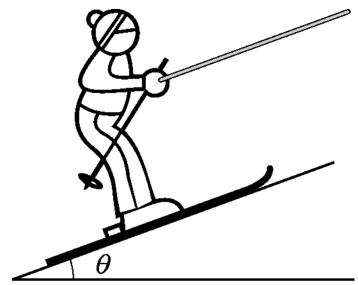


Mech. 3.

A skier of mass m will be pulled up a hill by a rope, as shown above. The magnitude of the acceleration of the skier as a function of time t can be modeled by the equations

$$\begin{aligned} a &= a_{max}\sin\frac{\pi t}{T} \qquad (0 < t < T) \\ &= 0 \qquad\qquad\qquad (t \geq T), \end{aligned}$$

where a_{max} and T are constants. The hill is inclined at an angle θ above the horizontal, and friction between the skis and the snow is negligible. Express your answers in terms of given quantities and fundamental constants.

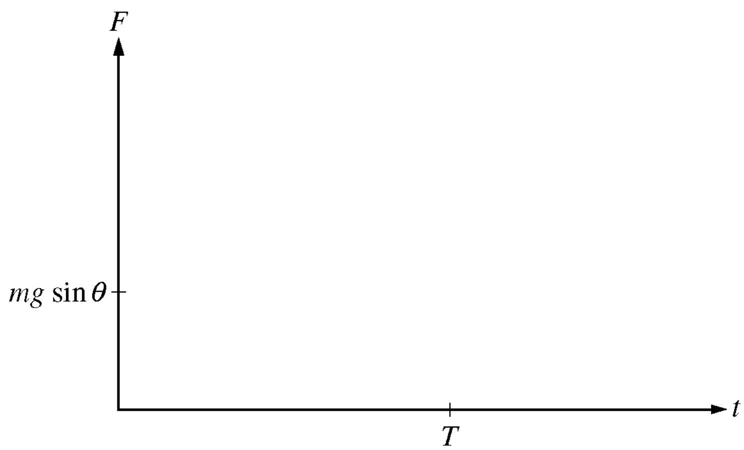
(a) Derive an expression for the velocity of the skier as a function of time during the acceleration. Assume the skier starts from rest.

(b) Derive an expression for the work done by the net force on the skier from rest until terminal speed is reached.

(c) Determine the magnitude of the force exerted by the rope on the skier at terminal speed.

(d) Derive an expression for the total impulse imparted to the skier during the acceleration.

(e) Suppose that the magnitude of the acceleration is instead modeled as $a = a_{max}e^{-\pi t/2T}$ for all $t > 0$, where a_{max} and T are the same as in the original model. On the axes below, sketch the graphs of the force exerted by the rope on the skier for the two models, from $t = 0$ to a time $t > T$. Label the original model F_1 and the new model F_2.



END OF EXAM

Question 3

15 points total | **Distribution of points**

(a) 4 points

For a correct relationship between velocity and acceleration — 1 point

$$\upsilon = \int a(t)\,dt \quad \textbf{OR} \quad \upsilon = \int_0^t a(t)\,dt \quad \textbf{OR} \quad \frac{d\upsilon}{dt} = a$$

For a correct substitution of the expression for acceleration into the integral relationship — 1 point

$$\upsilon = \int \left(a_{\max} \sin\frac{\pi t}{T}\right) dt \quad \textbf{OR} \quad \upsilon = \int_0^t \left(a_{\max} \sin\frac{\pi t}{T}\right) dt \qquad (0 < t < T)$$

For a correct evaluation of the integral, with an integration constant or correct limits — 1 point

$$\upsilon = -\frac{a_{\max}T}{\pi}\cos\frac{\pi t}{T} + C \quad \textbf{OR} \quad \upsilon = -\frac{a_{\max}T}{\pi}\cos\frac{\pi t}{T}\Big|_0^t \qquad (0 < t < T)$$

For a correct determination of the integration constant or evaluation between the limits — 1 point

$$\upsilon(0) = -\frac{a_{\max}T}{\pi} + C = 0 \;\Rightarrow\; C = \frac{a_{\max}T}{\pi} \quad \textbf{OR} \quad \upsilon = -\frac{a_{\max}T}{\pi}\left(\cos\frac{\pi t}{T} - 1\right) \qquad (0 < t < T)$$

$$\upsilon = \frac{a_{\max}T}{\pi}\left(1 - \cos\frac{\pi t}{T}\right) \qquad (0 < t < T)$$

(b) 2 points

For indicating that the work done by the net force is equal to the change in kinetic energy — 1 point

$$W = \frac{1}{2}m\left(\upsilon_f^2 - \upsilon_i^2\right)$$

For a correct substitution of velocity from (a) into the work-energy expression — 1 point

$$\upsilon_f = \upsilon_T = \frac{a_{\max}T}{\pi}(1 - \cos\pi) = \frac{2a_{\max}T}{\pi}$$

$$\upsilon_i = \upsilon_0 = \frac{a_{\max}T}{\pi}(1 - \cos 0) = 0$$

$$W = \frac{1}{2}m\left(\frac{2a_{\max}T}{\pi}\right)^2$$

$$W = \frac{2ma_{\max}^2T^2}{\pi^2}$$

Alternate solution (integral form) — *Alternate points*

$$W = \int F \cdot dx$$

For a correct substitution of the expression for force into the integral — *1 point*

$$W = \int ma_{\max} \sin\frac{\pi t}{T}\,dx$$

For a correct expression for dx in terms of time — *1 point*

Question 3 (continued)

