5. At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function W models the total amount of solid waste stored at the landfill. Planners estimate that W will satisfy the differential equation $\dfrac{dW}{dt} = \dfrac{1}{25}(W - 300)$ for the next 20 years. W is measured in tons, and t is measured in years from the start of 2010.

   (a) Use the line tangent to the graph of W at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \dfrac{1}{4}$).

   (b) Find $\dfrac{d^2W}{dt^2}$ in terms of W. Use $\dfrac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t = \dfrac{1}{4}$.

   (c) Find the particular solution $W = W(t)$ to the differential equation $\dfrac{dW}{dt} = \dfrac{1}{25}(W - 300)$ with initial condition $W(0) = 1400$.

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6. Let f be a function defined by $f(x) = \begin{cases} 1 - 2\sin x & \text{for } x \leq 0 \\ e^{-4x} & \text{for } x > 0. \end{cases}$

   (a) Show that f is continuous at $x = 0$.

   (b) For $x \neq 0$, express $f'(x)$ as a piecewise-defined function. Find the value of x for which $f'(x) = -3$.

   (c) Find the average value of f on the interval $[-1, 1]$.

WRITE ALL WORK IN THE EXAM BOOKLET.

END OF EXAM

Question 6

Let f be a function defined by $f(x) = \begin{cases} 1 - 2\sin x & \text{for } x \leq 0 \\ e^{-4x} & \text{for } x > 0. \end{cases}$

(a) Show that f is continuous at $x = 0$.

(b) For $x \neq 0$, express $f'(x)$ as a piecewise-defined function. Find the value of x for which $f'(x) = -3$.

(c) Find the average value of f on the interval $[-1, 1]$.

(a) $\lim\limits_{x \to 0^-} (1 - 2\sin x) = 1$

$\lim\limits_{x \to 0^+} e^{-4x} = 1$

$f(0) = 1$

So, $\lim\limits_{x \to 0} f(x) = f(0)$.

Therefore f is continuous at $x = 0$.

2 : analysis

(b) $f'(x) = \begin{cases} -2\cos x & \text{for } x < 0 \\ -4e^{-4x} & \text{for } x > 0 \end{cases}$

$-2\cos x \neq -3$ for all values of $x < 0$.

$-4e^{-4x} = -3$ when $x = -\dfrac{1}{4}\ln\left(\dfrac{3}{4}\right) > 0$.

Therefore $f'(x) = -3$ for $x = -\dfrac{1}{4}\ln\left(\dfrac{3}{4}\right)$.

$3 : \begin{cases} 2 : f'(x) \\ 1 : \text{value of } x \end{cases}$

(c) $\displaystyle\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx$

$\displaystyle = \int_{-1}^{0} (1 - 2\sin x)\,dx + \int_{0}^{1} e^{-4x}\,dx$

$\displaystyle = \Big[x + 2\cos x \Big]_{x=-1}^{x=0} + \Big[-\dfrac{1}{4}e^{-4x} \Big]_{x=0}^{x=1}$

$\displaystyle = (3 - 2\cos(-1)) + \left(-\dfrac{1}{4}e^{-4} + \dfrac{1}{4} \right)$

Average value $= \dfrac{1}{2}\displaystyle\int_{-1}^{1} f(x)\,dx$

$= \dfrac{13}{8} - \cos(-1) - \dfrac{1}{8}e^{-4}$

$4 : \begin{cases} 1 : \displaystyle\int_{-1}^{0} (1 - 2\sin x)\,dx \text{ and } \int_{0}^{1} e^{-4x}\,dx \\ 2 : \text{antiderivatives} \\ 1 : \text{answer} \end{cases}$