RICE UNIVERSITY

STAT 315

Probability and Statistics for Data Science

Homework 5 Solutions

1. **Given MGF, find μ, σ^2 (2.3-11)**
We have
$$M(t) = \frac{2}{5}e^t + \frac{1}{5}e^{2t} + \frac{2}{5}e^{3t} \ .$$
Now
$$M'(t) = \frac{2}{5}e^t + \frac{2}{5}e^{2t} + \frac{6}{5}e^{3t}$$
and
$$M''(t) = \frac{2}{5}e^t + \frac{4}{5}e^{2t} + \frac{18}{5}e^{3t}$$
Then
$$\mu = E(X) = M'(0) = \frac{2}{5} + \frac{2}{5} + \frac{6}{5} = 2$$
and
$$E(X^2) = M''(0) = \frac{2}{5} + \frac{4}{5} + \frac{18}{5} = 4.8$$
which means
$$\sigma^2 = E(X^2) - [E(X)]^2 = M''(0) - M'(0) = 0.8 \ .$$
Now, to determine the PMF, recall the definition of the MGF:
$$M(t) = E[e^{tx}] = \sum_{x \in S} e^{tx} f(x) \ .$$
From this, we can see
$$f(x) = \begin{cases} 2/5, & x = 1, 3, \\ 1/5, & x = 2 \end{cases}$$
which we can state more succinctly as
$$f(x) = \frac{|x-2|+1}{5}, \quad x = 1, 2, 3$$

2. **Must choice test and geometric distribution (2.3-13)**
The probability of getting a question correct on a random guess is $\frac{1}{5}$. Let X be the random variable corresponding to a correct guess on the kth question. Then $X \sim \text{Geom}(p = \frac{1}{5})$. So we have the PMF
$$f(k) = Pr(X = k) = (1-p)^{k-1}p = \left(\frac{4}{5}\right)^{k-1} \frac{1}{5}, \quad k = 1, 2, \ldots$$

Then

$$Pr(X = 4) = \left(\frac{4}{5}\right)^3 \frac{1}{5} = 0.1024 \ .$$

3. **Given MGF, find μ, σ, PDF, find a probability (2.3-19)**
We are given that the MGF of X is

$$M(t) = \frac{44}{120} + \frac{45}{120}e^t + \frac{20}{120}e^{2t} + \frac{10}{120}e^{3t} + \frac{1}{120}e^{5t} \ .$$

Note that this gives

$$f(x) = \begin{cases} 44/120, & x = 0, \\ 45/120, & x = 1, \\ 20/120, & x = 2, \\ 10/120, & x = 3, \\ 1/120, & x = 5. \end{cases}$$

(a)

$$\mu = E(X) = M'(0) = \frac{45}{120} + \frac{40}{120} + \frac{30}{120} + \frac{5}{120} = 1 \ .$$

Now

$$E(X^2) = M''(0) = \frac{45}{120} + \frac{80}{120} + \frac{90}{120} + \frac{25}{120} = 2$$

