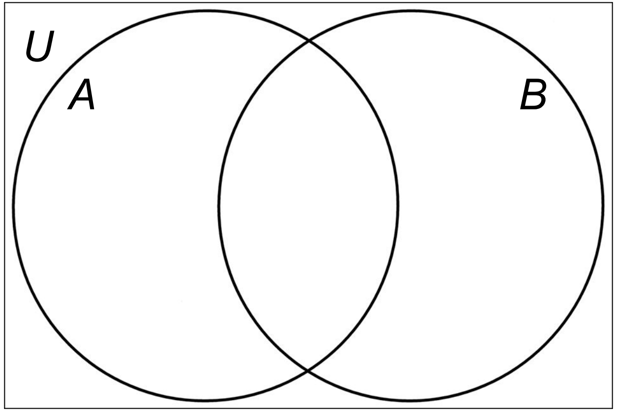
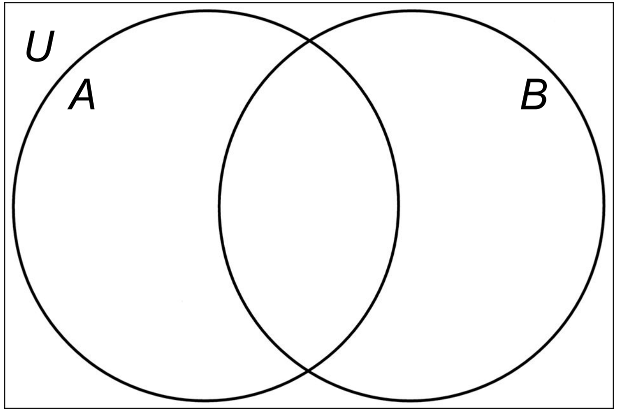
Warm-up problem

Example: Let *A* and *B* be two sets, and *U* be the universal set. Using Venn Diagrams, illustrate the following:

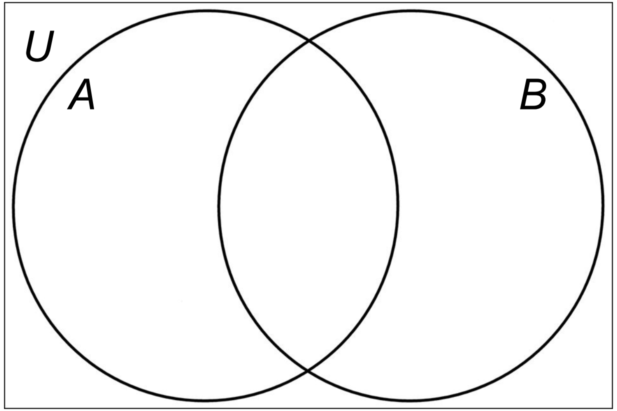
$A \cup B$



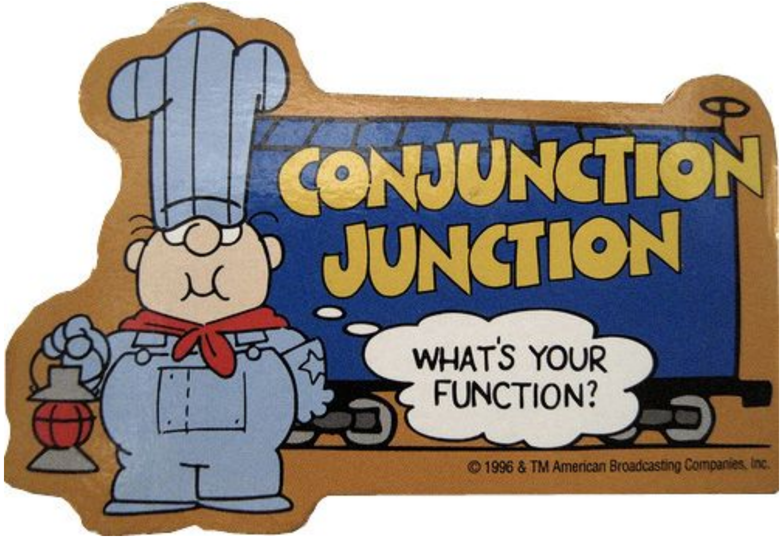
$A - B$



$U \setminus B$



Lecture 14: Functions



Announcements and reminders

- Homework 5 posted, **due Monday (8 Oct)**
- **Midterm 1: 6:30-8 PM, Tuesday 2 October**
  - Rachel (001) in HUMN 1B50
  - Tony (002) in DUAN G1B30

- Review (Q+A) Monday in class
- Quizlet 5 due Monday

What did we do last time?

- What are ***sets*** of things?
- What can we do with them? (combinations, manipulating them…)

Today:

- ***Functions!***
  - Things we can do to sets of stuff
  - Special kinds of functions
  - Special properties functions can have
  - Should we fear them?

Functions

Functions are everywhere in computer science and engineering.

```
In [7]: def Square(x_in):
   ...:
   ...:     x2_out = x_in * x_in
   ...:
   ...:     return (x2_out)
   ...:
```

A function is a routine that takes some kind of input, *does stuff*, and yields some kind of output, as well as possibly some side effects.

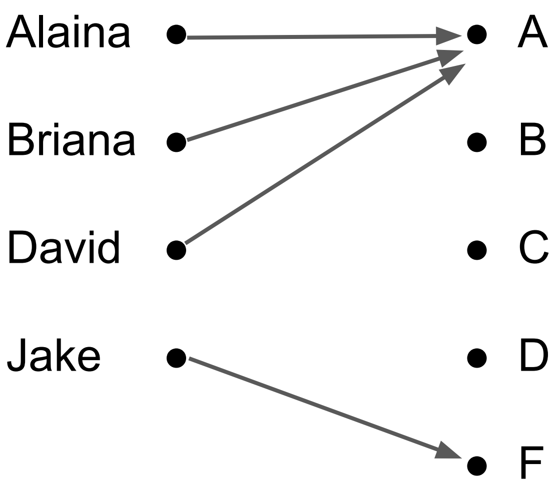
Computer science distinction: **function:** takes inputs, produces outputs
vs. **procedure:** takes inputs, produces side effects (but no outputs)

We will use this narrower definition of a function (inputs ⇒ outputs)

Functions

We will abstract this to the mathematical idea of a function.

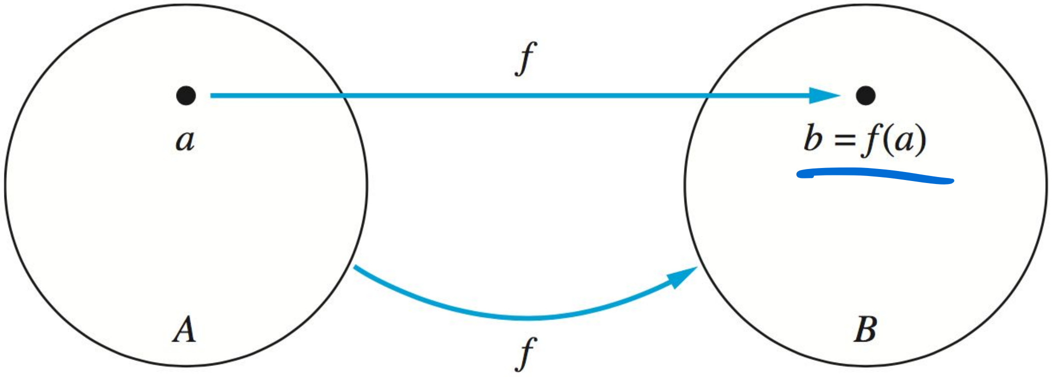
Example: Here is a function that assigns grades to Discrete Structures students.



The grade assignment function takes students {Alaina, Briana, David, Jake, … } and maps them to letter grades {A, B, C, D, F}.

Functions

The key idea behind functions is that **for each member of the input set *A* they produce exactly one member of the output set *B***.



Definition: Let A and B be nonempty sets. A **function** f from A to B is an assignment of exactly one element of B to each element of A.

We write $f(a) = b$ to indicate that b is the unique element of B corresponding to $a \in A$.

