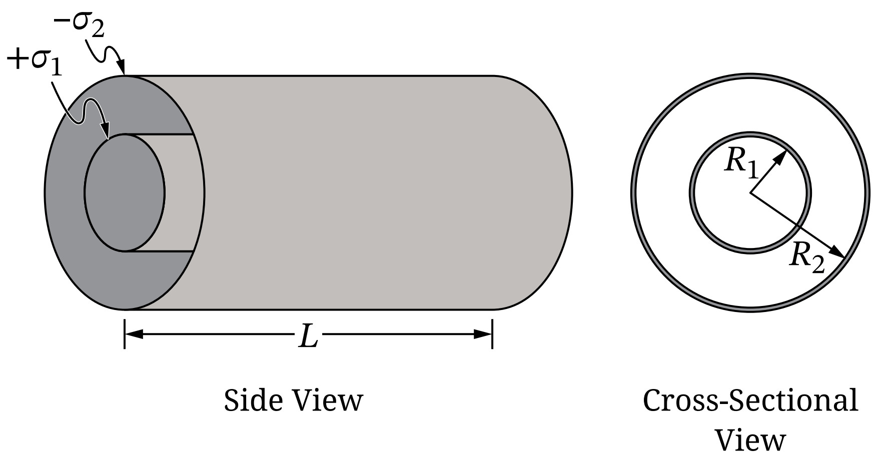
Question 1: Version J

1. An isolated, air-filled, charged capacitor consists of two conducting, coaxial, cylindrical shells that each have length L. The inner shell has radius R_1 and the outer shell has radius R_2, as shown in Figure 1, where $R_1 < R_2 \ll L$. The surface charge densities (amounts of charge per unit area) of the inner and outer shells are $+\sigma_1$ and $-\sigma_2$, respectively. The absolute values of the total charges on the shells are equal.



Note: Figures not drawn to scale.

Figure 1

A.

i. Using Gauss's law, **derive** an expression for the magnitude E of the electric field as a function of the radial distance r from the center of the capacitor for the region $R_1 < r < R_2$. Express your answer in terms of R_1, σ_1, r, and physical constants, as appropriate.

ii. **Derive** an expression for the absolute value $|\Delta V|$ of the potential difference between the outer and inner shells in terms of R_1, R_2, σ_1, and physical constants, as appropriate. Begin your derivation by writing a fundamental physics principle or an equation from the reference information.

iii. On the axes shown in Figure 2, **sketch** a graph of E as a function of r from $r = 0$ to a position that is outside the outer shell.

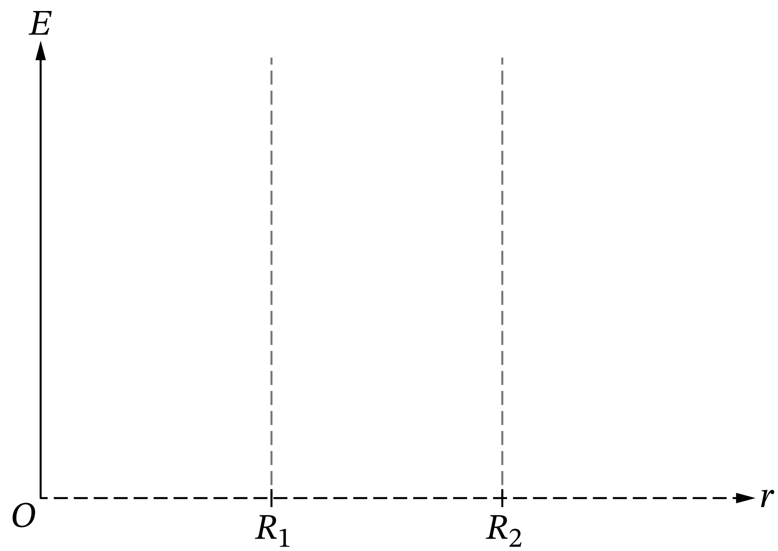
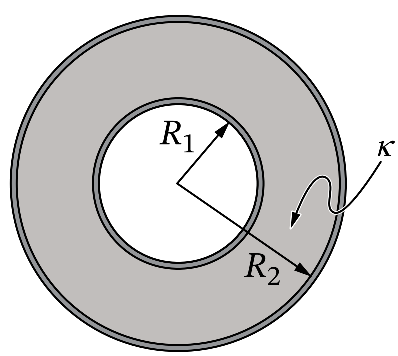


Figure 2

B. A material of dielectric constant κ is inserted into the isolated, charged capacitor such that the material fills the region $R_1 < r < R_2$, as shown in Figure 3.