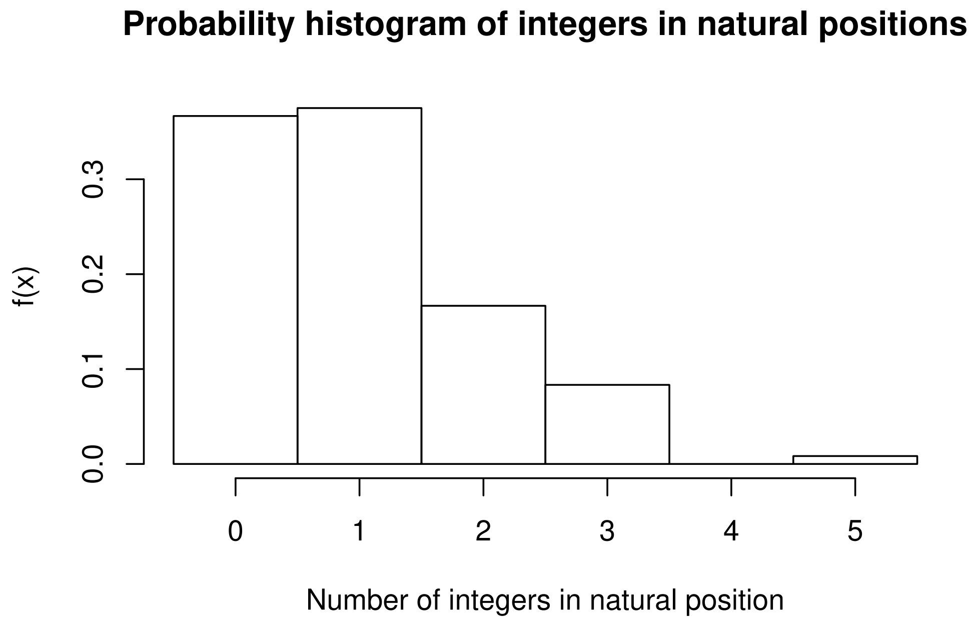
and thus

$$\sigma^2 = E(X^2) - [E(X)]^2 = 2 - 1 = 1 \ .$$

(b)

$$P(X \geq 1) = 1 - P(X < 1) = 1 - P(X = 0) = 1 - \frac{44}{120} = \frac{76}{120} \approx 0.637 \ .$$

(c) Below is the histogram for this probability distribution:



4. **Binary with $+1, -1$ outcomes, find PDF, μ, σ^2 (2.4-2)**
We have the PMF

$$f(x) = \begin{cases} 11/18, & x = -1, \\ 7/18, & x = 1. \end{cases}$$

Now,

$$E(X) = \sum_{x \in S} x f(x) = -1(\frac{11}{18}) + 1(\frac{7}{18}) = -\frac{2}{9} \approx -0.222 \ .$$

Furthermore,

$$E(X^2) = \sum_{x \in S} x^2 f(x) = 1(\frac{11}{18}) + 1(\frac{7}{18}) = 1$$

so

$$Var(X) = E(X^2) - [E(X)]^2 = 1 - \frac{4}{81} = \frac{77}{81} \approx 0.95 \ .$$

5. **Given MGF, name the distribution, mean, var (2.4-20)**

(a) i. We recognize this as the MGF of a Binomial distribution. Specifically, $X \sim \text{Binom}(n = 5, p = 0.7)$.

ii. We have

$$\mu = E(X) = np = 5(0.7) = 3.5$$

and

$$\sigma^2 = Var(X) = np(1 - p) = 5(0.7)(0.3) = 1.05 \ .$$

iii. The PMF of X is

$$f(x) = \binom{5}{x}(0.7)^x(0.3)^{5-x}, \quad x = 0, 1, 2, 3, 4, 5 \ .$$

Thus,

$$P(1 \le X \le 2) = P(X = 1) + P(X = 2) = 0.16065 \ .$$

(b) i. We recognize this as the MGF of a Geometric distribution. Specifically, $X \sim$ Geom$(p = 0.3)$.

ii. We have

$$\mu = E(X) = \frac{1}{p} = \frac{1}{0.3} \approx 3.33$$

and

$$\sigma^2 = Var(X) = \frac{1-p}{p^2} = \frac{0.7}{0.3^2} \approx 7.778 \ .$$

iii. The PMF of X is

$$f(x) = (0.3)(0.7)^{x-1}, \quad x = 1, 2, \dots \ .$$

Thus,

$$P(1 \le X \le 2) = P(X = 1) + P(X = 2) = 0.51 \ .$$

(c) i. We recognize this as the MGF of a Bernoulli distribution. Specifically, $X \sim$ Bern$(p = 0.55)$.

