Cluster manifolds and Painlevé equations

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based on joint paper with P. Gavrylenko and A. Marshakov
arXiv:1711.02063, arXiv:1804.10145

19 August 2018

Deatonomization of cluster integrable system

Newton polygon Δ

$\Downarrow$

Thurston diagram

$\Downarrow$

Bipartite graph on a torus

$\Downarrow$

Quiver $\mathcal{Q}$, X-cluster variety $\mathcal{X}_{\mathcal{Q}}$

$\Downarrow$

Integrable system. Casimir elements.

$\Downarrow$

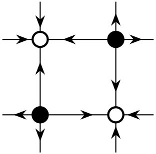
Group of discrete flows $G_\Delta \subset G_{\mathcal{Q}}$

$\Downarrow$

Deautomization, q-difference equations.

Main example

Bipartite graph on a torus (example):

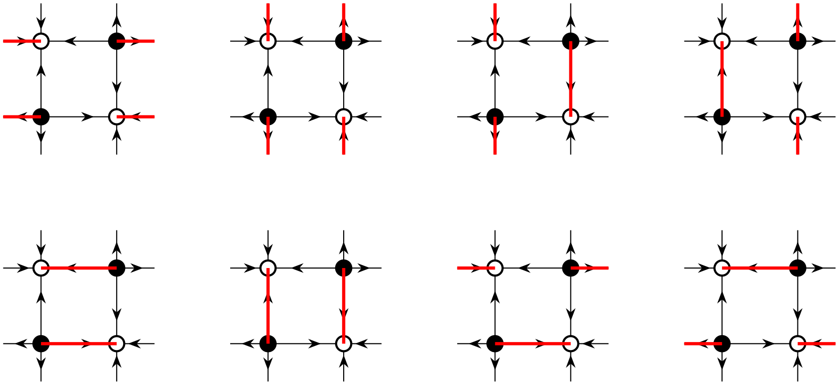


All edges are oriented from black to white.
To each edge e we assign the weight $w(e)$. In other words we have a discrete $GL(1)$ connection on a graph.

Dimer models

Dimer configurations is the set of edges with the property that every vertex is the endpoint of a unique edge in the set.

There are 8 dimer configurations for our bipartite graph



Weights of the configurations

The weight of each dimer configuration D is a product of weights of the edges

$$w(D) = \prod_{e \in D} w(e).$$

If we pick one dimer configuration D_0 then $D - D_0$ is cycle, $\partial(D - D_0) = 0$, for any D. Therefore, the weight $w(D_0)^{-1} w(D)$ is given by weights of elementary cycles — faces and A, B cycles on torus

$$\prod_{e \in \partial Face_i} w(e) = x_i, \qquad \prod_{e \in A-cycle} w(e) = \lambda, \qquad \prod_{e \in B-cycle} w(e) = \mu$$

Note that λ, μ depend on concrete choice of A, B cycles.

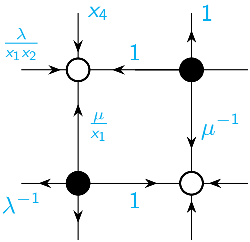
We have $\prod_i x_i = \prod_{e \in \partial \mathbb{T}^2} W(e) = 1$, since $\partial \mathbb{T}^2 = 0$.

Weigths of the edges in examle

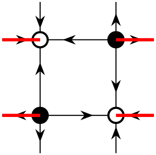
The conditions

$$\prod_{e\in\partial Face_i} w(e) = x_i, \qquad \prod_{e\in A-cycle} w(e) = \lambda, \qquad \prod_{e\in B-cycle} w(e) = \mu$$

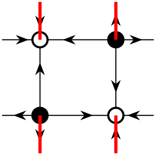
can be fulfilled by



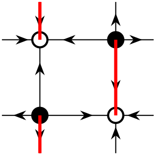
Weights of the dimer configurations



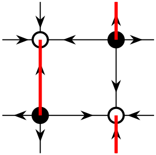
$x_1^{-1}x_2^{-1}$



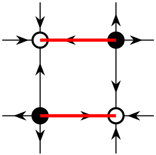
x_4



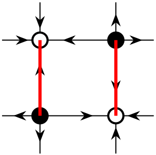
$\mu^{-1}x_4$



μx_1^{-1}



1



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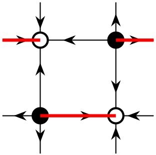
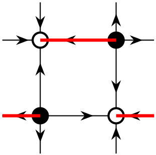


