# CSCI 2824: Discrete Structures

# Lecture 19: Algorithm Complexity and Matrix Operations

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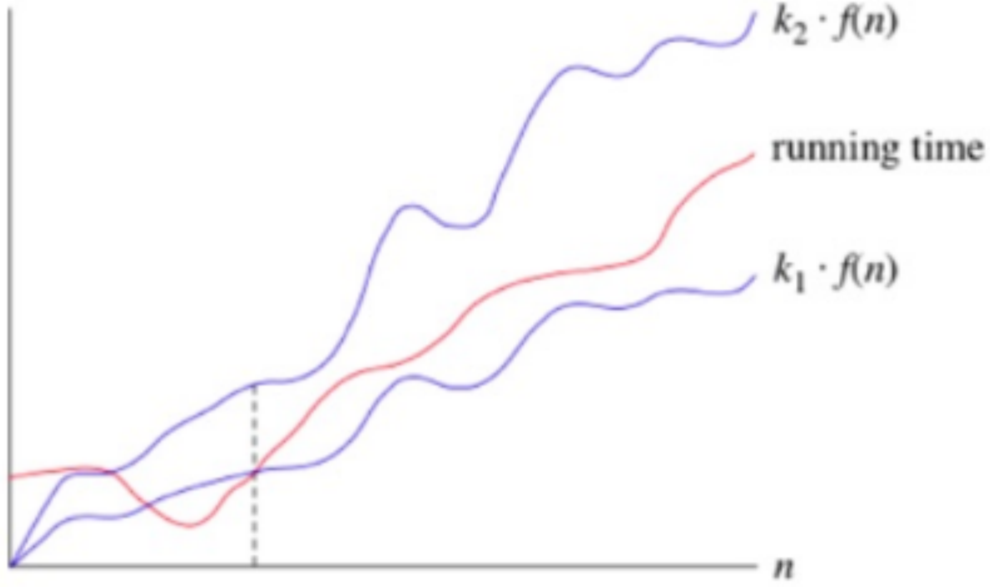
# Algorithm Complexity

## Growth of Functions:

**Def**: If $f(n)$ is both $\mathcal{O}(g(n))$ and $\Omega(g(n))$ then we say $f(n)$ is $\Theta(g(n))$ [read big-Theta of $g(n)$] and also that $f(n)$ is of *order* $g(n)$

/Users/rachel/Desktop/slide 1.png



# Algorithm Complexity

**Example**: Give a big-O estimate of $f(n) = n!$

**Example**: Give a big-O estimate of $g(n) = \log n!$

## Algorithm Complexity

**<u>Example</u>**: Show that $h(n) = 2n^2 + 5n\log n$ is $\Theta(n^2)$

## Algorithm Complexity

**Example** (Continued) : Show that $h(n) = 2n^2 + 5n \log n$ is $\Theta(n^2)$

# Matrices and Matrix Operations

Linear Algebra is the workhorse of computational science

In scientific and engineering code, 99% of computing time is spent on matrix operations

Although applications of matrices are incredibly broad, they were invented for a very simple purpose: to make solving systems of equations cleaner

$$\begin{aligned} 3x + 4y + 5z &= 1 \\ 2x + 8y + 3z &= 2 \\ 4x + 2y + 2z &= 3 \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} 3 & 4 & 5 \\ 2 & 8 & 3 \\ 4 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Rectangular thing is a **matrix**, tall skinny things are **vectors**

# Matrices and Matrix Operations

$$\begin{aligned} 3x + 4y + 5z &= 1 \\ 2x + 8y + 3z &= 2 \\ 4x + 2y + 2z &= 3 \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} 3 & 4 & 5 \\ 2 & 8 & 3 \\ 4 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Rectangular thing is a **matrix**, tall skinny things are **vectors**

**Def**: A matrix with $m$ rows and $n$ columns has **dimensions** $m \times n$

**Def**: A vector with $n$ entries has **length** $n$

**Notation**: Matrices are represented by capital letters, like $A$ and $M$.
Vectors are represented by lower case letters like $\mathbf{x}$ and $\mathbf{b}$

**Example**: The above **matrix equation** could be written as $A\mathbf{x} = \mathbf{b}$.

# Matrices and Matrix Operations

Matrices and vectors can be **added** and **multiplied** (but not divided)

