AMO TRAINING SESSIONS

2003 Australian Intermediate Mathematics Olympiad Problems

1. Whee Bin's Chinese restaurant serves dishes in three sizes: small dishes cost $\$x$, medium dishes cost $\$y$, and large dishes cost $\$z$, where $x < y < z$ and $x, y, z \in \mathbb{N}$.

   Last Friday, Giovanna, Peter and John ordered a total of 9 small dishes, 6 medium dishes and 10 large dishes.

   > John remarked: "This bill is exactly twice as much as when I was here 2 nights ago."
   > Peter remarked: "This bill is exactly three times as large as when I was here last night."
   > Giovanna said: "It is still a good price; the total is less than $100."

   The prices have not been changed this week. How many dollars was the bill?

2. Annie, Bruce and ken play a game of cards, each beginning and ending the game with a whole number of dollars. Annie starts with $3 for every $5 Bruce has, and finishes with twice as much money as Bruce. Ken starts with the same amount as Annie and ends with the same amount as Bruce.

   What is the least number of dollars that Bruce could have lost?

3. For how many integers between 1 and 2003 is the improper fraction $\dfrac{n^2 + 4}{n + 5}$ not in simplest form?

4. Side AB of $\triangle ABC$ is produced to P, so that $PB = AB$; BC is produced to Q, so that $QC = CB$; and CA is produced to R, so that $RA = AC$. The area of $\triangle ABC$ is 51.

   What is the area of $\triangle PQR$?

5. Rina sets off on her bike to Con's place. At exactly the same moment, Con sets off to Rina's place along the same straight road in his car. A while later, they pass each other (neither spotting the other) and shortly after, Con arrives at Rina's place to find that she is not there. Con waits 22 minutes and then heads back along the same road, arriving at his place at exactly the same time as Rina. Rina travelled at the same speed the whole time, whereas Con travelled 4 times as fast on the way to Rina's place and 5 times as fast on the way back.

   How many minutes did it take Rina to reach Con's place?

6. Faith designs the following logo for a telephone company. The logo consists of a T divided into 5 regions. Faith has to colour each region with one of three colours: red, green or gold. No two regions with a common edge can have the same colour; and each colour must be used at least once.
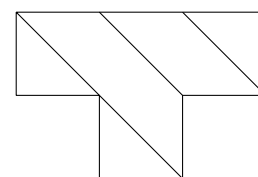
   How many possible colour schemes are available to Faith for her logo?

7. During a meeting of their school mathematics club, Andy, Becky, Chloe and Danny discover that their favourite number (a positive integer) is the same. The other club members would like to know the number. So, each of the four friends makes three statements about the number, at least one of which is true and at least one false.

Andy: (A1)   The number is less than 12.
        (A2)   Seven does not divide the number.
        (A3)   Five times the number is less than 70.

Becky: (B1)   Twelve times the number is greater than 1000.
        (B2)   Ten divides the number.
        (B3)   The number is greater than 100.

Chloe: (C1)   Four divides the number.
        (C2)   Eleven times the number is less than 1000.
        (C3)   Nine divides the number.

Danny: (D1)   The number is less than 20.
        (D2)   The number is prime.
        (D3)   Seven divides the number.

What is the number?

8. $ABCD$ is a trapezium with $AB \parallel DC$. A semicircle is constructed with centre on AB and with the other three sides of the trapezium tangent to it.

Given $AB = 1156$ and $AD = 784$, find BC.

9. A circle has diameter AB and chord PQ, with PQ perpendicular to AB and nearer to B, cutting AB at V; M is a point on AV; QM produced cuts the circle at R.

Show that $\angle RPA = \angle MPA$.

10. The 101 numbers $1, 2, \ldots, 101$ are split into two groups. Group A contains m of the numbers, and Group B contains the remaining $101 - m$ numbers. When the number 40 is moved from the group it is in to the other group, each group mean increases by $\frac{1}{2}$.

What is the value of m?

Investigation.

(a) Find an actual set of numbers in each group for which moving 40 from one group to the other increases each group mean by $\frac{1}{2}$.

Suppose we start with the numbers $1, 2, \ldots, k$.

(b) What relationships hold between k, m, the number x that is moved, and the amount y by which the mean increases?

(c) Is there an upper limit for the amount by which the mean can increase?