The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems IVwith some solutions

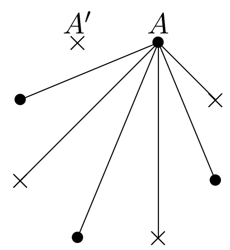
1. Suppose you and your partner attended a party with three other couples. Several handshakes took place. Noone shook hands with himself or herself or with their partner, and noone shook hands with the same person more than once. After all the handshaking was completed, you asked each person, including your partner, how many hands they had shaken. Each person gave a different answer.

   (i) How many hands did you shake?

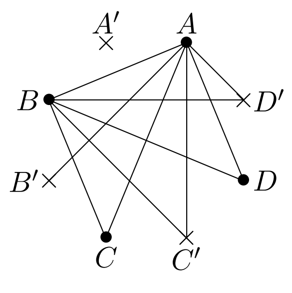
   (ii) How many hands did your partner shake?

   Solution. In all there are 4 couples, i.e. 8 people, 7 other than myself. Since noone shook hands with himself or herself or with their partner, and noone shook hands with the same person more than once, the most number of hands anyone shook is 6 hands, i.e. the possibilities for how many hands anyone shook are 0, 1, 2, 3, 4, 5 or 6 (7 possibilities). There were also 7 people asked how many hands they shook, and they all shook a different number, so the numbers of hands shaken by them, in some order, are 0, 1, 2,. . . , 7.

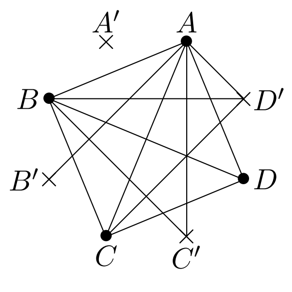
   Let us label one person from each couple A, B, C or D, and their partners A', B', C' and D', respectively, and represent these 8 people on a graph by •s or ×s and draw an edge between two such vertices if the two people represented shook hands. Without loss of generality, suppose A shook 6 hands. Then, so far we have the graph opposite. We observe that, except possibly for A', everyone shook at least one hand. But someone shook 0 hands. So the person who shook 0 hands is A'.



   So now we know how many hands A and A' shook, and who shook 0 and 6 hands. Each of the remaining 6 people is now known to have shaken at least one hand. One of them shook 5 hands. Without loss of generality, suppose B shook 5 hands. So other than having shaken A's hand, B shook the hands of C, C', D and D', who along with B have now shaken at least two hands, leaving B's partner B' as the only one who could have shaken 1 hand. So we now have what is represented in the new graph opposite.



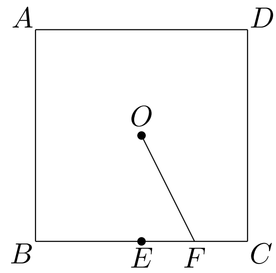
   So now we know who shook 0, 1, 5 and 6 hands. Someone shook 4 hands. Without loss of generality, suppose it was C. Then the two other hands C shook must be those of D and D', and since each of C, D and D', has shaken at least 3 hands, only C' could have shaken 2 hands. So we now have the graph opposite, and in fact now we are finished, since both D and D' have shaken 3 hands which was the only number of hands left to assign, one of whom is myself (since otherwise two people asked would have claimed to have shaken the same number of hands). So in fact, both my partner and I (D and D') shook 3 hands.



2. A paper rectangle $ABCD$ of area 1 is folded along a straight line so that C coincides with A. Prove that the area of the pentagon thus obtained is less than $3/4$. (Tournament, 1995)

3. In the centre of a square swimming pool is a boy, while his teacher (who cannot swim) is standing at one corner of the pool. The teacher can run three times as fast as the boy can swim, but the boy can run faster than the teacher. Can the boy escape from the teacher? (1987 Tournament)

Solution. Let the pool be represented by the diagrammed square $ABCD$, where initially the teacher is at A and the boy is at the centre of the pool O. Let the side length of the pool be x. Let us consider a couple of scenarios.



Scenario 1. The boy swims to C. Then the ratio of the distance run by the teacher to the distance swum by the boy is:

$$\frac{AB+BC}{OC} = \frac{2x}{(x/2)\sqrt{2}} = 2\sqrt{2} < 3,$$

so that in this scenario, the teacher catches the boy.