If f is a function from A to B, we write $f : A \to B$.

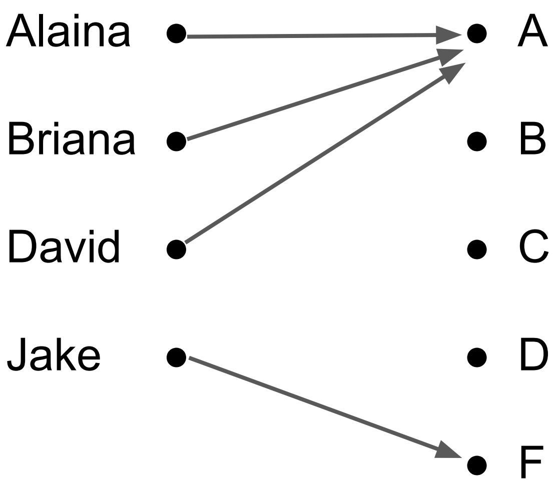
$f : A \to B$

(maps) set A to B

Definition: The **<u>domain</u>** of *f* is the set *A* that *f* maps **from**.

Definition: The **<u>codomain</u>** of *f* is the set of elements that *f* ***could*** **map to**.

Definition: The **<u>range</u>** of *f* is the set of elements that *f* ***actually does*** **map to**.



So in this example: *domain* = {Alaina, Briana, David, Jake}

codomain = {A, B, C, D, F}

range = {A, F}

Functions

Consider this `Square` function again.

```
In [7]: def Square(x_in):
   ...:
   ...:     x2_out = x_in * x_in
   ...:
   ...:     return (x2_out)
   ...:
```

Question: what are its domain, codomain and range? (assuming you are using it for its intended purpose…)

Domain = all real #'s all integers

Codomain = all nonneg. real #'s all nonneg. real #'s

Range = all nonneg. real #'s all perfect squares (0, 1, 4, …)

Functions

Consider this `Square` function again.

```
In [7]: def Square(x_in):
   ...:
   ...:     x2_out = x_in * x_in
   ...:
   ...:     return (x2_out)
   ...:
```

Question: what are its domain, codomain and range? (assuming you are using it for its intended purpose…)

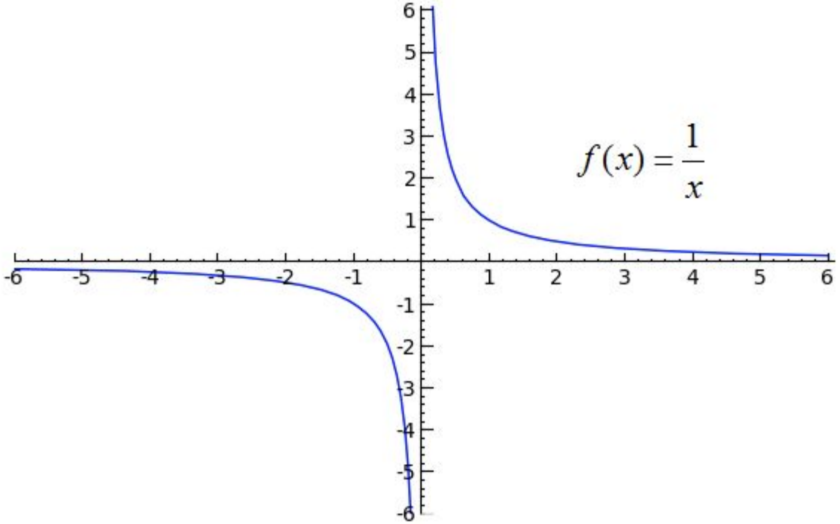
Answer: *domain* = set of all real numbers (don't worry about complex)
codomain = set of all non-negative real numbers
range = set of all non-negative real numbers

Functions

The choice of domain affects the nature of the function *f*

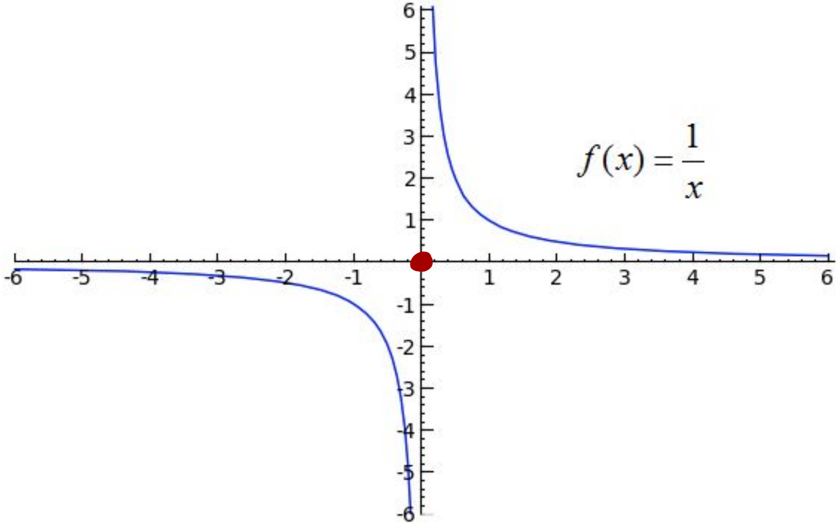
Example: Define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = 1/x$

Question: Is this a valid function?



Functions

The choice of domain affects the nature of the function *f*



Example: Define $f : \mathbf{R} \rightarrow \mathbf{R}$ by $f(x) = 1/x$

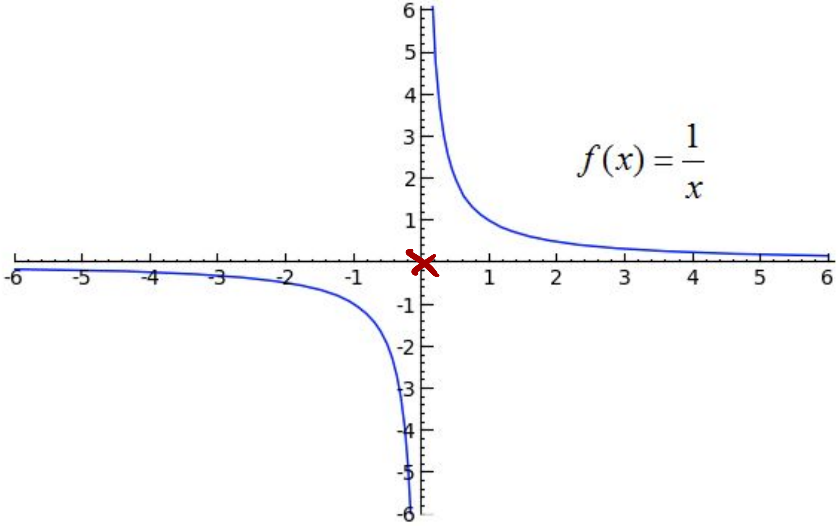
Question: Is this a valid function?

➢ **Answer:** No! Because *f* doesn't map every element of the domain into the codomain

Question: What if we change it to $f : (\mathbf{R} - \{0\}) \rightarrow \mathbf{R}$?

Functions

The choice of domain affects the nature of the function *f*



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➢ **Answer:** No! Because *f* doesn't map every element of the domain into the codomain

Question: What if we change it to $f : (\mathbf{R} - \{0\}) \to \mathbf{R}$?

➢ **Answer:** Yes! Now *f* maps every element of the domain to something in the range, so it's a function.

HERE

Functions

Consider this super unnecessary Python function to add two numbers

```
In [14]: # a function for adding
    ...: def Add(x_in, y_in):
    ...:
    ...:     sum_out = x_in + y_in
    ...:
    ...:     return (sum_out)
```

Question: What are the Add function's domain, codomain and range?

$$\mathrm{Add} : \underbrace{\mathbb{R} \times \mathbb{R}}_{\text{DOMAIN}} \to \underbrace{\mathbb{R}}_{\text{CODOMAIN}}$$

range = codomain = $\mathbb{R}$

saying that the fcn. "Add" maps this set to this set

Functions

Consider this super unnecessary Python function to add two numbers

```
In [14]: # a function for adding
    ...: def Add(x_in, y_in):
    ...:
    ...:     sum_out = x_in + y_in
    ...:
    ...:     return (sum_out)
```

Question: What are the `Add` function's domain, codomain and range?

