbb{Z}_n$-grading of B induced by the standard projection $\mathbb{Z} \to \mathbb{Z}_n$. By *Theorem 2.37*, there exists $\delta \in \mathrm{LND}(R)$ such that D is a quasi-extension of δ and $\delta^2 f = 0$. Then $\delta = 0$, which implies that $D = 0$. It follows that B is rigid. We have thus shown: (1) $\Rightarrow$ (2).

Suppose that $\delta \in \mathrm{LND}(R)$ is nonzero and $\delta^2 f = 0$. If D is the canonical quasi-extension of δ to B, then $D \neq 0$ and, by *Lemma 2.36*, D is locally nilpotent. We have thus shown: $\sim$(1) $\Rightarrow$ $\sim$(2). ☐

2.7 *G*-Critical Elements

Throughout this section, we assume that G is an additive abelian group and $B = \bigoplus_{g\in G} B_g$ is a G-graded commutative k-domain.

If $A = \bigoplus_{g\in G} A_g$ is a G-graded subalgebra, let $G(A)$ be the subgroup of G generated by $\{g \in G \mid A_g \neq 0\}$. If $H \subset G$ is a subgroup, set $B_H = \bigoplus_{g\in H} B_g$, noting that B_H is a G-graded subalgebra of B.

A nonzero G-homogeneous element x of B is ***G*-critical** if it satisfies the following equivalent conditions.

(a) There exists a G-graded subalgebra A of B such that $G(A) \neq G(B)$ and $B = A[x]$.
(b) There exists a subgroup H of $G(B)$ such that $\deg_G x \notin H$ and $B = B_H[x]$.
(c) There exists a subgroup H of G such that $\deg_G x \notin H$ and $B = B_H[x]$.

The main result concerning G-critical elements relative to locally nilpotent derivations is the following.

Theorem 2.39 ([83], Thm. 6.2) *Let $D \in \mathrm{LND}(B)$ be G-homogeneous.*

(a) *For every G-critical element $x \in B$, $D^2x = 0$.*
(b) *For every pair $x, y \in B$ of non-associated G-critical elements, either $Dx = 0$ or $Dy = 0$.*

Several preliminaries are required for its proof.

Theorem 2.40 ([83], Thm. 5.4 (b)) *Let x be a G-critical element of B. There exists a subgroup $H \subset G$ and prime integer p such that:*

$$B = B_H[x] \ , \ \deg_G x \notin H \text{ and } p \deg_G x \in H$$

In addition, $x^p \in B_H$ and $B = \bigoplus_{1 \le i \le p-1} B_H x^i$.

Proof Let $\tau = \deg_G x \in G$. There exists a subgroup H_0 of $G(B)$ satisfying $\tau \notin H_0$ and $B = B_{H_0}[x]$. Note that $G(B)$ is generated by $H_0 \cup \{\tau\}$. Since $\Gamma := \{n \in \mathbb{Z} \,|\, n\tau \in H_0\}$ is a proper subgroup of $\mathbb{Z}$, we may choose a prime number $p > 0$ such that $\Gamma \subset p\mathbb{Z}$. (It is possible that $\Gamma = \{0\}$.) Define the subgroup $H \subset G(B)$ by $H = H_0 + \langle p\tau \rangle$, noting that $\tau \notin H$ since $\Gamma \subset p\mathbb{Z}$. Since $H_0 \subset H$, we have $B_{H_0} \subset B_H$ and hence $B = B_H[x]$.

Since $p\tau \in H$, we see that $x^p \in B_H$. Let $\bar{\tau} = \tau + H \in G(B)/H$. Then $G(B)/H$ is generated by $\bar{\tau}$ and $\bar{\tau}$ has order p. Let $\pi : G(B) \to G(B)/H$ be the canonical homomorphism of the quotient group. It is clear that a $G(B)/H$-grading $B = \bigoplus_{k \in G(B)/H} S_k$ of B is defined by setting $S_k = \bigoplus_{g \in \pi^{-1}(k)} B_g$ for all $k \in G(B)/H$. We have $B_H x^i \subseteq S_{i\bar{\tau}}$ for $i = 0, \dots, p-1$, and $B = \sum_{i=0}^{p-1} B_H \cdot x^i$ because $B = B_H[x]$ and $x^p \in B_H$; so $B_H x^i = S_{i\bar{\tau}}$ for all $i = 0, \dots, p-1$ and consequently $B = \bigoplus_{i=0}^{p-1} B_H \cdot x^i$. □

Let $x \in B$ be a G-critical, element. A subgroup $H \subset G$ is a **G-critical subgroup associated to** x if $B = B_H[x]$, $\deg_G x \notin H$ and $p \deg_G x \in H$ for some prime integer p. *Theorem 2.40* asserts that a G-critical subgroup associated to a G-critical element always exists. In addition, the proof of *Theorem 2.40* shows the following result.

Lemma 2.41 ([83], Thm. 5.4(a)) *Let x be a G-critical element of B. If H_0 is any subgroup of $G(B)$ satisfying $\deg_G x \notin H_0$ and $B = B_{H_0}[x]$, then there exists a G-critical subgroup H associated to x with $H_0 \subset H$.*

Lemma 2.42 ([83], Lemma 5.6) *Assume that there exists a pair $x, y \in B$ of G-critical elements such that x and y are not associates and $x, y \notin B^*$. Let $H \subset G(B)$ be a G-critical subgroup associated to x, and let $K \subset G(B)$ be a G-critical subgroup associated to y. Then $x \in B_K$ and $y \in B_H$.*

Proof By *Theorem 2.40*, there exist prime integers p and q such that the decompositions

$$B = \bigoplus_{0 \le i < p} B_H \cdot x^i = \bigoplus_{0 \le j < q} B_K \cdot y^j$$

define gradings of B by $\mathbb{Z}_p$ and $\mathbb{Z}_q$, respectively. Let $a \in B_H$, $b \in B_K$ and non-negative integers i, j be such that $y = ax^i$ and $x = by^j$. Since B is an integral domain, and x and y are G-homogeneous, a and b are also G-homogeneous. We have:

$$y = a(by^j)^i = ab^i y^{ij}$$

Since $y \notin B^*$, it follows that either $ij = 0$, or $i = j = 1$ and $ab = 1$. However, the latter case cannot occur, since x and y are not associates. Therefore, $ij = 0$.

Suppose that $i = 0$. Then $y = a \notin B^*$. Since b is G-homogeneous, there exist $c \in B_H$ and $l \geq 0$ with $b = cx^l$. So $x = a^j cx^l$, which implies that either $l = 0$ or $l = 1$ (since $x \notin B^*$). If $l = 0$, then $x \in B_H$, a contradiction. Therefore, $l = 1$ and $a^j c = 1$, which implies that $j = 0$ (since $y = a \notin B^*$).

On the other hand, if $j = 0$, then the symmetric argument shows $i = 0$. Therefore, $i = j = 0$ in all cases. □

Lemma 2.43 ([83], Cor. 6.1) *Suppose that $D \in \mathrm{LND}(B)$ is G-homogeneous and $x \in B$ is a G-critical element with $Dx \neq 0$. For every G-critical subgroup $H \subset G(B)$ associated to x, we have*

$$Dx \in \ker D \subset B_H \quad \textit{and} \quad DB_H \subset B_H \cdot x^{p-1}$$

where $p = |G(B)/H|$.

Proof Recall from the proof of *Theorem 2.40* that the surjection $G(B) \to G(B)/H \cong \mathbb{Z}_p$ induces the $\mathbb{Z}_p$-grading of B given by $B = \bigoplus_{0 \leq i \leq p-1} B_H x^i$. Since D is G-homogeneous, it is also $\mathbb{Z}_p$-homogeneous. Therefore, part (b) of *Theorem 2.37* implies that $Dx \in \ker D$, and part (c) implies that $DB_H \subset B_H \cdot x^{p-1}$ and $\ker D \subset B_H$. □

We can now give the proof of *Theorem 2.40*.

Proof Assertion (a) follows from *Lemma 2.43*.

Note that assertion (b) is trivial if x or y is a unit. Assume that x, y are not units and let H (resp. K) be a G-critical subgroup associated to x (resp. to y). Let $p = |G(B)/H|$ and $q = |G(B)/K|$. Then *Lemma 2.43* gives $DB_H \subset B_H x^{p-1}$ and $DB_K \subset B_K y^{q-1}$. Also, *Lemma 2.42* implies that $x \in B_K$ and $y \in B_H$. Consequently, $Dx \in DB_K \subseteq B_K y^{q-1} \subseteq yB$, so $y \mid Dx$, and similarly $x \mid Dy$. Since D is locally nilpotent, *Principle 5* implies that $Dx = 0$ or $Dy = 0$. This proves part (b). □

The next result generalizes Thm. 5.1 of [174] and Thm. 2.6.3 of [248].

Corollary 2.44 ([83], Cor. 6.3) *Suppose B is finitely generated over B_0. In particular, let $B = B_0[x_1, \dots, x_n]$, where $x_i \in B_{d_i}$ and $d_i \neq 0$ for each i. Define subgroups $H_i = \langle d_1, \dots, \hat{d_i}, \dots, d_n \rangle$, $1 \leq i \leq n$. For every G-homogeneous $D \in \mathrm{LND}(B)$ the following conditions hold.*

- **(a)** *For each $i \in \{1, \dots, n\}$ such that $H_i \neq G(B)$, x_i is a G-critical element of B and $D^2 x_i = 0$.*
- **(b)** *For every choice of distinct $i, j \in \{1, \dots, n\}$ such that $H_i \neq G(B)$ and $H_j \neq G(B)$, either $Dx_i = 0$ or $Dx_j = 0$.*

Proof If $i \in \{1, \ldots, n\}$ is such that $H_i \neq G(B)$ then $\deg_G x_i \notin H_i$ and $B = B_{H_i}[x_i]$, so x_i is a G-critical element of B. Therefore, assertion (a) follows from *Theorem 2.39(a)*.

Consider distinct $i, j \in \{1, \ldots, n\}$ such that $H_i \neq G(B)$ and $H_j \neq G(B)$. Then x_i and x_j are G-critical elements of B. By *Lemma 2.41*, we may choose a G-critical subgroup H associated to x_i and satisfying $H_i \subset H$, and a G-critical subgroup K associated to x_j and satisfying $H_j \subset K$. Then $d_i \in H_j \subset K$ and $d_j \in H_i \subset H$, so:

$$d_i \in K \setminus H \quad \text{and} \quad d_j \in H \setminus K \tag{2.1}$$

Assume that $Dx_i \neq 0$ and $Dx_j \neq 0$. Then *Lemma 2.43* implies that $\ker D \subset B_H \cap B_K$. If $x_j = ux_i$ for some $u \in B^*$, then $u \in \ker D$, so $d_j - d_i = \deg_G u \in H \cap K$, which contradicts (2.1); so x_i and x_j are not associates. Now *Theorem 2.39(b)* implies that $Dx_i = 0$ or $Dx_j = 0$, a contradiction.

Therefore, either $Dx_i = 0$ or $Dx_j = 0$, and part (b) is proved. □

Observe that, in the notation of this corollary, $G(B) = \langle d_1, \ldots, d_n \rangle$. In addition, note that, in stating the corollary, we do *not* need to assume that $B_0 \subset \ker D$.

Corollary 2.44 immediately implies the following special case.

Corollary 2.45 ([174], Cor. 5.1) *Suppose that A is a commutative k-domain and $B = A[x, y] = A^{[2]}$. Assume that B is $\mathbb{Z}$-graded over A, where x, y are homogeneous,* $\deg x = a$, $\deg y = b$ *and* $\gcd(a, b) = 1$. *If $|a|$, $|b| \geq 2$, then for every homogeneous* $D \in \mathrm{LND}(B)$, *either $Dx = 0$ or $Dy = 0$.*

2.8 AB and ABC Theorems

Let B be a commutative k-domain. We say that $f, g \in B$ are **relatively prime** in B if and only if $fB \cap gB = fgB$. This generalizes the definition of relative primeness for elements in a UFD (at least for nonzero elements), and was introduced in [174]. The reader can easily verify the following three properties: Let $f, g \in B$ be relatively prime in B.

1. f and g are relative prime in $B^{[n]}$ for each $n \geq 0$.
2. f^m and g^n are relatively prime for every $m, n \geq 1$.
3. If $A \subset B$ is a factorially closed subring and $f, g \in A$, then f and g are relatively prime in A.

In fact, the first of these properties is a case of the following more general property, which is needed in *Chap. 10*. See [174], Lemma 3.4.

Lemma 2.46 *Let B be a commutative k-domain and $R \subset B$ a subalgebra such that B is a free R-module. Given $x, y \in R$, if x and y are relatively prime in R, then x and y are relatively prime in B.*

Proof We need to show that $xB \cap yB \subset xyB$. Let $u, v, w \in B$ be such that $w = xu = yv$. Since B is a free R-module, there exists a finitely generated free R-submodule $S \subset B$ with $u, v \in S$. Since $x, y \in R$, we also have $w \in S$.

Let $z_1, \dots, z_n \in B$ be such that $S = Rz_1 \oplus \cdots \oplus Rz_n$ and let

$$u = u_1 z_1 + \cdots + u_n z_n \quad \text{and} \quad v = v_1 z_1 + \cdots + v_n z_n$$

where $u_i, v_i \in R$. Since $xu = yv$, it follows that $xu_i = yv_i$ for each i. By hypothesis, x and y are relatively prime in R. Therefore, there exist $r_i \in R$, $1 \leq i \leq n$, such that $u_i = yr_i$ for each i. It follows that $u \in yB$, and thus $w \in xyB$. □

Theorem 2.47 (First AB Theorem) *Given positive $a, b \in \mathbb{Z}$, if $x, y \in B$ are relatively prime and $x^a + y^b = 0$, then $x, y \in ML(B)$.*

Proof If $x = 0$ or $y = 0$, then $x = y = 0$ and $x, y \in ML(B)$. So assume x and y are nonzero. Given $D \in \mathrm{LND}(B)$, apply D to the equation $x^a + y^b = 0$ to obtain:

$$\begin{aligned} ax^{a-1}Dx + by^{b-1}Dy = 0 &\Rightarrow ax^{a-1}yDx + by^bDy = 0 \\ &\Rightarrow ax^{a-1}yDx - bx^aDy = 0 \\ &\Rightarrow ayDx - bxDy = 0 \end{aligned}$$

If $z = ayDx = bxDy$, then $z \in xB \cap yB = xyB$. Let $h \in B$ be such that $z = xyh$. Then:

$$z = ayDx = bxDy = xyh \;\Rightarrow\; aDx = xh \text{ and } bDy = yh \;\Rightarrow\; Dx = Dy = 0$$

□

Theorem 2.48 (ABC Theorem [174], Thm. 6.1) *Suppose that $x, y, z \in B$ are pairwise relatively prime and satisfy $x^a + y^b + z^c = 0$ for integers $a, b, c \geq 2$. If $a^{-1} + b^{-1} + c^{-1} \leq 1$, then $k[x, y, z] \subset ML(B)$.*

Proof Assume that $k[x, y, z]$ is not contained in $ML(B)$, and let $D \in \mathrm{LND}(B)$ be given, where at least one of Dx, Dy, Dz is nonzero.

Clearly, if two of Dx, Dy, Dz are 0, then the third is also zero. So consider the case $Dz = 0$, but $Dx \neq 0$ and $Dy \neq 0$. Then $D(x^a + y^b) = 0$. By *Theorem 2.50(a)*, it follows that either $Dx = Dy = 0$ (which contradicts the assumption) or $x^a + y^b = 0$. If $x^a + y^b = 0$, then, since x^a and y^b are relatively prime, it follows that x and y are either 0 or invertible. In this case $Dx = Dy = 0$, again contradicting the assumption. Therefore, $\deg_D x, \deg_D y, \deg_D z > 0$.

Apply D to the equation $x^a + y^b + z^c = 0$ to obtain:

$$\begin{pmatrix} x & y & z \\ aDx & bDy & cDz \end{pmatrix} \begin{pmatrix} x^{a-1} \\ y^{b-1} \\ z^{c-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Define the matrix:

$$A = \begin{pmatrix} x & y & z \\ aDx & bDy & cDz \\ 0 & 0 & 1 \end{pmatrix}$$

If $\det A = bxDy - ayDx = 0$, then $\kappa := bxDy = ayDx$ belongs to xyB, since x and y are relatively prime. If $\kappa = xyh$ for $h \in B$, then $bDy = yh$, since B is a domain. But then $Dy = 0$, a contradiction. Therefore $\det A \neq 0$. Since

$$A \begin{pmatrix} x^{a-1} \\ y^{b-1} \\ z^{c-1} \end{pmatrix} = z^{c-1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

it follows that:

$$\det A \begin{pmatrix} x^{a-1} \\ y^{b-1} \\ z^{c-1} \end{pmatrix} = z^{c-1} \mathrm{adj}(A) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = z^{c-1} \begin{pmatrix} cyDz - bzDy \\ azDx - cxDz \\ bxDy - ayDx \end{pmatrix}$$

Since x, y, z are pairwise relatively prime in B, it follows that:

$$\begin{aligned} z^{c-1} \text{ divides } bxDy - ayDx &\Rightarrow (c-1)\deg_D z \leq \deg_D x + \deg_D y - 1 \\ y^{b-1} \text{ divides } azDx - cxDz &\Rightarrow (b-1)\deg_D y \leq \deg_D x + \deg_D z - 1 \\ x^{a-1} \text{ divides } cyDz - bzDy &\Rightarrow (a-1)\deg_D x \leq \deg_D y + \deg_D z - 1 \end{aligned}$$

Let $\sigma = \deg_D x + \deg_D y + \deg_D z$. The inequalities above show that:

$$\deg_D x \leq \frac{\sigma - 1}{a}\ , \quad \deg_D y \leq \frac{\sigma - 1}{b}\ , \quad \deg_D z \leq \frac{\sigma - 1}{c}$$

By addition, we obtain:

$$\sigma \leq \frac{\sigma - 1}{a} + \frac{\sigma - 1}{b} + \frac{\sigma - 1}{c} = (\sigma - 1)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

This implies:

$$1 < 1 + \frac{1}{\sigma - 1} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

□

The following well-known result is due to Davenport, dating to 1965. See [91] and [270]

Theorem 2.49 *Let nonzero $u, v \in k$ and $f, g \in k[t] = k^{[1]}$ be given, where f and g are not both constant, together with positive integers l and m. Then*

$$\deg_t(uf^l - vg^m) \geq \frac{1}{m}(lm - l - m)\deg_t f + 1$$

unless $uf^l = vg^m$ identically.[2]

We use Davenport's Theorem to prove the following result. Part (a) was first given in [281], Lemma 2, and part (b) is given in [174], Thm. 2.2.

Theorem 2.50 (Second AB Theorem) *Let $D \in \mathrm{LND}(B)$ be nonzero. Suppose $u, v \in \ker D$ and $x, y \in B$ are nonzero, and $a, b \in \mathbb{Z}$, $a, b \geq 2$. Assume $ux^a + vy^b \neq 0$.*

(a) *If $D(ux^a + vy^b) = 0$, then $Dx = Dy = 0$.*
(b) *If $D^2(ux^a + vy^b) = 0$ and a and b are not both 2, then $Dx = Dy = 0$.*

Proof If $A = \ker D$ and $t \in B$ is a local slice for D, then $B_{Dr} = A_{Dr}[t] = A_{Dr}^{[1]}$. Therefore, $B \subset K[t] = K^{[1]}$, where $K = \mathrm{frac}(A)$ and $\deg_t = \deg_D$

Since $a, b \geq 2$, we have $ab - a - b \geq 0$. If $Dx \neq 0$ or $Dy \neq 0$, then Davenport's Theorem implies:

$$\deg_D(ux^a + vy^b) \geq \frac{1}{b}(ab - a - b)\deg_D x + 1 \geq 1 \tag{2.2}$$

So $D(ux^a + vy^b) \neq 0$. This proves part (a).

For part (b), assume that $a \geq 3$ or $b \geq 3$. Then $ab - a - b \geq 1$.

Suppose that $Dx \neq 0$, so $\deg_D x \geq 1$. The first inequality of (2.2) implies that $\deg_D(ux^a + vy^b) \geq 2$, meaning that $D^2(ux^a + vy^b) \neq 0$, which contradicts the hypothesis of part (b). Therefore, one has that $Dx = 0$. After exchanging the roles of x and y, the same argument shows that $Dy = 0$. □

Remark 2.51 The condition of relative primeness in the First AB Theorem and the ABC Theorem is necessary. For example, in $k[t] = k^{[1]}$, if $x = -t^2$ and $y = t^3$, then $x^3 + y^2 = 0$, whereas $ML(k[t]) = k$.

2.9 Cables and Cable Algebras

The definitions in this section were introduced in [171].

[2] The condition *f and g are not both constant* is missing from Davenport's original formulation, but is necessary for the result to be valid.

2.9.1 Associated Rooted Tree

Given $D \in \mathrm{LND}(B)$, the **associated rooted tree** is the rooted tree $\mathrm{Tr}(B,D)$ whose vertex set is B, and whose (directed) edge set consists of pairs (f, Df), where $f \neq 0$. Equivalently, $\mathrm{Tr}(B,D)$ is the tree defined by the partial order on B: $a \leq b$ iff $D^n b = a$ for some $n \geq 0$. Let $A = \ker D$.

1. Given $a, b \in B$ with $b \neq 0$, b is a **predecessor** of a if and only if a is a **successor** of b if and only if $a < b$. Similarly, b is an **immediate predecessor** of a if and only if a is an **immediate successor** of b if and only if $Db = a$.
2. The **terminal vertices** of $\mathrm{Tr}(B,D)$ are those without predecessors, i.e., elements of $B \setminus DB$. If D has a slice, i.e., $DB = B$, then $\mathrm{Tr}(B,D)$ has no terminal vertices.
3. Every subtree X of $\mathrm{Tr}(B,D)$ has a unique root, denoted $\mathrm{rt}(X)$.
4. A subtree X of $\mathrm{Tr}(B,D)$ is **complete** if every vertex of X which is not terminal in $\mathrm{Tr}(B,D)$ has at least one predecessor in X.
5. A subtree X of $\mathrm{Tr}(B,D)$ is **linear** if every vertex of X has at most one immediate predecessor in X.
6. If B is graded by an abelian group G, then any G-homogeneous $b \in B$ is a **G-homogeneous vertex** of $\mathrm{Tr}(B,D)$. A subtree X of $\mathrm{Tr}(B,D)$ is **G-homogeneous** if every $b \in \mathrm{vert}(X)$ is G-homogeneous. If D is G-homogeneous, then the **full G-homogeneous subtree** is the subtree of $\mathrm{Tr}(B,D)$ spanned by the G-homogeneous vertices.

2.9.2 D-Cables

Definition 2.52 Given $D \in \mathrm{LND}(B)$, a ***D*-cable** is a complete linear subtree $\hat{s}$ of $\mathrm{Tr}(B,D)$ rooted at a nonzero element of $\ker D$. $\hat{s}$ is a **terminal** D-cable if it contains a terminal vertex. $\hat{s}$ is an **infinite** D-cable if it is not terminal.
We make several remarks and further definitions, assuming $D \in \mathrm{LND}(B)$. Recall that the image ideals of D are:

$$I_n = \ker D \cap D^n B \ (n \geq 0) \quad \text{and} \quad I_\infty = \cap_{n \geq 0} I_n$$

1. If $\hat{s}$ is a D-cable, then $\hat{s}$ is terminal if and only if its vertex set is finite, and $\hat{s}$ is infinite if and only if $\hat{s} \subset DB$.
2. A D-cable is denoted by $\hat{s} = (s_j)$, where $s_j \in B$ for $j \geq 0$ and $Ds_j = s_{j-1}$ for $j \geq 1$. It is rooted at $s_0 \in \ker D$, which is nonzero. For multiple D-cables $\hat{s}_1, \ldots, \hat{s}_n$, we write $\hat{s}_i = (s_i^{(j)})$ for $1 \leq i \leq n$ and $j \geq 0$.
3. The **length** of a D-cable $\hat{s}$ is the number of its edges (possibly infinite), denoted $\mathrm{length}(\hat{s})$. If $\hat{s} = (s_n)$ and $N = \mathrm{length}(\hat{s})$, then $s_0 \in I_N$, and if $\hat{s}$ is terminal, then s_N is its terminal vertex.

4. Every $b \in \ker D \setminus DB$ is a terminal vertex of $\mathrm{Tr}(B,D)$ and defines a terminal D-cable of length zero.
5. If B is graded by an abelian group G, then a D-cable is **G-homogeneous** if it is a G-homogeneous subtree of $\mathrm{Tr}(B,D)$.
6. Every nonzero vertex $b \in B$ belongs to a D-cable. If two D-cables $\hat{s} = (s_n)$ and $\hat{t} = (t_n)$ have $s_m = t_n$ for some $m, n \geq 0$, then $m = n$ and $s_i = t_i$ for all $i \leq m$. If $\hat{s}$ and $\hat{t}$ share an infinite number of vertices, then $\hat{s} = \hat{t}$.
7. Suppose that $B' \subset B$ is a subset with $DB' \subset B'$. If $\hat{s} \subset B$ is a D-cable and $\hat{s} \cap B'$ is infinite, then $\hat{s} \subset B'$.
8. Given D-cables $\hat{s}_1, \ldots, \hat{s}_n$ for $n \geq 0$, the notation $k[\hat{s}_1, \ldots, \hat{s}_n]$ (respectively $(\hat{s}_1, \ldots, \hat{s}_n)$) indicates the k-subalgebra of B (respectively, ideal of B) generated by the vertices of $\hat{s}_i$ for $1 \leq i \leq n$.
9. Extend D to a derivation D^* on $B[t] = B^{[1]}$ by $D^*t = 0$. If $\hat{s}(t) = (s_n(t))$ is a D^*-cable and $\alpha \in \ker D$ is such that $s_0(\alpha) \neq 0$, then $\hat{s}(\alpha) = (s_n(\alpha))$ is a D-cable rooted at $s_0(\alpha)$.

Example 2.53 Let $\Omega = k[x_0, x_1, x_2, \ldots]$ be the polynomial ring in a countably infinite set of variables x_i, and define $\Delta \in \mathrm{LND}(\Omega)$ by $\Delta x_i = x_{i-1}$ $(i \geq 1)$ and $\Delta x_1 = 0$. Then $\hat{x} = (x_j)_{j \geq 0}$ is an infinite Δ-cable, $x_0 \in I_\infty$, and $\Omega = k[\hat{x}]$. Relabel the variables x_i by $y_n^{(j)}$ so that $\Omega = k[x_0, y_n^{(j)} \mid n \geq 1, 1 \leq j \leq n]$. Define $\tilde{\Delta} \in \mathrm{LND}(\Omega)$ so that, for $n \geq 1$:

$$\tilde{\Delta} : y_n^{(n)} \to y_n^{(n-1)} \to \cdots \to y_n^{(1)} \to y_n^{(0)} := x_0 \to 0$$

Then $\hat{y}_n := (y_n^{(j)})_{0 \leq j \leq n}$ is a terminal $\tilde{\Delta}$-cable rooted at x_0 of length n for each $n \geq 1$. If $\tilde{I}_\infty$ is the core ideal for $\tilde{\Delta}$, then $x_0 \in \tilde{I}_\infty$ but it is easy to check that there is no infinite $\tilde{\Delta}$-cable rooted at x_0.

Since DB is an A-module, it is possible to define addition and scalar multiplication of infinite D-cables in certain cases, as described in the following result, which follows immediately from the definitions.

Lemma 2.54 *Given $D \in \mathrm{LND}(B)$, let $A = \ker D$.*

(a) *If $\hat{s} = (s_n)$ is an infinite D-cable and $a \in A$ is nonzero, then $a\hat{s} := (as_n)$ is an infinite D-cable.*
(b) *If $\hat{s} = (s_n)$ and $\hat{t} = (t_n)$ are infinite D-cables and $s_0 + t_0 \neq 0$, then $\hat{s} + \hat{t} := (s_n + t_n)$ is an infinite D-cable.*
(c) *If $\hat{s} = (s_n)$ and $\hat{t} = (t_n)$ are infinite D-cables and $m \in \mathbb{Z}$, $m \geq 1$, define the sequence $u_n \in B$ by $u_n = s_n$ if $n < m$ and $u_n = s_n + t_{n-m}$ if $n \geq m$. Then $\hat{u} := (u_n)$ is an infinite D-cable.*

The D-cable $\hat{u}$ in part (c) of *Lemma 2.54* is called the **m-shifted sum** of $\hat{s}$ and $\hat{t}$, and is denoted by $\hat{u} = \hat{s} +_m \hat{t}$.

2.9.3 Cable Algebras

Definition 2.55 Let B be a commutative k-domain.

(a) B is a **cable algebra** over k if there exist nonzero $D \in \mathrm{LND}(B)$ and a finite number of D-cables $\hat{s}_1, \dots, \hat{s}_n$ such that $B = A[\hat{s}_1, \dots, \hat{s}_n]$, where $A = \ker D$. In this case, we say that the pair (B, D) is a **cable pair**.
(b) B is a **monogenetic cable algebra** if $B = A[\hat{s}]$ for some cable pair (B, D) with $A = \ker D$ and some D-cable $\hat{s}$.
(c) B is a **simple cable algebra** over k if $B = k[\hat{s}]$ for some D-cable $\hat{s}$, where $D \in \mathrm{LND}(B)$ is nonzero. A simple cable algebra B is of **terminal type** if $\hat{s}$ can be chosen to be a terminal D-cable.

Note that B is a cable algebra if there exists nonzero $D \in \mathrm{LND}(B)$ for which B is finitely generated as an algebra over $\ker D$.

Example 2.56 Let B be a commutative k-domain, $D \in \mathrm{LND}(B)$ and $A = \ker D$. If

$$S \subset B \setminus (A \cup DB) \quad \text{and} \quad |S| = n \geq 1$$

then there exist terminal D-cables $\hat{s}_1, \dots, \hat{s}_n$ such that $A[S, D] = A[\hat{s}_1, \dots, \hat{s}_n]$. Let D' be the restriction of D to $A[S, D]$. Then $D' \neq 0$, $A[S, D]$ is a cable algebra, and $(A[S, D], D')$ is a cable pair.

Consider a nilpotent linear operator N on a finite-dimensional k-vector space V. Choose a basis $\{x_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq n_i\}$ of V so that the effect of N for fixed i is:

$$x_{i,n_i} \to x_{i,n_i-1} \to \cdots \to x_{i,2} \to x_{i,1} \to 0$$

This defines the Jordan form of N, which in turn gives a cable structure on the symmetric algebra $S(V)$. In particular, N induces a locally nilpotent derivation D on $S(V)$, and each sequence $x_{i,j}$ for fixed i is a D-cable $\hat{x}_i$, where $S(V) = k[\hat{x}_1, \dots, \hat{x}_m]$. In this sense, the cable algebra structure induced by a locally nilpotent derivation can be viewed as a generalization of Jordan block form for a nilpotent linear operator.

2.10 Exponential Automorphisms

Given an automorphism $\varphi \in \mathrm{Aut}_k(B)$, φ is an **exponential automorphism** if and only if $\varphi = \exp D$ for some $D \in \mathrm{LND}(B)$. It is natural to ask whether a given automorphism is exponential. A complete answer to this question, with detailed proofs, is given by van den Essen in Sect. 2.1 of [142]; see also the article of Gabriel and Nouazé [178], Sect. 3.5, and the book of Nowicki [333], 6.1.4. Here is a brief summary of van den Essen's treatment.

Given a ring homomorphism $f : B \to B$, define the map $E : B \to B$ by $E = f - I$, where I denotes the identity map. Then for any $a, b \in B$:

$$E(ab) = aEb + bEa + (Ea)(Eb) = aEb + f(b)Ea$$

We say that E is an f**-derivation** of B. E is said to be **locally nilpotent** if and only if to each $b \in B$, there exists a positive integer n with $E^n b = 0$.

In case E is locally nilpotent, define the map $\log(I + E) : B \to B$ by

$$\log(I + E) = \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} E^n$$

which is well-defined since E is a locally nilpotent f-derivation.

Proposition 2.57 (Prop. 2.1.3 of [142]) *Let $f : B \to B$ be a ring homomorphism, and set $E = f - I$.*

(a) *f is an exponential automorphism if and only if E is a locally nilpotent f-derivation.*
(b) *If E is a locally nilpotent f-derivation and $D = \log(I + E)$, then $D \in \mathrm{LND}(B)$ and $f = \exp D$.*

Of course, there may be simpler criteria showing that an automorphism φ is not an exponential automorphism. For example, $\exp D$ cannot have finite order when $D \neq 0$, since $(\exp D)^n = \exp(nD)$.

2.11 Transvectants and Wronskians

This section discusses two ways to construct kernel elements for a locally nilpotent derivation. The proofs for this section are elementary, and most are left to the reader. The assumption that B is a commutative k-domain continues.

2.11.1 Transvectants

Let nonzero $D \in \mathrm{Der}_k(B)$ be given. For each $n \in \mathbb{N}$, define a k-bilinear product $\phi_n^D : B \times B \to B$ by:

$$\begin{aligned} \phi_n^D(f, g) = [f, g]_n^D &= (f, Df, \cdots, D^n f) \cdot ((-1)^n D^n g, \cdots, -Dg, g) \\ &= \sum_{i=0}^{n} (-1)^{n-i} D^i f \, D^{n-i} g \end{aligned}$$

Observe that $\phi_0^D(f, g) = fg$. The product $[f, g]_n^D$ is called the nth **generalized transvectant** of f and g. These were introduced in [168].

We say that a k-linear map $E : B \to B$ is a ϕ_n^D**-derivation** of B if and only if, for all $f, g \in B$:

$$E[f,g]_n^D = [Ef,g]_n^D + [f,Eg]_n^D$$

The set of ϕ_n^D-derivations is denoted by $\mathrm{Der}_k(B,\phi_n^D)$.

Proposition 2.58 ([168], Prop. 2.1) *Let $D \in \mathrm{Der}_k(B)$, $D \neq 0$, $n \in \mathbb{N}$ and $f, g \in B$.*

(a) *ϕ_n^D is bilinear over $\ker D$*
(b) $[f,g]_n^D = (-1)^n [g,f]_n^D$
(c) *If n is odd, then $[f,f]_n^D = 0$*
(d) *If $E \in \mathrm{Der}_k(B)$ and $DE = ED$, then $E \in \mathrm{Der}_k(B,\phi_n^D)$*
(e) $D[f,g]_n^D = (-1)^n f D^{n+1} g + g D^{n+1} f$
(f) *If $f, g \in \ker D^{n+1}$, then $[f,g]_n^D \in \ker D$*

Consider the case that D is locally nilpotent. Given nonzero $f, g \in B$, let $n = \max\{\deg_D f, \deg_D g\}$. Then item (f) in this proposition implies that $[f,g]_n^D \in \ker D$. This gives an important way to construct kernel elements for elements of $\mathrm{LND}(B)$.

When $D \in \mathrm{LND}(B)$, transvectants can be used to define the Dixmier map associated to a local slice of D.

Proposition 2.59 ([168], Prop. 2.3) *Let $D \in \mathrm{LND}(B)$ be nonzero, let $t \in B$ be a local slice for D and let π_t be the associated Dixmier map.*

(a) *For all $f \in R$ and $n \geq 0$:*

$$[f,t^n]_n^D = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} D^i f (Dt)^{n-i} t^i$$

(b) *Given $n \geq 0$, if $\deg_D f \leq n$, then $[f,t^n]_n^D = n!(Dt)^n \pi_t(f)$*
(c) *Given $m, n \geq$ and $f, g \in B$, if $\deg_D f \leq n$ and $\deg_D g \leq m$, then:*

$$(n+m)![f,t^n]_n^D [g,t^m]_n^D = n!m![fg,t^{n+m}]_{n+m}^D$$

2.11.2 Wronskians

Given $D \in \mathrm{Der}_k(B)$ with $A = \ker D$, and given $\mathbf{f} = (f_1, \ldots, f_n) \in B^n$, let $D\mathbf{f} = (Df_1, \ldots, Df_n) \in B^n$. The **Wronskian** of $\mathbf{f}$ relative to D is:

$$W_D(\mathbf{f}) = \det \begin{pmatrix} \mathbf{f} \\ D\mathbf{f} \\ \vdots \\ D^{n-1}\mathbf{f} \end{pmatrix}$$

Observe that W_D is A-linear in each argument f_i.

Proposition 2.60 *Let M be a square matrix of order n with entries in B, and let $M_1, \ldots, M_n$ denote the rows of M. Then:*

$$D|M| = \sum_{i=1}^{n} \det \begin{pmatrix} M_1 \\ \vdots \\ DM_i \\ \vdots \\ M_n \end{pmatrix}$$

Corollary 2.61 (See [55], 7.3, ex. 8)

$$DW_D(\mathbf{f}) = \det \begin{pmatrix} \mathbf{f} \\ D\mathbf{f} \\ \vdots \\ D^{n-2}\mathbf{f} \\ D^n\mathbf{f} \end{pmatrix}$$

Corollary 2.62 *If $D^{n+1}f_i = 0$ for each i, then for $1 \le i \le n$:*

$$D^{n-i}W_D(\mathbf{f}) = \det \begin{pmatrix} \mathbf{f} \\ D\mathbf{f} \\ \vdots \\ \widehat{D^i\mathbf{f}} \\ \vdots \\ D^n\mathbf{f} \end{pmatrix}$$

Corollary 2.63 *If $D^nf_i = 0$ for each i, then $W_D(\mathbf{f}) \in A$.*

Corollary 2.64 *If $D^nf_i = 0$ for $i = 1, \ldots, n-1$, then:*

$$DW_D(\mathbf{f}) = D^nf_n \cdot W_D(f_1, \ldots, f_{n-1})$$

Proposition 2.65 *For any $g \in B$, $W_D(g\,\mathbf{f}) = g^nW_D(\mathbf{f})$.*

Proof From the generalized product rule:

$$D^n(fg) = \sum_{i=0}^{n} \binom{n}{i} D^if\, D^{n-i}g$$

Thus, the matrix

$$H = \begin{pmatrix} gf_1 & \cdots & gf_n \\ D(gf_1) & \cdots & D(gf_n) \\ \vdots & & \vdots \\ D^{n-1}(gf_1) & \cdots & D^{n-1}(gf_n) \end{pmatrix}$$

may be factored as $H = GF$, where

$$G = \begin{pmatrix} g & 0 & 0 & 0 & \cdots & 0 \\ Dg & g & 0 & 0 & \cdots & 0 \\ D^2g & 2Dg & g & 0 & \cdots & 0 \\ D^3g & 3D^2g & 3Dg & g & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ D^{n-1}g & c_1D^{n-2}g & c_2D^{n-3}g & c_3D^{n-4}g & \cdots & g \end{pmatrix}, \quad F = \begin{pmatrix} \mathbf{f} \\ D\mathbf{f} \\ \vdots \\ D^{n-1}\mathbf{f} \end{pmatrix}$$

and c_i denotes the binomial coefficient $\binom{n-1}{i}$. It follows that:

$$W_D(g\,\mathbf{f}) = |H| = |G| \cdot |F| = g^n W_D(\mathbf{f})$$

□

More generally, let $C_1, \ldots, C_n$ be columns of length $m \geq n \geq 2$ with entries in B such that:

$$D : C_n \to C_{n-1} \to \cdots \to C_1 \to 0$$

Corollary 2.66 *The $n \times n$ minor determinants of the $m \times n$-matrix $C = (C_1, \ldots, C_n)$ belong to* $\ker D$.

Proof Each minor determinant is a Wronskian of the form $W_D(c_{i_1,n}, \ldots, c_{i_n,n})$ and $D^n(c_{i_j,n}) = 0$ for each $j = 1, \ldots, n$. The result now follows by *Corollary 2.63*. □

The following variant of the Wronskian can also be used to construct kernel elements. It is especially useful when a derivation admits a large number of inequivalent local slices.

Suppose $D \in \mathrm{Der}_k(B)$ has local slices $z_{ij} \in B$ $(1 \leq i \leq n-1, 1 \leq j \leq n)$ which satisfy the following additional condition: If $\mathbf{z}_i = (z_{i1}, \ldots, z_{in})$ for $1 \leq i \leq n-1$, there exist $a_1, \ldots, a_n, y_1, \ldots, y_n \in B^D$ such that $D\mathbf{z}_i = a_i\mathbf{y}$, where $\mathbf{y} = (y_1, \ldots, y_n)$. Then for the matrix

$$M = \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n-1,1} & z_{n-1,2} & \cdots & z_{n-1,n-1} \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$$

we have $\det M \in B^D$. This fact is immediate from *Proposition 2.60*.

This method is used in *Chap. 6* below to construct certain homogeneous invariants for $\mathbb{G}_a^2$-actions of Nagata type.

2.12 Recognizing Polynomial Rings

Two criteria for polynomial rings over a field are given in *Sect. 2.3*. Suppose that K is an algebraically closed field and B is a UFD over K such that $\text{tr.deg}_K B = 1$.

1. If B is affine and $B^* = K^*$, then *Lemma 2.9* shows that $B = K^{[1]}$.
2. If B admits a degree function with values in $\mathbb{N}$ and $B_0 = K$, then *Lemma 2.12(b)* shows that $B = K^{[1]}$.

In addition, for any commutative k-domain B, the Slice Theorem implies that $B = A^{[1]}$ for a subring A if and only if there exists $D \in \text{LND}(B)$ with a slice and $A = \ker D$.

This section gives two additional criteria to determine if $B = A^{[1]}$. The first generalizes a result of Russell and Sathaye; the proof below is based on the theory of LNDs. In the second result, we describe a method for embedding cylinders as hypersurfaces of an affine variety.

Proposition 2.67 ([366], Lemma 2.3.1) *Let $A \subset B$ be a subalgebra such that B is finitely generated over A. If there exists a prime element $p \in A$ such that $pB \cap A = pA$, $B_p = A_p^{[1]}$ and A/pA is algebraically closed in B/pB, then $B = A^{[1]}$.*

Proof Since $B_p = A_p^{[1]}$, there exists surjective $d \in \text{LND}_{A_p}(B_p)$. Since B is finitely generated over A, there exists $n \geq 0$ such that $p^n d \in \text{LND}_A(B)$ and $p^n d$ is irreducible when restricted to B. Set $D = p^n d$.

Since d has a slice, there exists $r \in B$ with $Dr = p^n$. Since D is irreducible, D induces a nonzero quotient derivation $\bar{D} \in \text{LND}(B/pB)$. We have that $A/pA \subset \ker \bar{D} \subset B/pB$. Since A/pA is algebraically closed in B/pB, it follows that $\ker \bar{D} = A/pA$.

Let $\bar{r}$ denote the image of r in B/pB. If $n \neq 0$, then $\bar{D}\bar{r} = 0$ implies $\bar{r} \in A/pA$. In this case, if $r = a + pb$ for some $a \in A$ and $b \in B$, then $Db = p^{n-1}$. Continuing in this way, we obtain $s \in B$ with $Ds = 1$. By the Slice Theorem, it follows that $B = A[s] = A^{[1]}$. □

Proposition 2.68 *Let k be an algebraically closed field of characteristic zero, and let B be an affine k-domain. Given nonzero $D \in \mathrm{LND}(B)$, let $A = \ker D$ and let $I \subset A$ be the ideal $I = \sqrt{\mathrm{pl}(D)}$. Suppose that A is k-affine and $J \subset A$ is a prime ideal which satisfies:*

1. *I and J are co-maximal A-ideals*
2. *Every rank-one projective module over A/J is free*

Then $B/JB = (A/JA)^{[1]}$.

Proof Let $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$. The inclusion $A \to B$ induces the quotient map $\pi : X \to Y$. If $r \in B$ is a local slice of D, then $B_{Dr} = A_{Dr}[r]$, meaning that π is a trivial $\mathbb{A}^1$-fibration over the open set $Y \setminus \mathcal{V}(Dr)$. Therefore, π is a locally trivial $\mathbb{A}^1$-fibration over the open set $U = Y \setminus \mathcal{V}(I)$, since U is covered by such principal open sets.

By condition (1), we have $\mathcal{V}(J) \subset U$. Therefore, by restriction, π defines a locally trivial $\mathbb{A}^1$-fibration over $\mathcal{V}(J)$. By the fundamental result of Bass, Connell and Wright [14], any locally trivial $\mathbb{A}^n$-bundle over an affine variety is a vector bundle ($n \geq 0$). Therefore, condition (2) implies that π defines a trivial $\mathbb{A}^1$-fibration over $\mathcal{V}(J)$. $\square$

Chapter 3
Polynomial Rings

Locally nilpotent derivations are useful if rather elusive objects. Though we do not have them at all on "majority" of rings, when we have them, they are rather hard to find and it is even harder to find all of them or to give any qualitative statements. We do not know much even for polynomial rings.

Leonid Makar-Limanov, *Introduction* to [282]

This chapter investigates locally nilpotent derivations in the case B is a polynomial ring in a finite number of variables over a field k of characteristic zero. Equivalently, we are interested in the algebraic actions of $\mathbb{G}_a$ on $\mathbb{A}^n_k$.

3.1 Variables, Automorphisms, and Gradings

If $B = k^{[n]}$ for $n \geq 0$, then there exist $x_1, \ldots, x_n \in B$ such that $B = k[x_1, \ldots, x_n]$. Note that B cannot be generated over k by fewer than n elements Any such set $\mathbf{x} = \{x_1, \ldots, x_n\}$ is called a **system of variables** or a **coordinate system** for B over k. Any subset $\{x_1, \ldots, x_i\}$ is called a **partial system of variables** for B ($1 \leq i \leq n$). A polynomial $f \in B$ is called a **variable** or **coordinate function** for B if and only if f belongs to some system of variables for B.

The group $\text{Aut}_k(B)$ of algebraic k-automorphisms of $B = k^{[n]}$ is called the **general affine group** or **affine Cremona group** in dimension n, and is denoted $GA_n(k)$. This group may be viewed as an infinite-dimensional algebraic group over k. See [239].

G. Freudenburg, *Algebraic Theory of Locally Nilpotent Derivations*,
Encyclopaedia of Mathematical Sciences 136,
DOI 10.1007/978-3-662-55350-3_3

3.1.1 Linear Maps and Derivations

Let V be a vector space of finite dimension n over k. The symmetric algebra $S(V)$ is isomorphic to B as a k-algebra and is $\mathbb{N}$-graded: $S(V) = \bigoplus_{d\geq 0} S^d(V)$, where $S^d(V)$ is the vector space of **n-forms of degree d** in $S(V)$. In particular, $S^0(V) = k$ and $S^1(V) = V$. If $B = k[x_1, \ldots, x_n]$, then identifying V in $S(V)$ with the vector space $kx_1 \oplus \cdots \oplus kx_n$ in B gives an isomorphism $\alpha : S(V) \to B$.

A linear operator $L : V \to V$ induces both a k-algebra endomorphism $\phi_L : S(V) \to S(V)$ and a k-derivation $D_L : S(V) \to S(V)$. These, in turn, give $\alpha\phi_L\alpha^{-1} \in \mathrm{End}_k(B)$ and $\alpha D_L \alpha^{-1} \in \mathrm{Der}_k(B)$. Any $\phi \in \mathrm{End}_k(B)$ arising in this way is a **linear endomorphism** of B, and any $D \in \mathrm{Der}_k(B)$ arising in this way is a **linear derivation** of B. Given $D \in \mathrm{Der}_k(B)$, D is **linearlizable** if D is conjugate to a linear derivation by some element of $GA_n(k)$.

Note that both ϕ_L and D_L are homogeneous functions of degree 0 relative to the $\mathbb{N}$-grading of $S(V)$. In addition, observe that, if $I : V \to V$ is the identity, then $\phi_I : B \to B$ is the identity, whereas D_I is the **Euler derivation**: $D_I(f) = df$ for $f \in S^d(V)$. We also have:

Proposition 3.1 *A linear derivation of B is locally finite. If $D \in \mathrm{Der}_k(B)$ is a linear derivation induced by the linear operator $L : V \to V$, then D is locally nilpotent if and only if L is nilpotent.*

Proof Suppose that $D \in \mathrm{Der}_k(B)$ is linear, where $D = \beta D_L \beta^{-1}$ for some isomorphism $\beta : S(V) \to B$ and some linear operator L on V. By *Lemma 1.5*, D_L is locally finite, and therefore D is locally finite. In addition, *Corollary 1.11* implies that D_L is locally nilpotent if and only if L is nilpotent. Therefore, D is locally nilpotent if and only if L is nilpotent. □

If $L \in GL(V)$, then $\alpha\phi_L\alpha^{-1} \in GA_n(k)$, and this gives an algebraic embedding $GL(V) \to GA_n(k)$. The image of $GL(V)$ is denoted by $GL_n(k)$, the **general linear group** of order n, and elements of $GL_n(k)$ are called **linear automorphisms** of B. Suppose that $G \subset GL(V)$ is an algebraic subgroup. An algebraic representation $\rho : G \to GA_n(k)$ is **linearizable** if and only if ρ factors through a representation $\gamma : GL(V) \to GA_n(k)$, i.e., $\rho = \gamma\iota$, where $\iota : G \to GL(V)$ is the inclusion.

Example 3.2 Let $G \subset GL_2(k)$ act on $V = k^2$. Then G acts on the symmetric algebra $S(V) = \oplus_{d\geq 0} S^d(V) = k^{[2]}$ and this action restricts to each homogeneous summand $V_d = S^d(V) = k^{d+1}$. If $S(V) = k[x, y]$, then V_d has basis $x^i y^{d-i}$, $0 \leq i \leq d$, and is called the **G-module of binary forms of degree d**.

3.1.2 Triangular and Tame Automorphisms

Given a coordinate system $B = k[x_1, \ldots, x_n]$, an automorphism $F \in GA_n(k)$ is given by $F = (F_1, \ldots, F_n)$, where $F_i = F(x_i) \in B$. The **triangular automorphisms** or **Jonquières automorphisms** are those of the form $F = (F_1, \ldots, F_n)$, where

$F_i \in k[x_1, \ldots, x_i]$.[1] The triangular automorphisms form a subgroup, denoted $BA_n(k)$, which is the generalization of the Borel subgroup in the theory of finite-dimensional representations. Note that the subgroup $BA_n(k)$ depends on the underlying coordinate system.

The **tame subgroup** of $GA_n(k)$ is the subgroup generated by $GL_n(k)$ and $BA_n(k)$. Its elements are called **tame automorphisms** relative to the coordinate system $\mathbf{x}$. It is known that for $n \leq 2$, every element of $GA_n(k)$ is tame (see *Chap. 4*), whereas non-tame automorphisms exist in $GA_3(k)$. See [382, 383].

As to gradings of polynomial rings, we are mainly interested in $\mathbb{Z}^m$-gradings for some $m \geq 1$. In particular, suppose $B = k[x_1, \ldots, x_n]$. Given a homomorphism $\alpha : \mathbb{Z}^n \to \mathbb{Z}^m$ for $m \geq 1$, define the function $\deg_\alpha$ on the set of monomials by $\deg_\alpha(x_1^{e_1} \cdots x_n^{e_n}) = \alpha(e_1, \ldots, e_n)$. This defines a $\mathbb{Z}^m$-grading $B = \bigoplus_{i \in \mathbb{Z}^m} B_i$. For example, if $\alpha : \mathbb{Z}^n \to \mathbb{Z}$ is defined by $\alpha(e_1, \ldots, e_n) = \sum e_i$, then the induced grading is called the **standard $\mathbb{Z}$-grading** of B, relative to $\mathbf{x}$. Likewise, if $\alpha(e_1, \ldots, e_n) = e_1$, then B is graded according to its usual degree relative to x_1.

Remark 3.3 By considering the Jordan normal form of its defining matrix, we see that any linear $\mathbb{G}_a$-action on $\mathbb{A}^n$ is conjugate to a linear triangular $\mathbb{G}_a$-action. In addition, it is well-known that an action of a linear algebraic group G on $\mathbb{A}^n$ can be extended to a linear action on some larger affine space $\mathbb{A}^N$. Therefore, any algebraic $\mathbb{G}_a$-action on $\mathbb{A}^n$ extends to a linear triangular $\mathbb{G}_a$-action on some larger affine space $\mathbb{A}^N$.

3.2 Derivations of Polynomial Rings

3.2.1 Definitions

Let $B = k^{[n]}$. Given $D \in \mathrm{Der}_k(B)$, define the **corank** of D to be the maximum integer i such that there exists a partial system of variables $\{x_1, \ldots, x_i\}$ of B contained in $\ker D$. In other words, the corank of D is the maximal number of variables within the same system annihilated by D. Denote the corank of D by $\mathrm{corank}(D)$. The **rank** of D is $\mathrm{rank}(D) = n - \mathrm{corank}(D)$. By definition, the rank and corank are invariants of D, in the sense that these values do not change after conjugation by an element of $GA_n(k)$. The rank and corank were first defined in [159].

[1] Ernest Jean Philippe Fauque de Jonquières (1820–1901) was a career officer in the French navy, achieving the rank of vice-admiral in 1879. He learned advanced mathematics by reading works of Poncelet, Chasles, and other geometers. In 1859, he introduced the planar transformations $(x, y) \to \left(x, \frac{a(x)y+b(x)}{c(x)y+d(x)}\right)$, where $ad - bc \neq 0$. These were later studied by Cremona.

A k-derivation D of B is said to be **rigid** when the following condition holds: If $\text{corank}(D) = i$, and if $\{x_1, \ldots, x_i\}$ and $\{y_1, \ldots, y_i\}$ are partial systems of variables of B contained in $\ker D$, then $k[x_1, \ldots, x_i] = k[y_1, \ldots, y_i]$. This definition is due to Daigle [69].

We say that $D \in \text{Der}_k(B)$ is **quasi-linear** if and only if there exists a coordinate system $\mathbf{x} = (x_1, \ldots, x_n)$ and matrix $M \in \mathcal{M}_n(\ker D)$ such that $D\mathbf{x} = \mathbf{x}M$, where $D\mathbf{x} = (Dx_1, \ldots, Dx_n)$. Note that D is locally nilpotent if and only if M is a nilpotent matrix, since $D^i\mathbf{x} = \mathbf{x}M^i$. A family of quasi-linear locally nilpotent derivations is discussed in *Sect. 3.9.3*.

$D \in \text{Der}_k(B)$ is a **triangular derivation** of B if and only if $Dx_i \in k[x_1, \ldots, x_{i-1}]$ for $i = 2, \ldots, n$ and $Dx_1 \in k$. Note that triangularity depends on the choice of coordinates on B. By a **triangularizable derivation** of B we mean any $D \in \text{Der}_k(B)$ which is triangular relative to some system of coordinates on B, i.e., conjugate to a triangular derivation. As we will see, the triangular derivations form a large and important class of locally nilpotent derivations of polynomial rings. Several of the main examples and open questions discussed below involve triangular derivations.

For polynomial rings, other natural categories of derivations to study include the following: Let $D \in \text{Der}_k(B)$ for $B = k[x_1, \ldots, x_n] = k^{[n]}$.

1. D is a **monomial derivation** if each image Dx_i is a monomial in $x_1, \ldots, x_n$.
2. D is an **elementary derivation** if, for some j with $1 \leq j \leq n$, $Dx_i = 0$ for $1 \leq i \leq j$, and $Dx_i \in k[x_1, \ldots, x_j]$ if $j + 1 \leq i \leq n$.
3. D is a **nice derivation**[2] if $D^2 x_i = 0$ for each i.

These definitions depend on the underlying coordinate system. Note that any nice derivation is locally nilpotent, and that any elementary derivation is both triangular and nice. We also have:

Proposition 3.4 ([243]) *For the polynomial ring $B = k[x_1, \ldots, x_n] = k^{[n]}$, every monomial derivation $D \in \text{LND}(B)$ is triangular relative to some ordering of $x_1, \ldots, x_n$.*

Proof We may assume, with no loss of generality, that:

$$\deg_D(x_1) \leq \deg_D(x_2) \leq \cdots \leq \deg_D(x_n)$$

Given i, write $Dx_i = a x_1^{e_1} \cdots x_n^{e_n} \neq 0$ for $a \in k$ and $e_i \geq 0$. If $Dx_i \neq 0$, then $\deg_D(x_i) - 1 = \sum_{j=1}^{n} e_j \deg_D(x_j)$. Due to the ordering above, this is only possible if $e_j = 0$ for $j \geq i$. Therefore, $Dx_i \in k[x_1, \ldots, x_{i-1}]$ for every i. $\square$

We will see that triangular monomial derivations provide us with important examples.

[2] Van den Essen gives a more exclusive definition of a nice derivation. See [142], 7.3.12.

3.2.2 Partial Derivatives

To each system of variables $\mathbf{x} = (x_1, \ldots, x_n)$ on the polynomial ring $B = k[\mathbf{x}]$ we associate a corresponding system of **partial derivatives** ∂_{x_i} relative to $\mathbf{x}$, $1 \le i \le n$. In particular, $\partial_{x_i} \in \mathrm{Der}_k(B)$ is defined by $\partial_{x_i}(x_j) = \delta_{ij}$ (Kronecker delta). Another common notation for ∂_{x_i} is $\frac{\partial}{\partial x_i}$. Given $f \in B$, let $f_{x_i} = \partial_{x_i} f$.

Note that ∂_{x_i} is locally nilpotent for each i, since $B = A[x_i]$ for $A = k[x_1, \ldots, \widehat{x_i}, \ldots, x_n]$, and $\partial_{x_i}(A) = 0$. Note also that the meaning of ∂_{x_i} depends on the entire system of variables to which x_i belongs. For example, in the two-dimensional case, $k[x, y] = k[x, y+x]$, and $\partial_x(y+x) = 1$ relative to (x, y), whereas $\partial_x(y+x) = 0$ relative to $(x, y + x)$. In general, we will say $D \in \mathrm{LND}(B)$ is a **partial derivative** if and only if there exists a system of coordinates $(y_1, \ldots, y_n)$ on B relative to which $D = \partial_{y_1}$.

It is easy to see that, as a B-module, $\mathrm{Der}_k(B)$ is freely generated by $\{\partial_{x_1}, \ldots, \partial_{x_n}\}$, and that this is a basis of commuting derivations. In particular, given $D \in \mathrm{Der}_k(B)$:

$$D = \sum_{1 \le i \le n} D(x_i)\partial_{x_i}$$

To verify this expression for D, it suffices to check equality for each x_i, and this is obvious. Note that the rank of D is the minimal number of partial derivatives needed to express D in this form. Thus, elements of $\mathrm{Der}_k(B)$ having rank one are precisely those of the form $f\partial_{x_1}$ for $f \in B$, relative to some system of coordinates $(x_1, \ldots, x_n)$ for B.

Example 3.5 On the polynomial ring $B = k[x_1, \ldots, x_n] = k^{[n]}$, define the derivation:

$$D = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}$$

If $N = \prod_{i=1}^{n-1} i^i$, then

$$W_D(x_1^{n-1}, \ldots, x_n^{n-1}) = N \cdot \det \begin{pmatrix} x_1^{n-1} & \cdots & x_n^{n-1} \\ x_1^{n-2} & \cdots & x_n^{n-2} \\ \vdots & & \vdots \\ x_1 & \cdots & x_n \\ 1 & \cdots & 1 \end{pmatrix} = N \cdot \prod_{i>j}(x_i - x_j) \ ,$$

i.e., the Vandermonde determinant of $x_1, \ldots, x_n$ may be realized as a Wronskian. $\square$

The partial derivatives ∂_{x_i} also extend (uniquely) to the field $K = k(x_1, \ldots, x_n)$ by the quotient rule, although they are no longer locally nilpotent on all K:

$$\mathrm{Nil}(\partial_{x_i}) = k(x_1, \ldots, \widehat{x_i}, \ldots, x_n)[x_i] \ .$$

In this case, we see that $\mathrm{Der}_k(K)$ is a vector space over K of dimension n, with basis $\partial_{x_1}, \ldots, \partial_{x_n}$. More generally:

Proposition 3.6 *If L is a field of finite transcendence degree n over k, then* $\mathrm{Der}_k(L)$ *is a vector space over L of dimension n.*

Proof Suppose $k \subset k(x_1, \ldots, x_n) \subset L$ for algebraically independent x_i, and set $K = k(x_1, \ldots, x_n)$. Suppose $D \in \mathrm{Der}_k(L)$ and $t \in L$ are given, and let $P \in K[T] = K^{[1]}$ be the minimal polynomial of t over k. Suppose $P(T) = \sum_i a_i T^i$ for $a_i \in K$. Then $0 = D(P(t)) = P'(t)Dt + \sum_i D(a_i)t^i$. Since $P'(t) \neq 0$, this implies

$$Dt = -(P'(t))^{-1} \sum_i D(a_i)t^i$$

meaning that D is completely determined by its values on K. Conversely, this same formula shows that every $D \in \mathrm{Der}_k(K)$ can be uniquely extended to L.

In particular, the partial derivatives ∂_{x_i} extend uniquely to L. If $f \in K$ and $D \in \mathrm{Der}_k(L)$, then $Df = \partial_{x_1} f Dx_1 + \cdots + \partial_{x_n} f Dx_n$. We conclude that

$$\mathrm{Der}_k(L) = \mathrm{span}_L\{\partial_{x_1}, \ldots, \partial_{x_n}\} .$$

If $a_1\partial_{x_1} + \cdots a_n\partial_{x_n} = 0$ for $a_i \in L$, then evaluation at x_i shows that $a_i = 0$. Therefore, the partial derivatives are linearly independent over L, and the dimension of $\mathrm{Der}_k(L)$ equals n. □

Proposition 3.7 (Multivariate Chain Rule) *Suppose $D \in$* $\mathrm{Der}_k(K)$ *for $K = k(x_1, \ldots, x_n)$, and $f_1, \ldots, f_m \in K$. Then for any $g \in k(y_1, \ldots, y_m) = k^{(m)}$:*

$$D\big(g(f_1, , , f_m)\big) = \frac{\partial g}{\partial y_1}(f_1, \ldots, f_m) \cdot Df_1 + \cdots + \frac{\partial g}{\partial y_m}(f_1, \ldots, f_m) \cdot Df_m$$

Proof By the product rule, it suffices to assume $g \in k[y_1, \ldots, y_m]$. In addition, by linearity, it will suffice to show the formula in the case g is a monomial: $g = y_1^{e_1} \cdots y_m^{e_m}$ for $e_1, \ldots, e_m \in \mathbb{N}$.

From the product rule and the univariate chain rule, we have that:

$$\begin{aligned}\frac{\partial}{\partial x_j}(f_1^{e_1} \cdots f_m^{e_m}) &= \sum_i f_1^{e_1} \cdots \widehat{f_i^{e_i}} \cdots f_m^{e_m} \cdot \partial_{x_j}(f_i^{e_i}) \\ &= \sum_i e_i f_1^{e_1} \cdots f_i^{e_i - 1} \cdots f_m^{e_m} \cdot (f_i)_{x_j}\end{aligned}$$

Since $D = Dx_1\partial_{x_1} + \cdots + Dx_n\partial_{x_n}$, we have:

$$\begin{aligned}
D(f_1^{e_1}\cdots f_m^{e_m}) &= \sum_j \partial_{x_j}(f_1^{e_1}\cdots f_m^{e_m}) \cdot Dx_j \\
&= \sum_j \sum_i (f_i)_{x_j} \cdot e_i f_1^{e_1}\cdots f_i^{e_i-1}\cdots f_m^{e_m} \cdot Dx_j \\
&= \sum_i \sum_j (f_i)_{x_j} \cdot e_i f_1^{e_1}\cdots f_i^{e_i-1}\cdots f_m^{e_m} \cdot Dx_j \\
&= \sum_i e_i f_1^{e_1}\cdots f_i^{e_i-1}\cdots f_m^{e_m} \sum_j (f_i)_{x_j} \cdot Dx_j \\
&= \sum_i e_i f_1^{e_1}\cdots f_i^{e_i-1}\cdots f_m^{e_m} \cdot Df_i \\
&= \sum_i \frac{\partial g}{\partial y_i}(f_1^{e_1}\cdots f_m^{e_m}) \cdot Df_i
\end{aligned}$$

□

The use of partial derivatives also allows us to describe homogeneous decompositions of derivations relative to G-gradings of $B = k[x_1, \ldots, x_n] = k^{[n]}$.

Proposition 3.8 (See Prop. 5.1.14 of [142]) *Let G be an abelian group and $B = \bigoplus_{g\in G} B_g$ a G-grading such that x_i is G-homogeneous for $1 \le i \le n$. Every nonzero $D \in \mathrm{Der}_k(B)$ admits a unique decomposition $D = \sum_{g\in G} D_g$, where $D_g \in \mathrm{Der}_k(B)$ is G-homogeneous of degree g and $D_g = 0$ for all but finitely many $g \in G$.*

Proof There exist $f_1, \ldots, f_n \in B$ such that $D = \sum f_i\partial_{x_i}$. Since each x_i is G-homogeneous, each partial derivative ∂_{x_i} is a G-homogeneous derivation of B. Each coefficient function f_i admits a decomposition into G-homogeneous summands; suppose $f_i = \sum_{g\in G}(f_i)_g$. Then each summand $f_i\partial_{x_i}$ can be decomposed as a finite sum of G-homogeneous derivations, namely, $f_i\partial x_i = \sum_{g\in G}(f_i)_g\partial_{x_i}$. Therefore, $D = \sum_{i,g}(f_i)_g\partial_{x_i}$, and by gathering terms of the same degree, the desired result follows. □

Example 3.9 Let $G = \mathbb{Z}_2$ and define a G-grading on $\mathbb{C}[x, y, z] = \mathbb{C}^{[3]}$ by declaring that x, y, z are G-homogeneous with $\deg_G x = \deg_G z = 0$ and $\deg_G y = 1$. Then $\partial_x, \partial_y, \partial_z$ are G-homogeneous with $\deg_G \partial_x = \deg_G \partial_z = 0$ and $\deg_G \partial_y = 1$. Define $D \in \mathrm{LND}(\mathbb{C}[x, y, z])$ by:

$$D = (-3z^2)\partial_x + (3iz^2)\partial_y + 2(x - iy)\partial_z$$

Then $D = D_0 + D_1$ for $D_0 = (-3z^2)\partial_x + 2(x - iy)\partial_z$ and $D_1 = (3iz^2)\partial_y$. Note that, if $f = x^2 + y^2 + z^3$, then $\deg_G f = 0$ and $Df = 0$.

The preceding example can be used to show that *Proposition 3.8* does not generalize to affine rings: Let $B = \mathbb{C}[x,y,z]/(f)$ and let $\delta \in \mathrm{LND}(B)$ be the quotient derivation defined by D. In particular, $\delta \neq 0$. Since f is G-homogeneous, B inherits a non-trivial G-grading. However, it is shown in [83], Prop. 6.5, that $\Delta = 0$ for any G-homogeneous $\Delta \in \mathrm{LND}(B)$.

On the other hand, recall that when the group G is totally ordered and B is a G-graded affine k-domain, then any nonzero $D \in \mathrm{LND}(B)$ induces a nonzero G-homogeneous element of $\mathrm{LND}(B)$; see *Sect. 1.1.5*.

3.2.3 Jacobian Derivations

Let $B = k[x_1,\ldots,x_n] = k^{[n]}$. The **jacobian matrix** of $f_1,\ldots,f_m \in B$ is the $m \times n$ matrix of partial derivatives:

$$\mathcal{J}(f_1,\ldots,f_m) := \frac{\partial(f_1,\ldots,f_m)}{\partial(x_1,\ldots,x_n)} = \left((f_i)_{x_j}\right)$$

Note that the jacobian matrix depends on the underlying system of coordinates. When $m = n$, the **jacobian determinant** of $f_1,\ldots,f_n \in B$ is $\det \mathcal{J}(f_1,\ldots,f_n) \in B$.

Suppose $k[y_1,\ldots,y_m] = k^{[m]}$, and let $F : k[y_1,\ldots,y_m] \to k[x_1,\ldots,x_n]$ be a k-algebra homomorphism. Then the jacobian matrix of F is $\mathcal{J}(F) = \mathcal{J}(f_1,\ldots,f_m)$, where $f_i = F(y_i)$, and the jacobian determinant of F is $\det \mathcal{J}(F)$.[3] In addition, suppose $A = (a_{ij})$ is a matrix with entries a_{ij} in $k[y_1,\ldots,y_m]$. Then $F(A)$ denotes the matrix $\left(F(a_{ij})\right)$ with entries in $k[x_1,\ldots,x_m]$.

Given k-algebra homomorphisms

$$k[z_1,\ldots,z_l] \xrightarrow{G} k[y_1,\ldots,y_m] \xrightarrow{F} k[x_1,\ldots,x_n]$$

the **chain rule for jacobian matrices** is

$$\mathcal{J}(F \circ G) = F(\mathcal{J}(G)) \cdot \mathcal{J}(F)$$

where $(\cdot)$ on the right denotes matrix multiplication. This follows from the multivariate chain rule above. Note that if $\mathcal{J}(G)$ is a square matrix, then we have:

$$\det F\,(\mathcal{J}(G)) = F\,(\det \mathcal{J}(G))$$

Observe that the standard properties of determinants imply that:

$\det \mathcal{J}$ is a k-derivation of B in each one of its arguments.

[3] Some authors use DF to denote the jacobian matrix of F, but we prefer to reserve D for derivations.

In particular, suppose $f_1, \dots, f_{n-1} \in B$ are given, and set $\mathbf{f} = (f_1, \dots, f_{n-1})$. Then $\mathbf{f}$ defines $\Delta_{\mathbf{f}} \in \mathrm{Der}_k(B)$ via:

$$\Delta_{\mathbf{f}}(g) := \det \mathcal{J}(f_1, \dots, f_{n-1}, g) \quad (g \in B)$$

$\Delta_{\mathbf{f}}$ is called the **jacobian derivation** of B determined by $\mathbf{f}$.

Observe that the definitions of jacobian matrices and jacobian derivations also extend to the rational function field $K = k(x_1, \dots, x_n)$.

If $F = (f_1, \dots, f_n)$ is a system of variables for B, then:

$$\det \mathcal{J}(F) = \det \frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)} = \Delta_{\mathbf{f}}(f_n) \in k^*$$

This is easily seen from the chain rule: By definition, F admits a polynomial inverse F^{-1}, and $I = FF^{-1}$ implies that

$$1 = \det\big(F(\mathcal{J}(F^{-1})) \cdot \mathcal{J}(F)\big) = \det F(\mathcal{J}(F^{-1})) \det \mathcal{J}(F)$$

meaning $\det \mathcal{J}(F)$ is a unit of B.

In the other direction lurks the famous **Jacobian Conjecture**, which can be formulated in the language of derivations: Suppose $\mathbf{f} = (f_1, \dots, f_{n-1})$ for $f_i \in B$.

> If $\Delta_{\mathbf{f}}$ has a slice s, then $k[f_1, \dots, f_{n-1}, s] = B$. Equivalently, if $\Delta_{\mathbf{f}}$ has a slice, then $\Delta_{\mathbf{f}}$ is locally nilpotent and $\ker \Delta_{\mathbf{f}} = k[f_1, \dots, f_{n-1}]$.

See van den Essen [142], Chap. 2, for further details about the Jacobian Conjecture.

Let $B = \bigoplus_{i \in \mathbb{Z}} B_i$ be the standard $\mathbb{Z}$-grading relative to $(x_1, \dots, x_n)$. Given a system of variables $F = (f_1, \dots, f_n)$ for B, write $F = \sum_{i \in \mathbb{Z}} F_i$, where $F_i = ((f_1)_i, \dots, (f_n)_i)$. It is easy to check that $\det \mathcal{J}(F) = \det \mathcal{J}(F_1) \in k^*$. It follows that F_1 is a linear system of variables for B. We have thus shown:

$$F \in GA_n(k) \quad \Rightarrow \quad F_1 \in GL_n(k) \tag{3.1}$$

Following are several lemmas about jacobian derivations, which will be used to prove certain properties of locally nilpotent derivations of polynomial rings.

Lemma 3.10 *Suppose $K = k(x_1, \dots, x_n) = k^{(n)}$. Given $f_1, \dots, f_{n-1} \in K$, set $\mathbf{f} = (f_1, \dots, f_{n-1})$ and consider $\Delta_{\mathbf{f}} \in \mathrm{Der}_k(K)$.*

- **(a)** *$\Delta_{\mathbf{f}} = 0$ if and only if $f_1, \dots, f_{n-1}$ are algebraically dependent.*
- **(b)** *If $\Delta_{\mathbf{f}} \neq 0$, then $\ker \Delta_{\mathbf{f}}$ is the algebraic closure of $k(f_1, \dots, f_{n-1})$ in K.*
- **(c)** *For any $g \in K$, $\Delta_{\mathbf{f}}(g) = 0$ if and only if $f_1, \dots, f_{n-1}, g$ are algebraically dependent.*

Proof (Following [276]) To prove part (a), suppose $f_1, \dots, f_{n-1}$ are algebraically dependent. Let $P(t)$ be a polynomial with coefficients in the field $k(f_2, \dots, f_{n-1})$ of minimal degree such that $P(f_1) = 0$. Then:

$$0 = \Delta_{(P(f_1),f_2,\dots,f_{n-1})} = P'(f_1)\Delta_{(f_1,f_2,\dots,f_{n-1})} = P'(f_1)\Delta_{\mathbf{f}}$$

By minimality of degree, $P'(f_1) \neq 0$, so $\Delta_{\mathbf{f}} = 0$.

Conversely, suppose $f_1,\dots,f_{n-1}$ are algebraically independent, and choose $f_n \in K$ transcendental over $k(f_1,\dots,f_{n-1})$. Then for each i, x_i is algebraic over $k(f_1,\dots,f_n)$, and there exists $P_i \in k[y_1,\dots,y_{n+1}] = k^{[n+1]}$ such that $P_i(f_1,\dots,f_n,x_i) = 0$. Now $\partial P_i/\partial y_{n+1} \neq 0$, since otherwise P_i gives a relation of algebraic dependence for $f_1,\dots,f_n$. We may assume the degree of P_i is minimal in y_{n+1}, so that $\partial P_i/\partial y_{n+1}$ is nonzero when evaluated at $(f_1,\dots,f_n,x_i)$.

By the chain rule, for each i and each j,

$$0 = \partial_{x_j} P_i(f_1,\dots,f_n,x_i) = \sum_{1\le s\le n} (P_i)_s(f_s)_{x_j} + (P_i)_{n+1}(x_i)_{x_j}$$

where $(P_i)_s$ denotes $\frac{\partial P_i}{\partial y_s}(f_1,\dots,f_n,x_i)$. In matrix form, this becomes

$$0 = \begin{pmatrix} (P_i(f_1,\dots,f_n,x_i))_{x_1} \\ \vdots \\ (P_i(f_1,\dots,f_n,x_i))_{x_n} \end{pmatrix} = M \begin{pmatrix} (P_i)_1 \\ \vdots \\ (P_i)_n \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ (P_i)_{n+1} \\ \vdots \\ 0 \end{pmatrix}$$

where $M = \mathcal{J}(f_1,\dots,f_n)$. Let $e_i = (0,\dots,1,\dots 0) \in K^n$ be the standard basis vectors ($1 \le i \le n$). The image of M as a linear operator on K^n is spanned by $(P_1)_{n+1}e_1,\dots,(P_n)_{n+1}e_n$, and since $(P_i)_{n+1} \neq 0$ for each i, we conclude that M is surjective. Therefore, $\det M = \Delta_{\mathbf{f}}(f_n) \neq 0$. So part (a) is proved.

To prove (b), note first that, under the hypothesis $\Delta_{\mathbf{f}} \neq 0$, part (a) implies $f_1,\dots,f_{n-1}$ are algebraically independent. This means that the transcendence degree of $k(f_1,\dots,f_{n-1})$ equals $n-1$. Since $k(f_1,\dots,f_{n-1}) \subset \ker \Delta_{\mathbf{f}}$, we have that $\ker \Delta_{\mathbf{f}}$ is the algebraic closure of $k(f_1,\dots,f_{n-1})$ in K.

To prove (c), suppose first that $f_1,\dots,f_{n-1},g$ are algebraically independent. Then $f_1,\dots,f_{n-1}$ are algebraically independent, and $\ker \Delta_{\mathbf{f}}$ is an algebraic extension of $k(f_1,\dots,f_{n-1})$. Since g is transcendental over $k(f_1,\dots,f_{n-1})$, it is also transcendental over $\ker \Delta_{\mathbf{f}}$. Therefore, $\Delta_{\mathbf{f}}(g) \neq 0$.

Conversely, suppose that $f_1,\dots,f_{n-1},g$ are algebraically dependent. If $f_1,\dots,f_{n-1}$ are algebraically independent, the same argument used above shows that $g \in \ker \Delta_{\mathbf{f}}$. And if $f_1,\dots,f_{n-1}$ are algebraically dependent, then $\Delta_{\mathbf{f}}$ is the zero derivation, by part (a). $\square$

Lemma 3.11 (Lemma 6 of [276]) *Suppose* $K = k(x_1, \ldots, x_n) = k^{(n)}$ *and* $D \in \mathrm{Der}_k(K)$ *has* $\mathrm{tr.deg}_k(\ker D) = n-1$. *Then for any set* $\mathbf{f} = (f_1, \ldots, f_{n-1})$ *of algebraically independent elements of* $\ker D$, *there exists* $a \in K$ *such that* $D = a\Delta_{\mathbf{f}}$.

Proof First, $\ker D = \ker \Delta_{\mathbf{f}}$, since each is equal to the algebraic closure of $k(f_1, \ldots, f_{n-1})$ in K. Choose $g \in K$ so that $Dg \neq 0$. Define $a = Dg(\Delta_{\mathbf{f}} g)^{-1}$. Then $D = a\Delta_{\mathbf{f}}$ when restricted to the subfield $k(f_1, \ldots, f_{n-1}, g)$. Since $Dg \neq 0$, g is transcendental over $\ker D$, hence also over $k(f_1, \ldots, f_{n-1})$. Thus, K is an algebraic extension of $k(f_1, \ldots, f_{n-1}, g)$. By *Proposition 1.14* we conclude that $D = a\Delta_{\mathbf{f}}$ on all of K. □

Lemma 3.12 (Lemma 7 of [276]) *For* $n \geq 2$, *let* $K = k(x_1, \ldots, x_n) = k^{(n)}$. *Given* $f_1, \ldots, f_{n-1} \in K$ *algebraically independent, set* $\mathbf{f} = (f_1, \ldots, f_{n-1})$. *If* $\mathbf{g} = (g_1, \ldots, g_{n-1})$ *for* $g_i \in \ker \Delta_{\mathbf{f}}$, *then there exists* $a \in \ker \Delta_{\mathbf{f}}$ *such that* $\Delta_{\mathbf{g}} = a\Delta_{\mathbf{f}}$.

Proof If $\Delta_{\mathbf{g}} = 0$, we can take $a = 0$. So assume $\Delta_{\mathbf{g}} \neq 0$, meaning that $g_1, \ldots, g_{n-1}$ are algebraically independent. In particular, $g_i \notin k$ for each i.

Since $\mathrm{tr.deg.}_k \ker \Delta_{\mathbf{f}} = n-1$, the elements $f_1, \ldots, f_{n-1}, g_1$ are algebraically dependent. Let $P \in k[T_1, \ldots, T_n] = k^{[n]}$ be such that $P(\mathbf{f}, g_1) = 0$. The notation P_i will denote the partial derivative $\partial P/\partial T_i$. Then we may assume that $P_n(\mathbf{f}, g_1) \neq 0$; otherwise replace P by P_n. Likewise, by re-ordering the f_i if necessary, we may assume that $P_1(\mathbf{f}, g_1) \neq 0$. It follows that:

$$
\begin{aligned}
0 &= \Delta_{(P(\mathbf{f},g_1),f_2,\ldots,f_{n-1})} \\
&= \sum_{1 \leq i \leq n-1} P_i(\mathbf{f}, g_1)\Delta_{(f_i,f_2,\ldots,f_{n-1})} + P_n(\mathbf{f}, g_1)\Delta_{(g_1,f_2,\ldots,f_{n-1})} \\
&= P_1(\mathbf{f}, g_1)\Delta_{(f_1,f_2,\ldots,f_{n-1})} + P_n(\mathbf{f}, g_1)\Delta_{(g_1,f_2,\ldots,f_{n-1})}
\end{aligned}
$$

Thus, $\Delta_{(g_1,f_2,\ldots,f_{n-1})} = a\Delta_{\mathbf{f}}$ for some nonzero $a \in \ker \Delta_{\mathbf{f}}$.

If $n = 2$ we are done. Otherwise $n \geq 3$, and we may assume inductively that for some i with $1 \leq i \leq n-2$ we have

$$\Delta_{(g_1,\ldots,g_i f_{i+1}\cdots f_{n-1})} = b\Delta_{\mathbf{f}}$$

for some nonzero $b \in \ker \Delta_{\mathbf{f}}$. Then $g_1, \ldots, g_i, f_{i+1} \ldots, f_{n-1}$ are algebraically independent, since the derivation they define is nonzero. Choose $Q \in k[T_1, \ldots, T_n]$ with $Q(g_1, \ldots, g_i, f_{i+1}, \ldots, f_{n-1}, g_{i+1}) = 0$, noting that $Q_n \neq 0$ (otherwise Q is a dependence relation for $g_1, \ldots, g_i, f_{i+1}, \ldots, f_{n-1}$). By re-ordering the f_i if necessary, we may assume that $Q_{i+1}(g_1, \ldots, g_i, f_{i+1}, \ldots, f_{n-1}, g_{i+1}) \neq 0$. As above, we have

$$
\begin{aligned}
0 &= \Delta_{(g_1,\ldots,g_i,Q(*),f_{i+2},\ldots,f_{n-1})} \\
&= Q_{i+1}(*)\Delta_{(g_1,\ldots,g_i f_{i+1},\ldots,f_{n-1})} + Q_n(*)\Delta_{(g_1,\ldots,g_i,g_{i+1} f_{n+2},\ldots,f_{n-1})} \\
&= Q_{i+1}(*) \cdot b\Delta_{\mathbf{f}} + Q_n(*)\Delta_{(g_1,\ldots,g_i,g_{i+1} f_{n+2},\ldots,f_{n-1})}
\end{aligned}
$$

where $(*)$ denotes the input $(g_1, \ldots, g_i, f_{i+1}, \ldots, f_{n-1}, g_{i+1})$. Therefore,

$$\Delta_{(g_1,\ldots,g_{i+1} f_{i+2} \ldots f_{n-1})} = c\Delta_{\mathbf{f}}$$

for some nonzero $c \in \ker \Delta_{\mathbf{f}}$. By induction, the proof is complete. □

If $K = k(x_1, \ldots, x_n) = k^{(n)}$ and $D \in \mathrm{Der}_k(K)$, define the **divergence** of D by:

$$\mathrm{div}(D) = \sum_i \partial_{x_i}(Dx_i)$$

Lemma 3.13 *If $\Delta_{\mathbf{f}}$ is a jacobian derivation of $k^{(n)}$, then* $\mathrm{div}(\Delta_{\mathbf{f}}) = 0$.

Proof Given x_i, *Proposition* 2.60 implies that:

$$\partial_{x_i}\big(\Delta_{\mathbf{f}}(x_i)\big) = \sum_{j=1}^{n} \Delta_{(f_1,\ldots,(f_j)_{x_i},\ldots,f_{n-1})}(x_i)$$

Therefore:

$$\mathrm{div}(\Delta_{\mathbf{f}}) = \sum_{1 \le i,j \le n} \Delta_{(f_1,\ldots,(f_j)_{x_i},\ldots,f_{n-1})}(x_i)$$

Expanding these determinants, we see that

$$\mathrm{div}(\Delta_{\mathbf{f}}) = \sum_{\sigma \in S_n} \mathrm{sign}(\sigma)(f_1)_{y_1}(f_2)_{y_2} \cdots (f_j)_{y_j y_n} \cdots (f_{n-1})_{y_{n-1}}$$

where $\sigma = (y_1, \ldots, y_n)$ is a permutation of $(x_1, \ldots, x_n)$. Since $(f_j)_{y_j y_n} = (f_j)_{y_n y_j}$, terms corresponding to $(y_1, \ldots, y_j, \ldots, y_n)$ and $(y_1, \ldots, y_n, \ldots, y_j)$ cancel each other out, their signs being opposite. Therefore, the entire sum is 0. □

An additional fact about jacobian derivations is due to Daigle. It is based on the following result; the reader is referred to the cited paper for its proof.

Proposition 3.14 (Cor. 3.10 of [70]) *Let $f_1, \ldots, f_m \in B = k[x_1, \ldots, x_n] = k^{[n]}$ be given. Set $A = k[f_1, \ldots, f_m]$ and $M = \mathcal{J}(f_1, \ldots, f_m)$. Suppose $I \subset B$ is the ideal generated by the $d \times d$ minors of M, where d is the transcendence degree of A over k. If A is factorially closed in B, then* $\mathrm{height}(I) > 1$.

Corollary 3.15 (Cor. 2.4 of [70]) *Suppose $f_1, \ldots, f_{n-1} \in B = k[x_1, \ldots, x_n] = k^{[n]}$ are algebraically independent, and set $\mathbf{f} = (f_1, \ldots, f_{n-1})$. If $k[f_1, \ldots, f_{n-1}]$ is a factorially closed subring of B, then $\Delta_{\mathbf{f}}$ is irreducible, and* $\ker \Delta_{\mathbf{f}} = k[f_1, \ldots, f_{n-1}]$.

Proof Since $\Delta_{\mathbf{f}} \neq 0$, we have that $\ker \Delta_{\mathbf{f}}$ is equal to the algebraic closure of $k[f_1, \ldots, f_{n-1}]$ in B. By hypothesis, $k[f_1, \ldots, f_{n-1}]$ is factorially closed, hence also algebraically closed in B. Therefore $\ker \Delta_{\mathbf{f}} = k[f_1, \ldots, f_{n-1}]$.

Let I be the ideal generated by the image of $\Delta_{\mathbf{f}}$, namely,

$$I = (\Delta_{\mathbf{f}}(x_1), \ldots, \Delta_{\mathbf{f}}(x_n)) \ .$$

Since the images $\Delta_{\mathbf{f}}(x_i)$ are precisely the $(n-1) \times (n-1)$ minors of the jacobian matrix $\mathcal{J}(f_1, \ldots, f_{n-1})$, the foregoing proposition implies that height$(I) > 1$. Therefore, I is contained in no principal ideal other than B itself, and $\Delta_{\mathbf{f}}$ is irreducible. $\square$

This, of course, has application to the locally nilpotent case, as we will see. However, not all derivations meeting the conditions of this corollary are locally nilpotent. For example, it was pointed out in *Chap. 1* that $k[x^2 - y^3]$ is factorially closed in $k[x, y] = k^{[2]}$, but is not the kernel of any locally nilpotent derivation of $k[x, y]$.

Another key fact about Jacobians is given by van den Essen.

Proposition 3.16 (1.2.9 of [142]) *Let k be a field of characteristic zero and let $F = (F_1, \ldots, F_n)$ for $F_i \in k[x_1, \ldots, x_n] = k^{[n]}$. Then the rank of $\mathcal{J}(F)$ equals* $\mathrm{tr.deg}_k k(F)$. Here, the **rank** of the jacobian matrix is defined to be the maximal order of a nonzero minor of $\mathcal{J}(F)$.

Remark 3.17 It was observed that the jacobian determinant of a system of variables in a polynomial ring is always a unit of the base field. This fact gives a method to construct locally nilpotent derivations of polynomial rings, as follows. Let $B = k[x_1, \ldots, x_n] = k^{[n]}$ for $n \geq 2$. Given i with $1 \leq i \leq n-1$, let $K = k(x_1, \ldots, x_i)$, and suppose $f_{i+1}, \ldots, f_{n-1} \in B$ satisfy $K[x_{i+1}, \ldots, x_n] = K[f_{i+1}, \ldots, f_{n-1}, g]$ for some $g \in B$. Define $D \in \mathrm{Der}_k(B)$ by

$$D = \Delta_{(x_1, \ldots, x_i, f_{i+1}, \ldots, f_{n-1})}$$

and let E denote the extension of D to $K[x_{i+1}, \ldots, x_n]$. Since $E(f_j) = 0$ for each j and $E(g) \in K^*$, it follows that E is locally nilpotent. Therefore, D (being a restriction of E) is also locally nilpotent.

Example 3.18 Let $B = \mathbb{C}[x, y, z, u] = \mathbb{C}^{[4]}$, and define:

$$p = yu + z^2 \ , \ v = xz + yp \ , \ w = x^2u - 2xzp - yp^2$$

The **Vénéreau polynomials** are $f_n := y + x^n v$, $n \geq 1$. The preceding remark can be used to prove that f_n is an x-variable of B when $n \geq 3$.

First, define a $\mathbb{C}(x)$-derivation θ of $\mathbb{C}(x)[y, z, u]$ by

$$\theta y = 0 \ , \ \theta z = x^{-1}y \ , \ \theta u = -2x^{-1}z$$

noting that $\theta p = 0$. Then:

$$y = \exp(p\theta)(y)\ ,\ \ v = \exp(p\theta)(xz) \quad \text{and} \quad w = \exp(p\theta)(x^2u)$$

It follows that, for all $n \geq 1$:

$$\mathbb{C}(x)[y,z,u] = \mathbb{C}(x)[y,v,w] = \mathbb{C}[f_n,v,w]$$

Next, assume $n \geq 3$, and define a derivation d of B by $d = \Delta_{(x,v,w)}$. Since $\mathbb{C}(x)[y,v,w] = \mathbb{C}(x)[y,z,u]$, it follows from the preceding remark that d is locally nilpotent. And since $dx = dv = 0$, we have that $x^{n-3}vd$ is also locally nilpotent. In addition, it is easily checked that $dy = x^3$. Therefore:

$$\exp(x^{n-3}vd)(x) = x \quad \text{and} \quad \exp(x^{n-3}vd)(y) = y + x^{n-3}vd(y) = y + x^n v = f_n$$

Set $P_n = \exp(x^{n-3}vd)(z)$ and $Q_n = \exp(x^{n-3}vd)(u)$. Then $\mathbb{C}[x,f_n,P_n,Q_n] = \mathbb{C}[x,y,z,u]$.

The Vénéreau polynomials are further explored in *Sect. 10.3* below.

3.2.4 Homogenizing a Derivation

Suppose $B = k[x_1,\dots,x_n] = k^{[n]}$, and $D \in \mathrm{Der}_k(B)$ is given, $D \neq 0$. Set $A = \ker D$. Write $Dx_i = f_i(x_1,\dots,x_n)$ for $f_i \in B$, and set $d = \max_i \deg(Dx_i)$, where degrees are taken relative to the standard $\mathbb{Z}$-grading of B. The **homogenization** of D is the derivation $D^H \in \mathrm{Der}_k(B[w])$ defined by

$$D^H(w) = 0 \quad \text{and} \quad D^H(x_i) = w^d f_i(\tfrac{x_1}{w},\dots,\tfrac{x_n}{w})$$

where w is an indeterminate over B. Note that D^H is homogeneous of degree $d-1$, relative to the standard grading of $B[w]$, and $D^H \bmod (w-1) = D$ as derivations of B. In addition, if D is (standard) homogeneous to begin with, then $D^H(x_i) = Dx_i$ for every i.

