Групповые методы в динамике вихревых нитей

Group methods in the dynamics of vortex filaments

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OBJECT:
closed vortex filament with zero thickness

Closed curve which evolving in the space E_3:

$$\boldsymbol{z}(\tau,\xi) = \boldsymbol{z}_0 + R_0 \int\limits_0^{2\pi} [(\xi-\eta)/2\pi]\, \boldsymbol{j}(\tau,\eta) d\eta\,,$$

WHERE: 2π- periodical function $\boldsymbol{j}(\tau,\eta)$ satisfied to the **equation** for the continious Heisenberg spin chain

$$\partial_\tau \boldsymbol{j}(\tau,\xi) = \boldsymbol{j}(\tau,\xi) \times \partial_\xi^2 \boldsymbol{j}(\tau,\xi)\,. \tag{1}$$

and the **consraints**:

$$\Phi_k = \int\limits_0^{2\pi} j_k(\xi) d\xi = 0 \qquad k = 1,2,3, \tag{2}$$

$$\boldsymbol{j}^{\,2}(\xi) = 1\,. \tag{3}$$

LOCAL INDUCTION EQUATION

$$\partial_\tau \boldsymbol{z}(\tau,\xi) = \frac{1}{R_0}\,\partial_\xi \boldsymbol{z}(\tau,\xi) \times \partial_\xi^2 \boldsymbol{z}(\tau,\xi)\,.$$

Momentum and angular momenta:

$$\boldsymbol{p} = \frac{1}{2}\int \boldsymbol{r}\times\boldsymbol{\omega}(\boldsymbol{r})dV\,, \qquad \boldsymbol{s} = \frac{1}{3}\int \boldsymbol{r}\times\big(\boldsymbol{r}\times\boldsymbol{\omega}(\boldsymbol{r})\big)dV\,. \tag{4}$$

Vorticity $\quad \boldsymbol{\omega}(\boldsymbol{r}) = \Gamma\int\limits_0^{2\pi}\hat{\delta}(\boldsymbol{r}-\boldsymbol{z}(\xi))\partial_\xi \boldsymbol{z}(\xi)d\xi\,.\quad$ Circulation $\quad\Gamma$

$$\boldsymbol{p} = {R_0}^2\Gamma\boldsymbol{f}\,, \qquad \boldsymbol{f} = \frac{1}{2}\iint\limits_0^{2\pi}[\xi-\eta]\;\boldsymbol{j}(\xi)\times\boldsymbol{j}(\eta)d\xi d\eta\,. \tag{5}$$

The space-time symmetry group: $\quad E(3)\times E_\tau$

PROBLEM:
The standard formula for the energy leads to the divergences

$$\mathcal{E} = \frac{1}{8\pi} \iint \frac{\boldsymbol{\omega}(\boldsymbol{r})\boldsymbol{\omega}(\boldsymbol{r}')}{|\,\boldsymbol{r} - \boldsymbol{r}'|} dV dV' \,, \tag{6}$$

STANDARD APPROACH

to take into account non-zero thickness a and so on...

- WHAT ENERGY FOR THE DYNAMICAL SYSTEM FOR $a = 0$?
- WHAT THE EFFECTIVE MASS OF THE SYSTEM?

SUGGESTED APPROACH:

- We construct the non-stanard hamiltonian description
- We enlarge the space-time symmetry group $E(3) \times E_\tau \to \widetilde{\mathcal{G}}_3$, where $\widetilde{\mathcal{G}}_3$ is extended Galilei group.
- We use Cazimir functions of the algebra of the group $\widetilde{\mathcal{G}}_3$ to define the energy.

ADDITIONAL POSSIBILITIES:

We can calculate the inverse effective mass tensor

"OLD" DESCRIPTION:

The set $\mathcal{A}$

The variables $\{\, \boldsymbol{z}_0\,, \Gamma\,, \boldsymbol{j}(\xi)\,\}$ constrained by the conditions (2) and (3).

NEW VARIABLES:

STEPS:

- we use the equivalent variable $p = |\boldsymbol{p}|$ instead of the variable Γ
- we introduce the spherical coordinates (p, θ, φ) (p_3 axis $\parallel$ $\boldsymbol{f}$).
  After that **we add** the coordinates (θ, φ) as additional dynamical variables.
- we **replace** the spherical variables (p, θ, φ) with the Decart variables p_1, p_2 and p_3 and will use these quantities as new independent fundamental variables.