Answer: Assuming we are not dealing with complex numbers, can think of `Add` as

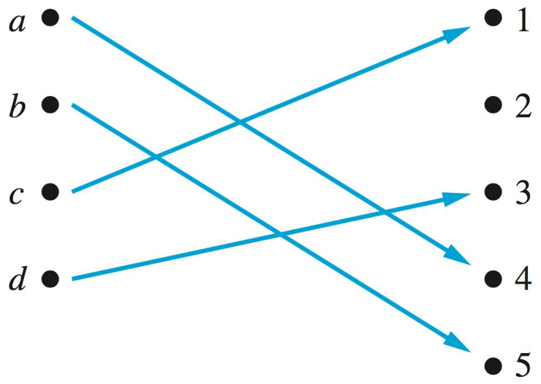
$$\text{Add} : (\mathbf{R} \times \mathbf{R}) \rightarrow \mathbf{R}$$

- *Domain* is the set of ordered pairs of real numbers, (x, y)
- *Codomain* is the set of real numbers
- *Range* is also the set of real numbers

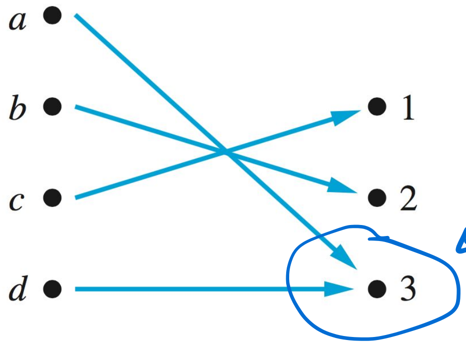
One-to-one and *Onto* Functions

Some functions never assign the same value in the range to more than one domain element. These functions are called **one-to-one** functions.

no two domain elts. map to same place



This **is** a one-to-one function



one-to-one

not 1-1 b/c two elts. of domain both map to 3

This **is not** a one-to-one function

Definition: A function *f* is <u>**one-to-one**</u>, or *injective*, if and only if *f(a)* = *f(b)* implies that *a* = *b* for all *a* and *b* in the domain of *f*.

One-to-one and *Onto* Functions

Example: Prove that $f(n) = n^3$ is one-to-one. (implicit: $f : \mathbf{R} \rightarrow \mathbf{R}$)

- **Strategy:** Assume $f(n) = f(m)$, and show it must be the case that $n = m$.

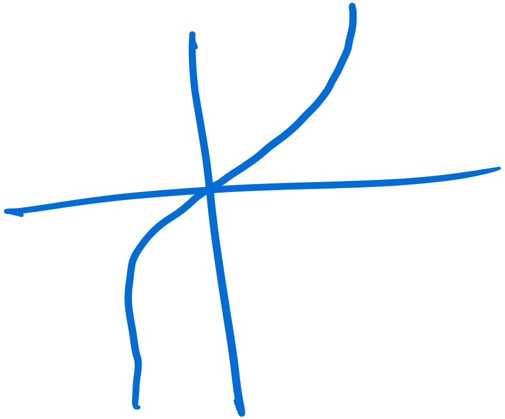
Proof: S'pose n & m are real #'s s.t. $f(n) = f(m)$

$\rightarrow n^3 = m^3$

$\rightarrow (n^3)^{\frac{1}{3}} = (m^3)^{\frac{1}{3}}$

$\rightarrow n = m$

$\therefore f$ is 1-1 ∎



One-to-one and *Onto* Functions

Example: Prove that $f(n) = n^3$ is one-to-one. (implicit: $f : \mathbf{R} \to \mathbf{R}$)

- **Strategy:** Assume $f(n) = f(m)$, and show it must be the case that $n = m$.

Proof:

Assume for $n, m \in \mathbf{R}$, $f(n) = f(m)$

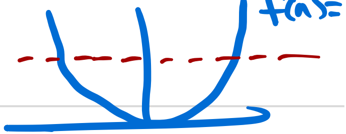
$\Rightarrow n^3 = m^3$

$\Rightarrow$ take the cube root of both sides to find $n = m$

$\Rightarrow$ so f must be one-to-one

$\square$

One-to-one and *Onto* Functions



Example: Prove that $f(n) = n^2$ is **not** one-to-one. (implicit: $f : \mathbf{R} \to (\mathbf{R} \geq 0)$)

- **Strategy:** Show that there is at least **one pair** of numbers that map to the same place.

One-to-one def: f is 1-1 iff. $\forall a \forall b [f(a) = f(b) \to a = b]$

To prove not true, prove the negative of this is true:

$\neg \forall a \forall b [\ldots] \equiv \exists a \exists b \neg [\ldots]$ (De Morgan's)

To show $\exists a \exists b \neg [f(a) = f(b) \to a = b]$ means

$a = 2$ & $b = -2$ both map to $f(2) = f(-2) = 4$, but $a \neq b$

One-to-one and *Onto* Functions

Example: Prove that $f(n) = n^2$ is **not** one-to-one. (implicit: $f : \mathbf{R} \rightarrow (\mathbf{R} \geq 0)$)

- **Strategy:** Show that there is at least **one pair** of numbers that map to the same place.

Proof:

Take $n = 2$ and $m = -2$.

$\Rightarrow$ Then $f(n) = f(m) = 4$

$\Rightarrow$ [at least] two numbers in the domain map to the same element in the range

$\Rightarrow$ so f must not be one-to-one

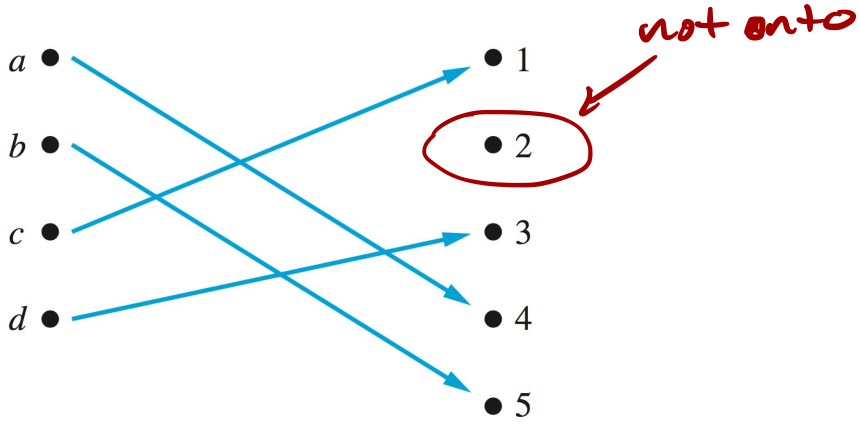
$\square$

One-to-one and *Onto* Functions

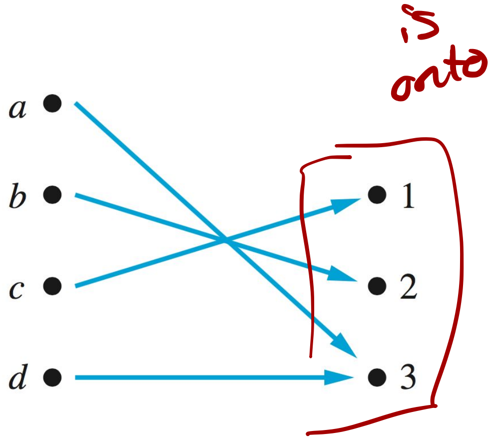
$\forall b \exists a [f(a)=b]$

$a \in A$
$b = B$

Some functions have the property that they actually map to every element in their codomain (i.e., the codomain = the range). These functions are called **onto** functions.