$\lambda x_1^{-1}x_2^{-1}$



λ^{-1}

Partition function $\mathcal{Z}(\lambda,\mu) = W(D_0)^{-1}\sum W(D)$, where

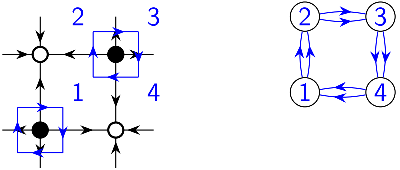
$$\mathcal{Z}(\lambda,\mu) = x_1^{-1}x_2^{-1}\lambda + \lambda^{-1} + \mu x_1^{-1} + \mu^{-1}x_4 + H,$$

where

$$H = 1 + x_1^{-1} + x_1^{-1}x_2^{-1} + x_4$$

Cluster structure

Quiver:



Poisson bracket is

$$\{x_1, x_2\} = 2x_1x_2, \{x_2, x_3\} = 2x_2x_3, \{x_3, x_4\} = 2x_3x_4, \{x_4, x_1\} = 2x_4x_1$$

Using the $(\mathbb{C}^\times)^3$ action $\mathcal{Z}(\lambda, \mu) \mapsto t_Z \cdot \mathcal{Z}(t_\lambda \cdot \lambda, t_\mu \cdot \mu)$ one can get

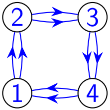
$$\mathcal{Z}(\lambda, \mu) = \lambda + z\lambda^{-1} + \mu + \mu^{-1} + H = 0$$

Casimirs and Hamiltonian:

$$1 = x_1x_2x_3x_4, \quad z = x_1x_3, \quad H = \sqrt{x_1x_2} + \frac{1}{\sqrt{x_1x_2}} + \sqrt{\frac{x_1}{x_2}} + z\sqrt{\frac{x_2}{x_1}}$$

Mutations

The discrete flows come from mutations of the quiver



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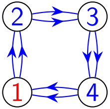

Mutation μ_1

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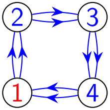

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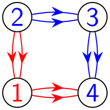

Reverse all incoming and outgoing arrows

$x_1' = x_1^{-1}$

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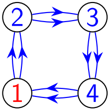

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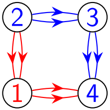

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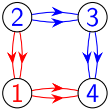

Complete cycles through mutation vertex
$x_4' = x_4(1 + x_1)^2$
$x_2' = x_2(1 + x_1^{-1})^{-2}$

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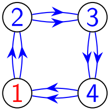

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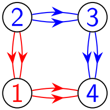

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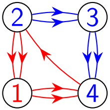

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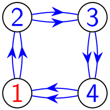

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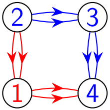

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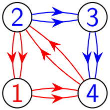

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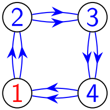

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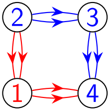

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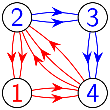

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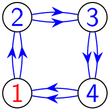

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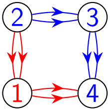

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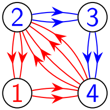

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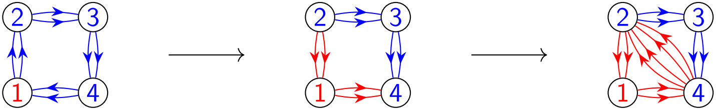

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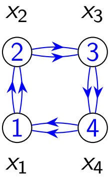
Formulas: $\mu_j\colon \epsilon_{ik} \mapsto -\epsilon_{ik},$ if $i=j$ or $k=j$, $\quad \epsilon_{ik} \mapsto \epsilon_{ik} + \frac{\epsilon_{ij}|\epsilon_{jk}|+\epsilon_{jk}|\epsilon_{ij}|}{2}$ otherwise.

