The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

2007 Australian Intermediate Mathematics Olympiad Problems with Solutions

1. Trevor's trailer has two wheels on its axle and carries a spare wheel. The three wheels are changed around from time to time. The three tyres have been worn for 25 000 km, 28 000 km and 31 000 km, respectively. How many thousand kilometres has Trevor's trailer travelled?

   Solution. Number the tyres 1, 2 and 3, so that the tyres have been worn 25 000 km, 28 000 km and 31 000 km, respectively, and let

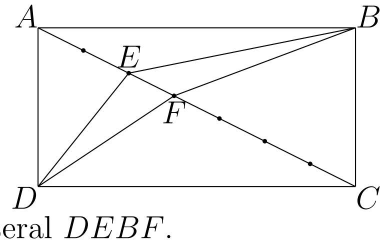
$$\begin{aligned} x &= \text{thousands of km travelled on tyres 1 and 2} \\ y &= \text{thousands of km travelled on tyres 1 and 3} \\ z &= \text{thousands of km travelled on tyres 2 and 3.} \end{aligned}$$

   Then

$$\begin{aligned} x+y &= 25 \\ x+z &= 28 \\ y+z &= 31 \\ \therefore 2(x+y+z) &= 25+28+31 \\ &= 84 \\ x+y+z &= 42 \end{aligned}$$

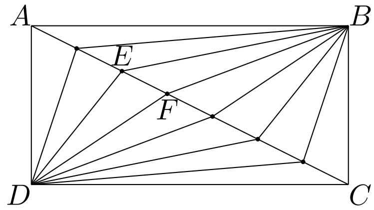
   The total distance travelled by Trevor's trailer is $x+y+z=42$ thousand km.

2. The rectangle shown has sides of length 28 and 15. The diagonal is divided into 7 equal parts.



   Find the area of the quadrilateral $DEBF$.

   Solution.
   Join all the marks on the diagonal to B and D as shown. The seven triangles on one side of the diagonal have equal bases and the same height. So they have the same area. Hence the rectangle is divided into 7 quadrilaterals of equal area.
   Therefore, the area of $DEBF$ is



$$\frac{28 \times 15}{7} = 4 \times 15 = 60.$$

3. When 113 744 and 109 417 are divided by a 3-digit positive integer N, the remainders are 119 and 292, respectively. Find N.

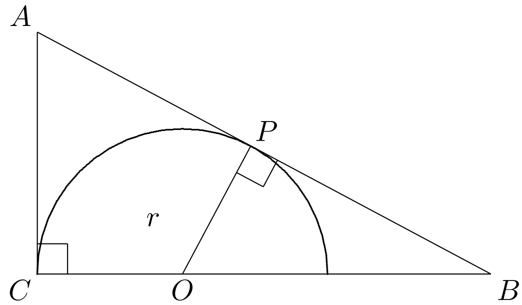
Solution. Dividing 113 744 by N gives remainder 119 $\implies N \mid (113\,744 - 119) = 113\,625$.
Dividing 109 417 by N gives remainder 292 $\implies N \mid (109\,417 - 292) = 109\,125$.
$\therefore N \mid \gcd(113\,625, 109\,125)$.
We find the gcd by the Euclidean Algorithm, via the following Division Table:

	113 625	109 125	
1	109 125	108 000	24
	4 500	1 125	
4	4 500		
	0		

$\therefore \gcd(113\,625, 109\,125) = 1\,125$ (the last non-zero remainder in the table).
Now N is a 3-digit factor of 1 125, and it must be greater than 292 (the larger remainder).
If the number formed by the last k digits of a number is divisible by 5^k, then the number itself is divisible by 5^k.
Since 1 125 ends in $125 = 5^3$, we have $5^3 \mid 1\,125$.
So $1\,125 = 5^3 \times 9 = 5^3 \times 3^2$.
The 3-digit factors of 1 125 are $1\,125/3 = 375$, $1\,125/5 = 225$ and $1\,125/9 = 125$. The only one of these larger than 292 is 375.
$\therefore N = 375$.

4. ABC is a triangle with $AB = 85$, $BC = 75$ and $CA = 40$. A semicircle is tangent to AB and AC and its diameter lies on BC. Find the radius of the semicircle.

Solution.
Firstly, we need to sketch the triangle, with semicircle centre at O on BC and tangent point at P on AB. Simce AC is tangent to the semicircle, $\angle ACB$ is a right-angle. Observe also that $\triangle ACB$ is an $8 : 15 : 17$ triangle, and that $8 : 15 : 17$ is a Pythagorean triad (which also confirms the right angle at C).



