HW7

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0.1 ### HW 7

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I worked by myself on this assignment.

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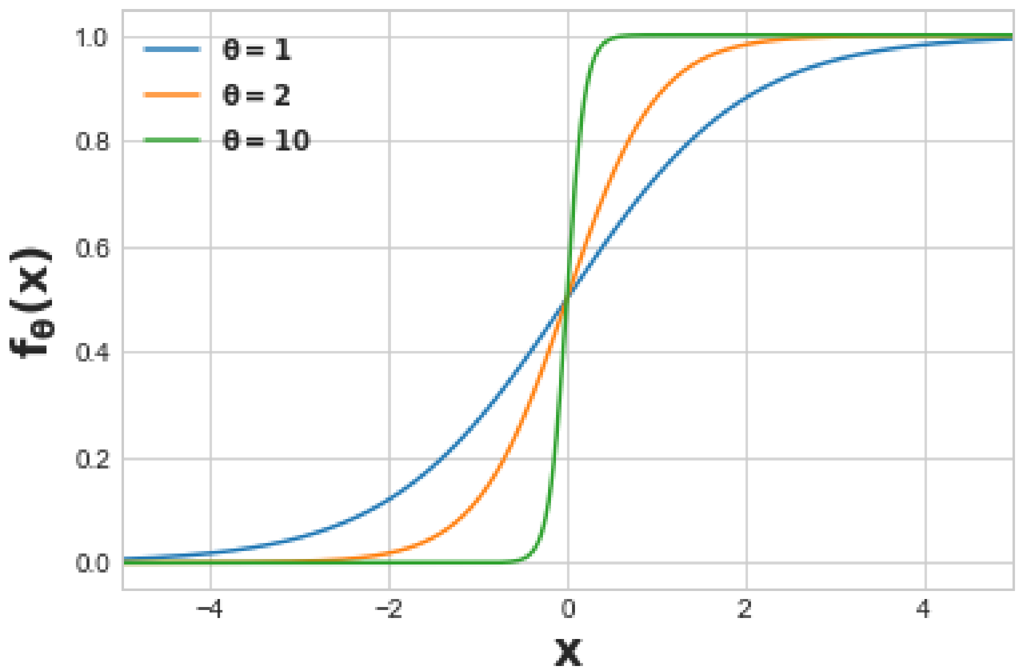
[1]:
```
'''GLOBAL IMPORTS'''

import numpy as np

%matplotlib inline
import matplotlib.pyplot as plt
plt.style.use('seaborn-whitegrid')
```

$f_\theta(x) = \frac{1}{1+e^{-\theta x}}, \ x \in [-5, 5].$

[2]:
```
f_theta = lambda x, theta: 1 / (1 + np.exp(-theta * x))
```

[3]:
```
interval = -5, 5
xs = np.linspace(interval[0], interval[1], num=8000)

plt.plot(xs, f_theta(xs, 1), label="$\\bf{\\theta=1}$")
plt.plot(xs, f_theta(xs, 2), label="$\\bf{\\theta=2}$")
plt.plot(xs, f_theta(xs, 10), label="$\\bf{\\theta=10}$")

plt.xlim([interval[0], interval[1]])
plt.xlabel("$\\bf{x}$", fontsize='xx-large')
plt.ylabel("$\\bf{f_{\\theta}(x)}$", fontsize='xx-large')

plt.legend()
```

[3]:
```
<matplotlib.legend.Legend at 0x109844e10>
```




```
[4]: # Helper functions for question #1
def generate_table(xs, ys):
    print("{:<3}| {:<23}| {:
↪<23}\n-----------------------------------------------------".format("i",␣
↪"x_i", "y_i"))
    i = 1
    for i in range(len(xs)):
        print("{:<3}| {:<23}| {:<23}".format(i, xs[i], ys[i]))
        i += 1

def get_ys(f, xs, theta):
    ys = []
    for x in xs:
        ys.append(f(x, theta))
    return ys
```

```
[5]: theta = 1

train_vec = np.linspace(interval[0], interval[1], num=7, endpoint=True)
train_ys  = get_ys(f_theta, train_vec, theta)

generate_table(train_vec, train_ys)
```

```
i  | x_i                    | y_i
-----------------------------------------------------
```

```
0   | -5.0                    | 0.0066928509242848554
1   | -3.333333333333333      | 0.03444519566621118
2   | -1.6666666666666665     | 0.15886910488091516
3   | 0.0                     | 0.5
4   | 1.666666666666667       | 0.8411308951190849
5   | 3.333333333333334       | 0.9655548043337889
6   | 5.0                     | 0.9933071490757153
```

$p_6(x) = \sum_{k=0}^{6} = c_k \cdot x^k$

[6]:
```python
vandermonde_matrix = np.vander(train_vec, increasing=True)
```

Now I have to solve for the vector that contains the coefficients.

