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CSCI 2824: Discrete Structures

Lecture 4: Propositional Logic and Applications

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Announcements and Reminders:

- Enroll in the class Moodle. https://moodle.cs.colorado.edu

- First homework (on Moodle) is due Friday 7 September at 12pm
  1 attempt per problem. infinite attempts for codeRunner. Clicking "Check" locks in your answer…don't do it!

- Enroll in the class Piazza: https://piazza.com/colorado/fall2018/csci2824

- Keep updated with the Schedule: https://goo.gl/DFuboZ

- CA office hours in CSEL (1st floor Engineering Center). A link to the schedule is on Piazza.

Definition: Let *p* and *q* be two propositions. The **conditional** "if *p* then *q*", denoted by $p \to q$, is false when *p* is true but *q* is false, and true otherwise.

- The conditional describes an *if-then* relationship between the two propositions.
- Think of the conditional $p \to q$ as defining a rule. What are the cases where the rule holds or where the rule is broken.

$$p \longrightarrow q$$

$q \longrightarrow p$ **converse**

$\neg p \longrightarrow \neg q$ **inverse**

$\neg q \longrightarrow \neg p$ **contrapositive**

p	q	$p \longrightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Definition: The compound propositions p and q are called **logically equivalent** if $p \Leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

- A tautology is when a statement is always true.

				conditional	converse	inverse	contrapositive
p	**q**	$\neg p$	$\neg q$	$p \rightarrow q$	$q \rightarrow p$	$\neg p \rightarrow \neg q$	$\neg q \rightarrow \neg p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Definition: The compound propositions p and q are called **logically equivalent** if $p \Leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

- A tautology is when a statement is always true.

Cond. $\Leftrightarrow$ contra   conv $\Leftrightarrow$ inv.

$p \rightarrow q$	$q \rightarrow p$	$\neg p \rightarrow \neg q$	$\neg q \rightarrow \neg p$	$(p \rightarrow q) \Leftrightarrow (\neg q \rightarrow \neg p)$	$(q \rightarrow p) \Leftrightarrow (\neg p \rightarrow \neg q)$
T	T	T	T	T	T
F	T	T	F	T	T
T	F	F	T	T	T
T	T	T	T	T	T