In order to give further properties of D^H relative to D, we first extend D to the derivation $\mathcal{D} \in \mathrm{Der}_k(B[w,w^{-1}])$ defined by $\mathcal{D}b = Db$ for $b \in B$, and $\mathcal{D}w = 0$. Note that $\ker \mathcal{D} = A[w,w^{-1}]$, and that if $D \in \mathrm{LND}(B)$, then $\mathcal{D} \in \mathrm{LND}(B[w,w^{-1}])$.

Next, define $\alpha \in \mathrm{Aut}_k(B[w,w^{-1}])$ by $\alpha(x_i) = \frac{x_i}{w}$ and $\alpha(w) = w$, noting that $\alpha\mathcal{D}\alpha^{-1} \in \mathrm{Der}_k(B[w,w^{-1}])$. In particular:

$$\alpha\mathcal{D}\alpha^{-1}(x_i) = \alpha\mathcal{D}(wx_i) = w\alpha(Dx_i) = wf_i(\tfrac{x_1}{w},\dots,\tfrac{x_n}{w})$$

Therefore, $w^{d-1} \cdot \alpha\mathcal{D}\alpha^{-1}(x_i) = D^H(x_i)$, that is, D^H equals the restriction of $w^{d-1}\alpha\mathcal{D}\alpha^{-1}$ to $B[w]$. From this we conclude that D^H has the following properties.

1. D^H is homogeneous of degree $d-1$ in the standard $\mathbb{Z}$-grading of $B[w]$.
2. $\ker(D^H) = \ker(\alpha\mathcal{D}\alpha^{-1}) \cap B[w] = \alpha(A[w, w^{-1}]) \cap B[w]$
3. If $p : B[w] \to B$ is evaluation at $w = 1$, then $p(\ker D^H) = \ker D$.
4. If D is irreducible, then D^H is irreducible.
5. If $D \in \mathrm{LND}(B)$, then $D^H \in \mathrm{LND}_w(B[w])$.

Since $D^H \equiv D$ modulo $(w-1)$, the assignment $D \mapsto D^H$ is an injective function from $\mathrm{LND}(B)$ into the subset of standard homogeneous elements of $\mathrm{LND}_w(B[w])$. This is not, however, a bijective correspondence, since D^H will never be of the form wE for $E \in \mathrm{LND}_w(B[w])$.

Homogenizations are used in *Chap.* 8 to calculate kernel elements of D, where property (3) above is especially important.

3.2.5 Other Base Rings

Observe that many of the definitions given for $k^{[n]}$ naturally generalize to the rings $A^{[n]}$ for non-fields A. In this case, we simply include the modifier **over** A. For example, if $B = A[x_1, \ldots, x_n]$, we refer to **variables of B over A** as those $f \in B$ such that $B = A[f]^{[n-1]}$. Likewise, **partial derivatives over A**, **jacobian derivations over A**, **linear derivations over A**, and **triangular derivations over A** are defined as elements of $\mathrm{Der}_A(B)$ in the obvious way.

For example, let A be a commutative k-domain and $A[x_1, \ldots, x_n] = A^{[n]}$. Given $D \in \mathrm{Der}_A(A[x_1, \ldots, x_n])$, the **divergence** of D over A is defined by

$$\mathrm{div}_A(D) = \sum_{i=1}^{n} \partial_{x_i}(Dx_i)$$

where $\partial_{x_i}(A) = 0$ and $\partial_{x_i}(x_j) = \delta_{ij}$ for each i, j. Nowicki [333] defines D to be **special** if $\mathrm{div}_A(D) = 0$. When D is locally nilpotent, we have:

Proposition 3.19 ([142], Prop. 1.3.51; [22], Prop. 2.8) $\mathrm{div}_A(D) = 0$ *for every* $D \in \mathrm{LND}_A(A[x_1, \ldots, x_n])$.

3.3 Locally Nilpotent Derivations of Polynomial Rings

One fundamental fact about locally nilpotent derivations of polynomial rings is the following, which is due to Makar-Limanov (Lemma 8 of [276]).

Theorem 3.20 (Makar-Limanov Theorem) *Let $D \in \mathrm{LND}(B)$ be irreducible, where $B = k^{[n]}$. Let $f_1, \dots, f_{n-1}$ be algebraically independent elements of* $\ker D$*, and set $\mathbf{f} = (f_1, \dots, f_{n-1})$. Then there exists $a \in \ker D$ such that $\Delta_{\mathbf{f}} = aD$. In particular, $\Delta_{\mathbf{f}} \in \mathrm{LND}(B)$.*
In case $n \leq 3$, even stronger properties hold; see *Theorem 5.9* below.

The proof below follows that of Makar-Limanov, using the lemmas proved earlier concerning jacobian derivations.

Proof Let S be the set of nonzero elements of $A = \ker D$, and let K be the field $S^{-1}A$. Then D extends to a locally nilpotent derivation $S^{-1}D$ of $S^{-1}B$. By *Principle 13*, we have that $K = \ker(S^{-1}D)$, and $S^{-1}B = K[r] = K^{[1]}$ for some local slice r of D. Therefore $(S^{-1}B)^* = K^*$.

Extend D to a derivation D' on all $\mathrm{frac}(B)$ via the quotient rule. (Note: D' is not locally nilpotent.) From *Corollary 1.29*, we have that $\ker D' = K$.

By *Lemma 3.11*, there exists $\eta \in \mathrm{frac}(B)$ such that $D' = \eta \Delta_{\mathbf{f}}$. Note that $\Delta_{\mathbf{f}}$ restricts to a derivation of B.

Suppose $\eta = b/a$ for $a, b \in B$ with $\gcd(a, b) = 1$. Write $\Delta_{\mathbf{f}} = c\delta$ for $c \in B$ and irreducible $\delta \in \mathrm{Der}_k(B)$. Then $aD = bc\delta$, and by *Proposition 2.3* we have that $(a) = (bc)$. Since $\gcd(a, b) = 1$, this means $b \in B^*$, so we may just as well assume $b = 1$. Therefore, $\Delta_{\mathbf{f}} = aD$. The key fact to prove is that $a \in \ker D$.

Let $g_1, \dots, g_n \in S^{-1}B$ be given, and consider the jacobian determinant $\det \mathcal{J}(g_1, \dots, g_n) \in \mathrm{frac}(B)$. We claim that $\det \mathcal{J}(g_1, \dots, g_n)$ is contained in the principal ideal $aS^{-1}B$ of $S^{-1}B$.

Since $S^{-1}B = K[r]$, each g_i can be written as a finite sum $g_i = \sum a_{ij} r^j$ for $a_{ij} \in K$ and $j \geq 0$. Therefore, $\det \mathcal{J}(g_1, \dots, g_n)$ is a sum of functions of the form $\det \mathcal{J}(a_1 r^{e_1}, \dots, a_n r^{e_n})$ for $a_i \in K$ and $e_i \geq 0$. By the product rule, for each i we also have:

$$\det \mathcal{J}(a_1 r^{e_1}, \dots, a_n r^{e_n}) =$$
$$a_i \det \mathcal{J}(a_1 r^{e_1}, \dots, r^{e_i}, \dots, a_n r^{e_n}) + r^{e_i} \det \mathcal{J}(a_1 r^{e_1}, \dots, a_i, \dots, a_n r^{e_n})$$

So $\det \mathcal{J}(g_1, \dots, g_n)$ may be expressed as a sum of functions of the form $q \det \mathcal{J}(b_1, \dots, b_n)$, where $q \in S^{-1}B$, and either $b_i \in K$ or $b_i = r^{e_i}$ for $e_i \geq 1$. If every $b_i \in K$, then $b_1, \dots, b_n$ are linearly dependent, and this term will be zero. Likewise, if $b_i = r^{e_i}$ and $b_j = r^{e_j}$ for $i \neq j$, then $b_1, \dots, b_n$ are linearly dependent, and this term is zero. Therefore, by re-ordering the b_i if necessary, any nonzero summand $q \det \mathcal{J}(b_1, \dots, b_n)$ is of the form $q \det \mathcal{J}(a_1, \dots, a_{n-1}, r^e) = q\Delta_{\mathbf{a}}(r^e)$, where $q \in S^{-1}B$, $a_i \in K$, $\mathbf{a} = (a_1, \dots, a_{n-1})$, and $e \geq 1$. By *Lemma 3.12*, there exists $h \in \ker \Delta_{\mathbf{f}} = K$ such that $\Delta_{\mathbf{a}} = h\Delta_{\mathbf{f}}$ for some $h \in K$. In particular, $\Delta_{\mathbf{a}}$ restricts to $S^{-1}B$. Since $\Delta_{\mathbf{f}}(y) \in aB$ for all $y \in B$, it follows that $q\Delta_{\mathbf{a}}(r^e) \in ahS^{-1}B = aS^{-1}B$ (since h is a unit). Since $\det \mathcal{J}(g_1, \dots, g_n)$ is a sum of such functions, we conclude that $\det \mathcal{J}(g_1, \dots, g_n) \in aS^{-1}B$ for *any* $g_1, \dots, g_n \in S^{-1}B$, as claimed.

In particular, if $B = k[x_1, \dots, x_n]$, then $1 = \det \mathcal{J}(x_1, \dots, x_n) \in aS^{-1}B$, implying that $a \in (S^{-1}B)^* = K^*$. But this means $a \in B \cap K = \ker D$. □

Makar-Limanov generalized this result in [282] to give a description of the locally nilpotent derivations of any commutative affine $\mathbb{C}$-domain. He writes that his goal is "to give a standard form for an lnd on the affine domains. This form is somewhat analogous to a matrix representation of a linear operator" (p. 2). The theorem he proves is the following.

Theorem 3.21 (Generalized Makar-Limanov Theorem) *Let I be a prime ideal of $B = \mathbb{C}^{[n]}$, and let R be the factor ring B/I, with standard projection $\pi : B \to R$. Given $D \in \mathrm{LND}(R)$, there exist elements $f_1, \ldots, f_{n-1} \in B$ and nonzero elements $a, b \in R^D$ such that, for every $g \in B$:*

$$aD(\pi(g)) = b\pi(\det \mathcal{J}(f_1, \ldots, f_{n-1}, g))$$

Another way to express the conclusion of this theorem is that $aD = b\Delta_{\mathbf{f}}/I$, where $\mathbf{f} = (f_1, \ldots, f_{n-1})$. The reader is referred to Makar-Limanov's paper for the general proof.

The Makar-Limanov Theorem implies the following.

Corollary 3.22 (Prop. 1.3.51 of [142]) *If $B = k^{[n]}$ and $D \in \mathrm{LND}(B)$, then* $\mathrm{div}(D) = 0$.

Proof Choose algebraically independent $f_1, \ldots, f_{n-1} \in \ker D$. There exists an irreducible $\delta \in \mathrm{LND}(B)$ and $c \in \ker D$ such that $D = c\delta$. According to the theorem above, there also exists $a \in \ker D$ such that $a\delta = \Delta_{\mathbf{f}}$. Therefore, $D = (c/a)\Delta_{\mathbf{f}}$, so by the product rule, together with *Lemma 3.13*, we have:

$$\mathrm{div}(D) = (c/a)\mathrm{div}(\Delta_{\mathbf{f}}) + \sum_i \partial_{x_i}(c/a)\Delta_{\mathbf{f}}(x_i) = 0 + \Delta_{\mathbf{f}}(c/a) = 0$$

□

The next two results are due to Daigle.

Lemma 3.23 (Prop. 1.2 of [70]) *Let B be a commutative k-domain and A a subalgebra such that B has transcendence degree 1 over A. If $D, E \in \mathrm{Der}_A(B)$, then there exist $a, b \in B$ for which $aD = bE$.*

Proof Let $K = \mathrm{frac}(A)$ and $L = \mathrm{frac}(B)$. By *Proposition 3.6*, the dimension of $\mathrm{Der}_K(L)$ as a vector space over L is equal to one. Therefore, if S is the set of nonzero elements of B, then $S^{-1}D$ and $S^{-1}E$ are linearly dependent over K, and consequently $aD = bE$ for some $a, b \in B$. □

Proposition 3.24 (Cor. 2.5 of [70]) *Suppose $B = k^{[n]}$, and $D \in \mathrm{LND}(B)$ has* $\ker D \cong k^{[n-1]}$. *If* $\ker D = k[f_1, \ldots, f_{n-1}]$ *and* $\mathbf{f} = (f_1, .., f_{n-1})$*, then $\Delta_{\mathbf{f}}$ is irreducible and locally nilpotent, and $D = a\Delta_{\mathbf{f}}$ for some $a \in \ker D$.*

Proof Let $A = \ker D$. Since A is factorially closed, the fact that $\Delta_{\mathbf{f}}$ is irreducible follows from *Corollary 3.15* above. By *Lemma 3.23*, there exist $a, b \in B$ such that $bD = a\Delta_{\mathbf{f}}$, since D and $\Delta_{\mathbf{f}}$ have the same kernel. We may assume $\gcd(a, b) = 1$. Then $\Delta_{\mathbf{f}}B \subset bB$, implying that b is a unit. So we may assume $b = 1$. The fact that $\Delta_{\mathbf{f}}$ is locally nilpotent and $a \in A$ now follows from *Principle 7*. □

In the other direction, we would like to know whether, if $\mathbf{f} = (f_1, \ldots, f_{n-1})$ for $f_i \in B$, the condition that $\Delta_{\mathbf{f}}$ is irreducible and locally nilpotent always implies $\ker \Delta_{\mathbf{f}} = k[f_1, \ldots, f_{n-1}]$. But this is a hard question. For example, the truth of this property for $n = 3$ would imply the truth of the two-dimensional Jacobian Conjecture!

To see this, we refer to Miyanishi's Theorem in *Chap. 5*, which asserts that the kernel of any nonzero locally nilpotent derivation of $k^{[3]}$ is isomorphic to $k^{[2]}$. Suppose $A = k[f, g]$ is the kernel of a locally nilpotent derivation of $k^{[3]}$. Let $u, v \in k[f, g]$ have the property that $\det \frac{\partial(u,v)}{\partial(f,g)}$ is a nonzero constant. We have

$$\Delta_{(u,v)} = \det \frac{\partial(u, v)}{\partial(f, g)} \Delta_{(f,g)}$$

which we know to be irreducible and locally nilpotent. If the above property were true, it would follow that $A = \ker \Delta_{(u,v)} = k[u, v]$.

Proposition 3.25 (Lemma 3 of [159]) *Suppose $B = k^{[n]}$ and $D \in \mathrm{Der}_k(B)$ is linear relative to the coordinate system $(x_1, \ldots, x_n)$ on B. Let V be the vector space $V = kx_1 \oplus \cdots \oplus kx_n$. Then $\mathrm{rank}(D)$ equals the rank of D as a linear operator on V.*

Proof Suppose that $\mathrm{corank}(D) = m$, and let η denote the nullity of D as a linear operator on V. Let $F = (f_1, \ldots, f_n)$ be a system of variables on B for which $f_1, \ldots, f_m \in \ker D$. Suppose that the standard $\mathbb{N}$-grading of B is given by $B = \oplus_{j\in\mathbb{N}} B_j$ and let $f_i = \sum_{j\in\mathbb{N}} (f_i)_j$. Since $Df_1 = \cdots = Df_m = 0$, we also have $D(f_1)_1 = \cdots = D(f_m)_1 = 0$. By (3.1), $(f_1)_1, \ldots, (f_m)_1$ are linearly independent. It follows that $\eta \geq m$.

Conversely, let $v_1, \ldots, v_\eta \in V$ be linearly independent vectors annihilated by D. Since $(v_1, \ldots, v_\eta)$ is a partial system of variables on B, it follows that $\eta \leq m$. □

In his thesis, Wang [414] (Lemma 2.3.5) gives the equivalent statement: With the notation and hypotheses of the proposition above:

$$\dim_k(V \cap \ker D) = \mathrm{corank}(D)$$

3.4 Slices in Polynomial Rings

The general topic of slices for locally nilpotent derivations is covered in *Chap. 10*. For polynomial rings, we have the following basic result.

Proposition 3.26 *Suppose $B = k^{[n]}$ and $D \in \mathrm{LND}(B)$ has $Ds = 1$ for $s \in B$.*

(a) *s is a variable of $B[w] = k^{[n+1]}$.*

(b) *If $B/sB = k^{[n-1]}$, then D is a partial derivative.*

Proof Let $A = \ker D \subset B$. By the Slice Theorem, $B = A[s]$ and $\pi_s(B) = A$, where π_s is the Dixmier map defined by s. Let $B[w] = B^{[1]}$ and extend D to $D^* \in$

LND($B[w]$) by setting $D^*w = 0$. Then $\ker D^* = A[w]$. Since w is transcendental over A, we have $A[w] \cong A[s] = B = k^{[n]}$. So there exist $g_1, \ldots, g_n \in B[w]$ such that $A[w] = k[g_1, \ldots, g_n]$. Therefore,

$$B[w] = A[s][w] = A[w][s] = k[g_1, \ldots, g_n, s] = k^{[n+1]}$$

and s is a variable of $B[w]$.

In addition, we have that $A \cong B/sB$ by the Slice Theorem. Thus, if $B/sB = k^{[n-1]}$, then $B = A[s]$ implies that s is a variable of B. □

Note that the condition of part (b) holds if s is a variable. Part (a) appears as part of the proof of Thm. 1.2 in [283]. But it clearly deserves to be highlighted. A crucial question is:

If $s \in B$ is a variable of $B[w]$, does it follow that follow that s is a variable of B?

A negative answer to this question would imply a negative solution to either the Embedding Problem or Cancellation Problem. A potential example of such phenomena is provided by the Vénéreau polynomial $f_1 \in \mathbb{C}^{[4]}$: This is known to be a variable of $\mathbb{C}^{[5]}$, but it is an open question whether it is a variable of $\mathbb{C}^{[4]}$. See *Chap. 10* for details.

In summary, suppose $D \in \text{LND}(k^{[n]})$ has a slice s. Then:

1. $\ker D$ is n-generated.
2. The trivial extension D^* of D to $k^{[n+1]}$ has $\ker D^*$ is n-generated.
3. If s is a variable of $k^{[n]}$, then $\ker D$ is $(n-1)$-generated.

The following result concerns systems of local slices in a ring; δ_{ij} denotes the Kronecker delta.

Proposition 3.27 *B is a commutative k-domain. Suppose that there exist $D_1, \ldots, D_n \in \text{LND}(B)$ and $s_1, \ldots, s_n \in B$ such that, for $1 \le i \le n$:*

1. $[D_i, D_j] = 0$
2. $D_i s_j = \delta_{ij}$

Then $B = A[s_1, \ldots, s_n] = A^{[n]}$, where $A = \bigcap_{1 \le i \le n} \ker D_i$.

Proof We proceed by induction on n. The case $n = 1$ follows from the Slice Theorem. Assume that $n \ge 2$ and that $B = C[s_1, \ldots, s_{n-1}] = C^{[n-1]}$, where $C = \bigcap_{1 \le i \le n-1} \ker D_i$. Note that $s_n \in C$. In addition, since $[D_i, D_n] = 0$ for $1 \le i \le n$, it follows that D_n restricts to C. Since $D_n s_n = 1$, the Slice Theorem implies $C = A[s_n] = A^{[1]}$. Therefore, $B = A[s_1, \ldots, s_n] = A^{[n]}$. □

As an application, we have the following.

Corollary 3.28 *Let $A = k[x_1, \ldots, x_n] = k^{[n]}$, and let $\bar{k}$ be the algebraic closure of k. If $y_1, \ldots, y_n \in A$ are such that $\bar{k}[x_1, \ldots, x_n] = \bar{k}[y_1, \ldots, y_n]$, then $k[x_1, \ldots, x_n] = k[y_1, \ldots, y_n]$.*

Proof Since $A = k[x_1,\dots,x_n]$, we have:

$$\bar{A} := \bar{k} \otimes_k A = \bar{k}[x_1,\dots,x_n] = \bar{k}[y_1,\dots,y_n] = \bar{k}^{[n]}$$

Define the jacobian derivations $D_1,\dots,D_n$ of $\bar{A}$ by:

$$D_i f = \frac{\partial(y_1,\dots,\hat{y}_i,\dots,y_n,f)}{\partial(x_1,\dots,x_n)} \quad (f \in \bar{A})$$

Note that D_i restricts to A. If $c_i = D_i y_i$, then $c_i \in \bar{k}^* \cap A$ for each i; see *Sect. 3.2.3*. Since k is algebraically closed in A, we have $\bar{k}^* \cap A = k^*$, so $c_i \in k^*$. In addition, $D(c_i^{-1}y_i) = 1$ and $D_i y_j = 0$ for $i \neq j$, and $[D_i, D_j] = 0$ for every i,j. Since each $c_i^{-1}y_i$ belongs to A, *Proposition 3.27* implies that $A = k[c_1^{-1}y_1,\dots,c_n^{-1}y_n] = k[y_1,\dots,y_n]$. □

3.5 Triangular Derivations and Automorphisms

Fix a coordinate system $B = k[x_1,\dots,x_n]$. Define subgroups $H_i, K_i \subset BA_n(k)$, $i = 1,\dots,n$, by:

$$H_i = \{h \in BA_n(k) | h(x_j) = x_j, 1 \leq j \leq n-i\}$$
$$K_i = \{g \in BA_n(k) | g(x_j) = x_j, i+1 \leq j \leq n\} = BA_i(k)$$

Then for each i, K_i acts on H_i by conjugation, and $BA_n(k) = H_i \rtimes K_{n-i}$.

Proposition 3.29 *Suppose $B = k^{[n]}$ and $D \in \mathrm{Der}_k(B)$ is triangular in some coordinate system. Then $D \in \mathrm{LND}(B)$. In addition, if $n \geq 2$, then $\mathrm{rank}(D) \leq n-1$.*

Proof We argue by induction on n for $n \geq 1$, the case $n = 1$ being obvious. For $n \geq 2$, note that since D is triangular, D restricts to a triangular derivation of $k[x_1,\dots,x_{n-1}]$. By induction, D is locally nilpotent on this subring. In particular, $Dx_n \in k[x_1,\dots,x_{n-1}] \subset \mathrm{Nil}(D)$, which implies $x_n \in \mathrm{Nil}(D)$. Therefore, D is locally nilpotent on all B.

Now suppose $n \geq 2$. If $Dx_1 = 0$ we are done, so assume $Dx_1 = c \in k^*$. Choose $f \in k[x_1]$ so that $Dx_2 = f'(x_1)$. Then $D(cx_2 - f(x_1)) = 0$, and $cx_2 - f(x_1)$ is a triangular variable of B. □

We next describe the factorization of triangular automorphisms into unipotent and semi-simple factors. (See [123] for a related result.)

Proposition 3.30 *Every triangular automorphism of $k^{[n]}$ is of the form $\exp T \circ L$, where L is a diagonal matrix and T is a triangular derivation.*

Proof If $F \in BA_n(k)$, then $F \circ L$ is unipotent triangular for some diagonal matrix L. So it suffices to assume F is unipotent, i.e., of the form

$$F = (x_1, x_2 + f_2(x_1), x_3 + f_3(x_1,x_2), \dots, x_n + f_n(x_1,\dots,x_{n-1}))$$

for polynomials f_i. We show by induction on n that the map $F - I = (0, f_2, \ldots, f_n)$ is locally nilpotent, the case $n = 1$ being obvious. (Observe that $(F - I)(c) = 0$ for $c \in k$.)

Let $A = k[x_1, \ldots, x_{n-1}]$, and suppose by induction that $F - I$ restricts to a locally nilpotent map on A. Then it suffices to show that $F - I$ is nilpotent at every polynomial of the form ax_n^t ($a \in A$). One easily obtains the formula:

$$(F - I)^m(ax_n^t) = (F - I)^m(a)x_n^t + (\text{lower } x_n \text{ terms})$$

By induction, $(F - I)^m(a) = 0$ for $m \gg 0$. Since the x_n-degree is thus lowered, we eventually obtain $(F - I)^M(ax_n^t) = 0$ for $M \gg 0$. It follows that $F - I$ is locally nilpotent on all B. Thus, *Proposition 2.57* implies $F = \exp D$ for $D = \log(I + (F - I)) \in \mathrm{LND}(B)$. □

Observe that, for triangular derivations D_1, D_2 of $B = k^{[n]}$, $D_1 + D_2$ is again triangular, hence locally nilpotent. In general, however, the triangular derivations D_1 and D_2 do not commute, and $\exp D_1 \exp D_2 \neq \exp(D_1 + D_2)$. Nonetheless, the product on the left is an exponential automorphism.

Corollary 3.31 *If D_1 and D_2 are triangular k-derivations of $B = k^{[n]}$, then there exists a triangular k-derivation E of B such that:*

$$\exp D_1 \exp D_2 = \exp E$$

Proof Since $\exp D_1 \exp D_2$ is triangular, it equals $\exp E \circ L$ for triangular E and diagonal L; see *Proposition 3.30*. It is clear that in this case $L = I$ (identity). □

See also the proof of Cor. 3 in [123].

The main theorem of this section is the following.

Theorem 3.32 *If $F \in BA_n(k)$ has finite order, then there exists $L \in GL_n(k)$ and a triangular $D \in \mathrm{LND}(B)$ such that $F = \exp(-D)L\exp D$.*

The linearizability of finite-order triangular automorphisms was first proved by Ivanenko in [219]. The proof presented below makes use of exponential automorphisms to give a shorter demonstration. Whether a general element of finite order in $GA_n(k)$ can be linearized remains an open problem.

The proof of the theorem is based on the following more general fact.

Proposition 3.33 *Let R be a UFD containing k, let $D \in \mathrm{LND}(R)$, and let $\lambda \in \mathrm{Aut}_k(R)$ have finite order $m \geq 2$. Set $A = \ker D$ and $\gamma = \exp D \circ \lambda$. Suppose the following properties hold.*

1. *$\lambda(a) \in A$ for all $a \in A$.*
2. *$\lambda(a) = a$ for all $a \in A^*$.*
3. *γ has finite order m*

Then there exists $E \in \mathrm{LND}(R)$ such that $\ker E = A$ and $\gamma = \exp(-E)\lambda\exp E$.

Proof Write $D = f\Delta$ for irreducible $\Delta \in \mathrm{LND}(R)$ and $f \in A$. Since $\ker\Delta = \ker(\lambda^{-1}\Delta\lambda) = A$ by hypothesis (1), we conclude from *Principle 12*, together with the fact that R is a UFD and Δ is irreducible, that $\lambda^{-1}\Delta\lambda = c\Delta$ for some $c \in A^*$. By hypothesis, $\lambda(c) = c$, and thus $\lambda^{-i}\Delta\lambda^i = c^i\Delta$ for each $i \in \mathbb{Z}$. It follows that for each $i \in \mathbb{Z}$, $\lambda^{-i}D\lambda^i = \lambda^{m-i}(f)c^i\Delta$. In particular, $D = \lambda^{-m}D\lambda^m = c^mD$, so $c^m = 1$.

Set $E = g\Delta$ for undetermined $g \in A$. Then:

$$\begin{aligned}\exp(-E)\lambda\exp(E) = (\exp D)\lambda &\Leftrightarrow \exp(-E)\exp(\lambda E\lambda^{-1}) = \exp D\\ &\Leftrightarrow \exp((\lambda(g)c^{-1} - g)\Delta) = \exp(f\Delta)\end{aligned}$$

So we need to solve for $g \in A$ which satisfies the equation $f = c^{-1}\lambda(g) - g$. We find a solution $g \in \mathrm{span}_{k[c]}\{f, \lambda(f), \lambda^2(f), \ldots, \lambda^{m-1}(f)\} \subset A$. (Note that $k[c]$ is a field.)

First, if $\gamma_i := \lambda^{-i}(\exp D)\lambda^i$, then:

$$1 = \gamma^m = (\exp D \circ \lambda)^m = \gamma_m\gamma_{m-1}\cdots\gamma_2\gamma_1$$

Since $\gamma_i = \exp(\lambda^{m-i}(f)c^i\Delta)$, it follows that

$$\exp(h\Delta) = 1 \quad \text{for} \quad h = \sum_{i=1}^{m}\lambda^{m-i}(f)c^i$$

Therefore, $h = 0$, and we may eliminate $\lambda^{m-1}(f)$ from the spanning set above.

Next, for undetermined coefficients $a_i \in k[c]$, consider $g = a_1f + a_2\lambda(f) + \cdots + a_{m-1}\lambda^{m-2}(f)$. Then $c^{-1}\lambda(g) - g$ equals:

$$-a_1f + (c^{-1}a_1 - a_2)\lambda(f) + \cdots + (c^{-1}a_{m-2} - a_{m-1})\lambda^{m-2}(f) + c^{-1}a_{m-1}\lambda^{m-1}(f)$$

Since $h = 0$, we have that $c^{-1}a_{m-1}\lambda^{m-1}(f)$ equals:

$$-c^{-2}a_{m-1}f - c^{-3}a_{m-1}\lambda(f) - \cdots - c^{-(m-1)}a_{m-1}\lambda^{m-3}(f) - a_{m-1}\lambda^{m-2}(f)$$

Combining these gives that $c^{-1}\lambda(g) - g$ equals:

$$\begin{aligned}&(-a_1 - c^{-2}a_{m-1})f + (c^{-1}a_1 - a_2 - c^{-3}a_{m-1})\lambda(f) + \cdots\\ &\cdots + (c^{-1}a_{m-3} - a_{m-2} - c^{-(m-1)}a_{m-1}\lambda^{m-3}(f) + (c^{-1}a_{m-2} - 2a_{m-1})\lambda^{m-2}(f)\end{aligned}$$

So we need to solve for a_i such that $M(a_1, a_2, \ldots, a_{m-1})^T = (1, 0, \ldots, 0)^T$ for:

$$M = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & -c^{-2}\\ c^{-1} & -1 & 0 & \cdots & 0 & -c^{-3}\\ 0 & c^{-1} & -1 & \cdots & 0 & -c^{-4}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & -c^{m-1}\\ 0 & 0 & 0 & \cdots & c^{-1} & -2 \end{pmatrix}_{(m-1)\times(m-1)}$$

It is easily checked that $|M| \neq 0$. For example, replace row 2 by c^{-1}(row 1) + (row 2); then replace row 3 by c^{-1}(row 2) + (row 3); and so on. Eventually, we obtain the non-singular upper-triangular matrix:

$$N = \begin{pmatrix} -1 & 0 & 0 & \cdots & -c^{-2} \\ 0 & -1 & 0 & \cdots & -2c^{-3} \\ 0 & 0 & -1 & \cdots & -3c^{-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -m \end{pmatrix}$$

Therefore, we can solve for g, and thereby conjugate γ to λ. □

Proof of Theorem 3.32 Let m be the order of F. We have that $BA_n(k) = H_1 \rtimes K_{n-1}$, so we can write $F = hg$ for $g \in K_{n-1}$ and $h \in H_1$. Then $1 = F^m = (gh)^m = g^m h'$ for some $h' \in H_1$, which implies $g^m = h' = 1$. By induction, there exists a triangular derivation D with $Dx_n = 0$ and $\tilde{g} := \exp(-D) g \exp D \in GL_n(k) \cap K_{n-1}$. Thus, $\exp(-D) F \exp D = \tilde{h}\tilde{g}$ for $\tilde{h} := \exp(-D) h \exp D \in H_1$. So it suffices to assume from the outset that $F = hg$ for linear $g \in K_{n-1}$ and $h \in H_1$.

If $h = (x_1, \ldots, x_{n-1}, ax_n + f(x_1, \ldots, x_{n-1}))$, then:

$$h = \exp\left(f \frac{\partial}{\partial x_n}\right) \circ (x_1, \ldots, x_{n-1}, ax_n)$$

Thus, $F = \exp(f \frac{\partial}{\partial x_n}) L$, where $L = (x_1, \ldots, x_{n-1}, ax_n) g \in GL_n(k)$. Note that L restricts to $\ker(\frac{\partial}{\partial x_n}) = k[x_1, \ldots, x_{n-1}]$. By *Proposition 3.33*, the theorem now follows. □

3.6 Group Actions on $\mathbb{A}^n$

3.6.1 Terminology

Given $f \in B = k^{[n]}$, the variety in $\mathbb{A}^n$ defined by f will be denoted by $\mathcal{V}(f)$. Likewise, if $I \subset B$ is an ideal, the variety defined by I is $\mathcal{V}(I)$.

The group of algebraic automorphisms of $\mathbb{A}^n$ is anti-isomorphic to $GA_n(k)$, in the sense that $(F_1 \circ F_2)^* = F_2^* \circ F_1^*$ in $GA_n(k)$ when F_1 and F_2 are automorphisms of $\mathbb{A}^n$. Thus, we identify these two groups with one another.

If an algebraic k-group G acts algebraically on affine space $X = \mathbb{A}^n$, we also define the **rank** of the G-action exactly as rank was defined for a derivation, i.e., the least integer $r \geq 0$ for which there exists a coordinate system $(x_1, \ldots, x_n)$ on $k[X]$ such that $k[x_{r+1}, \ldots, x_n] \subset k[X]^G$.

The G-action on $X = \mathbb{A}^n$ is a **linear action** if and only if G acts by linear automorphisms. The action is a **triangular action** if and only if G acts by triangular automorphisms. And the action is a **tame action** if and only if G acts by tame automorphisms. Similarly, the action is **linearizable** if it is conjugate to a linear action, and **triangularizable** if it is conjugate to a triangular action.

The case in which the ring of invariants is a polynomial ring over k is important. For example, if H is a normal subgroup of G, and if $k[X]^H = k^{[m]}$ for some m, then G/H acts on the affine space $\mathbb{A}^m$ defined by $k[X]^H$, and this action can be quite interesting. This is the idea behind the main examples of *Chaps. 7* and *10* below.

Following are some particulars when the group $\mathbb{G}_a$ acts on affine space. Let a $\mathbb{G}_a$-action on $\mathbb{A}^n$ be given by

$$\rho : \mathbb{G}_a \times \mathbb{A}^n \to \mathbb{A}^n \quad \text{where} \quad \rho(t, \mathbf{x}) = (F_1(t, \mathbf{x}), \ldots, F_n(t, \mathbf{x}))$$

for functions F_i, and $\mathbf{x} = (x_1, \ldots, x_n)$ for coordinate functions x_i on $\mathbb{A}^n$.

- ρ is algebraic if and only if $F_i \in k[t, x_1, \ldots, x_n] \cong k^{[n+1]}$ for each i.
- ρ is linear if and only if each F_i is a linear polynomial in $x_1, \ldots, x_n$ over $k[t]$.
- ρ is triangular if and only if $F_i \in k[t, x_1, \ldots, x_i]$ for each i.
- ρ is **quasi-algebraic** if and only if $F_i(t_0, \mathbf{x}) \in k[x_1, \ldots, x_n]$ for each $t_0 \in k$ and each i. (See [387].)
- If $k = \mathbb{C}$, then ρ is **holomorphic** if and only if each F_i is a holomorphic function on $\mathbb{C}^{n+1}$.

Note that $\exp(tD)$ is a linear algebraic $\mathbb{G}_a$-action if and only if D is a linear locally nilpotent derivation (i.e., given by a nilpotent matrix), and $\exp(tD)$ is a triangular $\mathbb{G}_a$-action if and only if D is a triangular derivation. In [398], Suzuki classified the quasi-algebraic and holomorphic $\mathbb{C}^+$-actions on $\mathbb{C}^2$, and the holomorphic $\mathbb{C}^*$-actions on $\mathbb{C}^2$.

3.6.2 Translations

The simplest algebraic $\mathbb{G}_a$-action on $X = \mathbb{A}^n$ is a **translation**, meaning that for some system of coordinates $(x_1, \ldots, x_n)$, the action is given by

$$t \cdot (x_1, \ldots, x_n) = (x_1 + t, x_2, \ldots, x_n) = \exp(t\partial_{x_1}) \ .$$

Clearly, a translation is fixed-point free, and admits a geometric quotient: $X/\mathbb{G}_a = X/\!\!/\mathbb{G}_a \cong \mathbb{A}^{n-1}$.

In case $n = 1$, the locally nilpotent derivations of $k[x]$ are those of the form $c\frac{d}{dx}$ for some $c \in k$ (*Principle 8*). So translations are the only algebraic $\mathbb{G}_a$-actions on the affine line: $t \cdot x = x + tc$.

3.6.3 Planar Actions

The simplest linear $\mathbb{G}_a$-action on the plane comes from the standard representation of $\mathbb{G}_a$ on $V = \mathbb{A}^2$ via matrices:

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \quad (t \in k)$$

The algebraic quotient $V/\!/\mathbb{G}_a$ is a line $\mathbb{A}^1$. If $\pi : V \to V/\!/\mathbb{G}_a$ is the quotient map, then the fiber $\pi^{-1}(\lambda)$ over any $\lambda \in V/\!/\mathbb{G}_a$ is the line $x = \lambda$, which is a single orbit if $\lambda \neq 0$, and a line of fixed points if $\lambda = 0$. In this case, the geometric quotient $V/\mathbb{G}_a$ does not exist.

More generally, a triangular action on $\mathbb{A}^2$ is defined by

$$t \cdot (x, y) = (x, y + tf(x)) = \exp(tD)$$

for any $f(x) \in k[x]$, where $D = f(x)\partial_y$. In case $k = \mathbb{C}$, define a planar $\mathbb{G}_a$-action by the orthogonal matrices

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \quad (t \in \mathbb{C}) \ .$$

This is not an algebraic action, although it is quasi-algebraic, locally finite, and holomorphic. It is the exponential of the locally finite derivation $x\partial_y - y\partial_x$ on $\mathbb{C}[x, y]$.

3.6.4 Theorem of Deveney and Finston

Deveney and Finston showed the following fundamental property of invariant rings for $\mathbb{G}_a$-actions on affine spaces.

Theorem 3.34 ([101]) *Over the ground field $\mathbb{C}$, the quotient field of the ring of invariants of an algebraic action of $\mathbb{G}_a$ on $\mathbb{A}^n$ $(n \geq 1)$ is ruled. Equivalently, if $D \in \mathrm{LND}(\mathbb{C}^{[n]})$ and $A = \ker D$, then $\mathrm{frac}(A)$ is a ruled field.*

Suppose that $D \in \mathrm{LND}(\mathbb{C}^{[n]})$ is given, where $1 \leq n \leq 4$ and $D \neq 0$. Then $\ker D$ is a polynomial ring: The case $n = 1$ is true because $\ker D$ is an algebraically closed subring of $\mathbb{C}$; the case $n = 2$ follows from results in *Chap. 4*; and the case $n = 3$ is the content of Miyanishi's Theorem in *Chap. 5*.

When $n = 4$, there are kernels which are not polynomial rings; see *Sect. 3.8* below for examples. However, these kernels are rational over $\mathbb{C}$. To see this, let $A = \ker D$ and let $s \in B$ be a local slice. By the Deveney and Finston result,

$\text{frac}(A) = L^{(1)}$ for a subfield $L \subset A$. Since $B_{Ds} = A_{Ds}^{[1]}$ we have:

$$\left(\mathbb{C}^{(2)}\right)^{(2)} = \text{frac}(\mathbb{C}^{[4]}) = (\text{frac}(A))^{(1)} = \left(L^{(1)}\right)^{(1)} = L^{(2)}$$

We can now invoke the cancellation theorem for fields to conclude that $L \cong \mathbb{C}^{(2)}$, and therefore $\text{frac}(A) = \mathbb{C}^{(3)}$.

In this way, Deveney and Finston obtain the following corollary.

Corollary 3.35 *Over the ground field $\mathbb{C}$, the quotient field of the ring of invariants of an algebraic action of $\mathbb{G}_a$ on $\mathbb{A}^4$ is rational. Equivalently, if $D \in \text{LND}(\mathbb{C}^{[4]})$ is nonzero and $A = \ker D$, then* $\text{frac}(A) \cong_{\mathbb{C}} \mathbb{C}^{(3)}$.

3.6.5 Proper and Locally Trivial $\mathbb{G}_a$-Actions

Proper $\mathbb{G}_a$-actions on complex affine varieties were studied in the 1976 paper of Fauntleroy and Magid [151], with particular attention to surfaces. This paper, together with the examples of Winkelman given in [421], motivated a series of papers on the subject dating from 1994 by Deveney and Finston [103–110] and by Deveney, Finston and Gehrke [111]. These papers study proper and locally trivial $\mathbb{G}_a$-actions on $\mathbb{C}^n$.

Suppose that $B = \mathbb{C}^{[n]}$ and $D \in \text{LND}(B)$, let $A = \ker D$ and let

$$\sigma : \mathbb{G}_a \times \mathbb{C}^n \to \mathbb{C}^n$$

denote the $\mathbb{G}_a$-action on $\mathbb{C}^n$ associated to D. In addition, let $B[t] = B^{[1]}$ and extend D to $B[t]$ by $Dt = 0$. The following result is from [111], Thm. 2.3.

Theorem 3.36 (Properness Criterion) *σ is proper if and only if:*

$$B[\exp(tD)B] = B[t]$$

Moreover, a proper $\mathbb{G}_a$-action on $\mathbb{C}^n$ is fixed-point free and its topological orbit space is Hausdorff.

The same paper also characterizes the locally trivial actions, as follows (see [111], Thm. 2.5, Thm. 2.8).

Theorem 3.37 (Local Triviality Criterion) *The following conditions are equivalent.*

1. *σ is locally trivial.*
2. *σ is proper and B is a flat extension of A.*
3. $\text{pl}(D) \cdot B = B$

In [103], Deveney and Finston asked if the ring of invariants for a locally trivial $\mathbb{G}_a$-action on $\mathbb{C}^n$ is finitely generated. In [110], they gave an affirmative answer to this question.

Theorem 3.38 ([110], Thm. 2.1) *Let X be a factorial affine variety over $\mathbb{C}$. For any locally trivial $\mathbb{G}_a$-action on X, the invariant ring $\mathbb{C}[X]^{\mathbb{G}_a}$ is finitely generated as a $\mathbb{C}$-algebra.*

Any fixed-point free $\mathbb{G}_a$-action on $\mathbb{C}^2$ or $\mathbb{C}^3$ is a translation, due to Rentschler and Kaliman, respectively (see *Chaps. 4* and *5*). In higher dimensions this is no longer the case. The examples in *Sect. 3.8* below show that there are fixed-point free $\mathbb{G}_a$-actions on $\mathbb{C}^4$ which are not proper; proper $\mathbb{G}_a$-actions on $\mathbb{C}^5$ which are not locally trivial; and locally trivial $\mathbb{G}_a$-actions on $\mathbb{C}^5$ which are not globally trivial. Each of these examples is triangular. In [221], Question 2, Jorgenson asked: Is there a triangular $\mathbb{G}_a$-action on $\mathbb{C}^4$ that is locally trivial but not equivariantly trivial? Recently, Dubouloz, Finston and Jaradat showed the following, which gives a negative answer to this question.

Theorem 3.39 ([130]) *A proper triangular $\mathbb{G}_a$-action on $\mathbb{C}^4$ is a translation.*

It is an open question whether every proper $\mathbb{G}_a$-action on $\mathbb{C}^4$ is a translation.

3.7 $\mathbb{G}_a$-Actions Relative to Other Group Actions

A special property belonging to a $\mathbb{G}_a$-action is, in many cases, equivalent to the condition that that the action can be embedded in a larger algebraic group action. For example, homogeneity for $\mathbb{Z}$-gradings equates to an action of $\mathbb{G}_a \rtimes \mathbb{G}_m$. Another important condition to consider is symmetry. The symmetric group S_n acts naturally on the polynomial ring $k[x_1, \ldots, x_n]$ by permutation of the variables x_i.

In the first case, suppose $D \in \mathrm{LND}(B)$ is homogeneous of degree d relative to some $\mathbb{Z}$-grading of B, where B is any affine k-domain. This is equivalent to giving an algebraic action of the group $\mathbb{G}_a \rtimes \mathbb{G}_m$ on $X = \mathrm{Spec}(B)$, where the action of $\mathbb{G}_m$ on $\mathbb{G}_a = \mathrm{Spec}(k[x])$ is given by $t \cdot x = t^d x$. This is further equivalent to giving $D \in \mathrm{LND}(B)$ and an action $\mathbb{G}_m \to \mathrm{Aut}_k(B)$, $t \to \lambda_t$, such that $\lambda_t^{-1} D \lambda_t = t^d D$ for all t. The homogeneous polynomials $f \in B_i$ are the semi-invariants $f \in B$ for which $t \cdot f = t^i f$ $(t \in \mathbb{G}_m)$.

Proposition 3.40 *Under the hypotheses above, if $s \in \mathbb{G}_m$ has finite order m not dividing d, then* $\exp D \circ \lambda_s$ *is conjugate to λ_s. In particular,*

$$(\exp D \circ \lambda_s)^m = 1 \ .$$

Proof

$$\exp\left(\tfrac{s^d}{1-s^d} D\right) (\exp D) \lambda_s \exp\left(-\tfrac{s^d}{1-s^d} D\right) = \lambda_s$$

□

In particular, this result shows that any action of a finite cyclic group on $k^{[n]}$ of the form given in the proposition can be embedded in a $\mathbb{G}_m$-action.

The second result of this section is about kernels of homogeneous derivations.

Proposition 3.41 *Suppose $D \in \mathrm{LND}(B)$, $D \neq 0$, is homogeneous relative to some $\mathbb{N}$-grading $\oplus_{i\in\mathbb{N}} B_i$ of $B = k^{[n]}$. If $\ker D$ is a polynomial ring and $B_0 \cap \ker D = k$, then $\ker D = k[g_1, \dots, g_{n-1}]$ for homogeneous g_i.*

This is immediately implied by the following more general fact about positive $\mathbb{Z}$-gradings, which is due to Daigle.

Proposition 3.42 (Lemma 7.6 of [68]) *Let $A = k^{[r]}$ for $r \geq 1$ and let $A = \bigoplus_{i\in\mathbb{Z}} A_i$ be a positive $\mathbb{Z}$-grading. If $A = k[f_1, \dots, f_m]$ for homogeneous $f_i \in A$, then there is a subset $\{g_1, \dots, g_r\}$ of $\{f_1, \dots, f_m\}$ with $A = k[g_1, \dots, g_r]$.*

Proof By Corollary 3.28, it suffices to assume that the field k is algebraically closed.

Let $M = \bigoplus_{i>0} A_i$. Then M is an ideal, and since $A_0 = k$, it is a maximal ideal of A. Since A is a polynomial ring, there exist $X_1, \dots, X_r \in A$ so that $A = k[X_1, \dots, X_r]$ and $M = (X_1, \dots, X_r)$. We may assume, without loss of generality, that $f_i \in M$ for $1 \leq i \leq m$.

Consider a subset $\{g_1, \dots, g_s\}$ of $\{f_1, \dots, f_m\}$ satisfying $A = k[g_1, \dots, g_s]$ and minimal with respect to this property; in particular, $\deg g_i > 0$ for all i. Let $R = k[T_1, \dots, T_s] = k^{[s]}$ with positive $\mathbb{Z}$-grading $R = \bigoplus_{i\in\mathbb{Z}} R_i$ determined by $\deg T_i = \deg g_i$. Then the surjective k-homomorphism $e : R \to A$, $e(\varphi) = \varphi(g_1, \dots, g_s)$, is homogeneous of degree zero, and $\ker e$ is a homogeneous ideal.

If $\mathfrak{m} = (T_1, \dots, T_s)$, then $e(\mathfrak{m}) \subset M$ and $e(\mathfrak{m}^2) \subset M^2$. We thus have a well-defined mapping of k-vector spaces $\bar{e} : R/\mathfrak{m}^2 \to A/M^2$, where $\{1, \bar{T}_1, \dots, \bar{T}_s\}$ is a basis of $R/\mathfrak{m}^2$ and $\{1, \bar{X}_1, \dots, \bar{X}_r\}$ is a basis of A/M^2.

Given $F \in \ker e$, write $F = \sum F_i$ for $F_i \in R_i$. Since $\ker e$ is a homogeneous ideal, $F_i \in \ker e$ for all i. In particular, $F_0 \in \ker e$. Since $R_0 = k$ by hypothesis, we see that $F_0 \in k$. But e is a k-map, so $F_0 = 0$. It follows that $\ker e \subset \mathfrak{m}$.

If $F \notin \mathfrak{m}^2$, then $F_i \notin \mathfrak{m}^2$ for some $i \geq 1$. Therefore, there exist $c_1, \dots, c_r \in k$ not all 0 such that $F_i \equiv c_1T_1 + \cdots + c_rT_r (\mathrm{mod}\, \mathfrak{m}^2)$. By degree considerations, it follows that, if $c_j \neq 0$ and $F_i = c_1T_1 + \cdots + c_rT_r + G$ for $G \in \mathfrak{m}^2$, then $G \in k[T_1, \dots, T_{j-1}, T_{j+1}, \dots, T_r]$. Therefore:

$$F_i - c_jT_j \in k[T_1, \dots, T_{j-1}, T_{j+1}, \dots, T_s]$$

But then

$$\varphi(F_i - c_jT_j) = -c_jg_j \in k[g_1, \dots, g_{j-1}, g_{j+1}, \dots, g_s]$$

contradicting the minimality of $\{g_1, \dots, g_s\}$.

Therefore, $\ker e \subset \mathfrak{m}^2$. Consequently, there is a well-defined surjection $A = R/\ker e \to R/\mathfrak{m}^2$, which implies that, if $P_i \in R$ is such that $e(P_i) = X_i$, $1 \leq i \leq r$, then $R/\mathfrak{m}^2$ has basis $\{1, \bar{P}_1, \dots, \bar{P}_r\}$. It follows that $r = s$. □

Corollary 3.43 *If $B = k^{[n]}$ and if $\mathbb{G}_m$ acts algebraically on $\mathbb{A}^n$ in such a way that $B^{\mathbb{G}_m} = k$, then the action is linearizable. Equivalently, for any positive $\mathbb{Z}$-grading of B there exists a system of homogeneous variables for B.*

Proof The action induces a $\mathbb{Z}$-grading of B for which elements of B_i are semi-invariants of weight i. In particular, $B_0 = B^{\mathbb{G}_m}$. If $f \in B_i$ and $g \in B_j$ for $i < 0$ and $j > 0$, then $f^j g^{-i} \in B_0$, a contradiction. Therefore, we can assume any non-constant semi-invariant has strictly positive weight. So the grading on B induced by the $\mathbb{G}_m$-action is an $\mathbb{N}$-grading: $B = \oplus_{i\in\mathbb{N}} B_i$.

Suppose $B = k[x_1, \ldots, x_n]$. Given i ($1 \le i \le n$), we can write $x_i = \sum_{j\in\mathbb{N}} f_{ij}$, where $f_{ij} \in B_j$. So B is generated as a k-algebra by finitely many homogeneous polynomials f_{ij}. By the preceding result, there exist homogeneous $g_1, \ldots, g_n \in B$ such that $B = k[g_1, \ldots, g_n]$, i.e., $(g_1, \ldots, g_n)$ is a system of semi-invariant variables for B. □

Next, let $B = k^{[n]}$ and consider the standard action of the symmetric group S_n on B relative to coordinates $(x_1, \ldots, x_n)$. Define $D \in \mathrm{Der}_k(B)$ to be **fully symmetric** if and only if $D\sigma = \sigma D$ for each $\sigma \in S_n$. To give $D \in \mathrm{LND}(B)$ fully symmetric is equivalent to giving an algebraic action of $\mathbb{G}_a \times S_n$ on B or on $\mathbb{A}^n$.

Example 3.44 $E = \sum_{i=1}^n \partial_{x_i}$ is fully symmetric and locally nilpotent, and $\ker E = k[x_1 - x_2, x_2 - x_3, \ldots, x_{n-1} - x_n]$. Note that E is a partial derivative. If $f \in \ker E \cap B^{S_n}$, then fE is also fully symmetric and locally nilpotent.

Proposition 3.45 *Let $\mathbb{Z}_2$ act on $B = k[x_1, \ldots, x_n]$ by transposing x_1 and x_2, and fixing $x_3, \ldots, x_n$. If $D \in$ LND(B) commutes with this $\mathbb{Z}_2$-action, then $D(x_1 - x_2) = 0$.*

Proof Let $\tau \in \mathbb{Z}_2$ transpose x_1 and x_2, fixing $x_3, \ldots, x_n$, and let $Dx_1 = F(x_1, x_2)$ for $F \in k[x_3, \ldots, x_n]^{[2]}$. Then $Dx_2 = D(\tau x_1) = \tau Dx_1 = F(x_2, x_1)$. This implies:

$$D(x_1 - x_2) = F(x_1, x_2) - F(x_2, x_1) \in (x_1 - x_2)B$$

By *Corollary 1.23*, we conclude that $D(x_1 - x_2) = 0$. □

Now suppose D is a fully symmetric locally nilpotent derivation. Then $D(x_i - x_j) = 0$ for all i, j, so $k[x_1 - x_2, x_2 - x_3, \ldots, x_{n-1} - x_n] \subset \ker D$. Consequently, the derivations fE above are the only fully symmetric locally nilpotent derivations.

Corollary 3.46 *If $D \in \mathrm{LND}(B)$ is fully symmetric and $D \neq 0$, then* $\mathrm{rank}(D) = 1$.

Remark 3.47 The conclusion of *Proposition 3.42* may fail to hold for more general polynomial algebras. For instance, we saw in *Example 1.27* that if $U = \mathbb{R}[x_1, x_2]/(x_1^2 + x_2^2 - 1)$ and $B = U[y_1, y_2]/(x_1 y_1 + x_2 y_2)$, then $B = U[s] = U^{[1]}$. But we also have $B = U[x_1 s, x_2 s]$, a homogeneous system of generators when $\deg x_i = 0$ and $\deg y_i = 1$ for each i, whereas $B \neq U[x_i s]$ for $i = 1, 2$.

3.8 Some Important Early Examples

This section illustrates the fact that the triangular derivations of polynomial rings already provide a rich source of examples.

In 1972, Nagata [325] published an example of a polynomial automorphism of $\mathbb{A}^3$ which, he conjectured, is not tame. Later, Bass embedded Nagata's automorphism as an element of a one-parameter subgroup of polynomial automophisms of $\mathbb{A}^3$, gotten by exponentiating a certain non-linear locally nilpotent derivation of $k[x,y,z]$. It was known at the time that every unipotent group of polynomial automorphisms of the plane is triangular in some coordinate system (see *Chap. 4*). In sharp contrast to the situation for the plane, Bass showed that the subgroup he constructed could not be conjugated to the triangular subgroup. Then Popov generalized Bass's construction to produce non-triangularizable $\mathbb{G}_a$-actions on $\mathbb{A}^n$ for every $n \geq 3$. These discoveries initiated the exploration of a new world of algebraic representations $\mathbb{G}_a \hookrightarrow GA_n(k)$.

Note that for some of the examples below, we exhibit, without explanation, the kernel of the derivation under consideration. Methods for calculating these kernels are discussed in *Chap. 8* below.

3.8.1 *Bass's Example ([12], 1984)*

The example of Bass begins with the linear derivation of $k[x,y,z]$ given by $\Delta = x\partial_y + 2y\partial_z$. Then $\ker \Delta = k[x,F]$, where $F = xz - y^2$. Note that $D := F\Delta$ is also a locally nilpotent derivation of $k[x,y,z]$, and the corresponding $\mathbb{G}_a$-action on $\mathbb{A}^3$ is:

$$\alpha_t := \exp(tD) = (x, y + txF, z + 2tyF + t^2xF^2)$$

Nagata's automorphism is α_1. The fixed point set of this action is the cone $F = 0$, which has an isolated singularity at the origin. On the other hand, Bass observed that any triangular automorphism $(x, y + f(x), z + g(x,y))$ has a cylindrical fixed point set, i.e., defined by $f(x) = g(x,y) = 0$, which (if non-empty) has the form $C \times \mathbb{A}^1$ for some variety C. In general, an affine variety X is called a **cylindrical variety** if $X = Y \times \mathbb{A}^1$ for some affine variety Y. Since a cylindrical variety can have no isolated singularities, it follows that α_t cannot be conjugated into $BA_3(k)$ relative to the coordinate system (x,y,z).

3.8.2 *Popov's Examples ([344], 1987)*

Generalizing Bass's approach, Popov pointed out that the fixed-point set of any triangular $\mathbb{G}_a$-action on $\mathbb{A}^n$ is a cylindrical variety, whereas the hypersurface defined by a non-degenerate quadratic form is not a cylindrical variety. So to produce non-triangularizable examples in higher dimensions, it suffices to find $D \in \mathrm{LND}(k^{[n]})$ such that $\ker D$ contains a non-degenerate quadratic form h; then $\exp(thD)$ is a non-triangularizable $\mathbb{G}_a$-action. In even dimensions, let $B = k[x_1, \dots, x_n, y_1, \dots, y_n]$, and define D by:

$$\begin{aligned}
&Dx_1 = 0,\ Dx_2 = x_1,\ Dx_3 = x_2,\ \dots ,Dx_n = x_{n-1}\\
&Dy_1 = y_2,\ Dy_2 = y_3,\ \dots ,Dy_{n-1} = y_n,\ Dy_n = 0
\end{aligned}$$

Then D is a triangular (linear) derivation, and $Dh = 0$ for the non-degenerate quadratic form $h = \sum_{i=1}^{n}(-1)^{i+1}x_iy_i$. For odd dimensions at least 5, start with D above, and extend D to $k[x_1, \dots, x_n, y_1, \dots, y_n, z]$ by $Dz = 0$. Then $h + z^2$ is a non-degenerate quadratic form annihilated by D.

3.8.3 Smith's Example ([386], 1989)

At the conclusion of his paper, Bass asked whether the $\mathbb{G}_a$-action he gave on $\mathbb{A}^3$ is **stably tame**, i.e., whether the action becomes tame when extended trivially to $\mathbb{A}^4$. M. Smith gave a positive answer to this question by first showing the following.

Lemma 3.48 (Smith's Formula) *Let $D \in \mathrm{LND}(B)$ for $B = k^{[n]}$ and let $f \in \ker D$ be given. Extend D to $B[w]$ by $Dw = 0$, and define $\tau \in GA_{n+1}(k)$ by $\tau = \exp(f\partial_w)$. Then:*

$$\exp(fD) = \tau^{-1}\exp(-wD)\tau\exp(wD)$$

Proof Since τ fixes B, $\tau D = D\tau$, so $\tau^{-1}(-wD)\tau = \tau^{-1}(-w)D = (f - w)D$. Applying the exponential now gives:

$$\begin{aligned}
\exp(fD)\exp(-wD) &= \exp((f - w)D)\\
&= \exp\big(\tau^{-1}(-wD)\tau\big)\\
&= \tau^{-1}\exp(-wD)\tau
\end{aligned}$$

□

Applying this lemma with $f = tF$ and $D = \Delta$ from Bass's example yields the following tame factorization for the example of Bass-Nagata. For $t \in \mathbb{G}_a$:

$$\begin{aligned}
\exp(tD) &= (x, y + txF, z + 2tyF + t^2xF^2, w)\\
&= (x, y, z, w - tF) \circ (x, y - wx, z - 2wy + w^2x, w)\\
&\quad \circ(x, y, z, w + tF) \circ (x, y + wx, z + 2wy + w^2x, w)
\end{aligned}$$

Lemma 3.49 *This $\mathbb{G}_a$-action on $\mathbb{A}^4$ is not triangularizable.*

Proof Note first that the rank of D on $k^{[4]}$ is clearly 2. Let $X \subset \mathbb{A}^4$ be the set of fixed points. Then $X = C \times \mathbb{A}^1$ for a singular cone C, and the singularities of X form a line. Suppose $k[x, y, z, w] = k[a, b, c, d]$ and that D is triangular in the latter system of coordinates, with $Da = 0$ and $Db \in k[a]$. The ideal defining X is (Db, Dc, Dd), and thus $X \subset \mathcal{V}(Db)$. If $Db \neq 0$, this is a union of parallel coordinate hyperplanes, implying $X \subset H$ for a coordinate hyperplane H. Since this is clearly impossible, $Db = 0$. We also have $X \subset \mathcal{V}(Dc)$, where $Dc \in k[a, b]$. If $Dc \neq 0$, this implies $X = Y \times \mathbb{A}^2$, where Y is a component of the curve in Spec$(k[a, b])$ defined by Dc. But this also cannot occur, since then the singularities of X would be of dimension 2. Thus, $Dc = 0$. But this would imply that the rank of D is 1, a contradiction. Therefore, D extended to $k[x, y, z, w]$ cannot be conjugated to a triangular derivation by any element of $GA_4(k)$. □

So in dimension 4 (and likewise in higher dimensions), there exist $\mathbb{G}_a$-actions which are tame but not triangularizable. It is an important open question whether every tame $\mathbb{G}_a$-action on $\mathbb{A}^3$ can be triangularized. It goes to the structure of the tame subgroup. Shestakov and Umirbaev [382, 383] have shown that the Nagata automorphism α_1 above is *not* tame as an element of $GA_3(k)$, thus confirming the conjecture of Nagata. In [428], Wright gives a structural description of $TA_3(k)$ as an amalgamation of three of its subgroups.

3.8.4 Winkelmann's Example 1 ([421], 1990)

In this groundbreaking paper, Winkelmann investigates $\mathbb{C}^+$-actions on $\mathbb{C}^n$ which are fixed-point free, motivated by questions about their quotients. In dimension 4, he defines $\exp(tD)$, where D is the triangular derivation on $B = \mathbb{C}[x, y, z, w]$ defined by:

$$Dx = 0\,,\ \ Dy = x\,,\ \ Dz = y\,,\ \ Dw = y^2 - 2xz - 1$$

$\exp(tD)$ defines a free algebraic $\mathbb{C}^+$-action on $\mathbb{C}^4$, but the orbit space (geometric quotient) is not Hausdorff in the natural topology (Lemma 8). In particular, D is not a partial derivative, i.e., the action is not a translation, since both the geometric and algebraic quotient for a translation of $\mathbb{C}^4$ is $\mathbb{C}^3$. Winkelmann calculates this kernel explicitly: $\ker D = \mathbb{C}[x, f, g, h]$, where:

$$f = y^2 - 2xz\,,\ \ g = xw + (1-f)y \quad \text{and} \quad xh = g^2 - f(1-f)^2$$

In particular, $\ker D$ is the coordinate ring of a singular hypersurface in $\mathbb{C}^4$. This implies $\operatorname{rank}(D) = 3$, since if the rank were 1 or 2, the kernel would be a polynomial ring (see *Chap. 4*).

Let $B[t] = B^{[1]}$ and consider the subring:

$$R = B[\exp(tD)B] = B[tx, ty + \frac{1}{2}t^2x, t(f-1)]$$

If $R = B[t]$, then setting $y = 1$ and $z = w = 0$ shows:

$$\mathbb{C}[x, tx, t + \frac{1}{2}t^2x] = \mathbb{C}[x, t]$$

However:

$$\mathbb{C}[x, tx, t + \frac{1}{2}t^2x] \cong \mathbb{C}[X, Y, Z]/(XZ - Y - \frac{1}{2}Y^2)$$

This ring is evidently not a UFD, and is therefore not isomorphic to $\mathbb{C}^{[2]}$, a contradiction. Therefore, $R \neq B[t]$ and the $\mathbb{G}_a$-action defined by D is not proper.

In [388], Snow gives the similar example

$$Ex = 0\,,\ Ey = x\,,\ Ez = y\,,\ Ew = 1 + y^2$$

and also provides a simple demonstration that the topological quotient is non-Hausdorff (Example 3.5). (It is easy to show that D and E are conjugate.) In [142], van den Essen considers E, and indicates that E does not admit a slice, a condition which is *a priori* independent of the fact that the corresponding quotient is not an affine space (Example 9.5.25). And in [111], Sect. 3, Deveney, Finston, and Gehrke consider E as well, showing that the associated $\mathbb{C}^+$-action $\exp(tE)$ on $\mathbb{C}^4$ is not proper.

3.8.5 Winkelmann's Example 2 ([421], 1990)

On $B = \mathbb{C}[u, v, x, y, z] = \mathbb{C}^{[5]}$, define the triangular derivation F by:

$$Fu = Fv = 0\,,\ Fx = u\,,\ Fy = v\,,\ Fz = 1 + (vx - uy)$$

Then $Fx, Fy, Fz \in \ker F$ and $(Fx, Fy, Fz) = (1)$, which implies $\exp(tF)$ is a locally trivial $\mathbb{C}^+$-action on $\mathbb{C}^5$. The kernel of F is presented in [111], namely:

$$\ker F = \mathbb{C}[u, v, vx - uy, x + x(vx - uy) - uz, y + y(vx - uy) - vz]$$

To see that the associated $\mathbb{C}^+$-action on $\mathbb{C}^5$ is not globally trivial, note that F is homogeneous of degree 0 relative to the $\mathbb{C}^*$-action $(\lambda u, \lambda^{-1}v, \lambda x, \lambda^{-1}y, z)$, $\lambda \in \mathbb{C}^*$. We thus have an action of $\mathbb{C}^+ \times \mathbb{C}^*$ on $\mathbb{C}^5$. The invariant ring of the $\mathbb{C}^*$-action is $B_0 = \mathbb{C}[uv, xy, vx, uy, z]$, the ring of degree-0 elements. Therefore F restricts to B_0. If F has a slice in B, then by homogeneity there exists a slice $s \in B_0$. But the ideal generated by the image of F restricted to B_0 equals $(vx+uy, uv, 1+vx-uy)$, which does not contain 1, meaning F has no slice in B_0. (The fixed-point set of the induced $\mathbb{C}^+$-action on $\mathrm{Spec}(B_0)$ is of dimension one.) Therefore, F has no slice in B.

3.8.6 *Example of Deveney and Finston ([104], 1995)*

Define δ on $B = \mathbb{C}[u, v, x, y, z] = \mathbb{C}^{[5]}$ by:

$$\delta u = \delta v = 0 \ , \ \delta x = u \ , \ \delta y = v \ , \ \delta z = 1 + uy^2$$

The authors show that $\exp(t\delta)$ is a proper $\mathbb{C}^+$-action on $\mathbb{C}^5$. To see this, let $B[t] = B^{[1]}$ and consider the subring:

$$R = B[\exp(t\delta)B] = B[tu, tv, t(1 + vy^2) + t^2uvy + \tfrac{1}{3}t^3uv^2]$$

Then $R = B[t]$, since:

$$t = \big(t(1 + vy^2) + t^2uvy + \tfrac{1}{3}t^3uv^2\big) - \big((tu)y^2 + (tu)(tv)y + \tfrac{1}{3}(tu)(tv)^2\big)$$

Therefore δ defines a proper action. Deveney and Finston show that $\ker\delta$ is isomorphic to the ring

$$\mathbb{C}[u_1, u_2, u_3, u_4, u_5]/(u_2u_5 - u_1^2u_4 - u_3^3 - 3u_1u_3)$$

which is the coordinate ring of a singular hypersurface $Y \subset \mathbb{C}^5$. If $p : \mathbb{C}^5 \to Y$ is the quotient morphism, then fibers of p over singular points of Y are two-dimensional, which implies that the action is not locally trivial.

3.9 Homogeneous Dependence Problem

In a remarkable paper [184] dating from 1876, Paul Gordan and Max Nöther investigated the vanishing of the Hessian determinant of an algebraic form, using the language of systems of differential operators. In particular, the question they consider is the following. Suppose $h \in \mathbb{C}[x_1, \ldots, x_n]$ is a homogeneous polynomial whose Hessian determinant is identically zero:

$$\det\left(\frac{\partial^2 h}{\partial x_i \partial x_j}\right)_{ij} = 0$$

Does it follow that h is degenerate, i.e., that $h \in \mathbb{C}[Tx_1, \ldots, Tx_{n-1}]$ for some $T \in GL_n(\mathbb{C})$? They prove that the answer is yes when $n = 3$ and $n = 4$, and garner some partial results for the case $n = 5$.

In the course of their proof, the authors consider changes of coordinates involving a parameter $\lambda \in \mathbb{C}$:

> Die Functionen $\Phi(x)$, gebildet für die Argumente $x + \lambda\xi$, sind unabhängig von λ:
>
> $$\Phi(x + \lambda\xi) = \Phi(x) \ . \quad \text{(p. 550)}$$ [4]

Here, x denotes a vector of coordinates $(x_1, \ldots, x_n)$, and ξ a vector of homogeneous polynomials. In modern terms, the association $\lambda \cdot x = x + \lambda\xi$ gives a $\mathbb{C}^+$-action on $\mathbb{C}^n$ (where $\lambda \in \mathbb{C}$), and the functions Φ are its invariants. The authors continue:

> Ist eine solche ganze Function Φ das Product zweier ganzen Functionen
>
> $$\Phi = \phi(x) \cdot \psi(x)$$
>
> so sind auch die Factoren selbst Functionen Φ. (p. 551)[5]

We recognize this as the property that the ring of invariants of a $\mathbb{C}^+$-action is factorially closed. In effect, Gordan and Nöther studied an important type of $\mathbb{C}^+$-action on $\mathbb{C}^n$, which we will now describe in terms of derivations.

Let $B = k[x_1, \ldots, x_n] = k^{[n]}$, and let $D \in \text{LND}(B)$ be given, $D \neq 0$. The **Homogeneous Dependence Problem** for locally nilpotent derivations asks:

> If D is standard homogeneous and has the property that $D^2 x_i = 0$ for each i, is the rank of D always strictly less than n? Equivalently, does there exist a linear form $L \in B$ with $DL = 0$, i.e., are the images Dx_i linearly dependent?

For such a derivation D, note that the $\mathbb{G}_a$-action is simply

$$\exp(tD) = (x_1 + tDx_1, \ldots, x_n + tDx_n)$$

and these are precisely the kinds of coordinate changes considered by Gordan and Nöther. Note also that, given i:

$$D \circ \exp D(x_i) = D(x_i + Dx_i) = Dx_i + D^2 x_i = Dx_i$$

On the other hand, $Dx_i \in \ker D$ means that $\exp D(Dx_i) = Dx_i$. Therefore, D and $\exp D$ commute. This in turn implies that, if we write $F = \exp D = x + H$, where $x = (x_1, \ldots, x_n)$ and $H = (Dx_1, \ldots, Dx_n)$, then $H \circ H = 0$. Herein lies the connection to the work of Gordan and Nöther.

In their paper, Gordan and Nöther effectively proved that the answer to the Homogeneous Dependence Problem is yes when $n = 3$ or $n = 4$. In fact, they

[4] "The functions $\Phi(x)$, constructed for the arguments $x + \lambda\xi$, are independent of λ."

[5] "If such an entire function Φ is a product of two entire functions $\Phi = \phi(x)\psi(x)$, then so also are the factors themselves functions Φ."

showed that in these cases there exist two independent linear forms, L and M, with $DL = DM = 0$, which implies that the rank of D is 1 when $n = 3$, and at most 2 when $n = 4$.

In the modern era, Wang proved in his 1999 thesis (Prop. 2.4.4) that if $D \in \mathrm{LND}(k[x_1, x_2, x_3])$ has the property that $D^2 x_i = 0$ for each i, then $\mathrm{rank}(D) \leq 1$ [414, 415]. So in the case of dimension 3, the homogeneity condition can be removed. A short proof of Wang's result is given in *Chap. 5* below. Wang further proved that, in dimension 4, the rank of a homogeneous derivation having $D^2 x_i = 0$ for each i could not equal 3 (Lemma 2.5.2). Then in 2000, Derksen constructed an example of such a derivation D in dimension 8 whose rank is 7, thereby showing that the stronger result of Gordan and Nöther (i.e., that the kernel contains two independent linear forms) does not generalize. In 2004, de Bondt found a way to construct counterexamples to the Homogeneous Dependence Problem in all dimensions $n \geq 6$ by using derivations of degree 4. So the Homogeneous Dependence Problem remains open only for the case $n = 5$. The examples of Derksen and de Bondt are discussed below.

At the time of their work, neither Wang nor Derksen seems to have been aware of the paper of Gordan and Nöther. Rather, it is an example of an important question resurfacing. The Gordan-Nöther paper was brought to the author's attention by van den Essen, and its existence was made known to him by S. Washburn. Van den Essen was interested in its connections to his study of the Jacobian Conjecture; see [33–35, 147] for a discussion of these connections, and some positive results for this conjecture. The article of DeBondt [93] gives a modern proof of the results of Gordan and Nöther, in addition to some partial results in dimension 5.

3.9.1 Construction of Examples

We construct, for each $N \geq 8$, a family of derivations D of the polynomial ring $k[x_1, \ldots, x_N]$ with the property that $D^2 x_i = 0$ for each i. The example of Derksen belongs to this family.

Given $m \geq 1$, let $B = k[s_1, \ldots, s_m] = k^{[m]}$ and let $\delta \in \mathrm{LND}(B)$ be such that $\delta^2 s_i = 0$ for each i (possibly $\delta = 0$). Let $u \in B^\delta = \ker \delta$ be given ($u \neq 0$). Extend δ to $B[t] = B^{[1]}$ by setting $\delta t = 0$.

Next, given $n \geq 3$, choose an $n \times n$ skew-symmetric matrix M with entries in $B[t]^\delta$, i.e., $M \in \mathcal{M}_n(B[t]^\delta)$ and $M^T = -M$. Also, let $\mathbf{v} \in (B[t]^\delta)^n$ be a nonzero vector in the kernel of M.

Next, let $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$, and z be indeterminates over $B[t]$, so that $B[t, \mathbf{x}, \mathbf{y}, z] = k^{[m+2n+2]}$. Note that $m + 2n + 2 \geq 9$. Extend δ to a locally nilpotent derivation of this larger polynomial ring by setting:

$$\delta \mathbf{x} = u\mathbf{v} \ , \ \delta \mathbf{y} = \mathbf{x}M \ , \ \delta z = u^{-1}\delta(\langle \mathbf{x}, \mathbf{y} \rangle)$$

Here, it is understood that for vectors $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$, the statement $\delta\mathbf{a} = \mathbf{b}$ means $\delta a_i = b_i$ for each i. In addition, $\langle \mathbf{a}, \mathbf{b} \rangle$ denotes the inner product of $\mathbf{a}$ and $\mathbf{b}$. Observe the **product rule for inner products**::

$$\delta(\langle \mathbf{a}, \mathbf{b} \rangle) = \langle \delta\mathbf{a}, \mathbf{b} \rangle + \langle \mathbf{a}, \delta\mathbf{b} \rangle$$

It is clear from the definition that $\delta^2\mathbf{x} = 0$. In addition:

$$\delta^2\mathbf{y} = \delta(\mathbf{x}M) = (\delta\mathbf{x})M = (u\mathbf{v})M = u(\mathbf{v}M) = 0$$

Further, since M is skew-symmetric, we have $0 = \langle \mathbf{x}, \mathbf{x}M \rangle = \langle \mathbf{x}, \delta\mathbf{y} \rangle$. Therefore:

$$\delta(\langle \mathbf{x}, \mathbf{y} \rangle) = \langle \delta\mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \delta\mathbf{y} \rangle = \langle \delta\mathbf{x}, \mathbf{y} \rangle - \langle \mathbf{x}, \delta\mathbf{y} \rangle \in \ker\delta$$

It follows that δz is a well-defined polynomial (since u divides $\delta\mathbf{x}$), and $\delta^2 z = 0$. In addition, if $F = uz - \langle \mathbf{x}, \mathbf{y} \rangle$, then $\delta F = 0$.

Since F does not involve t, the kernel element $t - F$ is a variable. It follows that

$$B[t, \mathbf{x}, \mathbf{y}, z]/(t - F) = B[\mathbf{x}, \mathbf{y}, z] = k^{[m+2n+1]}$$

and that the derivation $D := \delta \bmod (t - F)$ has the property that $D^2\mathbf{x} = D^2\mathbf{y} = D^2 z = 0$.

3.9.2 *Derksen's Example*

This example appears in [142], 7.3, Exercise 6. It uses the minimal values $m = 1$ and $n = 3$ from the construction above, so that $m + 2n + 1 = 8$. Derksen found this example by considering the exterior algebra associated to three linear derivations.

First, let δ be the zero derivation of $B = k[s] = k^{[1]}$, and choose $u = s$. The extension of δ to $k[s, t]$ is also zero. Choose:

$$\mathbf{v} = \begin{pmatrix} t^2 \\ s^2 t \\ s^4 \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} 0 & s^4 & -s^2 t \\ -s^4 & 0 & t^2 \\ s^2 t & -t^2 & 0 \end{pmatrix}$$

With these choices, we get the derivation D on the polynomial ring

$$k[s, x_1, x_2, x_3, y_1, y_2, y_3, z] = k^{[8]}$$

defined by $Ds = 0$,

$$D\mathbf{x} = \begin{pmatrix} sF^2 \\ s^3F \\ s^5 \end{pmatrix} \;, \quad D\mathbf{y} = \begin{pmatrix} 0 & s^4 & -s^2F \\ -s^4 & 0 & F^2 \\ s^2F & -F^2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and $Dz = F^2y_1 + s^2Fy_2 + s^4y_3$, where F is the quadratic form $F = sz - (x_1y_1 + x_2y_2 + x_3y_3)$.

Observe that D is homogeneous, of degree 4. To check that s is the only linear form in the kernel of D (up to scalar multiples), let V_i denote the vector space of forms of degree i in these 8 variables, and let $W \subset V_5$ denote the subspace generated by the monomials appearing in the image of $D : V_1 \to V_5$. Then it suffices to verify that the linear map $D : V_1 \to W$ has a one-dimensional kernel, and this is easily done with standard methods of linear algebra. We conclude that the rank of D is 7. □

3.9.3 De Bondt's Examples

Theorem 3.50 ([92]; [93], Cor. 3.3) *For $n \geq 3$, let*

$$B = k^{[2n]} = k[x_1, y_1, \ldots, x_n, y_n]$$

and define $D \in \mathrm{Der}_k(B)$ by

$$Dx_i = fgx_i - g^2y_i \quad \text{and} \quad Dy_i = f^2x_i - fgy_i$$

where $f = x_1y_2 - x_2y_1$ and $g = x_1y_3 - x_3y_1$. Then:

(a) *D is standard homogeneous of degree 4*
(b) $f, g \in \ker D$
(c) *$D^2x_i = D^2y_i = 0$ for each i*
(d) $\mathrm{rank}(D) = 2n$

Proof Let $R = k[a, b] = k^{[2]}$ and let $N \in \mathcal{M}_2(R)$ be given by:

$$N = \begin{pmatrix} ab & -b^2 \\ a^2 & -ab \end{pmatrix}$$

Then $N^2 = 0$.

Let $\mathcal{B} = R[x_1, y_1, \ldots, x_n, y_n] = k^{[2n+2]}$. Define R-linear $\mathcal{D} \in \mathrm{LND}_R(\mathcal{B})$ by:

$$\mathcal{D} = \begin{pmatrix} N & 0 & \cdots & 0 \\ 0 & N & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N \end{pmatrix}_{2n\times 2n}$$

Then for each i, we have:

$$\mathcal{D}x_i = abx_i - b^2y_i \ , \ \ \mathcal{D}y_i = a^2x_i - aby_i \ , \quad \text{and} \quad \mathcal{D}^2x_i = \mathcal{D}^2y_i = 0$$

In addition, for every pair i, j, we have

$$\mathcal{D}(x_iy_j) = x_i(a^2x_j - aby_j) + y_j(abx_i - b^2y_i) = a^2x_ix_j - b^2y_iy_j = \mathcal{D}(x_jy_i)$$

which implies $x_iy_j - x_jy_i \in \ker \mathcal{D}$ for each pair i, j.

Set $f = x_1y_2 - x_2y_1$ and $g = x_1y_3 - x_3y_1$. The crucial observation is that f and g are kernel elements not involving a or b. Thus, $(a - f, b - g, x_1, \ldots, y_n)$ is a triangular system of coordinates on $\mathcal{B}$. If $I \subset \mathcal{B}$ is the ideal $I = (a - f, b - g)$, then $B := \mathcal{B} \bmod I$ is isomorphic to $k^{[2n]}$, and we may take $B = k[x_1, y_1, \ldots, x_n, y_n]$. Since $a - f$ and $b - g$ belong to $\ker \mathcal{D}$, the ideal I is an integral ideal of $\mathcal{D}$, and we have that $D := \mathcal{D} \bmod I$ is well-defined, locally nilpotent and homogeneous on B.

It remains to show that $\mathrm{rank}(D) = 2n$. If $Dv = 0$ for a variable $v \in B$, then by homogeneity, there exists a linear form $L = \sum(a_ix_i + b_iy_i)$ for scalars a_i, b_i such that $DL = 0$. But then $\sum(a_iDx_i + b_iDy_i) = 0$. So it suffices to show that the images $Dx_1, Dy_1, \ldots, Dx_n, Dy_n$ are linearly independent.

To this end, define a vector of univariate polynomials

$$\mathbf{t} = (t, t^2, t^3, t^4 - 1, t^5 - 1, t^6, \ldots, t^{2n})$$

noting that $f(\mathbf{t}) = -t$ and $g(\mathbf{t}) = -t^2$. Then for each i, we have:

$$\deg_t Dy_i(\mathbf{t}) = 2i + 3 \quad \text{and} \quad \deg_t Dx_i(\mathbf{t}) = 2i + 4$$

Since these degrees are all distinct for $1 \leq i \leq n$, it follows that these polynomials are linearly independent. □

Note that de Bondt's derivations are quasi-linear, in addition to being nice derivations.

In order to exhibit an example in odd dimension $2n + 1$ for $n \geq 3$, let $k^{[2n+1]} = B[z]$, and extend D to this ring. In particular, Dz should satisfy: (1) $Dz \in \ker D$, (2) $\deg Dz = 5$, and (3) Dz is not in the span of $Dx_1, \ldots, Dy_n$. For example, $h = x_2y_3 - x_3y_2 \in \ker D$, so we may take $Dz = h(fx_n - gy_n)$. Then $Dz \in \ker D$ and $\deg Dz = 5$. Moreover, $\deg_t Dz(\mathbf{t}) = 2n + 7$, so Dz is independent of the other images.

Remark 3.51 The examples of de Bondt given above are for $n \geq 6$ and have $\deg D = 4$. In [93], Cor. 3.4, de Bondt also gives examples with $n \geq 10$ and $\deg D = 3$. It is an open question whether there exist homogeneous $D \in \mathrm{LND}(k^{[n]})$ with $\deg D = 2$ and $\mathrm{rank}(D) = n$.

3.9.4 Rank-4 Example in Dimension 5

In the notation of de Bondt's examples, consider the case $n = 2$: Let $\mathcal{B} = k[a, b, x_1, y_1, x_2, y_2] = k^{[6]}$ and $R = k[a, b]$. In this case, replace the matrix N with:

$$N' = \begin{pmatrix} ab^2 & -b^4 \\ a^2 & -ab^2 \end{pmatrix}$$

This defines an R-linear $\mathcal{D} \in \mathrm{LND}_R(\mathcal{B})$, namely:

$$\mathcal{D} = \begin{pmatrix} N' & 0 \\ 0 & N' \end{pmatrix}_{4\times 4}$$

Note that we still have $f = x_1y_2 - x_2y_1 \in \ker \mathcal{D}$. Set $E = \mathcal{D} \bmod (a - f)$ on $B = \mathcal{B} \bmod (a - f) = k^{[5]}$. Then E is standard homogeneous of degree 4, and satisfies:

$$E^2 b = E^2 x_1 = E^2 y_1 = E^2 x_2 = E^2 y_2 = 0$$

In addition, the rank of E is 4. To see this, it suffices to show that the images Ex_1, Ey_1, Ex_2, Ey_2 are linearly independent. As above, evaluate these polynomials at $\mathbf{t} = (1, t, t^2 - 1, t^3, t^4)$. Then:

$$Ex_1(\mathbf{t}) = t^4 - t^2 + 1\ ,\ Ey_1(\mathbf{t}) = t^7 - t^5 + t^3\ ,\ Ex_2(\mathbf{t}) = t^6 - t^4\ ,\ Ey_2(\mathbf{t}) = t^9 - t^7$$

Therefore, Ex_1, Ey_1, Ex_2, Ey_2 are linearly independent.

Chapter 4
Dimension Two

This chapter examines locally nilpotent R-derivations of $R[x, y] = R^{[2]}$ for certain rings R containing $\mathbb{Q}$. This set is denoted $\mathrm{LND}_R(R[x, y])$.

4.1 Background

We begin with the case R is a field, and here the main fact is due to Rentschler from 1968 [355].

Theorem 4.1 (Rentschler's Theorem) *Let $k[x, y] = k^{[2]}$ and let $D \in \mathrm{LND}(k[x, y])$ be nonzero. There exists $p(x) \in k[x]$ and a tame automorphism $\alpha \in GA_2(k)$ such that $\alpha D \alpha^{-1} = p(x)\partial_y$.*

Geometrically, this says that every planar $\mathbb{G}_a$-action is conjugate, by a tame automorphism, to a triangular action: $t \cdot (x, y) = (x, y + tp(x))$.[1] Rentschler also showed that his theorem implies Jung's Theorem, which appeared in 1942 [222], and which asserts that every plane automorphism is tame in the characteristic zero case. Rentschler's proof of Jung's Theorem is a compelling illustration of the importance of locally nilpotent derivations and $\mathbb{G}_a$-actions in the study of affine algebraic geometry.

Jung's Theorem was the predecessor of the well-known Structure Theorem for $GA_2(K)$, the group of algebraic automorphisms of the plane $\mathbb{A}^2_K$ for any field K. $Af_2(K)$ is the affine linear subgroup, whose elements are products of linear maps

[1]As noted in the *Introduction*, this description of the planar $\mathbb{G}_a$-actions was first given by Ebey in 1962 [133]. The statement about tame automorphisms is not explicit in his paper, but can be inferred from the proof.

G. Freudenburg, *Algebraic Theory of Locally Nilpotent Derivations*,
Encyclopaedia of Mathematical Sciences 136,
DOI 10.1007/978-3-662-55350-3_4

$L \in GL_2(K)$ and translations $T = (x + a, y + b)$, $a, b \in K$; and $BA_2(K)$ is the subgroup of triangular automorphisms $\alpha = (ax + b, cy + f(x))$, $a, c \in K^*, b \in K, f \in K[x]$.

Theorem 4.2 (Structure Theorem) *For any field K, $GA_2(K)$ has the amalgamated free product structure*

$$GA_2(K) = Af_2(K) *_B BA_2(K)$$

where $B = Af_2(K) \cap BA_2(K)$.

In his 1992 paper [427], Wright gives the following description of the evolution of this result.

> That GA_2 is generated by Af and BA was first proved by Jung [222] for k of characteristic zero. Van der Kulk [408] generalized this to arbitrary characteristic and proved a factorization theorem which essentially gives the amalgamated free product structure, although he did not state it in this language. Nagata [325] seems to be the first to have stated and proved the assertion as it appears above. The techniques in these proofs require that k be algebraically closed. However, it is not hard to deduce the general case from this (see [423]). Some fairly recent proofs have been given which use purely algebraic techniques, and for which it is not necessary to assume k is algebraically closed [112, 292]. (p. 283)

In this same paper, Wright gives another proof of the Structure Theorem, based on Serre's tree theory [378].

While the paper of Nagata mentioned by Wright dates to 1972, it should be noted that the Structure Theorem was stated without proof in the well-known 1966 paper of Shafarevich [380], Thm. 7. Another proof of Jung's Theorem was given by Engel [139]. Moreover, Jung's Theorem is implied by the Embedding Theorem of Abhyankar and Moh [3], and Suzuki [397].

Combining the Structure Theorem with Serre's theory gives a complete description of all planar group actions: If G is an algebraic group acting algebraically on $\mathbb{A}^2$, given by $\phi : G \to GA_2(K)$, then $\phi(G)$ is conjugate to a subgroup of either $GL_2(K)$ or $BA_2(K)$ (see [388]). The main fact used here comes from combinatorial group theory: Any subgroup of an amalgamated free product $A *_C B$ having bounded length can be conjugated into either A or B (Thm. 8 of [378]). If G is an algebraic subgroup of $GA_2(K)$, then it is of bounded degree; and Wright pointed out that, in $GA_2(K)$, bounded degree implies bounded length (see [239], Lemma 4.1 and Thm. 4.3). Earlier results on planar group actions appear in [30, 133, 195, 216, 294, 425].

In particular, planar actions of reductive groups can be conjugated to linear actions, and actions of unipotent groups can be conjugated to triangular actions. We thus recover Rentschler's Theorem from the Structure Theorem, and get the following description of planar $\mathbb{G}_a$-actions in any characteristic.

Theorem 4.3 *For any field K, an algebraic action of $\mathbb{G}_a$ on $\mathbb{A}^2_K$ is conjugate to a triangular action.*

Section 3 gives a proof of Rentschler's Theorem, Jung's Theorem, and the Structure Theorem (in the case k is of characteristic 0). *Section 4* then discusses $\mathrm{LND}_R(R[x, y])$ for integral domains R containing $\mathbb{Q}$, with particular attention to the case R is a UFD. In contrast to Rentschler's Theorem, it turns out that when R is not a field, most elements of $\mathrm{LND}_R(R[x,y])$ are not triangularizable, i.e., not conjugate to $f(x)\partial_y$ for $f \in R[x]$ via some R-automorphism. However, we show that if R is a highest common factor (HCF) ring, then it is still true that the kernel of a locally nilpotent R-derivation of $R[x,y]$ is isomorphic to $R^{[1]}$.

We are especially interested in how this theory applies to polynomial rings over k. We consider locally nilpotent derivations of $k[x_1,\ldots,x_n] \cong k^{[n]}$ of rank at most 2: If $Dx_1 = \cdots Dx_{n-2} = 0$ and $R = k[x_1,\ldots,x_{n-2}]$, then $D \in \mathrm{LND}_R(R[x_{n-1},x_n])$. Of particular importance is *Theorem 4.15*, which implies that a fixed-point free $\mathbb{G}_a$-action on $\mathbb{A}^n$ of rank at most 2 must be conjugate to a translation.

For a general affine k-domain R, the kernel of an element of $\mathrm{LND}_R(R[x,y])$ need not be isomorphic to $R^{[1]}$, or even finitely generated over R. A simple pathological example is the following.

Example 4.4 ([142], Exercise 9.5.9) Let $R = \mathbb{C}[t^2,t^3] \subset \mathbb{C}[t] = \mathbb{C}^{[1]}$ and define the R-derivation D on $R[x,y]$ by $Dx = t^2$ and $Dy = t^3$. Then $D \in \mathrm{LND}_R(R[x,y])$, but the kernel of D is not finitely generated over $\mathbb{C}$. To see this, note that D is the restriction of a derivation of $\mathbb{C}[t,x,y]$ whose kernel is $\mathbb{C}[t,f]$, where $f = y - tx$. Thus, $\ker D = \mathbb{C}[t,f] \cap R[x,y] = R[ft^2, f^2t^2, \ldots]$, which is not finitely generated.

Bhatwadekar and Dutta [27] studied $\mathrm{LND}_R(R[x,y])$ in the case that R is a noetherian integral domain containing $\mathbb{Q}$. They show that, even when R is normal, the kernel of such a derivation can be non-finitely generated over R. See *Sect. 4.4.3*.

