The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

Tournament of the Towns Questions II with Solution

1. What is the final digit of $7^{7^{7^{7^{7^{7^{7}}}}}}$?

Solution. Here's a start ...
Firstly, we will call an expression of the form

$$7^{7^{7^{7^{\cdot^{\cdot^{\cdot^{7}}}}}}}$$

a *tower* of 7s. Our problem has a tower of 7 7s. Observe that

$$7^4 = (7^2)^2 \equiv (-1)^2 \equiv 1 \pmod{10}.$$

Hence, *modulo* 10,

$$7^k \equiv \begin{cases} 1 & \text{if } k \equiv 0 \pmod 4 \\ 7 & \text{if } k \equiv 1 \pmod 4 \\ -1 & \text{if } k \equiv 2 \pmod 4 \\ -7 & \text{if } k \equiv 3 \pmod 4, \end{cases}$$

where k is a natural number. Thus to determine the last digit of a tower of 7 7s, we need to determine what a tower of 6 7s is congruent to *modulo* 4.
Now, $7 \equiv -1 \pmod 4$. Hence, *modulo* 4,

$$7^m \equiv \begin{cases} 1 & \text{if } m \text{ is } \textit{even} \\ -1 & \text{if } m \text{ is } \textit{odd}, \end{cases}$$

where m is a natural number. A tower of 5 7s is certainly odd. So, a tower of 6 7s is congruent to -1 *modulo* 4 (and $-1 \equiv 3 \pmod 4$). So, a tower of 7 7s is congruent to -7 *modulo* 10 (and $-7 \equiv 3 \pmod{10}$). Hence, a tower of 7 7s must end in a 3.