

HW12

December 5, 2020

Numerical Computation: HW12

Andrew Pickner I worked alone on this assignment.

```
[1]: import numpy as np
import pandas as pd

import scipy.integrate

%matplotlib inline
import matplotlib.pyplot as plt
plt.style.use('seaborn-whitegrid')
```

```
[2]: def forwardEuler(f, U_0, h, T):
    n_iters = int(round(float(T)/h))
    u = np.zeros((n_iters+1, len(U_0)))
    t = np.linspace(0, n_iters*h, len(u))
    u[0] = U_0
    for n in range(n_iters):
        u[n+1] = u[n] + h*f(t[n], u[n])
    return u, t
```

```
[3]: def rungeKutta2(f, U_0, h, T):
    n_iters = int(round(float(T)/h))
    u = np.zeros((n_iters+1, len(U_0)))
    t = np.linspace(0, n_iters*h, len(u))
    u[0] = U_0

    for n in range(n_iters):
        Y_star = u[n] + (h/2)*f(t[n], u[n])
        u[n+1] = u[n] + h*f((t[n] + (h/2)), Y_star)
    return u, t
```

```
[4]: # https://en.wikipedia.org/wiki/Runge-Kutta\_methods
def rungeKutta4(f, U_0, h, T):
    n_iters = int(round(float(T)/h))
```

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u = np.zeros((n_iters+1, len(U_0)))
t = np.linspace(0, n_iters*h, len(u))
u[0] = U_0

for n in range(n_iters):
    k1 = f(t[n], u[n])
    k2 = f(t[n] + (h/2), u[n] + (h * (k1/2)))
    k3 = f(t[n] + (h/2), u[n] + (h * (k2/2)))
    k4 = f(t[n] + h, u[n] + (h * k3))
    u[n+1] = u[n] + (h/6)*(k1 + (2*k2) + (2*k3) + k4)
return u, t

```

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[5]: def SIR_Demo(h, method, num_days=140, b=1/2, k=1/3, v=0):
    # If we guess that each infected would make a possibly infecting contact,
    ↪ every two days, then b = 1/2.
    # This is simply a guess.
    b = b
    # We have already estimated the average period of infectiousness at three,
    ↪ days, so that would suggest k = 1/3
    # This is simply a guess.
    k = k
    """Test case using a SIR model."""
    def f(t, u):
        S, I, R = u
        # ds/dt = -b * s(t) * i(t)
        # di/dt = b * s(t) * i(t) - k * i(t)
        # dr/dt = k * i(t)
        return np.array([-b*S*I + v*R, b*S*I - k*I, k*I - v*R])

    # time delta: change in time
    dt = h

    # time: total time in days
    T = num_days

    # S = S(t): is the number of susceptible individuals,
    S_0 = 7.9e6
    # I = I(t): is the number of infected individuals,
    I_0 = 10
    # R = R(t): is the number of recovered individuals,
    R_0 = 0

    # i(t) = I(t)/N : the infected fraction of the population,
    i_0 = 1.27e-6
    # s(t) = S(t)/N : the susceptible fraction of the population,
    s_0 = 1-i_0
    # r(t) = R(t)/N : the recovered fraction of the population,

```

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r_0 = 0

# initial condition vector
U_0 = [s_0, i_0, r_0]

# solving the system of ODEs using euler's method
u, t = method(f, U_0, dt, T)

S = u[:,0]
I = u[:,1]
R = u[:,2]
fig = plt.figure()
s1, i1, r1 = plt.plot(t, S, t, I, t, R)
fig.legend((s1, i1, r1), ('S', 'I', 'R'))
plt.xlabel('days')
plt.show()

# Consistency check:
N = S[0] + I[0] + R[0]
eps = 1E-12 # Tolerance for comparing real numbers
for n in range(len(S)):
    SIR_sum = S[n] + I[n] + R[n]
    if abs(SIR_sum - N) > eps:
        print('*** consistency check failed: S+I+R={} != {}'.
→format(SIR_sum, N))

```

0.0.1 SELECTING H:

[Euler's Method: SIR](#) Also where the infamous graph is referenced (I mention it a million times below).

[ODE Model: SIR](#)

[Background: SIR](#)

Other resources:

[Super dope write-up:](#) Kinda involving a lot of the computational and numerical theory, as well as explaining some of the constants and other disease-related aspects as well.

So, forward Euler is first order and its error is bound by $\mathcal{O}(h)$. Runge-Kutta 2 is second order (thus the 2), and its global error is bound by $\mathcal{O}(h^2)$ so it should do a little better than Euler's method (so long as h is small). Further, looking ahead towards RK4, following from the logic of RK2, this is obviously fourth order and I also assume its error is bound by $\mathcal{O}(h^4)$ so it should perform the best (again, assuming h is small). However, what I've gleaned is that we can theoretically make h as small as we want, but the computational cost grows as we do this. Thus, we need to find an h that is *sufficiently* small given what we are computing. Here, we are building a tool that can model

a disease within a population and I believe we are “calibrating” our tool by choosing this h so that it matches the graph on the website. Although I could let my code run for a long time with an incredibly small h , I’d rather not.

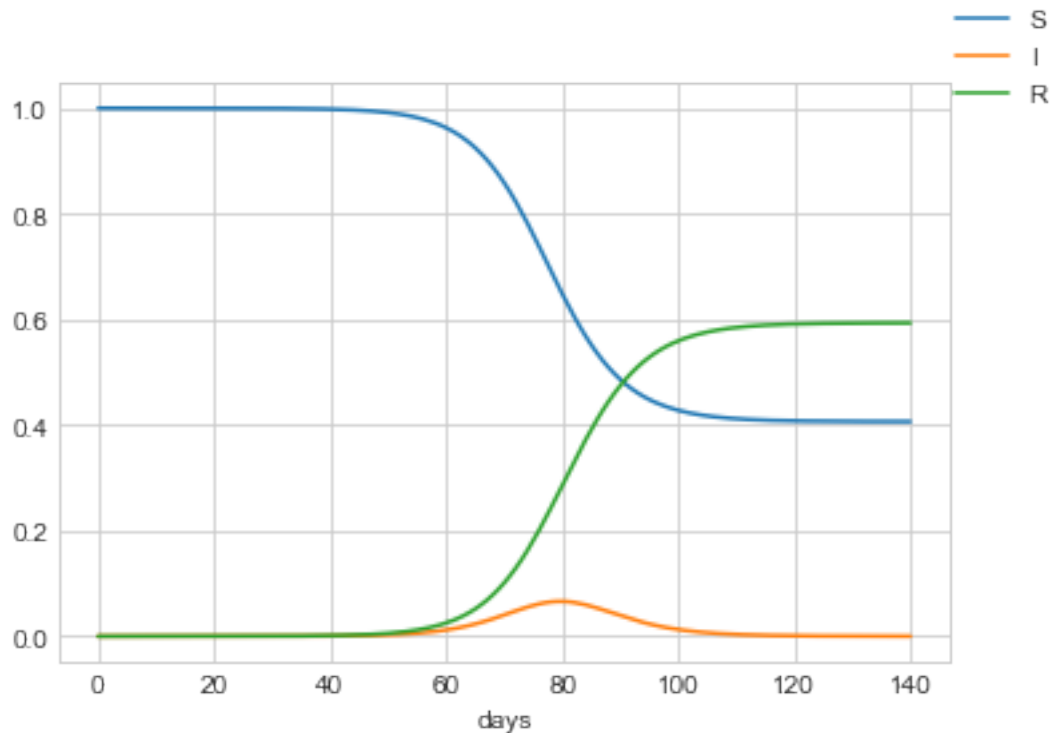
- Initially, I just wanted to get up and running with my code so I loop through 1-10 for testing purposes, but I figured this test *might* tell me something (once code was all debugged). Also, these h ’s are relatively plausible.
 - When $h = 1$, we see a *relatively* similar graph to the one on the website (Euler’s method).
 - As h grows, we see the graph become more jagged, and we see the infection curve shift to the right (unlike the graph on the website)
 - * Also interesting to note, although the graph does look a little jagged and it doesn’t match the website at all, RK2 at $h = 10$ is far better than Euler’s method. Important for future studies: when $h = 1$, RK2 also outperforms Euler’s method, and looks really close to the graph we’re trying to reproduce.
 - * In fact, RK2 holds on until around $h \approx 3$ and still very closely resembles the graph from the website.
 - * Finally, I skipped this study for RK4 because I had a few other ideas in mind.
- Then, because I knew it would be far better to have $h < 1$, I gave this a try similar to my last little experiment. I started at 0.1 and incremented by 0.1 until we hit 1. Of course rounding errors mess with this a little, it was still a helpful experiment for Euler’s method.
 - Right off the bat, it was clear our initial h value: 0.1 is the closest to the graph from the website we’ve seen yet.
 - For $h \leq 0.5$, we see incredibly similar graphs to that on the website.
 - As we see: $0.6 \leq h \leq 1$, the graph *looks* similar but the infected curve is slightly shifted to the right. (this analysis is all for Euler’s method...)
 - * Finally, as Euler’s method degrades as h approaches 1, we simply don’t see this same behavior for RK2. The difference in the RK2 graphs is extremely unnoticeable, and they look incredibly similar to the graph depicted on the website.
- Although not necessarily relevant to me finding my value for h , I thought of several *interesting* values for h I wanted to try.
 - 140, so we’d have one *bin* or iteration.
 - 70, same reason as above but we’d have 2.
 - 35, same reason as above but we’d have 4.
 - 17.5, same reason as above but we’d have 8.
 - 8.75, same reason as above but we’d have 16.
 - 4.375, same reason as above but we’d have 32.
 - 1, this is equivalent to 1 day, seemed like a good baseline test.
 - 10, also seemed like a good baseline test.
 - $(1/24) = 0.04167$, this is effectively equivalent to 1 hour. **This is what I began to lean towards relatively early on (more will be explained later)**
- Finally, I make h smaller exponentially to see if I could see anything interesting in the graphs. Unfortunately, nothing came to me as all of the graphs pretty much looked the same for each of the respective methods I ran this study on.
 - This is where I became pretty sure I would make $\boxed{h = \frac{1}{24}}$ for all the methods because it also works for each of them incredibly well.
 - * Although, I finally realized after restarting the kernel and running all several times that small h ’s would make RK2 and RK4 run longer...

The reason I like this decision is because 1 hour seems like a pretty reasonable time step and it **is** relatively small. Besides, $h = 0.5$ or simply 12 hours, seems like it just about begins to shift Euler's infection curve. However, I realized after I wrote this ginormous block of text that this isn't *really* what the question is asking... I *feel* like you're asking what the largest possible h is that it still matches the graph from the website.

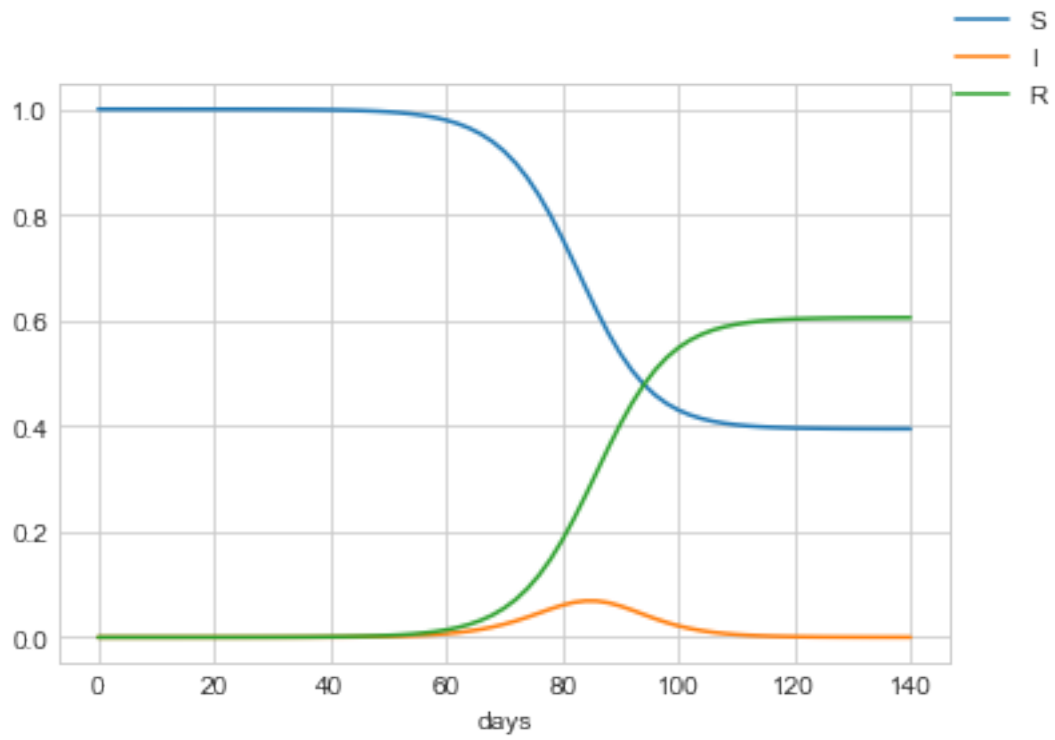
- **Euler's method:** $h \leq 0.5$, so h can be as large as 0.5 and still match the graph on the website.
- **RK2:** $h \leq 3$, so h can be as large as 3 and still match the graph on the website.
- **RK4:** $h \leq 6$, so h can be *almost* as large as 6 and still match the graph on the website.

```
[6]: for i in range(1, 11):  
      print(i)  
      SIR_Demo(i, forwardEuler)
```

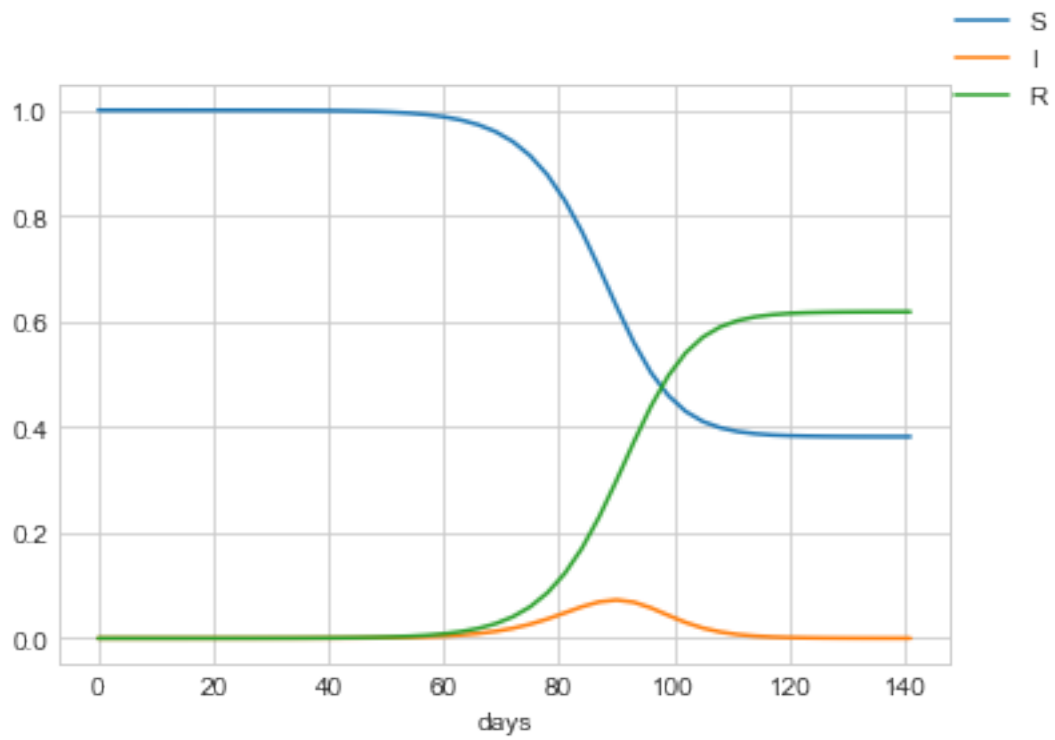
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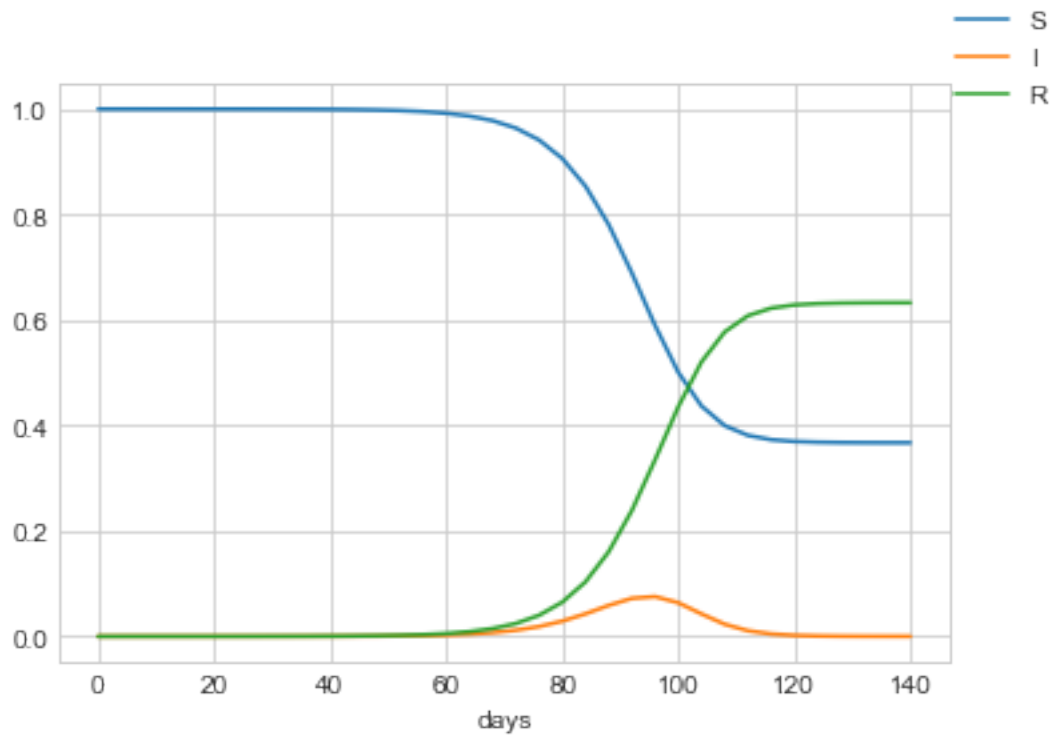
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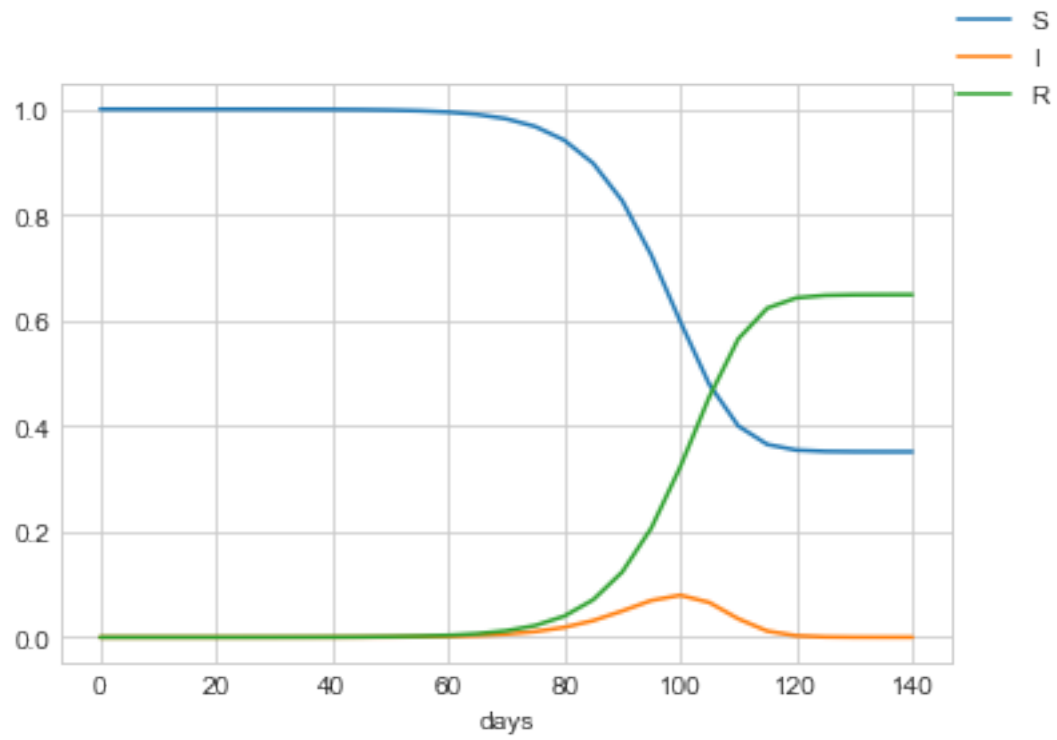
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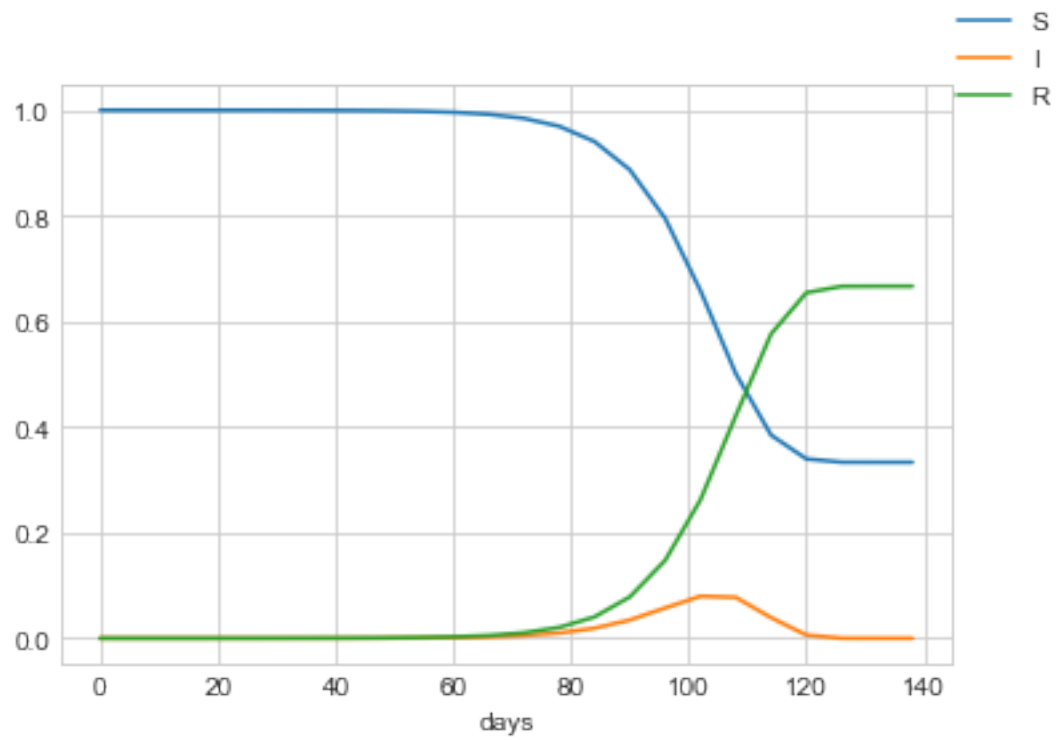
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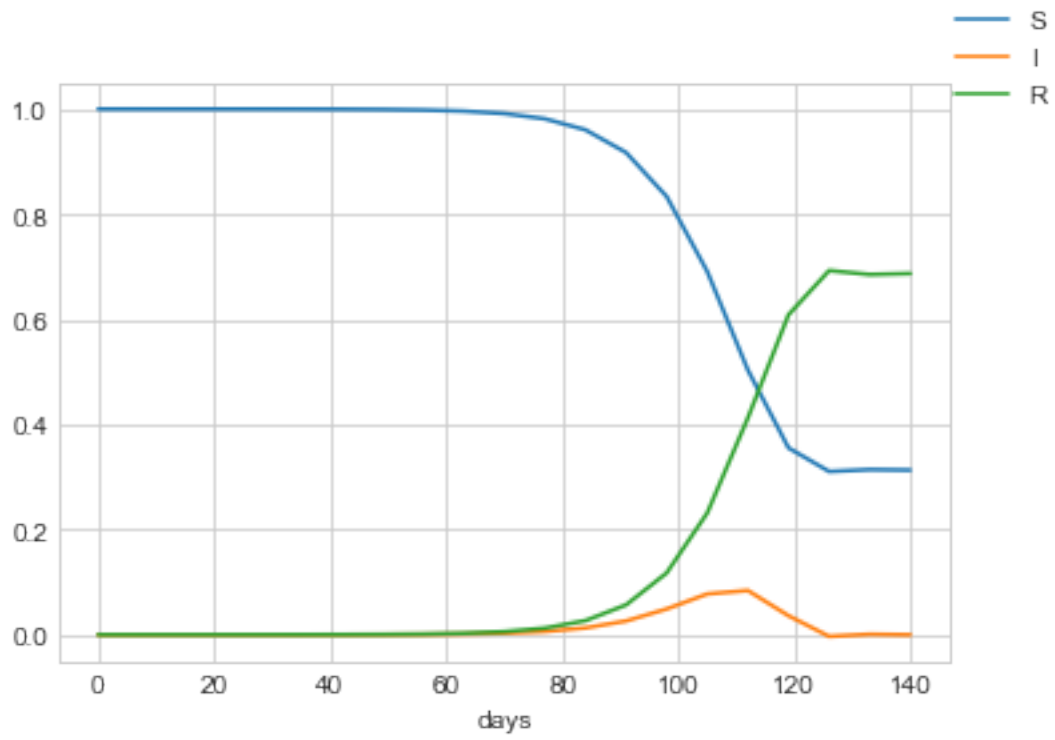
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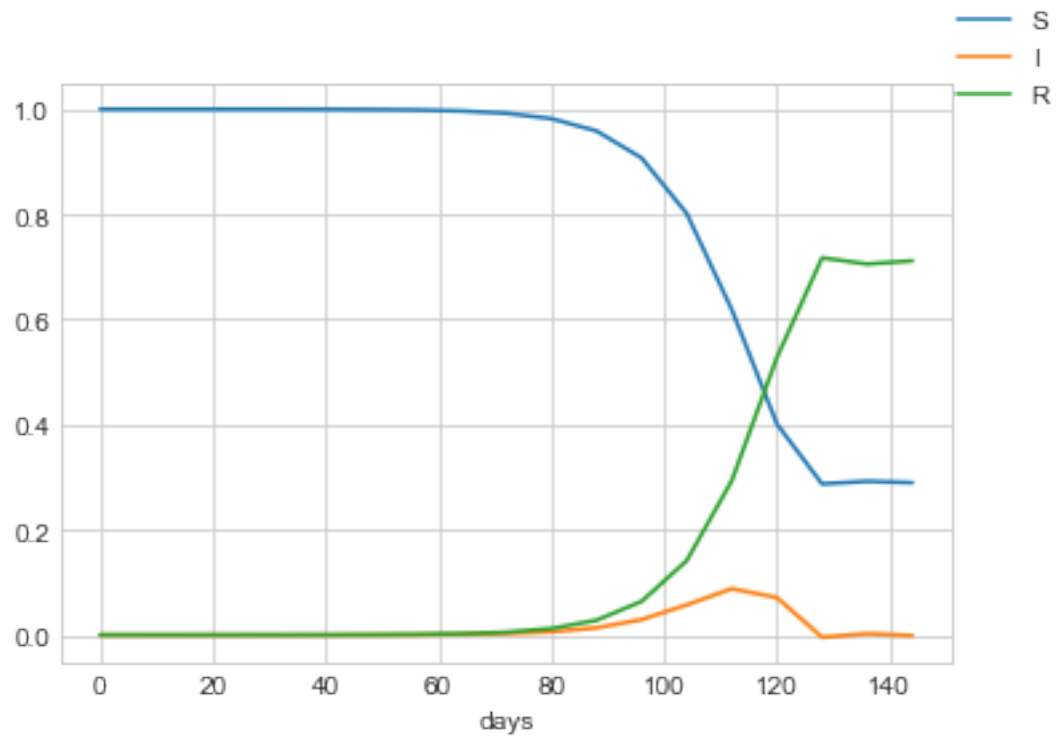
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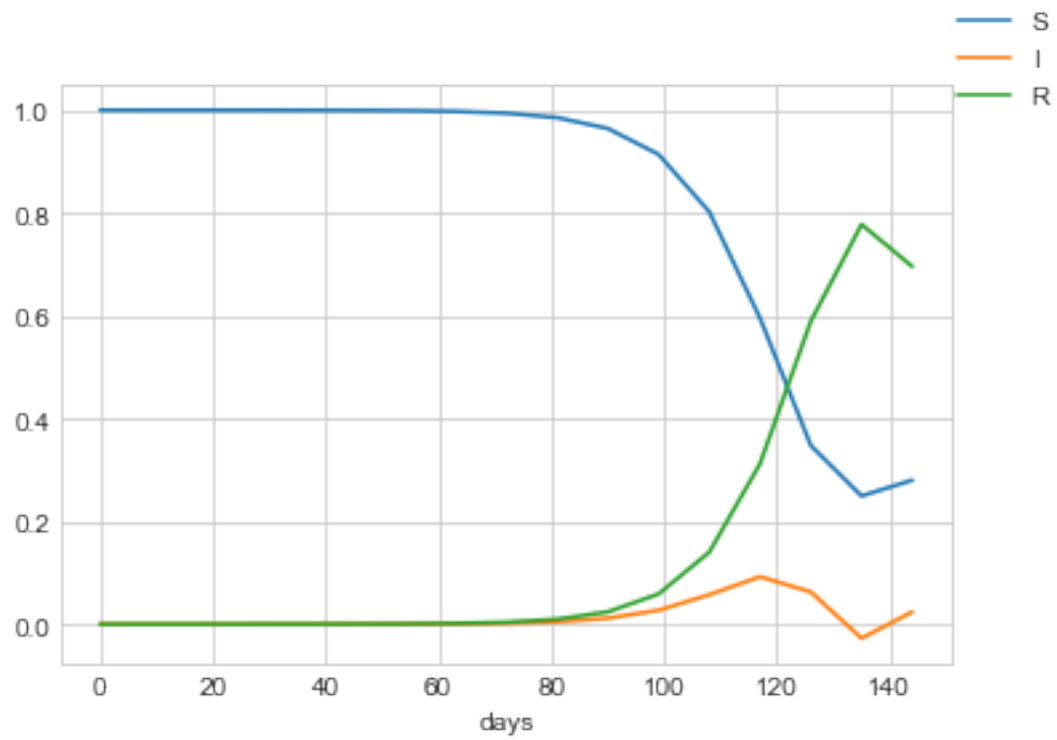
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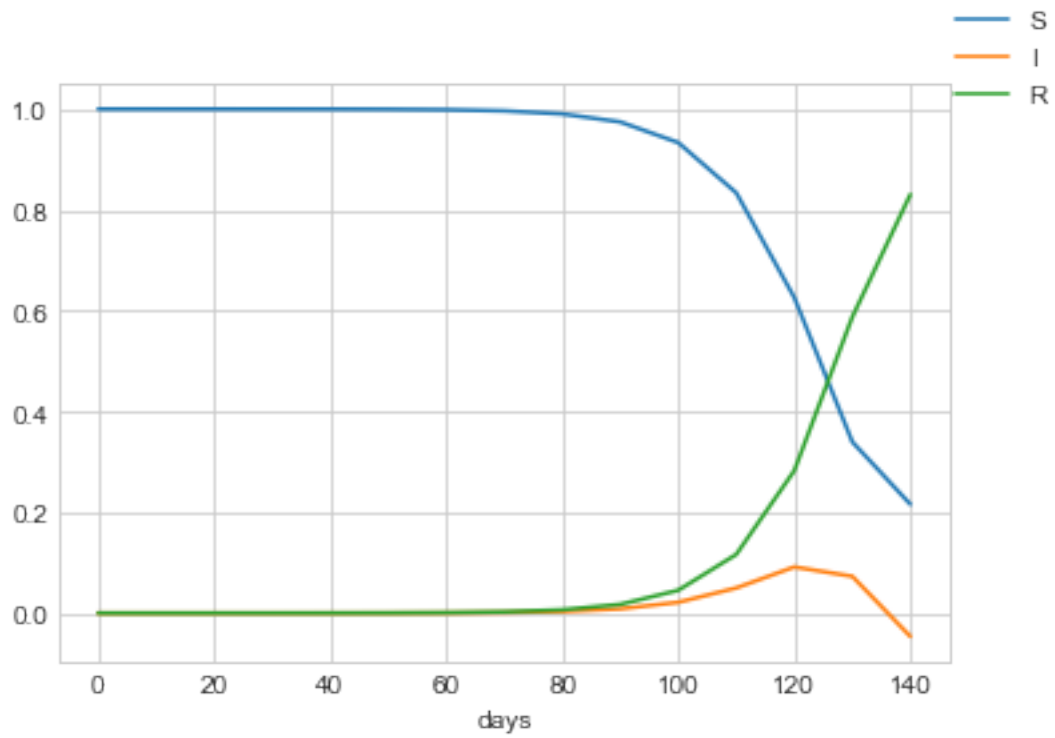
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9

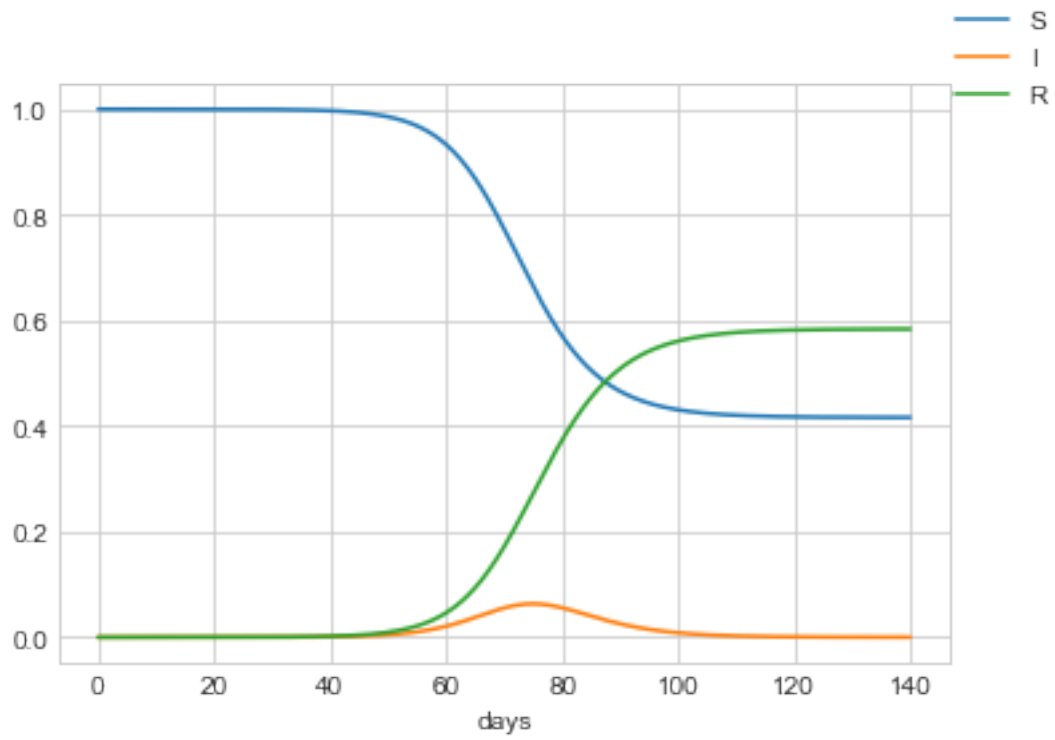


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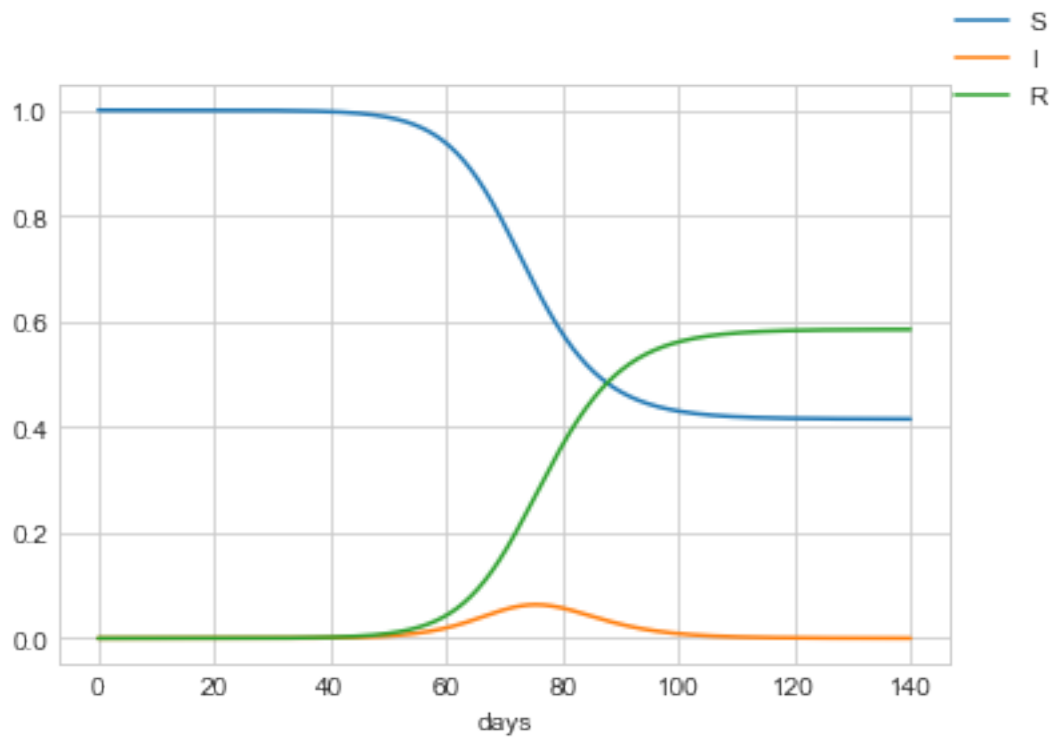