In what follows, the convention used for composing elements of $GA_2(k)$ is:

$$\begin{aligned}&\big(f_1(x,y), f_2(x,y)\big) \circ \big(g_1(x,y), g_2(x,y)\big)\\ &\qquad = \big(f_1(g_1(x,y), g_2(x,y)), f_2(g_1(x,y), g_2(x,y))\big)\end{aligned}$$

4.2 Newton Polygons

Let A be a commutative k-domain and $B = A[x,y] = A^{[2]}$. Let $B = \bigoplus_{g\in G} B_g$ be the G-grading for which $G = \mathbb{Z}^2$, $A = B_0$, $\deg_G x = (1,0)$ and $\deg_G y = (0,1)$. Given nonzero $f \in B$, recall that the G-support of f is $\mathrm{Supp}_G(f) = \{(i,j) \in G \mid f_{ij} \neq 0\}$. The **Newton polygon** of f relative to this G-grading is the convex hull of $\mathrm{Supp}_G(f) \cup \{(0,0)\}$ in $\mathbb{R}^2$, and is denoted by $\mathrm{Newt}_G(f)$.

The following result is well-known when the ring A is a field. The generalization given here, due to the author and D. Daigle, gives the form of the Newton polygon for a kernel element of a nonzero $D \in \mathrm{LND}(B)$ in the case A is any rigid affine k-domain. It is important to note that the theorem does *not* assume $DA = 0$.

Theorem 4.5 *Let A be a rigid affine k-domain and $B = A[x,y] = A^{[2]}$, and let $D \in \mathrm{LND}(B)$ be nonzero. Let $B = \bigoplus_{g\in G} B_g$ be the G-grading for which $G = \mathbb{Z}^2$, $A = B_0$, $\deg_G x = (1,0)$ and $\deg_G y = (0,1)$. For each nonzero $f \in \ker D \setminus A$, $\mathrm{Newt}_G(f)$ is a triangle with vertices $(0,0)$, $(m,0)$ and $(0,n)$, where $m,n \in \mathbb{N}$ and either $m|n$ or $n|m$.*

Proof Let $\lambda : G \to \mathbb{Z}$ be a $\mathbb{Z}$-linear map. Then λ induces a $\mathbb{Z}$-grading of B over A, where x, y are $\mathbb{Z}$-homogeneous, $\deg_{\mathbb{Z}} x = \lambda(1,0)$ and $\deg_{\mathbb{Z}} y = \lambda(0,1)$. Since B is affine, *Theorem 2.15* implies that $\deg_{\mathbb{Z}} D$ is defined, and since B is $\mathbb{Z}$-graded, the associated graded ring is $\mathrm{Gr}(B) = B$. Therefore, D induces a nonzero $\mathbb{Z}$-homogeneous $\bar{D} \in \mathrm{LND}(B)$ with $\bar{D}\bar{f} = 0$, where $\bar{f}$ is the highest-degree $\mathbb{Z}$-homogeneous summand of $f \in \ker D$. To be precise, we will call this the $(\mathbb{Z}, \lambda)$-grading of B.

If $f \in A[x]$ or $f \in A[y]$, then it is clear that the Newton polygon of f has the stated form. So assume that $f \notin A[x]$ and $f \notin A[y]$.

Let $(p, q) \in \mathrm{Newt}_G(f)$ be a vertex other than $(0, 0)$. By convexity, there exists a $\mathbb{Z}$-linear function $\lambda : G \to \mathbb{Z}$ such that $\lambda(p, q) > \lambda(u, v)$ for any $(u, v) \in \mathrm{Newt}_G(f)$ with $(u, v) \neq (p, q)$. Let $\bar{D}$ be the $(\mathbb{Z}, \lambda)$-homogeneous element of $\mathrm{LND}(B)$ defined by D, and let $\bar{f}$ be the highest-degree $(\mathbb{Z}, \lambda)$-homogeneous summand of f. By definition of λ, $\bar{f} = \alpha x^p y^q$ for some $\alpha \in A$. If $p > 0$ and $q > 0$, then $\bar{D}x = \bar{D}y = 0$. By *Corollary 2.26*, $\bar{D}(A) = 0$ as well, so $\bar{D} = 0$, a contradiction. Therefore, either $p = 0$ or $q = 0$. It follows that $\mathrm{Newt}_G(f)$ is a triangle with vertices $(0, 0)$, $(m, 0)$ and $(0, n)$ for $m, n \in \mathbb{N}$, and since f is not in $A[x]$ or $A[y]$, $m \geq 1$ and $n \geq 1$.

Set $d = \gcd(m, n)$ and define another $\mathbb{Z}$-linear map $\mu : G \to \mathbb{Z}$ by $\mu(1, 0) = n/d$ and $\mu(0, 1) = m/d$. Let Δ be the $(\mathbb{Z}, \mu)$-homogeneous element of $\mathrm{LND}(B)$ defined by D, and let F be the highest-degree $(\mathbb{Z}, \mu)$-homogeneous summand of f. If $\Delta x = 0$, then $\ker \Delta = A[x]$ by *Corollary 2.26*. But then $F \in A[x]$, which is not the case, since $y^n \in \mathrm{Supp}_{(\mathbb{Z},\mu)}(F)$. Therefore, $\Delta x \neq 0$, and similarly $\Delta y \neq 0$.

We have $\deg_{(\mathbb{Z},\mu)} x = n/d$ and $\deg_{(\mathbb{Z},\mu)} y = m/d$. If $m/d \geq 2$ and $n/d \geq 2$, then either $\Delta x = 0$ or $\Delta y = 0$ by *Corollary 2.44(b)*, a contradiction. Therefore, either $m/d = 1$ or $n/d = 1$. □

The situation in *Theorem 4.5* is depicted in *Fig. 4.1*. In cases where more variables are involved, one defines the Newton polytope of a polynomial. Some results on Newton polytopes over the field k are described in the *Appendix* to this chapter.

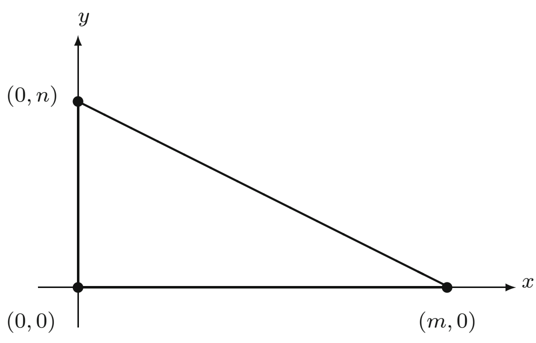

Fig. 4.1 Newton polygon of $f \in \ker D$ over the rigid ring A

4.3 Polynomial Ring in Two Variables Over a Field

If k is algebraically closed, then it is possible to give a short proof of Rentschler's Theorem without using Newton polygons. We do this in Sect. 4.3.1. We then give a second proof of Rentschler's Theorem using Newton polygons for any field k of characteristic zero. Rentschler's original proof of *Theorem 4.1* uses the Newton polygon of a derivation relative to the standard grading of $k[x, y]$, and deals first with the fixed-point free case. Other proofs along the same lines as Rentschler's proof are given in [124, 142]. The proofs presented here follow the ideas of Makar-Limanov in [276], using the theory of G-critical elements for a shorter exposition.

4.3.1 *Rentschler's Theorem: First Proof*

Assume that k is an algebraically closed field of characteristic zero.

Let $B = k[x, y] = k^{[2]}$, let $D \in \mathrm{LND}(B)$ be irreducible, and set $A = \ker D$. By *Corollary 2.13*, there exist $u, v \in B$ such that $B = k[u, v]$, $A = k[u]$ and $Dv = 1$. Note that D is homogeneous for the $\mathbb{Z}$-grading $B = \bigoplus_{i\geq 0} k[u]v^i$

Assume that $x \notin k[u]$ and $y \notin k[u]$, and let $\bar{x}$ and $\bar{y}$ be the highest-degree homogeneous summands of x and y, respectively. Then there exist nonzero $a, b \in A$ and integers $m, n \geq 1$ with $\bar{x} = av^m$ and $\bar{y} = bv^n$. If δ is the highest-degree homogeneous summand of $\partial/\partial x$, then $\delta \in \mathrm{LND}(B)$ and $\delta(\bar{x}) = 0$ (*Proposition 3.8* and *Corollary 1.31*). Therefore, $k[a, v] \subset \ker \delta$, which implies $a \in k^*$. Similarly, $b \in k^*$.

Let $d = \gcd(m, n)$ and put a $\mathbb{Z}$-grading on B by letting u, v be homogeneous, $\deg_{\mathbb{Z}} u = n/d$ and $\deg_{\mathbb{Z}} v = m/d$. Then D is $\mathbb{Z}$-homogeneous. Since $Dx \neq 0$ and $Dy \neq 0$, *Corollary 2.45* implies that either $m/d = 1$ or $n/d = 1$. If $m = d$ and $n = tm$ for $t \in \mathbb{Z}$, then the v-degree of $a^t y - bx^t$ is smaller than n, and $k[x, y] = k[x, a^t y - bx^t]$. Similarly, if $n = d$ and $m = nt$, then the v-degree of $b^t x - ay^t$ is smaller than m, and $k[x, y] = k[b^t x - ay^t, y]$.

Thus, after a sequence of triangular automorphisms of these two types, we obtain a tame automorphism α of B such that either $\alpha(x)$ or $\alpha(y)$ is in $k[u]$. If $\alpha(x) \in k[u]$, then $k[u, v] = k[x, y] = k[u, \alpha(y)]$ implies $\alpha(y) = cv + p(u)$ for $c \in k^*$ and $p \in k[u]$. Similarly, if $\alpha(y) \in k[u]$, then $k[u, v] = k[x, y] = k[u, \alpha(x)]$ implies $\alpha(x) = cv + p(u)$. So there exists tame $\beta \in GA_2(k)$ with $\beta(x) = u$ and $\beta(y) = v$. Therefore, $\beta^{-1} D\beta = \partial/\partial y$. □

4.3.2 *Rentschler's Theorem: Second Proof*

Assume that k is a field of characteristic zero.

Let $B = k[x, y] = k^{[2]}$, and let $B = \bigoplus_{g\in G} B_g$ be the G-grading for which $G = \mathbb{Z}^2$, $k = B_0$, $\deg_G x = (1, 0)$ and $\deg_G y = (0, 1)$. Let nonzero $f \in \ker D$ be given. By

Theorem 4.5, there exist $a, m \in \mathbb{N}$, $m \geq 1$, such that the vertex set of $\mathrm{Newt}_G(f)$ is either $\{(0,0),(m,0),(0,am)\}$ or $\{(0,0),(am,0),(0,m)\}$. In the latter case, let $\alpha \in GA_2(k)$ be the transposition $\alpha = (y,x)$, and replace D by $\alpha D\alpha$ and f by $\alpha(f)$. In this way, we can assume the vertex set of $\mathrm{Newt}_G(f)$ equals $\{(0,0),(m,0),(0,am)\}$. In this case, $f(x,0) \neq 0$, and we may assume that $f(x,0)$ is monic; otherwise, replace f by cf for some $c \in k^*$.

Assume that $a \neq 0$, and define a $\mathbb{Z}$-grading of B by declaring that x, y are $\mathbb{Z}$-homogeneous, $\deg_{\mathbb{Z}} x = a$ and $\deg_{\mathbb{Z}} y = 1$. Then $\deg_{\mathbb{Z}} f = am$. Let $\bar{f}$ and $\bar{D}$ denote the highest-degree homogeneous summands of f and D, respectively (see *Proposition 3.8*). Then $\bar{D} \in \mathrm{LND}(B)$ and $\bar{D}\bar{f} = 0$ (*Corollary 1.31*).

If $(i,j) \in \mathrm{Supp}_G(\bar{f})$, then $ai + j = \deg_{\mathbb{Z}} \bar{f} = \deg_{\mathbb{Z}} f = am$ implies that a divides j. Therefore, $\bar{f} \in k[x, y^a]$. Since $\bar{f}$ is $\mathbb{Z}$-homogeneous, there exists a G-homogeneous polynomial $F \in k[x,y]$ such that $\bar{f} = F(x, y^a)$. If K is an algebraically closed field containing k, then F factors as a product of linear polynomials over K. Let $\delta \in \mathrm{Der}_K(K[x,y])$ be the unique extension of $\bar{D}$ to $K[x,y]$. Then δ is locally nilpotent, $\delta \neq 0$, and $\delta\bar{f} = 0$ (see *Proposition 1.18*). If any two linear factors of F are linearly independent over K, then $\delta = 0$, a contradiction. Therefore, there exist $u, v \in K$ such that $F = (ux + vy)^m$. Then $\bar{f} = (ux + vy^a)^m$, and since $f(x,0)$ is monic, we may assume that $u = 1$. Since $0 = \delta(x + vy^a) = \bar{D}x + avy^{a-1}\bar{D}y$, we see that $v \in k(x,y) \cap K^* = k^*$.

Define $\alpha \in BA_2(k)$ and $D' \in \mathrm{LND}(A)$ by $\alpha = (x - vy^a, y)$ and $D' = \alpha D\alpha$. Then $\alpha(f) \in \ker D'$. The crucial observation is that $\deg_y \alpha(f) < \deg_y f$, whereas $\deg_x \alpha(f) = \deg_x f$.

If $D'x \neq 0$ and $D'y \neq 0$, then the same argument can be applied to D' and $\alpha(f)$ to lower the degree of the kernel element in either x or y. In this way, we obtain a tame automorphism γ such that either $\gamma D\gamma^{-1}(x) = 0$ or $\gamma D\gamma^{-1}(y) = 0$. By applying the transposition (y,x) in the latter case, we may assume γ is a tame automorphism with $\gamma D\gamma^{-1}(x) = 0$. Then $\gamma D\gamma^{-1}(y) \in k[x]$ by *Principle 6*. □

Recall that, for $f \in k[x,y]$, Δ_f denotes the derivation $\Delta_f h = f_x h_y - f_y h_x$ for $h \in k[x,y]$.

Corollary 4.6 *Let $D \in \mathrm{Der}_k(k[x,y])$ be given. Then $D \in \mathrm{LND}(k[x,y])$ if and only if D is of the form $D = \Delta_f$, where $f \in k[v]$ for some variable v of $k[x,y]$.*

Proof Assume D is locally nilpotent. If $D = 0$, then $D = \Delta_0$. If $D \neq 0$, then by Rentschler's Theorem, there exists a system of variables (u,v) for $k[x,y]$ relative to which $D = g(v)\partial_u$ for some $g(v) \in k[v]$. Since (u,v) is an automorphism of $k[x,y]$, $\frac{\partial(u,v)}{\partial(x,y)} = c$ for some $c \in k^*$. We may assume $c = -1$: otherwise replace v by $-c^{-1}v$. It follows that $\Delta_v v = 0$ and $\Delta_v u = 1$. Choose $f(v) \in k[v]$ so that $f'(v) = g(v)$. Then $\Delta_f = f'(v)\Delta_v$, which implies $\Delta_f u = Du = g(v)$ and $\Delta_f v = Dv = 0$. Therefore $D = \Delta_f$.

Conversely, if $f \in k[v]$ for some variable v, then (as above) there exists $u \in k[x,y]$ with $k[u,v] = k[x,y]$, $\Delta_v v = 0$ and $\Delta_v u = 1$. Since $\Delta_f = f'(v)\Delta_v$ and $f'(v) \in \ker \Delta_v$, it follows that Δ_f is locally nilpotent. □

4.3.3 Proof of Jung's Theorem

Every automorphism (i.e., change of coordinates) of $B = k[x, y]$ defines a pair of locally nilpotent derivations of B, namely, the partial derivatives relative to the new coordinate functions. Thus, in the characteristic zero case, Rentschler's Theorem can be used to study plane automorphisms. We use Rentschler's Theorem to generalize Jung's Theorem for any characteristic zero field k.

Theorem 4.7 *Let k be a field of characteristic zero. The group $GA_2(k)$ is generated by its linear and triangular subgroups, $GL_2(k)$ and $BA_2(k)$.*

Proof (Following Rentschler [355]) Given $(F, G) \in GA_2(k)$, we have $B = k[F, G]$ and the partial derivative $\partial_F \in \text{LND}(B)$ is given by $\partial_F(F) = 1$ and $\ker(\partial_F) = k[G]$. By Rentschler's Theorem, there exists tame $\varphi \in GA_2(k)$ such that $\varphi^{-1}\partial_F\varphi = f(x)\partial_y$ for some $f \in k[x]$. Since ∂_F has a slice, it is irreducible and $f \in B^* = k^*$.

Note that $\ker(\partial_y) = k[x] = k[\varphi^{-1}(G)]$, which implies $G = a\varphi(x)+b$ for $a, b \in k$, $a \neq 0$. In addition, $\partial_y(\varphi^{-1}(F)) = f^{-1}$, which implies $F = f^{-1}\varphi(y) + g(\varphi(x))$ for some $g \in k[x]$. We thus have

$$(F, G) = (f^{-1}\varphi(y) + g(\varphi(x)), a\varphi(x) + b)$$

which is the composition of φ with a triangular automorphism. Therefore (F, G) is tame. □

4.3.4 Proof of Structure Theorem

From Jung's Theorem we deduce the Structure Theorem for fields of characteristic zero. The theorem asserts that $GA_2(k)$ admits a kind of unique factorization property. Our proof follows Wright in [424], Prop. 7. See also [427] for another proof.

Generally, a group G is an **amalgamated free product** of two of its subgroups if and only if G is the homomorphic image of the free product of two groups. Specifically, if $f(G_1 * G_2) = G$ for groups G_1 and G_2 and epimorphism f, we write $G = A *_C B$, where $A = f(G_1)$, $B = f(G_2)$, and $C = A \cap B$. Equivalently, $G = A *_C B$ means that A, B and $C = A \cap B$ are subgroups of G satisfying the following condition.

> Let $\mathcal{A}$ and $\mathcal{B}$ be systems of nontrivial right coset representatives of A and B, respectively, modulo C. Then every $g \in G$ is uniquely expressible as $g = ch_1 \cdots h_n$, where $c \in C$, and the h_i lie alternately in $\mathcal{A}$ and $\mathcal{B}$.

Note that, in this case, if $C = \{1\}$, then G is the free product of A and B.

For the group $G = GA_2(k)$, consider subgroups:

$$
\begin{aligned}
A &= Af_2(k) \\
&= \{(a_1x + b_1y + c_1, a_2x + b_2y + c_2) \mid a_i, b_i, c_i \in k; a_1b_2 - a_2b_1 \neq 0\} \\
B &= BA_2(k) = \{(ax + b, cy + f(x)) \mid a, c \in k^*; b \in k; f(x) \in k[x]\} \\
C &= A \cap B = \{(ax + b, cx + dy + e) \mid a, d \in k^*; b, c, e \in k\}
\end{aligned}
$$

Suppose $\alpha = (a_1x + b_1y + c_1, a_2x + b_2y + c_2) \in A \setminus C$. Then $b_1 \neq 0$ and:

$$\alpha = (b_1x + c_1, b_2x + (a_2b_1 - b_2a_1)b_1^{-1}y + c_2) \cdot (a_1b_1^{-1}x + y, x)$$

Likewise, suppose $\beta = (ax + b, cy + f(x)) \in B \setminus C$. Write $f(x) = r + sx + x^2g(x)$ for some $r, s \in k$ and $g(x) \in k[x]$. Then:

$$\beta = (ax + b, cy + r + sx) \cdot (x, y + x^2 \cdot c^{-1}g(x))$$

Therefore, we may choose the following sets of nontrivial coset representatives for A and B, respectively, modulo C:

$$\mathcal{A} = \{(tx + y, x) | t \in k\} \quad \text{and} \quad \mathcal{B} = \{(x, y + x^2 \cdot f(x)) | f(x) \in k[x], f \neq 0\}$$

In addition, observe the semi-commuting relation for elements of C and $\mathcal{A}$:

$$(tx + y, x)(ax + b, cx + dy + e) = (dx + (tb + e), ay + b)(d^{-1}(ta + c)x + y, x)$$

Likewise, among elements of C and $\mathcal{B}$ we have

$$
\begin{aligned}
&(x, y + f(x))(ax + b, cx + dy + e) = (ax + b, cx + dy + e)(x, y + d^{-1}f(ax + b)) \\
&\quad = (ax + b, (c + ds)x + dy + (dr + e))(x, y + x^2h(x))
\end{aligned}
$$

where $d^{-1}f(ax + b) = r + sx + x^2h(x)$ for some $r, s \in k$. It follows that any element κ belonging to the subgroup generated by A and B can be expressed as a product

$$\kappa = ch_1 \cdots h_n \tag{4.1}$$

where $c \in C$, and the h_i lie alternately in $\mathcal{A}$ or $\mathcal{B}$. It remains to check the uniqueness of such a factorization.

Consider a product of the form

$$\varphi = \gamma_s\alpha_{s-1}\gamma_{s-1} \cdots \alpha_1\gamma_1 \quad (s \geq 1)$$

where, for each i, $\alpha_i = (t_ix + y, x) \in \mathcal{A}$, and $\gamma_i = (x, y + f_i(x)) \in \mathcal{B}$. For any $(F, G) \in GA_2(k)$, where $F, G \in k[x, y]$, define the **degree** of (F, G) to be $\max\{\deg F, \deg G\}$, where deg is the standard degree on $k[x, y]$. Set $d_i = \deg \gamma_i$ $(1 \leq i \leq s)$.

Lemma 4.8 *In the notation above, if* $\varphi = (F, G)$ *and* $s \geq 1$*, then* $\deg G > \deg F$ *and* $\deg \varphi = d_1 d_2 \cdots d_s$.

Proof For $s = 1$, this is clear. For $s > 1$, assume by induction that

$$\psi := \gamma_{s-1}\alpha_{s-2}\gamma_{s-2}\cdots\alpha_1\gamma_1 = (P, Q)$$

satisfies $\deg(\psi) = \deg Q = d_1 \cdots d_{s-1} > \deg P$. Then

$$\alpha_{s-1}\psi = (t_{s-1}P + Q, P) = (R, P)$$

where $\deg(\alpha_{s-1}\psi) = \deg R = d_1 \cdots d_{s-1} > \deg P$. Therefore:

$$\varphi = \gamma_s\alpha_{s-1}\psi = (R, P + f_s(R)) = (F, G)$$

Since $\deg R > \deg P$ and $\deg f_s \geq 2$, we see that $\deg(\varphi) = \deg G = (\deg f_s)(\deg R) = d_1 \cdots d_{s-1}d_s > \deg F$. □

In order to finish the proof of the Structure Theorem, we must show that the factorization (4.1) of κ above is unique. Due to basic properties of amalgamations, it suffices to assume $\kappa = (x, y)$, the identity. See, for example, [424], Prop. 1.

Suppose $(x, y) = ch_1 \cdots h_n$, where $c \in C$, and the h_i lie alternately in $\mathcal{A}$ or $\mathcal{B}$. By the preceding lemma, $1 = (\deg h_1) \cdots (\deg h_n)$. Since $\deg h_i > 1$ for each $h_i \in \mathcal{B}$, we conclude that $n \leq 1$. If $n = 1$, then $(x, y) = ch_1$ for $h_1 \in \mathcal{A}$, which is impossible since then $c = h_1^{-1} \in C$. Thus, $n = 0$, and $c = (x, y)$.

This completes the proof of the Structure Theorem. □

4.3.5 Remark About Fields of Positive Characteristic

Just after Rentschler's Theorem appeared, Miyanishi took up the question of $\mathbb{G}_a$-actions on the plane when the underlying field k is of positive characteristic. He proved the following.

Theorem 4.9 ([294]) *Let k be an algebraically closed field of positive characteristic p. Then any $\mathbb{G}_a$-action on $\mathbb{A}^2_k$ is conjugate to an action of the form*

$$t \cdot (x, y) = \left(x, y + tf_0(x) + t^p f_1(x) + \cdots + t^{p^n} f_n(x)\right)$$

where $t \in \mathbb{G}_a(k)$, $(x, y) \in k^2$, and $f_0(x), \ldots, f_n(x) \in k[x]$.
It seems that this result does not receive as much attention as it deserves, as it completely characterizes the planar $\mathbb{G}_a$-actions in positive characteristic.

4.4 Locally Nilpotent R-Derivations of $R[x, y]$

In this section, we consider $\text{LND}_R(R[x, y])$ for various rings R, where $R[x, y]$ denotes the polynomial ring in two variables over R.

4.4.1 Kernels in $R[x, y]$

Let R be an integral domain containing $\mathbb{Q}$. Consider the following property:

$$(\dagger)\ \ \ker D = R^{[1]} \text{ for every nonzero } D \in \text{LND}_R(R[x, y])$$

We have seen that, even if R is an affine (rational) integral domain containing $\mathbb{Q}$, it may fail to satisfy (†).

Recall that an integral domain R is a highest common factor ring, or HCF-ring, if and only if it has the property that the intersection of two principal ideals is again principal. Examples of HCF-rings are: a UFD, a valuation ring, or a polynomial ring over a valuation ring. These form a large and useful class of rings R, and we show that those containing $\mathbb{Q}$ also satisfy property (†).

Theorem 4.10 *Let R be an HCF-ring containing $\mathbb{Q}$. If $D \in \text{LND}_R(R[x, y])$ and $D \neq 0$, then there exists $P \in R[x, y]$ such that $\ker D = R[P]$.*
To prove this, we refer to the following well-known result from commutative algebra, due to Abhyankar, Eakin, and Heinzer (1972).

Proposition 4.11 (Prop. 4.8 of [2]) *Let R be an HCF-ring, and suppose A is an integral domain of transcendence degree one over R and that $R \subset A \subset R^{[n]}$ for some $n \geq 1$. If A is a factorially closed subring of $R^{[n]}$, then $A = R^{[1]}$.*

Proof of Theorem 4.10 Since $\text{tr.deg}_R R[x, y] = 2$ and $\text{tr.deg}_{\ker D} R[x, y] = 1$, we see that $\text{tr.deg}_R \ker D = 1$. Since $R \subset \ker D \subset R[x, y]$ and $\ker D$ is factorially closed, it follows from *Proposition 4.11* that there exists $P \in \ker D$ with $\ker D = R[P]$. □

Theorem 4.12 *Let R be an HCF-ring containing $\mathbb{Q}$, let $B = R[x, y] = R^{[2]}$, and let $K = \text{frac}(R)$. If $D \in \text{LND}_R(B)$ is irreducible, then there exist $P, Q \in B$ such that $K[P, Q] = K[x, y]$, $\ker D = R[P]$ and $DQ \in R$.*

Proof Continuing the notation above, we have $\ker D_K = K[P']$. Suppose $\ker D = R[P]$ for $P \in B$. Then $\ker D_K = K[P]$. To see this, note that $aP' \in B$ for some nonzero $a \in R$, which implies $aP' \in \ker D \subset K[P]$. But then $K[P'] \subset K[P] \subset \ker D_K$, so $K[P] = K[P']$.

Also by Rentshler's Theorem, there exists $Q' \in K[x, y]$ and $f(P) \in K[P]$ such that $K[P, Q'] = K[x, y]$ and $D_K = f(P)\partial/\partial Q'$ relative to this coordinate system. Since D is irreducible, D_K is also irreducible, meaning that $\deg_P f(P) = 0$. Therefore, $D_K Q' \in K^*$. Choose nonzero $b \in R$ so that $Q := bQ' \in B$. Then $K[P, Q] = K[x, y]$, and $DQ = D_K Q = bD_K Q' \in R$. □

To a large extent, the behavior of elements of $\mathrm{LND}_R(R[x, y])$ is governed by *Theorem 4.12* when R is an HCF-ring containing $\mathbb{Q}$. This leaves open the question:

If R satisfies (†), is R an HCF-ring?

Given $F \in R[x, y]$, define $F_x = \partial_x(F)$ and $F_y = \partial_y(F)$. If δ is any R-derivation of B, then $\delta F = F_x\delta x + F_y\delta y$. So if $\delta x = F_y$ and $\delta y = -F_x$, then $\delta F = 0$. Define $\Delta_F := F_y\partial_x - F_x\partial_y$. We ask:

For which $F \in R[x, y]$ is the induced R-derivation Δ_F locally nilpotent?

For HCF-rings we have the following answer.

Theorem 4.13 *Assume R is an HCF-ring containing $\mathbb{Q}$, $B = R[x, y] = R^{[2]}$, and $K = \mathrm{frac}(R)$. Given $F \in B$, Δ_F is locally nilpotent if and only if there exist $P, Q \in B$ such that $F \in R[P]$ and $K[P, Q] = K[x, y]$. Moreover, every $D \in \mathrm{LND}_R(B)$ equals Δ_F for some $F \in B$.*

Proof If $\Delta_F \in \mathrm{LND}_R(B)$, then $P, Q \in B$ with the stated properties exist, by *Theorem 4.12*. Conversely, if $F = \varphi(P)$ for some univariate polynomial φ, then $\Delta_F = \varphi'(P) \cdot \Delta_P$. It thus suffices to show Δ_P is locally nilpotent, since $\varphi'(P) \in \ker(\Delta_P)$. In $K[x, y]$, we have:

$$(\Delta_P)_K(P) = 0 \quad \text{and} \quad (\Delta_P)_K(Q) = P_yQ_x - P_xQ_y$$

Thus, $(\Delta_P)_K(Q) = \det \frac{\partial(P,Q)}{\partial(x,y)}$, and since (P, Q) defines a K-automorphism of $K[x, y]$, it follows that $(\Delta_P)_K(Q) \in K^*$. Therefore $(\Delta_P)_K$ is locally nilpotent, which implies Δ_P is also.

Finally, let $D \in \mathrm{LND}_R(B)$ be given. If $D = 0$, then $D = \Delta_0$. If $D \neq 0$, then there exist $P, Q \in B$ such that $\ker D = R[P]$, $K[P, Q] = K[x, y]$, and $DQ = g(P)$ for some $g \in R[P]$, according to *Theorem 4.12*. It follows that $\ker D_K = K[P]$ and $D_K = g(P)\partial_Q$. Choose $F \in R[P]$ so that $F'(P) = g(P)$. Then $(\Delta_F)_K = g(P)(\Delta_P)_K = g(P)\partial_Q = D_K$, which implies $D = \Delta_F$. □

In view of *Corollary 2.3*, we get a fairly complete description of the locally nilpotent R-derivations of $R[x, y]$ in the case R is a UFD.

Corollary 4.14 *Let R be a UFD, $K = \mathrm{frac}(R)$ and $B = R[x, y] = R^{[2]}$. Define the subset $\mathcal{L} \subset B$ by:*

$$\mathcal{L} = \{ P \in B \mid K[x, y] = K[P]^{[1]},\ \gcd_B(P_x, P_y) = 1\}$$

Then $\mathrm{LND}_R(R[x, y]) = \{f\Delta_P \mid P \in \mathcal{L}, f \in R[P]\}$. Moreover, the irreducible elements of $\mathrm{LND}_R(R[x, y])$ are precisely $\{\Delta_P \mid P \in \mathcal{L}\}$.

4.4.2 Case $(DB) = B$

Another key result about locally nilpotent R-derivations of $R[x, y]$ is the following. It concerns a larger class of rings than HCF-rings, but a smaller class of derivations than $\mathrm{LND}_R(R[x, y])$.

Theorem 4.15 *Let R be any commutative $\mathbb{Q}$-algebra, and let $B = R[x, y] = R^{[2]}$. Given $D \in \mathrm{LND}_R(R[x, y])$, the following conditions are equivalent.*

(1) *$(DB) = B$, where (DB) is the B-ideal generated by DB.*
(2) *There exists $s \in B$ with $Ds = 1$*

In addition, when these conditions hold, $\ker D = R^{[1]}$.
This result was proved by Daigle and the author in [78] (1998) for the case R is a UFD containing $\mathbb{Q}$. Since $k^{[n]}$ is a UFD, this could then be applied to questions about free $\mathbb{G}_a$-actions on affine space (see *Corollary 4.27* below). Subsequently, Bhatwadekar and Dutta showed that the result holds when R is a noetherian integral domain containing $\mathbb{Q}$ [27] (1997). Ultimately, Berson, van den Essen and Maubach proved the theorem in the general form above. Their work was motivated, in part, by certain questions relating to the Jacobian conjecture, questions in which one cannot always assume that the base ring R is even a domain.

The proof given here is for the case R is a UFD, and is a modified version of the one in [78]; see also Thm. 6.7 of [68]. We first need the following lemma.

Lemma 4.16 (Lemma 2.6 of [78]) *Suppose R is an integral domain containing $\mathbb{Q}$, and $F \in R[x, y]$. If Δ_F is locally nilpotent and if the ideal (F_x, F_y) contains 1, then* $\ker \Delta_F = R[F]$.

Proof Let K be the quotient field of R, and extend Δ_F to the locally nilpotent derivation δ on $K[x, y]$. By Rentschler's Theorem, $\ker \delta = K[G]$ for some $G \in K[x, y]$. Write $F = \varphi(G)$ for a univariate polynomial φ with coefficients in K. Choose $u, v \in R[x, y]$ so that $uF_x + vF_y = 1$. Then $1 = u\varphi'(G)G_x + v\varphi'(G)G_y$, which implies that $\varphi'(G)$ is a unit, and the degree of φ is 1. Therefore, $K[F] = K[\varphi(G)] = K[G]$.

We have $\ker \Delta_F = \ker \delta \cap R[x, y] = K[F] \cap R[x, y]$. Suppose that $K[F] \cap R[x, y]$ is not contained in $R[F]$. Choose $\lambda(T) \in K[T]$ (T an indeterminate) of minimal degree so that $\lambda(F) \in R[x, y]$ but $\lambda(F) \notin R[F]$. Since $\partial_x, \partial_y \in \mathrm{LND}_R(R[x, y])$, it follows that $\partial_x(\lambda(F)) = \lambda'(F)F_x$ and $\partial_y(\lambda(F)) = \lambda'(F)F_y$ belong to $R[x, y]$.

Choose $u, v \in R[x, y]$ such that $uF_x + vF_y = 1$. Then $\lambda'(F) = u\lambda'(F)F_x + v\lambda'(F)F_y \in R[x, y]$. By the assumption of minimality on the degree of λ, it follows that $\lambda'(F) \in R[F]$. This implies that the only coefficient of $\lambda(T)$ not in R is the degree-zero coefficient $\lambda(0)$, i.e., if $\mu(T) := \lambda(T) - \lambda(0)$, then $\mu(T) \in R[T]$. But then $\lambda(0) = \lambda(F) - \mu(F) \in R[x, y] \cap K = R$, a contradiction. □

Proof of Theorem 4.15 (UFD Case) The implication (2) ⇒ (1) is clear.

Conversely, suppose (1) holds. Since R is a UFD, there exists $P \in R[x, y]$ with $\ker D = R[P]$, and since D is irreducible, we may assume $D = \Delta_P$. By the preceding results of this section, we know that there exists $Q \in R[x, y]$ such that $DQ \in R$ and $DQ \neq 0$. Choose a minimal local slice ρ so that $\ker D[Q] \subset \ker D[\rho]$; this is possible, since $R[x, y]$ is a UFD, and thus satisfies the ACC on principal ideals. Then clearly $D\rho \in R$, so we may assume Q itself is a minimal local slice.

If $DQ \notin R^*$, there exists a prime element $q \in R$ dividing DQ. Set $\bar{R} = R/qR$, and let $\pi : R[x, y] \to \bar{R}[x, y]$ be the extension of the projection $R \to \bar{R}$ which sends $x \to x$ and $y \to y$. If $h = \pi(P)$, then $\Delta_h \pi = \pi \Delta_P = \pi D$, and by induction $\Delta_h^n \pi = \pi D^n$ for $n \geq 1$. Thus, given $\sigma \in \bar{R}[x, y]$, if $\sigma = \pi(\tau)$ for $\tau \in R[x, y]$, then $\Delta_h^n(\sigma) = \Delta_h^n(\pi(\tau)) = \pi D^n(\tau) = 0$ for n sufficiently large. Therefore, Δ_h is locally nilpotent.

Suppose $1 = uP_x + vP_y$ for $u, v, \in R[x, y]$. Then $1 = \pi(uP_x + vP_y) = \pi(u)h_x + \pi(v)h_y$. By *Lemma 4.16*, it follows that $\ker \Delta_h = \bar{R}[h] = \pi(R[P])$. Now $DQ \in qR$ means $0 = \pi(DQ) = \Delta_h \pi(Q)$. Therefore $\pi(Q) \in \pi(R[P])$, i.e., there exists $f(P) \in R[P]$ and $Q' \in R[x, y]$ such that $Q - f(P) = qQ'$, violating the condition that Q is minimal.

Therefore $DQ \in R^*$. Setting $s = (DQ)^{-1}Q$, we have $Ds = 1$. ◻

Following is a sketch for the proof of *Theorem 4.15* in its most general form. In [22], Berson, van den Essen, and Maubach show that *Theorem 4.15* holds with the added assumption that $\operatorname{div}_R(D) = 0$ (Thm. 3.5). In addition, they show (Prop. 2.8) that if Ω is an integral domain containing $\mathbb{Q}$, then each locally nilpotent Ω-derivation of $\Omega^{[n]}$ has divergence zero. So this already proves the theorem when the base ring is an integral domain containing $\mathbb{Q}$. Now use the following two facts: Let R be any $\mathbb{Q}$-algebra, N the nilradical of R, and $\delta \in \mathrm{LND}(R^{[n]})$ for some n. Then (1) $\operatorname{div}_R(\delta) \in N$; and (2) If the quotient derivation D/N has a slice in $(R/N)^{[n]}$, then D has a slice in $R^{[n]}$. The proof of (1) follows immediately from the domain case by considering $D \bmod \mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of R. The proof of (2) follows from Lemma 3.3.3 of van Rossum's thesis [409]. Finally, to complete the proof of the theorem, consider D/N on $(R/N)[x, y]$. Then $\operatorname{div}_{R/N}(D/N) = 0$ by (1), and therefore D/N has a slice. By (2), it follows that D itself has a slice.

Also in the paper of Berson, van den Essen, and Maubach is the following result, which is related to their investigation of the Jacobian Conjecture.

Proposition 4.17 (Thm. 4.1 of [22]) *Let R be any commutative $\mathbb{Q}$-algebra. If $D \in \mathrm{Der}_R(R[x, y])$ is surjective and has divergence zero, then $D \in \mathrm{LND}_R(R[x, y])$.*

This result was shown earlier by Stein [391] in the case R is a field,[2] and by Berson [20] (Thm. 3.6) in the case R is a commutative noetherian $\mathbb{Q}$-algebra.

It was stated in the *Introduction* that we would like to understand which properties of a locally nilpotent derivation come from its being a derivation, and which are special to the condition of being locally nilpotent. For this comparison, we quote the following result, which is due to Berson.

[2]The authors of the paper [22] mistakenly omitted the divergence condition when they quoted the result of Stein in their introduction.

Proposition 4.18 (Prop. 2.3 of [20]) *R is a UFD and $D \in \mathrm{Der}_R(R[x,y])$, where $R[x,y] = R^{[2]}$. If $D \neq 0$, then $\ker D = R[f]$ for some $f \in R[x,y]$.*
This generalizes the earlier result of Nagata and Nowicki in case R is a field ([327], Thm. 2.8). The reader should note that, for a derivation which is not locally nilpotent, it is possible that $\ker D = R$, i.e., $f \in R$. Nowicki gives an example of $d \in \mathrm{Der}_k(k[x,y])$ for a field k with $k[x,y]^d = k$ ([333], 7.3.1).

Another proof of Berson's result is given in [135], although the quotation of the Nagata-Nowicki theorem in the abstract of that paper is incorrect.

Remark 4.19 Let R be a commutative k-algebra and let B be an $\mathbb{A}^2$-fibration over R which is stably trivial. El Kahoui and Mustapha [137] showed that B is a trivial fibration if and only if there exists $D \in \mathrm{LND}_R(B)$ such that $(DB) = B$. Their result is closely related to *Theorem 4.15*.

Remark 4.20 The theorem of Abhyankar, Eakin, and Heinzer used in this section can also be used to prove the following.

Proposition 4.21 (Lemma 2 of [295]) *Suppose that the vector group $G = \mathbb{G}_a^{n-1}$ acts faithfully algebraically on $\mathbb{A}^n$. Then $k[\mathbb{A}^n]^G = k^{[1]}$.*

Proof Let $A \subset k^{[n]}$ be the ring of invariants $k[\mathbb{A}^n]^G$. Then A, being the ring of common invariants for successive commuting $\mathbb{G}_a$-actions, is factorially closed. In addition, since $\dim G = n-1$ and the action is faithful, the transcendence degree of A over k is 1. By *Proposition 4.11*, $A = k^{[1]}$. □

4.4.3 Two Theorems of Bhatwadekar and Dutta

In their paper [27], Bhatwadekar and Dutta studied $\mathrm{LND}_R(R[x,y])$ in the case R is a noetherian integral domain containing $\mathbb{Q}$. In the following result, they characterize irreducible elements whose kernels are polynomial rings.

Theorem 4.22 ([27], Thm. 4.7) *Let R be a noetherian integral domain containing $\mathbb{Q}$ and let $D \in \mathrm{LND}_R(R[x,y])$ be irreducible. Then $\ker D = R^{[1]}$ if and only if either Dx, Dy form a regular $R[x,y]$-sequence or $(Dx, Dy) = (1)$. If $(Dx, Dy) = (1)$ and $\ker D = R[P]$ for $P \in R[x,y]$, then $R[x,y] = R[P]^{[1]}$.*

In case the ring R is normal, they describe kernels as symbolic Rees algebras. If $I \subset R$ is an ideal, the induced symbolic Rees algebra is $\mathbb{Z}$-graded, and is denoted by $\mathcal{R}(I) = \bigoplus_{n \geq 0} I^{(n)} T^n$.

Theorem 4.23 ([27], Thm. 3.5) *Let R be a noetherian normal integral domain containing $\mathbb{Q}$.*

- **(a)** *Given nonzero $D \in \mathrm{LND}_R(R[x,y])$, $\ker D$ is a $\mathbb{Z}$-graded R-algebra, and if R is not a field, there exists an ideal $I \subset R$ of unmixed height one such that $\ker D \cong_R \mathcal{R}(I)$.*
- **(b)** *If $J \subset R$ is an ideal of unmixed height one, then there exists $\delta \in \mathrm{LND}_R(R[x,y])$ such that $\ker \delta \cong_R \mathcal{R}(J)$.*

In addition, they give a sufficient condition for a kernel to be finitely generated over R.

Corollary 4.24 ([27], Cor. 3.7) *Let R be a noetherian normal integral domain containing $\mathbb{Q}$. If $\mathrm{Cl}(R)/\mathrm{Pic}(R)$ is torsion, then $\ker D$ is finitely generated over R for every $D \in \mathrm{LND}(R[x, y])$.*

Here, $\mathrm{Cl}(R)$ denotes the class group of R and $\mathrm{Pic}(R)$ denotes the Picard group of R. They also give the following example.

Example 4.25 ([27], Example 3.6) Let $\mathbb{C}[x, y, z] = \mathbb{C}^{[3]}$ and

$$R = \mathbb{C}[x, y, z]_{(x,y,z)}/(F)$$

where $F = y^2z - x^3 + xz^2$ and $\mathbb{C}[x, y, z]_{(x,y,z)}$ denotes homogeneous localization at the ideal (x, y, z). Note that F defines a smooth elliptic curve in the projective plane. The authors show that there exists $D \in \mathrm{LND}_R(R^{[2]})$ whose kernel is isomorphic to $\mathcal{R}(P)$, where P is a height-one prime of R for which no symbolic power $P^{(n)}$ is principal $(n \geq 1)$. This implies $\ker D$ is not finitely generated over R.

4.4.4 Stable Tameness

For a commutative k-domain R and for $n \geq 1$, the group of R-automorphisms of $R^{[n]}$ is denoted by $GA_n(R)$. The definitions of the linear subgroup $GL_n(R)$ and triangular subgroup $BA_n(R)$ are analogous to those for the case when R is a field. The tame subgroup $TA_n(R)$ is generated by $GL_n(R)$ and $BA_n(R)$.

If R is not a field, then the theorems of Jung and Van der Kulk do not generalize to $GA_2(R)$. For example, let $a \in R$ be a non-unit and define $D \in \mathrm{LND}_R(R[x, y])$ by $D = a\partial_x - 2x\partial_y$. Then $D(ay + x^2) = 0$ and:

$$\exp((ay + x^2)D) = (x + a(ay + x^2), y - 2x(ay + x^2) - a(ay + x^2)^2)$$

Nagata [325] showed that $\exp((ay + x^2)D) \notin TA_2(R)$, so not all elements of $GA_2(R)$ are tame over R. However, M. Smith [386] showed that it factors as a product of tame automorphisms of $R[x, y, z]$ (see *Sect. 3.8.3*).

An element $\alpha \in GA_2(R)$ is **stably tame** if $\alpha \in TA_n(R)$ under the natural inclusion of $GA_2(R)$ in $GA_n(R)$ for some $n \geq 2$. Berson, van den Essen and Wright have shown the following.

Theorem 4.26 ([23], Thm. 4.10 and Thm. 4.6) *If R is regular, then every element of $GA_2(R)$ is stably tame. If R is a Dedekind domain containing $\mathbb{Q}$, then $GA_2(R) \subset TA_5(R)$.*

4.5 Rank-Two Derivations of Polynomial Rings

In working with polynomial rings, a natural invariant of a given derivation is its rank. Specifically, let B denote the polynomial ring in n variables over k for $n \geq 2$. From *Chap. 3*, recall that a k-derivation D of B has rank at most 2 if and only if there exist $x_1, \ldots, x_{n-2} \in B$ such that $B = k[x_1, \ldots, x_n]$ and $k[x_1, \ldots, x_{n-2}] \subset \ker D$. In this case, D is an R-derivation of $B = R^{[2]}$, where $R = k[x_1, \ldots, x_{n-2}]$, a UFD.

Suppose $D \in \mathrm{LND}(B)$ is of rank at most 2 on B, and also satisfies the condition $(DB) = B$. Geometrically, this implies that the corresponding $\mathbb{G}_a$-action on $\mathbb{A}^n$ is fixed-point free. By *Theorem 4.15*, there exists a slice $s \in B$ of D, i.e., $Ds = 1$. Suppose $\ker D = R[P]$ for $P \in B$. By the Slice Theorem, $B = R[P, s] = k[x_1, \ldots, x_{n-2}, P, s]$, meaning that P and s are variables of B, and $D = \partial_s$ relative to this coordinate system on B. So the rank of D is one in this case. We thus obtain:

Corollary 4.27 *For $n \geq 3$, every rank-two algebraic action of $\mathbb{G}_a$ on $\mathbb{A}^n$ has fixed points.*

4.5.1 *Variable Criterion*

Corollary 4.28 (Variable Criterion) *For $n \geq 2$, let $R = k^{[n-2]}$, $K = \mathrm{frac}(R)$, and $B = R[x, y] = k^{[n]}$. Given $P \in B$, the following are equivalent.*

(1) $K[x, y] = K[P]^{[1]}$ *and* $(P_x, P_y) = (1)$
(2) $B = R[P]^{[1]}$

Proof If (1) holds, then Δ_P has $(\Delta_P B) = B$. By *Theorem 4.15*, there exists $Q \in R[x, y]$ with $\Delta_P Q = 1$. By *Lemma 4.16*, $\ker \Delta_P = R[P]$. It follows from the Slice Theorem that $B = R[P, Q]$, so condition (2) holds.

Conversely, if (2) holds, then $B = R[P, Q]$ for some Q, and thus $K[P, Q] = K[x, y]$. Since (P, Q) defines a R-automorphism of $R[x, y]$, we have

$$\det \frac{\partial(P, Q)}{\partial(x, y)} = P_x Q_y - P_y Q_x \in R^*$$

implying that $(P_x, P_y) = (1)$. □

Example 4.29 In [53], Choudary and Dimca show that the hypersurfaces $X \subset \mathbb{C}^4$ defined by equations of the form $f = 0$ for

$$f = x + x^{d-1}y + y^{d-1}z + t^d \quad (d \geq 1)$$

are coordinate hyperplanes (i.e., f is a variable of $k[x, y, z, t]$ when $k = \mathbb{C}$). We show the same holds for any field k of characteristic zero. First, if $K = k(y, t)$, then f is a triangular K-variable of $K[x, z]$. Second,

$$(f_x, f_z) = (1 + (d-1)x^{d-2}y, y^{d-1}) = (1)$$

so the conditions of the Variable Criterion are satisfied. □

Even for $n = 3$, the rank-two case is already of interest. Here, we consider locally nilpotent R-derivations of $B = R[y, z]$, where $R = k[x]$. Note that R is not only a UFD in this case, but even a PID. This class of derivations includes all triangular derivations of B (in an appropriate coordinate system). The first examples of $D \in \mathrm{LND}(k[x, y, z])$ with $\mathrm{rank}(D) = 3$ were found in 1996; see *Chap. 5*. Such D cannot be conjugated to an element of $\mathrm{LND}_R(R[y, z])$.

Example 4.30 In his 1988 paper [387], Snow proved a special case of *Corollary 4.27*, namely, that a free triangular action of $\mathbb{G}_a$ on $\mathbb{A}^3$ is conjugate to a coordinate translation. In Example 3.4 of [388], Snow defines $D \in \mathrm{LND}_R(R[y, z])$ by $Dy = x^2$ and $Dz = 1 - 2xy$, where $R = k[x]$. This satisfies $(DB) = B$, so there exist $u, v \in B$ such that $B = R[u, v]$, $\ker D = R[u]$, and $Dv = 1$. To find such u, consider the image of z under the Dixmier map π_y relative to the local slice y:

$$\pi_y(z) \;=\; z - Dz\frac{y}{Dy} + \frac{1}{2}D^2z\frac{y^2}{(Dy)^2} \;=\; -x^{-2}(y - xF)$$

where $F = xz + y^2$. We see that, if $K = k(x)$, then $K[y, z] = K[y, F] = K[y - xF, F]$, so $u := y - xF$ is a variable of $K[y, z]$. It follows that $\ker D = R[y - xF]$. To find v, note that $0 = Du = Dy - xDF \Rightarrow DF = x$; and since $u \equiv y (\mathrm{mod}\, x)$ and $F \equiv y^2 (\mathrm{mod}\, x)$, x divides $F - u^2$, where $D(F - u^2) = x$. Thus, if $v = x^{-1}(F - u^2)$, then $Dv = 1$. Specifically, $v = z + 2yF - xF^2$. In other words, $D = \alpha(\partial_z)\alpha^{-1}$, where α is the automorphism of B defined by:

$$\alpha = (x, u, v) = (x, y - xF, z + 2yF - xF^2)$$

We recognize this as the Nagata automorphism of $k[x, y, z]$; see *Chap. 3*.

4.5.2 Applications to Line Embeddings

Set $B = k[x, y, z] = k^{[3]}$. In [1], Abhyankar introduced the algebraic embeddings $\theta_n : \mathbb{A}^1 \hookrightarrow \mathbb{A}^3$ defined by $\theta_n(t) = (t + t^{n+2}, t^{n+1}, t^n)$. For $n \leq 4$, it was shown by Craighero [65, 66] that θ_n is **rectifiable**, i.e., there exist $u, v, w \in k[x, y, z]$ such that $B = k[u, v, w]$ and $\theta_n^* : B \to k[t]$ is given by:

$$\theta_n^*(u) = \theta_n^*(v) = 0 \quad \text{and} \quad \theta_n^*(w) = t$$

Another proof of rectifiability for $n = 3, 4$ is given by Bhatwadekar and Roy in [29].

Note that, over the field $k = \mathbb{R}$, the image of θ_n is topologically a line (unknot), since this image is contained in the surface $y^n = z^{n+1}$, which is homeomorphic to a coordinate plane. The question is whether this curve can be straightened algebraically.

The proof of Bhatwadekar and Roy is based on the following result, which is due to Russell.

Proposition 4.31 (Prop. 2.2 of [363]) *Let $P \in k[x, y, z]$ have the form*

$$P = x + f(x, z)z + \lambda z^s y$$

where $f \in k[x, z]$, $s \in \mathbb{N}$ and $\lambda \in k^*$. Then P is a variable of $k[x, y, z]$.

Proof Set $R = k[z]$ and $K = k(z)$. Then P is a triangular K-variable of $K[x, y]$, and $(P_x, P_y) = (1 + f_x z, z^s) = (1)$. It follows from the Variable Criterion that P is a variable of $k[x, y, z]$. □

Example 4.32 Consider $\theta_3 = (t + t^5, t^4, t^3)$. We seek P of the form above such that $P(t + t^5, t^4, t^3) = t$, and one easily finds that $P := x - (x^2 - 2z^2)z + z^3 y$ works. By *Proposition 4.31*, P is a variable, and therefore, θ_3 is rectifiable. We find $u, v \in B$ so that $B = k[u, v, P]$ and the kernel of $\theta_3^* : B \to k[t]$ equals (u, v).

We first need Q with $\Delta_P Q = 1$. Note that

$$\ker \Delta_P = k[z, P] = k[z, x + zF + 2z^3] = k[z, x + zF]$$

where $F = z^2 y - x^2$. Therefore:

$$0 = \Delta_P(x + zF) = \Delta_P x + z\Delta_P F = z^3 + z\Delta_P F \quad \Rightarrow \quad \Delta_P F = -z^2$$

One then checks that $G := zy + 2xF + zF^2$ has $\Delta_P G = -z$. Likewise, $Q := -(y + 2xG + z^2 G^2)$ has $\Delta_P Q = 1$. Therefore, $B = k[z, P, Q]$.

Finally, suppose $\theta_3^*(Q) = f(t)$ for $f(t) \in k[t]$. If $u = z - P^3$ and $v = Q - f(P)$, then $B = k[u, v, P]$ and $\ker \theta_3^* = (u, v)$.

The foregoing method of finding Q is *ad hoc*. In [21], Berson and van den Essen give an algorithm for finding Q such that $R[x, y] = R[P, Q]$, given that $R[x, y] = R[P]^{[1]}$.

Next, consider θ_4. It is not hard to discover P of the form described in the proposition above which satisfies $P(t + t^6, t^5, t^4) = t$, namely:

$$P := x + (2x^3 z - x^2 - 2xz^4 - 5z^3)z - 4z^4 y$$

Since P is a variable of $k[x, y, z]$, θ_4 is rectifiable. However, to also find kernel generators for θ_4^* which are variables, as we did for θ_3^*, is far more complicated in this case. To this end, the interested reader is invited to implement the aforementioned algorithm of Berson and van den Essen.

It remains an open question whether, for any $n \geq 5$, θ_n can be rectified. However, it appears unlikely that a variable of the form described in the proposition can be used to straighten θ_5.

As a final remark about line embeddings, note that the rectifiable embeddings are precisely those defined by $\alpha(t, 0, 0)$, where $\alpha \in GA_3(k)$. If we could describe all locally nilpotent derivations of $k[x, y, z]$, then one would at least get a description of all the exponential embeddings $\alpha(t, 0, 0)$, where α is an exponential automorphism. The embeddings θ_3 and θ_4 are of this type. Consider the automorphism $\exp(D)$ of $\mathbb{A}^3$, where D is the $(2, 5)$ derivation of $k[x, y, z]$ discussed in *Chap. 5*: By setting $x = 0$, $y = t$, and $z = 0$, we obtain the rectifiable embedding $\phi : \mathbb{A}^1 \to \mathbb{A}^3$ defined by $\phi(t) = (t^5, t + t^{13}, t^9 + t^{21})$. It seems unlikely that a straightening automorphism for ϕ could be found directly from its definition.

4.6 Newton Polygon of a Derivation

The Newton polygon $\mathrm{Newt}(D)$ of a locally nilpotent derivation of $B = k[x, y]$ is defined in the following way. B admits a natural $\mathbb{Z}^2$-grading, namely,

$$B = \sum_{(i,j) \in \mathbb{Z}^2} B_{(i,j)}$$

where $B_{(i,j)} = k \cdot x^i y^j$ if $i, j \geq 0$, and otherwise $B_{(i,j)} = 0$. Note that the partial derivatives ∂_x and ∂_y are $\mathbb{Z}^2$-homogeneous, of degrees $(-1, 0)$ and $(0, -1)$, respectively. In addition, by *Proposition 3.8*, any $D \in \mathrm{LND}(B)$ can be decomposed as

$$D = \sum_{(i,j) \in \mathbb{Z}^2} D_{(i,j)}$$

where $D_{(i,j)}$ is $\mathbb{Z}^2$-homogeneous of degree (i, j), and $D_{(i,j)} = 0$ if $i < -1$ or $j < -1$. Define the **support** of D by

$$\mathrm{Supp}(D) = \{(i, j) \in \mathbb{Z}^2 \mid D_{(i,j)} \neq 0\} \subset \mathbb{Z}^2$$

and define the **Newton polygon** of D to be the convex hull of $\mathrm{Supp}(D) \cup \{(-1, -1)\}$ in $\mathbb{R}^2$. Note that every vertex of $\mathrm{Newt}(D)$ other than $(-1, -1)$ belongs to $\mathrm{Supp}(D)$.

Lemma 4.33 *If $D \neq 0$, then* $\mathrm{Newt}(D)$ *is equal to the triangle with vertices* $(-1, -1)$, $(\lambda, -1)$, *and* $(-1, \mu)$, *where:*

$$\lambda = \max\{\, i \mid D_{(i,-1)} \neq 0\} \cup \{-1\}\ , \quad \mu = \max\{j \mid D_{(-1,j)} \neq 0\} \cup \{-1\}$$

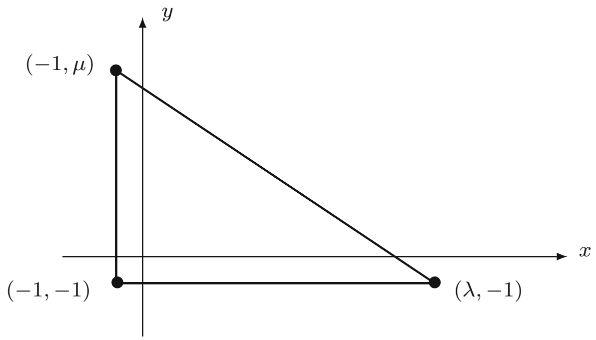

Fig. 4.2 Newton polygon of D

See *Fig. 4.2*. To prove Rentschler's Theorem from here, one shows that the Newton polygon can be reduced by a triangular automorphism. On the other hand, this lemma can be deduced from Rentschler's Theorem.

4.7 Automorphisms Preserving Lattice Points

In this chapter, we have considered rings $R[x, y]$, where the base ring R contains $\mathbb{Q}$. However, there is at least one other important ring of characteristic zero which should be mentioned in this chapter, namely, $\mathbb{Z}[x, y]$, the ring of bivariate polynomials with integral coefficients. Its group of automorphisms is $GA_2(\mathbb{Z}) = \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}[x, y])$, elements of which map the lattice points $\mathbb{Z}^2 \subset \mathbb{R}^2$ into themselves.

Regarding locally nilpotent derivations of $\mathbb{Z}[x, y]$, many of the properties which hold for k-domains carry over to these as well: $\mathrm{LND}(\mathbb{Z}[x, y]) = \mathrm{LND}_{\mathbb{Z}}(\mathbb{Z}[x, y])$, kernels are factorially closed and of the form $\mathbb{Z}[P]$, etc. However, one key difference is that $D \in \mathrm{LND}(\mathbb{Z}[x, y])$ does not generally give a well-defined automorphism by the exponential map. For example, if $D = \partial_x + x\partial_y$, then $D \in \mathrm{LND}(\mathbb{Z}[x, y])$, and $\exp D$ over $\mathbb{Q}$ is given by $\exp D = (x + 1, y + 2x + \frac{1}{2})$, which does not preserve lattice points. Similarly, the Dixmier map may not be defined. What we require is not only that D be locally nilpotent, but also that $D^n x, D^n y \in n!\mathbb{Z}[x, y]$ for all $n \geq 0$.

The group $SL_2(\mathbb{Z}) \subset GA_2(\mathbb{Z})$ plays a central role in the study of lattices and modular forms; see Serre [377]. Serre points out that $SL_2(\mathbb{Z}) = \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$, where $\mathbb{Z}_n$ denotes the cyclic group $\mathbb{Z}/n\mathbb{Z}$. Specifically, $\mathbb{Z}_4$ and $\mathbb{Z}_6$ are generated by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

respectively, and $\mathbb{Z}_2 = \{\pm I\}$.

As in the Structure Theorem over fields, the full automorphism group $GA_2(\mathbb{Z})$ also admits a description as an amalgamated free product, though this description is more complicated. In his paper [424], in the Corollary to Thm. 5, Wright describes the decomposition of the group $GA_2(K)$ for any principal ideal domain K. His main interest is in the case $K = k^{[1]}$, but the result holds for $\mathbb{Z}$ as well.

Appendix: Newton Polytopes

It is natural to define higher dimensional analogues of Newton polygons and investigate their properties relative to locally nilpotent derivations of polynomial rings. Such investigation is the subject of papers by Hadas, Makar-Limanov, and Derksen [97, 196, 197].

Let $f \in B = k^{[n]}$ be given. Relative to a coordinate system $B = k[x_1, \ldots, x_n]$, we define the **Newton polytope** of f, denoted $\mathrm{Newt}(f)$, as follows. Write:

$$f = \sum \alpha_e x_1^{e_1} \cdots x_n^{e_n} \ , \quad e = (e_1, \ldots, e_n) \in \mathbb{Z}^n \ , \ e_i \geq 0 \ , \ \alpha_e \in k$$

The **support** of f, denoted $\mathrm{Supp}(f)$, equals $\{e \in \mathbb{Z}^n | \alpha_e \neq 0\}$; and $\mathrm{Newt}(f)$ equals the convex hull of $\mathrm{Supp}(f) \cup \{0\}$ in $\mathbb{R}^n$. Note that this definition depends on the choice of coordinates for B.

For such polynomials, the following special property emerges.

Proposition 4.34 (Thm. 3.2 of [197]) *Suppose* $B = k[x_1, \ldots, x_n] = k^{[n]}$, $D \in \mathrm{LND}(B)$, $D \neq 0$, *and* $f \in \ker D$. *Then every vertex of* $\mathrm{Newt}(f)$ *is contained in a coordinate hyperplane. More precisely, if* $q = (q_1, \ldots, q_n) \in \mathbb{Q}^n$ *is a vertex of* $\mathrm{Newt}(f)$, *then* $q_i = 0$ *for at least one i.*

Proof Suppose $q \neq 0$. By convexity, there exists a hyperplane $H \subset \mathbb{R}^n$ such that $H \cap \mathrm{Newt}(f) = \{q\}$. We may suppose that H is defined by the equation:

$$\sum_{1 \leq i \leq n} a_i y_i = d \ , \quad y = (y_1, \ldots, y_n) \in \mathbb{R}^n \ , \ a_i, d \in \mathbb{Z} \ , \ d > 0$$

Since $0 \in \mathrm{Newt}(f)$, we conclude that $\sum a_i y_i \leq d$ for $y \in \mathrm{Newt}(f)$, with equality only at q.

Note that H determines a $\mathbb{Z}$-grading on B, namely, the degree of the monomial $x_1^{t_1} \cdots x_n^{t_n}$ is $\sum_{i=1}^n a_i t_i$. For this grading, write $B = \oplus_{i \in \mathbb{Z}} B_i$; then $f = \sum_{i=0}^d f_i$ for $f_i \in B_i$, and f_d is the monomial supported at q. In addition, if D' is the highest-degree homogeneous summand of D relative to this grading, then according to *Corollary 1.31*, $D' \in \mathrm{LND}(B)$ and $f_d \in \ker D'$. Since $D \neq 0$, $D' \neq 0$ as well. If each q_i were strictly positive, then since $\ker D'$ is factorially closed and f_d is a monomial, we would have that $x_i \in \ker D'$ for each i, implying $D' = 0$. Therefore, at least one of the q_i equals 0. □

Remark 4.35 Any variable f of $B = k^{[n]}$ is in the kernel of some nonzero $D \in \mathrm{LND}(B)$, for example, $D = \partial_g$ where (f, g) is a partial system of variables for

B. Thus, any variable possesses the property described in the theorem above. In fact, it is shown in [97] that, regardless of the characteristic of the field k, the Newton polytope of an invariant of an algebraic $\mathbb{G}_a$-action on $\mathbb{A}^n$ has all its vertices on coordinate hyperplanes (Thm. 3.1). Thus, this property applies to any variable, regardless of characteristic.

A second property of Newton polytopes involves its edges. Continuing the assumptions and notations above, an edge E of $\mathrm{Newt}(f)$ is called an **intrusive edge** if it is contained in no coordinate hyperplane of $\mathbb{R}^n$. (Such edges are called **trespassers** in [97].) If E is the intrusive edge joining vertices p and q, then $e(E)$, or just e, will denote the vector $e = p - q$ (or $q - p$, it doesn't matter).

Proposition 4.36 (Thm. 4.2 of [197]) *Suppose* $f \in \ker D$ *for* $D \in \mathrm{LND}(B)$ *and* $D \neq 0$. *Let* E *be an intrusive edge of* $\mathrm{Newt}(f)$, *where* $e = (e_1, \ldots, e_n) = p - q$ *for vertices* $p = (p_1, \ldots, p_n)$ *and* $q = (q_1, \ldots, q_n)$ *of* $\mathrm{Newt}(f)$.

(a) *There exist integers* $r \neq s$ *such that* $p_r = 0$, $q_s = 0$, *and either* $e \in q_r \cdot \mathbb{Z}^n$ *or* $e \in p_s \cdot \mathbb{Z}^n$

(b) $\min_{1 \leq i \leq n}\{e_i\} < 0$ *and* $\max_{1 \leq i \leq n}\{e_i\} > 0$

Proof (Following [197]) We first make several simplifying assumptions and observations.

- Given i, set $B_i = k[x_1, \ldots, \hat{x}_i, \ldots, x_n]$. Note that $f \notin \cup_{i=1}^n B_i$, since the Newton polytope of an element of B_i has no intrusive edges.
- Suppose this result holds for algebraically closed fields of characteristic zero, and let L denote the algebraic closure of k. If D_L denotes the extension of D to $B_L = L[x_1, \ldots, x_n]$, then $D_L \in \mathrm{LND}(B_L)$ and $f \in \ker(D_L)$. Note that the Newton polytope of f relative to D or D_L is the same. So from now on we assume k is algebraically closed.
- To simplify notation, for $u \in \mathbb{Z}^n$, x^u will denote the monomial $x_1^{u_1} \cdots x_n^{u_n}$.
- After a permutation of the variables, we may assume that $x_1, \ldots, x_\gamma \in \ker D$ and $x_{\gamma+1}, \ldots, x_n \notin \ker D$, where γ is an integer with $0 \leq \gamma \leq n$.

We now proceed with the proof.

To show (a), first choose a $\mathbb{Q}$-linear function $\lambda : \mathbb{Q}^n \to \mathbb{R}$ such that $\ker(\lambda) = \langle e \rangle$. (For example, choose a projection $\mathbb{Q}^n \to \mathbb{Q}^{n-1}$ mapping E to a single point, and then pick your favorite copy of $\mathbb{Q}^{n-1}$ in $\mathbb{R}$, such as $\mathbb{Q}[\zeta]$ for a real algebraic number ζ of degree $n-1$.) Thus, λ is constant on E, and we may further assume that $\lambda(E) = \max \lambda(\mathrm{Newt}(f)) \geq 0$. Such λ induces a grading $B = \oplus_{\rho \in \mathbb{R}} B_\rho$ in which the degree of the monomial x^u equals $\lambda(u)$. Elements of B_ρ will be called λ-homogeneous of λ-degree ρ.

As before, if D' is the highest λ-homogeneous summand of D, and if f' is the highest λ-homogeneous summand of f, then D' is locally nilpotent, $D'(f') = 0$, and $\mathrm{Newt}(f')$ is the convex hull of $0, p$ and q. So we might just as well assume $D = D'$ and $f = f'$.

Choose $\epsilon \in \mathbb{Q}^n$ which spans the kernel of λ. We may assume that $\epsilon \in \mathbb{Z}^n$, and that the entries of ϵ have no common factor. It follows that every λ-homogeneous element of B can be written as $x^\alpha P(x^\epsilon)$, where $\alpha \in \mathbb{Z}^n$ has non-negative entries, and P is a univariate polynomial over k. (Note that some entries of ϵ can be negative.) In particular, $f = x^\alpha P(x^\epsilon)$ for some α and some P. This implies that we may write

$$f = x^\alpha \prod_i (x^\epsilon - t_i) = x^w \prod_i (x^u - t_i x^v)$$

where $t_i \in k$, $u, v, w \in \mathbb{Z}^n$ have non-negative entries, and $u_i v_i = 0$ for each i. Moreover, since every x_i appears in f, it follows that $u_i + v_i + w_i > 0$ for each i.

Since e is an intrusive edge, f is divisible by at least one factor of the form $x^u - tx^v$, $t \in k$, meaning $x^u - tx^v \in \ker D$. If also $x^u - sx^v$ divides f for $s \neq t$, then we have $x^u, x^v \in \ker D$. Since $\ker D$ is factorially closed, $x^w \in \ker D$ as well. But then since $u_i + v_i + w_i > 0$, it would follow that $x_i \in \ker D$ for every i (again since $\ker D$ is factorially closed), which is absurd. Therefore $\ker D$ contains $x^u - tx^v$ for exactly one value of t, and $t \neq 0$. Altogether, this implies $f = cx^w(x^u - tx^v)^m$ for some $c \in k^*$ and positive integer m.

In the same way, if $g \in \ker D$ is any other λ-homogeneous element, then $g = dx^\omega(x^u - tx^v)^\mu$, where $d \in k$, $\omega \in \mathbb{Z}^n$ has non-negative entries, and μ is a non-negative integer. Since $\ker D$ is factorially closed, $x^\omega \in k[x_1, \ldots, x_\gamma]$. It follows that $\ker D = k[x_1, \ldots, x_\gamma, x^u - tx^v]$, and since the transcendence degree of $\ker D$ is $n-1$, we have $\gamma + 1 = n - 1$, or $\gamma = n - 2$.

Let $K = k(x_1, \ldots, x_{n-2})$. Then D extends to a locally nilpotent K-derivation of $R = K[x_{n-1}, x_n]$, so by Rentschler's Theorem $x^u - tx^v$ is a K-variable of R. As such, it must have a degree-one term in either x_{n-1} or x_n over K.

Let us assume $u_n = 1$ (so $v_n = 0$). Recalling that $\epsilon = u - v$, we have that $\epsilon_n = 1$. Now $f = x^w(x^u - tx^v)^m$, where $w_n = 0$. It follows that $p = mu + w$ and $q = mv + w$. Since $v_n = 0$, we conclude that $q_n = 0$ as well. Since $e = p - q$ is an integral multiple of ϵ, and $e_n = p_n$, we conclude that $e = p_n\epsilon$. The other cases are similar. This completes the proof of (a).

By the preceding result, both p and q have at least one 0 entry, and since e is an intrusive edge, we can find r and s with $p_r = 0$, $q_s = 0$, and $r \neq s$. Thus, $e_r = p_r - q_r = -q_r \leq 0$ and $e_s = p_s - q_s = p_s \geq 0$, and (b) follows. □

As noted in [197], these two results provide a quick way to determine that certain polynomials cannot be annihilated by a nonzero locally nilpotent derivation. For example, if $a, b, c \in \mathbb{Z}$ have $a, b, c > 1$ and $\gcd(a, b, c) = 1$, then $f = x^a + y^b + z^c$, $g = x^a y^b + z^c$, and $h = x^{a+1}y^b + x^a z^c + 1$ are elements of $k[x, y, z]$ not in the kernel of any nonzero $D \in \mathrm{LND}(k[x, y, z])$. In the first case, $\mathrm{Newt}(f)$ has the intrusive edge $(a, -b, 0)$. In the second case, $\mathrm{Newt}(g)$ has the intrusive edge $(a, b, -c)$. An in the third case, $\mathrm{Newt}(h)$ has vertices $p = (a+1, b, 0)$ and $q = (a, 0, c)$, so that $e = (1, b, -c)$ is an intrusive edge which belongs to neither $b\mathbb{Z}^3$ nor $c\mathbb{Z}^3$. Consequently, the surfaces in $\mathbb{A}^3$ defined by f, g and h are not stabilized by any $\mathbb{G}_a$-action on $\mathbb{A}^3$.

Example 4.37 Let $B = k[x, y, z] = k^{[3]}$, and define $G \in B$ by:

$$G = x^2z^3 - 2xy^2z^2 + y^4z + 2x^3yz - 2x^2y^3 + x^5$$

In *Chap.* 5 it is shown that G is irreducible, and that there exist nonzero $D \in \mathrm{LND}(B)$ with $DG = 0$. Since G is homogeneous in the standard sense, of degree 5, its support is contained in the plane H defined by $x_1 + x_2 + x_3 = 5$. Let Q be the convex hull of $\mathrm{Supp}(G)$, which is contained in H. It is easy to see that the vertex set of Q is

$$\{(2,0,3),(0,4,1),(2,3,0),(5,0,0)\}$$

so Q is a quadrilateral, and $\mathrm{Newt}(G)$ consists of a cone over Q. Thus, $\mathrm{Newt}(G)$ has 5 faces. Two of these faces lie in coordinate planes, and 3 do not (call these **intrusive faces**). There are two intrusive edges, namely, E_1 joining $(0,4,1)$ and $(2,0,3)$, and E_2 joining $(0,4,1)$ and $(2,3,0)$. They define vectors $e_1 = (-2,4,-2)$ and $e_2 = (-2,1,1)$.

One can also consider higher-dimensional faces of Newton polytopes for polynomials f annihilated by locally nilpotent derivations, and thereby get further conditions on $\mathrm{Newt}(f)$. The last section of [97] gives some results in this direction.

Remark 4.38 Define $f = x^2 + y^2 + z^2 \in \mathbb{C}[x,y,z]$, and $D \in \mathrm{LND}(\mathbb{C}[x,y,z])$ by:

$$Dx = -z\ ,\ \ Dy = -iz\ ,\ \ Dz = x + iy$$

Then $Df = 0$. Likewise, $f \in \mathbb{R}[x,y,z]$, but it is shown in *Chap.* 9 below that there is no nonzero $\delta \in \mathrm{LND}(\mathbb{R}[x,y,z])$ with $\delta f = 0$. This example points out the limitations of the information provided by $\mathrm{Newt}(f)$.

Chapter 5
Dimension Three

In the study of $\mathbb{G}_a$-actions on $\mathbb{A}^n$, the dimension-three case stands between the fully developed theory in dimension two, and the wide open possibilities in dimension four. The fundamental theorems of this chapter show that many important features of planar $\mathbb{G}_a$-actions carry over to dimension three: Every invariant ring is a polynomial ring; the quotient map is always surjective; and free $\mathbb{G}_a$-actions are translations. It turns out that none of these properties remains generally true in dimension four. In addition, there exist locally nilpotent derivations in dimension three of maximal rank 3, and these have no counterpart in dimension two. Such examples will be explored in this chapter.

As mentioned in the *Introduction*, a big impetus was given to the study of $\mathbb{G}_a$-actions by the appearance of Bass's 1984 paper, which showed that, in contrast to the situation for $\mathbb{A}^2$, there exist algebraic actions of $\mathbb{G}_a$ on $\mathbb{A}^3$ which cannot be conjugated to a triangular action. Since then, our understanding of the dimension three case has expanded dramatically, though it remains far from complete.

Parallel to developments for unipotent actions is the theorem of Koras and Russell, which asserts that every algebraic $\mathbb{C}^*$-action on $\mathbb{C}^3$ can be linearized [251]. Their result was then generalized to the following.

Every algebraic action of a connected reductive group G on $\mathbb{C}^3$ can be linearized.

See Popov [346]. On the other hand, the question whether every algebraic action of a finite group on $\mathbb{C}^3$ can be linearized remains open.

Throughout this chapter, B will denote the three dimensional polynomial ring $k[x, y, z]$ over k.

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5.1 Miyanishi's Theorem

In the late 1970s, Fujita, Miyanishi, and Sugie succeeded in proving the Cancellation Theorem for surfaces, which asserts that if Y is an affine surface over $\mathbb{C}$, and if $Y \times \mathbb{C}^n \cong \mathbb{C}^{n+2}$ for some $n \geq 0$, then $Y \cong \mathbb{C}^2$ [177, 307]. Key elements of their proof provided the foundation for Miyanishi's subsequent proof that the quotient Y of a nontrivial $\mathbb{C}^+$-action on $\mathbb{C}^3$ is isomorphic to $\mathbb{C}^2$. In each case, the crucial step is to show that the surface Y contains a cylinderlike open set, i.e., an open set of the form $K \times \mathbb{C}$ for some curve K. This is very close (in fact, equivalent) to saying that Y admits a $\mathbb{C}^+$-action.

The theorem of Miyanishi, which appeared in 1985, established the most important fact about locally nilpotent derivations and $\mathbb{G}_a$-actions in dimension three.

Theorem 5.1 (Miyanishi's Theorem) *Let k be a field of characteristic zero. The kernel of any nonzero locally nilpotent derivation of $k^{[3]}$ is a polynomial ring $k^{[2]}$.*

This theorem means that there exist $f, g \in k[x, y, z]$, algebraically independent over k, such that the kernel is $k[f, g]$. But in contrast to the situation for dimension 2, it is not necessarily the case that f and g form a partial system of variables, i.e., $k[x, y, z] \neq k[f, g]^{[1]}$ generally.

Miyanishi's geometric result is equivalent to the case $k = \mathbb{C}$ of the theorem. Applying Kambayashi's theorem on plane forms gives the proof for any field k of characteristic zero (see *Sect. 5.1.1* below). Miyanishi's first attempts to prove the theorem appeared in [298] (1980) and [299] (1981), but the arguments were incomplete. Then in 1985, Miyanishi sketched a correct proof for the field $k = \mathbb{C}$ in his paper [301]. Certain details for the complete proof were later supplied by Sugie's 1989 paper [395].

An independent proof of Miyanishi's Theorem was given by Kaliman and Saveliev [232]. Specifically, they show that if X is a smooth contractible affine algebraic threefold over $\mathbb{C}$ which admits a nontrivial $\mathbb{C}^+$-action, then the affine surface $S = X/\!\!/\mathbb{C}^+$ is smooth and contractible. If, in addition, X admits a dominant morphism from $\mathbb{C}^2 \times \Gamma$ for some curve Γ (for example, $X = \mathbb{C}^3$), then $S \cong \mathbb{C}^2$.

In the early 1990s, Zurkowski proved an important special case of Miyanishi's Theorem in the manuscript [431]. Zurkowski was apparently unaware of Miyanishi's result, as Miyanishi's work is not cited in the paper, and Zurkowski describes his result as "a step towards extending results of [Rentschler] to three dimensional space" (p. 3). Specifically, what Zurkowski showed was that, if k is an algebraically closed field of characteristic zero, and if $D \neq 0$ is a locally nilpotent derivation of $B = k[x, y, z]$ which is homogeneous relative to some positive $\mathbb{Z}$-grading on B, then $\ker D = k[f, g]$ for homogeneous $f, g \in B$. In his master's thesis, Holtackers [211] gives a more streamlined version of Zurkowski's proof. And in [36], Bonnet gives a third proof for the case $k = \mathbb{C}$, also quite similar to Zurkowski's. None of these three proofs was published.

A complete proof of Miyanishi's Theorem will not be attempted here, since both of the existing proofs use geometric methods which go beyond the scope of this book. Miyanishi's proof makes extensive use of the theory of surfaces with negative logarithmic Kodaira dimension, much of which was developed for the proof of cancellation for surfaces. The proof of Kaliman and Saveliev requires some algebraic topology, Phrill-Brieskorn theorems on quotient singularities, and a theorem of Taubes from gauge theory.

The section is organized in the following way. First, the reduction to the field $k = \mathbb{C}$ is given, using Kambayashi's Theorem. Then some general properties of the quotient map are discussed, followed by a description of Miyanishi's proof. Finally, we give a proof of Zurkowski's result which follows Zurkowski's algebraic arguments in the main.

5.1.1 Kambayashi's Theorem

To deduce the general case of Miyanishi's Theorem from the case $k = \mathbb{C}$ we need the following result, which was proved by Kambayashi in [238]; see also Shafarevich [380], Thm. 9.

Theorem 5.2 (Kambayashi's Theorem) *Let k and K be fields such that K is a separably algebraic extension of k. Suppose R is a commutative k-algebra for which $K \otimes_k R \cong K^{[2]}$. Then $R \cong k^{[2]}$.*

Kambayashi's proof relies on the Structure Theorem for $GA_2(k)$.

Now suppose k is any field of characteristic zero, and that $D \in \mathrm{LND}(B)$ for $B = k^{[3]}$, $D \neq 0$. Following is the argument given by Daigle and Kaliman in [84].

Let $k_0 \subseteq k$ be the subfield of k generated over $\mathbb{Q}$ by the coefficients of the polynomials Dx, Dy, and Dz, and set $B_0 = k_0[x, y, z]$. Then D restricts to $D_0 \in \mathrm{LND}(B_0)$. Since k_0 has the form $\mathbb{Q}(F)$ for a finite set F, it is isomorphic to a subfield of $\mathbb{C}$. If we assume $k_0 \subset \mathbb{C}$, then we may extend D_0 to $D' \in \mathrm{LND}(\mathbb{C}[x, y, z])$. By Miyanishi's Theorem for the complex field, together with *Proposition 1.18*, we have:

$$\mathbb{C} \otimes_{k_0} \ker D_0 = \ker D' = \mathbb{C}^{[2]}$$

By Kambayashi's Theorem, it follows that $\ker D_0 = k_0^{[2]}$. Therefore:

$$\ker D = k \otimes_{k_0} \ker D_0 = k^{[2]}$$

5.1.2 Properties of the Quotient Morphism

Let X be an affine threefold over $\mathbb{C}$ which is factorial, i.e., the coordinate ring $\mathcal{O}(X)$ is a UFD. Suppose X admits a nontrivial algebraic action of $\mathbb{C}^+$, and let $Y = X /\!/ \mathbb{C}^+$ denote the quotient of this action. By results of *Chap. 1*, Y is normal and $\mathrm{tr.deg.}_{\mathbb{C}}\mathcal{O}(Y) = 2$. By the Zariski Finiteness Theorem, it follows that Y is affine (see *Sect. 6.4*). From this, standard theory from commutative algebra implies that Y is regular in codimension 1, i.e., the singularities of Y form a finite set (see for example [259]).

Let $\pi : X \to Y$ be the corresponding quotient morphism. Recall that there exists a curve $\Gamma \subset Y$, determined by the image of a local slice for the corresponding locally nilpotent derivation, together with an open set $U = \pi^{-1}(Y \setminus \Gamma) \subset X$ such that $U \cong (Y \setminus \Gamma) \times \mathbb{C}$, and $\pi : U \to Y \setminus \Gamma$ is the standard projection.

Lemma 5.3 (Lemma 2.1 of [227]; Lemma 1 of [255]) *Under the above hypotheses:*

(a) *Every non-empty fiber of π is of dimension 1.*

(b) *If $C \subset Y$ is a closed irreducible curve, then $\pi^{-1}(C) \subset X$ is an irreducible surface.*

(c) *$Y \setminus \pi(X)$ is finite.*

Proof Let $B = \mathcal{O}(X)$ and $A = \mathcal{O}(X)^{\mathbb{C}^+} \subset B$. Assume that, for some $y \in Y$, the fiber $\pi^{-1}(y)$ has an irreducible component Z of dimension 2. Then Z is defined by a single irreducible function $f \in B$. Since Z is stable under the $\mathbb{C}^+$-action, f is a non-constant invariant function, i.e., f is an irreducible element of A. Any other invariant $g \in A$ is constant on Z, meaning that $g = fh + c$ for some $h \in B$ and $c \in \mathbb{C}$. Since A is factorially closed, $h \in A$ as well. In other words, $g = fh + c$ is an equation in A, which implies that $A/fA = \mathbb{C}$. But this is impossible, since A is an affine UFD of dimension 2 over $\mathbb{C}$, and $f \in A$ is irreducible. This proves (a).

By part (a), $\pi^{-1}(C)$ must have an irreducible component of dimension 2. The irreducible closed curve $C \subset Y$ is defined by an irreducible function $q \in \mathcal{O}(Y)$, which lifts to the irreducible function in $q\pi \in A$. Since A is factorially closed in B, this function is irreducible in B as well, meaning that $\pi^{-1}(C)$ is reduced and irreducible. This proves (b).

Let $W \subset Y$ denote the Zariski closure of $Y \setminus \pi(X)$. If $Y \setminus \pi(X)$ is not finite, then W is a curve, and $F := \pi^{-1}(W)$ is of dimension 2. This implies that $\pi(F)$ is of dimension 1, which contradicts the fact that $\pi(F)$ is contained in the finite set $W \cap \pi(X)$. Therefore, $Y \setminus \pi(X)$ is finite. □

5.1.3 Description of Miyanishi's Proof

Let Y be a smooth algebraic surface over $\mathbb{C}$. Then Y can be completed to a smooth projective surface V in such a way that the divisor at infinity, $D = V \setminus Y$, consists of smooth curves with simple normal crossings. Y is said to have **logarithmic Kodaira dimension** $\bar{\kappa}(Y) = -\infty$ if $|n(D + K_V)| = \emptyset$ for every $n > 0$. Here, K_V denotes the canonical divisor of V. The property $\bar{\kappa}(Y) = -\infty$ is independent of the completion V, and is thus an invariant of Y.

One of the more important facts relating to the logarithmic Kodaira dimension is the following. First, an open subset $U \subset Y$ is called a **cylinderlike open set** if $U \cong K \times \mathbb{C}$ for some curve K. Second, a **Platonic** $\mathbb{C}^*$**-fibration** is a surface of the form $\mathbb{C}^2/G \setminus \{0\}$, where G is a finite non-abelian subgroup of $GL_2(\mathbb{C})$ acting linearly on $\mathbb{C}^2$, and $\mathbb{C}^2/G$ denotes the quotient. If Y (a smooth surface which is not necessarily affine) has $\bar{\kappa}(Y) = -\infty$, then either (1) Y contains a cylinderlike open set, or (2) there exists a curve $\Gamma \subset Y$ such that $Y \setminus \Gamma$ is isomorphic to the complement of a finite subset of a Platonic $\mathbb{C}^*$-fibration. If Y is affine, then case (1) holds. This classification is due largely to Miyanishi and Tsunoda [310, 311]. See also Iitaka [218] for further details about the logarithmic Kodaira dimension.

Now suppose that a nontrivial algebraic $\mathbb{C}^+$-action on $X = \mathbb{C}^3$ is given, and let Y denote the quotient $X /\!\!/ \mathbb{C}^+$. As above, we conclude that Y is a normal affine surface, and that the set $Y' \subset Y$ of singular points of Y is finite.

Let $\pi : X \to Y$ be the quotient map, and let $X' = \pi^{-1}(Y')$. By the preceding lemma, X' is a union of finitely many curves. It is easy to show that there exists a coordinate plane $H \subset X$ which intersects X' in a finite number of points. If $H_0 = H \setminus X'$ and $Y_0 = Y \setminus Y'$, then π restricts to a dominant morphism of smooth surfaces $H_0 \to Y_0$. Since H_0 is the complement of a finite number of points in a plane, $\bar{\kappa}(H_0) = -\infty$. Because H_0 dominates Y_0, it follows that $\bar{\kappa}(Y_0) = -\infty$ as well.

Miyanishi next shows that, if Y' is non-empty, then Y is isomorphic to a quotient $\mathbb{C}^2/G$ for some nontrivial planar action of a finite group $G \subset GL_2(\mathbb{C})$. In particular, $Y' = \{0\}$. Let $X_0 = X \setminus \pi^{-1}(0)$. The topological universal cover for $Y_0 = Y \setminus \{0\}$ is $Z = \mathbb{C}^2 \setminus \{0\}$, where the general fiber of the covering map $p : Z \to Y_0$ consists of $|G|$ points. Therefore, the restriction $\pi : X_0 \to Y_0$ factors through Z, i.e., $X_0 \to Z \to Y_0$. But this is impossible, since there exist open sets $U \subset X$ and $V \subset Y$ such that $U \cong V \times \mathbb{C}$, and $\pi : U \to V$ is the standard projection. In particular, the fiber of π over an element of $V \cap Y_0$ is a single line. Therefore, Y must be smooth.

Miyanishi's argument next uses the fact that a smooth affine surface Y over $\mathbb{C}$ with $\bar{\kappa}(Y) = -\infty$ contains a cylinderlike open set. This part of the argument constitutes a key ingredient in the Fujita-Miyanishi-Sugie proof of the Cancellation Theorem for surfaces, which is discussed in *Chap. 10* below.

In summary, Y is an affine surface over $\mathbb{C}$ such that $\mathcal{O}(Y)$ is a UFD, $\mathcal{O}(Y)^* = \mathbb{C}^*$, and Y contains a cylinderlike open set. (We only needed smoothness to get at this latter condition.) By the Miyanishi characterization of the plane, it follows that $Y \cong \mathbb{C}^2$. This characterization is stated and proved in *Theorem 9.12* below.

The reader is referred to the original articles [301, 395] for further details.

Remark 5.4 In [50], the authors use techniques like those used in the proof of Miyanishi's Theorem to show that, when k is algebraically closed, every factorially closed subalgebra of $k^{[3]}$ is a polynomial ring over k.

5.1.4 Positive Homogeneous Case

Define rings:

$$\Omega_1 = k^{[2]} \quad \text{and} \quad \Omega_2 = k[x,y,z]/(x^5 + y^3 + z^2)$$

Theorem 5.5 *Let k be algebraically closed and let B be a rational UFD over k with* $\text{tr.deg}_k B = 3$. *Assume that B is not semi-rigid. If B has a positive $\mathbb{Z}$-grading over k and $D \in \text{LND}(B)$ is nonzero and $\mathbb{Z}$-homogeneous, then* $\ker D \in \{\Omega_1, \Omega_2\}$.

Proof Let deg denote the degree function associated to the $\mathbb{Z}$-grading of B. If $A = \ker D$, then A is a UFD by *Lemma 2.8* and $A = \bigoplus_i A_i$ is a $\mathbb{Z}$-graded subring of B. Let $n \geq 1$ be minimal so that $A_n \neq \{0\}$ and choose nonzero $f \in A_n$. Let $m \in \mathbb{N}$ be minimal so that $A_m \not\subset k[f]$ and choose nonzero $g \in A_m \setminus k[f]$. This is possible, since $A \neq k[f]$, being of transcendence degree 2 over k. By *Lemma 2.12*, $k[f]$ and $k[g]$ are factorially closed in A, which implies that $k[f]$ and $k[g]$ are algebraically closed in A, and that f and g are irreducible. Since $g \notin k[f]$, it follows that $k[f,g] = k^{[2]}$ and A is algebraic over $k[f,g]$. If $A = k[f,g]$, there is nothing further to prove. So assume that $A \neq k[f,g]$.

