eExercise 1

1.1

1. Find the solution to the following first-order differential equations:

$$y' + y = e^{-2x} \tag{1}$$

$$xy' - 2y = x^4 \tag{2}$$

Answer

1.(a)

$$y' + y = e^{-2x} \tag{3}$$

Solution to homogenous[齐次] pat,

> 先求齐次式, 求不定积分

$$y' + y = 0 \tag{4}$$

is

$$\frac{dy}{dx} + y = 0 \tag{5}$$

$$\frac{dy}{dx} = -y \tag{6}$$

$$\frac{dy}{y} = -dx \tag{7}$$

$$\frac{dy}{y} + dx = 0 \tag{8}$$

$$\frac{dy}{y} + dx = 0 \tag{9}$$

$$\int \frac{dy}{y} + \int dx = \int 0 \tag{10}$$

$$lny + x = lnk \tag{11}$$

$$ln\frac{y}{k} = -x \tag{12}$$

$$y = ke^{-x} \tag{13}$$

Solution to nonhomogenous e.g. found using variation method

$$y = K(x)e^{-x} \tag{14}$$

> 齐次方程通解, 通解带入原式, (前面求导+后面求导), 然后积分求出 $K(x)$

Thus Thus inserting in (1) leads to

$-x$ -2

$$y' + y = e^{-2x} \tag{15}$$

$$[K(x)e^{-x}]' + K(x)e^{-x} = e^{-2x} \tag{16}$$

$$[K(x)]'e^{-x} + K(x)[e^{-x}]' + K(x)e^{-x} = e^{-2x} \tag{17}$$

$$K'(x)e^{-x} - K(x)e^{-x} + K(x)e^{-x} = e^{-2x} \tag{18}$$

$$K'(x) = e^{-x} \tag{19}$$

$$K(x) = \int e^{-x} dx + k \tag{20}$$

Therefore

> $K(x)$ 代回原式

$$K(x) = -e^{-x} + k \tag{21}$$

Hence, the solution to (1) is

$$y = (-e^{-x} + k)e^{-x} = -e^{-2x} + ke^{-x}, (Check) \tag{22}$$

1.(b)