Distribution of points

(b) (continued)

$$W = \int_0^T ma_{\max} \sin\frac{\pi t}{T} \left(\frac{a_{\max}T}{\pi}\left(1 - \cos\frac{\pi t}{T}\right)\right)dt$$

$$W = \frac{ma_{\max}^2 T}{\pi} \int_0^T \left(\sin\frac{\pi t}{T} - \left(\sin\frac{\pi t}{T}\cos\frac{\pi t}{T}\right)\right) dt$$

$$W = \frac{ma_{\max}^2 T}{\pi} \int_0^T \left(\sin\frac{\pi t}{T} - \frac{1}{2}\sin\frac{2\pi t}{T}\right) dt$$

$$W = \frac{ma_{\max}^2 T^2}{\pi^2}\left(-\cos\frac{\pi t}{T} - \frac{1}{4}\cos\frac{2\pi t}{T}\right)\Bigg|_0^T$$

$$W = \frac{2ma_{\max}^2 T^2}{\pi^2}$$

(c) 1 point

Starting with Newton's second law:

$F_{net} = F_{rope} - mg\sin\theta = ma$

At terminal velocity, the net force and acceleration are zero:

$F_{rope} - mg\sin\theta = 0$

For a correct expression for the force — 1 point

$F_{rope} = mg\sin\theta$

(d) 2 points

$J = \int F\,dt$

For a correct substitution of force into the impulse-time relationship — 1 point

$$J = ma_{\max}\int_0^T \sin\frac{\pi t}{T}\,dt$$

$$J = \frac{ma_{\max}T}{\pi}\left(-\cos\frac{\pi t}{T}\right)\Bigg|_0^T$$

For evaluation at the limits of integration — 1 point

$$J = \frac{ma_{\max}T}{\pi}\left[-\cos\pi + \cos 0\right]$$

$$J = \frac{2ma_{\max}T}{\pi}$$

Alternate solution (impulse-momentum) — *Alternate points*

$J = \Delta p = m\upsilon_T$

Question 3 (continued)

Distribution of points

(d) (continued)

For a correct substitution of the velocity *1 point*

$J = \frac{ma_{max}T}{\pi}\left(1 - \cos\frac{\pi t}{T}\right)$

For setting $t = T$ *1 point*

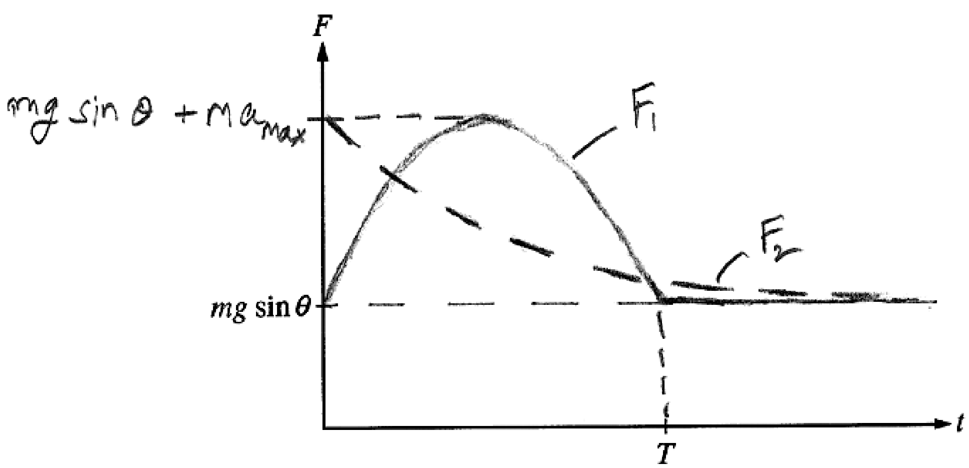
$J = \frac{ma_{max}T}{\pi}(1 - \cos\pi)$

$J = \frac{2ma_{max}T}{\pi}$

(e) 6 points

$F_1 = mg\sin\theta + ma_{max}\sin\left(\frac{\pi t}{T}\right) \quad (0 < t < T)$

$F_2 = mg\sin\theta + ma_{max}\, e^{-\pi t/2T}$



For a graph labeled F_1:

for starting at $mg\sin\theta$ 1 point

for half a sine wave with a maximum at $\sim T/2$ 1 point

for returning to original starting point at $t = T$ 1 point

for a horizontal line at the original starting point for $t > T$ 1 point

For a graph labeled F_2:

for starting on the vertical axis at a point above the starting point of F_1 (if there is no F_1 graph, this point was awarded if the F_2 graph starts above $mg\sin\theta$) 1 point

for an exponential decay graph 1 point