Lecture 20: Induction hits the weights - Proof by **Strong** Induction!

Announcements and reminders

- Homework 7 (Moodle) is posted and is due **Friday at 12 PM Noon**
- The CU final exam schedule is up. You must take your final exam during your scheduled final exam time.
  Tony's section: 7:30 - 10 PM, Sun 16 Dec
  Rachel's section: 1:30 - 4 PM, Wed 19 Dec
- **Feedback survey** on Moodle
  - We like teaching. We like data. Give us data about teaching!
  - Closes Friday 26 October

Extra credit opportunity!   Purely attendance-based.

People respond to incentives and we **know attendance correlates with course grade**

- Consider this carefully if the exams/homeworks are not going as well as you'd like
- Download **Arkaive** -- attendance app (or go to https://arkaive.com/login)
- Enroll in this course with enrollment code: **1KG4** (9 AM w/ Rachel)
R422 (11 AM w/ Tony)
- During first 15 minutes of each class, check in on Arkaive. You can get credit for attending either section, but not both.
- Remainder of course will be worth **2% pts extra credit** (*before* factored into any final curve)

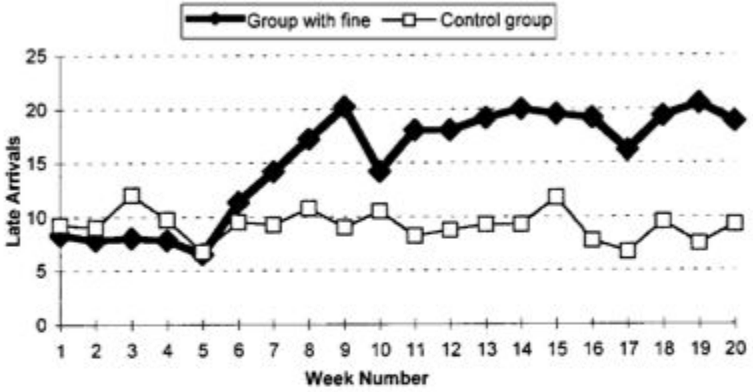


FIGURE 1.—Average number of late-coming parents, per week

Mathematical induction

S'pose you have an infinite line of dominos…

Now, we learn **strong induction:**

- If *all of the previous* dominoes falls, then *this* domino will fall
- **Base case:** Prove $P(1)$
- **Inductive step:** Prove: [if $P(\ell)$ for $1 \leq \ell \leq k$, then $P(k+1)$]

Strong induction

Example: Prove the Fundamental Theorem of Arithmetic. That is, prove that any integer $n \geq 2$ is either prime or can be written as the product of prime numbers.

Strong induction

Example: Prove the Fundamental Theorem of Arithmetic. That is, prove that any integer $n \geq 2$ is either prime or can be written as the product of prime numbers.

Strong induction

Example: Prove the Fundamental Theorem of Arithmetic. That is, prove that any integer $n \geq 2$ is either prime or can be written as the product of prime numbers.

Proof: (by strong induction)

- **Base case:** Let $n=2$. Then n is indeed prime ✔
- **Inductive step**: Assume that for each ℓ s.t. $2 \leq \ell \leq k$, ℓ is either prime or can be written as the product of primes. (this is the **inductive hypothesis**)

- **To show:** That $k+1$ is either prime or can be written as a product of primes.

  Two cases to consider:

  - Case 1: $k+1$ itself is prime
  - Case 2: $k+1$ is composite

Strong induction

- **Case 1:** $k+1$ itself is prime

  ⇒ we're done!

- **Case 2:** $k+1$ is composite

  If $k+1$ is not prime then it is composite and (from the definition of composite) can be written as a product of two integers a and b, with $a, b < k$

  But by the **induction hypothesis**, since a and $b < k$, they can be written as the product of primes.

  Multiply the prime factorization of a by the prime factorization of b, and we have the prime factorization of $k+1$

  Conclusion: We've shown that $k+1$ is either prime or can be written as the product of primes. Thus, we've proved the theorem by **strong induction**. □

Strong induction

Tougher Example: Prove: If *n* is a positive integer, then *n* can be written as the sum of distinct Fibonacci numbers.

