The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems with Solutions
Senior Paper: Years 11, 12
Northern Autumn 2007 (O Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. Pictures are taken of 100 adults and 100 children, with one adult and one child in each, the adult being the taller of the two. Each picture is reduced to $1/k$ of its original size, where k is a positive integer which may vary from picture to picture. Prove that it is possible to have the reduced image of each adult taller than the reduced image of every child. (3 points)

Solution. We will prove the general result for n pictures, by mathematical induction. To be explicit, the proposition $P(n)$ we wish to prove is:

> It is possible to find positive integers k_i, for $i = 1, \dots, n$, such that after scaling $\text{photo}_1, \dots, \text{photo}_n$ by factors $1/k_i, \dots, 1/k_n$, respectively, the image of each adult is taller than the image of each child.

For $n = 1$, the adult in the one picture is already larger than the child. So we need do nothing, i.e. with $k_1 = 1$, $P(1)$ holds.

Now we show that $P(m) \implies P(m+1)$ for $m \geq 1$: Assume $P(m)$ (the inductive hypothesis) holds. Then each of $\text{photo}_1, \dots, \text{photo}_m$ can be reduced so the reduced image of each adult is taller than the reduced image of each child. Let the minimum reduced height of the adults in these m photos be a and the maximum reduced height of the children in these m photos be c. Let the heights of the adult and child in the $(m+1)^{\text{st}}$ photo be b and d, respectively. We have two cases:

Case 1. $b > c$ and $a > d$. There is nothing to do, i.e. with $k_{m+1} = 1$, each adult image is taller than each child image.

Case 2. $c > b$ (so that $a > c > b > d$) or $d > a$ (so that $b > d > a > c$). We will show there exist positive integers p, q such that reducing the first m photos by a further factor p and reducing the $(m+1)^{\text{st}}$ photo by a factor q, we will have $b/q > c/p$ and $a/p > d/q$, or equivalently

$$br > c \text{ and } a > dr,$$

where $r = p/q$. It is sufficient to find a positive rational scale factor r such that br lies between a and c, since then $br > c$ and $a > br > dr$, e.g. choose br to be the midpoint of a and c:

$$\begin{aligned} br &= \frac{a+c}{2} \\ r &= \frac{a+c}{2b}. \end{aligned}$$

Thus with $p = a + c$ and $q = 2b$ we have $b/q > c/p$ and $a/p > d/q$, and hence by scaling photo_i by $1/(k_i p)$ for $i = 1, \dots, m$ and scaling photo_{m+1} by $1/q$, each adult image is taller than each child image.

Thus in either case $P(m+1)$ holds, if $P(m)$ holds.

Thus by the Principle of Mathematical Induction, $P(n)$ holds for all $n \in \mathbb{N}$.

In particular, $P(100)$ holds, i.e. there exist positive integers $k_1, \dots, k_{100}$, such that after scaling $\text{photo}_1, \dots, \text{photo}_{100}$ by factors $1/k_i, \dots, 1/k_{100}$, respectively, the image of each adult is taller than the image of each child.

2. Initially, the number 1 and two positive numbers x and y are written on a blackboard. In each step, we can choose two numbers on the blackboard, not necessarily different, and write their sum or difference on the blackboard. We can also choose a non-zero number on the blackboard and write its reciprocal on the blackboard. Is it possible to write on the blackboard, in a finite number of moves, the number

   (a) x^2; (2 points)

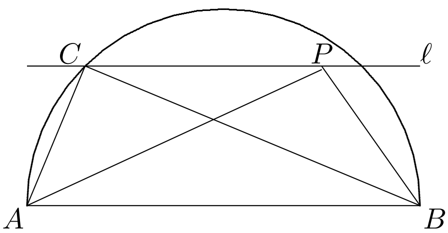
   Solution. Yes, it is possible. We start with 1, x and y on the board.
   We use 1 and x only and start by writing their sum $1 + x$ and their difference $1 - x$.
   Next we can write the reciprocals $1/(1 + x)$ and $1/(1 - x)$,
   followed by their sum $1/(1 + x) + 1/(1 - x) = 2/(1 - x^2)$,
   followed by its reciprocal $(1 - x^2)/2$.
   Adding $(1 - x^2)/2$ to itself we obtain $1 - x^2$.
   Finally the difference of 1 and $1 - x^2$ is x^2.

