Information and coding theory assignment

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1 Known equations

Define a probability mass function

$$\mathcal{P}(X = x, Y = y, Z = z)$$

Where

$$x \in \mathcal{X} = \{0, 1\}, y \in \mathcal{Y} = \{0, 1\}, z \in \mathcal{Z} = \{0, 1\}$$

by

$$\mathcal{P}(X = 0, Y = 0, Z = 0) = \mathcal{P}(X = 1, Y = 0, Z = 1) = 1/4$$
$$\mathcal{P}(X = 0, Y = 1, Z = 1) = \mathcal{P}(X = 1, Y = 1, Z = 0) = 1/4$$
$$\mathcal{P}(X = 0, Y = 1, Z = 0) = \mathcal{P}(X = 0, Y = 0, Z = 1) = 0$$
$$\mathcal{P}(X = 1, Y = 0, Z = 0) = \mathcal{P}(X = 1, Y = 1, Z = 1) = 0$$

then

$$\mathcal{P}(X = 0, Z = 0) = \mathcal{P}(X = 0, Y = 0, Z = 0) + \mathcal{P}(X = 0, Y = 1, Z = 0) = 1/4$$
$$\mathcal{P}(X = 0, Z = 1) = \mathcal{P}(X = 0, Y = 0, Z = 1) + \mathcal{P}(X = 0, Y = 1, Z = 1) = 1/4$$
$$\mathcal{P}(X = 1, Z = 0) = \mathcal{P}(X = 1, Y = 0, Z = 0) + \mathcal{P}(X = 1, Y = 1, Z = 0) = 1/4$$
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and

$$\mathcal{P}(X = 0, Y = 0) = \mathcal{P}(X = 0, Y = 0, Z = 0) + \mathcal{P}(X = 0, Y = 0, Z = 1) = 1/4$$
$$\mathcal{P}(X = 0, Y = 1) = \mathcal{P}(X = 0, Y = 1, Z = 0) + \mathcal{P}(X = 0, Y = 1, Z = 1) = 1/4$$
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$$\mathcal{P}(X = 1, Y = 1) = \mathcal{P}(X = 1, Y = 1, Z = 0) + \mathcal{P}(X = 1, Y = 1, Z = 1) = 1/4$$

and

$$\mathcal{P}(Y = 0, Z = 0) = \mathcal{P}(X = 0, Y = 0, Z = 0) + \mathcal{P}(X = 1, Y = 0, Z = 0) = 1/4$$

$$\mathcal{P}(Y=0,Z=1)=\mathcal{P}(X=0,Y=0,Z=1)+\mathcal{P}(X=1,Y=0,Z=1)=1/4$$
$$\mathcal{P}(Y=1,Z=0)=\mathcal{P}(X=0,Y=1,Z=0)+\mathcal{P}(X=1,Y=1,Z=0)=1/4$$
$$\mathcal{P}(Y=1,Z=1)=\mathcal{P}(X=0,Y=1,Z=1)+\mathcal{P}(X=1,Y=1,Z=1)=1/4$$

we can also find

$$\mathcal{P}(X=0)=\mathcal{P}(X=0,Z=0)+\mathcal{P}(X=0,Z=1)=1/2$$
$$\mathcal{P}(X=1)=\mathcal{P}(X=1,Y=0)+\mathcal{P}(X=1,Y=1)=1/2$$
$$\mathcal{P}(Y=0)=\mathcal{P}(Y=0,Z=0)+\mathcal{P}(Y=0,Z=1)=1/2$$
$$\mathcal{P}(Y=1)=\mathcal{P}(Y=1,Z=0)+\mathcal{P}(Y=1,Z=1)=1/2$$
$$\mathcal{P}(Z=0)=\mathcal{P}(X=0,Z=0)+\mathcal{P}(X=1,Z=0)=1/2$$
$$\mathcal{P}(Z=1)=\mathcal{P}(X=0,Z=1)+\mathcal{P}(X=1,Z=1)=1/2$$

2 Mutual independence

Thanks to the previous equations, we can find that

$$\mathcal{P}(X=0,Y=0,Z=0)=1/4$$
$$\mathcal{P}(X=0)\mathcal{P}(Y=0)\mathcal{P}(Z=0)=(1/2)^3=1/8$$