Let $a, b \in \mathbb{N}$ be such that $m = ad$ and $n = bd$ for $d = \gcd(m,n)$. Set $K = \text{frac}(A)$ and let $K_0 \subset K$ be the subfield:

$$K_0 = \{u/v \in K \mid u, v \in A_i, v \neq 0, i \in \mathbb{N}\}$$

By *Proposition 1.1*, K_0 is algebraically closed in K. Since $K_0 \neq K$, we have $\text{tr.deg}_k K_0 < \text{tr.deg}_k K = 2$. Since $f^a/g^b \in K_0$, we conclude that $\text{tr.deg}_k K_0 = 1$. Since $\text{frac}(B) = k^{(3)}$, Lüroth's Theorem implies that $K_0 = k(\zeta) = k^{(1)}$ for some $\zeta \in K_0$.

Let $\zeta = u/v$ for $u, v \in A_t$ with $\gcd(u,v) = 1$ for positive $t \in \mathbb{Z}$. Then there exist standard homogeneous $F, G \in k[X,Y] = k^{[2]}$ of the same degree r such that $\gcd(F(X,Y), G(X,Y)) = 1$ and $f^a G(u,v) = g^b F(u,v)$. Let $L_i, M_j \in k[X,Y]$, $1 \leq i,j \leq r$, be linear forms such that $F = L_1 \cdots L_r$ and $G = M_1 \cdots M_r$. If $\lambda \in A$ and $F(u,v), G(u,v) \in \lambda A$, then $L_i(u,v), M_j(u,v) \in \lambda A$ for some i,j. Since L_i and M_j are linearly independent, $u, v \in \lambda A$, which implies $\lambda \in k^*$. Therefore, $\gcd(F(u,v), G(u,v)) = 1$. It follows that f is the only prime divisor of L_1 and g is the only prime divisor of M_1. If $L_1 = f^{a_1}$ and $M_1 = g^{b_1}$ for $a_1 \leq a$ and $b_1 \leq b$, then $\deg f^{a_1} = \deg g^{b_1}$ implies $a_1 = a$ and $b_1 = b$. Since $(L_1, M_1) \in GL_2(k)$, it follows that $K_0 = k(f^a/g^b)$.

Let $p \in A$ be homogeneous and irreducible, and assume that $p \in A_l$ and $\gcd(f,p) = 1$ for positive $l \in \mathbb{Z}$. We have $p^m/f^l \in K_0$. Reasoning as above, we conclude that there exists a linear form $N \in k[X,Y]$ and positive $e \in \mathbb{Z}$ such that:

$$p^e = N(f^a, g^b) \tag{5.1}$$

Since $k[f,g] \neq A$, there exists minimal $\rho \in \mathbb{N}$ so that $A_\rho \setminus k[f,g] \neq \{0\}$; choose nonzero $h \in A_\rho \setminus k[f,g]$. By *Lemma 2.12*, h is irreducible. Therefore, by (5.1), $h^c = \alpha f^a + \beta g^b$ for some $c \geq 2$ and $\alpha, \beta \in k^*$. Since $\deg f \leq \deg g \leq \deg h$, we see that $a \geq b \geq c \geq 2$. Since B is not semi-rigid, $\mathrm{tr.deg}_k ML(B) \leq 1$ and $k[f,g,h] \not\subset ML(B)$. The ABC Theorem (*Theorem 2.48*) implies:

$$(a,b,c) \in \{(3,3,2),(4,3,2),(5,3,2)\}$$

If $k[f,g,h] \neq A$, there exists minimal $s \in \mathbb{N}$ so that $A_s \setminus k[f,g,h] \neq \{0\}$; choose nonzero $H \in A_s \setminus k[f,g,h]$. By *Lemma 2.12*, H is irreducible. Therefore, by (5.1), $H^e = \gamma f^a + \delta g^b$ for some $e \in \mathbb{N}$ and $\gamma, \delta \in k^*$. Since $\deg H \geq \deg h$, we must have $e \leq c = 2$, so $e = 2$. We thus have:

$$\alpha H^2 - \gamma h^2 = (\alpha\delta - \beta\gamma) g^b \neq 0$$

Since $H \notin k[h]$, we see that $\alpha\delta - \beta\gamma \neq 0$. But then g is the only prime divisor of $\sqrt{\alpha}\, H - \sqrt{\gamma}\, h$, which implies that $H \in k[g,h]$, a contradiction. Therefore, $A = k[f,g,h]$. Since the rings $k[x,y,z]/(x^3+y^3+z^2)$ and $k[x,y,z]/(x^4+y^3+z^2)$ are not UFDs, we conclude that $a = 5$ and $b = 2$, and that A is isomorphic to Ω_2 as a k-algebra.

So in all cases, $A \in \{\Omega_1, \Omega_2\}$. □

Corollary 5.6 *Suppose that $B = k^{[3]}$ has a positive $\mathbb{Z}$-grading over k. If $D \in$* LND(B) *is nonzero and $\mathbb{Z}$-homogeneous, then there exist homogeneous $u, v \in B$ such that* $\ker D = k[u,v]$.

Proof (Following Zurkowski) It suffices to assume that k is algebraically closed. The general case then follows from Kambayashi's Theorem; see *Sect. 5.1.1*. By *Proposition 3.42*, there exists a homogeneous system of variables $B = k[x_1,x_2,x_3]$. Set $A = \ker D$ and $H = \mathbb{Z}(A)$.

Consider first the case where $H \neq \mathbb{Z}(B)$. At least one of $\deg x_1$, $\deg x_2$, $\deg x_3$ does not belong to H; say $\deg x_1 \notin H$. Let $s \in B$ be an irreducible homogeneous local slice of D and let $\pi_s : B \to A_{Ds}$ be the associated Dixmier map. Then A_{Ds} is a homogeneous localization of A. If $\pi_s(x_1) \neq 0$, then $\deg x_1 = \deg \pi_s(x_1) \in H$, a contradiction. Therefore, $\pi_s(x_1) = 0$ and:

$$A_{Ds} = k[\pi_s(x_1), \pi_s(x_2), \pi_s(x_3)] = k[\pi_s(x_2), \pi_s(x_3)] = k^{[2]}$$

It follows that $Ds \in (A_{Ds})^* = k^*$, so $A = A_{Ds} = k^{[2]} = \Omega_1$ in this case.

Consider the case where $H = \mathbb{Z}(B)$. If $A = \Omega_2$, let $A = k[f,g,h]$, where $f^5+g^3+h^2 = 0$. Since $\gcd(\deg f, \deg g, \deg h) = 1$ in this case, it follows that $\deg f = 6$, $\deg g = 10$ and $\deg h = 15$. By applying $\partial/\partial x_i$ $(1 \leq i \leq 3)$ to the equation $f^5+g^3+h^2 = 0$, we obtain a second equation, and the two equations can be written in matrix form:

$$\begin{pmatrix} f & g & h \\ 5f_{x_i} & 3g_x & 2h_{x_i} \end{pmatrix} \begin{pmatrix} f^4 \\ g^2 \\ h \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

If the rows of the first matrix were proportional, then $f_{x_i} \in fB$, which would imply $f \in \ker \partial/\partial x_i$, and likewise $g, h \in \ker \partial/\partial x_i$. But then $A = \ker \partial/\partial x_i = k^{[2]}$, a contradiction. Therefore, the two rows of this matrix are not proportional, and the system can be solved over $\mathrm{frac}(B)$:

$$\begin{pmatrix} f^4 \\ g^2 \\ h \end{pmatrix} = \frac{u_i}{v_i} \begin{pmatrix} 2gh_{x_i} - 3hg_{x_i} \\ 5hf_{x_i} - 2fh_{x_i} \\ 3fg_{x_i} - 5gf_{x_i} \end{pmatrix} \tag{5.2}$$

where $u_i, v_i \in B$ are relatively prime. This means u_i divides f^4 and g^2, which have no common factor, so we may assume $u_i = 1$. The equality $v_i h = 3fg_{x_i} - 5gf_{x_i}$ gives:

$$\deg v_i + 15 \leq 16 - \deg x_i \;\Rightarrow\; \deg v_i \leq 1 - \deg x_i \;\Rightarrow\; \deg v_i = 0, \deg x_i = 1$$

Therefore, $v_i \in k^*$, $1 \leq i \leq 3$.

We may now combine the three matrix equations from (5.2) to obtain

$$\begin{pmatrix} f^4 \\ g^2 \\ h \end{pmatrix} dL = \begin{pmatrix} 2gdh - 3hdg \\ 5hdf - 2fdh \\ 3fdg - 5gdf \end{pmatrix}_{3\times 3} \tag{5.3}$$

where $dF = (F_{x_1}, F_{x_2}, F_{x_3}) \in B^3$ for $F \in B$, and $L = v_1x_1 + v_2x_2 + v_3x_3$.

Complete L to a linear system of variables $B = k[L, M, N]$ and let $Q \in GL_3(k)$ have rows L, M and N. If $\hat{d}F = (F_L, F_M, F_N)$, then $\hat{d}F \cdot Q = dF$. Applying Q^{-1} (on the right) to both sides of (5.3) yields:

$$\begin{pmatrix} f^4 \\ g^2 \\ h \end{pmatrix} (1, 0, 0) = \begin{pmatrix} 2g\hat{d}h - 3h\hat{d}g \\ 5h\hat{d}f - 2f\hat{d}h \\ 3f\hat{d}g - 5g\hat{d}f \end{pmatrix}_{3\times 3}$$

In particular:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2gh_M - 3hg_M \\ 5hf_M - 2fh_M \\ 3fg_M - 5gf_M \end{pmatrix}$$

But then $\partial/\partial M(h^2/g^3) = \partial/\partial M(f^5/h^2) = \partial/\partial M(g^3/f^5) = 0$. It follows that $f, g, h \in k[L, N]$, and therefore $A = k[L, N]$, a contradiction. Therefore, by *Theorem 5.5* we see that $A = \Omega_1$ when $H = \mathbb{Z}(B)$.

So in all cases, $A = \Omega_1$. By *Proposition 3.41*, there exist homogeneous $u, v, \in B$ with $A = k[u, v]$. □

Remark 5.7 Note that *Theorem 5.5* does not assume that the ring B is affine. The author is aware of no example where B satisfies the hypotheses of the theorem and $A = \Omega_2$.

5.1.5 Type of a Standard Homogeneous Derivation

Observe that *Corollary 5.6* does not claim $\deg u$ and $\deg v$ are relatively prime. Indeed, this may be false. For example, consider the grading of B given by $\deg x = 2$, $\deg y = 3$, and $\deg z = 4$, and let D denote the standard linear derivation $D = x\partial_y + y\partial_z$. Then D is homogeneous for this grading, and $\ker D = k[x, P]$ for $P = 2xz - y^2$, where $\deg x = 2$ and $\deg P = 6$.

There is, however, an important case in which the degrees of the generators are known to be coprime, due to Daigle.

Proposition 5.8 (Thm. 2.2 of [72]) *Suppose $D \in \mathrm{LND}(B)$ is homogeneous relative to some positive $\mathbb{Z}$-grading of $B = k[x, y, z]$, and $\ker D = k[f, g]$ for homogeneous f and g. If the integers $\deg x$, $\deg y$, and $\deg z$ are pairwise relatively prime, then $\deg f$ and $\deg g$ are also relatively prime.*

Note that this result applies to the standard homogeneous case. In this case, the pair $(\deg f, \deg g)$ is uniquely associated to D (up to order), giving rise to the following definition:

If $D \in \mathrm{LND}(B)$ is homogeneous in the standard grading of B, and $\ker D = k[f, g]$ for homogeneous polynomials f and g, then D is of **type** $(\deg f, \deg g)$, where $\deg f \leq \deg g$.

In particular, if type (e_1, e_2) occurs, then Daigle's result implies that $\gcd(e_1, e_2) = 1$, though only certain relatively prime pairs of integers can occur. The set of pairs which actually occur is known, and is given in the remark which concludes the paper [87]. In this paper, Daigle and Russell give a complete classification of affine rulings of weighted projective planes, and such rulings are closely related to homogeneous locally nilpotent derivations of $k^{[3]}$. In particular, they associate to standard homogeneous D the two projective plane curves defined by its homogeneous kernel generators f and g.

5.2 Other Fundamental Theorems

Given $f, g \in B$, $\Delta_{(f,g)}$ will denote the jacobian derivation:

$$\Delta_{(f,g)}h = \frac{\partial(f,g,h)}{\partial(x,y,z)}$$

Note that some authors use gradient notation to write:

$$\Delta_{(f,g)} = \nabla_f \wedge \nabla_g$$

If $D \in \mathrm{LND}(B)$ has $\ker D = k[f,g]$ for some $f, g \in B$, then we know from *Lemma 3.12* that $\Delta_{(f,g)}$ is locally nilpotent, $\ker \Delta_{(f,g)} = \ker D$, and there exist $a, b \in k[f,g]$ such that $a\Delta_{(f,g)} = bD$. In [70], Daigle proved the following stronger result.

Theorem 5.9 (Jacobian Formula) *Given $D \in \mathrm{LND}(B)$, $D \neq 0$, choose $f, g \in B$ such that $\ker D = k[f,g]$. Then $D = \lambda\Delta_{(f,g)}$ for some $\lambda \in \ker D$.*

Next, if $D \in \mathrm{LND}(B)$, let $\pi : \mathbb{A}^3 \to \mathbb{A}^2$ denote the quotient map, induced by the inclusion $\ker D \hookrightarrow B$. An important fact about π is due to Bonnet [37].

Theorem 5.10 (Bonnet's Theorem) *If k is a field of characteristic zero, and if $\pi : \mathbb{A}^3 \to \mathbb{A}^2$ is the quotient map associated to a nontrivial algebraic $\mathbb{G}_a$-action on $\mathbb{A}^3$, then π is surjective.*

Bonnet's proof is for $k = \mathbb{C}$, using a topological argument. The general form of the theorem was then deduced by Daigle and Kaliman [84], where surjectivity for the general case refers to π as a map from $\mathrm{Spec}(B)$ to $\mathrm{Spec}(\ker D)$. See Bonnet [37] for an example of a $\mathbb{G}_a$-action on $\mathbb{A}^4$ with quotient isomorphic to $\mathbb{A}^3$, but non-surjective quotient morphism.

In [227], Kaliman proved the following theorem for $k = \mathbb{C}$; the general case is deduced in [84].

Theorem 5.11 (Kaliman's Theorem) *Let k be a field of characteristic zero. Every free algebraic $\mathbb{G}_a$-action on $\mathbb{A}^3$ is a translation in a suitable polynomial coordinate system. Equivalently, if $B = k^{[3]}$ and $D \in \mathrm{LND}(B)$, and if $(DB) = B$ (the ideal generated by the image), then $Ds = 1$ for some $s \in B$.*

Special cases of this result were proved earlier in [78, 107, 254, 387]. Kaliman also gives a proof of Bonnet's Theorem in a more general setting. It should be noted that in dimension higher than 3, there exist free algebraic $\mathbb{G}_a$-actions which are not conjugate to a translation (see *Example 3.8.4*).

Also contained in the work of Bonnet, Daigle, and Kaliman is the following (see [84], Thm. 1).

Theorem 5.12 (Plinth Ideal Theorem) *Let k be a field of characteristic zero, $B = k^{[3]}$, $D \in \mathrm{LND}(B)$, and $A = \ker D$.*

(a) *B is faithfully flat as an A-module.*

(b) *The plinth ideal $A \cap DB$ of D is a principal ideal of A.*

In the language of *Chap.* 2, this implies that a locally nilpotent derivation of $k^{[3]}$ has a unique minimal local slice (up to equivalence), namely, we may choose $r \in B$ with $Dr = h$, where $A \cap DB = hA$. Then $B_h = A_h[r]$. Geometrically, this means that if $V \subset \mathbb{A}^2$ is the complement of the curve Γ defined by $h \in A$, and if $U \subset \mathbb{A}^3$ is the complement of the surface S defined by $h \in B$, then U is equivariantly isomorphic to $V \times \mathbb{A}^1$. In particular, the fiber of the quotient map $\pi : \mathbb{A}^3 \to \mathbb{A}^2$ lying over any point of V is a line (a single orbit, isomorphic to $\mathbb{G}_a$). Many of the remaining mysteries of the dimension three case thus lie hidden in the morphism $\pi : S \to \Gamma$. It was shown by Kaliman that every irreducible component of C of Γ is a polynomial curve, i.e., the coordinate ring $k[C]$ admits an embedding in $k^{[1]}$. See [227], Thm. 5.2 and [228], Thm. 10.1.

An additional fact concerning locally nilpotent derivations in dimension three is the following. The proof of this result uses a more general fact, which is stated and proved in the *Appendix* section at the end of this chapter.

Theorem 5.13 (Intersection of Kernels) *If $D, E \in \mathrm{LND}(B)$ are nonzero, then exactly one of the following three statements is true.*

1. $\ker D \cap \ker E = k$
2. *There exist $f, g, h \in B$ such that $\ker D = k[f, g]$, $\ker E = k[f, h]$, and $\ker D \cap \ker E = k[f]$.*
3. $\ker D = \ker E$

Proof Set $A_1 = \ker D$ and $A_2 = \ker E$, and assume $A_1 \cap A_2 \neq k$ and $A_1 \neq A_2$. We know that A_2 is factorially closed and isomorphic to $k^{[2]}$. Taking $S = A_2$ and $R = k$ in *Corollary 5.42* below, it follows that we can choose $f \in B$ such that $A_1 \cap A_2 = k[f]$ and $A_2 = k[f]^{[1]}$. By symmetry, $A_1 = k[f]^{[1]}$ as well. □

Corollary 5.14 *Suppose $D, E \in \mathrm{LND}(B)$ are nonzero and have distinct kernels. If $\ker D = k[f]^{[1]}$ and $Ef = 0$, then $\ker E = k[f]^{[1]}$.*

Proof By the preceding theorem, there exist $\tilde{f}, g, h \in B$ with $\ker D = k[\tilde{f}, g]$ and $\ker E = k[\tilde{f}, h]$. Since the kernels of D and E are distinct, $Eg \neq 0$. If $\tilde{f} = P(g)$ for $P \in k[f]^{[1]}$, then $0 = E\tilde{f} = P'(g)Eg$, which implies $P'(g) = 0$, i.e., $\tilde{f} \in k[f]$. Therefore $\ker E = k[\tilde{f}, h] \subset k[f, h] \subset \ker E$, so $\ker E = k[f, h]$. □

In calculating kernels, the following condition can be useful.

Proposition 5.15 (Kernel Criterion) *Suppose $a, b \in B = k[x, y, z]$ are such that $\Delta_{(a,b)}$ is locally nilpotent and nonzero. Then the following are equivalent.*

1. $k[a, b] = \ker \Delta_{(a,b)}$
2. *$\Delta_{(a,b)}$ is irreducible and* $\ker \Delta_{(a,b)} \subset k(a, b)$.

Proof The implication (1) implies (2) follows from *Proposition 3.24*. Conversely, assume (2) holds. By Miyanishi's Theorem, there exist $u, v \in B$ such that $\ker \Delta_{(a,b)} = k[u, v]$. It follows that:

$$\Delta_{(a,b)} = \frac{\partial(a,b)}{\partial(u,v)} \cdot \Delta_{(u,v)}$$

Since $\Delta_{(a,b)}$ is irreducible, $\frac{\partial(a,b)}{\partial(u,v)} \in k^*$, i.e., (a, b) is a Jacobian pair for $k[u, v]$. Since $k(a, b) = k(u, v)$, the inclusion $k[a, b] \hookrightarrow k[u, v]$ is birational. It is well known that the Jacobian Conjecture is true in the birational case, and we thus conclude $k[a, b] = k[u, v]$.[1] □

Another result in dimension three is the following, which was proved by Wang in his thesis [414]; see also [415]. Because of the Plinth Ideal Theorem, it is now possible to give a much shorter proof than that originally presented by Wang.

Theorem 5.16 (Wang's Theorem) *Let $B = k[x, y, z]$, and suppose $D \in \mathrm{LND}(B)$ is such that $D^2x = D^2y = D^2z = 0$. Then* $\mathrm{rank}(D) \leq 1$.

Proof It suffices to assume D is irreducible. Let $A = \ker D$, and let $I \subset A$ be the plinth ideal $I = DB \cap A$ of D. By the Plinth Ideal Theorem, $I = aA$ for some $a \in A$. Since $Dx, Dy, Dz \in I$ by hypothesis, we see that $DB \subset aB$. By irreducibility, $a \in B^*$. Therefore, $I = B$, and D has a slice s. It follows that $B = A[s]$, and since $A = k^{[2]}$, we conclude that the rank of D is 1. □

Recall that derivations with $D^2x = D^2y = D^2z = 0$ are called nice.

To conclude this section, we give the result of Kaliman and Saveliev in its full generality.

Theorem 5.17 ([232]) *Let X be a smooth contractible complex affine algebraic threefold with a nontrivial algebraic $\mathbb{C}^+$-action on it, and let $S = X /\!/ \mathbb{C}^+$ be its algebraic quotient.*

(a) *X is rational, and S is a smooth contractible surface.*

(b) *If X admits a dominant morphism from a threefold of the form $C \times \mathbb{C}^2$, then $S = \mathbb{C}^2$.*

(c) *If the action is free, then it is equivariantly trivial.*

Note that if both conditions (b) and (c) are satisfied, then $X = \mathbb{C}^3$.

In [255], Kraft also considers $\mathbb{C}^+$-actions on a smooth contractible threefold X. He gives a proof for the smoothness of the quotient map $\pi : X \to S$ under certain additional conditions. Using this smoothness property, together with some topological considerations due to Kaliman, he gives a short proof that a free

[1] The same reasoning yields yet another equivalent formulation of the two-dimensional Jacobian Conjecture: Given $a, b \in B$, if $\Delta_{(a,b)}$ is irreducible, then $\ker \Delta_{(a,b)} = k[a, b]$.

$\mathbb{C}^+$-action on X is a translation, under the additional assumption that the quotient S is smooth. This implies Kaliman's Theorem for $X = \mathbb{C}^3$.

Remark 5.18 There might be other classes of threefolds to which Miyanishi's Theorem could generalize. For example, $X = SL_2(k)$ is smooth and factorial, but when $k = \mathbb{C}$, it is not contractible. On the other hand, it seems likely that every nonzero $D \in \mathrm{LND}(k[X])$ has $\ker D = k^{[2]}$. In working with affine 3-space, the existence of three independent locally nilpotent derivations (namely, the partial derivatives) is of central importance. Geometrically, these are translations in three independent directions. Likewise, $SL_2(k)$ has four fundamental actions: Realize $\mathbb{G}_a$ as the subgroup of $SL_2(k)$ consisting of upper (respectively, lower) triangular matrices with ones on the diagonal. Then both left and right multiplication by elements of $\mathbb{G}_a$ define $\mathbb{G}_a$-actions, specifically:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b+ta \\ c & d+tc \end{pmatrix}, \quad \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+tc & b+td \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \begin{pmatrix} a+tb & b \\ c+td & d \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c+ta & d+tb \end{pmatrix}$$

These actions are conjugate to one another, fixed point free, and have $X/\!\!/\mathbb{G}_a = \mathbb{A}^2$; this quotient is calculated in *Chap. 6* below. In addition, we see that $ML(X) = k$. However, in contrast to the situation for three-dimensional affine space, the quotient morphism $\pi : X \to \mathbb{A}^2$ is not surjective, since $\pi^{-1}(0)$ is an empty fiber. For example, the quotient map for the first of these actions is given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to (a, c)$$

On the other hand, notice that the geometric quotient does exist: $X/\mathbb{G}_a = \mathbb{A}^2 \setminus \{(0,0)\}$.

5.3 Triangularizability and Tameness

5.3.1 Triangularizability

The non-triangularizable $\mathbb{G}_a$-action of Bass was introduced in *Chap. 3*. Specifically, it takes the form $\exp(tFD)$, where D is the basic linear derivation $D = x\partial_y + 2y\partial_z$ on $B = k[x, y, z]$, and $F = xz - y^2$. Subsequently, more general classes of non-triangularizable locally nilpotent derivations of $k^{[n]}$ for $n \geq 3$ appeared in [159, 344]. In [388], Snow speculated:

> Perhaps the fixed point set being 'cylindrical' is the only obstruction for a $\mathbb{C}$ action to be triangular. (p. 169)

However, Example 4.3 of [78], showed that this is not the case: The authors construct $D \in \mathrm{LND}(B)$ which is non-triangularizable of rank two, and whose corresponding set of fixed points is a line. In particular, D is irreducible, making this example quite different than that of Bass. Ultimately, Daigle produced:

Theorem 5.19 (Cor. 3.4 of [69]) *Let $D \in \mathrm{LND}(B)$ be irreducible of rank 2, where $B = k[x,y,z]$ and $\ker D = k[x,P]$ for some $P \in B$. The following are equivalent.*

1. *D is triangularizable*
2. *D is triangularizable over $k[x]$*
3. *there exists a variable Q of B such that $k(x)[P,Q] = k(x)[y,z]$*

Note that the difference between condition (3) of this theorem and the general rank-2 case is the requirement that Q is a k-variable of $k[x,y,z]$, rather than the weaker condition that Q is a $k(x)$-variable of $k(x)[y,z]$.

Example 5.20 (Example 3.5 of [69]) If $P = y + \frac{1}{4}(xz + y^2)$, define $D = \Delta_{(x,P)}$. Then D is an irreducible rank-two locally nilpotent derivation of B which is not triangularizable. Its fixed points are defined by the ideal $(DB) = (x, 1 + y^3)$, a union of three lines. Note that if T is the triangular derivation defined by $Tx = 0$, $Ty = x$, and $Tz = 1 + y^3$, then the fixed points of D and T agree in the strongest possible sense, in that they are defined by precisely the same ideals. See also van den Essen [142], 9.5.17.
It thus became evident that most elements of $\mathrm{LND}(B)$ are not triangularizable, and that one should focus on other aspects of the subject.

5.3.2 Tameness

Due to fundamental work of Shestakov and Umirbaev in [382, 383], there exist non-tame algebraic automorphisms of $B = k[x,y,z]$. In particular, their work shows that the Nagata automorphism is not tame; see *Example 3.8.1*. Following is an example of a tame automorphism of $\mathbb{A}^3$ which is not triangularizable.

Example 5.21 (Example 1.13 of [158]) Set $\lambda = (y,z,x) \in GA_3(k)$, and define $\alpha, \beta \in BA_3(k)$ by:

$$\alpha = (x, y, z + x^3 - y^2) \quad , \quad \beta = (x, y + x^2, z + x^3 + \tfrac{3}{2}xy)$$

If $\gamma = \alpha\lambda\beta\lambda^{-1}$, then γ is tame and:

$$\gamma = (x + z^2, y + z^3 + \tfrac{3}{2}xz, z - y^2 + x(x^2 - 3yz) + \tfrac{1}{4}z^2(3x^2 - 8yz))$$

As an automorphism of $\mathbb{A}^3$, the fixed point set of γ is the cuspidal cubic curve defined by $z = 0$ and $x^3 - y^2 = 0$, which has an isolated singularity at the origin. By Popov's criterion, γ cannot be conjugated into the triangular subgroup $BA_3(k)$.

However, using van den Essen's result (*Proposition* 2.57), it can also be shown that γ is not an exponential automorphism.

Following the work of Shestakov and Umirbaev, Karas [244] and other authors began to consider the multi-degree of an element $F \in GA_3(k)$ relative to tameness. If $F = (F_1, F_2, F_3)$ for $F_i \in B = k[x, y, z]$, then the **multi-degree** of F is $d_F = (\deg F_1, \deg F_2, \deg F_3) \in \mathbb{N}^3$, where deg is the standard degree function on B. For an exponential automorphism, suppose that $D \in \text{LND}(B)$ is standard homogeneous of degree $d > 0$, and set $F = \exp D$. Then:

$$d_F = (1 + d \cdot \deg_D x\,,\ 1 + d \cdot \deg_D y\,,\ 1 + d \cdot \deg_D z) \tag{5.4}$$

The most general results about multi-degrees are due to Kuroda. For example, he proves the following.

Theorem 5.22 ([268], Thm. 1.3) *Given $F \in GA_3(k)$, suppose that $d_F = (d_1, d_2, d_3)$ satisfies the following three conditions.*

(a) $\{3d_2 \neq 2d_3 \vee sd_1 \neq 2d_3\ \forall\ odd\ s \geq 3\} \wedge \{d_1 + d_2 \leq d_3 + 2\}$
(b) $\{\gcd(d_1, d_2, d_3) = \gcd(d_1, d_2) \leq 3\} \wedge \{d_1 + d_2 + d_3 \leq \text{lcm}(d_1, d_2) + 2\}$
(c) $\{d_1 \not| d_2\} \vee \{d_3 \notin d_1\mathbb{N} + d_2\mathbb{N}\}$

Then F is not in $TA_3(k)$.

In [407], Umirbaev gave defining relations for $TA_3(k)$. These relations were used by Wright in [428] to prove a structure theorem for $TA_3(k)$ analogous to the well-known amalgamated free product structure of $TA_2(k)$; see *Sect. 4.3.4*. Specifically, Wright shows that $TA_3(k)$ is the product of three subgroups amalgamated along pairwise intersections.

5.4 Homogeneous (2, 5) Derivation

In [159], Question 2 asked: Do there exist locally nilpotent derivations of the polynomial ring $k[x_1, \ldots, x_n]$ having maximal rank n? It was known at the time that, for $n = 1$, the answer is positive, and for $n = 2$ the answer is negative (by Rentschler's Theorem). But for $n \geq 3$ the answer was not known. The following example of a locally nilpotent derivation on $B = k[x, y, z]$ having rank three was found in 1996; it appeared in [162].

Define polynomials:

$$F = xz - y^2\ ,\ G = zF^2 + 2x^2yF + x^5\ ,\ R = x^3 + yF$$

Define $\Delta \in \text{Der}_k(B)$ by $\Delta = \Delta_{(F,G)}$. Observe that Δ is irreducible and homogeneous of degree 4 in the standard grading. It is easily checked that $\Delta^3 x = \Delta^7 y = \Delta^{11} z = 0$, and therefore Δ is locally nilpotent and $\deg_\Delta x = 2$, $\deg_\Delta y = 6$, $\deg_\Delta z = 10$. It follows from (5.4) that, if d_ϕ is the multi-degree of $\phi = \exp \Delta$, then $d_\phi = (9, 25, 41)$. By *Theorem* 5.22, ϕ is not tame.

In Bass's example (*Example 3.8.1*) we saw that the subring $k[x, F]$ is the kernel of the standard linear derivation of B. Therefore, by *Corollary 5.14*, we conclude that $\ker \Delta = k[F, g]$ for some homogeneous $g \in B$. In particular, $G \in k[F, g]$, and by considering degrees, we conclude that $\deg g$ is either 1 or 5. If $\deg g = 1$, then (by homogeneity) G is in the linear span of $\{g^5, g^3F, gF^2\}$, which implies g divides G. Since G is irreducible, we conclude $\deg g = 5$, and that G is in the linear span of g itself. Therefore, $\ker \Delta = k[F, G]$. Δ is called the **homogeneous (2, 5) derivation** of B.

Suppose that $h \in B$ is a variable of B and $\Delta h = 0$. By the implication (3.1), the linear summand h_1 of h is nonzero, and by homogeneity, $\Delta h_1 = 0$ as well. But it is clear that $k[F, G]$ can contain no polynomial of degree 1, so $\ker \Delta$ does not contain a variable. In other words, the rank of Δ is 3. This implies that Δ is not triangularizable, since any triangularizable derivation annihilates a variable. We have thus proved:

Theorem 5.23 *$\Delta \in \mathrm{LND}(B)$, $\ker \Delta = k[F, G]$, $\mathrm{rank}\,(\Delta) = 3$ and $\exp \Delta$ is not tame.*

The polynomial R is a minimal local slice of Δ, with $\Delta R = -FG$. Geometrically, this means that if $\pi : \mathbb{A}^3 \to \mathbb{A}^2$ is the quotient map for the corresponding $\mathbb{G}_a$-action, then π is an equivariant projection over the complement of the two lines of $\mathbb{A}^2$ defined by $FG = 0$. The fiber over the origin is the line of fixed points, defined by $x = y = 0$. Over points $(a, 0)$ for $a \neq 0$, the fiber consists of two lines (orbits) in the surface $G = 0$; and over $(0, b)$, the fiber consists of five lines (orbits) in the surface defined by $F = 0$.

Observe the relation $F^3 + R^2 = xG$. Applying Δ yields $2R(-FG) = G\Delta x$, and thus $\Delta x = -2FR$. Likewise, applying Δ to the equation $R = x^3 + yF$ gives $-FG = 3x^2(-2FR) + F\Delta y$, so $\Delta y = 6x^2R - G$. And Δz is gotten from $0 = \Delta F = z\Delta x + x\Delta z - 2y\Delta y$. In summary:

$$\Delta x = -2FR \ , \ \ \Delta y = 6x^2R - G \ , \ \ \Delta z = 2x(5yR + F^2)$$

Although Δ is not a triangularizable derivation, it can be realized as the quotient of a triangular derivation in dimension six.

Theorem 5.24 *Define a derivation T on $\mathcal{B} = k[u, v, w, x, y, z] = k^{[6]}$ by:*

$$Tu = Tv = 0\,,\ Tw = -uv\,,\ Tx = -2uw\,,\ Ty = 6wx^2 - v\,,\ Tz = 2x(u^2 + 5yw)$$

Define the ideal $I = (u - F, v - G, w - R)$. Then $TI \subset I$; $\mathcal{B}/I \cong k^{[3]}$; and $\Delta = T/I$.

Proof The latter two conclusions are obvious once it is shown that $TI \subset I$. To show this, define another derivation D of $\mathcal{B}$ by

$$D = \frac{\partial(u, v, f, g, h, \cdot)}{\partial(u, v, w, x, y, z)}$$

where:

$$f = u - (xz - y^2);\ \ g = v - (u^2z + 2ux^2y + x^5);\ \ h = w - (x^3 + uy)$$

Since $f, g, h \in \ker D$, $DI \subset I$, and it is clear that $D/I = \Delta$ on $\mathcal{B}/I \cong B$. (But D is not *a priori* locally nilpotent.) Direct calculation shows that, modulo I:

$$Dw \equiv Tw\,,\ Dx \equiv Tx\,,\ Dy \equiv Ty\,,\ Dz \equiv Tz$$

Thus, $0 = Df = Tf + \kappa$ for some $\kappa \in I$, implying $Tf \in I$. Likewise, $Tg, Th \in I$, so $TI \subset I$. $\square$

Note that, since $T(w - (x^3 + uy)) = 0$, the rank of T is 3. Geometrically, this result means that the triangular $\mathbb{G}_a$-action on $\mathbb{A}^6$ defined by T restricts to a $\mathbb{G}_a$-action on the coordinate threefold $X \subset \mathbb{A}^6$ defined by I, and this action is equivalent to Δ on $\mathbb{A}^3$.

We also have:

Theorem 5.25 ([162]) *Let Δ be the homogenous $(2,5)$ derivation of $B = k[x_1, x_2, x_3]$. Given $n \geq 4$, extend Δ to Δ° on $B[x_4, \ldots, x_n] = k^{[n]}$ by setting $\Delta^\circ x_i = x_{i-1}^5$, $4 \leq i \leq n$. Then Δ° is homogeneous and locally nilpotent of rank n.*

Remark 5.26 Consider the rank-4 derivation Δ° defined on $k[x_1, x_2, x_3, x_4]$, as above. At this writing, it is not known what $\ker \Delta^\circ$ is, or even whether this kernel is finitely generated. Existing algorithms are inconclusive, due to the size of calculations involved. Clearly, a method other than *brute force* is needed.

5.5 Local Slice Constructions

Local slice constructions were introduced in [161] in an effort to understand and generalize the $(2,5)$ example above, This procedure brought into view large new families of rank-3 elements of $\mathrm{LND}(B)$. Also working just after the appearance of the $(2,5)$ example, Daigle used a geometric approach, quite different from the method of local slice constructions, to find additional rank-3 examples. At the time, it appeared that the two methods produced the same examples.

The present section describes local slice constructions. The following section discusses some of the geometric theory developed by Daigle and Russell, and its connection to local slice constructions.

5.5.1 Definition and Main Facts

Given $D \in \mathrm{LND}(B)$, consider the following condition.

$(*)$ There exist $f, g, r \in B$ and $P \in k^{[1]}$ such that:

$$\ker D = k[f, g] \text{ and } Dr = g \cdot P(f) \neq 0$$

Note that this condition does not depend on any particular system of coordinates for B. To date, we know of no nonzero $D \in \mathrm{LND}(B)$ which fails to satisfy $(*)$.

Lemma 5.27 *Assume D satisfies $(*)$ and set $S = k[f] \setminus \{0\}$. For any local slice $r' \in B$ of D:*

$$D(r'/g) \in S \quad \Leftrightarrow \quad S^{-1}A[r'] = S^{-1}A[r]$$

Proof If $D(r'/g) \in S$, then $rDr' - r'Dr \in A$. Thus, for some nonzero $a, b \in k[f]$ we have $gar' - gbr \in A$. Since A is factorially closed, $ar' - br \in A$. Therefore, $S^{-1}A[r'] = S^{-1}A[r]$.

Conversely, suppose $S^{-1}A[r'] = S^{-1}A[r]$ for some $r' \in B$. Then $r' = cr + d$ for $c \in (S^{-1}A)^* = k(f)^*$ and $d \in S^{-1}A$. Thus:

$$D(r'/g) = cD(r/g) \in k(f)^* \cap (1/g)B = S$$

□

Now assume that D is irreducible and that D satisfies $(*)$ for some local slice r not belonging to gB. Since D is irreducible, we may assume $D = \Delta_{(f,g)}$ (Jacobian Formula). Let $\bar{B} = B/gB$, a domain, and let $\bar{D} = D \,(\mathrm{mod}\, gB) \in \mathrm{LND}(\bar{B})$, noting that $\bar{D}r = 0$. Since $\ker \bar{D}$ is the algebraic closure of $k[f]$ in $\bar{B}$, we conclude that there exists $\phi \in k[f]^{[1]}$ such that $\phi(r) \in gB$. If we choose ϕ to be of minimal r-degree, such that $\phi(r)$ is irreducible in $k[f, r]$, then ϕ is unique up to nonzero constant multiples. Suppose $h = g^{-1}\phi(r) \in B$.

Theorem 5.28 (Thm. 2 of [161]) *In the above notation:*

(a) $\Delta_{(f,h)} \in \mathrm{LND}(B)$
(b) $\Delta_{(f,h)}r = -h \cdot P(f)$
(c) *If $\Delta_{(f,h)}$ is irreducible, then* $\ker \Delta_{(f,h)} = k[f, h]$

Proof Let $\delta = \Delta_{(f,h)}$. Since $\Delta_{(f,gh)} = g \cdot \Delta_{(f,h)} + h \cdot \Delta_{(f,g)}$, it follows that:

$$g \cdot \delta = \Delta_{(f,\phi(r))} - h \cdot D = \phi'(r) \cdot \Delta_{(f,r)} - h \cdot D$$

Therefore, $g \cdot \delta r = -h \cdot Dr$, which implies $\delta r = -h \cdot (Dr/g) = -hP(f)$. So (b) is proved.

Since $\delta r \neq 0$, r is transcendental over $K = k(f, h)$, i.e., $K[r] \cong K^{[1]}$. Since $g = \phi(r)/h$, we have $g \in K[r]$; and since $k[f] \cap gB = (0)$, $\deg_r g \geq 1$ and $g \notin K[r]^*$.

We claim that g is irreducible in $K[r]$. Since $gh = \phi(f, r)$, it suffices to show that ϕ is irreducible in $K[r]$. However, ϕ was chosen to be irreducible in $k[f, r] \cong k^{[2]}$, hence it is also irreducible in $k[f, h, r] \cong k^{[3]}$. Since ϕ is not in K, it follows that ϕ is irreducible in $K[r]$. Consequently, g is also irreducible in $K[r]$.

Therefore, $g \cdot K[r]$ is a maximal ideal of $K[r]$ and:

$$g \cdot K[x, y, z] \cap K[r] = g \cdot K[r]$$

Set $T = \{g^n \cdot a(f) \mid n \geq 0,\ a \in k[f] \setminus \{0\}$, and let $A = \ker D = k[f,g]$. Then $T^{-1}A[r] = T^{-1}B$. Given $b \in B$, choose n so that $g^n b \in k(f)[g,r] \subset K[r]$. Then, using the above ideal equality inductively, we obtain $b \in K[r]$. Therefore, $B \subset K[r]$. Since δ is locally nilpotent on $K[r]$, part (a) is proved.

To prove (c), suppose δ is irreducible. Since $\ker D = k[f,g]$ and $\delta f = 0$, *Corollary 5.14* implies that $\ker \delta = k[f,\eta]$ for some $\eta \in B$. If $h = p(\eta)$ for $p \in k[f]^{[1]}$, then $\delta = p'(\eta)\Delta_{(f,\eta)}$. Since δ is irreducible, $p'(\eta) \in k^*$, implying that $h = a\eta + b(f)$ for some $a \in k^*$ and $b \in k[f]$. But then $k[f,h] = k[f,\eta]$. □

The procedure by which $\Delta_{(f,h)}$ is obtained from D is called a **local slice construction**. Specifically, we say $\Delta_{(f,h)}$ is obtained by local slice construction from the data $(f,g,r) \in B^3$. Note that local slice constructions do not require any homogeneity conditions.

Note also that when $\Delta_{(f,h)}$ is obtained from $\Delta_{(f,g)}$ using data (f,g,r), then $\Delta_{(f,g)}$ is obtained from $\Delta_{(f,h)}$ using data (f,h,r). To continue the process inductively, we may, by the lemma above, replace r with any r' for which $S^{-1}A[r'] = S^{-1}A[r]$.

It may also happen that the original derivation D admits a local slice r such that $Dr = fg$. Then $\Delta_{(f,h)}r = -fh$. Thus, to continue the process inductively, we may use data (h,f,r) instead of (f,h,r).

Example 5.29 Let D denote the standard linear locally nilpotent derivation of $B = k[x,y,z]$:

$$D = x\frac{\partial}{\partial y} + 2y\frac{\partial}{\partial z}$$

If $F,G,R \in B$ are defined as before, then the $(2,5)$-derivation Δ is obtained from D by a local slice construction with data (F,x,R). In particular, $\ker D = k[x,F]$ and $\ker \Delta = k[F,G]$, so F is a common kernel generator; $DR = xF$ and $\Delta R = -FG$, so R is a common local slice; and the algebraic relation between these four polynomials is $F^3 + R^2 = xG$.

5.5.2 *Examples of Fibonacci Type*

Using local slice constructions, we describe a sequence of homogeneous locally nilpotent derivations of B which plays a central role in the classification of the standard homogeneous elements of LND(B). In keeping with the notation of [161], define polynomials $F = xz - y^2$ and $r = -R = -(x^3 + yF)$; and inductively define H_n by [2]

$$H_0 = -y\ ,\ H_1 = x\ ,\ H_2 = F\ ,\ H_{n-1}H_{n+1} = H_n^3 + r^{a_n}$$

[2] The definition of H_1 was inadvertently omitted from the final printing of the original article [161].

where $a_n = \deg H_n$. The fact that $H_n \in B$ for all n was shown in [161]. The sequence of degrees a_n is given by every other element of the Fibonacci sequence, namely, $a_{n+1} = 3a_n - a_{n-1}$.

Define a sequence θ_n of derivations of B by $\theta_n = \Delta_{(H_n, H_{n+1})}$.

Theorem 5.30 (Sect. 4.2 of [161]) *For each $n \geq 0$:*

1. *θ_n is irreducible, locally nilpotent and homogeneous.*
2. $\ker \theta_n = k[H_n, H_{n+1}]$*, and θ_n is of type (a_n, a_{n+1}).*
3. *$\theta_n r = H_n H_{n+1}$*
4. *If $n \geq 1$, then θ_{n+1} is obtained from θ_n by a local slice construction using the data (H_n, H_{n-1}, r).*

For example, the partial derivative ∂_z equals θ_0, with kernel $k[x, y] = k[H_0, H_1]$; the standard linear derivation is θ_1, with kernel $k[x, F] = k[H_1, H_2]$; and the homogeneous $(2, 5)$ derivation is θ_2, with kernel $k[F, G] = k[H_2, H_3]$.

5.5.3 Type $(2\,, 4m + 1)$

Starting with any θ_n, there is a large derived family of standard homogeneous derivations D having $\ker D = k[H_n]^{[1]}$. To illustrate, start with the $(1, 2)$ example θ_1. For $m \geq 1$, set $r_m = x^{2m+1} + F^m y$, which is a homogeneous local slice of θ_1. Since $\theta_1 r_m = xF^m$, we can carry out a local slice construction with the data (F, x, r_m) to obtain a homogeneous locally nilpotent derivation with kernel $k[F, G_m]$, where G_m is homogeneous of degree $4m + 1$, namely, $G_m = zF^{2m} + 2x^{2m}F^m y + x^{4m+1}$.

5.5.4 Triangular Derivations

Let T denote any triangular derivation of $B = k[x, y, z]$. Then $\ker T = k[x, P]$ for $P \in B$ of the form $P = a(x)z + b(x, y)$. The polynomial $r = yP$ is a local slice of the partial derivative $\theta_0 = \partial_z$, with $\theta_0 r = a(x)y$. The derivation $T = \Delta_{(x,P)}$ is gotten by local slice construction using the data (x, y, r). In other words:

> The set of derivations obtained from a partial derivative by a single local slice construction is precisely the set of all triangular derivations of B.

Of course, this statement only makes sense in the context of a fixed coordinate system.

5.5.5 Rank Two Derivations

Proposition 5.31 *Every irreducible locally nilpotent derivation of $B = k^{[3]}$ of rank at most two can be transformed to a partial derivative by a sequence of local slice constructions.*

Proof Let $D \in \mathrm{LND}(B)$ be irreducible, with $\mathrm{rank}(D) \leq 2$, and suppose $Dx = 0$. Set $K = k(x)$. By *Theorem 4.12*, there exist $P, Q \in B$ such that $K[P, Q] = K[y, z]$, $\ker D = k[x, P]$, and $DQ \in k[x]$. Moreover, the ideal generated by the image of D is (P_y, P_z), and if $(P_y, P_z) = (1)$, then Q may be chosen so that $k[x, P, Q] = B$ and $D = \partial/\partial Q$ (*Theorem 4.15*).

We proceed by induction on $\deg_K P$.

Consider first the case $\deg_K P = 1$. If $P = ay + bz$ for $a, b \in k[x]$, then $(P_y, P_z) = (a, b)$. Since (a, b) is principal, and since D is irreducible, we conclude that $(a, b) = (1)$. Therefore, D is already a partial derivative in this case, as in the preceding paragraph.

Assume $\deg_K P > 1$. If $\deg_K Q \geq \deg_K P$, then the structure theory for $GA_2(K)$ implies that there exists $Q' \in K[y, z]$ such that $K[P, Q'] = K[y, z]$ and $\deg_K P > \deg_K Q'$; see *Sect. 4.3.4* above. Moreover, since $K[P, Q] = K[P, Q']$, we must have $\gamma Q' = \alpha Q + \beta(P)$ for some nonzero $\alpha, \gamma \in k[x]$ and some $\beta \in k[x, P]$. Thus, $\gamma DQ' = \alpha DQ \in k[x]$, which implies $DQ' \in k[x]$. So it is no loss of generality to assume $\deg_K P > \deg_K Q$. (Recall that $D = \Delta_{(x,P)}$ up to multiplication by elements of k^*, and we are therefore free to replace Q by Q' in the argument, since doing so does not affect the definition of D.) In addition, if QB is not a prime ideal of B, there exists $\ell \in k[x]$ dividing Q such that $(Q/\ell)B$ is prime. This is because Q is a K-variable. So it is no loss of generality to further assume Q is irreducible in B.

Observe that D satisfies condition $(*)$ in *Sect. 5.5.1*, since $r := PQ$ has $Dr = P \cdot DQ$ and $DQ \in k[x]$. Consider $D' := \Delta_{(x,Q)}$. By *Theorem 5.28*, D' is again locally nilpotent, and since $D'x = 0$, it is of rank at most two. Since Q is both irreducible and a K-variable, it follows that D' is irreducible. Therefore, $\ker D' = k[x, Q]$. Since $\deg_K Q < \deg_K P$, we may (by induction) assume that D' can be transformed into a partial derivative by a finite sequence of local slice constructions. Since D is obtained from D' by a single local slice construction, we conclude that D can be transformed into a partial derivative by a finite sequence of local slice constructions. □

5.6 Positive Homogeneous LNDs

In this section, we consider derivations of B which are homogeneous relative to a positive $\mathbb{Z}$-grading of B. For such a grading, there exists a homogeneous coordinate system (x, y, z) (*Corollary 3.43*), and we set $\omega = (\deg x, \deg y, \deg z)$. The subset of all homogeneous elements of $\mathrm{LND}(B)$ relative to ω will be denoted $\mathrm{LND}_\omega(B)$.

Recall that if $D \in \mathrm{LND}_\omega(B)$ and $D \neq 0$, then $\ker D = k[f,g]$ for homogeneous f and g.

Given ω, $\mathbb{P}^2_\omega$ will denote the **weighted projective plane** $\mathrm{Proj}(B)$ over the algebraic closure of k, and C_f, C_g will denote the projective curves defined by f and g. In case $\omega = (1,1,1)$, $\mathbb{P}^2$ will denote standard projective plane.

As we have seen, $\mathrm{LND}_\omega(B)$ is a large and interesting class of derivations, even for standard weights, and one would like to classify them in some meaningful way. A natural question is:

> Can every positive-homogeneous locally nilpotent derivation of B be obtained from a partial derivative via a finite sequence of local slice constructions?

Shortly after the appearance of the $(2,5)$ example, Daigle translated the problem of understanding homogeneous derivations into geometric language (Thm. 3.5 of [71]).[3]

Theorem 5.32 (Two Lines Theorem) *Suppose $\omega = (\omega_1, \omega_2, \omega_3)$ is a system of positive weights on B such that $\gcd(\omega_1, \omega_2, \omega_3) = 1$, and suppose $f, g \in B$ are homogeneous relative to ω, with $\gcd(\deg f, \deg g) = 1$. The following are equivalent.*

1. *There exists $D \in \mathrm{LND}_\omega(B)$ with $\ker D = k[f,g]$.*
2. *f and g are irreducible, and $\mathbb{P}^2_\omega \setminus (C_f \cup C_g)$ is a surface which is isomorphic to the complement of two lines in $\mathbb{P}^2$.*

In order to prove this, Daigle shows that the two conditions are each equivalent to a third, namely:

3. $B_{(fg)} = (A_{(fg)})^{[1]}$, where $A = k[f,g]$.

The subscript (fg) here denotes homogeneous localization. Specifically, since f and g are homogeneous, B_{fg} is a $\mathbb{Z}$-graded ring and $B_{(fg)}$ is the degree-zero component of B_{fg}. A key fact in showing the equivalence is that, when (3) holds, $A_{(fg)}$ is equal to the Laurent polynomial ring $k[t, t^{-1}]$, where $t = f^{\deg g}/g^{\deg f}$.

In some sense, the Two Lines Theorem replaces the problem of describing $\mathrm{LND}_\omega(B)$ by the following problem, which belongs to the theory of algebraic surfaces:

> Find all pairs of curves C_1, C_2 in $\mathbb{P}^2_\omega$ such that the complement of $C_1 \cup C_2$ is isomorphic to the complement of two lines in $\mathbb{P}^2$.

If $C_1, C_2 \subset \mathbb{P}^2_\omega$ is such a pair of curves, the isomorphism from the complement of two lines in $\mathbb{P}^2$ to $\mathbb{P}^2_\omega \setminus (C_1 \cup C_2)$ extends to a birational isomorphism $\sigma : \mathbb{P}^2 \to \mathbb{P}^2_\omega$ and σ can be factored into a finite succession of blow-ups and blow-downs.

In order to illustrate this idea, consider the locally nilpotent derivations θ_0, θ_1, and θ_2 defined in 5.5.2. These derivations are of Fibonacci Type, with degree type $(1,1)$, $(1,2)$, and $(2,5)$, respectively. In this case, the grading is the standard one, so

[3] The case $\gcd(\deg f, \deg g) \neq 1$ is also described in this paper.

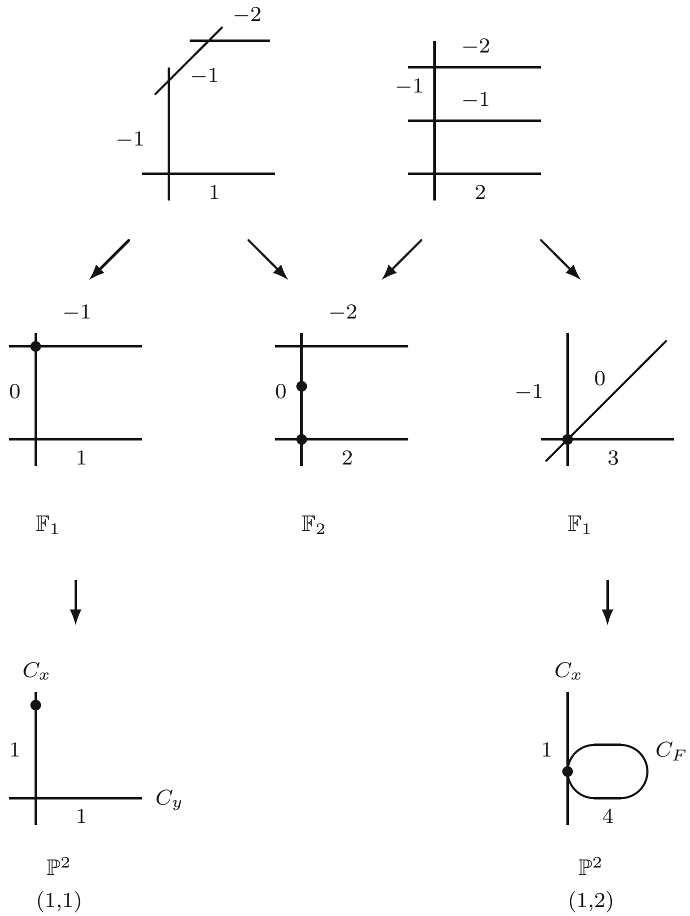

Fig. 5.1 Birational map π_1 of $\mathbb{P}^2$

$\omega = (1,1,1)$ and $\mathbb{P}^2_\omega = \mathbb{P}^2$. We have $\ker\theta_0 = k[x,y]$, and the two projective curves C_x, C_y are already lines. Likewise, $\ker\theta_1 = k[x,F]$ as above, and the complements of $C_x \cup C_y$ and $C_x \cup C_F$ in $\mathbb{P}^2$ are isomorphic. An explicit birational isomorphism of π_1 of $\mathbb{P}^2$ is given in *Fig. 5.1*, where the arrows ($\downarrow$) denote a blowing-down along a curve of self-intersection (-1) (so the inverse is a blowing up at the indicated point).

The numerical labels indicate the self-intersection number of the labeled curve, and the surfaces $\mathbb{F}_n$ are the **Hirzebruch surfaces** ($n \geq 1$); see [427] for a discussion of these surfaces.

Likewise, if $\ker\theta_2 = k[F,G]$ as above, then the complements of $C_x \cup C_F$ and $C_F \cup C_G$ in $\mathbb{P}^2$ are isomorphic, and *Fig. 5.2* illustrates the explicit isomorphism π_2. Note that C_F is smooth; C_G has a cusp; and the two curves intersect tangentially at this point. Note also that π_2 collapses C_F to a point, and maps C_x to C_G.

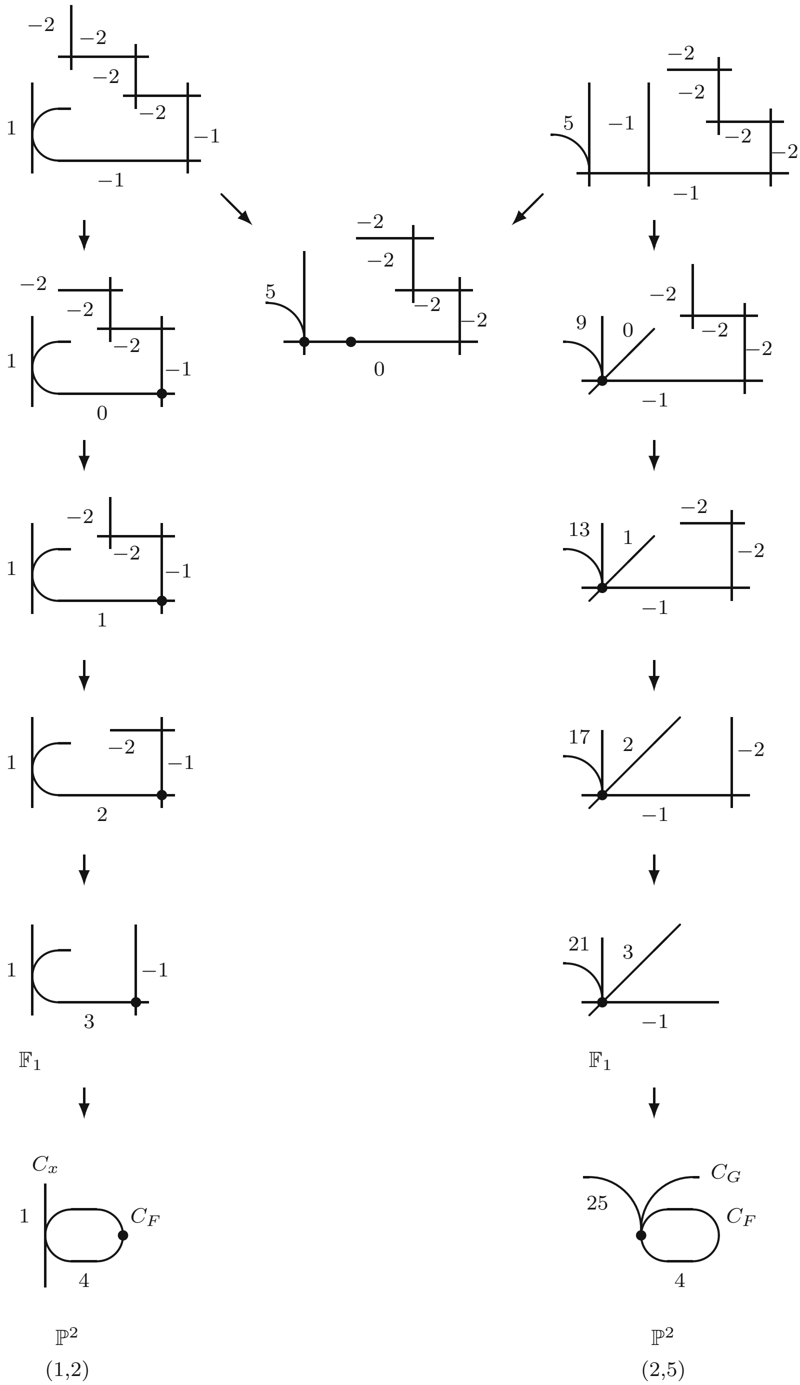

Fig. 5.2 Birational map π_2 of $\mathbb{P}^2$

A significant portion of this theory can be translated into a problem of combinatorics, using for example weighted dual trees. In this way, Daigle and Russell proved that finding all curves C_f and C_g which satisfy condition (2) of the Two Lines Theorem is equivalent to finding all affine rulings of the weighted projective plane $\mathbb{P}^2_\omega$ [86]; and then in [87], they give a complete description of all affine rulings of $\mathbb{P}^2_\omega$. Following is their definition of affine ruling, as found in [86].

Definition 5.33 Let X be a complete normal rational surface, and let Λ be a one-dimensional linear system on X without fixed components. Then Λ is an **affine ruling** of X if there exist nonempty open subsets $U \subset X$ and $\Gamma \subset \mathbb{P}^1$ such that $U \cong \Gamma \times \mathbb{A}^1$ and such that the projection morphism $\Gamma \times \mathbb{A}^1 \to \Gamma$ determines Λ.
Daigle writes: "There is a rich interplay between the theory of algebraic surfaces and homogeneous locally nilpotent derivations of $k[x, y, z]$" ([68], p. 35). Indeed, the results of Daigle and Russell are of broad significance in the study of algebraic surfaces, and the impressive geometric machinery and theory they develop has implications far beyond the study of $\mathbb{G}_a$-actions on $\mathbb{A}^3$. Their work effectively provides a complete classification of the positive homogeneous locally nilpotent derivations of B, where the local slice construction in the algebraic theory corresponds to a certain kind of birational modification of surfaces in the geometric theory. In particular, their work implies an affirmative answer to the question asked above.

Theorem 5.34 ([75]) *If ω is a positive system of weights on B, and if $D, E \in \mathrm{LND}_\omega(B)$ are irreducible, then D can be transformed to E via a finite sequence of local slice constructions.*
In the case of standard weights, the derivations of Fibonacci Type play a central role.

Theorem 5.35 *If $D \in \mathrm{LND}_\omega(B)$ for standard weights $\omega = (1, 1, 1)$, and $D \neq 0$, then, up to change of coordinates, $\ker D = k[H_n]^{[1]}$ for one of the polynomials H_n defined above.*
These two results are valid over an algebraically closed field k of characteristic zero. The second result, while unpublished, is due to Daigle, and can be proved using the results of [87, 88].

Remark 5.36 It is not surprising that some of the projective plane curves encountered here in the context of locally nilpotent derivations appeared in earlier work on classification of curves. For example, the quintic curve G used in the $(2, 5)$ example was studied by Yoshihara [429]. Yoshihara's example motivated the work of Miyanishi and Sugie in [308], who studied reduced plane curves D whose complement $\mathbb{P}^2 \setminus D$ has logarithmic Kodaira dimension $-\infty$. They remark: "So far, we have only one example for D of the second kind. That is, a quintic rational curve with only one cusp of multiplicity 2 which was obtained by H. Yoshihara." In his review of this paper (MR 82k:14013), Gizatulin asserts the existence of a family of curves C_i "of the second kind" whose degrees are the Fibonacci numbers $1, 2, 5, 13, \ldots$. In particular, C_3 is Yoshihara's quintic. It appears, however, that Gizatulin never published the details of his examples.

5.7 Generalized Local Slice Constructions

As we have seen, the study of $\mathrm{LND}_\omega(B)$ can be reduced to a problem in dimension two, where the tools of surface theory can be applied. What about the general case? As mentioned, local slice constructions do not require any kind of homogeneity, thus providing a point of departure for investigating the full set $\mathrm{LND}(B)$.

We are interested in subrings A of B which occur as the kernel of *some* $D \in \mathrm{LND}(B)$, rather than in the specific derivation D of which A is the kernel. In [161], Sect. 5, we define the graph Γ, where

$$\mathrm{vert}(\Gamma) = \{\ker D \mid D \in \mathrm{LND}(B), D \neq 0\}$$

and where two distinct vertices $\ker D$ and $\ker D'$ are joined by an edge if and only if D' can be obtained from D by a single local slice construction. Subsequently, Daigle in [74] generalized the graph Γ to a graph $\underline{\mathrm{KLND}}(\mathcal{B})$ defined for any integral domain $\mathcal{B}$ of characteristic zero, by first distinguishing in $\mathcal{B}$ certain subrings of codimension 2. The graph he defines is an invariant of the ring $\mathcal{B}$, and the group of automorphisms of $\mathcal{B}$ acts on it in a natural way. In case $\mathcal{B} = B = k^{[3]}$, Daigle's definition holds that neighboring vertices in $\underline{\mathrm{KLND}}(B)$ admit both a common kernel generator and a common local slice, and it turns out that $\Gamma = \underline{\mathrm{KLND}}(B)$ in this case.

This graph is related to Daigle's generalization of the local slice construction. According to Daigle:

> This generalization produces new insight into the local slice construction. In particular, we find that that process is essentially a two-dimensional affair and that it is intimately related to Danielewski surfaces. ([74], p. 1)

Here, a surface defined by a polynomial of the form $xy - \phi(z) \in k[x, y, z]$ is called a **special Danielewski surface** over k; these will be discussed in *Chap. 9* below.[4] If R is the coordinate ring of a special Danielewski surface, then any triple $(x, y, z) \in R^3$ such that $R = k[x, y, z]$ and $xy \in k[z] \setminus k$ is called a **coordinate system** of R. If $R \subset \mathcal{B}$ for some commutative k-domain $\mathcal{B}$, then $\mathcal{B}_R$ denotes localization of $\mathcal{B}$ at the nonzero elements of R.

Now suppose $\mathcal{B}$ is a k-affine UFD. Suppose there exists an element $w \in \mathcal{B}$ and subrings $R \subset A \subset \mathcal{B}$ which satisfy the following two conditions.

(i) $A = \ker D$ for some irreducible $D \in \mathrm{LND}(\mathcal{B})$
(ii) $A_R = K[Dw] = K^{[1]}$, where $K = R_R = \mathrm{frac}(R)$

Proposition 5.37 (Prop. 9.12.1 of [68]) *In the above notation, $\mathcal{B}_R$ is a special Danielewski surface over K, and there exists $\tilde{v} \in \mathcal{B}$ such that $(Dw, \tilde{v}, w)$ is a coordinate system of $\mathcal{B}_R$. Moreover, for any pair $u, v \in \mathcal{B}$ such that $A_R = K[u]$*

[4] Daigle refers to these simply as Danielewski surfaces.

and (u, v, w) *is a coordinate system of* $\mathcal{B}_R$*, the ring* $A' = K[v] \cap \mathcal{B}$ *is the kernel of a locally nilpotent derivation of* $\mathcal{B}$*.*
In this case, we say that A' is obtained from the triple (A, R, w) by a local slice construction. When $\mathcal{B} = B = k^{[3]}$, this procedure is equivalent to the local slice construction as originally defined.

Any subring R of $\mathcal{B}$ satisfying conditions (i) and (ii) above for some A and w will be called a **Daigle subring** of $\mathcal{B}$.

Example 5.38 (Ex. 9.14 of [68]) For the ring $\mathcal{B} = k[u, v, x, y, z] = k^{[5]}$, define elements

$$s = vx - uy \ , \ t = uz - x(s+1) \ , \ w = xt$$

and define subrings:

$$A = \ker \partial_z = k[u, v, x, y] \quad \text{and} \quad R = k[u, v, s]$$

Then $\partial_z w = ux$. Set $K = \text{frac}(R) = k(u, v, s)$. Then $A_R = K[x] = K[\partial_z w]$, so the triple (A, R, w) satisfies conditions (i) and (ii) above. Therefore, $\mathcal{B}_R$ is a Danielewski surface over K. In fact, $\mathcal{B}_R = K[x, z] = K[x, t] = K^{[2]}$. Therefore, the triple (x, t, w) is a coordinate system of $\mathcal{B}_R$. By the proposition, the ring $A' = K[t] \cap \mathcal{B}$ is the kernel of some $D \in \text{LND}(\mathcal{B})$, namely:

$$Du = Dv = 0 \ , \ Dx = u \ , \ Dy = v \ , \ \text{and } Dz = 1 + s$$

This is precisely the derivation of Winkelmann discussed in *Example 3.8.5*, where the kernel is given explicitly.

5.8 $\mathbb{G}_a^2$-Actions

Based on the following result of Kaliman [226], it is possible to describe all algebraic actions of $\mathbb{G}_a^2$ on $\mathbb{A}^3$, or equivalently, all commuting pairs $D, E \in \text{LND}(k[x, y, z])$.

Theorem 5.39 (Kaliman's Fiber Theorem) *Suppose* $f : \mathbb{C}^3 \to \mathbb{C}^1$ *is a polynomial function. If infinitely many fibers of* f *are isomorphic to* $\mathbb{C}^2$*, then* f *is a variable of* $\mathbb{C}[x, y, z]$*.*
This was generalized in [84] (Thm. 3) to all fields k of characteristic zero. This allows us to prove:

Proposition 5.40 *If* $B = k[x, y, z]$*, and if* $D, E \in \text{LND}(B)$ *are nonzero, commuting and have distinct kernels, then there exists a variable* $f \in B$ *such that* $\ker D \cap \ker E = k[f]$.

Proof Since D and E commute, E restricts to a nonzero locally nilpotent derivation on $A = \ker D = k^{[2]}$. By Rentschler's Theorem, there exists $f \in A$ such that $A = k[f]^{[1]}$ and $\ker D \cap \ker E = \ker (E|_A) = k[f]$. Therefore, the quotient map for the $\mathbb{G}_a^2$-action is of the form $F : \mathbb{A}^3 \to \mathbb{A}^1$, and is given by evaluation of the polynomial f.

The inclusions $k[f] \to A \to B$ give a factorization of F as the composition $H : \mathbb{A}^3 \to \mathbb{A}^2$ and $G : \mathbb{A}^2 \to \mathbb{A}^1$. By the Slice Theorem, there are open sets $U \subset \mathbb{A}^3$ and $V \subset \mathbb{A}^2$ such that $U = V \times \mathbb{A}^1$, $H(U) = V$, and $H : U \to V$ is a projection. Likewise, there exist open sets $V' \subset \mathbb{A}^2$ and $W \subset \mathbb{A}^1$ such that $V' = W \times \mathbb{A}^1$, $G(V') = W$, and $G : V' \to W$ is a projection. Thus, if $U' = H^{-1}(V \cap V')$ and $W' = G(V \cap V')$, it follows that $U' = W' \times \mathbb{A}^2$, $F(U') = W'$, and $F : U' \to W'$ is a projection. In particular, every fiber of F over a point of W' is isomorphic to $\mathbb{A}^2$. By the result of Kaliman, f is variable of B. □

It should be noted that, in his thesis [290] and later in [289], Maubach also recognized this application of Kaliman's result.

This proposition indicates that a rank-three $\mathbb{G}_a$-action on $\mathbb{A}^3$ cannot be extended to a $\mathbb{G}_a^2$-action. Nonetheless, there are actions of $\mathbb{G}_a^2$ on $\mathbb{A}^3$ which are not conjugate to a triangular action. For example, let $P, Q \in B$ be any pair such that $k(x)[P, Q] = k(x)[y, z]$. Define $k(x)$-derivations Δ_P and Δ_Q as in *Chap. 4*. Then:

$$\Delta_P = f(x)\partial_Q \quad \text{and} \quad \Delta_Q = g(x)\partial_P \qquad (f(x), g(x) \in k[x])$$

We see that Δ_P and Δ_Q commute on $k(x)[y, z]$, and restrict to B. Moreover, by *Theorem 5.19*, Δ_P is triangularizable if and only if Q is a variable of B, and likewise Δ_Q is triangularizable if and only P is a variable of B.

With a bit more work, one can show that such "neighboring pairs" of rank-two derivations provide a description of all $\mathbb{G}_a^2$-actions on $\mathbb{A}^3$.

Appendix: An Intersection Condition

The goal of this section is to prove the following fact. The theorem and its proof are due to the author and Daigle.

Theorem 5.41 *Let U be a UFD containing k, $R \subset U$ a k-subalgebra, $D \in \mathrm{LND}(U)$ nonzero and $A = \ker D$. Suppose that $S \subset U$ is a subring satisfying:*

1. *$S = R[u, v] \cong R^{[2]}$ for some $u, v \in U$*
2. *S is factorially closed in U*
3. *$R \subset S \cap A \subset S$, $R \neq S \cap A$, $S \cap A \neq S$*

There exists $w \in S$ such that $S \cap A = R[w]$ and $K[u, v] = K[w]^{[1]}$, where $K =$ $\mathrm{frac}(R)$.

Proof Let $\sigma \in S \cap A$, $\sigma \notin R$, be given, and write $\sigma = f(u, v)$ for $f \in R[u, v]$. Then:

$$0 = D\sigma = f_u Du + f_v Dv$$

Consider first the case $Du \neq 0$, $Dv \neq 0$. Set $t = \gcd(f_u, f_v) \in U$, and choose $a, b \in U$ such that $f_u = tb$ and $f_v = ta$. Since S is factorially closed, it follows that $a, b \in S$. Therefore, $aDv = -bDu$, and we conclude that a divides Du. Set $r = Du/a$.

Define $d \in \mathrm{Der}_R(S)$ by $ds = as_u - bs_v$ $(s \in S)$. Given $s \in S$, we have:

$$\begin{aligned} aDs &= a(s_u Du + s_v Dv) \\ &= as_u Du + s_v(aDv) \\ &= as_u Du - s_v(bDu) \\ &= (as_u - bs_v)Du \\ &= dsDu \end{aligned}$$

Therefore, $Ds = rds$ for all $s \in S$. Note that $d \neq 0$ and $r \neq 0$, since otherwise $S \subset A$. We conclude that, if neither Du nor Dv is zero, then D is a quasi-extension of d.

Consider next the case $Du = 0$ or $Dv = 0$. We may assume $Dv = 0$, in which case $Du \neq 0$ (otherwise $S \subset A$). In this case, let $d = \partial/\partial u$ and $r = Du$. Then for every $s \in S$, $Ds = s_u Du = rds$. So in either case, D is a quasi-extension of some nonzero d on S.

By *Lemma 2.35*, $d \in \mathrm{LND}(S)$. By *Theorem 4.12*, there exists $w \in S$ and $\alpha \in R[w]$ such that $d = \alpha\Delta_w$ and $\ker d = R[w]$, where Δ_w is the locally nilpotent R-derivation on $R[u, v]$ defined by $\Delta_w(h) = h_u w_v - h_v w_u$. Consequently, $R[w] \subset S \cap A$.

Conversely, let $\psi \in S \cap A$ be given. Then:

$$0 = D\psi = rd\psi \quad \Rightarrow \quad d\psi = 0 \quad \Rightarrow \quad \psi \in \ker d = R[w]$$

Therefore, $S \cap A = R[w]$. Moreover, *Theorem 4.12* shows that $K[u, v] = K[w]^{[1]}$. □

An immediate consequence of *Theorem 5.41* is the following.

Corollary 5.42 *If, in addition to the hypotheses of Theorem 5.41, R is a field, then there exists $w \in S$ such that:*

$$S \cap A = R[w] \quad \text{and} \quad S = R[w]^{[1]}$$

Another consequence is:

Corollary 5.43 (Thm. 2 of [160]) *Let D be a locally nilpotent k-derivation of the polynomial ring $k[x_1, \ldots, x_n]$, $n \geq 2$, and suppose that $k[x_1, x_2] \cap \ker D \neq k$. Then either $Dx_1 = Dx_2 = 0$, or there exists $g \in k[x_1, x_2]$ such that $k[x_1, x_2] = k[g]^{[1]}$ and $k[x_1, x_2] \cap \ker D \subset k[g] \subset \ker D$.*

Chapter 6
Linear Actions of Unipotent Groups

Invariant theory originally concerned itself with groups of vector space transformations, so the linear algebraic $\mathbb{G}_a$-actions were the first $\mathbb{G}_a$-actions to be studied. The action of $SL_2(\mathbb{C})$ on the vector space V_n of binary forms of degree n has an especially rich history, dating back to the mid-Nineteenth Century. The ring of SL_2-invariants, together with the ring of $\mathbb{G}_a$-invariants for the subgroup $\mathbb{G}_a \subset SL_2(\mathbb{C})$, were the focus of much research at the time. A fundamental result due to Gordan (1868) is that both $k[V_n]^{SL_2(\mathbb{C})}$ and $k[V_n]^{\mathbb{G}_a}$ are finitely generated [182]. Gordan calculated generators for these rings up to $n = 6$. These historical developments are discussed in *Sect. 6.3.1*.

This chapter investigates invariant rings of the form $k[V]^U$, where U is a unipotent linear algebraic group over k and V is a U-module. There are two main results which frame the discussion: (1) the theorem of Maurer and Weitzenböck, which asserts that (in the characteristic zero case) $k[V]^{\mathbb{G}_a}$ is finitely generated; and (2) the examples of Nagata and others, which show that $k[V]^U$ need not be finitely generated for the higher-dimensional vector groups $U = \mathbb{G}_a^m$.

After some discussion of Hilbert's Fourteenth Problem in *Sect. 1*, a proof of the Maurer-Weitzenböck Theorem is presented in *Sect. 2*, based on the Finiteness Theorem. *Section 3* discusses the question of finding generators and relations for the invariant rings $k[V]^{\mathbb{G}_a}$. The remainder of the chapter is devoted to discussion of the vector group actions of Nagata and others, in addition to some recent examples involving non-commutative unipotent groups.

G. Freudenburg, *Algebraic Theory of Locally Nilpotent Derivations*,
Encyclopaedia of Mathematical Sciences 136,
DOI 10.1007/978-3-662-55350-3_6

6.1 The Finiteness Theorem

In modern terminology, the famous **Fourteenth Problem of Hilbert** is as follows.

For fields $K \subset L$, if $B = K^{[n]}$ and $L \subset \mathrm{frac}(B) = K^{(n)}$, is $L \cap B$ finitely generated over K?

The main case of interest at the time of Hilbert was that of invariant rings for algebraic subgroups of $GL_n(\mathbb{C})$ acting on $\mathbb{C}^n$ as a vector space. But one can also consider the case of invariant rings of more general algebraic group actions on varieties:

For a field K, suppose the linear algebraic K-group G acts algebraically on an affine K-variety V. Is the invariant ring $K[V]^G$ finitely generated?

If the group G is reductive, then this question has a positive answer.

(Finiteness Theorem) If K is any field and G is a reductive K-group acting by algebraic automorphisms on an affine K-variety V, then the algebra of invariants $K[V]^G$ is finitely generated over K.

In general, however, this question has a negative answer: *Example 4.4* gives a $\mathbb{G}_a$-action on the variety $V = C \times \mathbb{A}^2$, where C is a cuspidal cubic plane curve, such that $k[V]^{\mathbb{G}_a}$ is not finitely generated. Note that this is example does not address Hilbert's Fourteenth Problem, since it does not fit the hypotheses of Hilbert's question.

In the late 1950s, Nagata published his celebrated counterexamples to Hilbert's Fourteenth Problem [320, 321]. One of these examples uses the unipotent group $\mathbb{G}_a^{13}$ acting linearly on $\mathbb{A}^{32}$, and Nagata proves that the invariant ring of this action is not finitely generated. In the language of derivations, this can be realized by 13 commuting linear triangular derivations D_i of the polynomial ring $K^{[32]}$ for which the subring $\cap_i \ker D_i$ is not finitely generated. Nagata's results are valid for any field K which is not an algebraic extension of a finite field.

The central idea in proving the Finiteness Theorem is due to Hilbert, whose original proof was for $SL_n(\mathbb{C})$. The full proof of the theorem represents the culmination of the efforts of many mathematicians over the past century, most recently for certain cases in positive characteristic. The case of finite groups was settled by E. Noether in her famous papers of 1916 ([329], characteristic 0) and 1926 ([330], positive characteristic).

In his lectures [323], Nagata formulated the following generalization of the Finiteness Theorem.

The following properties of the linear algebraic group G are equivalent: (a) For all algebraic actions of G on an affine algebraic variety X, the algebra $k[X]^G$ is finitely generated over k. (b) G is reductive.

That (a) implies (b) was proved in 1979 by Popov [342]. See also the Appendix to Chap. 1 of [319].

The Finiteness Theorem is the main tool used in the proof of the Maurer-Weitzenböck Theorem presented below. In fact, for this proof, we only need the Finiteness Theorem for the group $G = SL_2(k)$, which had already been established by Gordan in 1868. While we do not include a proof of the Finiteness Theorem, several accounts of the theorem and its proof can be found in the literature. See, for example, the book of van den Essen [142] for a proof in the case k is algebraically closed of characteristic zero (Chap. 9). The reader is also referred to the article of Humphreys [214], which provides an introductory survey of reductive group actions, and to the monograph of Popov [345], which gives a more extended treatment of the subject; each contains insightful historical background and a good list of pertinent references. The article of Mumford [318] is also required reading for anyone interested in the subject. Other standard references include [13, 113, 155, 319, 326, 328].

In view of the Finiteness Theorem, the question of finite generation for reductive groups has been replaced by other questions about invariant rings. For example, in what has come to be called **computational invariant theory**, the idea is to determine degree bounds for a system of generators, or to find algorithms which produce minimal generating sets, for invariant rings of reductive group actions; see [98].

The speech delivered by Hilbert in 1900 to the International Congress included 10 of his 23 famous problems; the speech and all the problems were later published in [205]. In contrast to its influence on mathematics in the following century, this speech bears the unassuming title *Mathematische Probleme*.[1] In 1902, the speech and problems appeared in English translation in [206]. In 1974, the American Mathematical Society sponsored a special Symposium on the mathematical consequences of Hilbert's problems. The volume [318] contains the proceedings of that symposium, as well as the English translation of Hilbert's speech. The purpose of the Symposium was "to focus upon those areas of importance in contemporary mathematical research which can be seen as descended in some way from the ideas and tendencies put forward by Hilbert in his speech" (from the Introduction). In particular, the volume contains one paper discussing each of the 23 problems, written by 23 of the most influential mathematicians of the day. The paper for Problem Fourteen was written by Mumford, *op. cit.*

6.2 Mauer-Weitzenböck Theorem

Theorem 6.1 (Maurer-Weitzenböck Theorem) *If k is a field of characteristic zero, and if V is a $\mathbb{G}_a$-module over k, then $k[V]^{\mathbb{G}_a}$ is finitely generated.*

6.2.1 Background

Both Maurer [291] in 1899, and Weitzenböck [418] in 1932, thought he had proved finite generation for any algebraic group acting linearly on $\mathbb{C}^n$, but each

[1] *Mathematical Problems.*

made essentially the same mistake. However, their proofs for actions of the one-dimensional groups were sound. In [39], V.4, A. Borel gives a detailed exposition of the history of this result, highlighting the contributions of Maurer to invariant theory. Borel writes:

> Maurer's next publication [291] is an unfortunate one, since he sketches what he claims to be a proof of a theorem on the finiteness of invariants for any (connected) linear Lie group, a statement we know to be false. However, it also contains some interesting results, with correct proofs, including one which nowadays is routinely attributed to Weitzenböck (although the latter refers to [Maurer] for it). (p. 111)

Regarding Weitzenböck's knowledge of Maurer's earlier paper, Borel writes:

> He views its results as valid. His goal is to give a full proof, rather than just a sketch. (p. 112)

It was Hermann Weyl who, in reviewing Weitzenböck's paper in 1932, found a gap. Borel continues:

> The theorem in that [one-dimensional] case is nowadays attributed to Weitzenböck, probably beginning with Weyl, but this seems unjustified to me. The proof is quite similar to Maurer's, to which Weitzenböck refers explicitly. In particular, in the most important case of a nilpotent transformation, there is the same reduction to a theorem of P. Gordan. It is true that Maurer limits himself to regular transformations. However, his argument extends trivially to the case where the given Lie algebra is commutative, spanned by one nilpotent transformation and several diagonalizable ones, with integral eigenvalues, and Maurer proved that the smallest regular algebra containing a given linear transformation is of that form. But, surely, this is not the reason for that misnomer. Simply, [Maurer's paper] had been overlooked. (p. 113)

Eventually, Seshadri gave a proof of the theorem in his 1962 paper [379], where he "brings out clearly the underlying idea of Weitzenböck's proof" (p. 404). Nagata included a proof of the Maurer-Weitzenböck Theorem in his classic *Lectures on the Fourteenth Problem of Hilbert* from 1965, based on Seshadri's ideas ([323], Chap. IV). Grosshans proved the Maurer-Weitzenböck Theorem in his book [188] in the context of more general group actions, and Tyc [406] gave a more algebraic version of Seshadri's proof in the case $k = \mathbb{C}$.

6.2.2 $\mathbb{G}_a$-Modules and Jordan Normal Form

Let $E_n \in \mathcal{M}_n(k)$ be the nilpotent matrix given by:

$$E_n = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{n\times n}$$

E_n is called the **elementary nilpotent matrix** of order n over k. Note that $E_n^n = 0$.

Let V be a vector space over k of dimension $n+1$, $n \geq 0$. An action of $\mathbb{G}_a$ on V is given by $\exp(tM)$, where $M \in \text{End}_k(V)$ is nilpotent.

Lemma 6.2 *The following conditions are equivalent.*

1. *The induced representation $\mathbb{G}_a \to GL(V)$ is irreducible.*
2. *The Jordan normal form of M is given by E_{n+1} in some basis for V.*
3. *$V \cong_{\mathbb{G}_a} V_n$, where V_n is the $\mathbb{G}_a$-module of binary forms of degree n.*

Proof That (1) and (2) are equivalent follows from the Jordan normal form of M. We show that (2) and (3) are equivalent.

Consider a non-trivial $\mathbb{G}_a$-action on $V_1 = k^2$. If $N \in \mathcal{M}_2(k)$ is nilpotent and nonzero, then the Jordan normal form of N is E_2. Therefore, the corresponding action of $\mathbb{G}_a$ on $k[V_1] = k[x, y]$ is given by $t \cdot (x, y) = (x, y + tx)$ in an appropriate basis $V_1 = kx \oplus ky$. This induces an action of $\mathbb{G}_a$ on $V_n = k^{n+1}$, the vector space of binary forms of degree n in x and y. A basis of V_n is given by the monomials $X_i = x^{n-i}y^i$ $(0 \leq i \leq n)$. The induced $\mathbb{G}_a$-action on V_n is therefore given by:

$$t \cdot X_i = x^{n-i}(y + tx)^i$$

Let D be the corresponding locally nilpotent derivation of $k[X_0, \ldots, X_n] = k^{[n+1]}$. Recall that, since $\exp(tD)(X_i) = X_i + tDX_i + \cdots$, we are looking for the degree-one coefficient relative to t, i.e., $d/dt(x^{n-i}(y + tx)^i)$ evaluated at $t = 0$. This is easily calculated, and we get $DX_i = ix^{n-(i-1)}y^{i-1} = iX_{i-1}$ if $i \geq 1$, and $DX_0 = 0$. As a linear operator on V_n, we see that D is conjugate to E_{n+1}. □

The vector space k^n with an irreducible $\mathbb{G}_a$-action is the **irreducible $\mathbb{G}_a$-module of dimension n**. (Tan [400] refers to these as **basic** $\mathbb{G}_a$-actions.) A linear $\mathbb{G}_a$-action which factors through a representation of $SL_2(k)$ is called **fundamental**. (Onoda [338] refers to these as **standard** $\mathbb{G}_a$-actions.)

If $M \in \mathcal{M}_n(k)$ is nilpotent, then its Jordan form is given by

$$M = \begin{pmatrix} E_{n_1} & 0 & \cdots & 0 \\ 0 & E_{n_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E_{n_r} \end{pmatrix}_{n \times n}$$

where the integers n_i satisfy:

$$n_1 \geq n_2 \geq \cdots \geq n_r \geq 1 \quad \text{and} \quad n_1 + n_2 + \cdots + n_r = n.$$

Taking all these considerations together, we have the following.

Proposition 6.3 *Let $V = k^n$ for $n \geq 1$.*

(a) *Every action of $\mathbb{G}_a$ on V is a direct sum of basic actions.*
(b) *Every action of $\mathbb{G}_a$ on V is fundamental.*
(c) *The number of conjugacy classes of $\mathbb{G}_a$-actions on V is equal to the number of partitions of the integer n.*

It should be noted that, in positive characteristic, there exist linear $\mathbb{G}_a$-actions on $\mathbb{A}^n$ which are not fundamental. An example is given by Fauntleroy in [149], though he also shows that the invariant ring for this example is finitely generated. This explains why Seshadri's proof does not work for arbitrary linear $\mathbb{G}_a$-actions in positive characteristic. The question of finite generation for algebraic $\mathbb{G}_a$-actions in positive characteristic remains open, even in the linear case.

For a nice overview of the classical theory of binary forms, see the lecture notes of Dixmier [115].

6.2.3 Proof of the Maurer-Weitzenböck Theorem

Let V be a $\mathbb{G}_a$-module. The proof of finite generation for $k[V]^{\mathbb{G}_a}$ exploits the fact that the action is the restriction of an $SL_2(k)$-action on V. It proceeds by showing that $k[V]^{\mathbb{G}_a}$ is isomorphic to the ring of invariants of an $SL_2(k)$-action on a larger polynomial algebra.

Proposition 6.4 (Thm. 1.2 of [187]) *Let G be a reductive k-group, and suppose $H \subset G$ is an algebraic subgroup for which the invariant ring $k[G]^H$ is finitely generated, where H acts on G by right multiplication. Then for any affine G-variety V, the corresponding ring of invariants $k[V]^H$ is finitely generated.*

Proof (Following Grosshans) Consider the action of the group $G \times H$ on the variety $G \times V$ defined by $(g,h) \cdot (a,v) = (gah^{-1}, g \cdot v)$, where $g \cdot v$ denotes the given action of G on V. We calculate the invariant ring $k[G \times V]^{G \times H}$ in two different ways.

First, since the action of $1 \times H$ on V (or, more properly, on $1 \times V$) is trivial, it follows that:

$$k[G \times V]^{1 \times H} = (k[G] \otimes k[V])^{1 \times H} = k[G]^H \otimes k[V]$$

See, for example, Lemma 1 (p. 7) of Nagata [323]. Since both $k[G]^H$ and $k[V]$ are affine rings (by hypothesis), $k[G]^H \otimes k[V]$ is also affine. And since G is reductive, it follows that

$$k[G \times V]^{G \times H} = (k[G] \otimes k[V])^{G \times H} = \left((k[G] \otimes k[V])^{1 \times H}\right)^{G \times 1}$$

is finitely generated.

Second, consider the equality:

$$k[G \times V]^{G\times H} = \left(k[G \times V]^{G\times 1}\right)^{1\times H}$$

Let $f \in k[G \times V]^{G\times 1}$. Then for all $(g, 1) \in G \times 1$ and $(a, v) \in G \times V$, we have:

$$f(a, v) = f((g, 1) \cdot (a, v)) = f(ga, v)$$

Since the action of G on itself by left multiplication is transitive, we conclude that f is a function of v alone. Therefore, f is also invariant under the action of $1 \times H$ if and only if $f(a, v) = f(a, h \cdot v)$ for all $h \in H$. It follows that $k[G \times V]^{G\times H} \cong k[V]^H$. □

Proposition 6.5 *Let $\mathbb{G}_a$ act on $SL_2(k)$ by right multiplication. Then:*

$$k[SL_2(k)]^{\mathbb{G}_a} \cong k^{[2]}$$

Proof By Kambayashi's Theorem, *Sect. 5.1.1*, it suffices to assume k is algebraically closed.

Let $B = k[SL_2(k)]$ and $A = B^{\mathbb{G}_a}$. Since B is a UFD, A is also a UFD, and $\text{tr.deg}_k A = 2$. As noted in *Remark 5.18*, there are four commuting $\mathbb{G}_a$-actions on $SL_2(k)$ defined by left and right multiplication, and the only functions invariant for each action are constants. Therefore, there are three commuting $\mathbb{G}_a$-actions on A whose common ring of invariants is k, which implies that $ML(A) = k$. By *Corollary 2.13*, $A \cong k^{[2]}$. □

Combining this proposition with the two which precede it, we get a proof of the Maurer-Weitzenböck Theorem.

