Hypersonic Vehicle Simulation

Randal W. Beard *†

November 15, 2023

Abstract

Cool stuff in this paper.

1 Introduction

2 Animation

3 Dynamics

The dynamics are adapted from the equations given in [1].

Let $\mathbf{p}_m^i \in \mathbb{R}^3$ be the position of the missile expressed in the inertial North-East-Down (NED) frame, and $\mathbf{p}_t^i \in \mathbb{R}^3$ be the inertial position of the target in the NED frame. Define the body frame of the hypersonic vehicle as the $\mathbf{x}_b$-axis pointing out the nose of the vehicle, the $\mathbf{z}_b$-axis pointing toward the belly of the hypersonic vehicle, and the $\mathbf{y}_b$-axis such that $(\mathbf{x}_b, \mathbf{y}_b, \mathbf{z}_b)$ form a right handed coordinate system. Let $R_b^i \in SO(3)$ represent the attitude of the body relative to the inertial frame, i.e.,

$$R_b^i = \begin{pmatrix} \mathbf{x}_b^i & \mathbf{y}_b^i & \mathbf{z}_b^i \end{pmatrix}.$$

Let $\mathbf{v}_m^b \in \mathbb{R}^3$ be the velocity of the missile expressed in the body frame, and let $\boldsymbol{\omega}_m^b \mathbb{R}^3$ be the angular velocity of the missile expressed in the body frame. Let $\boldsymbol{\delta} \in \mathbb{R}^3$ be the three rudder flaps that actuate the body frame along the x, y, and z axes.

Let $\mathbf{J} = \text{diag}([J_x, J_y, J_z])$ be the inertia matrix of the missile, where $\text{diag}(\mathbf{u})$ is the diagonal matrix with diagonal elements given by $\mathbf{u}$, then the missile

*This work has been funded by....

†R. W. Beard is with the Electrical and Computer Engineering Department at Brigham Young University, Provo, UT, 84602 USA. (e-mail: beard@byu.edu)

dynamics are given by

$$
\begin{aligned}
\dot{\mathbf{p}}^i_m &= R^i_b \mathbf{v}^b_m \\
\dot{\mathbf{v}}^b_m &= \mathbf{v}^b_m \times \boldsymbol{\omega}^b_m + \frac{1}{m}\mathbf{f} \\
\dot{R}^i_b &= R^i_b (\boldsymbol{\omega}^b_m)^\wedge \\
\dot{\boldsymbol{\omega}}^b_m &= \mathbf{J}^{-1}\left(-\boldsymbol{\omega}^b_m \times \mathbf{J}\boldsymbol{\omega}^b_m + \mathbf{M}\right)
\end{aligned}
$$

where

$$
\begin{pmatrix} a \\ b \\ c \end{pmatrix}^\wedge \triangleq \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}.
$$

Assuming the absence of wind, then $V_a = \left\|\mathbf{v}^b_m\right\|$ is the airspeed and the angle-of-attack α and the side-slip β are given by

$$
\alpha = \text{atan2}(\mathbf{v}^b_m \cdot \mathbf{e}_3, \mathbf{v}^b_m \cdot \mathbf{e}_1) \tag{1}
$$

$$
\beta = \sin^{-1}\left(\frac{\mathbf{v}^b_m \cdot \mathbf{e}_2}{V_a}\right), \tag{2}
$$

where

$$
\mathbf{e}_1 = (1,0,0)^\top, \qquad \mathbf{e}_2 = (0,1,0)^\top, \qquad \mathbf{e}_3 = (0,0,1)^\top.
$$

