The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems with Some Solutions
Senior Paper: Years 11, 12
Northern Autumn 2007 (A Level)

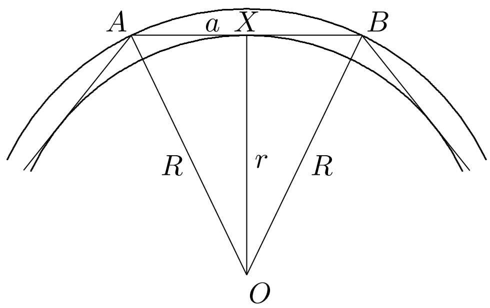
Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. (a) Each of Peter and Basil thinks of three positive integers. For each pair of his numbers, Peter writes down the greatest common divisor of the two numbers. For each pair of his numbers, Basil writes down the least common multiple of the two numbers. If both Peter and Basil write down the same numbers, prove that these three numbers are equal to one another. (2 points)

   (b) Can the analogous result be proved if each of Peter and Basil thinks of four positive integers instead? (2 points)

2. Let K, L, M and N be the midpoints of the sides AB, BC, CD and DA of a cyclic quadrilateral $ABCD$. Let P be the point of intersection of AC and BD. Prove that the circumradii of triangles PKL, PLM, PMN and PNK are equal to one another. (6 points)

Solution. More generally, consider the area between the circumcircle and incircle of a regular n-gon. Let $r = OX$ be the *inradius,* $R = OA$ be the *circumradius* and $a = AB$ be the side-length of the n-gon, as per the diagram.



Consider side AB of the n-gon; it is tangent to the incircle at a point X, so that OX is a radius of the incircle and $OX \perp AB$. So

$$\begin{aligned}
OXA = OXB &= 90^\circ, && \text{since } OX \perp AB\\
OA &= OB, && \text{radii of circumcircle}\\
OX &\text{ common} &&\\
\therefore \triangle OXA &\cong \triangle OXB, && \text{by the RHS Rule}\\
\therefore AX = BX &= \frac{a}{2}. &&
\end{aligned}$$

Hence, by Pythagoras' Theorem applied to $\triangle OXA$, we have $(a/2)^2 + r^2 = R^2$.
On the other hand, the area between the incircle and circumcircle is given by:

$$\begin{aligned}
\pi R^2 - \pi r^2 &= \pi(R^2 - r^2)\\
&= \pi\left(\frac{a}{2}\right)^2
\end{aligned}$$

which only depends on a and does not depend on n.
Thus if we call the side-length of the 7-gon a_7 and the side-length of the 17-gon a_{17}, then we are given that

$$\pi\left(\frac{a_7}{2}\right)^2 = \pi\left(\frac{a_{17}}{2}\right)^2$$

so that as a consequence $a_7^2 = a_{17}^2$, and hence $a_7 = a_{17}$ since they are both positive. Hence the sides of the 7-gon and 17-gon are equal.

3. Determine all finite arithmetic progressions in which each term is the reciprocal of a positive integer and the sum of all the terms is 1. (6 points)

4. Attached to each of a number of objects is a tag which states the correct mass of the object. The tags have fallen off and replaced on the objects at random. We wish determine if by chance all tags are in fact correct. We may use exactly once a horizontal lever which is supported at its middle. The objects can be hung from the lever at any point on either side of the support. The lever either stays horizontal or tilts to one side. Is this task always possible? (6 points)

   Solution. Let there be n objects and let the mass indicated by the tag on the i^{th} object be m_i, for $1 \le i \le n$. Choose arbitrary positive numbers $d_2 < d_3 < \cdots < d_n$ and let d_i be such that
$$d_1 m_1 = d_2 m_2 + d_3 m_3 + \cdots + d_n m_n.$$
   On one side of the support hang the i^{th} object at a distance d_1 from the support. On the other side of the support hang the other objects, where the i^{th} object is hung at distance d_i from the support for $2 \le i \le n$. Let the correct mass of the i^{th} object be M_i, for $1 \le i \le n$. Then $\{M1,M_2,\ldots,M_n\}$ is a permutation of $\{m_1,m_2,\ldots,m_n\}$, and hence by the Rearrangement Inequality,
$$d_2M_2 + d_3M_3 + \cdots + d_nM_n \le d_2m_2 + d_3m_3 + \cdots + d_nm_n = d_1m_1.$$
   We consider three cases:

   Case 1. All tags are correct. In this case we have equilibrium.

   Case 2. The tag on the 1st object is correct but those on some of the others are not. Then the Rearrangement Inequality is strict and we do not have equilibrium.

   Case 3. The tag on the 1st object is not correct. Then $m_1 < M_1 = m_j$ for some j such that $2 \le j \le n$. Hence $d_1M_1 = d1m_j > d_1m_1$, while
$$\begin{aligned} d_2m_2 + & d_3m_3 + \cdots + d_nm_n \\ & \ge d_2m_2 + d_3m_3 + \cdots + d_jm_{j-1}s + d_jm_{j+1} + \cdots + d_nm_n \\ & \ge d_2M_2 + d_3M_3 + \cdots + d_nM_n, \end{aligned}$$
   and again we do not have equilibrium.

5. The audience arranges n coins in a row. The sequence of heads and tails is chosen arbitrarily. The audience also chooses a number between 1 and n inclusive. Then the assistant turns one of the coins over, and the magician is brought in to examine the resulting sequence. By agreement with the assistant beforehand, the magician tries to determine the number chosen by the audience.

   (a) Prove that if this is possible for some n_1 and n_2, then it is possible for $n_1 n_2$. (4 points)

   (b) Determine all n for which this is possible. (4 points)

6. Let P and Q be two convex polygons. Let h be the length of the projection of Q onto a line perpendicular to a side P which is of length ℓ. Define $f(P, Q)$ to be the sum of the products $h\ell$ over all sides of P. Prove that $f(P, Q) = f(Q, P)$. (8 points)

7. There are n boxes, each containing either a red cube or a blue cube. Alex has a sum of money initially, and places bets on the colour of the cube in each box in turn. The bet can be anywhere from 0 up to everything he has at the time. After the bet has been placed, the box is opened. If Alex loses, his bet will be taken away. If he wins, he will get his bet back, plus a sum equal to the bet. Then he moves onto the next box, until he has bet on the last one, or until he runs out of money. What is the maximum factor by which he can guarantee to increase his money, if he knows that the exact number of blue cubes is

   (a) 1; (3 points)

   (b) some integer k, $1 < k \leq n$? (5 points)