[7]:
```python
cs = np.linalg.solve(vandermonde_matrix, train_ys)

print(cs)
```

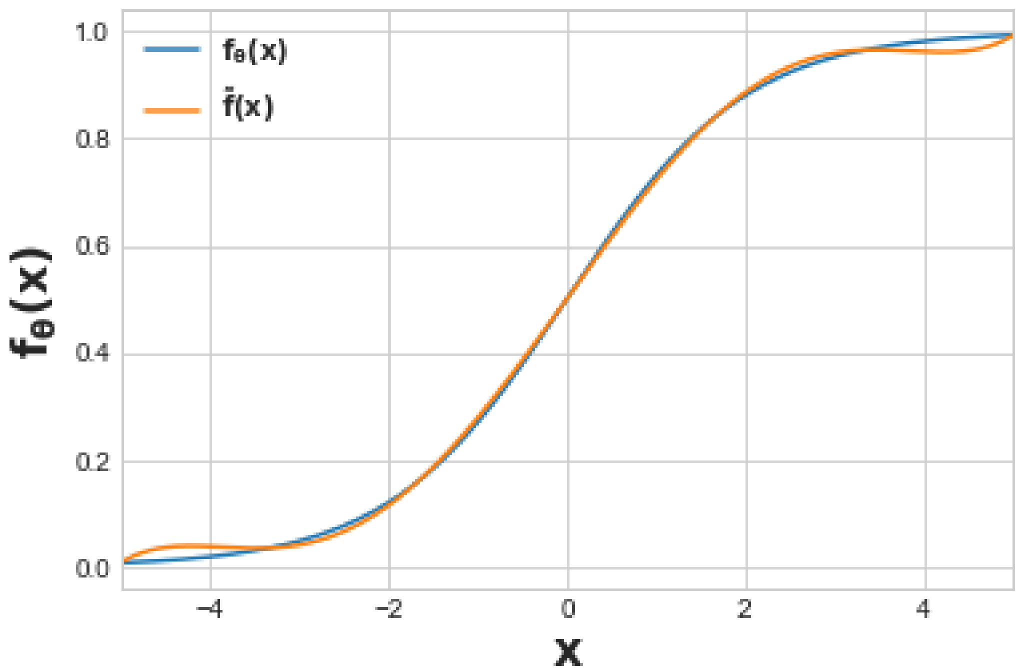
```
[ 5.00000000e-01  2.33084084e-01 -4.75283111e-17 -1.08321332e-02
  4.01333175e-18  2.18209080e-04 -8.09352585e-20]
```

And we get: $p_6(x) = -8.09352585e-20x^6 + 2.18209080e-04x^5 + 4.01333175e-18x^4 - 1.08321332e-02x^3 - 4.75283111e-17x^2 + 2.33084084e-01x + 0.5$

[8]:
```python
f_tilde = lambda x: (-8.09352585e-20 * (x ** 6)) + (2.18209080e-04 * (x ** 5))␣
↪+ (4.01333175e-18 * (x ** 4)) + (-1.08321332e-02 * (x ** 3)) + (-4.
↪75283111e-17 * (x ** 2)) + (2.33084084e-01 * x) + 0.5
```

[9]:
```python
plt.plot(xs, f_theta(xs, theta), label="$\\bf{f_{\\theta}(x)}$")
plt.plot(xs, f_tilde(xs), label="$\\bf{\\tilde{f}(x)}$")

plt.xlim([interval[0], interval[1]])
plt.xlabel("$\\bf{x}$", fontsize='xx-large')
plt.ylabel("$\\bf{f_{\\theta}(x)}$", fontsize='xx-large')

plt.legend()
```

[9]:
```
<matplotlib.legend.Legend at 0x10b9fc250>
```



[10]:
```
test_vec = np.linspace(interval[0], interval[1], num=101, endpoint=True)
test_ys  = get_ys(f_theta, test_vec, theta)


print("Mean: {}".format(np.mean(test_ys)))
print("Std.: {}".format(np.std(test_ys)))
```

```
Mean: 0.5000000000000001
Std.: 0.39019312637433506
```

