Warm-up problem

Example: Let the domain be the *natural numbers*, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. Determine the truth values of these statements.

1) $\forall x\ (x^2 \geq x)$

2) $\exists x\ (x^2 = x)$

Announcements/Reminders

Homework 2 (written) due **Friday at noon**.

- Graded partially on **style** and **neatness**. (just like in life)
- Multiple pages must be **stapled**. Do the problems **in order**.
- **No torn-out-of-notebook fringe crap.**
- **Write your name on the back of last page.**
- Do **not** try to cram too much into a small space. When in doubt, start a new page. Paper **literally** grows on trees.

Warm-up problem

Example: Let the domain be the *natural numbers*, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. Determine the truth values of these statements.

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Lecture 7: Nested quantifiers

What did we do last time?

Predicates and **propositional functions**

-- a more flexible framework to describe the truthiness of the world

Quantifiers:

- **Universal quantifier:** $\forall x\, P(x)$ means "for all x in my **domain**, $P(x)$"
- **Existential quantifier:** $\exists x\, P(x)$ means "there exists an x in my **domain**, $P(x)$"

Last time we finished up with these two nice rules for distributing quantifiers over compound propositional functions:

- $\forall x\, (P(x) \land Q(x)) \equiv \forall x\, P(x) \land \forall x\, Q(x)$
- $\exists x\, (P(x) \lor Q(x)) \equiv \exists x\, P(x) \lor \exists x\, Q(x)$
- And recall that distribution of $\forall$ over $\lor$ and $\exists$ over $\land$ did **not** work.

Now let's do negations!

Question: What is the negation of a universal? That is, what does $\neg \forall x\, P(x)$ mean?

Example: What is the negation of the following statement?

> "All CSCI 2824 instructors are super cool."

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Example: What is the negation of the following statement?

"All CSCI 2824 instructors are super cool."

Maybe: "It is not the case that all CSCI 2824 instructors are super cool."

Or more naturally: "There is a CSCI 2824 instructor who is not super cool."

Question: What is the negation of a universal? That is, what does $\neg\forall x\, P(x)$ mean?

Example: What is the negation of the following statement?

"All CSCI 2824 instructors are super cool."

Maybe: "It is not the case that all CSCI 2824 instructors are super cool."

Or more naturally: "There is a CSCI 2824 instructor who is not super cool."

Let our domain be the set of all CSCI 2824 instructors and let $P(x)$ represent "x is super cool".

The negated statement is then $\exists x\, \neg P(x)$, and we have the rule:

$\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$ (so pushing a $\neg$ through a $\forall$ turns it into a $\exists$)

Predicate and quantifier logic

Question: What is the negation of an existential? That is, what does $\neg \exists x\, P(x)$ mean?

Example: What is the negation of the following statement?

> "There exists a CSCI 2824 instructor who owns nine cats."

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Example: What is the negation of the following statement?

> "There exists a CSCI 2824 instructor who owns nine cats."

How about: "There are no CSCI 2824 instructors who own nine cats."

Or more naturally: "All CSCI 2824 instructors **do not** own nine cats."

Predicate and quantifier logic

Question: What is the negation of an existential? That is, what does $\neg \exists x\, P(x)$ mean?

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> "There exists a CSCI 2824 instructor who owns nine cats."

How about: "There are no CSCI 2824 instructors who own nine cats."

Or more naturally: "All CSCI 2824 instructors **do not** own nine cats."

Let our domain be the set of all CSCI 2824 instructors and let $P(x)$ represent "x owns nine cats".

The negated statement is then $\forall x\, \neg P(x)$, and we have the rule:

$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$ (so pushing a $\neg$ through a $\exists$ turns it into a $\forall$)

Collectively, these two are known as **DeMorgan's Laws for Quantifiers**

- $\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$
- $\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$

And we also had the **distribution laws** from last time

- $\forall x\, (P(x) \land Q(x)) \equiv \forall x\, P(x) \land \forall x\, Q(x)$
- $\exists x\, (P(x) \lor Q(x)) \equiv \exists x\, P(x) \lor \exists x\, Q(x)$

Armed with these rules, along with the **logical equivalences** for regular propositions, we can prove all kinds of equivalences of quantifier propositions.

