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CSCI 2824: Discrete Structures

Lecture 5: Propositional Equivalences

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Propositional Equivalences

A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a **<u>tautology</u>**. The compound propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

A compound proposition that is always false is called a **<u>contradiction</u>**.

A compound proposition that is neither a tautology nor a contradiction is called a **<u>contingency</u>**.

Example of a Tautology:

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

Example of a Contradiction:

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

Example: Show that $\neg(p \land q) \equiv \neg p \lor \neg q$

					s	t	$s \Leftrightarrow t$
p	***q***	$\neg p$	$\neg q$	$p \land q$	$\neg(p \land q)$	$\neg p \lor \neg q$	
T	T	F	F	T	F	F	T
T	F	F	T	F	T	T	T
F	T	T	F	F	T	T	T
F	F	T	T	F	T	T	T

tautology

Example: Show that $\neg(p \vee q) \equiv \neg p \wedge \neg q$

tautology!

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$ (s)	$\neg p \wedge \neg q$ (t)	$s \Leftrightarrow t$
T	T	F	F	T	F	F	T
T	F	F	T	T	F	F	T
F	T	T	F	T	F	F	T
F	F	T	T	F	T	T	T

match

We just Proved:

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Other Equivalences (from our book)

TABLE 7 Logical Equivalences Involving Conditional Statements.

★ $p \to q \equiv \neg p \vee q$ Relation by Implication (RBI)
★ $p \to q \equiv \neg q \to \neg p$ Contraposition
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

★ $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$ Biconditional
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

★ $p \oplus q \equiv (p \vee q) \wedge \neg(p \wedge q)$ Alternate definition of xor

Other Equivalences (from our book)

To show that two compound propositions are logically equivalent:

- Prove it with a Truth Table
- Use Equivalence Rules to go from one to the other.

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Example: Show that $p \rightarrow q \equiv \neg q \rightarrow \neg p$ (without a truth table)

$$p \rightarrow q \equiv \neg p \vee q \quad \text{Relation by Implication}$$

$$\equiv q \vee \neg p \quad \text{Commutativity}$$

$$\equiv \neg(\neg q) \vee \neg p \quad \text{double negation}$$

$$\equiv \neg q \rightarrow \neg p \quad \text{RBI (backwards)}$$

$\blacksquare$

$$p \rightarrow q \equiv \neg p \vee q$$

Example: Show that $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$ (without a truth table)

$$(p \to q) \vee (p \to r) \equiv (\neg p \vee q) \vee (\neg p \vee r) \quad \text{RBI}$$

$$\equiv \neg p \vee q \vee \neg p \vee r \quad \text{Associative}$$

$$\equiv \underbrace{\neg p \vee \neg p} \vee q \vee r \quad \text{Commutative}$$

$$\equiv \neg p \vee q \vee r \quad \text{idempotent}$$

$$\equiv \neg p \vee (q \vee r) \quad \text{associative}$$

$$\equiv p \to (q \vee r) \quad \text{RBI !}$$

A compound proposition is **satisfiable** if there is an assignment of truth values to its variables that makes it true. If there is no such case then we say it is **unsatisfiable** (i.e. a contradiction)

Example: Show that $\underline{(p \vee \neg q)} \wedge \underline{(\neg p \vee q)} \wedge \underline{(\neg p \vee \neg q)}$ is satisfiable.

T ∧ T ∧ T

1. $(p \vee \neg q) \wedge (\neg p \vee q)$ p, q must have same truth values

$\;\;\;\;$ T $\;$ F $\;\;\;\;\;\;\;\;$ F $\;$ T

2. add in $(\neg p \vee \neg q)$ p, q must be false.

Example: Show that $(p \to q) \land (p \to \neg q) \land (\neg p \to q) \land (\neg p \to \neg q)$ is not satisfiable.

Example: Sudoku puzzles can be written (and solved) as a satisfiability problems

Solving this:

1) First chain together the propositions with provided values: $p(1,1,5) \land p(1,2,3) \land p(1{,}5{,}7) \land \ldots$
2) Assert that every row contains every number: $$\bigwedge_{i=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{j=1}^{9} p(i,j,n)$$
3) Assert that every column contains every number: $$\bigwedge_{j=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{9} p(i,j,n)$$
4) Assert that every 3x3 block contains every number: $$\bigwedge_{r=0}^{2} \bigwedge_{s=0}^{2} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{3} \bigvee_{j=1}^{3} p(3r+i, 3s+j, n)$$

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

Let $p(i,j,n)$ denote the proposition that a number n is in the cell in row i and column j

9 rows, 9 columns, 9 numbers $=$
$9 \times 9 \times 9 = 729$ propositions

5) Assert that no cell contains more than one number: $$\bigwedge_{i=1}^{9} \bigwedge_{j=1}^{9} \bigwedge_{n=1}^{9} \bigwedge_{m=1, m\neq n}^{9} (\, p(i,j,n) \rightarrow \neg p(i,j,m))$$

6) String 1-5 together with conjunctions