```
[7]: x = 0.1
for i in range(0, 10):
    print(x)
    SIR_Demo(x, forwardEuler)
    x+=0.1
```

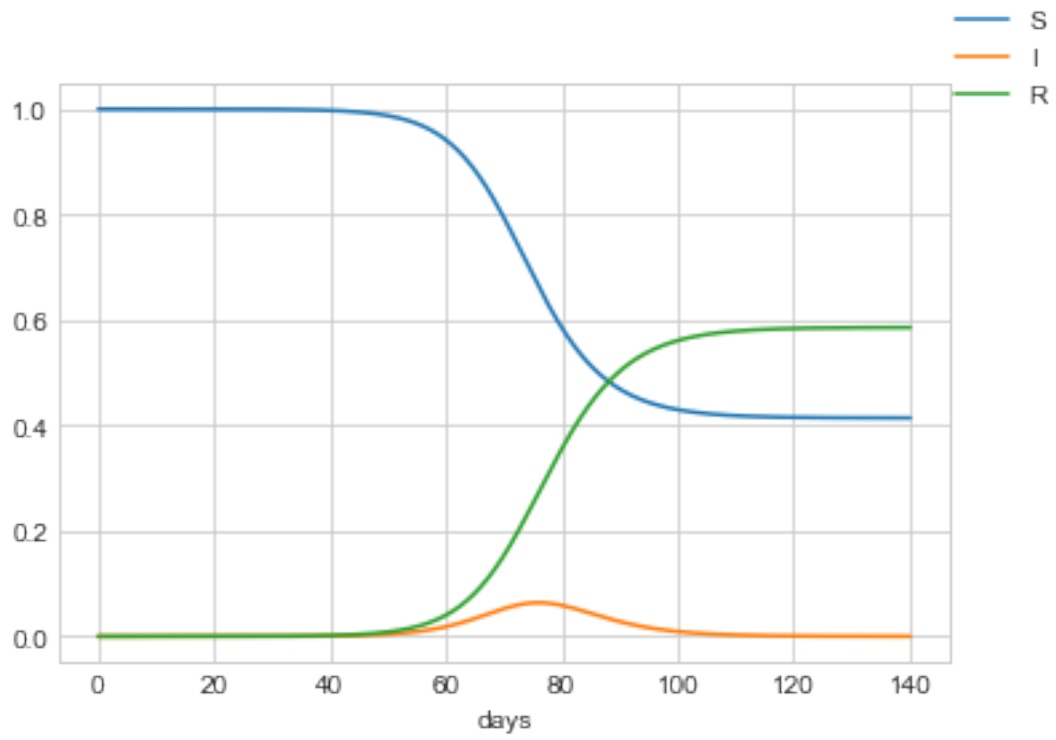
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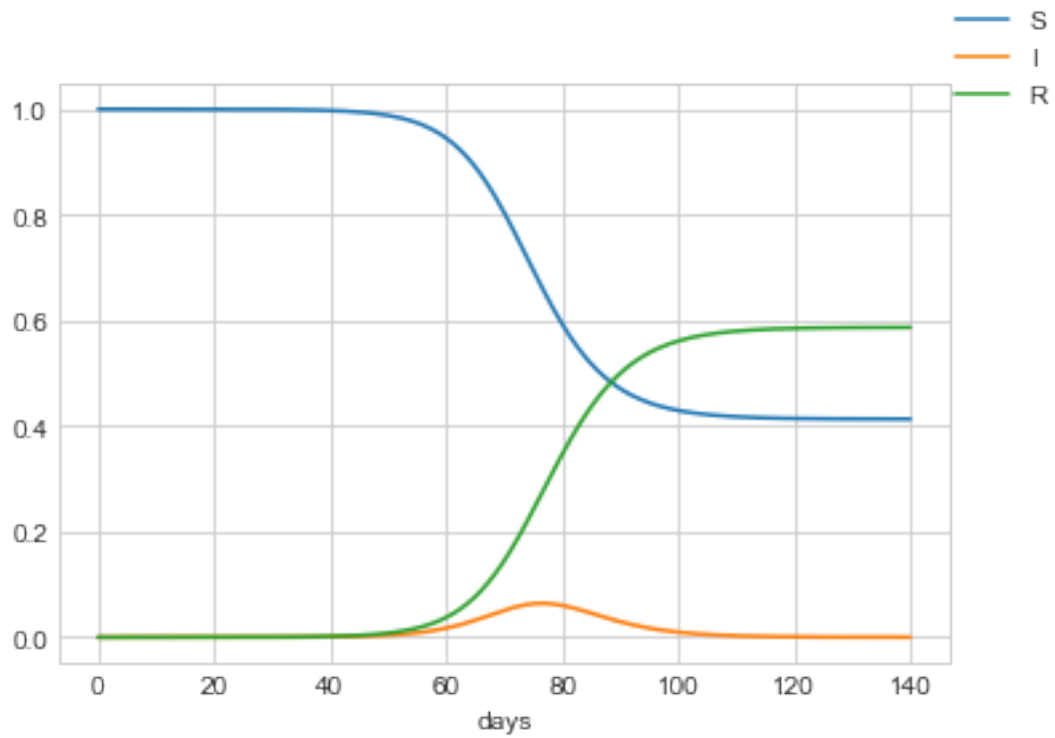
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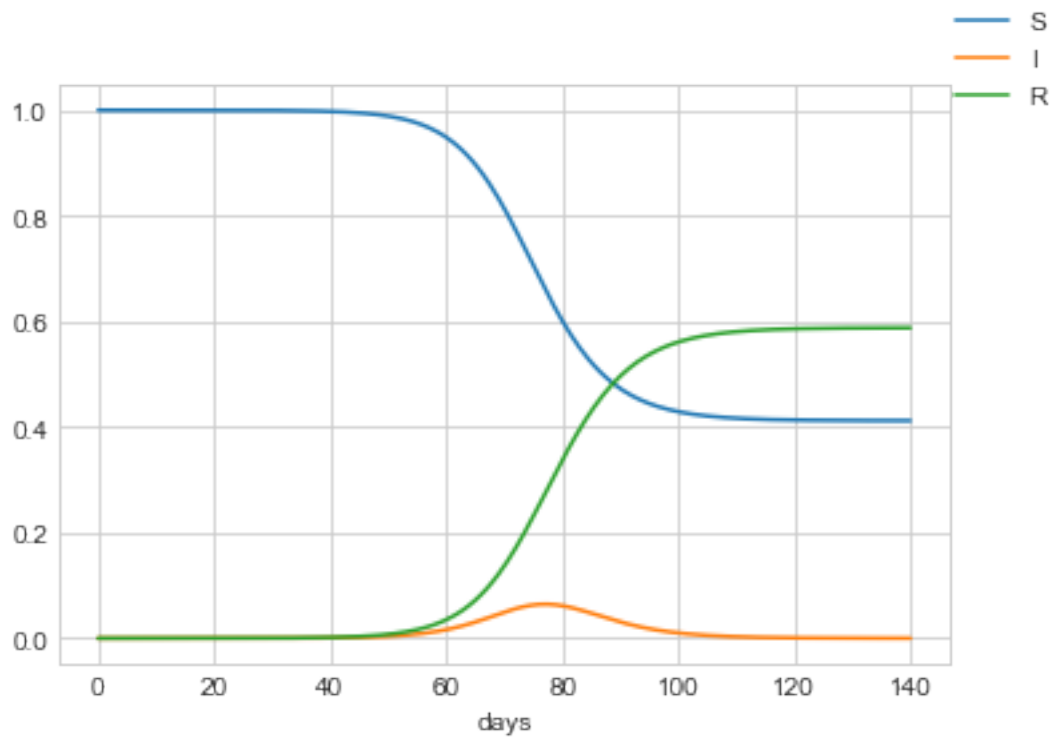
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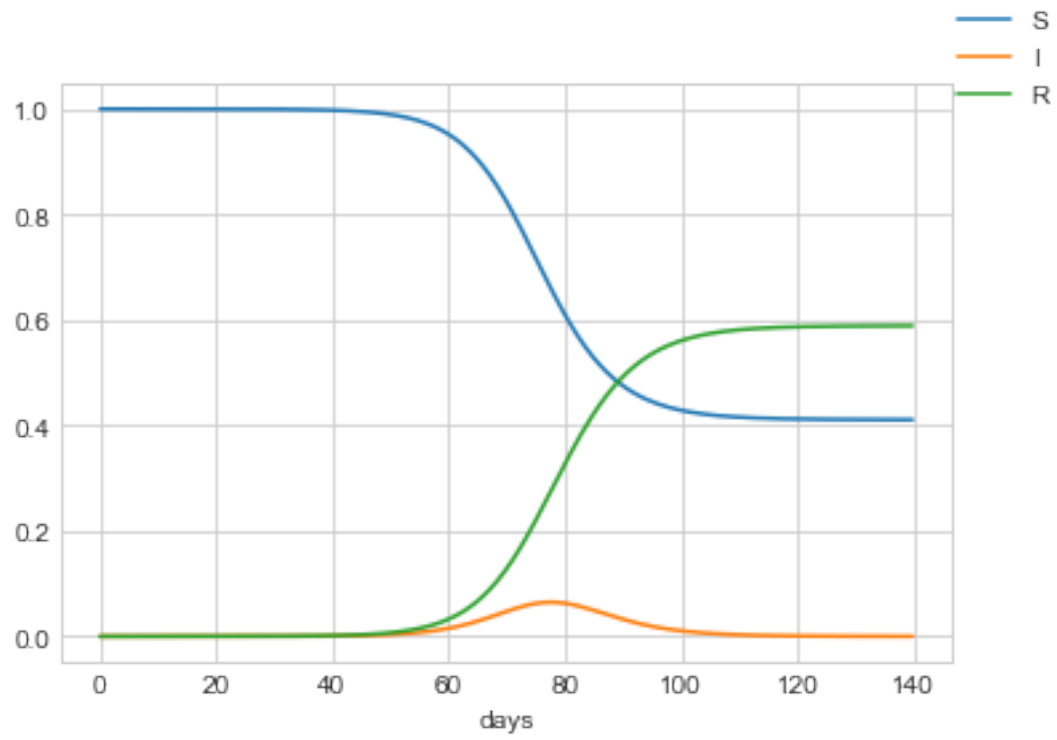
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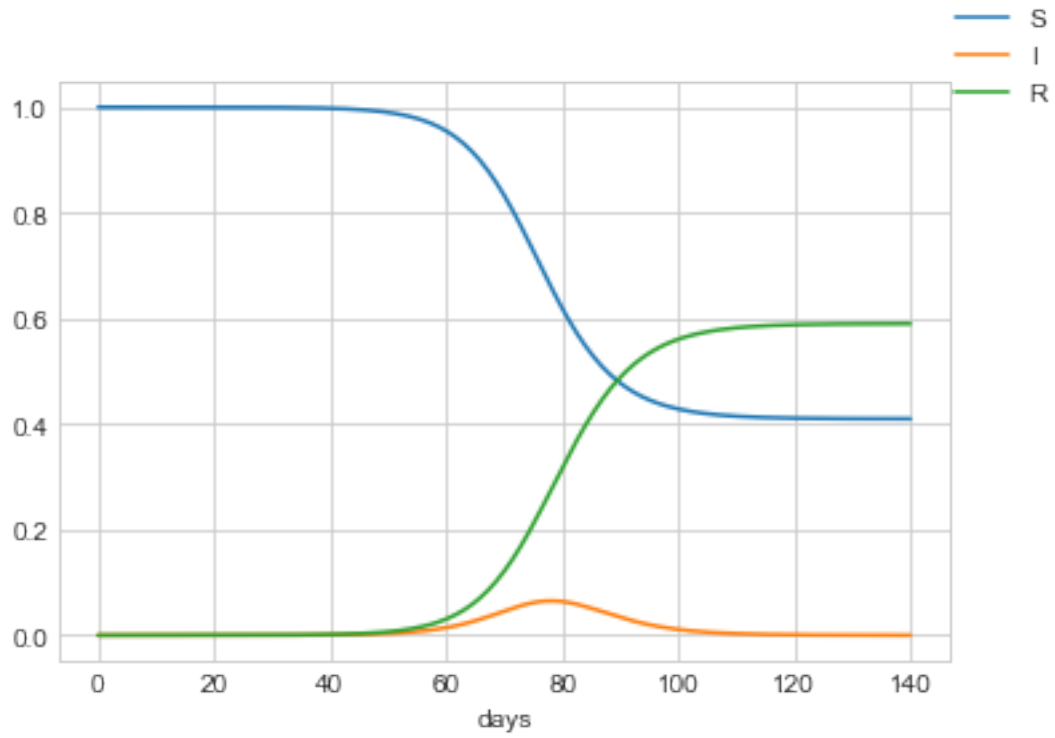
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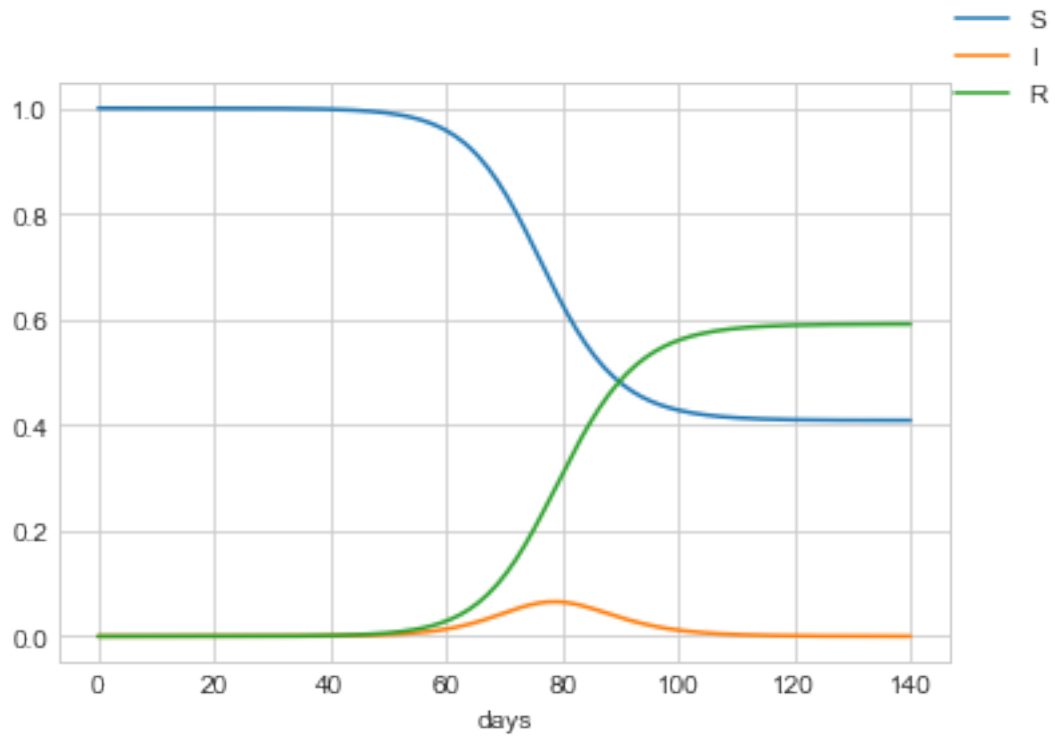
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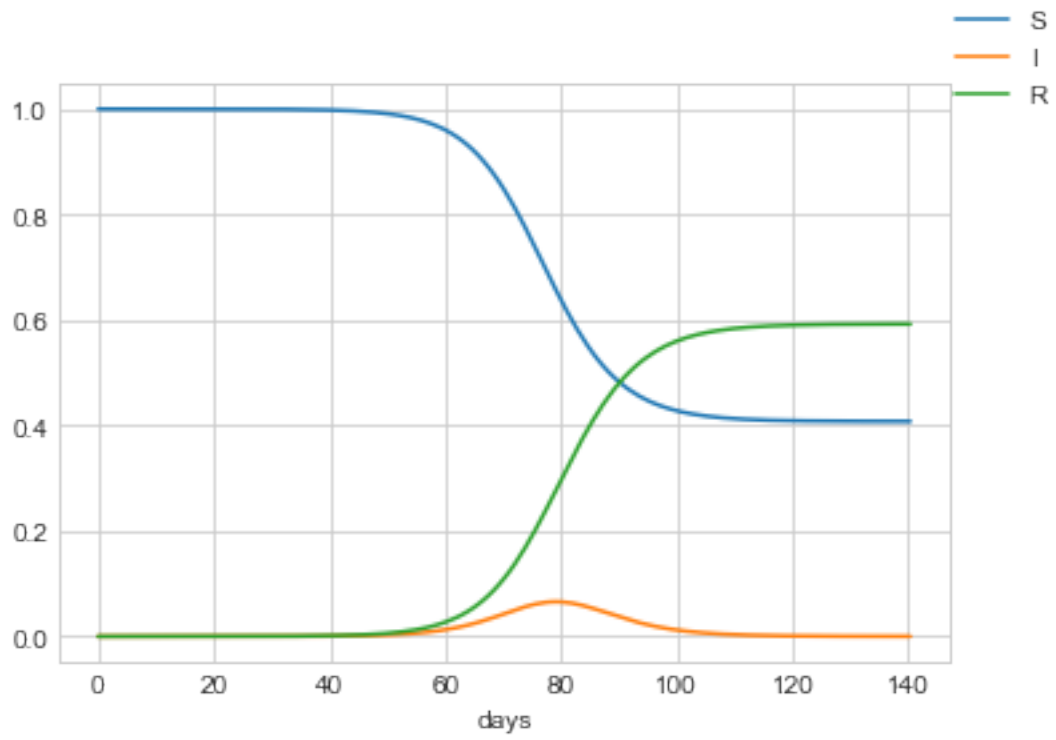
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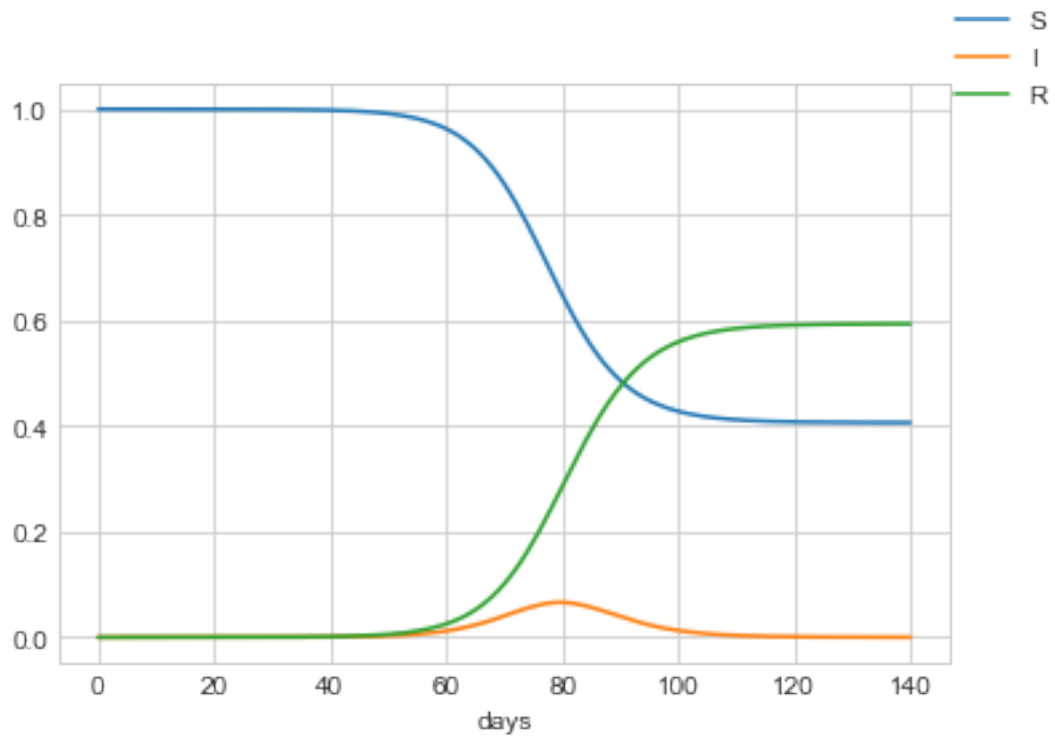
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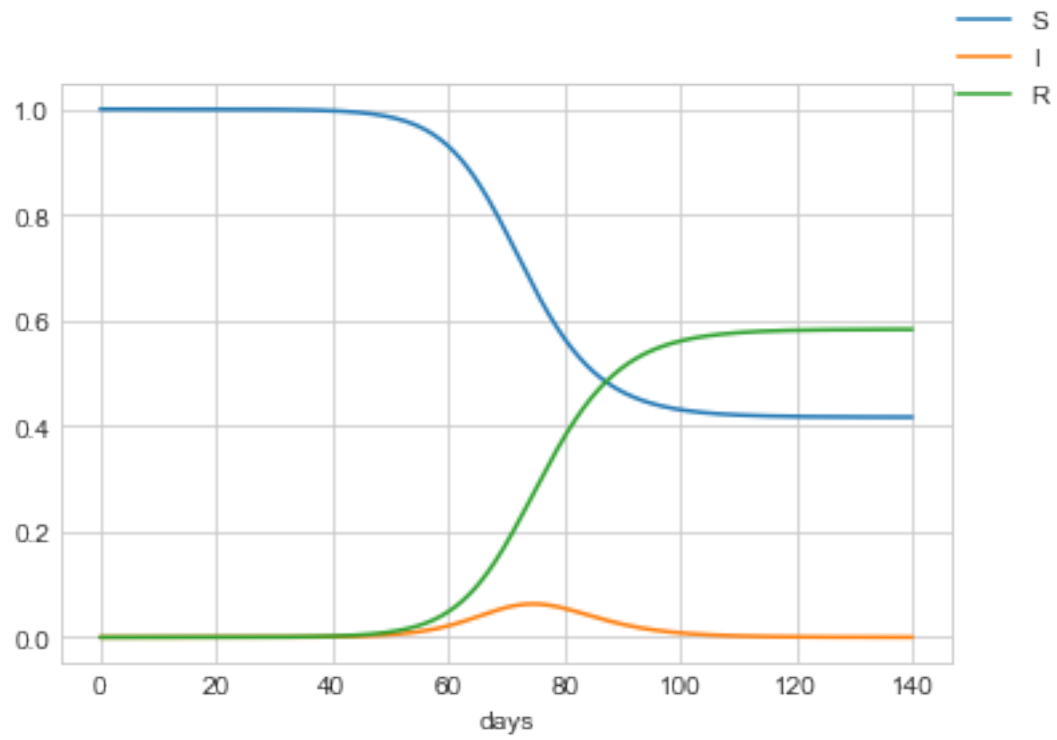


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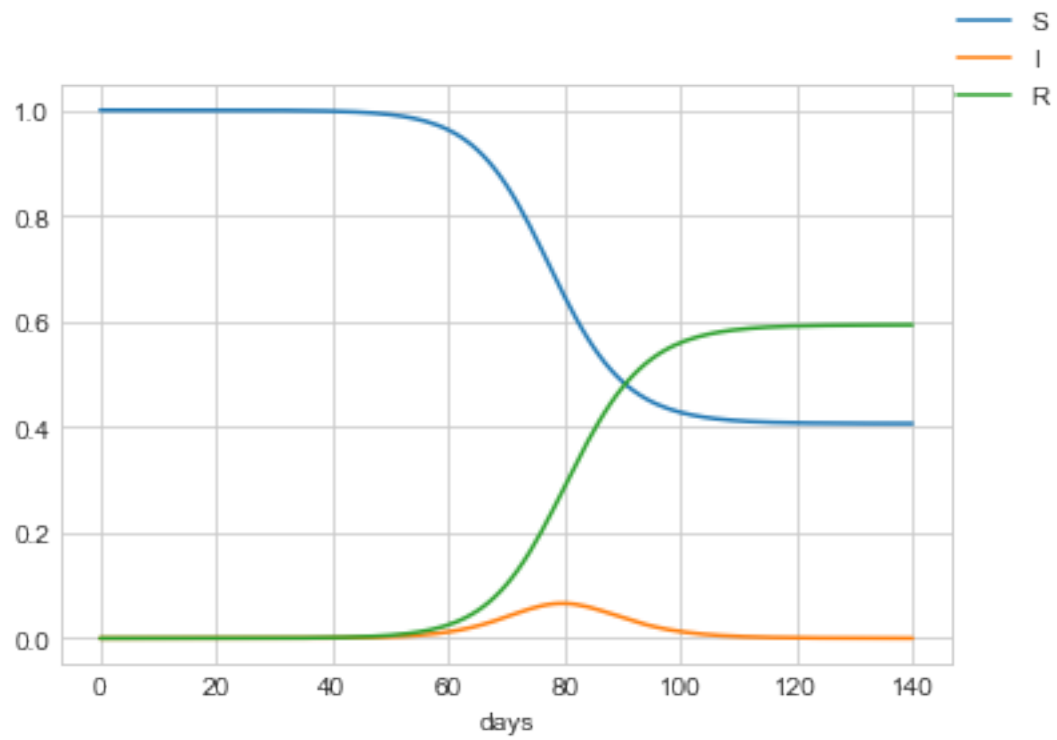


