Eta-function in modern investigations

Galina V. Voskresenskaya

Samara University
Samara, Russia

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Part 1.

Basic facts

Modular forms: some notions

$H = \{z \in \mathrm{C},\ Im(z) > 0\},\ \Gamma = SL_2(\mathrm{Z}),$
$\Gamma(N) \subset \tilde{\Gamma} \subset \Gamma,\ s \in \mathrm{Q} \bigcup \infty\ (cusp).$

$$\Gamma(N) = \{\begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix} \in \Gamma : a \equiv d \equiv 1(N), b \equiv c \equiv 0(N)\}.$$

"Slash " —operator:
$f \mid_k [\gamma] = (cz + d)^{-k} f(\frac{az+b}{cz+d}),$
$\gamma = \begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix} \in SL_2(\mathrm{Z}).$
$\chi(d)$ is **Dirichlet character** modulo N.

Def.

$f(z)$ is called a **modular form** on H if
1) $f\mid_k[\gamma] = \chi(d) f(z), \forall\, \gamma \in \tilde{\Gamma}$;
2) $f(z)$ is holomorphic on H and in cusps.

The condition 2) :

$$f(z) = \sum_{n=n_0}^{\infty} a(n) q^n, \;\; q = e^{2\pi i z}, -$$

Fourier series for $f(z)$ in ∞. ($f(z+1) = f(z)$).
If $n_0 \geq 0$, then $f(z)$ is holomorphic in ∞.
$s \in \mathbf{Q}, s = \alpha(\infty)$.
Fourier series for $f(z)$ in s is by the definition

$$(f\mid_k[\alpha])(z) = \sum_{n=n_0}^{\infty} a(n) q_N^n, \;\; q_N = e^{\frac{2\pi i z}{N}}.$$

Modular forms: some notions

$ord_s(f) = n_0 = \{min\, n : a(n) \neq 0\}$. If $s = \beta(s_1),\ \beta \in\ \tilde{\Gamma}$, then $ord_s(f) = ord_{s_1}(f)$.
If $n_0 \geq 0$, then $f(z)$ is holomorphic in s.
If $n_0 > 0$ for all cusps then $f(z)$ is called **cusp form.**
$M_k(\tilde{\Gamma}, \chi),\ \ S_k(\tilde{\Gamma}, \chi)$.
$\Gamma_0(N) = \left\{ \begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix} \in \Gamma : c \equiv 0 (mod\ N) \right\}$.
Hecke operators:
If

$$f(z) = \sum_{n=0}^{\infty} a(n) q^n \in M_k(\Gamma_0(N), \chi),$$

$$\mathbf{f(z)} \mid \mathbf{T_{p,k,\chi}} = \sum_{n=0}^{\infty} (a(pn) + \chi(p) p^{k-1} a(n/p)) q^n.$$

If $p \nmid n,\ \ a(n/p) = 0$.

Modular forms: some notions

$$\mathbf{f(z)} \mid \mathbf{T_{m,k,\chi}} = \sum_{n=0}^{\infty} \Big(\sum_{d \mid gcd(m,n)} (\chi(d) d^{k-1} a(mn/d^2)) q^n.$$

$\chi(n) = 0$, if $gcd(N, n) \neq 1$.
If $m \geq 2$, then $f(z) \mid T_{m,k,\chi} \in M_k(\Gamma_0(N), \chi)$.
If $f(z) \in S_k(\Gamma_0(N), \chi)$, then $f(z) \mid T_{m,k,\chi} \in S_k(\Gamma_0(N), \chi)$.
If $f(z)$ is an eigenform for all $T_{m,k,\chi}$, then
$a(mn) = a(n)a(m)$, if $gcd(m, n) = 1$.

$\eta-$ products

Dedekind's η-function:

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \; q = e^{2\pi i z}, \; z \in H.$$

$\eta-$**quotient**:

$$f(z) = \prod_{j} \eta^{t_j}(a_j z), \;\; a_j \in \mathrm{N}, \; t_j \in \mathbf{Z}.$$

If $t_j \in \mathrm{N} \; \forall j$, then $f(z)$ is called $\eta-$**product.**

Notations

$$\prod_{j=1}^{s} a_j^{t_j}$$

means

$$\prod_{j=1}^{s} \eta(a_j z)^{t_j}$$

Example

$1 \cdot 3 \cdot 5 \cdot 15$ means $\eta(z)\eta(3z)\eta(5z)\eta(15z)$.

Cohen-Oesterle formula

Let χ be Dirichlet character, $\chi(-1) = (-1)^k$, $f-$ its conducter. If $p|N$, let r_p denote the maximal power of p dividing N, let s_p denote the maximal power of p dividing f.

$$\lambda(r_p, s_p, p) = \begin{cases} p^{r'} + p^{r'-1}, & 2s_p \leq r_p = 2r', \\ 2p^{r'}, & 2s_p \leq r_p = 2r' + 1, \\ 2p^{r_p - s_p}, & 2s_p \geq r_p \end{cases}$$

$$\nu_k = \begin{cases} 0, & k \equiv 1 \ (mod\ 2), \\ -\frac{1}{4}, & k \equiv 2 \ (mod\ 4), \\ \frac{1}{4}, & k \equiv 0 \ (mod\ 4) \end{cases}$$

$$\mu_k = \begin{cases} 0, & k \equiv 1 \ (mod\ 3), \\ -\frac{1}{3}, & k \equiv 2 \ (mod\ 3), \\ \frac{1}{3}, & k \equiv 0 \ (mod\ 3) \end{cases}$$

