1 0 1 1 0 0 0 0 1 0 0 0   1 0 1 1 0 0 0 0 1 0 0 0   1 0 1 1 0 0 0 0 1 0 0 0   1 0 1 1 0 0 0 0 1 0 0 0   1 0 1 1 0 0 0 0 1 0 0 0

# CSCI 2824: Discrete Structures

# Lecture 10: Introduction to Proofs

Assume $a = b$ and neither are equal to 0.

$$ab = b^2$$
$$ab - a^2 = b^2 - a^2$$
$$a(b-a) = (b-a)(b+a)$$
$$a = b + a$$
$$a = a + a \text{ (since a=b)}$$
$$a = 2a$$

$$\therefore 1 = 2$$

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Department of Computer Science

1 0 1 1 0 0 0 0 1 0 0 0   1 0 1 1 0 0 0 0 1 0 0 0   1 0 1 1 0 0 0 0 1 0 0 0   1 0 1 1 0 0 0 0 1 0 0 0   1 0 1 1 0 0 0 0 1 0 0 0

# Introduction to Proofs

So how do we prove a statement of the form $\forall x\ (P(x) \to Q(x))$?

1. Prove $P(c) \to Q(c)$ for **arbitrary** $c$
2. Conclude $\forall x\ (P(x) \to Q(x))$ by universal generalization

This is what we really do, but we don't usually verbalize Step 2

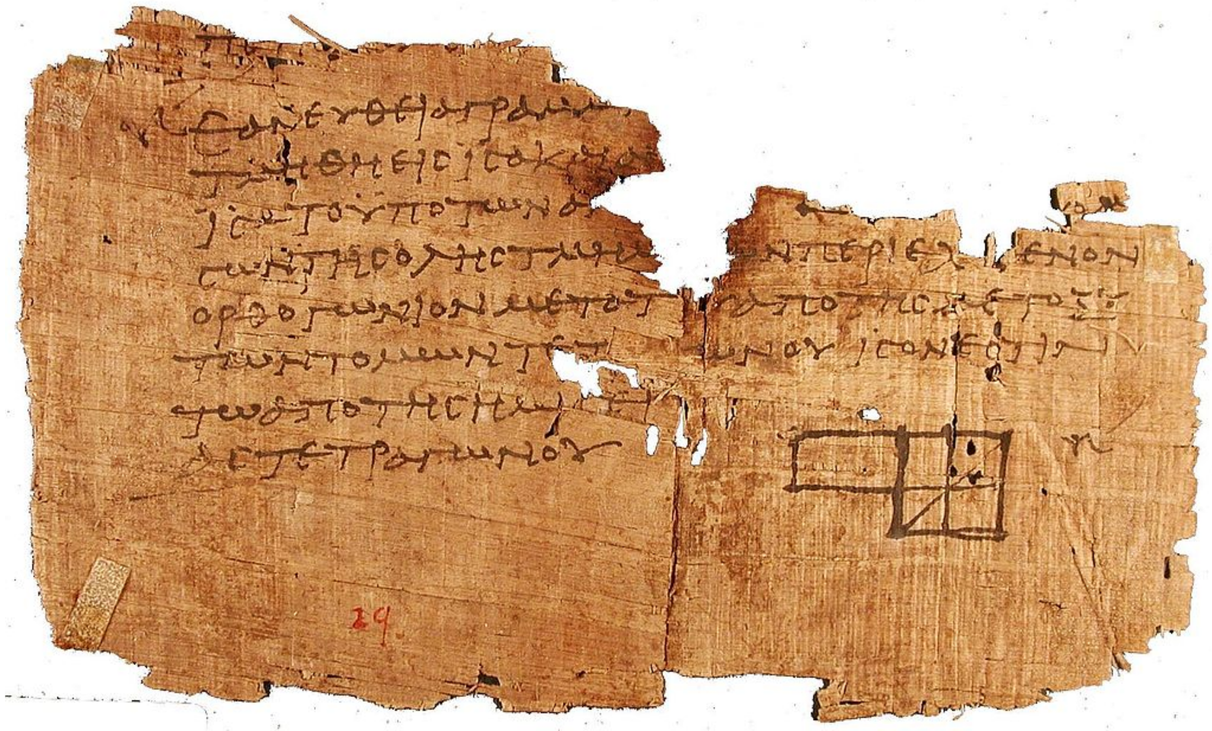
OK, so how to we prove $P(c) \to Q(c)$?