**Def**: The sum of matrices $A$ and $B$ is the matrix obtained by adding corresponding entries together

**Example**:

$$\begin{bmatrix} 3 & 4 & 5 \\ 2 & 8 & 3 \\ 4 & 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 0 \\ 5 & 1 & -3 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 5 \\ 7 & 9 & 0 \\ 5 & 3 & 4 \end{bmatrix}$$

**Note**: Only makes sense if $A$ and $B$ have the **same** dimensions

**Notation**: We refer the the entry in the $i^{\text{th}}$ row and $j^{\text{th}}$ column of the matrix $A$ as $a_{ij}$ or $A[i][j]$

# Matrices and Matrix Operations

**Complexity of Matrix Addition:**

Simple calculation. We add each pair of entries

There are $n \cdot n = n^2$ entries

So matrix addition requires $n^2$ additions (which is $\Theta(n^2)$)

# Matrices and Matrix Operations

**procedure** $\texttt{MatrixAddition}(A, B)$

$C := \mathbf{0}$ # Initialize $C$ to $n \times n$ zero matrix

**for** $i := 1$ to $n$

    **for** $j := 1$ to $n$

        $C[i][j] := A[i][j] + B[i][j]$

**return** $C$

Turn loop into summations and count number of basic operations

**Complexity**: $\sum_{i=1}^{n} \sum_{j=1}^{n} 1 = \sum_{i=1}^{n} n = n^2$

## Matrices and Matrix Operations

$$\begin{aligned} 3x + 4y + 5z &= 1 \\ 2x + 8y + 3z &= 2 \\ 4x + 2y + 2z &= 3 \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} 3 & 4 & 5 \\ 2 & 8 & 3 \\ 4 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Matrices can **multiply** vectors, resulting in a new vector.

Operation is defined directly from the analogy between matrix equation and system of equations

$$\begin{bmatrix} 3 & 4 & 5 \\ 2 & 8 & 3 \\ 4 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \Leftrightarrow \quad \begin{matrix} 3x + 4y + 5z \\ 2x + 8y + 3z \\ 4x + 2y + 2z \end{matrix}$$

## Matrices and Matrix Operations

$$\begin{bmatrix} 3 & 4 & 5 \\ 2 & 8 & 3 \\ 4 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3\cdot 1 + 4\cdot 2 + 5\cdot 3 \\ 2\cdot 1 + 8\cdot 2 + 3\cdot 3 \\ 4\cdot 1 + 2\cdot 2 + 2\cdot 3 \end{bmatrix} = \begin{bmatrix} 26 \\ 27 \\ 14 \end{bmatrix}$$

Think of it as putting the vector over the matrix, multiplying down the columns, and adding

$$\begin{array}{c} \begin{matrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \end{matrix} \\ \begin{bmatrix} 3 & 4 & 5 \\ 2 & 8 & 3 \\ 4 & 2 & 2 \end{bmatrix} \end{array} = \begin{bmatrix} 3\cdot 1 + 4\cdot 2 + 5\cdot 3 \\ 2\cdot 1 + 8\cdot 2 + 3\cdot 3 \\ 4\cdot 1 + 2\cdot 2 + 2\cdot 3 \end{bmatrix}$$

**Note**: Length of vector **must equal** num of columns in matrix

# Matrices and Matrix Operations

**Example**: Compute $A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 3 \\ -1 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

# Matrices and Matrix Operations

**Pseudocode and Complexity**: Let $A$ be $n \times n$ and $\mathbf{x}$ be length $n$

**procedure** $\texttt{MatVec}(A, x)$

$y := \mathbf{0}$ # Initialize $y$ to $n$-length zero vector

**for** $i := 1$ to $n$ # Do one row at a time

**for** $j := 1$ to $n$ # Loop over entries in $i^{\text{th}}$ row

$y[i] \; += \; A[i][j] * x[j]$ # Multiply and accumulate in y

**return** $y$

# Matrices and Matrix Operations

**procedure** $\texttt{MatVec}(A, x)$

$y := \mathbf{0}$ # Initialize $y$ to $n$-length zero vector

**for** $i := 1$ to $n$ # Do one row at a time

**for** $j := 1$ to $n$ # Loop over entries in $i^{\text{th}}$ row

$y[i] \ += \ A[i][j] * x[j]$ # Multiply and accumulate in y

**return** $y$

Could count additions and multiplications. Usually combine and count FLOPs (floating point operations) instead