Theorem

If k is an integer, and χ is a Dirichlet modulo N, $\chi(-1) = (-1)^k$, then

$$dim_{\mathbf{C}}(S_k(\Gamma_0(N),\chi)) - dim_{\mathbf{C}}(M_{2-k}(\Gamma_0(N),\chi)) =$$

$$\frac{(k-1)N}{12}\prod_{p|N}(1+p^{-1}) - \frac{1}{2}\prod_{p|N}\lambda(r_p, s_p, p) +$$

$$\nu_k \cdot \sum_{x:x^2+1\equiv 0(N)} \chi(x) + \mu_k \cdot \sum_{x:x^2+x+1\equiv 0(N)} \chi(x)$$

If $k > 2$, then $dim_{\mathbf{C}}(M_{2-k}(\Gamma_0(N),\chi)) = 0$. The left hand of side of the Theorem 1 reduces to $dim_{\mathbf{C}}(S_k(\Gamma_0(N),\chi))$. If $k \leq 0$, $dim_{\mathbf{C}}(S_k(\Gamma_0(N),\chi)) = 0$. The left hand of side of the Theorem 1 reduces to $-dim_{\mathbf{C}}(M_{2-k}(\Gamma_0(N),\chi))$.

Multiplicative η — products

$f(z)$	k	N	$\chi(d)$
$\eta(23z)\eta(z)$	1	23	$\left(\frac{-23}{d}\right)$
$\eta(22z)\eta(2z)$	1	44	$\left(\frac{-11}{d}\right)$
$\eta(21z)\eta(3z)$	1	63	$\left(\frac{-7}{d}\right)$
$\eta(20z)\eta(4z)$	1	80	$\left(\frac{-5}{d}\right)$
$\eta(18z)\eta(6z)$	1	108	$\left(\frac{-3}{d}\right)$
$\eta(16z)\eta(8z)$	1	128	$\left(\frac{-2}{d}\right)$
$\eta^2(12z)$	1	144	$\left(\frac{-1}{d}\right)$
$\eta^4(6z)$	2	36	1
$\eta^2(8z)\eta^2(4z)$	2	32	1
$\eta^2(10z)\eta^2(2z)$	2	20	1
$\eta(12z)\eta(6z)\eta(4z)\eta(2z)$	2	24	1
$\eta(15z)\eta(5z)\eta(3z)\eta(z)$	2	15	1
$\eta(14z)\eta(7z)\eta(2z)\eta(z)$	2	14	1
$\eta^2(9z)\eta^2(3z)$	2	27	1
$\eta^2(11z)\eta^2(z)$	2	11	1

Multiplicative η — products

$f(z)$	k	N	$\chi(d)$
$\eta^3(6z)\eta^3(2z)$	3	12	$\left(\frac{-3}{d}\right)$
$\eta^6(4z)$	3	16	$\left(\frac{-1}{d}\right)$
$\eta^2(8z)\eta(4z)\eta(2z)\eta^2(z)$	3	8	$\left(\frac{-2}{d}\right)$
$\eta^3(7z)\eta^3(z)$	3	7	$\left(\frac{-7}{d}\right)$
$\eta^2(6z)\eta^2(3z)\eta^2(2z)\eta^2(z)$	4	6	1
$\eta^4(5z)\eta^4(z)$	4	5	1
$\eta^8(3z)$	4	9	1
$\eta^4(4z)\eta^4(2z)$	4	8	1
$\eta^4(4z)\eta^2(2z)\eta^4(z)$	5	4	$\left(\frac{-1}{d}\right)$
$\eta^6(3z)\eta^6(z)$	6	3	1
$\eta^{12}(2z)$	6	4	1
$\eta^8(2z)\eta^8 z)$	8	2	1
$\eta^{24}(z)$	12	1	1
$\eta^3(8z)$	$\frac{3}{2}$	4	$\left(\frac{-4}{d}\right)$
$\eta(24z)$	$\frac{1}{2}$	576	$\left(\frac{12}{d}\right)$

Eta-quotients as modular forms

Let $f(z)$ is an $\eta-$quotient of the kind $\prod_j \eta(a_j z)^{t_j}$,
$k = \frac{1}{2}\sum_j t_j \in \mathbf{Z}$,
1)

$$\sum_j a_j t_j \equiv 0(mod\, 24);$$

2)

$$\sum_j \frac{N}{a_j} t_j \equiv 0(mod\, 24),$$

then $f(z)$ satisfies the following condition

$$f\left(\frac{az+b}{cz+b}\right) = \chi(d)(cz+d)^k f(z),$$

$$\forall \begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix}, \quad \chi(d) = \left(\frac{(-1)^k\, s}{d}\right), \quad s = \prod_j a_j^{t_j}.$$

Eta-quotients in cusps (A. Biagioli, 1990)

Let $m,\ n,\ N$ be natural numbers, $n|N,\ (m,n)=1$. If $f(z)$ satisfies the conditions of the previous theorem , then the order of vanishing of $f(z)$ at the cusp $\frac{m}{n}$ is

$$\frac{N}{24}\sum_j \frac{(n,a_j)^2 t_j}{(n,\frac{N}{n})n a_j}.$$

Multiplicative η — products have in each cusp zero of order 1.