   (b) xy? (2 points)

   Solution. Again, it is possible. We start with 1, x and y on the board.
   First we find $x + y$.
   From (a), we have a method of squaring any number that is on the board, so long as 1 is present.
   So, using (a), we may obtain x^2, y^2 and $(x + y)^2$.
   Next we find $(x + y)^2 - x^2$ and then $(x + y)^2 - x^2 - y^2 = 2xy$.
   Inverting $2xy$, we have $1/(2xy)$,
   which we add to itself to obtain $1/(xy)$,
   which finally we invert to obtain xy.

3. Give a construction by straight-edge and compass of a point C on a line ℓ parallel to a segment AB such that the product $AC \cdot BC$ is minimum. (4 points)

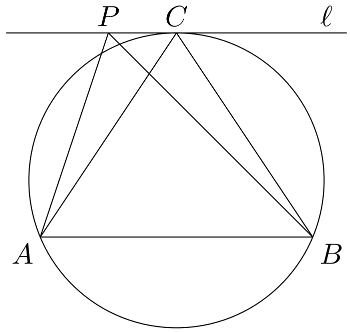
   Solution.
   Suppose first that a semicircle with diameter AB, drawn towards ℓ, intersects ℓ at C. Then $\angle BCA = 90°$, so that the area of $\triangle ABC$ is $\frac{1}{2}AC \cdot BC$. For any other point P on ℓ the area of $\triangle ABC$ is $\frac{1}{2}AP \cdot BP \sin(\angle BPA)$. All the triangles APB have the same area (each has base AB and the same perpendicular height, since $\ell \parallel AB$).
   So we have



$$\begin{aligned} \tfrac{1}{2}AC \cdot BC = \tfrac{1}{2}AP \cdot BP \sin(\angle BPA) &\leq \tfrac{1}{2}AP \cdot BP \\ AC \cdot BC &\leq AP \cdot BP \end{aligned}$$

   and hence as P moves along ℓ the expression $AP \cdot BP$ is minimised when $P = C$.
   [The ruler and compass construction of C is to first find the midpoint O of AB and then draw the circle with centre O and radius OA to intersect ℓ. Either point of intersection will do as the point C.]

Now suppose the semicircle drawn with diameter AB does not intersect ℓ. Let C be the point on ℓ equidistant from A and B, and draw the circumcircle of $\triangle ABC$. Then ℓ is tangent to the circumcircle. Hence any point P on ℓ other than C, is outside the circumcircle, so that $\angle APB < \angle ACB$ and these angles are acute since otherwise we would have the previous case where the semicircle drawn with diameter AB intersected ℓ. Hence $0 < \sin(\angle APB) < \sin(\angle ACB)$. As before, the triangles APB and ACB have the same area.



So we have

$$\sin(\angle ACB) > \sin(\angle APB) > 0$$
$$\tfrac{1}{2}AC \cdot BC \sin(\angle ACB) = \tfrac{1}{2}AP \cdot BP \sin(\angle APB)$$
$$\therefore AC \cdot BC < AP \cdot BP$$

and hence again as P moves along ℓ the expression $AP \cdot BP$ is minimised when $P = C$. [The ruler and compass construction of C is to find the perpendicular bisector of AB and produce it to intersect ℓ at C.]

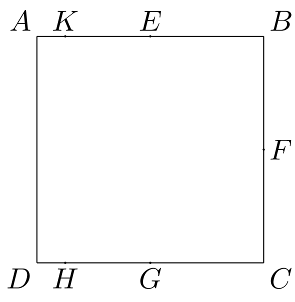
4. The audience chooses two of 29 cards, numbered from 1 to 29 respectively. The assistant of a magician chooses two of the remaining 27 cards, and asks a member of the audience to take them to the magician, who is in another room. The two cards are presented to the magician in arbitrary order. By an arrangement with the assistant beforehand, the magician is able to deduce which two cards the audience has chosen only form the two cards he receives. Explain how this may be done. (4 points)

   Solution. Arrange the numbers in order 1 to 29 in a circle so that 1 may be considered to follow 29. One possible strategy depends on whether the audience selects consecutive cards or non-consecutive cards. If the audience selects two consecutive cards, the assistant chooses the next two consecutive cards, e.g. if the audience selects 27 and 28, the assistant chooses 29 and 1. If the audience selects non-consecutive cards, the assistant chooses the next card in sequence for each of the cards, e.g. if the audience selects 1 and 3, the assistant chooses 2 and 4. The magician can then reverse this strategy to deduce the cards, i.e. if he receives consecutive cards 29 and 1, he knows that the preceding consecutive cards 27 and 28 were selected by the audience, and if he receives non-consecutive cards 2 and 4, then he knows that the cards 1 and 3 preceding the individual cards were selected by the audience.