The proof of *Proposition 6.4* gives a way to see the isomorphism between the ring $k[V]^{\mathbb{G}_a}$ and the ring of $SL_2(k)$-invariants on a larger polynomial algebra. The proof first shows that:

$$k[V]^{\mathbb{G}_a} = \left(k[SL_2(k) \times V]^{SL_2(k)}\right)^{\mathbb{G}_a} = k[SL_2(k) \times V]^{SL_2(k)\times\mathbb{G}_a}$$

The second calculation then shows:

$$\begin{aligned} k[SL_2(k) \times V]^{SL_2(k)\times\mathbb{G}_a} &= \left(k[SL_2(k) \times V]^{\mathbb{G}_a}\right)^{SL_2(k)} \\ &= \left(k[V] \otimes k[SL_2(k)]^{\mathbb{G}_a}\right)^{SL_2(k)} \\ &= (k[V] \otimes k[V_1])^{SL_2(k)} \\ &= k[V \times V_1]^{SL_2(k)} \end{aligned}$$

According to Derksen and Kemper, this isomorphism was proved in 1861 by M. Roberts [357]. See Example 2.5.2 of [98], and Example 3.6 of [44]. The invariants $k[V \oplus V_1]^{SL_2(k)}$ are called **covariants** of the $SL_2(k)$-action on V.

The main idea used in the proof of *Proposition 6.4* is the **transfer principle** or **adjunction argument**, also called the **Grosshans principle** in its more general form, which asserts that, if H is a closed subgroup of an algebraic group G, then $(k[G/H] \otimes k[X])^G \cong k[X]^H$ for any G-module X. In Chap. 2 of [188], Grosshans gives a nice historical outline of the transfer principle, followed by its proof and various applications; a statement and proof of the transfer principle in its most general form can be found in Popov [343]. Grosshans writes:

> Roughly speaking, it allows information on $k[G/H]$ to be transferred to W^H. For example, suppose that G is reductive and that $k[G/H]$ is finitely generated. Let $W = A$ be a finitely generated, commutative k-algebra on which G acts rationally. Then using the transfer principle and Theorem A, we see that A^H is finitely generated. The most important instance of this occurs when $H = U$ is a maximal unipotent subgroup of a reductive group G.... In the study of binary forms, H is taken to be a maximal unipotent subgroup of $SL(2, \mathbb{C})$. The transfer theorem in this context was proved by M. Roberts in 1871 [*sic*] and describes the relationship between "covariants", i.e., the algebra $(\mathbb{C}[G/H] \otimes A)^G$, and "semi-invariants", the algebra A^H. (From the Introduction to Chap. 2)

Specifically, any maximal unipotent subgroup U of $G = SL_2(k)$ is one-dimensional, i.e., $U = \mathbb{G}_a$. Thus, the Maurer-Weitzenböck Theorem is a special case of the following more general fact. See also Thm. 9.4 of [188].

Theorem 6.6 (Hadziev [198]) *Let G be a reductive group and let U be a maximal unipotent subgroup of G. Let A be a finitely generated, commutative k-algebra on which G acts rationally. Then A^U is finitely generated over k.*

The theorem of Hadziev was generalized by Grosshans in [187]. Another proof of the transfer principle is given in [44].

Remark 6.7 In [406], Tyc shows that, for any $\mathbb{G}_a$-module V over $\mathbb{C}$, $\mathbb{C}[V]^{\mathbb{G}_a}$ is a Gorenstein ring. The Gorenstein property was announced earlier (without proof) by Onoda in [338].

Remark 6.8 For *any* linear derivation $D \in \mathrm{Der}_k(k^{[n]})$, both $\ker D$ and $\mathrm{Nil}D$ are finitely generated; see 6.2.2 and 9.4.7 of Nowicki [333].

Remark 6.9 In the papers [148–150], Fauntleroy studies linear $\mathbb{G}_a$-actions from the geometric viewpoint, with particular attention to the case in which the ground field is of positive characteristic. For example, in the first of these papers, he shows that if the fixed point set of a linear $\mathbb{G}_a$-action on affine space is a hyperplane, then the ring of invariants is finitely generated. The recent article of Tanimoto [403] also gives some cases in which the ring of invariants of a linear $\mathbb{G}_a$-action on affine space is finitely generated for fields of positive characteristic.

6.3 Generators and Relations

This section discusses the question of finding generators and relations for $k[V]^{\mathbb{G}_a}$ when V is a $\mathbb{G}_a$-module.

We denote by A_n the ring of $\mathbb{G}_a$-invariant polynomials for n-forms, or equivalently, the kernel of the basic linear derivation D_n on $B_n = \mathbb{C}^{[n+1]}$. In particular, let $B_n = \mathbb{C}[x_0, \dots, x_n]$, and $D_n x_i = x_{i-1}$ for $i \geq 1$ and $Dx_0 = 0$. Note that D_n is homogeneous of degree 0 relative to the standard $\mathbb{Z}$-grading of B_n, and that A_n is a $\mathbb{Z}$-graded subring

Unlike the SL_2-invariants, the invariant rings A_n satisfy $A_n \subset A_{n+1}$ for each n. One difficulty of the subject is that many generators for A_n, typically found as the result of lengthy calculations, become superfluous in higher dimensions. Thus, existing algorithms for calculating these invariants are not progressive, that is, knowing generators for A_{n-1} may be of little use in finding generators of A_n. From another perspective, this is not surprising: A partial derivative of B_n restricts to A_n and its kernel is A_{n-1}. In general, we do not expect the generators of the kernel of a locally nilpotent derivation of a ring to belong to a minimal generating set for the ring.

6.3.1 Brief History

One goal of classical invariant theory was to calculate the invariants of the action of $SL_2(\mathbb{C})$ on the vector space of binary forms of degree n, together with its **semi-invariants**, which are the invariants of the subgroup $\mathbb{G}_a$. Interest in the invariants and semi-invariants of SL_2 dates back to at least the work of Boole, Cayley, Eisenstein, and Hesse. Cayley came to believe that the ring A_7 was not finitely generated. Subsequently, Gordan showed that both the invariant and semi-invariant rings must, in fact, be finitely generated, and calculated generators for these rings up to $n = 6$ [182]. Gordan's work inspired numerous attempts in the following decades to establish generating sets for these rings beyond $n = 6$, but most of these attempts resulted in proposed generating sets which were either incomplete or overdetermined, due to the size and complexity of the polynomials involved. For the case $n = 8$, Sylvester and Franklin (1879) and von Gall (1880) made important contributions [156, 412]. The reader is referred to [56, 258, 337, 339] for accounts of these developments from the Nineteenth Century.

Writing in 1906, Elliott [138] referred to "the old severe question" of finding minimal generating sets of these invariant and semi-invariant rings. In the intervening century, our knowledge of these generating sets has improved but little over what was known at the time. Indeed, the SL_2-invariants are currently known only for $n \leq 10$. The cases $n \leq 6$ were completed by Gordan in 1868, and the case $n = 8$ by Shioda in 1967 [384]; the case $n = 7$ was settled in 1986 by Dixmier and Lazard

Table 6.1 Known values of $\mu(n)$ and $\delta(n)$

n	2	3	4	5	6	7	8
$\mu(n)$	2	4	5	23	26	147	69
$\delta(n)$	2	4	3	18	15	30	12

Table 6.2 Brouwer-Popoviciu lower bounds

n	9	10	11	12
$\mu(n)$	≥ 476	≥ 510	Open	≥ 989
$\delta(n)$	≥ 22	≥ 21	Open	≥ 17

[117]; and the cases $n = 9, 10$ were completed in 2010 by Brouwer and Popoviciu [41, 42].

For invariants of the $\mathbb{G}_a$-action, the situation is even more opaque: These are known only for $n \leq 8$. Gordan gave generators for $n \leq 6$; the case $n = 8$ was done by Shioda; and the case $n = 7$ was completed by Cröni in 2002 [67].

Given $n \geq 2$, let $\mu(n)$ denote the minimal number of homogeneous generators of A_n as a $\mathbb{C}$-algebra, and let $\delta(n)$ be the highest degree occurring within a minimal generating set. As seen in *Table 6.1*, these two functions exhibit seemingly erratic behavior, at least based on the few values we know.

The first accurate calculation of a minimal generating set for A_7 is due to Cröni in 2002 [67]. In 2009, Bedratyuk, apparently unaware of Cröni's results, produced an equivalent generating set for A_7 [17]. In addition, Cerezo, Cröni and Bedratyuk each confirmed the results of Shioda for A_8 [16, 48, 67]. For $n = 9, 10, 12$, certain lower bounds are known. Cröni showed that $\mu(9) \geq 474$ and $\delta(9) \geq 20$. These bounds were improved by Brouwer and Popoviciu, who also gave bounds for $n = 10$ and $n = 12$ [40–42].[2] Their results are summarized in *Table 6.2*.

In 1879, Jordan showed that $\delta(n) \leq 2n^6$. This is still the best available upper bound for degrees, but is too large to be of practical use in calculating generators for A_n. Kraft and Weyman give a modern proof for Jordan's bound in [257].

Many of the results in *Tables 6.1* and *6.2* were originally found using the symbolic method, which Weyl called "the great war-horse of Nineteenth Century invariant theory" (see [258]). The reader is referred to [118, 258, 351] for details about the symbolic method and classical techniques for constructing invariants.

6.3.2 Quadratic and Cubic Invariants

Recall that, if $f, g \in \ker D_n^{m+1}$, then $D_n[f, g]_m^{D_n} = 0$; see *Sect. 2.11.1*. Therefore, the quadratic polynomial $[x_m, x_m]_m^{D_n}$ belongs to A for each $m \in 2\mathbb{N}$, $m \leq n$. It is well known that these are the only quadratic invariants of D_n; see [168], Cor. 3.3.

[2]In the preprint [271] posted in September, 2015, Lercier and Olive assert that the systems of invariants produced by Brouwer and Popviciu for A_9 and A_{10} are complete, thus giving $\mu(9) = 476$ and $\delta(9) = 22$, and $\mu(10) = 510$ and $\delta(10) = 21$ if their results are confirmed.

In Lecture XIX of Hilbert's 1897 course in invariant theory at Göttingen, Hilbert set out to explicitly identify all quadratic and cubic covariants of the SL_2-actions (equivalently, quadratic and cubic invariants for D_n for all $n \geq 0$). Hilbert stated, "Regarding the *covariants of degree three*, they all have odd weight", and gave what was thought to be a complete list [207] (pp. 62–63). This is clearly a mistake—for example, the generating set for A_4 calculated by Cayley includes a cubic of weight 6.

In Lecture XX, Hilbert succeeds in showing that A_n is rationally generated over $\mathbb{C}(x_0)$ by the quadratic and cubic invariants which he defined in Lecture XIX, namely,

$$[x_2, x_2]_2^{D_n},\ [x_1x_2, x_3]_3^{D_n},\ [x_4, x_4]_4^{D_n},\ [x_1x_4, x_5]_5^{D_n}, \cdots$$
$$\cdots, [x_1x_{n-1}, x_n]_n^{D_n} \text{ or } [x_n, x_n]_n^{D_n}$$

the latter depending on whether n is odd or even, respectively. This important fact was first shown by Stroh [394]. It means that $\text{frac}(A_n)$ is rational.

In general, work on cubic $\mathbb{G}_a$-invariants is sparsely represented in the literature. A terse symbolic description of these was given by Grace and Young in 1903 [185] (§260). In §6 of their paper [257], Kraft and Weyman offer a more detailed description of cubic invariants in terms of their symbolic representations, giving spanning sets for cubic invariants of a given weight for a binary form of a specified degree. An analysis of cubics of the type carried out by Kraft and Weyman is given by Hagedorn and Wilson in [199]. In it, the authors determine an explicit basis for a space of irreducible cubics complementary to the subspace of reducible cubics in symbolic form.

Theorem 6.2 of [168] gives a complete description of the cubic invariants of D_n for all $n \geq 0$ which does not rely on the symbolic method. It gives a basis for a space of cubic invariants complementary to the space of reducible cubics. This basis is ordered in such a way that cubics in A_n precede those in $A_{n+1} \setminus A_n$. With this description, one can immediately identify all cubic generators in A_n for any given value of n.

6.3.3 Rings A_n for Small n

The rings A_2 and A_4 are generated in degrees up to three, while A_3 requires a generator of degree 4. Their generators and relations are given as follows.

1. $A_2 = k[x_0, f_2]$, where $f_2 = [x_2, x_2]_2^{D_2} = 2x_0x_2 - x_1^2$
2. $A_3 = k[x_0, f_2, f_3, h]$, where

$$f_3 = [x_1x_2, x_3]_3^{D_3} = 3x_0^2x_3 - 3x_0x_1x_2 + x_1^3 \quad \text{and} \quad x_0^2h = f_2^3 + f_3^2$$

3. $A_4 = k[x_0, f_2, f_3, f_4, f_6]$, where

$$f_4 = [x_4, x_4]_4^{D_4} = 2x_0x_4 - 2x_1x_3 + x_2^2 \quad \text{and} \quad h = 3f_2f_4 + x_0f_6$$

In *Chap. 8*, we prove that the listed invariants are generators.

The rings A_5 and A_6 are considerably more complicated. As seen from *Table 6.1*, A_5 requires 23 generators in degrees up to 18, and A_6 requires 26 generators in degrees up to 15. A complete list of generators for A_5 is given in *Table 6.3*, and for A_6 in *Table 6.4*. These tables are found in the second *Appendix* to this chapter.

Cerezo's paper [48] is a lengthy and detailed hand-written treatise on the invariant rings A_n based on the geometric theory, and containing numerous examples. In it, he explicitly calculates a set of 23 generators of A_5. The generator of degree 18 involves more than eight hundred monomials with relatively prime integer coefficients on the order of 10^{10}, and requires eight pages to write. This is the SL_2-invariant which was famously discovered by Cayley and Faà di Bruno; see [115]. Cerezo's work on the invariants of linear $\mathbb{G}_a$-actions is contained in the three papers [47–49]. His work is not recognized as widely as it deserves to be, perhaps because these papers are unpublished.

Using classical methods, Bedratyuk calculated A_n for $n \leq 6$ in [19]. As to the more daunting ring A_7, the reader is referred to the thesis of Cröni [67]. Tables of generators for A_6 and A_8 are given by Olive in [336], p. 26 and p. 37. In the preprint [271], Lercier and Olive give tables of generators for A_9 (Appendix A) and A_{10} (Appendix B), though their results have not been published as of this writing.

6.3.4 Some Reducible $\mathbb{G}_a$-Modules

Given the positive integer n, recall that the number of distinct $\mathbb{G}_a$-actions on the vector space $V = k^n$ is equal to the number of partitions of n (*Proposition 6.3 (c)*). Let $\pi = \{n_1, \ldots, n_r\}$ be a partition of n, meaning that $\sum_i n_i = n$ and $n_1 \geq n_2 \geq \cdots \geq n_r \geq 1$. Let K_π denote the ring of invariants of the corresponding $\mathbb{G}_a$-action on V. Note that, in this notation, we have $A_n = K_{\{n+1\}}$.

For small values of n, many cases of K_π were calculated in the Nineteenth Century. Several of these can be found in [48], wherein Cerezo calculates K_π for the following two-term partitions:

$$\{2,2\}, \{3,2\}, \{3,3\}, \{4,2\}, \{4,3\}, \{4,4\}, \{5,2\}, \{5,3\}, \{5,4\}, \{5,5\}$$

More recently, Olive [336] calculated K_π for the partitions:

$$\{7,3\}, \{7,5\}, \{7,5,3\}, \{9,5,5\}$$

An important family of reducible actions is given by $n = lm$ for $l, m \geq 2$, with the partition $\pi = \{m, m, \ldots, m\}$. Let $V = \mathcal{M}_{l\times m}(k) \cong \mathbb{A}^n$ and let $\exp(tE_m)$ act on V by right multiplication. If $C = (C_1, \ldots, C_m)$ where C_i is the ith column of C, then $CE_m = (0, C_1, \ldots, C_{m-1})$. If $l \geq m$, then the $m \times m$ minor determinants of C are $\mathbb{G}_a$-invariants; see *Corollary 2.66*.

Consider the case $n = 2l$ and $\pi = \{2, 2, \dots, 2\}$. The invariant ring K_π was calculated by Weyl in [419], Chap. 2, § 7, namely, if $C_2 = (y_1, \dots, y_l)^T$ and $C_1 = (x_1, \dots, x_l)^T$, then:

$$K_{\{2,2,\dots,2\}} = k\left[x_r, [y_r, y_s]_1^{E_2} \mid 1 \le r, s \le l\right]$$

Richman [356] gave a characteristic-free proof of Weyl's result, under the assumption that the ground field is infinite. Weyl's result resurfaced as a conjecture in the book of Nowicki [333] and was subsequently reproved by several authors [18, 122, 246, 267].

In [18], Bedratyuk considers the case where $n = 3l$ and $\pi = \{3, 3, \dots, 3\}$. The author lists generators for K_π, namely, if $C_3 = (z_1, \dots, z_l)^T$, $C_2 = (y_1, \dots, y_l)^T$ and $C_1 = (x_1, \dots, x_l)^T$, then:

$$K_{\{3,3,\dots,3\}} = k\left[x_r, [y_r, y_s]_1^{E_3}, [z_r, z_s]_2^{E_3}, W_{E_3}(z_r, z_s, z_t) \mid 1 \le r, s, t \le m\right]$$

Here, $W_{E_3}(z_r, z_s, z_t)$ denotes the Wronskian (*Sect. 2.11*); it can be omitted if $n \le 2$. Bedratyuk states: "The proof follows from the description of generating elements of the algebra of covariants for n quadratic binary forms" (p. 3), but he does not provide details. In [417], Wehlau gives a complete proof of this theorem using different methods.

6.4 Linear Counterexamples to the Fourteenth Problem

In the statement of the Maurer-Weitzenböck Theorem, one cannot generally replace the group $\mathbb{G}_a$ with higher-dimensional vector groups $\mathbb{G}_a^n$. This section will discuss several examples of representations of $\mathbb{G}_a^n$ for which the ring of invariants is not finitely generated, beginning with the famous examples of Nagata. Counterexamples using smaller vector groups and smaller affine spaces were subsequently given by A'Campo-Neuen, Steinberg and Mukai. Non-abelian unipotent groups were used by Tanimoto, and by the author, to give linear counterexamples which reduce the dimension of the underlying affine space even further, and these are also discussed.

6.4.1 Examples of Nagata

The first counterexamples to Hilbert's Fourteenth Problem were presented by Nagata in 1958. Prior to the appearance of Nagata's examples, Rees [354] constructed a counterexample to Zariski's generalization of the Fourteenth Problem, which asks:

> Let R be a normal affine ring over a field K. If L is a field with $K \subseteq L \subseteq \mathrm{frac}(R)$, is $R \cap L$ an affine ring?

In Rees's example, $\mathrm{frac}(R \cap L)$ contains the function field of a non-singular cubic projective plane curve, and cannot therefore be a counterexample to Hilbert's problem. But Rees' example was very important in its own right, and indicated that counterexamples to Hilbert's problem might be found in a similar fashion.

Shortly thereafter, Nagata discovered two counterexamples to Hilbert's problem. In [320], he describes the situation as follows. (By "original 14-th problem", he means the specific case using fixed rings for linear actions of algebraic groups.)

> In 1958, the writer found at first a counter-example to the 14-th problem and then another example which is a counter-example to the original 14-th problem. This second example was announced at the International Congress in Edinburgh (1958). Though the first example is in the case where dim $K = 4$, in the second example dim K is equal to 13. Then the writer noticed that the first example is also a counter-example to the original 14-th problem. (p. 767)

How did Nagata find these examples? As Steinberg [392] points out, the heart of Nagata's method is to relate the structure of the ring of invariants to an interpolation problem in the projective plane, namely, that for each $m \geq 1$, there does not exist a curve of degree $4m$ having multiplicity at least m at each of 16 general points of the projective plane. Steinberg writes: "Nagata's ingenious proof of this is a tour de force but the results from algebraic geometry that he uses are by no means elementary" (p. 377).

The foundation of this geometric approach to the problem was laid by Zariski in the early 1950s. His idea was to look at rings of the form $R(D)$, where D is a positive divisor on some non-singular projective variety X, and $R(D)$ is the ring of rational functions on X with poles only on D. Mumford writes:

> In his penetrating article [430], Zariski showed that Hilbert's rings $K \cap k[x_1, \ldots, x_n]$ were isomorphic to rings of the form $R(D)$ for a suitable X and D; asked more generally whether all the rings $R(D)$ might not be finitely generated; and proved $R(D)$ finitely generated if dim $X = 1$ or $2 \ldots$ Unfortunately, it was precisely by focusing so clearly the divisor-theoretic content of Hilbert's 14th problem that Zariski cleared the path to counter-examples. [318]

In the example constructed by Rees, X is birational to $\mathbb{P}^2 \times C$ for an elliptic curve C; and in Nagata's examples, X is the surface obtained by blowing up $\mathbb{P}^2$ at 16 general points. For further details, the reader is referred to Mumford's article and Nagata's 1965 lectures on the subject [323].

It is well worth recording the positive result in Zariski's landmark 1954 paper, as mentioned by Mumford.

> **(Zariski's Finiteness Theorem)** For a field K, let A be an affine normal K-domain, and let L be a subfield of $\mathrm{frac}(A)$ containing K. If $\mathrm{tr.deg.}_K L \leq 2$, then $L \cap A$ is finitely generated over K.

In particular, Zariski's theorem can be applied to the case where X is an affine G-variety, $k[X]^G$ is normal and $\mathrm{tr\ deg}_k k[X]^G \leq 2$ to conclude that $k[V]^G$ is finitely generated. Note that *Example 4.4* shows the necessity of the hypothesis that A is normal in Zariski's result.

Another case to mention is the case where X is an affine G-variety and $k[X]$ is algebraic over $k[X]^G$. In this case, G is necessarily finite, and Noether's Theorem can be applied to conclude that $k[X]^G$ is finitely generated. Details of this reasoning are provided in the *Appendix* at the end of this chapter.

6.4.2 Examples of Steinberg and Mukai

In 1997, Steinberg [392] published a lucid exposition of Nagata's original constructions, and modified Nagata's approach to obtain linear counterexamples of reduced dimension. Subsequently, Mukai [312] generalized this geometric approach even further to give families of counterexamples, including some in yet smaller dimension.

Let the vector group $U = \mathbb{G}_a^n$ be represented on $\mathbb{A}^{2n}$ by:

$$(t_1, \ldots, t_n) \mapsto \begin{pmatrix} t_1E_2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & t_nE_2 \end{pmatrix} \quad \text{for} \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

This is called a **standard vector group representation**. Likewise, let the torus $\mathbb{G}_m^{2n}$ be represented on $\mathbb{A}^{2n}$ by:

$$(c_1, \ldots, c_{2n}) \mapsto \begin{pmatrix} c_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c_{2n} \end{pmatrix}$$

This is called a **standard torus representation**. We let $T = \mathbb{G}_m^{2n-1} \subset \mathbb{G}_m^{2n}$ denote the subgroup $T = \{(c_1, \ldots, c_{2n}) \mid c_1 \cdots c_{2n} = 1\}$. Since the standard actions of $\mathbb{G}_a^n$ and $\mathbb{G}_m^{2n}$ semi-commute (the torus normalizes the vector group), we obtain a representation of UT on $\mathbb{A}^{2n}$. The examples of Nagata, Steinberg, and Mukai each uses a subgroup $G \subset U$ or $G \subset UT$, acting on $\mathbb{A}^{2n}$ in the way described above.

Nagata showed that the invariant ring with respect to a general linear subspace $G \subset \mathbb{G}_a^n$ of codimension 3 was not finitely generated for $n = r^2 \geq 16$. In 1978, Choodnovsky [52] claimed that any $n \geq 10$ suffices in Nagata's formulation, but apparently never published a proof. In [340], Pommerening supplied some of the details needed to prove Choodnovsky's bound. Mukai's Theorem (below) confirms Choodnovsky's bound and shows that even the lower bound $n \geq 9$ can be used.

In Steinberg's paper, the main example is for $n = 9$, where he considers the codimension-3 subgroup $G = \mathbb{G}_a^6$ of U defined in the following way: In case char $k = 0$, choose $a_1, \ldots a_9 \in k$ such that $a_i \neq a_j$ for $i \neq j$, and $\sum a_i \neq 0$, and let $G \subset \mathbb{G}_a^9$ be the subgroup for which $\sum t_i = \sum a_i t_i = \sum a_i^3 t_i = 0$. In case char $k > 0$, choose distinct $a_1, \ldots, a_9 \in k$ so that $\prod a_i$ is neither 0 nor any root of 1, and let $G \subset \mathbb{G}_a^9$ be the subgroup for which $\sum t_i = \sum a_i t_i = \sum (a_i^2 - a_i^{-1}) t_i = 0$. Steinberg shows that the action of GT on $\mathbb{A}^{18}$ has non-finitely generated ring of invariants (Thm. 1.2), which implies (by the Finiteness Theorem) that the invariant ring of G is also non-finitely generated. The examples of Nagata and Steinberg are valid over any field k which is not a locally finite field. A **locally finite field** is an algebraic extension of a finite field.

Subsequently, Mukai proved the following result.

Theorem 6.10 (Mukai's Theorem) *Let $\mathbb{C}^n$ act on $\mathbb{C}^{2n}$ by the standard action. If $G \subset \mathbb{C}^n$ is a general linear subspace of codimension $r < n$, then the ring of G-invariant functions is finitely generated if and only if:*

$$\frac{1}{r} + \frac{1}{n-r} > \frac{1}{2}$$

(The proof of 'only if' for this theorem is given in [312]; for the proof of 'if', see [313, 315].) It follows that, if $S = \mathcal{O}(\mathbb{C}^{2n}) = \mathbb{C}^{[2n]}$, then S^G is not finitely generated if $\dim G = m \geq 3$ and $n \geq m^2/(m-2)$. Thus, there exist linear algebraic actions of $\mathbb{G}_a^3$ on $\mathbb{C}^{18}$ and of $\mathbb{G}_a^4$ on $\mathbb{C}^{16}$ whose rings of invariants are not finitely generated. At the time of their appearance, these were the smallest linear counterexamples to the Fourteenth Problem, both in terms of the dimension of the group which acts ($m = 3$), and the dimension of the space which is acted upon ($2n = 16$). Subsequently, both Tanimoto and the author found linear counterexamples using smaller affine spaces, namely, $\mathbb{A}^{13}$ and $\mathbb{A}^{11}$, respectively; these are discussed below. As we will see in the next chapter, even smaller counterexamples can be found if we consider more general (non-linear) actions.

The papers of Nagata, Steinberg, and Mukai are largely self-contained. In particular, Steinberg's two main lemmas (2.1 and 2.2) provide the crucial link between an interpolation problem in the projective plane and the structure of certain fixed rings. The group associated with a cubic curve plays an important role in this approach to the problem. Steinberg goes on to discuss the status of the classical geometric problem lying at the heart of this approach, which is of interest in its own right, described by him as follows:

> Find the dimension of the space of all polynomials (or curves) of a given degree with prescribed multiplicities at the points of a given finite set in general position in the plane, thus also determine if there is a curve, i.e., a nonzero polynomial, in the space and if the multiplicity conditions are independent. (p. 383)

The paper of Kuttler and Wallach [269] also gives an account of these ideas, in addition to generalizations of some of Steinberg's results. See also Mukai [315] and Roé [360].

6.4.3 Examples of A'Campo-Neuen and Tanimoto

In her paper [4] (1994), A'Campo-Neuen used a non-linear counterexample to Hilbert's Fourteenth Problem due to Roberts to construct a counterexample arising as the fixed ring of a linear action of $G = \mathbb{G}_a^{12}$ on $\mathbb{A}^{19}$. Her example is valid for any field k of characteristic 0. Apparently, this was the first linear counterexample to be published after those of Nagata, a span of 36 years!

As in the examples of Nagata, her example is gotten by restriction of a standard vector group action to a certain subgroup. In particular, given $(t_1, \ldots, t_{12}) \in G$, the G-action is defined explicitly by the lower triangular matrix

$$\begin{pmatrix} I & 0 \\ M^T & I \end{pmatrix}$$

of order 19, where the identities are of order 4 and 15 respectively, and M is the 4×15 matrix:

$$M = \begin{pmatrix} t_1 & t_2 & 0 & t_3 & t_4 & 0 & t_5 & t_6 & 0 & t_7 & t_8 & t_9 & t_{10} & t_{11} & 0 \\ t_{12} & t_1 & t_2 & 0 & 0 & 0 & 0 & 0 & 0 & t_{12} & t_7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_{12} & t_3 & t_4 & 0 & 0 & 0 & 0 & 0 & t_8 & t_9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t_{12} & t_5 & t_6 & 0 & 0 & 0 & 0 & t_{10} & t_{11} \end{pmatrix}$$

A'Campo-Neuen's proof is quite elegant, and is given in *Sect. 7.7* of the next chapter. This proof uses locally nilpotent derivations of polynomial rings, and is very different from the geometric proofs of Nagata, Steinberg, and Mukai.

In 2004, Tanimoto [401] followed the methods of A'Campo-Neuen to give two linear counterexamples to the Fourteenth Problem. His examples are based on the non-linear counterexamples of the author and Daigle [79], and the author [163]. In order to utilize these earlier examples, it was necessary to consider non-abelian unipotent group actions. In particular, Tanimoto gives a counterexample in which the group $\mathbb{G}_a^7 \rtimes \mathbb{G}_a$ acts linearly on $\mathbb{A}^{13}$, and another in which $\mathbb{G}_a^{18} \rtimes \mathbb{G}_a$ acts linearly on $\mathbb{A}^{27}$. Following are the particulars for the smaller of these two actions.

Let $\mu = (\mu_0, \ldots, \mu_6)$ denote an element of $V = \mathbb{G}_a^7$, and define a linear $\mathbb{G}_a$-action on V by ${}^t\mu = \exp(tD)(\mu)$, where:

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & E_3 & 0 \\ 0 & 0 & E_3 \end{pmatrix}_{7\times 7} \quad \text{for} \quad E_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

This action defines semi-addition on $\mathbb{G}_a^7 \rtimes \mathbb{G}_a$, namely:

$$(t, \mu)(t', \mu') = (t + t', \mu + {}^t\mu')$$

Tanimoto gives seven commuting linear triangular derivations $\Delta_0, \dots, \Delta_6$ of the polynomial ring

$$B = k[w, x, s_1, t_1, u_1, s_2, t_2, u_2, s_3, t_3, u_3, v_1, v_2] = k^{[13]}$$

whose exponentials give the linear action of $\mathbb{G}_a^7$ on $\mathbb{A}^{13}$, namely:

$$\begin{aligned}
&\Delta_0 = x\tfrac{\partial}{\partial v_2} + w\tfrac{\partial}{\partial v_1} \ , \ \Delta_1 = x\tfrac{\partial}{\partial s_3} - w\tfrac{\partial}{\partial s_2} \ , \ \Delta_2 = x\tfrac{\partial}{\partial t_3} - w\tfrac{\partial}{\partial t_2} \ , \\
&\Delta_3 = x\tfrac{\partial}{\partial u_3} - w\tfrac{\partial}{\partial u_2} \ , \ \Delta_4 = x\tfrac{\partial}{\partial s_2} + w\tfrac{\partial}{\partial s_1} \ , \ \Delta_5 = x\tfrac{\partial}{\partial t_2} + w\tfrac{\partial}{\partial t_1} \ , \\
&\Delta_6 = x\tfrac{\partial}{\partial u_2} + w\tfrac{\partial}{\partial u_1}
\end{aligned}$$

Combined with another derivation

$$\Delta = x\tfrac{\partial}{\partial s_1} + s_1\tfrac{\partial}{\partial t_1} + t_1\tfrac{\partial}{\partial u_1} + s_2\tfrac{\partial}{\partial t_2} + t_2\tfrac{\partial}{\partial u_2} + s_3\tfrac{\partial}{\partial t_3} + t_3\tfrac{\partial}{\partial u_3} + x\tfrac{\partial}{\partial v_1}$$

these eight induce the full action of $\mathbb{G}_a^7 \rtimes \mathbb{G}_a$ on $\mathbb{A}^{13}$. For the interested reader, Tanimoto also gives the action in matrix form.

6.4.4 Linear Counterexample in Dimension Eleven

The article [164] gives a family of linear counterexamples to the Fourteenth Problem in which, for each integer $n \geq 4$, the unipotent group $\Gamma_n = \mathbb{G}_a^n \rtimes \mathbb{G}_a$ acts on $W_n = \mathbb{A}^{2n+3}$ by linear transformations, and $k[W_n]^{\Gamma_n}$ is not finitely generated. The smallest of these is for the group $\mathbb{G}_a^4 \rtimes \mathbb{G}_a$ acting on $\mathbb{A}^{11}$. To date, this is the smallest affine space for which a linear counterexample is known to exist. The specific action in this case is described in the following theorem.

Theorem 6.11 ([164], Thm. 4.1) *Let*

$$B = k[w, x, s_1, s_2, t_1, t_2, u_1, u_2, v_1, v_2, z] = k^{[11]}$$

and define commuting linear triangular derivations T_1, T_2, T_3, T_4 on B by:

$$\begin{aligned}
T_1 &= x\tfrac{\partial}{\partial s_2} - w\tfrac{\partial}{\partial s_1} \ , \quad T_2 = x\tfrac{\partial}{\partial t_2} - w\tfrac{\partial}{\partial t_1} \\
T_3 &= x\tfrac{\partial}{\partial u_2} - w\tfrac{\partial}{\partial u_1} \ , \quad T_4 = x\tfrac{\partial}{\partial v_2} - w\tfrac{\partial}{\partial v_1}
\end{aligned}$$

Define a fifth linear triangular derivation Θ, which semi-commutes with the T_i:

$$\Theta = x\tfrac{\partial}{\partial s_1} + s_1\tfrac{\partial}{\partial t_1} + t_1\tfrac{\partial}{\partial u_1} + u_1\tfrac{\partial}{\partial v_1} + s_2\tfrac{\partial}{\partial t_2} + t_2\tfrac{\partial}{\partial u_2} + u_2\tfrac{\partial}{\partial v_2} + x\tfrac{\partial}{\partial z}$$

Let $\mathfrak{g}$ *be the Lie algebra generated by* T_1, T_2, T_3, T_4 *and* Θ*. Then the group* $\Gamma = \exp\mathfrak{g} \cong \mathbb{G}_a^4 \rtimes \mathbb{G}_a$ *acts on* $V = \mathbb{A}^{11}$ *by linear transformations and* $k[V]^\Gamma$ *is not finitely generated.*

The proof for this example is based on work of Kuroda involving non-linear $\mathbb{G}_a$-actions, and will be outlined in the next chapter.

6.5 Linear $\mathbb{G}_a^2$-Actions

Let $U \subset GL_n(k)$ denote the maximal unipotent subgroup consisting of upper triangular matrices with ones on the diagonal. Let $\mathfrak{u} \subset \mathfrak{gl}_n(k)$ denote the Lie algebra of U, i.e., upper triangular matrices with zeros on the diagonal. Note that $\mathfrak{u}$ is a nilpotent Lie algebra, consisting of nilpotent elements. Since $\mathbb{G}_a^2$ is a unipotent group, every rational representation $\mathbb{G}_a^2 \subset GL_n(k)$ can be conjugated to U. We consider the two-dimensional Lie subalgebras $\mathfrak{h}$ of $\mathfrak{u}$.

6.5.1 Actions of Nagata Type

In the setting of Mukai's Theorem, suppose that $n \geq 3$ and $G = \mathbb{G}_a^2$. In [316], the authors define the corresponding $\mathbb{G}_a^2$-action to be of **Nagata type**. For this action, Mukai's Theorem gives no indication whether the ring of invariants is finitely generated.

In his earlier paper [312], Mukai had already shown that the invariant ring of such an action is isomorphic to the total coordinate ring of the blow-up of $\mathbb{P}^{n-3}$ at n points ($n \geq 4$). The later paper then sketches how to use this fact to show their main result:

Theorem 6.12 (Cor. 1 of [316]) *The ring of invariants for a* $\mathbb{G}_a^2$*-action of Nagata type is finitely generated.*

See also [314]. In [46], Castravet and Tevelev give another proof of this result, using geometric methods similar to those of Mukai. Specifically, they show that if $\mathbb{G}_a^2$ acts on $\mathbb{C}^{2n}$ by an action of Nagata type, then the algebra of invariants is generated by 2^{n-1} invariant functions which they define explicitly using determinants (Thm. 1.1). The following is a simple construction of such invariants quite similar to that of Castravet and Tevelev.

Let $B = k[x_1, y_1, \ldots, x_n, y_n]$ for $n = 2m + 1$ and $m \geq 2$. Fix $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and consider the derivations D and E of B:

$$Dy_i = Ey_i = 0, \qquad Dx_i = y_i, \qquad \text{and} \quad Ex_i = \lambda_i y_i \ .$$

If $z_{ij} := x_i y_j - x_j y_i$ for $1 \leq i, j \leq n$, then $Dz_{ij} = 0$ and $Ez_{ij} = (\lambda_i - \lambda_j) y_i y_j$ for every pair i, j. Introduce a $\mathbb{Z}^2$-grading on B by declaring $\deg x_i = (-1, 1)$ and

$\deg y_i = (1, 0)$ for each i. Then D and E are homogeneous of degreee $(2, -1)$, and $\deg z_{ij} = (0, 1)$ for each i, j.

Proposition 6.13 *For each $d = 0, 1, \ldots, m$, there exist nonzero elements of degree $(1, d)$ in* $\ker D \cap \ker E$.

Proof Define $\mathbf{y} = (y_{m+1}, y_{m+2}, \ldots, y_{2m+1})$. Then there exist scalars $c_{rs} \in \mathbb{Q}[\lambda_{ij}]$ such that, if

$$\mathbf{z}_i = (c_{i,1}z_{i,m+1}, c_{i,2}z_{i,m+2}, \ldots, c_{i,m+1}z_{i,2m+1}) \quad (1 \le i \le m) \ ,$$

then $E\mathbf{z}_i = t_i y_i \mathbf{y}$ for some $t_i \in k$.

Now construct an $(m+1) \times (m+1)$ matrix K such that the i-th row equals $\mathbf{z}_i$ for $1 \le i \le m$, and such that the last row equals $\mathbf{y}$. Then $\deg(\det K) = (1, m)$ and $E(\det K) = 0$ (see *Sect. 2.11*). To construct invariants of degree $(1, d)$ for $d < m$, just consider minor determinants of K. $\square$

6.5.2 Actions of Basic Type

A rational representation $\mathbb{G}_a^2 \subset GL_n(k)$ is of **basic type** if it admits a restriction to the basic $\mathbb{G}_a$-action on $\mathbb{A}^n$. Specifically, this means that there exists $M \in \mathfrak{gl}_n(k)$ such that the representation of $\mathbb{G}_a^2$ is given by the exponential of the Lie algebra $kE_n + kM$, where $E_n \in \mathfrak{u}$ is the elementary nilpotent matrix in dimension n. In particular, M is nilpotent and commutes with E_n.

Let $Z(E_n, \mathfrak{u})$ denote the **centralizer** of E_n in $\mathfrak{u}$, i.e., elements of $\mathfrak{u}$ which commute with E_n under multiplication.

Proposition 6.14 *A basis of $Z(E_n, \mathfrak{u})$ is $\{E_n, E_n^2, \ldots, E_n^{n-1}\}$.*

Proof Recall first that E_n is the matrix with ones on the first super-diagonal and zeros elsewhere, and likewise E_n^i is the matrix with ones the ith super-diagonal with zeros elsewhere. Therefore, $E_n, E_n^2, \ldots, E_n^{n-1}$ are linearly independent. Let $\mathfrak{g} \subset \mathfrak{u}$ denote $\mathfrak{g} = kE_n + kE_n^2 + \cdots kE_n^{n-1}$.

Given $M \in \mathfrak{u}$, write

$$M = \begin{pmatrix} 0 & \alpha \\ 0 & A \end{pmatrix} = \begin{pmatrix} A' & \alpha' \\ 0 & 0 \end{pmatrix}$$

where A is the $(1, 1)$-minor submatrix of M, A' is the (n, n)-minor submatrix of M, α is the corresponding row matrix of length $(n-1)$, and α' is the corresponding column matrix of length $(n-1)$. Then by comparing elements of M lying on its superdiagonals, we conclude that $M \in \mathfrak{g}$ if and only if $A = A'$.

Write

$$E_n = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$$

where I is the identity matrix of order $(n-1)$, which is the $(n, 1)$-minor of E_n. Then

$$E_n M = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \alpha \\ 0 & A \end{pmatrix} = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$$

while

$$ME_n = \begin{pmatrix} A' & \alpha' \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & A' \\ 0 & 0 \end{pmatrix} .$$

Therefore, $M \in Z(E_n, \mathfrak{u})$ if and only if $A = A'$ if and only if $M \in \mathfrak{g}$. □

The subalgebra $\mathfrak{g} \subset \mathfrak{u}$ appearing in the proof above will be called the **superdiagonal algebra**. The corresponding unipotent Lie group $G \subset U$ will be called the **superdiagonal subgroup** of $GL_n(k)$. Note that $G \cong \mathbb{G}_a^{n-1}$.

Next, let $Z(E_n) = Z(E_n, \mathfrak{gl}_n(k))$ denote the full centralizer of E_n.

Corollary 6.15 *A basis for $Z(E_n)$ is $\{I, E_n, E_n^2, \ldots, E_n^{n-1}\}$.*

Proof Let $P \in \mathfrak{gl}_n(k)$ be given. The condition $PE_n = E_n P$ immediately implies P is upper triangular, and that its diagonal entries are equal. Thus, it is possible to write $P = cI + M$ for $M \in \mathfrak{u}$ and $c \in k$. Then $(I + M)E_n = E_n(I + M)$ implies $ME_n = E_n M$, so by the proposition, M is a linear combination of $E_n, E_n^2, \ldots, E_n^{n-1}$. Therefore, P is a linear combination of $I, E_n, E_n^2, \ldots, E_n^{n-1}$. □

An immediate consequence is the following.

Corollary 6.16 *If a rational representation $\mathbb{G}_a^2 \subset GL_n(k)$ admits a restriction to the basic $\mathbb{G}_a$-action on $\mathbb{A}^n$, then $\mathbb{G}_a^2 \subset G$, where G is the superdiagonal subgroup of $GL_n(k)$.*

The centralizer of a general element of $\mathfrak{u}$ can be similarly described, but the description is more complicated. For each postive integer j, let E_j denote the (upper triangular) elementary nilpotent matrix of order j. Given positive integers $i \geq j \geq r$, let $E_{(i,j,r)}$ denote the $i \times j$ matrix formed by E_j^r in the first j rows, and zeros elsewhere. Note that $E_{(j,j,1)} = E_j$.

Given $N \in \mathfrak{u}$, suppose N has Jordan block form

$$N = \begin{pmatrix} E_{s_1} & 0 & \cdots & 0 \\ 0 & E_{s_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E_{s_\lambda} \end{pmatrix}$$

where the integers s_i satisfy $s_1 \geq s_2 \geq \cdots \geq s_\lambda$ and $s_1 + \cdots + s_\lambda = n$. Then $Q \in Z(N, \mathfrak{u})$ if and only if Q has the block form

$$Q = \begin{pmatrix} Q_{(1,1)} & Q_{(1,2)} & Q_{(1,3)} & \cdots & Q_{(1,\lambda)} \\ 0 & Q_{(2,2)} & Q_{(2,3)} & \cdots & Q_{(2,\lambda)} \\ 0 & & Q_{(3,3)} & \cdots & Q_{(3,\lambda)} \\ \vdots & \vdots & \vdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & Q_{(\lambda,\lambda)} \end{pmatrix}$$

where $Q_{(i,j)}$ belongs to $\oplus_{1 \leq r < s_j} kE_{(s_i,s_j,r)}$. Details are left to the reader.

Remark 6.17 The variety defined by pairs of commuting nilpotent matrices of a fixed dimension has been studied. In particular, it was shown by Baranovsky [10] that this variety is irreducible (2001). The paper of Basili [11] contains a very nice historical survey of this problem, and an elementary proof of irreducibility (2003). And the article of Schröer [372] studies certain subvarieties of this variety relative to their irreducible components (2004).

Remark 6.18 From one point of view, the reason that the invariant ring in Nagata's example is not finitely generated is that, while the vector group itself has a very simple structure, its embedding in $GL(V)$ is complicated with respect to the coordinate lines in $GL(V)$ constituted by by all one-dimensional unipotent root subgroups relative to a fixed maximal torus T of $GL(V)$. This philosophy led to the Popov-Pommerening Conjecture. The conjecture claims that in the opposite case, where a unipotent subgroup H of $GL(V)$ is generated by some of these root subgroups (or equivalently, where H is normalized by T), the invariant algebra of H is finitely generated. This conjecture has been confirmed in many important special cases, but remains open in its full generality. It is one of the main problems in the invariant theory of linear actions of unipotent groups. For details, the reader is referred to [6, 188, 345].

Appendix 1: Finite Group Actions

The following fact is well-known, and is provided here for the readers' convenience. The statement of the proposition and the proof given here are due to Daigle (unpublished).

Proposition 6.19 *Suppose k is a field, and B is a finitely generated commutative k-domain. Let G be a group of algebraic k-automorphisms of B (i.e., G acts faithfully on B). Then the following are equivalent.*

(1) *G is finite*
(2) *B is integral over B^G*
(3) *B is algebraic over B^G*

Proof We first show that, for given $b \in B$, the following are equivalent.

(4) The orbit $\mathcal{O}_b$ is finite
(5) b is integral over B^G
(6) b is algebraic over B^G

(4) $\Rightarrow$ (5): If $\mathcal{O}_b$ is finite, define the monic polynomial $f(x) \in k[x]$ by

$$f(x) = \prod_{a \in \mathcal{O}_b} (x - a) .$$

Then $f \in B^G[x]$ and $f(b) = 0$, and (5) follows.

(5) $\Rightarrow$ (6): Obvious.

(6) $\Rightarrow$ (4): If $h(x) \in B^G[x]$ and $h(b) = 0$ for nonzero h, choose $a \in \mathcal{O}_b$, and suppose $a = g \cdot b$ for $g \in G$. Then

$$h(a) = h(g \cdot b) = g \cdot h(b) = g \cdot 0 = 0 ,$$

i.e., every $a \in \mathcal{O}_b$ is a root of h. Since B is a domain, the number of roots of h is finite, and (4) follows.

Therefore (4),(5), and (6) are equivalent.

(1) $\Rightarrow$ (2): Choose $b \in B$. Since G is finite, $\mathcal{O}_b$ is finite, and therefore b is integral over B^G. So B is integral over B^G.

(2) $\Rightarrow$ (3): Obvious.

(3) $\Rightarrow$ (1): Since $k \subset B^G$, we have that B is finitely generated over B^G. Write $B = B^G[x_1, \dots, x_n]$, and define a function

$$\begin{aligned} G &\to \mathcal{O}_{x_1} \times \cdots \times \mathcal{O}_{x_n} \\ g &\mapsto (g \cdot x_1, \dots, g \cdot x_n) \ , \end{aligned}$$

Now each element x_i is algebraic over B^G, meaning that each orbit $\mathcal{O}_{x_i}$ is finite. Therefore, the set $\mathcal{O}_{x_1} \times \cdots \times \mathcal{O}_{x_n}$ is also finite. In addition, the function above is injective, since the automorphism g of B is completely determined by its image on a set of generators. Therefore G is finite. $\square$

Appendix 2: Generators for A_5 and A_6

This section reproduces the tables of generators for A_5 and A_6 given by Grace and Young in 1903 [185], and also gives the reader a method to decode these invariants, which are expressed symbolically in the form of classical transvectants. *Table 6.3* gives generators for A_5 and *Table 6.4* gives generators for A_6. A similar table for A_8 is given in [336], and for A_9 and A_{10} in [271].

Table 6.3 Generators of A_5

Form	Degree	Order	Form	Degree	Order
$f_0 = x_0$	(1,0)	5	$f_{13} = (f_2, f_4^2)^3$	(6,13)	4
$f_2 = (f_0, f_0)^2$	(2,2)	6	$f_{14} = (f_2, f_4^2)^4$	(6,14)	2
$f_3 = (f_0, f_2)^1$	(3,3)	9	$f_{15} = (f_3, f_4^2)^4$	(7,15)	5
$f_4 = (f_0, f_0)^4$	(2,4)	2	$f_{16} = (f_0, f_4^3)^5$	(7,17)	1
$f_5 = (f_0, f_4)^1$	(3,5)	5	$f_{17} = (f_2, f_4^3)^5$	(8,19)	2
$f_6 = (f_0, f_4)^2$	(3,6)	3	$f_{18} = (f_2, f_4^3)^6$	(8,20)	0
$f_7 = (f_2, f_4)^1$	(4,7)	6	$f_{19} = (f_3, f_4^3)^6$	(9,21)	3
$f_8 = (f_2, f_4)^2$	(4,8)	4	$f_{20} = (f_3, f_4^4)^8$	(11,27)	1
$f_9 = (f_3, f_4)^2$	(5,9)	7	$f_{21} = (f_0^2, f_4^5)^{10}$	(12,30)	0
$f_{10} = (f_4, f_4)^2$	(4,10)	0	$f_{22} = (f_3, f_4^5)^9$	(13,32)	1
$f_{11} = (f_0, f_4^2)^3$	(5,11)	3	$f_{23} = (f_0 f_3, f_4^7)^{14}$	(18,45)	0
$f_{12} = (f_0, f_4^2)^4$	(5,12)	1			

Table 6.4 Generators of A_6

Form	Degree	Order	Form	Degree	Order
$f_0 = x_0$	(1,0)	6	$F_{13} = (f_4, F_8)^1$	(5,13)	4
$f_2 = (f_0, f_0)^2$	(2,2)	8	$F_{14} = (f_4, F_8)^2$	(5,14)	2
$f_3 = (f_0, f_2)^1$	(3,3)	12	$F_{15} = (f_6, F_8)^1$	(6,15)	6
$f_4 = (f_0, f_0)^4$	(2,4)	4	$F_{17} = (f_5, F_8)^2$	(6,15)	6
$f_5 = (f_0, f_4)^1$	(3,5)	8	$F_{18} = (F_8, F_8)^2$	(6,18)	0
$f_6 = (f_0, f_4)^2$	(3,6)	6	$F_{19} = (f_0, F_8^2)^3$	(7,19)	4
$f_7 = (f_2, f_4)^1$	(4,7)	10	$F_{20} = (f_0, F_8^2)^4$	(7,20)	2
$F_6 = (f_0, f_0)^6$	(2,6)	0	$F_{21} = (f_4, F_8^2)^3$	(8,23)	2
$F_8 = (f_0, f_4)^4$	(3,8)	2	$F_{22} = (f_5, F_8^2)^4$	(9,25)	4
$F_9 = (f_0, F_8)^1$	(4,9)	6	$F_{23} = (f_0, F_8^3)^5$	(10,29)	2
$F_{10} = (f_0, F_8)^2$	(4,10)	4	$F_{24} = (f_0, F_8^3)^6$	(10,30)	0
$F_{11} = (f_2, F_8)^1$	(5,11)	0	$F_{25} = (f_5, F_8^3)^6$	(12,35)	2
$F_{12} = (f_4, f_4)^4$	(4,12)	8	$F_{26} = (f_5, F_8^4)^8$	(15,45)	0

On the polynomial ring $k[x_0, \ldots, x_n]$, D_n is the basic linear derivation, defined by $D_n x_i = x_{i-1}$ for $i \geq 1$ and $D_n x_0 = 0$. U_n is the up operator, a linear locally nilpotent derivation defined by:

$$U_n x_i = (i+1)(n-i)x_{i+1} \ (0 \leq i \leq n-1) \quad \text{and} \quad U_n x_n = 0$$

Given $f, g \in A_n = \ker D_n$ and $i \geq 1$, the symbol $(f, g)^i$ denotes:

$$(f, g)^i = [U_n^i f, U_n^i g]_i^{D_n}$$

See *Sect.* 2.11.1 for the definition of the transvectant on the right side of this equation. The theory developed in *Sect.* 7.4 shows that $(f, g)^i \in A_n$.

Recall that the degree of x_i equals $(1, i)$. If f is homogeneous of degree (r, s), then the **order** of f (relative to n) is $nr - 2s$. Elements in A_n of order 0 are precisely the $SL_2(k)$-invariants. If g is homogeneous of degree (u, v), then $(f, g)^i$ is homogeneous of degree $(r + u, s + v + i)$.

Example 6.20 To illustrate, we calculate $f_8 = (f_2, f_4)^2$ in the table for A_5. First:

$$f_2 = (f_0, f_0)^2 = [x_2, x_2]_2^{D_n} = 2x_0x_2 - x_1^2$$

and

$$f_4 = (f_0, f_0)^4 = [x_4, x_4]_4^{D_n} = 2x_0x_4 - 2x_1x_3 + x_2^2$$

(Note that these are equalities up to an integer multiple.) Next, we find that

$$U_5^2 f_2 = 12x_0x_4 + 3x_1x_3 - 4x_2^2 \quad \text{and} \quad U_5^2 f_4 = 10x_1x_5 - 16x_2x_4 + 9x_3^2$$

(again, up to integer multiples). Therefore:

$$\begin{aligned} f_8 &= (f_2, f_4)^2 = [U_5^2 f_2, U_5^2 f_4]_2^{D_5} \\ &= -150x_0^2x_3x_5 + 48x_0^2x_4^2 + 150x_0x_1x_2x_5 + 54x_0x_1x_3x_4 - 152x_0x_2^2x_4 \\ &\quad +60x_0x_2x_3^2 - 50x_1^3x_5 + 50x_1^2x_2x_4 - 57x_1^2x_3^2 + 32x_1x_2^2x_3 - 8x_2^4 \end{aligned}$$

Chapter 7
Non-Finitely Generated Kernels

The Maurer-Weitzenbröck Theorem for linear $\mathbb{G}_a$-actions on affine space does not generalize to non-linear $\mathbb{G}_a$-actions. In 1990, Paul Roberts [359] gave the first examples of non-affine invariant rings for $\mathbb{G}_a$-actions on an affine space. These examples involved actions of $\mathbb{G}_a$ on $\mathbb{A}^7$ over a field k of characteristic zero, and are counterexamples to Hilbert's Fourteenth Problem. Subsequent examples of $\mathbb{G}_a$-actions of non-finite type were constructed in Freudenburg [163] and in Daigle and Freudenburg [79] for $\mathbb{A}^6$ and $\mathbb{A}^5$, respectively.

Since the appearance of Roberts' examples, an abundance of other $\mathbb{G}_a$-actions of non-finite type have been discovered, for example in [164, 168, 247, 262]. Indeed, it appears that "most" invariant rings $k[\mathbb{A}^n]^{\mathbb{G}_a}$ are of non-finite type. We are thus motivated to identify positive structural properties of these rings. The most fundamental of these properties is Winkelmann's Theorem, which implies that $k[\mathbb{A}^n]^{\mathbb{G}_a}$ is always quasi-affine; see *Sect. 7.11*.

From the point of view of classical invariant theory, a structural description of a ring of invariants involves determination of a minimal set of generators of the ring as a k-algebra, together with a minimal set of generators for the ideal of their relations. However, for an infinite set of generators, or even a large finite set of generators, such a description can be complicated, and the choice of generating set can seem arbitrary. Kuroda [262] used SAGBI basis techniques to show that an infinite system of invariants constructed by Roberts for the action on $\mathbb{A}^7$ generates the invariant ring as a k-algebra. Tanimoto [402] used the same techniques to identify generating sets for the actions on $\mathbb{A}^6$ and $\mathbb{A}^5$, but Tanimoto's generating sets are not minimal. In Freudenburg and Kuroda [171], cable algebras are introduced to describe the structure of these non-affine rings, and a minimal set of generators and relations is given for the action on $\mathbb{A}^5$. From *Sect. 2.9*, recall that the condition for a ring to be a cable algebra is a finiteness condition: B is a cable algebra if B is generated by a finite number of D-cables over $\ker D$ for some nonzero $D \in \mathrm{LND}(B)$.

In this chapter, we begin by exploring the examples of Roberts and some of the rich theory which has flowed from them. We then introduce two families of

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derivations with kernels of non-finite type. The first of these, due to Kuroda, is discussed in *Sect. 7.2*. This is a family of elementary monomial derivations on $k^{[n]}$ for $n \geq 7$ with kernels of non-finite type which includes the examples of Roberts and many of its successors. The second family, due to the author, is discussed in *Sect. 7.5*. It is a family of quotients of a $\mathbb{G}_a$-module which give triangular derivations on $k^{[n]}$ for $n \geq 5$ with kernels of non-finite type. This family includes the earlier examples on $k^{[5]}$ and $k^{[6]}$ [79, 163]. The proof of non-finite generation for these examples relies on properties of the down operator, which is introduced in *Sect. 7.4*.

In addition to questions of finite or non-finite generation, we also address structural questions about some of these kernels. *Theorem 7.1* shows that the smallest of the rings constructed by Roberts is a cable algebra. *Theorem 7.6* shows that the kernel of the triangular derivation of $k^{[5]}$ appearing in [79] is a monogenetic cable algebra, and gives an explicit description of its cable generators and cable relations.

7.1 Roberts' Examples

In the mid-1980s, Roberts was studying the examples of Rees and Nagata from a point of view somewhat different than that presented above. The main idea of this approach is to consider a ring R which is the symbolic blow-up of a prime ideal P in a commutative Noetherian ring A. What this means is that R is isomorphic to a graded ring of the form $\oplus_{n\geq 0}P^{(n)}$, where $P^{(n)}$ denotes the nth symbolic power of P, defined as:

$$P^{(n)} = \{x \in A \mid xy \in P^n \text{ for some } y \notin P\}$$

Rees used symbolic blow-ups in constructing his counterexample to Zariski's Problem. In his 1985 paper [358], Roberts writes:

> In a few nice cases the symbolic blow-up of P is a Noetherian ring or, equivalently, a finitely generated A-algebra. In general, however, $\oplus P^{(n)}$ is not Noetherian. The first example of this is due to Rees.

It was in this paper that Roberts constructed a new counterexample to Zariski's problem similar to Rees's example, but having somewhat nicer properties. Subsequently, in a 1990 paper [359] Roberts constructed an important new counterexample to Hilbert's Fourteenth Problem along similar lines. In the latter paper, Roberts gave the following description of these developments.

> In his example, Rees takes R to be the coordinate ring of the cone over an elliptic curve and shows that if P is the prime ideal corresponding to a point of infinite order then the ring $\oplus_{n\geq 0}P^{(n)}$ is not finitely generated and is a counterexample to Zariski's problem. Shortly thereafter Nagata gave a counterexample to Hilbert's original problem, and, in fact gave a counterexample which was a ring of invariants of a linear group acting on a polynomial ring, which is the special case which motivated the original problem. In his example a similar construction to that of Rees was used in which P was not prime, but was the ideal

defining sixteen generic lines through the origin in affine space of three dimensions. The proof was based on the existence of points of infinite order on elliptic curves.

But this did not totally end the story. Rees's example uses a ring which is not regular, and Nagata's uses an ideal which is not prime; Cowsick then asked whether there were examples in which the ring was regular and the ideal prime. Such an example was given in Roberts [358]. However, this still did not totally finish the problem, since this example was based on that of Nagata and made crucial use of the fact that when the ring was completed the ideal broke up into pieces and did not remain prime.

Roberts proceeded to construct an example of a prime ideal in a complete regular local ring (a power series ring in seven variables) whose symbolic blow-up is not finitely generated.

Explicitly, Roberts takes k to be any field of characteristic 0 and $B = k^{[7]} = k[X, Y, Z, S, T, U, V]$, and defines a graded $k[X, Y, Z]$-module homomorphism $\phi : B \to B$. He proves that the kernel of ϕ is not finitely generated over k. This construction is then "completed" to give the example in terms of symbolic blow-ups.

In Roberts' paper, ϕ is defined explicitly by its effect on monomials in S, T, U, V. Though Roberts does not use the language of derivations, one recognizes from his description of these images that ϕ is equivalent to the triangular k-derivation $\mathcal{D}$ of B defined by

$$\mathcal{D}_t = X^{t+1}\partial_S + Y^{t+1}\partial_T + Z^{t+1}\partial_U + (XYZ)^t\partial_V$$

where $t \geq 2$. According to Roberts, this example originated in his study of Hochster's Monomial Conjecture, which had been proved for any field of characteristic 0. The conjecture asserted that for any local ring of dimension 3 with system of parameters X, Y, Z, and for any non-negative integer t, the monomial $X^tY^tZ^t$ is not in the ideal generated by the monomials X^{t+1}, Y^{t+1}, and Z^{t+1}. In Lemma 3 of [359], Roberts showed the existence of a sequence in $\ker \mathcal{D}_t$ of the form:

$$F_{(t,i)} = XV^i + (\text{terms of lower degree in } V)\ , \ i \geq 0$$

According to Roberts, the proof of this lemma "is considerably more difficult, and is in fact the main result of the paper". By combining this lemma with homogeneity conditions, he concluded that $\ker \mathcal{D}_t$ is not finitely generated over k for each $t \geq 2$.

Define an action of the cyclic group $\mathbb{Z}_3 = \langle \alpha \rangle$ on B by:

$$\alpha(X, Y, Z, S, T, U, V) = (Z, X, Y, U, S, T, V)$$

Then α, $\mathcal{D}_t$ and the partial derivative $\partial/\partial V$ commute pairwise with each other. Therefore, α and $\partial/\partial V$ restrict to $\ker \mathcal{D}_t$. We denote the restriction of $\partial/\partial V$ to $\ker \mathcal{D}_t$ by δ_t.

Given $t \geq 2$, define $H_t \in \ker \mathcal{D}_t$ by $H_t = Y^{t+1}S - X^{t+1}T$. In [262], Thm 3.3, Kuroda shows:

$$\ker \mathcal{D}_t = k[H_t, \alpha H_t, \alpha^2 H_t, F_{(t,i)}, \alpha F_{(t,i)}, \alpha^2 F_{(t,i)}\,]_{i \geq 0}$$

The sequence of elements $F_{(t,i)}$ is not unique. The author and Kuroda showed that, in case $t = 2$, the elements $F_{(2,i)}$ can be chosen to form δ_2-cables, i.e., $\delta_2 F_{(2,i)} = F_{(2,i-1)}$ for $i \geq 0$. Therefore, $\ker \mathcal{D}_2$ is a cable algebra.

Theorem 7.1 ([171], Thm. 8.2) *There exists an infinite δ_2-cable $\hat{P}$ rooted at X, and for any such $\hat{P}$ we have:*

$$ker\,\mathcal{D}_2 = k[H_2, \alpha H_2, \alpha^2 H_2, \hat{P}, \alpha\hat{P}, \alpha^2\hat{P}]$$

It was Derksen [94] who first investigated the connection between counterexamples to Hilbert's problem and derivations (see *Proposition 1.14*), but the derivations he uses are not locally nilpotent. In particular, he constructs a derivation of the polynomial ring in 32 variables whose kernel coincides with the fixed ring of Nagata's example.

It was shortly after its appearance that A'Campo-Neuen [4] recognized that Roberts' example could be realized as the invariant ring of an algebraic (but non-linear) $\mathbb{G}_a$-action on $\mathbb{A}^7$. This was recognized independently by Deveney and Finston in [102] at about the same time, and they give a different proof that $\ker \mathcal{D}_2$ is not finitely generated.

It was observed by van den Essen and Janssen [144] that, because of Roberts' example, there exist locally nilpotent derivations with non-finitely generated kernels in all dimensions higher than 7 as well. For example, extend $\mathcal{D}_t$ to $k[x_1, \dots, x_n]$ ($n \geq 8$) by setting $\mathcal{D}x_i = 0$ for $8 \leq i \leq n$. This extended derivation has kernel equal to $\ker \mathcal{D}_t[x_8, \dots, x_n]$, which is also not finitely generated. Another family of counterexamples in higher (odd) dimensions was given by Kojima and Miyanishi in [247]. They consider the triangular derivations λ on polynomial rings $k[x_1, \dots, x_n, y_1, \dots, y_n, z]$ defined by $\lambda(x_i) = 0$, $\lambda(y_i) = x_i^{t+1}$, and $\lambda(z) = (x_1 \cdots x_n)^t$. They prove that, for each $n \geq 3$ and $t \geq 2$, the kernel of λ is not finitely generated. Since Roberts' examples are included in this family as the case $n = 3$, their paper provides a new proof for Roberts' example as well.

In [163], the author gave a triangular derivation in dimension 6 with non-finitely generated kernel, and then used this to give an independent proof that $\ker \mathcal{D}_2$ is not finitely generated. Other proofs for Roberts' example may be found in [142, 262, 286]. It should be noted that Kurano showed $\ker \mathcal{D}_1$ is generated by nine elements over k [260].

7.2 Kuroda's Examples

In [262], Kuroda considers generic triangular monomial derivations on $k^{[m+n]}$ having rank at most n. Theorem 1.3 shows that, if $n \geq 4$ and $m \geq n - 1$, and if the exponents appearing in the defining monomials satisfy certain inequalities, then the kernel of the derivation is not finitely generated. The statement of Thm. 1.3 is somewhat technical, and the reader is referred to the paper for its precise statement.

The examples of Roberts and of Kojima and Miyanishi are special cases of this theorem.

In case $n = 4$ and $m = 3$, Kuroda's result is as follows.

Theorem 7.2 ([262], Thm. 1.4) *Let* (e_{ij}), $1 \leq i \leq 4$, $1 \leq j \leq 3$, *be an array of integers which satisfies:*

1. $e_{jj} > e_{ij} \geq 0$ *for* $i \neq j$
2. $\dfrac{e_{11} - e_{41}}{m_1} + \dfrac{e_{22} - e_{42}}{m_2} + \dfrac{e_{33} - e_{43}}{m_3} \leq 1$ *where:*

$$m_j = min\{e_{jj} - e_{ij} \mid 1 \leq i \leq 3, i \neq j\}$$

Let $B = k[x_1, x_2, x_3, y_1, y_2, y_3, y_4] = k^{[7]}$ *and define* $D \in LND(B)$ *by:*

$$Dx_1 = Dx_2 = Dx_3 = 0\ ,\ Dy_i = x_1^{e_{i1}} x_2^{e_{i2}} x_3^{e_{i3}}\ \ (1 \leq i \leq 4)$$

Then $\ker D$ *is not finitely generated as a k-algebra.*

According to Kuroda (p. 503), if we impose the restriction that D is irreducible and $e_{ij} \leq 10$ for all i, j, then there exist 2,450,001 derivations which satisfy the conditions of this theorem!

As a special case, consider the array:

$$(e_{ij}) = \begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \\ s & s & s \end{pmatrix} \qquad r, s \in \mathbb{N}$$

The induced derivation of $R = k[X, Y, Z, S, T, U, V] = k^{[7]}$ is given by:

$$\mathcal{D}_{(r,s)} = X^r \partial_S + Y^r \partial_T + Z^r \partial_U + (XYZ)^s \partial_V$$

The derivations $\mathcal{D}_{(r,s)}$ are the **generalized Roberts derivations**.

Condition (1) of *Theorem 7.2* is equivalent to $0 < s < r$, and condition (2) is equivalent to $2r \leq 3s$. We thus obtain the following.

Theorem 7.3 *If* $1 \leq s < r \leq \frac{3}{2}s$, *then* $\ker \mathcal{D}_{(r,s)}$ *is not finitely generated as a k-algebra.*

7.3 Non-finiteness Criterion

Given $(m, n) \in \mathbb{N}^2$ with $m \geq 1$, define the rectangular array:

$$\mathcal{R}_{(m,n)} = \{(a, b) \in \mathbb{N}^2 \mid 1 \leq a \leq m\, ,\ 0 \leq b \leq n\}$$

Lemma 7.4 *Given $(m,n) \in \mathbb{N}^2$ with $m \geq 1$, let $\langle \mathcal{R}_{(m,n)} \rangle$ be the submonoid of $\mathbb{N}^2$ generated by $\mathcal{R}_{(m,n)}$. For each $\ell \in \mathbb{N}$ with $\ell \geq 1$, the set $\langle \mathcal{R}_{(m,n)} \rangle \cap (\{\ell\} \times \mathbb{N})$ is finite.*

Proof Suppose that $\sum_{1 \leq i \leq r} (u_i, v_i) = (\ell, N)$ for $(u_i, v_i) \in \mathcal{R}_{(m,n)}$ and some $r, N \in \mathbb{N}$. Since $u_i \geq 1$ for each i, we see that $r \leq \sum_{1 \leq i \leq r} u_i = \ell$. Since $v_i \leq n$ for each i, we see that $N = \sum_{1 \leq i \leq r} v_i \leq rn \leq \ell n$. Therefore, the intersection of $\langle \mathcal{R}_{(m,n)} \rangle$ with $\{\ell\} \times \mathbb{N}$ is bounded, hence finite. □

The following is given in Lemma 2.1 of [79].

Lemma 7.5 (Non-finiteness Criterion) *Let B be a commutative k-domain and let $B = \bigoplus_{i \in \mathbb{Z}} B_i$ be a positive $\mathbb{Z}$-grading over k. Let $\delta \in LND(B)$ be homogeneous, and set $A = \ker \delta$. Given homogeneous $\alpha \in A \setminus \delta B$, let $\tilde{\delta}$ be the extension of δ to $B[T] = B^{[1]}$ defined by $\tilde{\delta} T = \alpha$. Suppose β_n is a sequence of nonzero elements of $\ker \tilde{\delta}$ having leading T-coefficients $b_n \in B$. If $\deg b_n$ is bounded, but $\deg_T \beta_n$ is not bounded, then $\ker \tilde{\delta}$ is not finitely generated over k.*

Proof Let $M[T]$ be the extension to $B[T]$ of the maximal ideal $M = \bigoplus_{i>0} B_i$ of B. Set $m = \deg \alpha - \deg \delta$, and for every integer n, define $B[T]_n = \sum_{i \in \mathbb{N}} B_{n-mi} T^i$. Then $\bigoplus_{n \in \mathbb{Z}} B[T]_n$ is a $\mathbb{Z}$-grading of $B[T]$, and $\tilde{\delta}$ is homogeneous.

Let $\tilde{A} = \ker \tilde{\delta}$. If $\phi \in \tilde{A}$ is homogeneous, then $\phi = \sum \phi_i T^i$ for homogeneous $\phi_i \in B$. Since $\tilde{\delta}(\phi) = 0$, it follows from the product rule that $\delta(\phi_{i-1}) = -i\alpha\phi_i$ for $i > 0$. Thus, $\phi_i \notin k^*$ for $i > 0$, since otherwise $\alpha = \delta(-i^{-1}\phi_i^{-1}\phi_{i-1})$, which is in the image of δ. So if $i > 0$, then $\phi_i \in M$, since each ϕ_i is homogeneous. Since also $\phi_0 \in k + M$, we conclude that $\phi \in k + M[T]$. Since $\tilde{A}$ is a $\mathbb{Z}$-graded subring, it follows that $\tilde{A} \subset k + M[T]$.

The $\mathbb{N}$-grading of B extends to an $\mathbb{N}^2$-grading of $B[T]$ by letting T be homogeneous of degree $(0,1)$. Let $G(\tilde{A}) \subset \mathbb{N}^2$ be the submonoid generated by degrees in $\tilde{A}$. The hypotheses imply that, for some $\ell \in \mathbb{N}$ with $\ell \geq 1$, the set $G(\tilde{A}) \cap (\{\ell\} \times \mathbb{N})$ is infinite.

Let $k[f_1, \ldots, f_N] \subset \tilde{A}$ be a finitely generated subring. We may assume that $f_i \in M[T]$ for each i, which implies that, if $\deg f_i = (u_i, v_i)$, then $u_i \geq 1$. Therefore, there exists $(m,n) \in \mathbb{N}^2$ such that $m \geq 1$ and:

$$G(k[f_1, \ldots, f_N]) \subset \langle \mathcal{R}_{(m,n)} \rangle$$

By *Lemma 7.4*, the set $G(k[f_1, \ldots, f_N]) \cap (\{\ell\} \times \mathbb{N})$ is finite, whereas $G(\tilde{A}) \cap (\{\ell\} \times \mathbb{N})$ is infinite. Therefore, $k[f_1, \ldots, f_N] \neq \tilde{A}$. □

7.4 Down Operator

Let V_n be the SL_2-module of binary n-forms. The action of $SL_2(k)$ on V_n does not respect the vector space inclusions $V_0 \subset V_1 \subset \cdots \subset V_n$. However, the corresponding $\mathbb{G}_a$-action does respect this sequence of inclusions, and it makes sense to study the $\mathbb{G}_a$-actions from this point of view. Toward this end, the down operator Δ of the infinite polynomial ring $\Omega = k[x_0, x_1, x_2, \ldots]$ was introduced in [168]. The overarching goal of this approach is to describe a homogeneous generating set for $\ker \Delta$, and to give a framework for understanding the $\mathbb{G}_a$-invariant rings A_n.

Theorem 7.10 below is one of the main tools used in this chapter in proving non-finite generation for certain invariant rings.

7.4.1 Properties

The **down operator** $\Delta \in \mathrm{LND}(\Omega)$ is defined by:

$$\Delta x_i = x_{i-1} \quad (i \geq 1) \quad \text{and} \quad \Delta x_0 = 0$$

Let $\Gamma = \ker \Delta$. Define a $\mathbb{Z}^2$-grading of Ω by letting x_i be homogeneous of degree $(1, i)$, and write:

$$\Omega = \bigoplus_{(r,s)\in\mathbb{Z}^2} \Omega_{(r,s)}$$

Then Δ is homogeneous and Γ is a graded subring. If $\Gamma_{(r,s)} = \Gamma \cap \Omega_{(r,s)}$, then:

$$\Gamma = \bigoplus_{(r,s)\in\mathbb{Z}^2} \Gamma_{(r,s)}$$

Define ideals $\Omega_+ \subset \Omega$ and $\Gamma_+ \subset \Gamma$ by:

$$\Omega_+ = \sum_{i\geq 0} x_i\Omega \quad \text{and} \quad \Gamma_+ = \Gamma \cap \Omega_+$$

Define the map $\theta : \Omega \to \Gamma$ by:

$$\theta(f) = \sum_{i\geq 0}(-1)^i \Delta^i(f) x_i$$

Then θ is a map of Γ-modules, and if $f \neq 0$ and $d = \deg_\Delta(f)$, then $\theta(f) = [f, x_d]_d^\Delta$, so $\theta(\Omega) \subset \Gamma$, as asserted.

Lemma 7.6 ([168], Lemma 3.1) *If $r \geq 1$ and $f \in \Gamma_{(r,s)}$, then:*

$$\theta\left(\frac{\partial f}{\partial x_0}\right) = rf$$

Consequently, $\theta(\Omega) = \Gamma_+$.

Proof For all $n \geq 0$, we have:

$$[\partial/\partial x_n, \Delta] = \frac{\partial}{\partial x_{n+1}} \tag{7.1}$$

Therefore, for all $i \geq 0$:

$$\Delta^i \left(\frac{\partial f}{\partial x_0} \right) = (-1)^i \frac{\partial f}{\partial x_i} \tag{7.2}$$

Thus, by Euler's Lemma, it follows that:

$$rf = \sum_{i \geq 0} x_i \frac{\partial f}{\partial x_i} = \sum_{i \geq 0} x_i (-1)^i \Delta^i \left(\frac{\partial f}{\partial x_0} \right) = \theta \left(\frac{\partial f}{\partial x_0} \right)$$

□

Theorem 7.7 ([168], Thm. 3.1) *The sequence of Γ-modules*

$$\Omega \xrightarrow{\theta} \Omega_+ \xrightarrow{\Delta} \Omega_+ \rightarrow 0$$

is exact.

Proof We need to show $\operatorname{im} \Delta = \Omega_+$ and $\operatorname{im} \theta = \Gamma_+$. The second of these equalities is given by *Lemma 7.6*.

In order to prove the first equality, let $m \geq 0$ and $g \in \Omega$ be given, and set $n = \deg_\Delta g$. By *Proposition 2.58(e)*, we have:

$$\Delta[x_{m+n+1}, g]_n = g\Delta^{n+1} x_{m+n+1} = g x_m$$

Therefore, the ideal $x_m \Omega$ is in the image of Δ for each $m \geq 0$, which implies that $\Delta\Omega = \Omega_+$. □

7.4.2 Operator Relations

Given $n \geq 0$, define $E_n, U_n \in \mathrm{Der}_k(\Omega)$ as follows. For each $i \geq 0$:

$$E_n x_i = (n - 2i)x_i \quad \text{and} \quad U_n x_i = (i + 1)(n - i)x_{i+1}$$

E_n is the n**th Euler derivation** and U_n is the n**th up operator** for Ω. E_n and U_n are homogeneous of degrees $(0, 0)$ and $(0, 1)$, respectively. Note that, for each $f \in \Omega_{(r,s)}$, we have: $E_n f = (nr - 2s)f$. In addition, for each $n \geq 0$:

$$[\partial/\partial x_n, U_n] = \begin{cases} 0 & n = 0 \\ n\,\partial/\partial x_{n-1} & n \geq 1 \end{cases} \tag{7.3}$$

Let $k[\Delta, U_n]$ denote the subalgebra of $\mathrm{End}_k(\Omega)$ generated by Δ and U_n, noting that $k[\Delta, U_n]$ is not unitary. The following relations are easily verified:

$$[\Delta, U_n] = E_n \ , \quad [\Delta, E_n] = -2\Delta \ , \quad [U_n, E_n] = 2U_n \tag{7.4}$$

We see that, for each $n \geq 0$, $k[\Delta, U_n]$ is isomorphic to $\mathfrak{sl}_2(k)$ as a Lie algebra.

Define $\delta \in \mathrm{Der}_k(k[\Delta, U_n])$ by $\delta = \mathrm{ad}(\Delta)$, the adjoint operator. Then δ is a locally nilpotent derivation of $k[\Delta, U_n]$, since by (7.4), δ acts on the algebra generators by:

$$\delta : U_n \to E_n \to -2\Delta \to 0$$

The reader can easily verify that, for all $m \geq 1$:

$$\delta(U_n^m) = mU_n^{m-1}E_n - m(m-1)U_n^{m-1}$$

In addition, for all $l, m \geq 1$:

$$\Delta^l U_n^m - \Delta^{l-1}U_n^m\Delta = \delta(\Delta^{l-1}U_n^m) = \Delta^{l-1}\delta(U_n^m)$$

It follows that, if $I : \Omega \to \Omega$ is the identity, then:

$$\Delta^l U_n^m = \Delta^{l-1}U_n^{m-1}(U_n\Delta + mE_n - m(m-1)I) \tag{7.5}$$

Therefore, given $f \in \Gamma_{(r,s)}$, Eq. (7.5) implies:

$$\Delta^l U_n^m f = m(nr - 2s - m + 1)\Delta^{l-1}U_n^{m-1}f \tag{7.6}$$

In this way, we obtain the following key integration property for Δ.

Lemma 7.8 ([168], Lemma 3.2) *If $f \in \Gamma_{(r,s)}$ and $n \geq 1$, then*

$$\Delta^m U_n^m(f) = c_1 \cdots c_m f$$

where $c_i = i(nr - 2s - i + 1)$, $1 \leq i \leq m$.

7.4.3 Restriction to B_n

Given $n \geq 0$, $B_n = k[x_0, \dots, x_n]$ and $A_n = \Gamma \cap B_n$ are graded subrings of Ω, the derivations Δ, U_n and E_n restrict to B_n, and Δ and U_n are locally nilpotent on B_n. Therefore, $\deg_{U_n}$ is defined on B_n.

Lemma 7.9 ([168], Lemma 4.4)

(a) $\deg_{U_n} f = nr - 2s \geq 0$ *for every nonzero* $f \in B_n \cap \Gamma_{(r,s)}$
(b) $B_n \cap \Gamma_{(r,s)} = \{0\}$ *if* $nr - 2s < 0$

Proof If $N = \deg_{U_n} f$, then $U_n^{N+1} f = 0$ and $U_n^N f \neq 0$. From Eq. (7.6), we have $0 = \Delta U_n^{N+1} f = c_{N+1} U_n^N f$, where $c_{N+1} = (N+1)(nr-2s-N)$. Therefore $0 = c_{N+1}$ implies $nr - 2s = N$. This proves part (a), and part (b) follows from part (a).

The mapping $\Delta : \Omega_+ \to \Omega_+$ is a surjection, while its restriction $\Delta : B_n \to B_n$ is not. The following result was known classically. It is an important tool in calculating generators for the rings A_n.

Theorem 7.10 ([47], Cor. 2.3; [168], Prop. 4.1) *Given integers* $n \geq 1$ *and* $r, s \geq 0$, *the mapping*

$$\Delta : B_n \cap \Omega_{(r,s+1)} \to B_n \cap \Omega_{(r,s)}$$

is surjective if $2s < rn$, *and injective if* $2s \geq rn$.

Proof Consider first the case that $2s < rn$. Given ℓ with $0 \leq \ell \leq s$, set:

$$(B_n \cap \Omega_{(r,s)})^{(\ell)} = \ker \Delta^{\ell+1} \cap (B_n \cap \Omega_{(r,s)})$$

This gives a nested sequence of subspaces of $B_n \cap \Omega_{(r,s)}$ with:

$$(B_n \cap \Omega_{(r,s)})^{(0)} = B_n \cap \Gamma_{(r,s)} \quad \text{and} \quad (B_n \cap \Omega_{(r,s)})^{(s)} = B_n \cap \Omega_{(r,s)}$$

We show by induction on ℓ that Δ surjects onto $(B_n \cap \Omega_{(r,s)})^{(\ell)}$ for each $\ell = 0, \dots, s$.

Let nonzero $f \in B_n \cap \Gamma_{(r,s)}$ be given. Then $U_n f \in B_n \cap \Omega_{(r,s+1)}$, since $\deg U_n = (0, 1)$. By *Lemma 7.8*, we have $\Delta U_n f = (nr-2s)f$ and $nr-2s > 0$. This establishes the basis for induction.

Given ℓ with $1 \leq \ell \leq s$, assume that Δ surjects onto $(B_n \cap \Omega_{(r,s)})^{(\ell-1)}$. Let $g \in (B_n \cap \Omega_{(r,s)})^{(\ell)}$ be given, and assume that $\Delta^\ell g \neq 0$. By *Lemma 7.8*, we have

$$\Delta^{\ell+1} U_n^{\ell+1} (\Delta^\ell g) = c \Delta^\ell g$$

where $c \in \mathbb{Z}$ is given by:

$$c = \prod_{1 \leq i \leq \ell} i(nr - 2s + 2\ell - i + 1) > 0$$

Define:

$$h = \Delta U_n^{\ell+1}\Delta^\ell g - cg \in (B_n \cap \Omega_{(r,s)})^{(\ell-1)}$$

By the inductive hypothesis, there exists $\eta \in B_n \cap \Omega_{(r,s+1)}$ such that $\Delta\eta = h$. It follows that:

$$cg = \Delta U_n^{\ell+1}\Delta^\ell g - \Delta\eta = \Delta(U_n^{\ell+1}\Delta^\ell g - \eta)$$

By induction, we conclude that Δ surjects onto $(B_n \cap \Omega_{(r,s)})^{(\ell)}$. Therefore, $\Delta : B_n \cap \Omega_{(r,s+1)} \to B_n \cap \Omega_{(r,s)}$ is surjective if $2s < nr$.

Consider next the case $2s \geq rn$. By *Lemma 7.9(b)*:

$$nr - 2(s+1) < 0 \quad \Rightarrow \quad B_n \cap \Gamma_{(r,s+1)} = \{0\}$$

Therefore, the restriction of Δ to $B_n \cap \Omega_{(r,s+1)}$ is injective in this case. □

Remark 7.11 In classical terminology, given $f \in \Omega_{(r,s)}$, the **degree** of f is r and the **weight** of f is s. In addition, for fixed n, if $f \in B_n$, then the **order** of f equals $nr - 2s$. Elements of order 0 in A_n are precisely the SL_2-invariants in B_n.

7.5 Quotients of a $\mathbb{G}_a$-Module

Theorem 7.10 affords a relatively easy way to show that certain $\mathbb{G}_a$-actions on affine space are of non-finite type. Given $n \geq 0$, let V_n be the irreducible $\mathbb{G}_a$-module of dimension $n+1$. The following result generalizes Thm. 3.1 of [164].

Theorem 7.12 ([168], Thm. 8.2) *Let n, N, λ, μ be positive integers such that $3 \leq n \leq N$ and $2\lambda = n\mu$. Let x_0, y_0, z_0 denote the unique linear invariants for V_N, V_1, V_0, respectively, and consider the $\mathbb{G}_a$-module $V_N \oplus V_1 \oplus V_0$. If X is the $\mathbb{G}_a$-subvariety defined by $x_0 - z_0^\lambda = y_0 - z_0^\mu = 0$, then $X \cong \mathbb{A}^{N+2}$ and $k[X]^{\mathbb{G}_a}$ is not finitely generated.*

Note that the special case $\mu = 1$ is particularly simple, requiring only one hypersurface: If X is the $\mathbb{G}_a$-subvariety of $V_N \oplus V_1$ defined by $x_0 - y_0^\lambda = 0$, then $k[X]^{\mathbb{G}_a}$ is not finitely generated.

In order to prove this theorem, define the sequence of integers k_r ($r \geq 0$) by:

$$k_r = \begin{cases} nr/2 & nr \text{ even} \\ (nr+1)/2 & nr \text{ odd} \end{cases}$$

Define the index set $J_n = \{(0,0)\} \cup \{(r,s) \mid r \geq 1,\ k_{r-1} + 1 \leq s \leq k_r\}$.

Lemma 7.13 ([168], Thm. 8.1) *There exists a sequence $w_{(r,s)} \in B_n \cap \Omega_{(r,s)}$ for $(r,s) \in J_n$ such that $w_{(0,0)} = 1$, and for $r \geq 1$:*

$$\Delta w_{(r,s)} = \begin{cases} w_{(r,s-1)} & k_{r-1}+2 \leq s \leq k_r \\ x_0 w_{(r-1,k_{r-1})} & s = k_{r-1}+1 \end{cases}$$

Proof Given $(r,s) \in J_n$, set $W_{(r,s)} = B_n \cap \Omega_{(r,s)}$. Using lexicographical ordering on J_n, assume that the sequence $w_{(i,j)} \in W_{(i,j)}$ has been constructed up to $(i,j) = (r-1, k_{r-1})$, where $r \geq 1$. By *Theorem 7.10*, each mapping in the following sequence of maps is surjective:

$$x_0 W_{(r-1,k_{r-1})} \subset W_{(r,k_{r-1})} \xleftarrow{\Delta} W_{(r,k_{r-1}+1)} \xleftarrow{\Delta} \cdots \xleftarrow{\Delta} W_{(r,k_r-1)} \xleftarrow{\Delta} W_{(r,k_r)}$$

We may thus extend the sequence $w_{(i,j)}$ to $(i,j) = (r, k_r)$. □

Proof of Theorem 7.12. Let $k[V_0] = k[z_0] = k^{[1]}$, $k[V_1] = k[y_0, y_1] = k^{[2]}$ and:

$$k[V_N] = k[x_0, \dots, x_N] = k^{[N+1]}$$

Then:

$$k[V_N \oplus V_1 \oplus V_0] = k[x_0, \dots, x_N, y_0, y_1, z_0] = k^{[N+4]}$$

In addition, $V_n \oplus V_1 \oplus V_0$ is a submodule of $V_N \oplus V_1 \oplus V_0$. We have:

$$k[V_n \oplus V_1 \oplus V_0] = k[x_0, \dots, x_n, y_0, y_1, z_0] = k^{[n+4]}$$

and

$$k[V_n \oplus V_1 \oplus V_0]^{\mathbb{G}_a} = \ker \mathcal{D}$$

where $\mathcal{D}$ is the triangular linear derivation of $k[V_n \oplus V_1 \oplus V_0]$ defined by:

$$\mathcal{D} = \left(\sum_{i=1}^{n} x_{i-1} \frac{\partial}{\partial x_i} \right) + y_0 \frac{\partial}{\partial y_1}$$

Let D be the restriction of $\mathcal{D}$ to $k[V_n]$, i.e., $Dx_i = x_{i-1}$ for $1 \leq i \leq n$ and $Dx_0 = 0$, and let $w_{(r,s)} \in k[V_n]$, $(r,s) \in J_n$, be the sequence defined in *Lemma 7.13*. Given $m \geq 1$, the lemma implies that:

$$x_0^{2i} \mid D^{in+j} w_{(2m,nm)} \quad (0 \leq i \leq m-1 \,,\, 0 \leq j \leq n-1) \tag{7.7}$$

Given $m \geq 1$, define $F_m(x_0, \ldots x_n, y_0, y_1, z_0) = [w_{(2m,nm)}, y_1^{nm}]_{nm}^{\mathcal{D}}$. Then *Proposition 2.58(f)* implies that $F_m \in \ker \mathcal{D}$ and *Proposition 2.59(a)* shows that:

$$F_m = (nm)! \sum_{l=0}^{nm} \frac{(-1)^l}{l!} \mathcal{D}^l w_{(2m,nm)} (\mathcal{D}y_1)^{nm-l} y_1^l$$
$$= (nm)! \left(\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \frac{(-1)^{in+j}}{(in+j)!} D^{in+j} w_{(2m,nm)} y_0^{nm-(in+j)} y_1^{in+j} \right)$$
$$+ (-1)^{nm} x_0^{2m} y_1^{nm}$$

Substitute $x_0 = z_0^\lambda$ and $y_0 = z_0^\mu$ in the term $D^{in+j} w_{(2m,nm)} y_0^{nm-(in+j)} y_1^{in+j}$. Equation (7.7) implies that the resulting term is divisible by:

$$z_0^{2\lambda i + \mu(nm-(in+j))} = z_0^{\mu(nm-j)}$$

In addition, substituting $x_0 = z_0^\lambda$ in the last term $x_0^{2m} y_1^{nm}$ yields $z_0^{2\lambda m} y_1^{nm} = z_0^{\mu nm} y_1^{nm}$. Since $j \leq n-1$, we have $\mu(nm-j) \geq \mu(nm-n+1)$. Therefore, there exists $G_m \in k[x_1, \ldots, x_n, y_1, z_0]$ such that:

$$F_m(z_0^\lambda, x_1, \ldots, x_n, z_0^\mu, y_1, z_0) = (-1)^{nm} z_0^{\mu(nm-n+1)} G_m$$

The coefficient of y_1^{nm} in G_m equals $z_0^{\mu(n-1)}$, which does not depend on m.