We have

$$\begin{aligned}
\angle OPB &= \angle ACB = 90^\circ \\
\angle PBO &= \angle CBA, && \text{(common angle)} \\
\therefore \triangle OPB &\sim \triangle ACB, && \text{by the AA Rule.} \\
\therefore \frac{OP}{OB} &= \frac{AC}{AB} \\
\frac{r}{75 - r} &= \frac{40}{85} = \frac{8}{17} \\
17r &= 8(75 - r) \\
&= 8 \cdot 75 - 8r \\
25r &= 8 \cdot 75 \\
r &= 8 \cdot 3 = 24
\end{aligned}$$

5. Find $x+y$ where x and y are non-zero solutions of the system of equations

$$
\begin{aligned}
y^2x &= 15x^2+17xy+15y^2\\
x^2y &= 20x^2+3y^2.
\end{aligned}
$$

Solution. Assume x and y are non-zero. Then we may divide the given equations through by xy to obtain:

$$y = 15\left(\frac{x}{y}\right)+17+15\left(\frac{y}{x}\right) \tag{1}$$

$$x = 20\left(\frac{x}{y}\right)+3\left(\frac{y}{x}\right) \tag{2}$$

Now let $a=y/x$ and divide (1) by (2). Then

$$
\begin{aligned}
a = \frac{y}{x} &= \frac{\frac{15}{a}+17+15a}{\frac{20}{a}+3a}\\
a\left(\frac{20}{a}+3a\right) &= \frac{15}{a}+17+15a\\
20a+3a^3 &= 15+17a+15a^2\\
3a+3a^3 &= 15+15a^2\\
a(1+a^2) &= 5(1+a^2)\\
(a-5)(1+a^2) &= 0
\end{aligned}
$$

Now $1+a^2\neq 0$. So $a-5=0$, i.e. $a=5$. So, adding (1) and (2) we have

$$
\begin{aligned}
x+y &= \frac{15}{a}+17+15a+\frac{20}{a}+3a\\
&= \frac{35}{a}+17+18a\\
&= 7+17+90 = 114
\end{aligned}
$$

6. When a positive integer N is written in base 4 it has three digits. When $3N$ is written in base 6 it also has three digits and has the same middle digit as N to base 4. Find the decimal sum of all such numbers N.

Solution. Let the base 4 representation of N be $(abc)_4$ and the base 6 representation of $3N$ be $(dbe)_6$. Then in base ten we have:

$$N = (abc)_4 = 4^2a+4b+c = 16a+4b+c$$

$$3N = (dbe)_6 = 6^2d+6b+e = 36d+6d+e \tag{3}$$

where $1\le a\le 3$, $0\le b\le 3$, $0\le c\le 3$, $1\le d\le 5$, $0\le e\le 5$.
From (3), we deduce $3\,|\,e$, and hence $e=3e'$ where $e'\in\{0,1\}$, so that (3) reduces to:

$$N = 12d+2b+e'.$$

$$
\begin{aligned}
\therefore\ 12d+2b+e' &= N\\
&= 16a+4b+c\\
12d+e' &= 16a+2b+c
\end{aligned}
$$

Now we find bounds for d:

$$
\begin{aligned}
12d+e' &= 16a+2b+c\\
&\le 16\cdot 3+2\cdot 3+3 = 57\\
\therefore\ 12d &\le 57\\
d &\le 4
\end{aligned}
$$

Also,

$$\begin{aligned} 12d + e' = 16a + 2b + c &\geq 16 \cdot 1 \\ 12d &\geq 16 - e' \\ &\geq 15 \\ d &\geq 2 \end{aligned}$$

Now let us enumerate the possibilities, specifying d and e' first, then determining feasible a, b, c and hence determining N:

d	e'	$12d + e'$	a	b	c	$N = 12d + e' + 2b$
4	1	49	3	0	1	49
4	0	48	3	0	0	48
3	1	37	2	2	1	41
			2	1	3	39
3	0	36	2	2	0	40
			2	1	2	38
2	1	25	1	3	3	31
2	0	24	1	3	2	30
			Total			316

Thus the required sum of all integers N with the required property is 316.

7. $x^2 - 19x + 94$ is a perfect square where x is an integer. Find the largest possible value of x.