Strong induction

Tougher Example: Prove: If *n* is a positive integer, then *n* can be written as the sum of distinct Fibonacci numbers.

Proof: (by strong induction)

- **Base case:** Let $n=1$. Then $n = F_0 + F_1 = 0 + 1 = 1$ ✔
- **Inductive step**: Assume that for each ℓ s.t. $1 \le \ell \le k$, ℓ can be written as the sum of distinct Fibonacci numbers. (inductive hypothesis)

- **To show:** That $k+1$ can be written as the sum of distinct Fibonacci numbers.

  Two cases to consider:

  - Case 1: $k+1$ itself is a Fibonacci number
  - Case 2: $k+1$ is not a Fibonacci number

Strong induction

- **Case 1:** If k+1 itself is a Fibonacci number, then for some $m \in \mathbf{N}$, we have

  $k+1 = F_m = F_{m-1} + F_{m-2}$ ✔

- **Case 2:** If k+1 is not a Fibonacci number, then we can let F_j be the largest Fibonacci number that is less than k+1.

  ⇒ So we have: $F_j < k+1 < F_{j+1}$

  ⇒ Subtract F_j everywhere to find: $0 < (k+1) - F_j < F_{j+1} - F_j = F_{j-1} < k+1$

  But now $(k+1) - F_j \leq k$, so by the **inductive hypothesis**, $(k+1) - F_j$ can be written as the sum of distinct Fibonacci numbers. Furthermore, since $(k+1) - F_j < F_{j-1}$, we know that F_j is not part of that sum.

  Thus, adding F_j to that sum expresses k+1 as the sum of distinct Fibonacci numbers.

  Thus, we've proved the claim by **strong induction**. □

Strong induction

Example: Consider a game in which two players take turns removing any positive number of matches they want from one of two piles. The player who removes the last match wins the game. Prove that if the two piles contain the same number of matches initially, then the second player can always guarantee a win.

Strong induction

Example: Consider a game in which two players take turns removing any positive number of matches they want from one of two piles. The player who removes the last match wins the game. Prove that if the two piles contain the same number of matches initially, then the second player can always guarantee a win.

Solution:

Let $P(n)$ = proposition that Player 2 can always win if each pile starts out with n matches

Base case: If n=1, then Player 1 is forced to take 1 match from 1 of the piles. Then Player 2 can take the last match from the other pile ✔

Inductive step: Suppose that Player 2 can always win if each pile starts out with ℓ matches, where $1 \leq \ell \leq k$ (inductive hypothesis)

To show: Player 2 can always win if the piles start out with k+1 matches

Strong induction

Example: Consider a game in which two players take turns removing any positive number of matches they want from one of two piles. The player who removes the last match wins the game. Prove that if the two piles contain the same number of matches initially, then the second player can always guarantee a win.

Solution:

Strong induction

Example: Consider a game in which two players take turns removing any positive number of matches they want from one of two piles. The player who removes the last match wins the game. Prove that if the two piles contain the same number of matches initially, then the second player can always guarantee a win.

Solution:

There are, again, two cases, depending on how many matches Player 1 takes (call it r):

Case 1: Player 1 takes $r = k+1$ matches (all) from one pile

Case 2: Player 1 takes $1 \leq r \leq k$ matches from one pile

Strong induction

Example: Consider a game in which two players take turns removing any positive number of matches they want from one of two piles. The player who removes the last match wins the game. Prove that if the two piles contain the same number of matches initially, then the second player can always guarantee a win.

Solution:

Case 1: Player 1 takes $r = k+1$ matches (all) from one pile

⇒ Player 2 can take all of the matches from the other pile and win!

Strong induction

Example: Consider a game in which two players take turns removing any positive number of matches they want from one of two piles. The player who removes the last match wins the game. Prove that if the two piles contain the same number of matches initially, then the second player can always guarantee a win.