Example: Prove that $\neg\forall x\,(P(x) \rightarrow Q(x)) \equiv \exists x\,(P(x) \land \neg Q(x))$

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Solution:

$\neg\forall x\,(P(x) \rightarrow Q(x)) \equiv \exists x\,\neg(P(x) \rightarrow Q(x))$ (DeMorgan)

$\equiv \exists x\,\neg(\neg P(x) \lor Q(x))$ (relation by implication)

$\equiv \exists x\,(\neg\neg P(x) \land \neg Q(x))$ (DeMorgan)

$\equiv \exists x\,(P(x) \land \neg Q(x))$ (double negation)

Example: Prove that $\neg\forall x\,(P(x) \rightarrow Q(x)) \equiv \exists x\,(P(x) \land \neg Q(x))$

Solution:

$$
\begin{aligned}
\neg\forall x\,(P(x) \rightarrow Q(x)) &\equiv \exists x\,\neg(P(x) \rightarrow Q(x)) && \text{(DeMorgan)}\\
&\equiv \exists x\,\neg(\neg P(x) \lor Q(x)) && \text{(relation by implication)}\\
&\equiv \exists x\,(\neg\neg P(x) \land \neg Q(x)) && \text{(DeMorgan)}\\
&\equiv \exists x\,(P(x) \land \neg Q(x)) && \text{(double negation)}
\end{aligned}
$$

Let's read it out loud to see if it makes sense.

- "It is not the case that for all *x*, if *P(x)* then *Q(x)*."
- "There is some *x* such that *P(x)* and not *Q(x)*."

FYOG: Come up with a specific example (domain and propositional functions *P* and *Q*) to illustrate this equivalence.

FYOG: Determine whether $\forall x\, (P(x) \rightarrow Q(x)) \equiv \forall x\, P(x) \rightarrow \forall x\, Q(x)$

Translating from English into logical expressions

Example: Translate the following into symbols.

"Every student in CSCI 2824 has passed Calculus 1."

Translating from English into logical expressions

Example: Translate the following into symbols.

> "Every student in CSCI 2824 has passed Calculus 1."

But we might want to consider an even larger domain. S'pose the domain is all students at CU (instead of just the CSCI 2824 students).

Let *D(x)* represent "is a CSCI 2824 student".

Then our statement becomes $\forall x\ (D(x) \rightarrow C(x))$

Note: This does **not** work as a translation: $\forall x\ (D(x) \land C(x))$

> (That would say that all students at CU are in CSCI 2824 and have passed Calculus 1.)

Example: Let the domain be the set of all CU students, and translate:

"Every student in CSCI 2824 is either taking Data Structures, or has already passed it."

Example: Let the domain be the set of all CU students, and translate:

"Every student in CSCI 2824 is either taking Data Structures, or has already passed it."

Solution:

- Let $D(x)$ represent "is a CSCI 2824 student".
- Let $P(x)$ represent "has passed Data Structures".
- Let $T(x)$ represent "is taking Data Structures".

Then our statement becomes $\forall x\, (D(x) \rightarrow (P(x) \vee T(x)))$

Although it might be more accurate with real-world experience to write it as
$\forall x\, (D(x) \rightarrow (P(x) \oplus T(x)))$ (with the exclusive or, because most people don't take the class again just for fun…)

FYOG: Let the domain be all CSCI 2824 students, and translate:

“There exists a CSCI 2824 student who has taken Calculus 3 but not Differential Equations.”

FYOG: Let the domain be all CU students, and translate:

“There exists a student in CSCI 2824 who has taken Calculus 3 but not Differential Equations.”

FYOG: Find the negations of the previous two statements in both symbols and in plain English.

Nested quantifiers: Where things **really** get interesting, because we include multiple quantifiers for a propositional function.