$$\mu_j : x_j \mapsto x_j^{-1}, \qquad x_i \mapsto x_i \left(1 + x_j^{\mathrm{sgn}\,\epsilon_{ij}}\right)^{\epsilon_{ij}}, \quad i \neq j. \qquad \{x_i', x_k'\} = \epsilon_{ik}' x_i' x_k'$$

We have to find all combinations of mutations, permutations of vertices and simultaneous inversions of edges, that preserve quiver. Example:



Cluster automorphisms (group $G_{\mathcal{Q}}$)

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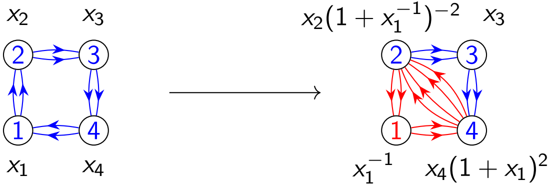

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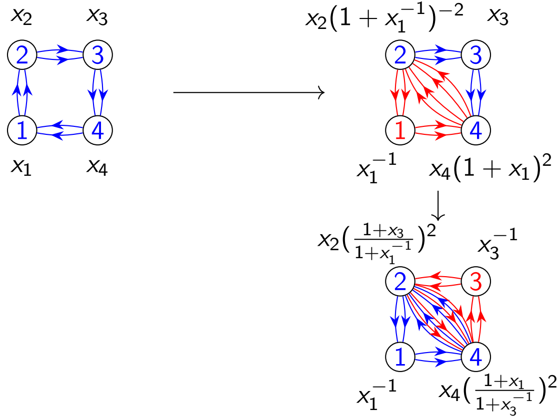

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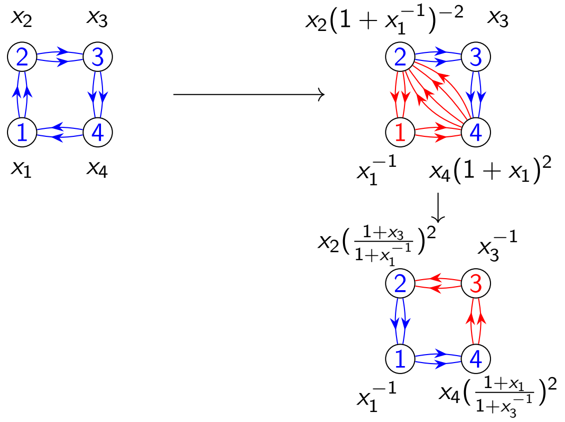

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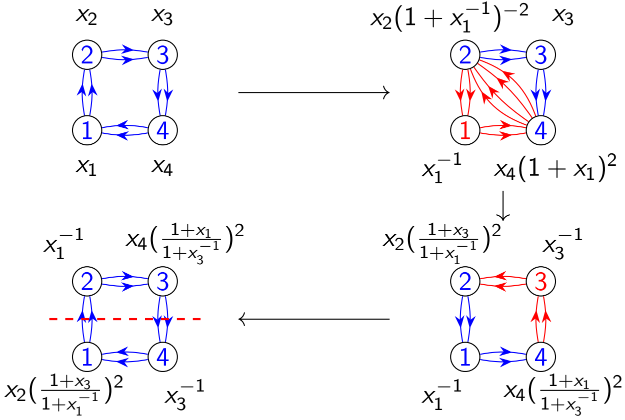

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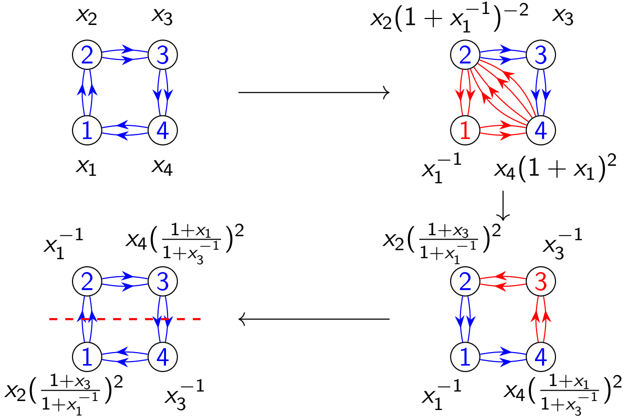


$T:(x_1,x_2,x_3,x_4)\mapsto\left(x_2(\frac{1+x_3}{1+x_1^{-1}})^2, x_1^{-1}, x_4(\frac{1+x_1}{1+x_3^{-1}})^2, x_3^{-1}\right)$. Rational functions with nonnegative integer coefficients. Laurent phenomenon in τ variables.