**Complexity**: $\sum_{i=1}^{n} \sum_{j=1}^{n} 2 = \sum_{i=1}^{n} 2n = 2n^2$

## Matrices and Matrix Operations

We can also multiply matrices together

**Example:**

$$\begin{bmatrix} 3 & 4 & 5 \\ 2 & 8 & 3 \\ 4 & 2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 14 & 16 \\ 25 & 14 & 21 \\ 4 & 6 & 10 \end{bmatrix}$$

**Process:** Think of it as multiple matrix-vector products

$$AB = A\,[\mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \mathbf{b}_3] = [A\mathbf{b}_1 \;\; A\mathbf{b}_2 \;\; A\mathbf{b}_3]$$

**Question:** Which dims of $A$ and $B$ must match for this to work?

# Matrices and Matrix Operations

$$\begin{bmatrix} 3 & 4 & 5 \\ 2 & 8 & 3 \\ 4 & 2 & 2 \end{bmatrix} \begin{bmatrix} \mathbf{-1} & 0 & 1 \\ \mathbf{3} & 1 & 2 \\ \mathbf{1} & 2 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{14} & 14 & 16 \\ \mathbf{25} & 14 & 21 \\ \mathbf{4} & 6 & 10 \end{bmatrix}$$

**Process:** Think of it as multiple matrix-vector products

$$AB = A\,[\mathbf{b}_1\ \ \mathbf{b}_2\ \ \mathbf{b}_3] = [A\mathbf{b}_1\ \ A\mathbf{b}_2\ \ A\mathbf{b}_3]$$

$$\begin{bmatrix} 3 & 4 & 5 \\ 2 & 8 & 3 \\ 4 & 2 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} =$$

## Matrices and Matrix Operations

$$\begin{bmatrix} 3 & 4 & 5 \\ 2 & 8 & 3 \\ 4 & 2 & 2 \end{bmatrix} \begin{bmatrix} -1 & \mathbf{0} & 1 \\ 3 & \mathbf{1} & 2 \\ 1 & \mathbf{2} & 1 \end{bmatrix} = \begin{bmatrix} 14 & \mathbf{14} & 16 \\ 25 & \mathbf{14} & 21 \\ 4 & \mathbf{6} & 10 \end{bmatrix}$$

**Process:** Think of it as multiple matrix-vector products

$$AB = A\,[\mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \mathbf{b}_3\,] = [A\mathbf{b}_1 \;\; A\mathbf{b}_2 \;\; A\mathbf{b}_3\,]$$

$$\begin{bmatrix} 3 & 4 & 5 \\ 2 & 8 & 3 \\ 4 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} =$$

# Matrices and Matrix Operations

$$\begin{bmatrix} 3 & 4 & 5 \\ 2 & 8 & 3 \\ 4 & 2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 & \mathbf{1} \\ 3 & 1 & \mathbf{2} \\ 1 & 2 & \mathbf{1} \end{bmatrix} = \begin{bmatrix} 14 & 14 & \mathbf{16} \\ 25 & 14 & \mathbf{21} \\ 4 & 6 & \mathbf{10} \end{bmatrix}$$

**Process**: Think of it as multiple matrix-vector products

$$AB = A\,[\mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \mathbf{b}_3] = [A\mathbf{b}_1 \;\; A\mathbf{b}_2 \;\; A\mathbf{b}_3]$$

$$\begin{bmatrix} 3 & 4 & 5 \\ 2 & 8 & 3 \\ 4 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} =$$

## Matrices and Matrix Operations

$$AB = A\,[\mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \mathbf{b}_3\,] = [A\mathbf{b}_1 \;\; A\mathbf{b}_2 \;\; A\mathbf{b}_3\,]$$

**Question**: Which dims of $A$ and $B$ must match for this to work?

**Question**: What are the dimensions of $C = AB$?

**Summary**: If $A$ is $m \times n$ and $B$ is $n \times k$ then $AB$ is $m \times k$

# Matrices and Matrix Operations

**Example**: Find the **Complexity**: $C = AB$ where $A$ and $B$ are both $n \times n$

# Matrices and Matrix Operations

**Summary:**

- Matrix addition is $\Theta(n^2)$
- Matrix-vector multiplication is $\Theta(n^2)$
- Matrix-Matrix multiplication is $\Theta(n^3)$

What we've done:

- ❖ Complexity of Algorithms
- ❖ Matrix Operations

Next:

**Induction!**