Example: Consider the domain of all real numbers. What does the following statement mean?

$$\forall x \; \exists y \; (x + y = 0)$$

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$$\forall x \; \exists y \, (x + y = 0)$$

Solution: It probably helps to say it out loud:

- "For all x, there exists y such that $x + y = 0$."
- What do you think? True or false?

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Example: Consider the domain of all real numbers. What does the following statement mean?

$$\forall x \ \exists y \ (x + y = 0)$$

Solution: It probably helps to say it out loud:

- "For all x, there exists y such that $x + y = 0$."
- What do you think? True or false?
- This is **true**. It is expressing the fact that all real numbers have an **additive inverse** (a number you can add to it to get 0).

Nested quantifiers: Where things **really** get interesting, because we include multiple quantifiers for a propositional function.

Example: How can we express the law of **commutation of addition**?
(That is, $x + y = y + x$.)

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(That is, $x + y = y + x$.)

Solution: S'pose the domain is all real numbers.

- Then we could use: $\forall x\ \forall y\ (x + y = y + x)$

What happens if you swap the order of $\forall x$ and $\forall y$?

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(That is, $x + y = y + x$.)

Solution: S'pose the domain is all real numbers.

- Then we could use: $\forall x\ \forall y\ (x + y = y + x)$

What happens if you swap the order of $\forall x$ and $\forall y$?

- Then we would have instead: $\forall y\ \forall x\ (x + y = y + x)$
- Turns out, nothing changes. You still loop over all of the combinations of x's and y's.

Let's go back to the previous example: $\forall x \; \exists y \; (x + y = 0)$

Question: What happens here if we swap the order of $\forall x$ and $\exists y$?

Let's go back to the previous example: $\forall x \ \exists y \ (x + y = 0)$

Question: What happens here if we swap the order of $\forall x$ and $\exists y$?

Answer: A lot happens!

- The original statement was "For every x, there exists some y such that $x + y = 0$".
- The new one is "There exists some y such that for every x, $x + y = 0$".
- Can you think of such a number?

Let's go back to the previous example: $\forall x \ \exists y \ (x + y = 0)$

Question: What happens here if we swap the order of $\forall x$ and $\exists y$?

Answer: A lot happens!

- The original statement was "For every x, there exists some y such that $x + y = 0$".
- The new one is "There exists some y such that for every x, $x + y = 0$".
- Can you think of such a number?
- Nope, me neither!
- In fact, after switching the order, the statement - which was originally a fundamental rule in algebra - becomes **false**.

Rules for switching quantifiers:

- Okay to swap $\forall x$ and $\forall y$
- Okay to swap $\exists x$ and $\exists y$ (**FYOG:** check that this is true!)
- Generally, *not* okay to swap $\forall x$ and $\exists y$

Example: Consider the domain of all real numbers. How can we express the fact that all numbers have a **multiplicative inverse**? (That is, a number we can multiply the original by to get 1.)

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Solution: First off, is this even true? Do *all* real numbers have a multiplicative inverse?

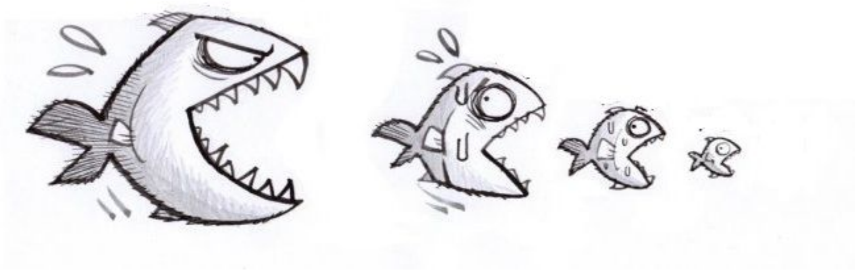
- Answer: No, but all **nonzero** numbers do!
- How can we say this with quantifiers?
- First, in plain English:

  "For all x that aren't 0, there exists some number y such that $xy = 1$."

