CALCULUS BC
SECTION II, Part B
Time—45 minutes
Number of problems—3

No calculator is allowed for these problems.

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4. Consider the curve given by $x^2 + 4y^2 = 7 + 3xy$.

   (a) Show that $\dfrac{dy}{dx} = \dfrac{3y - 2x}{8y - 3x}$.

   (b) Show that there is a point P with x-coordinate 3 at which the line tangent to the curve at P is horizontal. Find the y-coordinate of P.

   (c) Find the value of $\dfrac{d^2y}{dx^2}$ at the point P found in part (b). Does the curve have a local maximum, a local minimum, or neither at the point P? Justify your answer.

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5. A population is modeled by a function P that satisfies the logistic differential equation

$$\frac{dP}{dt} = \frac{P}{5}\left(1 - \frac{P}{12}\right).$$

   (a) If $P(0) = 3$, what is $\displaystyle\lim_{t \to \infty} P(t)$?
   If $P(0) = 20$, what is $\displaystyle\lim_{t \to \infty} P(t)$?

   (b) If $P(0) = 3$, for what value of P is the population growing the fastest?

   (c) A different population is modeled by a function Y that satisfies the separable differential equation

$$\frac{dY}{dt} = \frac{Y}{5}\left(1 - \frac{t}{12}\right).$$

   Find $Y(t)$ if $Y(0) = 3$.

   (d) For the function Y found in part (c), what is $\displaystyle\lim_{t \to \infty} Y(t)$?

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Question 4

Consider the curve given by $x^2 + 4y^2 = 7 + 3xy$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{3y - 2x}{8y - 3x}$.

(b) Show that there is a point P with x-coordinate 3 at which the line tangent to the curve at P is horizontal. Find the y-coordinate of P.

(c) Find the value of $\dfrac{d^2y}{dx^2}$ at the point P found in part (b). Does the curve have a local maximum, a local minimum, or neither at the point P? Justify your answer.

(a) $2x + 8yy' = 3y + 3xy'$
$(8y - 3x)y' = 3y - 2x$
$$y' = \frac{3y - 2x}{8y - 3x}$$

$2 : \begin{cases} 1 : \text{implicit differentiation} \\ 1 : \text{solves for } y' \end{cases}$

(b) $\dfrac{3y - 2x}{8y - 3x} = 0;\ 3y - 2x = 0$

When $x = 3,\ 3y = 6$
$\qquad\qquad\quad y = 2$

$3^2 + 4{\cdot}2^2 = 25$ and $7 + 3{\cdot}3{\cdot}2 = 25$

Therefore, $P = (3, 2)$ is on the curve and the slope is 0 at this point.

$3 : \begin{cases} 1 : \dfrac{dy}{dx} = 0 \\ 1 : \text{shows slope is 0 at } (3, 2) \\ 1 : \text{shows } (3, 2) \text{ lies on curve} \end{cases}$

(c) $\dfrac{d^2y}{dx^2} = \dfrac{(8y - 3x)(3y' - 2) - (3y - 2x)(8y' - 3)}{(8y - 3x)^2}$

At $P = (3, 2)$, $\dfrac{d^2y}{dx^2} = \dfrac{(16 - 9)(-2)}{(16 - 9)^2} = -\dfrac{2}{7}$.

Since $y' = 0$ and $y'' < 0$ at P, the curve has a local maximum at P.

$4 : \begin{cases} 2 : \dfrac{d^2y}{dx^2} \\ 1 : \text{value of } \dfrac{d^2y}{dx^2} \text{ at } (3, 2) \\ 1 : \text{conclusion with justification} \end{cases}$