Lecture 5: Logical equivalence

Announcements and Reminders

- ❒ Enroll in the class Moodle
  - Find it at: https://moodle.cs.colorado.edu
  - Search for “Cox/Wong” and you should be able to find it
- ❒ First homework (on Moodle) is due **Today! at 12pm Noon**
- ❒ Enroll in the class Piazza: https://piazza.com/colorado/fall2018/csci2824
- ❒ Keep track of everything via the schedule: https://goo.gl/DFuboZ
- ❒ CA office hours in CSEL (1st floor Engineering Center)
  - A link to the schedule is on Piazza

What did we do last time?

- **Truth tables:** How can we keep track of our propositions? How do their truth values relate to one another?
- **Solving riddles using truth tables**: Knights and Knaves

Today:

- **Logical equivalence** (do these statements mean the same thing?)

  … And **constructing arguments/proofs** using logical equivalences

Fond memory:

The **conditional:** If *p*, then *q*.

- $p \to q$
- The occurrence of *p* implies the occurrence of *q*.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

There are three other **conditionals** closely related to our old friend $p \to q$

- The **converse**: $q \to p$
- The **inverse**: $\neg p \to \neg q$
- The **contrapositive**: $\neg q \to \neg p$

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A natural question is then: **How are each of these related to $p \to q$** ?

A Demonstrative and Painful Example:
If it is snowing, then I crash my bike.

- **Converse**:
- **Inverse**:
- **Contrapositive**:

There are three other **conditionals** closely related to our old friend $p \to q$

- The **converse**: $q \to p$
- The **inverse**: $\neg p \to \neg q$
- The **contrapositive**: $\neg q \to \neg p$

A natural question is then: **How are each of these related to $\boldsymbol{p \to q}$** ?

A Demonstrative and Painful Example:
If it is snowing, then I crash my bike.

- **Converse**: If I crash my bike, then it is snowing.
- **Inverse**: If it is not snowing, then I do not crash my bike.
- **Contrapositive**: If I do not crash my bike, then it is not snowing.

Let’s check using a truth table.

				Conditional	Converse	Inverse	Contrapositive
p	q	$\neg p$	$\neg q$	$p \to q$	$q \to p$	$\neg p \to \neg q$	$\neg q \to \neg p$
T	T	F	F				
T	F	F	T				
F	T	T	F				
F	F	T	T				

Let's check using a truth table.

Definition: Two propositions are **logically equivalent** if they have the same truth values for all combinations of their constituents.

				Conditional	Converse	Inverse	Contrapositive
p	q	$\neg p$	$\neg q$	$p \to q$	$q \to p$	$\neg p \to \neg q$	$\neg q \to \neg p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Let's check using a truth table.

$\equiv$

Definition: Two propositions are **logically equivalent** if they have the same truth values for all combinations of their constituents.

Thus, the conditional $p \to q$ and its contrapositive $\neg q \to \neg p$ are logically equivalent.

We write: $\boxed{p \to q \equiv \neg q \to \neg p}$

				Conditional	Converse	Inverse	Contrapositive
p	q	$\neg p$	$\neg q$	$p \to q$	$q \to p$	$\neg p \to \neg q$	$\neg q \to \neg p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Example: Wason selection task

Consider the following four cards. They have letters on one side and numbers on the other. Suppose I tell you the following rule:

If a card has an odd number, then its letter is a vowel.

A	B	1	2

Question: What card(s) do you need to turn over in order to verify that the given rule is true?

Example: Wason selection task

Consider the following four cards. They have letters on one side and numbers on the other. Suppose I tell you the following rule:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

If a card has an odd number, then its letter is a vowel.

A	B	1	2

Question: What card(s) do you need to turn over in order to verify that the given rule is true?

Answer: Say *p* = card has an odd number and *q* = card has a vowel letter.

Then…

Rule to Check: $p \to q \equiv \neg q \to \neg p$

conson. → even

Example: Wason selection task

If a card has an odd number then its letter is a vowel.

A	B	1	2

Question: What card(s) do you need to turn over in order to verify that the given rule is true?

Answer: Say p = card has an odd number and q = card has a vowel letter.

