PHY2018, Problem Set 1.

Q1 is due 11am, Monday 30 September.
Q2 is due 1pm, Thursday 3 October.

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1. **Thermodynamics of a non-interacting Bose gas.** In Lecture 3, I proposed a "hack" for how to calculate the chemical potential $\mu = k_B T \ln z$ of an ideal Bose gas. The chemical potential is treated differently above and below the critical temperature for BEC: Above T_c, we ignored the ground-state population, found z by solving $\bar{N}_{\text{ex}}(z, T) = \bar{N}$, using the continuum limit expression, $\bar{N}_{\text{ex}} = C\beta^{-\alpha}\Gamma(\alpha)g_\alpha(z)$. Below T_c, we set $z = 1$, and found the condensed atom number with $\bar{n}_0 = \bar{N} - \bar{N}_{\text{ex}}(1, T)$. In this problem, I ask you to use numerical methods to solve the full $\bar{N} = \bar{n}_0 + \bar{N}_{\text{ex}}$ equation at all temperatures.

(i) First, recreate the "hack" solution, and make a plot for μ versus T. You should be able to recreate something like Fig. 3 in the posted typeset notes "Bose and Fermi distributions." But instead of E_F, use $k_B T_c$ as the energy scale to make your axes dimensionless. **Plot $\mu/k_B T_c$ for the range $T/T_c = 0 \to 2$.**

(ii) Next, find the perturbative correction below T_c in the following way: Solve for z using $\bar{n}_0 = z/(1-z)$ using the condensate number found from (i), which is $n_0 = \bar{N} - \bar{N}_{\text{ex}}(z = 0)$. **Plot** $\mu/k_B T_c$ found in this way, for the range $T = 0$ to $T = 0.99T_c$. **Does μ diverge as $T \to T_c$? Why or why not?** { *You will need to choose a total atom number for this part and subsequent ones: I suggest* 10^2, *since these corrections vanish as* $N \to \infty$. }

(iii) Now solve the self-consistent equation $\bar{N} = \bar{n}_0(z) + \bar{N}_{\text{ex}}(z, T)$. Make a **plot for $\mu/k_B T_c$ that compares three solutions (i), (ii), and (iii),** for the range $T/T_c = 0 \to 0.99$. *Hint: Since numerical methods usually require an initial guess to solve nonlinear equations, you can use part (ii) as an initial guess for part (iii).*

(iv) Above T_c, a perturbative correction to μ is to solve $\bar{N} = \bar{n}_0(z_{\text{part(i)}}) + \bar{N}_{\text{ex}}(z, T)$. The full solution is to let z vary in both terms, $\bar{N} = \bar{n}_0(z) + \bar{N}_{\text{ex}}(z, T)$. Make a **plot for $\mu/k_B T_c$ that compares three solutions,** for the range $T/T_c = 1.01 \to 1.30$.

(v) Finally, let's not consider **what happens at $T = T_c$**, variable total atom number $\bar{N}$. Here, both of the perturbative methods considered above fail, so we have to use the self-consistent method. **Plot fugacity z versus** $\log_{10}(N)$, for $N = 10^2 \to 10^4$. Notice that as $N \to \infty$, the approximation $z \approx 1$ gets better. In retrospect, we could have called part (i) the "thermodynamic limit" solution.

(vi) I notice that $z = 1 - 2N^{-1/3}/\ln N$ seems to be a good approximation to $z(N)$ at T_c. Add this to your plot as well. Try to figure out, analytically, **why this works so well.**

2. **Gaussian Ansatz.** Consider the GP energy functional for N particles of mass m and scattering length a:

$$E(\Phi) = \int d^3\mathbf{r} \left[\frac{\hbar^2}{2m} |\nabla\Phi|^2 + U(\mathbf{r})|\Phi|^2 + \frac{g}{2}|\Phi|^4 \right].$$

For a simple harmonic oscillator potential $U = \frac{1}{2}m\omega^2 r^2$, let's assume that the ground state of the GP equation is a Gaussian, $\Phi(\mathbf{r}) = \sqrt{N}\phi(\mathbf{r})$, where

$$\phi = \frac{1}{\pi^{3/4}\sigma^{3/2}} e^{-r^2/2\sigma^2}.$$

{ *This problem is considered in §6.2.1 of your textbook. Please read that section before working the problem, so that you understand the significance of the work.* }

(i) Find the energy per particle E/N as a function of σ, ω, N, m, and fundamental constants.

(ii) The energy can be written in a nondimensional form

$$\epsilon = As^{-2} + Bs^2 + \frac{\chi}{2}s^{-3},$$

where $\epsilon = E/N\hbar\omega$, $s = \sigma/\ell_{ho}$, and $\ell_{ho} = \sqrt{\hbar/m\omega}$. Give the values of A, B, and χ.

(iii) Find an expression for the s which minimizes the energy.

(iv) Solve for s in the limit of large Na. Compare the scaling to the Thomas-Fermi radius for a BEC in the same limit,

$$R_{\rm TF} = \ell_{\rm ho} \left(\frac{15Na}{\ell_{\rm ho}} \right)^{1/5}$$
