Announcements and Reminders

- Homework 4 (written) is available and **due Friday at 12 pm (noon)**

- **Midterm 1: 6:30-8 PM, Tuesday 2 October**

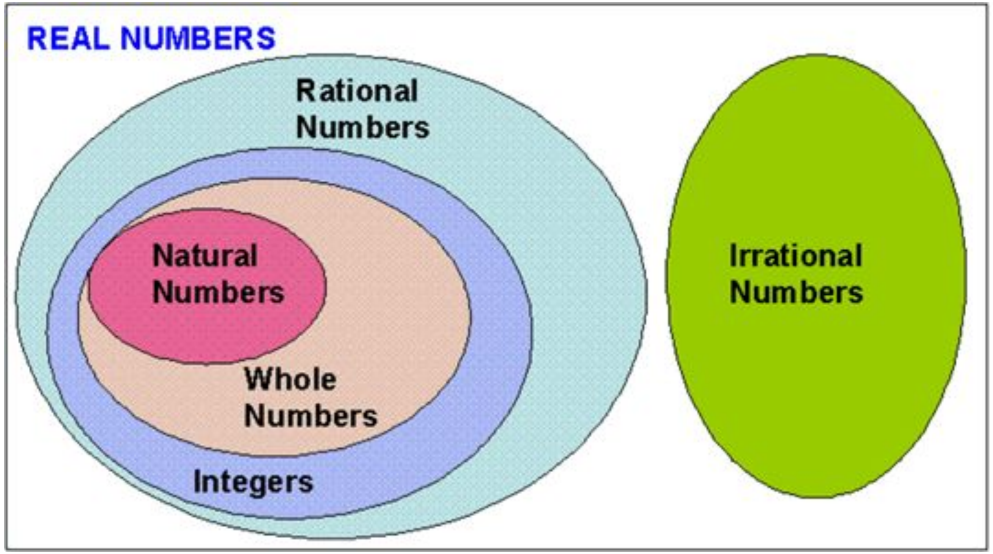
  Rachel (001) in HUMN 1B50

  Tony (002) in DUAN G1B30

  - **Review:**
    - Concept guide
    - Written homework
    - “All Moodle problems” set
    - Workgroup worksheets
    - Lecture slides and examples

Quizlet 05 DUE MONDAY

Lecture 13: More on Sets and Set Operations



What did we do last time?

- We learned what **<u>sets</u>** are,
- some special sets (empty set, singleton set, power set),

Today:

- how to cook up larger sets (unions, power sets) from smaller ones,
- and how to cook up smaller sets (intersections, subsets) from larger ones
- doing stuff with sets (proofs and manipulations)

Cartesian products

$\{\underbrace{\{1,3\}}, \underbrace{\{1,4\}}, \ldots \neq A \times B$

would use sets instead of ordered pairs

- Sometimes we want to talk about things that come from sets, but have a defined order.
- **Example:** The point in the *xy*-plane given by (3,5) is distinctly different from (5,3).

We can generate all ordered pairs from two sets using the *Cartesian product*:

Definition: Let *A* and *B* be sets. The **Cartesian product** of *A* and *B*, written *A* x *B*, is the set of all ordered pairs (*a*, *b*) such that *a* ∈ *A* and *b* ∈ *B*.

$A = \{1, 2\}$

$B = \{3, 4\}$

$A \times B = \{(1,3), (1,4), (2,3), (2,4)\}$

picking 1st elt. from the first set

second elt. from the second set

Note: $A \times B \neq B \times A$

Cartesian products

- Sometimes we want to talk about things that come from sets, but have a defined order.
- **Example:** The point in the *xy*-plane given by (3,5) is distinctly different from (5,3).

We can generate all ordered pairs from two sets using the *Cartesian product*:

Definition: Let A and B be sets. The **<u>Cartesian product</u>** of A and B, written $A \times B$, is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$.

So in quantifier-speak:

$$A \times B = \{(a, b) \mid a \in A \land b \in B\}$$

In plain English: $A \times B$ is the set of all *ordered* pairs you can generate by <u>pulling the first thing in the pairs from A</u> and the <u>second thing from B</u>

Sets and set operations

Example: Consider the sets A = {Jake, David} and B = {1, 2, 3}. What is A x B?