Painleveé equations

Set $x_1x_2x_3x_4 = q$, (since $x_1x_2x_3x_4 \neq 1 \implies$ no integrable system.) $z = x_1x_3$

$$T : (x_1, x_2, x_3, x_4) \mapsto \left(x_2(\frac{1+x_3}{1+x_1^{-1}})^2, x_1^{-1}, x_4(\frac{1+x_1}{1+x_3^{-1}})^2, x_3^{-1} \right)$$

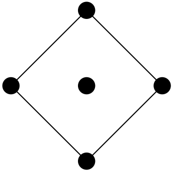
$$T : (x_1, x_2, z, q) \mapsto \left(x_2(\frac{x_1+z}{x_1+1})^2, x_1^{-1}, qz, q \right)$$

Casimir z becomes "time", so introduce $x_i = x_i(z)$, $T : x_i(z) \mapsto x_i(qz)$.

$$x_1(qz)x_1(q^{-1}z) = \left(\frac{x_1(z)+z}{x_1(z)+1} \right)^2$$

This is q-Painlevé III_3 equation, or $P(A_7^{(1)'})$.

Directions of the generalization



4 boundary points, internal points on one line $\longrightarrow$

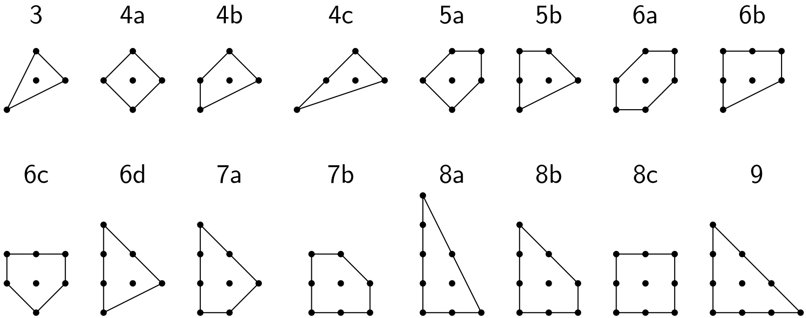
Non-autonomous discrete Hirota equations. Integrable system is relativistic Toda

One internal point $\downarrow$

q-difference Painlevé equations

? General Newton polygons

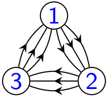
Newton polygons with one internal point



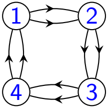
Same area $\Leftrightarrow$ same quivers, except for 4_a, 4_c vs 4_b.