Define the triangular derivation $\mathfrak{d}$ on $k[x_1, \ldots, x_N, z_0] = k^{[N+1]}$ by:

$$\mathfrak{d} = z_0^\lambda \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} + \cdots + x_{n-1} \frac{\partial}{\partial x_n} + \cdots + x_{N-1} \frac{\partial}{\partial x_N}$$

The conditions $2\lambda = n\mu$ and $n \geq 3$ insure that $\lambda > \mu$, which implies that z_0^μ is not in the image of $\mathfrak{d}$. Extend $\mathfrak{d}$ to $\hat{\mathfrak{d}}$ on $k[x_1, \ldots, x_N, y_1, z_0] = k[X] = k^{[N+2]}$ by setting $\hat{\mathfrak{d}} y_1 = z_0^\mu$. Then each polynomial G_m ($m \geq 1$) is in the kernel of $\hat{\mathfrak{d}}$. By the Non-Finiteness Criterion (*Lemma 7.5*), it follows that $\ker \hat{\mathfrak{d}} = k[X]^{\mathbb{G}_a}$ is not finitely generated. □

In this theorem, the case $n = N = 3$, $\lambda = 3$, $\mu = 2$ yields the counterexample in dimension 5 which first appeared in [79]. This is discussed in *Sect. 7.6*. The case $n = N = 4$, $\lambda = 2$, $\mu = 1$ yields the counterexample in dimension 6 first given in [164]. This example was used to construct a linear representation of the unipotent group $\mathbb{G}_a^4 \rtimes \mathbb{G}_a$ on $\mathbb{A}^{11}$ with non-finitely generated ring of invariants. This is discussed in *Sect. 7.8*.

7.6 Family of Examples in Dimension Five

7.6.1 Derivations $D_{(r,s)}$

Let $B = k[a,x,y,z,v] = k^{[5]}$. Given integers $r,s \geq 0$, let $D_{(r,s)}$ be the triangular monomial derivation of B defined by:

$$D_{(r,s)} = a^r\partial_x + x\partial_y + y\partial_z + a^s\partial_v$$

In [79] it was shown that $\ker D_{(3,2)}$ is not finitely generated as a k-algebra. Van den Essen [143] gives another proof of this result. More generally, we have:

Theorem 7.14 *If $1 \leq s < r \leq \frac{3}{2}s$, then $\ker D_{(r,s)}$ is not finitely generated as a k-algebra.*

A proof of this theorem is given in *Sect.* 7.6.2 below.

Observe that, if $r \geq s$, then $D_{(r,s)}(x - a^{r-s}v) = 0$ and:

$$B = k[a, x - a^{r-s}v, y, z, v]$$

Let κ be the automorphism of B fixing a,y,z,v and mapping x to $x - a^{r-s}v$. Then:

$$\kappa D_{(r,s)}\kappa^{-1} = a^s\partial_v + (x + a^{r-s}v)\partial_y + y\partial_z$$

Let $\mathcal{O} = k[a,b] = k^{[2]}$. Given positive integers r,s, define the triangular $\mathcal{O}$-derivation $d_{(r,s)}$ of $\mathcal{O}[X,Y,Z] = \mathcal{O}^{[3]}$ by:

$$d_{(r,s)} = a^s\partial_X + (a^rX + b)\partial_Y + Y\partial_Z$$

Then *Theorem* 7.14 is equivalent to:

Theorem 7.15 *If $1 \leq r \leq \frac{1}{2}s$, then $\ker d_{(r,s)}$ is not finitely generated as a k-algebra.*

7.6.2 Proof of Theorem 7.14

Given (r,s) with $r > s \geq 1$ and $2r \leq 3s$, let $\mathcal{D}_{(r,s)}$ be the generalized Roberts derivation on $R = k[X,Y,Z,S,T,U,V] = k^{[7]}$:

$$\mathcal{D}_{(r,s)} = X^r\partial_S + Y^r\partial_T + Z^r\partial_U + (XYZ)^s\partial_V$$

This derivation has many symmetries. For example, consider the faithful action of the torus $\mathbb{G}_m^3$ on R defined by:

$$(\lambda,\mu,\nu)\cdot(X,Y,Z,S,T,U,V) = (\lambda X, \mu Y, \nu Z, \lambda^r S, \mu^r T, \nu^r U, (\lambda\mu\nu)^s V)$$

This torus action commutes with $\mathcal{D}_{(r,s)}$.

Similarly, there is an action of the symmetric group S_3 on R, with orbits $\{X, Y, Z\}$, $\{S, T, U\}$, and $\{V\}$. Specifically, S_3 is generated by $\sigma = (Z, X, Y, U, S, T, V)$ and $\tau = (X, Z, Y, S, U, T, V)$. Again, this action commutes with $\mathcal{D}_{(r,s)}$.

It is easily checked that S_3 acts on $\mathbb{G}_m^3$ by conjugation, and we thus obtain an action of $\mathbb{G}_m^3 \rtimes S_3$ on R. Specifically, $\tau(\lambda, \mu, \nu)\tau = (\lambda, \nu, \mu)$ and $\sigma(\lambda, \mu, \nu)\sigma^{-1} = (\nu, \lambda, \mu)$.

Since the $\mathbb{G}_m^3$-action has only constant invariants, consider instead the subgroup H of $\mathbb{G}_m^3$ defined by $\lambda\mu\nu = 1$. H is a two-dimensional torus, and the group $G := H \rtimes S_3$ acts on R. The invariant ring of H is generated by monomials, namely:

$$R^H = k[XYZ,\ STU,\ X^rZ^rT,\ X^rY^rU,\ Y^rZ^rS,\ Y^rSU,\ X^rTU,\ Z^rST,\ V]$$

Since H is normal in G, the fixed ring of the G-action is

$$R^G = (R^H)^{S_3} = k[x, s, t, u, v]$$

where:

$$x = XYZ\,,\ \ s = X^rY^rU + X^rZ^rT + Y^rZ^rS\,,$$
$$t = X^rTU + Y^rSU + Z^rST\,,\ \ u = STU\,,\ \ v = V$$

The main point to observe is that R^G is a polynomial ring: $R^G \cong k^{[5]}$. Since the action of G commutes with $\mathcal{D}_{(r,s)}$, it follows that $\mathcal{D}_{(r,s)}$ restricts to a locally nilpotent derivation of R^G. In particular:

$$\mathcal{D}_{(r,s)}x = 0\,,\ \mathcal{D}_{(r,s)}s = 3x^r\,,\ \mathcal{D}_{(r,s)}t = 2s\,,\ \mathcal{D}_{(r,s)}u = t\,,\ \mathcal{D}_{(r,s)}v = x^s \tag{7.8}$$

We see that the restriction of $\mathcal{D}_{(r,s)}$ to R^G is equivalent to $D_{(r,s)}$.

According to Lemma 2.2 in Kuroda [262], there exists a positive integer e and sequence of homogeneous elements $\alpha_m \in \ker \mathcal{D}_{(r,s)}$ such that the leading V-term of α_m is X^eV^m, $m \geq 0$. By homogeneity, we have $(\lambda, \mu, \nu)\cdot\alpha_m = \lambda^e\alpha_m$ for each $m \geq 0$.

Let β_m denote the product of all elements in the orbit of α_m under the action of S_3. Then $\{\beta_m\} \subset R^G \cap \ker \mathcal{D}_{(r,s)}$ and β_m has leading V-term $(XYZ)^{2e}V^{6m}$. It follows that $(R^G)^{\mathcal{D}_{(r,s)}}$ contains a sequence whose leading v-terms are $x^{2e}v^{6m}$, $m \geq 0$. The Non-Finiteness Criterion implies that $(R^G)^{\mathcal{D}_{(r,s)}}$ is not finitely generated. Therefore, $\ker D_{(r,s)}$ is not finitely generated. □

Remark 7.16 Theorem 7.12 can be used to give an independent proof that $\ker \mathcal{D}_{(3,2)}$ is not finitely generated. From (7.8) we see that $\mathcal{D}_{(3,2)}$ restricted to $R^G = k^{[5]}$ is equivalent to $D_{(3,2)}$. Therefore, $R^{G\times\mathbb{G}_a} = (R^G)^{\mathbb{G}_a}$ is not finitely generated. Suppose that $R^{\mathbb{G}_a}$ is finitely generated. Then by the Finiteness Theorem, $(R^{\mathbb{G}_a})^G = R^{\mathbb{G}_a\times G}$ would be finitely generated, a contradiction. Therefore, $R^{\mathbb{G}_a}$ is not finitely generated.

7.6.3 Kernel of $D_{(3,2)}$

Let $D_{(3,2)} \in \text{LND}(k^{[5]})$ be as defined in *Sect. 7.6.1* and let $A = \ker D_{(3,2)}$. The article [171] gives a detailed analysis of the structure of A, showing that A is a monogenetic cable algebra and calculating its cable relations. This leads to an explicit construction of a generating cable for A. These results are given in *Theorem 7.17*, *Theorem 7.18* and *Theorem 7.19* below. The proofs for these theorems, which rely heavily on properties of the down operator, are lengthy and somewhat technical, and the reader is referred to the cited paper for these proofs.

Define a $\mathbb{Z}^2$-grading of $B = k[a,x,y,z,v] = k^{[5]}$ by declaring that a,x,y,z,v are homogeneous and:

$$\deg(a,x,y,z,v) = ((1,0),(3,1),(3,2),(3,3),(2,1))$$

Then D is a homogeneous derivation of degree $(0,-1)$ and A is a graded subring of B.

Note that the partial derivative ∂_v commutes with D, and therefore restricts to A. Let ∂ denote the restriction of ∂_v to A and define $h \in \ker \partial$ by:

$$h = 9a^6z^2 - 18a^3xyz + 6x^3z + 8a^3y^3 - 3x^2y^2$$

Theorem 7.17 ([171], Thm. 5.1) *There exists an infinite homogeneous ∂-cable $\hat{s}$ rooted at a, and for any such ∂-cable, $A = k[h,\hat{s}]$. Moreover, this is a minimal generating set for A over k.*

Next, let Ω be the infinite polynomial ring over k and let Δ be the down operator. Let $\Omega[t] = \Omega^{[1]}$ and extend Δ to $\tilde{\Delta}$ to $\Omega[t]$ by $\tilde{\Delta}t = 0$. Let $\hat{s} = (s_n)$ be an infinite ∂-cable rooted at a and consider the surjection $\Omega[t] \to A$ defined by $t \mapsto h$ and $x_n \mapsto s_n$.

Theorem 7.18 ([171], Thm. 7.1) *There exists an ideal $\mathcal{I} = (\hat{\Theta}_4, \hat{\Theta}_6, \hat{\Theta}_8, \dots)$ in $\Omega[t]$ generated by quadratic homogeneous $\tilde{\Delta}$-cables $\hat{\Theta}_n$ such that:*

$$A \cong \Omega[t]/\mathcal{I}$$

The cables $\hat{\Theta}_n$ depend on the choice of ∂-cable $\hat{s}$.

The relations in *Theorem 7.18* are then used to explicitly construct a ∂-cable $\hat{\sigma} = (\sigma_n)$. Let A_a be the localization of A at a, and define a sequence $\sigma_n \in A_a$ as follows:

$$\sigma_0 = a\ ,\ \sigma_1 = av - x\ ,\ \sigma_2 = \tfrac{1}{2}(av^2 - 2xv + 2a^2y)$$

and:

$$\sigma_3 = \tfrac{1}{6}(av^3 - 3xv^2 + 6a^2yv - 6a^4z)$$

Given $n \geq 4$, suppose that $\sigma_0, \dots, \sigma_{n-1}$ are known.

1. If $n \equiv 2, 4 \pmod 6$, then: $\sigma_n = \dfrac{1}{2a} \displaystyle\sum_{i=1}^{n-1} (-1)^{i+1} \sigma_i \sigma_{n-i}$

2. If $n \equiv 3, 5 \pmod 6$, then: $\sigma_n = \dfrac{1}{na} \displaystyle\sum_{i=1}^{n-1} (-1)^{i} i \sigma_i \sigma_{n-i}$

3. If $n \equiv 0 \pmod 6$, then:

$$\sigma_n = \frac{1}{n(n+1)a} \sum_{i=1}^{n-1} (-1)^{i+1} \left(3i(i-1) - n(n-2)\right) \sigma_i \sigma_{n-i}$$

4. If $n \equiv 1 \pmod 6$, then:

$$\sigma_n = \frac{1}{n(n-1)a} \sum_{i=1}^{n-1} (-1)^{i} \left((i-1)(i-2) - (n-1)(n-3)\right) i \sigma_i \sigma_{n-i}$$

Theorem 7.19 ([171], Thm. 7.6) *$\sigma_n \in A$ for each $n \geq 0$ and $\hat{\sigma} = (\sigma_n)$ is a ∂-cable rooted at a.*

For example, the next three σ_n are given by:

$$\begin{aligned}
4!\sigma_4 &= av^4 - 4xv^3 + 12a^2yv^2 - 24a^4zv + (24a^3xz - 12a^3y^2) \\
5!\sigma_5 &= av^5 - 5xv^4 + 20a^2yv^3 - 60a^4zv^2 + (120a^3xz - 60a^3y^2)v \\
&\quad + (24a^5yz - 72x^2a^2z + 36xa^2y^2) \\
6!\sigma_6 &= av^6 - 6xv^5 + 30a^2yv^4 - 120a^4zv^3 + (360a^3xz - 180a^3y^2)v^2 \\
&\quad + (144a^5yz - 432a^2x^2z + 216a^2xy^2)v \\
&\quad - \tfrac{1}{7}(720a^7z^2 - 432a^4xyz + 360a^4y^3 - 864ax^3z + 432ax^2y^2)
\end{aligned}$$

7.7 Proof for A'Campo-Neuen's Example

We can now give a proof that the fixed ring of A'Campo-Neuen's linear action of $\mathbb{G}_a^{12}$ on $\mathbb{A}^{19}$ is not finitely generated. The matrix form of this action was given in *Sect. 6.4.3*. A'Campo-Neuen's proof is quite elegant, and we follow it here, emphasizing the role of commuting locally nilpotent derivations.

Let $Q = k[w, x, y, z] = k^{[4]}$. The proof uses (repeatedly) the fact that, given $f \in k[x, y, z]$, the triangular Q-derivation of $Q[\lambda, \mu] = Q^{[2]}$ defined by

$$\theta = w\partial_\lambda + f(x, y, z)\partial_\mu$$

has $\ker\theta = Q[w\mu - f\lambda] = Q^{[1]}$. This equality is an immediate consequence of the results of *Chap. 4*. Let $\mathcal{B} = Q[\mathbf{s}, \mathbf{t}, \mathbf{u}, \mathbf{v}] = k^{[19]}$, where:

$$\mathbf{x} = (x, y, z) \ , \ \mathbf{s} = (s_1, s_2, s_3) \ , \ \mathbf{t} = (t_1, t_2, t_3)$$
$$\mathbf{u} = (u_1, u_2, u_3) \ , \ \mathbf{v} = (v_1, v_2, v_3, v_4, v_5, v_6)$$

Define commuting linear derivations $D_1, \dots, D_{12} \in \mathrm{LND}_Q(\mathcal{B})$ as follows:

$$\begin{array}{lll} D_1 = w\partial_{v_5} + z\partial_{v_6} & D_2 = w\partial_{v_4} + z\partial_{v_5} & D_3 = w\partial_{v_3} + y\partial_{v_4} \\ D_4 = w\partial_{v_2} + y\partial_{v_3} & D_5 = w\partial_{v_1} + x\partial_{v_2} & D_6 = w\partial_{u_2} + z\partial_{u_3} \\ D_7 = w\partial_{u_1} + z\partial_{u_2} & D_8 = w\partial_{t_2} + y\partial_{t_3} & D_9 = w\partial_{t_1} + y\partial_{t_2} \\ D_{10} = w\partial_{s_2} + x\partial_{s_3} & D_{11} = w\partial_{s_1} + x\partial_{s_2} & \end{array}$$

and

$$D_{12} = x\partial_{s_1} + y\partial_{t_1} + z\partial_{u_1} + x\partial_{v_1} \ .$$

Given i, $1 \le i \le 12$, let $\mathcal{B}_i = (\cdots((\mathcal{B}^{D_1})^{D_2})\cdots)^{D_i}$. Then successive use of the kernel calculation above shows that:

$$\begin{aligned}
\mathcal{B}_1 &= k^{[18]} = Q[\mathbf{s}, \mathbf{t}, \mathbf{u}, v_1, v_2, v_3, v_4, v_5'] \ , \ \text{where } v_5' = zv_5 - wv_6 \\
\mathcal{B}_2 &= k^{[17]} = Q[\mathbf{s}, \mathbf{t}, \mathbf{u}, v_1, v_2, v_3, v_4'] \ , \ \text{where } v_4' = z^2v_4 - wv_5' \\
\mathcal{B}_3 &= k^{[16]} = Q[\mathbf{s}, \mathbf{t}, \mathbf{u}, v_1, v_2, v_3'] \ , \ \text{where } v_3' = yz^2v_3 - wv_4' \\
\mathcal{B}_4 &= k^{[15]} = Q[\mathbf{s}, \mathbf{t}, \mathbf{u}, v_1, v_2'] \ , \ \text{where } v_2' = y^2z^2v_2 - wv_3' \\
\mathcal{B}_5 &= k^{[14]} = Q[\mathbf{s}, \mathbf{t}, \mathbf{u}, V] \ , \ \text{where } V = xy^2z^2v_1 - wv_2' \\
\mathcal{B}_6 &= k^{[13]} = Q[\mathbf{s}, \mathbf{t}, u_1, u_2', V] \ , \ \text{where } u_2' = zu_2 - wu_3 \\
\mathcal{B}_7 &= k^{[12]} = Q[\mathbf{s}, \mathbf{t}, U, V] \ , \ \text{where } U = z^2u_1 - wu_2' \\
\mathcal{B}_8 &= k^{[11]} = Q[\mathbf{s}, t_1, t_2', U, V] \ , \ \text{where } t_2' = yt_2 - wt_3 \\
\mathcal{B}_9 &= k^{[10]} = Q[\mathbf{s}, T, U, V] \ , \ \text{where } T = y^2t_1 - wt_2' \\
\mathcal{B}_{10} &= k^{[9]} = Q[s_1, s_2', T, U, V] \ , \ \text{where } s_2' = xs_2 - ws_3 \\
\mathcal{B}_{11} &= k^{[8]} = Q[S, T, U, V] \ , \ \text{where } S = x^2s_1 - ws_2'
\end{aligned}$$

Finally, the effect of D_{12} on $\mathcal{B}_{11}$ is

$$D_{12}S = x^3 \ , \ D_{12}T = y^3 \ , \ D_{12}U = z^3 \ , \ D_{12}V = (xyz)^2$$

and this is just the Roberts derivation $\mathcal{D}_{(3,2)}$ on $R = k^{[7]}$ extended to $k^{[8]}$ by $D_{12}(w) = 0$. Therefore, $\mathcal{B}_{12} = R^{\mathcal{D}_{(3,2)}}[w] = (R^{\mathcal{D}_{(3,2)}})^{[1]}$, and this implies $\mathcal{B}_{12}$ is not finitely generated. $\square$

Remark 7.20 Tanimoto [404] gives the following generalization of A'Campo-Neuen's method. For $n \geq 2$, let δ be an elementary monomial derivation of $k^{[n]}$ which does not have a slice. Then there exist integers m, N and a linear representation of $G = \mathbb{G}_a^m$ on $X = \mathbb{A}^N$ such that $k[X]^G \cong (\ker\delta)^{[1]}$ (Cor. 1.4 of [404]).

7.8 Proof for Linear Example in Dimension Eleven

This section gives a proof of *Theorem 6.11*.

Let $A = k[x, s, t, u, v, z] = k^{[6]}$, and define the derivation d on A by:

$$dx = 0\ ,\ ds = x^2\ ,\ dt = s\ ,\ du = t\ ,\ dv = u\ ,\ dz = x\ .$$

This derivation corresponds to the $\mathbb{G}_a$-action on $\mathbb{A}^6$ defined in *Theorem 7.12* for the case $n = N = 4$, $\lambda = 2$ and $\mu = 1$. The theorem implies that A^d is not finitely generated. Recall from *Sect. 6.4.4* that:

$$B = k[w, x, s_1, s_2, t_1, t_2, u_1, u_2, v_1, v_2, z] = k^{[11]}$$

The common ring of constants for the commuting derivations T_1, T_2, T_3, T_4 of B is:

$$(((B^{T_4})^{T_3})^{T_2})^{T_1} = k[w, x, xs_1 + ws_2, xt_1 + wt_2, xu_1 + wu_2, xv_1 + wv_2, z] \cong k^{[7]}$$

This is due to the simple fact that, for any base ring R, if γ is the R-derivation of $R[X, Y] = R^{[2]}$ defined by $\gamma X = a$ and $\gamma Y = -b$ for some $a, b \in R$ (not both 0), then $R[X, Y]^\gamma = R[bX + aY] \cong R^{[1]}$. This fact is an easy consequence of the results in *Chap. 4*. Applying this four times in succession gives the equality above.

Recall from *Sect. 6.4.4* that Θ is a fifth linear triangular derivation of B which semi-commutes with the T_i. In particular, in the Lie algebra of k-derivations of B, we have relations:

$$[T_1, \Theta] = T_2\ ,\quad [T_2, \Theta] = T_3\ ,\quad [T_3, \Theta] = T_4\ ,\quad [T_4, \Theta] = 0 \tag{7.9}$$

Let $\mathfrak{g}$ be the Lie algebra $k[T_1, T_2, T_3, T_4, \Theta]$ and let $\mathfrak{h} \subset \mathfrak{g}$ be the subalgebra $k[T_1, T_2, T_3, T_4]$. noting that $\operatorname{ad}\Theta$ is the down operator on $\mathfrak{h}$. The group $G = \exp\mathfrak{g}$ acts linearly on $\mathbb{A}^{11}$. Let $H \subset G$ denote the subgroup $H = \exp\mathfrak{h} \cong \mathbb{G}_a^4$. Then H is normal in G and $G \cong \mathbb{G}_a^4 \rtimes \mathbb{G}_a$.

It follows that Θ restricts to the subring $B^H = k^{[7]}$, and is given by:

$$\Theta w = 0$$
$$\Theta x = 0$$
$$\Theta(xs_1 + ws_2) = x^2$$
$$\Theta(xt_1 + wt_2) = xs_1 + ws_2$$

$\Theta(xu_1 + wu_2) = xt_1 + wt_2$
$\Theta(xv_1 + wv_2) = xu_1 + wu_2$
$\Theta z = x$

This is the same as the derivation d extended to the ring $A[w] = A^{[1]}$ by $dw = 0$. Therefore,

$$B^G = (B^H)^\Theta \cong A[w]^d = A^d[w] \cong (A^d)^{[1]}$$

and this ring is not finitely generated over k. $\square$

Remark 7.21 Using *Theorem 7.12*, the same reasoning yields, for each integer $r \geq 4$, a linear action of the group $\mathbb{G}_a^r \rtimes \mathbb{G}_a$ on $\mathbb{A}^{2r+3}$ for which the ring of invariants is not finitely generated.

7.9 Locally Trivial Examples

Working over the field $k = \mathbb{C}$, Deveney and Finston [109] show how to use known counterexamples to the Fourteenth Problem in order to construct a class of locally trivial $\mathbb{G}_a$-actions on factorial affine varieties Y such that $k[Y]^{\mathbb{G}_a}$ is not finitely generated. Similarly, Jorgenson [221] used Rees's counterexample to the Zariski Problem to construct an example of a normal affine variety X of dimension 4 which admits a locally trivial $\mathbb{G}_a$-action such that $k[X]^{\mathbb{G}_a}$ is not finitely generated. His paper includes the following result.

Theorem 7.22 (Thm. 3.1 of [221]) *Let A be a normal affine k-domain, where k is an algebraically closed field of characteristic 0. Let L be a field such that $k \subset L \subset$ frac(A), and set $R = L \cap A$. Then there exists a normal affine $\mathbb{G}_a$-variety $X \subset \mathbb{A}^n$ (for some $n \geq 1$) such that the $\mathbb{G}_a$-action on X is locally trivial and $R = k[X]^{\mathbb{G}_a}$.*

7.10 Some Positive Results

To date, the following question is open.

Is the kernel of every locally nilpotent derivation of $k^{[4]}$ finitely generated?

Maubach showed that the answer is positive for triangular monomial derivations of $k^{[4]}$ [287]. Then Daigle and Freudenburg showed that the kernel of any triangular derivation of $k^{[4]}$ is of finite type. More generally, their paper includes the following.

Theorem 7.23 ([81], Thm. 1.1) *Let k be an algebraically closed field of characteristic zero, and let R be a k-affine Dedekind domain or a localization of such a ring. The kernel of any triangular R-derivation of $R[x, y, z]$ is finitely generated as an R-algebra.*
The proof of this result relies on Sathaye's generalized theory of Newton-Puiseux expansions, in particular, a certain property of polynomials given in [371].

In [25], Daigle and Bhatwadekar showed that the triangular condition can be dropped in the statement of this theorem. Let R be a commutative noetherian domain containing $\mathbb{Q}$. Given an integer $n \geq 0$, define the condition:

$FG(n)$: $\forall\, D \in \mathrm{LND}_R(R^{[n]})$, $\ker D$ is finitely generated as an R-algebra

Note that $R = k^{[2]}$ does not satisfy $FG(3)$; see *Theorem 7.15*.

Theorem 7.24 ([25], Thm. 4.15) *R satisfies $FG(3)$ if and only if R is a Dedekind domain or a field.*
They further show that, if R is a commutative noetherian $\mathbb{Q}$-domain which is not a field, then R does not satisfy $FG(n)$ for $n \geq 4$ (Cor. 4.16). We observe:

Corollary 7.25 *If $D \in LND(k^{[4]})$ and $rank(D) \leq 3$, then $\ker D$ is finitely generated over k.*

Finite generation notwithstanding, the kernel of a trianglar derivation of $k^{[4]}$ is generally very complicated. In Daigle and Freudenburg [80], the authors show the following.

Theorem 7.26 *For each integer $n \geq 3$, there exists a triangular derivation of $k^{[4]}$ whose kernel, though finitely generated, cannot be generated by fewer than n elements.*
The construction of such derivations is rather involved, and the reader is referred to the article for details.

Following are two other positive results concerning special classes of derivations.

Theorem 7.27 (van den Essen and Janssen [144]) *Let D be an elementary derivation of $B = k[x_1, \ldots, x_n]$ for which $Dx_1 = \cdots = Dx_i = 0$ and $Dx_j \in k[x_1, \ldots, x_i]$ for $j > i$.*

(a) *If either $i \leq 2$ or $n - i \leq 2$ then $\ker D$ is finitely generated.*
(b) *If $1 \in (DB)$, then $\ker D$ is a polynomial ring.*

Theorem 7.28 (Khoury [245]) *For $n \leq 6$, the kernel of every elementary monomial derivation of $k^{[6]}$ is generated by at most 6 elements.*

In his thesis [290], Maubach asked the following question, which is still open at this writing. Define a monomial derivation D on $B = k[x, y, z, u, v]$ by

$$D = x\partial_y + y\partial_z + z\partial_u + u^2\partial_v \ .$$

Is the kernel of D finitely generated?

7.11 Winkelmann's Theorem

We have seen that, in general, the ring of invariants of an algebraic group acting on an affine variety need not be the coordinate ring of an affine variety. However, a fundamental result of Winkelmann shows that, if the invariant ring is normal, then it is quasi-affine, that is, isomorphic to the coordinate ring of a Zariski-open subset of an affine variety.

Theorem 7.29 ([422], Thm. 1) *Let K be a field and R a commutative K-domain integrally closed in its field of fractions. The following properties are equivalent.*

1. *There exists a reduced irreducible quasi-affine K-variety U such that $R \cong K[U]$.*
2. *There exists a reduced irreducible affine K-variety V and a regular $\mathbb{G}_a$-action on V such that $R \cong K[V]^{\mathbb{G}_a}$.*
3. *There exists a reduced irreducible K-variety W and a subgroup $G \subset Aut_K(W)$ such that $R \cong K[W]^G$.*

To illustrate this theorem, Winkelmann considers the dimension-five counterexample $D_{(3,2)}$ defined and discussed in this chapter (see *Theorem 7.14*). Let $V \subset \mathbb{A}^6$ be the affine subvariety defined by points $(w_1, \dots, w_6)$ such that:

$$w_3 = w_2w_4 - w_1w_5 \quad \text{(a coordinate hypersurface)} \quad \text{and} \quad w_1^2w_6 = w_2^3 + w_3^2$$

Then the set of singularities $\mathrm{Sing}(V)$ of V is defined by $w_1 = w_2 = w_3 = 0$, and $\ker D_{(3,2)} \cong k[V \setminus \mathrm{Sing}(V)]$ (Lemma 12).

Appendix: Nagata's Problem Two

Nagata posed the following question in his 1959 paper [320] as "Problem 2".

> Let K be a subfield of the field $k(x_1, \dots, x_n)$ such that tr. $\deg_{.k}K = 3$. Is $K \cap k[x_1, \dots, x_n]$ always finitely generated?

The example in Nagata's paper has tr. $\deg_{.k}K = 4$, as does the example of Steinberg (Thm. 1.2 of [392]). Also, the kernels of the derivations $D_{(r,s)}$ in *Sect. 7.14* have tr. $\deg_k(\ker D_{(r,s)}) = 4$. On the other hand, Rees's counterexample to Zariski's Problem has fixed ring of transcendence degree three; see *Sect. 6.4*.

Nagata's question was answered in the negative by Kuroda in [261, 263]. Kuroda's first counterexamples are subfields of $k^{(4)}$. Let γ and δ_{ij} be integers $(1 \le i \le 3, 1 \le j \le 4)$ such that $\gamma \ge 1$ and:

$$\delta_{ij} \ge 1 \text{ if } 1 \le j \le 3 \quad \text{and} \quad \delta_{i,4} \ge 0 \text{ if } 1 \le i \le 3$$

Let $k(\Pi)$ denote the subfield of $k(\mathbf{x}) = k(x_1, x_2, x_3, x_4) = k^{(4)}$ generated by:

$$\Pi_1 = x_4^{\gamma} - x_1^{-\delta_{1,1}} x_2^{\delta_{1,2}} x_3^{\delta_{1,3}} x_4^{\delta_{1,4}}$$

$$\Pi_2 = x_4^{\gamma} - x_1^{\delta_{2,1}} x_2^{-\delta_{2,2}} x_3^{\delta_{2,3}} x_4^{\delta_{2,4}}$$

$$\Pi_2 = x_4^{\gamma} - x_1^{\delta_{3,1}} x_2^{\delta_{3,2}} x_3^{-\delta_{3,3}} x_4^{\delta_{3,4}}$$

Let $k[\mathbf{x}] = k[x_1, x_2, x_3, x_4]$.

Theorem 7.30 (Kuroda [261], Thm 1.1) *If*

$$\frac{\delta_{1,1}}{\delta_{1,1} + \min\{\delta_{2,1}, \delta_{3,1}\}} + \frac{\delta_{2,2}}{\delta_{2,2} + \min\{\delta_{3,2}, \delta_{1,2}\}} + \frac{\delta_{3,3}}{\delta_{3,3} + \min\{\delta_{1,3}, \delta_{2,3}\}} < 1$$

then $k(\Pi) \cap k[\mathbf{x}]$ is not finitely generated over k.
Kuroda also shows that $k(\Pi) \cap k[\mathbf{x}]$ cannot be the kernel of any locally nilpotent derivation of $k[\mathbf{x}]$.

Nonetheless, there does exist $D \in \mathrm{Der}_k(k[\mathbf{x}])$ with $\ker D = k(\Pi) \cap k[\mathbf{x}]$. For the simplest symmetric example, take $k(\Pi) = k(f, g, h)$ for:

$$f = x_4 - x_1^{-1} x_2^3 x_3^3 \ , \ \ g = x_4 - x_1^3 x_2^{-1} x_3^3 \ , \ \ h = x_4 - x_1^3 x_2^3 x_3^{-1}$$

The jacobian derivation $\Delta_{(f,g,h)} \in \mathrm{Der}_k(k(\mathbf{x}))$ restricts to $k[\mathbf{x}]$, namely, $\Delta_{(f,g,h)} = 4x_1x_2x_3D$, where:

$$Dx_i = x_i(5x_i^4 - \varphi) \qquad \text{for } \varphi = x_1^4 + x_2^4 + x_3^4 \quad (1 \le i \le 3)$$
$$Dx_4 = -20(x_1x_2x_3)^3$$

That $\ker D = k(\Pi) \cap k[\mathbf{x}]$ can be proved using [264].

Kuroda's second family of examples have members which are subfields L of $K = k(x_1, x_2, x_3) = k^{(3)}$, i.e., K is an algebraic extension of L, but $L \cap k[x_1, x_2, x_3]$ is not finitely generated. These appear in [263]. By the result proved in *section "Appendix 1: Finite Group Actions"* of *Chap. 6* together with the Finiteness Theorem, it follows that $L \cap k[x_1, x_2, x_3]$ cannot be the ring of invariants of any algebraic group action on $\mathbb{A}^3$.

Given positive integers γ and δ_{ij} $(i, j = 1, 2)$, let $k(H)$ denote the subfield of K generated by:

$$H_1 = x_1^{\delta_{2,1}} x_2^{-\delta_{2,2}} - x_1^{-\delta_{1,1}} x_2^{\delta_{1,2}}$$

$$H_2 = x_3^{\gamma} - x_1^{-\delta_{1,1}} x_2^{\delta_{1,2}}$$

$$H_3 = 2x_1^{\delta_{2,1}-\delta_{1,1}} x_2^{\delta_{1,2}-\delta_{2,2}} - x_1^{-2\delta_{1,1}} x_2^{2\delta_{1,2}}$$

Theorem 7.31 (Kuroda [263], Thm. 1.1) *If*

$$\frac{\delta_{1,1}}{\delta_{1,1}+\delta_{2,1}}+\frac{\delta_{2,2}}{\delta_{2,2}+\delta_{1,2}}<\frac{1}{2}$$

then $K(H)\cap k[x_1,x_2,x_3]$ *is not finitely generated over k.*
Kuroda uses the theory of locally nilpotent derivations in his proofs. See also [265, 266]. These examples bear study in the effort to decide whether kernels of locally nilpotent derivations of $k^{[4]}$ are finitely generated.

Chapter 8
Algorithms

We have seen that the invariant ring of a $\mathbb{G}_a$-action on an affine variety need not be finitely generated as a k-algebra. But in many cases, most notably in the linear case, the invariant ring is known to be finitely generated, and in these cases it is desirable to have effective means of calculating invariants. In this chapter, we consider constructive invariant theory for $\mathbb{G}_a$-actions.

The main idea of classical invariant theory was to produce invariants and determine their syzygies using a fundamental theorem of Gordan. Any algorithm to construct a finite generating set for a ring of invariants must incorporate these two ingredients, namely, a method to construct new invariants from a given set of invariants, and a method to recognize whether, at any given step, the invariants so constructed generate the ring of invariants.

For $\mathbb{G}_a$-actions, one standard method for constructing invariants exploits the fact that, if A is the kernel of a locally nilpotent derivation D of the k-domain B, then A is factorially closed in B. If $x, f_1, \dots, f_m \in A$ and $P(f_1, \dots, f_m) = xh$ for some $P \in k^{[m]}$ and $h \in B$, then $h \in A$. Thus, one gets new invariants from a given set of invariants by considering their ideal of relations modulo x. This procedure was understood and used in the Nineteenth Century; see [368], §192, and [351], §15.2. Another method for constructing invariants used in the Nineteenth Century was the symbolic method. The tables in *section "Appendix 2: Generators for A_5 and A_6"* of *Chap. 6* express covariants of the binary quintic and sextic symbolically using classical transvectants.

In the modern era, the explicit determination of the rings of invariants for basic $\mathbb{G}_a$-actions was first taken up by Fauntleroy (1977) [148], who considered these actions over an algebraically closed field (in any characteristic). However, it was later shown by Tan (1989) that "the finite sets claimed to be generating sets in [Fauntleroy's paper] are not generating sets"[400]. Tan's paper gives an algorithm for calculating generators for the rings of invariants of the basic $\mathbb{G}_a$-actions, again in the case k is an algebraically closed field. Tan then illustrates his algorithm by calculating several examples.

G. Freudenburg, *Algebraic Theory of Locally Nilpotent Derivations*,
Encyclopaedia of Mathematical Sciences 136,
DOI 10.1007/978-3-662-55350-3_8

For a field of characterstic 0, Cerezo (1988) gave an algorithm to find invariants for any $\mathbb{G}_a$-module V [49], which he describes as follows:

> First a basis of the field of invariant rational functions is given; then enough invariants are computed to separate the separable orbits, and we can describe the stratification; finally, the system obtained is completed by an integral closure argument. (From the Abstract)

Bedratyuk (2010) proposed an algorithm based on classical theory to calculate the kernel of any linear derivation of a polynomial ring [19].

Based on Tan's ideas, van den Essen (1993) developed an algorithm to calculate rings of invariants for more general $\mathbb{G}_a$-actions [141]. For any field k of characteristic zero, and for any finitely generated commutative k-domain B, the algorithm of van den Essen calculates $\ker D$ for any $D \in \mathrm{LND}(B)$, under the assumption that $\ker D$ is finitely generated. Thus, the algorithm already provides a method to calculate a set of generators for the ring of invariants of a linear $\mathbb{G}_a$-action on $\mathbb{A}^n$ in the characteristic zero case, even when the underlying field is not algebraically closed. Van den Essen's algorithm relies on the theory and computation of Gröbner bases. The algorithm can also be used to show finite generation, since termination of the algorithm in a finite number of steps means that a generating set has been calculated, and this will be a finite set.

Despite their merits, these algorithms, in their current forms, lack the efficiency needed to be computationally feasible and effective in higher dimensions. One difficulty is that the algorithms do not predict the number of steps required to calculate a given kernel, making them impractical in certain cases. For example, using van den Essen's algorithm, it can happen that a finitely generated kernel can be computed easily by *ad hoc* methods, whereas the algorithm runs for several days on a computer algebra system without reaching a conclusion in attempting to calculate the same kernel. Such an example is the homogeneous $(2, 5)$ example in dimension three, which was discussed in *Chap. 5*. The difficulty is that, in order to capture all relations in a set of invariants, one typically needs Buchberger's algorithm, and the ensuing calculations can blow up even for small cases.

In an effort to address some of these difficulties, Maubach (2001) found an algorithm to compute generators of the kernel of any k-derivation of the polynomial ring $k[x_1, \dots, x_n]$ up to a certain predetermined degree bound. It is based on the idea that a homogeneous derivation D restricts to a vector space mapping on the subspace of forms of fixed degree d. Then one can calculate the kernel of each such restriction by linear algebra, rather than using Gröbner bases. Of course, one cannot be sure to get all kernel elements in this way. But in his thesis [290] of 2003, Maubach points out that one can use an abbreviated form of the van den Essen algorithm (which he calls the Kernel-Check Algorithm) to see whether a given set of kernel elements is a generating set. He describes the situation as follows.

> The major drawback of the Essen algorithm is that in practice it is not very fast for most locally nilpotent derivations. The major drawback of the homogeneous algorithm is that it cannot answer the question if found generators are sufficient. However, if we use the homogeneous algorithm to compute generators, and then use the kernel-check algorithm . . . to decide if these actually generate the whole kernel, then generally this is a fast way (p. 42).

Maubach discusses an example for which he made calculations using the MAGMA computer algebra system. His algorithm calculated generators up to certain degree within 22 s, and an additional 2 s were used in the Kernel-Check Algorithm to verify that these generated the entire kernel. Applying only the van den Essen algorithm used 65 min.

In addition to the kernel algorithm, van den Essen gives in his book [142] two additional algorithms related to locally nilpotent derivations. The first is the Image Membership Algorithm. Assuming again that B is a finitely generated commutative k-domain, $D \in \mathrm{LND}(B)$, and $\ker D$ is finitely generated, this algorithm decides whether a given element $a \in B$ belongs to the image DB, and if so gives $b \in B$ with $Db = a$. The second is the Extendibility Algorithm. For the polynomial ring $B = k[x_1, \ldots, x_n]$, this gives a way of deciding, by means of locally nilpotent derivations, whether a given set of polynomials $f_1, \ldots, f_{n-1} \in B$ forms a partial system of variables, i.e., whether there exists $f_n \in B$ such that $B = k[f_1, \ldots, f_{n-1}, f_n]$. If a positive conclusion is reached, then the image algorithm can be used to find f_n. These two algorithms are discussed in *Sects.* 8.2 and 8.5 below.

In [367], de Salas gives an algorithm for computing invariants for unipotent group actions over an algebraically closed field, but only relative to subvarieties where the quotient exists. His description is the following.

> The general theory of invariants is reduced, firstly, to cases where G is either geometrically reductive or unipotent. Furthermore, computation of invariants by the action of a smooth, connected group is reduced to computation of the invariants by the action of the additive group. Indeed, if $B \subset G$ is a Borel subgroup and E is a linear representation of G, then it is known that $E^G = E^B$ and hence computation of the invariants is reduced to the case of a solvable group. It is therefore enough to give a method for computing invariants for the additive group $\mathbb{G}_a$ and the multiplicative group $\mathbb{G}_m$. However the latter is very simple and one is therefore reduced to the case of the additive group $\mathbb{G}_a$. (From the Introduction)

In his 2015 thesis [5], Alhajjar gives an algorithm to calculate the degree modules $\mathcal{F}_n$ of a locally nilpotent derivation of an affine k-domain B, employing what he terms a "twisted embedding technique". *Section* 8.6 below introduces an algorithm to calculate degree modules which is very different than the one given by Alhajjar, one which is modeled on the van den Essen Kernel Algorithm. Once $\mathcal{F}_n$ is known, the Generating Principle can be applied to determine if $B = k[\mathcal{F}_n]$ (*Principle* 15). So the algorithm also gives a way to calculate a set of algebra generators for a non-rigid affine k-domain B, and a way to find the canonical factorization of the associated quotient morphism.

Several examples using these techniques are given in *Sect.* 8.7.

8.1 van den Essen's Kernel Algorithm

Van den Essen's algorithm to compute the kernel of a locally nilpotent derivation applies to affine rings where the kernel of the derivation is also known to be affine. We follow van den Essen's exposition of the algorithm, found in §1.4 of his book [142].

8.1.1 Description of the Algorithm

Suppose $B = k[b_1, \dots, b_n]$, a finitely generated commutative k-domain. Let $D \in \mathrm{LND}(B)$ be nonzero, and assume that $A := \ker D$ is known to be finitely generated. Select a local slice $r \in B$, and set $f = Dr$. Then we know that $B_f = A_f[r] = A_f^{[1]}$. Moreover, if D_f denotes the extension of D to B_f, then $D_f(s) = 1$ for $s = r/f$, and by the Slice Theorem,

$$\ker D_f = A_f = k[\pi_s(b_1), \dots, \pi_s(b_n), 1/f]$$

where π_s is the Dixmier map on B_f. Choose $e_i \geq 0$ so that $f^{e_i}\pi_s(b_i) \in A$. Then A is an algebraic extension of the subring

$$A_0 := k[f^{e_1}\pi_s(b_1), \dots, f^{e_n}\pi_s(b_n), f]$$

and $A_0 \subset A \subset A_0[1/f]$.

Given $m \geq 1$, suppose that subrings $A_0 \subset A_1 \subset \cdots \subset A_{m-1} \subset A$ are known. Define:

$$A_m = \{h \in B \,|\, fh \in A_{m-1}\}$$

Since every $h \in A_m$ is algebraic over A, it follows that $A_m \subset A$.

Theorem 8.1 *In the notation above:*

(a) *A_m is a finitely generated k-subalgebra of A for every $m \geq 0$.*
(b) *A is finitely generated if and only if $A = A_m$ for some m.*
(c) *If $A_m = A_{m+1}$ for some $m \geq 0$, then $A = A_m$.*

Proof Part (a) is proved by induction on m, the case $m = 0$ being clear. Given $m \geq 1$ assume $A_{m-1} = k[g_1, \dots, g_l]$. Let J be the ideal in $k[Y] = k[Y_1, \dots, Y_l] = k^{[l]}$ of polynomials P such that $P(g_1, \dots, g_l) \in fB$. By the Hilbert Basis Theorem, there exist $P_1, \dots, P_s \in k[Y]$ such that $J = (P_1, \dots, P_s)$. Let $h_1, \dots, h_s$ be elements of A for which $P_i(g_1, \dots, g_l) = fh_i$; then $A_{m-1}[h_1, \dots, h_s] \subset A_m$, and we wish to see that the reverse inclusion also holds. But this is clear: If h is a generator of A_m, then there exists $F \in J$ with $F(g_1, \dots, g_l) = fh$. Choosing $Q_1, \dots, Q_s \in k[Y]$ with $F = \sum Q_iP_i$, we have $fh = \sum Q_i(g_1, \dots, g_l)fh_i$, implying $h = \sum Q_i(g_1, \dots, g_l)h_i$. So (a) is proved.

To prove (b), suppose $A = k[t_1, \dots, t_N]$. Since $A \subset A_0[1/f]$, there exists a non-negative integer m such that $\{f^m t_1, f^m t_2, \dots, f^m t_N\} \subset A_0$. For every j, $1 \leq j \leq N$, we see that $f^{m-1}t_j \in A_1, f^{m-2}t_j \in A_2$, and so on, until finally we arrive at $t_j \in A_m$. Therefore, $A \subset A_m$, meaning $A = A_m$.

To prove (c), assume $A_m = A_{m+1}$. If $h \in A_{m+2}$, then $fh \in A_{m+1} = A_m$, so in fact $A_{m+2} = A_{m+1}$. By induction, we have that $A_M = A_m$ for all $M \geq m$. Since every element of A belongs to A_M for some $M \geq 0$, the conclusion of (c) follows. □

This result provides the theoretical basis for the **van den Essen Kernel Algorithm**. The algorithm is carried out in the following three steps.

1. Use the Dixmier map to write down the initial subring A_0.
2. Given A_m for $m \geq 0$, calculate A_{m+1}.
3. Decide if $A_{m+1} = A_m$. If so, stop; if not, repeat Step 2 for A_{m+1}.

Step 2 is achieved by calculating a set of generators $P_1, \ldots, P_s$ of the ideal $J \subset k[Y_1, \ldots, Y_l]$. Suppose $A_m = k[g_1, \ldots, g_l]$, and let $\bar{B}$ denote B modulo fB. If $\bar{g_i}$ is the residue class of g_i in $\bar{B}$, then:

$$J = \{ P \in k[Y] \mid P(\bar{g_1}, \ldots, \bar{g_l}) = \bar{0}\}$$

Then one can use standard Gröbner basis calculations to find generators of J; see, for example the Relation Algorithm in Appendix C of van den Essen's book. Once we find $J = (P_1, \ldots, P_s)$, we have that $P_i(g_1, \ldots, g_l) = fh_i$ for some $h_i \in B$. Since B is a domain, the h_i are uniquely determined. If B is a polynomial ring, we have that $h_i = f^{-1}P_i(g_1, \ldots, g_l)$. In more general rings, one can again use Gröbner bases in order to calculate the h_i explicitly; see p. 39 of van den Essen's book for details.

As for Step 3, one uses another standard Gröbner basis calculation known as the Membership Algorithm. This algorithm will decide whether the generator h_i of A_{m+1} belongs to the subalgebra A_m, and if so, it computes a polynomial $Q_i \in k[Y]$ so that $h_i = Q_i(g_1, \ldots, g_l)$. The Membership Algorithm is described in Appendix C of van den Essen's book.

8.1.2 Kernel Check Algorithm

As remarked by Maubach, "One of the great strengths of the [van den Essen] algorithm is to be able to determine if one has sufficient generators" ([290], p. 32). Maubach gives an abbreviated version of the algorithm, to be used for checking a given finite set of kernel elements, and calls this the **Kernel-Check Algorithm**. Its output is simply yes or no, depending on whether or not the given set generates the kernel over k. The algorithm proceeds in the following two steps: $B = k[b_1, \ldots, b_n]$, $D \in \mathrm{LND}(B)$, and $f \in \ker D$ are as above. A set $\{f_1, \ldots, f_m\} \subset \ker D$ is given.

1. Find generators $P_1, \ldots, P_s$ for the ideal

$$J = \{ P \in k^{[m]} \mid P(f_1, \ldots, f_m) \in fB\} \ .$$

2. If $f^{-1}P_i(f_1, \ldots, f_m) \in k[f_1, \ldots, f_m]$ for each i, then $\ker D = k[f_1, \ldots, f_m]$, and the output is yes; otherwise the output is no.

8.1.3 Generalized van den Essen Kernel Algorithm

The initial ring in van den Essen's Kernel Algorithm is:

$$A_0 := k[f^{e_1}\pi_s(b_1), \dots, f^{e_n}\pi_s(b_n), f]$$

However, the reader will note that the only properties of this ring used in the remainder of the algorithm is that it is affine and that A is algebraic over A_0. Therefore, one may replace A_0 in the algorithm by any affine subring $R \subset A$ such that A is algebraic over R. We call this the **Generalized van den Essen Kernel Algorithm**.

It is often less computationally expensive to use a transcendence basis for A other than the one which generates A_0, as this requires calculating images of the Dixmier map. For example, Tan's algorithm for finding kernels of the basic linear derivations uses Stroh's transcendence basis for these kernels. Examples which use this simplified algorithm are included in *Sect. 8.7* below.

8.2 Image Membership Algorithm

Based on the Kernel Algorithm, the Image Membership Algorithm decides whether $a \in DB$ for given $D \in \mathrm{LND}(B)$ and $a \in B$. In addition, if $a \in DB$, then the algorithm gives $b \in B$ for which $Db = a$. Following is a brief description of the algorithm.

Continuing the notation above, we assume $D \in \mathrm{LND}(B)$ is given, and $A = \ker D$ is finitely generated. Suppose $a \in B$ is given. We continue to assume r is a local slice, $Dr = f$, and $s = r/f$. Then $B_f = A_f[s]$ and $D = d/ds$ on B_f. The degree of a as a polynomial in s equals $m := \deg_D(a)$. By integration, there exists $\beta \in B_f$ of degree $m+1$ such that $D\beta = a$. Thus, $g := f^{m+1}\beta \in B$ and $Dg = f^{m+1}a$. For the sake of computations, van den Essen gives the explicit formula:

$$\beta = \sum_{0 \le i \le m} \frac{(-1)^i}{(i+1)!} D^i(a) s^{i+1}$$

Since A is finitely generated, it can be represented as $A = k[X_1, \dots, X_N]/I$ for some positive integer N, where $k[X_1, \dots, X_N] = k^{[N]}$, and $I \subset k[X_1, \dots, X_N]$ is an ideal. If $A = k[f_1, \dots, f_l]$ has been computed, then I can be found by the Relation Algorithm. Suppose $I = (H_1, \dots, H_t)$, and suppose $F, F_i, G \in k[X_1, \dots, X_N]$ are such that $\bar{F} = f$, $\bar{F}_i = f_i$, and $\bar{G} = g$, where $\bar{(\cdot)}$ denotes congruence class modulo I. Let $J \subset k[X_1, \dots, X_N, Y_1, \dots, Y_N] = k^{[2N]}$ be the ideal:

$$J = (Y_1 - F_1, \dots, Y_l - F_l, F^{m+1}, H_1, \dots, H_t)$$

Finally, we let $\tilde{G}$ be the normal form of G relative to an appropriately chosen Gröbner basis of J.

Proposition 8.2 (1.4.15 of [142]) *In the above notation, $a \in DB$ if and only if $\tilde{G} \in k[Y_1, \ldots, Y_N]$. In this case, the element $b := (g - \tilde{G}((f_1, \ldots, f_l))/f^{m+1}$ satisfies $Db = a$.*
For proofs and further details about the algorithm, the reader is referred to van den Essen [142], Sect. 1.4.

8.3 Criteria for a Derivation to be Locally Nilpotent

In both the Kernel Algorithm and Image Algorithm, one uses $\deg_D(q)$ for elements $q \in B$, but there is no way to predict how large this might be for arbitrary choice of q. Motivated by this consideration, van den Essen [141] posed the **Recognition Problem**:

> Let B be a finitely generated k-algebra and $D \in \mathrm{Der}_k(B)$. Give an algorithm to decide if D is locally nilpotent.

As a step towards achieving this, he gives the following; see [141] and [142], 1.4.17.

Proposition 8.3 (Partial Nilpotency Criterion) *Let $B = k[b_1, \ldots, b_n]$, and suppose $D \in Der_k(B)$ has the property that the transcendence degree of B over $\ker D$ equals 1. Given a transcendence basis $x_1, \ldots, x_{t-1}$ of $\ker D$, set*

$$N = \max_i \{[frac(B) : k(x_1, \ldots, x_{t-1}, b_i)] | Db_i \neq 0\}$$

which is finite. Then D is locally nilpotent if and only if $D^{N+1}(b_i) = 0$ for every i.
The proof is a direct application of *Corollary 1.30*. The key to using this criterion is to find the $t-1$ algebraically independent kernel elements.

In the case $B = k[x, y] = k^{[2]}$, van den Essen gives a complete solution to the Recognition Problem.

Proposition 8.4 (1.3.52 of [142]) *Let $B = k[x, y]$, and let $D \in Der_k(B)$ be given, $D \neq 0$. Set:*

$$d = \max\{\deg_x(Dx), \deg_x(Dy), \deg_y(Dx), \deg_y(Dy)\}$$

Then D is locally nilpotent if and only if $D^{d+2}x = D^{d+2}y = 0$.

Proof If the generators x and y belong to $\mathrm{Nil}(D)$, then D is locally nilpotent (*Principle 2*).

Conversely, suppose D is locally nilpotent. We may assume that D is irreducible, since the value of d will only increase when D is multiplied by a kernel element, whereas $\deg_D(x)$ and $\deg_D(y)$ will not change.

By Rentschler's Theorem, there exist $P, Q \in B$ such that $B = k[P, Q]$, $\ker D = k[P]$, and $DQ \in k[P]$. Since D is irreducible, we may write $D = P_y\partial_x - P_x\partial_y$ (*Corollary 4.6*). Observe that:

$$\deg_y P = [k(x,y) : k(x,P)] = [k(P,Q) : k(x,P)] = \deg_Q x = \deg_D(x)$$

and

$$\deg_x P = [k(x,y) : k(P,y)] = [k(P,Q) : k(P,y)] = \deg_Q y = \deg_D(y)$$

In each case, the last equality is due to the fact that Q is a local slice. Thus:

$$\begin{aligned}
&\deg_x(Dx) = \deg_x(P_y) \le \deg_x(P) = \deg_D(y)\\
&\deg_y(Dx) = \deg_y(P_y) = \deg_y(P) - 1 = \deg_D(x) - 1\\
&\deg_x(Dy) = \deg_x(P_x) = \deg_x(P) - 1 = \deg_D(y) - 1\\
&\deg_y(Dy) = \deg_y(P_x) \le \deg_y(P) = \deg_D(x)
\end{aligned}$$

The desired result now follows from the definition of $\deg_D$. □

For homogeneous derivations in dimension 3, a similar bound was given by Holtackers.

Proposition 8.5 ([211], Thm. 3.1) *Let $B = k[x,y,z] = k^{[3]}$ and let $D \in \mathrm{Der}_k(B)$ be standard homogeneous, $D \neq 0$. Let e denote the greatest integer $e \le \frac{1}{4}(\deg D + 3)^2 + 1$. Then D is locally nilpotent if and only if $D^{e+1}x = D^{e+1}y = D^{e+1}z = 0$.*

Proof As above, if the generators x, y, and z belong to $\mathrm{Nil}(D)$, then D is locally nilpotent.

Conversely, assume D is locally nilpotent, and that $\ker D = k[f,g]$ for homogeneous f and g. It is no loss of generality to assume D is irreducible. By the Jacobian Formula, we may also assume that $D = \Delta_{(f,g)}$.

A general formula for field extensions is:

$$[k(x_1,\ldots,x_n) : k(F_1,\ldots,F_n)] \le (\deg F_1)\cdots(\deg F_n)$$

(See [142], Prop. B.2.7, and [211], Prop. 1.12.) Set:

$$N = [k(x,y,z) : k(f,g,x)] \le (\deg f)(\deg g)$$

By the Partial Nilpotency Criterion, $D^{N+1}x = 0$.

Since f and g are homogeneous,

$$\deg D + 1 = \deg(Dx) = \deg(f_y g_z - f_z g_y) = \deg f + \deg g - 2 \le d$$

where $d = \max\{\deg Dx, \deg Dy, \deg Dz\}$. Thus, setting $a = \deg f$ and $b = \deg g$, we wish to maximize the quantity $ab+1$ subject to the condition $a+b = \deg D + 3$. Viewing $ab+1$ as quadratic in a, the result follows immediately. □

In fact, Holtackers gives a more general formula, namely, for the weighted-homogeneous case.

Note that, in the general dimension 3 case, the obstruction is that the degrees $\deg(f), \deg(g)$ are not always uniformly bounded by a polynomial function of d, where:

$$d = \max\{\deg(f_y g_z - f_z g_y)\,,\ \deg(f_x g_z - f_z g_x)\,,\ \deg(f_x g_y - f_y g_x)\}$$

8.4 Maubach's Algorithm

This algorithm calculates generators of the kernel of a homogeneous derivation up to a certain predetermined degree. The derivation involved need not be locally nilpotent. In addition, Maubach describes a procedure for using the homogeneous algorithm to calculate kernel elements of a non-homogeneous derivation. This is accomplished by homogenizing the given derivation. In the present treatment, we will content ourselves with a brief description of these ideas. The reader is referred to [288, 290] for the proofs.

8.4.1 Homogeneous Algorithm

Let B be a commutative k-domain graded by the additive semi-group $I = \mathbb{N}^q$ for some $q \geq 1$. In addition, if $B = \oplus_{\alpha \in I} B_\alpha$, suppose that each B_α is a vector space of finite dimension over k, and $B_0 = k$.

For elements of I, declare $\beta \leq \alpha$ in I if and only if $\beta_i \leq \alpha_i$ for each i, $1 \leq i \leq q$. Likewise, declare $\beta < \alpha$ if and only if $\beta \leq \alpha$ and $\beta \neq \alpha$. Define:

$$B_{\leq\alpha} = \sum_{\beta \leq \alpha} B_\beta \quad \text{and} \quad B_{<\alpha} = \sum_{\beta < \alpha} B_\beta$$

Let $D \in \mathrm{Der}_k(B)$ be homogeneous relative to this grading, $D \neq 0$. Given $\alpha \in I$, we say that a subset $\mathcal{S} = \{F_1, \dots, F_s\} \subset B_{\leq\alpha}$ is a **good set for** α (relative to D) if it satisfies:

(1) each $F_i \in B_\beta$ for some $\beta \leq \alpha$
(2) $k[\mathcal{S}] \cap B_{\leq\alpha} = \ker D \cap B_{\leq\alpha}$
(3) $F_i \notin k[F_1, \dots, \hat{F}_i, \dots, F_s]$ for each i

Likewise, $\mathcal{S}$ is a **good set for** $< \alpha$ when:

(1)′ each $F_i \in B_\beta$ for some $\beta < \alpha$
(2)′ $k[\mathcal{S}] \cap B_{<\alpha} = \ker D \cap B_{<\alpha}$
(3)′ $F_i \notin k[F_1, \dots, \hat{F}_i, \dots, F_s]$ for each i

Note that, since $A_0 = k$, a good set for 0 is the empty set.

Given a degree bound α, the algorithm calculates finite sets $\mathcal{S}_\beta \subset B_\beta$ such that their union gives a good set for α. The main tool is the following induction step, which, given a good set $\mathcal{S}$ for $< \alpha$, calculates a set $\mathcal{S}_\alpha \subset B_\alpha$ such that $\mathcal{S}_\alpha \cup \mathcal{S}$ is a good set for α; this will be the set of "kernel generators up to degree α", in the sense of condition (2) for good sets above.

Proposition 8.6 (3.17 and 3.18 of [290]) *Let $\alpha \in I$ be given.*

(a) *Suppose $\{\mathcal{S}_\beta \mid \beta < \alpha\}$ is a collection of sets such that, for each $\beta < \alpha$, $\cup_{\gamma \le \beta} \mathcal{S}_\gamma$ is a good set for β. Then $\cup_{\beta < \alpha} \mathcal{S}_\beta$ is a good set for $< \alpha$.*

(b) *Suppose $\{\mathcal{S}_\beta \mid \beta < \alpha\}$ is a collection of sets such that, for each $\beta < \alpha$, $\cup_{\beta < \alpha} \mathcal{S}_\beta$ is a good set for $< \alpha$. Then a finite set $\mathcal{S}_\alpha \subset B_\alpha$ such that $\cup_{\beta \le \alpha} \mathcal{S}_\beta$ is a good set for α can be constructed.*

By this result, a good set for α can be calculated inductively from the empty set, which is a good set for 0.

The key step is to construct the set $\mathcal{S}_\alpha$ in part (b) of the proposition. To this end, let $A = \ker D$ and write $A = \oplus_{\alpha \in I} A_\alpha$. Let $\mathcal{S} = \cup_{\beta < \alpha} \mathcal{S}_\beta$, as in part (b), which is a good set for $< \alpha$. If $\mathcal{S} = \{F_1, \ldots, F_s\}$, then

$$k[\mathcal{S}] \cap A_\alpha = \sum_{e \in E} k \cdot F_1^{e_1} \cdots F_s^{e_s}$$

where $E = \{e \in \mathbb{N}^s | F_1^{e_1} \cdots F_s^{e_s} \in A_\alpha\}$. Choose a subset $J \subset E$ so that the set $\mathcal{B} = \{F_1^{e_1} \cdots F_s^{e_s} | e \in J\}$ is a basis for the vector space $k[\mathcal{S}] \cap A_\alpha$. Then $\mathcal{B}$ is in the kernel K of the linear map $D : B_\alpha \to B_{\alpha+\delta}$ (where δ is the degree of D), and we take $\mathcal{S}_\alpha$ to be the completion of $\mathcal{B}$ to a basis of K.

In addition, Maubach shows:

Proposition 8.7 (3.1.10 of [290]) *Suppose that the preceding algorithm produces the set $\mathcal{S} = \{F_1, \ldots, F_s\}$ which is a good set for α ($\alpha \in I$), and assume that it is verified that $\ker D = k[\mathcal{S}]$. If also $\ker D = k[g_1, \ldots, g_t]$, then $s \le t$.*

8.4.2 Application to Non-homogeneous Derivations

Next, Maubach shows how to apply the homogeneous algorithm to non-homogeneous derivations in the case B is a polynomial ring.

Suppose $B = k[x_1, \ldots, x_n] = k^{[n]}$ and $B[w] = k^{[n+1]}$. Recall from *Chap. 3* that every $D \in \mathrm{Der}_k(B)$ admits a homogenization $D^H \in \mathrm{Der}_k(B[w])$, and that if $p : B[w] \to B$ is the evaluation map $p(f(w)) = f(1)$, then $p(\ker D^H) = \ker D$. So the idea is simply to apply the homogeneous algorithm to D^H to produce a subset $\mathcal{S} \subset \ker(D^H)$ of generators up to a certain degree. Then $p(\mathcal{S}) \subset \ker D$. If $k[\mathcal{S}] = \ker(D^H)$, then $k[p(\mathcal{S})] = \ker D$.

Maubach points out one pitfall of this approach: It may happen that $\ker(D^H)$ is not finitely generated, but $\ker D$ is finitely generated. He gives the following example.

Define D on $k[x,s,t,u,v] = k^{[5]}$ by:

$$D = \partial_s + (sx^2)\partial_t + (tx^2)\partial_u + x\partial_v$$

Since D has a slice s, $\ker D$ is finitely generated. On the other hand, D^H on $k[w,x,s,t,u,v]$ is given by:

$$D^H = w^3\partial_s + (sx^2)\partial_t + (tx^2)\partial_u + (xw^2)\partial_v$$

and $\ker(D^H)$ is not finitely generated. This is because $D^H \equiv D_{(3,2)}$ modulo $(x-1)$, where $D_{(3,2)}$ is the triangular derivation of $k[w,s,t,u,v]$ defined by

$$D_{(3,2)} = w^3\partial_s + s\partial_t + t\partial_u + w^2\partial_v$$

and it was shown in *Chap.* 7 that the kernel of $D_{(3,2)}$ is not finitely generated.

8.5 Extendibility Algorithm

The next algorithm is also due to van den Essen, and addresses the **Extendibility Problem**:

> Let $B = k[x_1,\dots,x_n] = k^{[n]}$, and suppose $f_1,\dots,f_{n-1} \in B$. Give an algorithm to decide if $(f_1,\dots,f_{n-1})$ constitutes a partial system of coordinates, i.e., if there exists $f_n \in B$ such that $(f_1,\dots,f_n) \in GA_n(k)$; and if so, find f_n. (3.2.10 of [142])

Set $\mathbf{f} = (f_1,\dots,f_{n-1})$. If $f_1,\dots,f_{n-1}$ can be extended to an automorphism $(f_1,\dots,f_n)$, then $\Delta_{\mathbf{f}} \in \mathrm{LND}(B)$, where:

$$\Delta_{\mathbf{f}}(f_n) \in k^* \quad \text{and} \quad \ker \Delta_{\mathbf{f}} = k[f_1,\dots,f_{n-1}]$$

In other words, $\Delta_{\mathbf{f}}$ is a partial derivative in the appropriate coordinate system. By the Slice Theorem, the converse is also true: If $\Delta_{\mathbf{f}}$ is locally nilpotent and admits a slice s, and if $\ker \Delta_{\mathbf{f}} = k[f_1,\dots,f_{n-1}]$, then $B = k[f_1,\dots,f_{n-1},s]$. This equivalence is the basis of the **extendibility algorithm**, described here in four steps.

1. Check that the f_i are algebraically independent, which amounts to verifying that at least one image $\Delta_{\mathbf{f}}(x_i)$ is nonzero.
2. If $\Delta_{\mathbf{f}} \neq 0$, use the Partial Nilpotency Criterion to see if $\Delta_{\mathbf{f}}$ is locally nilpotent. This requires calculating each degree

$$[k(x_1,\dots,x_n) : k(f_1,\dots,f_{n-1},x_i)]$$

such that $\Delta_{\mathbf{f}} x_i \neq 0$. (Van den Essen indicates how to calculate these in the case k is algebraically closed.)

3. If $\Delta_{\mathbf{f}}$ is locally nilpotent, check whether $\Delta_{\mathbf{f}}$ has a slice: Use the image membership algorithm to see if 1 belongs to the image of $\Delta_{\mathbf{f}}$. If so, the membership algorithm will produce a slice s, and by the Slice Theorem, $B = \ker \Delta_{\mathbf{f}}[s]$.
4. Finally, decide whether $\ker \Delta_{\mathbf{f}} = k[f_1, \ldots, f_{n-1}]$ by the Kernel-Check Algorithm. Alternatively, use 3.2.1 of van den Essen [142] to check whether $(f_1, \ldots, f_{n-1}, s)$ is an automorphism of B.

8.6 Algorithm to Compute Degree Modules

Let B be a commutative k-domain, $D \in \mathrm{LND}(B)$ non-zero and $A = \ker D$, and let $\{\mathcal{F}_n\}_{n \geq 0}$ be the degree modules of B defined by A. Fix a local slice $r \in B$ and integer $n \geq 1$, and set $f = Dr \in A$. For each $n \geq 0$, define the free A-submodule $\mathcal{G}_n(r) \subset \mathcal{F}_n$ by:

$$\mathcal{G}_n(r) = A + Ar + \cdots Ar^n$$

Suppose that M_0 is an A-submodule of B such that $\mathcal{G}_n(r) \subset M_0 \subset \mathcal{F}_n$. Inductively, define the ascending chain $M_0 \subset M_1 \subset M_2 \subset \cdots$ of A-submodules of B by:

$$M_i = \{h \in B \,|\, fh \in M_{i-1}\} = \{h \in B \,|\, f^i h \in M_0\} \quad (i \geq 1)$$

Then $f M_{i+1} \subset M_i \subset M_{i+1} \subset \mathcal{F}_n$ for each $i \geq 0$, since $\mathcal{F}_n$ is factorially closed.

Lemma 8.8 $\mathcal{F}_n = \cup_{i \geq 0} M_i$

Proof It must be shown that, to each $h \in \mathcal{F}_n$, there exists $s \geq 0$ such that $h \in M_s$. Let $s \geq 0$ be such that $f^s h \in A[r]$. Then:

$$f^s h \in A[r] \cap \mathcal{F}_n = \mathcal{G}_n(r) \subset M_0$$

By definition of the modules M_i, we see that $h \in M_s$, and the lemma is proved. □

Theorem 8.9 *The following conditions are equivalent.*

1. $fB \cap M_s = fM_s$ *for some* $s \geq 0$.
2. $M_s = M_{s+1}$ *for some* $s \geq 0$.
3. *The ascending chain* $M_0 \subset M_1 \subset M_2 \subset \cdots$ *stabilizes.*
4. $\mathcal{F}_n = M_s$ *for some* $s \geq 0$.

If A is a noetherian ring, then these conditions are valid.

Proof (1) $\Leftrightarrow$ (2): This follows by definition of the modules M_i.

(2) $\Rightarrow$ (3): Assume that $M_s = M_{s+1}$ for some $s \geq 0$. If $h \in M_{s+2}$, then:

$$fh \in M_{s+1} = M_s \quad \Rightarrow \quad h \in M_{s+1} = M_s$$

Therefore, $M_s = M_{s+2}$. By induction, we obtain that $M_s = M_S$ for all $S \geq s$.

(3) $\Rightarrow$ (4): Assume that, for some $s \geq 0$, $M_s = M_S$ for all $S \geq s$. By *Lemma 8.8*, it follows that $\mathcal{F}_n = \cup_{i \geq 0} M_i = M_s$.

(4) $\Rightarrow$ (2): Assume that $\mathcal{F}_n = M_s$ for some $s \geq 0$. Then $M_{s+1} \subset \mathcal{F}_n = M_s \subset M_{s+1}$ implies that $M_s = M_{s+1}$.

We have thus shown that conditions (1)–(4) are equivalent. Assume that A is a noetherian ring. By *Theorem 1.47*, there exists a finite module basis $\{z_1, \dots, z_t\}$ for $\mathcal{F}_n$, where $t \geq 1$. By *Lemma 8.8*, there exists $s \geq 0$ such that $\{z_1, \dots, z_t\} \subset M_s$. Therefore, $\mathcal{F}_n = M_s$, and condition (4) is validated. □

Theorem 8.9 gives the theoretical basis for an algorithm to calculate the degree modules $\mathcal{F}_n$ in the case where A is noetherian. Suppose that $\{x_1, \dots, x_m\}$ is a module basis for M_i and let $\{X_1, \dots, X_m\}$ be a basis for the free A-module of rank m. Define $\rho : A^m \to M_i$ by $\rho(X_j) = x_j$. Then $K := \rho^{-1}(fB \cap M_i)$ is a submodule of A^m. Since A is noetherian, K is finitely generated. A basis for K can be calculated by standard methods; see for example [134], Chap. 15. Suppose that $\{Y_1, \dots, Y_l\}$ is a module basis for K, and let $s_1, \dots, s_l \in B$ be such that $\rho(Y_j) = fs_j$. Then $M_{i+1} = M_i + As_1 + \cdots + As_l$.

8.7 Examples

The purpose of this section is to illustrate the use of the techniques outlined in this chapter via several examples, with emphasis on the van den Essen algorithm applied to locally nilpotent derivations of polynomial rings. We begin with some of the basic linear derivations, and here the generalized kernel algorithm reduces to Tan's algorithm.

8.7.1 Basic Linear Derivations

Let $\Omega = k[x_0, x_1, x_2, \dots]$ be the infinite polynomial ring, let Δ be the down operator on Ω and let $A = \ker \Delta$. As in *Sect. 6.3*, we let $B_n = k[x_0, \dots, x_n] = k^{[n+1]}$ and $A_n = A \cap B_n$. Recall from *Sect. 6.3* that the Stroh transcendence basis for A_n over $k(x_0)$ is given by:

$$T_2 = [x_2, x_2]_2^{\Delta} \ , \ T_3 = [x_1x_2, x_3]_3^{\Delta} \ , \ T_4 = [x_4, x_4]_4^{\Delta} \ ,$$
$$T_5 = [x_1x_4, x_5]_5^{\Delta} \ , \cdots , \ T_n = [x_1x_{n-1}, x_n]_n^{\Delta} \text{ or } [x_n, x_n]_n^{\Delta}$$

the latter depending on whether n is odd or even, respectively; see *Sect. 2.11.1*. Set $T_1 = x_0$. The first few polynomials T_i are as follows.

$T_1 = x_0$
$T_2 = 2x_0x_2 - x_1^2$
$T_3 = 3x_0^2x_3 - 3x_0x_1x_2 + x_1^3$

$$T_4 = 2x_0x_4 - 2x_1x_3 + x_2^2$$
$$T_5 = 5x_0^2x_5 - 5x_0x_1x_4 + x_0x_2x_3 + 2x_1^2x_3 - x_1x_2^2$$
$$T_6 = 2x_0x_6 - 2x_1x_5 + 2x_2x_4 - x_3^2$$

Over the field $k(x_0)$, it is easy to see that $T_2, \dots, T_n$ is a triangular system of variables for $k(x_0)[x_1, \dots, x_n] = k(x_0)^{[n]}$. Therefore, $\mathrm{frac}(A_n) = k(T_1, \dots, T_n) = k^{(n)}$.

Let $n \geq 2$ be given. In order to calculate A_n, we assume that A_{n-1} is known, beginning with $A_1 = k[T_1]$. For the first step of the generalized kernel algorithm, let $(A_n)_0 = A_{n-1}[T_n]$. We consider algebraic relations modulo x_0, where $\Delta x_1 = x_0$. In these examples, the notation $\bar{f}$ for $f \in B$ will denote the image of f in $B_n/x_0B_n = k[x_1, \dots, x_n]$.

Example 8.10 For $n = 2$ we have $(A_2)_0 = A_1[T_2] = k[T_1, T_2]$, where $\bar{T}_2 = -x_1^2$. Since $\bar{T}_2$ is transcendental over k, we conclude that:

$$A_2 = (A_2)_0 = k[T_1, T_2]$$

Example 8.11 For $n = 3$ we have $(A_3)_0 = A_2[T_3] = k[T_1, T_2, T_3]$, where $\bar{T}_3 = x_1^3$. Therefore, $T_2^3 + T_3^2 \in x_0A$. We find that $T_2^3 + T_3^2 = x_0^2h$ for $h \in A$ given by:

$$h = 9x_0^2x_3^2 - 18x_0x_1x_2x_3 + 8x_0x_2^3 + 6x_1^3x_3 - 3x_1^2x_2^2$$

Since $\bar{h}$ is transcendental over $k[\bar{T}_2, \bar{T}_3]$, we conclude that:

$$A_3 = (A_3)_1 = k[T_1, T_2, T_3, h]$$

Note that A_3 is not a polynomial ring. Geometrically, it is the coordinate ring of the singular hypersurface $X^2U - Z^2 - Y^3 = 0$ in $\mathbb{A}^4$.

Example 8.12 For $n = 4$ we have $(A_4)_0 = A_3[T_4] = k[T_1, T_2, T_3, h, T_4]$, where $\bar{T}_4 = -2x_1x_3 + x_2^2$ and $\bar{h} = 3x_1^2(2x_1x_3 - x_2^2)$. The ideal of relations modulo x_0 among these five polynomials is given by $(X^3 + Y^2, 3XW - Z)$. Therefore, there exists $r \in A_4$ such that $3T_2T_4 - h = x_0r$ and:

$$(A_4)_1 = (A_4)_0[r] = k[T_1, T_2, T_3, T_4, r]$$

In particular, we find that:

$$r = 12x_0x_2x_4 - 6x_1^2x_4 - 9x_0x_3^2 + 6x_1x_2x_3 - 2x_2^3$$

Since $\bar{r}$ is transcendental over $k[\bar{T}_2, \bar{T}_3, \bar{T}_4]$, we conclude that:

$$A_4 = (A_4)_1 = k[T_1, T_2, T_3, T_4, r]$$

We see from the relations above that:

$$A_4 \cong k[X_1, X_2, X_3, X_4, X_5]/(X_2^3 + X_3^2 - X_1^2(3X_2X_4 - X_1X_5))$$

See Onoda [338].

Remark 8.13 For the basic action of $\mathbb{G}_a$ on $\mathbb{A}^4$, Tan [400] shows that in characteristic 0 and 3, the invariant ring is minimally generated by four polynomials, whereas in characteristic 2, the invariant ring is 3-generated, i.e., is a polynomial ring. The book of Grosshans [188] includes a calculation of A_n for $n \leq 4$ (pp. 56–58). For an alternate calculation of A_4, see Nowicki [333] (Example 6.8.4).

Any attempt to calculate A_n for $n \geq 5$ directly using the Tan or van den Essen algorithm is unlikely to succeed, given the size of the generators and complexity of their relations. In particular, the required Gröbner basis calculations become too large. Cerezo and Bedratyuk applied their algorithms to calculate A_n for $n \leq 5$ and $n \leq 6$, respectively, thus confirming the results of Gordan from the Nineteenth Century [19, 47]. Olive, and Lercier and Olive, used a modified version of Gordan's algorithm to calculate A_6 and A_8, and A_9 and A_{10} [271, 336].

By considering its associated Poincarè series, Onoda showed that A_5 is not a complete intersection ([338], Cor. 3.5). In [121], Drensky and Genov give an algorithm to calculate the Hilbert series of the invariant ring of a linear action of $\mathbb{G}_a$ on a vector space.

8.7.2 Examples in Dimension Four

Recall that every kernel of a locally nilpotent derivation of $k^{[3]}$ is a polynomial ring, though these are not generally coordinate subrings. This is no longer true for $k^{[4]}$, as *Example 8.11* shows. Kernels of triangular derivations of $k^{[4]}$ are always finitely generated but can require an arbitrarily large number of generators (*Corollary 7.25*, *Theorem 7.26*). For a locally nilpotent derivation of $k^{[4]}$ having rank 4, it is not known if the kernel is finitely generated. The third example of this section calculates the kernel of a particular rank-4 locally nilpotent derivation, showing that it is finitely generated and that its coordinate ring is a hypersurface of $\mathbb{A}^4$.

Example 8.14 Let $B = k[x, y, z, u] = k^{[4]}$ and define the triangular derivation T of B by $T := \Delta_{(x,f,g)}$, where $f = xz - y^2$ and $g = xu - yz$. Then y is a local slice with $Ty = x^2$. Let $A = \ker T$ and set $A_0 = k[x, f, g]$. Since f and g are already algebraically independent modulo x, the kernel algorithm terminates after one step:

$$A = A_0 = k[x, f, g] = k^{[3]}$$

Example 8.15 It is shown in [80] that, in dimension 4, there exists for any integer $n \geq 3$ a triangular derivation of $B = k[x, y, z, u] = k^{[4]}$ whose kernel cannot be generated by fewer than n elements. The proof is based on the kernel algorithm.

In the cited paper, an explicit example is constructed whose kernel is generated by seven elements but which cannot be generated by fewer than six elements. The following is a simpler example of the same type of construction.

Let $f = x^2z + xy + y^4$ and $g = x^2u + y^6$. Then $\Delta_{(x,f,g)}$ is a **twin triangular derivation** of $k^{[4]}$, meaning that it restricts to a triangular derivation on both $k[x, y, z]$ and $k[x, y, u]$. In addition, $\Delta_{(x,f,g)}$ is divisible by x^2. If

$$\delta = x^{-2}\Delta_{(x,f,g)}$$

then δ is also twin triangular and $\delta y = x^2$. Let $A = \ker\delta$. Given $b \in B$, let $\bar{b}$ denote the image of b in B/xB.

At the first stage of the kernel algorithm, $A_0 = k[x, f, g]$. Then $\bar{f} = y^4$ and $\bar{g} = y^6$. Thus, $f^3 - g^2 \in xB$, and we set $h = x^{-1}(f^3 - g^2)$ to obtain $A_1 = k[x, f, g, h]$. Direct calculation shows $\bar{h} = 3y^9$. The ideal of relations $J \subset k[X, Y, Z]$ between y^4, y^6, and $3y^9$ is:

$$J = (X^3 - Y^2, 81X^9 - Z^4, 9Y^3 - Z^2) = (X^3 - Y^2, 9Y^3 - Z^2)$$

Set $\ell = x^{-1}(9g^3 - h^2)$, so that $A_2 = k[x, f, g, h, \ell]$. Direct calculation shows that $\bar{\ell} = 12uy^{15} - 18y^{17}z - 18y^{15}$, which is transcendental over $\bar{A}_2$. This implies that the algorithm terminates: $A = k[x, f, g, h, \ell]$. From the relations calculated above, we see that:

$$A \cong k[X, Y, Z, U, V]/(XU + Y^3 + Z^2, XV + Z^3 + U^2)$$

One sees easily that A is not a polynomial ring, and therefore requires at least four generators. We conjecture that A cannot be generated by fewer than five elements, i.e., that the given set of generators is a minimal set. To prove this, one would show first that x can be included in any minimal set of generators; then, according to Lemma 1 of [80], the quotient ring $A/xA = k[y^4, y^6, y^9][t]$ cannot be generated by three elements, where t is an indeterminate over $k[y]$. Note that the only system of integer weights relative to which δ is homogeneous is $(3, 1, -2, 0)$, meaning that the result of Maubach (*Proposition 8.7*) does not apply: The homogeneous algorithm requires non-negative weights.

Example 8.16 Let $B = k[X, Y, Z] = k^{[3]}$ and let D be the $(2, 5)$ derivation, which is defined in *Sect. 5.4*, and let $F, G, R \in B$ be as defined in that section. Then $\ker D = k[F, G]$ and $DR = -FG$, and the plinth ideal of D is $FG \cdot \ker D$.

Let $B[u] = B^{[1]} = k^{[4]}$ and extend D to $\theta \in \mathrm{LND}(B[u])$ by $\theta u = G$. Then θ is homogeneous in the standard $\mathbb{Z}$-grading of $B[u]$ and $\deg\theta = 4$. Set $A = \ker\theta$. Suppose that $v \in B[u]$ is a variable with $\theta v = 0$, and let $v_1 \neq 0$ be its linear summand. By homogeneity, $\theta v_1 = 0$. However, θ does not annihilate any nonzero linear polynomial in x, y, z, u, since their images are linearly independent. Therefore, no such variable v exists, and $\mathrm{rank}(\theta) = 4$. Thus, we have no knowledge, *a priori*, that A is finitely generated. We will show, however, that this is the case, calculating A by *ad hoc* methods combined with the Kernel Check Algorithm.

We have that $G \in A$ is the image of the local slice u. Set $H = R + uF \in A$, where $\deg H = 3$, and observe that:

$$G \equiv x^5 \pmod{F} \quad \text{and} \quad H \equiv x^3 \pmod{F}$$

Therefore, there exists $K \in B[u]$ such that $G^3 - H^5 = FK$, where $\deg K = 13$, and since A is factorially closed in $B[u]$ we have $K \in A$.

Let $\pi : B[u] \to B[u]/GB[u]$ be the standard surjection, noting that:

$$\pi(B[u]) = \pi(B)[\pi(u)] = \pi(B)^{[1]}$$

Consider the subring $\Gamma = k[\pi(F), \pi(H), \pi(K)]$. Since $k[\pi(F)] \subset \pi(B)$ and $\pi(F)\pi(K) + \pi(H)^5 = 0$, we conclude that:

$$\text{tr.deg}_k \Gamma = 2 \tag{8.1}$$

Let $J \subset k[X_1, X_2, X_3] = k^{[3]}$ be the prime ideal:

$$J = \{ P \in k^{[3]} \mid P(F, H, K) \in GB[u] \}$$

Then $\Gamma = k[X_1, X_2, X_3]/J$. From the equality (8.1) we conclude that J is a height-one prime, and therefore principal. Since $X_1X_3 + X_2^5 \in J$ and $(X_1X_3 + X_2^5)$ is a prime ideal, we conclude that $J = (X_1X_3 + X_2^5)$.

It follows that $G^{-3}P(F, H, K) \in k[F, H, K]$ for any $P \in J$. By the Kernel Check Algorithm and the relations above, we conclude that:

$$A = k[F, G, H, K] \cong k[X_1, X_2, X_3, X_4]/(X_1X_3 + X_4^3 + X_2^5)$$

This is called the **homogeneous** $(2, 3, 5)$ **derivation** in dimension four.

Example 8.17 Let $B = k[x_1, x_2, y_1, y_2] = k^{[4]}$ and define $T \in \text{LND}(B)$ by:

$$T = x_1 \frac{\partial}{\partial y_1} + x_2 \frac{\partial}{\partial y_2}$$

Let $A = \ker T$ and define $g \in A$ by $g = x_1y_2 - x_2y_1$. Then y_1 is a local slice with image x_1, and it is immediate from the Kernel Check Algorithm that $A = k[x_1, x_2, g]$. Since y_1 and y_2 are local slices, we see that $B = B_1 = k[\mathcal{F}_1]$. We calculate $\mathcal{F}_1$.

Define $M_0 \subset \mathcal{F}_1$ by $M_0 = A + Ay_1 + Ay_2$. Suppose that $x_1w \in M_0$ for $w \in \mathcal{F}_1$, and write $x_1w = a_0 + a_1y_1 + a_2y_2$ for $a_i \in A$. Then $x_1Dw = a_1x_1 + a_2x_2$, which implies that $a_2 \in x_1B \cap A = x_1A$ and $a_0 + a_1y_1 \in x_1B$.

Let $p : B \to B/x_1B$ be the canonical surjection, let $\bar{b} = p(b)$ for $b \in B$, and let $\bar{A} = p(A) = k[\bar{x}_2, \bar{g}]$, noting that $\bar{g} + \bar{x}_2\bar{y}_1 = 0$. If $\bar{A}X \oplus \bar{A}Y$ is the free $\bar{A}$-module of rank 2, then:

$$\bar{A} + \bar{A}\bar{y}_1 = \bar{A}X \oplus \bar{A}Y/\bar{A}(\bar{g}X + \bar{x}_2Y)$$

Therefore, $\bar{a}_0 + \bar{a}_1\bar{y}_1 = 0$ implies $a_0 + a_1y_1 = \alpha(g + x_2y_1) = \alpha x_1y_2$ for some $\alpha \in A$. So $w \in M_0$ and $x_1B \cap M_0 = x_1M_0$. By *Theorem 8.9* we conclude that $M_0 = \mathcal{F}_1$.

8.7.3 Vector Group Action

The kernel algorithm can also be applied successively to vector group actions.

Example 8.18 Let L be the elementary nilpotent matrix of order 5, which defines a linear transformation of the vector space $V = k^5$. Then $M := L^2$ is also a nilpotent linear transformation of V, and M commutes with L. Let D_L and D_M be the induced locally nilpotent derivations on the symmetric algebra $S(V) = k^{[5]}$ (see *Sect. 3.1.1*). Since D_L and D_M commute, D_M restricts to $S(V)^{D_L}$, and for the associated $\mathbb{G}_a^2$-action on $S(V)$ we have $S(V)^{\mathbb{G}_a^2} = (S(V)^{D_L})^{D_M}$.

Suppose that $S(V) = k[x_0, \dots, x_4]$ and $D_L(x_i) = x_{i-1}$ for $1 \le i \le 4$ and $D_L(x_0) = 0$. Then $S(V)^{D_L} = A_4 = k[T_1, T_2, T_3, T_4, r]$, as calculated in *Example 8.12* above. The action of D_M on $S(V)$ is given by

$$x_4 \to x_2 \to x_0 \to 0 \quad \text{and} \quad x_3 \to x_1 \to 0$$

and it restricts to A_4. Direct calculation shows:

$$D_M(T_2) = 2T_1^2\ ,\ D_M(T_3) = 0\ ,\ D_M(T_4) = 2T_2\ ,\ D_M(r) = 6T_1T_4$$

Define $H \in A_4^{D_M}$ by:

$$H = \tfrac{1}{4}[T_4, T_4]_2^{D_M} = 2T_1^2T_4 - T_2^2$$

Then $\{T_1, T_3, H\}$ is a transcendence basis for $A_4^{D_M}$. Modulo T_1, we see that $T_3 \equiv x_1^3$ and $H \equiv -x_1^4$. Therefore, $T_3^4 + H^3 = T_1K$ for some $K \in S(V)$, and since $A_4^{D_M}$ is factorially closed, $K \in A_4^{D_M}$. It is easily checked (with a computer algebra system) that $K \pmod{T_1}$ is transcendental over $k[T_3, H] \pmod{T_1}$. Therefore:

$$S(V)^{\mathbb{G}_a^2} = A_4^{D_M} = k[T_1, T_3, H, K]$$

8.8 Canonical Factorization for $(2, 5)$ Action

This section works out the canonical factorization of the $\mathbb{G}_a$-action on $\mathbb{A}^3$ induced by the homogeneous $(2, 5)$ derivation D of $k[x, y, z] = k^{[3]}$; see [157]. The first step is to find the degree modules and degree resolution for D. See *Sect. 1.6*.

8.8.1 Homogeneous $(2, 5)$ Derivation

For the standard $\mathbb{Z}$-grading of $B = k[x, y, z]$, the homogeneous elements $F, G, R, S \in B$ are defined by:

$$F = xz - y^2 \ , \quad G = zF^2 + 2x^2yF + x^5 \ , \quad R = x^3 + yF \ , \quad S = x^2y + zF$$

The homogeneous $(2, 5)$ derivation $D \in \mathrm{LND}(B)$ of B has $\ker D = k[F, G]$; see *Sect. 5.4*. Observe the relations

$$F^3 + R^2 = xG \ , \quad x^2R + FS = G \ , \quad xS - yR = F^2$$

and the images

$$DR = -FG \ , \quad Dx = -2FR \ , \quad DS = x(5xG - 4F^3)$$

and:

$$Dy = 6x^2R - G \ , \quad Dz = 2x(5yR + F^2)$$

In particular, R is a local slice of D, $\deg_D x = 2$, $\deg_D S = 5$, $\deg_D y = 6$ and $\deg_D z = 10$.

Let $A = \ker D$ and let $\{\mathcal{F}_n\}_{n \geq 0}$ be the associated degree modules.

8.8.2 Degree Modules $\mathcal{F}_n$

Define the submodule $N_0 = \mathcal{G}_{10}(R) \subset \mathcal{F}_{10}$, that is:

$$N_0 = A + AR + AR^2 + AR^3 + AR^4 + AR^5 + AR^6 + AR^7 + AR^8 + AR^9 + AR^{10}$$

Since $R^2 = xG - F^3$, we see that $N_0 \subset N_1 \subset \mathcal{F}_{10}$, where:

$$N_1 = A + AR + Ax + AxR + Ax^2 + Ax^2R + Ax^3 + Ax^3R + Ax^4 + Ax^4R + Ax^5$$

Since $x^3 = R - yF$, we see that $N_1 \subset N_2 \subset \mathcal{F}_{10}$, where:

$$N_2 = A + AR + Ax + AxR + Ax^2 + Ax^2R + Ay + Ax^3R + Axy + Ax^4R + Ax^2y$$

Since $x^2R = G - FS$, we see that $N_2 \subset N_3 \subset \mathcal{F}_{10}$, where:

$$N_3 = A + AR + Ax + AxR + Ax^2 + AS + Ay + AxS + Axy + Ax^2S + Ax^2y$$

Since $x^2y = S - zF$, we see that $N_3 \subset M \subset \mathcal{F}_{10}$, where:

$$M = A + AR + Ax + AxR + Ax^2 + AS + Ay + AxS + Axy + Ax^2S + Az$$

Note that $F^2M \subset N_1 \subset M$.