Part 2.

Dedekind eta-function and group representations

Frame-shape correspondence

Let Φ be a representation of a finite group G in a vector space of over $\mathbb{C}$ such as the characteristic polynomial of an element $g \in G$ has the form

$$P_g(x) = \prod_{j=1}^{s} (x^{a_j} - 1)^{t_j}, \; a_j \in \mathrm{N}, \; t_j \in \mathbf{Z}.$$

Frame-shape correspondence

$$g \to \eta_g(z) = \prod_{j=1}^{s} \eta(a_j z)^{t_j}$$

The symbol $\prod_{j=1}^{r} a_j^{t_j}$ is called Frame-shape.

$M\eta P$–groups: the idea

Basic fact

If g is an element in G such that $\eta_g(z)$ associated with g by a representation is a multiplicative $\eta-$product then $\eta_h(z), h = g^k$, are also multiplicative $\eta-$products.

Example

$G \cong \ Z_{15} \cong < g > .$

If $g \leftrightarrow \ \eta(15z)\eta(5z)\eta(3z)\eta(z)$, then

$g^2, \ g^4, g^7, \ g^8, g^{11}, \ g^{13}, g^{14} \leftrightarrow$

$\eta(15z)\eta(5z)\eta(3z)\eta(z),$

$g^3, \ g^6, g^9, g^{12} \leftrightarrow \ \eta^4(5z)\eta^4(z),$

$g^5, g^{10} \leftrightarrow \ \eta^6(3z)\eta^6(z), e \leftrightarrow \ \eta^{24}(z).$

$M\eta P$–groups: the definition

Definition

A group G is called $M\eta P-$group, if there is a faithful representation Φ such that $\eta_{g,\Phi}(z)$ is a multiplicative $\eta-$product $\forall g \in G$.

There are $M\eta P-$subgroups in each group (may be trivial).

Classification problems

1)To classify all $M\eta P-$groups;
2)to describe $M\eta P-$subgroups in various groups and the corresponding representations.

Theorem

Finite simple group G is $M\eta P-$group iff G is a subgroup in M_{24}.

The proof in G.V.,
" Finite simple groups and multiplicative η — products ",
Zapiski POMI, 375, 71–91, 2010.

The group $L_2(7)$.

	1A	2A	3A	4A	7A	7B
χ_1	1	1	1	1	1	1
χ_2	3	- 1	0	1	b_7	$\overline{b_7}$
χ_3	3	- 1	0	1	$\overline{b_7}$	b_7
χ_4	6	2	0	0	- 1	- 1
χ_5	7	- 1	1	- 1	0	0
χ_6	8	0	- 1	0	1	1

$$b_7 = \zeta_7 + \zeta_7^3 + \zeta_7^5.$$

$\Phi_1 = 3T_1 \oplus 3T_5.$

$2A$	$3A$	$4A$	$7A$
$\eta^{12}(2z)$	$\eta^6(3z)\eta^6(z)$	$\eta^6(4z)$	$\eta^3(7z)\eta^3(z)$

$\Phi_2 = T_1 \oplus T_5 \oplus 2T_6.$

$2A$	$3A$	$4A$	$7A$
$\eta^{12}(2z)$	$\eta^8(3z)$	$\eta^6(4z)$	$\eta^3(7z)\eta^3(z)$

$\Phi_3 = 5T_1 \oplus 2T_4 \oplus T_5.$

$2A$	$3A$
$\eta^8(2z)\eta^8(z)$	$\eta^6(3z)\eta^6(z)$

$4A$	$7A$
$\eta^4(4z)\eta^2(2z)\eta^4(z)$	$\eta^3(7z)\eta^3(z)$

Classes of conjugate elements in M_{24}

element	weight	level		element	weight	level
1^{24}	12	1		12^2	1	144
$1^8 \cdot 2^8$	8	2		$3 \cdot 21$	1	63
$1^6 \cdot 3^6$	6	3		$2^4 \cdot 4^4$	4	8
$1^4 \cdot 5^4$	5	4		$2^2 \cdot 10^2$	2	20
$1^3 \cdot 7^3$	3	7		$1 \cdot 2 \cdot 7 \cdot 14$	2	14
$1^2 \cdot 11^2$	2	11		$1 \cdot 3 \cdot 5 \cdot 15$	2	15
$1 \cdot 23$	1	23		$2 \cdot 4 \cdot 6 \cdot 12$	2	24
2^{12}	6	4		$1^2 \cdot 2^2 \cdot 3^2 \cdot 6^2$	4	6
3^8	4	9		$1^4 \cdot 2^2 \cdot 4^4$	5	4
4^6	3	16		$1^2 \cdot 2 \cdot 4 \cdot 8^2$	3	8
6^4	2	36				

$M\eta P$–groups: results

1

Metacyclic $M\eta P-$groups of the kind $< a, b : a^m = e,\ b^s = e, b^{-1}ab = a^r >$ where the intersection of the subgroups $< a >$ and $< b >$ is trivial are described.