$A \times B = \{$(Jake, 1), (Jake, 2), (Jake, 3),
(David, 1), (David, 2), (David, 3)$\}$

in general, has: $\boxed{|A \times B| = |A| \cdot |B|}$ elts in it

Note: A x B ≠ B x A. Why?

Example: What is the Cartesian product A x B x C, where A = {a, b}, B = {x, y} and C = {m, n}?

$\leftarrow$ ordered triplets!

(d, e, f)

$d \in A$ $\quad e \in B$ $\quad f \in C$

$A \times B \times C = \{$

Sets and set operations

Example: Consider the sets A = {Jake, David} and B = {1, 2, 3}. What is A x B?

Note: $A \times B \neq B \times A$. Why?

Example: What is the Cartesian product $A \times B \times C$,
where $A = \{a, b\}$, $B = \{x, y\}$ and $C = \{m, n\}$?

Answer:

$A \times B \times C = \{ (a, x, m), (a, x, n), (a, y, m), (a, y, n), (b, x, m), (b, x, n), (b, y, m), (b, y, n) \}$

Sets and set operations

Two or more sets can be combined in different ways

Definition: Let A and B both be sets. The **union** of the sets A and B, denoted $A \cup B$, is the set that contains all elements that are either in A, or in B, or in both.

$$A \cup B = \{x \mid x \in A \lor x \in B\}$$

all x in domain   such that $x \in A$ or $x \in B$ (inclusive)

Example: Consider the set C of all computer science majors at CU and the set A of all applied math majors at CU.

- Then the set $C \cup A$ is the set of all CU students who are majoring in either computer science, applied math, or *double-majoring*.

Sets and set operations

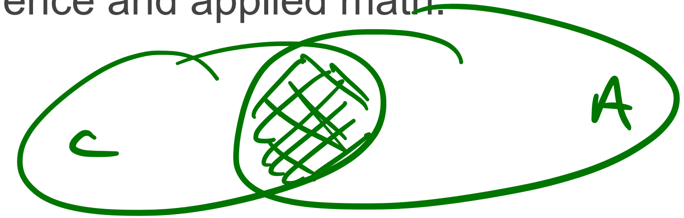
Two or more sets can be combined in different ways

Definition: Let A and B both be sets. The **intersection** of the sets A and B, denoted $A \cap B$, is the set that contains all elements that are both in A and in B.

$$A \cap B = \{x \mid x \in A \land x \in B\}$$

Example: Consider the set C of all computer science majors at CU and the set A of all applied math majors at CU.

- Then the set $C \cap A$ is the set of all CU students who are *double-majoring* in computer science and applied math.