- Then… the rule is the conditional $p \rightarrow q$.
- This only has a truth value of F if p is T but q is F.
- So we definitely need to check the card where p is T. **So need to flip over the 1.**

Example: Wason selection task

If a card has an odd number then its letter is a vowel.

A	B	1	2

Question: What card(s) do you need to turn over in order to verify that the given rule is true?

Answer: Say p = card has an odd number and q = card has a vowel letter.

- But the rule $p \to q$ is logically equivalent to another rule: $\neg q \to \neg p$
- So we need to check the card where $\neg q$ is T to verify that $\neg p$ is also true.
- **So we need to flip over B as well as the 1.**

Example: Wason selection task

- Experiment designed by Peter Wason in the 1960s.
- Demonstrates that if-then relationships may be intuitive to us, but contrapositives are not.
- If you did not immediately get it correct, you are in good company - **less than 10%** of Wason's original sample got it correct.

Example: Wason selection task

- Experiment designed by Peter Wason in the 1960s.
- Demonstrates that if-then relationships may be intuitive to us, but contrapositives are not.
- If you did not immediately get it correct, you are in good company - **less than 10%** of Wason's original sample got it correct.
- Interestingly, later work found it might be **context-specific**.
  - **For example:** If you are drinking a beer, you must be over 21.
  - Most participants successfully answer these "social rules" versions.

Logical equivalence

- We have found that $p \to q \equiv \neg q \to \neg p$. So. Who cares?
- Turns out, this can be **very** useful in proving things.
- Mathematical arguments/proofs:
  - progressing from a set of assumptions to useful/interesting conclusions
  - **logical equivalences** link the steps together
- To prove $p \to q$, you might suppose p is true, then work your way forward to show that it must be the case that q is true.
- **But** it might be easier to suppose that q is *false*, then work your way toward showing that it must be the case that p is also false.
  - And because $p \to q \equiv \neg q \to \neg p$, either way is valid.

Logical equivalence

$\ldots, -2, -1, 0, 1, 2, 3, \ldots$

Example: Suppose n is an integer. Prove that if $\underbrace{n^2 \text{ is even}}_{p}$, then $\underbrace{n \text{ must be even}}_{q}$.

$p \longrightarrow q$

Try the direct route first.

S'pose n^2 is even

$\rightarrow$ that means $n^2 = 2k$, where k is some integer

$\rightarrow$ uhh.... $n = \sqrt{2k}$? STUCK

Logical equivalence

Example: Suppose n is an integer. Prove that if n^2 is even, then n must be even.

Try the direct route first.

- Say n^2 is even.
- Then $n^2 = 2k$ for some integer k
- Then… Um. Well shoot.

Logical equivalence

Example: Suppose n is an integer. Prove that if $\underbrace{n^2 \text{ is even}}_{p}$, then $\underbrace{n \text{ must be even}}_{q}$.

$\neg p$: n^2 is odd

$\neg q$: n is odd

$\neg q \to \neg p$

Try the direct route first.

- Say n^2 is even.
- Then $n^2 = 2k$ for some integer k
- Then… Um. Well shoot.

Okay, let's try the contrapositive then: Prove that if n is odd, then n^2 is odd.

S'pose n is odd

$\to n = 2k+1$ for some integer k

$\to n^2 = (2k+1)^2 = \underbrace{4k^2 + 4k}_{2(2k^2+2k)} + 1 = 2\underbrace{(2k^2+2k)}_{\text{some integer! call it } \ell} + 1$

$\to n^2 = 2\ell + 1$, which is odd

$\blacksquare$ QED

Logical equivalence

Example: Suppose n is an integer. Prove that if n^2 is even, then n must be even.

Try the direct route first.

- Say n^2 is even.
- Then $n^2 = 2k$ for some integer k
- Then… Um. Well shoot.

Okay, let's try the contrapositive then: Prove that if n is odd, then n^2 is odd.

- S'pose n is an odd integer.
- Then for some integer k, $n = 2k+1$
- Then $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = \underbrace{2(2k^2 + 2k) + 1}_{\text{even number + 1 = odd number}}$
- Done!