$$xy' - 2y = x^4 \tag{23}$$

Solutions to homogeneous equations

> 求齐次方程通解

$$xy' - 2y = 0 \tag{24}$$

$$x \cdot \frac{dy}{dx} - 2y = 0 \tag{25}$$

$$x \cdot \frac{dy}{dx} = 2y \tag{26}$$

$$x \cdot dy = 2y \cdot dx \tag{27}$$

$$\frac{dy}{y} = 2\frac{dx}{x} \tag{28}$$

$$\int \frac{dy}{y} = 2\int \frac{dx}{x} + \int 0 \tag{29}$$

$$lny = 2lnx + lnk \tag{30}$$

$$lny = lnx^2 + lnk \tag{31}$$

$$lny = ln(k \cdot x^2) \tag{32}$$

$$y = kx^2 \tag{33}$$

Solution to nonhomogenous e.g. found using variation method

$$y = K(x)x^2 \tag{34}$$

Using parameter variation method, the solution to nonhomogenous e.g. is found

> 齐次方程通解, 通解带入原式, (前面求导+后面求导), 然后积分求出 $K(x)$

$$xy' - 2y = x^4 \tag{35}$$

$$x[K(x)x^2]' - 2 \cdot K(x)x^2 = x^4 \tag{36}$$

$$x\{[K(x)]'x^2 + K(x)[x^2]'\} - 2 \cdot K(x)x^2 = x^4 \tag{37}$$

$$x\{K'(x)x^2 + K(x)2x\} - 2 \cdot K(x)x^2 = x^4 \tag{38}$$

$$K'(x)x^3 + 2 \cdot K(x)x^2 - 2 \cdot K(x)x^2 = x^4 \tag{39}$$

$$K'(x)x^3 = x^4 \tag{40}$$

$$K'(x) = x \tag{41}$$

$$\int K'(x) = \int x + \int 0 \tag{42}$$

$$K(x) = \frac{1}{2}x^2 + k \tag{43}$$

Thus

$$y = K(x)x^2 \tag{44}$$

$$y = (\frac{1}{2}x^2 + k)x^2 \tag{45}$$

$$y = \frac{1}{2}x^4 + k \cdot x^2, (Check) \tag{46}$$

2. Verify that $y = e^{-x}$ is a solution to the differential equation $y'' + 2y' + y = 0$. What is special about this equation? Use method of parameter variation to find the second solution.

Answer

To verify $y = e^{-x}$

$$y' = -e^{-x} \tag{47}$$

$$y'' = e^{-x} \tag{48}$$

Thus,

$$y'' + 2y' + y = 0 \tag{49}$$

$$e^{-x} + 2(-e^{-x}) + e^{-x} = 0 \tag{50}$$

The special thing about the differential equation is that its characteristic e.g. has repeat roots:

$$\lambda^2 + 2\lambda + 1 = 0 \tag{51}$$

Parameters variations method can be applied to

Find the general (complete) solution; i.e.

$$y = K(x)e^{-x} \tag{52}$$

Substitution in the differential equation to obtains

$$K''(x)e^{-x} - K'(x)e^{-x} - K'(x)e^{-x} + K(x)e^{-x} + 2[K'(x)e^{-x} - K(x)e^{-x}] + K(x)e^{-x} = 0 \tag{53}$$

$$K''(x)e^{-x} = 0 \tag{54}$$

$$K''(x) = 0 \tag{55}$$

$$K'(x) = A, (constant) \tag{56}$$

$$K(x) = Ax + B \tag{57}$$

Thus,

$$y = (Ax + B)e^{-x} \tag{58}$$

$$y = Axe^{-x}{}_{second\ from P_o V_o} + Be^{-x}{}_{original}, complete\ solution \tag{59}$$

3. Verify that $y = -cos(x)$ is a solution to the differential equation $y'' - y = 2cos(x)$. Find the general solution of the equation.

```
line:     \ldots
diagonal: \ddots
vertical: \vdots
```

$$line \ldots \tag{60}$$

$$diagonal \ddots \tag{61}$$

$$vertical \vdots \tag{62}$$

LaTeX markup...	...results in:	...is used for:
`\begin{matrix}` `\alpha& \beta^{*}\\` `\gamma^{*}& \delta` `\end{matrix}`	$\begin{matrix} \alpha & \beta^{*} \\ \gamma^{*} & \delta \end{matrix}$	plain matrix
`\begin{bmatrix}` . . . `\end{bmatrix}`	$\begin{bmatrix} \alpha & \beta^{*} \\ \gamma^{*} & \delta \end{bmatrix}$	bracketed matrix; typically represents the matrix itself
`\begin{Bmatrix}` . . . `\end{Bmatrix}`	$\begin{Bmatrix} \alpha & \beta^{*} \\ \gamma^{*} & \delta \end{Bmatrix}$	braced matrix
`\begin{pmatrix}` . . . `\end{pmatrix}`	$\begin{pmatrix} \alpha & \beta^{*} \\ \gamma^{*} & \delta \end{pmatrix}$	parenthesized matrix
`\begin{vmatrix}` . . . `\end{vmatrix}`	$\begin{vmatrix} \alpha & \beta^{*} \\ \gamma^{*} & \delta \end{vmatrix}$	vertical bar matrix; typically represents the determinant
`\begin{Vmatrix}` . . . `\end{Vmatrix}`	$\begin{Vmatrix} \alpha & \beta^{*} \\ \gamma^{*} & \delta \end{Vmatrix}$	double-vertical bar matrix
`\begin{smallmatrix}` . . . `\end{smallmatrix}`	$\begin{smallmatrix} \alpha & \beta^{*} \\ \gamma^{*} & \delta \end{smallmatrix}$	small matrix; can be used inline

Answer

Considering $y = -\cos(x)$

$$\begin{pmatrix} -3a_1 + 2a_0 = 0 \\ 2a_1 = 4 \\ 2b = 1 \\ -2c_1 + bc_2 = 1 \\ -2c_2 - bc_1 = 0 \end{pmatrix} \to \{\} \tag{63}$$

4. Find the particular solution[特解] of the differential equation $y'' - 3y' + 2y = 4x + e^{3x} + sin(2x)$. What is the general solution to the equation?

