CSCI3656: Numerical Computation
Homework 6: Due 5pm Friday, Oct. 16

Turn in your own writeup that includes your code. List any resources you used including collaborating with others. You shouldn't need to use the symbolic toolbox. Submit a PDF on Canvas by Friday, Oct. 16 at 5pm.

Consider the following nonlinear system of equations with two equations and two unknowns. The math problem can be stated as follows. Given $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ defined as

$$\begin{aligned} f_1(x_1, x_2) &= x_1^3 - x_2^3 + x_1, \\ f_2(x_1, x_2) &= x_1^2 + x_2^2 - 1. \end{aligned} \tag{1}$$

Find r_1 and r_2 such that $f_1(r_1, r_2) = 0$ and $f_2(r_1, r_2) = 0$.

1. Note that all the points such that $f_2 = 0$ define a circle of radius 1 centered at the origin. Make a plot that shows (i) all the points that satisfy $f_1 = 0$ and (ii) all the points that satisfy $f_2 = 0$. Identify the points on the plot that satisfy both $f_1 = 0$ and $f_2 = 0$.

2. By hand, calculate the 2×2 Jacobian matrix of the system (f_1, f_2).

3. Use Newton's method for systems to find the two solutions to the system of equations $(f_1 = 0, f_2 = 0)$. Try several (10 or so) different initial guesses. Make a table of the answer that Newton's method gives—something like:

Initial guess $(x_1^{(0)}, x_2^{(0)})$	Newton's answer (r_1, r_2)
####, ####	####, ####

   The superscript in the column heading indicates the iteration number, i.e., 0 means the initial guess. Check the plot you made in problem 1 to see whether the answers you're getting make sense.

4. Find a starting point where Newton's method fails. Why did it fail?

5. BONUS (10%): Plot the iterates from Newton's method in (x_1, x_2) space. Interpret what you see (related to the solution that Newton's method gives).

6. BONUS (10%): Set up a grid on the (x_1, x_2) space; you can use `meshgrid` in Matlab/Numpy. For each point in the grid, compute the Newton direction

   $$p = -J(x_1, x_2)^{-1} f(x_1, x_2), \tag{2}$$

   where J is the Jacobian matrix and $f = (f_1, f_2)$. Make a `quiver` plot of the Newton directions on the grid. How does this relate to the solution that Newton's method gives?

7. BONUS (20%): Derive a fixed point iteration for nonlinear systems. (That is: show how to transform the nonlinear system problem to a fixed point problem and then write the iteration). Implement your method. Solve the same two-equation system with your fixed point method. Anything interesting?