The force and torque models are given by

$$
\begin{aligned}
\mathbf{f} &= \frac{1}{2}\rho V_a^2 S \left(\mathbf{f}_\alpha \alpha + \mathbf{f}_\beta \beta\right) \\
\mathbf{M} &= \frac{1}{2}\rho V_a^2 S L \left(\mathbf{m}_\alpha \alpha + \mathbf{m}_\beta \beta + \text{diag}(\mathbf{m}_\delta)\boldsymbol{\delta}\right),
\end{aligned}
$$

where the rudder dynamics are given by

$$
\dot{\boldsymbol{\delta}} = \text{diag}(\boldsymbol{\tau})^{-1}(\boldsymbol{\delta}^c - \boldsymbol{\delta}),
$$

with $\boldsymbol{\delta}^c$ being the commanded rudder angles, and where

$$
\begin{aligned}
\mathbf{f}_\alpha &= \begin{pmatrix} 0, & 0, & c_z^\alpha \end{pmatrix}^\top \\
\mathbf{f}_\beta &= \begin{pmatrix} 0, & c_y^\beta, & 0 \end{pmatrix}^\top \\
\mathbf{m}_\alpha &= \begin{pmatrix} m_x^\alpha, & m_y^\alpha, & 0 \end{pmatrix}^\top \\
\mathbf{m}_\beta &= \begin{pmatrix} m_x^\beta, & 0, & m_z^\beta \end{pmatrix}^\top \\
\mathbf{m}_\delta &= \begin{pmatrix} m_x^\delta, & m_y^\delta, & m_z^\delta \end{pmatrix}^\top \\
\boldsymbol{\tau} &= \begin{pmatrix} \tau_x, & \tau_y, & \tau_z \end{pmatrix}^\top.
\end{aligned}
$$

The parameters listed in [1] are given in Table 1.

We will assume that the initial speed is $V_{a0} = 2000\ m/s$.

Table 1: Parameters for hypersonic vehicle. It seems like this last value should be negative

Parameter	Value	Parameter	Value
m	$1200\ kg$	c_z^α	-57.15
J_x	$100\ kg \cdot m^2$	c_y^β	-56.32
J_y	$5700\ kg \cdot m^2$	m_x^α	0.45
J_z	$5800\ kg \cdot m^2$	m_y^α	-28.15
ρ	$1.1558\ kg/m^3$	m_x^β	-0.38
S	$0.43\ m^2$	m_z^β	$\cancel{-}+27.3$
L	$0.6\ m$	m_x^δ	2.13
$\tau_x,\ \tau_y,\ \tau_z$	$0.1\ s$	m_z^δ	-26.6
		m_y^δ	-27.9
		c_z^β	0.081
		$c_z^{\delta_y}$	-5.75
		c_y^α	0.091
		$c_y^{\delta_z}$	5.6

4 Engagement Geometry

The engagement geometry is shown in Figure 4. If the position of the target is given by $\mathbf{p}_t$, then the line-of-sight vector is given by

$$\boldsymbol{\ell} \triangleq \mathbf{p}_t - \mathbf{p}_m.$$

We assume that the line-of-sight vector is measured in the body frame, and then internally resolved in the inertial frame. The range to the target is given by

$$r = \|\boldsymbol{\ell}\|,$$

and, as shown in Figure 4, the elevation and azimuth angles of $\boldsymbol{\ell}$ are given respectively by

$$q_1 = -\sin^{-1}\left(\frac{\boldsymbol{\ell}_z^i}{\sqrt{\boldsymbol{\ell}_x^{i2} + \boldsymbol{\ell}_y^{i2}}}\right)$$
$$q_2 = \operatorname{atan2}(\ell_y^i, \boldsymbol{\ell}_x^i).$$

