

CSCI 2824: Discrete Structures

Lecture 5: Propositional Equivalences

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Propositional Equivalences

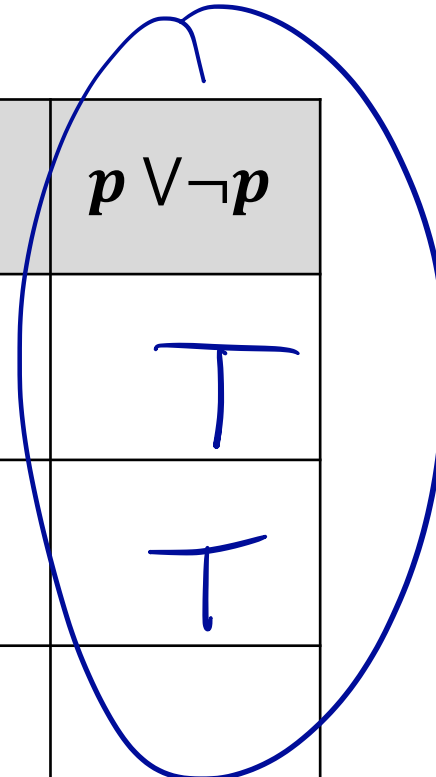
A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a **tautology**. The compound propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

A compound proposition that is always false is called a **contradiction**.

A compound proposition that is neither a tautology nor a contradiction is called a **contingency**.

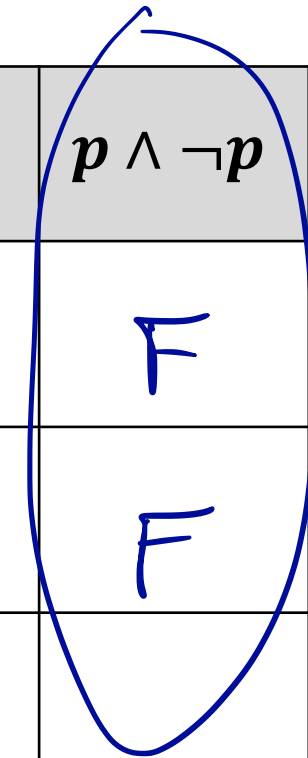
Example of a Tautology:

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T



Example of a Contradiction:

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F



Example: Show that $\neg (p \wedge q) \equiv \neg p \vee \neg q$

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

tautology

Example: Show that $\neg(p \vee q) \equiv \neg p \wedge \neg q$

		s t					
p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$	$s \Leftrightarrow t$
T	T	F	F	T	F	F	T
T	F	F	T	T	F	F	T
F	T	T	F	T	F	F	T
F	F	T	T	F	T	T	T

match

tautology!

We just Proved:

De Morgan's Laws

$$\neg (p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg (p \vee q) \equiv \neg p \wedge \neg q$$

Other Equivalences (from our book)

TABLE 7 Logical Equivalences Involving Conditional Statements.

★ $p \rightarrow q \equiv \neg p \vee q$ Relation by Implication (RBI)

★ $p \rightarrow q \equiv \neg q \rightarrow \neg p$ Contraposition

$$p \vee q \equiv \neg p \rightarrow q$$

$$p \wedge q \equiv \neg(p \rightarrow \neg q)$$

$$\neg(p \rightarrow q) \equiv p \wedge \neg q$$

$$\bullet (p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$$

$$\bullet (p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$$

$$\bullet (p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$$

$$\bullet (p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

★ $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$ Biconditional

$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$

$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$

$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

★ $p \oplus q \equiv (p \vee q) \wedge \neg(p \wedge q)$ Alternate definition of xor

Other Equivalences (from our book)

To show that two compound propositions are logically equivalent:

- Prove it with a Truth Table
- Use Equivalence Rules to go from one to the other.



TABLE 6 Logical Equivalences.

<i>Equivalence</i>	<i>Name</i>
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Example: Show that $p \rightarrow q \equiv \neg q \rightarrow \neg p$ (without a truth table)

$$p \rightarrow q \equiv \neg p \vee q$$

Relation by Implication

$$\equiv q \vee \neg p$$

Commutativity

$$\equiv \neg(\neg q) \vee \neg p$$

double negation

$$\equiv \neg q \rightarrow \neg p$$

RBI (backwards)



$$\boxed{p \rightarrow q \equiv \neg p \vee q}$$

Example: Show that $(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$ (without a truth table)

$$\begin{aligned}(p \rightarrow q) \vee (p \rightarrow r) &\equiv (\neg p \vee q) \vee (\neg p \vee r) && \text{RBI} \\&\equiv \neg p \vee q \vee \neg p \vee r && \text{Associative} \\&\equiv \underbrace{\neg p \vee \neg p} \vee q \vee r && \text{Commutative} \\&\equiv \neg p \vee q \vee r && \text{idempotent} \\&\equiv \neg p \vee (q \vee r) && \text{associative} \\&\equiv p \rightarrow (q \vee r) && \text{RBI!}\end{aligned}$$

A compound proposition is **satisfiable** if there is an assignment of truth values to its variables that makes it true. If there is no such case then we say it is **unsatisfiable** (i.e. a contradiction)

Example: Show that $\underbrace{(p \vee \neg q)}_T \wedge \underbrace{(\neg p \vee q)}_T \wedge \underbrace{(\neg p \vee \neg q)}_T$ is satisfiable.

1. $(\underbrace{p}_T \vee \underbrace{\neg q}_F) \wedge (\underbrace{\neg p}_F \vee \underbrace{q}_T)$ p, q must have same truth values
2. add in $(\neg p \vee \neg q)$ p, q must be false.

Example: Show that $(p \rightarrow q) \wedge (p \rightarrow \neg q) \wedge (\neg p \rightarrow q) \wedge (\neg p \rightarrow \neg q)$ is not satisfiable.

Example: Sudoku puzzles can be written (and solved) as a satisfiability problems

Solving this:

- 1) First chain together the propositions with provided values: $p(1, 1, 5) \wedge p(1, 2, 3) \wedge p(1, 5, 7) \wedge \dots$

- 2) Assert that every row contains every number: $\bigwedge_{i=1}^9 \bigwedge_{n=1}^9 \bigvee_{j=1}^9 p(i, j, n)$

- 3) Assert that every column contains every number:

$$\bigwedge_{j=1}^9 \bigwedge_{n=1}^9 \bigvee_{i=1}^9 p(i, j, n)$$

- 4) Assert that every 3x3 block contains every number:

$$\bigwedge_{r=0}^2 \bigwedge_{s=0}^2 \bigwedge_{n=1}^9 \bigvee_{i=1}^3 \bigvee_{j=1}^3 p(3r + i, 3s + j, n)$$

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

Let $p(i, j, n)$ denote the proposition that a number n is in the cell in row i and column j

9 rows, 9 columns, 9 numbers =
 $9 \times 9 \times 9 = 729$ propositions

- 5) Assert that no cell contains more than one number:

$$\bigwedge_{i=1}^9 \bigwedge_{j=1}^9 \bigwedge_{n=1}^9 \bigwedge_{m=1, m \neq n}^9 (p(i, j, n) \rightarrow \neg p(i, j, m))$$

- 6) String 1-5 together with conjunctions