Lecture 19: Proof by (weak) Induction

Announcements and reminders

- Homework 7 (Moodle) is posted and is due **Friday at 12 PM Noon**
- The CU final exam schedule is up. You must take your final exam during your scheduled final exam time.
  Tony's section: 7:30 - 10 PM, Sunday 16 Dec
  Rachel's section: 1:30 - 4 PM, Wednesday 19 Dec

Extra credit opportunity!   Purely attendance-based.

People respond to incentives and we **know attendance correlates with course grade**

- Consider this carefully if the exams/homeworks are not going as well as you'd like
- Download **Arkaive** -- attendance app (or go to https://arkaive.com/login)
- Enroll in this course with enrollment code: **1KG4** (9 AM w/ Rachel)
  R422 (11 AM w/ Tony)
- During first 15 minutes of each class, check in on Arkaive. You can get credit for attending either section, but not both.
- Remainder of course will be worth **2% pts extra credit** (*before* factored into any final curve)

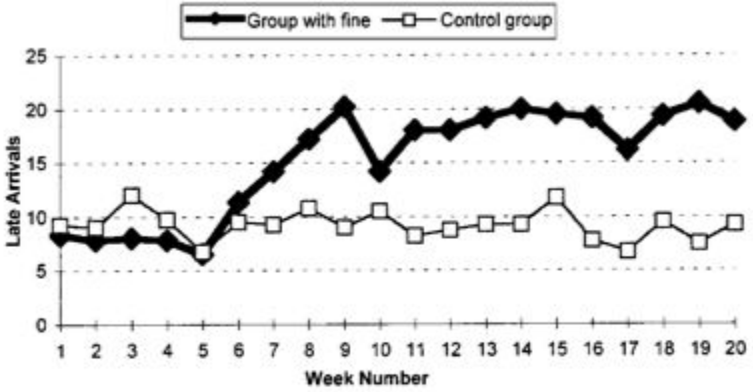

FIGURE 1.—Average number of late-coming parents, per week

What did we do last time?

- Finished off our **complexity** unit

Today:

- Back to some **proofs**. Oh boy!
- In particular, we look at ***proof by induction*****,** which is like a **run-away proof by cases**.

Mathematical induction

- Mathematical induction is a common and powerful way to prove properties of
  - Natural numbers
  - Sets
  - Relations
  - Trees and graphs (we'll get to these in a few weeks!)

- Let's start with a motivating example though, with (almost) no math!

Mathematical induction

S’pose you have an infinite line of dominos. (set up “correctly”, i.e., not too far apart)

Prove that if you tip over the first domino, then the rest of them will fall.

Mathematical induction

Your argument could go something like this:

First: The first domino falls (because you knocked it over)

Then: Whenever domino number k falls, the one after it numbered $k+1$ also will fall (because domino k knocks it over).

Therefore: We conclude that all of the dominoes will fall over.

Mathematical induction

Your argument could go something like this:

Base case: The first domino falls (because you knocked it over)

Inductive step: Whenever domino number *k* falls, the one after it numbered *k*+1 also will fall (because domino *k* knocks it over).

Therefore: We conclude that all of the dominoes will fall over.

Mathematical induction

Your argument could go something like this:

That kind of argument is the crux of **induction**. To prove a property holds for all natural numbers k, we argue as follows:

1. The property is true for $k = 0$ (or $k = 1$, or some other **base case**)
2. If the property is true for some natural number k, then it is true for natural number $k + 1$

(note that this is asking us to prove a conditional: [property true for k] → [true for k+1])

Mathematical induction

Example: Prove that $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$

Mathematical induction

Example: Prove that $1+2+\cdots+n=\dfrac{n(n+1)}{2}$

Base case: This formula holds for n=1: 1 = (1)(1+1)/2 = 1 ✔

Induction step: This requires a proof of the conditional

[formula works for n=k] → [formula works for n=k+1]

In general, the hypothesis of this conditional, "property holds for n=k", is known as the **inductive hypothesis**

