The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

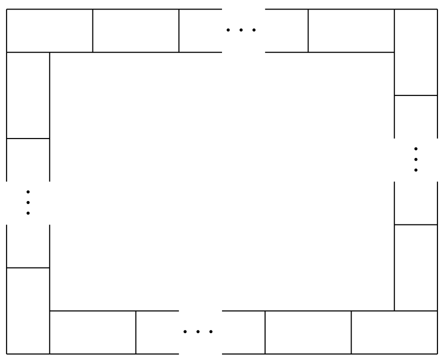
Tournament of the Towns Problems with Solution to Problem 1
Junior Paper: Years 8, 9, 10
Northern Autumn 2010 (O Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. In a multiplication table, the entry in the i^{th} row and the j^{th} column is the product ij. From an $m \times n$ subtable with both m and n odd, the interior $(m-2)\times(n-2)$ rectangle is removed, leaving behind a frame of width 1. The squares of the frame are painted alternately black and white.

   Prove that the sum of the numbers in the black squares is equal to the sum of the numbers in the white squares. (4 points)

Solution. Without loss of generality, assume the lefthand top corner is a black square. Then, since the side dimensions are odd, all the corner squares are black. If we negate the values of each of the white squares, then we must show that the sum of the total value of the black squares and the total value of the white squares cancels to zero. Now imagine adjacent pairs of squares of the $m \times n$ frame are covered by dominoes as shown in the diagram, and consider the value covered by each domino.



Let the top and bottom rows of the frame be t and b, respectively, and the leftmost and rightmost columns of the the frame be ℓ and r, respectively, so that $t = b + m - 1$ and $r = \ell + n - 1$. Suppose a left-right domino in row t, covers squares with coordinates (t, j') and $(t, j'+1)$, then the value covered by each such domino is

$$tj' - t(j'+1) = -t,$$

noting that the left square (t, j') is black.
The up-down oriented dominoes of column r, covering squares with coordinates (i', r) and $(i'+1, r)$, say, where square (i', r) is black, each "have value"

$$i'r - (i'+1)r = -r.$$

Similarly, the bottom row left-right dominoes each have a value of b (it's the same calculation as for the top row dominoes, with b replacing t, except that this time each right square covered by a domino is black, so that the sign is opposite). Also, the up-down oriented dominoes of column ℓ each have value ℓ (swap ℓ for r and change the sign in the column r calculation).
Finally, there are $\frac{n-1}{2}$ left-right dominoes in each of rows t and b, and $\frac{m-1}{2}$ up-down dominoes in each of columns ℓ and r. So the total value covered by the dominoes is:

$$\begin{aligned}\frac{n-1}{2}\cdot -t + \frac{n-1}{2}\cdot b + \frac{m-1}{2}\cdot -r + \frac{m-1}{2}\cdot \ell &= -\frac{n-1}{2}(t-b) + \frac{m-1}{2}(\ell - r)\\ &= -\frac{n-1}{2}(m-1) + \frac{m-1}{2}(n-1)\\ &= 0.\end{aligned}$$

Thus the total value of the black squares of the frame is equal to the total (unsigned) value of the white squares of the frame, as required.

Alternatively, assume the colouring of the squares and negate the values of the white squares as above, but consider the value of the entire $m \times n$ subtable.
Observe that the product of the sum of the (signed) i coordinates and the the sum of the j coordinates expanded gives the sum of the sum of the (signed) values of the entire $m \times n$ subtable. The sum S_i of the (signed) i coordinates is

$$\begin{aligned} S_i &= \ell - (\ell + 1) + \cdots + r, && \text{(summed forwards)} \\ &= r - (r - 1) + \cdots + \ell, && \text{(summed backwards)} \\ \therefore 2S_i &= (\ell + r) - (\ell + r) + \cdots + (\ell + r) \\ &= \ell + r \\ \therefore S_i &= \frac{\ell + r}{2}. \end{aligned}$$

Similarly, the sum S_j of the (signed) j coordinates is $\frac{t+b}{2}$.
Hence the "value" of the entire $m \times n$ subtable is

$$S_i S_j = \frac{\ell + r}{2} \cdot \frac{t + b}{2},$$

which is the value of the centre square of the $m \times n$ subtable. But the interior $(m-2) \times (n-2)$ rectangle, i.e. the $m \times n$ subtable minus the frame of width 1, has the same square at its centre, and hence the same "value".
It follows therefore that the frame of width 1 has value 0, and hence the total value of the black squares of the frame is equal to the total (unsigned) value of the white squares of the frame, as before.

2. In a quadrilateral $ABCD$ with an incircle, $AB = CD$, $BC < AD$ and $BC \parallel AD$.

   Prove that the bisector of $\angle C$ bisects the area of $ABCD$. (4 points)

3. A $1 \times 1 \times 1$ cube is placed on an 8×8 chessboard so that its bottom face coincides with a square of the chessboard. The cube rolls over a bottom edge so that the adjacent face now lands on the chessboard. In this way, the cube rolls around the chessboard, landing on each square at least once.

   Is it possible that a particular face of the cube never lands on the chessboard? (4 points)

4. In a school, more than 90% of the students know both English and German, and more than 90% of the students know both English and French.

   Prove that more than 90% of the students who know both German and French also know English. (4 points)

5. A circle is divided by $2N$ points into $2N$ arcs of length 1. These points are joined in pairs to form N chords. Each chord divides the circle into two arcs, the length of each being an even integer.

   Prove that N is even. (4 points)