## Warm-up problem

**Example:** Prove the biconditional by proving both directions:

An integer $n$ is even if and only if $n$ - 1 is odd. $\equiv$

$\Rightarrow$/ $n$ even $\rightarrow$ $n-1$ odd

$\Leftarrow$/ $n-1$ odd $\rightarrow$ $n$ even

Proof:

$\Rightarrow$/ Direct proof: If $n$ even, then $n-1$ odd

S'pose $n$ is even

$\rightarrow \exists$ some integer $k$ s.t. $n = 2k$

$\rightarrow n-1 = 2k-1 = 2(k-1)+1$, which is odd ✓

$\Leftarrow$/ If $n-1$ odd, then $n$ is even

Direct proof

S'pose $n-1$ odd

$\rightarrow \exists$ some integer $l$ s.t. $n-1 = 2l+1$

$\rightarrow n = 2l+1+1 = 2l+2 = 2(l+1)$, which is even ✓ $\blacksquare$

# Warm-up problem

**Example:** Prove the biconditional by proving both directions:

An integer *n* is even if and only if *n* - 1 is odd.

**Proof:**

( ⇒, direct ) S'pose *n* is an even integer.

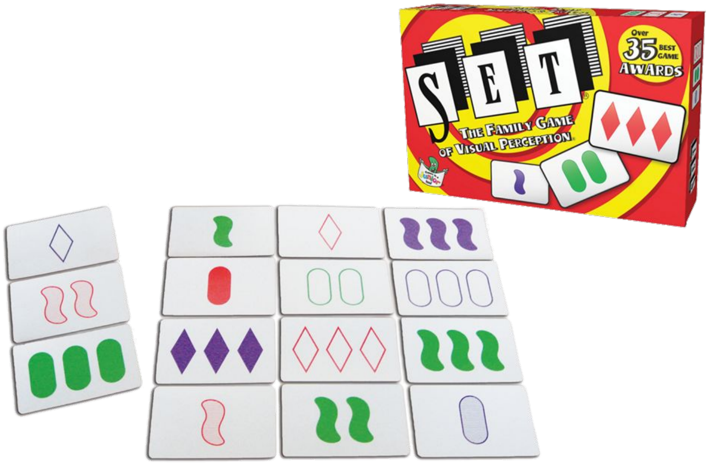
1. Then $n = 2a$, where *a* is some integer
2. Then $n - 1 = 2a - 1 = 2(a - 1) + 1$, which is odd
3. Thus, if *n* is even, then *n* - 1 is odd. ✔

( ⇐, direct ) S'pose *n* - 1 is an odd integer.

1. Then $n - 1 = 2a + 1$, where *a* is some integer
2. Then $n = 2a + 2 = 2(a + 1)$, which is even
3. Thus, if *n* - 1 is odd, then *n* is even. ✔

□

# Lecture 12: Sets and Set Operations

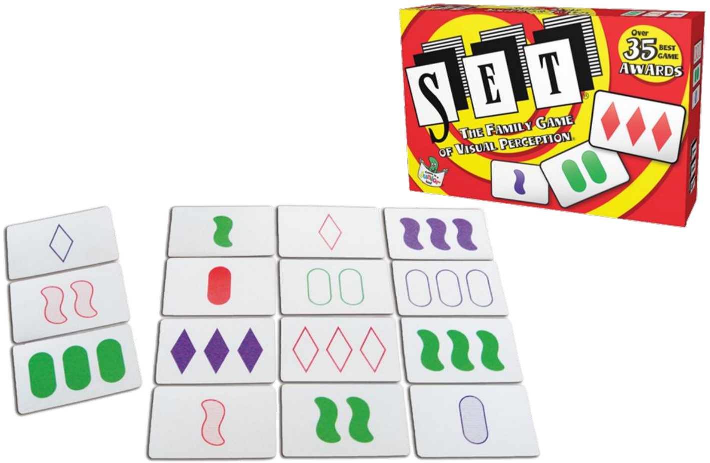



## Announcements and reminders

- Homework 4 posted, **due Friday at 12 pm (noon)**
- **Midterm 1: 6:30-8 PM, Tuesday 2 October**
  - Rachel (001) in HUMN 1B50
  - Tony (002) in DUAN G1B30

# What did we do last time?

- Proofs!  Lots and lots of proofs.
- Proof strategies, proof techniques…
- … and applying our logical equivalences, rules for inference and propositional logic (you know, everything we've done so far) to proving things.