This **is not** an onto function



This **is** an onto function

Definition: A function *f* is **onto**, or *surjective*, if and only if for every element *b* ∈ *B*, there is an element *a* ∈ *A* such that *f(a)* = *b*.

codomain = range (everything you can map to gets mapped to)

One-to-one and *Onto* Functions

Example: Prove that our Add function is onto,
where $Add : (\mathbf{Z} \times \mathbf{Z}) \rightarrow \mathbf{Z}$ is:

$Add(m, n) = m + n$

```
In [14]: # a function for adding
    ...: def Add(x_in, y_in):
    ...:
    ...:     sum_out = x_in + y_in
    ...:
    ...:     return (sum_out)
```

Strategy: Show that any arbitrary element of the codomain has at least one element of the domain that maps to it. (by construction)

need to find some m & $n \in \mathbb{Z}$ s.t. $\underbrace{m + n}_{Add(m,n)} = b$

Proof: Let $b \in \mathbb{Z}$ be an arbitrary integer

One way: pick $n = 0$
& $m = b$

$\Rightarrow \exists\, m$ & n s.t. $Add(m,n) = b$, no matter what b (in the codomain is)

$\Rightarrow Add(\cdot,\cdot)$ is onto! ∎

One-to-one and *Onto* Functions

Example: Prove that our Add function is onto,
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```

Strategy: Show that any arbitrary element of the codomain has at least one element of the domain that maps to it.

Proof:

S'pose b is any integer (i.e., an element of the codomain)

⇒ Then we need to find integers m and n such that $m + n = b$.

⇒ Let n be any integer.

⇒ Let $m = b - n$ □

One-to-one and *Onto* Functions

Example: Prove that our `Add` function is onto,
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Question: Is $Add(m, n) = m + n$ one-to-one?

One-to-one and *Onto* Functions

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    ...:
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```
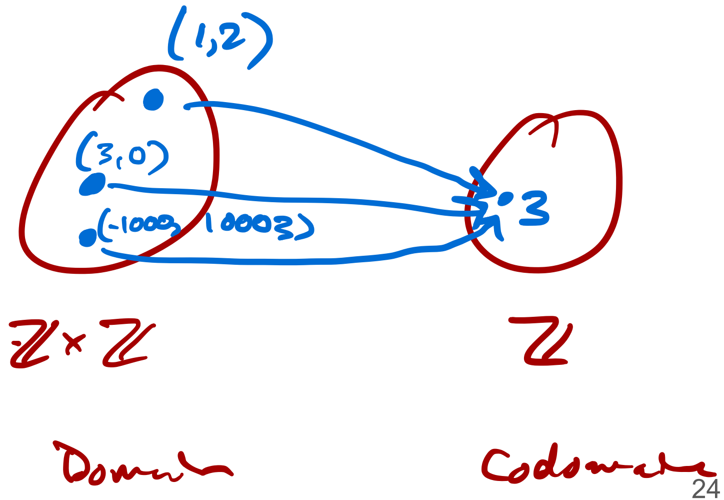

Question: Is $Add(m, n) = m + n$ one-to-one?

Answer: Nope.

- S'pose $f(m, n) = m + n = 3$.
- There are LOTS of m and n that add up to 3.

$$f(1,2) = f(3,0) = f(-1000, 1003) = \ldots = 3$$



One-to-one and *Onto* Functions

Two starting points:
1) think about 0 ✓
2) think about neg. #'s ✓

Example: Prove that $f(n) = n^2$ where $f : \mathbf{Z} \rightarrow \mathbf{Z}$ is not onto.

Strategy: Show that there is at least one element of the codomain that is not mapped to.

any negative $\mathbb{Z}$ is not mapped to!

One-to-one and *Onto* Functions

Example: Prove that $f(n) = n^2$ where $f : \mathbf{Z} \to \mathbf{Z}$ is not onto.

Strategy: Show that there is at least one element of the codomain that is not mapped to.

Proof: Any negative number is not mapped to.

- e.g., nothing maps to -1 (there is no $n \in \mathbf{Z}$ s.t. $f(n) = n^2 = -1$)

Let's make it more interesting: What if we redefine it to $f : \mathbf{Z} \to \mathbf{N}$?

One-to-one and *Onto* Functions

Example: Prove that $f(n) = n^2$ where $f : \mathbf{Z} \to \mathbf{Z}$ is not onto.

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Let's make it more interesting: What if we redefine it to $f : \mathbf{Z} \to \mathbf{N}$?

Answer: Still no - not every natural number is a perfect square.

- e.g., nothing maps to 5 (there is no $n \in \mathbf{Z}$ s.t. $f(n) = n^2 = 5)$

$\neg \exists n \in \mathbb{Z}$ s.t. ...

$\nexists$ also okay!

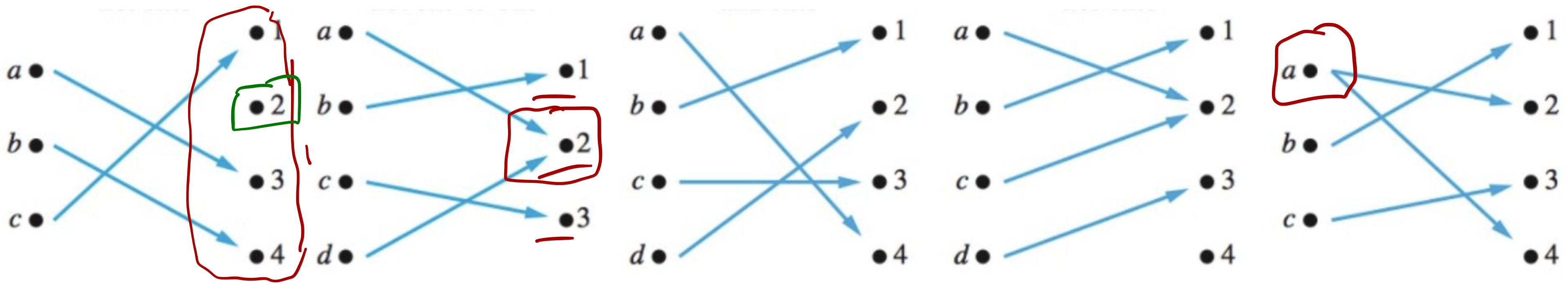
$n \notin \mathbb{Z}$ s.t.

<u>not onto</u>

One-to-one and *Onto* Functions

Example: Classify these functions as one-to-one (1-1), onto, both or neither.

R. S. T. U. V.



R.: ☒ 1-1? ○ ~~onto?~~ bc 2 isn't mapped to

S.: ☐ ~~1-1?~~ ⊗ onto?

T.: ☑ 1-1? ✓ onto?

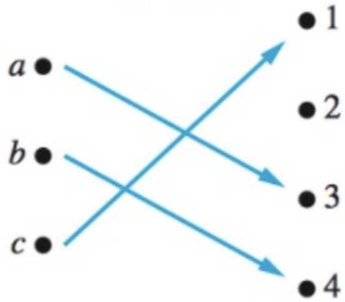
U.: ☐ 1-1? ○ onto?

V.: ☐ 1-1? ○ onto?

One-to-one and *Onto* Functions

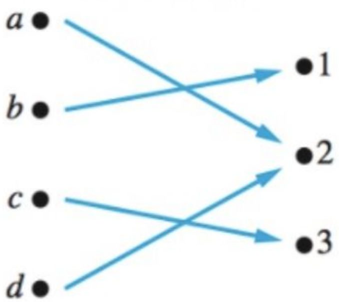
Example: Classify these functions as one-to-one (1-1), onto, both or neither.

R.



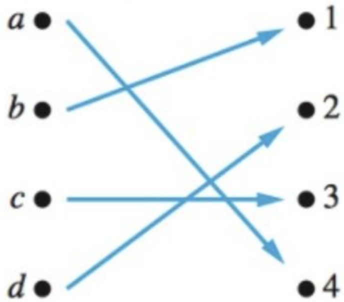
1-1
not onto

S.



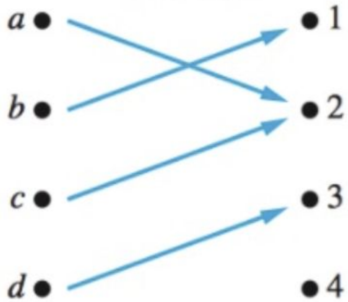
not 1-1
onto

T.