```
[8]: interesting_list = [(1/24), 1, 4.375, 8.75, 10, 17.5, 35, 70, 140]
for item in interesting_list:
    print(item)
    SIR_Demo(item, forwardEuler)
```

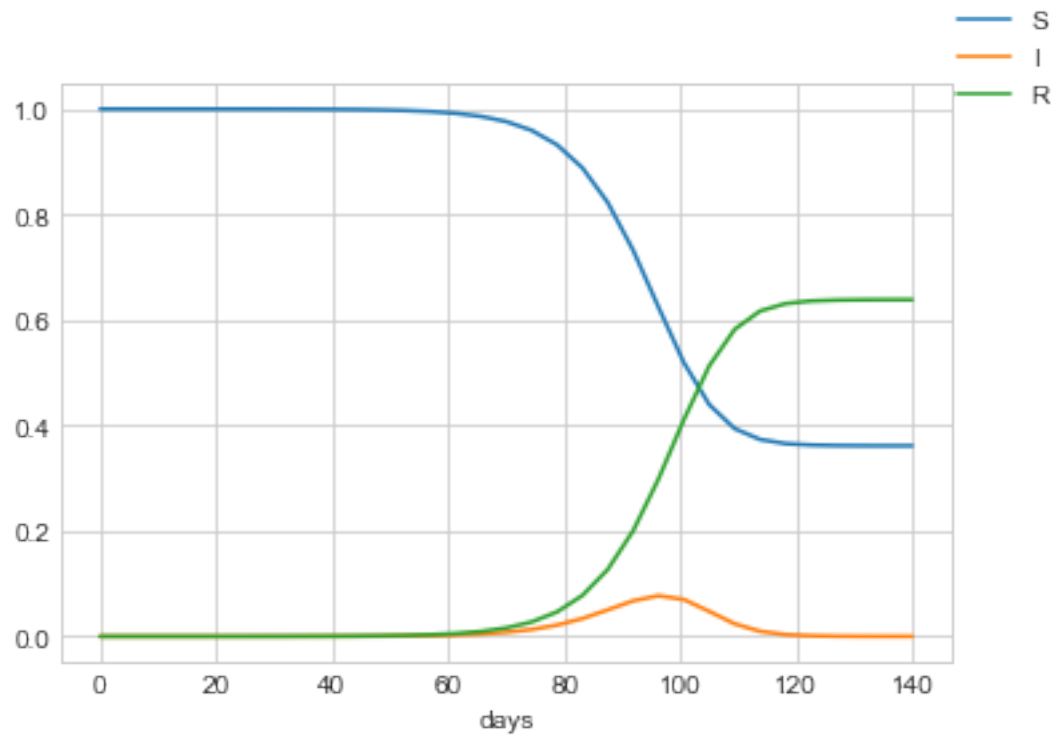
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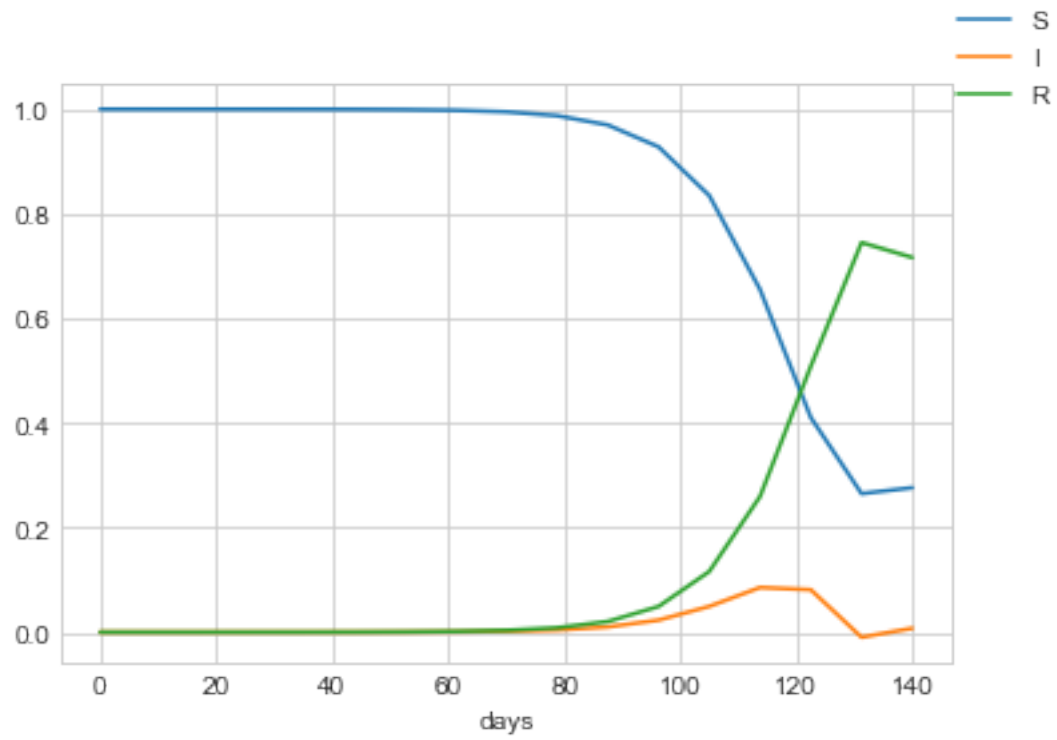
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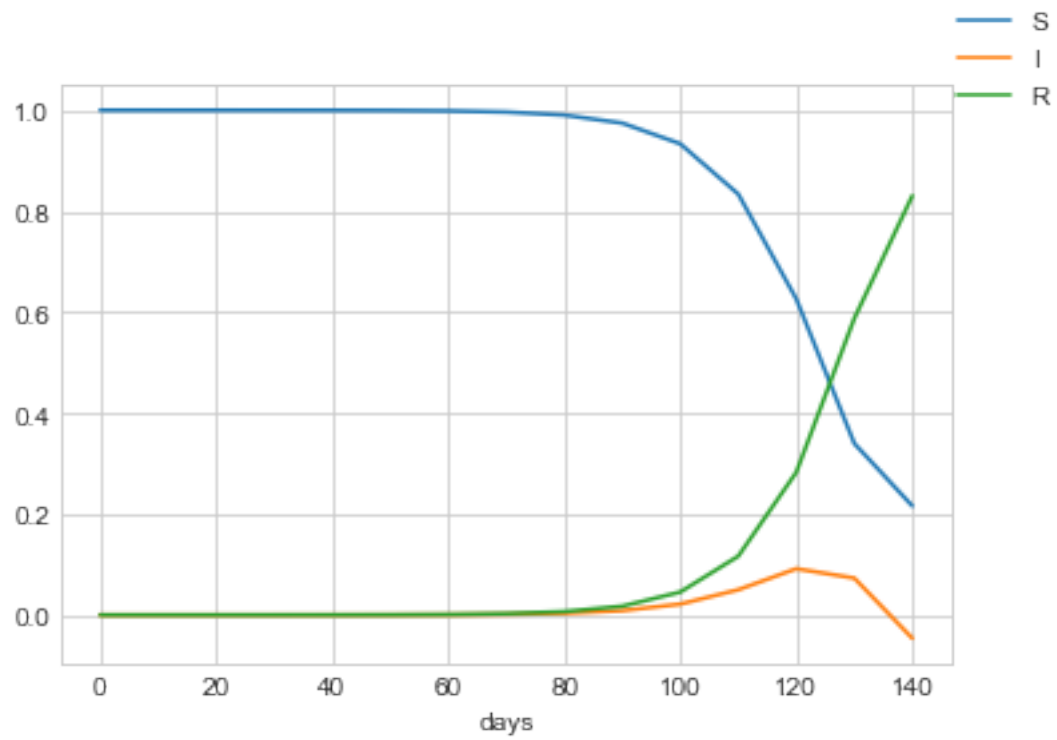
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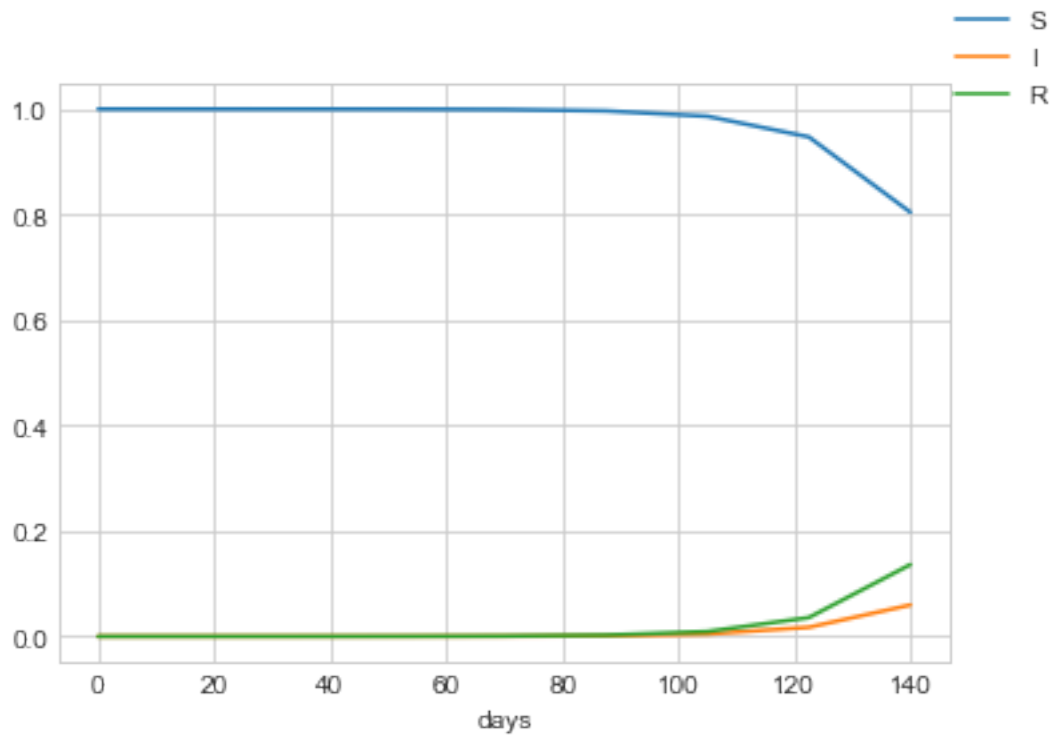
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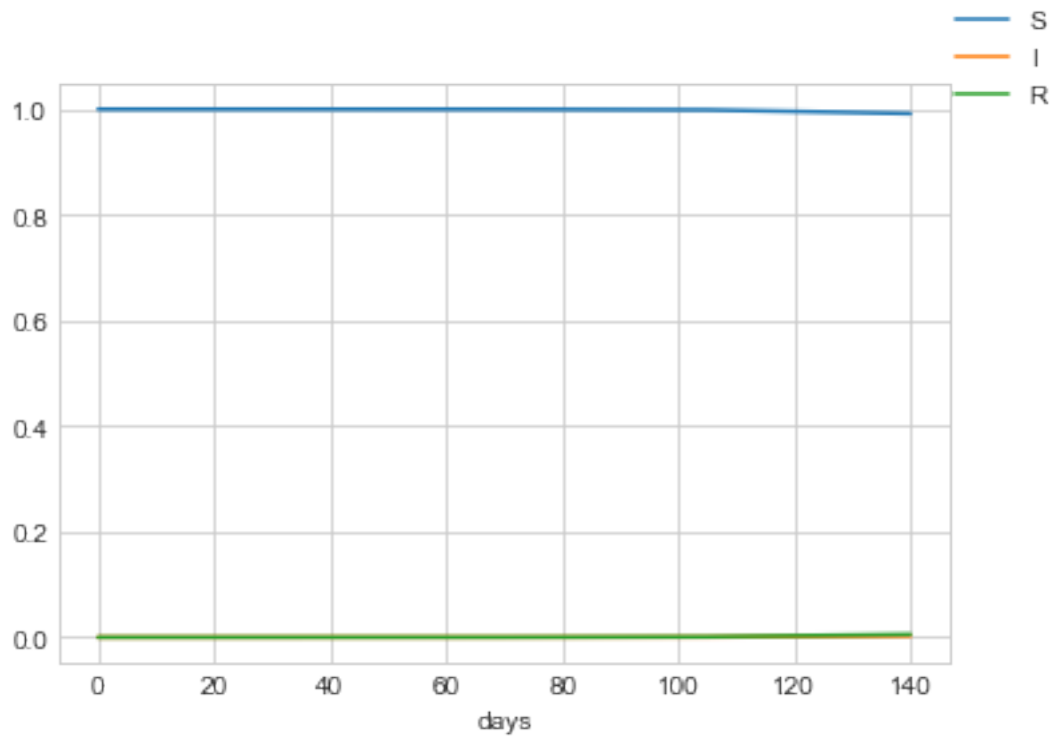
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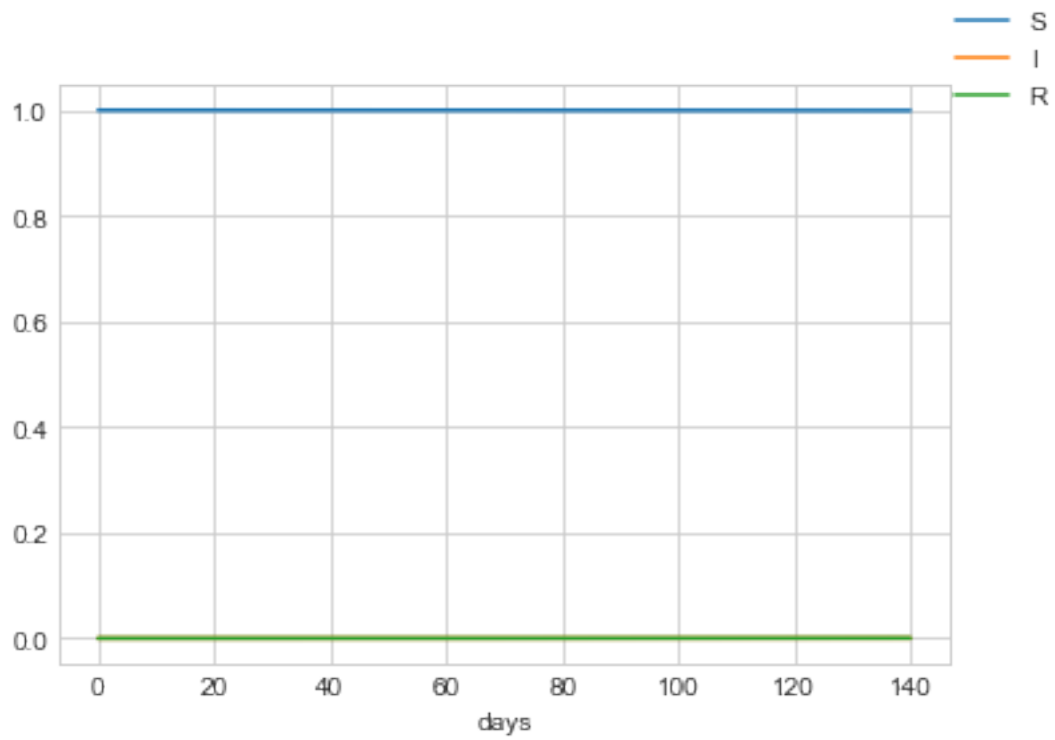
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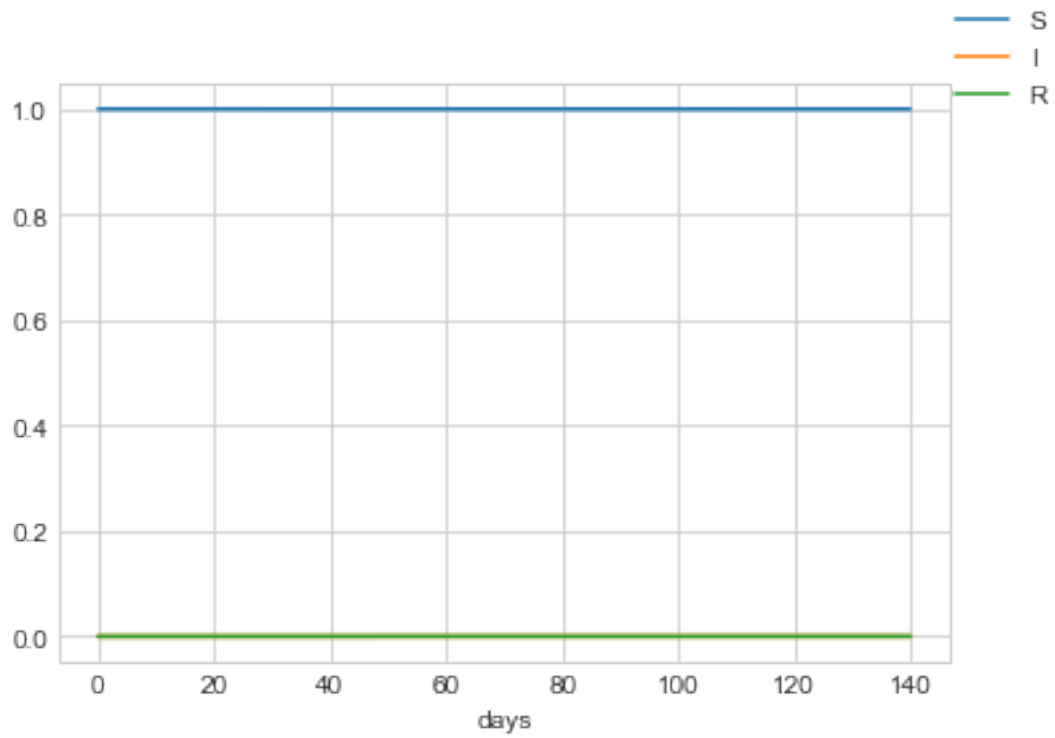
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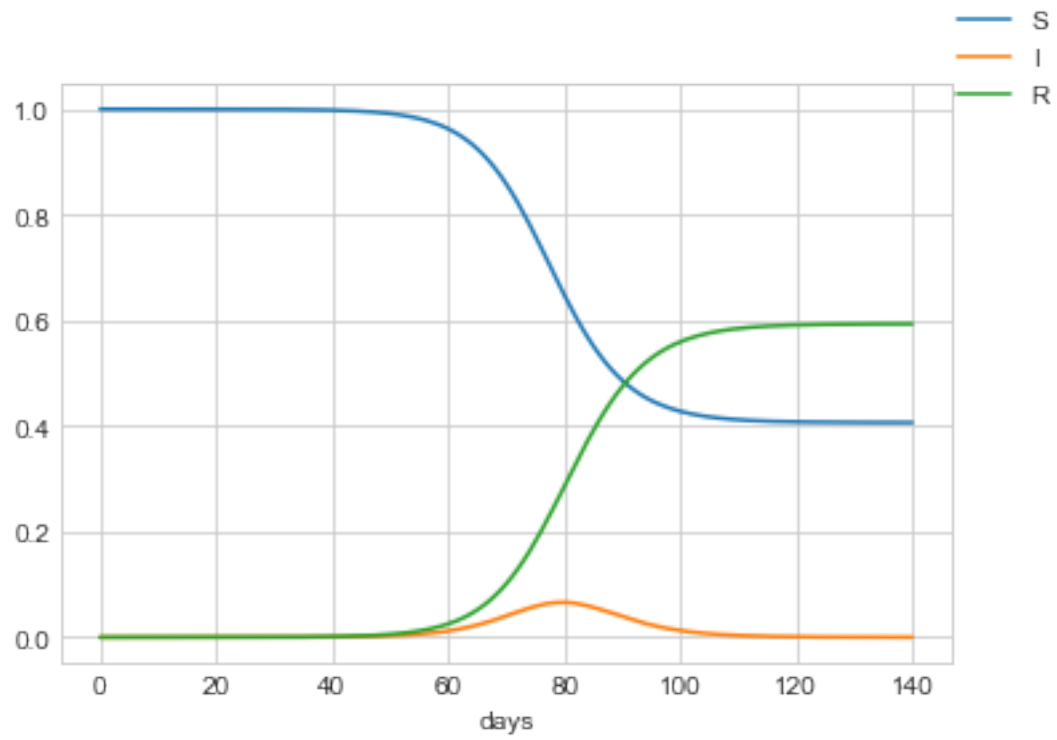


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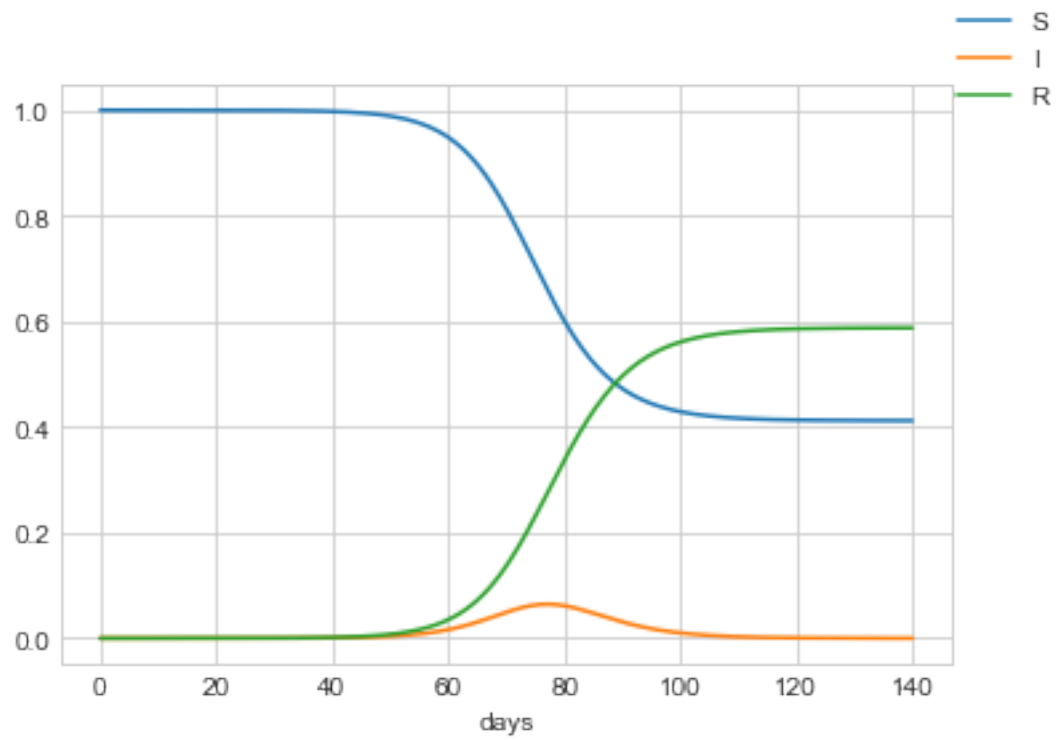