- Note that "that aren't 0" is a *condition* that we need to satisfy in order to move on to the second part of this statement. ⇒ suggests we will need to use a conditional!

- So maybe: $\forall x\, ((x \neq 0) \rightarrow \exists y\, (xy = 1))$

Example: How can you express the fact that there are an infinite number of natural numbers? (Again, that's $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$)



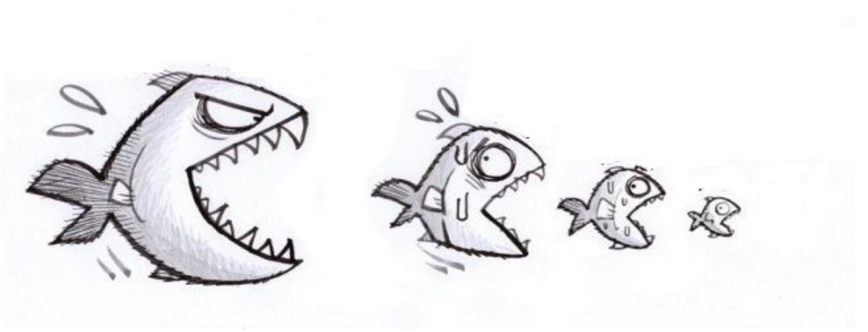
Example: How can you express the fact that there are an infinite number of natural numbers? (Again, that's ℕ = {0, 1, 2, 3, …})

Solution:

- S'pose our domain is the set of natural numbers.
- So maybe:

$\forall x \ \exists y \ (y > x)$

(there's always a bigger fish)



FYOG: How could you express the fact that if you multiply two negative numbers together, the result is a positive number?

FYOG: How could you express the fact that the real numbers have a **multiplicative identity**? That is, there is a real number out there such that if you multiply any real number by this special one, the result is the original number.

(Note: this is a long-winded way of saying that the number 1 exists and is neat.)

Playing with those translations of cool math laws is fun, but how about some non-mathy translations?

Example: Translate the statement "You can fool some of the people all of the time."

Playing with those translations of cool math laws is fun, but how about some non-mathy translations?

Example: Translate the statement "You can fool some of the people all of the time."

Solution:

- Let *F(p, t)* represent "you can fool person *p* at time *t*"
- Let the domain for *p* be all people
- Let the domain for *t* be all times
- Then we have $\exists p\ \forall t\ F(p, t)$

Example: Translate the statement “You can’t fool all of the people all of the time.”

Example: Translate the statement "You can't fool all of the people all of the time."

Solution:

- "It is not the case that for every person, for all times, they can be fooled."
- In logical symbols:

$$\neg(\forall p\ \forall t\ F(p, t))$$

- What if we push the negation through?

$$\neg(\forall p\ \forall t\ F(p, t)) \equiv \exists p\ \neg(\forall t\ F(p, t)) \equiv \exists p\ \exists t\ \neg F(p, t)$$

- So "there exists some person for some time that can't be fooled" (but that reads a bit more awkwardly)

Predicate and quantifier logic

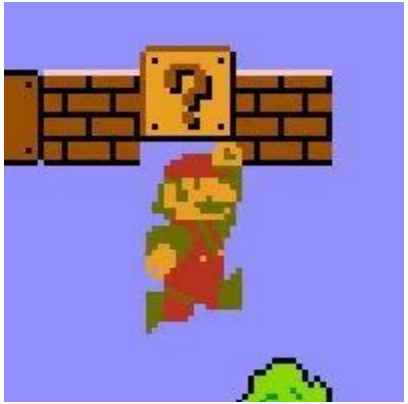
Recap:

- We can represent propositions with quantifiers and predicates.
- We can translate statements from English to symbolic logical statements with quantifiers.
- We can prove and derive logical equivalences.

Next time:

- We get serious about our ***proofs***

Bonus material!



