Announcements and Reminders

HW 2 posted! Piazza / Resources & Calendar

- ❒ Enroll in the class Moodle
    - Find it at: https://moodle.cs.colorado.edu
    - Search for “Cox/Wong” and you should be able to find it
- ❒ First homework (on Moodle) is due **~~Friday~~** TODAY! **at 12pm Noon**
- ❒ Enroll in the class Piazza: https://piazza.com/colorado/fall2018/csci2824
- ❒ Keep track of everything via the schedule: https://goo.gl/DFuboZ
- ❒ CA office hours in CSEL (1st floor Engineering Center)
    - A link to the schedule is on Piazza (/ Resources / Course information)

Lecture 4: Propositional logic and applications

What did we do last time?

- **Connectives**: *and*, *or*, *if-then (conditional)*, *if-and-only-if (biconditional)*
- **Truth tables**: got more complex, with compound propositions
- **Solving riddles using truth tables**: Knights and Knaves

Today:

- **Propositional satisfiability** (can these statements possibly be true?)
- **Necessary/sufficient conditions:** Is this enough information to draw conclusions?

Propositional satisfiability

Definition: A compound proposition is **satisfiable** if there is an assignment (at least one) of truth values to its constituent propositions that makes it true. If there is no such case, then the compound proposition is **unsatisfiable** (i.e., a **contradiction**).

more later!

Propositional satisfiability

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Example: $p \wedge \neg p$ is unsatisfiable.

F

1 T T F F 2 F T F 3

Example: Show that $(p \vee \neg q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$ is satisfiable.

→ need some TVs for p, q s.t. this comp. prop. is T

→ each component disjunction must be T

→ 1 & 2 tell us p & q have same TV

→ 3 tells us that p = q = F so that $\neg p \vee \neg q = \neg F \vee \neg F = T \vee T$

→ $\boxed{p = q = F}$ make the comp. prop. T, so it is satisfiable

b/c there's at least 1 comb. of p, q to make ~~it~~ T

Propositional satisfiability

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Example: $p \land \neg p$ is unsatisfiable.

Example: Show that $(p \lor \neg q) \land (\neg p \lor q) \land (\neg p \lor \neg q)$ is satisfiable.

- To show that a compound proposition is satisfiable, we only need to demonstrate that there is ***one*** combination of truth values for p and q that makes this statement true.
- The first two conjuncts tell us that p and q must have the same truth values.
- The last one tells us that they must be F (otherwise this conjunct would be F).

Propositional satisfiability

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- The first two conjuncts tell us that p and q must have the same truth values.
- The last one tells us that they must be F (otherwise this conjunct would be F).

Solution: p=F and q=F works.

$a \to b$ is T unless $a = T$ & $b = F$

Propositional satisfiability

Example: Show that $\underset{①}{(p \to q)} \wedge \underset{②}{(p \to \neg q)} \wedge \underset{③}{(\neg p \to q)} \wedge \underset{④}{(\neg p \to \neg q)}$ is **un**satisfiable.

- To show that a compound proposition is unsatisfiable, we would need to demonstrate that ***for all*** combinations of truth values for *p* and *q* that makes this statement ~~true~~. FALSE Which means a truth table.
- ***Or*** we can construct a logical argument.

① & ②: if $p = T$, then one of these would be $T \to F$ (p; q or ¬q)

So $p = F$

③ & ④: $p = F$, so $\neg p = T$, & ~~at least~~ one of ③ or ④ is $T \to F$ (¬p; q or ¬q)

⇒ at least one of ①-④ is F

⇒ the whole comp. prop. must be F regardless of TVs for p & q

Propositional satisfiability

Example: Show that $(p \to q) \land (p \to \neg q) \land (\neg p \to q) \land (\neg p \to \neg q)$ is **un**satisfiable.

- To show that a compound proposition is unsatisfiable, we would need to demonstrate that ***for all*** combinations of truth values for p and q that makes this statement true. Which means a truth table.
- ***Or*** we can construct a logical argument.