Solution:

Case 2: Player 1 takes $1 \leq r \leq k$ matches from one pile

⇒ There are $k - r + 1$ matches left in this pile, and $k+1$ matches in the other pile

⇒ Player 2 should take r matches out of the other pile, leaving $k - r + 1$ matches in each pile

⇒ Now there are $k - r + 1 < k+1$ matches in each pile, and it's Player 1's turn

⇒ By the **inductive hypothesis**, there must be a strategy for Player 2 to guarantee a win

⇒ Thus, we've proved (by **strong induction**) that Player 2 can always win the game if the piles start out with equal integer numbers of matches. ◇

Induction: *bad* examples

(Bad) Example: "Prove" that $6n = 0$ for all $n \geq 0$

Base case: Let $n = 0$. Then $6n = 6(0) = 0$ ✔

Induction step: Assume that $6\ell = 0$ for all ℓ s.t. $0 \leq \ell \leq k$

To show: $6(k+1) = 0$

⇒ Write $k+1 = a + b$, where a and b are integers s.t. $0 \leq a, b \leq k$

⇒ By **induction hypothesis**, $6a = 6b = 0$

⇒ $6(k+1) = 6a + 6b = 0$ "✔"

Induction: *bad* examples

(Bad) Example: "Prove" that $6n = 0$ for all $n \geq 0$

Base case: Let $n = 0$. Then $6n = 6(0) = 0$ ✔

Induction step: Assume that $6\ell = 0$ for all ℓ s.t. $0 \leq \ell \leq k$

To show: $6(k+1) = 0$

⇒ Write $k+1 = a + b$, where a and b are integers s.t. $0 \leq a, b \leq k$

⇒ By **induction hypothesis**, $6a = 6b = 0$

⇒ $6(k+1) = 6a + 6b = 0$ "✔"

Mistake: We relied on our ability to write n as the sum of two non-negative integers. What happens when k=0?

⇒ $0 \leq a, b \leq k = 0$, so $a = b = 0$… so it turns out we can't write $k+1 = a+b$

Induction: *bad* examples

(Bad) Example: "Prove" that all Fibonacci numbers are even

Base case: Let $n = 0$. $F_0 = 0$, which is even ✔

Induction step: Assume that F_ℓ is even for all ℓ s.t. $0 \le \ell \le k$

To show: F_{k+1} is even

⇒ $F_{k+1} = F_k + F_{k-1}$

⇒ By **induction hypothesis**, F_k and F_{k-1} are both even

⇒ F_{k+1} is the sum of two evens, and therefore even "✔"

Induction: *bad* examples

(Bad) Example: "Prove" that all Fibonacci numbers are even

Base case: Let $n = 0$. $F_0 = 0$, which is even ✔

Induction step: Assume that F_ℓ is even for all ℓ s.t. $0 \le \ell \le k$

To show: F_{k+1} is even

⇒ $F_{k+1} = F_k + F_{k-1}$

⇒ By **induction hypothesis**, F_k and F_{k-1} are both even

⇒ F_{k+1} is the sum of two evens, and therefore even "✔"

Mistake: We only showed F_0 is even, but then we used $F_{k+1} = F_k + F_{k-1}$ in our proof

Induction: *bad* examples

(Bad) Example: "Prove" that all Fibonacci numbers are even

Mistake: We only showed F_0 is even, but then we used $F_{k+1} = F_k + F_{k-1}$ in our proof

⇒ If you think of our "proof" as a line of dominoes, each domino falling in this case requires **two** previous ones to fall

⇒ Here, we only did one base case, so we only knocked over **one** domino

Rule of thumb: If your proof requires going back *s* steps, then you need *s* base cases.

Strong induction

Example (a real one this time): Prove that for any integer $n \geq 1$, there exist numbers a, $b \geq 1$ such that $5^n = a^2 + b^2$

Strong induction

Example (a real one this time): Prove that for any integer $n \geq 1$, there exist numbers a, $b \geq 1$ such that $5^n = a^2 + b^2$

Important and sorta philosophical note: Proofs almost never "just plain work out".

- Usually: flail around, figure out what you need, …
- Then go back and put it all together in the right order.

Strong induction

Example (a real one this time): Prove that for any integer $n \geq 1$, there exist numbers a, $b \geq 1$ such that $5^n = a^2 + b^2$

Important and sorta philosophical note: Proofs almost never “just plain work out”.