Logical equivalence

Caveat (warning):

- Sometimes it is easier to prove the direct conditional.
  Sometimes it is easier to prove the contrapositive.
  Both are valid, but which is easier may depend on the situation.
- Don't just give up if one doesn't work. **Try hard.**

Some special cases:

Example: Tony is either wearing shoes, or he is not wearing shoes.

What can you say about this proposition?

Logical equivalence

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What can you say about this proposition?

This is a compound proposition, although maybe a silly one.

- Let p = Tony is wearing shoes.
- Then we have the proposition $p \vee \neg p$.
- Let’s look at a potentially silly truth table.

HERE

Some special cases:

Example: Tony is either wearing shoes, or he is not wearing shoes.

What can you say about this proposition?

This is a compound proposition, although maybe a silly one.

- Let p = Tony is wearing shoes.
- Then we have the proposition $p \vee \neg p$.
- Let's look at a potentially silly truth table.

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

Definition: A compound proposition that is always true is called a **tautology**.

Some other examples of tautologies:

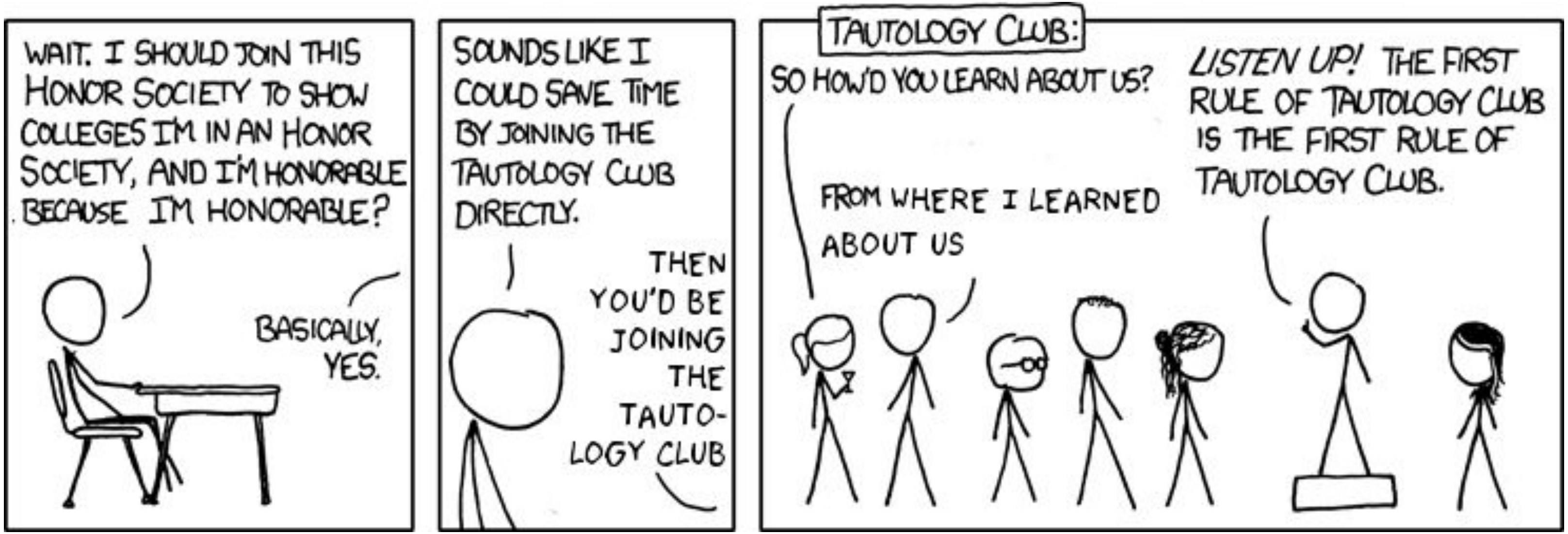
Remember: $p \to q$ is only F when $p = T$, $q = F$

a. $p \to p$

b. $\boxed{((p \to q) \wedge (q \to r)) \to (p \to r)}$

(b) has *three* constituent propositions (p, q and r).