So, s'pose the formula works for n = k (**inductive hypothesis**)

$$\Rightarrow \quad 1+2+\cdots+k=\frac{k(k+1)}{2}$$

$$\Rightarrow \quad 1+2+\cdots+k+(k+1)=\frac{k(k+1)}{2}+(k+1)$$

$$=\frac{k(k+1)}{2}+\frac{2(k+1)}{2}=\frac{(k+1)(k+2)}{2}$$

Mathematical induction

So if we let *P(n)* be the property that we're trying to prove, where *n* is some natural number, then an inductive argument goes like this:

1. Base case: Verify that *P(0)* holds (or *P(1)*, or *P(whatever)*)

2. Induction step: $(\forall k \geq 0)$ **if** $P(k)$ **then** $P(k+1)$

3. Conclusion: $(\forall n \geq 0)\ P(n)$

There are two slightly different kinds of inductive argument...

Mathematical induction

There are two slightly different kinds of inductive argument:

If the argument is of the form

- Verify that $P(1)$ is true
- Assume $P(k)$ is true, and show that $P(k+1)$ must be true   (the proof part)

then we call it **weak induction**, or **ordinary induction**.

If the argument is of the form

- Verify that $P(1)$ is true
- Assume $P(k)$ is true for all $k = 1, 2, \ldots, n$, and show $P(n+1)$

then we call it **strong induction**.

Mathematical induction

Example: Propose a formula for the sum of the first *n* odd positive integers. Then prove it using induction.

Exploration:

Conjecture:

Mathematical induction

Example: Propose a formula for the sum of the first *n* odd positive integers. Then prove it using induction.

Exploration: $1 = 1$

$1 + 3 = 4$

$1 + 3 + 5 = 9$

Conjecture: $1 + 3 + 5 + \ldots + (2n - 1) = n^2$

Now it's show time!

Mathematical induction

Example: Propose a formula for the sum of the first *n* odd positive integers. Then prove it using induction.

To show: $1 + 3 + 5 + \ldots + (2n - 1) = n^2$

Proof: (by weak induction)

Base case:

Inductive step:

Mathematical induction

Example: Propose a formula for the sum of the first n odd positive integers. Then prove it using induction.

To show: $1 + 3 + 5 + \ldots + (2n - 1) = n^2$

Proof: (by weak induction)

Base case: $n=1$, $1 = 1^2$ ✔

Inductive step: S'pose the formula holds for $n=k$ (that is, $1 + 3 + 5 + \ldots + (2k - 1) = k^2$)

$\Rightarrow$ $1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) = k^2 + (2k + 1)$

$\Rightarrow$ $1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) = (k + 1)^2$

Conclusion: We've shown that the formula holds for $n=k+1$, thus, by induction, we've proved that the formula is true in general. ⍂

Mathematical induction

Example: Prove that if n is an integer and $n \geq 4$, then $2^n < n!$

Base case:

Inductive step:

Conclusion:

Mathematical induction

Example: Prove that if n is an integer and $n \geq 4$, then $2^n < n!$

Base case: $n=4$, we have $2^{(4)} = 16$ and $4! = 24$, so it is true that $2^4 < 4!$ ✔

Inductive step: S'pose the formula holds for $n=k$ $(2^k < k!)$

$\Rightarrow\ 2^k < k!$

$\Rightarrow\ 2^k \cdot 2 < k! \cdot 2$

$\Rightarrow\ 2^{k+1} < k! \cdot 2 < k! \cdot (k+1)$ ($k+1$ definitely > 2, since $k \geq n \geq 4$)

$\Rightarrow\ 2^{k+1} < (k+1)!$

Conclusion: We've shown that the inequality holds for $n=k+1$, thus, by induction, we've proved that the inequality. ⍂

Mathematical induction

Example: (geometric progressions) Prove that when $r \neq 1$,

$$a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r - 1}$$

Base case:

Inductive step:

Mathematical induction

Example: (geometric progressions) Prove that when $r \neq 1$,

$$a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r - 1}$$

Base case: $n=0$, $\quad a = \dfrac{ar^1 - a}{r-1} = \dfrac{a(r-1)}{r-1} = a \quad$ ✔

Inductive step: S'pose the formula holds for $n=k$. Then…

$$\Rightarrow \quad a + ar + ar^2 + \cdots + ar^k + ar^{k+1} = \frac{ar^{k+1} - a}{r-1} + ar^{k+1}$$

$$= \frac{ar^{k+1} - a}{r-1} + \frac{ar^{k+1}(r-1)}{r-1}$$

$$= \frac{ar^{k+1} - a + ar^{k+2} - ar^{k+1}}{r-1} = \frac{ar^{k+2} - a}{r-1}$$

Conclusion:
Thus, by induction,
the formula is true ⧄

Mathematical induction

Example: Prove that if $n \geq 1$ is an integer, then $n^3 - n$ is divisible by 3

Mathematical induction

Example: Prove that if $n \geq 1$ is an integer, then $n^3 - n$ is divisible by 3

Base case: $n=1 \Rightarrow n^3-n = 1^3-1 = 0$, and $0=3(0)$ so 1^3-1 is divisible by 3.

Inductive step: S'pose k^3-k is divisible by 3, for some integer $k > 1$ (**inductive hypothesis**)

To show: $(k+1)^3 - (k+1)$ must also be divisible by 3.

$$
\begin{aligned}
\Rightarrow \quad (k+1)^3 - (k+1) &= (k^3 + 3k^2 + 3k + 1) - k - 1 \\
&= k^3 - k + 3k^2 + 3k \\
&= 3(\text{some integer}) + 3k^2 + 3k \qquad \leftarrow \text{by } \textbf{inductive hypothesis} \\
&= 3(\text{some integer} + k^2 + k)
\end{aligned}
$$

Therefore, $(k+1)^3 - (k+1)$ is divisible by 3, and the hypothesis is true by **weak induction** □

Mathematical induction

Example: Let F_n by the n^{th} Fibonacci number. Prove that $\sum_{k=0}^{n} F_k^2 = F_n F_{n+1}$

Mathematical induction

Example: Let F_n by the n^{th} Fibonacci number. Prove that $\sum_{k=0}^{n} F_k^2 = F_n F_{n+1}$

Base case: $n=0 \Rightarrow \sum_{k=0}^{0} F_k^2 = F_0^2 = 1^1 = 1$, and $F_0 \cdot F_1 = 1 \cdot 1$

Inductive step: S'pose the formula is true for some $m > 0$: $\sum_{k=0}^{m} F_k^2 = F_m F_{m+1}$ (ind. hypothesis)

To show: formula holds for $m+1$: $\sum_{k=0}^{m+1} F_k^2 = F_{m+1} F_{m+2}$

Mathematical induction

(... continued…)

Inductive step: S'pose the formula is true for some *m* > 0: $\sum_{k=0}^{m} F_k^2 = F_m F_{m+1}$ (ind. hypothesis)

To show: formula holds for *m*+1: $\sum_{k=0}^{m+1} F_k^2 = F_{m+1} F_{m+2}$

$$\sum_{k=0}^{m+1} F_k^2 = F_{m+1}^2 + \sum_{k=0}^{m} F_k^2$$

$$= F_{m+1}^2 + F_m F_{m+1}$$ ← by induction hypothesis

$$= F_{m+1}(F_{m+1} + F_m)$$

$$= F_{m+1} F_{m+2}$$ ← by definition of Fibonacci sequence

Therefore, the formula holds in general, by **weak induction** □

Mathematical induction

Recap:

- **Math induction** -- the proof technique that's a proof by cases on steroids
  - **Base case:** show that the hypothesis is true for the first case
  - **Inductive step:** name your inductive hypothesis
    ⇒ S'pose true at k, **To Show** true at k+1

Next time:

- We make our induction ***strong***

Bonus material!