Quivers and their automorphism groups

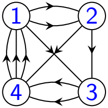
S_3



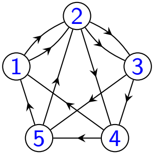
$Dih_4 \ltimes W(A_1^{(1)})$



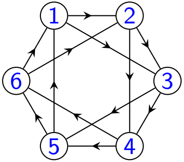
$W(A_1^{(1)})$



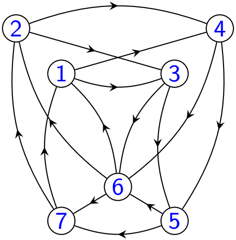
$\tilde{W}((A_1+A_1)^{(1)})$



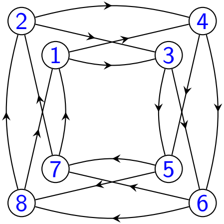
$\tilde{W}((A_1+A_2)^{(1)})$



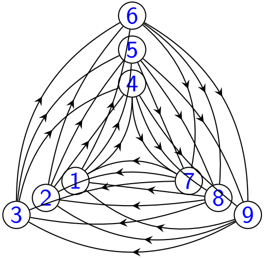
$\tilde{W}(D_4^{(1)})$



$\tilde{W}(D_5^{(1)})$



$\tilde{W}(E_6^{(1)})$

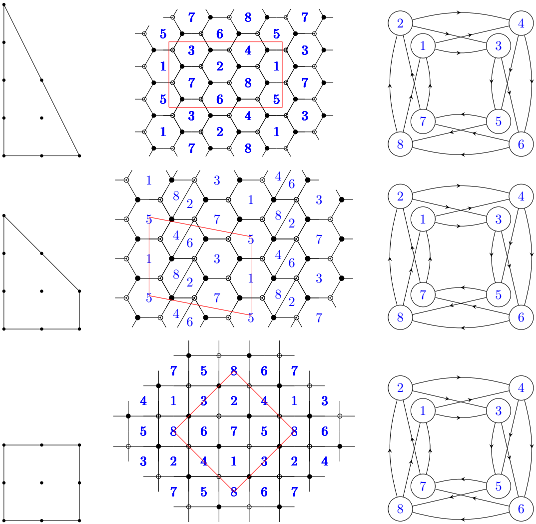


$8_{a,b,c}$ case — figures

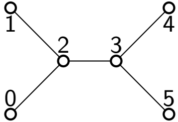
8_a

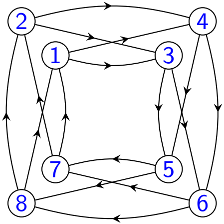
8_b

8_c



$8_{a,b,c}$ case — $q-PVI$

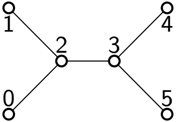


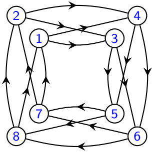


The generators of the group $G_{\mathcal{Q}}$:

$$s_0 = (1,2), \;\; s_1 = (5,6), \;\; s_2 = (1,5)\circ\mu_5\circ\mu_1, \;\; s_3 = (3,7)\circ\mu_3\circ\mu_7,$$
$$s_4 = (3,4), \;\; s_5 = (7,8), \;\; \pi = (1,7,5,3)(2,8,6,4), \; \sigma = (1,7)(2,8)(3,5)(4,6)\circ\varsigma.$$

$8_{a,b,c}$ case — $q-PVI$





Action on cluster coordinates

$$s_0\colon (x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)\mapsto(x_2,x_1,x_3,x_4,x_5,x_6,x_7,x_8)$$

$$s_1\colon (x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)\mapsto(x_1,x_2,x_3,x_4,x_6,x_5,x_7,x_8)$$

$$s_2\colon (x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)\mapsto\left(x_5^{-1},x_2,x_3\frac{1+x_5}{1+x_1^{-1}},x_4\frac{1+x_5}{1+x_1^{-1}},x_1^{-1},x_6,x_7\frac{1+x_1}{1+x_5^{-1}},x_8\frac{1+x_1}{1+x_5^{-1}}\right),$$

$$s_3\colon (x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)\mapsto\left(x_1\frac{1+x_3}{1+x_7^{-1}},x_2\frac{1+x_3}{1+x_7^{-1}},x_7^{-1},x_4,x_5\frac{1+x_7}{1+x_3^{-1}},x_6\frac{1+x_7}{1+x_3^{-1}},x_3^{-1},x_8\right),$$

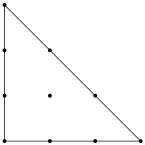
$$s_4\colon (x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)\mapsto(x_1,x_2,x_4,x_3,x_5,x_6,x_7,x_8)$$

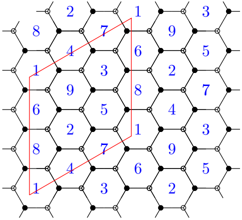
$$s_5\colon (x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)\mapsto(x_1,x_2,x_3,x_4,x_5,x_6,x_8,x_7).$$

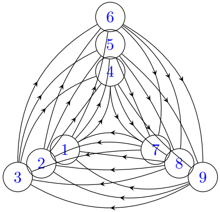
This tranformations belong to $Aut(X_{\mathcal{Q}})$ and generate affine Weyl group $W(D_5^{(1)})$.

q-Painlevé equations (relation to Sakai theory)

