## Contents

I	Foundations			5
1	Line	ear Moo	dels	7
	1.1	Stocha	astic processes	7
	1.2	The co	ovariance world	9
		1.2.1	Second-order stationary processes	9
		1.2.2	Spectral representation	13
		1.2.3	Wold decomposition	17
	1.3	Linear	r processes	19
		1.3.1	What are linear Gaussian processes?	19
		1.3.2	ARMA models	20
			Prediction	23
		1.3.4	Estimation	25
			1.3.4.1 Time domain	25
			1.3.4.2 Frequency domain	27
	1.4	The m	nultivariate cases	29
		1.4.1	Time domain	29
		1.4.2	Frequency domain	31
	1.5	Nume	erical examples	32
	Exe	cises		33
2	Linear Gaussian State Space Models			37
	2.1		ng, smoothing, and forecasting	40
	2.2	Maxin	num likelihood estimation	46
		2.2.1	Newton-Raphson	46
		2.2.2	<u> </u>	47
	2.3		thing splines and the Kalman smoother	49
	2.4		ptotic distribution of the MLE	51
	2.5		ng data modifications	53
	2.6		ural component models	54
	2.7	State-s	space models with correlated errors	57
		2.7.1	ARMAX models	58
		2.7.2	Regression with autocorrelated errors	60
	Ever	rises		60

ii			CONTENTS
3	Bevo	ond Linear Models	65
	3.1	Nonlinear non-Gaussian data	66
	3.2	Volterra series expansion	72
	3.3	Cumulants and higher-order spectra	73
	3.4	Bilinear models	76
	3.5	Conditionally heteroscedastic models	77
	3.6	Threshold ARMA models	81
	3.7	Functional autoregressive models	82
	3.8	Linear processes with infinite variance	83
	3.9	Models for counts	85
		3.9.1 Integer valued models	85
		3.9.2 Generalized linear models	87
	3.10	Numerical examples	88
		rcises	93
4		chastic Recurrence Equations	95
	4.1	The Scalar Case	97
		4.1.1 Strict stationarity	97
		4.1.2 Weak stationarity	102
		4.1.3 GARCH(1, 1)	106
		4.1.3.1 Strict stationarity	106
		4.1.3.2 Weak stationarity	108
	4.2	The Vector Case	111
		4.2.1 Strict stationarity	113
		4.2.2 Weak stationarity	115
		4.2.3 $GARCH(p, q)$	118
	4.3	Iterated random function	122
		4.3.1 Strict stationarity	122
		4.3.2 Weak stationarity	125
	Exer	rcises	127
II	M	arkovian Models	137
5	Mar	rkov Models: Construction and Definitions	139
	5.1	Markov chains: Past, future, and forgetfulness	139
	5.2	Kernels	140
	5.3	Homogeneous Markov chain	142
	5.4	Canonical representation	144
	5.5	Invariant measures	145
	5.6	Observation-driven models	148
	5.7	Iterated random functions	149
	5.8	MCMC methods	158
	5.0	5.8.1 Metropolis-Hastings algorithm	158
		5.8.2 Gibbs sampling	161
		1 C	

CC	NTE	NTS	iii
	Exe	rcises	163
6	Stab	173	
	6.1	Uniform ergodicity	174
		6.1.1 Total variation distance	174
		6.1.2 Dobrushin coefficient	175
		6.1.3 The Doeblin condition	177
		6.1.4 Examples	178
	6.2	V-geometric ergodicity	181
		6.2.1 V-total variation distance	181
		6.2.2 V-Dobrushin coefficient	182
		6.2.3 Drift and minorization conditions	183
		6.2.4 Examples	189
	6.3	Some proofs	195
	6.4	Endnotes	197
	Exe	rcises	198
7	Sam	205	
	7.1	8	206
		7.1.1 Dynamical system and ergodicity	206
		7.1.2 Markov chain ergodicity	213
	7.2	Central limit theorem	221
	7.3	Deviation inequalities for additive functionals	228
		7.3.1 Rosenthal type Inequality	228
		7.3.2 Concentration inequality Some proofs	231
	7.4	235	
	Exe	241	
8		rence for Markovian Models	251
	8.1		251
	8.2	MLE: Consistency and asymptotic normality	256
		8.2.1 Consistency	256
		8.2.2 Asymptotic normality	258
	8.3	Observation-driven models	265
	8.4	Bayesian inference	274
	8.5	Some proofs	283
	8.6		286
	Exe	286	
II	I S	tate Space and Hidden Markov Models	297
9	Non	-Gaussian and Nonlinear State Space Models	299
	9.1	Definitions and basic properties	299
		9.1.1 Discrete-valued state space HMM	299
		9.1.2 Continuous-valued state-space models	307