5. A square of side-length 1 cm is cut into three convex polygons. Is it possible that the diameter of each of them does not exceed

   (a) 1 cm; (1 point)

   Solution.
   Let $ABCD$ be the square (with side-length 1 cm). Let E, F and G be the midpoints of AB, BC and CD, respectively, and let H and K be on AB and CD, respectively, such that $HD = KA = \frac{1}{8}$ cm (see diagram at right). Suppose $ABCD$ has been cut into three convex polygons, such that the diameter of each does not exceed 1 cm. By the Pigeon Hole Principle, two of the vertices of the square belong to the same polygon. They cannot be opposite vertices as otherwise the diameter will be $\sqrt{2}$ cm.



Hence, without loss of generality, we may assume A and D belong to the first polygon. This leaves two cases (neither B nor C can belong to the first polygon, since then the first polygon would contain two opposite vertices):

Case 1. B and C belong to the same (say the 2nd) polygon. Now $EC > 1$ and $ED > 1$. So E can be in neither of the first two polygons. It must be in the 3rd polygon. However, now $HA > 1$, $HB > 1$ and $HE > 1$. So H does not belong to any of the three polygons, and so we have a contradiction in this case.

Case 2. B and C belong to the 2nd and 3rd polygons, respectively. Now $HA > 1$, $HB > 1$ implies H belongs to the 3rd polygon, and $KB > 1$, $KC > 1$ implies K belongs to the 2nd polygon. Since $FA > 1$, F does not belong to the 1st polygon. Also,

$$HF^2 = (\tfrac{1}{2})^2 + (\tfrac{7}{8})^2 = \tfrac{4^2+7^2}{8^2} = \tfrac{65}{64}$$

e.e. $HF > 1$; so F does not belong to the 3rd polygon. Similarly, $KF^2 = \frac{65}{64}$; so F does not belong to the 2nd polygon. So F does not belong to any of the three polygons, and again we have a contradiction.

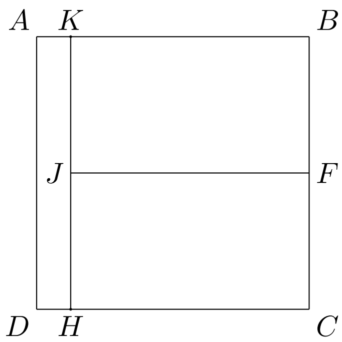
Hence it is not possible to cut a square of side-length 1 cm into three convex polygons, such that the diameter of each does not exceed 1 cm.

(b) 1.01 cm; (1 point)

Solution.
Cut square $ABCD$ (of side-length 1 cm) into three rectangles as in the diagram at right, where F, K and H are as in (a) and J is the midpoint of KH. The diameter of a rectangle is a diagonal. $HA^2 = 1^2 + (\frac{1}{2})^2 = \frac{65}{64}$, and as in (a), $HF^2 = KF^2 = \frac{65}{64}$. Now $\frac{1}{64} < \frac{1}{50} < \frac{201}{10000}$ and hence

$$\tfrac{65}{64} = 1 + \tfrac{1}{64} < \tfrac{10201}{10000} = (\tfrac{101}{100})^2$$

$$HA = HF = KF = \sqrt{\tfrac{65}{64}} = \tfrac{\sqrt{65}}{8} < 1.01$$



So the diameter of each rectangle is less than 1.01 cm as required, and hence it is possible to cut a square of side-length 1 cm into three convex polygons, such that the diameter of each does not exceed 1.01 cm.

(c) 1.001 cm? (2 points)

Solution. The same argument as (a) shows it is not possible to cut a square of side-length 1 cm into polygons, each of whose diameters is less than 1.001 cm, since $\sqrt{65}/8 > 1.001$.