- Phase space is a submanifold of $X_{\mathcal{Q}}$ with fixed Casimirs.
  In this case it has dimansion 2. Conjecturally it is Sakai surface: $\mathbb{P}^1 \times \mathbb{P}^1$ blown up in 8 points. Positions of the blow up points depend on type of the quiver and values the Casimirs.
- Automorphisms of quivers — automorphisms of Sakai surface that can move blown-up points.
- Affine Weyl group acts on $X_{\mathcal{Q}}$. Conjecturally this is full cluster mapping class group $G_{\mathcal{Q}}$.
- Affine Weyl group gives Painlevé equation and its Bäcklund transformations.



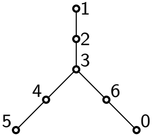




The generators of the group $G_{\mathcal{Q}}$:

$s_1 = (2,3),\;\; s_2 = (1,2),\;\; s_4 = (4,5),\;\; s_5 = (5,6),\;\; s_6 = (7,8),\;\; s_0 = (8,9),$
$s_3 = (4,7) \circ \mu_1 \circ \mu_4 \circ \mu_7 \circ \mu_1,\; \pi = (1,4,7)(2,5,8)(3,6,9),\; \sigma = (1,7)(2,8)(3,9) \circ \varsigma.$

9 case



$$s_1 : (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \mapsto (x_1, x_3, x_2, x_4, x_5, x_6, x_7, x_8, x_9)$$

$$s_2 : (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \mapsto (x_2, x_1, x_3, x_4, x_5, x_6, x_7, x_8, x_9)$$

$$\begin{aligned} s_3 : (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \mapsto \Big(& \frac{x_1}{x_4 x_7} \frac{1+x_4+x_1^{-1}}{1+x_1+x_7^{-1}}, x_1 x_2 \frac{1+x_4+x_1^{-1}}{1+x_1+x_7^{-1}}, x_1 x_3 \frac{1+x_4+x_1^{-1}}{1+x_1+x_7^{-1}}, \\ & \frac{x_4}{x_1 x_7} \frac{1+x_7+x_4^{-1}}{1+x_4+x_1^{-1}}, x_4 x_5 \frac{1+x_7+x_4^{-1}}{1+x_4+x_1^{-1}}, x_4 x_6 \frac{1+x_7+x_4^{-1}}{1+x_4+x_1^{-1}}, \\ & \frac{x_7}{x_1 x_4} \frac{1+x_1+x_7^{-1}}{1+x_7+x_4^{-1}}, x_7 x_8 \frac{1+x_1+x_7^{-1}}{1+x_7+x_4^{-1}}, x_7 x_9 \frac{1+x_1+x_7^{-1}}{1+x_7+x_4^{-1}} \Big). \end{aligned}$$

$$s_4 : (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \mapsto (x_1, x_2, x_3, x_5, x_4, x_6, x_7, x_8, x_9)$$

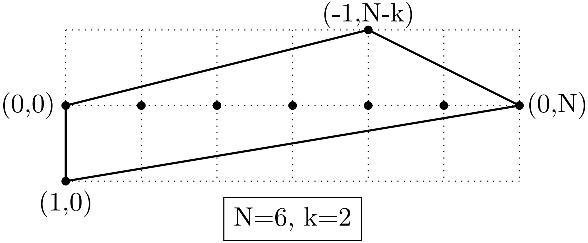
$$s_5 : (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \mapsto (x_1, x_2, x_3, x_4, x_6, x_5, x_7, x_8, x_9)$$

$$s_6 : (x_1, x_2, x_3, x_4, x_5, x_6, x_8, x_7, x_9) \mapsto (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)$$

$$s_0 : (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \mapsto (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_8)$$

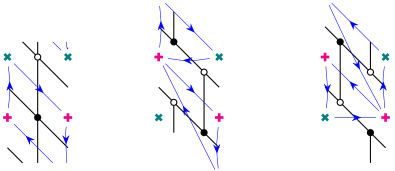
4 boundary points, hyperelliptic curves (Toda family)

Classification: $Y^{N,k}$ polygons with $0 \leq k \leq N$ (see picture below) and $L^{1,2N-1,2}$ polygons (which we omit in the talk):