## Today:

We will learn about ***sets*** of things!

## Sets and set operations

"$a \in A$" = "a is an element of A"

**Definition:** A **set** is a collection of objects, usually called **elements** or **members** of the set. A set is said to **contain** its elements.

- We write $a \in A$ to denote that $a$ is an element of set $A$.
- We write $a \notin A$ to denote that $a$ is not an element of set $A$.

$a \notin A$ = "a is NOT an elt. of A"

**Notation:** For sets with a small number of elements, that we can actually list, we write the set with its members inside curly braces: $A = \{a, b, c, d\}$

# Sets and set operations

**Definition:** A **<u>set</u>** is a collection of objects, usually called **elements** or **members** of the set. A set is said to **contain** its elements.

- We write $a \in A$ to denote that $a$ is an element of set $A$.
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**Notation:** For sets with a small number of elements, that we can actually list, we write the set with its members inside curly braces: $A = \{a, b, c, d\}$

**Example:** The set of all vowels is $V = \{a, e, i, o, u\}$

**Example:** The set of all prime numbers less than 10 is $P = \{2, 3, 5, 7\}$

**Example:** The set of all positive integers less than 100: $A = \{1, 2, \ldots, 98, 99\}$

## Sets and set operations

**Example:** The set of all positive integers less than 100: $A = \{1, 2, \ldots, 98, 99\}$

- This is an example of using the **roster method** for conveying what elements are in the set $A$.
- Equally popular and sometime more compact is the **builder method**:

$$A = \{x \in \mathbf{Z}^+ \mid x < 100\}$$

define a set of rules ~~for~~ inclusion in the set

$\mathbb{Z}^+$ = positive integers

set A = all of the positive integers... such that ... the integers are < 100

Like a quantified statement:

Domain = ? all pos. integers

$\forall x \, [(x < 100) \rightarrow (x \in A)]$

Also include −2:

$\rightsquigarrow \forall x \, [((x > 0) \wedge (x < 100)) \vee (x == -2)) \rightarrow (x \in A)]$

Domain = $\mathbb{Z}$ (integers)

## Sets and set operations

**Example:** The set of all positive integers less than 100: $A = \{1, 2, \ldots, 98, 99\}$

- This is an example of using the **roster method** for conveying what elements are in the set $A$.
- Equally popular and sometime more compact is the **builder method**:

$$A = \{x \in \mathbf{Z}^+ \mid x < 100\}$$

Note: this looks a lot like a quantifier statement!
But it's inside curly braces because it's a set.

Popular sets:

| | | |
|---|---|---|
| $\mathbf{N} = \{0, 1, 2, \ldots\}$ | natural numbers | $(\mathbb{N})$ |
| $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ | integers | $(\mathbb{Z})$ |
| $\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$ | positive integers | $(\mathbb{Z}^+)$ |
| $\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, q \neq 0\}$ | rational numbers | $(\mathbb{Q})$ |
| $\mathbf{R}$ = the set of **real** numbers $(\mathbb{R})$ | | |

p & q have no common factors

## Sets and set operations

$A = \{1,1,2,2\} = \{1,2\} = B = \{1,2\}$

Sets can have pretty much anything in them. Even other sets

PYTHON:

A=[1,1,2,2,2,2]

set(A) = {1,2}

**Example:** $A = \{\mathbf{N}, \mathbf{Q}, \mathbf{Z}^+\}$

**Fun fact:** Sets have no ordering.

- So $\{1, 2, x, y\} = \{x, 1, y, 2\}$ (or however you want to rearrange it)

**Fun fact:** It doesn't matter if an element of a set is repeated

$\{1,1,1\} = \{1\}$

- So $\{1, 2, x, y\} = \{1, 1, 1, 2, x, y\}$

**Definition:** Two sets are **equal** if and only if they contain the same elements. So if $A$ and $B$ are sets, we say $A$ and $B$ are equal if and only if

$$\forall x\, (x \in A \Leftrightarrow x \in B)$$

## Sets and set operations

$a \le b \rightsquigarrow A \subseteq B$

**Definition:** The set $A$ is a **<u>subset</u>** of another set $B$ if and only if every element of $A$ is also an element of $B$. We use the notation $A \subseteq B$ to indicate that $A$ is a subset of $B$.