2

abelian groups;

3

If G is an $M\eta P-$group of order $p^k,\ p-$ odd prime, then G is one of the following groups:
$S(3) \cong Z_3,\ S(3) \cong Z_3 \times Z_3,\ Z_9,$
$S(3) \cong < a, b, c : a^3 = e, b^3 = e, c^3 = e, ab = bac, ac = ca, bc = cb >,$
$S(5) \cong Z_5,\ S(7) \cong Z_7,\ \ S(11) \cong Z_{11},\ \ S(23) \cong Z_{23}.$

4

$M\eta P-$ groups of odd order are subgroups in following groups:
$G_1 \cong < a, b, c : a^3 = b^3 = c^3 = e,\ ab = bac,\ ac = ca,\ bc = cb >$,
$G_2 \cong < a, b : a^{21} = b^3 = e,\ b^{-1}ab = a^4 >$,
$G_3 \cong < a, b : a^{23} = b^{11} = e,\ b^{-1}ab = a^{10} >$,
$G_4 \cong < a, b : a^{11} = b^5 = e,\ b^{-1}ab = a^5 >$,
$G_5 \cong Z_9,\ \ G_6 \cong Z_{15}$.

5

All groups of order 16 and 24 are $M\eta P-$groups.

6

$M\eta P-$groups in $SL(5, \mathrm{C})$.

$SL(5, \mathrm{C})$ and $\eta-$products.

Theorem.

Let Ad be the adjoint representation of the group $SL(5, C)$ and $g \in SL(5, C)$, $\;ord(g) \neq 3, 6, 9, 21$, is such that the characteristic polynomial of the operator $Ad(g)$ is of the form $P_g(x) = \prod_{j=1}^{s}(x^{a_j} - 1)^{t_j}, \;\; a_j \in \mathrm{N}, \; t_j \in \mathrm{N}.$
Then $\eta_g(z) = \prod_{j=1}^{s} \eta^{t_j}(a_j z)$ is a multiplicative $\eta-$product of the weight $k(g) > 1$, and all multiplicative $\eta-$products of the weight $k(g) > 1$ can be obtained by this way.
If $ord(g) = 3, 6, 9, 21$ then by this way we can obtain all multiplicative $\eta-$products of the weight $k(g) > 1$. Moreover in this correspondence there are five modular forms which are not multiplicative $\eta-$products:

$$\eta^4(3z)\eta^{12}(z), \; \eta^7(3z)\eta^3(z), \; \eta^2(6z)\eta^6(2z),$$

$$\eta^2(9z)\eta(3z)\eta^3(z), \; \eta(21z)\eta^3(z).$$

$M\eta P-$groups in $SL(5,C)$.

The maximal finite subgroups of $SL(5,\mathrm{C})$ whose elements g have characteristic polynomials of the form $\prod_{j=1}^{s}(x^{a_j}-1)^{t_j}$ in adjoint representation and the corresponding cusp forms $\eta_g(z)=\prod_{j=1}^{s}\eta^{t_j}(a_j z)$ are multiplicative $\eta-$products, are the direct products of the group Z_5 (which is generated by the scalar matrix) and one of the following groups: $S_4,\ A_4\times\mathrm{Z}_2,\ \mathrm{Q}_8\times\mathrm{Z}_3,\ D_4\times\mathrm{Z}_3$, the binary tetrahedral group, the metacyclic group of order 21, D_6, the metacyclic group of order 12: $<S,\ T\ :\ S^3=T^2=(ST)^2>$, all groups of order 16, $\mathrm{Z}_3\times\mathrm{Z}_3,\ \mathrm{Z}_{15},\ \mathrm{Z}_{14},\ \mathrm{Z}_{11},\ \mathrm{Z}_{10},\ \mathrm{Z}_9$.

Problem .

To investigate $g\in SL(n,\mathrm{C})$ such that $P_g(x)=\prod_{j=1}^{r}(x^{a_j}-1)^{t_j},\ \ a_j\in\mathrm{N},\ t_j\in\mathrm{N},\ \ \Phi=Ad$, and associated functions $\eta_{g,\Phi}(z)$.

Ramanujan characters: motivation

If $f(z) \in S_k(\Gamma_0, \chi)$ is a Hecke eigenform than

$$L_f(s) = \prod_p \left(1 - \frac{a(p)}{p^s} + \frac{\psi(p)}{p^{2s}}\right)^{-1}$$

$$\psi(p) = a^2(p) - a(p^2).$$

Ramanudjan characters for M_{24}

G.Mason considers M_{24} in the article
"M_{24} and certain automorphic forms". Contemporary Math., v.45,1985,223-244.
M_{24} is considered as a permutation group on 24 letters.

$$g = 1^{j_1} \cdot 2^{j_2} ... r^{j_r}$$

$$k = k(g) = \frac{1}{2} \cdot (number \ \ of \ \ cycles \ \ g)$$

$N = N(g)$ = product of the longest and the shortest cycle.

$$\chi_p(g) = \left(\frac{(-1)^k 1^{j_1} \cdot 2^{j_2} ... r^{j_r}}{p} \right)$$

Ramanujan character

$$\psi_p(g) = \begin{cases} p^{k(g)-1}\chi_p(g), & (ord(g), p) = 1, \\ 0, & (ord(g), p) = p. \end{cases}$$

Theorem (Mason)

For M_{24} ψ_p^2 is a character; $\psi_p, \quad p \neq 3,$ is a character.