iv			CONTENTS
		9.1.3 Conditionally linear Gaussian models	309
		9.1.4 Switching processes with Markov regimes	312
	9.2	Filtering and smoothing	314
		9.2.1 Discrete-valued state-space HMM	315
		9.2.2 Continuous-valued state-space HMM	322
	9.3	Endnotes	326
	Exer	rcises	327
10	Part	icle Filtering	335
		Importance sampling	335
		Sequential importance sampling	343
		Sampling importance resampling	348
		10.3.1 Algorithm description	349
		10.3.2 Resampling techniques	349
	10.4	Particle filter	351
		10.4.1 Sequential importance sampling	351
		10.4.2 Auxiliary sampling	353
	10.5	Long-term stability	355
		10.5.1 Deviation inequalities	355
		10.5.2 Long-term stability	357
	10.6	Endnotes	363
	Exer	cises	364
11	Part	icle Smoothing	375
	11.1	Poor man's smoother algorithm	376
	11.2	FFBSm algorithm	379
	11.3	FFBSi algorithm	381
	11.4	Convergence of the FFBSm and FFBSi algorithms	383
		11.4.1 Exponential deviation inequality	386
		11.4.2 Asymptotic normality	388
		Smoothing functionals	392
		Particle independent Metropolis-Hastings	394
	11.7	Particle Gibbs	400
		Endnotes	404
	Exer	rcises	407
12	Inference for Nonlinear State Space Models		417
	12.1	Monte Carlo maximum likelihood estimation	419
		12.1.1 Particle approximation of the likelihood function	419
		12.1.2 Particle stochastic gradient	422
		12.1.3 Particle Monte Carlo EM algorithms	424
		12.1.4 Particle stochastic approximation EM	428
	12.2	Bayesian analysis	430
		12.2.1 Gaussian linear state space models	431
		12.2.2 Gibbs sampling for NLSS model	435

CONTI	ENTS	V
	12.2.3 Particle marginal Markov chain Monte Carlo	440
	12.2.4 Particle Gibbs algorithm	443
12.3	3 Endnotes	445
Exe	rcises	447
13 Asy	mptotics of the MLE for NLSS	453
13.1	Strong consistency of the MLE	454
	13.1.1 Forgetting the initial distribution	454
	13.1.2 Approximation by conditional likelihood	456
	13.1.3 Strong consistency	457
	13.1.3.1 Misspecified models	457
	13.1.3.2 Identifiability and well specified models	458
	13.1.4 Identifiability of mixture densities	464
13.2	2 Asymptotic normality	466
	13.2.1 Convergence of the observed information	471
	13.2.2 Limit distribution of the MLE	473
	3 Endnotes	473
Exe	rcises	474
IV A	Appendices	481
Append	dix A Some Mathematical Background	483
	Some measure theory	483
	Some probability theory	485
Appen	lix B Martingales	489
B.1	Definitions and elementary properties	489
B.2	Limit Theorems	491
	dix C Stochastic Approximation	497
	Robbins-Monro algorithm: Elementary results	498
	Stochastic gradient	501
	Stepsize selection and averaging	502
C.4	The Kiefer-Wolfowitz procedure	502
Appen	dix D Data Augmentation	505
D.1	8	505
D.2		508
D.3	E	509
	D.3.1 Stochastic approximation EM	510
D.4		511
D.5	$\varepsilon$	513
	D.5.1 Convergence of perturbed dynamical systems	514
	D.5.2 Convergence of the MCEM algorithm	516