```
[9]: x = 1
     for i in range(15):
         x = (2**(-i))
         print(x)
         SIR_Demo(x, forwardEuler)
```

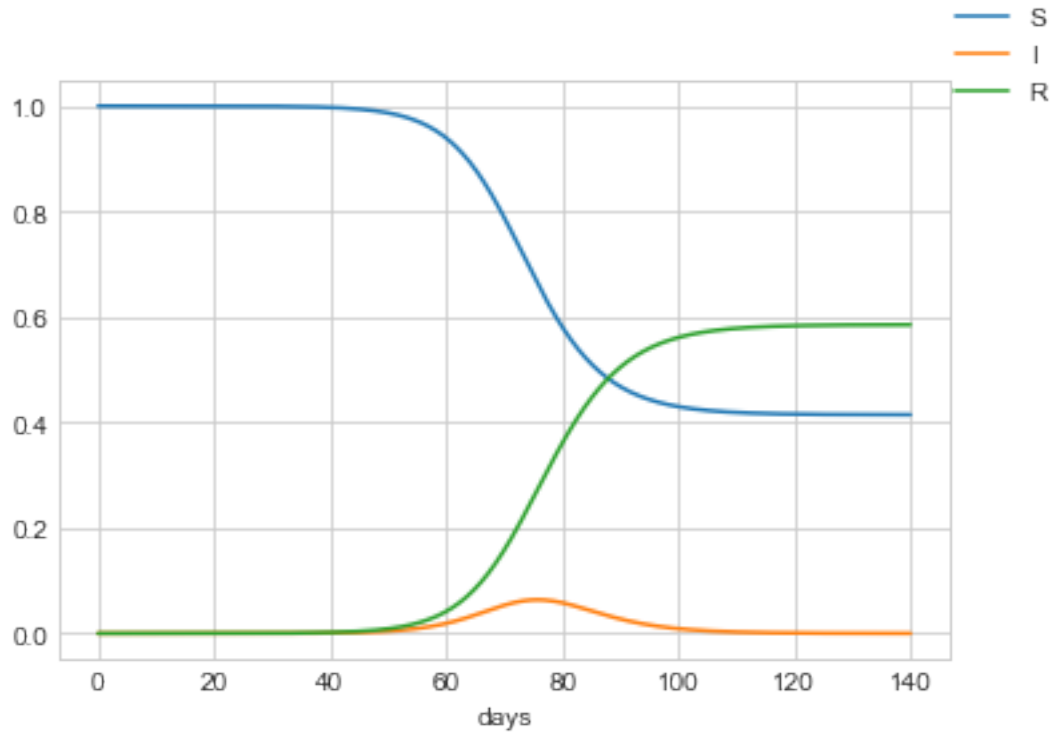
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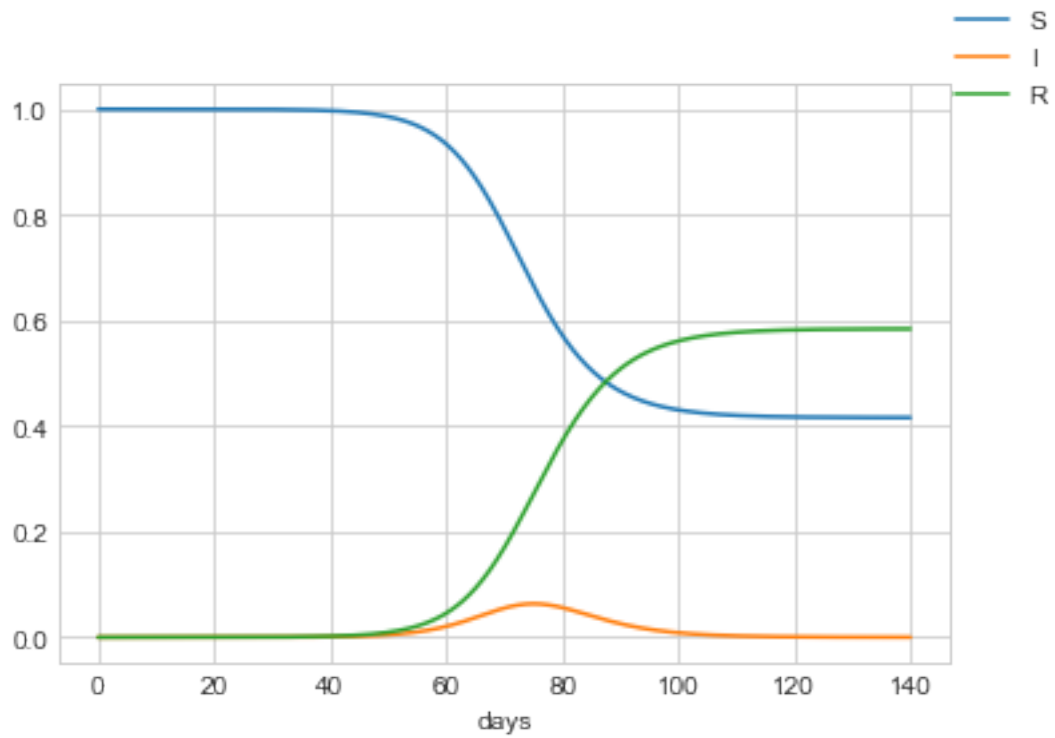
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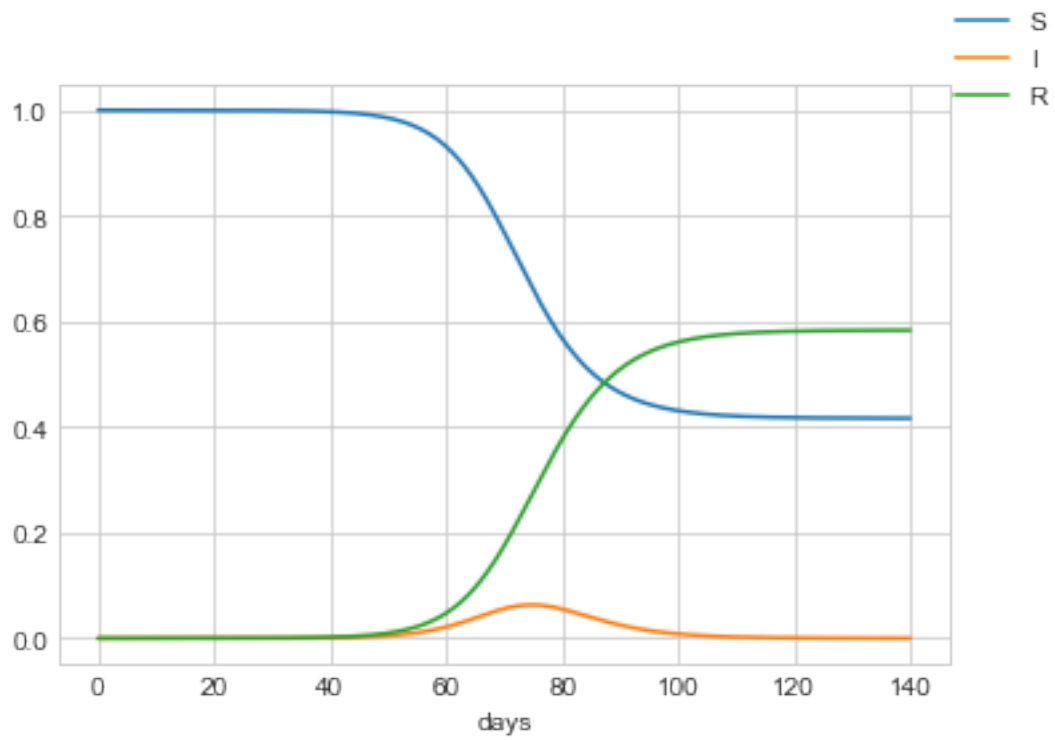
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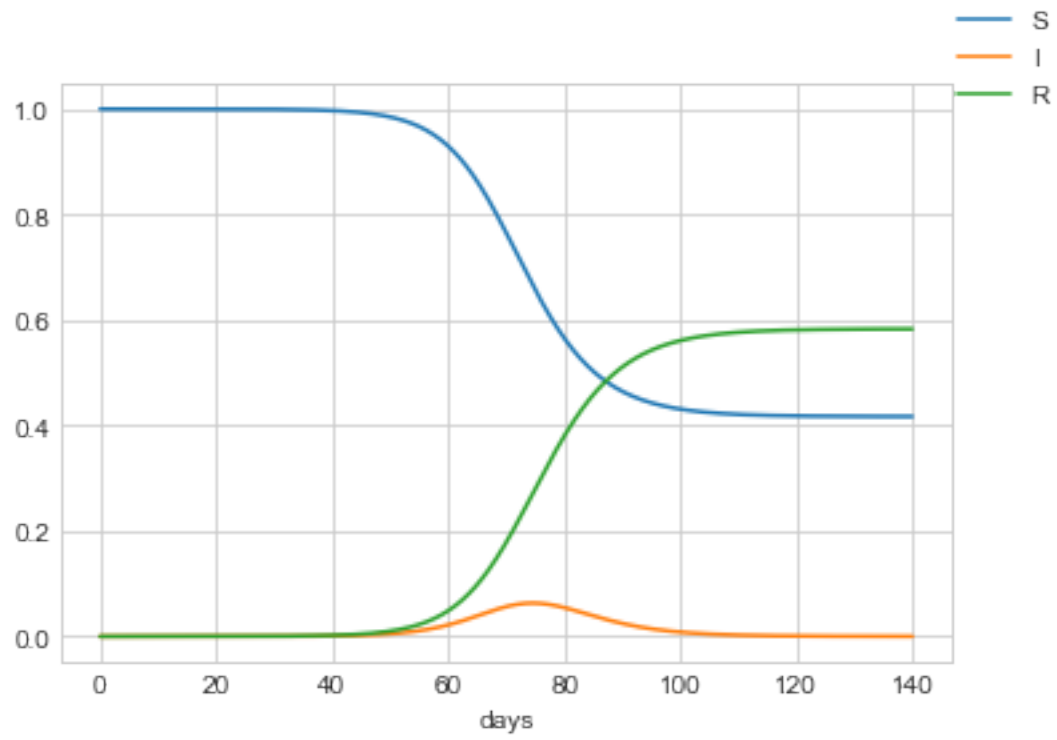
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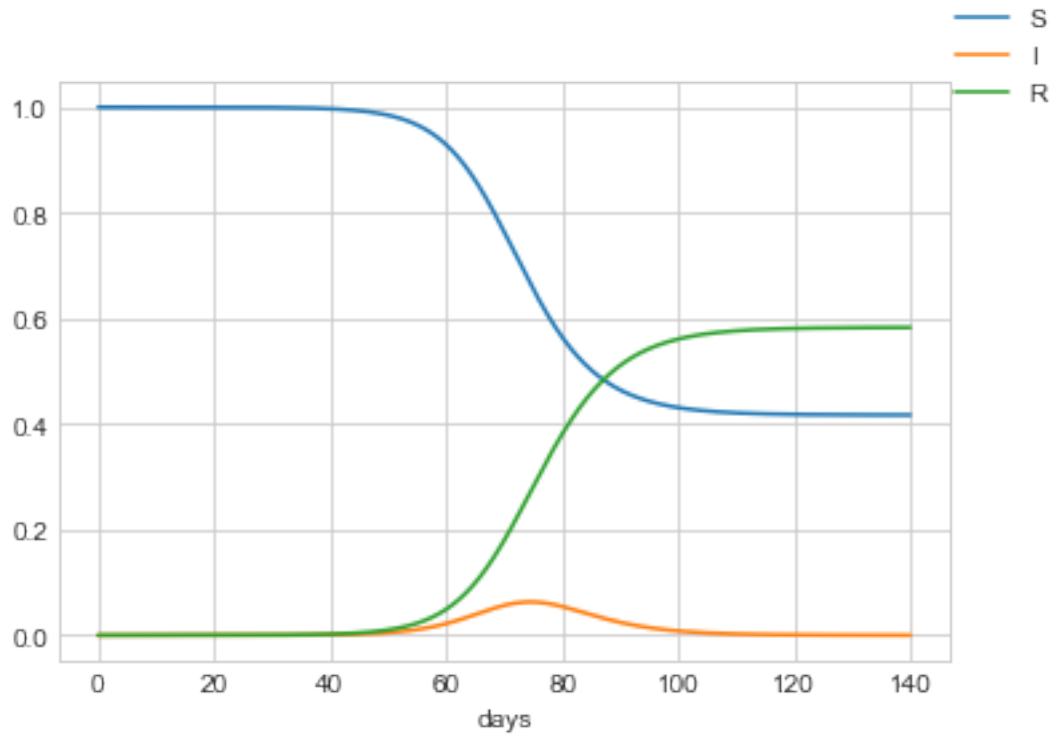
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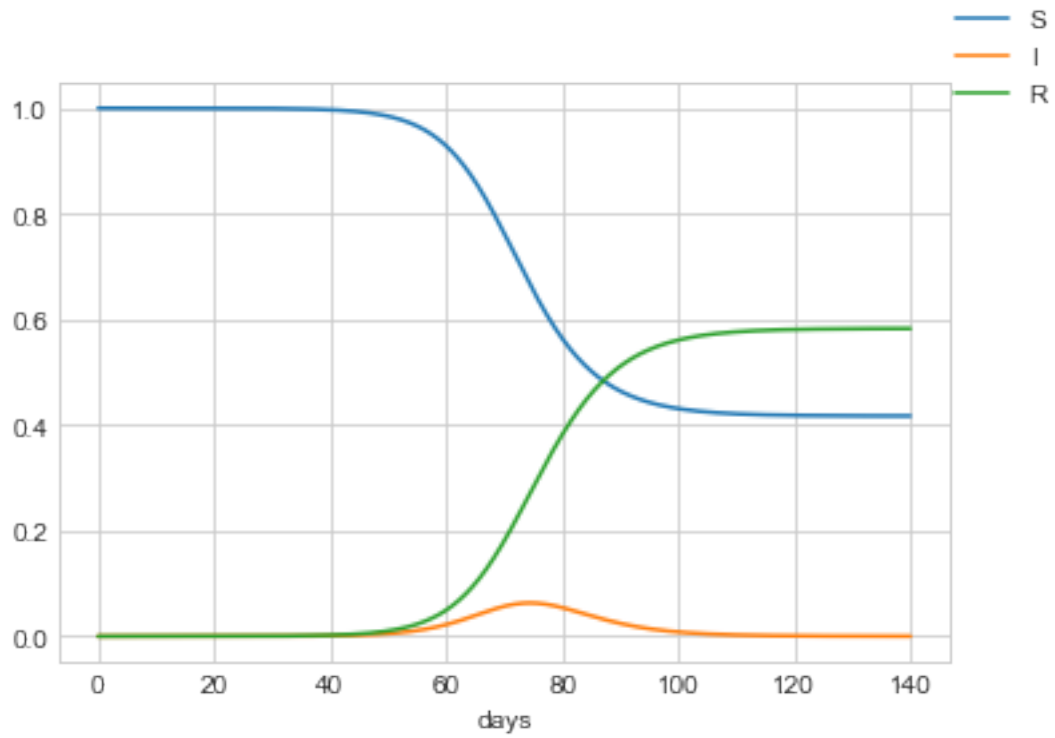
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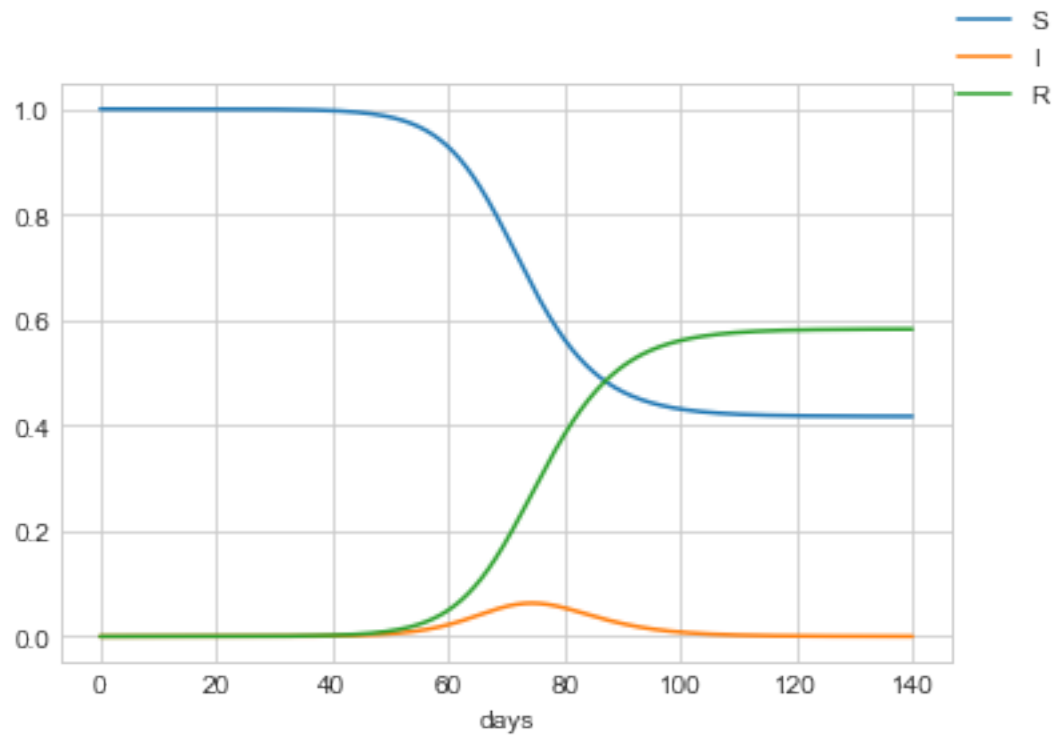
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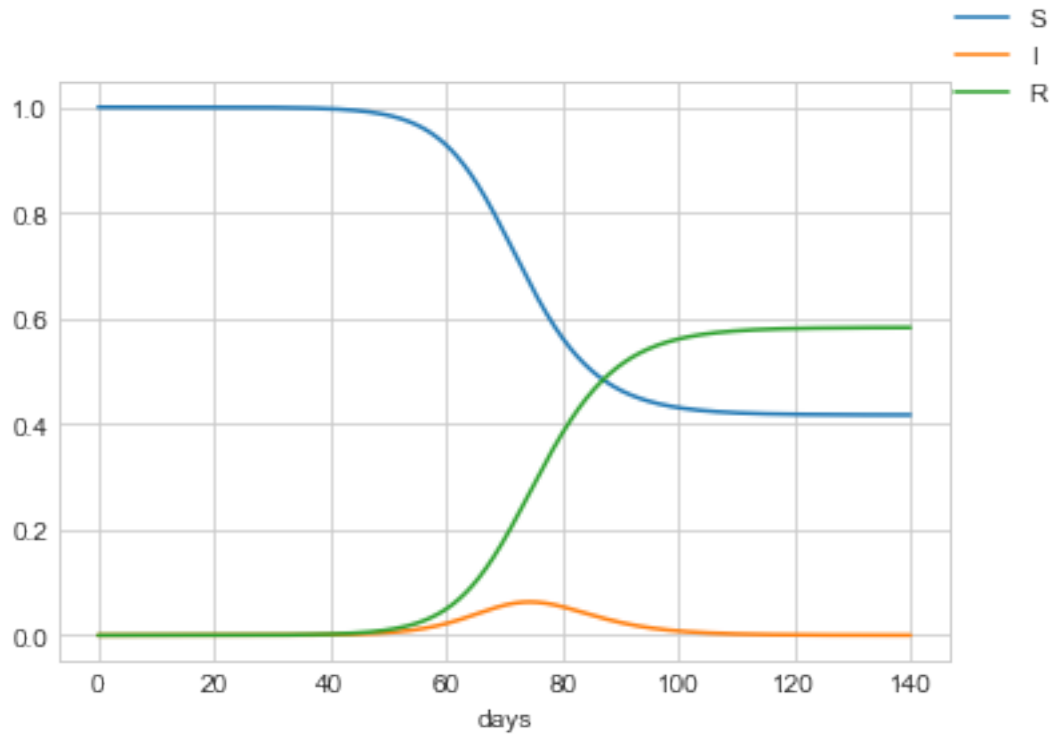
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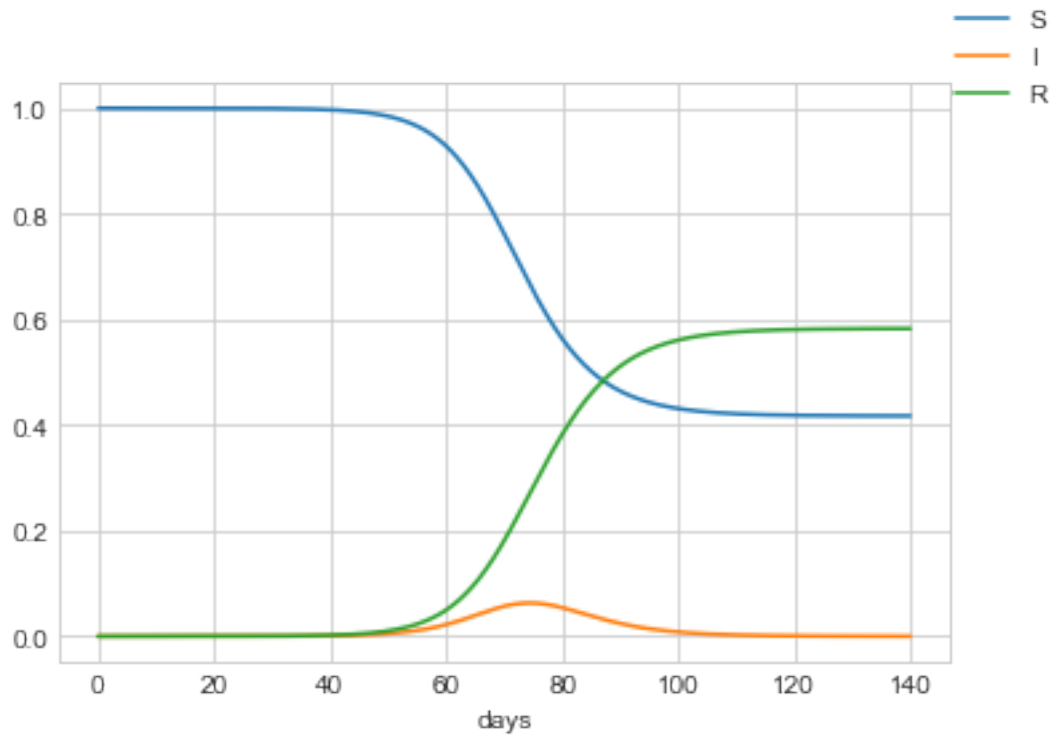
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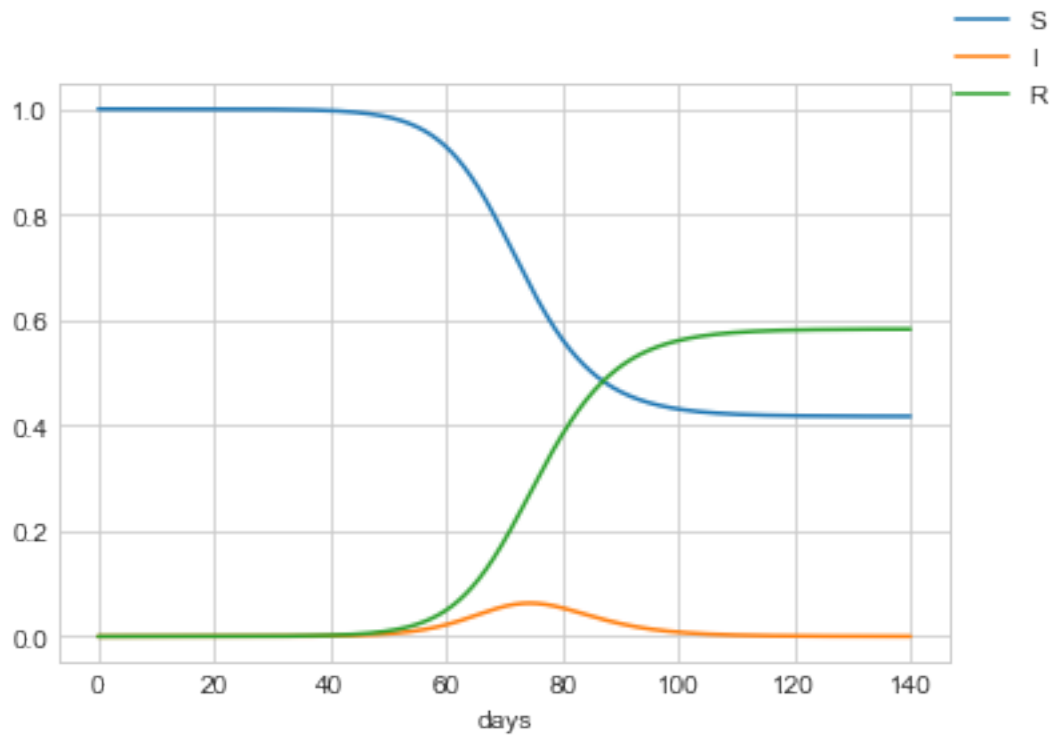
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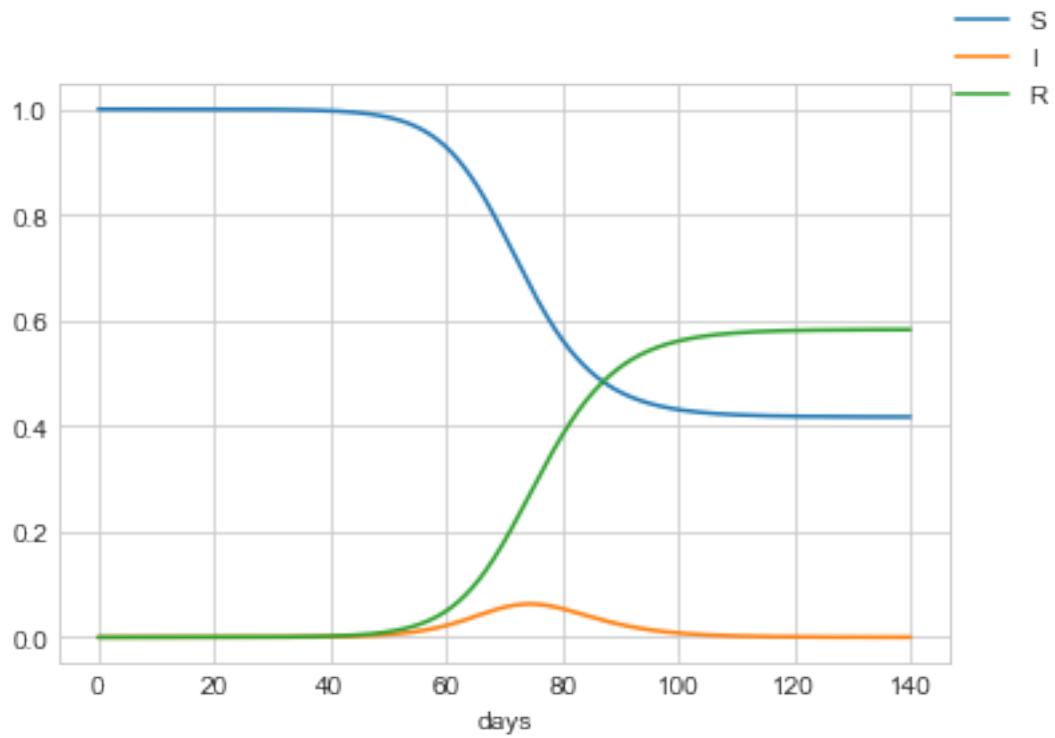
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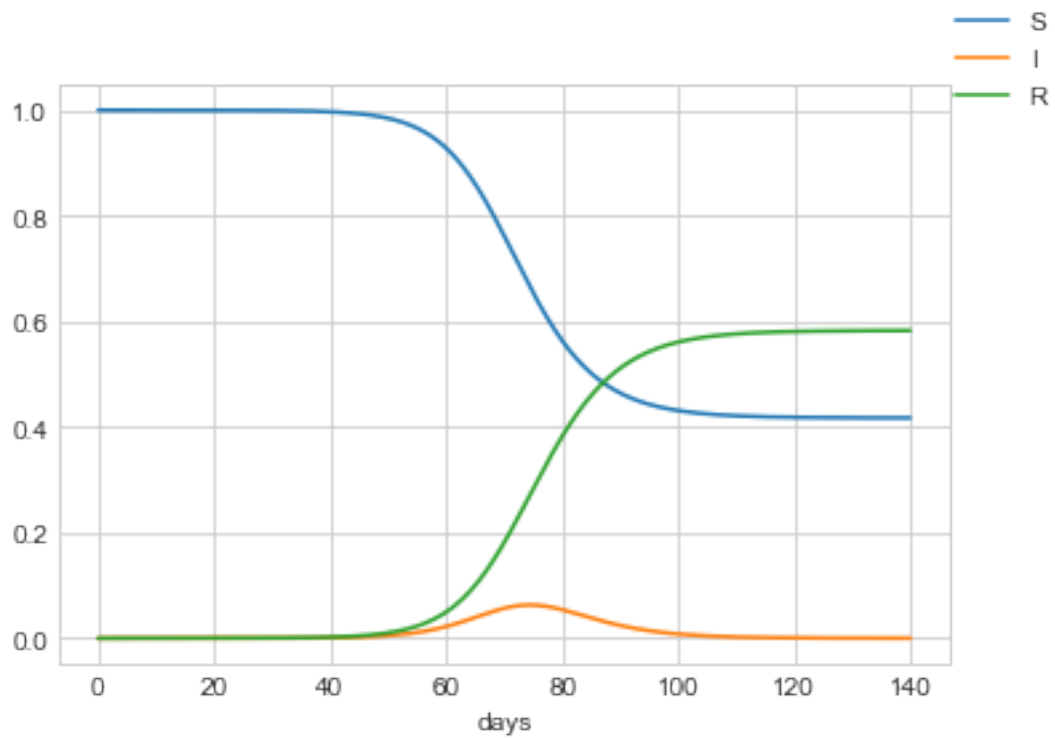
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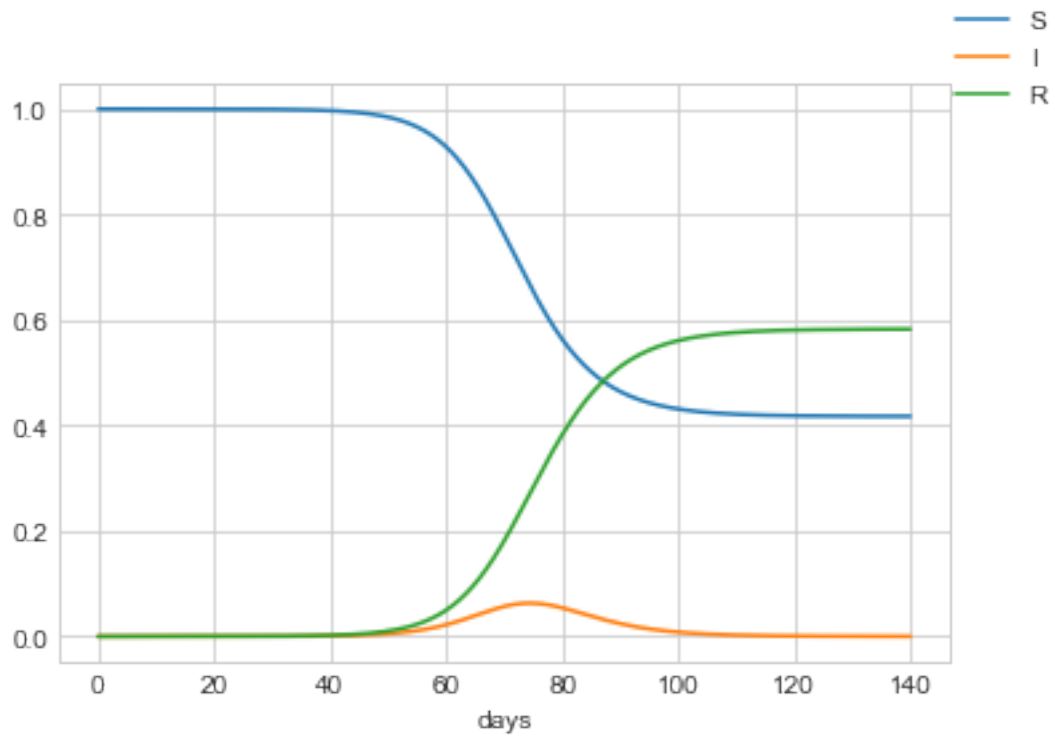
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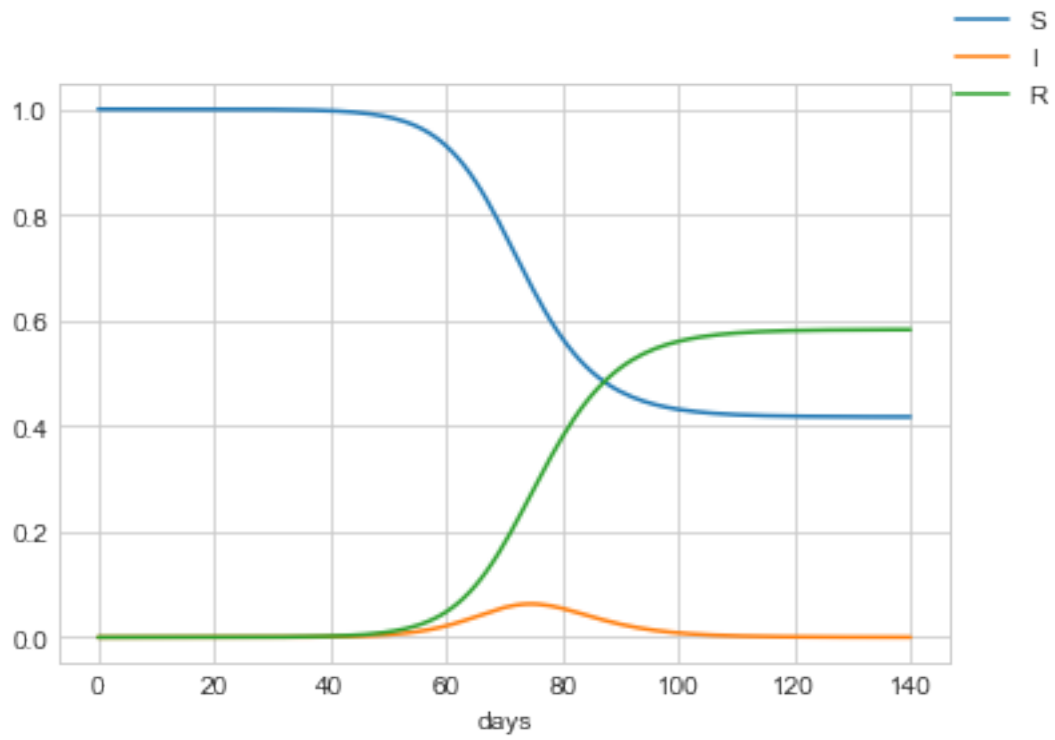


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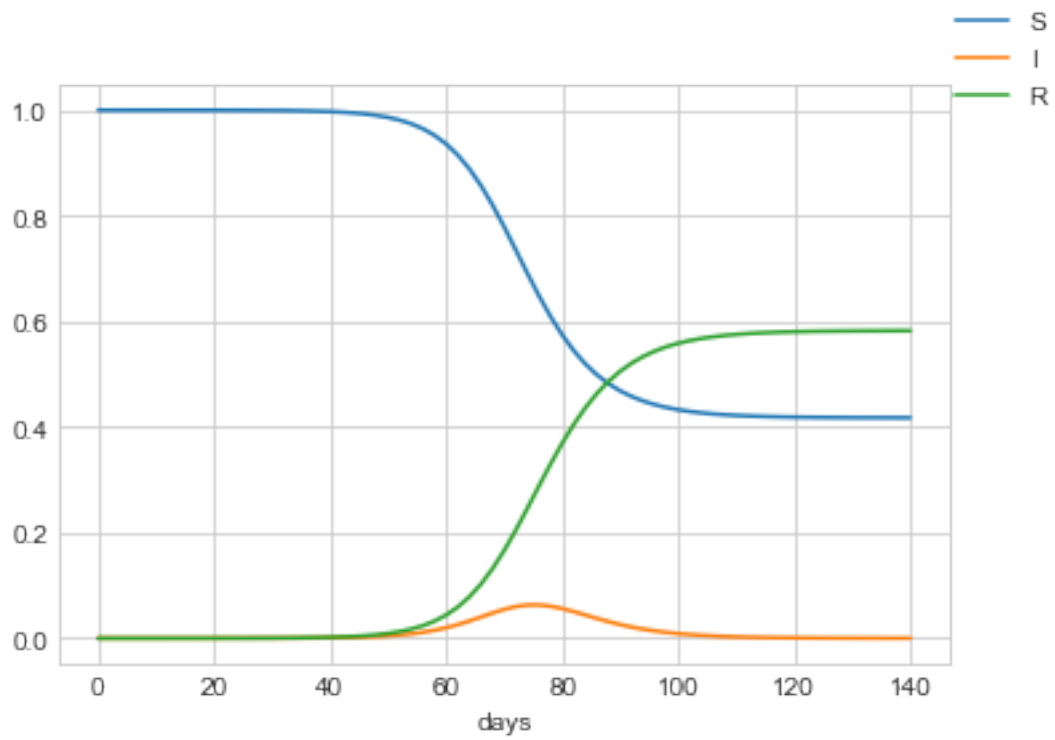


```
[10]: for i in range(1, 11):  
       print(i)  
       SIR_Demo(i, rungeKutta2)
```

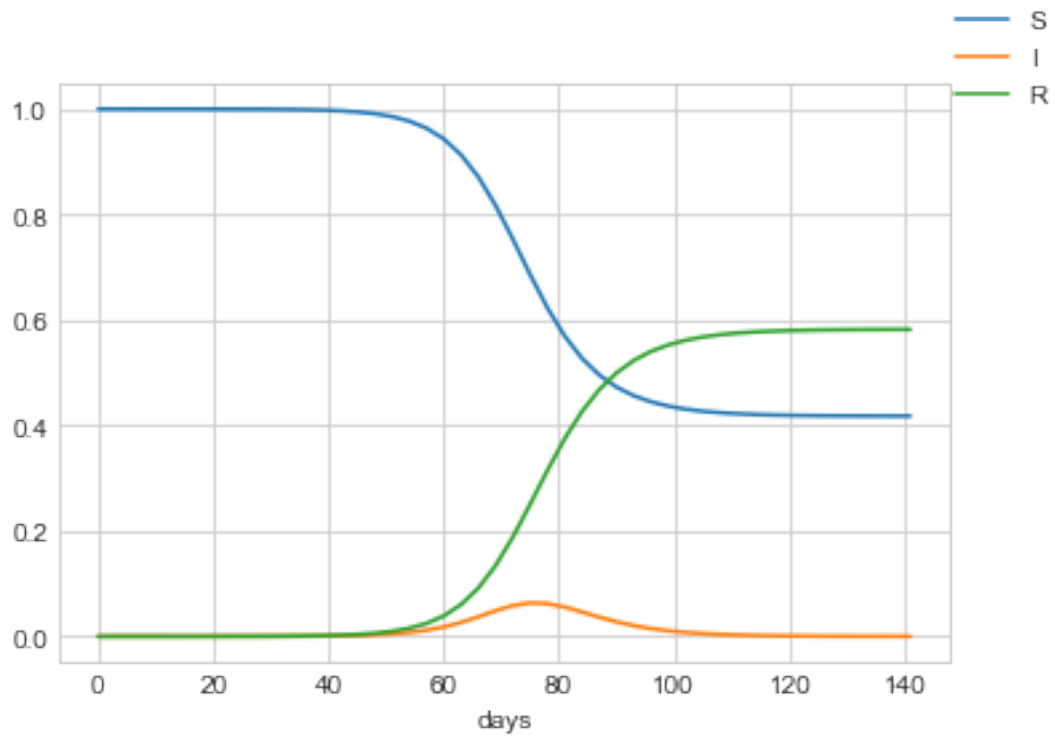
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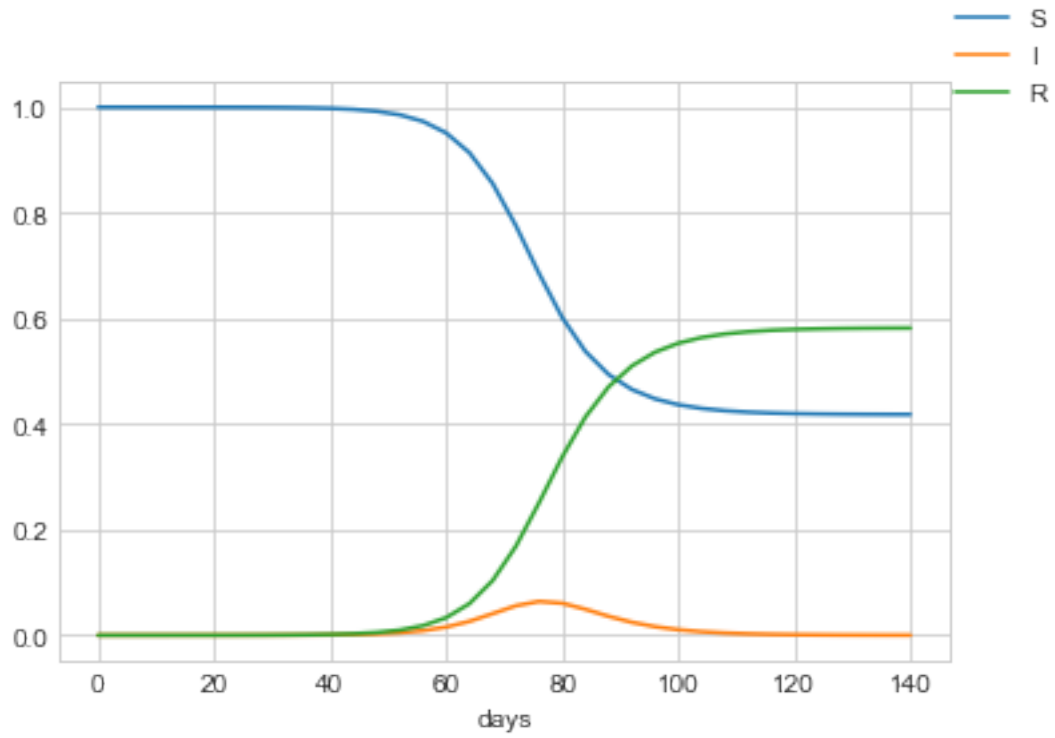
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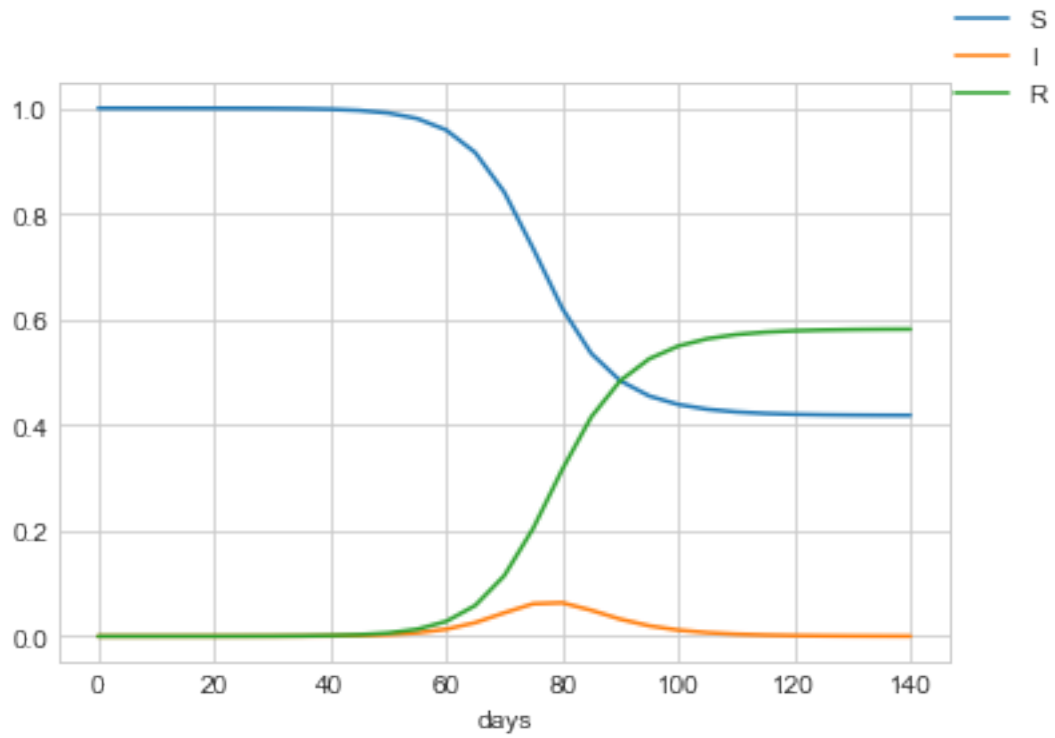
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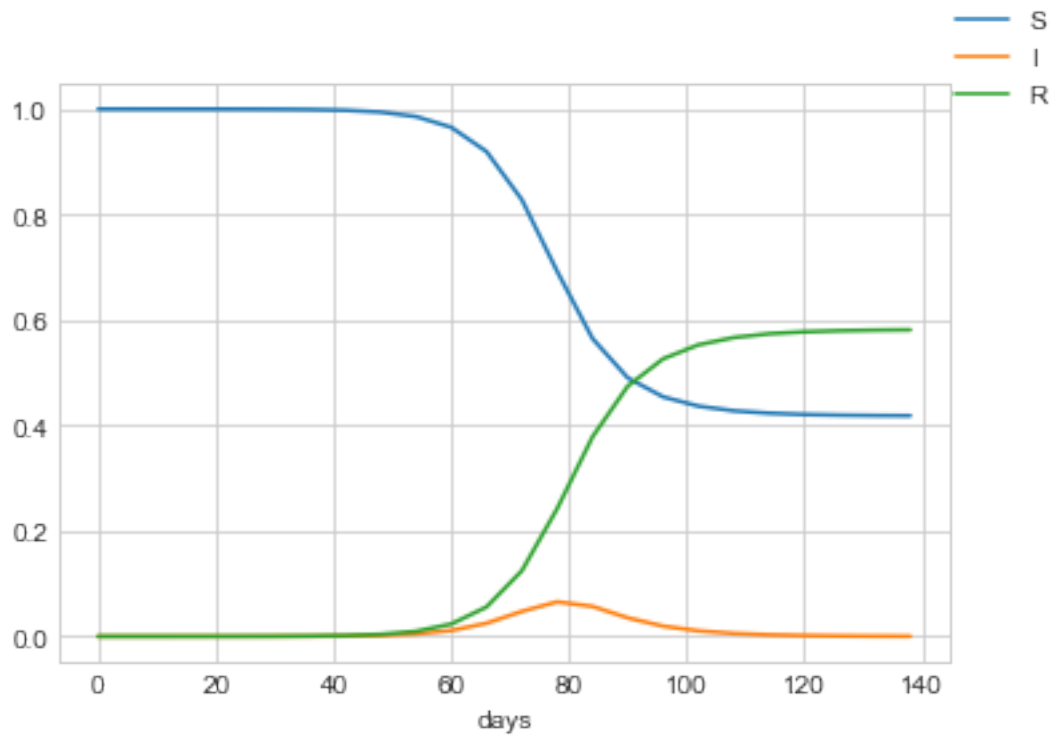
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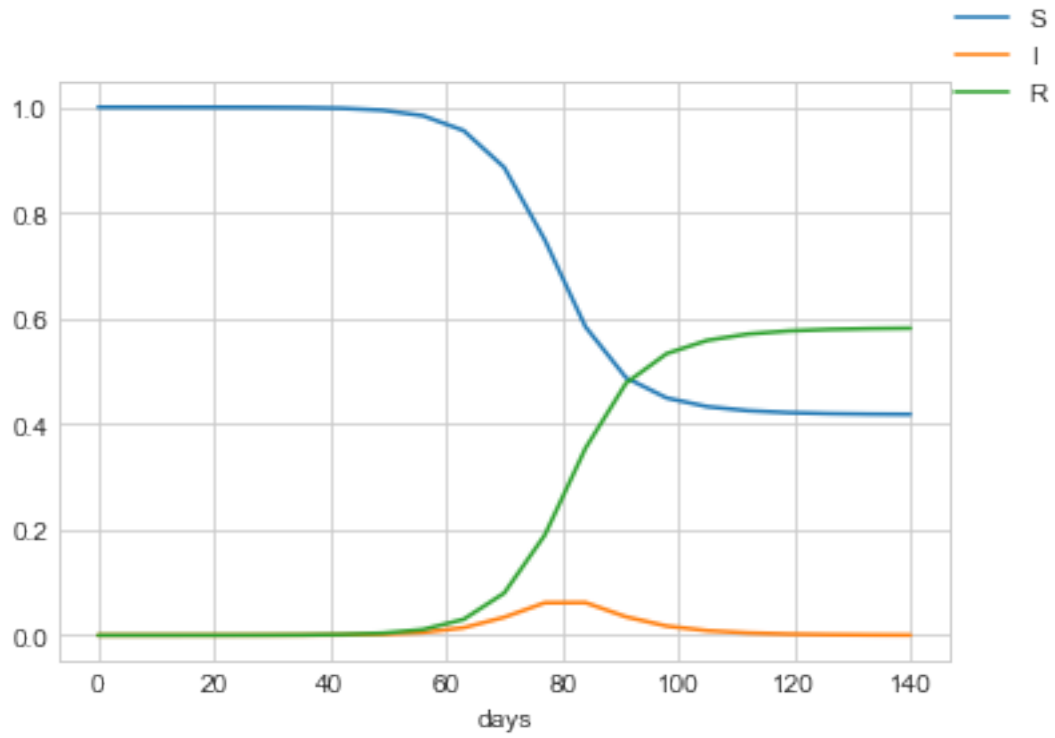
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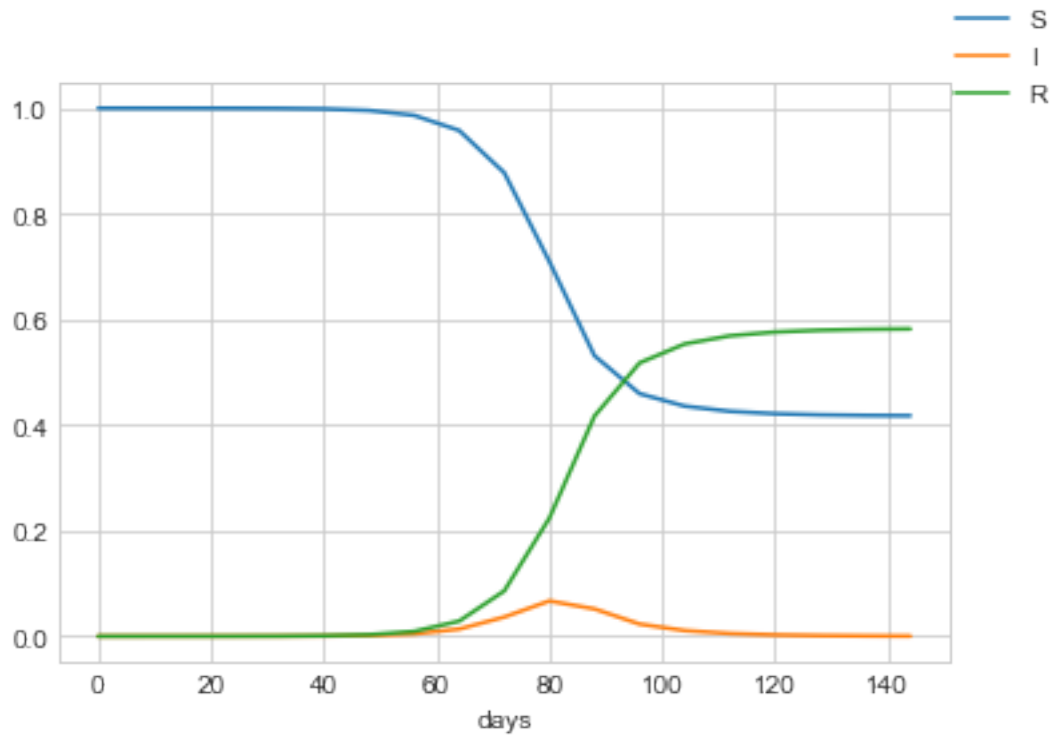
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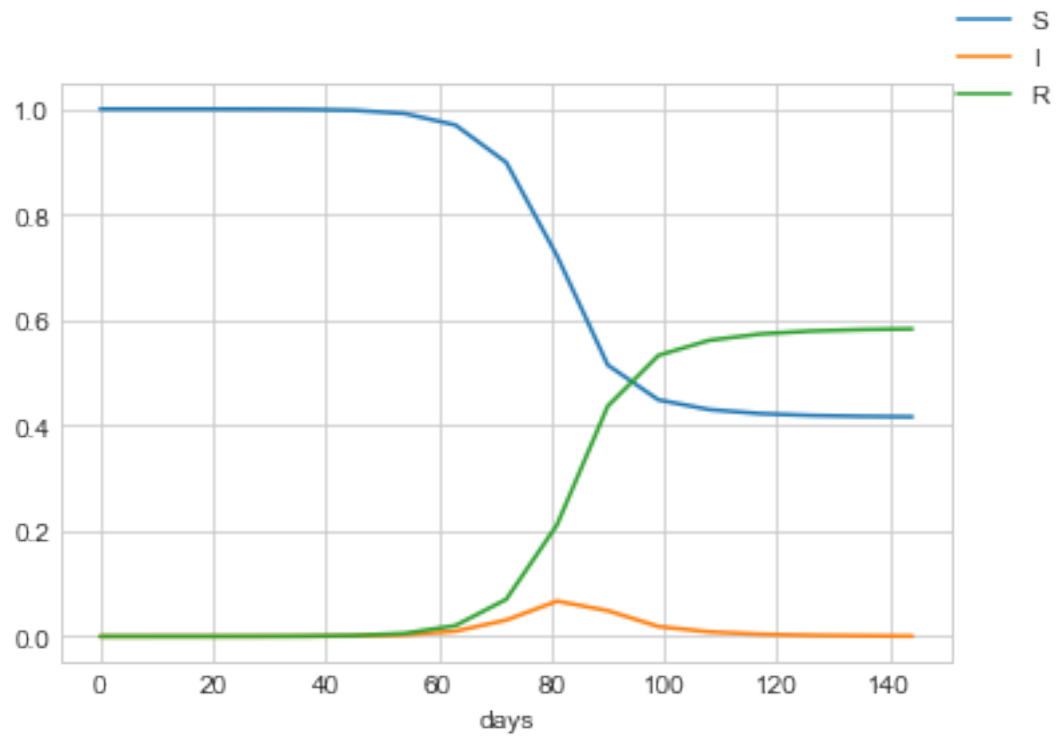
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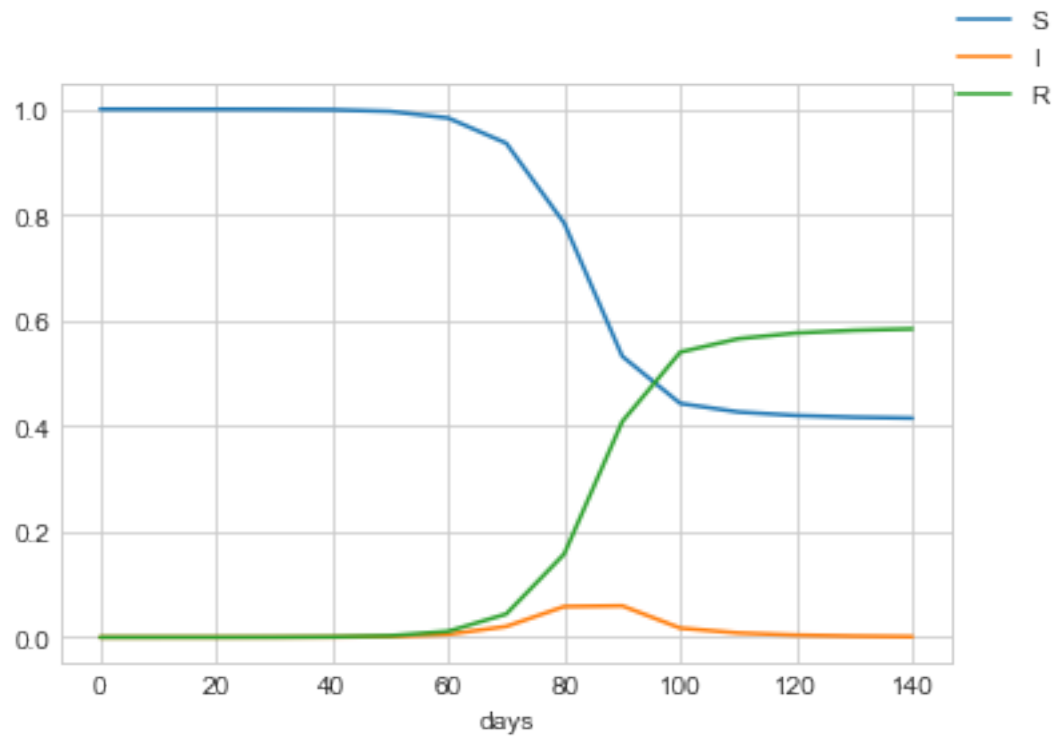
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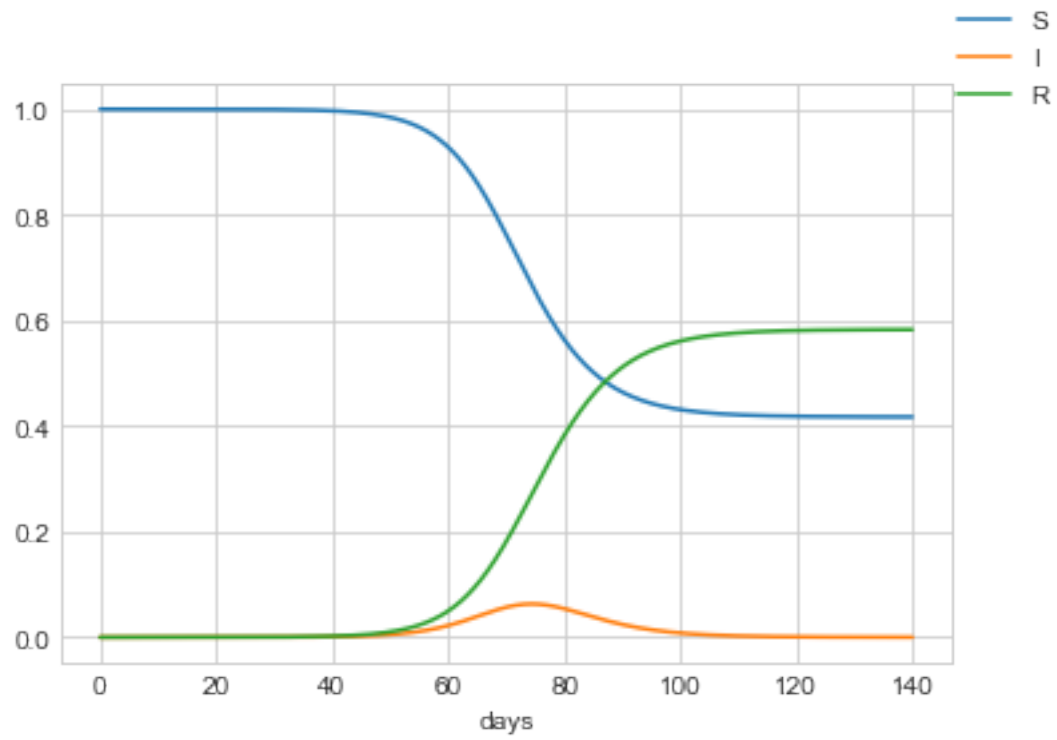


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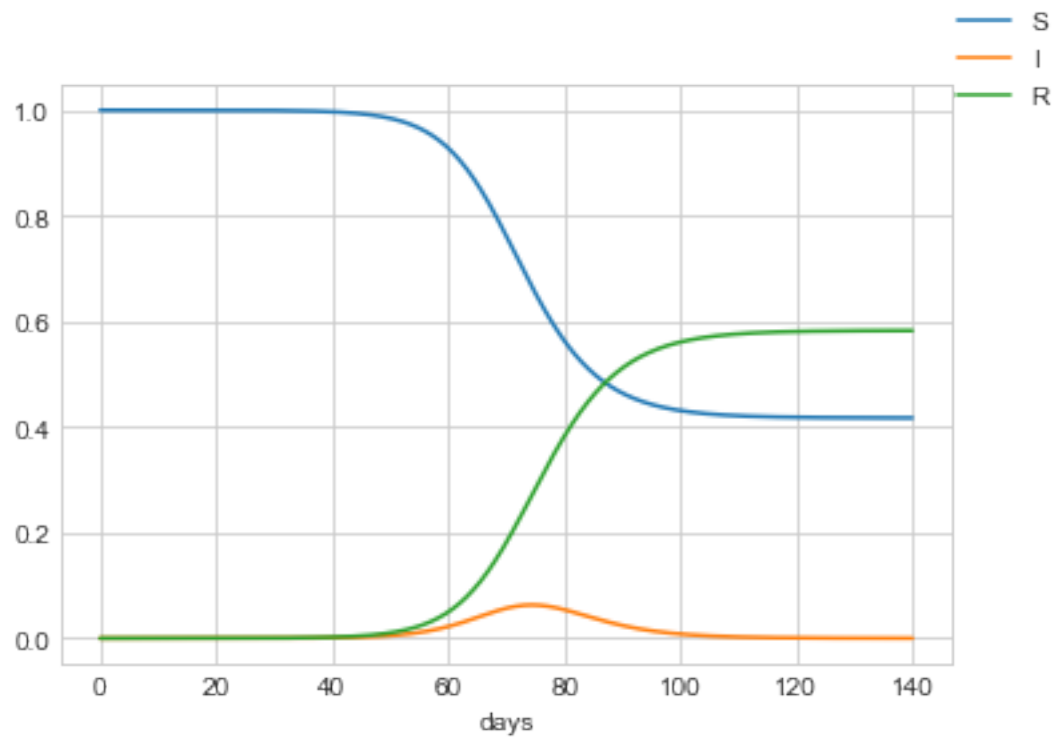


