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1 0 1 1 0 0 0 0 1 0 0 0

CSCI 2824: Discrete Structures

Lecture 15: Functions and Cardinality

* Homework 5 (on Moodle)

Due Monday Oct. 8th

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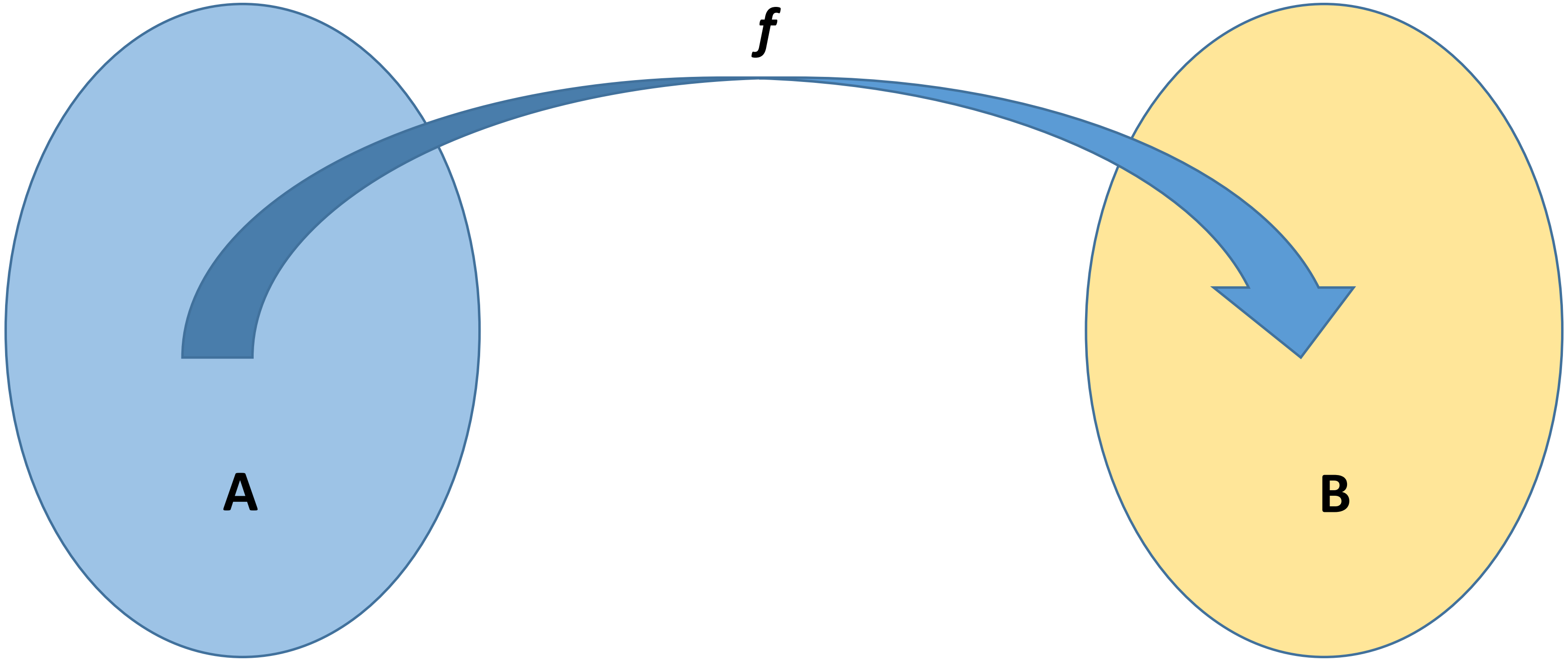
Functions

Let A and B be nonempty sets. A **function** f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f: A \to B$.



Functions

Functions are everywhere in computer science and engineering.

```
In [7]: def Square(x_in):
   ...:
   ...:     x2_out = x_in * x_in
   ...:
   ...:     return (x2_out)
   ...:
```

A function is a routine that takes some kind of input, *does stuff*, and yields some kind of output, as well as possibly some side effects.

Computer science distinction:	**function:**	takes inputs, produces outputs
vs.	**procedure:**	takes inputs, produces side effects (but no outputs)

We will use this narrower definition of a function (inputs ⇒ outputs)