**Question:** How can we use quantifiers to denote $A \subseteq B$?

**Example:** Finish the sentences:

a. The set of all integers **Z** is a subset of …

b. The set of all rational numbers **Q** is a subset of …

c. The set of all CSCI 2824 students is a subset of …

## Sets and set operations

**Definition:** The set $A$ is a <u>**subset**</u> of another set $B$ if and only if every element of $A$ is also an element of $B$. We use the notation $A \subseteq B$ to indicate that $A$ is a subset of $B$.

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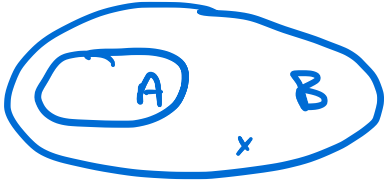



$$\forall x\ ((x \in A) \rightarrow (x \in B))$$

Boulder $\subseteq$ Colorado

**Example:** Finish the sentences:

a. The set of all integers **Z** is a subset of … **Q, R** $\quad \mathbb{Z} \subseteq \mathbb{Q}$

b. The set of all rational numbers **Q** is a subset of … **R**

c. The set of all CSCI 2824 students is a subset of … {all CU students}
{all people}

## Sets and set operations

**Strategy:** To show $A \subseteq B$, we must show that every element of $A$ is also an element of $B$

To show $A \nsubseteq B$, we must find at least one element of $A$ that is not an element of $B$

**Question:** How do you feel about the following?

$$\underline{\{1, 2, 3, 4\}} \subseteq \underline{\{1, 2, 3, 4\}}$$

also in

## Sets and set operations

$a < b \rightsquigarrow A \subset B$

NOT possibly equal

**Strategy:** To show $A \subseteq B$, we must show that every element of $A$ is also an element of $B$

To show $A \nsubseteq B$, we must find at least one element of $A$ that is not an element of $B$

**Question:** How do you feel about the following?

$2 \in \mathbb{Z}$ & $2 \in \mathbb{Q}$

$$\{1, 2, 3, 4\} \subseteq \{1, 2, 3, 4\}$$

**Definition:** If we want to emphasize the fact that $A \subseteq B$ but $A \neq B$, we say that $A$ is a **proper subset** of $B$ and we write $A \subset B$.

**Example:** The set of all even integers is a **proper subset** of the integers.

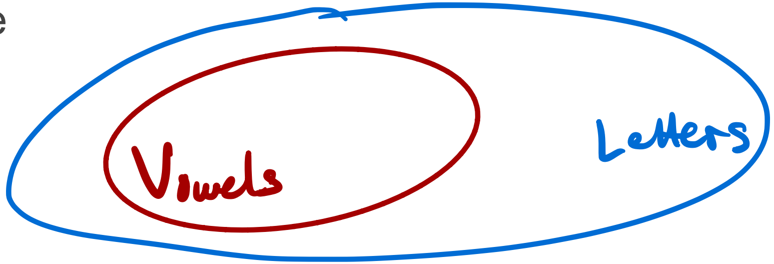
$$\{x \in \mathbf{Z} \mid x \text{ is even}\} \subset \mathbf{Z}$$

OR $\mathbb{Z}^+ \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

# Sets and set operations

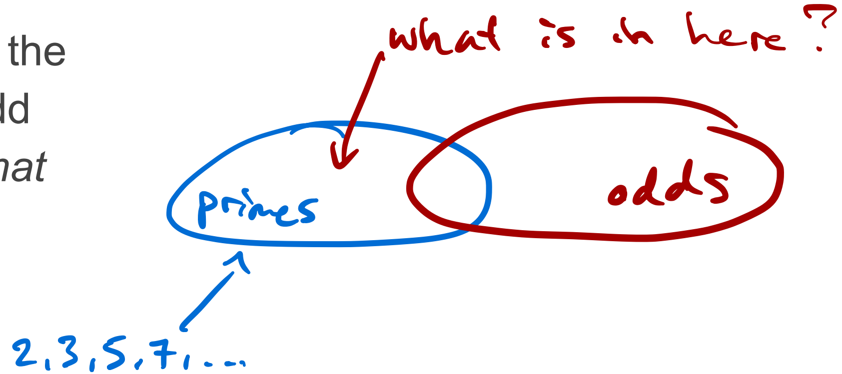
**Venn Diagrams:** drawing a picture can help when thinking about sets and subsets