[11]:
```
# Helper functions for question #2
def get_errors(xs, ys, f_tilde):
    error = []
    for i in range(len(xs)):
        error.append(abs(ys[i] - f_tilde(xs[i])))
    return error
```

[12]:
```
errors = get_errors(test_vec, test_ys, f_tilde)
print("Error (theta = 1): ", max(errors))
```

```
Error (theta = 1):  0.025163854763386617
```

1

```
[13]: theta = 10

train_vec = np.linspace(interval[0], interval[1], num=7, endpoint=True)
train_ys  = get_ys(f_theta, train_vec, theta)

generate_table(train_vec, train_ys)
```

```
i   | x_i                    | y_i
----------------------------------------------------
0   | -5.0                   | 1.928749847963918e-22
1   | -3.333333333333333     | 3.338237795365011e-15
2   | -1.6666666666666665    | 5.777748185595394e-08
3   | 0.0                    | 0.5
4   | 1.666666666666667      | 0.9999999422225181
5   | 3.333333333333334      | 0.9999999999999967
6   | 5.0                    | 1.0
```

2

```
[14]: vandermonde_matrix = np.vander(train_vec, increasing=True)
cs = np.linalg.solve(vandermonde_matrix, train_ys)

print(cs)
```

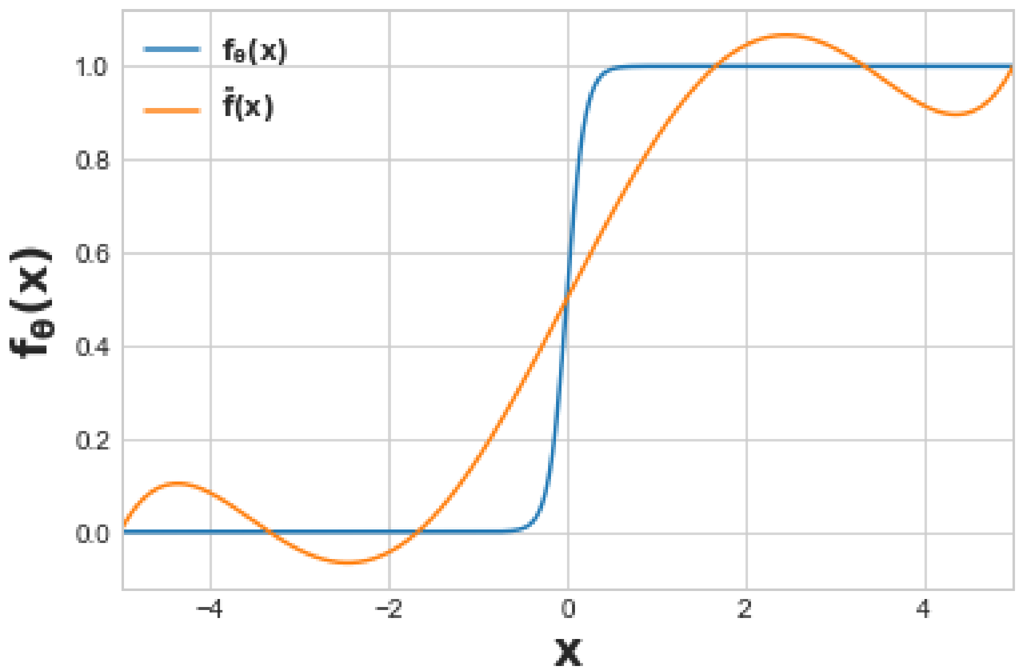
```
[ 5.00000000e-01  3.69999948e-01 -8.08372189e-17 -2.69999932e-02
  6.60579119e-18  6.47999813e-04 -1.34892097e-19]
```