Answer

The particular solution, y_p, has an expression (shape), similar to the excitation function, i.e.

$$y_p = a_0 + a_1 x \tag{64}$$

5. Find the solution to $y'' + y = sin(5t)$. If the input to this differential equation, $sin(5t)$, (providing the forced oscillation) is replaced by $sin(t)$, find the solution.

   Note:

$$y' = \frac{d}{dx} y \; and \; y'' = \frac{d^2}{dx^2} y \tag{65}$$

   (derivatives are with respect to x)

$$y' = \frac{d}{dt} y \; and \; y'' = \frac{d^2}{dt^2} y \tag{66}$$

   (derivatives are with respect to t)

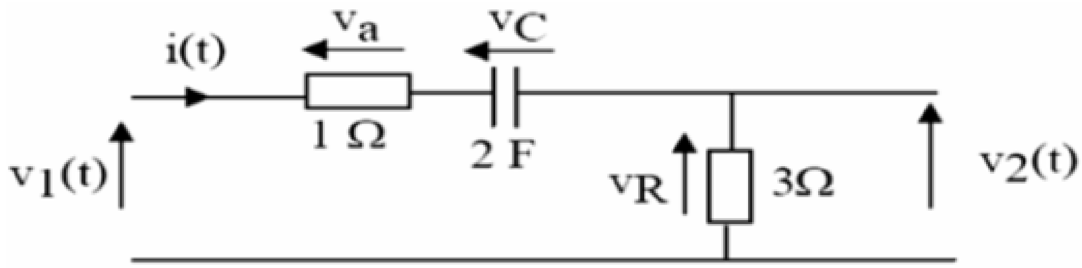
Answer

Exercise 2

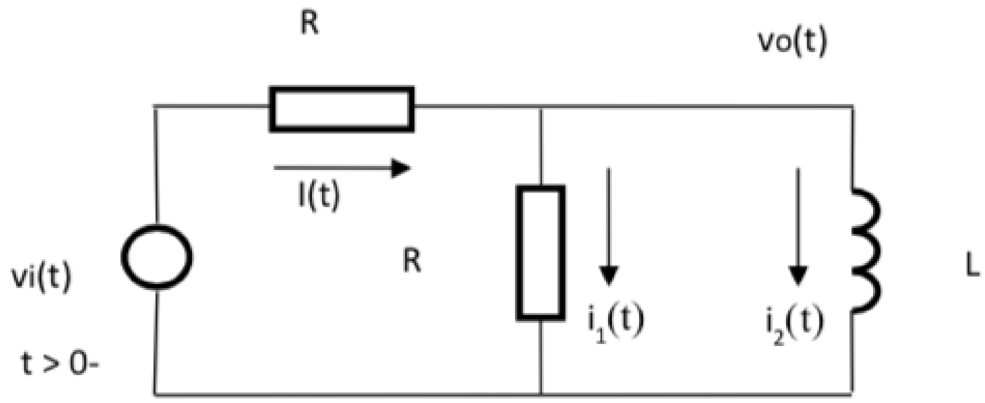
P1 Find the relationship between the output voltage and input voltage in the following circuits both in the time-domain and in the Laplace-domain. Assuming the initial conditions for C and L are zero and the excitation is impulsive find the behaviour of the output voltage.

P2 If the impulsive excitation is replaced by a sinusoidal excitation, find the frequency domain voltage transfer function for each case.