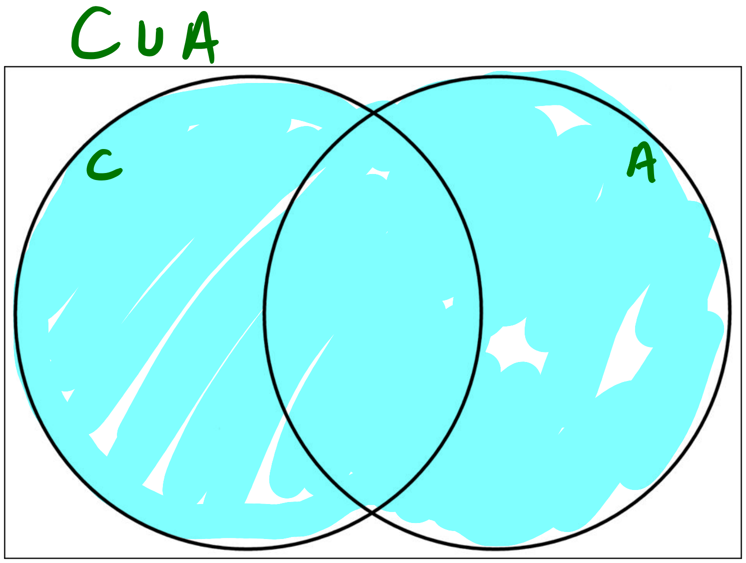


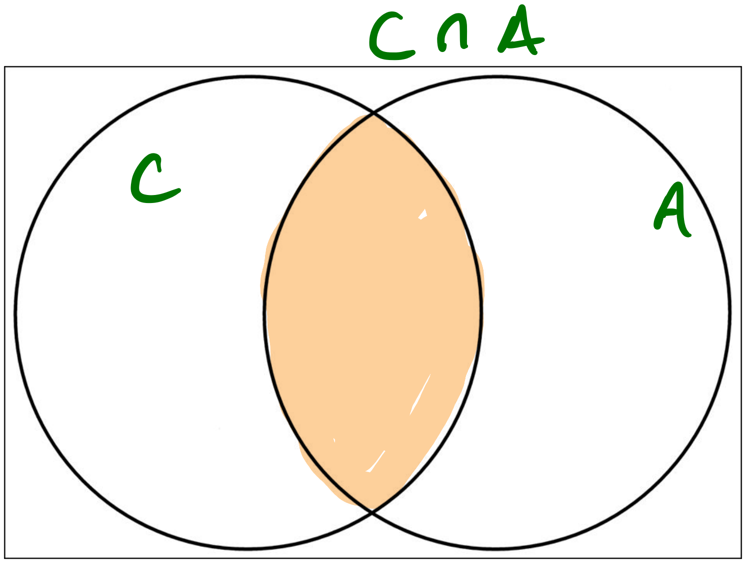
Sets and set operations

Two or more sets can be combined in different ways

Example: Consider the set C of all computer science majors at CU and the set A of all applied math majors at CU.

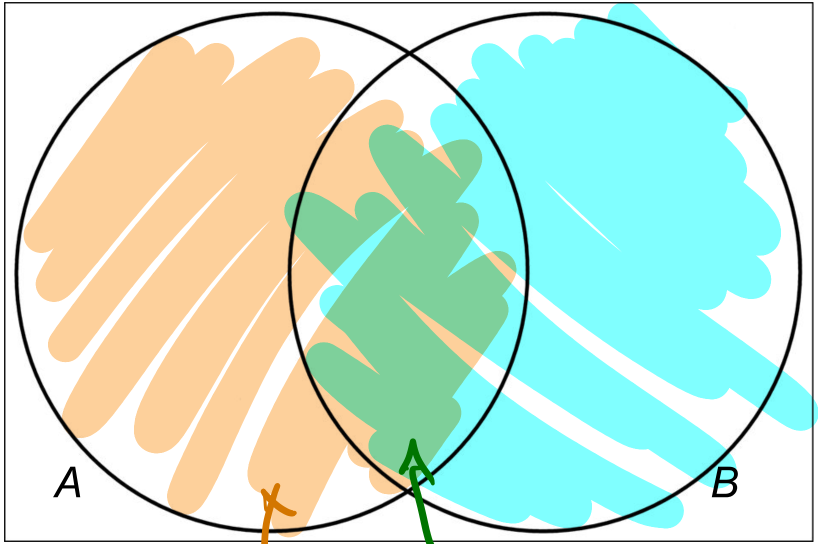
- With two Venn diagrams, what do the sets $C \cup A$ and $C \cap A$ look like?





Sets and set operations

Question: How many elements are in the set $A \cup B$?