Scenario 2. The boy swims to E. Then the ratio of the distance run by the teacher to the distance swum by the boy is:

$$\frac{AB+BE}{OE} = \frac{x+x/2}{x/2} = 3,$$

so that in this scenario, the teacher only just catches the boy.

What happens if the boy heads towards a point F between E and C? Suppose $EF = ax$ and that for some a such that $0 < a < \frac{1}{2}$ the boy can escape then $AB + BF > 3\,OF$ where

$$AB + BF = (1 + \tfrac{1}{2} + a)x \text{ and } OF = x\sqrt{(\tfrac{1}{2})^2 + a^2}$$

i.e.

$$\begin{aligned}
(1 + \tfrac{1}{2} + a)x &> 3x\sqrt{(\tfrac{1}{2})^2 + a^2} \\
3 + 2a &> 3\sqrt{1 + 4a^2} \\
(3 + 2a)^2 &> 9(1 + 4a^2) \\
9 + 12a + 4a^2 &> 9(1 + 4a^2) \\
12a &> 8 \cdot 4a^2 \\
8a^2 - 3a &< 0 \\
a(a - \tfrac{3}{8}) &< 0 \\
a - \tfrac{3}{8} &< 0, \qquad \text{since } a \text{ was assumed positive} \\
a &< \tfrac{3}{8}
\end{aligned}$$

Thus the assumptions imply $0 < a < \frac{3}{8}$. Checking, we see that $0 < a < \frac{3}{8}$ implies

$$(1 + \tfrac{1}{2} + a)x > 3x\sqrt{(\tfrac{1}{2})^2 + a^2}$$

so the boy does indeed escape if the boy heads towards E where $EF = ax$ and $0 < a < \frac{3}{8}$ (i.e. the argument above is reversible). So the boy can escape.

4. A schoolgirl forgot to write a multiplication sign between two 3-digit numbers and wrote them as one number. This 6-digit result proved to be 3 times as great as the product. Find the numbers. (1984 Tournament)

5. Eight students were asked to solve 8 problems (the same set for each of the students).

   (i) Each problem was solved by 5 students. Prove that one can find two students so that each of the problems was solved by at least one of them.

   (ii) If each problem was solved by 4 students, then it is possible that no such pair of students exists. Prove this.

   (1996 Tournament)

6. Prove that from any sequence of 1996 numbers $a_1, a_2, \ldots, a_{1996}$ one can choose one or several consecutive numbers so that their sum differs from an integer by less than 0.001. (1996 Tournament)

7. Sixty children participate in a summer camp. Among any 10 of the children, there are three or more who live in the same block. Prove that there must be 15 or more children from the same block. (1994 Tournament)

8. $ABCD$ is a convex quadrilateral inscribed in a circle with centre O, and with mutually perpendicular diagonals. Prove that the broken line AOC divides the quadrilateral into two parts of equal area. (1981 Tournament)

9. Prove that if $a, b, c > 0$ such that

$$a^2 + b^2 - ab = c^2,$$

then $(a - c)(b - c) \leq 0$. (1996 Tournament)

Solution. Without loss of generality, assume $a \leq b$. Then

$$\begin{aligned}
& a^2 \leq ab \leq b^2 \\
\Longrightarrow\ & a^2 - ab \leq 0 \leq b^2 - ab \\
\Longrightarrow\ & c^2 = a^2 - ab + b^2 \leq b^2 \\
& \text{and } a^2 \leq a^2 + b^2 - ab = c^2 \\
\Longrightarrow\ & a^2 \leq c^2 \leq b^2 \\
\Longrightarrow\ & a \leq c \leq b, \qquad \text{since } a, b, c > 0 \\
\Longrightarrow\ & a - c \leq 0 \\
& \text{and } b - c \geq 0 \\
\Longrightarrow\ & (a - c)(b - c) \leq 0
\end{aligned}$$

10. Prove that in any set of 17 distinct natural numbers it is possible to find either a set of 5 numbers such that four of them divide the fifth, or a set of 5 numbers none of which divides the others. (1983 Tournament)