The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

Tournament of the Towns Problems
Senior Paper: Years 11, 12
Northern Autumn 2006 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. Ann draws a circle and represents each of her friends by a chord in this circle. If two of her friends know each other they are represented by chords which intersect inside or on the circle. If they do not know each other, they are represented by chords which do not intersect inside or on the circle. Ann believes that it is always possible to do so. Is she right? (4 points)

2. The points D, E and F are on the sides BC, CA and AB of an acute triangle ABC. If AD, BE and CF are the angle bisectors of the triangle DEF, prove that they are the altitudes of triangle ABC. (6 points)

3. The n^{th} digit after the decimal point of a real number $a = 0.12457\ldots$ is equal to the first digit of $n\sqrt{2}$. Prove that a is an irrational number. (6 points)

4. Can a prism be cut into a number of nonintersecting pyramids such that each pyramid has its base on one base of the prism and its vertex on the other base of the prism. (6 points)

5. For any positive integer n, let
$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \frac{a_n}{b_n},$$
where $\frac{a_n}{b_n}$ is an irreducible fraction.
Prove that there exist infinitely many positive integers n such that $b_{n+1} < b_n$. (7 points)

6. The 52 playing cards of a standard deck are arranged face-up in a row. The first card must be the Ace of Spades. Each subsequent card must match the preceding one either in suit or in rank. The last card must either be an Ace or a Spade. Prove that the number of possible arrangements is a multiple of
   (a) 12!; (3 points)
   (b) 13!. (5 points)

7. The positive real numbers $x_1, \ldots, x_k$ are such that
$$2(x_1^2 + \cdots + x_k^2) < x_1 + \cdots + x_k < \tfrac{1}{2}(x_1^3 + \cdots + x_k^3).$$
   (a) Prove that $k > 50$. (3 points)
   (b) Give an example of such numbers for some value of k. (3 points)
   (c) Determine the minimum value of k for which such an example arises. (3 points)