Question: So how many rows would its truth table have?

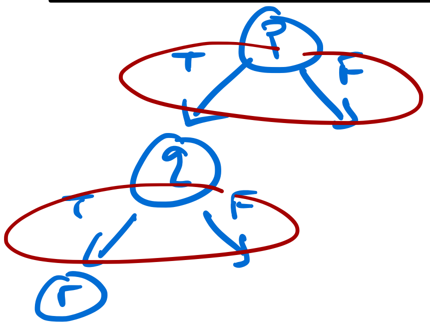


Some other examples of tautologies:

a. $p \rightarrow p$

b. $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

FYOG: Work out the 8-row truth table for this one and show that it is a tautology



(b) has *three* constituent propositions (p, q and r).

Question: So how many rows would its truth table have?

Answer: We need a row in our truth tables for each possible combination of the constituent propositions.

$$(2 \text{ poss.})^{3 \text{ prop.}} = 2^3 = 8$$

Recall that for our compound propositions with just p and q, (like $p \rightarrow q$) we had 4 rows.

- That's because 4 = (2 possible truth values)$^{(2 \text{ propositions})}$

In general, with n constituent propositions, we need 2^n rows in our truth table.

Logical equivalence

not a comp. prop

contrapositive

Example: We have recently learned that $p \rightarrow q \equiv \neg q \rightarrow \neg p$

- One way to create a tautology is to take two compound propositions that we know are logically equivalent, and join them with a biconditional, $\Leftrightarrow$
- So I propose that the following is a tautology: $(p \rightarrow q) \Leftrightarrow (\neg q \rightarrow \neg p)$ ← is a comp. prop.

Let's check with a truth table!

FYOG

Logical equivalence

Example: Check that $p \to q \Leftrightarrow \neg q \to \neg p$ is a tautology using a truth table.

p	q	$p \to q$	$\neg q \to \neg p$	$(p \to q) \Leftrightarrow (\neg q \to \neg p)$
T	T			
T	F			
F	T			
F	F			

Logical equivalence

Example: Check that $p \to q \Leftrightarrow \neg q \to \neg p$ is a tautology using a truth table.

also a taut! $\to \neg p \to \neg p$

All T, so $(p \to q) \Leftrightarrow (\neg q \to \neg p)$ **is a tautology.**

p	q	$p \to q$	$\neg q \to \neg p$	$(p \to q) \Leftrightarrow (\neg q \to \neg p)$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

all T, therefore this comp. prop. is a tautology

Logical equivalence

Aside from the opposite of a tautology… Haha!

Okay, so tautologies are cool. But **what is the opposite of a tautology**?

- Tautologies are *true* for all combinations of constituent truth values, so…
- **Definition:** a **contradiction** is a compound proposition that is **false** for all possible combinations of constituent proposition truth values.

→ col. in TT is all False

Logical equivalence

Okay, so tautologies are cool. But **what is the opposite of a tautology**?

- Tautologies are *true* for all combinations of constituent truth values, so…
- **Definition:** a **contradiction** is a compound proposition that is **false** for all possible combinations of constituent proposition truth values.

Example: Today it will rain and today it will not rain ($p \land \neg p$)

Other examples: Take any tautology and negate the whole thing.

"Tony is wearing shoes or Tony is not wearing shoes." $s \lor \neg s$ (tautology)

⇒ "Tony is *not* wearing shoes *and* Tony *is* wearing shoes." $\neg(s \lor \neg s)$

$\neg s \land s$

Logical equivalence

Okay, so tautologies are cool. But **what is the opposite of a tautology**?

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- **Definition:** a **contradiction** is a compound proposition that is **false** for all possible combinations of constituent proposition truth values.

Example: Today it will rain and today it will not rain ($p \land \neg p$)

Other examples: Take any tautology and negate the whole thing.

"Tony is wearing shoes or Tony is not wearing shoes."

⇒ "Tony is *not* wearing shoes *and* Tony *is* wearing shoes."

Definition: A compound proposition that is neither a tautology or a contradiction is a **contingency**.

any comp. prop. is exactly one of a taut., a contra., or a conting.

