University of Colorado **Boulder**

CSCI 2824: Discrete Structures

Fall 2018 Tony Wong

Lecture 6: Predicates and Quantifiers

Man, those logic classes are coming in handy.

Announcements and reminders

- Homework 2 (written) is due **Friday 14 Sept** (posted to Piazza, and course calendar)

What did we do last time?

- **Logical equivalence**
  (do these statements mean the same thing?)
- And **constructing arguments/proofs** using
  logical equivalences

Today:

- We talk **predicate logic**

  → a more flexible framework for constructing
  arguments and proofs!

Predicate logic

Consider the following statements:

- All of my dogs love ham.
- Corky is one of my dogs.

You can probably figure out that Corky loves ham…

… but propositional logic lacks the flexibility to show this easily. You would need to do something like

All of my dogs love ham = Corky loves ham ∧ Citra loves ham ∧ Archer loves ham

This gets ridiculous as the number of dogs you are talking about grows.
… And lots of dogs love ham!

Predicate logic

Two new constructs in **predicate logic** help us achieve this flexibility:

predicates and **quantifiers**

Consider the statement $x > 3$.

- x is a variable or a placeholder
- > 3 is a **predicate**

Let $P(x)$ represent $x > 3$

We call $P(x)$ a **propositional function**. When we assign a value to x, then $P(x)$ becomes a proposition and has a truth value.

Example: $P(4)$ is T, and $P(1)$ is F.

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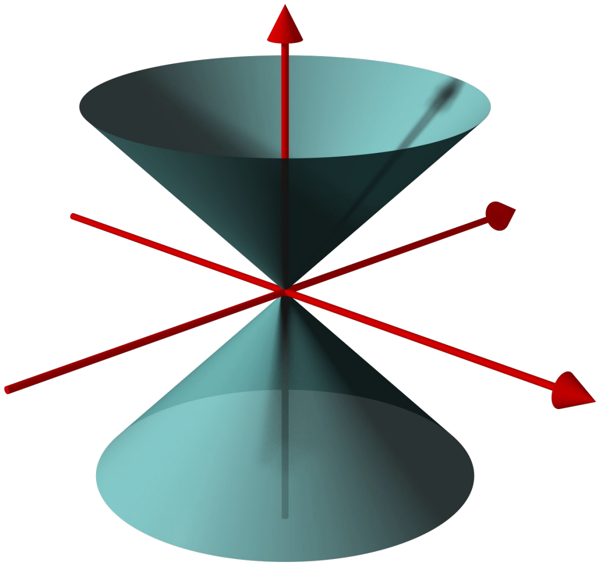
Predicate logic

Propositional functions can have multiple variables.

Example: Let $Q(x,y)$ represent $x + 1 = y$

- What is the truth value of $Q(1,2)$?
- What is the truth value of $Q(3,2)$?



Example: Let $R(x,y,z)$ represent $x^2 + y^2 = z^2$

- What is the truth value of $R(1,1,1)$?
- What is the truth value of $R(3,4,5)$?

Predicate logic

We see propositional functions in computer science all the time.

- **If / Then statements**
  - `if ( x > 10 ) then BREAK`
- **While loops**
  - `while ( i ≤ 10 ) do [stuff]`
- **Error checking**
  - `assert ( len(variable) ≥ 7 )`

Predicate logic

When using predicates, we need to think about what values we input

Example: * All of my dogs love ham.
* Corky is one of my dogs.

- Let *P(x)* represent "*x* loves ham"
- *P(Corky)* is T
- What about *P(7)*?

Definition: The set of values we intend to plug into the propositional function is called the **domain of discourse** or just the **domain**.

Example (cont.)

Okay, so we can set the **domain** to {*Corky, Citra, Archer*}

How can we say “All of my dogs love ham” ?

Of course we could say

$$P(Corky) \land P(Citra) \land P(Archer)$$

but that still runs into problems if you are dealing with many dogs.

Example (cont.)

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but that still runs into problems if you are dealing with many dogs.

Instead, we introduce the **universal quantifier**: $\forall$

- $\forall x\, P(x)$ means "for all x in my domain, $P(x)$"
- So for our example, $\forall x\, P(x)$ means "for all dogs in the set {*Corky, Citra, Archer*} (i.e., all of my dogs), those dogs love ham" $\Rightarrow$ So, "all of my dogs love ham"
- Note: the quantifier $\forall$ turns $\forall x\, P(x)$ into a proposition

Predicate logic

Question: When is the proposition $\forall x\, P(x)$ true?

Question: When is the proposition $\forall x\, P(x)$ false?

FYOG: What is the truth value of $\forall x\, (x^2 \geq x)$ if the domain is the real numbers?