- Usually: flail around, figure out what you need, …
- Then go back and put it all together in the right order.

→ Tackle this one *without any base cases* to begin,

→ then go back and do at the end once we know what we need

Strong induction

Example (a real one this time): Prove that for any integer $n \geq 1$, there exist numbers a, $b \geq 1$ such that $5^n = a^2 + b^2$

Important and sorta philosophical note: Proofs almost never "just plain work out".

- Usually: flail around, figure out what you need, …
- Then go back and put it all together in the right order.

→ Tackle this one *without any base cases* to begin,

→ then go back and do at the end once we know what we need

Probable inductive step: Suppose that for each ℓ s.t. $1 \leq \ell \leq k$, there exist integers a, $b \geq 1$ s.t. $5^\ell = a^2 + b^2$ (← inductive hypothesis)

To show: There exist integers a, $b \geq 1$ s.t. $5^{k+1} = a^2 + b^2$

Strong induction

To show: There exist integers a, $b \geq 1$ s.t. $5^{k+1} = a^2 + b^2$

Strong induction

To show: There exist integers a, $b \geq 1$ s.t. $5^{k+1} = a^2 + b^2$

Note that $5^{k+1} = 5^2 \cdot 5^{k-1}$

Since k-1 ≤ k, the **induction hypothesis** tells us that there exist integers c and d s.t. $5^{k-1} = c^2 + d^2$

$$\Rightarrow \quad 5^{k+1} = 5^2 \cdot 5^{k-1}$$

$$= 5^2 \cdot (c^2 + d^2)$$

$$= 5^2 \cdot c^2 + 5^2 \cdot d^2$$

$$= (5c)^2 + (5d)^2$$

⇒ Let $a = 5c$ and $b = 5d$, and we've found the ~~droids~~ integers we're looking for, such that $5^{k+1} = a^2 + b^2$ ◇

Strong induction

Base cases: We went back to k-1, which is **two** steps back from showing k+1 (as we do in the inductive step)

⇒ Need **two** base cases:

Base case 1: Let $n = 1$. Then $5^1 = 1^2 + 2^2 = 1 + 4$ ✔

Base case 2: Let $n = 2$. Then $5^2 = 25 = 3^2 + 4^2 = 9 + 16$ ✔

Now armed with our base cases **on homework and out in the wild** you would go back and **rewrite the whole thing** (having done all of this work so far on scratch paper, right?) neatly and in order. That is, the base cases would come before the inductive step, and the conclusion from the previous slide.

Only in the interest of time do I not tackle the rewrite here.

Strong induction

Example (a delicious one this time): It used to be the case that you could buy McDonald’s chicken nuggets in 4, 6, 9 and 20 piece boxes. Prove that back in the good ol’ days, for any integer $n \geq 12$, you could buy exactly n nuggets.

Strong induction

Example (a delicious one this time): It used to be the case that you could buy McDonald's chicken nuggets in 4, 6, 9 and 20 piece boxes. Prove that back in the good ol' days, for any integer $n \geq 12$, you could buy exactly n nuggets.

Proof: (by strong induction)

Base cases:

- If $n = 12$, then we can obtain 12 delicious nugs by $3 \cdot \mathbf{4}$
- If $n = 13$, we have $13 = \mathbf{9} + \mathbf{4}$
- If $n = 14$, we have $14 = \mathbf{6} + 2 \cdot \mathbf{4}$
- If $n = 15$, we have $15 = \mathbf{9} + \mathbf{6}$

Strong induction

Example (a delicious one this time): It used to be the case that you could buy McDonald's chicken nuggets in 4, 6, 9 and 20 piece boxes. Prove that back in the good ol' days, for any integer $n \geq 12$, you could buy exactly n nuggets.

Proof: (by strong induction)

Inductive step: S'pose that we can buy exactly ℓ nuggets for $12 \leq \ell \leq k$ (induct. hypoth.)

To show: We can buy exactly k+1 nugs.