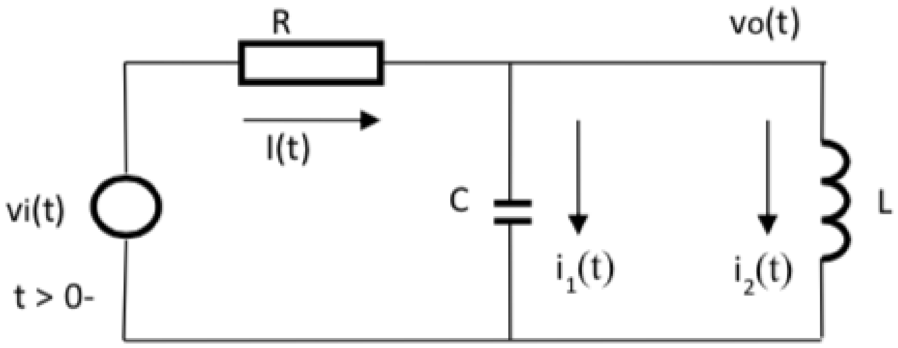
P5 Then find the mag. and phase of each transfer function.



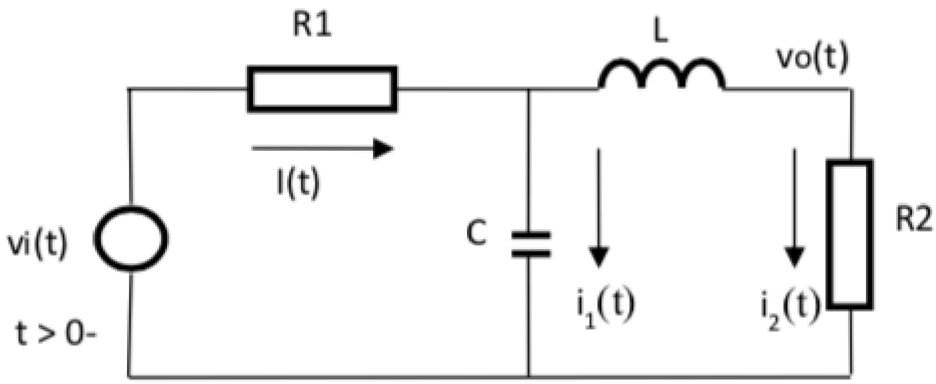
i(t)
va
vC
1 Ω
2 F
v1(t)
vR
3Ω
v2(t)