Predicate logic

Question: When is the proposition $\forall x\, P(x)$ true?

- **Answer**: It is true when *P(x)* is true for all *x* in the domain.
- **Example**: Let the domain be all integers. $\forall x\, (x^2 \geq 0)$ is **true**.
    - (Any integer squared is non-negative.)

Question: When is the proposition $\forall x\, P(x)$ false?

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Question: When is the proposition $\forall x\, P(x)$ false?

- **Answer**: It is false if there is **any** x in the domain such that $P(x)$ is false.
- **Example**: Let the domain be all integers. $\forall x\, (x^2 > x)$ is **false**.
    - The case where $x = 0$ "breaks" the universal statement that *for all* integers x, $x^2 > x$
    - A case that demonstrates the breaking of a universal statement is called a **<u>counterexample</u>**.

FYOG: What is the truth value of $\forall x\, (x^2 \geq x)$ if the domain is the real numbers?

Predicate logic

Important take-aways:

- To **disprove** a universal proposition, all you need is **one** specific counterexample that makes the statement not work.
    - “All computer science instructors have gray hair.”

Predicate logic

Important take-aways:

- To **disprove** a universal proposition, all you need is **one** specific counterexample that makes the statement not work.
    - “All computer science instructors have gray hair.”

- To **prove** a universal proposition, **one** specific example **does nothing**. Usually, you need to work much, much harder.
    - Even if you find a hundred computer science instructors with gray hairs, you have not proved that *all* of us have gray hair.
    - **Never** try to prove a universal statement by demonstrating its truth for a set of specific examples and then asserting *P(x)* to be true in general.
        - (that’s why we have gray hairs)

Predicate logic

So that's how you talk about **all things**

What if we just want to talk about **something**?

- Often, we just want to show that *something* is possible. In mathematical terms, that means showing that *there exists* an element in the domain that has some certain property in which we are interested.
  - "Are there any flights that will get me home on time?"
  - "Does there exist a green men's size 9 Croc in this store?"

Predicate logic

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  - "Are there any flights that will get me home on time?"
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- For this, we introduce the **existential quantifier**: $\exists$
  - $\exists x\, P(x)$ means "there exists an x in the domain, $P(x)$"
  - **Example**: $\exists x\, P(x)$ could mean "there exists a dog [in my domain] such that that dog loves ham"

Predicate logic

Question: When is the statement $\exists x\, P(x)$ true?

Question: When is the statement $\exists x\, P(x)$ false?

Predicate logic

Question: When is the statement $\exists x\, P(x)$ true?

- **Answer**: It is true if you can find at least one x in the domain such that $P(x)$ is true.
- **Example**: Let the domain be the integers. $\exists x\, (x^2 > x)$ is **true**.
  - $x = 2$ works, because $2^2 = 4 > 2$ ✔

Question: When is the statement $\exists x\, P(x)$ false?

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- **Example**: Let the domain be the integers. $\exists x\, (x^2 > x)$ is **true**.
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Question: When is the statement $\exists x\, P(x)$ false?

- **Answer**: It is false if there is no x in the domain that makes $P(x)$ is true.
- **Example**: Let the domain be the integers. $\exists x\, (x^2 < 0)$ is **false**.

Predicate logic

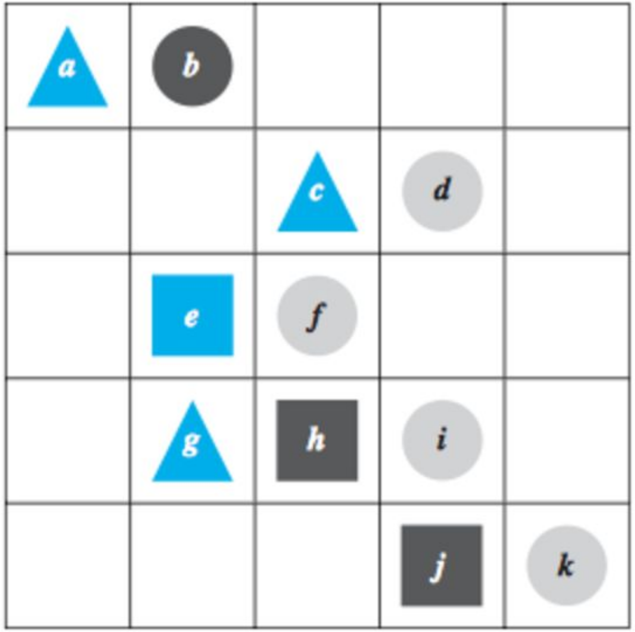
FYOG: Use the **Tarski world** below to determine the truth values for the following propositions.