```
[11]: x = 0.1
      for i in range(0, 10):
          print(x)
          SIR_Demo(x, rungeKutta2)
          x+=0.1
```

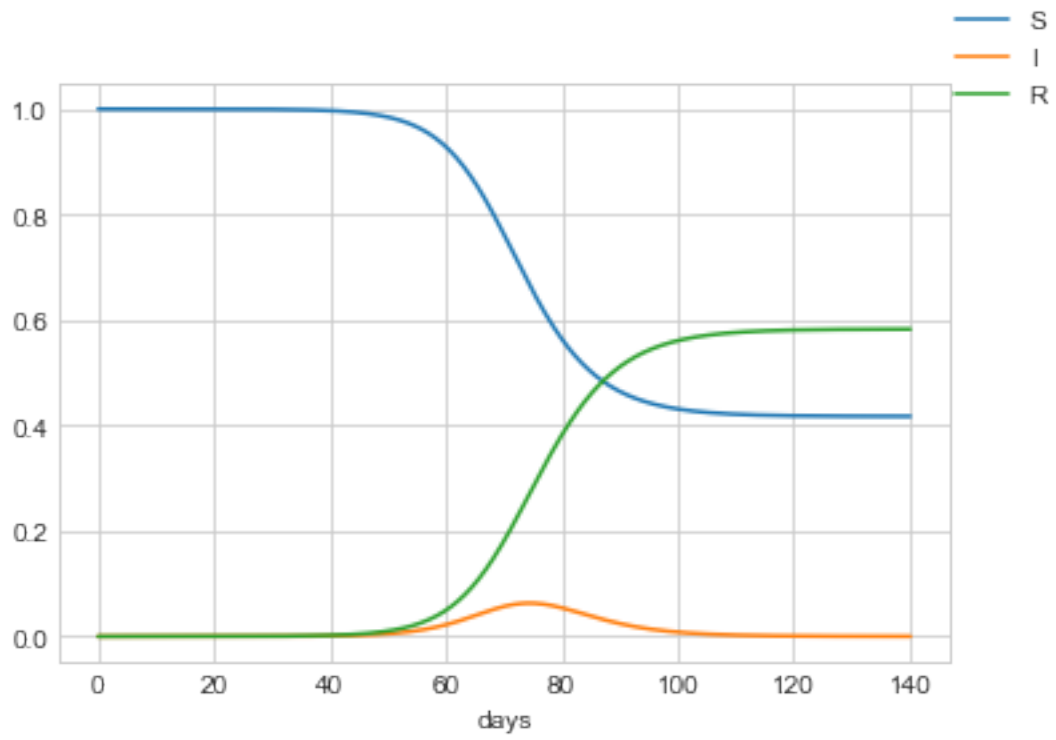
0.1



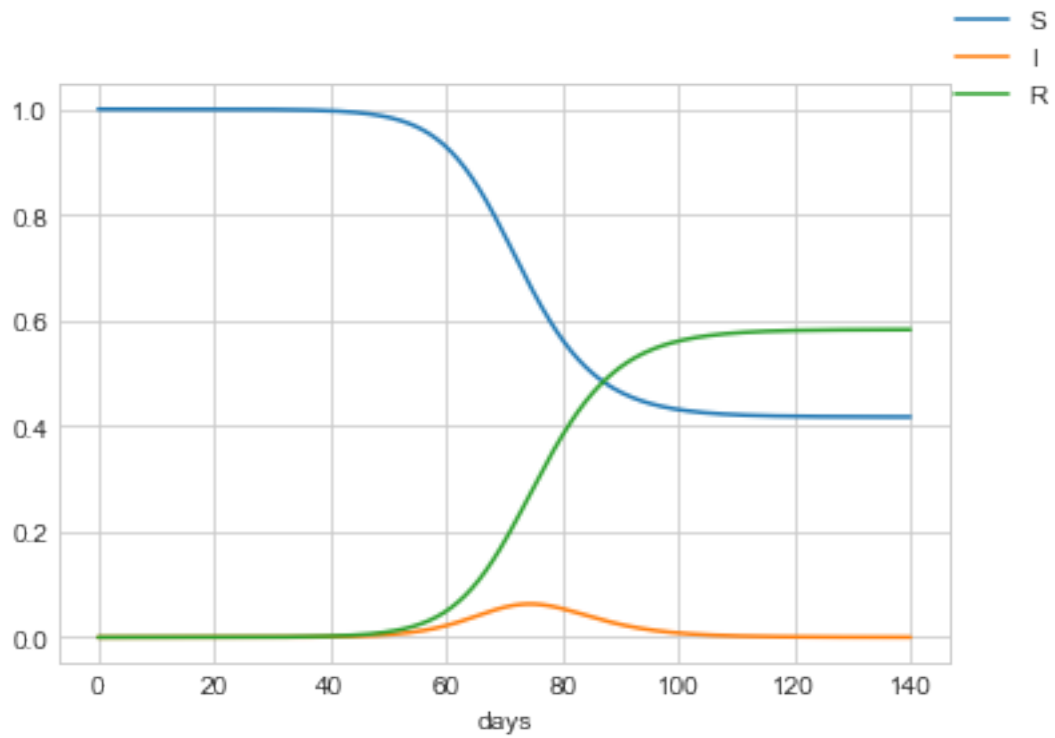
0.2



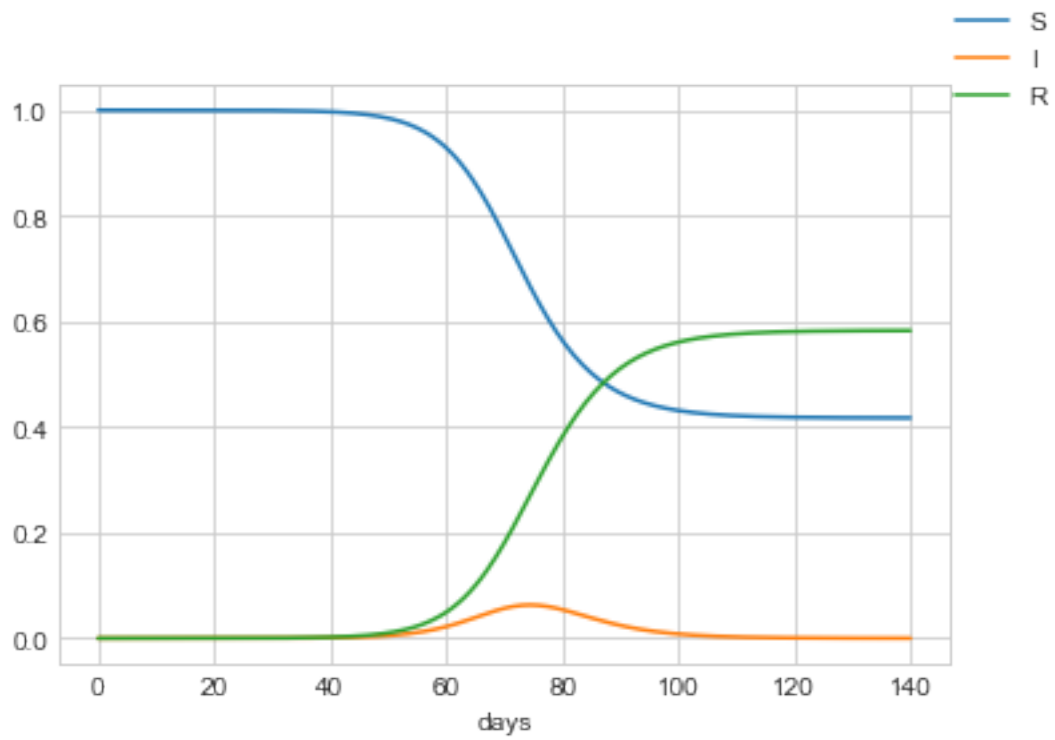
0.30000000000000004



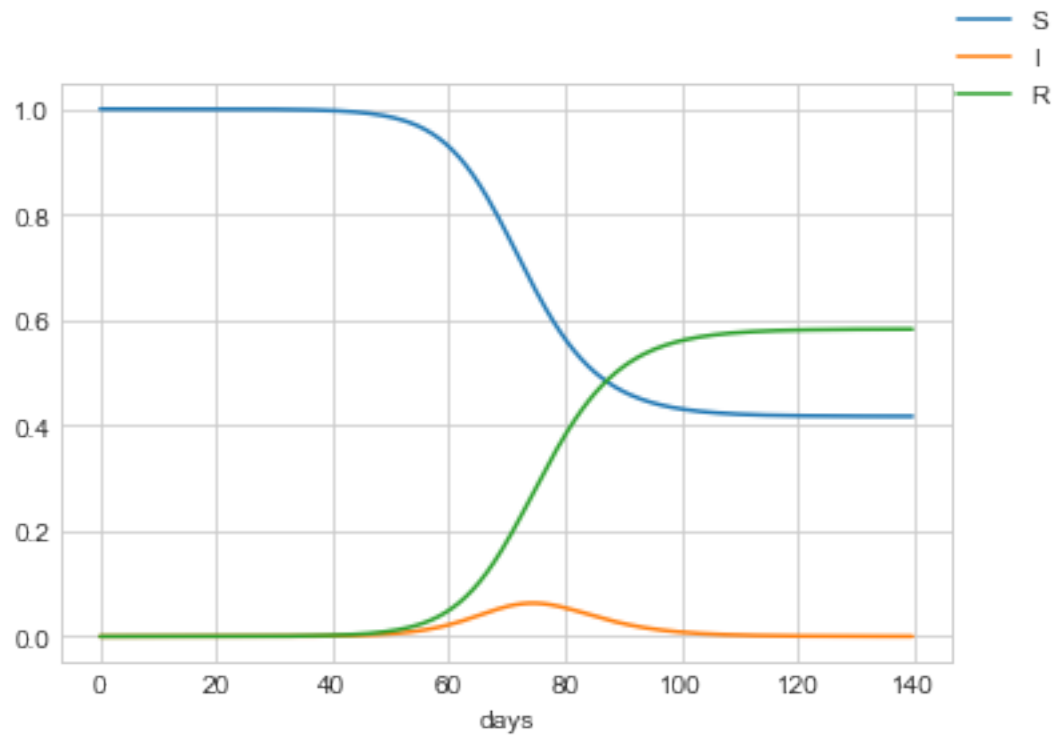
0.4



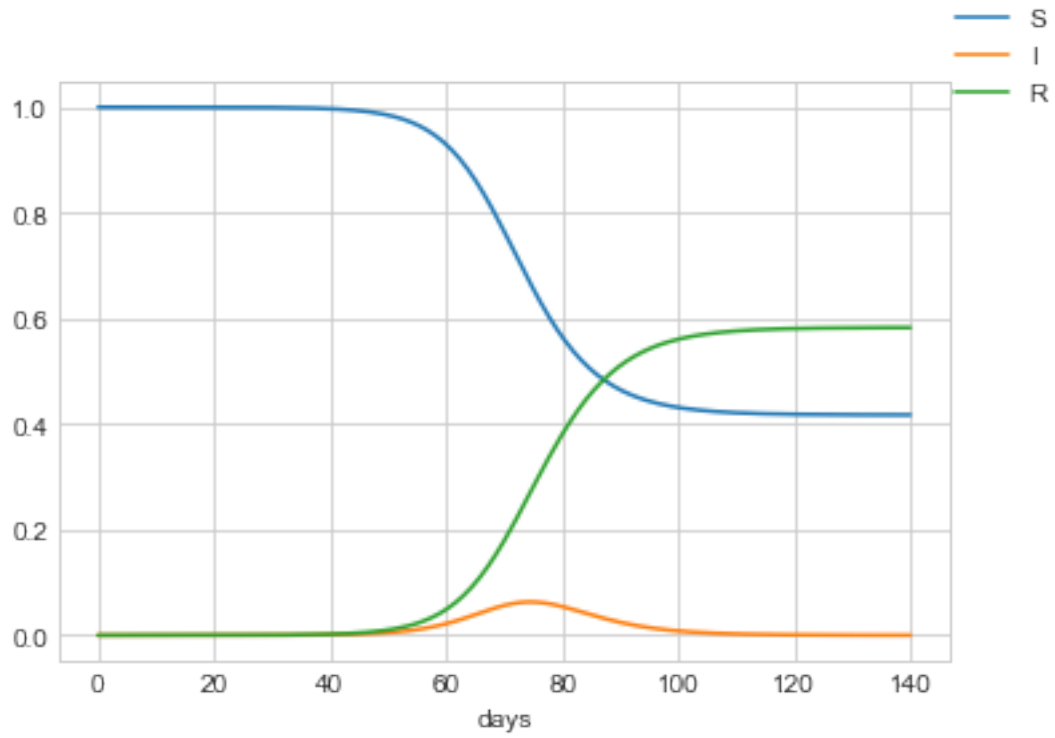
0.5



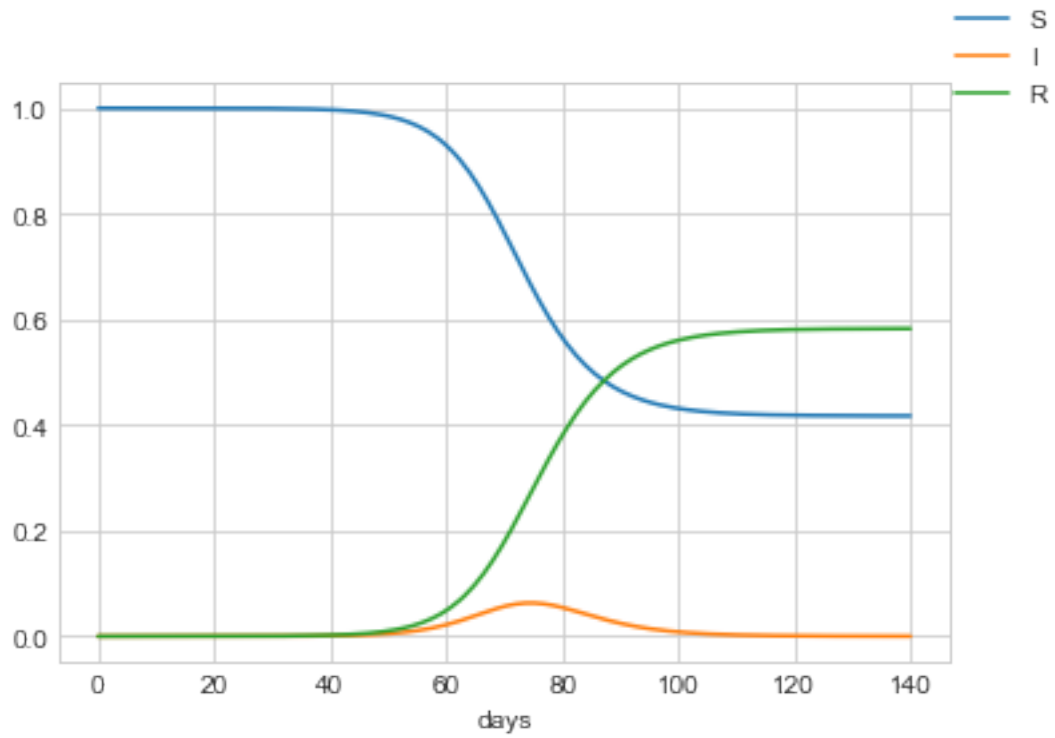
0.6



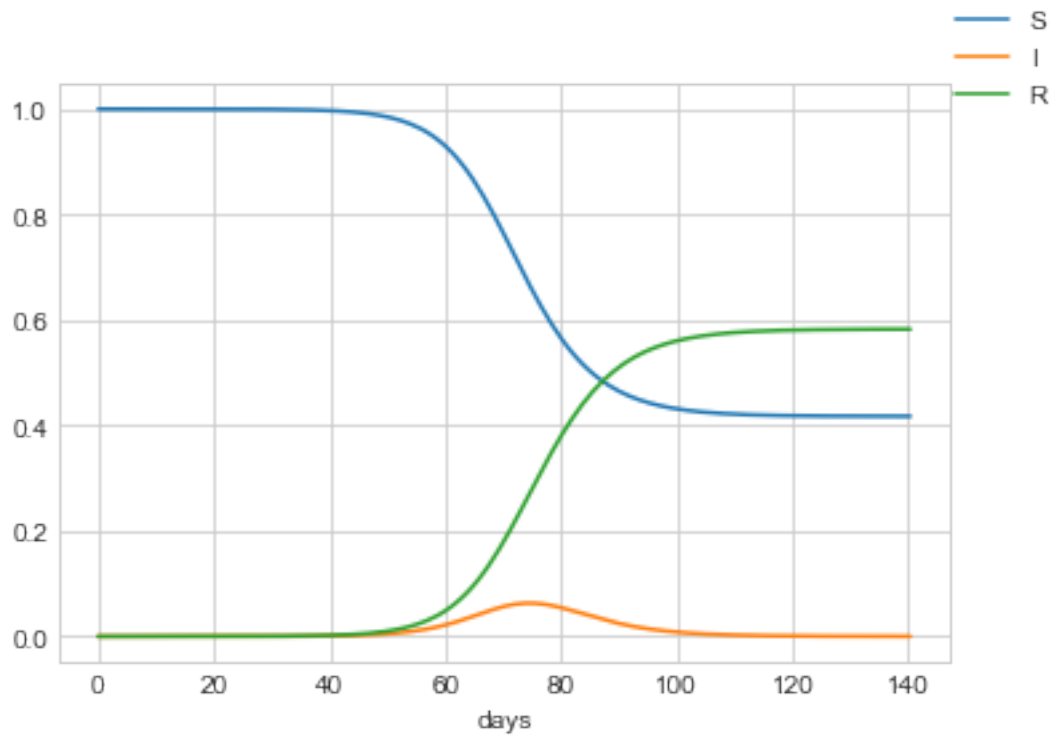
0.7



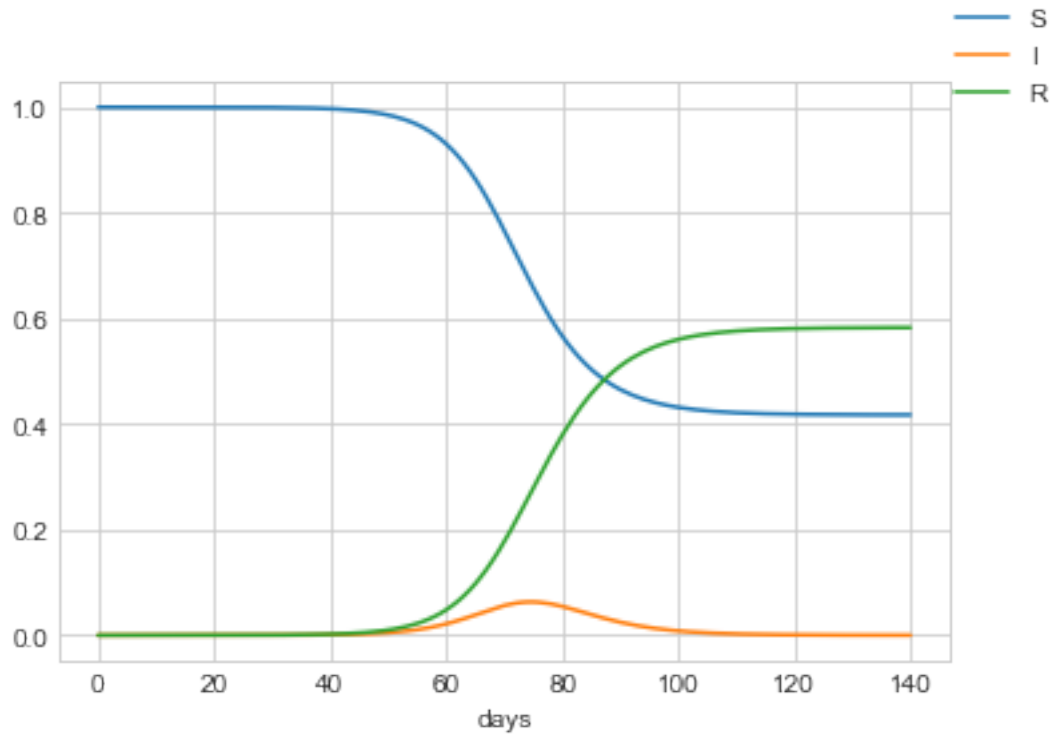
0.7999999999999999



0.8999999999999999

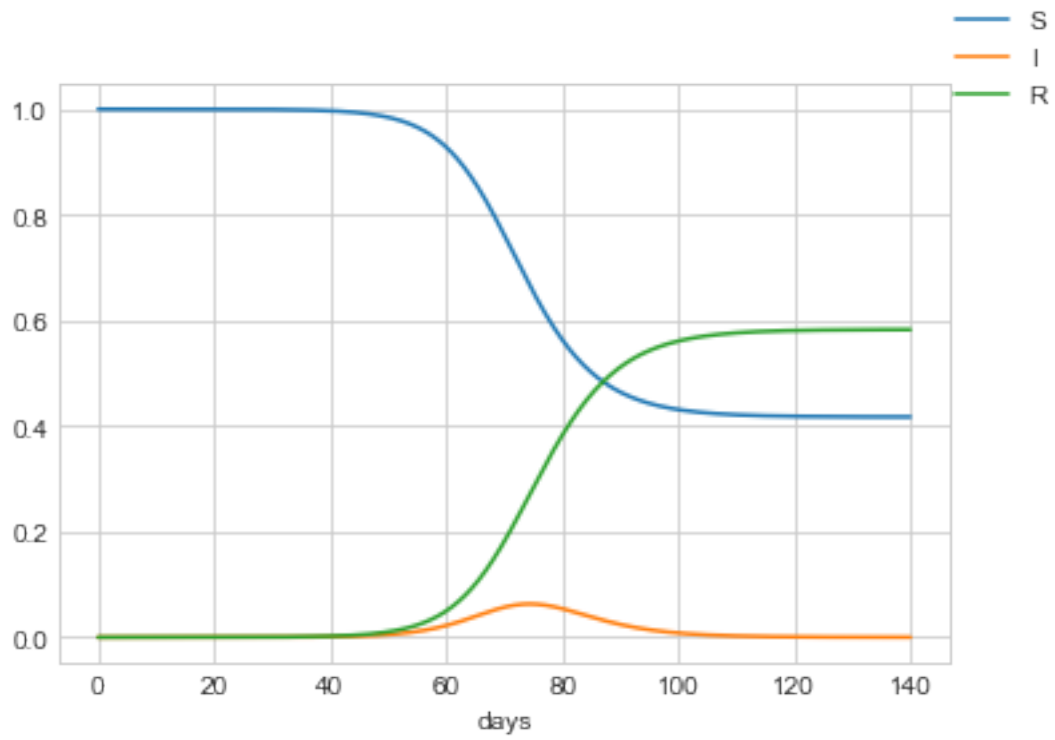


0.9999999999999999

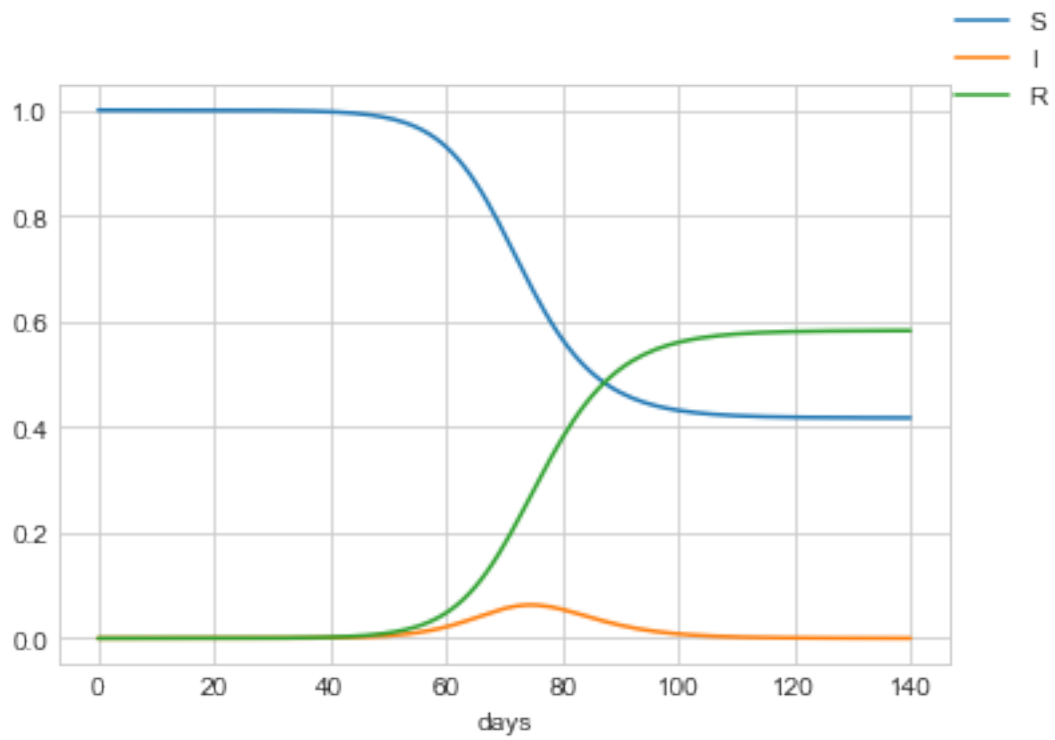