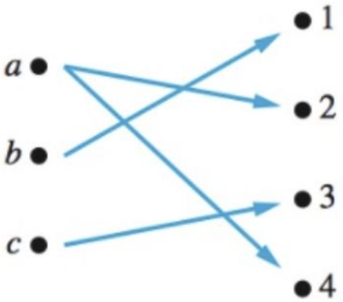
1-1
onto

U.



not 1-1
not onto

V.



not a
function

Inverse Functions

Nice things happen when a function is both 1-1 and onto.

- This is called a ***bijective*** function.
  The function can be called a ***bijection*** (special kind of function that is 1-1 and onto).

Inverse Functions

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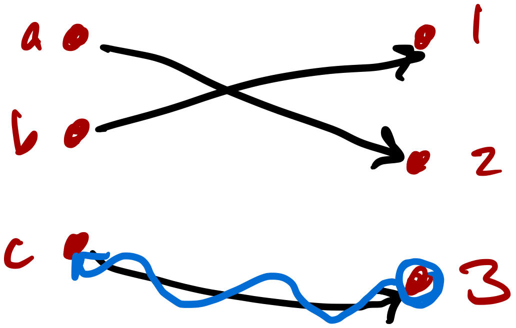
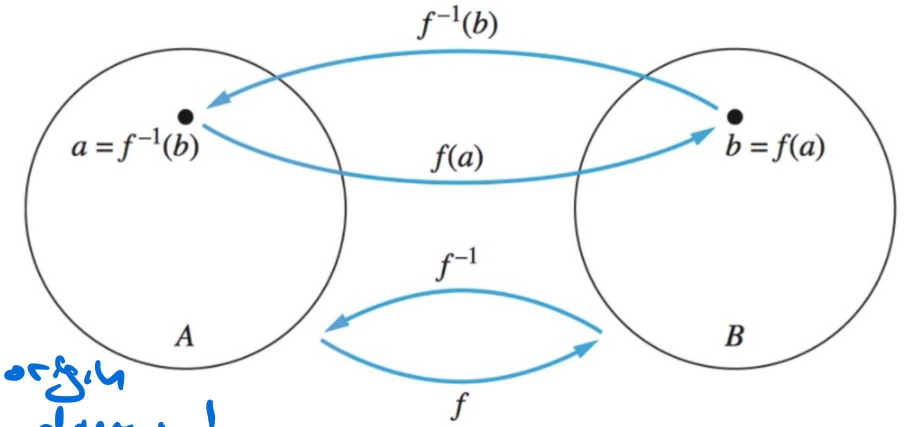




can trace arrows back to origin in the domain!

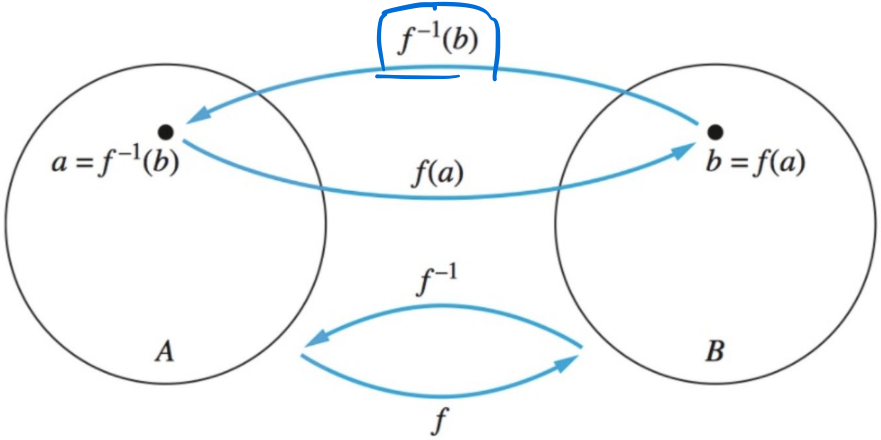
If $f : A \to B$ is 1-1 and onto, then:

- f maps to each of the elements of B (because f is onto)
- But f is 1-1 as well, so each element of B has a unique element in A that maps to it.

Inverse Functions

This implies that there is a unique **one-to-one correspondence** between elements in *A* and elements in *B*.

When this happens, we can go back and forth between *A* and *B* via *f* and an **inverse** function f^{-1}



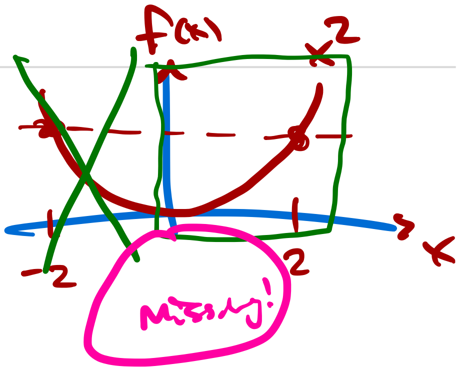
Definition: Let *f* be a 1-1 and onto function from *A* to *B*. Then there exists an **inverse function**, f^{-1}, such that $f^{-1}(b) = a$ when $f(a) = b$.

Inverse Functions

Example: The inverse of $f(x) = x^3$, where $f : \mathbf{R} \to \mathbf{R}$ is $f^{-1}(y) = y^{1/3}$.

- f cubes stuff, and f^{-1} "un-cubes" stuff.

Inverse Functions



Example: The inverse of $f(x) = x^3$, where $f : \mathbf{R} \rightarrow \mathbf{R}$ is $f^{-1}(y) = y^{1/3}$.

- *f* cubes stuff, and f^{-1} "un-cubes" stuff.

Example: $f(x) = x^2$ where $f : \mathbf{R} \rightarrow \mathbf{R}$ does ***not*** have an inverse.

- One reason: *f* is not 1-1. e.g., $f(-2) = f(2) = 4$
- Another reason: *f* is not onto. e.g., there is no *x* in the domain s.t. $f(x) = -1$

Question: could we redefine *f* such that it does have an inverse?

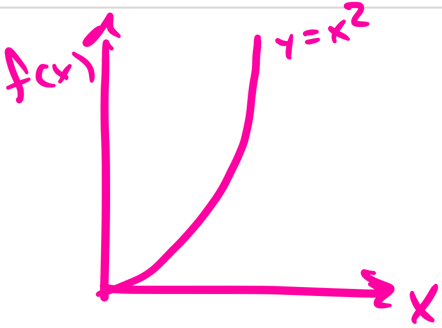
Fix 1-1: restrict domain to only $\mathbb{R} \geq 0$ (real #'s that are greater than or equal to 0)

Fix onto: restrict codomain also to $\mathbb{R} \geq 0$

Inverse Functions

Example: The inverse of $f(x) = x^3$, where $f : \mathbf{R} \to \mathbf{R}$ is $f^{-1}(y) = y^{1/3}$.

- f cubes stuff, and f^{-1} “un-cubes” stuff.



Example: $f(x) = x^2$ where $f : \mathbf{R} \to \mathbf{R}$ does ***not*** have an inverse.

- One reason: f is not 1-1. e.g., $f(-2) = f(2) = 4$
- Another reason: f is not onto. e.g., there is no x in the domain s.t. $f(x) = -1$

Question: could we redefine f such that it does have an inverse?

Answer: sure!