Solution. Since $x^2 - 19x + 94$ is a perfect square, we have

$$x^2 - 19x + 94 = k^2,$$

for some integer $k \geq 0$, or equivalently

$$x^2 - 19x + 9 - k^2 = 0. \tag{4}$$

$$\therefore x = \frac{19 \pm \sqrt{\Delta}}{2}$$

where Δ is the discriminant of (4). Now, x is an integer, and so

$$\Delta = (-19)^2 - 4 \cdot 1 \cdot (9 - k^2) = 4k^2 - 15$$

must itself be a perfect square, i.e.

$$\begin{aligned} 4k^2 - 15 &= \ell^2, && \text{for some integer } \ell \geq 0 \\ 4k^2 - \ell^2 &= 15 \\ (2k - \ell)(2k + \ell) &= 15 \end{aligned}$$

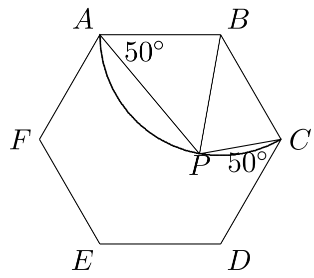
Now $15 = 1 \times 15 = 3 \times 5$ and these are the only factorisations of 15 into two factors. So we have

$$\begin{aligned} &2k - \ell = 1 \text{ and } 2k + \ell = 15 \Longrightarrow 4k = 16 \Longrightarrow k = 4 \text{ or} \\ &2k - \ell = 3 \text{ and } 2k + \ell = 5 \Longrightarrow 4k = 8 \Longrightarrow k = 2. \end{aligned}$$

If $k = 4$ then $x = (19 \pm \sqrt{49})/2 = 13$ or 6.
If $k = 2$ then $x = (19 \pm \sqrt{1})/2 = 10$ or 9.
$\therefore$ the largest value x can be is 13.

8. A point P is marked inside a regular hexagon $ABCDEF$ so that $\angle BAP = \angle DCP = 50°$. Find $\angle ABP$.

Solution.
The internal angles of a regular hexagon are $120°$. (Either use the fact that a regular n-gon has angles of size $180(n-2)/n°$ $(= 120°$, if $n = 6)$, or use the property that one can form a regular hexagon from six equilateral triangles, in a fairly obvious way.)



$\therefore \angle ABC = 120°$ and $\angle BCP = 120° - 50° = 70°$.
Now $\angle APC = 360° - (50 + 70 + 120)° = 120°$, since angles of quadrilateral $ABCP$ sum to $360°$.

Observe $\text{ext}\angle ABC = 360° - \text{int}\angle ABC = 360° - 120° = 240°$ (where $\text{ext}\angle$ denotes the *external* angle as opposed to $\text{int}\angle \equiv \angle$ (*internal*) angle).
Now, $\text{ext}\angle ABC = 2\angle APC$ and $BA = BC \implies P$ lies on the circle centred at B, and passing through A and C.
$\therefore BA = BP$ (radii of same circle) and so $\triangle ABP$ is isosceles.
$\therefore \angle APB = \angle BAP = 50°$.
$\therefore \angle ABP = 180° - (50 + 50)° = 80°$.

9. Find a prime p with the property that for some larger prime q, both $2q - p$ and $2q + p$ are prime. Prove that there is only one such p.

Solution. First observe that $p \neq 2$, since $p = 2 \implies 2q + p = 2q + 2 = 2(q + 1)$ which is not prime since we are given $q > p \implies q > 0$.
If $p = 3$, we can find prime quadruples $(p, q, 2q - p, 2q + p)$, e.g.

$$(3, 5, 7, 13), \quad (3, 7, 11, 17), \quad (3, 13, 23, 29), \quad (3, 17, 31, 37), \quad \ldots$$

Now suppose p is a prime, $p > 3$ and has the required property, i.e. there is $q > p$ such that $p, q, 2q - p, 2q + p$ are all prime.
Consider these primes modulo 6.
Then $2 \nmid p$ and $2 \nmid q \implies p, q \not\equiv 0, 2, 4 \pmod 6$.
Also $3 \nmid p$ and $3 \nmid q \implies p, q \not\equiv 3 \pmod 6$.
$\therefore p, q \equiv 1$ or $5 \pmod 6$ (and note $5 \equiv -1 \pmod 6$).
So we have two cases to consider: $p \equiv q \equiv \pm 1 \pmod 6$ or $-p \equiv q \equiv \pm 1 \pmod 6$.

Case 1. $p \equiv q \equiv \pm 1 \pmod 6$. Here $2q + p \equiv \pm 3 \pmod 6 \implies 3 \mid 2q + p$ so that $2q + p$ is not prime, since $2q + p > p > 3$.

Case 2. $-p \equiv q \equiv \pm 1 \pmod 6$. Here $2q - p \equiv \pm 3 \pmod 6 \implies 3 \mid 2q - p$ so that $2q - p$ is not prime, since $2q - p > q > p > 3$.