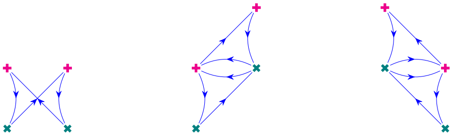


Bipartite graphs and quivers for $Y^{N,k}$

The bipartite graphs can be constructed from the building blocks. N_0 type 0 blocks, N_1 type I blocks and N_{-1} type $-\mathrm{I}$ blocks on the torus with $N = N_0 + N_1 + N_{-1}$, $k = N_1 - N_{-1}$. The order of blocks can be arbitrary.

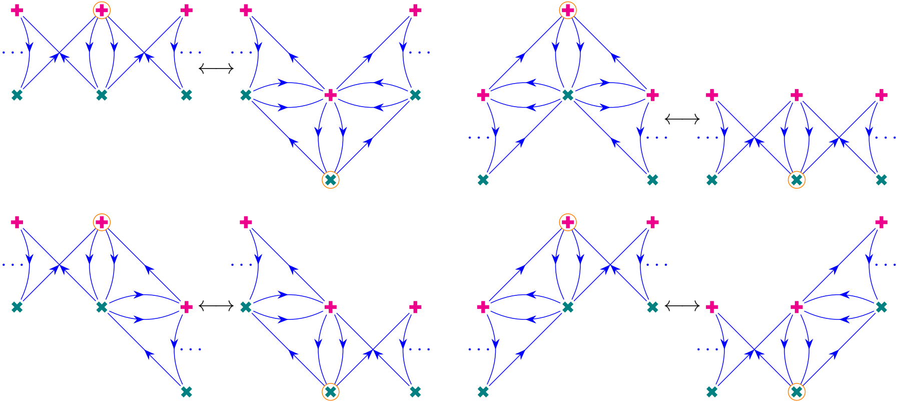


Quivers for $Y^{N,k}$ theories can be glued from blocks of three types 0, 1, -1.



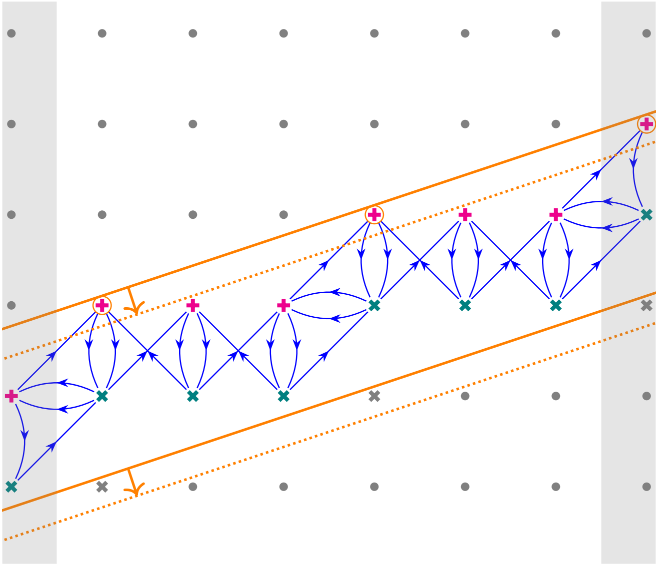
Quivers and mutations

It is convenient to draw mutations on the integer lattice



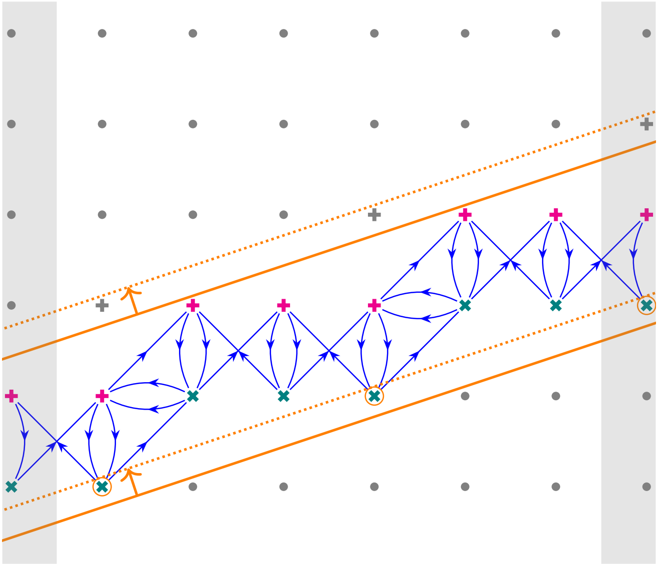
Quivers should be periodic with period (N, k).
There are in total 3^N quivers, but only $N + 1$ of them are inequivalent.