(a) $\forall x\, (\text{Triangle}(x) \rightarrow \text{Blue}(x))$

(b) $\forall x\, (\text{Blue}(x) \rightarrow \text{Triangle}(x))$

(c) $\exists y\, (\text{Square}(y) \land \text{RightOf}(\text{d}, y))$

(d) $\exists y\, (\text{Square}(y) \land \text{Gray}(y))$



Predicate logic

Special cases: The Empty Domain

- If the domain is empty, then $\forall x\, P(x)$ is **true**.
- Nothing qualifies as an x, so there is nothing to make the statement false.
- This is called a **vacuously true** statement.

  Example: “All of the DeLoreans I have owned are also time machines.”

- If the domain is empty, then $\exists x\, P(x)$ is **false**.
- Nothing qualifies as an x, so certainly nothing can make $P(x)$ true.

  Example: “One of the DeLoreans I have owned was a time machine.”

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Scope of quantifiers

Quantifiers have the *narrowest scope* of all logical operands.

Example: $\forall x\ (P(x) \land Q(x))$ is not the same as $\forall x\ P(x) \land Q(x)$

- In the first one, $\forall x$ is applied to the compound propositional function $P(x) \land Q(x)$, but in the second one $\forall x$ is only applied to $P(x)$.

- Note that the second one is not even a proposition anymore because $Q(x)$ does not have a truth value unless we assign a value to x.

Logical equivalence involving quantifiers

Predicate/quantifier statements: **logically equivalent** if and only if they have the same truth value no matter which predicates you use and which domain they are defined over.

Example: Are these equivalent?

$$\forall x\, (P(x) \land Q(x)) \equiv \forall x\, P(x) \land \forall x\, Q(x)$$

Intuition: let …

domain = all of my dogs, *P(x)* = "loves ham", *Q(x)* = "has a short tail"

What do you think?

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Example: Are these equivalent? $\forall x\,(P(x) \land Q(x)) \equiv \forall x\,P(x) \land \forall x\,Q(x)$

They are! Let's prove it.

- Let domain = "all of my dogs"
- Let *i* index the set of all of my dogs. Like, $x_1 = Citra$, $x_2 = Corky$, $x_3 = Archer$, …

Then …

$$\forall x\,P(x) \equiv P(x_1) \land P(x_2) \land P(x_3) \land P(x_4) \land \ldots$$

and

$$\forall x\,P(x) \land \forall x\,Q(x) \equiv (P(x_1) \land P(x_2) \land P(x_3) \land \ldots) \land (Q(x_1) \land Q(x_2) \land Q(x_3) \land \ldots)$$

Rearrange the order (associativity):

$$\forall x\,P(x) \land \forall x\,Q(x) \equiv (P(x_1) \land Q(x_1)) \land (P(x_2) \land Q(x_2)) \land (P(x_3) \land Q(x_3)) \land \ldots$$

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Then …

$$\forall x\, P(x) \equiv P(x_1) \land P(x_2) \land P(x_3) \land P(x_4) \land \ldots$$

and

$$\forall x\, P(x) \land \forall x\, Q(x) \equiv (P(x_1) \land P(x_2) \land P(x_3) \land \ldots) \land (Q(x_1) \land Q(x_2) \land Q(x_3) \land \ldots)$$

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Logical equivalence involving quantifiers

Cool! So you can distribute $\forall$ across conjunctions ($\wedge$)

Can you do the same thing with disjunctions? ($\vee$)

Example: Are these equivalent?

$$\forall x\, (P(x) \vee Q(x)) \equiv \forall x\, P(x) \vee \forall x\, Q(x)$$

Logical equivalence involving quantifiers

Cool! So you can distribute $\forall$ across conjunctions ($\wedge$)

Can you do the same thing with disjunctions? ($\vee$)

Example: Are these equivalent?

$$\forall x\ (P(x) \vee Q(x)) \equiv \forall x\ P(x) \vee \forall x\ Q(x)$$

Answer: No!

Counterexample:

Logical equivalence involving quantifiers

A Slightly Different Example: Are these equivalent?

$$\exists x\, (P(x) \lor Q(x)) \equiv \exists x\, P(x) \lor \exists x\, Q(x)$$

Logical equivalence involving quantifiers

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$$\exists x\, (P(x) \lor Q(x)) \equiv \exists x\, P(x) \lor \exists x\, Q(x)$$

Answer: They are! *Demonstration* for the domain {*a*, *b*, *c*} (**not** a proof!)