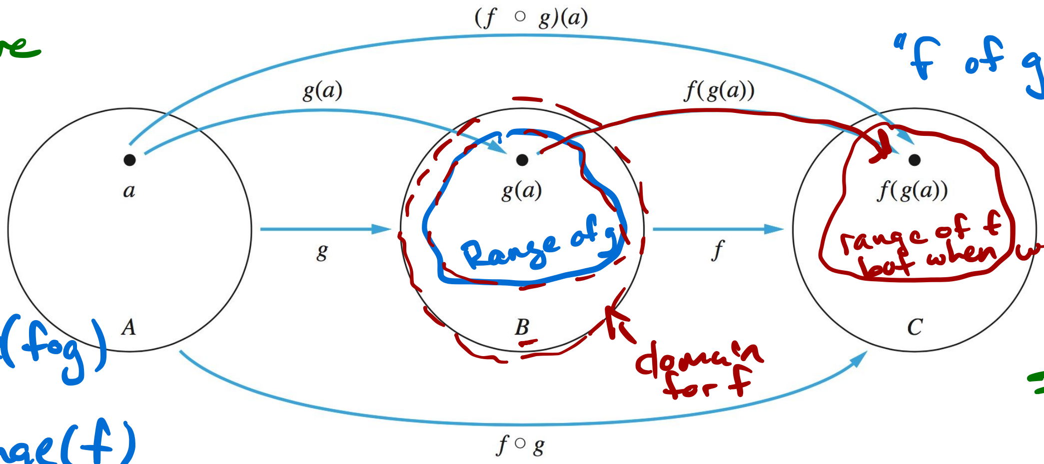
- The problem was those pesky negative numbers. So redefine $f : (\mathbf{R} \geq 0) \to (\mathbf{R} \geq 0)$
- Then $f^{-1}(y) = y^{1/2}$

Composition of Functions

Definition: Let g be a function from set A to set B, and let f be a function from set B to set C. The **composition** of f and g, denoted $f \circ g$, is defined for $a \in A$ by $(f \circ g)(a) = f(g(a))$.

Question: How are range of f & range of fog related?

"f of g" : actually a fcn. itself!



we plug in range of g (not the whole domain of f) = range of fog

☐ range(f) ⊆ range(fog)

~~☐~~ range(fog) ⊆ range(f)

Example: What are the domain, codomain and range of $(f \circ g)(a)$?

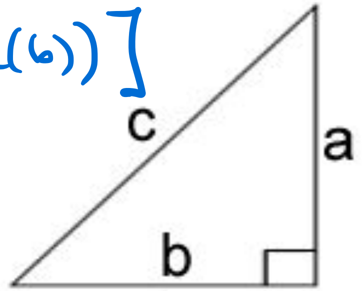
- Domain = domain of g Codomain = codomain of f
- Range = range of f if you use the range of g as the domain of f

range(g) ⊆ codomain(g)

Composition of Functions

We use composition of functions all the time in the wild.

Example: $c = \sqrt{a^2 + b^2}$ Sqrt [Add(square(a) , square(b))]



$a^2 + b^2 = c^2$

```
In [14]: # a function for adding
    ...: def Add(x_in, y_in):
    ...:
    ...:     sum_out = x_in + y_in
    ...:
    ...:     return (sum_out)
    ...:
    ...: # a function for squaring
    ...: def Square(x_in):
    ...:
    ...:     x2_out = x_in * x_in
    ...:
    ...:     return (x2_out)
    ...: c = pow( Add( Square(3), Square(4) ) , 0.5)
    ...: print(c)
    ...:
5.0
```

Some important functions

⌊x⌋

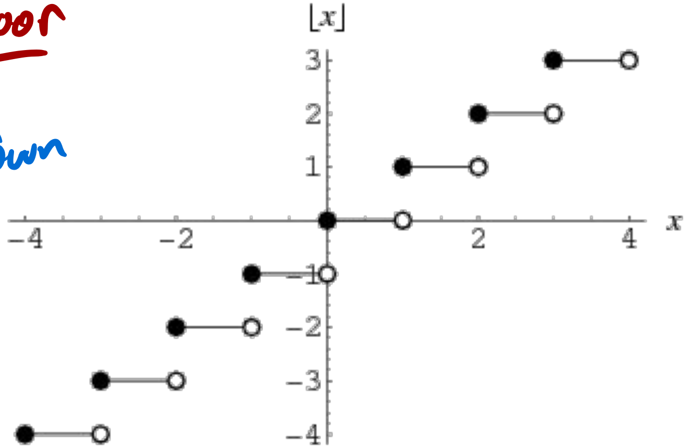
Definition: The **floor** function, denoted ⌊*x*⌋, assigns to the real number *x* the largest integer that is less than or equal to *x*. The **ceiling** function, denoted ⌈*x*⌉, assigns to the real number *x* the smallest integer that is greater than or equal to *x*.

⌈x⌉

Remark: Both of these come up frequently in CS applications.

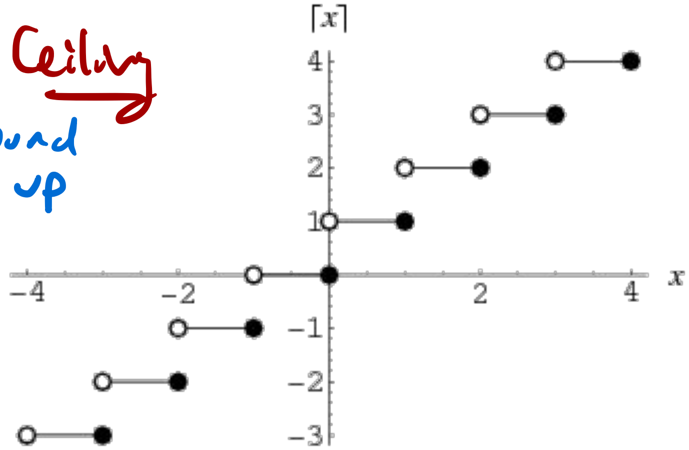
Floor

round down



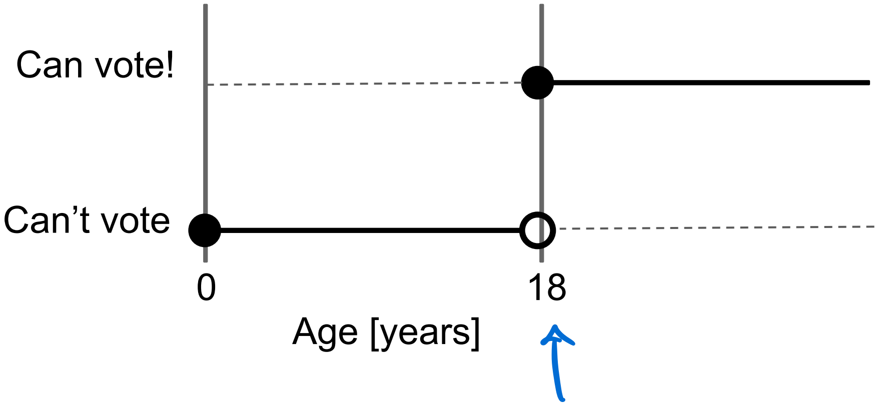
Ceiling

round up



Some important functions

Example: Being able to vote follows a floor function:



More examples:

$\lfloor 3.5 \rfloor = 3$ $\lfloor 5 \rfloor = 5$ $\lfloor -3.5 \rfloor = -4$

$\lceil 3.5 \rceil = 4$ $\lceil 5 \rceil = 5$ $\lceil -3.5 \rceil = -3$

HERE

Some important functions

Example: Prove or disprove that $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$.

FALSE!

the most this can be ceiling'd is just less than 1

each of these could go up by just less than 1

Counterexple:

Take $x = 1.1$ & $y = 1.1$

$\longrightarrow \lceil x+y \rceil = \lceil 1.1 + 1.1 \rceil = \lceil 2.2 \rceil = 3$

$\neq$

$\longrightarrow \lceil x \rceil + \lceil y \rceil = \lceil 1.1 \rceil + \lceil 1.1 \rceil = 2 + 2 = 4$

Some important functions

Example: Prove or disprove that $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$.

Turns out, this is *not* true. We just need to find a single counterexample that breaks it.

Some important functions

Example: Prove or disprove that $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$.

Turns out, this is *not* true. We just need to find a single counterexample that breaks it.