```
[12]: for item in interesting_list:  
       print(item)  
       SIR_Demo(item, rungeKutta2)
```

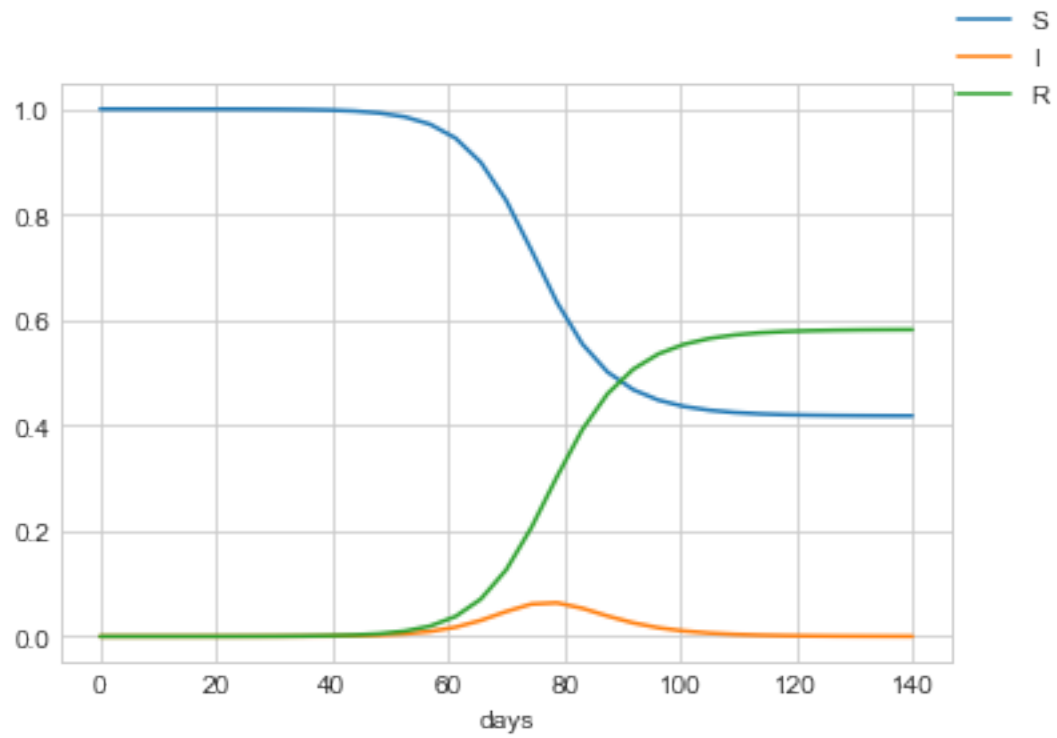
0.041666666666666664



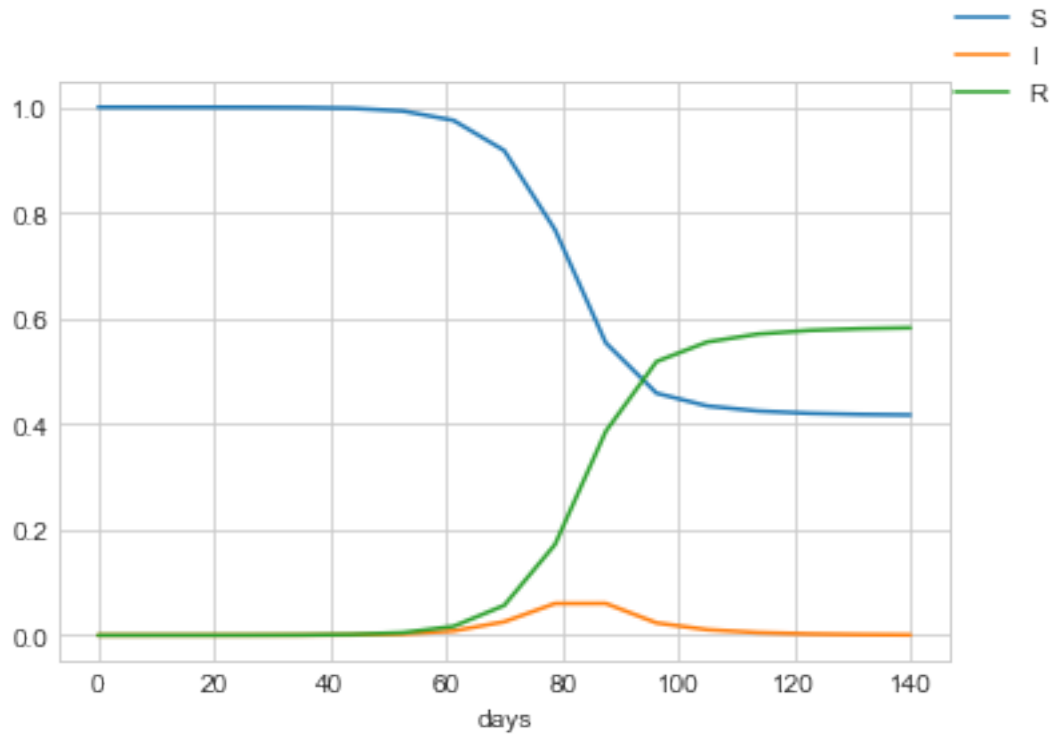
1



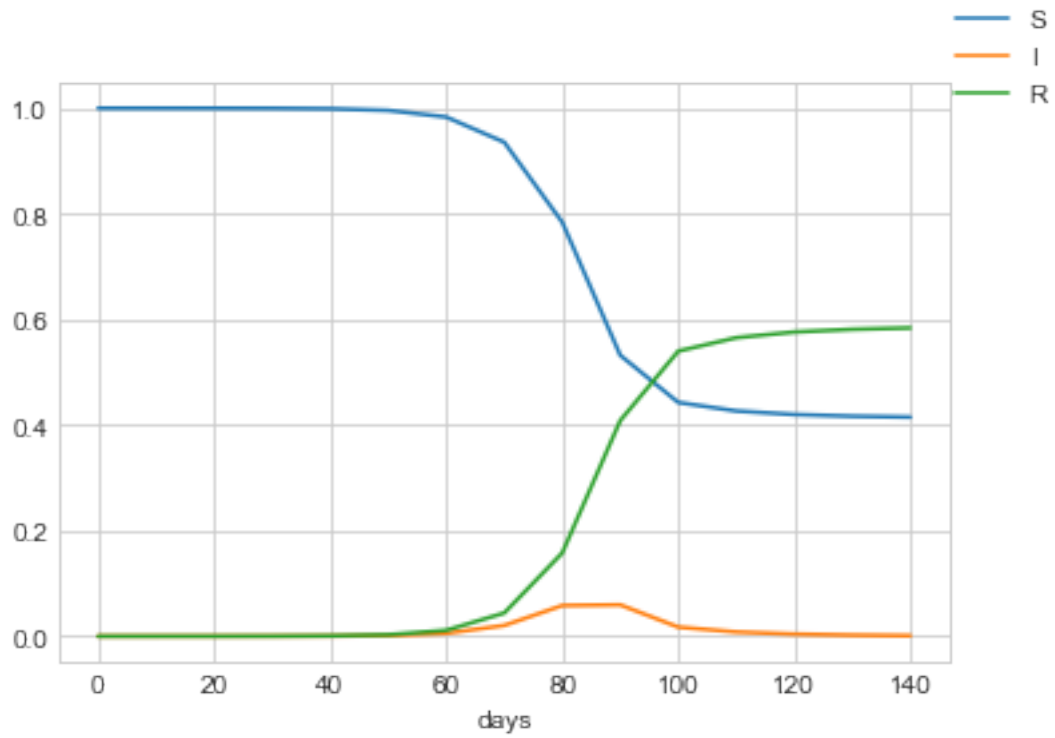
4.375



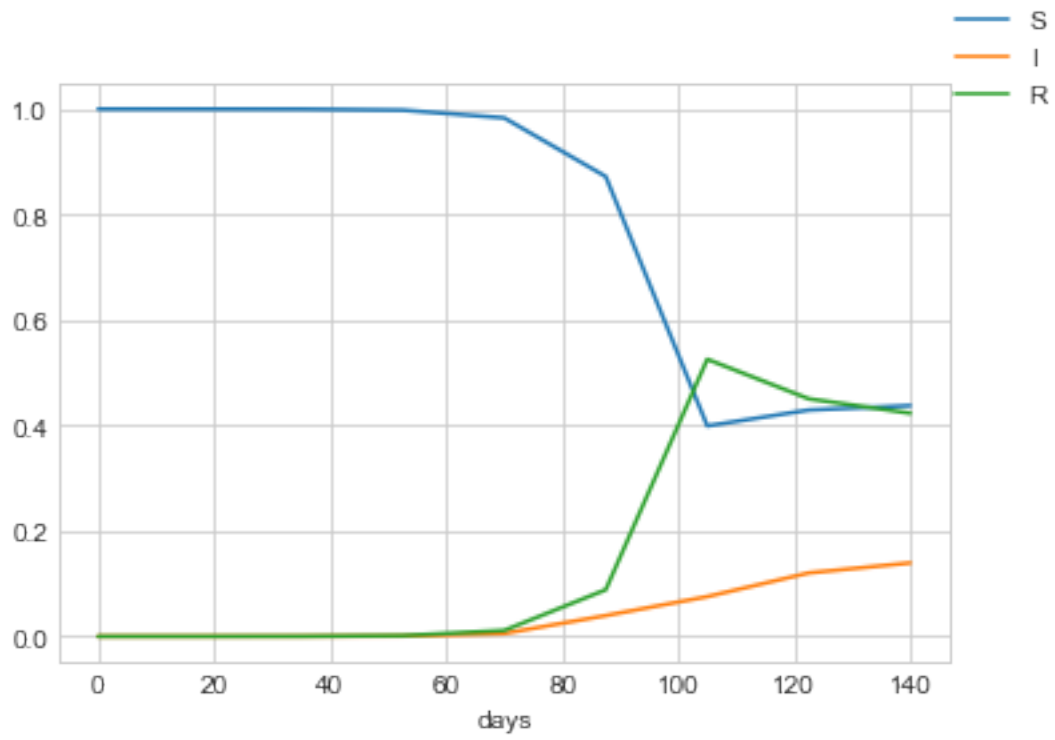
8.75



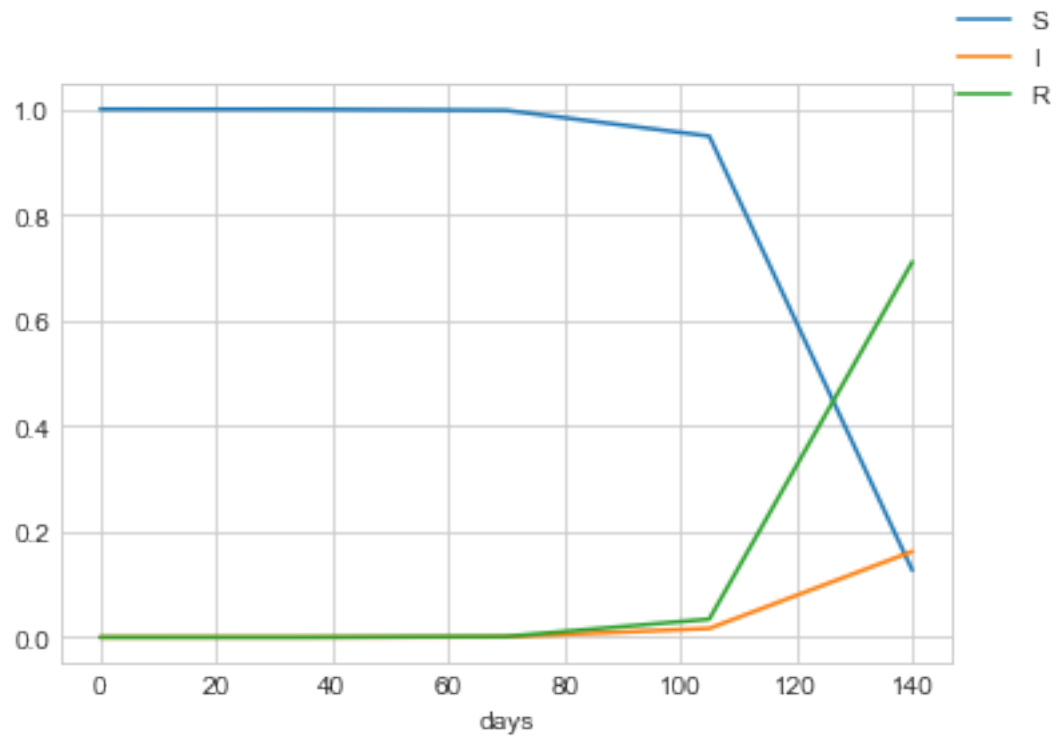
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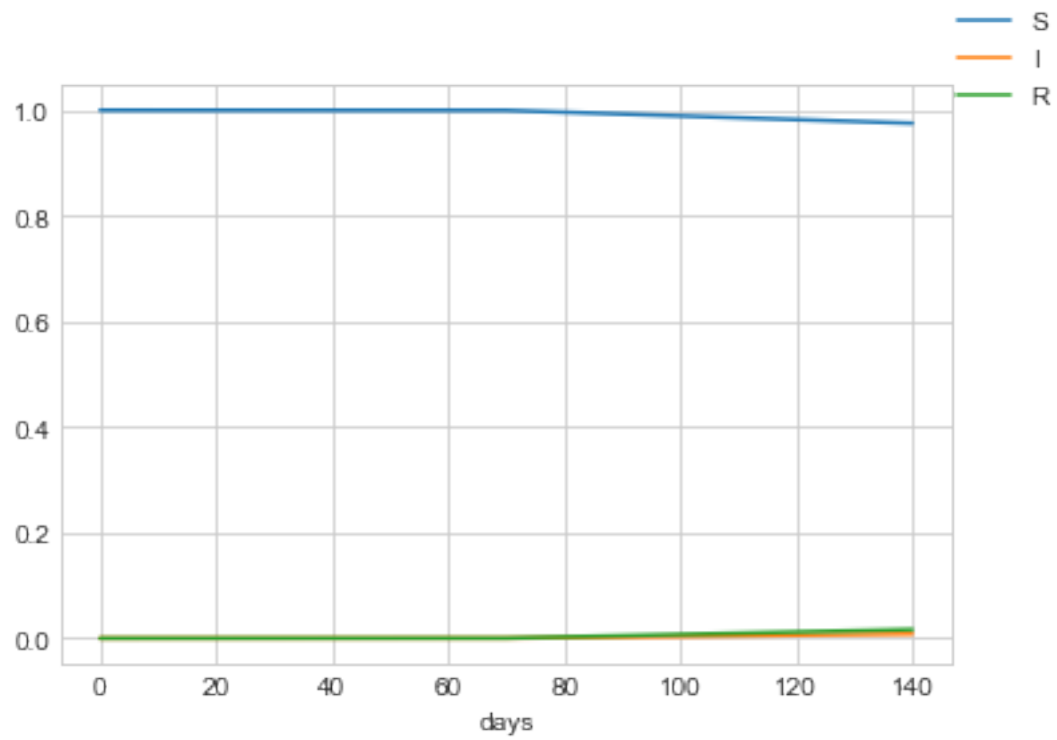
17.5



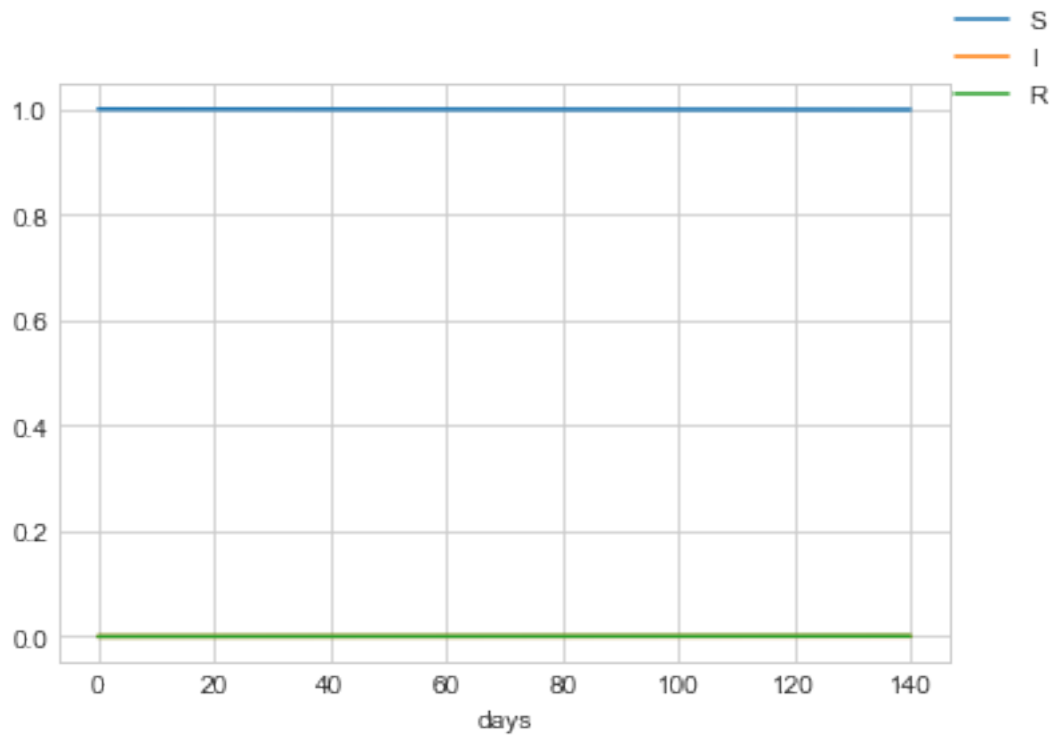
35



70

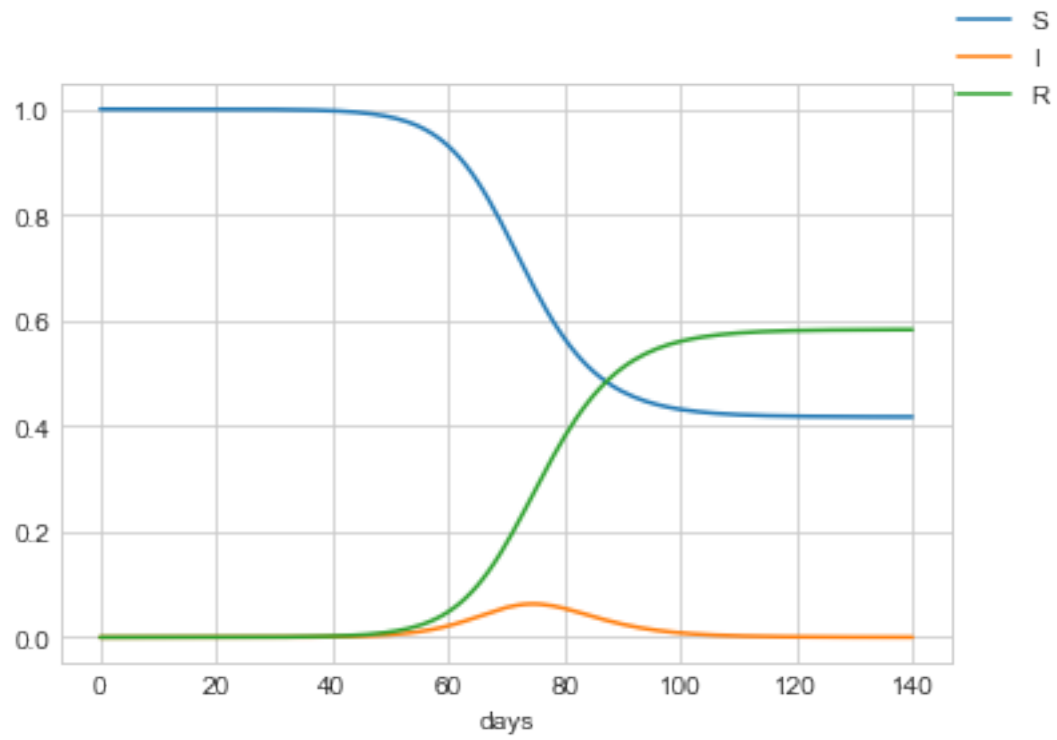


140

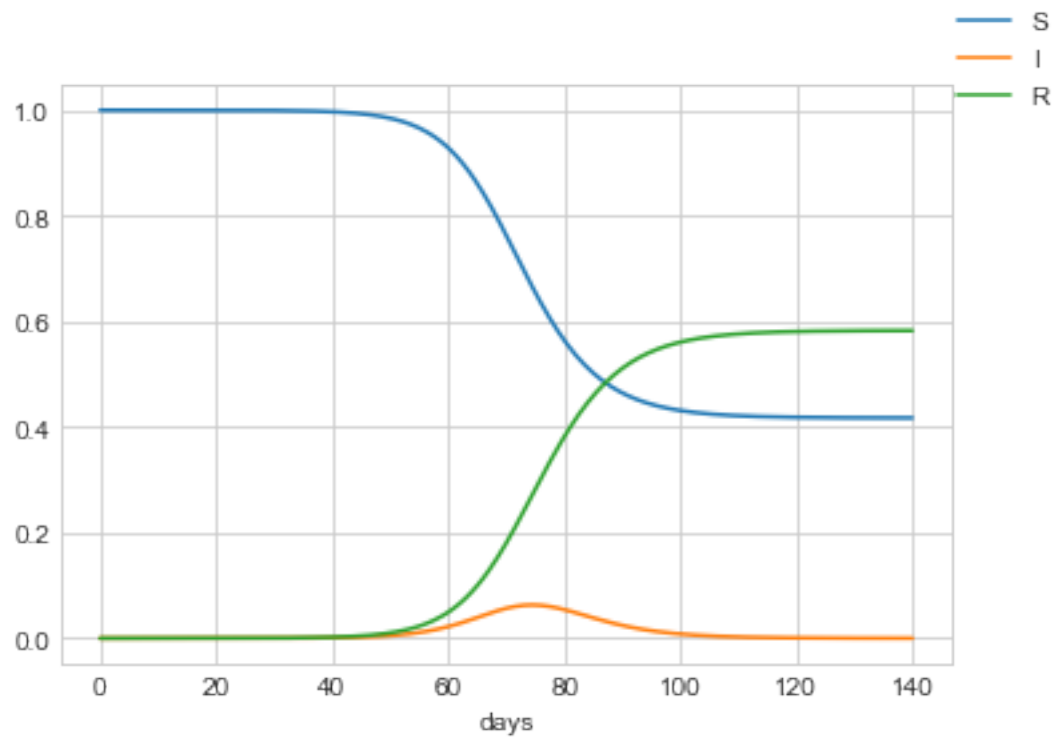