ii. We have

$$\mu = E(X) = p = 0.55$$

and

$$\sigma^2 = Var(X) = p(1-p) = 0.55(0.45) = 0.2475 \ .$$

iii. The PMF of X is

$$f(x) = (0.55)^x(0.45)^{1-x}, \quad x = 0, 1 \ .$$

Thus,

$$P(1 \le X \le 2) = P(X = 1) = 0.55 \ .$$

(d) i. This MGF does not belong to a named family of distributions, but we can easily see that the PMF is

$$f(x) = \begin{cases} 0.3, & x = 1, \\ 0.4, & x = 2, \\ 0.2, & x = 3, \\ 0.1, & x = 4. \end{cases}$$

ii. We have

$$\mu = E(X) = \sum_{x=1}^{4} x f(x) = 2.1$$

and

$$E(X^2) = \sum_{x=1}^{4} x^2 f(x) = 5.3$$

so

$$\sigma^2 = Var(X) = E(X^2) - [E(X)]^2 = 0.89 \ .$$

iii. We have

$$P(1 \le X \le 2) = P(X = 1) + P(X = 2) = 0.7 \ .$$

(e) i. We recognize this as the MGF of a discrete uniform distribution. Specifically, $X \sim \text{Unif}(m = 10)$.

ii. We have

$$\mu = E(X) = \frac{m+1}{2} = 5.5$$

and

$$\sigma^2 = Var(X) = \frac{m^2 - 1}{12} = 8.25 \ .$$

iii. The PMF of X is

$$f(x) = 0.1, \quad x = 1, 2, 3 \ldots, 9, 10 \ .$$

Thus,

$$P(1 \le X \le 2) = P(X = 1) + P(X = 2) = 0.2 \ .$$

Exam Type Question 1 (2.3-17)

a) First note that we must have a minimum of 2 coin flips before we can witness a heads-tails event. Thus $S_X = 2, 3, 4, \ldots$. If you draw a tree diagram, you will see that after 2 flips, the probability of heads-tails is $\frac{1}{4}$. After 3 flips, there are 2 branches in the tree that have heads-tails outcomes out of a total of 8 (2^3) possible sequences of coin flips. After 4 flips, there are 3 branches out of a total of 16 which end with a heads-tail outcome. The pattern continues and we can see that

$$f(x) = \frac{x-1}{2^x}, \quad x = 2, 3, 4, \ldots \ .$$

To prove this is a proper PMF, we must show that it sums to 1 over its support. Note that, for $k < |1|$,

$$S_1 = \sum_{x=2}^{\infty} (x-1)k^x = k^2 + 2k^3 + 3k^4 + 4k^5 + \ldots$$

and

$$kS_1 = k^3 + 2k^4 + 3k^5 + 4k^6 + \ldots$$

which means

$$S_1 - kS_1 = (1-k)S_1 = k^2 + k^3 + k^4 + \ldots = \sum_{x=2}^{\infty} k^x = \frac{k^2}{1-k}$$

where the latter equality comes from the formula for a geometric series (starting at index $x = 2$). Thus,

$$S_1 = \sum_{x=2}^{\infty} (x-1)k^x = \frac{k^2}{(1-k)^2} \ .$$

Applying this formula with $k = \frac{1}{2}$, we see

$$\sum_{x=2}^{\infty} f(x) = \sum_{x=2}^{\infty} (x-1)(\frac{1}{2})^x = \frac{(\frac{1}{2})^2}{(1-\frac{1}{2})^2} = 1 \ .$$

b)

$$M(t) = E(e^{tx}) = \sum_{x=2}^{\infty} e^{tx}\frac{x-1}{2^x} = \sum_{x=2}^{\infty} x\left(\frac{e^t}{2}\right)^x - \sum_{x=2}^{\infty}\left(\frac{e^t}{2}\right)^x \ .$$

Recall that we can only distribute this infinite series if both sums converge. Note that

$$x\left(\frac{e^t}{2}\right)^x = \frac{\partial}{\partial t}\left(\frac{e^t}{2}\right)^x \ .$$

Remark: Notice that this is the partial derivative with respect to t, not x.