Again, we get: $p_6(x) = -1.34892097e-19x^6 + 6.47999813e-04x^5 + 6.60579119e-18x^4 - 2.69999932e-02x^3 - 8.08372189e-17x^2 + 3.69999948e-01x + 0.5$

```
[15]: f_tilde = lambda x: (-1.34892097e-19 * (x ** 6)) + (6.47999813e-04 * (x ** 5))␣
↪+ (6.60579119e-18 * (x ** 4)) + (-2.69999932e-02 * (x ** 3)) + (-8.
↪08372189e-17 * (x ** 2)) + (3.69999948e-01 * x) + 0.5
```

```
[16]: plt.plot(xs, f_theta(xs, theta), label="$\\bf{f_{\\theta}(x)}$")
plt.plot(xs, f_tilde(xs), label="$\\bf{\\tilde{f}(x)}$")

plt.xlim([interval[0], interval[1]])
plt.xlabel("$\\bf{x}$", fontsize='xx-large')
plt.ylabel("$\\bf{f_{\\theta}(x)}$", fontsize='xx-large')

plt.legend()
```

```
[16]: <matplotlib.legend.Legend at 0x10bb17550>
```



3

[17]:
```
test_ys = get_ys(f_theta, test_vec, theta)

print("Mean: {}".format(np.mean(test_ys)))
print("Std.: {}".format(np.std(test_ys)))
```

```
Mean: 0.5
Std.: 0.4899989875597514
```

4

[18]:
```
errors = get_errors(test_vec, test_ys, f_tilde)
print("Error (theta = 10): ", max(errors))
```

```
Error (theta = 10):  0.34230156759928765
```

The error grows. Although it is true that increasing n will increase error, the error that is caused in this case is due to θ. In this question, similar to #4, we use a polynomial of degree 6. As θ grows, so does our error thus our approximation isn't as good. The **EC** is the part that showcases how changing n impacts the error.

[19]:
```
# Helper functions to make looping through different n's simpler
def get_polynomial_ys(xs, cs):
```

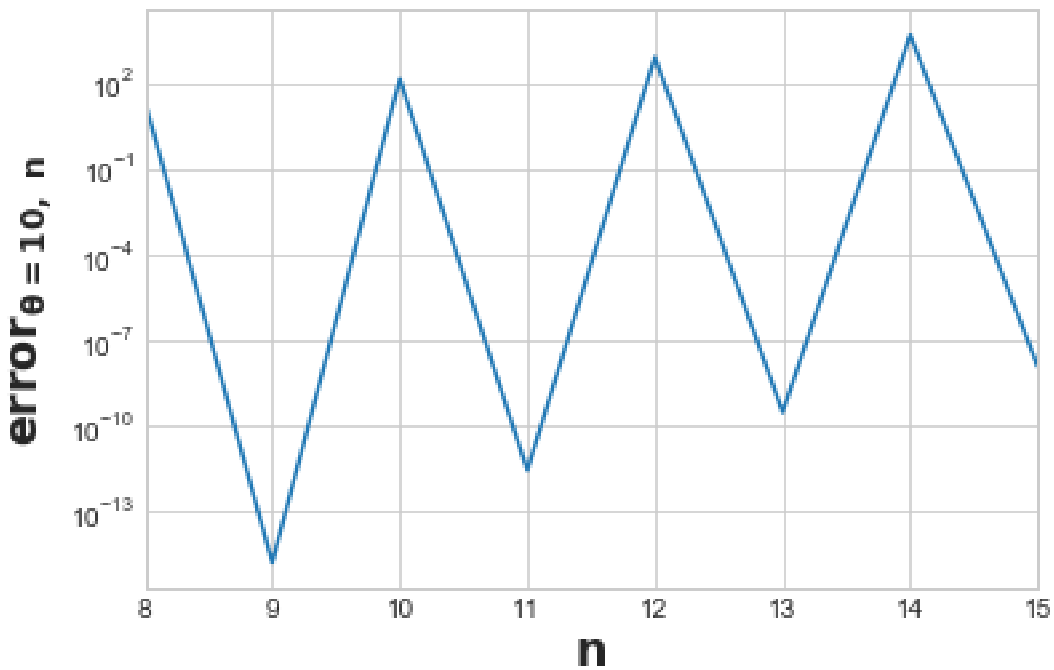
```
    ys = []
    for x in xs:
        y = 0
        for i in range(len(cs)-1):
            y += cs[i] * (x ** i)
        ys.append(y)
    return ys

def get_errors2(ys, poly_ys):
    error = []
    for i in range(len(ys)):
        error.append(abs(ys[i] - poly_ys[i]))
    return error
```