Functions

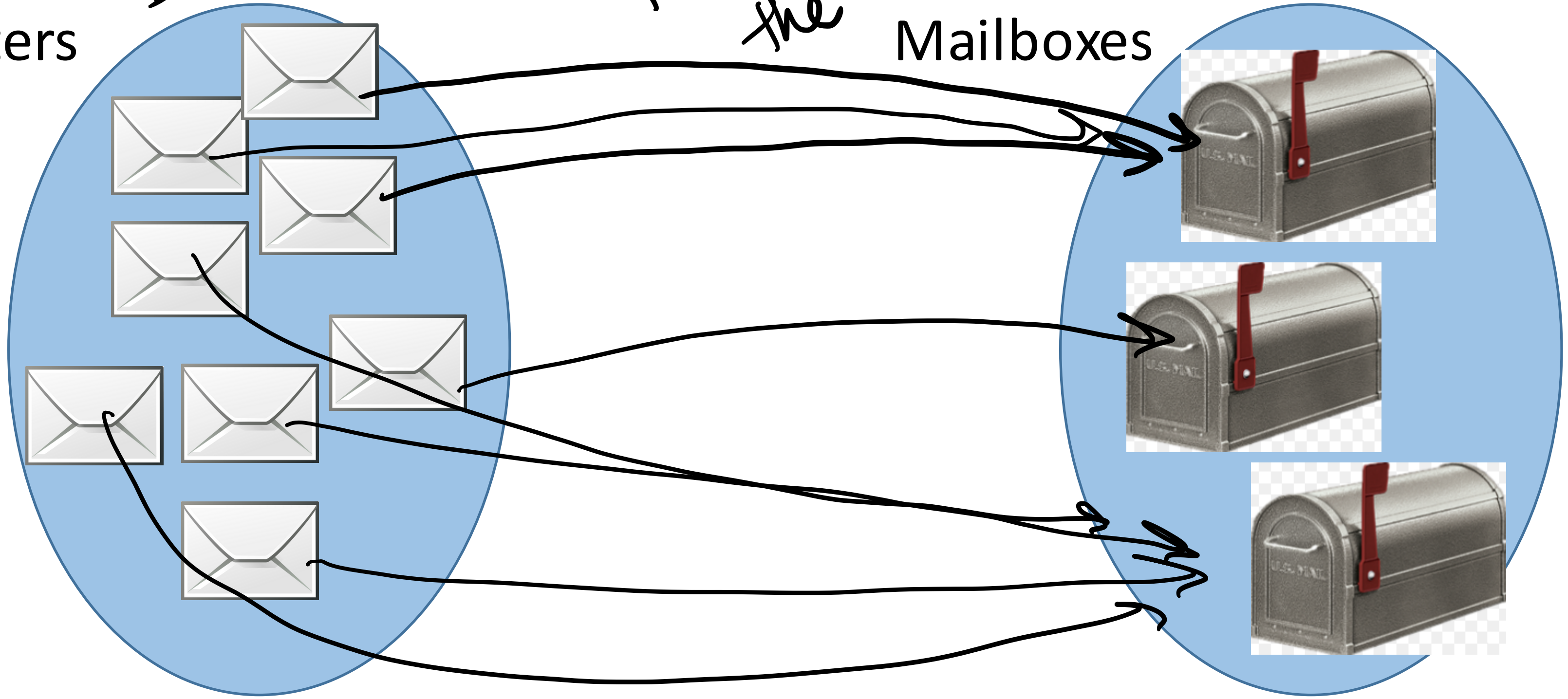
Examples of Functions:

- Let $\boxed{f : \mathbb{Z} \to \mathbb{Z}}$ assign the cube of an integer to this integer. $\}$ $f(x) = x^3$