Lemma 8.19 $FB \cap M = FM$

Proof Let $p_F : B \to B/FB$ be the canonical surjection. Let $\bar{b} = p_F(b)$ for $b \in B$, $\bar{A} = p_F(A)$, $\bar{M} = p_F(M)$ and $\bar{B} = p_F(B)$. Since $F = xz - y^2$, we see that:

$$k[\bar{x}, \bar{z}] = k^{[2]} \quad \text{and} \quad \bar{B} = k[\bar{x}, \bar{z}] \oplus k[\bar{x}, \bar{z}]\bar{y} \tag{8.2}$$

We have:

$$\bar{G} = \bar{x}^5 \ , \ \bar{R} = \bar{x}^3 \ , \ \bar{S} = \bar{x}^2\bar{y} \ , \ \bar{A} = k[\bar{x}^5]$$

Define:

$$\mathcal{O} = k[\bar{x}] = \bar{A} \oplus \bar{A}\bar{x} + \oplus\bar{A}\bar{x}^2 \oplus \bar{A}\bar{x}^3 \oplus \bar{A}\bar{x}^4 \tag{8.3}$$

Then:

$$\begin{aligned}\bar{M} &= \bar{A} + \bar{A}\bar{x}^3 + \bar{A}\bar{x} + \bar{A}\bar{x}^4 + \bar{A}\bar{x}^2 + \bar{A}\bar{x}^2\bar{y} + \bar{A}\bar{y} + \bar{A}\bar{x}^3\bar{y} + \bar{A}\bar{x}\bar{y} + \bar{A}\bar{x}^4\bar{y} + \bar{A}\bar{z} \\ &= \mathcal{O} + \mathcal{O}\bar{y} + \bar{A}\bar{z}\end{aligned}$$

From (8.2) and (8.3) it follows that $\bar{M}$ is a free $\bar{A}$-module of rank 11. Since $FB \cap A = FA$, we conclude that $FB \cap M = FM$. □

Lemma 8.20 $GB \cap M = GM$

Proof Let $p_G : B \to B/GB$ be the canonical surjection and let $\bar{D}$ be the locally nilpotent derivation of B/GB induced by D. Let $\bar{b} = p_G(b)$ for $b \in B$, $\bar{A} = p_G(A)$ and $\bar{N}_1 = p_G(N_1)$. Then $\bar{A} = k[\bar{F}]$. Define $\mathcal{R} = k[\bar{F}, \bar{R}]$. Since $xG = F^3 + R^2$, we have $\bar{F}^3 + \bar{R}^2 = 0$. In addition, note that $\mathcal{R} \subset \ker \bar{D}$ and $\bar{D}\bar{x} \neq 0$. We thus have:

$$\mathcal{R} = \bar{A} \oplus \bar{A}\bar{R} \quad \text{and} \quad \mathcal{R}[\bar{x}] = \mathcal{R}^{[1]}$$

Therefore,

$$\begin{aligned}\bar{N}_1 &= \bar{A} + \bar{A}\bar{R} + \bar{A}\bar{x} + \bar{A}\bar{x}\bar{R} + \bar{A}\bar{x}^2 \\ &\qquad + \bar{A}\bar{x}^2\bar{R} + \bar{A}\bar{x}^3 + \bar{A}\bar{x}^3\bar{R} + \bar{A}\bar{x}^4 + \bar{A}\bar{x}^4\bar{R} + \bar{A}\bar{x}^5 \\ &= \mathcal{R} \oplus \mathcal{R}\bar{x} \oplus \mathcal{R}\bar{x}^2 \oplus \mathcal{R}\bar{x}^3 \oplus \mathcal{R}\bar{x}^4 \oplus \bar{A}\bar{x}^5\end{aligned}$$

is a free $\bar{A}$-module of rank 11. Since $GB \cap A = GA$, we conclude that:

$$GB \cap N_1 = GN_1 \tag{8.4}$$

Suppose that $Gw \in M$ for some $w \in B$. Then $F^2Gw \in F^2M \subset N_1$, so $F^2Gw \in GB \cap N_1 = GN_1$ by (8.4). Therefore, $F^2w \in N_1 \subset M$, so $F^2w \in F^2B \cap M = F^2M$ by *Lemma 8.19*, and $w \in M$. □

Proposition 8.21 $M = \mathcal{F}_{10}$.

Proof By *Theorem 8.9*, it will suffice to show $FG \cdot B \cap M = FG \cdot M$. This follows immediately from *Lemma 8.19* and *Lemma 8.20*. □

Note that, by degree considerations, $\mathcal{F}_{10}$ is a free A-module of rank 11. Therefore:

$$\begin{aligned}
\mathcal{F}_0 &= A \\
\mathcal{F}_1 &= A + AR \\
\mathcal{F}_2 &= A + AR + Ax \\
\mathcal{F}_3 &= A + AR + Ax + AxR \\
\mathcal{F}_4 &= A + AR + Ax + AxR + Ax^2 \\
\mathcal{F}_5 &= A + AR + Ax + AxR + Ax^2 + AS \\
\mathcal{F}_6 &= A + AR + Ax + AxR + Ax^2 + AS + Ay \\
\mathcal{F}_7 &= A + AR + Ax + AxR + Ax^2 + AS + Ay + AxS \\
\mathcal{F}_8 &= A + AR + Ax + AxR + Ax^2 + AS + Ay + AxS + Axy \\
\mathcal{F}_9 &= A + AR + Ax + AxR + Ax^2 + AS + Ay + AxS + Axy + Ax^2S
\end{aligned}$$

The higher degree modules can now be found from the following.

Theorem 8.22 $B = \bigoplus_{i \geq 0} \mathcal{F}_9 \cdot z^i$. *Consequently, each* $\mathcal{F}_n$ *is a free* A*-module, and* B *is a free* A*-module.*

In order to prove this theorem, define a non-empty set $V \subset B$ to be a D**-set** if the restriction $\deg_D : V \to \mathbb{N} \cup \{-\infty\}$ is injective. A D**-basis** of an A-module is a free basis which is a D-set.

We need two obvious but useful lemmas.

Lemma 8.23 *Let $V \subset B$ be a D-set.*

(a) *The elements of V are linearly independent over A.*
(b) *If $b \in B$ and $\deg_D b > \deg_D v$ for all $v \in V$, then $\bigcup_{i\geq 0} Vb^i$ is a D-set.*

Lemma 8.24 *Let $M \subset B$ be a finitely generated A-module, and let $b \in B$ satisfy $\deg_D b > \deg_D f$ for all $f \in M$. Define $N = \sum_{i\geq 0} Mb^i$.*

(a) *If M is a free A-module, then N is a free A-module.*
(b) *If M admits a D-basis $\{v_1, \dots, v_m\}$, then N admits a D-basis given by:*

$$\{v_i b^j \mid 1 \leq i \leq m\,,\, j \geq 0\}$$

Let $p : B \to B/zB = k[x, y]$ be the standard surjection, and set $\bar{A} = p(A)$ and $\bar{\mathcal{F}}_n = p(\mathcal{F}_n)$.

Lemma 8.25 $\bar{\mathcal{F}}_9 = k[x, y]$ *and* $\bar{\mathcal{F}}_9$ *is a free* $\bar{A}$*-module of rank 10.*

Proof We have $\bar{A} = k[y^2, x^5 + 2x^2y^3]$. We see that $\bar{A} \subset \bar{A}[y]$ is an integral extension of degree 2, and that $\bar{A}[y] \subset k[x, y]$ is an integral extension of degree 5. Therefore, $k[x, y]$ is a free $\bar{A}$-module of rank 10 with basis:

$$\mathcal{B} = \{1, x, x^2, x^3, x^4, y, xy, x^2y, x^3y, x^4y\}$$

Observe that $\mathcal{F}_9$ contains the set

$$\mathcal{C} := \{1, x, x^2, R - yF, x(R - yF), y, xy, S, xS, x^2S\}$$

and that $p(\mathcal{C}) = \mathcal{B}$. Therefore, $\bar{\mathcal{F}}_9 = k[x, y]$. $\square$

Proof of Theorem 8.22 Set $N = \sum_{i\geq 0} \mathcal{F}_9 \cdot z^i$. By *Lemma 8.24*, we see that $N = \bigoplus_{i\geq 0} \mathcal{F}_9 \cdot z^i$. Consider the descending chain of submodules:

$$B \supset N + zB \supset N + z^2B \supset \cdots$$

By *Lemma 8.25*, $B = N + zB$. Since $N + zN = N$, if follows that $N + z^nB = B$ for every $n \geq 0$. Given non-zero $f \in B$, choose an integer $n > \deg_D f$ and write $f = \sum_{0\leq i\leq d} a_i z^i + z^n b$ for $0 \leq d < n$, $a_i \in \mathcal{F}_9$ and $b \in B$. If $b \neq 0$, then, since the degrees of $a_0, a_1 z, \dots, a_d z^d$ are distinct, we see that $\deg_D f = 10n + \deg b$, a contradiction. Therefore, $b = 0$ and $f \in N$. $\square$

The foregoing calculations allow us to calculate the associated graded ring for D:

$$\mathrm{Gr}_D(B) = A[FGt, F^2Gt^2, F^4G^3t^5, F^5G^3t^6, F^8G^5t^{10}] \subset A[t] = A^{[1]}$$

In particular, $\mathrm{Gr}_D(B)$ is finitely generated as a k-algebra.

8.8.3 Degree Resolution

Results above give the degree resolution of B induced by D.

1. $B_0 = \mathcal{F}_0 = A = k[F, G] = k^{[2]}$
2. $B_1 = k[\mathcal{F}_1] = A[R] = k[F, G, R] = k^{[3]}$
3. $B_2 = k[\mathcal{F}_2] = B_1[x] = k[F, G, R, x]$ where $xG = F^3 + R^2$
4. $B_4 = B_3 = B_2$
5. $B_5 = k[\mathcal{F}_5] = B_2[S] = k[F, R, x, S]$ where $F(xS - F^2) = R(R - x^3)$
6. $B_6 = k[\mathcal{F}_6] = B_5[y] = k[F, x, S, y]$ where $x(S - x^2y) = F(F + y^2)$
7. $B_9 = B_8 = B_7 = B_6$
8. $B_{10} = k[\mathcal{F}_{10}] = B_6[z] = B$

We see that $\mathcal{N}_B(A) = \{n_0, \ldots, n_5\} = \{0, 1, 2, 5, 6, 10\}$ and $\text{index}_B(A) = 5$. See *Sect. 1.6*. Observe that B_0, B_1, B_2, B_{10} are UFDs, whereas neither B_5 nor B_6 is a UFD.

8.8.4 Fixed Points

Let $X_i = \text{Spec}(B_{n_i})$ for $i = 0, \ldots, 5$.

1. $X_1^{\mathbb{G}_a} = \mathcal{V}(FG) \subset X_1$, which defines two planes in $\mathbb{A}^3$.
2. $X_2^{\mathbb{G}_a} = \mathcal{V}(F) \subset X_2$, which defines a cone.
3. $X_3^{\mathbb{G}_a} = \mathcal{V}(F, R) \subset X_3$, which defines a plane.
4. $X_4^{\mathbb{G}_a} = \mathcal{V}(F, x) \subset X_4$, which defines a plane.
5. $X_5^{\mathbb{G}_a} = \mathcal{V}(x, y) \subset X_5$, which defines a line in $\mathbb{A}^3$.

8.8.5 Affine Modifications

We describe each ring $B_{n_{i+1}}$ as a $\mathbb{G}_a$-equivariant affine modification of B_{n_i}, $1 \leq i \leq 4$.

1. $B_2 = B_1[G^{-1}J_1]$ for $J_1 = G \cdot B_1 + (F^3 + R^2) \cdot B_1$
2. $B_5 = B_2[F^{-1}J_2]$ for $J_2 = F \cdot B_2 + (G - x^2R) \cdot B_2$
3. $B_6 = B_5[F^{-1}J_5]$ for $J_5 = F \cdot B_5 + (R - x^3) \cdot B_5$
4. $B_{10} = B_6[F^{-1}J_6]$ for $J_6 = F \cdot B_6 + (S - x^2y) \cdot B_6$

8.8.6 Canonical Factorization

Let $X = \operatorname{Spec}(B) = \mathbb{A}^3$ and $Y = \operatorname{Spec}(A) = \mathbb{A}^2$, and let $\pi : X \to Y$ be the quotient morphism. Over points $p \in Y$ defined by $F = \alpha$, $G = \beta$, the fiber $\pi^{-1}(p)$ is a line which is a single orbit if $\alpha, \beta \neq 0$; a union of five lines which are orbits if $\alpha = 0$, $\beta \neq 0$; a union of two lines which are orbits if $\alpha \neq 0$, $\beta = 0$; and a line of fixed points if $\alpha = \beta = 0$.

Let $\pi_i : X_{i+1} \to X_i$ be the morphism induced by the inclusion $B_{n_i} \to B_{n_{i+1}}$, $0 \leq i \leq 4$. The canonical factorization of π is given by:

$$X = X_5 \xrightarrow{\pi_4} X_4 \xrightarrow{\pi_3} X_3 \xrightarrow{\pi_2} X_2 \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0 = Y$$

We consider each mapping π_i individually.

1. For π_0, we have $X_0 = \mathbb{A}^2$, $X_1 = X_0 \times \mathbb{A}^1$ and π_0 is projection on the first factor.
2. For π_1, let $W_1 = \mathcal{V}(G) \subset X_1$ and $W_2 = \mathcal{V}(G) \subset X_2$. Since $(B_1)_G = (B_2)_G$, the mapping
$$\pi_1 : X_2 \setminus W_2 \to X_1 \setminus W_1$$
is an isomorphism. We find that $W_1 = \mathbb{A}^2$ and $\pi_1(W_2) = C$, where C is the cuspidal cubic curve $\mathcal{V}(G, F^3 + R^2)$ in W_1, and that $W_2 = C \times \mathbb{A}^1$, where the restriction of π_1 to W_2 is projection on the first factor. The image of π_1 excludes $W_1 \setminus C$.
3. For π_2, let $V_2 = X_2^{\mathbb{G}_a} = \mathcal{V}(F) \subset X_2$ and $W_3 = \mathcal{V}(F) \subset X_3$. Since $(B_2)_F = (B_5)_F$, the mapping
$$\pi_2 : X_3 \setminus W_3 \to X_2 \setminus V_2$$
is an isomorphism. We find that $\pi_2(W_3) = Z$, where Z is the union of two lines $\mathcal{V}(F, G, R)$ and $\mathcal{V}(F, G - x^5, R - x^3)$ in V_2, and that $W_3 = Z \times \mathbb{A}^1$, where the restriction of π_2 to W_3 is projection on the first factor. The image of π_2 excludes $W_2 \setminus Z$.
4. For π_3, let $V_3 = X_3^{\mathbb{G}_a} = \mathcal{V}(F, R) \subset X_3$ and $V_4 = X_4^{\mathbb{G}_a} = \mathcal{V}(F, x) = \mathcal{V}(F, R) \subset X_4$. Since $(B_5)_F = (B_6)_F$ and $(B_5)_R = (B_6)_R$, the mapping
$$\pi_3 : X_4 \setminus V_4 \to X_3 \setminus V_3$$
is an isomorphism. We find that $\pi_3(V_4) = L$, where L is the line $\mathcal{V}(F, R, x) \subset X_3$, and that $V_4 = L \times \mathbb{A}^1$ (a plane), where the restriction of π_3 to V_4 is projection on the first factor. The image of π_3 excludes $V_3 \setminus L$.

5. For π_4, let $X_5^{\mathbb{G}_a} = \mathcal{V}(F,x) = \mathcal{V}(x,y) \subset X_5$. Since $(B_6)_F = B_F$ and $(B_6)_x = B_x$, the mapping

$$\pi_4 : X_5 \setminus V_5 \to X_4 \setminus V_4$$

is an isomorphism. We find that $\pi_4(V_5) = P$, where P is the point $\mathcal{V}(F,x,S,y)$, and that $V_5 = P \times \mathbb{A}^1$ (a line), where the restriction of π_4 to V_5 is projection on the first factor. The image of π_4 excludes $V_4 \setminus P$.

Observe that the affine modification $\pi_4 : X_5 \to X_4$ differs from the first three in that its exceptional locus is one-dimensional.

8.9 Winkelmann's Example

In [421], Winkelmann gave the first examples of free $\mathbb{G}_a$-actions on affine space which are not translations. We analyze the smallest of these examples, which is in dimension four. See *Sect. 3.8.4*.

Let $B = k[x,y,z,u] = k^{[4]}$, and define $F \in B$ by $F = 2xz - y^2$. Define $D \in \mathrm{LND}(B)$ by:

$$Dx = 0 \ , \ Dy = x \ , \ Dz = y \ , \ Du = F + 1$$

Let $A = \ker D$, and let $\pi : X \to Y$ be the induced quotient morphism, where $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$. Since $xB + yB + (F+1)B = B$, the induced $\mathbb{G}_a$-action on $\mathbb{A}^4$ is fixed-point free.

Let $\mathcal{F}_n$ be the A-module $\mathcal{F}_n = \ker D^{n+1}$ and let $B_n = k[\mathcal{F}_n]$, $n \geq 0$. Since $x,y,z,u \in \mathcal{F}_2$, we see that $B = B_2$, so the degree resolution of A is given by:

$$A = B_0 \subset B_1 \subset B_2 = B$$

Hence, if $X_i = \mathrm{Spec}(B_i)$, then the canonical factorization of π is given by:

$$\mathbb{A}^4 = X = X_2 \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0 = Y$$

8.9.1 *The Plinth Ideal*

Observe the following.

1. Since y and u are local slices, $G := uDy - yDu \in A$.
2. If $T = yu - 2z(F+1)$, then $DT = G$.
3. Since u and T are local slices, $H := uDT - TDu \in A$.

4. The kernel of D is $A = k[x, F, G, H]$. The prime relation for this ring is:

$$xH = G^2 + F(F + 1)$$

Lemma 8.26 *$pl(D) = (x, F + 1, G)$.*

Proof Let $\sigma : B \to B/xB$ be the standard surjection, let $\bar{A} = \sigma(A)$ and let $\bar{H} = \sigma(H)$. Then:

$$\bar{A} = k[y^2, y(y^2 - 1), \bar{H}] = k[y^2, y(y^2 - 1)]^{[1]}$$

Let $J \subset A$ be the ideal $J = (x, F + 1, G)$, and let $f \in I_1$ be given. Since $J \subset I_1$ and $A/J = k[H]$, it suffices to assume $f \in k[H]$.

Write $f = P(H)$ for $P \in k^{[1]}$ and let $L \in B$ be such that $DL = f$. Then:

$$yDL - LDy = yP(H) - xL \in A \quad \Rightarrow \quad yP(\bar{H}) \in \bar{A}$$

If $P(\bar{H}) \neq 0$, choose $\lambda \in k$ so that $P(\lambda) \neq 0$. Then:

$$yP(\lambda) \in k[y^2, y(y^2 - 1)] \quad \Rightarrow \quad k[y^2, y(y^2 - 1)] = k[y]$$

But this is clearly a contradiction. Therefore, $P(\bar{H}) = 0$, which implies $f = P(H) = 0$. Therefore, $J = I_1$. □

8.9.2 The Mapping π_0

Let $W_0 \subset Y$ be defined by $W_0 = \mathrm{Spec}(A/I_1)$. *Lemma 8.26* implies that $A/I_1 = k[H] \cong k^{[1]}$, so $W_0 \cong \mathbb{A}^1$. Let $V = Y \setminus W_0$. Then there is an open set $U \subset X_1$ such that $U = V \times \mathbb{A}^1$ and π_0 is projection on the first factor. In particular, let D_1 be the restriction of D to B_1. Since y, u, T are local slices, we have:

$$(B_1)_x = A_x[y] \ , \quad (B_1)_{F+1} = A_{F+1}[u] \ , \quad (B_1)_G = A_G[T]$$

Therefore, $U = U_x \cup U_{F+1} \cup U_T$, where $U_x = \{x \neq 0\}$, $U_{F+1} = \{F + 1 \neq 0\}$ and $U_T = \{T \neq 0\}$.

Lemma 8.26 further implies that $\mathcal{F}_1 = A + Ay + Au + AT$. Relations in this module are given by:

$$yG - xT = F(F + 1) \ , \ xu - y(F + 1) = G \ , \ uG - (F + 1)T = H$$

The ring B_1 is given by $B_1 = k[x, y, u, F, N]$, where $N = z(F + 1)$. The prime relation in this ring is $2xN = (F + y^2)(F + 1)$.

The closed set $W_1 = X_1 \setminus U$ is the set of fixed points $X_1^{\mathbb{G}_a}$, defined by the ideal:

$$I_1 B_1 = xB_1 + (F+1)B_1$$

We find that $W_1 \cong \mathbb{A}^3$. Therefore, $\pi_0 : W_1 \cong \mathbb{A}^3 \to W_0 \cong \mathbb{A}^1$ and $\pi_0(W_1) = p$, where $p \in Y$ is the point defined by the A-ideal $(x, F+1, G, H)$. Therefore, $\pi_0^{-1}(W_0 \setminus \{p\}) = \emptyset$.

8.9.3 The Mapping π_1

Let $W \subset X = \mathbb{A}^4$ be defined by the ideal $xB + (F+1)B$. Then $W = W_+ \cup W_-$, where $W_+ = \{x = 0, y = 1\} \cong \mathbb{A}^2$ and $W_- = \{x = 0, y = -1\} \cong \mathbb{A}^2$. Since $(B_1)_x = B_x$ and $(B_1)_{F+1} = B_{F+1}$, we see that π_1 maps $X \setminus W$ isomorphically to $X_1 \setminus W_1$.

Using coordinate functions (y, u, N) on $W_1 = \mathbb{A}^3$, we find that $\pi_1(W_+) = L_+ \cong \mathbb{A}^1$ and $\pi_1(W_-) = L_- \cong \mathbb{A}^1$, where $L_+ = \{y = 1, N = 0\}$ and $L_- = \{y = -1, N = 0\}$.

8.9.4 Summary

The canonical factorization of π splits as follows:

$$X \setminus W \underset{\sim}{\xrightarrow{\pi_1}} X_1 \setminus X_1^{\mathbb{G}_a} \cong V \times \mathbb{A}^1 \xrightarrow{\pi_0} V$$

and:

$$W \cong \mathbb{A}^2 \cup \mathbb{A}^2 \xrightarrow{\pi_1} X_1^{\mathbb{G}_a} \cong \mathbb{A}^3 \xrightarrow{\pi_0} W_0 \cong \mathbb{A}^1$$

where $\pi_1(W) = \mathbb{A}^1 \cup \mathbb{A}^1$ and $\pi_0(X_1^{\mathbb{G}_a}) = p$. As an affine modification, we find that $B = B_1[(F+1)^{-1}J]$, where $J = (F+1)B_1 + NB_1$.

Chapter 9
Makar-Limanov and Derksen Invariants

In 1994, a meeting entitled “Workshop on Open Algebraic Varieties” was held at McGill University. This meeting was organized by Peter Russell, who at the time was working with Mariusz Koras to solve the Linearization Problem for $\mathbb{C}^*$-actions on $\mathbb{C}^3$. A key remaining piece of their work was to decide whether certain hypersurfaces in $\mathbb{C}^4$ were algebraically isomorphic to $\mathbb{C}^3$. The simplest such threefold $X \subset \mathbb{C}^4$ is given by zeros of the polynomial $f \in k[x, y, z, t]$ defined by:

$$f = x + x^2y + z^2 + t^3$$

X resembles $\mathbb{C}^3$ in many ways, and had resisted previous attempts to discern its nature: It is a smooth contractible factorial affine threefold which admits a dominant morphism from $\mathbb{C}^3$. Moreover, X is diffeomorphic to $\mathbb{R}^6$ (see Dimca [114] and Kaliman [225]). By way of comparison, every normal affine surface which is homeomorphic to $\mathbb{C}^2$ is isomorphic to $\mathbb{C}^2$ (see [254]).

One participant in the McGill meeting was Leonid Makar-Limanov, who announced in the meeting a proof that X is not isomorphic to $\mathbb{C}^3$. The proof was rather lengthy, but the main idea was ingeneously simple: Show that $Dx = 0$ for every $D \in \mathrm{LND}(k[X])$. Since no such non-constant regular function exists for $\mathbb{C}^3$, it follows that $X \ncong \mathbb{C}^3$. This is an example of an **exotic affine space**, which is a complex algebraic variety diffeomorphic to $\mathbb{R}^{2n}$ for some $n \geq 1$ but not isomorphic to $\mathbb{C}^n$ as a complex algebraic variety.

This breakthrough did not entirely complete the proof of linearization for $\mathbb{C}^*$-actions, but provided the crucial new idea which allowed this to happen. A revised version of Makar-Limanov’s original paper was published in 1996 [278]. In their 1997 paper [230], Makar-Limanov and Kaliman dealt with the full class of Koras-Russell threefolds, allowing completion of the linearization proof. Following is the abstract of that paper.

G. Freudenburg, *Algebraic Theory of Locally Nilpotent Derivations*,
Encyclopaedia of Mathematical Sciences 136,
DOI 10.1007/978-3-662-55350-3_9

> P. Russell and M. Koras classified all smooth affine contractible threefolds with hyperbolic $\mathbb{C}^*$-action and quotient isomorphic to that of the corresponding linear action on the tangent space at the unique fixed point. It is not clear from their description whether there exist nontrivial Russell-Koras threefolds that are isomorphic to $\mathbb{C}^3$. They showed that this question arises naturally in connection with the problem of linearizing a $\mathbb{C}^*$-action on $\mathbb{C}^3$. We prove that none of the nontrivial Russell-Koras threefolds are isomorphic to $\mathbb{C}^3$.

Two papers of Koras and Russell [250, 251] give details for their proof of linearization for $\mathbb{C}^*$-actions on $\mathbb{C}^3$, and the article [229] provides an overview of their work.

These ideas led Makar-Limanov to formulate the definition and basic theory of a subtle new invariant for an algebraic variety V, which he called the ring of absolute constants of the variety, or AK invariant. This is now more commonly known as the Makar-Limanov invariant $ML(V)$, as defined in *Chap. 2*.

Shortly after Makar-Limanov's proof emerged, Derksen defined another invariant using $\mathbb{G}_a$-invariant subrings, and used this invariant to give another proof that the Russell cubic threefold X is not an affine space. For the k-algebra B, the **Derksen invariant** $\mathcal{D}(B)$ of B is the subalgebra of B generated by the sets

$$\{\ker D \mid D \in \mathrm{LND}(B)\,,\ D \neq 0\}$$

i.e., $\mathcal{D}(B)$ is the smallest subalgebra of B containing the kernel of every nonzero locally nilpotent derivation of B. For example, if $B = k[x_1, \ldots, x_n] = k^{[n]}$, then $x_i \in \mathcal{D}(B)$ for each i, and thus $\mathcal{D}(B) = B$. Note that, if B is rigid, then $\mathcal{D}(B) = k[\emptyset] = k$.

Section 9.1 studies Danielewski surfaces and their Makar-Limanov invariants, and *Sect. 9.2* discusses Pham-Brieskorn surfaces. *Section 9.3* gives a proof that the Derksen invariant of the Russell cubic threefold X is a proper subring of the coordinate ring $k[X]$. This proof is based on the theory of G-critical elements, and may be viewed as a more concise version of Makar-Limanov's original proof. *Section 9.4* discusses various algebraic characterizations of the affine plane originating in the work of Swan and Miyanishi, while *Sects. 9.5* and *9.6* examine locally nilpotent derivations of special Danielewski surfaces. *Section 9.7* gives examples showing the independence of the Makar-Limanov and Derksen invariants, and *Sect. 9.8* provides an overview of developments in the classification of affine surfaces relative to their $\mathbb{G}_a$-actions.

9.1 Danielewski Surfaces

In 1989, it was shown by Danielewski [89] that if W_n is the complex surface defined by $x^n z = y^2 - y$ for $n \geq 1$, then W_1 and W_2 are not isomorphic as algebraic varieties, or even homeomorphic in the Euclidean topology, but the cylinders $W_m \times \mathbb{C}$ and $W_n \times \mathbb{C}$ are isomorphic for all pairs $m, n \geq 1$. Then in [153], Fieseler showed that $W_m \not\cong W_n$ if $m \neq n$. These examples and their connection to the Zariski Cancellation Problem are discussed in *Chap. 10*.

In this section, we consider a more general class of surfaces. Given the integer $d \geq 1$, define a set of polynomials $\mathcal{S}_d \subset k[x, y] = k^{[2]}$ by:

$$\mathcal{S}_d = \{P \in k[x, y] \mid d = \deg_y P = \deg_y P(0, y)\}$$

Define $\mathcal{S} = \cup_{d\geq 1}\mathcal{S}_d$. Given $n \geq 0$ and $P \in \mathcal{S}$, define the subring $\mathbf{D}_{(n,P)}$ of $k[x, x^{-1}, y]$ by:

$$\mathbf{D}_{(n,P)} = k[x, y, z] \quad \text{where} \quad x^n z = P(x, y)$$

Since the polynomial $X^n Z - P(X, Y)$ in $k[X, Y, Z] = k^{[3]}$ is irreducible, $D_{(n,P)}$ is an integral domain.

Definition 9.1 A **Danielewski surface** is any surface S isomorphic to $\mathrm{Spec}(\mathbf{D}_{(n,P)})$ for $n \geq 0$ and $P \in \mathcal{S}$.[1]

Note that we may assume $\deg_x P(x, y) < n$: If the monomial $x^s y^t$ appears in P and $s \geq n$, then for $\tilde{P} = P - x^s y^t$ we have $x^n(z - x^{s-n} y^t) = \tilde{P}(x, y)$. It follows that $\mathbf{D}_{(n,P)}$ and $\mathbf{D}_{(n,\tilde{P})}$ are isomorphic. In particular, if $n = 1$, then we can assume that $P \in k[y] \setminus k$.

A Danielewski surface is not rigid, owing to the fact that the derivation $x^n \partial_y$ on $k[x, y]$ extends uniquely to $\mathbf{D}_{(n,P)}$. Therefore, either $\mathbf{D}_{(n,P)}$ is semi-rigid or $ML(\mathbf{D}_{(n,P)}) = k$. The latter case is non-empty, since, for example, $\mathbf{D}_{(n,y)}$ and $\mathbf{D}_{(0,P)}$ define coordinate planes in $\mathbb{A}^3$. In addition, if $P \in k[y] \setminus k$, then $ML(\mathbf{D}_{(1,P)}) = k$, since these rings are symmetric in x and z. Accordingly, we define a Danielewski surface S to be **special** if and only if $ML(S) = k$. These surfaces are important for a number of reasons, including the fact that they have a relatively large automorphism group.

Makar-Limanov calculated both $ML(S)$ and $\mathrm{Aut}_k(S)$ in the case $k[S] = \mathbf{D}_{(n,P)}$ and $P \in k[y] \setminus k$, and gives conditions as to when two such Danielewski surfaces are isomorphic; see [59, 277, 280]. In line with his work, we have:

Theorem 9.2 ([5], Prop. 6.16) *If $n, d \geq 2$ and $P \in \mathcal{S}_d$, then:*

$$ML(\mathbf{D}_{(n,P)}) = k[x]$$

Proof Let $G = \mathbb{Z}^2$ and define a total order $\preceq$ on G by lexicographical ordering. Define a G-grading of $B = k[x, x^{-1}, y]$ over k by letting x and y be homogeneous with $\deg_G x = (-1, 0)$ and $\deg_G y = (0, -1)$. Given $f \in B$, let $\bar{f}$ denote the highest-degree homogeneous summand of f.

Let $R = \mathbf{D}_{(n,P)}$ and let $D \in \mathrm{LND}(R)$ be nonzero. Then $\deg_G$ restricts to R and $\deg_G z = (n, -d)$. If $A = \{f \in R \mid \deg_G f \preceq (0, 0)\}$, then $A = k[x, y]$ and $R = A[z]$. Therefore, by *Lemma 1.2* we have:

$$\mathrm{Gr}(R) = \bar{R} = A[\bar{z}] = k[x, y, \bar{z}]$$

[1]There is no general agreement on this nomenclature. We follow here the definition found in [5].

Since $z = -x^{-n}P(x,y)$ in B, we see that $\bar{z} = -x^{-n}\bar{P}$, i.e., $x^d\bar{z} = \bar{P}(x,y)$. We thus obtain:

$$\mathrm{Gr}(\mathbf{D}_{(n,P)}) = \mathbf{D}_{(n,\bar{P})}$$

By *Theorem 2.15*, the induced homogeneous derivation $\bar{D}$ of $\bar{R}$ is nonzero and locally nilpotent. We first show that $ML(\bar{R}) = k[x]$.

Define subgroups of $G(\bar{R}) = \mathbb{Z}^2$ by:

$$H_1 = \langle \deg_G y, \deg_G \bar{z} \rangle = n\mathbb{Z} \times \mathbb{Z} \quad \text{and} \quad H_2 = \langle \deg_G x, \deg_G \bar{z} \rangle = \mathbb{Z} \times d\mathbb{Z}$$

Since $n, d \geq 2$, we see that H_1 and H_2 are proper subgroups of $G(\bar{R})$. *Cor. 2.44(b)* implies that either $\bar{D}x = 0$ or $\bar{D}y = 0$.

If $\bar{D}y = 0$, set $Q(x,y) = \bar{P}(x,y) - \bar{P}(0,y)$. Then $Q \in x\bar{R}$ and we have $\bar{D}(x^nz - Q(x,y)) = 0$. But then $\bar{D}x = 0$, meaning that $\bar{D} = 0$, a contradiction. Therefore, $\bar{D}y \neq 0$, which implies $\bar{D}x = 0$. Since $\bar{R} \subset k[x, x^{-1}, y]$, we see that y is transcendental over $k[x]$, meaning that $k[x]$ is algebraically closed in $\bar{R}$. It follows that $ML(\bar{R}) = k[x]$.

Given $f \in \ker D, \bar{f} \in \ker \bar{D} = k[x]$. We may assume that $\bar{f} \notin k$; otherwise, replace f by $f - \bar{f}$. We have:

$$\deg_G f = \deg_G \bar{f} \preceq (0,0) \quad \Rightarrow \quad f \in A = k[x,y]$$

Since $\bar{f} \notin k$, we have $\bar{f} = x^c$ for $c \geq 1$. Suppose that x^ay^b is a non-constant monomial appearing in f. Then:

$$\deg_G(x^ay^b) \preceq \deg_G(x^c) \quad \Rightarrow \quad (-a,-b) \preceq (-c,0) \quad \Rightarrow \quad c \leq a$$

Therefore, x divides f, which implies $Dx = 0$. Since y is transcendental over $k[x]$, it follows that $k[x]$ is algebraically closed in R and $ML(R) = k[x]$. □

As a consequence of this result, we see that, if $ML(\mathbf{D}_{(n,P)}) = k$, then $n = 1$ or $\deg_y P = 1$. In this case, if $n \geq 2$, then $x^nz = p(x)y + q(x)$, where $p(0) \neq 0$. Since $\gcd(x^n, p(x)) = 1$, we see that $X^nZ - p(X)Y + q(X)$ is a variable of $k[X,Y,Z] = k^{[3]}$. Therefore, the special Danielewski surfaces are given (up to isomorphism) by the rings $\mathbf{D}_{(1,P)}$ for $P \in k[y] \setminus k$.

The following result and its proof are found in Prop. 2.3 (a) of Daigle [74].

Proposition 9.3 *A special Danielewski surface S is normal.*

Proof We may assume that S is defined by $\mathbf{D}_{(1,P)}$, where $P \in k[y]$ and $d = \deg P \geq 1$. In this case, y is integral over $A := k[x,z] \cong k^{[2]}$ of degree d, and we have:

$$\mathbf{D}_{(1,P)} = A \oplus Ay \oplus \cdots \oplus Ay^{d-1} \tag{9.1}$$

Given $\beta \in k(x)[y] \cap k(z)[y]$, let $F, G \in k[X,Y,Z] = k^{[3]}, f \in k[X]$ and $g \in k[Z]$ be such that:

$$\beta = f(x)^{-1}F(x,y,z) = g(z)^{-1}G(x,y,z)$$

By (9.1), we may assume that $\deg_Y F < d$ and $\deg_Y G < d$. Since

$$g(z)F(x,y,z) = f(x)G(x,y,z)$$

in $\mathbf{D}_{(1,P)}$, the decomposition (9.1) implies that

$$g(Z)F(X,Y,Z) = f(X)G(X,Y,Z)$$

in $k[X,Y,Z]$. In particular, $F = fQ$ for some $Q \in k[X,Y,Z]$, so $\beta = Q(x,y,z) \in \mathbf{D}_{(1,P)}$. Therefore:

$$k(x)[y] \cap k(z)[y] = \mathbf{D}_{(1,P)}$$

Since $k(x)[y] = k(x)^{[1]}$ and $k(z)[y] = k(z)^{[1]}$ are normal domains, it follows that $\mathbf{D}_{(1,P)}$ is also a normal domain. □

Note that, in certain cases, $\mathbf{D}_{(n,P)}$ is a UFD, for example, when $n = 1$ and $P \in k[y] \setminus k$ is irreducible.

Corollary 9.4 *If B is the coordinate ring of a Danielewski surface over k and $D \in \mathrm{LND}(B)$, $D \neq 0$, then $\ker D \cong k^{[1]}$.*

Proof Let $B = \mathbf{D}_{(n,P)}$ for $P \in \mathcal{S}_d$, $n \geq 0$ and $d \geq 1$. If B is semi-rigid, then $n, d \geq 2$ and *Theorem 9.2* implies $\ker D = k[x]$. Otherwise, $ML(B) = k$ and *Proposition 9.3* and *Lemma 2.34* imply that $\ker D = k^{[1]}$. □

These results immediately imply the following corollary.

Corollary 9.5 *Let $P \in k[y]$, where $\deg P \geq 2$. For each $n \geq 2$, $\mathbf{D}_{(1,P)}$ and $\mathbf{D}_{(n,P)}$ are not isomorphic as k-algebras.*

Remark 9.6 For the ring $\mathbf{D}_{(n,P)}$ with $n \geq 0$ and $P \in \mathcal{S}$, define $\delta \in \mathrm{LND}(\mathbf{D}_{(n,P)})$ by $\delta x = 0$ and $\delta y = x^n$, noting that $\ker \delta = k[x]$. In his thesis, Alhajjar points out the sequence of subrings

$$k[x] \subset \mathbf{D}_{(0,P)} \subset \mathbf{D}_{(1,P)} \subset \cdots \subset \mathbf{D}_{(n,P)}$$

where $\mathbf{D}_{(i,P)} = k[x,y,x^{n-i}z]$. This is an example of what he calls an exponential chain, which differs from the degree resolution; see [5], Prop. 5.7. The derivation δ restricts to each subring in this chain. Consequently, if $X_i = \mathrm{Spec}(\mathbf{D}_{(i,P)})$ and $Y = \mathrm{Spec}(k[x])$, then the quotient map $\pi : X_n \to Y$ for the corresponding $\mathbb{G}_a$-action on X_n factors into an equivariant sequence of dominant morphisms

$$X_n \xrightarrow{\pi_n} X_{n-1} \to \cdots \to X_1 \xrightarrow{\pi_1} X_0 = \mathbb{A}^2 \xrightarrow{\pi_0} Y = \mathbb{A}^1$$

where the morphisms $\pi_n, \ldots, \pi_1$ are birational and π_0 is a standard projection.

9.2 Pham-Brieskorn Surfaces

For $n \geq 1$, let $x_0, \dots, x_n$ be coordinate functions on $\mathbb{A}^{n+1}$. Given positive integers a_i, $0 \leq i \leq n$, the corresponding **Pham-Brieskorn variety** is the hypersurface of $\mathbb{A}^{n+1}$ defined by:

$$x_0^{a_0} + x_1^{a_1} + \cdots + x_n^{a_n} = 0$$

These hypersurfaces have been of interest in topology and algebraic geometry for decades; see for example the survey of Seade [374].

In particular, given positive integers a_0, a_1, a_2, consider the corresponding Pham-Brieskorn surface with coordinate ring:

$$A = k[x_0, x_1, x_2]/(x_0^{a_0} + x_1^{a_1} + x_2^{a_2})$$

If $a_i = 1$ for any i, then $A \cong k^{[2]}$, so assume that $a_i \geq 2$ for each i. Working over the field $k = \mathbb{C}$, Kaliman and Zaidenberg [236] considered $\mathbb{G}_a$-actions on these surfaces. If $a_i = a_j = 2$ for $i \neq j$, then A corresponds to a special Danielewski surface and $ML(A) = \mathbb{C}$. Kaliman and Zaidenberg showed that this is the only case in which A is not rigid. Their result was generalized in [174], Thm. 7.1, and is further generalized by:

Theorem 9.7 *Let*

$$A = k[x_0, x_1, x_2]/(x_0^{a_0} + x_1^{a_1} + x_2^{a_2})$$

where $a_i \geq 2$ for each i and at most one a_i equals 2.

(a) *A is rigid.*
(b) *Let B be an integral domain containing A which is free as an A-module. If $a_0^{-1} + a_1^{-1} + a_2^{-1} \leq 1$, then $A \subset ML(B)$.*

Proof We first prove part (b). If $i, j \in \{0, 1, 2\}$ and $i \neq j$, then A is a free module over $k[x_i, x_j] = k^{[2]}$. By *Lemma 2.46*, we see that x_i and x_j are relatively prime in A. Since B is a free A-module, x_i and x_j are also relatively prime in B.

If $a_0^{-1} + a_1^{-1} + a_2^{-1} \leq 1$, then $A \subset ML(B)$ by the ABC Theorem (*Theorem 2.48*). This proves part (b), which implies part (a) in the case:

$$a_0^{-1} + a_1^{-1} + a_2^{-1} \leq 1$$

In order to prove part (a), it suffices to assume that $a_0^{-1} + a_1^{-1} + a_2^{-1} > 1$, which includes only a finite number of cases. If we assume $a_0 \leq a_1 \leq a_2$, then $a_0 = 2$, $a_1 = 3$ and $a_2 \in \{3, 4, 5\}$. Define $R \subset A$, $f \in R$ and $n \in \mathbb{Z}$ as follows.

1. If $a_2 = 5$, let $R = k[x_0, x_1], f = x_0^2 + x_1^3$ and $n = 5$.
2. If $a_2 = 4$, let $R = k[x_0, x_2], f = x_0^2 + x_2^4$ and $n = 3$.
3. If $a_2 = 3$, let $R = k[x_1, x_2], f = x_1^3 + x_2^3$ and $n = 2$.

In each case, $R \cong k^{[2]}$ is a $\mathbb{Z}$-graded ring, f is homogeneous and $\gcd(\deg f, n)$ equals 1. In addition, we see by the Second AB Theorem (*Theorem 2.50*) that $\delta^2 f \neq 0$ for every nonzero $\delta \in \mathrm{LND}(R)$. From *Corollary 2.38*, it follows that A is rigid. This completes the proof of part (a). $\square$

In [174], § 8, the authors consider Pham-Brieskorn threefolds, and show that certain families of these are rigid. One important case outside the reach of the algebraic techniques used in this paper is the affine **Fermat cubic threefold** $X \subset \mathbb{A}^4$ defined by:

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$$

Using geometric techniques, Cheltsov, Park and Won succeeded to show that a certain family of threefolds consists of rigid members, and this family includes X; see [51].

9.3 Koras-Russell Threefolds

In this section, we prove the result of Makar-Limanov stated in the chapter's introduction. As mentioned, Makar-Limanov's original proof was rather long and technical, and relied on jacobian derivations. Eventually, he streamlined his arguments and wrote a shorter proof, which appeared in [281]. These proofs were given for the field $k = \mathbb{C}$.

In his thesis [95], Derksen introduced the definition of the invariant $\mathcal{D}(R)$, and showed that $\mathcal{D}(R) \neq R$ for the ring R which Makar-limanov had considered. His proof follows the ideas of Makar-Limanov, placing them in a more geometric framework.

In [58], Crachiola extends the definition of the Makar-Limanov invariant to any field, stating that his purpose is to place the Makar-Limanov invariant in a characterstic free environment. Rather than using locally nilpotent derivations, he defines a class of exponential maps and uses the intersection of their fixed rings to define the invariant. The main results of Crachiola's paper assume that the underlying field is algebraically closed.

The coordinate ring of the **Russell cubic threefold** over the field k is defined by:

$$R_0 = k[x, y, z, t]/(x + x^2y + z^2 + t^3)$$

In this section, we use *Theorem 2.39* to show that $\mathcal{D}(R_0) \neq R_0$. The proof given here first appeared in [83]. The main implication is that $R_0 \neq k^{[3]}$.

A **Koras-Russell threefold of the first kind** is defined by

$$R = k[x, y, z, t]/(x + x^d y + z^u + t^v) \tag{9.2}$$

where $d, u, v \geq 2$ and u and v are relatively prime. We treat this more general case.

Lemma 9.8 ([83], Lemma 8.1) *Given* $d, u, v \geq 2$ *with* $\gcd(u, v) = 1$*, define:*

$$Q = k[x, y, z, t]/(x^d y + z^u + t^v).$$

Let $G = \mathbb{Z}^2$ *and put a* G*-grading on* Q *by declaring that* x, y, z, t *are* G*-homogeneous and:*

$$\deg_G(x, y, z, t) = ((-1, 0), (d, -uv), (0, -v), (0, -u)).$$

Then for every nonzero G*-homogeneous* $D \in LND(Q)$*, either* $\ker D = k[x, z]$ *or* $\ker D = k[x, t]$.

Proof Since the subgroups

$$\langle \deg_G y, \deg_G z, \deg_G t\rangle = d\mathbb{Z} \times \mathbb{Z}$$
$$\langle \deg_G x, \deg_G y, \deg_G t\rangle = \mathbb{Z} \times u\mathbb{Z}$$
$$\langle \deg_G x, \deg_G y, \deg_G t\rangle = \mathbb{Z} \times v\mathbb{Z}$$

are proper subgroups of $G(Q) = \mathbb{Z}^2$, *Corollary 2.44(b)* implies that at least two of Dx, Dz, Dt must be 0.

Suppose that $Dz = Dt = 0$. Then $D(x^d y) = 0$ implies $D = 0$. Therefore, if $D \neq 0$, then either $Dx = Dz = 0$ or $Dx = Dt = 0$. Since there exist $D_1, D_2 \in \text{LND}(Q)$ with $\ker D_1 = k[x, z]$ and $\ker D_2 = k[x, t]$, it follows that either $\ker D = k[x, z]$ or $\ker D = k[x, t]$. □

Theorem 9.9 ([83], Thm. 8.2) *Let* R *be the ring*

$$R = k[x, y, z, t]/(x + x^d y + z^u + t^v)$$

where $d, u, v \geq 2$ *and* $\gcd(u, v) = 1$.

(a) $\mathcal{D}(R) = k[x, z, t]$
(b) $ML(R) = k[x]$

Proof Let $G = \mathbb{Z}^2$ and define a total order $\preceq$ on G by lexicographical ordering. Define a G-grading on $B = k[x, x^{-1}, z, t]$ such that x, z, t are homogeneous and:

$$\deg_G(x, z, t) = ((-1, 0), (0, -v), (0, -u))$$

If $A = \{f \in B \mid \deg(f) \preceq (0, 0)\}$, then $A = k[x, z, t] \subset R$. The degree function $\deg_G$ on B restricts to R, where $\deg_G y = (d, -uv)$. According to *Lemma 1.2*, we have:

$$\text{Gr}(R) = \bar{R} = A[\bar{y}] = k[x, z, t, \bar{y}]$$

Since $y = -x^{-d}(x + z^u + t^v)$ in B, we see that $\bar{y} = -x^{-d}(z^u + t^v)$, i.e., $x^d\bar{y} + z^u + t^v = 0$.

Let nonzero $D \in \mathrm{LND}(R)$ and $f \in \ker D$ be given. By *Theorem 2.15*, the induced homogeneous derivation $\bar{D}$ of $\bar{R}$ is nonzero and locally nilpotent. Therefore, by *Lemma 9.8*, $\ker \bar{D} \subset A$. Since $\bar{f} \in \ker \bar{D}$, we see that:

$$\deg_G f = \deg_G \bar{f} \preceq (0,0)$$

Therefore, $f \in A$, and $\mathcal{D}(R) \subset A$. The reverse inclusion follows from the observation that the derivations

$$D_1 = vt^{v-1}\frac{\partial}{\partial y} - x^d\frac{\partial}{\partial t} \quad \text{and} \quad D_2 = uz^{u-1}\frac{\partial}{\partial y} - x^d\frac{\partial}{\partial z} \tag{9.3}$$

are elements of $\mathrm{LND}(R)$. This proves part (a).

For part (b), suppose that $Dx \neq 0$. Choose $f, g \in \ker D$ which are algebraically independent. By the foregoing result, $f, g \in A = k[x,z,t] = k^{[3]}$. Let $f_1, g_1 \in A$ and $f_2, g_2 \in k[z,t]$ be such that:

$$f = xf_1 + f_2 \quad \text{and} \quad g = xg_1 + g_2$$

Note that f_2 and g_2 are algebraically independent in R. Otherwise, there is a bivariate polynomial P over k with $P(f_2, g_2) = 0$. But then $P(f,g) \in xR$, which implies $Dx = 0$, a contradiction. In addition, note that $\deg_G(xf_1) \prec \deg_G f_2$, which implies that $\deg_G f = \deg_G f_2$. Similarly, $\deg_G g = \deg_G g_2$.

Continuing the notation of the preceding proof, let $\bar{D}$ be the associated derivation of $\bar{R}$. We have

$$\bar{f} = \bar{f_2} = f_2(\bar{z}, \bar{t}) \quad \text{and} \quad \bar{g} = \bar{g}_2 = g_2(\bar{z}, \bar{t})$$

and these images are algebraically independent elements of $\ker \bar{D}$: Since $k[z,t] \subset R$ is a G-graded subring of B, the restriction $\mathrm{gr} : k[z,t] \to k[\bar{z}, \bar{t}]$ is an algebra isomorphism. Since $k[\bar{z}, \bar{t}]$ is the algebraic closure of $k[\bar{f}, \bar{g}] \subset \ker \bar{D}$, it follows that $k[\bar{z}, \bar{t}] \subset \ker \bar{D}$. But then $0 = \bar{D}(\bar{x}^d\bar{y} + \bar{z}^u + \bar{t}^v) = \bar{D}(\bar{x}^d\bar{y})$, which implies $\bar{D} = 0$, a contradiction.

So the only possibility is that $Dx = 0$.

Conversely, we see that, for D_1 and D_2 as in (9.3), $\ker D_1 \cap \ker D_2 = k[x]$. This proves part (b). □

Define $\varphi : \mathrm{LND}_{k[x]}(k[x,z,t]) \to \mathrm{LND}(R)$ by letting $\varphi(\theta)$ be the extension of $x^d\theta$ to R. Then φ is well-defined and injective. If θ is irreducible, then the results of *Chap. 4* show that there exists a $k(x)$-variable $P \in k[x][z,t]$ such that $\theta = \Delta_P$. In this case, we see that, for the derivations D_1, D_2 defined in (9.3),

$$\varphi(\Delta_P) = P_z D_1 - P_t D_2$$

and $\varphi(\Delta_P)$ is irreducible.

Corollary 9.10 *Let R be the Koras-Russell threefold defined in (9.2). The following conditions are equivalent.*

1. *$D \in LND(R)$ is irreducible.*
2. *There exists a $k(x)$-variable $P \in k[x][z,t]$ such that $D = P_z D_1 - P_t D_2$.*

In this case, $\ker D = k[x, P]$.

Proof Assume that $D \in \mathrm{LND}(R)$ is irreducible. Since $x^d y \in A$ and $Dx = 0$, there exists $n \geq 0$ such that $x^n D$ restricts to $A = k^{[3]}$. In addition, $x^n D \in \mathrm{LND}(R)$. Let $\delta \in \mathrm{LND}(A)$ denote the restriction of $x^n D$ to A. Since $\ker(x^n D) = \ker D \subset A$, it follows that $\ker D = \ker \delta$. Since $\delta x = 0$, the rank of δ is at most 2. The results of *Chap. 4* show that there exists a $k(x)$-variable $P \in k[x][z,t]$ such that $\ker D = k[x, P]$. Since $\varphi(\Delta_P)(P) = 0$, we conclude that $D = \varphi(\Delta_P)$, up to a constant multiple. □

Remark 9.11 Kaliman's Fiber Theorem (*Theorem 5.39*) provides an alternate way to show that the Russell cubic threefold X is not an affine space. Consider the mapping $f : X \to \mathbb{A}^1$ induced by the inclusion $k[x] \to R_0$. Given nonzero $\lambda \in k$, the fiber over λ is a plane. The Fiber Theorem asserts that, if $X = \mathbb{A}^3$, then every fiber is a plane. However, the fiber of f over $\lambda = 0$ is of the form $C \times \mathbb{A}^1$, where C is the cuspidal cubic curve $z^2 + t^3 = 0$ in the (z,t)-plane. Therefore, $X \not\cong \mathbb{A}^3$.

9.4 Characterizing $k[x, y]$ by LNDs

A general problem of commutative algebra is to give conditions which imply that a given ring is a polynomial ring. In 1971, C.P. Ramanujam characterized the affine plane over $\mathbb{C}$ as the only nonsingular algebraic surface that is contractible and simply connected at infinity [352]. The first algebraic characterization was given by Miyanishi in 1975: If K is an algebraically closed field (of any characteristic), and if X is an affine factorial surface over K with trivial units which admits a nontrivial $\mathbb{G}_a$-action, then $X = \mathbb{A}^2_K$ [296].

Several equivalent conditions for a ring to be $k[x,y]$ are given in the next theorem. Its proof is based on the lemmas about UFDs in *Sect. 2.3*.

Theorem 9.12 *Let k be an algebraically closed field of characteristic zero, and suppose B is a UFD of transcendence degree 2 over k. The following conditions are equivalent.*

1. *B is affine, $B^* = k^*$ and $B_f = R^{[1]}$ for some $f \in B$ and $R \subset B_f$*
2. *B is affine, $B^* = k^*$ and B is not rigid*
3. *$ML(B) = k$*
4. *There exist a degree function $\deg : B \to \mathbb{Z} \cup \{-\infty\}$ and nonzero $D \in LND(B)$ such that $\deg f > 0$ for every $f \in B^D \setminus k$*
5. *$B = k^{[2]}$*

Geometrically, condition (1) says that the surface $S = \text{Spec}(B)$ contains a cylinderlike open set. Notice that neither (3) nor (4) assumes, *a priori*, that B is affine. The implication (1) $\Rightarrow$ (5) is due to Swan [399], 1979. The implication (2) $\Rightarrow$ (5) is Miyanishi's 1975 result [296]. The implication (3) $\Rightarrow$ (5) was shown by Makar-Limanov in 1998 [276], Lemma 19, under the additional assumption that B is k-affine. See also [304], Thm. 2.6; [305], Thm. 2.21; and [136], Thm. 3.1.

Proof That (5) implies the other four conditions is clear, and the implication (3) $\Rightarrow$ (5) was proved in *Corollary 2.13*. We will show:

$$(1) \Rightarrow (2) \Rightarrow (5) \quad \text{and} \quad (4) \Rightarrow (5)$$

(1) $\Rightarrow$ (2). Since B is affine and B_f is not rigid, *Lemma 2.23 (a)* implies that B is not rigid.
(2) $\Rightarrow$ (5). Since B is not rigid, there exists nonzero $D \in \text{LND}(B)$, and by *Proposition 2.2*, we may assume D is irreducible (since B is a UFD). By *Lemma 2.10*, D has a slice y. By the Slice Theorem, $B = A[y]$, where $A = \ker D$. Since B is affine, we see that A is also affine. Therefore, A is an affine UFD of transcendence degree 1 over k, and the units of A are trivial, since $B^* = k^*$. By *Lemma 2.9*, $A = k^{[1]}$.
(4) $\Rightarrow$ (5). We may assume D is irreducible. By *Lemma 2.10*, D has a slice y. By the Slice Theorem, $B = B^D[y]$. So it will suffice to show $B^D = k^{[1]}$.

Since $D \neq 0$, tr. $\deg_{\cdot k} B^D = 1$, meaning $B^D \neq k$. By hypothesis, the restriction of deg to B^D takes values in $\mathbb{N}$ and, for $f \in B^D$, $\deg f = 0$ if and only if $f \in k$. By *Lemma 2.12(b)*, $B^D = k^{[1]}$. □

Note that these results may no longer be true without the assumption that the field k is algebraically closed, as seen in the following two examples.

Example 9.13 Consider $R = k[x, y, z]$, where $xz = P(y)$ for $P \in k^{[1]}$. Then R is the coordinate ring of a special Danielewski surface and $ML(R) = k$. In addition, R is a UFD if and only if $P(y)$ is irreducible. So if $P(y)$ is irreducible and of degree at least 2, then R is a UFD. However, R is not a polynomial ring in this case: If $\bar{k}$ is the algebraic closure of k, then $\bar{k} \otimes_k R$ is not a UFD.

Example 9.14 Define $f \in k[x, y, z] = k^{[3]}$ by $f = xz - y^2$ and define the subfield $K \subset k(x, y, z)$ by $K = k(f)$. Define $B = K \otimes_k k[x, y, z] = K[x, y, z]$. Then B is a UFD (being the localization of a UFD) of transcendence degree 2 over K with $B^* = K^*$. Moreover, $ML(B) = K$, since the triangular derivations

$$D = x\partial_y + 2y\partial_z \quad \text{and} \quad E = z\partial_y + 2y\partial_x$$

on $k[x, y, z]$ extend to $D_K, E_K \in \text{LND}(B)$. Since $\ker D = k[x, f]$, we have $\ker D_K = K[x]$. If $B = K^{[2]}$, then by Rentschler's Theorem, $B = K[x, y]$, since $D_K y = x$. However, since $z \notin K[x, y]$, it follows that $B \neq K^{[2]}$.

Example 9.15 *Theorem 9.12* gives another way to show that the Pham-Brieskorn surface S defined by $x_0^2 + x_1^3 + x_2^5 = 0$ over $\mathbb{C}$ is rigid. This was proved in *Theorem 9.7*. It is well-known that $A = \mathbb{C}[S]$ is a UFD. Observe that A has a positive $\mathbb{Z}$-grading for which x_0, x_1, x_2 are homogeneous and $\deg x_0 = 15$, $\deg x_1 = 10$ and $\deg x_2 = 6$. If A were not rigid, then *Theorem 9.12(4)* would imply that $S \cong \mathbb{C}^2$, which is absurd since S has a singular point.

Remark 9.16 Recall Kambayahsi's Theorem (*Sect. 5.1.1*), which says that if R is a commutative k-algebra and $K \otimes_k R = K^{[2]}$ (K a separable algebraic field extension of k), then $R = k^{[2]}$. Thus, for non-algebraically closed fields k, one could give a characterization of $k[x, y]$ similar to the one above by replacing the condition "k is algebraically closed" with the condition "$\bar{k} \otimes_k B$ is a UFD, where $\bar{k}$ is the algebraic closure of k".

Remark 9.17 Makar-Limanov points out that his characterization of the plane does not generalize to rings of transcendence degree three. For example, if $X = SL_2(\mathbb{C})$, then $k[X]$ is a UFD, and we saw in *Remark 5.18* that $ML(X) = \mathbb{C}$. Kaliman posed the following question.

> Let X be a smooth contractible algebraic $\mathbb{C}$-variety of dimension 3, with $ML(X) = \mathbb{C}$. Is X isomorphic to $\mathbb{C}^3$? (Problem 1, p. 7 of [175])

In [300, 302], Miyanishi gives an algebraic characterization of affine 3-space. See also [236].

9.5 Characterizing Danielewski Surfaces by LNDs

In § 2 of [74], Daigle gives two characterizations of the special Danielewski surfaces in terms of their locally nilpotent derivations. These are stated and applied below; the reader is referred to the article for proofs. To paraphrase, the first of these says that an affine surface S is a special Danielewski surface if and only if $k[S]$ admits two distinct locally nilpotent derivations having a common local slice, and whose kernels are polynomial rings. The second asserts that, if B is a UFD of transcendence degree 2 over k which admits a *k-simple* derivation, then $B = k[S]$ for a special Danielewski surface S. § 4 of the paper investigates of the graph of kernels $\underline{\mathrm{KLND}}(\mathcal{B})$, where $\mathcal{B}$ is a commutative k-domain of transcendence degree 2 over k. (This graph is discussed in *Chap. 5*.) A third characterization is given by Daigle and Kolhatkar in [85].

It should be noted that Bandman and Makar-Limanov proved earlier that any smooth hypersurface S of $\mathbb{C}^3$ such that $ML(S) = \mathbb{C}$ is a special Danielewski surface [9]. It should also be noted that, unlike the characterizations of $k[x, y]$ in the preceding section, Daigle's characterizations do not require the underlying field to be algebraically closed.

9.5.1 Three Characterizations

Theorem 9.18 ([74], Thm. 2.5) *Let R be a commutative k-domain, and let $D_1, D_2 : R \to R$ be locally nilpotent k-derivations which satisfy:*

1. $\ker D_1 = k^{[1]}$ *and* $\ker D_2 = k^{[1]}$ *but* $\ker D_1 \neq \ker D_2$
2. *There exists* $y \in R$ *such that* $D_i y \in \ker D_i \setminus k$ *for each* i

Set $x_1 = D_1 y$ *and* $x_2 = D_2 y$*. Then* $\ker D_1 = k[x_1]$*,* $\ker D_2 = k[x_2]$*, and R is isomorphic to the ring* $k[X_1, X_2, Y]/(X_1X_2 - \phi(Y))$ *for some* $\phi \in k[Y]$*.*
Observe that x_1, x_2 and y are algebraically dependent in R, and the equation $X_1X_2 - \phi(Y)$ is their (essentially unique) dependence relation.

For the second characterization, Daigle gives the following definition. Let B be any commutative k-domain of transcendence degree 2 over k. Then $D \in \mathrm{Der}_k(B)$ is ***k*-simple** if and only if D is locally nilpotent, irreducible, and there exists $y \in B$ such that $\ker D = k[Dy]$.

Theorem 9.19 ([74], Thm. 2.6) *Let B be a UFD of transcendence degree 2 over k. If B admits a k-simple derivation, then B is the coordinate ring of a special Danielewski surface over k.*

Daigle writes, "This work started as an attempt to understand the process known as the local slice construction" (p. 37). Regarding the locally nilpotent derivations of $k[x, y, z] = k^{[3]}$, he writes that "a crucial rôle is played by the polynomials $f \in k[x, y, z]$ whose generic fiber is a Danielewski surface", i.e., $k(f)[x, y, z]$ is a special Danielewski surface over the field $k(f)$ (p. 77).

The third characterization, due to Daigle and Kolhatkar, is as follows.

Theorem 9.20 ([85], Thm. 9.9) *Let S be an affine surface over a field k of characteristic zero. If S is a complete intersection and $ML(S) = k$, then S is isomorphic to a hypersurface in $\mathbb{A}^3$ of the form $XZ = P(Y)$ for some non-constant $P \in k[Y]$.*

See also Daigle [76].

9.5.2 Application to Embedding Questions

In [172], we use Daigle's first characterization, together with the derivations of Fibonacci type defined in *Chap. 5*, to construct non-equivalent embeddings of certain special Danielewski surfaces in $\mathbb{C}^3$. Two embeddings $i, j : S \hookrightarrow \mathbb{C}^3$ of a surface S are **equivalent** if there exists $\alpha \in GA_3(\mathbb{C})$ with $j = \alpha \circ i$. Otherwise, the embeddings are **non-equivalent**.

Let $\{H_n\} \subset \mathbb{C}[x, y, z]$ be the sequence of polynomials defined in *Sect. 5.5.2*. Then $\mathbb{C}[H_n, H_{n+1}]$ is the kernel of a locally nilpotent derivation θ_n of $\mathbb{C}[x, y, z]$ of Fibonacci type. Recall that these have a common local slice r, and satisfy:

$$H_{n-1}H_{n+1} = H_n^3 + r^{d_n} \quad (d_n \in \mathbb{N}) \quad \text{and} \quad \theta_n r = -H_n H_{n+1}$$

Given $a \in \mathbb{C}$, let $(Y_n)_a \subset \mathbb{C}^3$ be the surface defined by the fiber $H_n - a$.

Theorem 9.21 (Thm. 6 of [172]) *Let the integer $n \geq 3$ be given.*

(a) *For each $a \in \mathbb{C}^*$, $(Y_n)_a$ is isomorphic to the special Danielewski surface in $\mathbb{A}^3$ defined by $xz = y^{d_n} + a^3$.*
(b) *The zero fiber $(Y_n)_0$ is not a Danielewski surface.*

Proof Let θ_{n-1}, θ_n be the locally nilpotent derivations of $B = \mathbb{C}[x, y, z]$ whose kernels are $\mathbb{C}[H_{n-1}, H_n]$ and $\mathbb{C}[H_n, H_{n+1}]$, respectively. Let $\bar{\theta}_{n-1}, \bar{\theta}_n$ be the induced locally nilpotent derivations on the quotient ring:

$$\bar{B} := B \bmod (H_n - a)$$

Then it is easy to check that $\ker \bar{\theta}_{n-1} = \mathbb{C}[\bar{H}_{n-1}]$ and $\ker \bar{\theta}_n = \mathbb{C}[\bar{H}_{n+1}]$. In addition, $\theta_{n-1} r = H_{n-1}H_n$ and $\theta_n r = H_n H_{n+1}$, implying $\bar{\theta}_{n-1}(\bar{r}) = a\bar{H}_{n-1}$ and $\bar{\theta}_n(\bar{r}) = a\bar{H}_{n+1}$. Moreover, we have $H_{n-1}H_{n+1} = H_n^3 + r^{d_n}$, so that in $\bar{B}$:

$$\bar{H}_{n-1}\bar{H}_{n+1} = a^3 + \bar{r}^{d_n}$$

Using the theorem of Daigle, we conclude that $\bar{B}$ is isomorphic to the ring $B/(xz - y^{d_n} - a^3)$, and (a) is proved.

Now consider the locally nilpotent derivation $\Delta := \theta_{n-1} \,(\bmod\, H_n)$ on the ring $B \bmod H_n$. Since $H_{n-2}H_n = H_{n-1}^3 + r^{d_{n-1}}$, it follows that $\ker \Delta = \mathbb{C}[\bar{H}_{n-1}, \bar{r}]$, where $\bar{H}_{n-1}^3 + \bar{r}^{d_{n-1}} = 0$. In particular, $\ker \Delta$ is not a polynomial ring. However, the kernel of a locally nilpotent derivation of the coordinate ring of a Danielewski surface over $\mathbb{C}$ is always a polynomial ring (*Corollary 9.4*). Therefore, $\mathrm{Spec}(B \bmod H_n)$ is not a Danielewski surface, and (b) is proved. □

Corollary 9.22 (Cor. 2 of [172]) *Let $n \geq 3$ and $a \in k^*$ be given. Let $Z \subset \mathbb{A}^3$ be the Danielewski surface defined by $xz = y^{d_n} - 1$, and let $(Y_n)_a \subset \mathbb{A}^3$ be the surface defined by $H_n = a$. Then Z and $(Y_n)_a$ are isomorphic as algebraic varieties, but their embeddings in $\mathbb{A}^3$ are non-equivalent.*

Question 9.23 In case S is the special Danielewski surface defined by $xz = y^2$, do there exist non-equivalent embeddings of S in $\mathbb{A}^3$?

Remark 9.24 The paper [172] gives non-equivalent embeddings for certain non-special Danielewski surfaces. In addition, this paper gives the first example of two smooth algebraic hypersurfaces in $\mathbb{C}^3$ which are algebraically non-isomorphic, but holomorphically isomorphic. Locally nilpotent derivations are a central tool in the exposition.

9.6 LNDs of Special Danielewski Surfaces

In [73], Daigle describes completely the locally nilpotent derivations of a special Danielewski surface. The following theorem, which is the main result of his paper, gives this description.

9.6.1 *Transitivity Theorem*

Let $B = k[x_1, x_2, y]$, where $x_1x_2 = \phi(y)$ for some univariate polynomial ϕ. Define $\delta \in \mathrm{LND}(B)$ by $\delta x_1 = 0$ and $\delta y = x_1$. Given $f \in k[x_1]$, let Δ_f denote the exponential automorphism $\Delta_f = \exp(f\delta)$. Note that:

$$\Delta_{f+g} = \Delta_f \Delta_g \quad \text{and} \quad \Delta_f \delta \Delta_f^{-1} = \delta$$

In addition, let τ be the automorphism of B interchanging x_1 and x_2, and let G denote the subgroup of $\mathrm{Aut}_k(B)$ generated by τ and all $\Delta_f, f \in k[x_1]$. Finally, set

$$\mathrm{KLND}(B) = \{\ker D \mid D \in \mathrm{LND}(B), D \neq 0\}$$

noting that G acts on this set by $\alpha \cdot A = \alpha(A)$.

Theorem 9.25 (Transivity Theorem) *The action of G on KLND(B) is transitive.*
Since a plane is a special Danielewski surface, this result is, in fact, a generalization of Rentschler's Theorem, where the derivation δ plays the role of a partial derivative.

Corollary 9.26 *Given nonzero $D \in LND(B)$, there exists $\theta \in G$ and $f \in k[x_1]$ such that $\theta D \theta^{-1} = f(x_1)\delta$.*
As in the proof of Jung's Theorem, this result implies a kind of tameness for the automorphism group of the surface.

9.6.2 *Example Over the Reals*

We saw in *Chap. 4* that the Newton polytope of a polynomial can, in some cases, indicate that the polynomial is not in the kernel of a locally nilpotent derivation. However, the Newton polytope pays no attention to the underlying field, and is therefore inadequate in many situations to make such determination. This is illustrated in the following example.

For this section, let $f \in \mathbb{Q}[X, Y, Z] = \mathbb{Q}^{[3]}$ be the polynomial:

$$f = X^2 + Y^2 + Z^2$$

Then there is a nonzero locally nilpotent derivation T of $\mathbb{C}[X,Y,Z]$ with $Tf = 0$, namely:

$$TX = -Z \ , \ TY = -iZ \ , \ TZ = X + iY$$

However, there is no nonzero $D \in \text{LND}(\mathbb{R}[X,Y,Z])$ with $Df = 0$. This fact is a consequence of the following result, which is the main result of this section.

Theorem 9.27 *Let $\mathbb{C}[X,Y,Z] = \mathbb{C}^{[3]}$ and let $f = X^2 + Y^2 + Z^2$. If*

$$B = \mathbb{C}[X,Y,Z]/(f) \quad \textit{and} \quad B' = \mathbb{R}[X,Y,Z]/(f)$$

then $ML(B') = B'$, while $ML(B) = \mathbb{C}$.

In particular, this result implies $ML(\mathbb{C} \otimes_{\mathbb{R}} B') \neq \mathbb{C} \otimes_{\mathbb{R}} ML(B')$. Note that Spec B is a special Danielewski surface over $\mathbb{C}$. However, this result shows that Spec B' is not a Danielewski surface over $\mathbb{R}$. See Example 2.2 of [76] for another proof.

In order to prove the theorem, a preliminary result is needed. Write $B = \mathbb{C}[x,y,z]$ and $B' = \mathbb{R}[x,y,z]$, where x, y, and z denote the congruence classes modulo f of X, Y, and Z, respectively. Put a $\mathbb{Z}$-grading on B and B' by declaring that x, y, z are homogeneous of degree one. Call this the **standard grading** of each ring. Write $B = \oplus B_j$ and $B' = \oplus (B')_j$ accordingly. Then $B_1 = \mathbb{C}x \oplus \mathbb{C}y \oplus \mathbb{C}z$ and $(B')_1 = \mathbb{R}x \oplus \mathbb{R}y \oplus \mathbb{R}z$. We will say that $D \in \text{LND}(B)$ is **linear** if it is homogeneous of degree 0, i.e., $D : B_1 \to B_1$. Likewise, say $D' \in \text{LND}(B')$ is **linear** if it is homogeneous of degree 0, i.e., $D' : (B')_1 \to (B')_1$.

Define $x_1 = x + iz$ and $x_2 = x - iz$. Then:

$$B = \mathbb{C}[x_1, x_2, y], \qquad x_1 x_2 + y^2 = 0 \quad \text{and} \quad B_1 = \mathbb{C}x_1 \oplus \mathbb{C}x_2 \oplus \mathbb{C}y$$

Let δ be the **standard linear derivation** of B, namely, $\delta x_1 = 0$ and $\delta y = x_1$. Let G be the group described in the Transivity Theorem, and let Γ be the subgroup of G generated by the involution τ and all Δ_t for $t \in \mathbb{C}$. Note that Γ may be viewed as a subgroup of the orthogonal group $O_3(\mathbb{C})$.

The following is a corollary to the Transivity Theorem.

Corollary 9.28 *Let $D \in LND(B)$ be given. The following are equivalent.*

1. *D is irreducible and homogeneous*
2. *$D = \gamma\delta\gamma^{-1}$ for some $\gamma \in \Gamma$*
3. *D is linear*

Proof The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (1) are clear. Assume D is irreducible and homogeneous. By the Transivity Theorem, we have $D = \alpha(h\delta)\alpha^{-1}$ for some $\alpha \in G$ and $h \in \mathbb{C}[x_1]$. Since D is irreducible, $h \in \mathbb{C}^*$, so we can assume $D = \alpha\delta\alpha^{-1}$.

Let $\gamma \in \Gamma$ be given, along with $t \in \mathbb{C}^*$ and integer $n \geq 1$. Set $T = \Delta_f(\gamma\delta\gamma^{-1})\Delta_f^{-1}$ for $f = tx_1^n$, and suppose that T is homogeneous. There exist $a, b, c, a', b', c' \in \mathbb{C}$ such that:

$$L := \gamma\delta\gamma^{-1}(x_1) = ax_1 + bx_2 + cy \text{ and } M := \gamma\delta\gamma^{-1}(y) = a'x_1 + b'x_2 + c'y$$

We have:

$$T(x_1) = \Delta_f(L) = ax_1 + b(x_2 + 2tyx_1^n + t^2x_1^{2n+1}) + c(y + tx_1^{n+1})$$

If either $b \neq 0$ or $c \neq 0$, then by homogeneity, $1 = n+1$, a contradiction. Therefore $b = c = 0$, so $\gamma\delta\gamma^{-1}(x_1) = ax_1$. But this implies $\gamma\delta\gamma^{-1}(x_1) = 0$.

In the same way $b' = c' = 0$, so $\gamma\delta\gamma^{-1}(y) = a'x_1$. Since any $\mathbb{C}$-derivation of B is determined by its images on x_1 and y, we conclude that $\gamma\delta\gamma^{-1} = a'\delta$. Therefore, $T = \Delta_f(a'\delta)\Delta_f^{-1} = a'\delta$.

The other possibility is that $n = 0$, and then $\Delta_f \in \Gamma$ already.

By induction, we conclude that D is, in all cases, conjugate to δ by some element of Γ. □

Proof of Theorem 9.27. Let $D' \in \mathrm{LND}(B')$ be given, $D' \neq 0$. To prove the result, it suffices to assume D' is homogeneous and irreducible. If $D \in \mathrm{LND}(B)$ is the natural extension of D' to B, then D is homogeneous.

Write $D = hE$ for homogeneous $h \in \ker D$ and irreducible homogeneous $E \in \mathrm{LND}(B)$. By *Corollary 9.28*, E is linear and $\ker D = \ker E = \mathbb{C}[L]$ for some $L \in B_1$. Therefore:

$$\ker D = \mathbb{C} \otimes_{\mathbb{R}} \ker D' = \mathbb{C}[L] = \mathbb{C}^{[1]} \quad \Rightarrow \quad \ker D' = \mathbb{R}^{[1]}$$

(Any form of the affine line over the field of real numbers is trivial; see [362].) If $\ker D' = \mathbb{R}[F]$, then we may assume F is homogeneous. We have:

$$\ker D = \mathbb{C}[L] = \mathbb{C}[F] \quad \Rightarrow \quad F = \alpha L + \beta \quad (\alpha \in \mathbb{C}^*, \beta \in \mathbb{C})$$

By homogeneity, $\beta = 0$ and $F = \alpha L \in B'$, so we may assume $F = L$. By completing L to a basis for $(B')_1$, we see that E restricts to a linear derivation E' of B'.

We can thus view $\exp(E')$ as an element of the orthogonal group $O_3(\mathbb{R})$. Let $o(3,\mathbb{R})$ denote the real Lie algebra corresponding to $O_3(\mathbb{R})$. Then $M \in o(3,\mathbb{R})$ if and only if $M + M^T = 0$. Since $E' \in o(3,\mathbb{R})$, we conclude that $E' + (E')^T = 0$. Therefore E' has the form:

$$E' = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \quad (a,b,c \in \mathbb{R})$$

The characteristic polynomial of E' is $|E' - \lambda I| = -(\lambda^3 + (a^2+b^2+c^2)\lambda)$. Since E' is also a nilpotent matrix, its only eigenvalue (in $\mathbb{C}$) is 0. Therefore, $a^2+b^2+c^2 = 0$, implying $a = b = c = 0$, a contradiction. □

Remark 9.29 Geometrically, it is easy to see that the real unit sphere $S^2 \subset \mathbb{R}^3$ defined by $f = 1$ is rigid: The orbits are closed, hence compact (since S^2 is compact), hence of dimension 0.

9.7 Examples of Crachiola and Maubach

As observed, for a polynomial ring B, we have $\mathcal{D}(B) = B$ and $ML(B) = k$. In proving that the coordinate ring R of the Russell threefold discussed above is not a polynomial ring, we showed both $\mathcal{D}(R) \neq R$ and $ML(R) \neq k$. In [63], Crachiola and Maubach show that the Derksen and Makar-Limanov invariants are independent of one another.

Specifically, they first construct an affine noetherian $\mathbb{C}$-domain S of dimension two for which $ML(S) = \mathbb{C}$ and $\mathcal{D}(S) \neq S$. This ring is defined by

$$\begin{aligned} S &= \mathbb{C}[x^2, x^3, y^3, y^4, y^5, x^{i+1}y^{j+1}] \quad (i, j \in \mathbb{N}) \\ &= \mathbb{C}[x^2, x^3, y^3, y^4, y^5, xy, x^2y, xy^2, x^2y^2, xy^3, x^2y^3, xy^4, x^2y^4] \; . \end{aligned}$$

Similarly, they construct a ring S' with the property that $ML(S') \neq \mathbb{C}$, but $\mathcal{D}(S') = S'$. In particular, let A be any commutative $\mathbb{C}$-domain of transcendence degree one over $\mathbb{C}$, other than $\mathbb{C}^{[1]}$. Recall that any such ring is rigid. By *Proposition 2.28*, we have that $ML(A[x_1, \ldots, x_n]) = ML(A) = A \neq \mathbb{C}$. On the other hand, when $n \geq 2$, the partial derivatives relative to A show that $\mathcal{D}(A[x_1, \ldots, x_n]) = A[x_1, \ldots, x_n]$. So we may take $S' = A[x_1, \ldots, x_n]$ for $n \geq 2$.

9.8 Further Results in Classification of Surfaces

One method in the study of algebraic surfaces is to use $\mathbb{G}_a$-actions and the Makar-Limanov invariant to classify surfaces. This section will survey some of these results.

In this chapter, we have seen characterizations for the plane over an algebraically closed field, in addition to two characterizations of the special Danielewski surfaces due to Daigle. These results are based on the existence of nontrivial $\mathbb{G}_a$-actions on the surface. Recall that, if X is a factorial affine surface over an algebraically closed field k, then every irreducible element of $\mathrm{LND}(k[X])$ has a slice (*Lemma 2.10*). If X is not rigid and $k[X]$ has trivial units, then $X = \mathbb{A}^2$; this is Miyanishi's characterization of the plane.

A series of similar and related results have been published which aim to classify certain normal affine surfaces X which admit a nontrivial $\mathbb{G}_a$-action. In case X admits at least two independent $\mathbb{G}_a$-actions (i.e., $ML(X) = k$), then even more can be said. Dubouloz defines X to be an **ML-surface** if $ML(X) = k$ [126]. Apart from a plane, we have seen such surfaces in the form of special Danielewski surfaces S, defined by $xz = f(y)$ for non-constant f. This surface admits two independent (conjugate) $\mathbb{G}_a$-actions.

Some of the earliest work in this direction was done by Gizatullin [181] (1971), who studied surfaces which are **geometrically quasihomogeneous**. By definition, such a surface has an automorphism group with a Zariski open orbit whose complement is finite. See also [90, 180]. Two papers of Fauntleroy and Magid from the 1970s study affine and quasi-affine surfaces with free $\mathbb{G}_a$-actions [151, 152]. The 1983 paper of Bertin [24] is about surfaces which admit a $\mathbb{G}_a$-action.

According to Dubouloz [126], the 2001 paper [9] of Bandman and Makar-Limanov represents the rediscovery of a link between nonsingular ML-surfaces and Gizatullin's geometrically quasihomogeneous surfaces. Dubouloz writes:

> More precisely, they have established that, on a nonsingular ML-surface V, there exist at least two nontrivial algebraic $\mathbb{C}_+$-actions that generate a subgroup H of the automorphism group $\mathrm{Aut}(V)$ of V such that the orbit $H \cdot v$ of a general closed point $v \in V$ has finite complement. By Gizatullin, such a surface is rational and is either isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$ or can be obtained from a nonsingular projective surface $\tilde{V}$ by deleting an ample divisor of a special form, called a *zigzag*. This is just a linear chain of nonsingular rational curves. Conversely, a nonsingular surface V completable by a zigzag is rational and geometrically quasihomogeneous. In addition, if V is not isomorphic to $\mathbb{C}^* \times \mathbb{A}^1$ then it admits two independent $\mathbb{C}_+$-actions. More precisely, Bertin showed that if V admits a $\mathbb{C}_+$-action then this action is unique unless V is completable by a zigzag. (From the Introduction to [126])

In this paper from 2004, Dubouloz generalizes these earlier results by giving the following geometric characterization of the ML-surfaces in terms of their boundary divisors.

Theorem 9.30 ([126]) *Let V be a normal affine surface over $\mathbb{C}$ that is not isomorphic to $\mathbb{C}^* \times \mathbb{A}^1$. Then V is completable by a zigzag if and only if $ML(V) = \mathbb{C}$.*
In particular, Dubouloz has removed the condition that V be nonsingular.

The characterization given by Bandman and Makar-Limanov was for smooth affine rational ML-surfaces embedded in $\mathbb{C}^3$ as a hypersurface. Their conclusion is that these must be Danielewski surfaces, given by equations $xz = y^m - 1$ in a suitable coordinate system (x, y, z), with $m \geq 1$.

One type of surface which draws attention is a **homology plane**, defined to be a smooth algebraic surface X over $\mathbb{C}$ whose homology groups $H_i(X; \mathbb{Z})$ are trivial for $i > 0$. For example, the affine plane $\mathbb{A}^2$ over $\mathbb{C}$ is the unique homology plane X with $\bar{\kappa}(X) = -\infty$ (see [306]). Similarly, X is a **$\mathbb{Q}$-homology plane** if X is a smooth algebraic surface defined over $\mathbb{C}$ such that $H_i(X; \mathbb{Q}) = (0)$ for $i > 0$. Finally, the definition of a **log $\mathbb{Q}$-homology plane** X coincides with that of a $\mathbb{Q}$-homology plane, except that X is permitted to have certain kinds of singular points (at worst quotient singularities). Homology planes share many properties with the affine plane. One motivation to study them comes from their connection to the plane Jacobian Conjecture. See §3.3 of [306], as well as [309], for details about homology planes.

In [284] Masuda and Miyanishi considered the case of a surface X which is a $\mathbb{Q}$-homology plane. In this case X must be affine and rational. If X is also not rigid, then the orbits are the fibers of an $\mathbb{A}^1$-fibration $X \to \mathbb{A}^1$, which implies that $\bar{\kappa}(X) = -\infty$. In the strongest case, X is a $\mathbb{Q}$-homology plane which is an ML-surface, and then the authors conclude that X is isomorphic to the quotient of one of the surfaces $xz = y^m - 1$ under a suitable free action of the cyclic group $\mathbb{Z}_m$.

Subsequently, Gurjar and Miyanishi [194] extended these results to the case in which X is a log $\mathbb{Q}$-homology plane. Their main result in this regard is that if X is a log $\mathbb{Q}$-homology plane, then $ML(X) = \mathbb{C}$ if and only if the fundamental group at infinity, $\pi_1^{\infty}(X)$, is a finite cyclic group.

Another paper classifying ML-surfaces is due to Daigle and Russell [88] (2004). They work over an algebraically closed field k, and if $k = \mathbb{C}$ then the class of surfaces they consider is the class of log $\mathbb{Q}$-homology planes with trivial Makar-Limanov invariant. Specifically, they consider the class $\mathcal{M}_0$ of normal affine surfaces U over k satisfying: (i) $ML(U) = k$ and (ii) $\mathrm{Pic}(U_s)$ is a finite group, where U_s denotes the smooth part of U, and $\mathrm{Pic}(U_s)$ denotes its Picard group. They show that every $U \in \mathcal{M}_0$ can be realized as an open subset of some weighted projective plane, and give precise conditions as to when any two such surfaces are isomorphic (Thm. A and Thm. B). In particular, surfaces U and U' belonging to this class are isomorphic if and only if the equivalence class of the weighted graphs at infinity and the resolution graphs of singularities are the same for U and U'. The authors also classify the $\mathbb{G}_a$-actions on these surfaces. For many of these surfaces, the analogue of Daigle's Transitivity Theorem for Danielewski surfaces does not hold. However, Theorem C indicates that the number of orbits in the set

$$\{\ker D \mid D \in \mathrm{LND}(\mathcal{O}(U)), D \neq 0\}$$

under the action of $\mathrm{Aut}_k\mathcal{O}(U)$ is at most 2. This theorem also gives necessary and sufficient conditions for the action to be transitive. Some of Daigle and Russell's results are based on their earlier papers [86, 87].

In Thm. 0.3 of [127] (2005), Dubouloz gives a description of normal affine surfaces with trivial Makar-Limanov invariant which generalizes previous results obtained by Daigle and Russell and by Miyanishi and Masuda. Other work on ML-surfaces may be found in Daigle [76] (2008) and in Daigle and Kolhatkar [85] (2012).

The reader is referred to the monograph of Miyanishi [306] for an excellent overview of progress in the classification of open algebraic surfaces. $\mathbb{G}_a$-actions and locally nilpotent derivations constitute one of the major themes of his exposition.

Chapter 10
Slices, Embeddings and Cancellation

The **Zariski Cancellation Problem** can be viewed as a descendant of Zariski's cancellation question for fields; see *Sect. 1.1.2*. It can be stated as follows.[1]

> Let K be a field. If X and Y are affine K-varieties and $X \times \mathbb{A}^n_K \cong_K Y \times \mathbb{A}^n_K$ for $n \geq 0$, is $X \cong_K Y$? Equivalently, if A and B are affine K-domains and $A^{[n]} \cong_K B^{[n]}$, is $A \cong_K B$?

In 1972, Abhyankar, Eakin and Heinzer [2] gave an affirmative answer to this question when $\dim_K X = \dim_K Y = 1$. In general, however, the answer to this question is negative. In 1972, Hochster [209] considered the tangent bundle $\mathcal{T}(S^2)$ of the real 2-sphere, S^2. $\mathcal{T}(S^2)$ is locally trivial, and it was known (for topological reasons) that $\mathcal{T}(S^2)$ is not trivial. Hochster showed that $\mathcal{T}(S^2)$ is stably trivial. Combining this with the observation that its coordinate ring is a formally real domain, he concluded:

$$\mathcal{T}(S^2) \times \mathbb{R}^1 \cong_{\mathbb{R}} S^2 \times \mathbb{R}^3 \quad \text{but} \quad \mathcal{T}(S^2) \not\cong_{\mathbb{R}} S^2 \times \mathbb{R}^2$$

In 1989, Danielewski [89] found a pair of smooth complex surfaces V and W such that $V \times \mathbb{C}^1 \cong W \times \mathbb{C}^1$ but $V \not\cong W$. Danielewski used a topological invariant, namely, the first homology at infinity, to distinguish these two surfaces. So V and W are not even homeomorphic. The results of Danielewski were generalized to other surfaces by Fieseler in [153], whose work highlights the role of $\mathbb{C}^+$-actions on these surfaces. See also Wilkens [420].

Notably, the surfaces of Danielewski and Fieseler are not factorial. In [60], Crachiola showed that, if X and Y are factorial affine surfaces over an algebraically closed field K, then the condition $X \times \mathbb{A}^1 \cong_K Y \times \mathbb{A}^1$ implies $X \cong_K Y$.

[1]Zariski's original question for fields is nowadays referred to as the Birational Cancellation Problem.

G. Freudenburg, *Algebraic Theory of Locally Nilpotent Derivations*,
Encyclopaedia of Mathematical Sciences 136,
DOI 10.1007/978-3-662-55350-3_10

Examples of non-cancellation for pairs of higher dimensional varieties were constructed by Finston and Maubach [154]. These examples involve factorial (singular) affine threefolds over $\mathbb{C}$ which admit a locally trivial $\mathbb{G}_a$-action. In [125], Dubouloz constructs smooth complex affine surfaces X and Y such that $X \ncong_{\mathbb{C}} Y$ and $X \times \mathbb{C} \ncong_{\mathbb{C}} Y \times \mathbb{C}$, but $X \times \mathbb{C}^2 \cong_{\mathbb{C}} Y \times \mathbb{C}^2$. See also [129, 131].

A special case of the Zariski Cancellation Problem is the **Cancellation Problem for Affine Spaces**.

Let K be a field. If X is an affine K-variety and $X \times \mathbb{A}^n_K \cong_K \mathbb{A}^{m+n}_K$ for $m, n \geq 0$, does it follow that $X \cong_K \mathbb{A}^m_K$? Equivalently, if A is an affine K-domain and $A^{[n]} \cong_K K^{[m+n]}$, is $A \cong_K K^{[m]}$?

In 1979, Fujita, Miyanishi and Sugie [177, 307] gave an affirmative answer when $m = 2$ and the characteristic of K is 0; and in 1981, Russell [364] extended their result to fields of positive characteristic. See also [191].

In 1987, Asanuma [7] used Segre's non-equivalent line embeddings in $\mathbb{A}^2_K$ over a field K of positive characteristic to construct a family of threefolds X such that $X \times \mathbb{A}^1_K \cong_K \mathbb{A}^4_K$. In 2014, N. Gupta [189] showed that $X \ncong_K \mathbb{A}^3$ for most members of this family. Her technique was to show that, for these threefolds, the Makar-Limanov invariant $ML(X)$ is non-trivial.[2] In [190], Gupta generalized these constructions: Given a field K of positive characteristic and integer $d \geq 3$, she gives a K-variety $X \ncong_K \mathbb{A}^d$ such that $X \times \mathbb{A}^1 \cong_K \mathbb{A}^{d+1}$.