$|A \cup B| = |A| + |B| - |A \cap B|$

Sets and set operations

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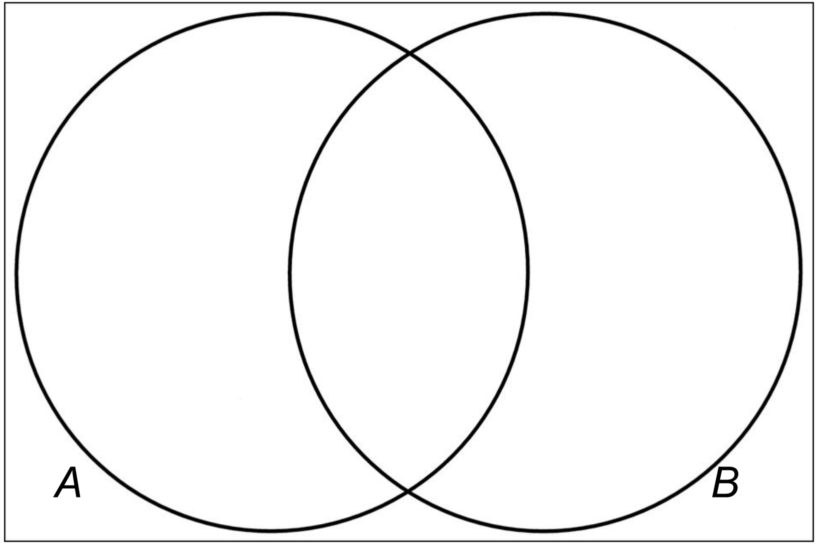


Answer: $|A \cup B| = |A| + |B| - |A \cap B|$

Things that are in both A and B (i.e., in $A \cap B$) are counted twice, by both $|A|$ and $|B|$.

So we need to remove them

Sets and set operations

$A - B$

Definition: Let A and B be sets. The **difference** of A and B, denoted $A - B$ or $A \backslash B$, is the set that contains all elements that are in A but not in B.

The difference of A and B is also called the ***complement*** *of B with respect to A*.

Question: How do we represent $A - B$ in set builder logic?

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$

Question: What is the difference of the set of all $\underbrace{\text{positive integers less than 10}}_{A}$ and the set of all $\underbrace{\text{prime numbers}}_{P}$?

$$A - P = \{1, \underline{2}, \underline{3}, 4, \underline{5}, 6, \underline{7}, 8, 9\} - \{2, 3, 5, 7, 11, \ldots\} = \{1, 4, 6, 8, 9\}$$

Sets and set operations

Definition: Let A and B be sets. The **difference** of A and B, denoted $A - B$ or $A \setminus B$, is the set that contains all elements that are in A but not in B.

The difference of A and B is also called the ***complement*** *of B with respect to A*.

Question: How do we represent $A - B$ in set builder logic?

$$A\text{-}B = \{x \mid x \in A \land x \notin B\}$$

Question: What is the difference of the set of all positive integers less than 10 and the set of all prime numbers?

$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ $\Rightarrow$ $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ $\Rightarrow$ $A\text{-}P = \{1, 4, 6, 8, 9\}$

$P = \{2, 3, 5, 7, 11, \ldots\}$

Warm-Up / Review Problem

Valid or invalid?

$N \to S$

N

$\therefore S$

Modus Ponens } $\boxed{\text{VALID}}$

NOT SOUND

Any integer that is negative is a perfect square. −5 is a negative integer. Therefore, −5 is a perfect square.

F

Announcements and reminders

- Homework 5 posted, **due Monday (8 Oct)**
- **Midterm 1: 6:30-8 PM, Tuesday 2 October**
  - Rachel (001) in HUMN 1B50
  - Tony (002) in DUAN G1B30

- Review (Q+A) Monday in class
- Quizlet 5 due Monday

HERE

Sets and set operations

Python has some nice functionality that can help you convert lists of elements into sets, and perform some operations on them.

```
In [8]: mylist = [1,2,3,1,4]

In [9]: myset = set(mylist)

In [10]: print(myset)
{1, 2, 3, 4}
```

If/when the time comes, you should feel free to explore these functions for manipulating sets...

… he said with a
knowing grin.

```
In [15]: A = set([1,2,3,4])

In [16]: B = set([3,4,5,6])

In [17]: print(set.intersection(A,B))
{3, 4}

In [18]: print(set.union(A,B))
{1, 2, 3, 4, 5, 6}

In [19]: print(set.difference(A,B))
{1, 2}
```

A - B

```
In [20]: print(set.difference(B,A))
{5, 6}
```

Sets and set operations

Definition: The **universal set**, denoted typically by *U*, is the set containing all elements within the domain of discourse.