**Example:** Draw a Venn Diagram relating the set of all vowels to the set of all letters in the English alphabet.




$\text{vowels} \subset \text{Letters}$

**Example:** Draw a Venn Diagram relating the set of all prime numbers and the set of odd numbers. *(prime numbers are numbers that are only divisible by 1 and itself)*



## Sets and set operations

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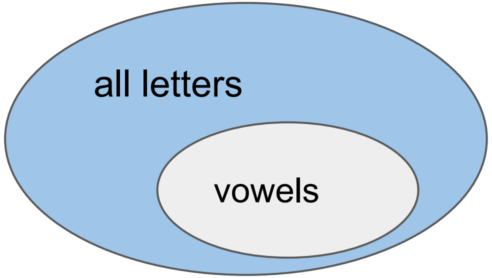


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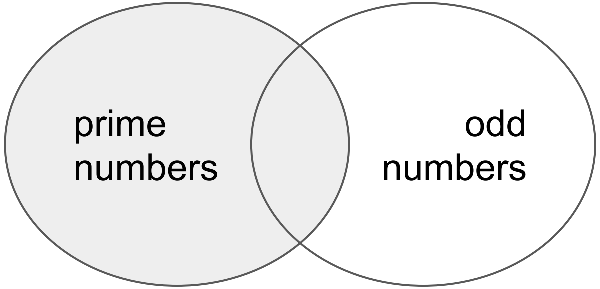
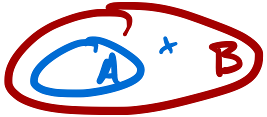


# Concept Check!




**Definition:** If we want to emphasize the fact that $A \subseteq B$ but $A \neq B$, we say that $A$ is a **proper subset** of $B$ and we write $A \subset B$.

**Definition:** The set $A$ is a **subset** of another set $B$ if and only if every element of $A$ is also an element of $B$. We use the notation $A \subseteq B$ to indicate that $A$ is a subset of $B$.

S'pose we know $A \subset B$. Then what must be true?

- [ ] ① There is stuff in $A$ that is not in $B$
- [x] ② There is stuff in $B$ that is not in $A$
- [x] ③ Everything in $A$ must also be in $B$

## Three special sets

- **the empty set:** the set with no elements, written as either ∅ or { }   $\phi$   $\{\ \}$
- **the singleton set:** a set with only one element, for example:

  {42}     or     {Tony}     or     {∅} $\neq \phi$

- **the power set:** $P(S)$ is the set of all subsets of set $S$   $P(s)$
  - Confused? That's normal.

## Three special sets

$\mathcal{P}(\{\phi, 1\}) = \{\phi, \{\phi\}, \{1\}, \{\phi, 1\}\}$

- **the empty set:** the set with no elements, written as either ∅ or { }
- **the singleton set:** a set with only one element, for example:

  {42} or {Tony} or {∅}

- **the power set:** *P*(*S*) is the set of all subsets of set *S*
  - Confused? That's normal.

the elements of a Power set are all sets themselves

$\{0\} \subseteq \{0,1,2\}$

**Example:** *P*($\underline{\{0, 1, 2\}}$) = {∅, {0}, {1}, {2}, {0, 1}, {0, 2}, {1, 2}, $\underbrace{\{0, 1, 2\}}_{S}$}

Never see: ~~$1 \in \mathcal{P}(S)$~~

**Every nonempty set has at least two subsets:**

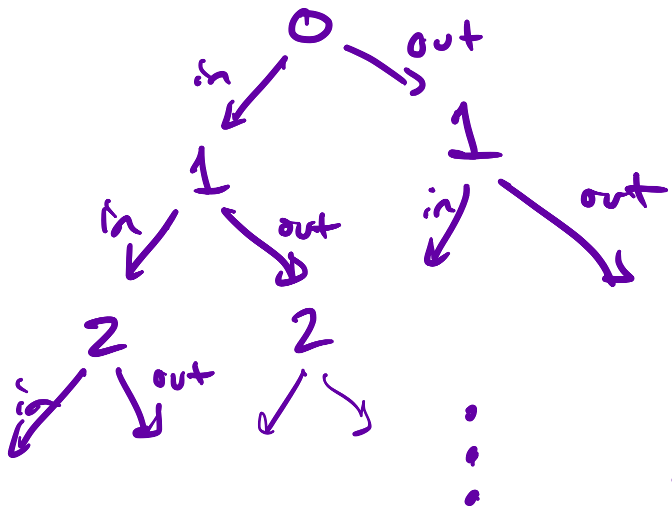
**Theorem:** For every set *S*, ∅ ⊆ *S* and *S* ⊆ *S*.