Domain

function is the delivery of the mail

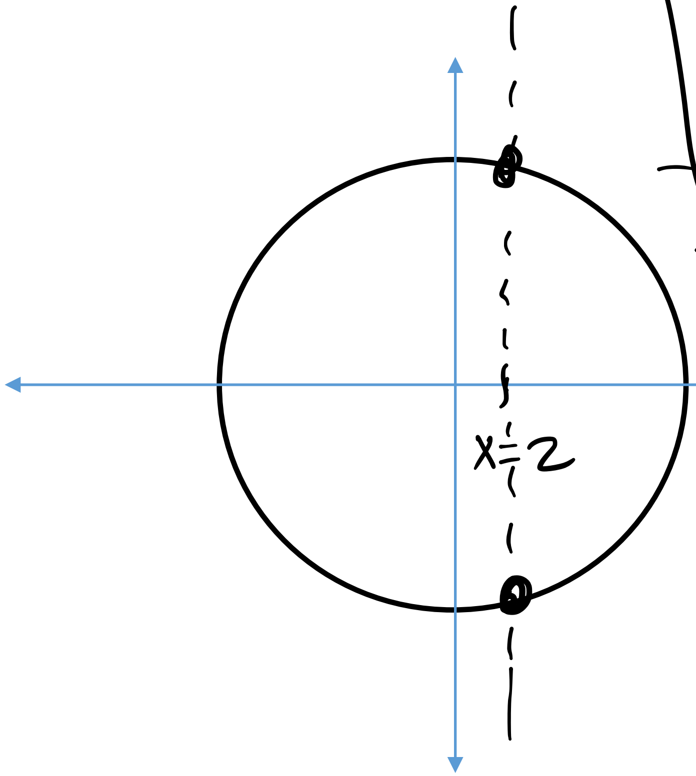
- Letters



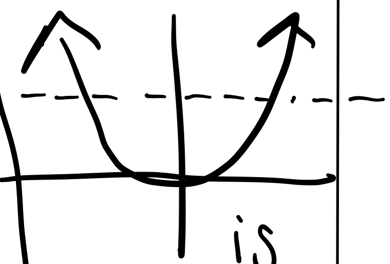
Functions

Examples of **NOT** Functions

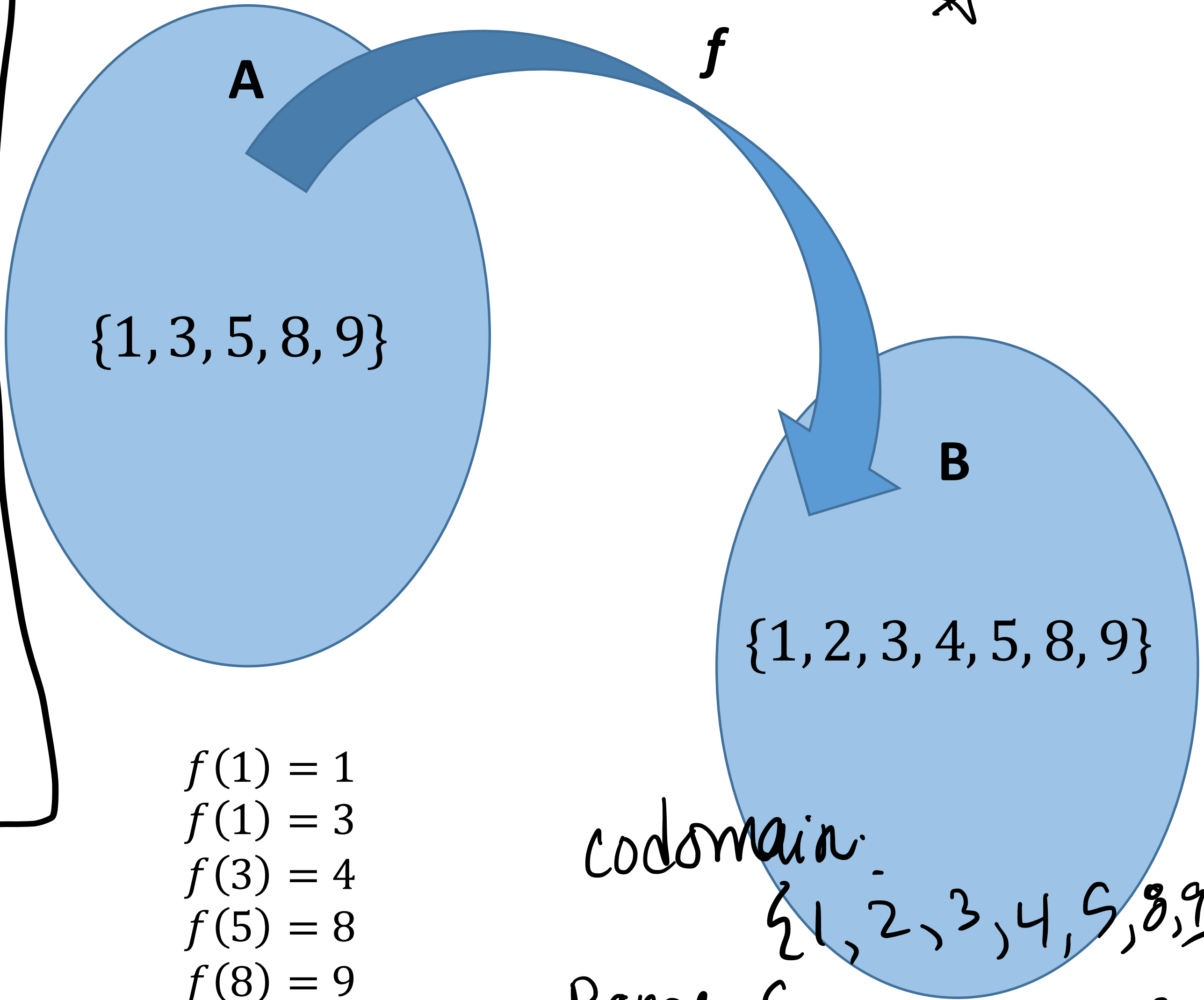
$x^2 + y^2 = 16$



Contrast with $f(x)=x^2$



is a function



$f(1) = 1$
$f(1) = 3$
$f(3) = 4$
$f(5) = 8$
$f(8) = 9$
$f(9) = 9$

codomain: $\{1,2,3,4,5,8,9\}$

Range: $\{1,3,4,8,9\}$

Functions

If f is a function from A to B, we say that A is the **domain** of f and B is the **codomain** of f. If $f(a) = b$, we say that b is the **image** of a and a is the **preimage** of b. The **range**, or image, of f is the set of all images of elements of A. Also, if f is a function from A to B, we say f maps A to B.

Example: Let $f: \mathbb{Z}^+ \to \mathbb{Z}^+$ assign the cube of an integer to this integer. What is the domain, the codomain, and the range of this function?

domain: $\mathbb{Z}^+$

codomain: $\mathbb{Z}^+$

Range: 1, 8, 27, ... perfect cubes!

Range $f \subseteq$ Codomain f

domain: Set that f maps things from

codomain: set that f could map to

Range: set that f actually maps to

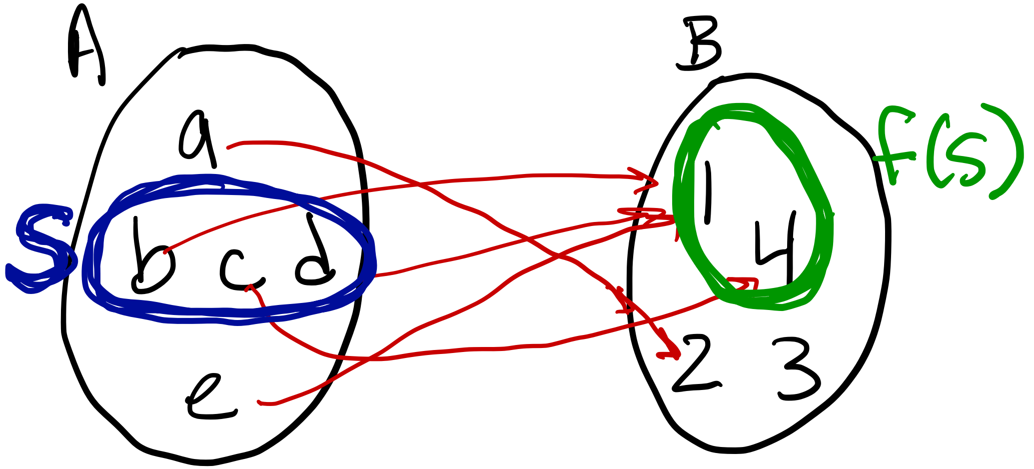
Functions

Let f be a function from A to B and let S be a subset of A. The image of S under the function f is the subset of B that consists of the images of the elements of S. We denote the image of S by $f(S)$, so

$$f(S) = \{t \mid \exists s \in S \ (t = f(s))\}$$

Example: Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$ with $f(a) = 2, f(b) = 1, f(c) = 4, f(d) = 1$, and $f(e) = 1$. The image of the subset $S = \{b, c, d\}$ is the set $f(S) = \{1, 4\}$.