- You can think about the universal set as the set containing all elements under consideration.

Example: if the domain of discourse is all CU students, then *U* is the set of all CU students.

↳ Example: C = set of all CSCI majors
S = set of all seniors

$C \cup S$ = all CSCI majors and/or seniors $\subseteq U$

$U - C$ = set of everyone at CU that isn't a CSCI major
(student)

Sets and set operations

Definition: The **universal set**, denoted typically by U, is the set containing all elements within the domain of discourse.

- You can think about the universal set as the set containing all elements under consideration.

Example: if the domain of discourse is all CU students, then U is the set of all CU students.

Definition: Let U be the universal set. The **complement** of the set A, denoted $\bar{A}$, is the set U - A. Also see: $\bar{A} = A^c$

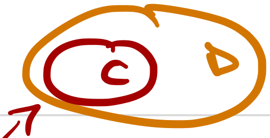
An element belongs to $\bar{A}$ if and only if $x \notin A$, so $\bar{A} = \{x \in U \mid x \notin A\}$, or just $\{x \mid x \notin A\}$

$\bar{C} = U - C$

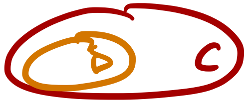
Sets and set operations

$x \geq y$ & $x \leq y$ $\longrightarrow$ $x = y$

$x \geq 3$ & $x \leq 3$ $\longrightarrow$ $x = 3$



Sometimes, you want to show that two complicated sets are equal.



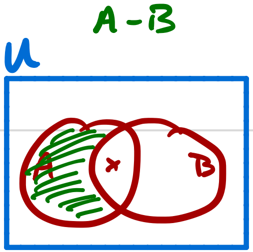
Strategy: To show that sets $C = D$, show that $C \subseteq D$ and $D \subseteq C$.

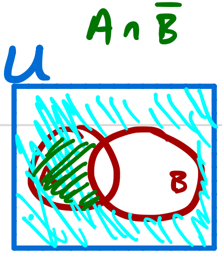
(Why does this work? Think about what must be true if $x \leq y$ and $y \leq x$…)

1. (⇒) To show $C \subseteq D$, assume that $x \in C$ and show that this implies $x \in D$.
2. (⇐) To show $D \subseteq C$, assume that $x \in D$ and show that this implies $x \in C$.

Sets and set operations

Example: Prove that $A - B = A \cap \bar{B}$





Strategy: To prove that two sets C and D are equal, we must

1. (⇒) Prove that $C \subseteq D$
2. (⇐) Prove that $D \subseteq C$

Proof: (⇒) Need to show that $A - B \subseteq A \cap \bar{B}$

S'pose x is an arbitrary elt. of $A - B$

→ $x \in A \wedge x \notin B$ (def. of set diff.)

→ $x \in A \wedge x \in \bar{B}$ (def. of complement)

→ $x \in A \cap \bar{B}$ (def. of intersection)

→ if $x \in A - B$, then $x \in A \cap \bar{B}$

→ $A - B \subseteq A \cap \bar{B}$ ✓

Sets and set operations

Example: Prove that $A - B = A \cap \bar{B}$

Strategy: To prove that two sets *C* and *D* are equal, we must

1. (⇒) Prove that $C \subseteq D$
2. (⇐) Prove that $D \subseteq C$

Proof: (⇒)

1. S'pose *x* is an arbitrary element in *A - B*
2. $\Rightarrow x \in A \ \wedge \ x \notin B$
3. $x \notin B \ \Rightarrow \ x \in \bar{B}$
4. $\Rightarrow \ (x \in A) \wedge (x \in \bar{B})$
5. $\Rightarrow \ x \in A \cap \bar{B}$
6. Since *x* was any arbitrary element in *A - B*, we have shown $A - B \subseteq A \cap \bar{B}$

Sets and set operations

Example: Prove that $A - B = A \cap \bar{B}$

Strategy: To prove that two sets *C* and *D* are equal, we must

1. (⇒) Prove that $C \subseteq D$
2. (⇐) Prove that $D \subseteq C$

Proof: (⇐) Need to show: $\underline{A \cap \bar{B} \subseteq A - B}$

S'pose x is an arbitrary elt. of $A \cap \bar{B}$

→ $x \in A \wedge x \in \bar{B}$ (def. of intersection)

→ $x \in A \wedge x \notin B$ (def. of complement)

→ $x \in A - B$ (def. of set diff.)