# Power sets… $P(\emptyset) = \{\emptyset\}$ // $P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$

**Question:** How many elements does $\boldsymbol{P}(S)$ have?

$\boldsymbol{P}(\{0, 1, 2\}) = \{\varnothing, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$

⇒ Each of $n$ elements has 2 possibilities: in or out




elt 1 in/out

elt. n in/out

$2 \cdot 2 \cdots 2 = 2^{\wedge}$

elt. 2 in/out

# Power sets… and cardinality

**Question:** How many elements does $\boldsymbol{P}(S)$ have?

$\boldsymbol{P}(\{0, 1, 2\}) = \{\varnothing, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$

$\Rightarrow$ Each of $n$ elements has 2 possibilities: in or out

**Answer:** If $S$ has $n$ elements, then $\boldsymbol{P}(S)$ has $2^n$ elements. (2 x 2 x 2 x … x 2 ($n$ times))

**Definition:** The number of elements in a set is called its **cardinality**. If a set's cardinality is a finite number, then we say the set is **finite**. We denote the cardinality of a set $A$ by $|A|$.

# Power sets… and cardinality

**Question:** How many elements does $\boldsymbol{P}(S)$ have?

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**Answer:** If $S$ has $n$ elements, then $\boldsymbol{P}(S)$ has $2^n$ elements. (2 x 2 x 2 x … x 2 ($n$ times))

**Definition:** The number of elements in a set is called its **<u>cardinality</u>**. If a set's cardinality is a finite number, then we say the set is **<u>finite</u>**. We denote the cardinality of a set $A$ by $|A|$.

**Example:** What is the cardinality of the English alphabet? $\longrightarrow$ $26 = |\{\text{letters}\}|$ finite set

**Example:** What is the cardinality of the English vowels? $\longrightarrow$ $|V| = 5$ finite set

**Example:** What is the cardinality of the natural numbers? $\rightarrow$ **We'll come back to this!**

$\hookrightarrow |\{0, 1, 2, \ldots\}| = \ldots \infty?$ infinite set!

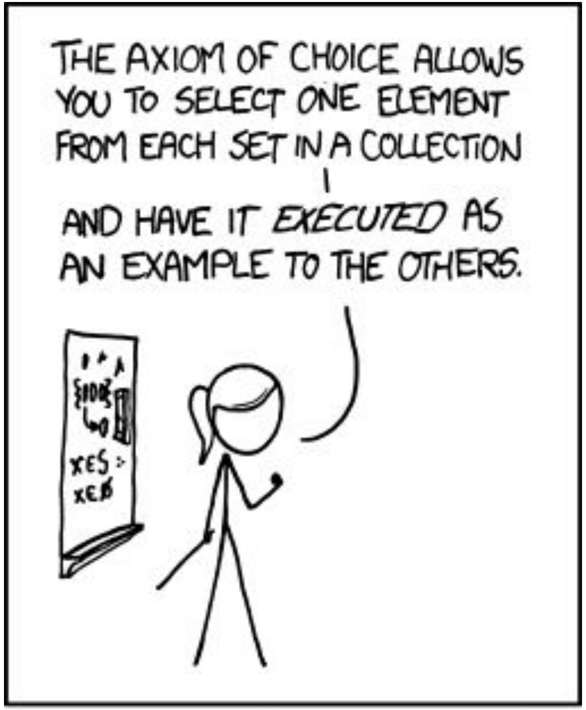
# Sets and set operations

**Recap:**

- We learned what **sets** are,
- some special sets (empty set, singleton set, power set),
- how to cook up larger sets (power sets) from smaller ones,
- and how to cook up smaller sets (subsets) from larger ones

**Next time:**

- More on sets, and *infinite* sets!




MY MATH TEACHER WAS A BIG BELIEVER IN PROOF BY INTIMIDATION.

# Bonus material!