5. At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function $W$ models the total amount of solid waste stored at the landfill. Planners estimate that $W$ will satisfy the differential equation $\dfrac{dW}{dt} = \dfrac{1}{25}(W - 300)$ for the next 20 years. $W$ is measured in tons, and $t$ is measured in years from the start of 2010.

(a) Use the line tangent to the graph of $W$ at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \dfrac{1}{4}$).

(b) Find $\dfrac{d^2W}{dt^2}$ in terms of $W$. Use $\dfrac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t = \dfrac{1}{4}$.

(c) Find the particular solution $W = W(t)$ to the differential equation $\dfrac{dW}{dt} = \dfrac{1}{25}(W - 300)$ with initial condition $W(0) = 1400$.

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**WRITE ALL WORK IN THE EXAM BOOKLET.**
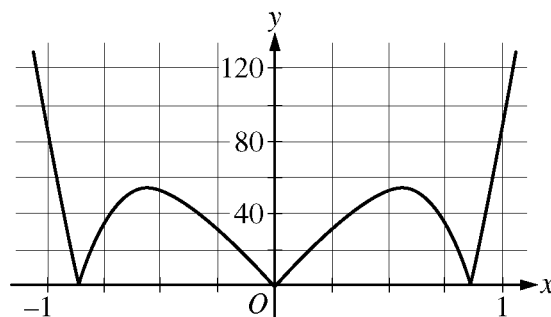
Graph of $y = \left| f^{(5)}(x) \right|$

6. Let $f(x) = \sin\left(x^2\right) + \cos x.$ The graph of $y = \left| f^{(5)}(x) \right|$ is shown above.

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0,$ and write the first four nonzero terms of the Taylor series for $\sin\left(x^2\right)$ about $x = 0.$

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0.$ Use this series and the series for $\sin\left(x^2\right),$ found in part (a), to write the first four nonzero terms of the Taylor series for $f$ about $x = 0.$

(c) Find the value of $f^{(6)}(0).$

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for $f$ about $x = 0.$ Using information from the graph of $y = \left| f^{(5)}(x) \right|$ shown above, show that $\left| P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right) \right| < \frac{1}{3000}.$

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**WRITE ALL WORK IN THE EXAM BOOKLET.**

**END OF EXAM**

## Question 5

At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function $W$ models the total amount of solid waste stored at the landfill. Planners estimate that $W$ will satisfy the differential equation $\dfrac{dW}{dt} = \dfrac{1}{25}(W - 300)$ for the next 20 years. $W$ is measured in tons, and $t$ is measured in years from the start of 2010.

(a) Use the line tangent to the graph of $W$ at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \dfrac{1}{4}$).

(b) Find $\dfrac{d^2W}{dt^2}$ in terms of $W$. Use $\dfrac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t = \dfrac{1}{4}$.

(c) Find the particular solution $W = W(t)$ to the differential equation $\dfrac{dW}{dt} = \dfrac{1}{25}(W - 300)$ with initial condition $W(0) = 1400$.

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(a) $\left. \dfrac{dW}{dt} \right|_{t=0} = \dfrac{1}{25}(W(0) - 300) = \dfrac{1}{25}(1400 - 300) = 44$

The tangent line is $y = 1400 + 44t$.

$W\left(\dfrac{1}{4}\right) \approx 1400 + 44\left(\dfrac{1}{4}\right) = 1411$ tons

$2 : \begin{cases} 1 : \dfrac{dW}{dt} \text{ at } t = 0 \\ 1 : \text{answer} \end{cases}$

(b) $\dfrac{d^2W}{dt^2} = \dfrac{1}{25}\dfrac{dW}{dt} = \dfrac{1}{625}(W - 300)$ and $W \geq 1400$

Therefore $\dfrac{d^2W}{dt^2} > 0$ on the interval $0 \leq t \leq \dfrac{1}{4}$.

The answer in part (a) is an underestimate.

$2 : \begin{cases} 1 : \dfrac{d^2W}{dt^2} \\ 1 : \text{answer with reason} \end{cases}$

(c) $\dfrac{dW}{dt} = \dfrac{1}{25}(W - 300)$

$\displaystyle\int \dfrac{1}{W - 300}\, dW = \int \dfrac{1}{25}\, dt$

$\ln|W - 300| = \dfrac{1}{25}t + C$

$\ln(1400 - 300) = \dfrac{1}{25}(0) + C \Rightarrow \ln(1100) = C$

$W - 300 = 1100e^{\frac{1}{25}t}$

$W(t) = 300 + 1100e^{\frac{1}{25}t}, \quad 0 \leq t \leq 20$

$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } W \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables