# Lecture 15: Sequences

## EXAMS

- solutions (w/ rubrics) will be posted this afternoon
- CHECK THEM FIRST before asking me things
- if you want me to regrade anything, submit it IN WRITING w/ your exam by next Friday
- I will regrade everything - be careful what you wish for

# Announcements and reminders

Quizlet 6
due Wednesday 8 AM

- ~~Homework 5 posted, due **Monday 8 Oct at 12 PM noon**~~

- Homework 6 (written) is posted ~~(or will be soon!)~~ and is due Friday 12 Oct at 12 PM Noon
  - Staple it!
  - Tear-the-“chit”-off it!
  - Write-neatly-on it!


- The CU final exam schedule is up. You must take your final exam during your scheduled final exam time.

  **Tony’s section: 7:30 - 10 PM, Sunday 16 Dec**
  **Rachel’s section: 1:30 - 4 PM, Wednesday 19 Dec**

# What did we do last time?

- We learned about ***functions***…
  - What are they?
  - Special kinds of functions (floor, ceiling, inverse, composition… )
  - Special properties functions can have (onto, 1-1… )
  - That we shouldn't fear them!

## Today:

- We talk ***sequences***
  - What are they?
  - Special kinds of sequences…
  - Different ways to define them…

## Sequences

A **sequence** is a discrete structure used to store an ordered list of elements.
It is one of the most fundamental structures in computer science.

**Examples:**

- 0, 1, 2, 3, 4, 5, …
- 2, 4, 8, 16, 32, …
- 1, 4, 78, 109, 4, 25, …

Perfectly okay for sequence to begin not at 0:

$\{b_n\}_{n=5}^{\infty} = b_5, b_6, b_7, \ldots$

**Definition:** A **sequence** is a function that maps from a subset of the integers (usually {0, 1, 2, 3, … } or {1, 2, 3, … }) to a set *S*.

We use the notation $\{a_n\}$ to represent a sequence where $a_n$ represents the $n^{th}$ individual term in the sequence.

$\{a_n\}_{n=0}^{\infty} = a_0, a_1, a_2, \ldots$

# Sequences

sequence's indices

| $n$ | $E_n$ |
|---|---|
| 0 | 2 |
| 1 | 4 |
| 2 | 6 |
| 3 | 8 |
| 4 | ⋮ |
| ⋮ | |

Def. of countability:

$E_n = (n+1) \cdot 2$

$(n \in \mathbb{N})$

**Fun fact:** This very definition of a sequence implies that if we can define a ***set*** in terms of a sequence, then that set must be countable.

**Example, rebooted:** Positive Even integers

$E_0 \quad E_1 \quad E_2$

$\{E_n\}_{n=0}^{\infty} = 2, 4, 6, 8, \ldots$

**Definition:** A **sequence** is a function that maps from a subset of the integers (usually {0, 1, 2, 3, … } or {1, 2, 3, … }) to a set *S*.

We use the notation $\{a_n\}$ to represent a sequence where $a_n$ represents the $n^{th}$ individual term in the sequence.

# Sequences

**Example:** Consider the sequence $\{a_n\}$,

where $\boxed{a_n = \frac{1}{n}}$

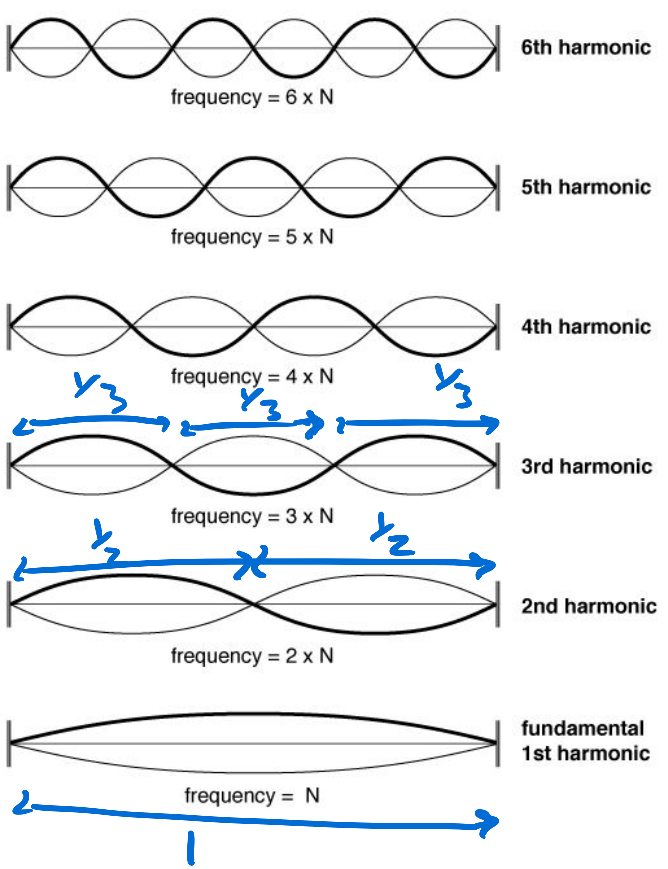
- In this case, the sequence starts with $n = 1$ (do you see why?)

- The first three terms are:

  $a_1 = 1$ $\qquad$ $a_2 = ½$ $\qquad$ $a_3 = ⅓$

  $\frac{1}{1}$ $\qquad$ $a_2 = \frac{1}{(2)}$ $\qquad$ $\frac{1}{3}$

- This is one of several sequence patterns that is so common, we have a special name for it: ***the harmonic sequence***




**wave image of the harmonic series**

# Sequences

**Definition:** A **geometric progression** is a sequence of the form

$a, ar, ar^2, ar^3, ar^4, ar^5, \ldots$

$r$ = ratio btw. subsequent terms in seq.

where the **initial term** $a$ and the $\boxed{\textbf{common ratio } r}$ are real numbers.

Note that this can be rewritten to show the pattern explicitly as:

$\underbrace{ar^0}_{n=0}, \overbrace{ar^1}^{n=1}, ar^2, ar^3, ar^4, \overbrace{ar^5}^{n=5}, \ldots$

$a_0 = a$

$a_1 = ar$

$a_2 = ar^2 \ldots$

$a_n = ar^n$

**Example:** What are $a$ and $r$ in the following geometric sequence? What is the next term?

$3 \xrightarrow{\times 2} 6 \xrightarrow{\times 2} 12 \xrightarrow{\times 2} 24 \xrightarrow{\times 2} \ldots$ $\boxed{48}$

$a_0 = \boxed{a = 3}$

$a_1 = ar = 3r = 6$

$\Rightarrow \boxed{r = 2}$

$a_4 = ar^4 = 3 \cdot 2^4 = \boxed{48}$

# Sequences

**Example:** What are *a* and *r* in the following geometric sequence? What is the next term?

3, 6, 12, 24, …

# Sequences

**Example:** What are $a$ and $r$ in the following geometric sequence? What is the next term?

3, 6, 12, 24, …

**Solution:** We know a geometric sequence follows the pattern: $a, ar, ar^2, ar^3, ar^4, \ldots$ , so we can use the $a_0$ and $a_1$ terms to pick out $a$ and $r$.

- $a_0 = a = 3$
- $a_1 = ar = 3r = 6 \Rightarrow r = 2$

- The next term would be $a_4 = ar^4 = 3 \times 2^4 = 3 \times 16 = 48$

# Sequences

**Definition:** Another common sequence is an **arithmetic progression**, which has the form

$$a, a + d, a + 2d, a + 3d, a + 4d, a + 5d, \ldots$$

d = difference between subsequent terms in the sequence

where the **initial term** *a* and the **common difference** *d* are real numbers.

Note that this can be rewritten to show the pattern explicitly as:

$a_0 = a$

$*a_1 = a + d$

$a_2 = a + 2d \cdots a_n = a + nd$

$$\underbrace{a + 0d}_{a_0}, a + 1d, a + 2d, a + 3d, a + 4d, a + 5d, \ldots$$

**Example:** What are *a* and *d* in the following arithmetic sequence? What is the next term?

$\overset{+4}{\curvearrowright}$ $\overset{+4}{\curvearrowright}$ $\overset{+4}{\curvearrowright}$ $\overset{+4}{\curvearrowright}$

5, 9, 13, 17, … $\boxed{21}$

$a_0$ $a_1$ $a_2$ $a_3$ $a_4$

$a_0 = \boxed{a = 5}$

$a_1 = a + d = 9$

$\rightarrow 5 + d = 9 \rightarrow \boxed{d = 4}$

$\boxed{21}$

# Sequences

**Example:** What are *a* and *d* in the following arithmetic sequence? What is the next term?

$2 \xrightarrow{+7} 9 \xrightarrow{+7} 16 \xrightarrow{+7} 23, \ldots$

$\boxed{d=7}$

$\boxed{a=2}$

# Sequences

**Example:** What are *a* and *d* in the following arithmetic sequence? What is the next term?

2, 9, 16, 23, …

**Solution:** We know an arithmetic sequence follows the pattern $a, a + d, a + 2d, a + 3d, a + 4d, a + \boxed{5}d, \ldots$ so we can use the $a_0$ and $a_1$ terms to pick out $a$ and $d$.

- $a_0 = a = 2$
- $a_1 = a + d = 2 + d = 9 \quad \Rightarrow d = 7$

FYOG: What continuous form is the geometric progression a gen. of?

- The next term would be $a_4 = a + 4d = 2 + 4 * 7 = 30$

**Fun fact:** This is the discrete generalization of a linear function $f(x) = a + dx$

## Sequences

Sometimes we don't give an explicit formula for the terms $a_n$

Instead, we might define the sequence in terms of a **recurrence relation**, where the later terms are a function of the previous ones, and specifying the first few terms.

**Example:** Let $\underbrace{a_0 = 1,\ a_1 = 3,}_{\text{initial conditions}}$ and $\underbrace{a_n = 2a_{n-1} - a_{n-2}}_{\text{recurrence relation}}$

in general, need k initial conditions if recurrence relation goes back k steps (here, we went back 2 steps. so needed 2 IC's)

$a_0 = 1$
$a_1 = 3$

$a_2 = 2a_{(2-1)} - a_{(2-2)} = 2a_1 - a_0 = 2 \cdot 3 - 1 = 5$

$a_3 = 2a_{3-1} - a_{3-2} = 2 \cdot a_2 - a_1 = 2 \cdot 5 - 3 = 7$

$a_4 = 2a_{4-1} - a_{4-2} = 2a_3 - a_2 = 2 \cdot 7 - 5 = 9$

$\vdots$

$a_n = \underline{2n+1}$ ← closed form solution to the recurrence relation

$n^{th}$ term in seq., as fcn. of n only (must also work for initial conditions)

# Sequences

Sometimes we don't give an explicit formula for the terms $a_n$

Instead, we might define the sequence in terms of a **recurrence relation**, where the later terms are a function of the previous ones, and specifying the first few terms.

**Example:** Let $a_0 = 1$, $a_1 = 3$, and $a_n = 2a_{n-1} - a_{n-2}$

Then $a_2 = 2a_1 - a_0 = 2(3) - 1 = 5$
$a_3 = 2a_2 - a_1 = 2(5) - 3 = 7$
$a_4 = 2a_3 - a_2 = 2(7) - 5 = 9$ … and so on…

So the sequence is: 1, 3, 5, 7, 9, …

If we want a ***closed form*** version of $a_n$, we can figure it out:

$$a_n = 2n + 1$$

$a_n = 2n + 1$ is called a ***solution*** of the recurrence relation defined above.

## Sequences

$a_{n-1} = 4^{n-1}$ & $a_{n-2} = 4^{n-2}$

**Example:** Show that $a_n = 4^n$ is a solution to the recurrence $\boxed{a_n = 8a_{n-1} - 16a_{n-2}}$

think of as a fcn of n

LHS

RHS

**Strategy:** Plug $a_n$ into the recurrence and show both sides are equal

$$\underbrace{8a_{n-1} - 16a_{n-2}}_{\text{RHS}} = 8 \cdot 4^{n-1} - 16 \cdot 4^{n-2}$$

$$= 8 \cdot 4^n \cdot 4^{-1} - 16 \cdot 4^n \cdot 4^{-2}$$

$$= \cancel{8}^{2} \cdot \frac{4^n}{\cancel{4}} - \cancel{16} \cdot \frac{4^n}{\cancel{4^2}}$$

⋮

$$= 2 \cdot 4^n - 4^n$$

$$= 4^n$$

$$= \underbrace{a_n}_{\text{LHS}}$$

# Sequences

**Example:** Show that $a_n = 4^n$ is a solution to the recurrence $a_n = 8a_{n-1} - 16a_{n-2}$

**Strategy:** Plug $a_n$ into the recurrence and show both sides are equal

$a_n = 4^n$ $\qquad a_{n-1} = 4^{n-1}$ $\qquad a_{n-2} = 4^{n-2}$

We have:

$$\begin{aligned}
8a_{n-1} - 16a_{n-2} &= 8(4^{n-1}) - 16(4^{n-2}) \\
&= 8(4^n)(4^{-1}) - 16(4^n)(4^{-2}) \\
&= 8(4^n)(1/4) - 16(4^n)(1/16) \\
&= 2(4^n) - 4^n \\
&= 4^n \\
&= a_n \quad ✔
\end{aligned}$$

# Sequences

$a_n = 2n+1 \rightarrow a_{n-1} = 2(n-1)+1 = 2n\underbrace{-2+1}_{-1}$

$a_n = 2n+1 \rightarrow a_{n-2} = 2(n-2)+1 = 2n\underbrace{-4+1}_{-3}$

Sometimes we don't give an explicit formula for the terms $a_n$

Instead, we might define the sequence in terms of a **recurrence relation**, where the later terms are a function of the previous ones, and specifying the first few terms.

**Example:** Let $a_0 = 1$, $a_1 = 3$, and $a_n = 2a_{n-1} - a_{n-2}$

(Check that it fits the initial conditions too!)

Show that $\boxed{a_n = 2n+1}$ is a solution to this recurrence relation.

$$2a_{n-1} - a_{n-2} = 2(\underbrace{2n-1}_{a_{n-1}}) - (\underbrace{2n-3}_{a_{n-2}})$$

$$= 4n-2-2n+3$$

$$= \underbrace{2n+1}$$

$$= a_n \checkmark$$

Initial conditions:

$a_0 = 2(0)+1 = 1 = 1 \checkmark$

$a_1 = 2(1)+1 = 3 = 3 \checkmark$

fits ICs ✓

# Sequences

**FYOG:** Show that $a_n = n\,4^n$ is also a solution to the recurrence $a_n = 8a_{n-1} - 16a_{n-2}$

**FYOG:** Determine a recurrence relation with solution $a_n = n + (-1)^n$

**FYOG:** Determine a recurrence relation with solution $a_n = n^2 + n$

**FYOG:** Determine a recurrence relation with solution $a_n = n!$

# Sequences

Probably the most famous sequence is the ***Fibonacci sequence***:

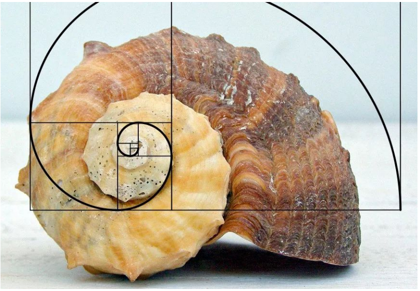
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …

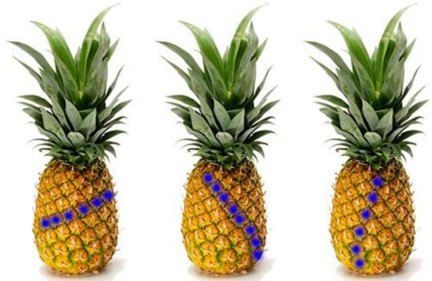
$F_3$

It has the recurrence relation: $F_0 = 0$, $F_1 = 1$, $\boxed{F_n = F_{n-1} + F_{n-2}}$

Lots of cool stuff hidden in the Fibonacci sequence

- Ratio of the terms approaches the **golden ratio:** $\dfrac{1+\sqrt{5}}{2}$
- Pattern found all over the place in nature




# Sequences

$F_n = F_{n-1} + F_{n-2}$

Probably the most famous sequence is the Fibonacci sequence:

$$\underset{F_0}{0},\ 1,\ 1,\ \underset{F_3}{2},\ 3,\ 5,\ \underset{F_6}{8},\ 13,\ 21,\ \underset{F_9}{34},\ 55,\ \ldots$$

Notice any interesting patterns?

➢ **Parity pattern:** even, odd, odd, even, odd, odd, even, …

⇒ Could we find the sequence of just the *even* Fibonacci numbers?

⇒ Seems like the evens are the Fibonacci numbers with indices that are multiples of 3.

subsequence: $F_0 = 0$, $F_3 = 2$, $F_6 = 8$, …

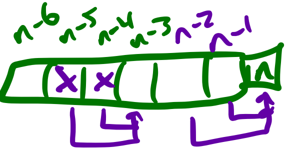
$F_n = F_{n-1} + F_{n-2}$

**Tougher Example:** Determine a recurrence relation for the even Fibonacci numbers, $\{E_n\}$

$E_0 = F_0 = 0$, $\quad E_1 = F_3 = 2$, $\quad E_2 = F_6 = 8$, $\quad E_3 = F_9 = 34$, $\quad \ldots$ $\qquad E_0 = 0$ & $E_1 = 2$

**Strategy:** We want a recurrence of the form $\boxed{F_n = a\,F_{n-3} + b\,F_{n-6}}$ $\iff$ $E_n = a\,E_{n-1} + b\,E_{n-2}$

- As long as we start it off with the first two evens, this recurrence will stick to the even Fibonaccis




$$F_n = F_{n-1} + F_{n-2}$$

$$= (F_{n-2} + F_{n-3}) + (F_{n-3} + F_{n-4})$$

put the $F_{n-3}$ to the side b/c our recurrence should contain them!

$$= 2F_{n-3} + F_{n-2} + F_{n-4}$$

$$= 2F_{n-3} + (F_{n-3} + F_{n-4}) + (F_{n-5} + F_{n-6})$$

$$= 3F_{n-3} + F_{n-6} + \boxed{F_{n-4} + F_{n-5}}$$

$$= 3F_{n-3} + F_{n-6} + \underbrace{F_{n-3}}$$

* From Fib. recurrence *

$$F_n = 4F_{n-3} + F_{n-6}$$

**Tougher Example:** Determine a recurrence relation for the even Fibonacci numbers, $\{E_n\}$

$$E_0 = F_0 = 0, \quad E_1 = F_3 = 2, \quad E_2 = F_6 = 8, \quad E_3 = F_9 = 34, \quad \ldots$$

**Strategy:** We want a recurrence of the form $F_n = a\,F_{n-3} + b\,F_{n-6}$

- As long as we start it off with the first two evens, this recurrence will stick to the even Fibonaccis

$$\begin{aligned}
F_n &= F_{n-1} + F_{n-2} \\
&= (F_{n-2} + F_{n-3}) + (F_{n-3} + F_{n-4}) \\
&= ((F_{n-3} + F_{n-4}) + F_{n-3}) + (F_{n-3} + (F_{n-5} + F_{n-6})) \\
&= 3F_{n-3} + F_{n-6} + F_{n-4} + F_{n-5} \\
&= 3F_{n-3} + F_{n-6} + F_{n-3} \\
&= 4F_{n-3} + F_{n-6}
\end{aligned}$$

**Solution:** $\boxed{E_0 = 0, E_1 = 2,}$ and $\underline{E_n = 4\,E_{n-1} + E_{n-2}}$

# Sequences

Solving recurrence relations using **iteration**:

Not simplifying algebra usually helps

**Example:** Let $a_n = a_{n-1} + 3$, with $a_1 = 2$. Find a closed form solution to this recurrence.

not starting w/ n=0 is totally okay!

**Solution:** (using iteration)

We write out the first few terms by plugging successively back into the recurrence:

$$a_1 = 2 \longrightarrow = 2 + 0 \cdot 3$$

$$a_2 = a_1 + 3 = 2 + 3 = 2 + 1 \cdot 3$$

$$a_3 = a_2 + 3 = (2+3) + 3 = 2 + 2 \cdot 3$$

$$a_4 = a_3 + 3 = \underbrace{(2 + 2 \cdot 3)}_{a_3} + 3 = 2 + 3 \cdot 3$$

$$\vdots$$

$$\boxed{a_n = 2 + (n-1) \cdot 3}$$

have to plug this into the recurrence relation to prove it's a solution

## Sequences

Solving recurrence relations using **iteration**:

**Example:** Let $a_n = a_{n-1} + 3$, with $a_1 = 2$. Find a closed form solution to this recurrence.

**Solution:** (using iteration)

We write out the first few terms by plugging successively back into the recurrence:

$a_1 = 2$

$a_2 = a_1 + 3 = 2 + 3$

$a_3 = a_2 + 3 = (2 + 3) + 3 = 2 + 3 \cdot 2$

$a_4 = a_3 + 3 = (2 + 3 \cdot 2) + 3 = 2 + 3 \cdot 3$

…

$a_n = 2 + 3 \cdot (n-1)$

# Countable and uncountable sets

HERE!

**Fun fact:** This very definition of a sequence implies that if we can define a ***set*** in terms of a sequence, then that set must be countable.

**Example, rebooted:** Even integers

**Reminder:**
THIS happened

**Definition:** A **sequence** is a function that maps from a subset of the integers (usually {0, 1, 2, 3, … } or {1, 2, 3, … }) to a set $S$.

We use the notation $\{a_n\}$ to represent a sequence where $a_n$ represents the $n^{th}$ individual term in the sequence.

## Countable and uncountable sets

**Theorem:** If $A$ and $B$ are countable sets, then $A \cup B$ is countable as well.

**Proof:** (by cases)

3 cases: (i) $A$ and $B$ are both finite

(ii) $A$ is infinite and $B$ is finite (without loss of generality, covers the case of $B$ being infinite and $A$ finite)

(iii) $A$ and $B$ both infinite

# Countable and uncountable sets

**Theorem:** If $A$ and $B$ are countable sets, then $A \cup B$ is countable as well.

**Proof:** (by cases)

Case (i): $A$ and $B$ are both finite

⇒ We can write the elements of $A$ as $A = \{a_1, a_2, \dots, a_n\}$ and the elements of $B$ as $B = \{b_1, b_2, \dots, b_m\}$ for some positive integers $n$ and $m$ (equal to $|A|$ and $|B|$, resp.)

⇒ We can write the elements of $A \cup B$ as $\{a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_m\}$, which has $n+m$ elements

## Countable and uncountable sets

**Theorem:** If $A$ and $B$ are countable sets, then $A \cup B$ is countable as well.

| $\mathbb{N}$ | $A$ |
|---|---|
| $0 \to$ | $a_1$ |
| $1 \to$ | $a_2$ |
| $2 \to$ | $a_3$ |
| $\vdots$ | $\vdots$ |

Upshot: Since $A$ is countable, we can write it as a sequence!

**Proof:** (by cases)

Case (ii): $A$ infinite and $B$ finite

⇒ Because $A$ is countably infinite, we can write its elements as an infinite sequence:

$a_1, a_2, \ldots, a_n, a_{n+1}, \ldots$ and because $B$ is finite we can write its elements as

$B = \{b_1, b_2, \ldots, b_m\}$ for some positive integer $m$ (equal to $|B|$)

⇒ We can write the elements of $A \cup B$ as an infinite sequence as:

$\{c_n\}$ $b_1, b_2, \ldots, b_m, a_1, a_2, \ldots, a_n, \ldots$

$$c_i = \begin{cases} b_i & \text{if } 1 \le i \le m \\ a_{i-m+1} & \text{if } i > m \end{cases}$$

## Countable and uncountable sets

**Theorem:** If $A$ and $B$ are countable sets, then $A \cup B$ is countable as well.

could be useful for a particular proof by contradiction...

**Proof:** (by cases)

<u>Case (iii): *A* and *B* both infinite</u>

⇒ Because $A$ and $B$ are countably infinite, we can write their elements as infinite sequences:

$a_1, a_2, \ldots, a_n, a_{n+1}, \ldots$ and $b_1, b_2, \ldots, b_n, b_{n+1}, \ldots$

⇒ We can write the elements of $A \cup B$ as an infinite sequence by alternating between the sequences $\{a_n\}$ and $\{b_n\}$:

$\{c_n\} = a_1, b_1, a_2, b_2, a_3 \ldots, a_n, b_n, \ldots$

$n = 0 \quad 1 \quad 2 \quad 3 \quad 4$

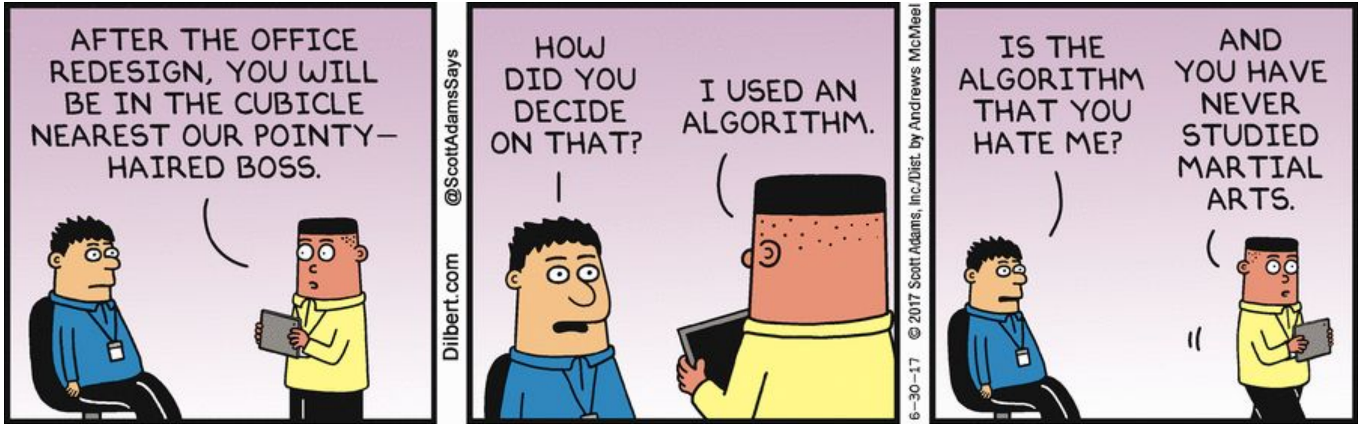
$c_{2n} = a_n$

$c_{2n+1} = b_n$

# Recap

We learned about ***sequences…***

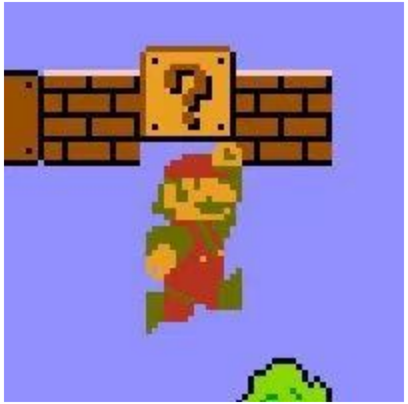
* What are they? * Special kinds of sequences * Nifty ways to define sequences




**Next time:**

- we learned about *functions*
- we learned about *sequences*…
- ***algorithms*** are basically doing a function to all the elements along a sequence
- … and pretty much the backbone of computational science.

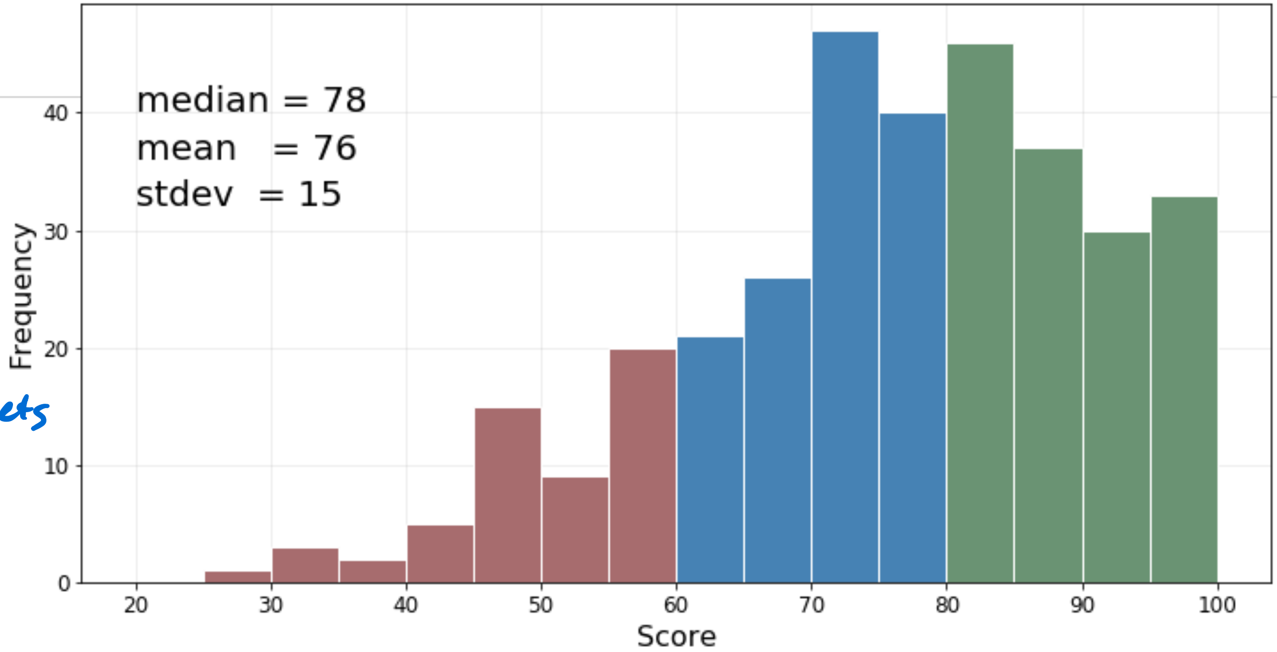
# Bonus material!



green = keep doing what you're doing

blue = decent-pretty good, but could swing depending on HW/quizlets

red = might want to try some new study strategies




- solutions w/ rubrics posted Friday PM
- CHECK THEM before asking Tony/Rachel questions
- Regrade requests IN WRITING by Friday next week
- Will regrade everything (so think first, & check all solutions)