Ramanujan character defined by a representation

Let Φ be a representation of a finite group G in a vector space of an even dimension over $\mathbb{C}$ such as the characteristic polynomial of an element $g \in G$ has the form

$$P_g(x) = \prod_{j=1}^{s} (x^{a_j} - 1)^{t_j},\ a_j \in \mathrm{N},\ t_j \in \mathrm{Z}.$$

Frame-shape correspondence

$$g \to \eta_g(z) = \prod_{j=1}^{s} \eta(a_j z)^{t_j}$$

Ramanujan character

$$\psi_p(g) = \begin{cases} p^{k(g)-1}\chi_p(g), & (ord(g), p) = 1, \\ 0, & (ord(g), p) = p. \end{cases}$$

Eta-products for M_{24}.

All $\eta-$products associated with elements of M_{24} are Hecke eigenforms and ψ_p appears in Euler product.
If we consider $\Phi = T_1 \oplus T_{23}$, then $\eta_{g,\Phi}(z)$ are as above.

class	modular form
$2A$	$\eta^8(2z)\eta^8(z)$
$2B$	$\eta^{12}(2z)$
$3A$	$\eta^6(3z)\eta^6(z)$
$3B$	$\eta^8(3z)$
$4A$	$\eta^4(4z)\eta^4(2z)$
$4B$	$\eta^4(4z)\eta^2(2z)\eta^4(z)$
$4C$	$\eta^6(4z)$
$5A$	$\eta^4(5z)\eta^4(z)$
$6A$	$\eta^2(6z)\eta^2(3z)\eta^2(2z)\eta^2(z)$
$6B$	$\eta^4(6z)$

$7A$	$\eta^3(7z)\eta^3(z)$
$8A$	$\eta^2(8z)\eta(4z)\eta(2z)\eta^2(z)$
$10A$	$\eta^2(10z)\eta^2(2z)$
$11A$	$\eta^2(11z)\eta^2(z)$
$12A$	$\eta(12z)\eta(6z)\eta(4z)\eta(2z)$
$12B$	$\eta^2(12z)$
$14A$	$\eta(14z)\eta(7z)\eta(2z)\eta(z)$
$15A$	$\eta(15z)\eta(5z)\eta(3z)\eta(z)$
$21A$	$\eta(21z)\eta(3z)$
$23A$	$\eta(23z)\eta(z)$

Ramanujan characters and Weyl characters

Let us consider a simple Lie group G_0 of an even rang r.
Let G be a finite subgroup in G_0 such that the characteristic polynomial of $Ad(g)$ has the form $P_g(x) = \prod_{j=1}^{s}(x^{a_j} - 1.)^{t_j}$, $ch_{(p-1)\rho}$ is Weyl character of the irreducible representation of G_0 with the leading weight $(p-1)\rho$, where ρ is a half-sum of positive roots of $Lie(G_0)$. $\forall g \in G$ and prime p that is coprime with the $ord(g)$ we have

$$\psi_p(g) = \left(\frac{-1}{p}\right)^{\frac{dim G_0}{2}} p^{\frac{r}{2}-1} ch_{(p-1)\rho}.$$

Sets of $\eta-$functions which define groups.

$(G, \Phi)-$**set** of $\eta-$functions:

$$S = \{\eta_{g,\Phi}(z)\}_{\{g\in G\}};$$

Reduced $(G, \Phi)-$**set** of $\eta-$functions is obtained by omitting in S the same forms.

Example.

$(S_3, 4T_{reg}) =$
$\{\eta_e(z),\ \eta_a(z),\ \eta_{a^2}(z),\ \eta_b(z),\ \eta_{ab}(z),\ \eta_{a^2b}(z)\} =$

$$\{\eta^{24}(z),\ \eta^8(3z),\ \eta^8(3z),\ \eta^{12}(2z),\ \eta^{12}(2z),\ \eta^{12}(2z).\}$$

$$(S_3, 4T_{reg})_{red} = \{\eta^{24}(z),\ \eta^8(3z),\ \eta^{12}(2z).\}$$

Def.

$\eta-$products $f_1, ..., f_t$ are called
$G-$**connected** if $\{f_1, ..., f_t\}$ is $(G, \Phi)_{red}$-set;
$\eta-$products $f_1, ..., f_t$ are called $G-$**dependent** if
$\{f_1, ..., f_t\} \subset S, \quad S \; is \; a \; (G, \Phi)_{red}$-set.

Def.

Let S_1- be a reduced $(G_1, \Phi_1)-$set,
S_2- reduced $(G_2, \Phi_2)-$set,
the intersection of S_1 and S_2 is a maximal set S, with the following properties:
1) each reduced $(H_j, \Psi_j)-$ set V_j, which is included in S_1 and in S_2, contains S;
2) S is a reduced $(H, \Psi)-$set for a group H and a faithful representation Ψ.

Def.

Let $S_1 = \{f_1, ..., f_s\}-$ be a reduced $(G_1, \Phi_1)-$set, $S_2 = \{g_1, ..., g_t\}-$ reduced $(G_2, \Phi_2)-$set, set, **the union** of the sets S_1 and S_2 is a set U , which we obtain from $S = \{f_i g_j\},\ i = \overline{1, s},\ j = \overline{1, t}$, by deleting the same forms.