→ $\underline{A \cap \bar{B} \subseteq A - B}$ & since $A - B \subseteq A \cap \bar{B}$ too, we know they must be equal ☒

Sets and set operations

Example: Prove that $A - B = A \cap \bar{B}$

Strategy: To prove that two sets *C* and *D* are equal, we must

1. ($\Rightarrow$) Prove that $C \subseteq D$
2. ($\Leftarrow$) Prove that $D \subseteq C$

Proof: ($\Leftarrow$)

1. S'pose *x* is an arbitrary element in $A \cap \bar{B}$
2. $\Rightarrow$ $(x \in A) \wedge (x \in \bar{B})$
3. $x \in \bar{B}$ $\Rightarrow$ $x \notin B$
4. $(x \in A) \wedge (x \notin B) \Rightarrow (x \in A - B)$
5. Since *x* was any arbitrary element in $A \cap \bar{B}$, we have shown $A \cap \bar{B} \subseteq A - B$

Since $(A \cap \bar{B} \subseteq A - B)$ and $(A - B \subseteq A \cap \bar{B})$

it must be the case that $A - B = A \cap \bar{B}$ $\square$

Sets and set operations

We have a bunch of nice set identities similar to our logical equivalences for propositional functions.

Let's look closer at one of the familiar-looking ones!

TABLE 1 Set Identities.

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

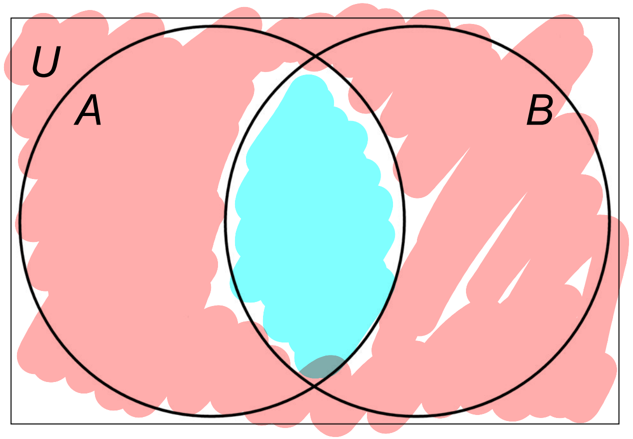
Sets and set operations

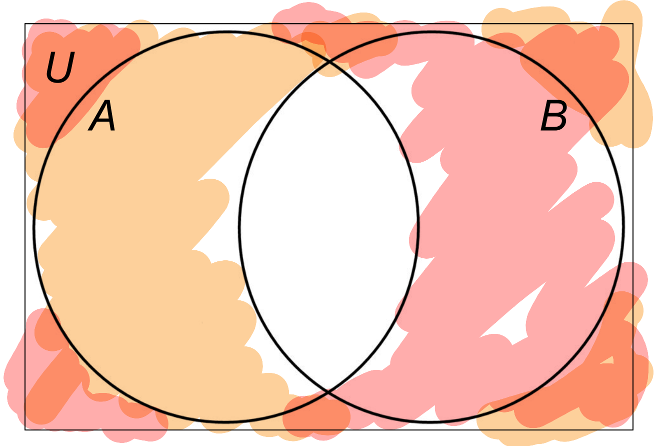
$\overline{(A \cap B)}$ "=" $\bar{A} \,\bar{\cap}\, \bar{B}$ "=" $\bar{A} \cup \bar{B}$

Symmetry between sets and propositional logic

- Turns out, there is nice symmetric between ***sets***, with ⋃, ⋂ and ‾‾ ,
  and ***propositional logic***, with ⋁, ⋀ and ¬

Example: one of DeMorgan’s laws: $\underline{\overline{A \cap B}} = \underline{\bar{A} \cup \bar{B}}$