So in either case, $p > 3$ leads to a contradiction, and hence there is no p with the required property for $p > 3$.
Thus the only prime p with the property is $p = 3$ (for which there are many prime quadruples $(p, q, 2q - p, 2q + p)$ where $q > p$).

10. In a triangle ADC, $DC = 65$ and altitudes DB and CE have lengths 33 and 63, respectively. Prove that the lengths of AB and AE cannot both be integers.

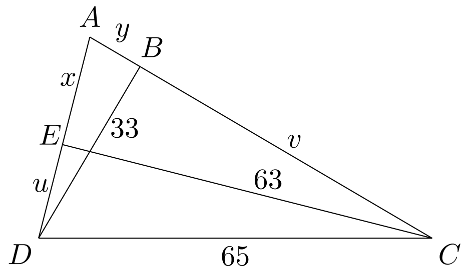
Investigation

Find AB and AE.

In a triangle $A'D'C'$, $D'C' = 65k$ and altitudes $D'B'$ and $C'E'$ have lengths $33k$ and $63k$, respectively. Is there a value for k so that $A'B'$ and $A'E'$ are integers? If not, explain why. If so, find all such values of k.

Solution.

Let $u = DE$, $v = BC$, $x = AE$ and $y = AB$. Then by Pythagoras' Theorem we have



$$\begin{aligned} u^2 &= 65^2 - 63^2 \\ &= (65-63)(65+63) = 2 \cdot 128 = 2^8 \\ \therefore u &= 2^4 = 16 \\ v^2 &= 65^2 - 33^2 \\ &= (65-33)(65+33) = 32 \cdot 98 = 2^6 \cdot 7^2 \\ \therefore v &= 2^3 \cdot 7 = 56 \end{aligned}$$

Also, by Pythagoras' Theorem,

$$33^2 + y^2 = (x+16)^2 = x^2 + 32x + 16^2 \tag{5}$$

$$63^2 + x^2 = (y+56)^2 = y^2 + 112y + 56^2 \tag{6}$$

Adding (5) and (6) and cancelling $x^2 + y^2$ from both sides gives

$$\begin{aligned} 33^2 + 63^2 &= 32x + 112y + 16^2 + 56^2 \\ 32x + 112y &= (33^2 - 16^2) + (63^2 - 56^2) \\ &= 17 \cdot 49 + 7 \cdot 119 = 2 \cdot 7^2 \cdot 17 \end{aligned}$$

$$16x + 56y = 7^2 \cdot 17 \tag{7}$$

Now suppose $x, y \in \mathbb{Z}$. Then 2 divides the LHS of (7) but does not divide the RHS of (7) – a contradiction. So $AB = y$ and $AE = x$ cannot both be integers.

Investigation

$$\begin{aligned} \angle ABD &= \angle AEC = 90^\circ \\ &\angle A \text{ (common)} \\ \therefore \triangle ABD &\sim \triangle AEC, && \text{by the AA Rule} \\ \therefore \frac{x+16}{33} &= \frac{y+56}{63} \\ 21(x+16) &= 11(y+56) \\ 21x - 11y &= 11 \cdot 56 - 21 \cdot 16 \\ &= 8 \cdot 7(11-6) = 7 \cdot 40 \\ 21x - 11y &= \tfrac{7}{8} \cdot 320 && && (8) \\ 2x + 7y &= \tfrac{7}{8} \cdot 119, && (7) \times \tfrac{1}{8} && (9) \\ x - 81y &= \tfrac{7}{8} \cdot 870, && (8) - 10\,(9) && (10) \\ 169y &= \tfrac{7}{8} \cdot 1859, && (9) - 2\,(10) \\ y &= \tfrac{7}{8} \cdot 11 = 77/8 && && (11) \\ x &= \tfrac{7}{8}(870 + 891) = 147/8, && (10) + 81\,(11) && (12) \end{aligned}$$

In $\triangle A'D'C'$, the first applications of Pythagoras' Theorem with $33, 63, 65$ replaced by $33k, 63k, 65k$, respectively, result in $D'E' = 16k$ and $C'B' = 56k$.
$\therefore \triangle D'B'C' \sim \triangle DBC$ and $\triangle D'E'C' \sim \triangle DEC$.
$\therefore \angle B'C'D' = \angle BCD$ and $\angle E'D'C' = \angle EDC$.
$\therefore \triangle A'B'D' \sim \triangle ABD$ and $\triangle A'E'C' \sim \triangle AEC$.
$\therefore A'B' = \frac{77}{8}k$ and $A'E' = \frac{147}{8}k$.
So $A'B', A'E' \in \mathbb{Z}$ if and only if $8 \mid k \in \mathbb{N}$.