$f(a) = 2$
$f(b) = 1$
$f(c) = 4$
$f(d) = 1$
$f(e) = 1$



Functions: One–to–One, Onto

A function f is said to be **one-to-one**, or an injunction, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be ***injective*** if it is one-to-one.

A function f from A to B is said to be **onto**, or a surjection, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function is said to be ***surjective*** if it is onto.

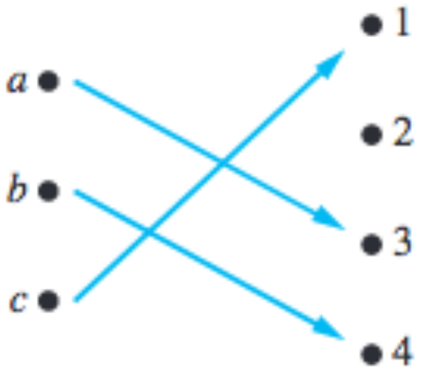
The function f is a one-to-one correspondence, or a **bijection**, if it is both one-to-one and onto.

Functions: One–to–One, Onto

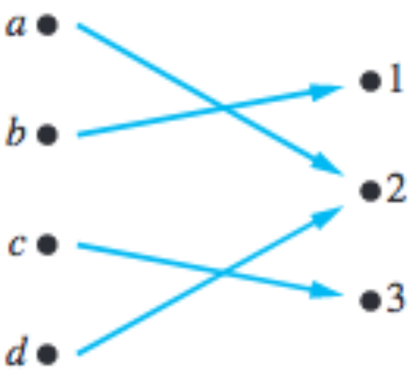
Example: Classify each of these functions as one-to-one, onto, both, or neither.

Domain

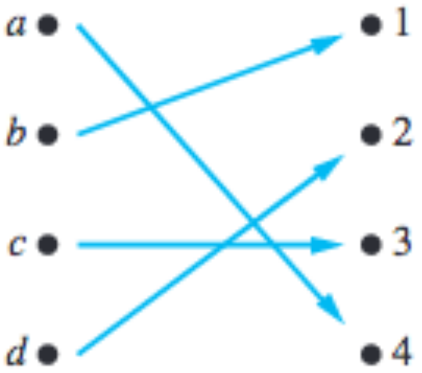
Codomain



- one-to-one (1–1)
- not onto



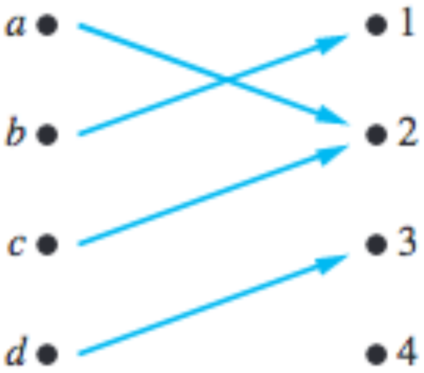
- not 1-1
- onto



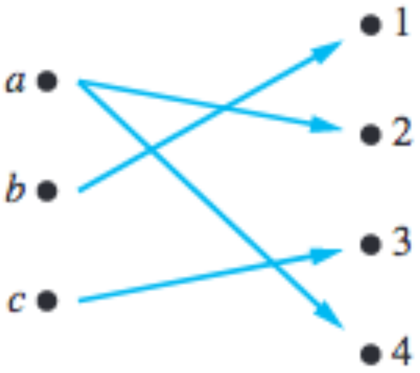
- 1-1

<u>AND</u>

- onto



- not 1-1
- not onto

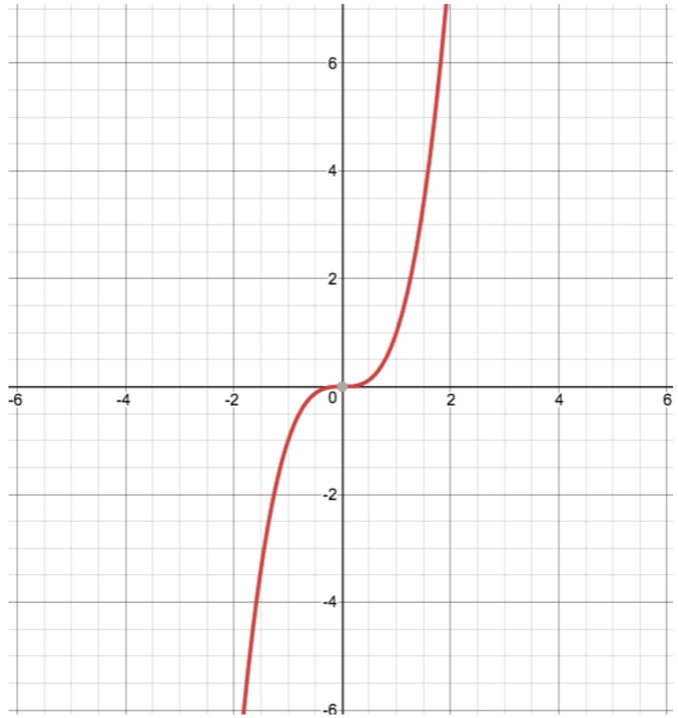


Not a function

Functions

A function f whose domain and codomain are subsets of the set of real numbers is called ***increasing*** if $f(x) \leq f(y)$, and ***strictly increasing*** if $f(x) < f(y)$, whenever $x < y$ and x and y are in the domain of f. Similarly, f is called ***decreasing*** if $f(x) \geq f(y)$, and ***strictly decreasing*** if $f(x) > f(y)$, whenever $x < y$ and x and y are in the domain of f.