Logical equivalence

Different - but the same - ways of writing compound propositions

$\neg$ p q

Example: *It is not the case that Gary is boring and rides a motorcycle.* $\neg(p \wedge q)$

- Another way to express this:

  Either Gary is not boring or Gary does not ride a motorcycle.

  $\neg p \vee \neg q$

Logical equivalence

Different - but the same - ways of writing compound propositions

Example: *It is not the case that Gary is boring and rides a motorcycle.*

- Another way to express this:

  Either Gary is not boring or Gary does not ride a motorcycle.

- Let's break it down. Define:

  p = *Gary is boring* and q = *Gary rides a motorcycle*

Logical equivalence

Different - but the same - ways of writing compound propositions

Example: *It is not the case that Gary is boring and rides a motorcycle.*

- Another way to express this:

  Either Gary is not boring or Gary does not ride a motorcycle.

- Let's break it down. Define:

  p = *Gary is boring* and q = *Gary rides a motorcycle*

- Then the original compound proposition is: $\neg(p \land q)$
- And the revised (more normal-sounding) version is: $\neg p \lor \neg q$

→ Are they logically equivalent?

To the ~~Batmobile~~!!

truth table...mobile!!

Logical equivalence

p, q → $2^2 = 4$ rows

Example: (1) *It is not the case that Gary is boring and Gary rides a motorcycle.*
(2) *Either Gary is not boring or Gary does not ride a motorcycle.*

Are (1) and (2) logically equivalent? That is, $\neg(p \wedge q) \equiv \neg p \vee \neg q$?

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

invert

same!
logically equivalent

Logical equivalence

Example: (1) *It is not the case that Gary is boring and Gary rides a motorcycle.*
(2) *Either Gary is not boring or Gary does not ride a motorcycle.*

Are (1) and (2) logically equivalent? That is, $\neg(p \land q) \equiv \neg p \lor \neg q$?

p	q	$\neg p$	$\neg q$	$p \land q$	$\neg(p \land q)$	$\neg p \lor \neg q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Slightly Different Example: *It is not the case that Gary is either boring or rides a motorcycle.*

$\neg(p \vee q)$

Another way to express this:

Gary is neither boring nor rides a motorcycle.

$\neg p \wedge \neg q$

FYOG: Show that these two compound propositions are logically equivalent.

Hint: The first one is $\neg(p \vee q)$ and the second one is $\neg p \wedge \neg q$.

Find $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Logical equivalence

Those two previous examples combine to give us a powerful pair of logical manipulations that we can use to rearrange or simplify compound propositions.

De Morgan's Laws:

$\neg(p \land q) \equiv \neg p \lor \neg q$

$\neg(p \lor q) \equiv \neg p \land \neg q$

Think of these a the *distribution rules* for compound propositions. If you distribute the negation into a conjunction/disjunction of two propositions, then you get a disjunction/conjunction (the carrot flips direction) of the negated propositions.

This begs the question: Are there other manipulations like De Morgan's laws?

Logical equivalence

There sure are!

Consider the conditional: $p \to q$

- For example: *If it snows, then Tony crashes his bike*.
- Or rephrased: *Either it is not snowing or Tony crashes his bike.*

$p \to q$

$\overbrace{\text{If you steal my cookies}}^{p}$, then $\underbrace{\text{you're in big trouble}}_{q}$.

You do not steal my cookies or you're in big trouble

$$p \to q \stackrel{?}{\equiv} \neg p \vee q$$

Logical equivalence

There sure are!

Consider the conditional: $p \rightarrow q$

- For example: *If it snows, then Tony crashes his bike*.
- Or rephrased: *Either it is not snowing or Tony crashes his bike.*

> This is $\neg p \vee q$, known as **relation by implication**:
>
> $$p \rightarrow q \equiv \neg p \vee q$$

(RBI)

(**FYOG:** work out the truth table for this one)

… And many more…!

Logical equivalence

NB: On homework and exams, Table 6 (p. 27 of Rosen) and the following 4 are the only ones you can/should invoke:

$p \rightarrow q \equiv \neg p \lor q$ **relation by implication (RBI)**

$p \rightarrow q \equiv \neg q \rightarrow \neg p$ **contraposition**

$p \Leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p)$ **definition of biconditional**

$p \oplus q \equiv (p \lor q) \land \neg(p \land q)$ **alt. definition of xor**

TABLE 6 Logical Equivalences.