(Dis)proof:

- Note that each of $\lceil x \rceil$ and $\lceil y \rceil$ will round up to x+1 and y+1 if x and y are just a tiny bit larger than an integer.
- But $\lceil x + y \rceil$ will only round up once, to x+y+1, if x and y are only a tiny bit larger
- So try: $x = 1.1$ and $y = 2.1$
- Then: $\lceil x + y \rceil = \lceil 1.1 + 2.1 \rceil = \lceil 3.2 \rceil = 4$

  But $\lceil x \rceil + \lceil y \rceil = \lceil 1.1 \rceil + \lceil 2.1 \rceil = 2 + 3 = 5$

□

Countable and uncountable sets

So $|\mathbf{N}| = |\mathbf{Z}| = |\mathbf{Q}| = |\mathbf{R}| = \infty$, and we're done, right?

naturals integers rationals real

Countable and uncountable sets

A is countable if can find a 1-1 fcn. from $\mathbb{N} \to A$

$$\mathbb{N} \xrightarrow{f} A$$

So |**N**| = |**Z**| = |**Q**| = |**R**| = ∞, and we're done, right?

Wrong!

- Turns out, it is useful to break these up into cases of ***how infinite*** a set is.
- These are described roughly as follows:
  - **Countably infinite** (or **countable**): We could count up each member of the set if we had infinite time.
  - **Uncountably infinite** (or **uncountable**): We could never count even list each element of the set, even if we had infinite time.

Definition: A set *A* is called **countable** or **countably infinite** if it is not finite and there is a one-to-one function between each element of *A* and the natural numbers. *A* is called **uncountable** if it is infinite and not countable. (Finite sets are **countable**.)

Countable and uncountable sets

Example: Show that the set of positive even integers is countably infinite.

We need to find a one-to-one map

from the positive even integers $\quad \mathbf{E} = \{2, 4, 6, 8, \ldots\}$

to the natural numbers $\quad \mathbf{N} = \{0, 1, 2, 3, \ldots\}$

row #'s $\mathbb{N}$ A

$n \in \mathbb{N}$

take $f(n) = 2(n+1)$

(Could prove f is 1-1 by s'posing $f(n) = f(m)$ & show n must $= m$.)

$\mathbb{N}$	E
0	2
1	4
2	6
3	8
⋮	⋮

Since there exists a 1-1 fcn. from $\mathbb{N}$ to E, the set E is countable

Countable and uncountable sets

Example: Show that the set of positive even integers is countably infinite.

We need to find a one-to-one map
from the positive even integers **E** = {2, 4, 6, 8, …}
to the natural numbers **N** = {0, 1, 2, 3, …}

Well, if we divide all the elements in **E** by 2, we get {1, 2, 3, 4, …}

… and notice that those are each a corresponding element of **N**, but + 1

So we could establish a pattern as:

N		**E**
0	⇔	2
1	⇔	4
2	⇔	6
	…	

Or define the relationship:

$f(n) = 2(n + 1)$

Countable and uncountable sets

Example: Show that the set of all integers is countable.

Need a map from the set of natural numbers {0, 1, 2, 3, …}
to the set of all integers { …, -2, -1, 0, 1, 2, … }

$$f : \mathbb{N} \to \mathbb{Z}$$

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ \boxed{-\lceil \frac{n}{2} \rceil} & \text{if } n \text{ is odd} \end{cases}$$

$\mathbb{N}$	$\mathbb{Z}$
0	0
1	-1
2	1
3	-2
4	2
5	-3
⋮	⋮

Countable and uncountable sets

Example: Show that the set of all integers is countable.

Need a map from the set of natural numbers $\{0, 1, 2, 3, \ldots\}$
to the set of all integers $\{ \ldots, -2, -1, 0, 1, 2, \ldots \}$

The natural way to line them up might be:

N		**Z**
0	⇔	0
1	⇔	1
2	⇔	-1
3	⇔	2
4	⇔	-2
5	⇔	3
	…	

Does this give rise to a relationship $f(n)$ we can define?

Yes! Easiest to break into cases:

$$f(n) = \begin{cases} -n/2 & \text{if } n \text{ is even} \\ \underline{(n+1)}/2 & \text{if } n \text{ is odd} \end{cases}$$

Countable and uncountable sets

(num, den)

Example: Show that the set of positive rational numbers is countable.

$\mathbb{Q} > 0$

$$\begin{array}{cccccc} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \cdots \\ \frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \cdots \\ & \vdots & & & & \end{array}$$

$f_1 : \mathbb{N} \to$ Row 1

$f_1(n) = \frac{1}{n+1}$

f_1 is 1-1 from $\mathbb{N}$ to Row 1.

So, Row 1 is countable!

$f_2 : \mathbb{N} \to$ Row 2

$f_2(n) = \frac{2}{n+1}$

Row 2 also countable!

All rows are countable, & # rows is countable (if $n \in \mathbb{N}$, $f(n) = n+1$ is the numerator for that row)

$\to$ All of $\{\mathbb{Q} > 0\}$ is a countable # of countable sets (rows), $\therefore$ is countable!

Countable and uncountable sets

Example: Show that the set of positive rational numbers is countable.

$$
\begin{array}{ccccc}
\frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \cdots \\
\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \cdots \\
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \cdots \\
 & \vdots & & &
\end{array}
$$

Countable and uncountable sets

Example: Show that the set of positive rational numbers is countable.

$$\begin{array}{ccccc} \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \cdots \\[2ex] \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \cdots \\[2ex] \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \cdots \\ & & \vdots & & \end{array}$$

$p + q = 2$

$p + q = 3$

$p + q = 4$

$\cdots$

Countable and uncountable sets

Example: Show that the set of real numbers is uncountable.

Countable and uncountable sets

$(0, 1)$ ← figure out from context

Example: Show that the set of real numbers is uncountable.

Let's just look at the interval [0, 1]

Note: intervals like [0, 1] or (0, 1) are **always** assumed to be subsets of $\mathbb{R}$ unless otherwise stated.

We will try for a **proof by contradiction**.

So s'pose we have a list of all of the real numbers between 0 and 1.

0.	4	8	7	5	4	6	9	0	1	…
0.	9	1	9	6	5	5	3	4	8	…
0.	0	9	7	4	3	3	7	9	9	…
0.	1	2	3	4	5	6	7	8	9	…
0.	2	5	3	0	0	4	2	1	7	…

(and so on…)

Countable and uncountable sets

Example: Show that the set of real numbers is uncountable.

So s'pose we have a list of all of the real numbers between 0 and 1.

0.	4	8	7	5	4	6	9	0	1	…
0.	9	1	9	6	5	5	3	4	8	…
0.	0	9	7	4	3	3	7	9	9	…
0.	1	2	3	4	5	6	7	8	9	…
0.	2	5	3	0	0	4	2	1	7	…

(and so on…)

Let's **construct** a number *m* that **can't be on the list**.

Strategy: Let the k^{th} digit of *m* depend on the k^{th} digit of the k^{th} element in our list, according to a rule that ensures *m* will be different from the k^{th} list element.

Countable and uncountable sets

Example: Show that the set of real numbers is uncountable.

So s'pose we have a list of all of the real numbers between 0 and 1.

0.	4	8	7	5	4	6	9	0	1	…
0.	9	1	9	6	5	5	3	4	8	…
0.	0	9	7	4	3	3	7	9	9	…
0.	1	2	3	4	5	6	7	8	9	…
0.	2	5	3	0	0	4	2	1	7	…

(and so on…)

$m = 0.\underline{1}\ \underline{9}\ \underline{1}\ \underline{1}\ \underline{1}\ \ldots$

Rule:

- If the k^{th} digit of the k^{th} digit of the k^{th} number in our list is a 1, then the k^{th} digit of m is a 9.
- If the k^{th} digit of the k^{th} digit of the k^{th} number is not a 1, then the k^{th} digit of m is a 1.

Countable and uncountable sets

Example: Show that the set of real numbers is uncountable.

So s'pose we have a list of all of the real numbers between 0 and 1.

0.	**4**	8	7	5	4	6	9	0	1	…
0.	9	**1**	9	6	5	5	3	4	8	…
0.	0	9	**7**	4	3	3	7	9	9	…
0.	1	2	3	**4**	5	6	7	8	9	…
0.	2	5	3	0	**0**	4	2	1	7	…

(and so on…)

m = 0. 1 9 1 1 1 …

Rule: - If the k^{th} digit of the k^{th} digit of the k^{th} number in our list is a 1, then the k^{th} digit of m is a 9.