```
[13]: x = 1
      for i in range(15):
          x = (2**(-i))
          print(x)
          SIR_Demo(x, rungeKutta2)
```

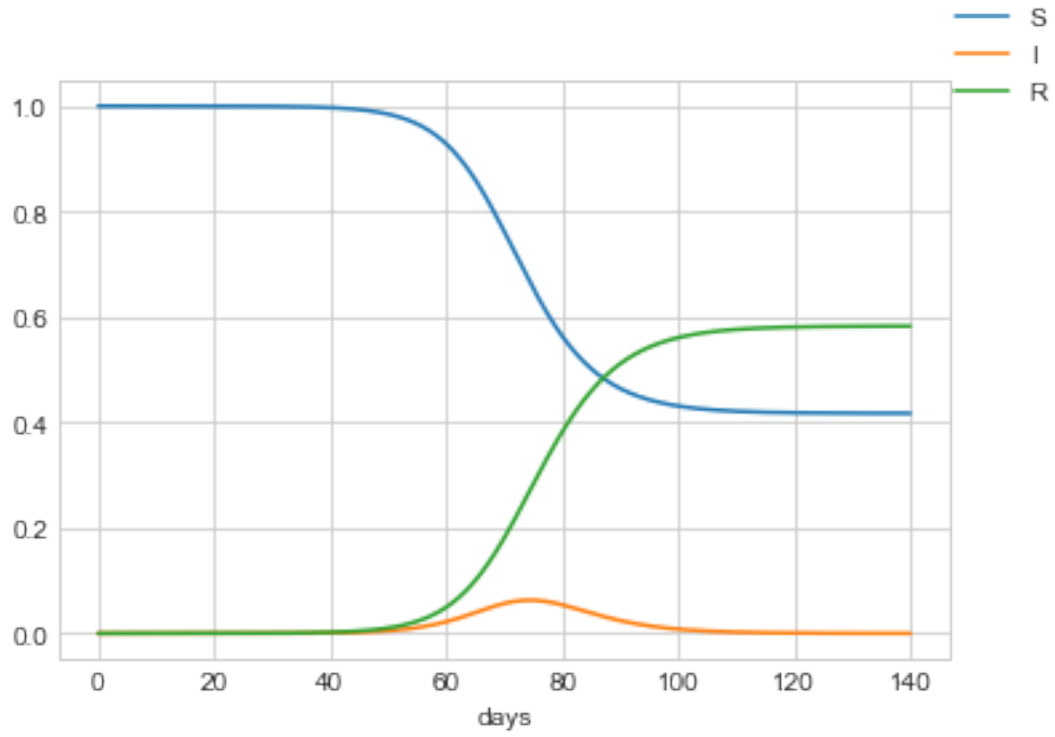
1



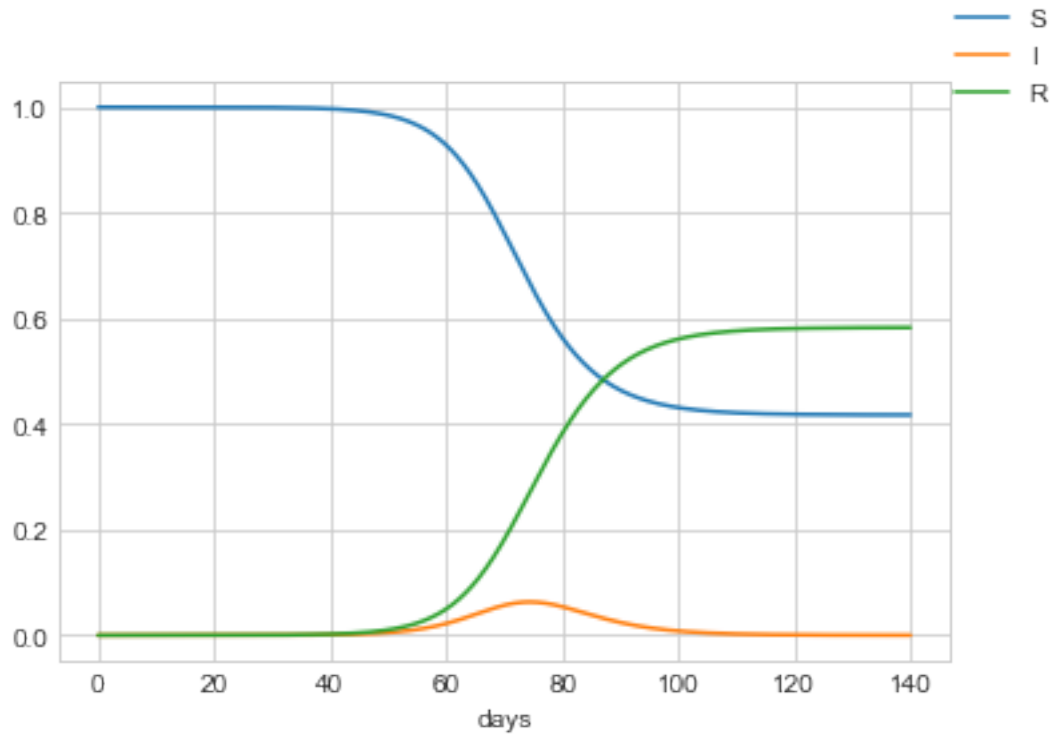
0.5



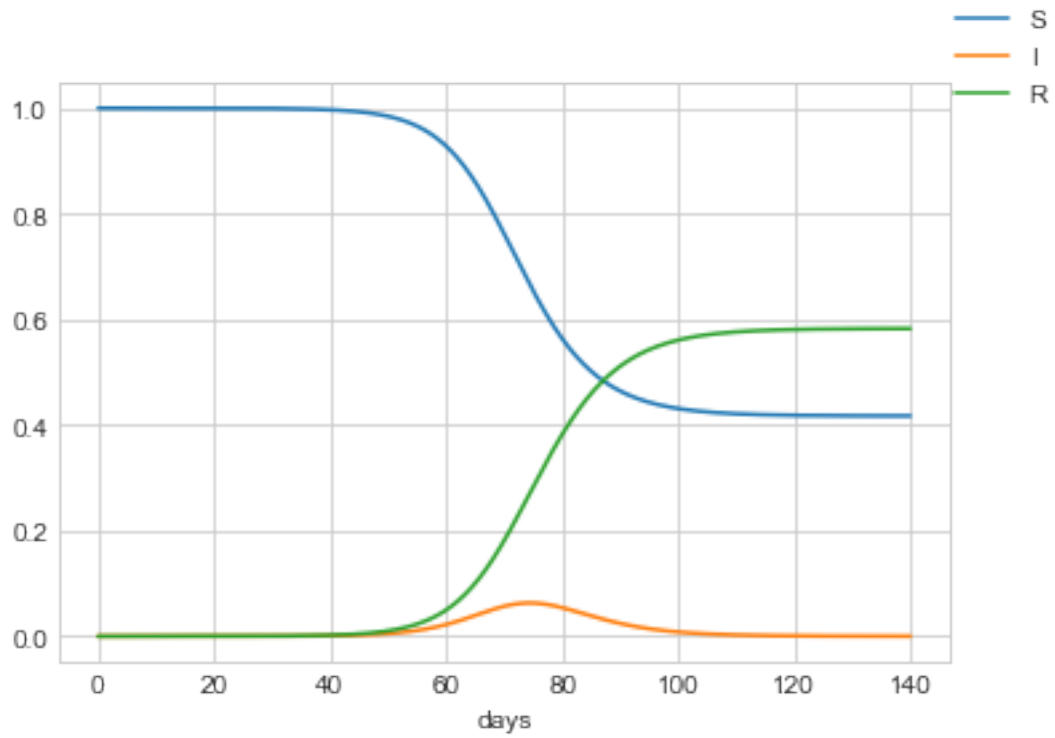
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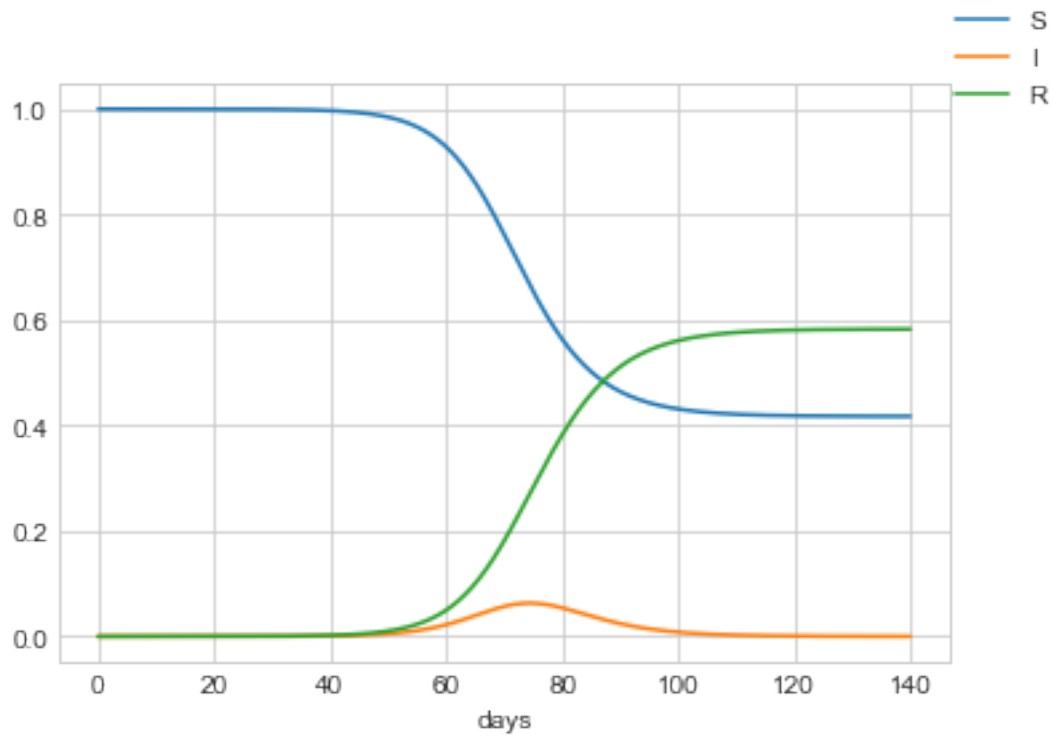
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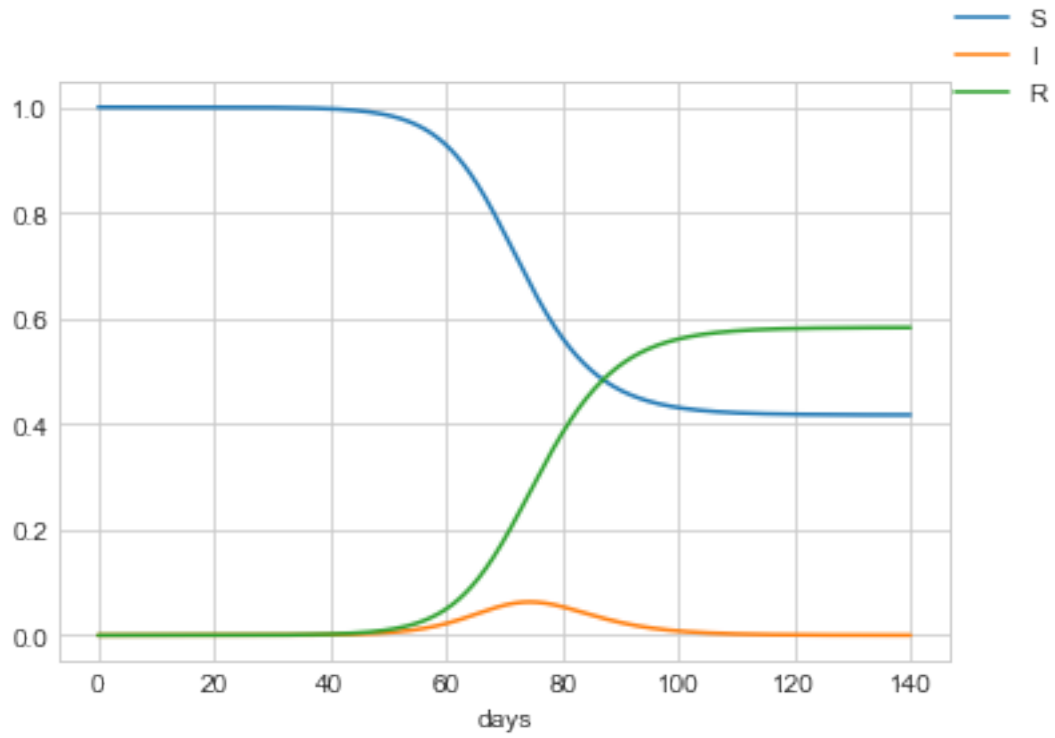
0.0625



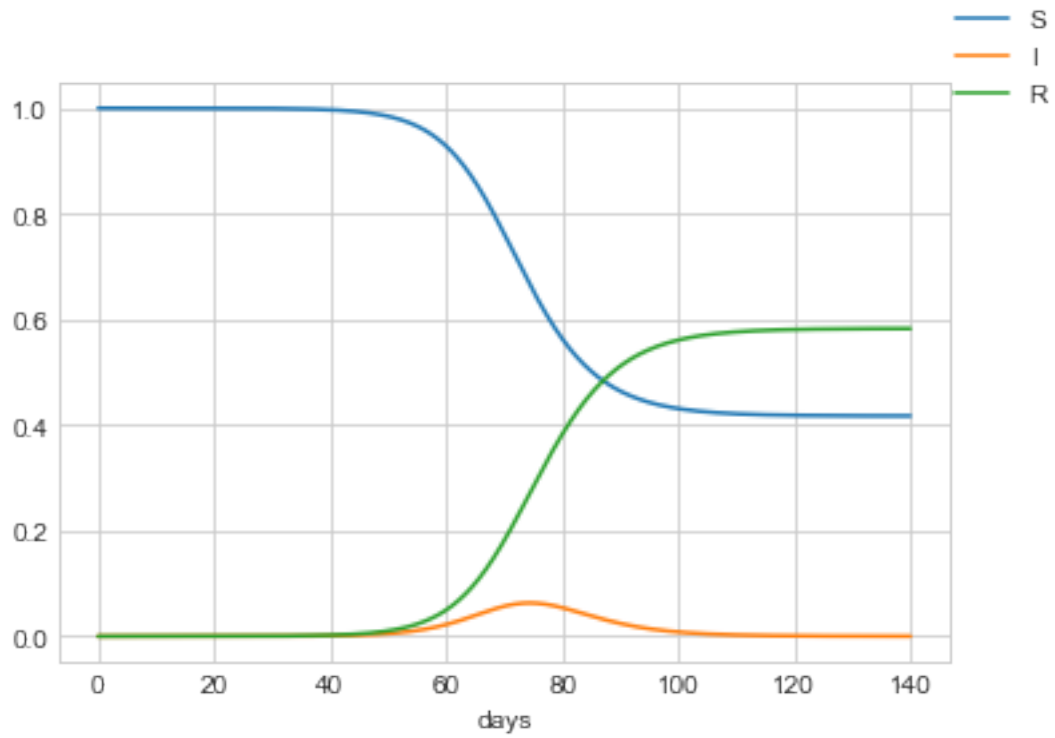
0.03125



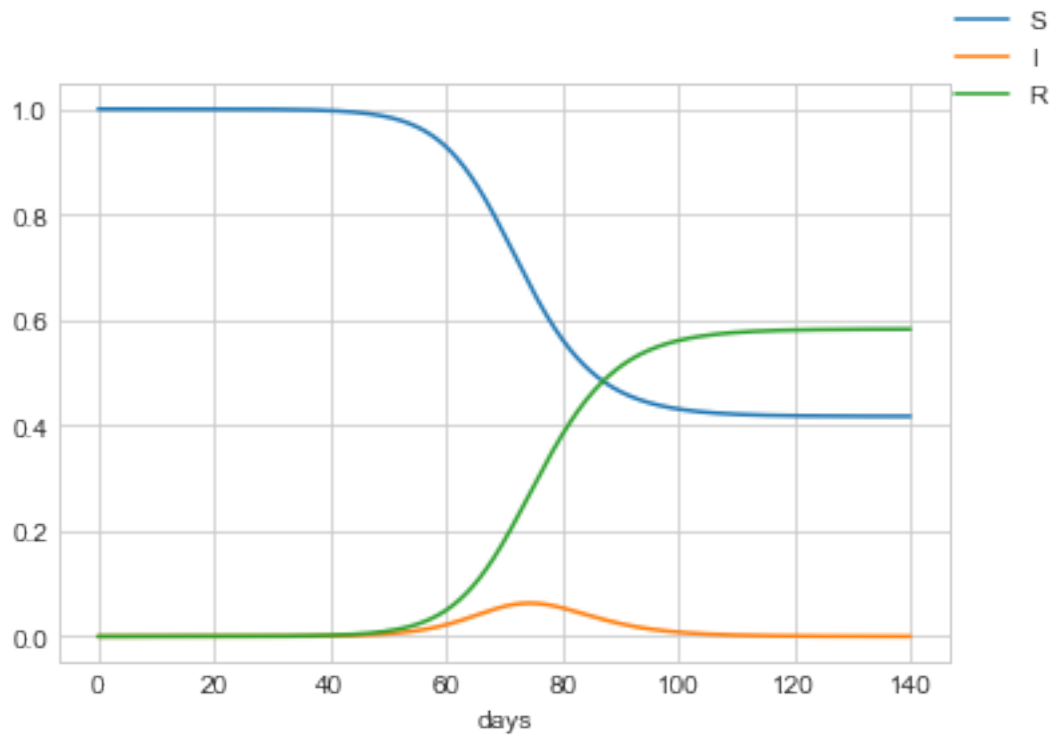
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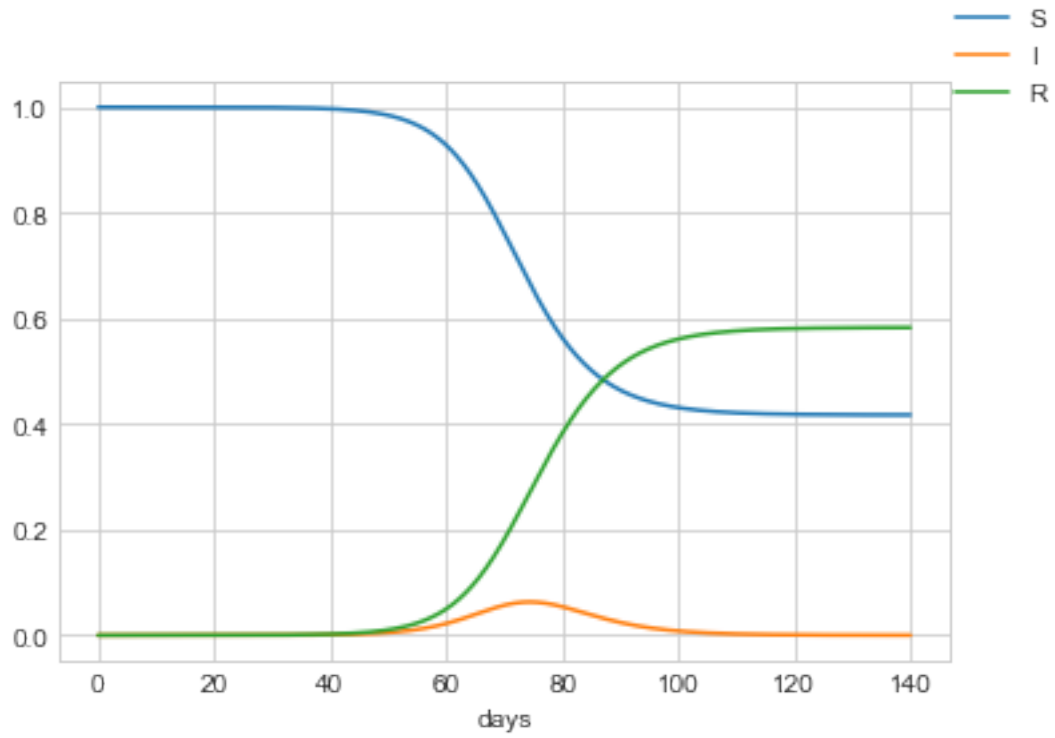
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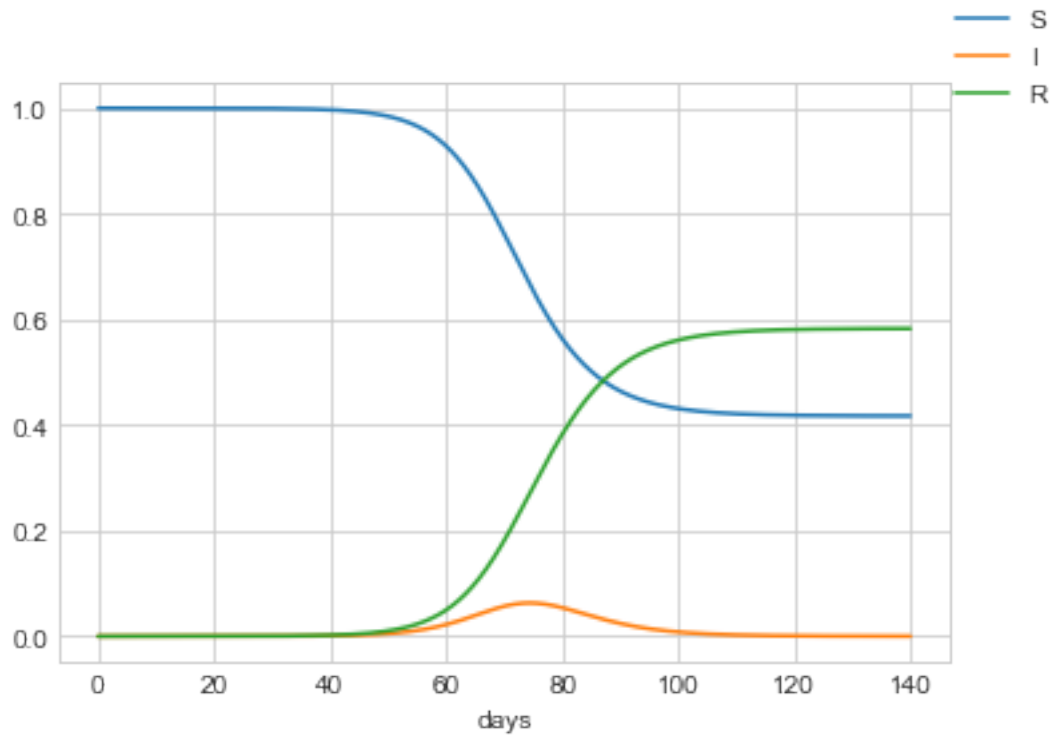
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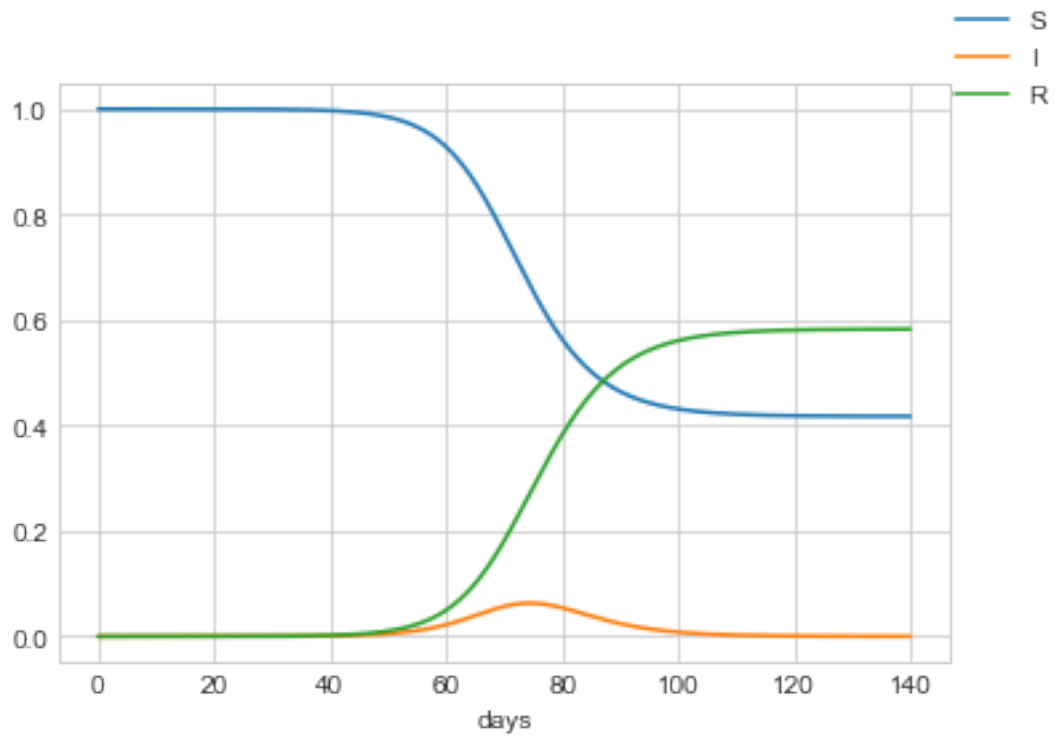
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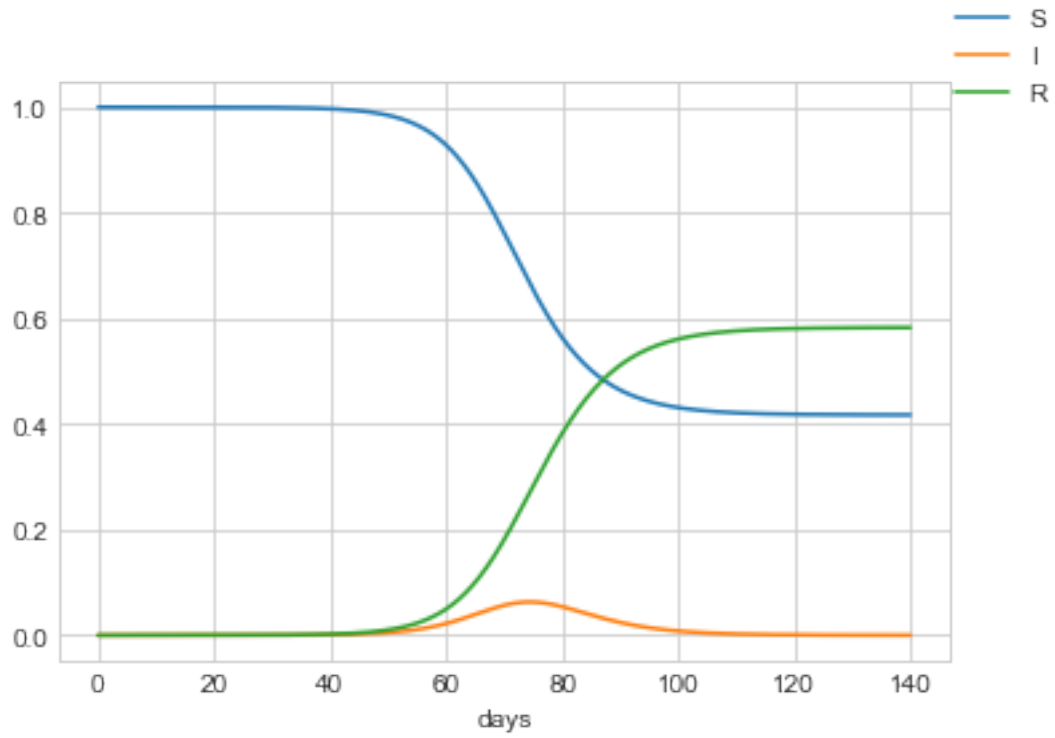
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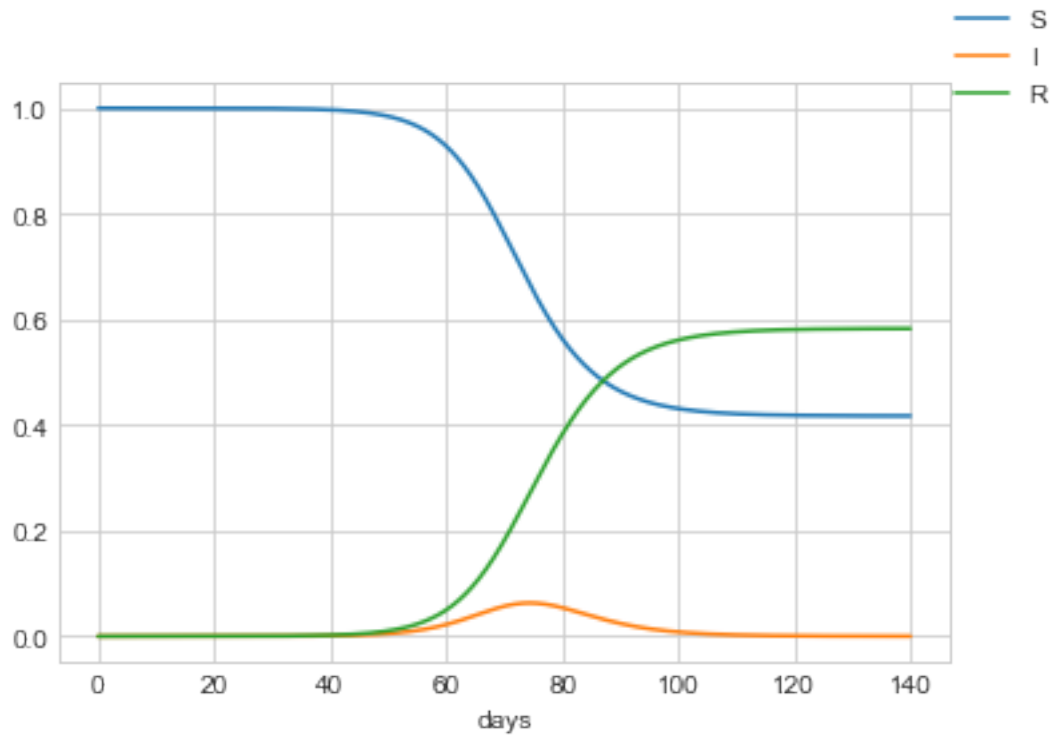
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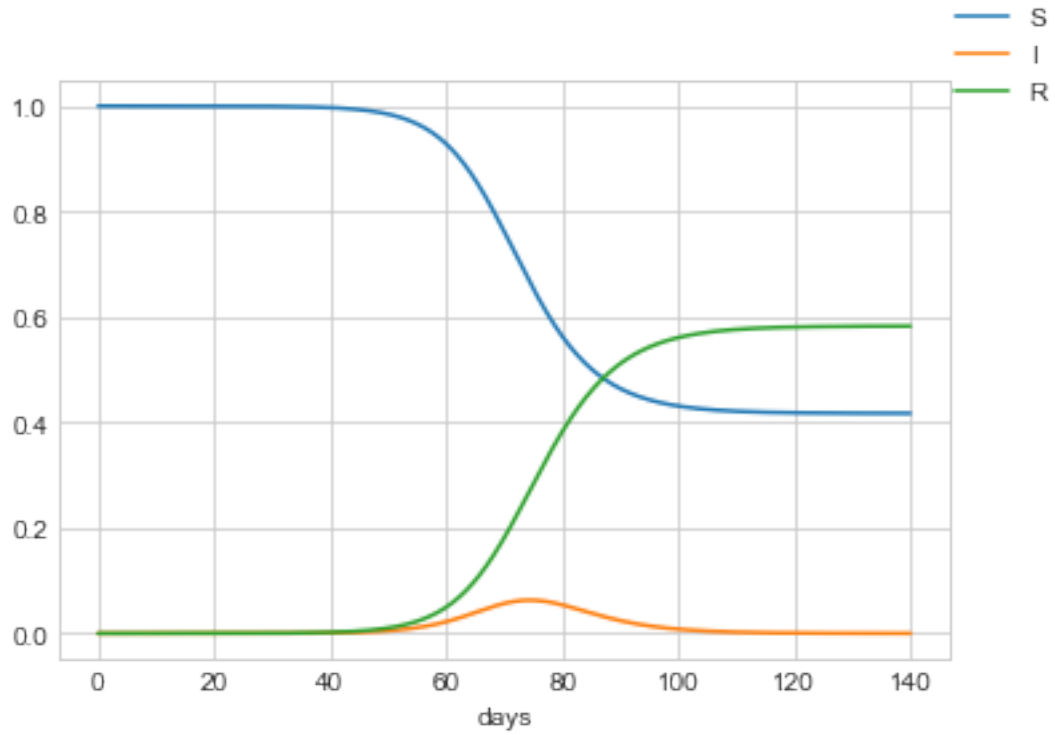
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0.0001220703125

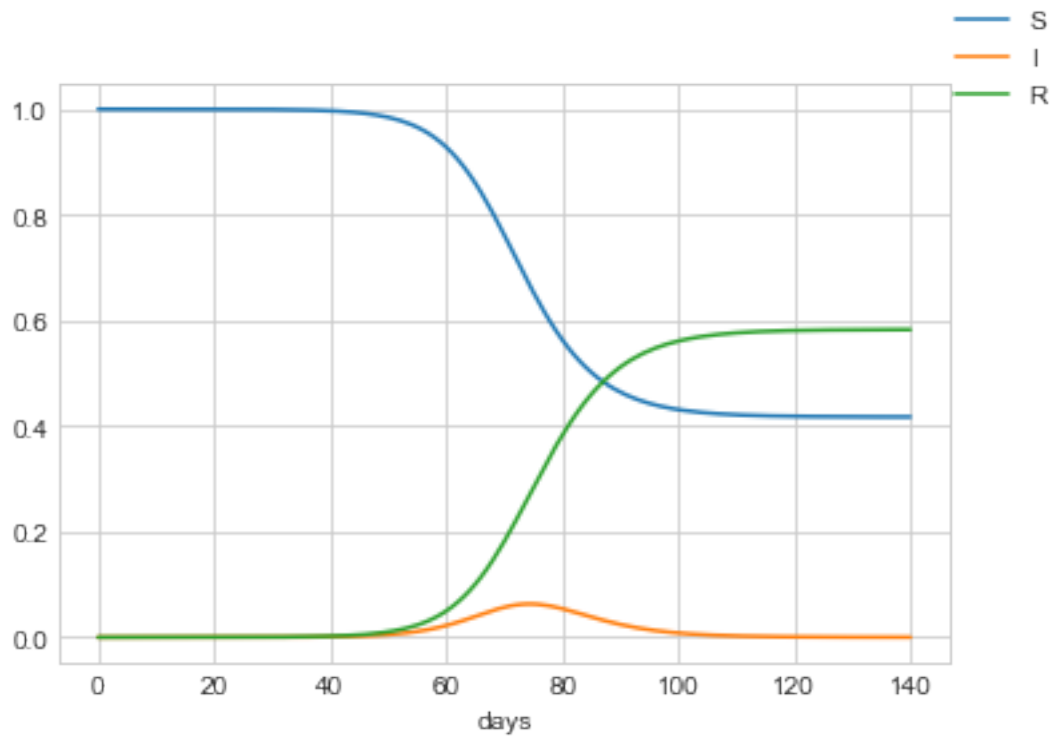


6.103515625e-05

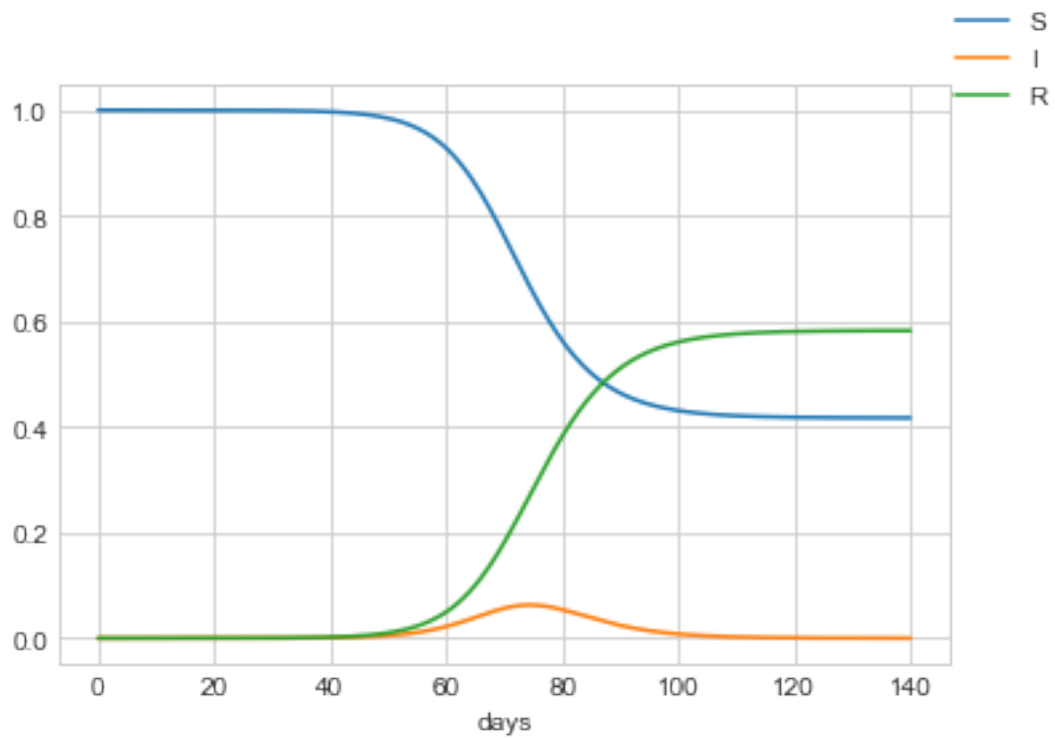


