The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

Tournament of the Towns Questions III with Some Solutions

1. Find all integer solutions to the equation

$$y^k = x^2 + x$$

where k is a natural number greater than 1. (3 points)

Solution. First write

$$y^k = x(x+1)$$

Since $\gcd(x, x+1) = 1$, we must have $x = a^k, x+1 = b^k$ and $y = ab$, for some integers a, b. Since $k > 1$, the only pairs of consecutive integers that can be written as kth powers are -1 and 0 or 0 and 1. Thus we have $x = -1$ or 0, and in either case $y = 0$ and $k > 1$ is arbitrary.

2. Find all solutions of

$$2^n + 7 = x^2$$

in which n and x are both integers. Prove that there are no other solutions. (4 points)

3. A set of 1989 numbers is given. It is known that the sum of any 10 of them is positive. Prove that the sum of all of these numbers is positive. (Folklore, 3 points)

4. Find the positive integer solutions of the equation

$$x + \cfrac{1}{y + \cfrac{1}{z}} = \frac{10}{7}$$

(G. Galperin, 3 points)

Solution. First write

$$x + \cfrac{1}{y + \cfrac{1}{z}} = 1 + \frac{3}{7}$$

and observe that since y, z are positive integers

$$0 < \cfrac{1}{y + \cfrac{1}{z}} < 1$$

and so since x is also a positive integer

$$x = 1 \quad \text{and} \quad \cfrac{1}{y + \cfrac{1}{z}} = \frac{3}{7}$$

and hence

$$y + \frac{1}{z} = \frac{7}{3} = 2 + \frac{1}{3}.$$

Arguing as before, we necessarily have

$$y = 2 \quad \text{and} \quad \frac{1}{z} = \frac{1}{3}.$$

Thus $x = 1, y = 2, z = 3$ is the only solution among the positive integers.

5. We define $N!!$ to be $N(N-2)(N-4)\ldots 5.3.1$ if N is odd and $N(N-2)(N-4)\ldots 6.4.2$ if N is even. For example, $8!! = 8.6.4.2$ and $9!! = 9.7.5.3.1$. Prove that $1986!! + 1985!!$ is divisible by 1987. (V. V. Proizvolov, Moscow, 5 points)

6. The numbers 2^{1989} and 5^{1989} are written out one after the other (in decimal notation). How many digits are written altogether? (G. Galperin, 3 points)

7. For which natural number k does
$$\frac{k^2}{1.001^k}$$
attain its maximum value? (4 points)

8. For any natural number $n \geq 2$ prove the inequality
$$\sqrt{2\sqrt{3\sqrt{4\ldots\sqrt{(n-1)\sqrt{n}}}}} < 3.$$
(V. Proizvolov, Moscow, 5 points)