- A function that is either strictly increasing or strictly decreasing must be one-to-one.



Functions

domain codomain

Example: Prove that $f(n) = n^3$ is one-to-one. (implicit: $f\colon \mathbb{R} \to \mathbb{R}$)

If $f(a) = f(b)$, then $a = b$

Pf: Assume $f(n) = f(m)$ for real numbers n, m.

$\Rightarrow n^3 = m^3$

$\Rightarrow n = m$

$\Rightarrow f$ must be 1-1

$\blacksquare$

Alternate way:

$n^3 - m^3 = 0$

$(n-m)\underline{(n^2 + nm + m^2)} = 0$

$\Rightarrow n - m = 0$

$\Rightarrow n = m$

$\ln n^3 = \ln m^3$

$3 \ln n = 3 \ln m$

$\ln n = \ln m$

Functions

Example: Prove that $f(n) = n^2$ is NOT one-to-one. (implicit: f: $\mathbb{R} \to (\mathbb{R} \geq 0)$)

Disprove with a counterexample.

Consider $n = -2$ and $m = 2$

$$f(n) = (-2)^2 = 4$$

$$f(m) = 2^2 = 4$$

So $f(n) = f(m)$ but $n \neq m$

$\Rightarrow$ Not 1-1 $\blacksquare$

Functions

Example: Prove that $f(n) = n^2$ is NOT onto. (where: $f\colon \mathbb{Z} \to \mathbb{N}$)

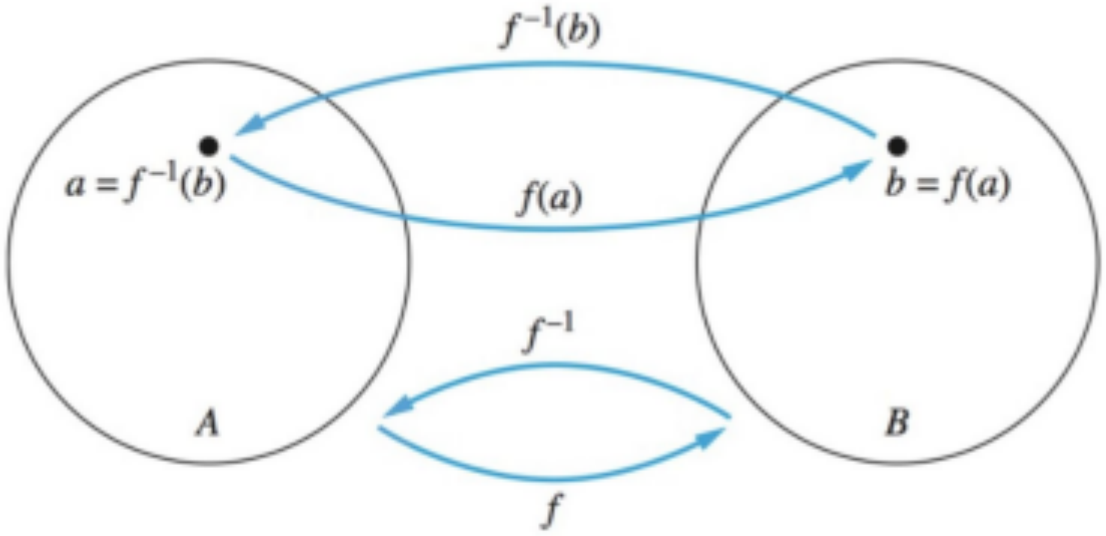
Consider the codomain element 3.

There is no such $z \in \mathbb{Z}$ such that

$f(z) = 3$.

Functions – Inverse Functions

Let f be a one-to-one and onto function from A to B. Then there exists an inverse function, f^{-1}, such that $f^{-1}(b) = a$ when $f(a) = b$.



If $f: A \rightarrow B$ is 1-1 and onto, then:

- f maps to each of the elements of B (because f is onto)
- But f is 1-1 as well, so each element of B has a unique element in A that maps to it.

There is a unique one-to-one correspondence between elements in A and elements in B. When this happens, we can go back and forth between A and B via f and f^{-1}.

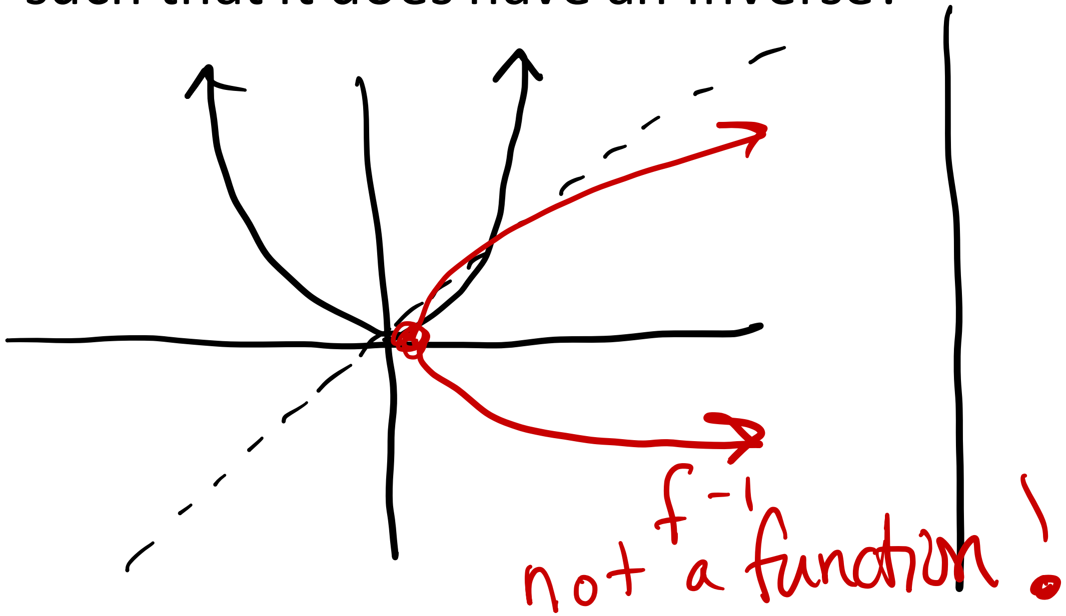
Functions – Inverse Functions

Example: The inverse of $f(x) = x^3$, where $f\colon \mathbb{R} \to \mathbb{R}$ is $f^{-1}(y) = y^{1/3}$.

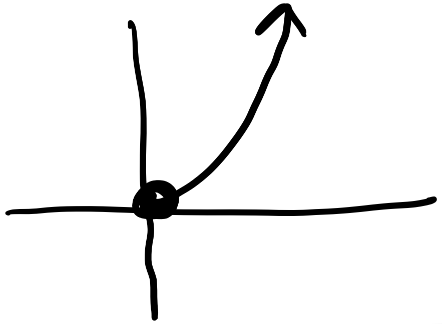
f: cubes numbers

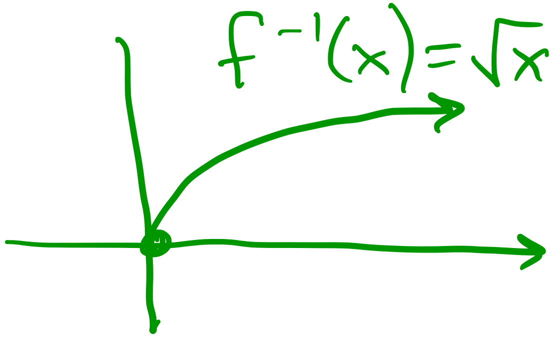
f^{-1}: cube roots numbers.

Example: The $f(x) = x^2$, where $f\colon \mathbb{R} \to \mathbb{R}$ does not have an inverse. Could we redefine f such that it does have an inverse?



$f(x) = x^2$ $\quad f\colon (\mathbb{R}_{\geq 0} \to \mathbb{R})$

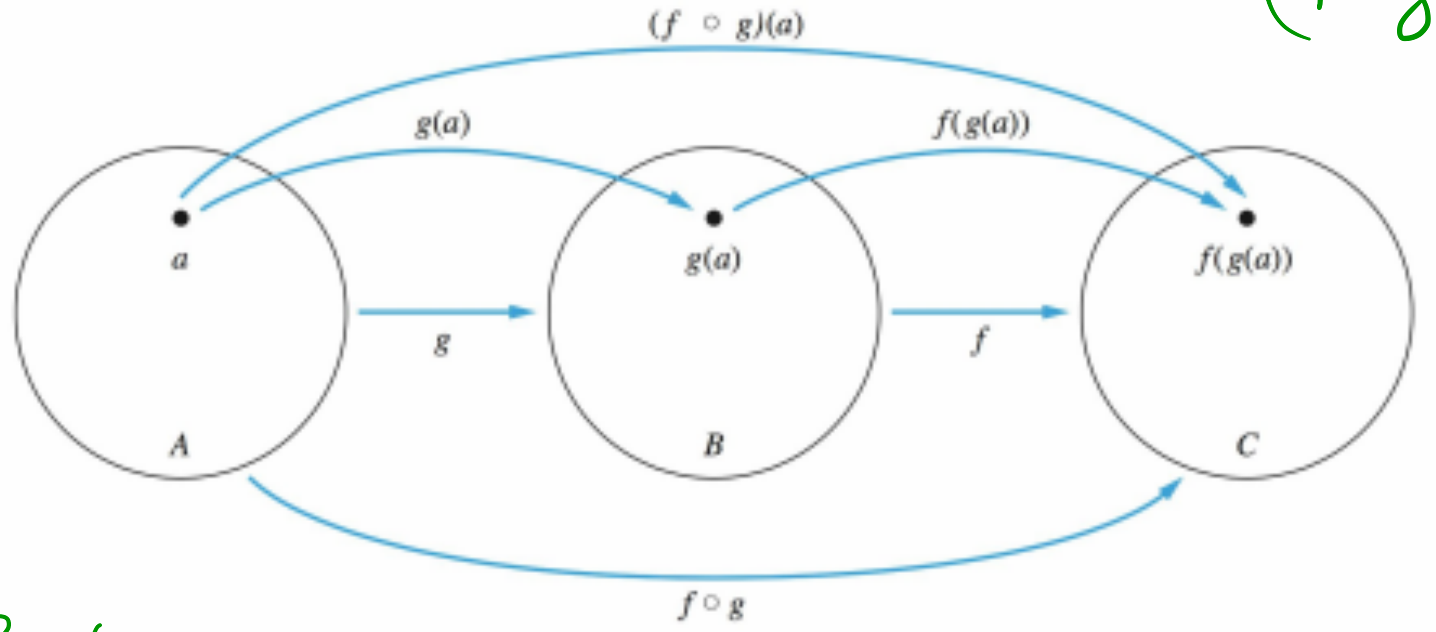




Functions – Composition of Functions

Let g be a function from set A to set B, and let f be a function from set B to set C. The composition of f and g, denoted $f \circ g$, is defined for $a \in A$ by $(f \circ g)(a) = f\big(g(a)\big)$.