NEW CONSTRAINT:

$$\Phi_0(p_1, p_2, p_1\,; \boldsymbol{j}) \equiv (\boldsymbol{p}\boldsymbol{f})^2 - \boldsymbol{p}^2\boldsymbol{f}^2 = 0\,. \tag{7}$$

The set Ω

The one-to-one correspondence $\mathcal{A} \quad \longleftrightarrow \quad \Omega$ holds.

We enlarge the space-time symmetry group $E(3) \times E_\tau$ by means of addition of Galilei boosts

$$p_j \longrightarrow \tilde{p_j} = p_j + cv_j\,, \qquad c, v_j = const, \quad j = 1,2,3\,.$$

(The one-parameter (m_0) central extension for the standard Galilei group $\mathcal{G}_3$is fulfilled.)

The symmetry group for our theory is an extended Galilei group $\widetilde{\mathcal{G}}_3$.

We introduce variables

$$q_i = m_0 z_{0i} + \tau t_0 p_i\,, \qquad i = 1,2,3\,,$$

Finally, the variables $\boldsymbol{j}(\xi)$, $\boldsymbol{q}$, $\boldsymbol{p}$, – the new fundamental variables.

The curve $\boldsymbol{z}(\tau,\xi)$:

$$\boldsymbol{z}(\tau,\xi) = \frac{1}{m_0}\left(\boldsymbol{q} - \tau t_0 \boldsymbol{p}\right) + R_0 \int\limits_0^{2\pi} [\xi - \eta]\, \boldsymbol{j}(\tau,\eta) d\eta\,.$$

The energy of the vortex filament

Lee algebra of group $\widetilde{\mathcal{G}}_3$ has three Cazimir functions:

$$\hat{C}_1 = m_0 \hat{I}\,, \quad \hat{C}_2 = \left(\hat{M}_i - \sum_{k,j=1}^{3} \epsilon_{ijk} \hat{P}_j \hat{B}_k \right)^2, \quad \hat{C}_3 = \hat{H} - (1/2m_0) \sum_{i=1}^{3} \hat{P}_i^{\,2}\,,$$

where

$\hat{I}$ – unit operator,

$\hat{M}_i$ – generator of rotations,

$\hat{H}$ – generator of time translations,

$\hat{P}_i$ – generator of space translations,

$\hat{B}_i$ – Galilean boosts.

The value $\hat{C}_3$ can be interpreted as an ”internal energy of the system”

Hamiltonian structure

- Phase space $\mathcal{H} = \mathcal{H}_j \times \mathcal{H}_3$.
  The space $\mathcal{H}_3$ is the phase space of a free structureless $3D$ particle.
  The space $\mathcal{H}_j$ is parametrized by the 2π-periodical functions $j_k(\xi)$, where $k = 1, 2, 3$.

- Poisson structure:

$$
\begin{aligned}
\{p_i\,, q_j\} &= m_0\,\delta_{ij}\,, \qquad i, j = 1, 2, 3\,, \\
\{j_a(\xi), j_b(\eta)\} &= \beta\,\epsilon_{abc} j_c(\xi)\hat{\delta}(\xi - \eta)\,, \qquad \epsilon_{123} = 1\,.
\end{aligned}
\tag{8}
$$

  where $\beta = -2/\mathcal{E}_0 t_0$.

- Constraints: $\Phi_k = 0$, where $k = 0, \ldots, 3$. The functions Φ_k were defined in the eq. (2) and (7);

- Hamiltonian

$$
H = H_0 + \sum_{k=0}^{3} l_k \Phi_k\,,
$$

  where the function H_0

$$
H_0(p_1, p_2, p_3\,; \boldsymbol{j}) = \frac{1}{2m_0} \sum_{i=1}^{3} p_i^2 + \frac{\mathcal{E}_0}{2\pi} \int_0^{2\pi} \big(\partial_\xi \boldsymbol{j}(\xi)\big)^2 d\xi\,.
$$

The expression for the energy of our system:

$$\mathcal{E} = H\Big|_{\Omega} = \frac{1}{2m_0}\left(\boldsymbol{p}\,\boldsymbol{n}_f\right)^2 + \frac{\mathcal{E}_0}{2\pi}\int\limits_0^{2\pi}\left(\partial_\xi \boldsymbol{j}(\xi)\right)^2 d\xi\,,$$

where vector $\boldsymbol{n}_f = \boldsymbol{f}/|\boldsymbol{f}|$.

The inverse effective mass tensor $(1/m_{\text{eff}})_{ik}$:

$$\left(\frac{1}{m_{\text{eff}}}\right)_{ik} \equiv \frac{\partial\mathcal{E}}{\partial p_i \partial p_k} = \frac{1}{m_0}(\boldsymbol{n}_f)_i(\boldsymbol{n}_f)_k\,.$$