Equivalence	***Name***
$p \land \mathbf{T} \equiv p$ $p \lor \mathbf{F} \equiv p$	Identity laws
$p \lor \mathbf{T} \equiv \mathbf{T}$ $p \land \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \lor p \equiv p$ $p \land p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \lor q \equiv q \lor p$ $p \land q \equiv q \land p$	Commutative laws
$(p \lor q) \lor r \equiv p \lor (q \lor r)$ $(p \land q) \land r \equiv p \land (q \land r)$	Associative laws
$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$ $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$	Distributive laws
$\neg(p \land q) \equiv \neg p \lor \neg q$ $\neg(p \lor q) \equiv \neg p \land \neg q$	De Morgan's laws
$p \lor (p \land q) \equiv p$ $p \land (p \lor q) \equiv p$	Absorption laws
$p \lor \neg p \equiv \mathbf{T}$ $p \land \neg p \equiv \mathbf{F}$	Negation laws

Logical equivalence

So far, if we wanted to show that two compound propositions are logically equivalent, then we need to use a truth table.

Truth tables with more than a few propositions can become unwieldy.

These logical equivalences provide an elegant - and potentially much simpler! - alternative.

- Can construct a chain of logical equivalences starting from the first compound proposition and leading to the second one.
- This is *exactly* how we construct mathematically sound arguments.

Example: Show that $p \to q \equiv \neg q \to \neg p$ without using a truth table.

$p \to q \equiv \neg p \vee q$ RBI

$\equiv q \vee \neg p$ commutativity

$\equiv \neg(\neg q) \vee \neg p$ double negation

$\equiv \neg q \to \neg p$ RBI (backwards)

QED

Logical equivalence

Example: Show that $p \to q \equiv \neg q \to \neg p$ without using a truth table.

Solution:

$p \to q \equiv \neg p \lor q$ (relation by implication)

$\equiv q \lor \neg p$ (commutativity)

$\equiv \neg\neg q \lor \neg p$ (double negation)

$\equiv \neg q \to \neg p$ (relation by implication again, but used backwards)

Done!

FYOG: Show that $(p \to q) \land (p \to r) \equiv p \to (q \land r)$ without using a truth table.

Recap:

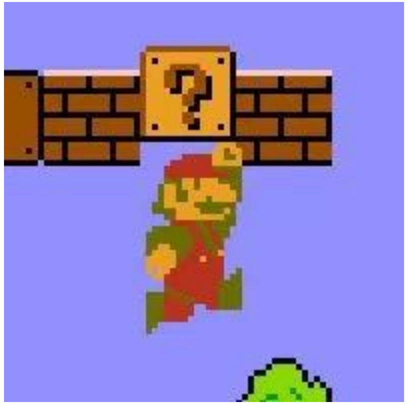
- Today, we learned about…
  - **Logical equivalence** (do these statements mean the same thing?)
  - And **constructing arguments/proofs** using logical equivalences

Next time:

- We talk **predicate logic**

  → a more flexible framework for constructing
  arguments and proofs!

Bonus material!



FYOG: Show that $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$ without using a truth table.

This one is a lot easier to work from the right to the left. So we start with the right-hand side:

Proof:

$p \rightarrow (q \wedge r) \equiv \neg p \vee (q \wedge r)$ relation by implication

$\equiv (\neg p \vee q) \wedge (\neg p \vee r)$ distribution

$\equiv (p \rightarrow q) \wedge (p \rightarrow r)$ relation by implication

FYOG: Prove $p \to q \equiv \neg p \vee q$ using a truth table.

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg p \vee q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

FYOG: Prove $((p \to q) \wedge (q \to r)) \to (p \to r)$ is a tautology using a truth table.

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$((p \to q) \wedge (q \to r)) \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

The column for $((p \to q) \wedge (q \to r)) \to (p \to r)$ is all True, so it is a tautology.