Solution:

- The first two conjuncts tell us that p=F
  (if it were T, then we would have (T→T) ∧ (T→F), the second of which is F)
- If p=F, then the third conjunct tells us that q=T
  (otherwise, we would have ¬F→F, or T→F, which is F)
- But if p=F and q=T, then the fourth construct is ¬F → ¬T, or T→ F, which is F

⇒ a **contradiction**. So we conclude that the proposition must be **unsatisfiable.**

Propositional satisfiability

FYOG: Show that $(p \Leftrightarrow q) \land (\neg p \Leftrightarrow q)$ is unsatisfiable.

Necessary and sufficient conditions

Example: Let *n* be a natural number (0, 1, 2, 3, …). It is **sufficient** that *n* $\underbrace{\text{be divisible by 12}}_{r}$ for $\underbrace{n \text{ to be divisible by 6}}_{s}$.

How could we represent this claim using a conditional?

- Let *r* = "*n* is divisible by 12" and *s* = "*n* is divisible by 6"

$$\boxed{r \to s}$$

Knowledge that n is divs. by 12 is enough for you to conclude that n is also divis. by 6

Necessary and sufficient conditions

Example: Let n be a natural number (0, 1, 2, 3, …). It is **<u>sufficient</u>** that n be divisible by 12 for n to be divisible by 6.

How could we represent this claim using a conditional?

- Let r = "n is divisible by 12" and s = "n is divisible by 6"
- This statement is telling us that **under the condition that** n is divisible by 12, then it must be divisible by 6.

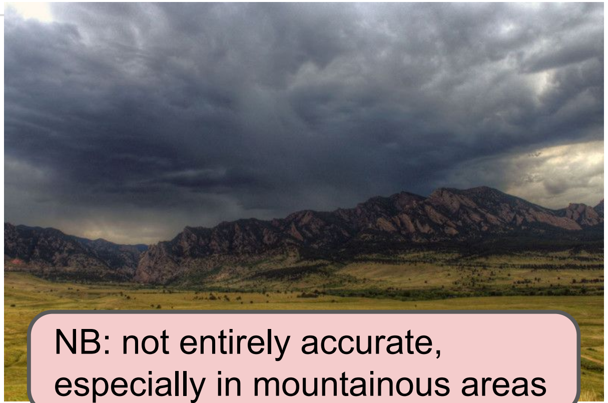
Answer: For a **sufficient condition**, the condition goes at the front of the conditional:

$$r \rightarrow s$$

the conclusion of cond.

hypothesis of conditional

Necessary and sufficient conditions



NB: not entirely accurate, especially in mountainous areas

Example: It is <u>**necessary**</u> for warm surface air to start up convection in order for a severe summer thunderstorm to occur.

How could we represent this claim using a conditional?

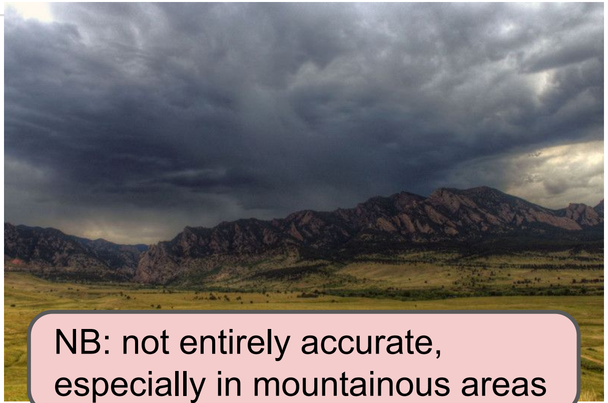
- Let *t* = "severe summer thunderstorm occurs" and *w* = "warm surface air spurs convection"

Example: It is necessary to $\underbrace{\text{break eggs}}_{b}$ in order to $\underbrace{\text{make an omelette}}_{m}$.

$\cancel{b \to m}$

$m \to b$

Necessary and sufficient conditions



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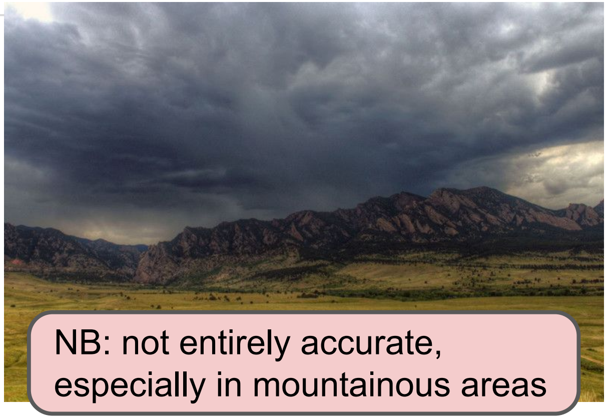
How could we represent this claim using a conditional?