U is a $(G_1 \times G_2, T)_{red}-$ set, the representation T is defined by the formula :

$$T(g) = \Phi_1(g_1) \oplus \Phi_2(g_2),$$

$$g = g_1 g_2,\ g_1 \in G_1,\ g_2 \in G_2.$$

Example.

Let

$S_1 = \{\eta^{24}(z),\ \eta^{8}(3z),\ \eta^{12}(2z)\}-$
a $(\mathrm{S}_3, \Phi_1)_{red}-$ set, $\Phi_1 = 4T_{reg}$;
$S_2 = \{\eta^{24}(z),\ \eta^{6}(4z),\ \eta^{12}(2z)\}-$
$(D_4, \Phi_2)_{red}-$ set, $\Phi_2 = 3T_{reg}$.

$$S = \{\eta^{24}(z),\ \eta^{12}(2z)\},$$

$$U = \{\eta^{48}(z),\ \eta^{6}(4z)\eta^{24}(z),\ \eta^{12}(2z)\eta^{24}(z),$$
$$\eta^{8}(3z)\eta^{24}(z),\ \eta^{6}(4z)\eta^{12}(2z),$$
$$\eta^{6}(4z)\eta^{8}(3z),\ \eta^{8}(3z)\eta^{12}(2z),\ \eta^{24}(2z)\}.$$

Problem .

To investigate this notions to give an example of G-independent $\eta-$products.

Problem .

To investigate the following phenomenon:
the sets $S_1, ..., S_t$ can be considered as $(G, \Phi_1)_{red}-, ..., (G, \Phi_t)_{red}-$sets only for the unique group $G.$

Theorem.

Let p be odd prime, $p \neq 3, 7$.
Then D_p is defined by the the set

$$\{\eta^{24p}(z),\ \ \eta^{24}(pz),\ \ \eta^{12p}(2z)\}.$$

The groups D_3 and D_7 are defined by the following sets :
$D_3 \cong S_3 : \{\eta^{24}(z),\ \eta^{8}(3z),\ \eta^{12}(2z)\} \bigwedge$
$\{\eta^{24}(z),\ \eta^{3}(6z)\eta^{6}(z),\ \eta^{11}(2z)\eta^{2}(z)\};$
$D_7 : \{\eta^{168}(z),\ \eta^{24}(7z),\ \eta^{84}(2z)\} \bigwedge$
$\{\eta^{24}(z),\ \eta^{3}(7z)\eta^{3}(z),\ \eta^{10}(2z)\eta^{4}(z)\}$

Part 3.

Structure theorems

The idea of cutting

Let $f(z) \in S_l(\Gamma_0(N)), l, k,$ —natural even numbers, $k > l$, then

$$S_k(\Gamma_0(N)) \cong f(z) \cdot M_{k-l}(\Gamma_0(N)) \oplus V,$$

$dim\ \ V$ depends on N, l and sometimes on k modulo 12.

Classical fact

$$\Delta(z) = \eta^{24}(z) = \frac{E_4(z)^3 - E_6(z)^2}{1728}.$$

$$S_k(\Gamma) = \Delta(z) \cdot M_{k-12}(\Gamma).$$

Analogous theorems can be proved for all levels corresponding to multiplicative eta-functions .

The structure theorems: model example

Theorem .
Let $N \equiv 5 \pmod{12}$. Then

$$S_k(\Gamma_0(N)) = V_1 \oplus V_2 \oplus V_3,$$

where

$$V_1 = \eta^4(Nz)\eta^4(z) \cdot M_{k-4}(\Gamma_0(N)),$$
$$V_2 = < \eta^4(Nz)\eta^4(z) >^{\perp} \cdot G(z),$$
$$V_3 = S_2(\Gamma_0(N)) \cdot H(z),$$

$$G(z) = E_4^{\frac{k-4}{4}}(z), H(z) = E_6^{\frac{k-2}{6}}(z), \text{ if } k \equiv 0 \pmod 4,$$
$$G(z) = E_6^{\frac{k-4}{6}}(z), H(z) = E_4^{\frac{k-4}{4}}(z), \text{ if } k \equiv 2 \pmod 4.$$

The exact cutting: even weight

Theorem .

Let k, l — even numbers, $k > l$. Then

$$S_k(\Gamma_0(N)) \cong f(z) \cdot M_{k-l}(\Gamma_0(N)) \Leftrightarrow$$

1) $f(z)$ — multiplicative η — product,
or
2) $N = 17, k \equiv 2 \pmod 4, k \geq 6, l = 2;$
$N = 19, k \equiv 2 \pmod 6, k \geq 8, l = 2.$

The exact cutting: common case

Theorem.

Let $N \neq 3, 17, 19$.
Let χ be a quadratic character modulo N such that $\chi(-1) = -1$, k, l are positive integers. Then

$$S_k(\Gamma_0(N), \chi^k) = f(z) \cdot M_{k-l}(\Gamma_0(N), \chi^{k-l}),$$

where $f(z) \in S_l(\Gamma_0(N), \chi^l)$ iff $f(z)$ is a multiplicative eta-product.
Here χ^l is trivial if l is even.