For fields of characteristic zero, the Cancellation Problem for Affine Spaces is open for $m \geq 3$.

The following related question about locally nilpotent derivations of polynomial rings is of fundamental importance.

Slice Problem. If $D \in \mathrm{LND}(k^{[n]})$ has a slice, is $\ker D \cong k^{[n-1]}$?

The Slice Theorem (*Theorem 1.26*) asserts that, when $D \in \mathrm{LND}(B)$ has a slice (B a commutative k-domain), $A = \ker D$ satisfies $B = A[s]$, $D = d/ds$, and $\pi_s(B) = A$ for the Dixmier map π_s determined by s. Conversely, if $B = A[s] = A^{[1]}$, then the derivation $D = d/ds$ of B defined by $DA = 0$ and $Ds = 1$ is locally nilpotent and $\ker D = A$. So, for affine rings over k, the Slice Problem is a special case of the Cancellation Problem for Affine Spaces.

Another fundamental problem for affine algebraic geometry is the Embedding Problem. Over the field K, an algebraic embedding $g : \mathbb{A}^m_K \to \mathbb{A}^n_K$ is **rectifiable** if and only if there exists a system of coordinate functions $f_1, \ldots, f_n$ on $\mathbb{A}^n_K$ such that $g(\mathbb{A}^m_K)$ is defined by the ideal $(f_{m+1}, \ldots, f_n)$.

Embedding Problem. Is every closed algebraic embedding $g : \mathbb{A}^m_K \to \mathbb{A}^n_K$ rectifiable?

The answer is known to be affirmative if $n > 2m + 1$ [65, 220, 223, 390], or if $m = 1$ and $n = 2$ and the characteristic of K is 0 [3, 397]. The latter result is the celebrated Abhyankar-Moh-Suzuki (AMS) Theorem, also called the Epimorphism

[2] For any affine variety, the Makar-Limanov invariant can be defined as the intersection of all rings of $\mathbb{G}_a$-invariants. This generalization was introduced by Crachiola and Makar-Limanov; see [61].

Theorem. For fields of positive characteristic, non-rectifiable embeddings of $\mathbb{A}^1_K$ in $\mathbb{A}^2_K$ were given by Segre [375] (see the Introduction to [179]), and Shastri used knots to give non-rectifiable embeddings of $\mathbb{R}^1$ in $\mathbb{R}^3$. There are presently no known counterexamples to the Embedding Problem for the field $k = \mathbb{C}$, though potential counterexamples have been constructed, some of which are discussed in this chapter.

The following concerns a special case of the Embedding Problem.

Abhyankar-Sathaye Conjecture. Let k be an algebraically closed field of characteristic zero. Given $n \geq 1$, every closed algebraic embedding of $\mathbb{A}^{n-1}$ in $\mathbb{A}^n$ is rectifiable.

Popov writes:

> Every orbit of a morphic unipotent algebraic group action on $\mathbb{A}^n$ is the image of a closed embedding of some $\mathbb{A}^d$ in $\mathbb{A}^n$. In particular, orbits of dimension $n-1$ are the hypersurfaces of the sought-for type. Such actions, with a view of getting an approach to the Abhyankar-Sathaye conjecture, have been the object of study during the last decade. ([349], p. 2)

The connection between unipotent actions and the Abhyankar-Sathaye Conjecture motivated the following conjecture, due to Maubach [289].

Commuting Derivations Conjecture. Let k be an algebraically closed field of characteristic zero. Given $n \geq 2$, suppose that $D_1, \dots, D_{n-1} \in \mathrm{LND}(k^{[n]})$ are linearly independent over $k^{[n]}$ and $[D_i, D_j] = 0$ for each i, j. Then there exists a variable $f \in k^{[n]}$ such that:

$$\bigcap_{1 \leq i \leq n-1} \ker D_i = k[f]$$

Geometrically, the conjecture asserts that, for any action of $G = \mathbb{G}_a^{n-1}$ on $X = \mathbb{A}^n$, $Y = X/\!\!/G = \mathbb{A}^1$ and $X = \mathbb{A}^{n-1} \times Y$ equivariantly. The reader is referred to the article of Popov [349] for further discussion of these conjectures.

A related conjecture is the following.

Dolgachev-Weisfeiler Conjecture. If $\varphi : \mathbb{A}^n \to \mathbb{A}^m$ is a flat morphism of affine spaces in which every fiber is isomorphic to $\mathbb{A}^{n-m}$, then φ is a trivial fibration. [119]

In his 2001 thesis, Vénéreau constructed a family of fibrations $\varphi_n : \mathbb{C}^4 \to \mathbb{C}^2$ ($n \geq 1$) whose status relative to the Dolgachev-Weisfeiler Conjecture could not be determined. These examples attracted wide interest in the intervening years, and have been investigated in several papers, for example, [165, 233, 234, 237]. An earlier example of an affine fibration, due to Bhatwadekar and Dutta, appeared in 1992 [26], and it turns out that this example is quite similar to the fibration φ_1 of Vénéreau. Bhatwadekar and Dutta asked if their fibration is trivial, a question which remains open.

Section 10.1 uses the theory of locally nilpotent derivations to give a proof for Danielewski's example. This section also gives proofs of the Abhyankar, Eakin, Heinzer Theorem and of Crachiola's Theorem for algebraically closed fields of characteristic zero. *Section 10.2* explores the fascinating constructions of Asanuma, who used embeddings of affine spaces to construct torus actions on $\mathbb{A}^n$. The purpose of *Sect. 10.2* is to give a self-contained treatment of the main constructions and proofs in Asanuma's work by translating them into the language of locally nilpotent derivations. *Section 10.3* considers the examples of Vénéreau and Bhatwadekar-Dutta and their relation to the various conjectures discussed above. *Section 10.4* discusses elements of $\mathrm{LND}_R(R[X,Y,Z])$ with a slice for various rings R.

For a survey of the Cancellation Problem, Embedding Problem and related topics, see Kraft [254], Russell [365] and Gupta [191].

10.1 Zariski Cancellation Problem

10.1.1 Danielewski's Example

Let $B = k[x,y,z] = k^{[3]}$ and define hypersurfaces $V, W \subset \mathbb{A}^3$ by:

$$V = \mathcal{V}(xz - (y^2+1)) \quad \text{and} \quad W = \mathcal{V}(x^2z - (y^2+1))$$

Then V and W are smooth normal surfaces. Note that, if $k = \mathbb{C}$, then $V = \mathbb{C} \otimes_{\mathbb{R}} S^2$, the complexification of the real 2-sphere.

In the notation of *Sect. 9.1*, if $P = y^2 + 1$, then:

$$k[V] = \mathbf{D}_{(1,P)} \quad \text{and} \quad k[W] = \mathbf{D}_{(2,P)}$$

By *Corollary 9.5*, we conclude that:

$$V \not\cong_k W$$

However, their cylinders are isomorphic, as seen in the next result.

Theorem 10.1 $V \times \mathbb{A}^1 \cong_k W \times \mathbb{A}^1$

Proof Let $A = k[V] = k[x,y,z]$, where $xz - y^2 = 1$, and let $B = A[t] = A^{[1]}$. Define $D \in \mathrm{LND}(B)$ by:

$$Dx = 0 \quad Dy = x \ , \ \ Dz = 2y \ , \ \ Dt = 3z$$

Let $R = \ker D$. If $s = \frac{1}{2}(xt - yz)$, then $Ds = xz - y^2 = 1$, so s is a slice for D. By the Slice Theorem, we have $B = R[s] = R^{[1]}$.

If $\pi_s : B \to R$ is the Dixmier map defined by s, then π_s is surjective, and $B = k[x, Y, Z, T]$, where:

$$x = \pi_s(x) \ , \ \ Y = \pi_s(y) \ , \ \ Z = \pi_s(z) \ , \ \ T = \pi_s(t)$$

By direct calculation, we see that $Z \in xB \cap R = xR$. If $Z = x\tilde{Z}$ for $\tilde{Z} \in R$, then:

$$1 = \pi_s(xz - y^2) = xZ - Y^2 = x^2\tilde{Z} - Y^2$$

In addition:

$$0 = \pi_s(s) = xT - YZ = x(T - Y\tilde{Z}) \quad \Rightarrow \quad T = Y\tilde{Z}$$

Therefore, $R = k[x, Y, Z, T] = k[x, Y, \tilde{Z}] \cong k[W]$. □

Remark 10.2 This example shows that, in general, $ML(B^{[1]}) \neq ML(B)$. Since $ML(V) = k$, it follows that $ML(W \times \mathbb{A}^1) = ML(V \times \mathbb{A}^1) = k$, even though $ML(W) = k[x]$. So there exists $D \in \mathrm{LND}(k[W]^{[1]})$ with $Dx \neq 0$. Makar-Limanov constructs such an example in [279]. Let $k[W]^{[1]} = k[x, y, z, t]$, where $x^2z = y^2 - 1$, and define D by the jacobian determinant:

$$Df = \frac{\partial(x^2z - y^2, t^2x + 2ty + xz, t^3x + 3t^2y + 3txz + yz, f)}{\partial(x, y, z, t)} \ , \ f \in k[W][t]$$

So one would like to have conditions under which $ML(B^{[1]}) = ML(B)$. Makar-Limanov conjectured in [279] that this is the case whenever B is a UFD.

10.1.2 Abhyankar-Eakin-Heinzer Theorem

The Abhyankar-Eakin-Heinzer Theorem appeared in 1972.

Theorem 10.3 ([2], Thm. 3.3) *Let K be a field, and let R and S be integral K-domains of transcendence degree one over K. If $R^{[n]} \cong_K S^{[n]}$ for some $n \geq 0$, then $R \cong_K S$. If R is not of the form $L^{[1]}$ for an algebraic extension field L of K, then any isomorphism $\alpha : R^{[n]} \to S^{[n]}$ satisfies $\alpha(R) = S$.*

Miyanishi (1973) and Kang (1987) also gave proofs of this result [240, 295]. The following proof is for algebraically closed fields of characteristic zero, and is due to Makar-Limanov (1998) [276]. The proof of Miyanishi uses the theory of $\mathbb{G}_a$-actions in any characteristic, and the proof of Makar-Limanov is based on the theory of locally nilpotent derivations.

Proof Assume that $K = k$ is algebraically closed of characteristic zero, and let $\alpha : R^{[n]} \to S^{[n]}$ be a k-algebra isomorphism. Since $ML(R)$ is algebraically closed in R, either $ML(R) = R$ or $ML(R) = k$. By *Theorem 2.28*, we have $ML(R^{[n]}) = ML(R)$.

If $ML(R^{[n]}) = R$, then $ML(S^{[n]}) = S$ by symmetry. Since α maps the Makar-Limanov invariant onto itself, we conclude that $\alpha(R) = S$.

If $ML(R^{[n]}) = k$, then *Theorem 2.28* implies that $ML(R) = k$. By *Corollary 1.28*, we have $ML(R) \neq R$ if and only if $R = k^{[1]}$. But then $ML(S^{[n]}) = k$ and $S = k^{[1]}$ as well. □

Remark 10.4 Most of the heavy lifting for this proof was done already in *Theorem 2.28*, aided in no small measure by the theorems of Seidenberg and Vasconcelos. In [276], Makar-Limanov used the techniques he pioneered, notably involving the ring of absolute constants, to give a proof of the Abhyankar-Eakin-Heinzer Theorem for the field $K = \mathbb{C}$. This approach was adapted by Crachiola [57] and by Crachiola and Makar-Limanov [61] to give proofs in the case K is an algebraically closed field. The proof given above first appeared in [170]. It follows the ideas of Makar-Limanov, but the proof of *Theorem 2.28* is considerably shorter than the proofs given in [57, 61, 276] precisely because it invokes the theorems of Seidenberg and Vasconcelos.

10.1.3 Crachiola's Theorem

Theorem 10.5 ([60], Cor. 3.2) *Let K be an algebraically closed field, and let A, B be affine UFDs of dimension 2 over K. If $A^{[1]} \cong_K B^{[1]}$, then $A \cong_K B$.*

Proof (Characteristic $K = 0$) Let $k = K$ be an algebraically closed field of characteristic 0, let $R = A[x] = A^{[1]}$ and $S = B[y] = B^{[1]}$, and let $\alpha : R \to S$ be a k-algebra isomorphism. Then α restricts to an isomorphism $\alpha : ML(R) \to ML(S)$.

Suppose that $ML(A) = A$. Then $ML(R) = ML(A) = A$ by the Semi-Rigidity Theorem (*Theorem 2.24*). Therefore:

$$A = ML(A) = ML(R) \cong_k ML(S) \subset ML(B)$$

It follows that $\text{tr.deg}_k ML(B) = 2$ and $ML(B) = B$. Therefore:

$$A \cong_k ML(S) = B$$

By symmetry, if $ML(B) = B$, then $A \cong_k B$ as well.

Suppose that $ML(A) \neq A$ and $ML(B) \neq B$. Let $D \in \text{LND}(A)$ and $E \in \text{LND}(B)$ be irreducible. By *Lemma 2.10*, there exist $s \in A$ and $t \in B$ such that $Ds = 1$ and $Et = 1$, implying $A = A^D[s]$ and $B = B^E[t]$. Therefore:

$$(A^D)^{[2]} = A^D[s, x] = R \cong_k S = B^E[t, y] = (B^E)^{[2]}$$

Since $\text{tr.deg}_k A^D = \text{tr.deg}_k B^E = 1$, the Abhyankar-Eakin-Heinzer Theorem implies $A^D \cong_k B^E$. It follows that:

$$A = A^D[s] \cong_k B^E[t] = B$$

□

As a corollary, we obtain the following special case of the cancellation theorem for surfaces.

Corollary 10.6 *Let K be a perfect field. If B is a commutative K-domain such that $B^{[1]} \cong_K K^{[3]}$, then $B \cong_K K^{[2]}$.*

Proof Let L be the algebraic closure of K, and set $B_L = L \otimes_K B$. Then:

$$B_L^{[1]} \cong_L L \otimes_K K^{[3]} = L^{[3]}$$

We may suppose that $B_L \subset L[X,Y,Z] = L^{[3]}$ and that $B_L[T] = L[X,Y,Z]$ for some $T \in L[X,Y,Z]$. Since B_L is a factorially closed subring of a UFD, we see that B_L is a UFD containing L (*Lemma 2.8*). In addition, we have $B_L \cong_L L[X,Y,Z]/(T)$, which implies that B_L is of finite type over L. By *Theorem 10.5*, $B_L \cong_L L^{[2]}$. By Kambayashi's Theorem (*Theorem 5.2*), all forms of the affine plane over a perfect field are trivial, so $B \cong_K K^{[2]}$. □

Question 10.7 Let K be an algebraically closed field, and let A, B be affine UFDs over K of dimension 2 over K. Does the condition $A^{[n]} \cong_K B^{[n]}$ for some $n \geq 0$ imply $A \cong_K B$?

Remark 10.8 In 1975, Miyanishi [296] showed that, if S is a surface over an algebraically closed field which contains a cylinderlike open set and which satisfies $S \times \mathbb{A}^1 \cong \mathbb{A}^3$, then $S \cong \mathbb{A}^2$. Subsequently, Miyanishi and Sugie gave a geometric characterization of affine surfaces which contain a cylinderlike open set [307]. Based on these results, Fujita [177] in 1979 gave his proof of the cancellation theorem for surfaces for fields of characteristic zero. The theorem featured by Fujita in his paper is as follows.

> Let S be a surface defined over a field of characteristic zero such that $S \times \mathbb{A}^1 \cong \mathbb{A}^3$. Then $S \cong \mathbb{A}^2$. (Thm., p. 106)

The more general version of the cancellation theorem for surfaces (for which proofs using LNDs have not yet succeeded) appears as a corollary towards the end of Fujita's paper. It takes the following form.

> Let S be an affine surface. Suppose that $S \times V \cong \mathbb{A}^2 \times V$ for some algebraic variety V. Then $S \cong \mathbb{A}^2$. (Cor. (3.3))

Using $\bar{\kappa}$ for the logarithmic Kodaira dimension and $H.(\ ;\mathbb{Z})$ for integral homology, Fujita writes as justification: "Because both $\bar{\kappa}$ and $H.(\ ;\mathbb{Z})$ are cancellation invariants" (p. 109). These two invariants characterize $\mathbb{A}^2$.

In 2002, Gurjar [192] gave a topological proof that, if S is a complex surface and $S \times \mathbb{C} = \mathbb{C}^3$, then $S = \mathbb{C}^2$. In his 2004 thesis, Crachiola used the Semi-Rigidity Theorem to prove that, if K is an algebraically closed field and A is an affine K-domain with $A^{[1]} = K^{[3]}$, then $A = K^{[2]}$ ([57], Example 4.8). Crachiola and Makar-Limanov gave another version of this argument in [62].

10.1.4 Hamann's Theorem

In her 1973 doctoral thesis [200] and in the subsequent article [201], Hamann shows the following.

Theorem 10.9 ([201], Thm. 2.8) *Let R be a commutative $\mathbb{Q}$-algebra. If B is an R-algebra such that $B^{[n]} \cong_R R^{[n+1]}$ for some $n \geq 0$, then $B \cong_R R^{[1]}$.*

Hamann also shows that the same conclusion holds for commutative rings R which are seminormal, and gives examples to show that the result may fail to hold for other commutative rings. See also Hochster [210], Section 9, for a discussion of Hamann's Theorem and examples.

10.1.5 Hochster's Example

Recall that an integral domain A is **formally real** if, given $m \geq 1$, the only solution to $\sum_{1 \leq i \leq m} a_i^2 = 0$ for $a_i \in A$ is $a_i = 0$ for each i. The field $\mathbb{R}$ of real numbers is a formally real field.

Lemma 10.10 *Let A be a formally real domain.*

(a) *Any subring of frac(A) is a formally real domain.*
(b) *Given $n \geq 0$, if $p_1, \dots, p_m \in A^{[n]}$ and $\sum_{1 \leq i \leq m} p_i^2 \in A$, then $p_i \in A$ for each i.*
(c) *$A^{[n]}$ is formally real for all $n \geq 0$.*
(d) *If B is a ring with $A \subset B \subset A^{[n]}$ for some $n \geq 0$, then B is formally real.*

Proof Part (a) follows easily from the definition.

For part (b), it suffices to prove the case $n = 1$. Let $A[t] = A^{[1]}$ and suppose that $\sum_{1 \leq i \leq m} p_i(t)^2 \in A$ for nonzero $p_i(t) \in A[t]$. Let $d_i = \deg_t p_i$ and let $a_i \in A$ be the coefficient of t^{d_i} in $p_i(t)$, noting that $a_i \neq 0$ for each i.

Suppose that $\max_{1 \leq i \leq m} d_i \neq 0$. Then we may assume that:

$$d_1 \geq d_2 \geq \cdots \geq d_m \geq 1$$

In this case, there exists $1 \leq j \leq m$ such that $d_1 = \cdots = d_j$ and:

$$(a_1 t^{d_1})^2 + \cdots (a_j t^{d_j})^2 = 0 \quad \Rightarrow \quad a_1^2 + \cdots a_j^2 = 0$$

But then $a_i = 0$ for $1 \leq i \leq j$, a contradiction. Therefore, $\max_{1 \leq i \leq m} d_i = 0$, and part (b) is proved.

If $\sum_{1 \leq i \leq m} P_i^2 = 0$ for $P_i \in A^{[n]}$, then part (b) implies $P_i \in A$ for each i. But then $P_i = 0$ for each i, and part (c) is proved. Part (d) is implied by parts (a) and (c). $\square$

Given $n \geq 1$, let A be the coordinate ring of S^n, the real sphere of dimension n:

$$A = \mathbb{R}[x_0, \dots, x_n]/(x_0^2 + \cdots + x_n^2 - 1)$$

Since frac$(A) = \mathbb{R}^{(n)}$, *Lemma 10.10* implies that A is a formally real domain.

Let $\mathcal{T}(S^n)$ be the algebraic tangent bundle of S^n. Its coordinate ring is:

$$B = A[y_0, \dots, y_n]/(x_0 y_0 + \cdots + x_n y_n)$$

Lemma 10.11 *For each $n \geq 1$:*

(a) $B^{[1]} \cong_A A^{[n+1]}$
(b) *B is a formally real domain.*
(c) *If $\varphi \in Aut_{\mathbb{R}}(B)$, then $\varphi(A) = A$.*
(d) *$LND(B) = LND_A(B)$*

Proof Let $A[X_0, \dots, X_n] = A^{[n+1]}$. Define $D \in \text{LND}_A(A^{[n+1]})$ by $DX_i = x_i$, $0 \leq i \leq n$. If $s = x_0X_0 + \cdots + x_nX_n$, then $Ds = 1$ and:

$$K := \ker D \cong_A A[X_0, \dots, X_n]/(s) = B$$

Therefore, by the Slice Theorem, $A[X_0, \dots, X_n] = K[s] = K^{[1]} \cong_A B^{[1]}$. This proves part (a). Part (b) follows from part (a) and *Lemma 10.10(d)*.

For part (c), we have:

$$x_0^2 + \cdots + x_n^2 = 1 \quad \Rightarrow \quad \varphi(x_0)^2 + \cdots + \varphi(x_n)^2 = 1$$

By part (a), the latter equation is in $B \subset A^{[n+1]}$, and *Lemma 10.10(b)* implies that $\varphi(x_i) \in A$ for each i. Therefore, $\varphi(A) \subset A$. By the same argument, $\varphi^{-1}(A) \subset A$. It follows that $\varphi(A) = A$.

For part (d), let $D \in \text{LND}(B)$ be given. Part (c) shows $\exp(tD)(A) = A$, and this implies $D(A) \subset A$. But A is a rigid ring: The orbits of a $\mathbb{G}_a$-action on S^n are closed, and therefore compact, which implies they are of dimension zero. Therefore, $D(A) = 0$. □

For the next result, note that S^n is a subvariety of $\mathcal{T}(S^n)$ via the ideal $y_0B + \cdots + y_nB$.

Lemma 10.12 *Given $n \geq 1$, every $\mathbb{G}_a$-action on $\mathcal{T}(S^{2n})$ has a fixed point in S^{2n}.*

Proof Let M be an orientable topological manifold, and let $\chi(M)$ denote its Euler characteristic. A well-known theorem from topology asserts that, if M admits a nonvanishing (tangent) vector field, then $\chi(M) = 0$. See, for example, [215]. Since $\chi(S^{2n}) = 2$ and $\chi(S^{2n-1}) = 0$, it follows that there are no nonvanishing vector fields on spheres S^{2n} of even dimension.

Suppose that $N \geq 1$. In coordinates $(a_0, ., ., , a_N, b_0, \dots, b_N)$ on $\mathcal{T}(S^N)$, the subvariety $S^N \subset \mathcal{T}(S^N)$ is given by $b_0 = \cdots b_N = 0$. Given $D \in \text{LND}(B)$, *Lemma 10.11(d)* implies that $DA = 0$. If $p \in S^N$ is given by $p = (a_0, \dots, a_N, 0, \dots, 0)$, then the orbit $\mathbb{G}_a \cdot p$ of p under the $\mathbb{G}_a$-action induced by D is given by:

$$f(t,p) = (a_0, \dots, a_N, f_0(t,p), \dots, f_N(t,p))$$

where $f_i(t,p) \in t\mathbb{R}[t]$ and for each i. Therefore, $\mathbb{G}_a \cdot p$ is contained in $T_p(S^N)$, the tangent space to S^N at p, and the mapping $p \mapsto f'(0,p)$ defines a tangent vector field on S^N. Note that p is a fixed point for the $\mathbb{G}_a$-action if and only if $f'(0,p) = 0$.

If $N = 2n$, then the theorem from topology cited above implies that $f'(0, p) = 0$ for some $p \in S^{2n}$, and such a point p is a fixed point of the $\mathbb{G}_a$-action induced by D. □

We can now prove the following generalization of Hochster's result [209].

Theorem 10.13 *Given $n \geq 1$,*

$$\mathcal{T}(S^{2n}) \times \mathbb{R}^1 \cong_{\mathbb{R}} S^{2n} \times \mathbb{R}^{2n+1}$$

but there is no subvariety $Z \subset \mathcal{T}(S^{2n})$ such that $\mathcal{T}(S^{2n}) \cong_{\mathbb{R}} Z \times \mathbb{R}^1$. In particular, $\mathcal{T}(S^{2n}) \ncong_{\mathbb{R}} S^{2n} \times \mathbb{R}^{2n}$.

Proof The first assertion is implied by *Lemma 10.11(a)*. For any algebraic $\mathbb{R}$-variety Z, the cylinder $Z \times \mathbb{R}^1$ admits a free $\mathbb{G}_a$-action. Therefore, the second assertion is implied by *Lemma 10.12*. □

Remark 10.14 In [210], Hochster remarks that the only proof that he knows of non-triviality for $\mathcal{T}(S^2)$ depends on the topological fact that there is no continuous nonvanishing vector field on S^2 (p. 62). Similarly, Eisenbud [134] writes: "It is a remarkable fact, still lacking a simple algebraic proof" that the algebraic tangent bundle $\mathcal{T}(S^n)$ to the real n-sphere is trivial if and only if $n = 0, 1, 3$ or 7 (p. 485).

10.2 Asanuma's Torus Actions

In a remarkable paper published in 1999, Asanuma showed how non-rectifiable embeddings $\mathbb{R}^1 \hookrightarrow \mathbb{R}^3$ (e.g., knots) could be used to construct non-linearizable algebraic actions of real tori $\mathbb{R}^*$ and $(\mathbb{R}^*)^2$ on $\mathbb{R}^5$ [8]. These were the first examples in which a commutative reductive k-group admits a non-linearizable algebraic action on affine space, where k is a field of characteristic 0. Later, using completely different methods, the author and Moser-Jauslin found a non-linearizable action of the circle group $S^1 = SO_2(\mathbb{R})$ on $\mathbb{R}^4$ [173]. It remains an open question whether, over the field $k = \mathbb{C}$, every algebraic action of a commutative reductive group on $\mathbb{C}^n$ can be linearized; for a discussion of the linearization problem for complex reductive groups, see [254].

The main result Asanuma proves is this.

Theorem 10.15 (Cor. 6.3 of [8]) *Let K be an infinite field. If there exists a non-rectifiable closed embedding $\mathbb{A}^m_K \to \mathbb{A}^n_K$, then there exist non-linearizable faithful algebraic $(K^*)^r$-actions on $\mathbb{A}^{1+n+m}_K$ for each $r = 1, \dots, 1 + m$.*
To put his theorem to use, Asanuma employs an earlier result of Shastri.

Theorem 10.16 ([381]) *Every open knot type admits a polynomial representation in $\mathbb{R}^3$.*

Therefore, for topological reasons, there exist non-rectifiable algebraic embeddings of $\mathbb{R}^1$ into $\mathbb{R}^3$. For example, Shastri gives the following polynomial parametrization of the trefoil knot:

$$\phi(v) = (v^3 - 3v, v^4 - 4v^2, v^5 - 10v)$$

Note that the complexification $\phi_{\mathbb{C}}$ of ϕ defines an algebraic embedding of $\mathbb{C}^1$ into $\mathbb{C}^3$. It is presently unknown whether its image can be conjugated to a coordinate line by an algebraic automorphism of $\mathbb{C}^3$. However, Kaliman proved that the image of any algebraic embedding of $\mathbb{C}^1$ in $\mathbb{C}^3$ can be conjugated to a coordinate line by a *holomorphic* automorphism of $\mathbb{C}^3$; see [224]. Thus, there is no topological obstruction, or even analytic obstruction, to rectifying $\phi_{\mathbb{C}}$.

Asanuma uses a purely algebraic approach in his constructions. The main idea in his paper is to associate a certain Rees algebra to an embedding $\phi : K^m \hookrightarrow K^n$, namely, the Rees algebra of the ideal $\ker(\phi^* : K^{[n]} \to K^{[m]})$.

In the present treatment, we consider only embeddings $\phi : k^1 \hookrightarrow k^3$. We associate to ϕ a certain triangular derivation D on $k^{[5]}$, and show that D induces a torus action in the manner of *Theorem 10.30* (which appears in the *Appendix* to this chapter). Geometrically, the fixed-point set L is isomorphic to a line k^1, and the quotient Q is isomorphic to k^3. The main fact is that the canonical embedding of L into Q is precisely the embedding ϕ. It should be noted that van den Essen and van Rossum also recognized derivations implicit in Asanuma's work; see [146].

10.2.1 Derivation Associated to an Embedding

Let $\phi : k^1 \to k^3$ be an algebraic embedding, given by $\phi(v) = (f(v), g(v), h(v))$. Specifically, this means ϕ is injective and $\phi'(v) \neq 0$ for all v (see van den Essen [142], Cor. B.2.6). Let ϕ^* denote the corresponding ring homomorphism $\phi^* : k[x, y, z] \to k[v]$, given by $\phi^*(p(x, y, z)) = p(\phi(v))$. Since ϕ is an embedding, ϕ^* is surjective, i.e., there exists $F \in k[x, y, z]$ such that $\phi^*(F) = v$.

Set $\mathbf{x} = (x, y, z)$, and let $B = k^{[5]} = k[u, v, \mathbf{x}]$. Define a triangular derivation D on B by

$$Du = 0 \ , \ Dv = -u \ , \ D\mathbf{x} = \phi'(v) \tag{10.1}$$

and let $A = \ker D$. Observe that $D(u\mathbf{x} + \phi(v)) = 0$, and thus $F(u\mathbf{x} + \phi(v)) \in A$. Observe also that (by Taylor's Formula) there exists $s \in B$ such that:

$$F(u\mathbf{x} + \phi(v)) = us + F(\phi(v)) = us + v$$

Therefore, $0 = D(us + v) = uDs - u$, which implies $Ds = 1$, i.e., D has a slice and $B = A[s]$.

From the Slice Theorem, we conclude that the kernel of D is generated by the images under π_s of the system of variables $u, v, \mathbf{x}$, namely:

$$\ker D = k[u, v + us, \mathbf{x} - \textstyle\sum_{i \geq 1} \frac{1}{i!}\phi^{(i)}(v)u^{i-1}s^i\,] \tag{10.2}$$

So the kernel, which is of transcendence degree four over k, is generated by 5 polynomials. In general, it is unknown whether this number can always reduced to four, or equivalently, whether s is a variable of B. Recall that s is known to be a variable of $B[t] = k^{[6]}$ (*Proposition 3.26*).

For example, using Shastri's parametrization of the trefoil knot, given above, it is easy to calculate the slice s explicitly. As Shastri points out (p. 14), if $F = yz - x^3 - 5xy + 2z - 7x$, then $F(\phi(v)) = v$, and thus:

$$s = u^{-1}(F(u\mathbf{x} + \phi(v)) - v)$$

Of course, when ϕ is rectifiable, D is conjugate to a partial derivative. In this case, finding a system of variables for the kernel is equivalent to finding a system of coordinates in $\mathbb{A}^3$ relative to which $\phi(\mathbb{A}^1)$ is a coordinate line.

Theorem 10.17 *Suppose $\phi : \mathbb{A}^1 \to \mathbb{A}^3$ is a rectifiable embedding, and let D be the induced triangular derivation (10.1) of $k[u, v, \boldsymbol{x}]$. If $k[x, y, z] = k[F, G, H]$, where $\phi^*(F) = v$ and $\ker \phi^* = (G, H)$, then:*

$$\ker D = k[u, F(u\boldsymbol{x} + \phi(v)), \tfrac{1}{u}G(u\boldsymbol{x} + \phi(v)), \tfrac{1}{u}H(u\boldsymbol{x} + \phi(v))]$$

In particular, D is conjugate to a partial derivative.

Proof Set $A = \ker D$, and let w denote the slice of D derived from F, i.e., $v + uw = F(u\mathbf{x} + \phi(v))$. Set:

$$m = \tfrac{1}{u}G(u\mathbf{x} + \phi(v)) \ , \ n = \tfrac{1}{u}H(u\mathbf{x} + \phi(v))$$

Then $m, n \in A$, since A is factorially closed.

Let $R \subset A$ denote the subring:

$$\begin{aligned} R &= k[u\mathbf{x} + \phi(v)] \\ &= k[F(u\mathbf{x} + \phi(v)), G(u\mathbf{x} + \phi(v)), H(u\mathbf{x} + \phi(v))] \\ &= k[v + uw, um, un] \end{aligned}$$

Set $\mathbf{X} = \pi_w(\mathbf{x})$. Since π_w fixes elements of A, and since $\pi_w(\phi(v)) = \phi(v + uw)$, we see that:

$$u\mathbf{X} + \phi(v + uw) = \pi_w(u\mathbf{x} + \phi(v)) = u\mathbf{x} + \phi(v)$$

Therefore, $u\mathbf{X} + \phi(v + uw) \in R^3$ implies $u\mathbf{X} \in R^3$. Note that the R-ideal $uB \cap R$ is generated over R by um and un. Therefore we can write $u\mathbf{X} = (um)\mathbf{p} + (un)\mathbf{q}$ for some $\mathbf{p},\mathbf{q} \in R^3$. But then

$$\mathbf{X} = m\mathbf{p} + n\mathbf{q} \in k[u, v + uw, m, n]^3$$

which implies:

$$A = \pi_w(B) = k[u, v + uw, \mathbf{X}] \subset k[u, v + uw, m, n] \subset A$$

□

10.2.2 Two-Dimensional Torus Action

Let $G = G_1 \times G_2$ denote the two-dimensional torus $\mathbb{G}_m^2$, where each $G_i \cong \mathbb{G}_m$. Define an algebraic action of G on $\mathbb{A}^5$ in the following way.

- G_1 acts as in *Theorem 10.30*, namely, the action of $\theta \in G_1$ is given by $\exp(\lambda D)|_{\lambda=(1-\theta)s}$. Observe that $B^{G_1} = A$.
- G_2 acts linearly, namely, the action of $t \in G_2$ is given by $t(u, v, \mathbf{x}) = (t^{-1}u, v, t\mathbf{x})$.

If the G_2-action is extended to $B[\lambda]$ by declaring that $t \cdot \lambda = t\lambda$, then λD is homogeneous of degree 0 relative to the G_2-action. It follows that the actions of G_1 and G_2 commute (see *Sect. 3.7*), and we thus obtain an action of the torus G on $\mathbb{A}^5$. Explicitly, this action is given by:

$$(\theta, t)(u, v, \mathbf{x}) = \left(t^{-1}u, v + (1-\theta)us, t\left(\mathbf{x} - \sum_{i \geq 1} \tfrac{1}{i!}\phi^{(i)}(v)u^{i-1}(1-\theta)^i s^i\right)\right) \tag{10.3}$$

Theorem 10.18 *For the G-action above:*

1. $B^G = k[u\boldsymbol{x} + \phi(v)] \cong k^{[3]}$
2. *The fixed point set Fix(G) is the line defined by the ideal* $I = (u, \boldsymbol{x})$.
3. *The canonical embedding* $Fix(G) \hookrightarrow Spec(B^G)$ *is equivalent to the embedding* $\phi : k^1 \hookrightarrow k^3$.

Proof First, $B^G = (B^{G_2})^{G_1} = k[v, u\mathbf{x}]^{G_1}$. Now $k[v, u\mathbf{x}] \cong k^{[4]}$, and the derivation D restricts to a mapping $\delta : k[v, u\mathbf{x}] \to uk[v, \mathbf{x}]$, where $\ker \delta = k[v, u\mathbf{x}]^{G_1}$.

Define the triangular derivation Δ on $k[v, u\mathbf{x}]$ by $\Delta v = -1$ and $\Delta(u\mathbf{x}) = \phi'(v)$. Then $\delta = u\Delta$. Since $k[v, u\mathbf{x}] = k[v, u\mathbf{x} + \phi(v)]$, it follows that $\ker \delta = \ker \Delta = k[u\mathbf{x} + \phi(v)]$. This proves (1), and (2) is obvious.

As for (3), the canonical embedding of the fixed points into the quotient is induced by the composition $B^G \hookrightarrow B \to B/I$. The image of $p(u\mathbf{x} + \phi(v)) \in B^G$ equals $p(\phi(v)) (\mathrm{mod}\, I)$, and (3) now follows. □

10.2.3 One-Dimensional Torus Action

Let $H \subset G$ denote the one-dimensional torus consisting of pairs $(t, t^{-1}) \in G$.

Theorem 10.19 *For the H-action above:*

1. $B^H = k[u\boldsymbol{x} + \phi(v), s] \cong k^{[4]}$
2. *The fixed point set* $Fix(H)$ *is the surface defined by the ideal*

$$J = (u, \boldsymbol{x} - \phi'(v)s)$$

and $Fix(H) \cong k^2$.

3. *The canonical embedding* $Fix(H) \hookrightarrow Spec(B^H)$ *is equivalent to the embedding* $1 \times \phi : k^2 \hookrightarrow k^4$.

Proof Note that, since D and H commute, D restricts to B^H, and H restricts to $\ker D$. Denote the restriction of D to B^H by d. Then $\ker d = (\ker D)^H = (B^{G_1})^H = B^G$ (since $G = G_1 \times H$). In addition, since $s \in B^H$ and $ds = 1$, it follows that $B^H = (\ker d)[s] = B^G[s]$. So (1) follows by *Theorem 10.18* (see the *Appendix* to this chapter).

For (2), note first the condition $t^{-1}u = u$ for all $t \in k^*$ implies $u = 0$. Second, $v = v + (1 - t^{-1})us$, which imposes no new condition, since $u = 0$. Similarly, since $u = 0$, we see from the formula (10.3) that:

$$\mathbf{x} = t\Big(\mathbf{x} - \phi'(v)(1 - t^{-1})s\Big) \;\Rightarrow\; (1 - t)\mathbf{x} = -t\phi'(v)(1 - t^{-1})s = (1 - t)\phi'(v)s$$

Item (2) now follows by cancelling $(1 - t)$ on each side.

As for (3), the canonical embedding of the fixed points into the quotient is induced by the composition $B^H \hookrightarrow B \to B/J$. Since $B = A[s]$, we see from line (10.2) that B/J is generated by the class of v and s, so $B/J \cong k^{[2]}$. Since the image of $p(u\mathbf{x} + \phi(v), s) \in B^H$ equals $p(\phi(v), s) \mod J$, (3) now follows. □

Corollary 10.20 *If the embedding* $\phi : \mathbb{A}^1 \hookrightarrow \mathbb{A}^3$ *is not rectifiable, then the induced algebraic actions of* $H = \mathbb{G}_m$ *and* $G = \mathbb{G}_m^2$ *on* $\mathbb{A}^5$ *are not linearizable.*

Proof If the $\mathbb{G}_m$-action were linearizable, it would be conjugate to the induced tangent space action, in which case the fixed points would be embedded in the quotient as a linear subspace. In particular, this embedding would be rectifiable, contradicting part (3) of *Theorem 10.19*. Therefore, the $\mathbb{G}_m$-action cannot be linearizable. Since it is a restriction of the $\mathbb{G}_m^2$-action, the $\mathbb{G}_m^2$-action is also not linearizable. □

Remark 10.21 There exist non-rectifiable holomorphic embeddings of $\mathbb{C}$ into $\mathbb{C}^n$ for all $n \geq 2$. By methods similar to Asanuma, Derksen and Kutzschebauch use these embeddings to show that, for every nontrivial complex reductive Lie group G, there exists an effective non-linearizable holomorphic action of G on some affine space. See their article [99] for details.

Question 10.22 Let $\phi : \mathbb{R}^1 \hookrightarrow \mathbb{R}^3$ denote the trefoil knot embedding given above, and let D be the triangular derivation of $\mathbb{R}^{[5]}$ associated to ϕ. Is $\ker D$ a polynomial ring over $\mathbb{R}$?
If the answer here is negative, Asanuma's constructions provide a counterexample to the Cancellation Problem for Affine Spaces over the field of real numbers.

The reader is referred to the article of Bhatwadekar and Roy [29] for further discussion of embedded lines. See also [167] for methods to obtain families of polynomial knot embeddings in $\mathbb{R}^3$.

10.3 Examples of Bhatwadekar-Dutta and Vénéreau

The Vénéreau polynomials f_n were first defined explicitly by Vénéreau in his 2001 thesis [411]. They evolved out of his work with Kaliman and Zaidenberg on questions related to cancellation and embedding problems in the affine setting. Their papers [234, 237] discuss the origin and importance of these polynomials.

Vénéreau showed that each inclusion $\mathbb{C}[x,f_n] \to \mathbb{C}^{[4]}$ defines an affine fibration $\mathbb{C}^4 \to \mathbb{C}^2$, and that f_n is an x-variable of B for $n \geq 3$. In 2013, Lewis [272] succeeded in showing that f_2 is also an x-variable. So it is only f_1 for which the status relative to the Embedding Problem and Dolgachev-Weisfeiler Conjecture is unknown.

Recall from *Example 3.18* that, if $B = \mathbb{C}[x,y,z,u] = \mathbb{C}^{[4]}$, then

$$f_1 = y + xv_1 \quad \text{where} \quad p_1 = yu + z^2 \quad \text{and} \quad v_1 = xz + yp_1$$

Similarly, the polynomial in the earlier example of Bhatwadekar and Dutta ([26], Example 4.13) can be realized as:

$$\tilde{f} = y + x\tilde{v} \quad \text{where} \quad \tilde{p} = yu + z^2 + z \quad \text{and} \quad \tilde{v} = xz + y\tilde{p}$$

The authors show that the inclusion $\mathbb{C}[x,\tilde{f}] \to B$ induces an affine fibration $\mathbb{C}^4 \to \mathbb{C}^2$, and ask whether this is a trivial fibration (Question 4.14).

10.3.1 Family of Affine Fibrations of $\mathbb{A}^4$

Let B be a commutative k-domain and A a subalgebra. B is a **stably polynomial algebra over** A if and only if $B^{[n]} = A^{[m+n]}$ for some $m, n \geq 0$. If $B = k^{[m]}$, then $f \in B$ is a **stable variable** or **stable coordinate** of B if and only if f is a variable of $B^{[n]} = k^{[m+n]}$ for some $n \geq 0$ if and only if $f \notin k$ and B is a stably polynomial algebra over $k[f]$.

In [82], the authors consider a class of polynomials which includes both f_1 and $\tilde{f}$. The following result is a special case of [82], Prop. 1.

Theorem 10.23 *Let $B = k[x,y,z,u] = k^{[4]}$, where k be a field of characteristic zero. Suppose that $v, w \in B$ are such that:*

$$k[x, x^{-1}, y, z, u] = k[x, x^{-1}, y, v, w]$$

Given $\phi \in k[x, v]$, define $f = y + x\phi$ and $A = k[x,f]$.

(a) *B is a stably polynomial algebra over A, i.e., $B^{[n]} = A^{[n+2]}$ for some $n \geq 0$.*
(b) *If $a \in A$ is such that $A/aA = k^{[1]}$, then $B/aB = (A/aA)^{[2]} = k^{[3]}$.*

The proof given in [82] relies on the theory of affine fibrations, and the reader is referred to the article for the proof of this theorem.

Define $p = yu + z^2 + cz$ for $c \in k$, and define $\theta \in \mathrm{LND}(k[x, x^{-1}, y, z, u])$ by:

$$\theta x = \theta y = 0 \ , \ \ \theta z = x^{-1}y \ , \ \ \theta u = -x^{-1}(2z + c)$$

Note that $\theta p = 0$. Set:

$$v = \exp(p\theta)(xz) = xz + yp \ , \ \ w = \exp(p\theta)(x^2u) = x^2u - xp(2z + c) - yp^2$$

Then $k[x, x^{-1}, y, z, u] = k[x, x^{-1}, y, v, w]$. Define:

$$f = y + xv \quad \text{and} \quad A = k[x,f] \tag{10.4}$$

For any $c \in k$, *Theorem 10.23* implies that:

(i) $B^{[n]} = A^{[n+2]}$ for some $n \geq 0$
(ii) $B/(f - \lambda(x)) \cong \mathbb{A}^3$ for each $\lambda(x) \in k[x]$

In the next section, we show $B^{[1]} = A^{[3]}$ for any such f. Note that, if $c = 0$, then $f = f_1$ (Vénéreau's polynomial), and if $c = 1$, then $f = \tilde{f}$ (the polynomial of Bhatwadekar and Dutta).

10.3.2 Stable Coordinates in $k^{[4]}$

Theorem 10.24 (See [165]) *Let $B = k[x,y,z,u] = k^{[4]}$ and $B[t] = B^{[1]}$, and let $f \in B$ and $A = k[x,f]$ be as in (10.4). Then $B[t] = A^{[3]}$.*

Proof To simplify notation, given $h_1, \dots, h_5 \in B[t]$, define:

$$\partial(h_1, h_2, h_3, h_4, h_5) = \frac{\partial(h_1, h_2, h_3, h_4, h_5)}{\partial(x, y, z, u, t)}$$

Note that ∂ is a derivation in each of its arguments.

Define $\delta \in \mathrm{LND}(B[t])$ by $\delta = -y\frac{\partial}{\partial z} + (2z + c)\frac{\partial}{\partial u}$ and $V, W \in B[t]$ by:

$$V = \exp(t\delta)(z) = z - yt \ , \ \ W = \exp(t\delta)(u) = u + (2z + c)t - yt^2$$

Then $p \in \ker\delta$ and $p = yW + V^2 + cV$, and if $T = xt + p$, then $v = xV + yT$. Define $V_1 = V - vT$ and $V_2 = V_1 + fT^2$. Then there exists $T_1 \in B[t]$ such that the following relations hold:

$$v = xV_1 + fT \quad \text{and} \quad T = xT_1 + fW + V_2^2 + cV_2$$

Since $1 = \partial(x, y, V, W, t)$, we have:

$$x = \partial(x, y, V, W, xt) = \partial(x, y, V, W, xt + yW + V^2 + cV) = \partial(x, y, V, W, T)$$

Therefore,

$$\begin{aligned}
x^2 &= \partial(x, y, xV, W, T) \\
&= \partial(x, y, v - yT, W, T) \\
&= \partial(x, y, v, W, T) \\
&= \partial(x, y + xv, v, W, T) \\
&= \partial(x, f, v, W, T) \\
&= \partial(x, f, v - fT, W, T) \\
&= \partial(x, f, xV_1, W, T) \\
&= x\partial(x, f, V_1, W, T)
\end{aligned}$$

which implies:

$$\begin{aligned}
x &= \partial(x, f, V_1, W, T) \\
&= \partial(x, f, V_1 + fT^2, W, T) \\
&= \partial(x, f, V_2, W, T) \\
&= \partial(x, f, V_2, W, T - (fW + V_2^2 + cV_2)) \\
&= \partial(x, f, V_2, W, xT_1) \\
&= x\partial(x, f, V_2, W, T_1)
\end{aligned}$$

So $\partial(x, f, V_2, W, T_1) = 1$. We have $k(x, f, V_2, W, T_1) = k(x, y, z, u, t)$, and it is well-known that the Jacobian Conjecture holds in the birational case. Therefore, $A[V_2, W, T_1] = B[t]$. ☐

Using different techniques, Lewis [273], Cor. 5, gives another system of coordinates $k[x, y, z, u, t] = k[x, f_1, U_1, U_2, U_3]$ for the Vénéreau polynomial f_1.

10.4 *R*-Derivations of $R[X, Y, Z]$ with a Slice

In 1968, Raynaud [353] showed that, if R is the ring defined by

$$R = \mathbb{C}[a, b, c, x, y, z]/(ax + by + cz - 1)$$

then the unimodular row (a, b, c) cannot be completed to an invertible matrix over R. Another proof of this fact was given by Suslin in [396], Thm. 2.8.

Theorem 10.25 ([145]; [166], Prop. 3.1) *Define:*

$$R = \mathbb{C}[a, b, c, x, y, z]/(ax + by + cz - 1)$$

Let $R[X, Y, Z] = R^{[3]}$ *and* $s = aX + bY + cZ$. *If* $A = R[X - xs, Y - ys, Z - zs]$, *then:*

$$R^{[3]} \cong_R A^{[1]} \quad \text{and} \quad A \ncong_R R^{[2]}$$

Proof Define $D \in \mathrm{LND}_R(R[X, Y, Z])$ by $DX = x$, $DY = y$ and $DZ = z$. Then $Ds = 1$ and:

$$A := \ker D = R[\pi_s(X), \pi_s(Y), \pi_s(Z)] = R[X - xs, Y - ys, Z - zs]$$

Therefore, by the Slice Theorem, $R[X, Y, Z] = A[s] = A^{[1]}$.

Assume that $A = R[f, g]$ for $f, g \in R[X, Y, Z]$, and let α be the R-automorphism of $R[X, Y, Z]$ defined by $X \to f$, $Y \to g$, $Z \to s$. Since α is invertible, its jacobian determinant is a unit of R:

$$\det \begin{pmatrix} f_X & f_Y & f_Z \\ g_X & g_Y & g_Z \\ s_X & s_Y & s_Z \end{pmatrix} = \det \begin{pmatrix} f_X & f_Y & f_Z \\ g_X & g_Y & g_Z \\ a & b & c \end{pmatrix} \in R^*$$

Modulo the ideal (X, Y, Z), this yields

$$\det \begin{pmatrix} f_X(0) & f_Y(0) & f_Z(0) \\ g_X(0) & g_Y(0) & g_Z(0) \\ a & b & c \end{pmatrix} \in R^*$$

contradicting the fact that the row (a, b, c) cannot be completed to an invertible matrix over R.

Therefore, $A \ncong_R R^{[2]}$. □

The idea to apply the theory of locally nilpotent derivations to these types of rings was introduced by Van den Essen and Van Rossum in [145].

This example and Hochster's example show that, if R is a commutative k-domain and D is a locally nilpotent R-derivation of $R[X,Y,Z]$ with a slice, then $\ker D \neq R^{[2]}$ in general. This contrasts the case for $R[X,Y]$, in which the kernel of a locally nilpotent R-derivation with a slice is always equal to $R[T]$ for some $T \in R[X,Y]$; see *Theorem 4.15*. What we can say is that the kernel of a locally nilpotent R-derivation of $R[X,Y,Z]$ with a slice always has the structure of an affine fibration.

An algebra A over a ring R is an $\mathbb{A}^m$**-fibration over** R if and only if A is finitely generated as an R-algebra, flat as an R-module, and there exists $m \geq 0$ such that for every $\mathfrak{p} \in \mathrm{Spec}(R)$, $\kappa(\mathfrak{p}) \otimes_R A \cong \kappa(\mathfrak{p})^{[m]}$. Here, $\kappa(\mathfrak{p})$ denotes the residue field $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. An $\mathbb{A}^m$-fibration is **trivial** if and only if $A = R^{[m]}$.

Theorem 10.26 ([166], Thm. 1.1) *Let R be a commutative k-domain and let $R[X,Y,Z] = R^{[3]}$. The kernel of a locally nilpotent R-derivation of $R[X,Y,Z]$ with a slice is an $\mathbb{A}^2$-fibration over R.*

The reader is referred to the cited article for the proof, which relies on the Cancellation Theorem for Surfaces.

For some classes of rings, the kernel of a locally nilpotent R-derivation of $R[X,Y,Z]$ with a slice is always a trivial fibration. For example, Derksen, van den Essen and van Rossum showed:

Theorem 10.27 ([96]) *Let R be a Dedekind domain containing $\mathbb{Q}$ and let $R[X,Y,Z] = R^{[3]}$. If $D \in LND_R(R[X,Y,Z])$ has a slice, then $\ker D = R^{[2]}$.*

For other rings, notably $R = k^{[n]}$ for $n \geq 2$, it is not known if the conclusion of this theorem holds. For $R = k^{[2]}$, the constructions of Bhatwadekar-Dutta and Vénéreau provide examples of $D \in \mathrm{LND}_R(R[X,Y,Z])$ with a slice, but it is unknown whether $\ker D = R^{[2]}$. These examples are not triangular over R. Even the case of triangular R-derivations is open; see *Sect. 11.15*.

Bhatwadekar and Daigle proved the following result related to *Theorem 10.27*.

Theorem 10.28 ([25], Thm. 2) *Let R be a noetherian domain containing $\mathbb{Q}$ and let $R[X,Y,Z] = R^{[3]}$. The following are equivalent.*

1. *$\ker D$ is finitely generated over R for every $D \in LND_R(R[X,Y,Z])$.*
2. *R is a Dedekind domain or a field.*

Appendix: Torus Action Formula

Any affine ring $B = A[s]$ (s transcendental over A) has an obvious locally nilpotent derivation D with a slice, namely, $DA = 0$ and $Ds = 1$. Likewise, there are obvious actions of $\mathbb{G}_a$ and $\mathbb{G}_m$: If $t \in \mathbb{G}_a$, then ${}^t a = a$ for $a \in A$ and ${}^t s = s + t$; and if $t \in \mathbb{G}_m$, then $t \cdot a = a$ for $a \in A$ and $t \cdot s = t^n s$ for some $n \in \mathbb{Z}$. An explicit formula for the $\mathbb{G}_a$-action is given by $\exp(tD)$. In this case, there is also an explicit formula for the action of the torus $\mathbb{G}_m$ in terms of D.

Given $b \in B$, if we write $b = P(s)$ for some $P \in A^{[1]}$, then of course $t \cdot b = P(t^n s)$. This relies on our ability to write b as a polynomial in s over A, which is achieved in the following way.

Lemma 10.29 *Let B be a commutative k-domain, and let $D \in LND(B)$ have a slice $s \in B$. Then for $b \in B$:*

$$b = \sum_{n \geq 0} \frac{1}{n!} \pi_s(D^n b) s^n$$

Proof For each $i \geq 0$, define the function $F_i : B \to A$ as follows: Given $b \in B$, suppose $b = \sum_{n \geq 0} a_i s^i$ for $a_i \in A$. Then $F_i(b) = a_i$.

Recall that the kernel of π_s is the ideal sB (*Corollary 1.26*), and that $\pi_s(a) = a$ for all $a \in A$. Therefore, $a_0 = \pi_s(a_0) = \pi_s(b)$. Moreover, it is clear that for $i \geq 1$:

$$F_{i-1}(Db) = ia_i = iF_i(b) \quad \Rightarrow \quad F_i(b) = \frac{1}{i} F_{i-1}(b) \quad (i \geq 1)$$

By induction, it follows that, for all $n \geq 0$, $F_n(b) = \frac{1}{n!} \pi_s(D^n b)$. □

However, it is possible to give a more direct formula for the torus action which does not rely on finding the coefficients of an element over the kernel A.

Specifically, let $\rho : B \times \mathbb{G}_m \to B$ denote the $\mathbb{G}_m$-action on B defined by $\rho(a, t) = a$ for $a \in A$, and $\rho(s, t) = t^n s$. Given $t \in \mathbb{G}_m$, let $\rho_t : B \to B$ denote the restriction of ρ to $B \times \{t\}$. Let λ denote an indeterminate over B, and extend D to $B[\lambda]$ via $D\lambda = 0$. Then D (extended) is locally nilpotent on $B[\lambda]$, and $\exp(\lambda D)$ is a well-defined automorphism of $B[\lambda]$.

Theorem 10.30 *In the notation above, $\rho_t = exp(-\lambda D)|_{\lambda=(1-t^n)s}$.*

Proof Given $t \in \mathbb{G}_m$, define $\beta_t = \exp(-\lambda D)|_{\lambda=(1-t^n)s}$. The main fact to show is that, given $u, v \in \mathbb{G}_m$, $\beta_u \beta_v = \beta_{uv}$, i.e., this defines an action β of $\mathbb{G}_m$ on B. Since the fixed ring of β is A, and since $\beta_t(s) = \rho_t(s)$ for all t, it will then follow that $\beta = \rho$.

Introduce a second indeterminate μ. Given $Q \in B[\mu]$, let $\epsilon_Q : B[\lambda, \mu] \to B[\lambda, \mu]$ denote evaluation at Q, i.e., $\epsilon_Q(f) = f$ for $f \in B[\mu]$ and $\epsilon_Q(\lambda) = Q$. Likewise, if $R \in B[\lambda]$, let δ_R denote evaluation at R, i.e., $\delta_R(g) = g$ for $g \in B[\lambda]$ and $\delta_R(\mu) = R$. Note that if α is any automorphism of $B[\lambda, \mu]$ such that $\alpha(B[\mu]) \subset B[\mu]$ and $\alpha(\lambda) = \lambda$, then $\alpha \epsilon_Q \alpha^{-1} = \epsilon_{\alpha(Q)}$. A similar formula holds for δ_R.

Let $u, v \in \mathbb{G}_m$ be given. Then:

$$\beta_u \beta_v = \epsilon_s \circ \exp(-(1 - u^n)\lambda D) \circ \delta_s \circ \exp(-(1 - v^n)\mu D)$$

If $\alpha = \exp(-(1 - u^n)\lambda D)$, then:

$$\alpha \circ \delta_s = \delta_{\alpha(s)} \circ \alpha = \delta_{(s-(1-u^n)\lambda)} \circ \alpha$$

Therefore:

$$\begin{aligned}\beta_u\beta_v &= \epsilon_s \circ \delta_{(s-(1-u^n)\lambda)} \circ \exp(-(1-u^n)\lambda D) \circ \exp(-(1-v^n)\mu D)\\ &= \epsilon_s \circ \delta_{(s-(1-u^n)\lambda)} \circ \exp\Big(-\big((1-u^n)\lambda + (1-v^n)\mu\big)D\Big)\end{aligned}$$

It follows that:

$$\begin{aligned}\epsilon_s \circ \delta_{(s-(1-u^n)\lambda)}\Big((1-u^n)\lambda + (1-v^n)\mu\Big) &= \epsilon_s\Big((1-u^n)\lambda\\ &\qquad +(1-v^n)(s-(1-u^n)\lambda)\Big)\\ &= \epsilon_s\Big((1-v^n)s + v^n((1-u^n)\lambda\Big)\\ &= (1-v^n)s + v^n(1-u^n)s\\ &= (1-u^nv^n)s\\ &= \epsilon_s\Big((1-u^nv^n)\lambda\Big)\end{aligned}$$

Finally:

$$\beta_u\beta_v = \epsilon_s \circ \exp(-(1-u^nv^n)\lambda D) = \beta_{uv}$$

□

Remark 10.31 The reader is warned that some authors would write

$$\exp(-(1-t^n)sD)$$

in place of the evaluation notation used in this formula. To do so is technically incorrect, but might be accepted as a convenient abuse of notation: Since s is not in the kernel of D, $(1-t^n)sD$ is not locally nilpotent, and its exponential is *not* an algebraic automorphism of B.

Example 10.32 The simple action of $\mathbb{G}_m$ on $\mathbb{A}^n$ given by

$$t(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}, t^N x_n)$$

is of the form $\exp(-\lambda D)|_{\lambda=(1-t^N)x_n}$, where $D = \frac{\partial}{\partial x_n}$.

Suppose $D \in \mathrm{LND}(k[x_1, \dots, x_n])$ has a slice s, and suppose also that D is nice, i.e., $D^2 x_i = 0$ for each i. In this case, the induced $\mathbb{G}_m$-action on $\mathbb{A}^n$ defined by *Theorem 10.30* is also called **nice**, and has the simple form:

$$t \cdot x_i = x_i - (1-t)sDx_i$$

Question 10.33 Can every nice $\mathbb{G}_m$-action on $\mathbb{A}^n$ be linearized?

Chapter 11
Epilogue

> *It is by the solution of problems that the investigator tests the temper of his steel; he finds new methods and new outlooks, and gains a wider and freer horizon. . . . for he who seeks for methods without having a definite problem in mind seeks for the most part in vain.*
>
> David Hilbert, *Mathematical Problems*

Many open questions, ranging from specific cases to broader themes, have already been posed and discussed in the foregoing chapters. A solution to the Embedding Problem or Cancellation Problem for complex affine spaces would reverberate across the whole of algebra, and we have seen how locally nilpotent derivations might play a role in their solution. Following are several additional directions for future inquiry.

11.1 Kernels for Polynomial Rings

Given $n \geq 2$, let $D \in \mathrm{LND}(k^{[n]})$ and let $A = \ker D$.

Question 11.1 Is A always non-rigid?

Question 11.2 Is A always rational?
A partial answer to (2) is given by the theorem of Deveney and Finston, which shows that the field of fractions K of $\ker D$ is ruled when $k = \mathbb{C}$, and that K is rational when $n = 4$. See *Sect. 3.6.4*.

The following question is due to Miyanishi (Section 1.2 of [306]).

Question 11.3 Are all projective modules over A free?

G. Freudenburg, *Algebraic Theory of Locally Nilpotent Derivations*,
Encyclopaedia of Mathematical Sciences 136,
DOI 10.1007/978-3-662-55350-3_11

Bhatwadekar, Gupta and Lokhande [28] showed that the answer to this question is generally negative. In particular, Example 5.6 of their paper shows that the answer is negative for the example of Winkelman in dimension 4, which is discussed in *Sect. 3.8.4* above.

In his 1984 paper [12], Bass asked the following question.

Question 11.4 If a unipotent group G acts on $\mathbb{A}^n_k$, can the action be rationally triangularized, i.e., can we write $k(x_1, \dots, x_n) = k(y_1, \dots, y_n)$ so that each subfield $k(y_1, \dots, y_i)$ is G-invariant?
This problem remains generally open. It was considered by Deveney and Finston in their papers [100, 101], where they gave several positive results. Working over the field $k = \mathbb{C}$, they observed that a $\mathbb{G}_a$-action is rationally triangularizable if and only if the quotient field of the invariant ring is a pure transcendental extension of $\mathbb{C}$; that any $\mathbb{G}_a$-action on $\mathbb{A}^n$ becomes rationally triangularizable if one more variable is added; and that any $\mathbb{G}_a$-action on $\mathbb{A}^n$ for $n \leq 4$ is rationally triangularizable. Recent progress by Popov on this question is represented in [341], which studies Bass's question and generalizes several of the results of Deveney and Finston.

11.2 Freeness Conjecture

Let $B = k^{[n]}$ for some $n \geq 2$. Given $D \in \text{LND}(B)$, let $\mathcal{F}_i$, $i \geq 0$, be the associated degree modules, where $A = \mathcal{F}_0 = \ker D$. In some cases, B is a free A-module. For example, if D has a slice, then B is a free A-module, even if the slice is not a variable of B. For $n = 2$, Rentschler's Theorem shows that there exist $x, y \in B$ such that $B = k[x, y]$ and $A = k[x]$. It follows that B is a free A-module when $n = 2$, as is $\mathcal{F}_i$ for each $i \geq 0$. For $n \geq 4$, this is no longer the case, as seen in *Example 8.17*.

What about the case $n = 3$? For example, *Theorem 8.22* shows that, if A is the kernel of the $(2, 5)$ derivation D of $B = k^{[3]}$, then B is a free A-module, and each $\mathcal{F}_i$ is a free A-module. In [157], it is conjectured that B is always a free A-module. More precisely:

Conjecture *Let $B = k^{[3]}$. Given $D \in LND(B)$, if $A = \ker D$, then the following equivalent conditions hold.*

1. *B is a free A-module with basis $\{Q_i\}_{i \geq 0}$ such that* $\deg_D Q_i = i$.
2. *Each degree module $\mathcal{F}_n$ is a free A-module with basis $\{Q_i\}_{0 \leq i \leq n}$ such that* $\deg_D Q_i = i$.
3. *Each image ideal $I_n = D^n \mathcal{F}_n \subset A$ is principal.*

Recall that the Plinth Ideal Theorem asserts that I_1 is principal; see *Sect. 5.2*.

11.3 Local Slice Constructions in Dimension Three

Let $B = k^{[3]}$ with a positive $\mathbb{Z}$-grading. As discussed in *Chap. 5*, Daigle showed that, given any two nonzero homogeneous elements of $\mathrm{LND}(B)$, there exists a finite sequence of local slice constructions which transforms one into the other. We ask to what extent this result generalizes to other elements of $\mathrm{LND}(B)$. In particular:

Question 11.5 Can every irreducible locally nilpotent derivation of $B = k^{[3]}$ be obtained from a partial derivative via a finite sequence of local slice constructions? Equivalently, does every connected component of the graph of kernels of B contain a vertex corresponding to a partial derivative of B?

11.4 Tame $\mathbb{G}_a$-Actions in Dimension Three

For $n \geq 1$, let $BA_n(k)$ and $TA_n(k)$ denote the triangular and tame subgroups of $GA_n(k)$, respectively. Recall that a $\mathbb{G}_a$-action $\rho : \mathbb{G}_a \to GA_n(k)$ is tame if and only if $\rho(\mathbb{G}_a) \subset TA_n(k)$. Recall from *Chap. 3* that Bass's $\mathbb{G}_a$-action on $\mathbb{A}^3$ is not tame, and that it becomes tame, but not triangularizable, when extended to $\mathbb{A}^4$.

Let $\rho : \mathbb{G}_a \to GA_3(k)$ denote Bass's $\mathbb{G}_a$-action on $\mathbb{A}^3$, where $\rho(1)$ is the Nagata automorphism; see *Example 3.8.1*. Then $\rho(\mathbb{G}_a) \cap TA_3(k) = \{1\}$. On the other hand, any triangular $\mathbb{G}_a$-action $\tau : \mathbb{G}_a \to GA_3(k)$ has:

$$\tau(\mathbb{G}_a) \subset BA_3(k) \subset TA_3(k)$$

The following question is open.

Question 11.6 Is every *tame* $\mathbb{G}_a$-action on $\mathbb{A}^3$ conjugate to a triangular action?
In other words, given nonzero $E \in \mathrm{LND}(B)$, does $\exp E \in TA_3(k)$ imply that E is conjugate to a triangular derivation? No doubt, an important tool in answering this question will be Wright's Structure Theorem for $TA_3(k)$ [428], which describes $TA_3(k)$ as the product of three subgroups amalgamated along pairwise intersections.

11.5 Fundamental Problem for Cable Algebras

The following question was posed in [171]. Suppose that B is an affine k-domain and $D \in \mathrm{LND}(B)$, and let I_∞ be the core ideal of D.

Question 11.7 If $I_\infty \neq (0)$, does B contain an infinite D-cable?
Example 2.53 shows that, without the assumption that B is affine, the answer to this question is generally negative.

11.6 Nilpotency Criterion

Let B be a finitely generated k-domain. Given $D \in \mathrm{Der}_k(B)$, does $D \in \mathrm{LND}(B)$? To date, the author knows of no specific example where the answer to this question is not known. This question was discussed in *Chap.* 8, where van den Essen's Partial Nilpotency Criterion was given. That criterion is based on finding a transcendence basis for the kernel of D, which leaves the seemingly weaker question: Given $D \in \mathrm{LND}(B)$, if tr. $\deg._k B \geq 2$, can we construct even one non-constant kernel element? Experience indicates that when B is equipped with a degree function and the degree of $f \in B$ is known, then it should be possible to determine an integer N such that $D^N f = 0$, thereby yielding $D^{N-1} f \in B^D$. At present such bounds are not generally known.

11.7 Calculating the Makar-Limanov Invariant

In their paper [231], Kaliman and Makar-Limanov describe methods for calculation of $ML(A)$ for affine domains A over $\mathbb{C}$. They use the following.

Proposition 11.8 ([231], Cor. 2.1) *Let A be an affine $\mathbb{C}$-domain. Suppose that $a, b \in A$ are algebraically independent, and that $p \in \mathbb{C}^{[2]}$ is non-constant, irreducible and not a variable. Then for every nonzero $D \in LND(A)$, $D(p(a, b)) = 0$ implies $Da = Db = 0$.*

Notice that the conditions on the polynomial p in this proposition are precisely *equivalent* to the condition that $\delta p \neq 0$ for every nonzero $\delta \in \mathrm{LND}(\mathbb{C}[x, y])$: Suppose $\delta f = 0$ for nonzero $\delta \in \mathrm{LND}(B)$ and non-constant $f \in B$. By Rentschler's Theorem, $f \in \mathbb{C}[z]$ for some variable z. Thus, if f is irreducible, then f is a linear function of z, hence a variable.

We ask whether this result generalizes in the following way.

Question 11.9 Let B be an affine $\mathbb{C}$-domain, and let $p \in \mathbb{C}^{[m]}$, $m \geq 2$, be such that $\delta p \neq 0$ for all nonzero $\delta \in \mathrm{LND}(\mathbb{C}^{[m]})$. Suppose that there exist algebraically independent $a_1, \ldots, a_m \in B$ and nonzero $D \in \mathrm{LND}(B)$ such that $D(p(a_1, \ldots, a_m)) = 0$. Does this imply $Da_i = 0$ for each $i = 1, \ldots, n$?

11.8 Maximal Subalgebras

For a commutative k-domain B, we have seen that $\mathrm{Der}_k(B)$ forms a Lie algebra over k, while $\mathrm{LND}(B)$ does not. Nonetheless, it is natural to study the Lie subalgebras $\mathfrak{g}$ of $\mathrm{LND}(B)$, by which we mean subalgeras $\mathfrak{g} \subset \mathrm{Der}_k(B)$ contained in $\mathrm{LND}(B)$. For example, we have mainly been interested in one-dimensional subalgebras, generated by a single element $D \in \mathrm{LND}(B)$. This induces the larger subalgebra $A \cdot D$, where $A = \ker D$. Likewise, if $D_1, \ldots, D_n \in \mathrm{LND}(B)$ commute, then their

k-span is a subalgebra of finite dimension. We might also ask: What are the **maximal** subalgebras of LND(B), i.e., subalgebras $\mathfrak{m}$ of LND(B) with the property that if $\mathfrak{m} \subseteq \mathfrak{g}$ for another subalgebra $\mathfrak{g}$ of LND(B), then $\mathfrak{m} = \mathfrak{g}$. For the polynomial ring $B = k[x_1, \ldots, x_n]$, a natural candidate is the triangular subalgebra:

$$\mathfrak{T} = k\partial_{x_1} \oplus k[x_1]\partial_{x_2} \oplus \cdots \oplus k[x_1, \ldots, x_{n-1}]\partial_{x_n}$$

Question 11.10 Is $\mathfrak{T}$ is a maximal subalgebra?
Note that we have seen earlier that $\mathrm{Der}_k(B) = B\partial_{x_1} \oplus B\partial_{x_2} \oplus \cdots \oplus B\partial_{x_n}$.

11.9 Invariants of a Sum

This question will be stated for polynomial rings, though it could also be stated more generally. Let $k[\mathbf{x}] = k^{[n]}$ and $k[\mathbf{y}] = k^{[m]}$ be polynomial rings, where $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_m)$, and let $D \in \mathrm{LND}(k[\mathbf{x}])$ and $E \in \mathrm{LND}(k[\mathbf{y}])$ be given. Extend D and E to locally nilpotent derivations of the polynomial ring $k[\mathbf{x}, \mathbf{y}] = k^{[n+m]}$ by setting $D\mathbf{y} = 0$ and $E\mathbf{x} = 0$. Then $DE = ED = 0$, which implies that $D + E$ is locally nilpotent. Describe $\ker(D + E)$ in terms of $\ker D$ and $\ker E$. In particular:

Question 11.11 Is $\ker(D+E)$ finitely generated if both $\ker D$ and $\ker E$ are finitely generated?
Notice that $\ker(D + E)$ contains cross elements, for example, if $r \in k[\mathbf{x}]$ is a local slice of D and $s \in k[\mathbf{y}]$ is a local slice of E, then $rEs - sDr$ belongs to $\ker(D + E)$.

Note that $\ker D$ and $\ker E$ are subalgebras of $\ker(D + E)$. However, $\ker(D+E)$ is, in general, strictly larger than $\ker D \otimes_k \ker E$, since the latter is the invariant ring of the $\mathbb{G}_a^2$-action defined by the commutative Lie algebra $kD+kE$ (see Nagata [323], Lemma 1). Thus, the transcendence degree of $\ker D \otimes_k \ker E$ over k generally equals $n - 2$.

This question is motivated by the linear locally nilpotent derivations D. We saw that, due to the Jordan form of a matrix, such derivations can always be written as a sum $D = D_1 + \cdots + D_t$, where the polynomial ring $k[\mathbf{x}]$ decomposes as $k[\mathbf{x}_1, \ldots, \mathbf{x}_t]$ for $\mathbf{x}_i = (x_{i1}, \ldots, x_{ij_i})$, and where D_i restricts to the basic derivation of $k[\mathbf{x}_i]$, with $D_i\mathbf{x}_j = 0$ when $i \neq j$.

11.10 Finiteness Problem for Extensions

Suppose B is a finitely generated commutative k-domain, and suppose $D \in \mathrm{LND}(B)$ is such that $\ker D$ is finitely generated. If D is extended to a derivation D^* on the ring $B[t] = B^{[1]}$, then D^* is locally nilpotent if and only $D^*t \in B$ (*Principle 6*). Assuming this is the case:

Question 11.12 What conditions on D^*t guarantee that $\ker D^*$ is also finitely generated?
This question is also motivated by the linear locally nilpotent derivations of polynomial rings, in particular, the basic (triangular) ones. A good understanding of this problem, together with the preceding problem, might yield a proof of the Maurer-Weitzenböck Theorem which does not rely on the Finiteness Theorem for reductive groups.

11.11 Geometric Viewpoint

It might be profitable to think about a $\mathbb{G}_a$-action as a point belonging to a variety or scheme. For example, if $B = k[x_1, \ldots, x_n] = k^{[n]}$, then the set of all linear k-derivations of B is an affine space, since we may view a linear element $D \in \mathrm{Der}_k(B)$ as a matrix: $D \in M_n(k) = \mathbb{A}^{n^2}$. The condition that $D = (x_{ij})$ be locally nilpotent (a nilpotent matrix) is that $(D^n)_{ij} = 0$ for $1 \leq i,j \leq n$, where the functions $(D^n)_{ij} \in k[x_{ij} | 1 \leq i,j \leq n]$ are the component functions of D^n. These conditions can be written down explicitly: If $f(\lambda) = \sum_{i=0}^{n} f_i \lambda^i$ is the characteristic polynomial of D, then the condition that D be nilpotent is precisely that D belong to the subvariety $X := \mathcal{V}(f_0, \ldots, f_{n-1}) \subset \mathbb{A}^{n^2}$. X is the **variety of nilpotent matrices** in dimension n^2. Note that $GL_n(k)$ acts on X by conjugation. In addition, since the polynomials defining X are algebraically independent, $\dim X = n^2 - n$. For instance, when $n = 2$, write:

$$D = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

Then the subvariety of nilpotents is $V(x_{11} + x_{22}, x_{11}x_{22} - x_{12}x_{21}) \subset \mathbb{A}^4$, which is a special Danielewski surface.

Now suppose more generally that the polynomial ring B is graded by $B = \oplus_{i \geq 0} B_i$, where each B_i is a finite dimensional vector space over k. Given homogeneous $D \in \mathrm{Der}_k(B)$, set $d_i = \deg Dx_i$, $1 \leq i \leq n$. Then $D \in B_{d_1} \times \cdots \times B_{d_n}$, which is an affine space, and we may ask:

Question 11.13 What conditions imply that D is locally nilpotent?

Likewise, given $N > 0$, we may consider triangular $\delta \in \mathrm{Der}_k(B)$ such that $\deg \delta(x_i) \leq N$ for each i, $1 \leq i \leq n$. In this case, is the condition "$\ker \delta$ is finitely generated" an open condition?

In the same vein, notice that a $\mathbb{G}_a^m$-action on $\mathbb{A}^n$ may be specified by an $m \times n$ matrix over $k[x_1, \ldots, x_n]$. If the action is linear, the entries of the matrix are linear forms.

11.12 Russell Cubic Threefold

The Russell cubic threefold $X \subset \mathbb{C}^4$ is defined by $x + x^2y + z^2 + t^3 = 0$. This is an example of an exotic affine space, meaning that X is diffeomorphic to $\mathbb{R}^6$ but not isomorphic to $\mathbb{C}^3$ as an algebraic variety.

If $Y \subset \mathbb{C}^4$ is the singular threefold defined by the equation $x^2y + z^2 + t^3 = 0$, then $\mathbb{C}[Y]$ is the graded ring naturally associated to $\mathbb{C}[X]$; see *Sect.* 9.3. We also saw in *Chap.* 8 that $\mathbb{C}[Y]$ is the invariant ring of the basic linear action of $\mathbb{G}_a$ on $\mathbb{C}^4$. We ask:

Question 11.14 Does there exist $D \in \mathrm{LND}(\mathbb{C}^{[4]})$ such that $\ker D \cong \mathbb{C}[X]$?
In [175], Kaliman asked whether $X \times \mathbb{C}^1$ is an exotic affine space, or equivalently:

Question 11.15 Is $X \times \mathbb{C}^1$ isomorphic to $\mathbb{C}^4$?
Dubouloz [128] showed that $ML(X \times \mathbb{C}^1) = \mathbb{C}$, whereas $ML(X) = \mathbb{C}^{[1]}$.

11.13 Extending $\mathbb{G}_a$-Actions

Question 11.16 Does there exist a non-triangularizable $\mathbb{G}_a$-action on $\mathbb{C}^n$ which extends to an action of some reductive group G, for example, $G = SL_2(\mathbb{C})$? If such an action can be found, can the $\mathbb{G}_a$-action be chosen to be fixed-point free?
A similar question was posed by Bass in [13]:

Question 11.17 If a reductive group G acts on $\mathbb{A}^n$, and if $U \subset G$ is a maximal unipotent subgroup, can the action of U be triangularized "as a step toward linearizing the action of G"? (p. 5)
Note that a non-triangularizable $\mathbb{G}_a$-action on $\mathbb{C}^3$ cannot be extended to an action of a reductive group G, since $\mathbb{G}_a \subset G_0$, the connected component of the identity. G_0 is again reductive, and every algebraic action of a connected reductive group on $\mathbb{C}^3$ can be linearized; see *Chap.* 5.

11.14 Variable Criterion

In *Chap.* 4, locally nilpotent derivations were used to prove the Variable Criterion for polynomial rings (*Corollary* 4.28). Specifically, it asserts that if $F \in R[x,y]$ is a variable over frac(R), and $(F_x, F_y) = (1)$, then F is an R-variable, where $R = k^{[n]}$. Give necessary and sufficient conditions that $F \in R[x,y,z]$ be an R-variable of $R[x,y,z]$. In particular, we must have $(F_x, F_y, F_z) = (1)$. Such considerations arose in connection to the Vénéreau polynomials in *Chap.* 10. In [370], Sathaye gives the following case.

Let k be an algebraically closed field of characteristic zero and let $f \in k[x,y,z] = k^{[3]}$ be such that $(f_x, f_y, f_z) = (1)$. If the generic fiber of f is a plane, i.e.,

$$k(f) \otimes_{k[f]} k[x,y,z] \cong k(f)^{[2]}$$

then f is a variable of $k[x,y,z]$.

11.15 Free $\mathbb{G}_a$-Actions on Affine Spaces

In [103], Deveney and Finston ask several question about fixed point free $\mathbb{G}_a$-actions on complex affine spaces, including the following, which are still open.

Question 11.18 Is every proper (respectively, locally trivial) $\mathbb{G}_a$-action on $\mathbb{C}^4$ globally trivial?

Question 11.19 Does every fixed point free (respectively, proper) $\mathbb{G}_a$-action on $\mathbb{C}^n$ have a finitely generated ring of invariants?
To these we add the following two questions.

Question 11.20 Is every globally trivial $\mathbb{G}_a$-action on $\mathbb{C}^n$ a translation?

Question 11.21 Let $R = \mathbb{C}^{[2]}$ and $R[X,Y,Z] = R^{[3]}$, and let D be a triangular R-derivation:

$$DX \in R \ , \ DY \in R[X] \ , \ DZ \in R[X,Y]$$

If D has a slice, is $\ker D = R^{[2]}$?

For each of these questions, we give an example whose status relative to the given question is not known.

Example 11.22 Let D be the homogeneous $(2,5)$ derivation of $\mathbb{C}[x,y,z] = \mathbb{C}^{[3]}$; see *Sect.* 5.4. The kernel of D is $\mathbb{C}[F,G]$, where F and G are homogeneous polynomials of degree 2 and 5, respectively, and $\mathrm{pl}(D) = (FG)$. Extend D to the derivation D_1 of $B = \mathbb{C}[x,y,z,u]$ by setting $D_1 u = 1 + F$. It is easy to check that the image of D_1 generates the unit ideal, so the induced $\mathbb{G}_a$-action on $\mathbb{C}^4$ is fixed point free. In addition, if $R = x^3 - yF$, then $DR = -FG$ and $D_1(R + uG) = G$. So $\mathrm{pl}(D_1)$ contains G and $1 + F$. Define $H \in \ker D_1$ by $H = (1+F)R + FGu$. Then $H^2 + (1+F)^2 F^3 = GK$ for some $K \in B$, and since $\ker D_1$ factorially closed, we have $K \in \ker D_1$. Arguing as in *Example* 8.16, it can be shown that:

$$\ker D_1 = \mathbb{C}[F,G,H,K] = \mathbb{C}[X_1,X_2,X_3,X_4]/(X_2X_4 - (1+X_1)^2 X_1^3 - X_3^2)$$

Since this is the coordinate ring of a singular hypersurface, the induced $\mathbb{G}_a$-action on $\mathbb{C}^4$ cannot have a slice.

If D is extended by $D_2 u = 1 + G$, then D_2 is very similar to D_1.

Question 11.23 Is the $\mathbb{G}_a$-action on $\mathbb{C}^4$ induced by D_1 or D_2 proper?

Example 11.24 This example first appeared in [157]. Let $R = \mathbb{C}[x, y] = \mathbb{C}^{[2]}$ and $B = R[z, u] = R^{[2]}$, and define the triangular R-derivation:

$$\delta = y^2\partial_z - (x + 2yz)\partial_u$$

Then $\ker\delta = R[v]$ for $v = xz + yp$ and $p = yu + z^2$. Note that $\ker\delta$ contains the Vénéreau polynomial $f = y + xv$; see *Example 3.18*. It is shown in [157] that $f^2 \notin \delta B$. On the other hand, we find that $\delta r = f^3$, where:

$$r = yz - 3yvp - 3xv^2p - v^3w \quad \text{and} \quad w = x^2u - 2xzp - yp^2$$

Extend δ to Δ on $B[t] = B^{[1]} = R^{[3]}$ by $\Delta t = 1 + f + f^2$. If $s = (1 - f)t + r$, then $\Delta s = 1$. Note that

$$\ker\Delta = R[z - y^2s, u + (x + 2yz)s - y^3s^2, f^3t - (1 + f + f^2)r]$$

and by *Theorem 10.26*, $\ker\Delta$ is an $\mathbb{A}^2$-fibration over R.

Question 11.25 Is the $\mathbb{G}_a$-action on $\mathbb{C}^5$ defined by Δ a translation? Is $\ker\Delta = R[P, Q]$ for some $P, Q \in B[t]$?

Example 11.26 Let $B = \mathbb{C}[a, x, y, z, v] = \mathbb{C}^{[5]}$ and let $D = D_{(3,2)}$ be the triangular derivation of B defined in *Sect. 7.6*, together with $A = \ker D$ and the element $h \in A$ (defined in *Sect. 7.6.3*). Recall that A is not finitely generated over $\mathbb{C}$. Note that h belongs to the ideal (DB). However, using the techniques of [171], it can be shown that $h \notin DB$.

Extend D to the triangular derivation $\tilde{D}$ on $B[u] = B^{[1]}$ by setting $\tilde{D}u = 1 + h$. Specifically, we have:

$$\tilde{D} = a^3\partial_x + x\partial_y + y\partial_z + a^2\partial_v + (1 + h)\partial_u$$

The $\mathbb{G}_a$-action on $\mathbb{C}^6$ defined by $\tilde{D}$ is proper. In order to see this, let $\tilde{B} = B[\exp(t\tilde{D})(B)]$ and note that:

$$\begin{aligned}
\exp(t\tilde{D})(v) &= v + ta^2 \\
\exp(t\tilde{D})(a^3z) &= a^3z + ta^3y + \tfrac{1}{2}t^2a^3x + \tfrac{1}{6}t^3a^6 \\
\exp(t\tilde{D})(xy) &= xy + t(a^3y + x^2) + \tfrac{3}{2}t^2a^3x + \tfrac{1}{2}t^3a^6 \\
\exp(t\tilde{D})(u) &= u + t(1 + 9a^6z^2 - 18a^3xyz + 6x^3z + 8a^3y^3 - 3x^2y^2)
\end{aligned}$$

The first equation shows $ta^2 \in \tilde{B}$, which gives $t^2a^3x \in \tilde{B}$ from the second equation. Using this in the third equation shows $tx^2 \in \tilde{B}$, which gives $t \in \tilde{B}$ from the fourth equation. Therefore, by the Properness Criterion, the action is proper.

Question 11.27 Is $\ker \tilde{D}$ finitely generated?

Note that there is no obvious way to apply the Non-Finiteness Criterion to this example, since the criterion requires an $\mathbb{N}$-grading whose ring of degree-0 elements is $\mathbb{C}$. However, if we want $\tilde{D}$ to be homogeneous, then we must have $\deg h = 0$.

11.16 Wood's Question

Let n be an integer with $n \geq 2$. Recall that $D \in \mathrm{LND}(k^{[n]})$ is elementary if, for some j with $1 \leq j \leq n$, $k[x_1, \dots, x_j] \subset A$ and $Dx_i \in k[x_1, \dots, x_j]$ for $i = j+1, \dots, n$. For example, the Roberts derivation of $k^{[7]}$ is elementary, where $\deg Dx_i \leq 6$ for each i. A **quadratic elementary derivation** is an elementary derivation D such that each Dx_i is a quadratic form. Melanie Wood (private communication, 2013) asked the following.

Question 11.28 Is the kernel of a quadratic elementary derivation always finitely generated as a k-algebra?

11.17 Popov's Questions

The following questions of V. Popov appear in [175].

Working over an algebraically closed field k of characteristic zero, Popov defines the subgroup $GA_n^*(k)$ of $GA_n(k)$ of automorphisms of jacobian determinant 1 (the "volume-preserving" elements). A subgroup G of $GA_n^*(k)$ is called **∂-generated** if G is generated by exponential automorphisms, and **finitely ∂-generated** if there exists a finite set of elements $d_1, \dots, d_N \in \mathrm{LND}(k^{[n]})$, $N \geq 1$, such that G is generated by:

$$\{\exp(fd_i) \,|\, f \in \ker d_i, 1 \leq i \leq N\}$$

He goes on to give several examples of finitely ∂-generated subgroups, namely, any connected semisimple algebraic subgroup of $GA_n^*(k)$, the group of translations, and the triangular subgroup of $GA_n^*(k)$.

Question 11.29 Is $GA_n^*(k)$ ∂-generated? If yes, is it finitely ∂-generated?

Clearly, $GA_n^*(k)$ is ∂-generated for $n = 1$, and it follows from the Structure Theorem that this is true for $n = 2$ as well. Note that for $n = 3$, Nagata's automorphism is not tame, but it is exponential.

A second question posed by Popov is the following.

Question 11.30 Let $D, E \in \mathrm{LND}(k^{[n]})$, and let $G \subset GA_n(k)$ be the minimal closed subgroup containing the groups:

$$\{\exp(tD) \,|\, t \in k\} \quad \text{and} \quad \{\exp(tE) \,|\, t \in k\}$$

When is G of finite dimension?

Here, "closed" means closed with respect to the structure of $GA_n(k)$ as an infinite dimensional algebraic group, as defined by Shafarevich in [380]. See also Kambayashi [239].

Other important results of Popov on unipotent group actions and automorphisms of affine varieties can be found in [347, 348, 350].

11.18 Bass's Question

The following question of Bass [12] is still open.

Question 11.31 If k is an algebraically closed field, is the automorphism group $GA_n(k)$ generated by one-parameter subgroups, i.e., by images of algebraic homomorphisms from $\mathbb{G}_a$ and $\mathbb{G}_m$?

11.19 Two Commuting Nilpotent Matrices

Suppose that the unipotent group G of dimension m acts linearly on the affine space $V = \mathbb{A}^n$ in such a way that the ring of invariants $k[V]^G$ is not finitely generated.

Question 11.32 What is the minimal dimension $m = \mu$ which can occur in this situation?

Question 11.33 What is the minimal dimension $n = \nu$ which can occur in this situation?

The examples of Mukai show that $\mu \leq 3$, whereas the theorem of Maurer-Weitzenböck implies $\mu \geq 2$. Likewise, the example of [164] shows that $\nu \leq 11$, while Zariski's Theorem implies $\nu \geq 5$.

In particular, the first question reduces to a single case when G is abelian.

Question 11.34 Is the ring of invariants of a linear $\mathbb{G}_a^2$-action on $\mathbb{A}^n$ always finitely generated? Equivalently, if M and N are commuting nilpotent matrices, is $\ker M \cap \ker N$ finitely generated?

11.20 $\mathbb{G}_a$-Actions in Positive Characteristic

Let K be any field. For $n \leq 3$, the ring of invariants of a $\mathbb{G}_a$-action on $\mathbb{A}_K^n$ is of finite type, due to a fundamental theorem of Zariski. It is not known if the ring of invariants of a $\mathbb{G}_a$-action on $\mathbb{A}_K^4$ is always of finite type. According to the classical Mauer-Weitzenböck Theorem, if the characteristic of K is zero, then $K[\mathbb{A}_K^n]^{\mathbb{G}_a}$ is of finite type when $\mathbb{G}_a$ acts on $\mathbb{A}_K^n$ by linear transformations. It is not known if this is true for all fields.

The famous examples of Nagata are characteristic free, requiring only that the ground field K is not locally finite (i.e., an algebraic extension of a finite field). Thus, for such fields K, there exists an algebraic K-group G which acts on $\mathbb{A}^n_K$ by linear transformations for some n in such a way that $K[\mathbb{A}^n_K]^G$ is not finitely generated over K. On the other hand, if K is a finite field, then any algebraic K-group G is finite, and by the classical theorem of E. Nöther, $K[\mathbb{A}^n_K]^G$ is finitely generated for any G-action on $\mathbb{A}^n_K$.

The following question is open.

Question 11.35 Let K be a field of positive characteristic, and suppose that $\mathbb{G}_a$ acts algebraically on $\mathbb{A}^n_K$ for some $n \geq 4$. Is $K[\mathbb{A}^n_K]^{\mathbb{G}_a}$ finitely generated as a K-algebra? This question applies to all $\mathbb{G}_a$-actions, not just the linear ones.

It is natural to consider $\mathbb{G}_a$-actions in positive characteristic induced by examples in zero characteristic where the invariant ring is known to be of non-finite type. This is done in [260] and [132]. In particular, the authors reformulate the examples of Roberts in dimension 7 and of Daigle and Freudenburg in dimension 5 to give examples of $\mathbb{G}_a$-actions in positive characteristic, and they show that the resulting invariant rings are finitely generated.

Fauntleroy [149] showed that the invariants of a representation of $\mathbb{G}_a$ of codimension one in positive characteristic is finitely generated. In [405], Tanimoto classifies the representations of $\mathbb{G}_a$ of codimension two in positive characterstic, but even for these specific actions, it is unkown whether the invariant rings are finitely generated.

11.21 Kronecker's Paradox

The Prussian mathematician Leopold Kronecker famously remarked:

> Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk.

See Weber [416]. A direct translation reads:

> The whole numbers were made by dear God, all else is the work of man.

Question 11.36 On which day of creation did God bring forth the integers?

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