The rotation angle from the line-of-sight (los) frame to the inertial frame is

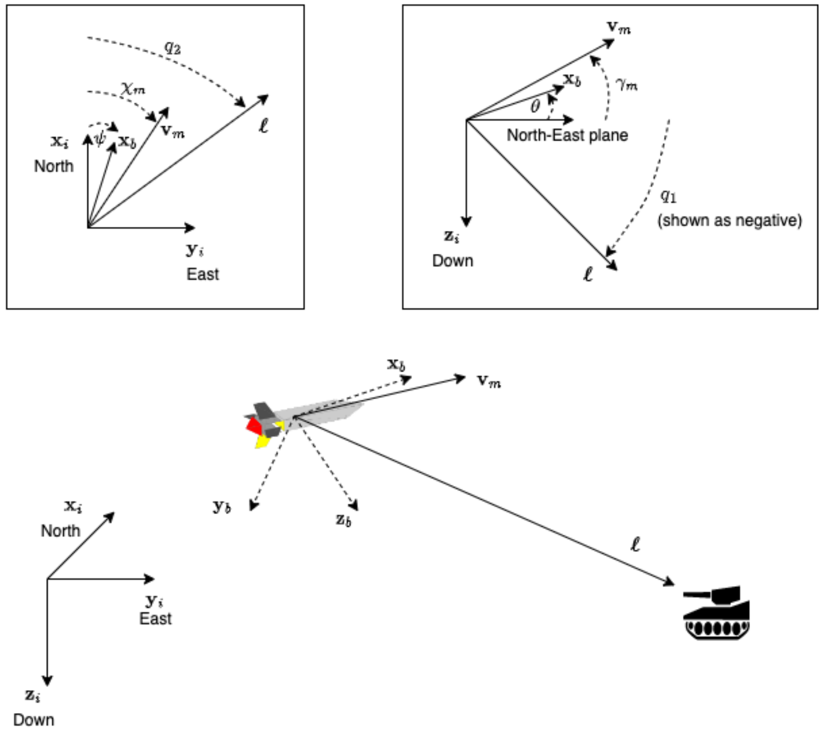

Figure 1: The engagement geometry. $\{\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i\}$ is the inertial frame, $\{\mathbf{x}_b, \mathbf{y}_b, \mathbf{z}_b\}$ is the body frame with yaw angle ψ and pitch angle θ. The velocity of the hypersonic vehicle is denoted $\mathbf{v}_M$ with azimuth ψ_M and elevation θ_M angles measured relative to the inertial frame. The line-of-sight vector from the hypersonic vehicle to the target is denoted $\boldsymbol{\ell}$ with azimuth and elevation angle denoted as q_2 and q_1 respectively.

given by

$$
\begin{aligned}
R^i_\ell &= R_z(q_2)R_y(q_1) \\
&= \begin{pmatrix} \cos q_2 & -\sin q_2 & 0 \\ \sin q_2 & \cos q_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos q_1 & 0 & \sin q_1 \\ 0 & 1 & 0 \\ -\sin q_1 & 0 & \cos q_1 \end{pmatrix} \\
&= \begin{pmatrix} \cos q_1 \cos q_2 & -\sin q_2 & \sin q_1 \cos q_2 \\ \cos q_1 \sin q_2 & \cos q_2 & \sin q_1 \sin q_2 \\ -\sin q_1 & 0 & \cos q_1 \end{pmatrix}
\end{aligned}
$$

Therefore, the light-of-sight vector in the inertial frame is given by

$$
\begin{aligned}
\boldsymbol{\ell}^i &= \begin{pmatrix} \cos q_1 \cos q_2 & -\sin q_2 & \sin q_1 \cos q_2 \\ \cos q_1 \sin q_2 & \cos q_2 & \sin q_1 \sin q_2 \\ -\sin q_1 & 0 & \cos q_1 \end{pmatrix} \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} r \cos q_1 \cos q_2 \\ r \cos q_1 \sin q_2 \\ -r \sin q_1 \end{pmatrix}.
\end{aligned}
$$

Given that

$$\dot{R}^i_\ell = R^i_\ell \boldsymbol{\omega}^{\ell\wedge}_{\ell/i},$$

we get that the angular velocity of line-of-sight vector in the line-of-sight frame is

