5. Consider the differential equation $\frac{dy}{dx} = 1 - y$. Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(1) = 0$. For this particular solution, $f(x) < 1$ for all values of x.

   (a) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(0)$. Show the work that leads to your answer.

   (b) Find $\lim_{x \to 1} \frac{f(x)}{x^3 - 1}$. Show the work that leads to your answer.

   (c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = 1 - y$ with the initial condition $f(1) = 0$.

$$f(x) = \begin{cases} \dfrac{\cos x - 1}{x^2} & \text{for } x \neq 0 \\ -\dfrac{1}{2} & \text{for } x = 0 \end{cases}$$

6. The function f, defined above, has derivatives of all orders. Let g be the function defined by $g(x) = 1 + \int_0^x f(t)\, dt$.

   (a) Write the first three nonzero terms and the general term of the Taylor series for $\cos x$ about $x = 0$. Use this series to write the first three nonzero terms and the general term of the Taylor series for f about $x = 0$.

   (b) Use the Taylor series for f about $x = 0$ found in part (a) to determine whether f has a relative maximum, relative minimum, or neither at $x = 0$. Give a reason for your answer.

   (c) Write the fifth-degree Taylor polynomial for g about $x = 0$.

   (d) The Taylor series for g about $x = 0$, evaluated at $x = 1$, is an alternating series with individual terms that decrease in absolute value to 0. Use the third-degree Taylor polynomial for g about $x = 0$ to estimate the value of $g(1)$. Explain why this estimate differs from the actual value of $g(1)$ by less than $\frac{1}{6!}$.

WRITE ALL WORK IN THE PINK EXAM BOOKLET.

END OF EXAM

Question 6

$$f(x) = \begin{cases} \dfrac{\cos x - 1}{x^2} & \text{for } x \neq 0 \\ -\dfrac{1}{2} & \text{for } x = 0 \end{cases}$$

The function f, defined above, has derivatives of all orders. Let g be the function defined by $g(x) = 1 + \int_0^x f(t)\,dt.$

(a) Write the first three nonzero terms and the general term of the Taylor series for $\cos x$ about $x = 0$. Use this series to write the first three nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Use the Taylor series for f about $x = 0$ found in part (a) to determine whether f has a relative maximum, relative minimum, or neither at $x = 0$. Give a reason for your answer.

(c) Write the fifth-degree Taylor polynomial for g about $x = 0$.

(d) The Taylor series for g about $x = 0$, evaluated at $x = 1$, is an alternating series with individual terms that decrease in absolute value to 0. Use the third-degree Taylor polynomial for g about $x = 0$ to estimate the value of $g(1)$. Explain why this estimate differs from the actual value of $g(1)$ by less than $\frac{1}{6!}$.

(a) $\cos(x) = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{4!} - \cdots + (-1)^n \dfrac{x^{2n}}{(2n)!} + \cdots$

$f(x) = -\dfrac{1}{2} + \dfrac{x^2}{4!} - \dfrac{x^4}{6!} + \cdots + (-1)^{n+1} \dfrac{x^{2n}}{(2n+2)!} + \cdots$

$3 : \begin{cases} 1 : \text{terms for } \cos x \\ 2 : \text{terms for } f \\ \quad 1 : \text{first three terms} \\ \quad 1 : \text{general term} \end{cases}$

(b) $f'(0)$ is the coefficient of x in the Taylor series for f about $x = 0$, so $f'(0) = 0$.

$\dfrac{f''(0)}{2!} = \dfrac{1}{4!}$ is the coefficient of x^2 in the Taylor series for f about $x = 0$, so $f''(0) = \dfrac{1}{12}$.

Therefore, by the Second Derivative Test, f has a relative minimum at $x = 0$.

$2 : \begin{cases} 1 : \text{determines } f'(0) \\ 1 : \text{answer with reason} \end{cases}$

(c) $P_5(x) = 1 - \dfrac{x}{2} + \dfrac{x^3}{3 \cdot 4!} - \dfrac{x^5}{5 \cdot 6!}$

$2 : \begin{cases} 1 : \text{two correct terms} \\ 1 : \text{remaining terms} \end{cases}$

(d) $g(1) \approx 1 - \dfrac{1}{2} + \dfrac{1}{3 \cdot 4!} = \dfrac{37}{72}$

Since the Taylor series for g about $x = 0$ evaluated at $x = 1$ is alternating and the terms decrease in absolute value to 0, we know $\left| g(1) - \dfrac{37}{72} \right| < \dfrac{1}{5 \cdot 6!} < \dfrac{1}{6!}$.

$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{explanation} \end{cases}$