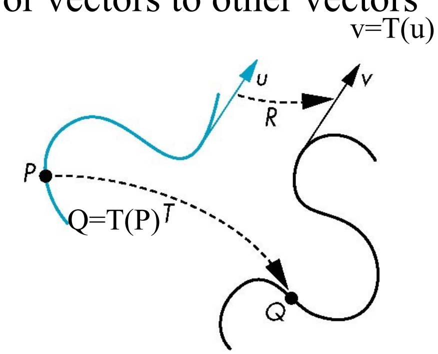
More Geometry

Objectives

- Introduce standard transformations
  - Rotation
  - Translation
  - Scaling
  - Shear
- Derive homogeneous coordinate transformation matrices
- Learn to build arbitrary transformation matrices from simple transformations

General Transformations

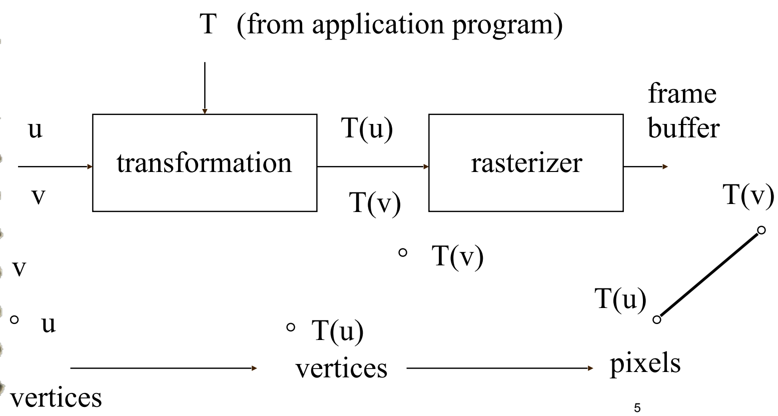
A transformation maps points to other points and/or vectors to other vectors



Affine Transformations

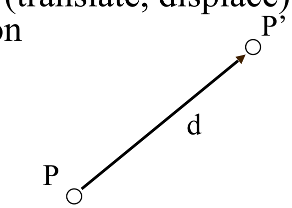
- Line preserving
- Characteristic of many physically important transformations
  - Rigid body transformations: rotation, translation
  - Scaling, shear
- Importance in graphics is that we need only transform endpoints of line segments and let implementation draw line segment between the transformed endpoints

Pipeline Implementation



Translation

- Move (translate, displace) a point to a new location



- Displacement determined by a vector d
  - Three degrees of freedom
  - P'=P+d

Translation Using Representations

Using the homogeneous coordinate representation in some frame

$\mathbf{p}=[\ x\ y\ z\ 1]^T$

$\mathbf{p}'=[x'\ y'\ z'\ 1]^T$

$\mathbf{d}=[dx\ dy\ dz\ 0]^T$

Hence $\mathbf{p}' = \mathbf{p} + \mathbf{d}$ or

$x'=x+d_x$

$y'=y+d_y$

$z'=z+d_z$

note that this expression is in four dimensions and expresses point = vector + point

Translation Matrix

We can also express translation using a 4 x 4 matrix **T** in homogeneous coordinates

p'=**Tp** where

$$\mathbf{T} = \mathbf{T}(d_x, d_y, d_z) = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This form is better for implementation because all affine transformations can be expressed this way and multiple transformations can be concatenated together

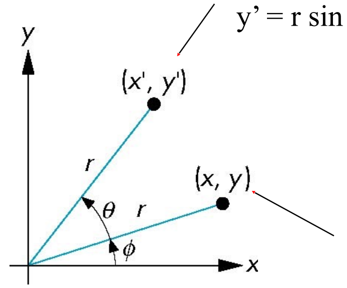
Rotation (2D)

Consider rotation about the origin by θ degrees

– radius stays the same, angle increases by θ

$x' = r \cos(\phi + \theta)$

$y' = r \sin(\phi + \theta)$



$$\boxed{\begin{array}{c} x' = x \cos\theta - y \sin\theta \\ y' = x \sin\theta + y \cos\theta \end{array}}$$

$x = r \cos \phi$

$y = r \sin \phi$

Rotation about the z axis

- Rotation about z axis in three dimensions leaves all points with the same z
  - Equivalent to rotation in two dimensions in planes of constant z

$$x' = x \cos\theta - y \sin\theta$$
$$y' = x \sin\theta + y \cos\theta$$
$$z' = z$$

  - or in homogeneous coordinates

$$\mathbf{p}' = \mathbf{R}_z(\theta)\mathbf{p}$$

Rotation Matrix

$$\mathbf{R} = \mathbf{R}_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation about x and y axes

- Same argument as for rotation about *z* axis
  - For rotation about *x* axis, *x* is unchanged
  - For rotation about *y* axis, *y* is unchanged

$$\mathbf{R} = \mathbf{R}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R} = \mathbf{R}_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scaling

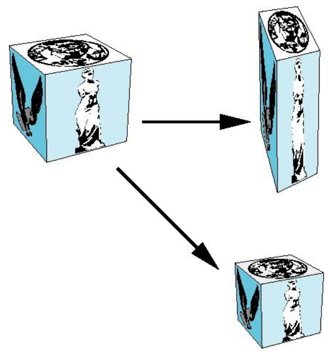
Expand or contract along each axis (fixed point of origin)