Modular forms as η — polynomials

Classical fact

Each element $f(z) \in M_k(\Gamma)$ is a polynomial of $E_4(z)$ and $E_6(z)$.

$$E_4(z) = \frac{\eta^{16}(z)}{\eta^8(2z)} + 2^8 \cdot \frac{\eta^{16}(2z)}{\eta^8(z)} \quad ,$$

$$E_6(z) = \frac{\eta^{24}(z)}{\eta^{12}(2z)} - 2^5 \cdot 3 \cdot 5 \cdot \eta^{12}(2z)$$
$$-2^9 \cdot 3 \cdot 11 \cdot \frac{\eta^{12}(2z)\eta^8(4z)}{\eta^8(z)} + 2^{13} \cdot \frac{\eta^{24}(4z)}{\eta^{12}(2z)}.$$

Modular forms as eta–polynomials.

1) Each element $f(z)$ from $M_{4l}(\Gamma_0(2))$ is a homogeneous polynomial of functions $\eta^{-8}(z)\eta^{16}(2z)$ and $\eta^{16}(z)\eta^{-8}(2z)$. If $f(z)$ is a cusp form, then $f(z)$ is divided by $\eta^{8}(z)\eta^{8}(2z)$.

2) Each element $f(z)$ from $M_{3l}\left(\Gamma_0(3),\ \left(\frac{-3}{d}\right)^l\right)$ is a homogeneous polynomial of functions $\eta^{-3}(z)\eta^{9}(3z)$ $\eta^{9}(z)\eta^{-3}(3z)$. If $f(z)$ is a cusp form, $f(z)$ is divided by $\eta^{6}(z)\eta^{6}(3z)$.

3) Each element $f(z)$ from $M_{l}\left(\Gamma_0(4),\ \left(\frac{-1}{d}\right)^l\right)$ is a homogeneous polynomial of functions $\eta^{-4}(z)\eta^{10}(2z)\eta^{-4}(4z)$; $\eta^{-4}(2z)\eta^{8}(4z)$. If $f(z)$ is a cusp form, then $f(z)$ is divided by $\eta^{4}(z)\eta^{2}(2z)\eta^{4}(4z)$ and its weight more than 4; if $f(z)$ is a cusp form of the weight more than 5, then it is also divided by $\eta^{12}(2z)$.

Each modular form $f(z) \in M_k(\Gamma_0(N),\ \chi^k)$ is a homogeneous polynomial of functions $f_1(z), \ldots, f_s(z)$; if $f(z)$ is a cusp form then it is divided by $g(z)$. The values $N,\ \chi,\ f_1(z), \ldots, f_s(z)$,
$g(z)$ are given in the following table.

Table.

N	χ	$f_1(z), \ldots, f_s(z)$	$g(z)$
36	$\left(\frac{-3}{d}\right)$	$3^{-1}\cdot 9^3$; $4^3\cdot 12^{-1}$; $6^{-1}\cdot 18^3$; $12^{-1}\cdot 36^3$; $6^{-2}\cdot 12\cdot 18^6\cdot 36^{-3}$; $1^{-2}\cdot 2\cdot 3^6\cdot 6^{-3}$	6^4
32	$\left(\frac{-1}{d}\right)$	$4^{-2}\cdot 8^4$; $4^4\cdot 8^{-2}$; $2^4\cdot 4^{-2}$; $1^{-1}\cdot 2^{10}\cdot 4^{-4}$	$4^2\cdot 8^2$
27	$\left(\frac{-3}{d}\right)$	$3^3\cdot 9^{-1}$; $1^3\cdot 3^{-1}$; $3^{-1}\cdot 9^3$	$3^2\cdot 9^2$
24	$\left(\frac{-6}{d}\right)$	$1\cdot 3^{-1}\cdot 4\cdot 6\cdot 8^{-1}\cdot 24$; $1^{-1}\cdot 2\cdot 3\cdot 8\cdot 12\cdot 24^{-1}$; $2^{-4}\cdot 4^8$; $6^{-4}\cdot 12^8$; $12^{-4}\cdot 24^8$; $4^{-2}\cdot 8^4\cdot 12^{-2}\cdot 24^4$; $4^{-4}\cdot 8^8$	$2\cdot 4\cdot 6\cdot 12$
20	$\left(\frac{-5}{d}\right)$	$1\cdot 2\cdot 4^{-1}\cdot 5^{-1}\cdot 10\cdot 20$; $1^{-1}\cdot 2\cdot 4\cdot 5\cdot 10\cdot 20^{-1}$; $2^{-4}\cdot 4^8$; $10^{-4}\cdot 20^8$; $2^{-2}\cdot 4^4\cdot 10^{-2}\cdot 20^4$	$2^2\cdot 10^2$
15	$\left(\frac{-15}{d}\right)$	$1^2\cdot 3^{-1}\cdot 5^{-1}\cdot 15^2$; $1^{-1}\cdot 3^2\cdot 5^2\cdot 15^{-1}$; $1^3\cdot 3^{-1}\cdot 5^3\cdot 15^{-1}$	$1\cdot 3\cdot 5\cdot 15$
14	$\left(\frac{-7}{d}\right)$	$1^2\cdot 2^{-1}\cdot 7^2\cdot 14^{-1}$; $1^{-1}\cdot 2^2\cdot 7^{-1}\cdot 14^2$; $1^5\cdot 2^{-3}\cdot 7^{-3}\cdot 14^5$	$1\cdot 2\cdot 7\cdot 14$

Part 4.