Action of the automorphism group



N=6, k=2

Action of the automorphism group



N=6, k=2

Equations

Mutable "+"-variables labelled by the points of integer lattice: $x_{(n,m)}$. They satisfy periodicity condition and Y-system in order to be compatible with mutations:

$$\frac{x_{(n,m+1)}x_{(n,m-1)}}{x^2_{(n,m)}} = \frac{(1+x_{(n+1,m)})(1+x_{(n-1,m)})}{(1+x_{(n,m)})^2}, \quad x_{(n,m)} = x_{(n+N,m+k)}$$

One can move from Y-system to T-system (from X-clusters to A-clusters):

$$x_{(n,m)} = z_0^{1/N} q^{(kn-Nm+N)/N^2} \frac{\tau_{(n-1,m-1)}\tau_{(n+1,m-1)}}{\tau^2_{(n,m-1)}}, \quad \tau_{(n,m)} = \tau_{(n+N,m+k)}$$

$$\tau_{(n,m+1)}\tau_{(n,m-1)} = \tau^2_{(n,m)} + z_0^{1/N} q^{(kn-Nm)/N^2} \tau_{(n+1,m)}\tau_{(n-1,m)}$$

And after some change of labeling:

$$\tau_j\left(qz\right)\tau_j\left(q^{-1}z\right) = \tau_j(z)^2 + z^{1/N}\tau_{j+1}\left(q^{k/N}z\right)\tau_{j-1}\left(q^{-k/N}z\right), \quad j \in \mathbb{Z}/N\mathbb{Z}$$

Quantization

- In addition to non-autonomous parameter q one may add quantum deformation p:
$$x_i x_j = p^{-2\epsilon_{ij}} x_j x_i$$
- All groups $G_{\mathcal{Q}}$ remain the same.
- There are quantum mutations
$$\mu_j: \quad x_j \mapsto x_j^{-1}, \quad x_i^{1/|\epsilon_{ij}|} \mapsto x_i^{1/|\epsilon_{ij}|}\left(1 + p x_j^{\operatorname{sgn}\epsilon_{ij}}\right)^{\operatorname{sgn}\epsilon_{ij}}, \; i \neq j$$
- And so there are quantum deformations of all systems. For example, quantum q-Painlevé III_3:
$$\begin{cases} x_1(q^{-1}z)^{1/2}\, x_1(qz)^{1/2} = \dfrac{x_1(z) + pz}{x_1(z) + p}, \\ x_1(z) x_1(q^{-1}z) = p^4 x_1(q^{-1}z) x_1(qz). \end{cases}$$

What I did not mention

1. Solution of the autonomous equations in terms of theta functions,
2. Solution of the nonautonomous equations in terms of
   - 5d Nekrasov partition functions with $\epsilon_1 + \epsilon_2 = 0$.
   - topological string amplitues.
   - q-deformed conformal blocks for special integer central charge.
3. Solutions of *quantum nonautonomous* equations.

Thank you for your attention!

Some numerology

B is the number of boundary integer points of Δ.
$B-3$ is the number of Casimirs for the Poisson bracket.
$B-3$ is the rank of a free abelian group of discree flows in $G_{\mathcal{Q}}$

I is the number of interior integer points of Δ.
$2I$ is the rank of the Poisson bracket of $X_{\mathcal{Q}}$
I is the number of commuting Hamiltonians $H_1, \dots, H_I$.
I is the genus of a spectral curve.

$S = I + B/2 - 1$ is the area of Δ
$2S$ is the number of vertices in quiver $\mathcal{Q}$.