Cross-Sectional View

Figure 3

Derive an expression for the capacitance C of the capacitor with the material inserted in terms of L, R_1, R_2, κ, and physical constants, as appropriate. Begin your derivation by writing a fundamental physics principle or an equation from the reference information.

Question 2: Version J

2. A rotating, circular, conducting loop of area A and resistance R is in an external uniform magnetic field of magnitude B that is directed in the $-z$-direction. At time $t = 0$, the magnetic field is perpendicular to the plane of the loop, as shown in Figure 1. The loop is rotating with constant angular speed ω and period T about the dashed line that is along the diameter of the loop. The value of the magnetic flux through the loop as a function of time t is $\Phi = BA\cos(\omega t)$.

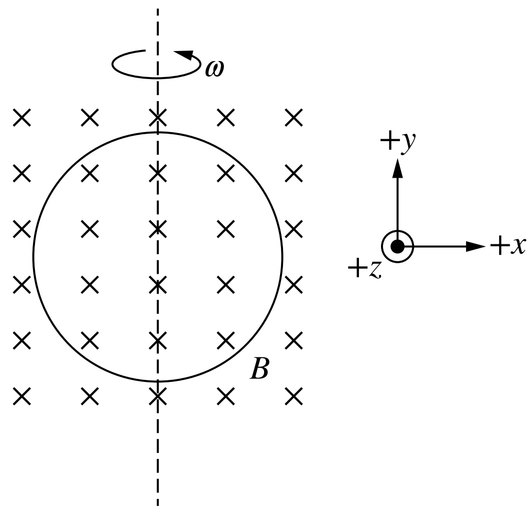


Figure 1

A. The absolute value of the induced emf in the loop is $|\varepsilon|$. The partially completed bar chart in Figure 2 shows a bar that represents $|\varepsilon|$ at $t = \frac{3}{4}T$. In Figure 2, **draw** bars to represent $|\varepsilon|$ at times $t = 0$, $\frac{1}{4}T$, and $\frac{1}{2}T$ relative to $|\varepsilon|$ shown at $\frac{3}{4}T$. If $|\varepsilon| = 0$, **write** a “0” in that column.

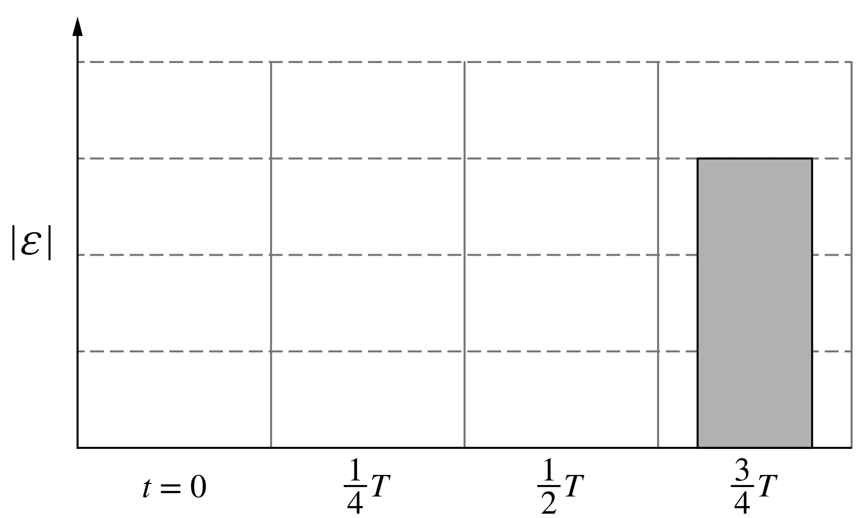


Figure 2

Question 1: Mathematical Routines (MR) — 10 points

A (i) For a multistep derivation that includes the equation $\oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{enc}}}{\varepsilon_0}$ — **Point A1**

Scoring Note: Vector notation is not required for this point to be earned.