$x' = s_x x$

$y' = s_y y$

$z' = s_z z$

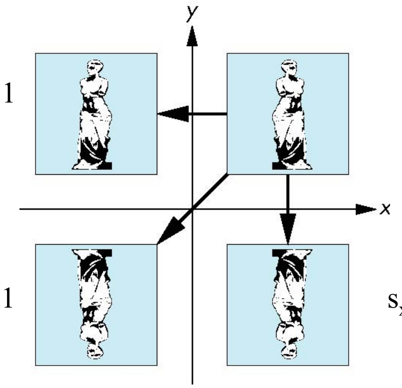
$\mathbf{p}' = \mathbf{S}\mathbf{p}$



$$\mathbf{S} = \mathbf{S}(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Reflection

corresponds to negative scale factors



Inverses

- Although we could compute inverse matrices by general formulas, we can use simple geometric observations
  - Translation: $\mathbf{T}^{-1}(d_x, d_y, d_z) = \mathbf{T}(-d_x, -d_y, -d_z)$
  - Rotation: $\mathbf{R}^{-1}(\theta) = \mathbf{R}(-\theta)$
    - Holds for any rotation matrix
    - Note that since $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$

    $\mathbf{R}^{-1}(\theta) = \mathbf{R}^{T}(\theta)$
  - Scaling: $\mathbf{S}^{-1}(s_x, s_y, s_z) = \mathbf{S}(1/s_x, 1/s_y, 1/s_z)$

Concatenation

- We can form arbitrary affine transformation matrices by multiplying together rotation, translation, and scaling matrices
- Because the same transformation is applied to many vertices, the cost of forming a matrix **M**=**ABCD** is not significant compared to the cost of computing **Mp** for many vertices **p**
- The difficult part is how to form a desired transformation from the specifications in the application

Order of Transformations

- Note that matrix on the right is the first applied
- Mathematically, the following are equivalent

$$\mathbf{p}' = \mathbf{ABCp} = \mathbf{A}(\mathbf{B}(\mathbf{Cp}))$$

- Note many references use column matrices to represent points. In terms of column matrices

$$\mathbf{p}'^{T} = \mathbf{p}^{T}\mathbf{C}^{T}\mathbf{B}^{T}\mathbf{A}^{T}$$

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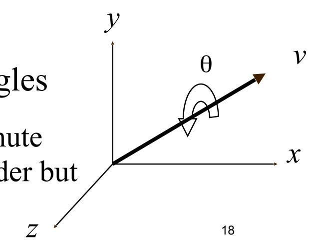
General Rotation About the Origin

A rotation by θ about an arbitrary axis can be decomposed into the concatenation of rotations about the *x*, *y*, and *z* axes

$$\mathbf{R}(\theta) = \mathbf{R}_z(\theta_z)\,\mathbf{R}_y(\theta_y)\,\mathbf{R}_x(\theta_x)$$

$\theta_x\ \theta_y\ \theta_z$ are called the Euler angles

Note that rotations do not commute
We can use rotations in another order but with different angles



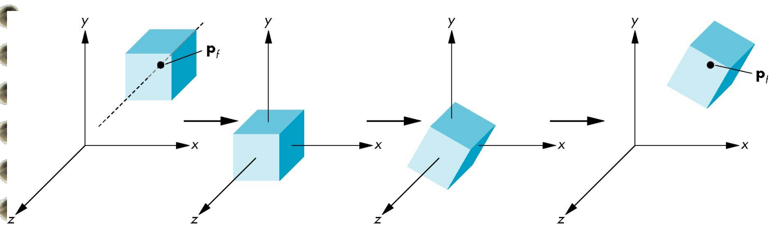
Rotation About a Fixed Point other than the Origin

Move fixed point to origin

Rotate

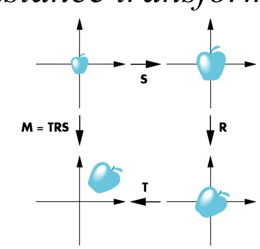
Move fixed point back

$\mathbf{M} = \mathbf{T}(p_f)\,\mathbf{R}(\theta)\,\mathbf{T}(-p_f)$



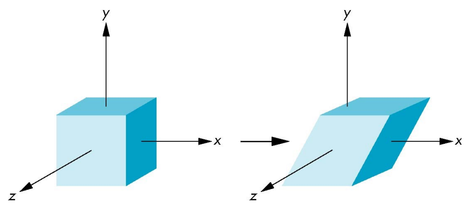
Instancing

- In modeling, we often start with a simple object centered at the origin, oriented with the axis, and at a standard size
- We apply an *instance transformation* to its vertices to
  - Scale
  - Orient
  - Locate



Shear

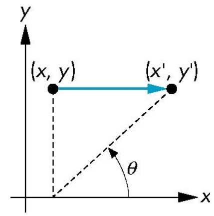
- Helpful to add one more basic transformation
- Equivalent to pulling faces in opposite directions



Shear Matrix

Consider simple shear along *x* axis

$$x' = x + y \cot\theta$$
$$y' = y$$
$$z' = z$$



$$\mathbf{H}(\theta) = \begin{bmatrix} 1 & \cot\theta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$