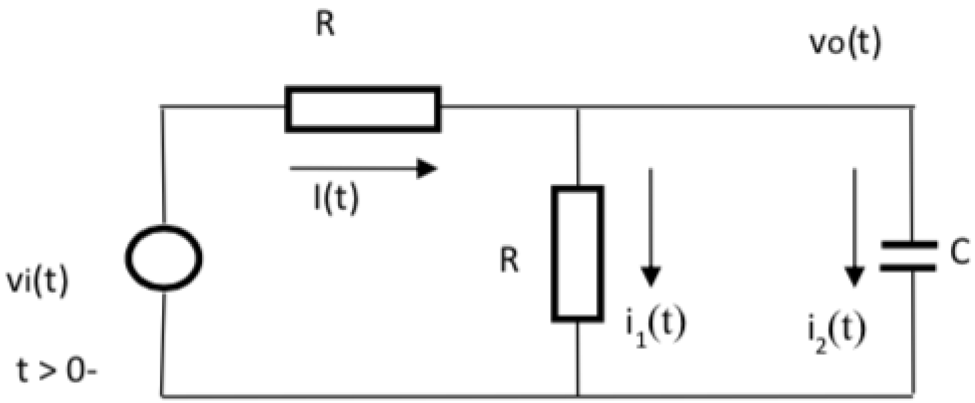


R
vo(t)
I(t)
R
vi(t)
t > 0-
i1(t)
i2(t)
L

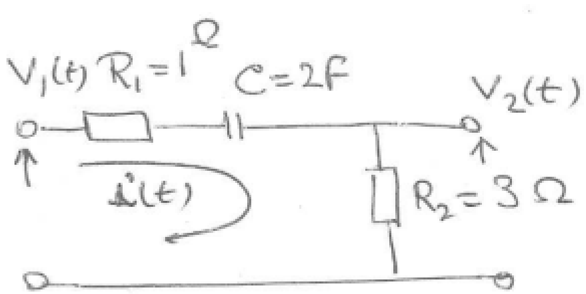


R
vo(t)
I(t)
vi(t)
C
L
i1(t)
i2(t)
t > 0-





Answer:



writing voltage around the

> 电容是 1/C 积分 i(t)

$$+R_i(t) + \frac{1}{C}\int i(t)dt + R_2 i(t) = V_i(t) \tag{67}$$

$$-V_i(t) + R_i(t) + \frac{1}{C}\int i(t)dt + R_2 i(t) = 0 \tag{68}$$

where

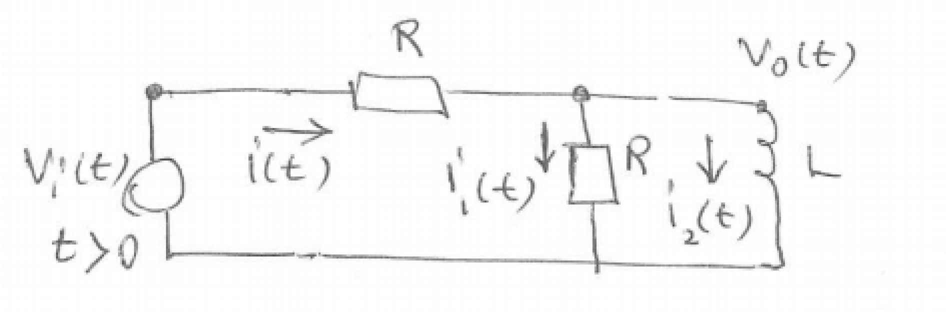
$$i(t) = \frac{V_2(t)}{R_2} \tag{69}$$

Differentiate $-V_i(t) + R_i(t) + \frac{1}{C}\int i(t)dt + R_2 i(t) = 0$ with respect to i

> 同时求导

$$-\frac{dV_i(t)}{dt} + R_1\frac{di(t)}{dt} + \frac{1}{C}i(t) + R_2\frac{di(t)}{dt} = 0 \tag{70}$$

P2 If the impulsive excitation is replaced by a sinusoidal excitation, find the frequency domain voltage transfer function for each case.

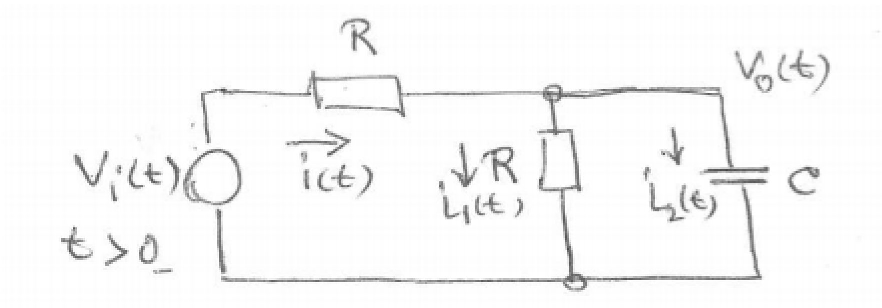


$$i(t) = i_1(t) + i_2(t) \tag{71}$$

Therefore,

$$i(t) \tag{72}$$

P5 Then find the mag. and phase of each transfer function.



This is from the circuit

$$i(t) = i_1(t) + i_2(t) \tag{73}$$

therefore

> 并联 1/R, 1/L，C

$$i(t) = \frac{V_o(t)}{R} + C\frac{dV_o(t)}{dt} \tag{74}$$

Exercise 3

Exercise 4