Conditional

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

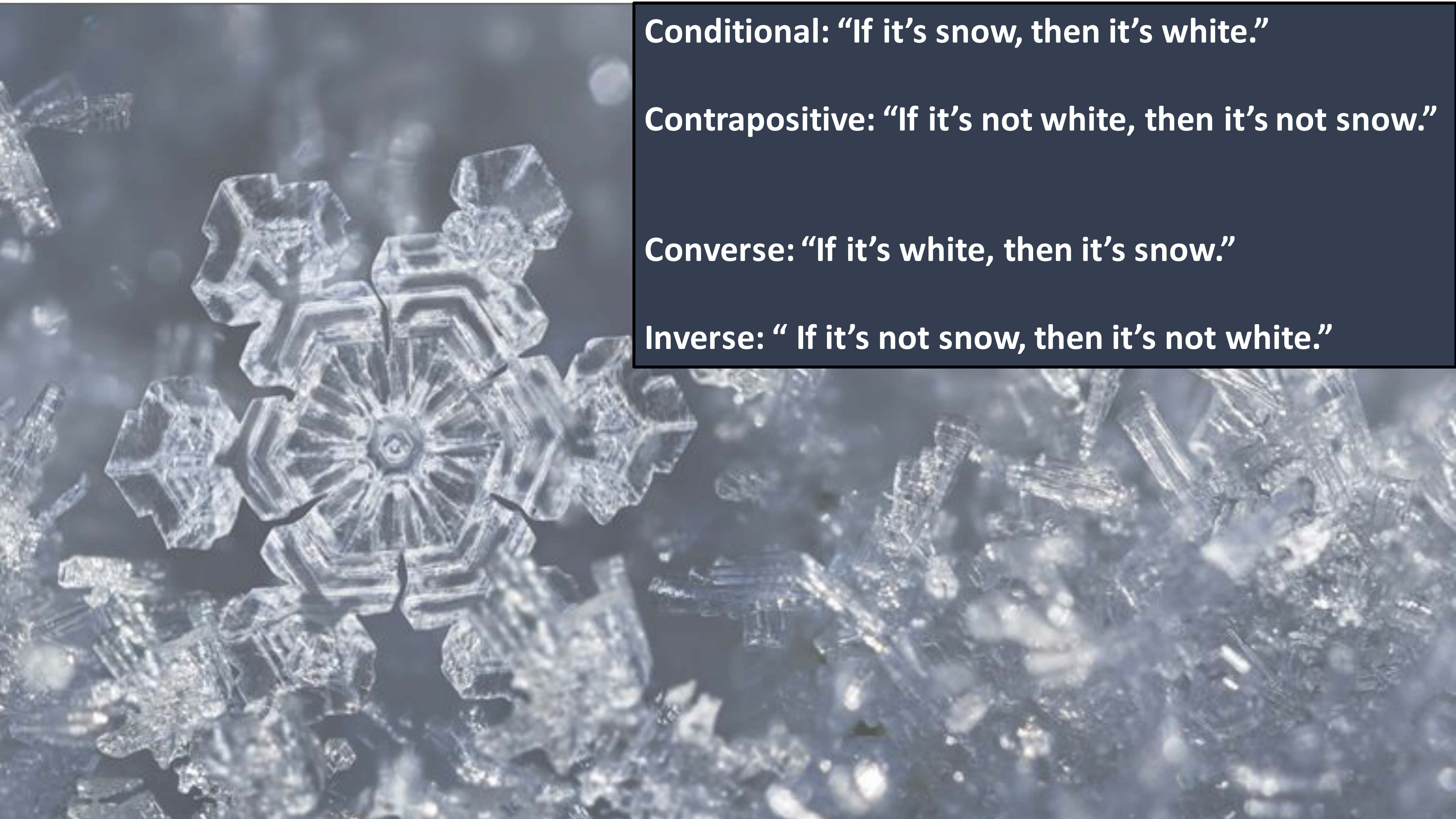
Biconditional

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

The conditional and the contrapositive are logically equivalent.

The converse and inverse are logically equivalent.

$(p \to q) \Leftrightarrow (\neg q \to \neg p)$	$(p \to q) \Leftrightarrow (\neg q \to \neg p)$



Conditional: “If it’s snow, then it’s white.”

Contrapositive: “If it’s not white, then it’s not snow.”

Converse: “If it’s white, then it’s snow.”

Inverse: “ If it’s not snow, then it’s not white.”

On an island there are two types of people:
Knights who always tell the truth, and Knaves who always lie.

Example: On the island you encounter two people, who we'll call A and B. A tells you that "I am a Knave or B is a Knight." Use a truth table to determine what type of people A and B are.

Let p: A is a knight.

Let q: B is a knight.

How can we represent A's comment symbolically?

$\neg p \vee q$

New Strategy: We'd like to know the combinations of truth values of p and q that ensure that statements made by A and B are consistent with their nature as Knights or Knaves. (i.e. we don't want A to be a Knight but utter a False statement.)

In this example, one way to accomplish this is to test that p (the statement that A is a Knight) is equivalent in truth value to the statement that he uttered (i.e. $\neg p \vee q$)

$$p \Leftrightarrow (\neg p \vee q)$$

Example: On the island you encounter two people, who we'll call A and B. A tells you that "I am a Knave or B is a Knight." Use a truth table to determine what type of people A and B are.

p	q	$\neg p$	$\neg p \vee q$	$p \Leftrightarrow (\neg p \vee q)$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	F
F	F	T	T	F

★ A, B are both Knights

On an island there are two types of people:
Knights who always tell the truth, and Knaves who always lie.

Example: On the island you encounter two people, who we'll call A and B. Person A tells you that "B and I are of opposite types." Person B tells you that "A is a knave and I am a knight." Use a truth table to determine what type of people A and B are.

p: A is a Knight.

q: B is a Knight.

A's statement: $p \oplus q$

B's statement: $\neg p \wedge q$

We want A's and B's nature to be consistent with their statements.

So $[p \Leftrightarrow (p \oplus q)]$
$\wedge [q \Leftrightarrow (\neg p \wedge q)]$

Example: (continued) On the island you encounter two people, who we'll call A and B. Person A tells you that "B and I are of opposite types." Person B tells you that "A is a knave and I am a knight." Use a truth table to determine what type of people A and B are.

p	**q**	$p \oplus q$	$\neg p$	$\neg p \wedge q$	$p \Leftrightarrow (p \oplus q)$	$q \Leftrightarrow (\neg p \wedge q)$	$p \Leftrightarrow (p \oplus q) \wedge q \Leftrightarrow (\neg p \wedge q)$
T	T	F	F	F	F	F	F
T	F	T	F	F	F	T	T ☆
F	T	T	T	T	F	T	F
F	F	F	T	F	T	T	T ☆

A: cannot determine nature   B: is a knave.

Application: Necessary and Sufficient Conditions

Example: Let n be a natural number. It is **sufficient** that n be divisible by 12 for n to be divisible by 6

Let $\underbrace{r = n \text{ is divisible by 12}}$ and $\underbrace{s = n \text{ is divisible by 6}}$

How could we represent this claim using a conditional?

$r \rightarrow s$

Application: Necessary and Sufficient Conditions

Example: Let n be a natural number. It is **necessary** that n^2 be divisible by 9 for n to be divisible by 6

Let $q = n^2$ *is divisible by 9* and $s = n$ *is divisible by 6*

How could we represent this claim using a conditional?

Example: Wason selection task

Consider the following four cards. They have letters on one side and numbers on the other. Suppose I tell you the following rule:

If a card has an odd number, then its letter is a vowel.

A	**B**	**1**	**2**

Question: What card(s) do you need to turn over in order to verify that the given rule is true?

Logical equivalence

- We have found that $p \to q \equiv \neg q \to \neg p$. So. Who cares?
- Turns out, this can be **very** useful in proving things.
- Mathematical arguments/proofs:
    - progressing from a set of assumptions to useful/interesting conclusions
    - **logical equivalences** link the steps together
- To prove $p \to q$, you might suppose p is true, then work your way forward to show that it must be the case that q is true.
- **But** it might be easier to suppose that q is *false*, then work your way toward showing that it must be the case that p is also false.
    - And because $p \to q \equiv \neg q \to \neg p$, either way is valid.

Example: Suppose *n* is an integer. Prove that if n^2 is even, then *n* must be even.