The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems
Junior Paper: Years 8, 9, 10
Northern Spring 2010 (O Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. There are some pears, plums and apples in six baskets. The number of plums in each basket is equal to the number of apples in all other baskets together, while the number of apples in each basket is equal to the number of pears in all other baskets together.

   Prove that the total number of items of fruit is divisible by 31. (3 points)

2. Lillebror and Karlsson have a square cake to cut up. Karlsson chooses a point at the cake (not on an edge). Next Lillebror makes a straight cut from that point to an edge (in any direction). Then Karlsson makes another straight cut from the point he first chose, to the edge of the cake, perpendicular to Lillebror's cut. The smaller of two pieces goes to Lillebror. Lillebror wants to have at least a quarter of the cake. Can Karlsson prevent him from having such a piece of cake? (3 points)

3. An angle is drawn and a compass is the only available tool.

   (a) What is the smallest number of circles which must be constructed to determine accurately whether the given angle is acute? (2 points)

   (b) How could you determine whether the given angle is equal to $31°$? (It is allowed to construct as many circles as needed.) (2 points)

4. Each Olympiad participant has acquaintance with at least three other participants. Prove that it is possible to choose a group with an even number of more than two participants, that can be arranged around a circular table in such a way that each participant has acquaintance with both his/her neighbours. (5 points)

5. There are 101 numbers written on a board: $1^2, 2^2, \ldots, 101^2$. At each step, two numbers are erased and replaced by (the absolute value of) their difference. What is the smallest number that can be obtained as the result of 100 steps? (5 points)

Tournament of the Towns Problems
Senior Paper: Years 11, 12
Northern Spring 2010 (O Level)

1. 2010 ships loaded with bananas, lemons and pineapples travel from South America to Russia. The number of bananas on each ship is equal to the number of lemons on all other ships combined, while the number of lemons on each ship is equal to the number of pineapples on all other ships combined.

   Prove that the total number of items of fruit is divisible by 31. (3 points)

2. The following is known about a function $f(x)$:

   > Any straight line in the x-y plane has the same number of intersections with the graph of $y = f(x)$ as with the graph of the parabola $y = x^2$.

   Prove that $f(x)$ is equal to x^2. (4 points)

3. Is it possible to glue some regular hexagons over the surface of a regular octahedron, without overlaps or gaps?

   Note. A regular octahedron has 6 vertices, all of its faces are equilateral triangles, and each vertex is incident with 4 faces.

4. Baron Munchausen asks that a non-constant polynomial $P(x)$ with non-negative integer coefficients be chosen, and asks to be told only the values $P(2)$ and $P\big(P(2)\big)$. The Baron claims that he will be able to identify the polynomial $P(x)$ given only these two values.

   Isn't the Baron mistaken? (5 points)

5. There is a needle lying in a plane. One is allowed to rotate the needle 45° around either of its ends.

   Is it possible, after some rotations, for the needle to end up in its initial position, but with its ends swapped?

   Note. Consider a needle to be a line segment. (6 points)

Tournament of the Towns Problems
Junior Paper: Years 8, 9, 10
Northern Spring 2010 (A Level)

1. There is a piece of cheese. It is possible to choose any positive number (not necessarily an integer) $a \neq 1$ and cut up the piece of cheese in the ratio $1 : a$ according to its mass. Then, one can keep cutting up any resulting pieces, in the same ratio any of the existing pieces.

   Is it possible to follow this process in such a way that all the resulting pieces of cheese could be put into two piles of equal mass after a finite number of cuttings? (3 points)

2. Let ABC be a triangle in which M is the midpoint of AC and point P lies on BC, AP meets BM at O, and $BO = BP$.

   Find the ratio $OM : PC$. (4 points)

3. There are 999 numbers located on the circumference of a circle. Each of them is equal to either 1 or -1, and both possible numbers occur. Someone computes all products of 10 consecutive numbers and adds them.

   (a) What is the least sum which can be obtained? (3 points)

   (b) What is the greatest sum which can be obtained? (3 points)

4. The sum of the digits of a positive integer n is equal to 100.

   Is it possible that the sum of the digits of n^3 is equal to 100^3? (6 points)

5. (a) Three knights are riding along a circular road in an anti-clockwise direction. There is a single point on the road where a knight may overtake another.

   Is it possible that they can ride for an arbitrarily long period of time with pairwise distinct constant speeds? (3 points)

   (b) What is the answer if there are ten knights instead? (5 points)

6. A polygonal curve in the plane is not closed, has no self-intersections and consists of 31 edges (neighbouring segments do not lie on the same straight line). Someone has drawn all lines containing these edges. This results in 31 lines (some of which may be coincident).

   What is the minimum possible number of distinct lines? (8 points)

7. There are fleas in some squares of a 10×10 chessboard (no more than a single flea in a square). Once a minute each flea jumps from its square to an adjacent one (that is, to a square with which it has a common side). All fleas jump simultaneously. Each flea keeps the same direction of jumping unless on the next jump it would leave the board, in which case it changes to the opposite direction. During a whole hour, there was not a minute when any square was occupied by two or more fleas.

   Find the maximum possible number of fleas on the board. (11 points)

Tournament of the Towns Problems
Senior Paper: Years 11, 12
Northern Spring 2010 (A Level)

1. Is it possible to arrange all lines in the plane into pairs of perpendicular lines such that each line belongs to a single pair? (3 points)

2. (a) A piece of cheese is given. One may choose any irrational number $a > 0$ and cut it into pieces of ratio $1 : a$ according to mass, and then cut any of the resulting pieces in the same ratio and so on.

   Is it possible to follow this process in such a way that all the resulting pieces of cheese could be put into two piles of equal mass after a finite number of cuttings? (2 points)

   (b) What is your answer if a is a positive rational number such that $a \neq 1$ instead? (2 points)

3. Can one obtain the integer 2010 by repeated application of functions sin, cos, tan, cot, arcsin, arccos, arctan, arccot to the integer 1? (Each function may be applied an arbitrary number of times.) (6 points)

4. There are 5000 movie amateurs invited to a meeting. Each of them watched at least one movie. The participants should be separated into groups of two different kinds. In a group of the first kind, all the members should have seen the same movie. In a group of the second kind, each member should have seen a movie that no other member of the group has seen.

   Prove that it is possible to split all the fans into exactly 100 such groups. (A group consisting of a single person is allowed.) (6 points)

5. Thirty three knights ride around a ring road in an anti-clockwise direction.

   Could they travel indefinitely having different constant speeds, if there is only one place on the road where these knights can overtake each other? (7 points)

6. A quadrilateral $ABCD$ is circumscribed around a circle with centre I. Points M and N are midpoints of sides AB and CD, respectively. Also, $IM/AB = IN/CD$.

   Prove that $ABCD$ is a trapezoid or a parallelogram. (8 points)

7. We are given a positive integer. It is permitted to insert plus signs in an arbitrary way between the digits of the number and calculate the sum. For example, from the number 123456789 it is possible to obtain $12345 + 6 + 789 = 13140$. It is then permitted to do the same operation with the newly obtained number and so on.

   Prove that starting from an arbitrary positive integer, one can obtain a single-digit integer after not more than 10 such procedures. (9 points)