- Direct or Conditional Proof
- Contrapositive Proof
- Proof by Contradiction

# Introduction to Proofs

**Direct Proof**: We want to show that $p \to q$ is true.

**Strategy**:
- Assume $p$ is true.
- Proceed through rules of inference, mathematical facts, axioms, etc. as necessary until we find that $q$ is true.



A fragment from Euclid's *Elements*

# Introduction to Proofs

**Example**: If $a$ divides $b$ and $b$ divides $c$, then $a$ divides $c$.

## Introduction to Proofs

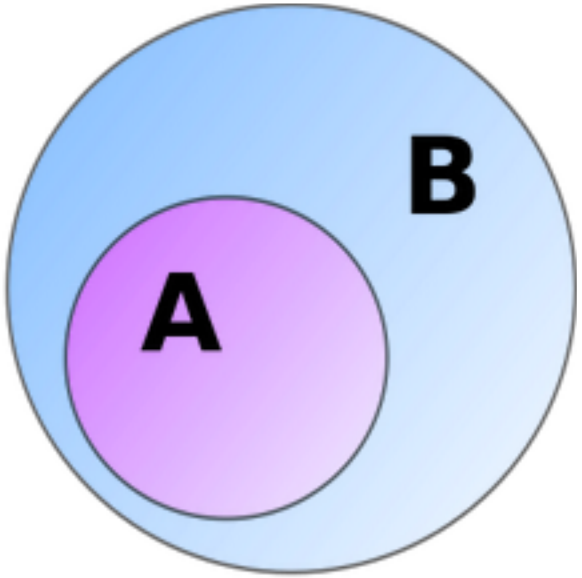
**Example**: If n is a four-digit palindrome then n is divisible by 11.

# Introduction to Proofs

**Contraposition Proof**: We want to show that $p \to q$ is true. If that is difficult to show directly, we apply a direct proof to prove the logically equivalent contrapositive statement $\neg q \to \neg p$

**Strategy**:

- Assume $\neg q$ is true.
- Proceed through rules of inference, mathematical facts, axioms, etc. as necessary until we find that $\neg p$ is true.




If $p \to q$, then $\neg q \to \neg p$

# Introduction to Proofs

**Example**: If $n = ab$, where $a$ and $b$ are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

# Introduction to Proofs

**Example**: If $x^2(y+3)$ is even, then $x$ is even or $y$ is odd.

# Introduction to Proofs

**Proof by Contradiction**: We want to show that $p \rightarrow q$ is true. Assume $p$ is true and $\neg q$ is true, then derive a contradiction. Alternatively, when proving $p$ is true, assume ¬p and derive a contradiction.

**Strategy**:

- Assume $p$ and $\neg q$, derive a contradiction.
- Alternatively, when proving $p$ is true, assume $\neg p$, then derive a contradiction.

# Introduction to Proofs

**Example**: Prove that if $3n + 2$ is odd, then $n$ is odd.

# Introduction to Proofs

We wanted to prove $p \to q$

The argument form that we just used looked as follows

$$((p \wedge \neg q) \to \mathbf{F}) \to (p \to q)$$

| $p$ | $q$ | $\neg q$ | $p \wedge \neg q$ | $(p \wedge \neg q) \to \mathbf{F}$ | $p \to q$ | $((p \wedge \neg q) \to \mathbf{F}) \to (p \to q)$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $F$ | $F$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $T$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $F$ | $F$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $F$ | $T$ | $T$ | $T$ |

The argument is a **tautology** so it is valid

# Introduction to Proofs

**Example**: Prove that $\sqrt{2}$ is irrational.

# Introduction to Proofs

**Example**: Integer $n$ is even if and only if $3n + 5$ is odd.

- **Proving a biconditional**: $p \leftrightarrow q$ is logically equivalent to $(p \rightarrow q) \vee (q \rightarrow p)$

# Introduction to Proofs