$(f \circ g)(x) \neq (g \circ f)(x)$



$f(x) = \sqrt{x}$

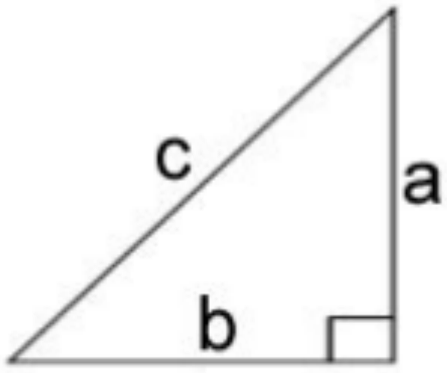
$g(x) = x^2 - 1$

$(f \circ g)(x) = f(g(x)) = \sqrt{x^2 - 1}$

Functions – Composition of Functions

Example: $c = \sqrt{a^2 + b^2}$

```
In [14]: # a function for adding
    ...: def Add(x_in, y_in):
    ...:
    ...:     sum_out = x_in + y_in
    ...:
    ...:     return (sum_out)
    ...:
    ...: # a function for squaring
    ...: def Square(x_in):
    ...:
    ...:     x2_out = x_in * x_in
    ...:
    ...:     return (x2_out)
    ...: c = pow( Add( Square(3), Square(4) ) , 0.5)
    ...: print(c)
    ...:
5.0
```

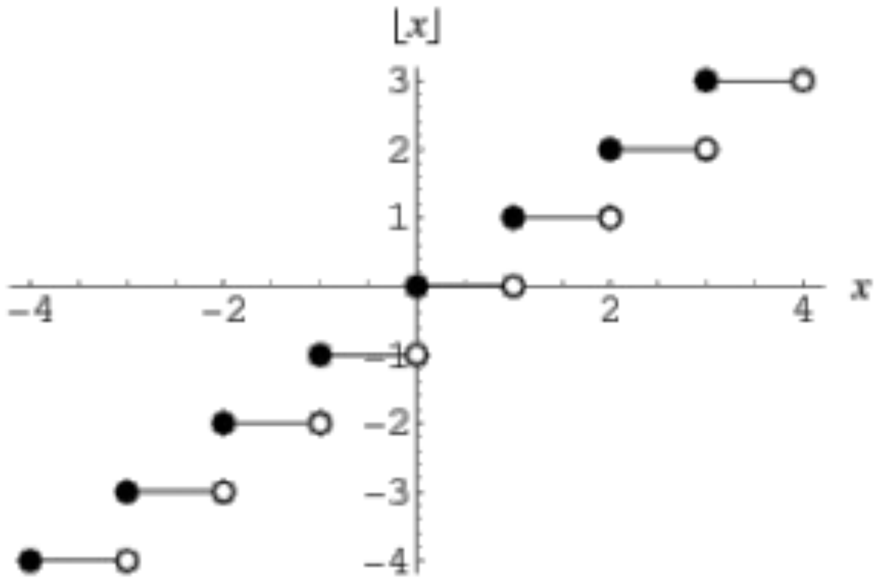


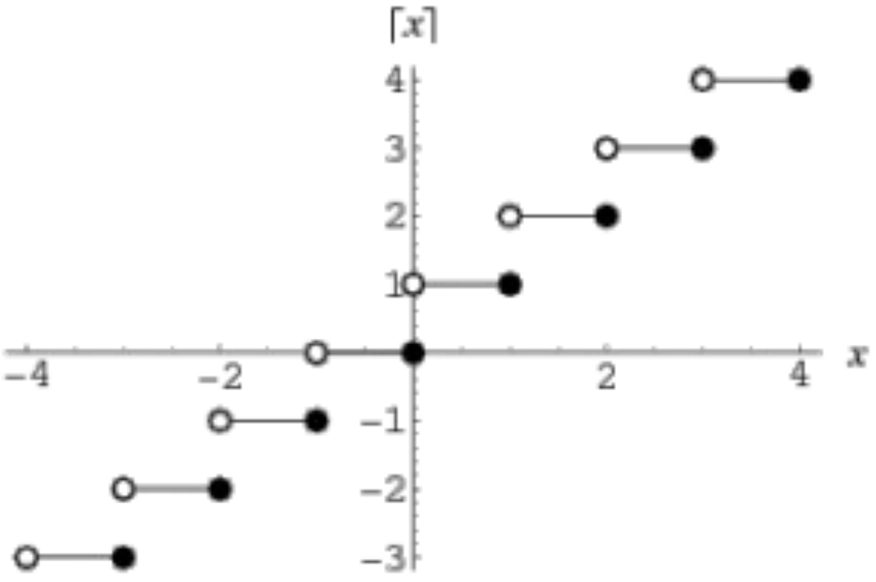
$a^2 + b^2 = c^2$

Functions

Definition: The **floor** function, denoted $\lfloor x \rfloor$, assigns to the real number x the largest integer that is less than or equal to x. The **ceiling** function, denoted $\lceil x \rceil$, assigns to the real number x the smallest integer that is greater than or equal to x.