- Let *t* = "severe summer thunderstorm occurs" and *w* = "warm surface air spurs convection"
- This statement is telling us that **under the condition that** a thunderstorm occurs, then it must be the case that warm surface air has spurred convection.
- Note that it is ***not*** saying that if warm surface air starts up convection, then thunderstorms will always occur. (we don't get thunderstorms everyday, right?)

Necessary and sufficient conditions



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Answer: For a **necessary condition**, the condition goes at the end of the conditional:

$\cancel{t \to w}$ $m \to b$

Necessary and sufficient conditions

Example: We saw earlier two examples in translating to symbols:

a. *If it snows, then I crash my bicycle riding home.*

b. *I crash my bicycle only if it snows.*

c s

CRASH SNOW

In (*a*), is this a necessary or sufficient (or neither) condition? What is the condition?

$S \to C$ snow is a sufficient cond. for a crash

In (*b*), is this a necessary or sufficient (or neither) condition? What is the condition?

$C \to S$

necessary condition

Necessary and sufficient conditions

Example: We saw earlier two examples in translating to symbols:

a. *If it snows, then I crash my bicycle riding home.*

b. *I crash my bicycle only if it snows.*

What if I tell you that *It is both necessary* ***and*** *sufficient that it snows for me to crash my bicycle* ?

$$\underbrace{(c \rightarrow s)}_{\text{nec.}} \land \underbrace{(s \rightarrow c)}_{\text{suff.}} \Rightarrow s \leftrightarrow c$$

biconditional

I crash $\underbrace{\text{"if \& only if"}}_{\text{iff}}$ it snows

Propositional satisfiability

Note that we could do these examples using truth tables, but they often are too large to be of any practical use.

Example: Sudoku puzzles can be written (and solved) as satisfiability problems.

Turns out, we will see that the truth table for Sudoku puzzles would require 2^{729} rows, which is more than the number of atoms in the universe.
(10^{78} - 10^{82}, which is about 2^{259}-2^{272} atoms)

FYOG, check this out!

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

Propositional satisfiability

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Goal: fill in the missing values such that each row, each column, and each 3x3 sub-square uses each of the numbers 1-9 exactly once (and no cell contains more than one number).

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

Propositional satisfiability

Representing Sudoku:

Let $p(i, j, n)$ represent that proposition that n occurs at row i and column j

- For example, in this puzzle, $p(3, 5, 4) = \text{T}$ and $p(1, 3, 2) = \text{F}$
- There are 9 rows, 9 columns, and 9 numbers.

  ⇒ total of 9 x 9 x 9 = 729 propositions

  … good luck with your truth table…

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

Propositional satisfiability

Notation:

$$\bigwedge_{j=1}^{4} p_j = p_1 \wedge p_2 \wedge p_3 \wedge p_4$$

$$\bigvee_{j=1}^{4} p_j = p_1 \vee p_2 \vee p_3 \vee p_4$$

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

Propositional satisfiability

Row constraint:

- Row *i* contains a particular *n*: $\bigvee_{j=1}^{9} p(i,j,n)$
- Row *i* contains all *n*: $\bigwedge_{n=1}^{9} \bigvee_{j=1}^{9} p(i,j,n)$
- All rows contains all *n*: $\boxed{\bigwedge_{i=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{j=1}^{9} p(i,j,n)}$

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

Propositional satisfiability

Column constraint:

- Column *j* contains a particular *n*: $\bigvee_{i=1}^{9} p(i,j,n)$
- Column *j* contains all *n*: $\bigwedge_{n=1}^{9} \bigvee_{i=1}^{9} p(i,j,n)$
- All columns contains all *n*: $\bigwedge_{j=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{9} p(i,j,n)$

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

Propositional satisfiability

3x3 squares constraint:

