The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Selected 1997–1998 Tournament of the Towns Problems with Solutions

1. Prove that the equation $x^2 + y^2 - z^2 = 1997$ has infinitely many solutions in integers x, y and z. (Junior O Level Autumn, N Vassiliev, 3 points)

   Solution. Rearranging $x^2 + y^2 - z^2 = 1997$, we have

   $$\begin{aligned} y^2 - z^2 &= 1997 - x^2 \\ (y-z)(y+z) &= 1997 - x^2 \end{aligned}$$

   If we let $y - z = 1$, then $y + z = 1997 - x^2$ and ...

   $$\begin{aligned} y - z &= 1 && && (1) \\ y + z &= 1997 - x^2 && && (2) \\ 2y &= 1998 - x^2, && \text{adding (1) and (2).} && (3) \\ 2z &= 1996 - x^2, && \text{taking (1) from (2).} && (4) \end{aligned}$$

   We now see that we can satisfy (3) and (4), if x is even. Thus put $x = 2t$ then

   $$\begin{aligned} 2y &= 1998 - 4t^2 \\ y &= 999 - 2t^2 \\ 2z &= 1996 - 4t^2 \\ z &= 998 - 2t^2 \end{aligned}$$

   i.e. $(x, y, z) = (2t, 999 - 2t^2, 998 - 2t^2)$ is a solution of $x^2 + y^2 - z^2 = 1997$ for each $t \in \mathbb{Z}$, and so there are infinitely many integer triples (x, y, z) satisfying the equation.

2. Prove that the equation

   $$xy(x-y) + yz(y-z) + zx(z-x) = 6$$

   has infinitely many solutions in integers x, y and z.
   (Senior O Level Autumn, N Vassiliev, 4 points)

   Solution. Let $f(x, y, z) = xy(x-y) + yz(y-z) + zx(z-x)$. Observe that if we let $x = y$ then $f(x, y, z) = 0$. Thus if we consider $f(x, y, z)$ to be a polynomial in x, we have by the Factor Theorem that $x - y$ is a factor of $f(x, y, z)$. Similarly, $f(x, y, z) = 0$ if $x = z$ and if $y = z$. So, we have that $x - z$ and $y - z$ are factors of $f(x, y, z)$. Hence $(x-y)(x-z)(y-z)$ divides $f(x, y, z)$.
   Now observe that all the terms of $(x-y)(x-z)(y-z)$ and of $f(x, y, z)$ are of degree three. So we must have:

   $$f(x, y, z) = k(x-y)(x-z)(y-z)$$

for some constant k. Comparing the coefficients of x^2y in the expansions of $f(x, y, z)$ and $(x-y)(x-z)(y-z)$ (they are both 1), we see $k = 1$. Therefore,

$$f(x, y, z) = (x-y)(x-z)(y-z).$$

Now observe that with

$$x - y = 2 \ldots\ldots\ldots (1), \qquad x - z = 3 \ldots\ldots\ldots (2), \qquad y - z = 1 \ldots\ldots\ldots (3)$$

we satisfy $f(x, y, z) = 6$ and moreover $x - y = 2$ follows from $x - z = 3$ and $y - z = 1$, since $(1) = (2) - (3)$. So we may ignore (1), leaving two equations in three unknowns. Observe that if we choose $z = t$, an arbitrary integer, then (2) and (3) give:

$$x = 3 + t, \quad y = 1 + t.$$

So the triple $(x, y, z) = (3 + t, 1 + t, t)$ is a solution of $f(x, y, z) = 6$ for any integer t. Thus there are infinitely many integer triples (x, y, z) satisfying $f(x, y, z) = 6$.
Alternative Approach. We could have also factorised $f(x, y, z)$ this way:

$$\begin{aligned}
f(x, y, z) &= xy(x-y) + yz(y-z) + zx(z-x) \\
&= x^2y - xy^2 + y^2z - yz^2 + z^2x - zx^2 \\
&= x^2(y-z) - y^2(x-z) + z^2(x-y) \\
&= x^2(y-z) - y^2\big((x-y) + (y-z)\big) + z^2(x-y) \\
&= (x^2 - y^2)(y-z) + (z^2 - y^2)(x-y) \\
&= (x-y)(x+y)(y-z) + (z-y)(z+y)(x-y) \\
&= (x-y)(y-z)\big((x+y) - (z+y)\big) \\
&= (x-y)(y-z)(x-z)
\end{aligned}$$

3. For every three-digit number, we take the product of its three digits and then we add all these products together. What is the result?

(Junior O Level Spring, G Galperin, 4 points)