$$
\begin{aligned}
\boldsymbol{\omega}^\ell_{\ell/i} &= (R^{i\top}_\ell \dot{R}^i_\ell)^\vee \\
&= \left(\dot{q}_1 R_y^\top R_z^\top R_z R_y^{'} + \dot{q}_2 R_y^\top R_z^\top R_z^{'} R_y\right)^\vee \\
&= \left(\dot{q}_1 R_y^\top R_y^{'} + \dot{q}_2 R_y^\top R_z^\top R_z^{'} R_y\right)^\vee \\
&= \left(\dot{q}_1 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + \dot{q}_2 R_y^\top \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} R_y \right)^\vee \\
&= \begin{pmatrix} 0 & -\dot{q}_2 \cos q_1 & \dot{q}_1 \\ \dot{q}_2 \cos q_1 & 0 & \dot{q}_2 \sin q_1 \\ -\dot{q}_1 & -\dot{q}_2 \sin q_1 & 0 \end{pmatrix}^\vee \\
\implies \boldsymbol{\omega}^\ell_{\ell/i} &= \begin{pmatrix} -\dot{q}_2 \sin q_1 \\ \dot{q}_1 \\ \dot{q}_2 \cos q_1 \end{pmatrix}.
\end{aligned}
$$

The velocity of the line-of-sight vector expressed in the line-of-sight frame is therefore

$$
\begin{aligned}
\mathbf{v}^\ell_{\ell/i} &= \frac{d}{dt}\boldsymbol{\ell}^\ell + \boldsymbol{\omega}^\ell_{\ell/i} \times \boldsymbol{\ell}^\ell \\
&= \begin{pmatrix} \dot{r} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & -\dot{q}_2 \cos q_1 & \dot{q}_1 \\ \dot{q}_2 \cos q_1 & 0 & \dot{q}_2 \sin q_1 \\ -\dot{q}_1 & -\dot{q}_2 \sin q_1 & 0 \end{pmatrix} \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \dot{r} \\ r\dot{q}_2 \cos q_1 \\ -r\dot{q}_1 \end{pmatrix}.
\end{aligned}
$$

Similarly, the acceleration of the line-of-sight vector expressed in the line-of-sight frame is

$$
\begin{aligned}
\mathbf{a}^{\ell}_{\ell/i} &= \frac{d}{dt}\mathbf{v}^{\ell}_{\ell/i} + \boldsymbol{\omega}^{\ell}_{\ell/i} \times \mathbf{v}^{\ell}_{\ell/i} \\
&= \begin{pmatrix} \ddot{r} \\ \dot{r}\dot{q}_2\cos q_1 + r\ddot{q}_2\cos q_1 - r\dot{q}_2\dot{q}_1\sin q_1 \\ -\dot{r}\dot{q}_1 - r\ddot{q}_1 \end{pmatrix} + \begin{pmatrix} 0 & -\dot{q}_2\cos q_1 & \dot{q}_1 \\ \dot{q}_2\cos q_1 & 0 & \dot{q}_2\sin q_1 \\ -\dot{q}_1 & -\dot{q}_2\sin q_1 & 0 \end{pmatrix} \begin{pmatrix} \dot{r} \\ r\dot{q}_2\cos q_1 \\ -r\dot{q}_1 \end{pmatrix} \\
&= \begin{pmatrix} \ddot{r} - r\dot{q}_2^2\cos^2 q_1 - r\dot{q}_1^2 \\ r\ddot{q}_2\cos q_1 + 2\dot{r}\dot{q}_2\cos q_1 - 2r\dot{q}_1\dot{q}_2\sin q_1 \\ -r\ddot{q}_1 - 2\dot{r}\dot{q}_1 - r\dot{q}_2^2\cos q_1\sin q_1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & r\cos q_1 \\ 0 & -r & 0 \end{pmatrix} \begin{pmatrix} \ddot{r} \\ \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} -r\dot{q}_2^2\cos^2 q_1 - r\dot{q}_1^2 \\ 2\dot{r}\dot{q}_2\cos q_1 - 2r\dot{q}_1\dot{q}_2\sin q_1 \\ -2\dot{r}\dot{q}_1 - r\dot{q}_2^2\cos q_1\sin q_1 \end{pmatrix}
\end{aligned}
$$

