

2013 AP[®] CALCULUS BC FREE-RESPONSE QUESTIONS

6. A function f has derivatives of all orders at $x = 0$. Let $P_n(x)$ denote the n th-degree Taylor polynomial for f about $x = 0$.
- (a) It is known that $f(0) = -4$ and that $P_1\left(\frac{1}{2}\right) = -3$. Show that $f'(0) = 2$.
- (b) It is known that $f''(0) = -\frac{2}{3}$ and $f'''(0) = \frac{1}{3}$. Find $P_3(x)$.
- (c) The function h has first derivative given by $h'(x) = f(2x)$. It is known that $h(0) = 7$. Find the third-degree Taylor polynomial for h about $x = 0$.
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STOP

END OF EXAM

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2013 SCORING GUIDELINES

Question 6

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- (b) It is known that $f''(0) = -\frac{2}{3}$ and $f'''(0) = \frac{1}{3}$. Find $P_3(x)$.
- (c) The function h has first derivative given by $h'(x) = f(2x)$. It is known that $h(0) = 7$. Find the third-degree Taylor polynomial for h about $x = 0$.

(a) $P_1(x) = f(0) + f'(0)x = -4 + f'(0)x$

$$P_1\left(\frac{1}{2}\right) = -4 + f'(0) \cdot \frac{1}{2} = -3$$

$$f'(0) \cdot \frac{1}{2} = 1$$

$$f'(0) = 2$$

$$2 : \begin{cases} 1 : \text{uses } P_1(x) \\ 1 : \text{verifies } f'(0) = 2 \end{cases}$$

(b) $P_3(x) = -4 + 2x + \left(-\frac{2}{3}\right) \cdot \frac{x^2}{2!} + \frac{1}{3} \cdot \frac{x^3}{3!}$

$$= -4 + 2x - \frac{1}{3}x^2 + \frac{1}{18}x^3$$

$$3 : \begin{cases} 1 : \text{first two terms} \\ 1 : \text{third term} \\ 1 : \text{fourth term} \end{cases}$$

- (c) Let $Q_n(x)$ denote the Taylor polynomial of degree n for h about $x = 0$.

$$h'(x) = f(2x) \Rightarrow Q_3'(x) = -4 + 2(2x) - \frac{1}{3}(2x)^2$$

$$Q_3(x) = -4x + 4 \cdot \frac{x^2}{2} - \frac{4}{3} \cdot \frac{x^3}{3} + C; \quad C = Q_3(0) = h(0) = 7$$

$$Q_3(x) = 7 - 4x + 2x^2 - \frac{4}{9}x^3$$

$$4 : \begin{cases} 2 : \text{applies } h'(x) = f(2x) \\ 1 : \text{constant term} \\ 1 : \text{remaining terms} \end{cases}$$

OR

$$h'(x) = f(2x), \quad h''(x) = 2f'(2x), \quad h'''(x) = 4f''(2x)$$

$$h'(0) = f(0) = -4, \quad h''(0) = 2f'(0) = 4, \quad h'''(0) = 4f''(0) = -\frac{8}{3}$$

$$Q_3(x) = 7 - 4x + 4 \cdot \frac{x^2}{2!} - \frac{8}{3} \cdot \frac{x^3}{3!} = 7 - 4x + 2x^2 - \frac{4}{9}x^3$$