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CSCI 2824: Discrete Structures

Lecture 16: Sequences

- Written HW 6 Due Friday at noon

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Cantor's diagonalization argument.

List all numbers between [8, 9]

1^{st} $8.\underline{0}1020304 \ldots\ldots$

2^{nd} $8.1\underline{1}1111121 2 1 \ldots$

3^{rd} $8.38\underline{7}593342 \ldots$

4^{th} $8.7764\ldots$

$\vdots$

want to show that we have not listed m.

$m = 8.7767\ldots$

this number can't have already been listed.

If the k^{th} digit is a 7, we set the k^{th} digit of m to 6 otherwise its a 7.

If we can't list something, we can't count it.

$\Rightarrow$ Real numbers are uncountable.

Theorem: If A and B are countable sets, then the union $A \cup B$ is also countable.

e.g. E: set of even natural numbers
O: set of odd natural numbers

$E \cup O = \mathbb{N}$ which is countable.

$\mathbb{R}$ are uncountable.
$\mathbb{Q}$ are countable.

What about irrational numbers? uncouble.

Sequences

A **sequence** is a function from a subset of the set of integers (usually either the set {0, 1, 2, . . .} or the set {1, 2, 3, . . .}) to a set S. We use the notation a_n to denote the image of the integer n. We call a_n a *term* of the sequence.

n is the index

function notation

$f(n) = \frac{1}{n+5}$

Example: Write out the first 5 terms of the sequence $a_n = \frac{1}{n+5}$

$a_0 = \frac{1}{5}$

$a_1 = \frac{1}{6}$

$a_2 = \frac{1}{7}$

$a_3 = \frac{1}{8}$

$a_4 = \frac{1}{9}$

Sequences

A **geometric sequence** (or geometric progression) is a sequence of the form:

$$a, ar, ar^2, ar^3, \ldots, ar^n, \ldots$$

where the initial term a and the common ratio r are real numbers.

$$\boxed{r = \frac{a_{n+1}}{a_n}}$$

$a_0 \quad a_1 \quad a_2 \quad a_3$

Example: Find the n^{th} term of the following sequence: $1, -\frac{3}{4}, \frac{9}{16}, -\frac{27}{64}, \ldots$

$$r = \frac{a_1}{a_0} = \frac{-3/4}{1} = -3/4 \qquad a = 1$$

$$r = \frac{a_2}{a_1} = \frac{9/16}{-3/4} = -3/4$$

$a_0 = 1$

$a_1 = -\frac{3}{4}$

$a_2 = \frac{9}{16} = \frac{3^2}{4^2}$

$a_3 = -\frac{27}{64} = -\frac{3^3}{4^3} \ldots$

$a_n = (-1)^n \left(\frac{3}{4}\right)^n$

$$\boxed{a_n = (-3/4)^n}$$

Sequences

An **arithmetic sequence** (or arithmetic progression) is a sequence of the form:

$$a, a + d, a + 2d, a + 3d, \ldots, a + nd, \ldots$$

where the initial term a and the common difference d are real numbers.

Example: Find the n^{th} term of the following sequence: $8, 3, -2, -7, -12, \ldots$

common difference: $d = -5$

first term: $a = 8$

$a_n = 8 - 5n$

start at

a_0

start with a_1

$a_1 = 8$

$a_2 = 3$

$a_3 = -2$:

$\boxed{a_n = 8 - 5(n-1)}$

Sequences

typo!

A recurrence relation for a sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer. A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

Example: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation

$$\boxed{a_n = a_{n-1} + a_{n-2} + a_{n-3}}$$

for $n = 3, 4, 5, \ldots$ and suppose that $a_0 = 3, a_1 = -1, a_2 = 4$. What are a_3, a_4, and a_5?

$$a_3 = a_2 + a_1 + a_0 = 4 + -1 + 3 = 6$$

$$a_4 = a_3 + a_2 + a_1 = 6 + 4 - 1 = 9$$

$$a_5 = a_4 + a_3 + a_2 = 9 + 6 + 4 = 19$$

$3, -1, 4, 6, 9, 19, 34 \ldots$

Sequences – Fibonacci

The Fibonacci sequence, $f_0, f_1, f_2, \ldots$, is defined by the initial conditions $\boxed{f_0 = 0, f_1 = 1,}$ and the recurrence relation

$$\underline{f_n = f_{n-1} + f_{n-2}}$$

for $n = 2, 3, 4, \ldots$

Example: Write out the first 10 terms of the Fibonacci sequence.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

Sequences

defining a_n with just n and numbers.

We say that we have solved the recurrence relation together with the initial conditions when we find an explicit formula, called a **closed formula**, for the terms of the sequence.

Example: Let $a_0 = 1, a_1 = 3,$ and $\boxed{a_n = 2a_{n-1} - a_{n-2}}$. Find a closed-form version of a_n.

$a_0 = 1$

$a_1 = 3$

$a_2 = 2a_1 - a_0 = 2 \cdot 3 - 1 = 5$

$a_3 = 2a_2 - a_1 = 2 \cdot 5 - 3 = 7$

$a_4 = 2a_3 - a_2 = 2 \cdot 7 - 5 = 9$

$a_n = 2n - 1$

or

$\boxed{a_n = 2n + 1}$

* **Sequences** closed form $a^m \cdot a^n = a^{m+n}$

Example: Show that $\boxed{a_n = 4^n}$ is a solution to $a_n = 8a_{n-1} - 16a_{n-2}$.

$$8a_{n-1} - 16a_{n-2} = 8 \cdot 4^{n-1} - 16 \cdot 4^{n-2}$$
$$= 2 \cdot 4^1 \cdot 4^{n-1} - 4^2 \cdot 4^{n-2}$$
$$= 2(4^n) - (4^n)$$
$$= 4^n$$
$$= a_n \quad \checkmark$$

Sequences

Example: Determine a recurrence relation for the even Fibonacci numbers.

$$\underline{0}, 1, 1, \underline{2}, 3, 5, \underline{8}, 13, 21, \underline{34}, 55, 89, \underline{144}$$

f_0 f_3 f_6 f_9 f_{12}

fibonacci: $f_n = f_{n-1} + f_{n-2}$

just evens: $f_n = a f_{n-3} + b f_{n-6}$

Sequences

Example: Determine a recurrence relation for the even Fibonacci numbers.

$$
\begin{aligned}
f_n &= f_{n-1} + f_{n-2} \\
&= (f_{n-2} + f_{n-3}) + (f_{n-3} + f_{n-4}) \\
&= f_{n-2} + 2f_{n-3} + f_{n-4} \\
&= f_{n-3} + f_{n-4} + 2f_{n-3} + f_{n-4} \\
&= 3f_{n-3} + 2f_{n-4} \\
&= 3f_{n-3} + f_{n-4} + f_{n-4} \\
&= 3f_{n-3} \; (+ f_{n-4} + (f_{n-5}) + f_{n-6})
\end{aligned}
$$

Sequences

Example: Determine a recurrence relation for the even Fibonacci numbers.

$$= 3f_{n-3} + \underbrace{f_{n-4} + f_{n-5}} + f_{n-6}$$

$$= 3f_{n-3} + f_{n-3} + f_{n-6}$$

$$\boxed{= 4f_{n-3} + f_{n-6}}$$

$E_2 = 4(2) + 0$
$= 8$

$E_3 = 4 \cdot 8 + 2$
$= 34$ ✓

represent just even fibonaccis

0	2	8	34	144 .....
$n=0$	$n=1$	$n=2$	$n=3$	$n=4$

E_n

$$\boxed{E_n = 4E_{n-1} + E_{n-2}}$$