- If the k^{th} digit of the k^{th} digit of the k^{th} number is not a 1, then the k^{th} digit of m is a 1.

Countable and uncountable sets

Claim: Our constructed number *m* can't already be on the list.

Argument:

1. *m* is not the 1st number because their 1st digits are different.
2. *m* is not the 2nd number because their 2nd digits are different.
3. *m* is not the 3rd number because their 3rd digits are different.

(and so on…)

⇒ Thus we have constructed a number *m* that can't be on the list. ✗

⇒ This contradicts our assumptions that the real numbers between 0 and 1 are countable.

⇒ Thus the real numbers between 0 and 1 are uncountable.

⇒ All real numbers are uncountable.

Upshot: is any continuous interval of reals is uncountable

□ (this proof is called **Cantor's diagonal argument**)

Countable and uncountable sets

A non-exhaustive summary:

Countable	**Uncountable**
N (natural numbers)	[0, 1] (or **any** interval of **R**)
Z (integers)	**R**
$\mathbb{Q}$ (the rational numbers)	**R**-$\mathbb{Q}$ (the irrational numbers)
{1.234, π} (**finite** sets of anything)	

Functions

Recap: We learned about ***functions***…

- What are they?
- Special kinds of functions (floor, ceiling, inverse, composition…)
- Special properties functions can have (onto, 1-1…)
- How we can use functions to see how large a **set** is…

Next time:

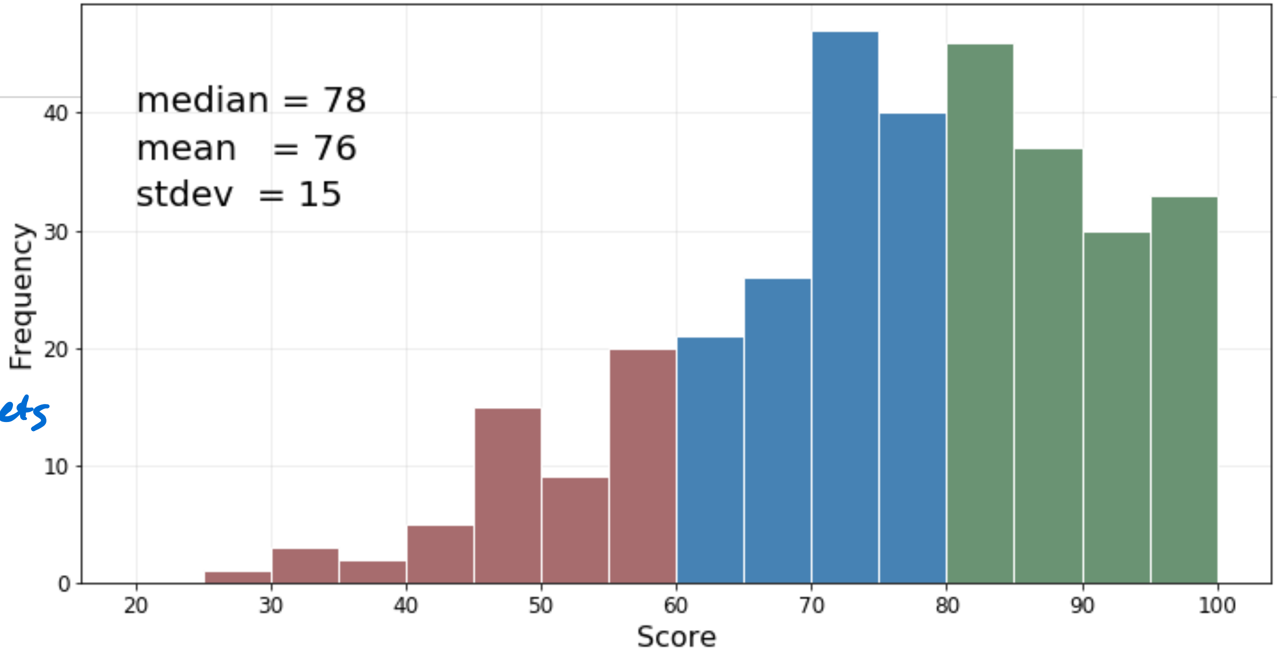
- We learn about ***sequences!***

… no, not that →

(green) = keep doing what you're doing

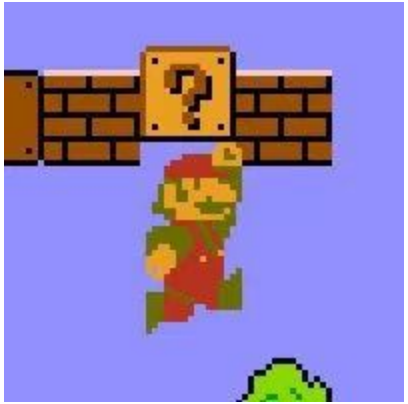
(blue) = decent-pretty good, but could swing depending on HW/quizlets

(red) = might want to try some new study strategies



- solutions w/ rubrics posted Friday PM
- CHECK THEM before asking Tony/Rachel questions
- Regrade requests IN WRITING by Friday next week
- Will regrade everything (so think first, & check all solutions)

Bonus material!



Wacky Things (AKA lin. alg.)

$$f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z} \quad (\mathbb{Z}^2)$$

$$f(a,b) = (b,a)$$

Some important functions

- **Example:** Prove that if x is a real number, then $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + ½ \rfloor$

- **Typical strategy:** (1) Write x as $x = n + \epsilon$, where n is an integer and $0 \leq \epsilon < 1$.
  (2) Plug in stuff and "do math"
  (3) Wealth and fame!

- **Proof:** We notice that depending on what ϵ is, the $\lfloor x + ½ \rfloor$ term might do different things.

  $\Rightarrow$ so let's split this up into two cases:

  (i) $0 \leq \epsilon < ½$

  (ii) $½ \leq \epsilon < 1$

Some important functions

- **Example:** Prove that if x is a real number, then $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$

Proof (cont.)

- **Case (i):** S'pose $x = n + \epsilon$, where $0 \leq \epsilon < \frac{1}{2}$
  - Then $\lfloor 2x \rfloor = \lfloor 2(n + \epsilon) \rfloor = \lfloor 2n + 2\epsilon \rfloor$

    And we know that $0 \leq 2\epsilon < 1$

    $\Rightarrow \lfloor 2x \rfloor = \lfloor 2n + 2\epsilon \rfloor = \lfloor 2n \rfloor = 2n$ *(on the left-hand side)*
  - Similarly, $\lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor = \lfloor n + \epsilon \rfloor + \lfloor n + \epsilon + \frac{1}{2} \rfloor = \lfloor n \rfloor + \lfloor n \rfloor = n + n = 2n$

    $0 \leq \epsilon < \frac{1}{2}$, so $0 \leq \epsilon + \frac{1}{2} < 1$
  - Since we proved both the LHS and the RHS = $2n$, we've proved they are equal ✔

Some important functions

- **Example:** Prove that if x is a real number, then $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + ½ \rfloor$

Proof (cont.)

- **Case (ii):** S'pose $x = n + \epsilon$, where $½ \leq \epsilon < 1$
  - Then $\lfloor 2x \rfloor = \lfloor 2(n + \epsilon) \rfloor = \lfloor 2n + 2\epsilon \rfloor$

    And we know that $1 \leq 2\epsilon < 2$

    $\Rightarrow \lfloor 2x \rfloor = \lfloor 2n + 2\epsilon \rfloor = \lfloor 2n + 1 \rfloor = 2n + 1$
  - Similarly, $\lfloor x \rfloor + \lfloor x + ½ \rfloor = \lfloor n + \epsilon \rfloor + \lfloor n + \epsilon + ½ \rfloor = \lfloor n \rfloor + \lfloor n+1 \rfloor = n + n + 1 = 2n + 1$
  - Since we proved both the LHS and the RHS = $2n+1$, we've proved they are equal ✔

$½ \leq \epsilon < 1$, so $1 \leq \epsilon+½ < 3/2$

□