$$f(0) = 0$$
$$f'(0) = 1$$
$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \text{ for all } n \geq 1$$

6. A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(a) Show that the first four nonzero terms of the Maclaurin series for f are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, and write the general term of the Maclaurin series for f.

(b) Determine whether the Maclaurin series described in part (a) converges absolutely, converges conditionally, or diverges at $x = 1$. Explain your reasoning.

(c) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t)\,dt$.

(d) Let $P_n\left(\frac{1}{2}\right)$ represent the nth-degree Taylor polynomial for g about $x = 0$ evaluated at $x = \frac{1}{2}$, where g is the function defined in part (c). Use the alternating series error bound to show that

$$\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| < \frac{1}{500}.$$

STOP

END OF EXAM

Question 6

(a) $f(0) = 0$ $f'(0) = 1$ $f''(0) = -1(1) = -1$ $f'''(0) = -2(-1) = 2$ $f^{(4)}(0) = -3(2) = -6$ The first four nonzero terms are $0 + 1x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}.$ The general term is $\frac{(-1)^{n+1}x^n}{n}.$	$3 : \begin{cases} 1 : f''(0),\ f'''(0), \text{ and } f^{(4)}(0) \\ 1 : \text{verify terms} \\ 1 : \text{general term} \end{cases}$
(b) For $x = 1,$ the Maclaurin series becomes $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$ The series does not converge absolutely because the harmonic series diverges. The series alternates with terms that decrease in magnitude to 0, and therefore the series converges conditionally.	2 : converges conditionally with reason
(c) $\int_0^x f(t)\,dt = \int_0^x \left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots + \frac{(-1)^{n+1}t^n}{n} + \cdots\right)dt$ $= \left[\frac{t^2}{2} - \frac{t^3}{3 \cdot 2} + \frac{t^4}{4 \cdot 3} - \frac{t^5}{5 \cdot 4} + \cdots + \frac{(-1)^{n+1}t^{n+1}}{(n+1)n} + \cdots\right]_{t=0}^{t=x}$ $= \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \cdots + \frac{(-1)^{n+1}x^{n+1}}{(n+1)n} + \cdots$	$3 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$
(d) The terms alternate in sign and decrease in magnitude to 0. By the alternating series error bound, the error $\left\lvert P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)\right\rvert$ is bounded by the magnitude of the first unused term, $\left\lvert -\frac{(1/2)^5}{20}\right\rvert.$ Thus, $\left\lvert P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)\right\rvert \le \left\lvert -\frac{(1/2)^5}{20}\right\rvert = \frac{1}{32 \cdot 20} < \frac{1}{500}.$	1 : error bound