- Let r indicate the row block and let c indicate the column block (0, 1 or 2)
- Given r and c, how do we sum over the rows and columns within that 3x3 block?
  - Rows: block 0 goes from row 1 to 3
    block 1 goes from row 4 to 6
    block 2 goes from row 7 to 9
    $\Rightarrow 3r + 1$ to $3r + 3$
    $\Rightarrow 3r + i, \quad i = 1, 2, 3; \; r = 0, 1, 2$
  - Same goes for the columns:
    $\Rightarrow 3c + j, \quad j = 1, 2, 3; \; c = 0, 1, 2$
  - And we need each block to contain $n = 1, \ldots, 9$

$$\bigwedge_{r=0}^{2} \bigwedge_{c=0}^{2} \bigwedge_{n=1}^{9} \bigvee_{j=1}^{3} \bigvee_{i=1}^{3} p(i, j, n)$$

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

Propositional satisfiability

“Each cell contains only one number” constraint:

$$\bigwedge_{i=1}^{9}\bigwedge_{j=1}^{9}\bigwedge_{n=1}^{9}\bigwedge_{\substack{m=1\\ m\neq n}}^{9}(p(i,j,n)\rightarrow\neg p(i,j,m))$$

What were we given? constraint:

$p(1,1,5) \land p(1,2,3) \land p(1,5,7) \land \ldots$
$\ldots \land p(9,8,7) \land p(9,9,9)$

So chain together these 5 constraints (red boxes) with conjunctions and see (or rather, have a computer determine) what unique set of truth values satisfy them.

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

Propositional logic and applications

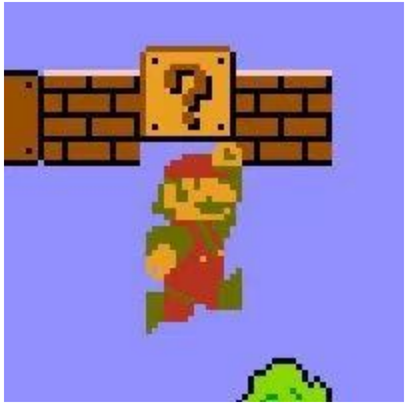
Recap:

- Today, we learned about…
  - **Propositional satisfiability** (can these statements possibly be true?)
  - **Necessary/sufficient conditions:** Is this enough information to draw conclusions?

Next time:

- We talk ***logical equivalence*** (do these statements mean the same thing?)

Bonus material!



Propositional satisfiability

FYOG: Show that $(p \Leftrightarrow q) \wedge (\neg p \Leftrightarrow q)$ is unsatisfiable.

S'pose it is satisfiable.

p is either T or F…

1. S'pose *p* is T. Then the first biconditional means that *q* must also be T
2. But if *q* is T, then the second biconditional means that ¬*p* is T, so *p* is F
3. But this can't be the case, because we originally supposed that *p* is T.
4. So *p* must be false if this thing is satisfiable.
5. If *p* is F, then the first biconditional means that *q* must also be F
6. But if *q* is F, then the second biconditional means that ¬*p* is F, so *p* is T
7. But we already know that can't be the case (from #4)
8. Since *p* can't be T or F in order to make this satisfiable, and those are the only options, the compound proposition must not be satisfiable.

Necessary and sufficient conditions

Example: Let n be a natural number (0, 1, 2, 3, …). It is <u>**necessary**</u> that n^2 be divisible by 9 for n to be divisible by 6.

How could we represent this claim using a conditional?

- Let q = "n^2 is divisible by 9" and s = "n is divisible by 6"

Necessary and sufficient conditions

Example: Let n be a natural number (0, 1, 2, 3, …). It is **<u>necessary</u>** that n^2 be divisible by 9 for n to be divisible by 6.

How could we represent this claim using a conditional?

- Let q = "n^2 is divisible by 9" and s = "n is divisible by 6"
- This statement is telling us that if a number is divisible by 6, then it must be the case that the number-squared is divisible by 9.
- Note that it is ***not*** saying that if a number-squared is divisible by 9, then the number itself must be divisible by 6.

Necessary and sufficient conditions

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Answer: For a **necessary condition**, the condition goes at the end of the conditional:

$$s \rightarrow q$$