[20]:

```
theta = 10

exp_range = 8, 15
n = [x for x in range(exp_range[0], exp_range[1] + 1)]
err = []

for i in n:
    train_vec = np.linspace(interval[0], interval[1], num=i, endpoint=True)
    train_ys  = get_ys(f_theta, train_vec, theta)
    vandermonde_matrix = np.vander(train_vec, increasing=True)
    cs = np.linalg.solve(vandermonde_matrix, train_ys)
    poly_ys = get_polynomial_ys(train_vec, cs)
    errors = get_errors2(train_ys, poly_ys)
    err.append(max(errors))
plt.semilogy(n, err)

plt.xlim([exp_range[0], exp_range[1]])
plt.xlabel("$\\bf{n}$", fontsize='xx-large')
plt.ylabel("$\\bf{error_{\\theta=10,\\ n}}$", fontsize='xx-large')
```

[20]:

```
Text(0, 0.5, '$\\bf{error_{\\theta=10,\\ n}}$')
```

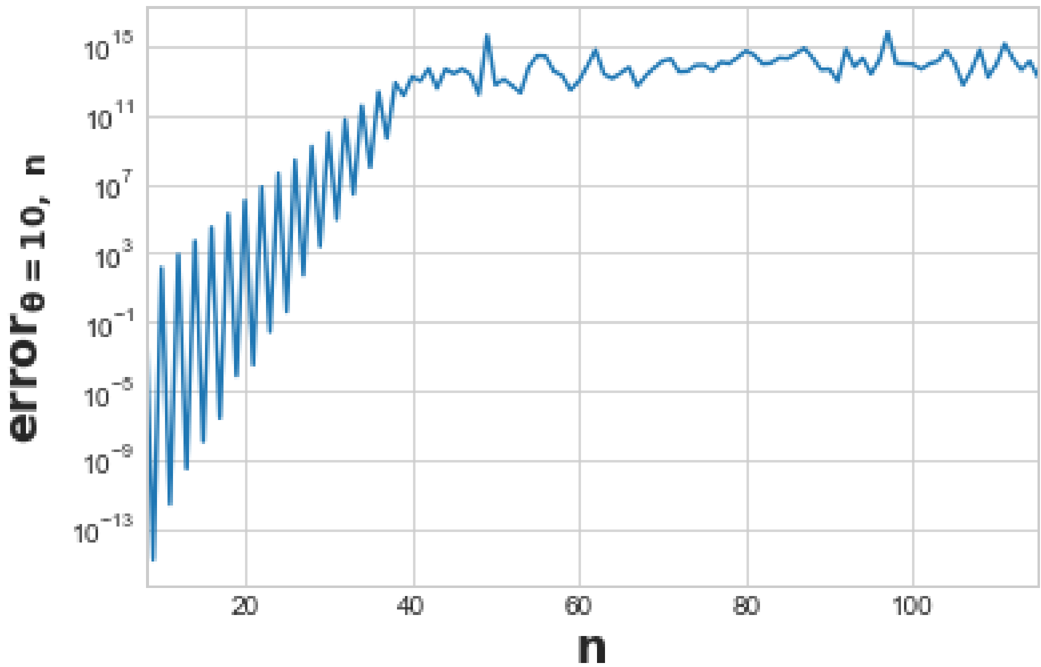


I did all 4 steps in one cell to make it easier to loop through $n = 8, \ldots, 15$. Although this experiment helped and made it seem to converge to ∞, I did it with 115 trials below, and it looks to converge at 10^{13}. Obviously as we take more and more evenly spaced points, our vandermonde matrix becomes likely more and more ill-conditioned which explains the larger error; also, the approximation typically really sucks near both the endpoints as can be seen in the plot I made for **#5**.

[21]:
```
theta = 10

exp_range = 8, 115
n = [x for x in range(exp_range[0], exp_range[1] + 1)]
err = []

for i in n:
    train_vec = np.linspace(interval[0], interval[1], num=i, endpoint=True)
    train_ys  = get_ys(f_theta, train_vec, theta)
    vandermonde_matrix = np.vander(train_vec, increasing=True)
    cs = np.linalg.solve(vandermonde_matrix, train_ys)
    poly_ys = get_polynomial_ys(train_vec, cs)
    errors = get_errors2(train_ys, poly_ys)
    err.append(max(errors))
plt.semilogy(n, err)

plt.xlim([exp_range[0], exp_range[1]])
plt.xlabel("$\\bf{n}$", fontsize='xx-large')
```

```
plt.ylabel("$\\bf{error_{\\theta=10,\\ n}}$", fontsize='xx-large')
```

[21]: Text(0, 0.5, '$\\bf{error_{\\theta=10,\\ n}}$')




```
[ ]:
```