We also know that the acceleration of the line-of-sight vector is the acceleration of the target minus the acceleration of the hypersonic vehicle, therefore

$$
\mathbf{a}^{\ell}_{\ell/i} = \mathbf{a}^{\ell}_{t/i} - \mathbf{a}^{\ell}_{m/i},
$$

which implies that

$$
\begin{aligned}
\begin{pmatrix} \ddot{r} \\ \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & r\cos q_1 \\ 0 & -r & 0 \end{pmatrix}^{-1} \left(- \begin{pmatrix} -r\dot{q}_2^2\cos^2 q_1 - r\dot{q}_1^2 \\ 2\dot{r}\dot{q}_2\cos q_1 - 2r\dot{q}_1\dot{q}_2\sin q_1 \\ -2\dot{r}\dot{q}_1 - r\dot{q}_2^2\cos q_1\sin q_1 \end{pmatrix} + \mathbf{a}^{\ell}_{t/i} - \mathbf{a}^{\ell}_{m/i} \right) \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{-1}{r} \\ 0 & \frac{1}{r\cos q_1} & 0 \end{pmatrix} \left(- \begin{pmatrix} -r\dot{q}_2^2\cos^2 q_1 - r\dot{q}_1^2 \\ 2\dot{r}\dot{q}_2\cos q_1 - 2r\dot{q}_1\dot{q}_2\sin q_1 \\ -2\dot{r}\dot{q}_1 - r\dot{q}_2^2\cos q_1\sin q_1 \end{pmatrix} + \mathbf{a}^{\ell}_{t/i} - \mathbf{a}^{\ell}_{m/i} \right) \\
&= \begin{pmatrix} r\dot{q}_2^2\cos^2 q_1 + r\dot{q}_1^2 \\ -2\frac{\dot{r}}{r}\dot{q}_1 - \dot{q}_2^2\cos q_1\sin q_1 \\ -2\frac{\dot{r}}{r}\dot{q}_2 + 2\dot{q}_1\dot{q}_2\tan q_1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{-1}{r} \\ 0 & \frac{1}{r\cos q_1} & 0 \end{pmatrix} \left(\mathbf{a}^{\ell}_{t/i} - \mathbf{a}^{\ell}_{m/i} \right)
\end{aligned}
$$

We can also write a force balance equation for the hypersonic vehicle to describe how the force impact the inertial flight path and course angles. As shown in Figure 4, the velocity vector of the hypersonic vehicle expressed in the inertial frame is given by $\mathbf{v}^i_m$, which implies that that flight path angle and course angle are given by

$$
\begin{aligned}
\gamma_m &= -\sin^{-1}\left(\frac{\mathbf{v}^i_{mz}}{\sqrt{\mathbf{v}^{i2}_{mx} + \mathbf{v}^{i2}_{my}}} \right) \\
\chi_m &= \operatorname{atan2}(\mathbf{v}^i_{my}, \mathbf{v}^i_{mx}).
\end{aligned}
$$

If $V_m = \left\| \mathbf{v}^i_{m/i} \right\|$ is the magnitude of the velocity, then we can write

$$
\mathbf{v}^i_{m/i} = \begin{pmatrix} V_m\cos\chi_m\cos\gamma_m \\ V_m\sin\chi_m\cos\gamma_m \\ -V_m\sin\gamma_m \end{pmatrix}.
$$