For a correct substitution of the area of an appropriate Gaussian surface with nonzero flux for the region $R_1 < r < R_2$ (e.g., $2\pi r\ell$) — **Point A2**

For a correct expression for the enclosed charge (e.g., $\sigma_1\left(2\pi R_1\ell\right)$) — **Point A3**

Example Response

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{enc}}}{\varepsilon_0}$$

$$E\left(2\pi r\ell\right) = \frac{\sigma_1\left(2\pi R_1\ell\right)}{\varepsilon_0}$$

$$E = \frac{\sigma_1 R_1}{\varepsilon_0 r}$$

(ii) For substituting the expression for E from part A (i) into $\Delta V = -\int_a^b \vec{E} \cdot d\vec{r}$ — **Point A4**

Scoring Notes:
- Vector notation is not required for this point to be earned.
- The sign of ΔV is not considered for this point to be earned.

For an attempt to solve the integral for $\left|\Delta V\right|$ that includes correct limits (e.g., $\left|\Delta V\right| = \frac{\sigma_1 R_1}{\varepsilon_0}\int_{R_1}^{R_2}\frac{1}{r}dr$) — **Point A5**

Scoring Note: This point may be earned regardless of the order of the limits of integration; for example, $\frac{\sigma_1 R_1}{\varepsilon_0}\int_{R_2}^{R_1}\frac{1}{r}dr$.

Example Response

$$\left|\Delta V\right| = \left|-\int_a^b \vec{E} \cdot d\vec{r}\right|$$

$$\left|\Delta V\right| = \int_{R_1}^{R_2}\frac{\sigma_1 R_1}{r\varepsilon_0}dr = \frac{\sigma_1 R_1}{\varepsilon_0}\int_{R_1}^{R_2}\frac{1}{r}dr$$

$$\left|\Delta V\right| = \frac{\sigma_1 R_1}{\varepsilon_0}\ln\left(r\right)\Big|_{R_1}^{R_2} = \frac{\sigma_1 R_1}{\varepsilon_0}\ln\left(\frac{R_2}{R_1}\right)$$

(iii) For sketching a graph that is zero for both $0 < r < R_1$ and $r > R_2$ — **Point A6**

For sketching a graph that is decreasing and concave up for $R_1 < r < R_2$ — **Point A7**

Scoring Note: The curve does not have to intersect the vertical dashed lines for this point to be earned.

Example Response

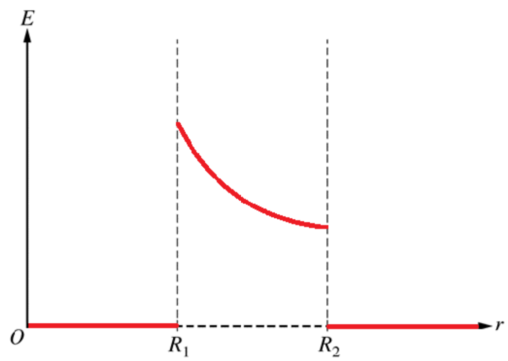


Figure 2

B	For a multistep derivation that includes the equation $C = \dfrac{Q}{\Delta V}$	**Point B1**
	For indicating that C with the dielectric material inserted is C without the dielectric material inserted multiplied by κ (e.g., $C = \kappa \dfrac{Q}{\Delta V}$)	**Point B2**
	For substitutions of both the total charge Q and the potential difference ΔV that are consistent with part A	**Point B3**

Example Response

Without the dielectric material

$$C = \frac{Q}{\Delta V}$$

$$C = \frac{\sigma_1 (2\pi L R_1)}{\left(\dfrac{\sigma_1 R_1}{\varepsilon_0} \ln\left(\dfrac{R_2}{R_1}\right)\right)}$$

$$C = \frac{2\pi L \varepsilon_0}{\ln\left(\dfrac{R_2}{R_1}\right)}$$

With the dielectric material

$$C = \kappa \frac{Q}{\Delta V}$$

$$C = \frac{2\pi L \kappa \varepsilon_0}{\ln\left(\dfrac{R_2}{R_1}\right)}$$