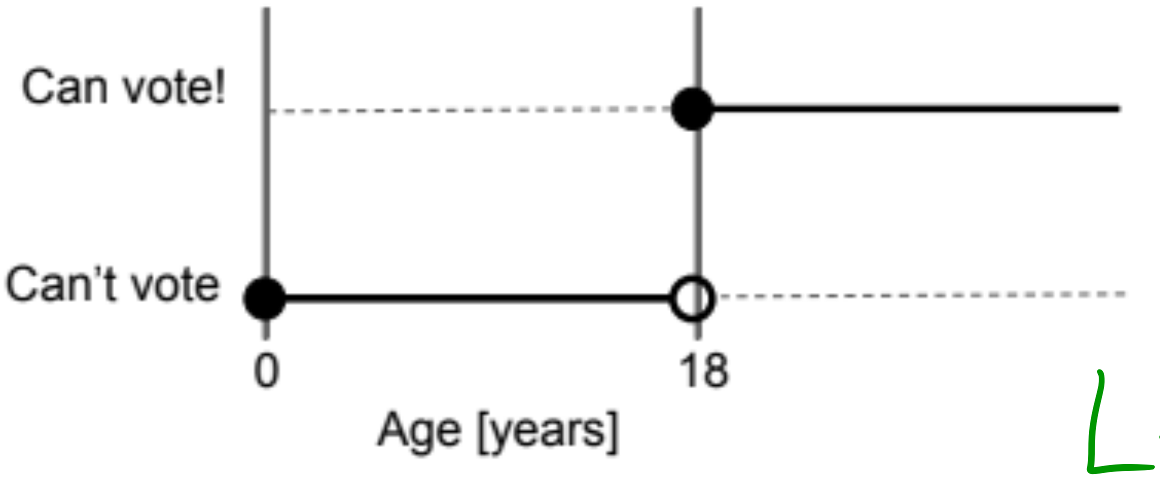
Remark: Both of these come up frequently in CS applications.





Functions

Example: Being able to vote is an example of a floor function.



$\lfloor \text{floor} \rfloor$

More examples:

$\lfloor 3.5 \rfloor = 3$	$\lfloor 5 \rfloor = 5$	$\lfloor -3.5 \rfloor = -4$
$\lceil 3.5 \rceil = 4$	$\lceil 5 \rceil = 5$	$\lceil -3.5 \rceil = -3$

$\lceil \text{ceiling} \rceil$

Functions

Example: Prove or disprove that $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$.

$x = 4.01$

$y = 5.01$

Disproven with a counterexample.

$\lceil x + y \rceil = \lceil 4.01 + 5.01 \rceil = 10$

$\lceil x \rceil + \lceil y \rceil = \lceil 4.01 \rceil + \lceil 5.01 \rceil = 5 + 6 = 11$

$10 \neq 11$

$x = -3.01$

$y = -5.02$

Cardinality of Sets

A set A is called **countable** or **countably infinite** if it is not finite and there is a one-to-one function between each element of A and the natural numbers. A is called **uncountable** if it is infinite and not countable.

- ***Countably infinite***: we could count up each member of the set if we had infinite time.

- ***Uncountably infinite***: we could never count or list each element of the set, even if we had infinite time.

$\mathbb{N} = \{0, 1, 2, \ldots\}$

Cardinality of Sets

Example: Is the set of positive even integers countably infinite? or uncountably infinite?

$$\begin{array}{cccccc} 2, & 4, & 6, & 8, & 10, & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 0, & 1, & 2, & 3, & 4, & \ldots \end{array}$$

positive even integers

natural numbers

$$\boxed{f(n) = 2(n+1)}$$

Cardinality of Sets

Example: Is the set of all integers countably infinite? or uncountably infinite?

$\mathbb{Z}$: 0 , 1 , −1 , 2 , −2 , 3 , −3 , 4 , −4 , ...

$\mathbb{N}$: 0 , 1 , 2 , 3 , 4 , 5 , 6 , . . .

$\mathbb{N}$	$\mathbb{Z}$
0	0
1	1
2	−1
3	2

$\mathbb{N}$	$\mathbb{Z}$
4	−2
5	3
6	−3

$$f(n) = \begin{cases} -\frac{n}{2} & n \text{ even} \\ \frac{n+1}{2} & n \text{ odd} \end{cases}$$

Cardinality of Sets

Rational number : $\frac{p}{q}$ ← integers, $q \neq 0$

Example: Is the set of positive rational numbers countably infinite? or uncountably infinite?

$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$	$\frac{6}{1}$	$\cdots$	} countable
$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\frac{5}{2}$	$\frac{6}{2}$	$\cdots$	} countable
$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{6}{3}$	$\cdots$	
$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$	$\frac{6}{4}$	$\cdots$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		

The set of Rational numbers is countable.

Cardinality of Sets

Repeating decimal as a fraction → calc 2 sum of a geometric series

Example: Is the set of real numbers countably infinite? or uncountably infinite?

look at the interval [0,1]

first 0.2 5 7 8 9 1 3 2 5 4 . . . .

second 0. 3 4 1 2 3 4 7 7 8 9 . . . .

third 0. 4 7 8 9 2 1 0 0 2 3 . . . .

fourth 0. 5 6 7 8 9 1 2 3 4 5 . . .

fifth 0. 1 1 1 1 1 1 1 1 1 1 . . .

0. 1 2 1 2 1 3 4 5 6 7 . . .

⋮

construct m 0. 1 1 1 1 0 1

$$[0.00001, 0.00002]$$

- ❖ We just learned about functions: one-to-one, onto, injective, surjective, bijective functions, inverse functions, composition of functions

- ❖ We talked about different sizes of infinity and the cardinality of sets.

Next: Sequences!