Sets and set operations

$A = \{x \mid x \in A\}$

$\overline{(A \cap B)}$ $\bar{A} \cup \bar{B}$

Example: Prove (one of) De Morgan's laws for sets, using **builder notation** $\overline{A \cap B} = \bar{A} \cup \bar{B}$ (remember, builder notation looked a lot like quantifiers, so this will look at lot like our old proofs from Ch 1)

$\overline{A \cap B}$

$= \{x \mid x \in \overline{A \cap B}\}$ def. of a set

$= \{x \mid x \notin A \cap B\}$ def. of complement

$= \{x \mid \neg(x \in A \cap B)\}$ def. of not-in a set

$= \{x \mid \neg(x \in A \wedge x \in B)\}$ def. of intersection

$= \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$ De Morgan's

$= \{x \mid x \notin A \vee x \notin B\}$ def. of not-in

$= \{x \mid x \in \bar{A} \vee x \in \bar{B}\}$ def. of complement

$= \{x \mid x \in \bar{A} \cup \bar{B}\}$ def. of union

$= \bar{A} \cup \bar{B}$ $\blacksquare$

To prove 2 sets $=$:

① show they are subsets of one another

② do something like this

③ set identities

Sets and set operations

Example: Prove (one of) De Morgan's laws for sets, using **builder notation** $\overline{A \cap B} = \bar{A} \cup \bar{B}$ (remember, builder notation looked a lot like quantifiers, so this will look at lot like our old proofs from Ch 1)

Proof: $\overline{A \cap B}$ $= \{ x \mid x \notin A \cap B \}$ — Definition of complement

$= \{ x \mid \neg(x \in A \cap B) \}$ — Definition of not-in

$= \{ x \mid \neg(x \in A \land x \in B) \}$ — Definition of intersection

$= \{ x \mid \neg(x \in A) \lor \neg(x \in B) \}$ — De Morgan's

$= \{ x \mid (x \notin A) \lor (x \notin B) \}$ — Definition of not-in

$= \{ x \mid (x \in \bar{A}) \lor (x \in \bar{B}) \}$ — Definition of complement

$= \{ x \mid x \in \bar{A} \cup \bar{B} \}$ — Definition of union

$= \bar{A} \cup \bar{B}$

$\square$

Sets and set operations

$\neg(A \vee (B \wedge C))$
$= \neg A \wedge \neg(B \wedge C)$
$= \neg A \wedge (\neg B \vee \neg C)$

Example: Use set identities to prove

$$\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$$

$\overline{A \cup (B \cap C)} = \overline{A} \cap \overline{(B \cap C)}$ De Morgan's

$= \overline{A} \cap (\overline{B} \cup \overline{C})$ De Morgan's

$= (\overline{B} \cup \overline{C}) \cap \overline{A}$ commutative

$= (\overline{C} \cup \overline{B}) \cap \overline{A}$ commutative

$= (\overline{C} \cap \overline{A}) \cup (\overline{B} \cap \overline{A})$ distributive

?

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$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
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Sets and set operations

Example: Use set identities to prove

$$\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$$

Proof:

$\overline{A \cup (B \cap C)} = \overline{A} \cap \overline{(B \cap C)}$	De Morgan's
$= \overline{A} \cap (\overline{B} \cup \overline{C})$	De Morgan's
$= (\overline{B} \cup \overline{C}) \cap \overline{A}$	commutative
$= (\overline{C} \cup \overline{B}) \cap \overline{A}$	commutative

FYOG: Use set identities to prove

$$(A \cup \overline{B}) \cap \overline{(B \cap A)} = \overline{B}$$

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Sets and set operations

Recap:

- We learned what <u>**sets**</u> are,
- some special sets (empty set, singleton set, power set),
- how to cook up larger sets (unions, power sets) from smaller ones,
- and how to cook up smaller sets (intersections, subsets) from larger ones

Next time:

- More on ***infinite*** sets and …
- doing stuff with sets of things
  to get other sets of things
  (we call the stuff we do ***functions***!)

Bonus material!