FYOG: Come up with a specific example (domain and propositional functions *P* and *Q*) to illustrate this equivalence. $(\neg \forall x\ (P(x) \rightarrow Q(x)) \equiv \exists x\ (P(x) \land \neg Q(x))$

Consider:

Domain = all planets in our solar system

P(x) = planet *x* has a moon (at least one)

Q(*x*) = planet *x* is a gas giant (like Jupiter for example)

Then the left side is saying: *It is not the case that if a planet has a moon, then it is a gas giant.*

And the right side is saying: *There exists a planet that has a moon and is not a gas giant.* (which is true - Earth and Mars are rocky planets with moons!)

FYOG: Determine whether $\forall x\, (P(x) \rightarrow Q(x)) \equiv \forall x\, P(x) \rightarrow \forall x\, Q(x)$

They are ***not*** logically equivalent. Here's a counterexample:

Let the domain for *x* be all integers.

Let *P(x)* = "*x* is divisible by 2" and let *Q(x)* = "*x* is not a number"

Since we're plugging in integers, *Q*(*x*) is always False

Taking *x* = 2 (for example), we see that **the left-hand side is False**, because *x* = 2 "breaks" this rule that if *x* is divisible by 2 (2 indeed is divisible by 2), then *x* is not a number.

Now on the right-hand side, the hypothesis of the conditional is "every integer is divisible by 2", which is False.

But remember the truth table for the conditional -- F → T and F → F are both True conditionals (because they don't break the rule). So **the right-hand side is True**.

FYOG: Let the domain be all CSCI 2824 students, and translate:

“There exists a CSCI 2824 student who has taken Calculus 3 but not Differential Equations.”

$\exists x\, (C(x) \wedge \neg D(x))$ (C(x) = “x has taken Calc 3” and D(x) = “x has taken Diff Eq”)

FYOG: Let the domain be all CU students, and translate:

“There exists a student in CSCI 2824 who has taken Calculus 3 but not Differential Equations.”

$\exists x\, (S(x) \wedge C(x) \wedge \neg D(x))$ (S(x) = “x is in CSCI 2824”, and the above definitions)

FYOG: Find the negations of the previous two statements in both symbols and in plain English.

$\neg(\exists x\, (C(x) \wedge \neg D(x))) \equiv \forall x\, \neg(C(x) \wedge \neg D(x)) \equiv \forall x\, (\neg C(x) \vee D(x))$

There does not exist a CSCI 2824 student who has taken Calc 3 but not Diff Eq

$\neg(\exists x\, (S(x) \wedge C(x) \wedge \neg D(x))) \equiv \forall x\, \neg(S(x) \wedge C(x) \wedge \neg D(x)) \equiv \forall x\, (\neg S(x) \vee \neg C(x) \vee D(x))$

There does not exist a CU student who is in CSCI 2824 and has taken Calc 3 but not Diff Eq

Rules for switching quantifiers:

- Okay to swap $\exists x$ and $\exists y$ (**FYOG:** check that this is true!)

Consider $\exists x\ \exists y\ (x + y = 10)$, where the domain for x and y is all integers.

→ “There exists an integer x and an integer y such that $x + y = 10$”

This is a true proposition because we have $x = 1$ and $y = 9$ (for example; and many others!)

Does anything change if we swap the order to $\exists y\ \exists x\ (x + y = 10)$?

→ “There exists an integer y and an integer x such that $x + y = 10$”

Absolutely nothing changes.

FYOG: How could you express the fact that if you multiply two negative numbers together, the result is a positive number?

Let the domain for *x* and *y* be all real numbers. Then this statement is "for every *x* and every *y*, if *x* and *y* are both < 0, the *xy* > 0". This could be:

$\forall x\ \forall y\ [\ ((x < 0)\ \wedge\ (y < 0)) \rightarrow (xy > 0)\ ]$

FYOG: How could you express the fact that the real numbers have a **multiplicative identity**? That is, there is a real number out there such that if you multiply any real number by this special one, the result is the original number.

Let the domain for *x* and *y* be all real numbers. Then this statement is "there exists some number *x* such that any number *y* multiplied by *x* just gives you *y* back". This could be:

$\exists x\ \forall y\ (xy = y)$