So,

$$M(t) = \sum_{x=2}^{\infty} \frac{\partial}{\partial t}\left(\frac{e^t}{2}\right)^x - \sum_{x=2}^{\infty}\left(\frac{e^t}{2}\right)^x = \frac{\partial}{\partial t}\sum_{x=2}^{\infty}\left(\frac{e^t}{2}\right)^x - \sum_{x=2}^{\infty}\left(\frac{e^t}{2}\right)^x .$$

Recall that the interchange of differentiation and summation of an infinite series requires convergence of the summand derivatives. It is important to realize that this type of operation can not be performed in all cases.

Now, for $t < \ln(2)$, $\sum_{x=2}^{\infty}\left(\frac{e^t}{2}\right)^x$ is a convergent geometric series (with index starting at $x = 2$), so

$$\begin{aligned}
M(t) &= \frac{\partial}{\partial t}\left((\frac{e^t}{2})^2 \cdot \frac{1}{1-\frac{e^t}{2}}\right) - \left((\frac{e^t}{2})^2 \cdot \frac{1}{1-\frac{e^t}{2}}\right) = \frac{\partial}{\partial t}\left(\frac{e^{2t}}{2}\cdot\frac{1}{2-e^t}\right) - \left(\frac{e^{2t}}{2}\cdot\frac{1}{2-e^t}\right) \\
&= \frac{e^{2t}}{2-e^t} + \frac{e^{3t}}{2(2-e^t)^2} - \frac{e^{2t}}{2(2-e^t)} = \frac{e^{2t}}{2(2-e^t)} + \frac{e^{3t}}{2(2-e^t)^2} \\
&= \frac{e^{2t}}{(2-e^t)^2} = \frac{e^{2t}}{(e^t-2)^2}
\end{aligned}$$

as desired.

c) We have

$$M'(t) = \frac{-4e^{2t}}{(e^t-2)^3}$$

so

$$E(X) = M'(0) = 4 .$$

Furthermore,

$$M''(t) = \frac{4e^{2t}(e^t+4)}{(e^t-2)^4}$$

so

$$E(X^2) = M''(0) = 20$$

and thus

$$Var(X) = E(X^2) - [E(X)]^2 = 20 - 16 = 4 .$$

d) i)

$$P(X \le 3) = P(X=2) + P(X=3) = f(2) + f(3) = \frac{1}{2}$$

ii)

$$P(X \ge 5) = 1 - P(X \le 4) = 1 - (P(X \le 3) + P(X = 4)) = 1 - (\frac{1}{2} + \frac{3}{16}) = \frac{5}{16}$$

iii)

$$P(X = 3) = f(3) = \frac{1}{4}$$

Exam Type Question 2 (2.3-18)
We wish to show that the geometric distribution has the "memoryless" property, which essentially means that your waiting time for a success does not depend on how much you have already waited. In mathematical terms, this means that, given $k, j \in \mathbb{Z}^+$,

$$P(X > k + j | X > k) = P(X > j) \ .$$

Let $X \sim \text{Geom}(p)$. So it has PMF

$$f(x) = p(1-p)^{x-1} \ .$$

Then,

$$P(X > x) = \sum_{k=x+1}^{\infty} f(k) = \sum_{k=x+1}^{\infty} p(1-p)^{k-1} = p\sum_{k=x}^{\infty}(1-p)^k = p\frac{(1-p)^x}{1-(1-p)} = (1-p)^x \ .$$

With this formula, we see that

$$\begin{aligned} P(X > k + j | X > k) &= \frac{P((X > k + j) \cap (X > k))}{P(X > k)} = \frac{P(X > k + j)}{P(X > k)} \\ &= \frac{(1-p)^{k+j}}{(1-p)^k} = (1-p)^j = P(X > j) \end{aligned}$$

as desired.

Appendix

Below is the code used to create the histogram in Exercise 3:

```
#Permutations of Integers
natpos <- c(rep(0,44), rep(1,45), rep(2,20), rep(3,10), rep(5,1))
hist(natpos,breaks=-1:5+0.5, freq=FALSE,
     xlab="Number of integers in natural position", ylab="f(x)",
     main="Probability histogram of integers in natural positions")
```