Thus

$$\mathcal{P}(X=0,Y=0,Z=0)\neq\mathcal{P}(X=0)\mathcal{P}(Y=0)\mathcal{P}(Z=0)$$

X,Y and Z are not mutually independent

3 Pairwise independence

Thanks to the previous equations we can find that

$$\mathcal{P}(X=0,Y=0)=\mathcal{P}(X=0)\mathcal{P}(Y=0)=1/4$$
$$\mathcal{P}(X=0,Y=1)=\mathcal{P}(X=0)\mathcal{P}(Y=1)=1/4$$
$$\mathcal{P}(X=1,Y=0)=\mathcal{P}(X=1)\mathcal{P}(Y=0)=1/4$$
$$\mathcal{P}(X=1,Y=1)=\mathcal{P}(X=1)\mathcal{P}(Y=1)=1/4$$

$$\mathcal{P}(X=0,Z=0)=\mathcal{P}(X=0)\mathcal{P}(Z=0)=1/4$$
$$\mathcal{P}(X=0,Z=1)=\mathcal{P}(X=0)\mathcal{P}(Z=1)=1/4$$
$$\mathcal{P}(X=1,Z=0)=\mathcal{P}(X=1)\mathcal{P}(Z=0)=1/4$$
$$\mathcal{P}(X=1,Z=1)=\mathcal{P}(X=1)\mathcal{P}(Z=1)=1/4$$

$$\mathcal{P}(Y=0,Z=0)=\mathcal{P}(Y=0)\mathcal{P}(Z=0)=1/4$$

$$\mathcal{P}(Y=0,Z=1)=\mathcal{P}(Y=0)\mathcal{P}(Z=1)=1/4$$

$$\mathcal{P}(Y=1,Z=0)=\mathcal{P}(Y=1)\mathcal{P}(Z=0)=1/4$$

$$\mathcal{P}(Y=1,Z=1)=\mathcal{P}(Y=1)\mathcal{P}(Z=1)=1/4$$

Thus
X,Y and Z are pairwise independent.

4 Conditional independence

For any random variables X1,X2 and X3 X1 is independent of X2 conditioning on X3 if and only if

$$\mathcal{P}(x1,x2,x3)\mathcal{P}(x2)=\mathcal{P}(x1,x2)\mathcal{P}(x2,x3)$$

In our case, for any correspondence between X1, X2, X3 and X,Y, Z $\mathcal{P}(x1,x2)\mathcal{P}(x2,x3)$ will always be equal to 1/8 On the other hand, $\mathcal{P}(x1,x2,x3)\mathcal{P}(x2)$ will be equal to 0 for some value of x,y and z, for example when X=Y=Z=1. Thus, for any possible permutation of X,Y,and Z as X1, X2, and X3, none of them is conditionally independent.

5 Equivalent definitions of Conditional independence

Let's prove that the jumping form and shortened form are equivalent to the symetric form.

5.1 Jumping form

for $\mathcal{P}(x)>0,\mathcal{P}(y)>0$

$$\mathcal{P}(x,y,z)=\mathcal{P}(x)\mathcal{P}(y|x)\mathcal{P}(z|y)$$

$$\Leftrightarrow$$

$\mathcal{P}(x,y,z)=\mathcal{P}(x)\frac{\mathcal{P}(x,y)}{\mathcal{P}(x)}\frac{\mathcal{P}(y,z)}{\mathcal{P}(y)}$ using the formulae $\mathcal{P}(z|x,y)=\frac{\mathcal{P}(x,y,z)}{\mathcal{P}(x,y)}$

$$\Leftrightarrow$$

$$\mathcal{P}(x,y,z)\mathcal{P}(y)=\mathcal{P}(x,y)\mathcal{P}(y,z)$$

QED

5.2 Shortened form

for $\mathcal{P}(x,y) > 0$

$$\mathcal{P}(z|x,y) = \mathcal{P}(z|y)$$

$$\Leftrightarrow$$

$\frac{\mathcal{P}(x,y,z)}{\mathcal{P}(x,y)} = \frac{\mathcal{P}(y,z)}{\mathcal{P}(y)}$ using the formulae $\mathcal{P}(z|x,y) = \frac{\mathcal{P}(x,y,z)}{\mathcal{P}(x,y)}$

$$\Leftrightarrow$$

$$\mathcal{P}(x,y,z)\mathcal{P}(y) = \mathcal{P}(x,y)\mathcal{P}(y,z)$$

QED