6. Let f be the function given by $f(x) = e^{-x^2}$.

   (a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

   (b) Use your answer to part (a) to find $\lim\limits_{x \to 0} \dfrac{1 - x^2 - f(x)}{x^4}$.

   (c) Write the first four nonzero terms of the Taylor series for $\int_0^x e^{-t^2}\,dt$ about $x = 0$. Use the first two terms of your answer to estimate $\int_0^{1/2} e^{-t^2}\,dt$.

   (d) Explain why the estimate found in part (c) differs from the actual value of $\int_0^{1/2} e^{-t^2}\,dt$ by less than $\dfrac{1}{200}$.

WRITE ALL WORK IN THE PINK EXAM BOOKLET.

END OF EXAM

Question 6

Let f be the function given by $f(x) = e^{-x^2}$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Use your answer to part (a) to find $\lim_{x \to 0} \frac{1 - x^2 - f(x)}{x^4}$.

(c) Write the first four nonzero terms of the Taylor series for $\int_0^x e^{-t^2}\, dt$ about $x = 0$. Use the first two terms of your answer to estimate $\int_0^{1/2} e^{-t^2}\, dt$.

(d) Explain why the estimate found in part (c) differs from the actual value of $\int_0^{1/2} e^{-t^2}\, dt$ by less than $\frac{1}{200}$.

(a) $e^{-x^2} = 1 + \frac{(-x^2)}{1!} + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \cdots + \frac{(-x^2)^n}{n!} + \cdots$ $= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots + \frac{(-1)^n x^{2n}}{n!} + \cdots$	$3: \begin{cases} 1: \text{two of } 1, -x^2, \frac{x^4}{2}, -\frac{x^6}{6} \\ 1: \text{remaining terms} \\ 1: \text{general term} \end{cases}$
(b) $\frac{1 - x^2 - f(x)}{x^4} = -\frac{1}{2} + \frac{x^2}{6} + \sum_{n=4}^{\infty} \frac{(-1)^{n+1} x^{2n-4}}{n!}$ Thus, $\lim_{x \to 0}\left(\frac{1 - x^2 - f(x)}{x^4}\right) = -\frac{1}{2}$.	1 : answer
(c) $\int_0^x e^{-t^2}\, dt = \int_0^x \left(1 - t^2 + \frac{t^4}{2} - \frac{t^6}{6} + \cdots + \frac{(-1)^n t^{2n}}{n!} + \cdots\right) dt$ $= x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots$ Using the first two terms of this series, we estimate that $\int_0^{1/2} e^{-t^2}\, dt \approx \frac{1}{2} - \left(\frac{1}{3}\right)\left(\frac{1}{8}\right) = \frac{11}{24}$.	$3: \begin{cases} 1: \text{two terms} \\ 1: \text{remaining terms} \\ 1: \text{estimate} \end{cases}$
(d) $\left\lvert \int_0^{1/2} e^{-t^2}\, dt - \frac{11}{24} \right\rvert < \left(\frac{1}{2}\right)^5 \cdot \frac{1}{10} = \frac{1}{320} < \frac{1}{200}$, since $\int_0^{1/2} e^{-t^2}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)^{2n+1}}{n!(2n+1)}$, which is an alternating series with individual terms that decrease in absolute value to 0.	$2: \begin{cases} 1: \text{uses the third term as the error bound} \\ 1: \text{explanation} \end{cases}$