$\Rightarrow$ $k+1 = (k-3) + 4$

$\Rightarrow$ By the **inductive hypothesis**, we can buy exactly $k-3 < k+1$ nugs

$\Rightarrow$ So we just add a **4**-piece to that, and we have k+1 nugs!

Strong induction

FYOG (a morbid one this time): Once upon a time, half the inhabitants of an island died all at once, at the hands of *Strong Induction!!*

On an island lived 1,000 very mathematically-inclined natives. 500 of them had blue eyes and 500 of them had brown eyes. The natives followed a super weird religion, the tenets of which were:

1. Nobody is allowed to discuss one another's eye color.
2. Each day, at high noon, everyone must gather at the center of the island to silently gaze at one another in the presence of the Oracle (a statue of their deity)
3. If any person should ever discover that s/he has blue eyes, s/he must immediately commit suicide in the name of the Oracle.

Strong induction

FYOG (a morbid one this time): Once upon a time, half the inhabitants of an island died all at once, at the hands of *Strong Induction!!*

So everything was going fine, until one fateful day at the look-at-each-other meeting, the Oracle suddenly became animated and spoke the horrible, tragic words: *There are blue-eyed natives on this island!*

Exactly 599 days later, at high noon, every blue-eyed native committed suicide.

So. How can giving such a small piece of seemingly obvious information make such a big difference? They all *knew* there were blue-eyed people already! (you know, from their daily look-at-each-other meetings)

Strong induction

FYOG (a morbid one this time): Once upon a time, half the inhabitants of an island died all at once, at the hands of *Strong Induction!!*

So everything was going fine, until one fateful day at the look-at-each-other meeting, the Oracle suddenly became animated and spoke the horrible, tragic words: *There are blue-eyed natives on this island!*

Exactly 599 days later, at high noon, every blue-eyed native committed suicide.

So. How can giving such a small piece of seemingly obvious information make such a big difference? They all *knew* there were blue-eyed people already! (you know, from their daily look-at-each-other meetings)

Answer: Strong Induction.

(historical note: this is why strong induction is so strong)*

*no, it's not

Mathematical induction

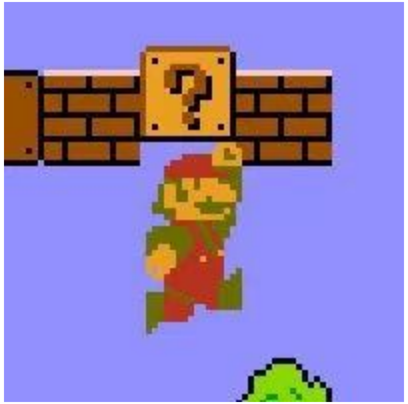
Recap:

- ***Strong* induction** -- the proof technique that's a proof by cases on **for real** steroids
  - **Base case:** show that the hypothesis is true for the first case
  - **Inductive step:** name your inductive hypothesis
    $\Rightarrow$ S'pose true at all stages $j \leq k$, **To Show** true at k+1

Next time:

- We talk **re-re-re-re-recursively**

Bonus material!



FYOG: the morbid-est strong induction

Base case: Let n = # blue-eyed folks = 1. On day 0 (right after the Oracle opened his big, stupid mouth), the one blue-eyed person would look around, see no other blue-eyed folks, realized **they** must be the only blue-eyed person, and kill themself on the spot.

Base case: Let n = 2 (not strictly necessary, but it helps). On day 0, the two blue-eyed people would look around, see each other, and see everyone else has brown eyes. So each of them would conclude that there is either 1 blue-eyed person (each of them thinks the other is the only one), or there are two blue-eyed people (they are, and the other person whose eyes they can see). But then day 1 would roll around, and each of them would see that the other hasn't killed themself yet, so they'll both realize that there must be two blue-eyed people - the person who they each can see has blue eyes, and *themself!* And so they'll both off themselves on the spot.

… When n = 500, on the 499th day, each of the blue-eyed people would look around, see that the other 499 blue-eyed people haven't yet killed themselves, realized that they must also have blue eyes (otherwise the 499 people would have had enough time to figure out they have blue eyes), and all kill themselves on the spot.