Arithmetic results

Let $\mathrm{K} = \mathrm{Q}(\sqrt{-\mathrm{D}})$ be an imaginary quadratic field with discriminant $-D$, and let O_K be its ring of algebraic integers. Let M be a nontrivial ideal in O_K and let $I(M)$ denote the group of fractional ideals prime to M. A Hecke Grossencharacter ϕ of weight $k \geq 2$ with modulus M is a homomorphism

$$\phi : I(M) \to \mathrm{C}^*$$

such that for each $\alpha \in K^*,\ \alpha \equiv 1(mod M)$ we have

$$\phi(\alpha O_K) = \alpha^{k-1}.$$

Let

$$\omega_\phi(n) = \frac{\phi((n))}{n^{k-1}}$$

for every integer n coprime to M.

$$\Psi(z) = \sum_{\mathfrak{a}} \phi(\mathfrak{a}) q^{N(\mathfrak{a})},$$

where the sum is over the integral ideals $\mathfrak{a}$ that are prime to M.

Dummit, Kisilevsky and McKay have found for 16 of 28 multiplicative η-products of the integer weight L- functions with grossen-characters of imaginary quadratic fields which are equal to the Mellin transformations of this forms. They have proved that for other 12 multiplicative η-products of the integer weight this correspondence is impossible.

Example.

$\Psi(z) = \eta^6(4z),\ \ K = Q(\sqrt{-1}),\ \ M = (2),$
$\Psi(z) \in S_3(\Gamma_0(16)), (\frac{-4}{\cdot}).$

We present the analogous formulas where instead of the ring of integers of an imaginary quadratic field we consider orders in the algebra of quaternions and the Cayley algebra.

Let $\mathbf{H}$ be the algebra of quaternions over $\mathbf{Q}$ and Γ_4 is the lattice of the Hurwitz quaternions :

$$\alpha = \frac{a+bi+cj+dk}{2},$$

$$a \equiv b \equiv c \equiv d \pmod 2, a, b, c, d \in \mathbf{Z}.$$

Then

$$\frac{1}{12} \sum_{\alpha \in \Gamma_4 \subset \mathbf{H}} \alpha^6 e^{2\pi i z N(\alpha)} = \eta^8(z)\eta^8(2z).$$

Furthermore

$$\frac{1}{8} \sum_{\substack{\alpha \in \mathbf{H}, \\ a+b+c+d \equiv 1 \pmod 2}} \alpha^4 e^{2\pi i z N(\alpha)}$$

$$= \eta^{12}(2z),$$

where the summation is taken over such quaternions

Theorem. Let Ca be the Cayley algebra. Then we can construct in Ca the order on which the bilinear form

$$< \alpha, \beta >= \alpha\bar{\beta} + \beta\bar{\alpha}$$

defines the structure of the even unimodular lattice of the type Γ_8 where the root system E_8 is closed under the multiplication in Cayley algebra.
Then the sum

$$\frac{1}{12} \sum_{\alpha \in \Gamma_8 \subset Ca} \alpha^8 e^{2\pi i z N\alpha}$$

over all elements of this order is equal to the cusp form $\eta^{24}(z)$.

Shimura sums.

Let a(n) be an arithmetic function and c a positive integer ($a(x) = 0$, if x is not integer).
Then for $m \geq 1$ the Shimura sum :

$$Sh(m, a, c) = \sum_{j=1}^{m-1} a\left(\frac{m^2 - j^2}{c}\right).$$

$$Let\ f(z)\ be\ \ \eta(18z)\eta(6z)=\sum_{n=1}^{\infty}a(n)q^n\in S_1(108,\chi).$$

$1)\ If\ p\ is\ inert\ in\ K=\mathrm{Q}(\sqrt{-3}),\ then$

$$p=-2Sh(p^2,a,1)-1.$$

$2)\ If\ p\ splits\ in\ K=\mathrm{Q}(\sqrt{-3}),\ then$

$$p=(2Sh(p,a,1)+a^2(p))^2-2Sh(p^2,a,1)$$
$$-a^2(p^2)-2Sh(p,a,1).$$

$In\ particular,\ if\ |a(p)|=1,\ then$

$$p=4Sh^2(p,a,1)-2Sh(p^2,a,1)+2Sh(p,a,1)+1;$$

$if\ |a(p)|=2,\ then$

$$p=4Sh^2(p,a,1)-2Sh(p^2,a,1)+14Sh(p,a,1)-5.$$

Shimura sums

$$Let\ f(z)\ be\ \ \eta^2(12z) = \sum_{n=1}^{\infty} a(n)q^n \in S_1(144, \chi).$$

1) $If\ p\ is\ inert\ in\ K = \mathrm{Q}(\sqrt{-3}),\ then$

$$p = -2Sh(p^2, a, 1) - 1.$$

2) $If\ p\ splits\ in\ K = \mathrm{Q}(\sqrt{-3}),\ then\ p = l^2 + 3m^2, where$
$(\frac{l}{3})\ l = Sh(p, a, 1) + 2.$

$$p = (2Sh(p, a, 1) + a^2(p))^2 - 2Sh(p^2, a, 1) -$$

$$a^2(p^2) - 2Sh(p, a, 1).$$

Open problems

To classify all $M\eta P$— groups.

Investigate the structure of $S_k(\mathbf{\Gamma_0}, \chi)$.

Ken Ono' problem: describe the spaces generated by eta-polynomials.

Describe the spaces with bases formed by eta-quotients .

Investigate Shimura sums.

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