which implies that

$$
\begin{aligned}
\mathbf{v}_{m/i}^{\ell} &= R_i^{\ell}\mathbf{v}_{m/i}^{i} \\
&= \begin{pmatrix} \cos q_1 \cos q_2 & \cos q_1 \sin q_2 & -\sin q_1 \\ -\sin q_2 & \cos q_2 & 0 \\ \sin q_1 \cos q_2 & \sin q_1 \sin q_2 & \cos q_1 \end{pmatrix} \begin{pmatrix} V_m \cos\chi_m \cos\gamma_m \\ V_m \sin\chi_m \cos\gamma_m \\ -V_m \sin\gamma_m \end{pmatrix} \\
&= V_m \begin{pmatrix} \cos q_1 \cos\gamma_m \cos(q_2 - \chi_m) + \sin q_1 \sin\gamma_m \\ -\cos\gamma_m \sin(q_2 - \chi_m) \\ \sin q_1 \cos\gamma_m \cos(q_2 - \chi_m) - \cos q_1 \sin\gamma_m \end{pmatrix}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\mathbf{a}_{m/i}^{\ell} &= \frac{d}{dt}\mathbf{v}_{m/i}^{\ell} + \boldsymbol{\omega}_{\ell/i}^{\ell\wedge}\mathbf{v}_{m/i}^{\ell} \\
&= \frac{d}{dt}R_{\ell}^{i\top}\mathbf{v}_{m/i}^{i} + R_{\ell}^{i\top}\dot{R}_{\ell}^{i}R_{\ell}^{i\top}\mathbf{v}_{m/i}^{i} \\
&= -R_{\ell}^{i\top}\dot{R}_{\ell}^{i}R_{\ell}^{i\top}\mathbf{v}_{m/i}^{i} + R_{\ell}^{i\top}\dot{\mathbf{v}}_{m/i}^{i} + R_{\ell}^{i\top}\dot{R}_{\ell}^{i}R_{\ell}^{i\top}\mathbf{v}_{m/i}^{i} \\
&= R_{\ell}^{i\top}\dot{\mathbf{v}}_{m/i}^{i} \\
&= V_m R_{\ell}^{i\top} \begin{pmatrix} -\sin\chi_m \cos\gamma_m & -\cos\chi_m \sin\gamma_m \\ \cos\chi_m \cos\gamma_m & -\sin\chi_m \sin\gamma_m \\ 0 & -\cos\gamma_m \end{pmatrix} \begin{pmatrix} \dot{\chi}_m \\ \dot{\gamma}_m \end{pmatrix},
\end{aligned}
$$

where we have assumed that $\dot{V}_m = 0$.

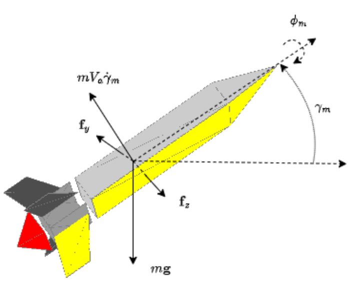

Figure 2: The longitudinal forces acting on the hypersonic vehicle when the flight path angle γ_m is changing.

The longitudinal forces acting on the hypersonic vehicle are shown in Figure 4. Summing the forces along the longitudinal axis gives

$$
mV_m\dot{\gamma}_m = mg\cos\gamma + f_z\cos\phi + f_y\sin\phi,
$$

from which we obtain

$$
\dot{\gamma}_m = \frac{g}{V_m}\cos\gamma + \frac{f_z\cos\phi + f_y\sin\phi}{mV_m}.
$$

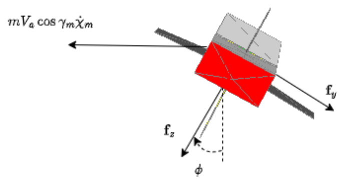

Figure 3: The lateral forces acting on the hypersonic vehicle when the course angle χ_m is changing.

The lateral forces acting on the hypersonic vehicle are shown in Figure 4. Summing the forces along the lateral axis gives

$$mV_m \cos\gamma_m \dot{\chi}_m = f_z \sin\phi + f_y \cos\phi,$$

from which we obtain

$$\dot{\chi}_m = \frac{f_z \sin\phi + f_y \cos\phi}{mV_m \cos\gamma_m}.$$

If we employ a roll stabilization loop that ensures that the roll angle $\phi = 0$, then

$$\begin{pmatrix} \dot{\chi}_m \\ \dot{\gamma}_m \end{pmatrix} = \frac{1}{V_m} \begin{pmatrix} \frac{f_y}{m\cos\gamma_m} \\ g\cos\gamma_m + \frac{f_z}{m} \end{pmatrix}.$$

Therefore

References

[1] Z. Li, X. Zhang, H. Zhang, and F. Zhang, "Three-dimensional cooperative integrated guidance and control with fixed-impact time and azimuth constraints," *Aerospace Science and Technology*, vol. 142, no. A, pp. 1–27, November 2023.