Recall the original idea behind the existential quantifier: $\exists x\, P(x) \equiv P(a) \lor P(b) \lor P(c)$

Logical equivalence involving quantifiers

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Recall the original idea behind the existential quantifier: $\exists x\, P(x) \equiv P(a) \lor P(b) \lor P(c)$

LHS: $\exists x\, (P(x) \lor Q(x)) \equiv (P(a) \lor Q(a)) \lor (P(b) \lor Q(b)) \lor (P(c) \lor Q(c))$

$\equiv (P(a) \lor P(b) \lor P(c)) \lor (Q(a) \lor Q(b) \lor Q(c))$
(associativity)

RHS: $\exists x\, P(x) \lor \exists x\, Q(x) \equiv (P(a) \lor P(b) \lor P(c)) \lor (Q(a) \lor Q(b) \lor Q(c))$

Logical equivalence involving quantifiers

Yet Another Similar But Different Example: Are these equivalent?

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Logical equivalence involving quantifiers

Yet Another Similar But Different Example: Are these equivalent?

$$\exists x\,(P(x) \wedge Q(x)) \equiv \exists x\,P(x) \wedge \exists x\,Q(x)$$

Answer: No - (**FYOG**, think about it!)

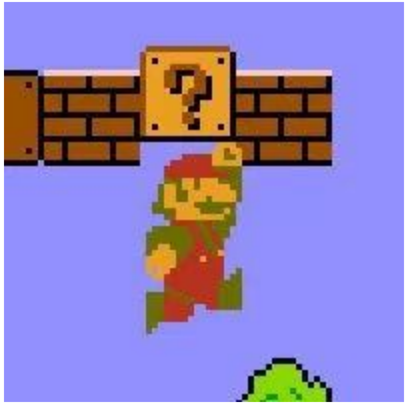
To recap some of the two rules we found to be true:

1. $\forall x\ (P(x) \land Q(x)) \equiv \forall x\ P(x) \land \forall x\ Q(x)$ distribution of universal quantifier ($\forall$) over conjunctions

2. $\exists x\ (P(x) \lor Q(x)) \equiv \exists x\ P(x) \lor \exists x\ Q(x)$ distribution of existential quantifier ($\exists$) over disjunctions

… next time: ***negations!***

Bonus material!



FYOG: What is the truth value of $\forall x\,(x^2 \geq x)$ if the domain is the real numbers?

False - consider x = ½ (or any other fraction between $0 < x < 1$)

Then we would have x^2 = ¼ , which is $< x$

Thus, we have a **counterexample**, proving the statement false *for all x* in the real numbers.

FYOG: Use the **Tarski world** below to determine the truth values for the following propositions.

(a) $\forall x\,(\text{Triangle}(x) \rightarrow \text{Blue}(x))$

This would only be False if there were some shape where Triangle(x) is True but Blue(x) is False. But the only 3 triangles are indeed blue, so this statement is True.

(b) $\forall x\,(\text{Blue}(x) \rightarrow \text{Triangle}(x))$

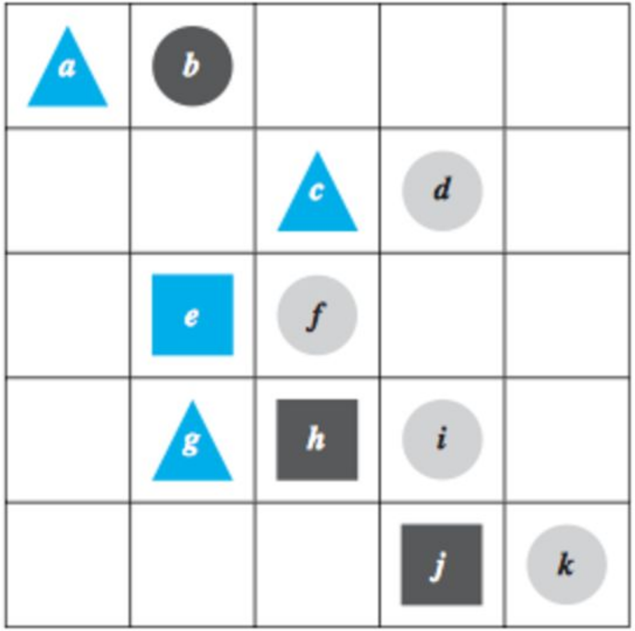
False - consider the blue square *e*

(c) $\exists y\,(\text{Square}(y) \land \text{RightOf}(d, y))$

False - no square exists to the right of shape *d*

(d) $\exists y\,(\text{Square}(y) \land \text{Gray}(y))$

True - *h* and *j* are both gray squares



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Answer: No - (**FYOG**, think about it!)

This is False, which we can demonstrate with a counterexample.

Let *P*(*x*) represent "*x* is an even integer" and let *Q*(*x*) represent "*x* is an odd integer", and suppose the domain is all integers.

Then the right-hand side is certainly True - there exist integers that are even, and there exist integers that are odd.

But the left-hand side is certainly False - there do not exist any integers that are even *and* odd.