```
[14]: x = 1
      for i in range(15):
          x = (2**(-i))
          print(x)
          SIR_Demo(x, rungeKutta4)
```

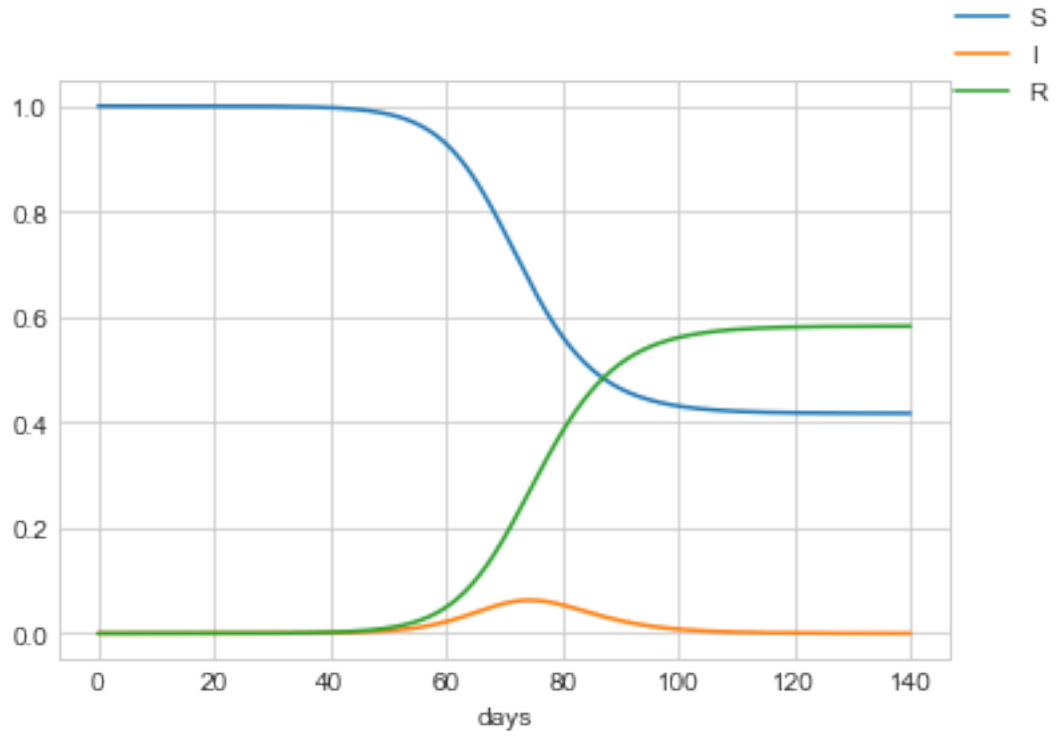
1



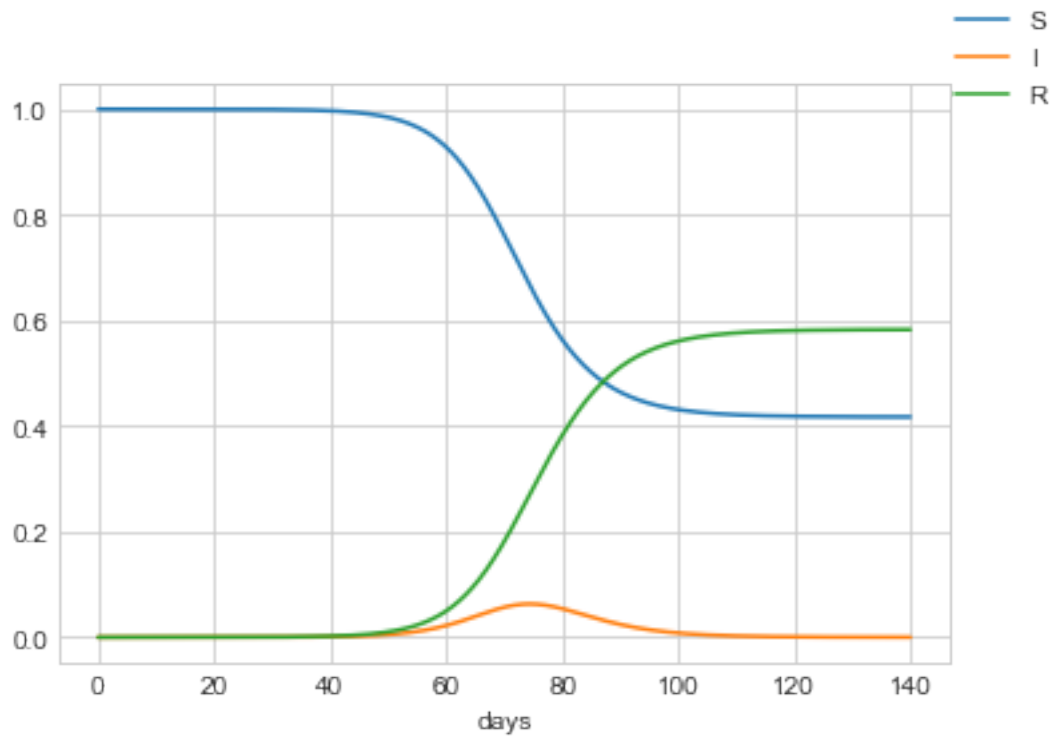
0.5



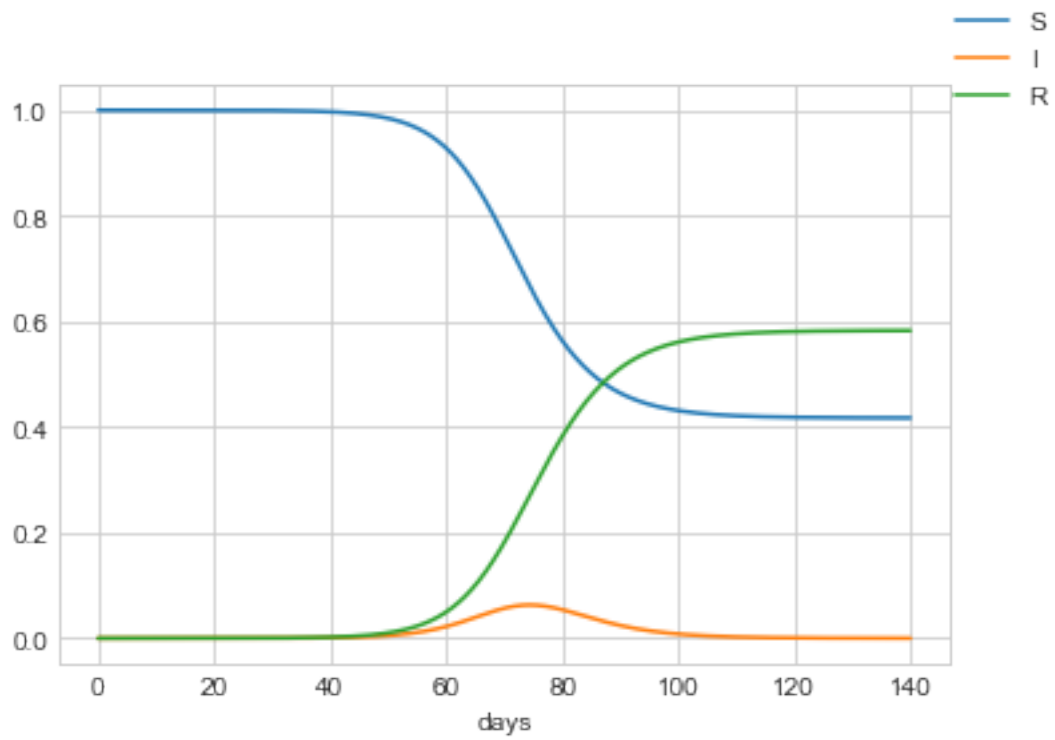
0.25



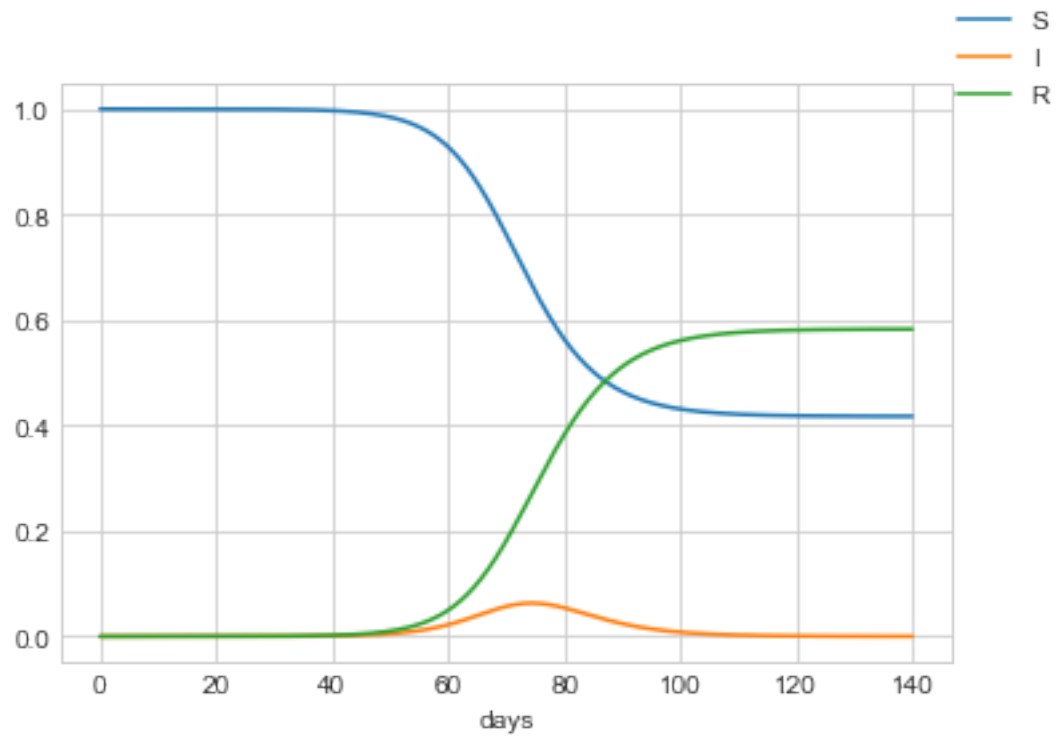
0.125



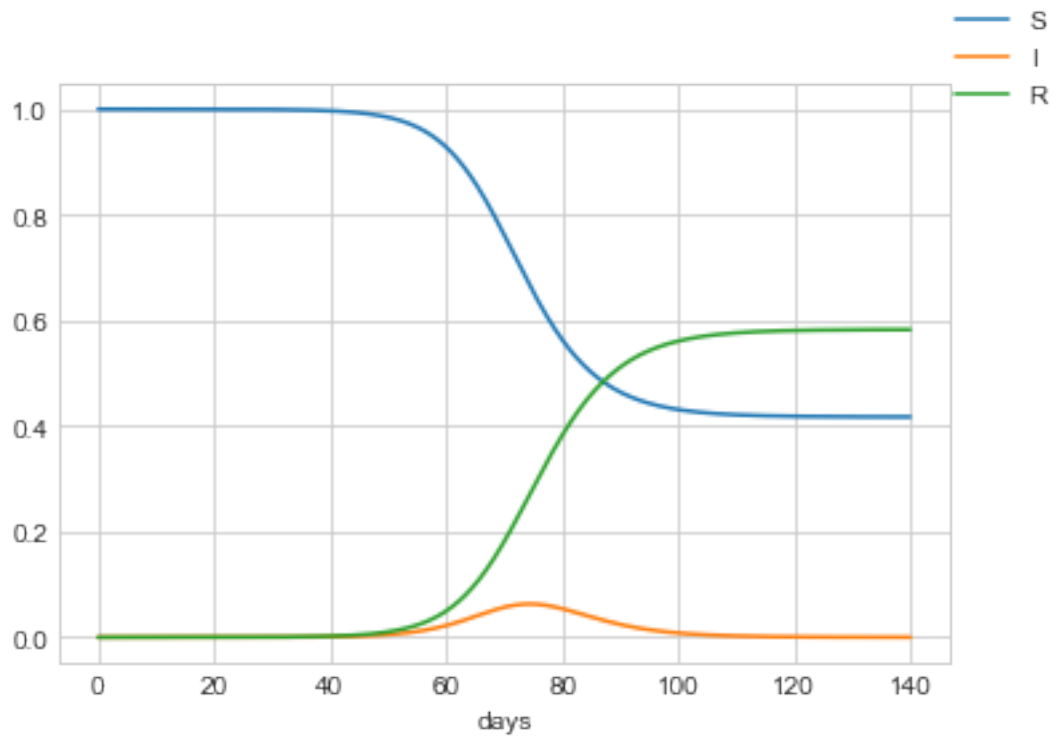
0.0625



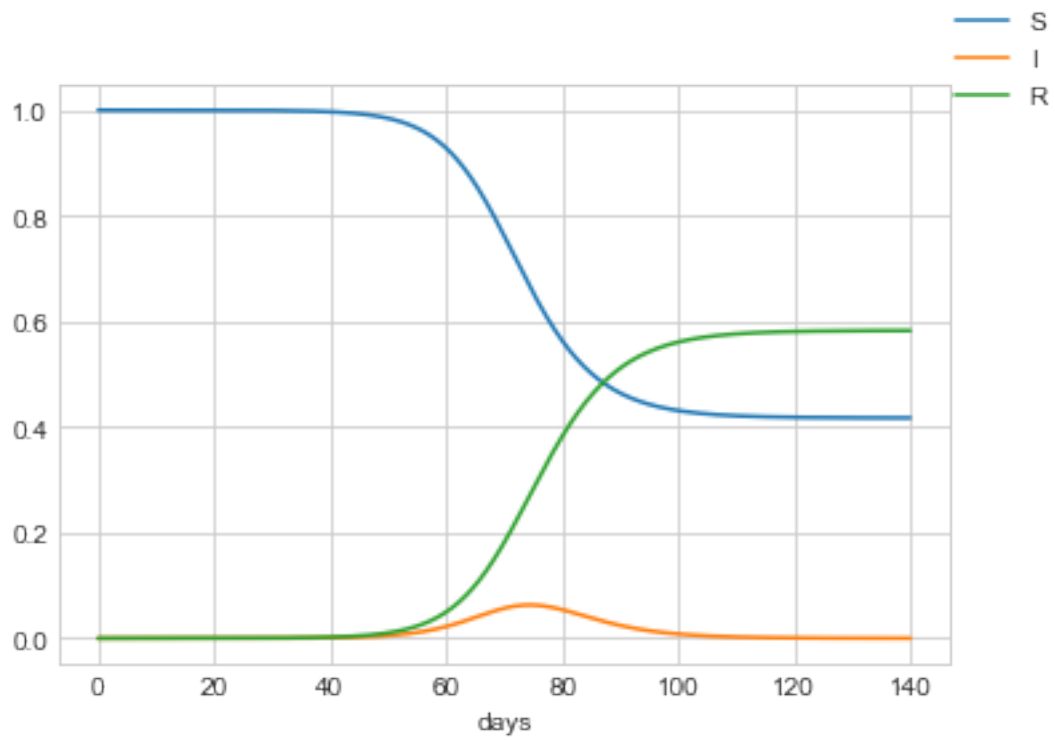
0.03125



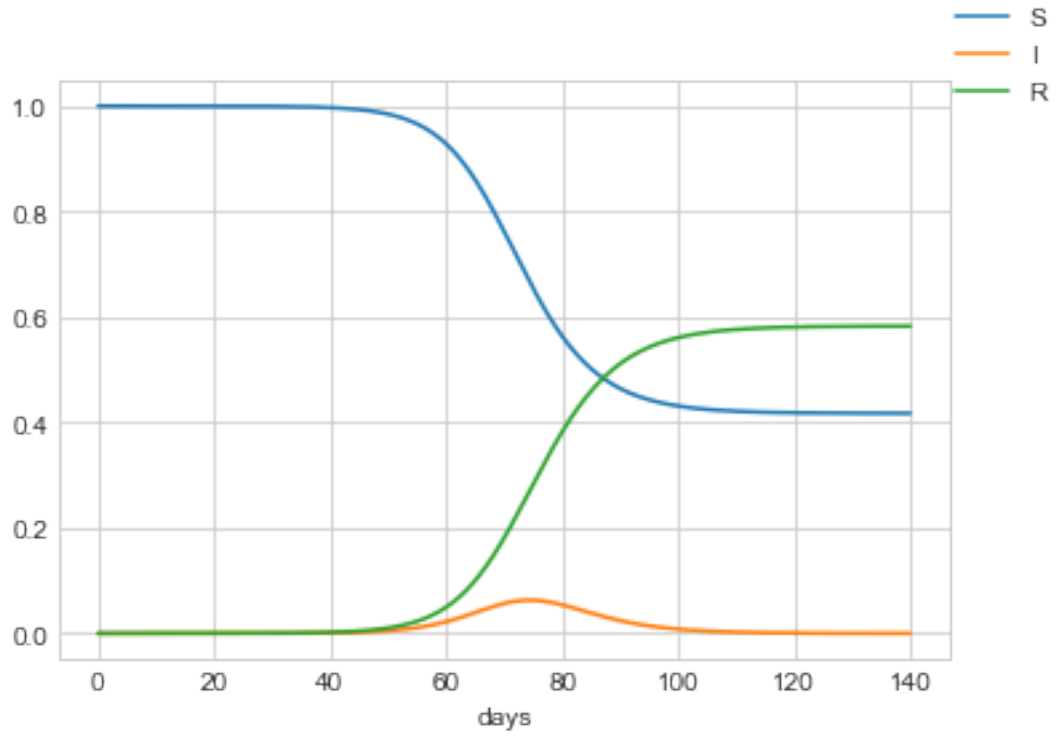
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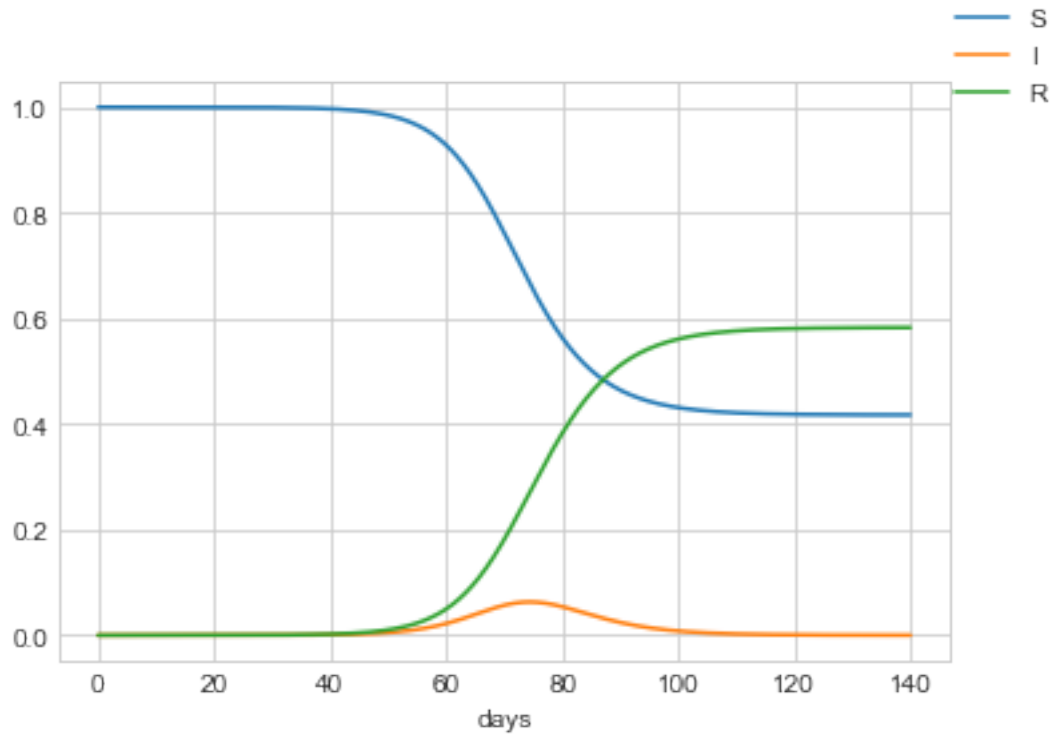
0.0078125



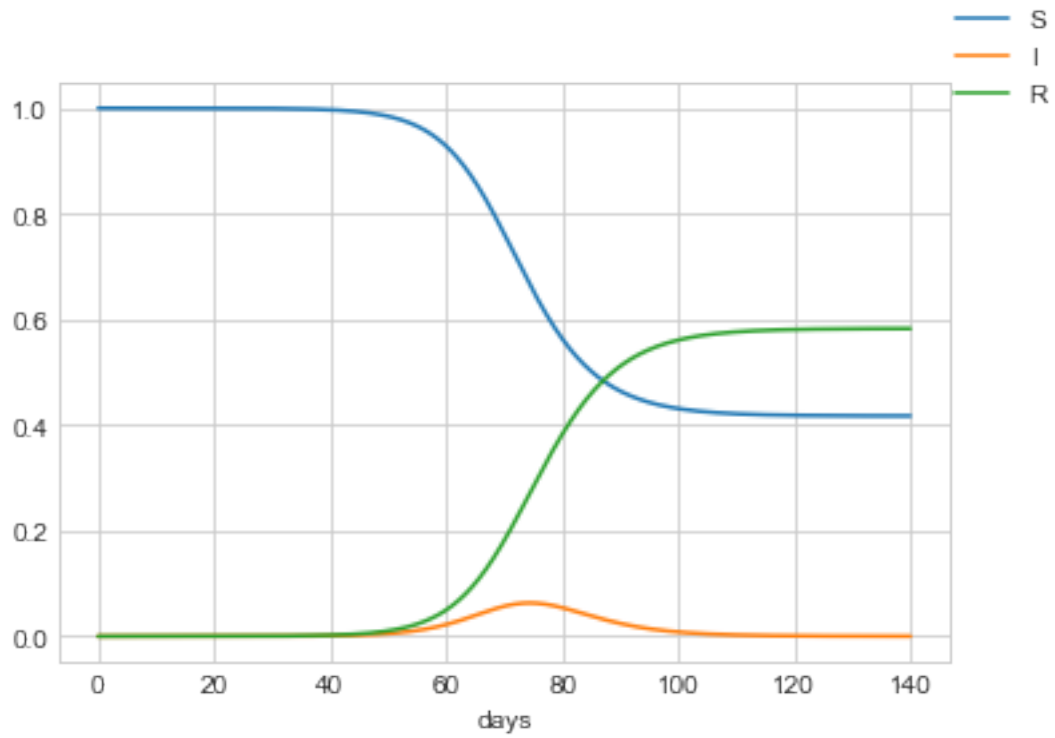
0.00390625



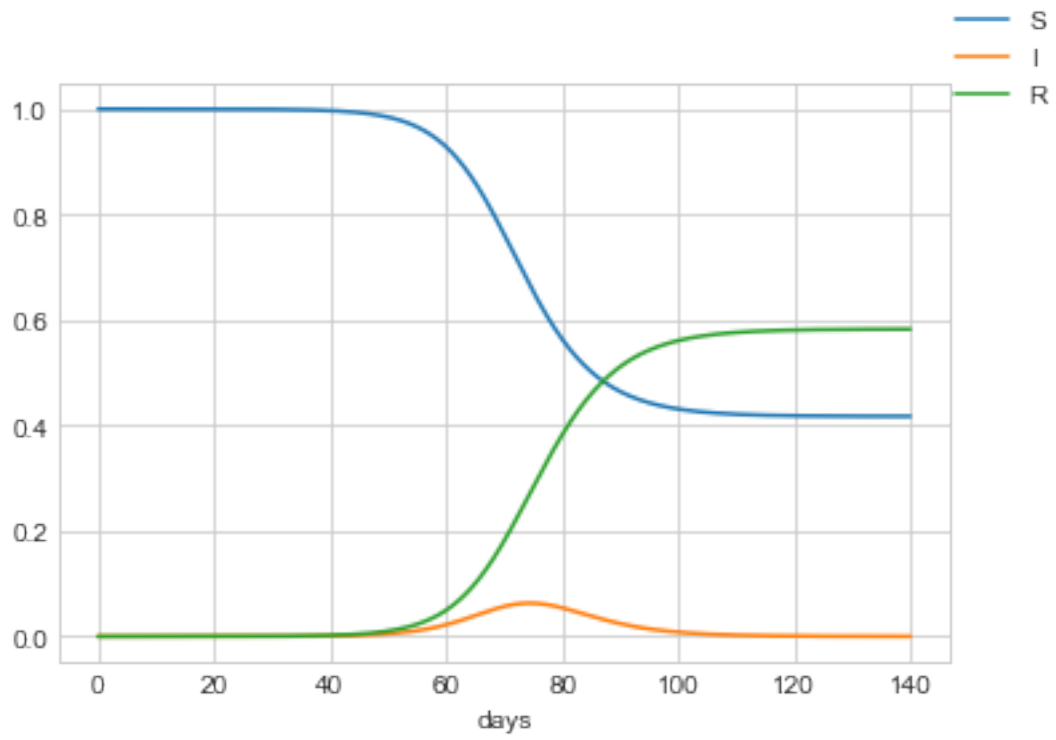
0.001953125



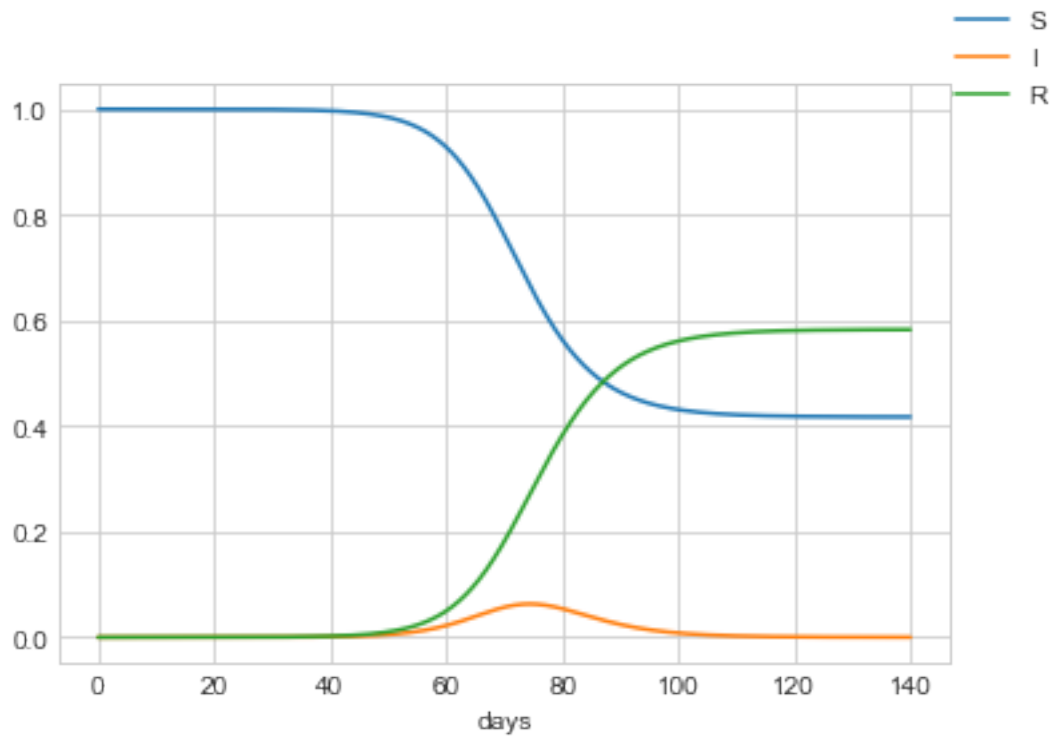
0.0009765625



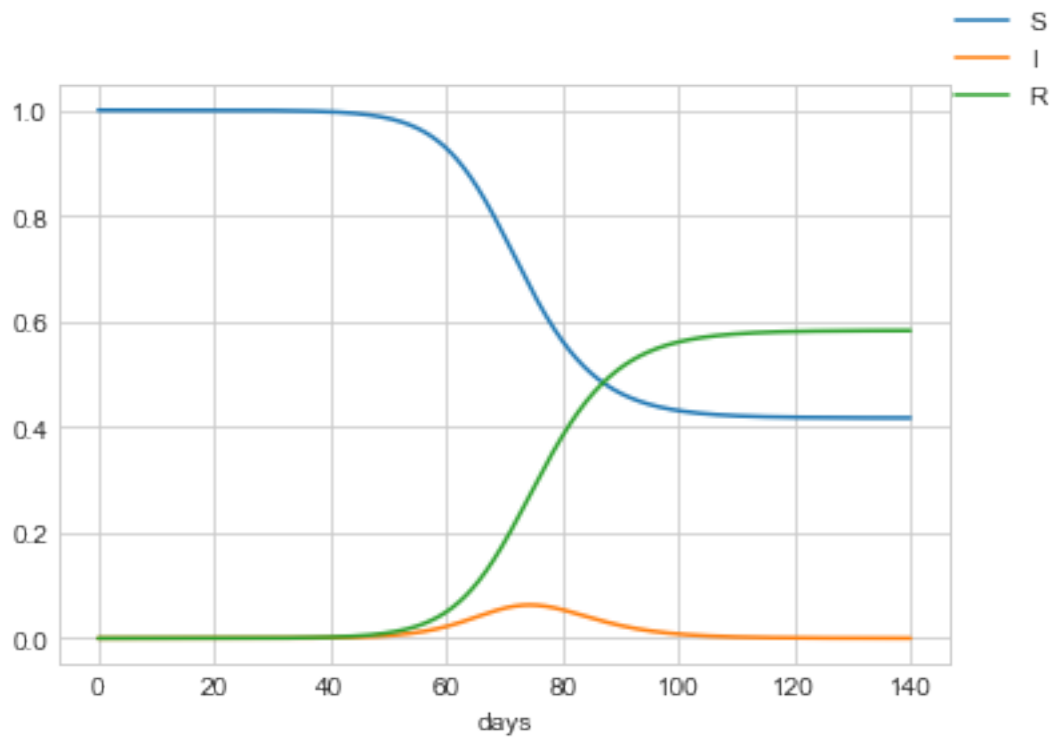
0.00048828125



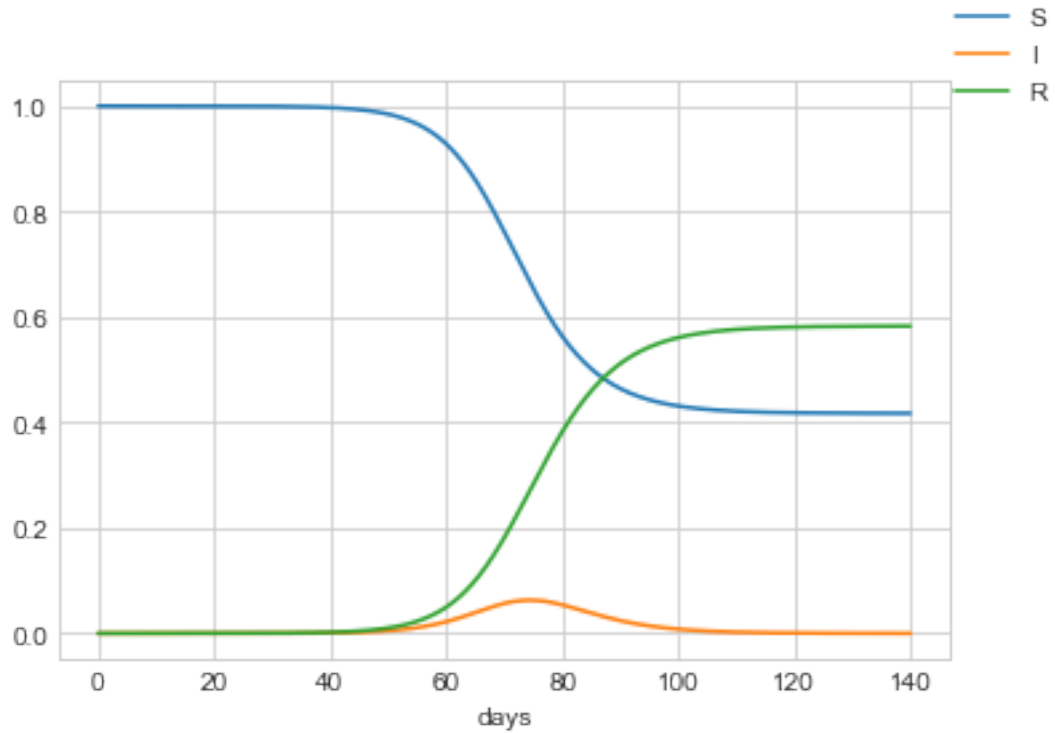
0.000244140625



0.0001220703125

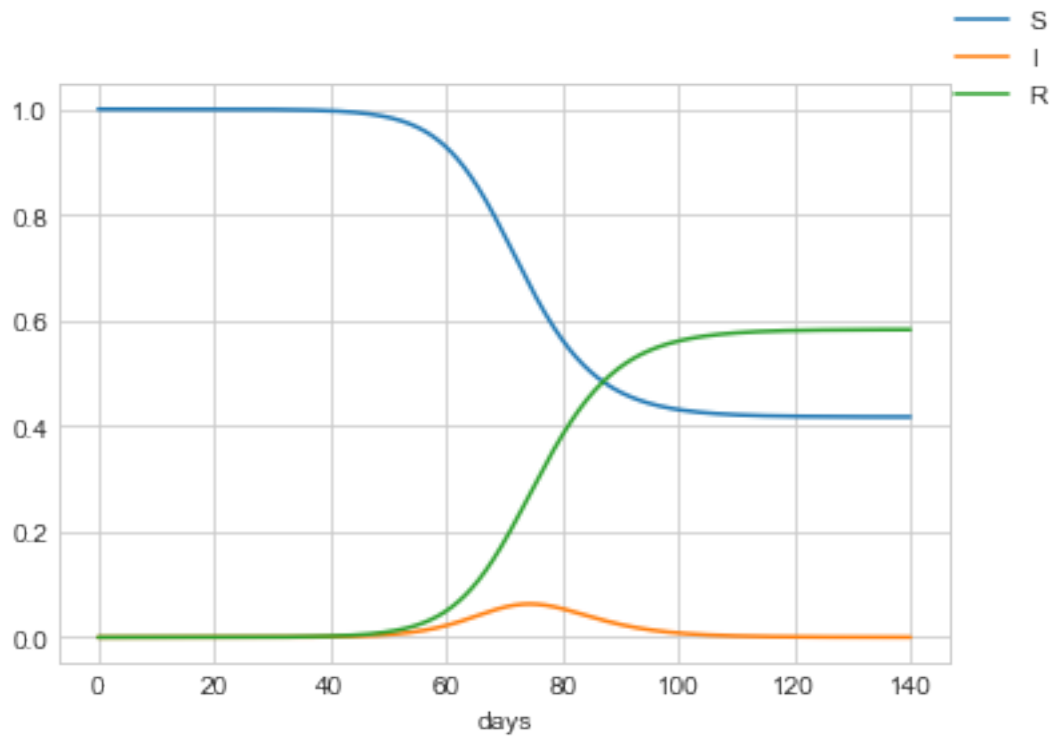


6.103515625e-05

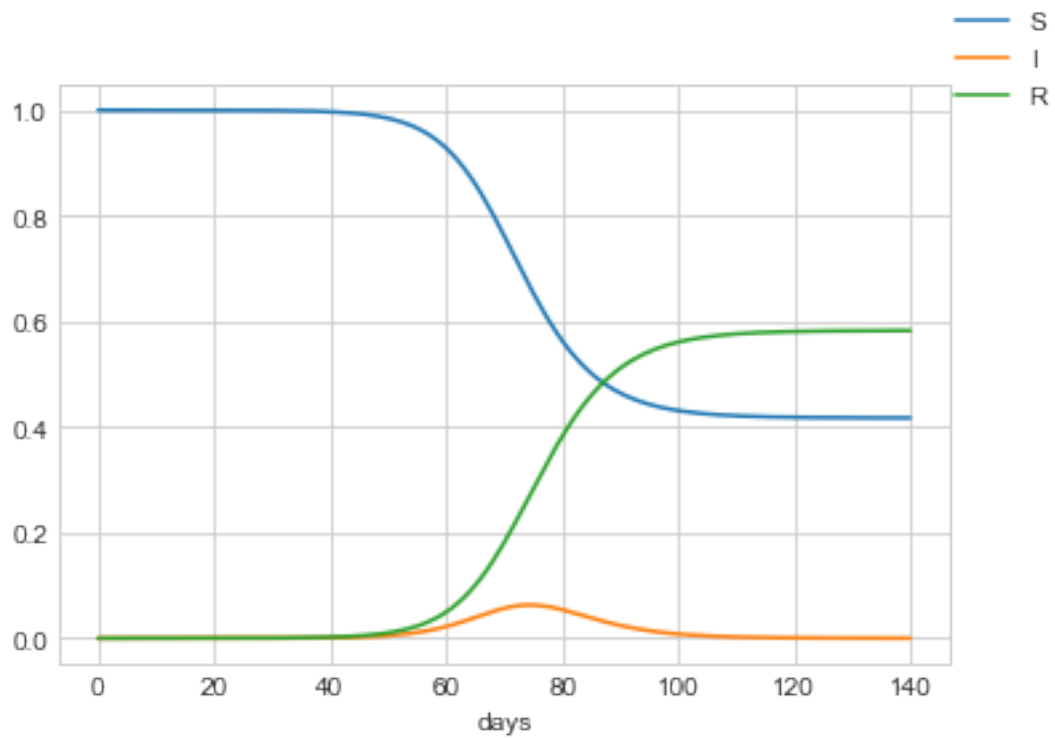


```
[15]: for item in interesting_list:
      print(item)
      SIR_Demo(item, rungeKutta4)
      SIR_Demo(6, rungeKutta4)
```

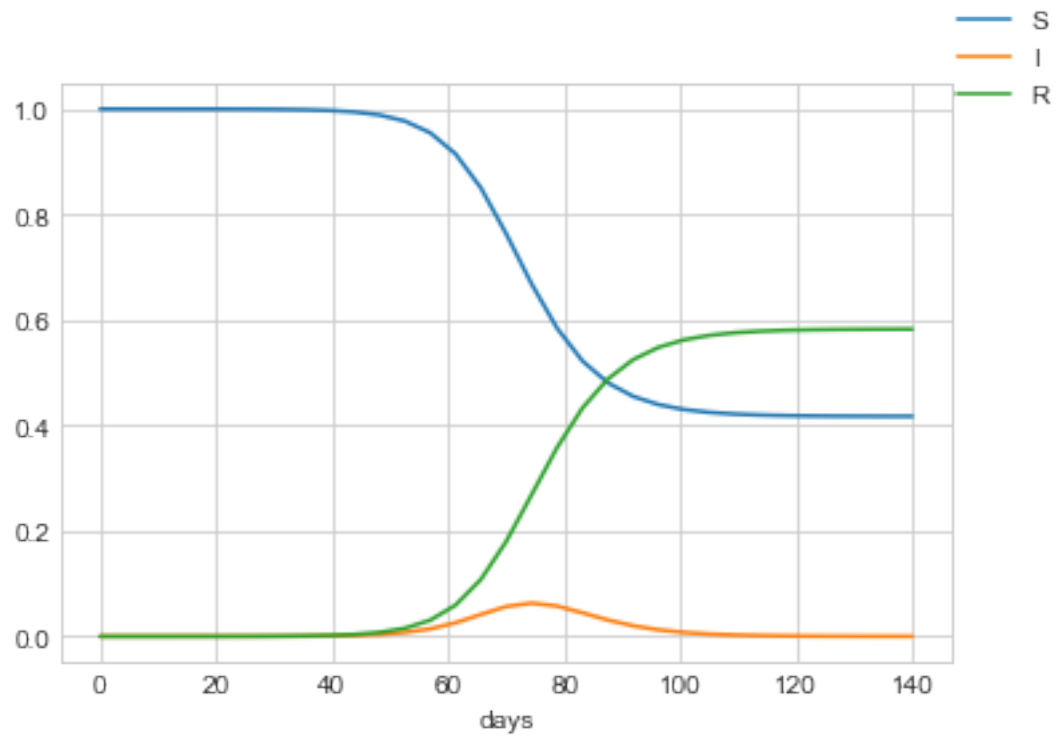
0.041666666666666664



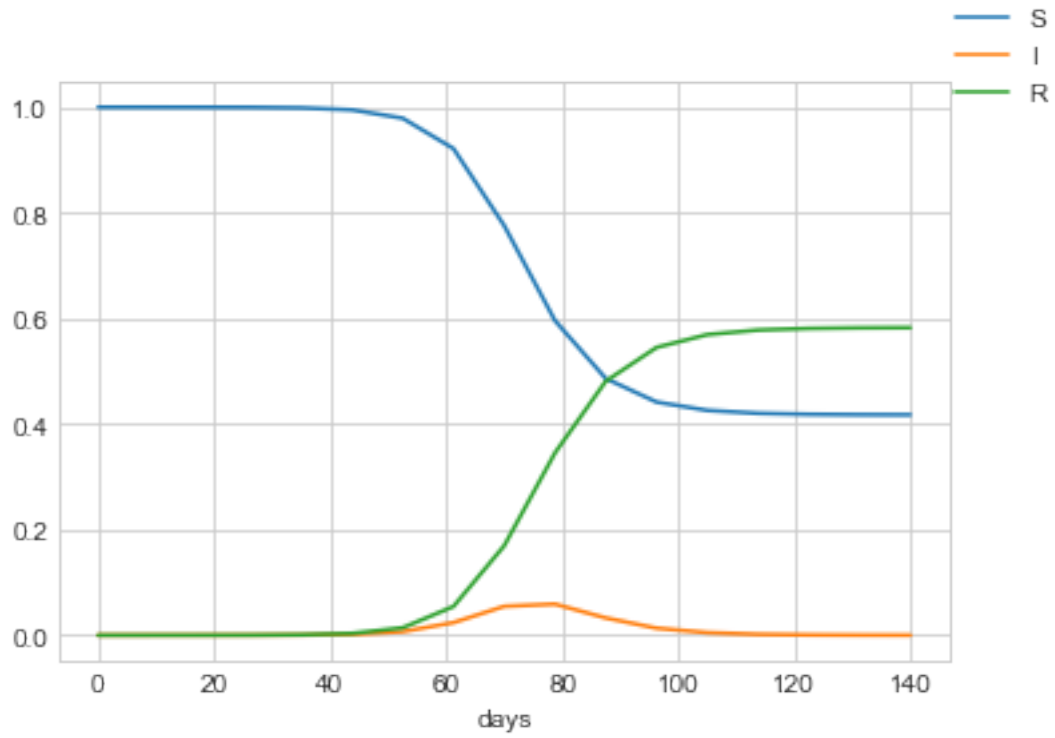
1



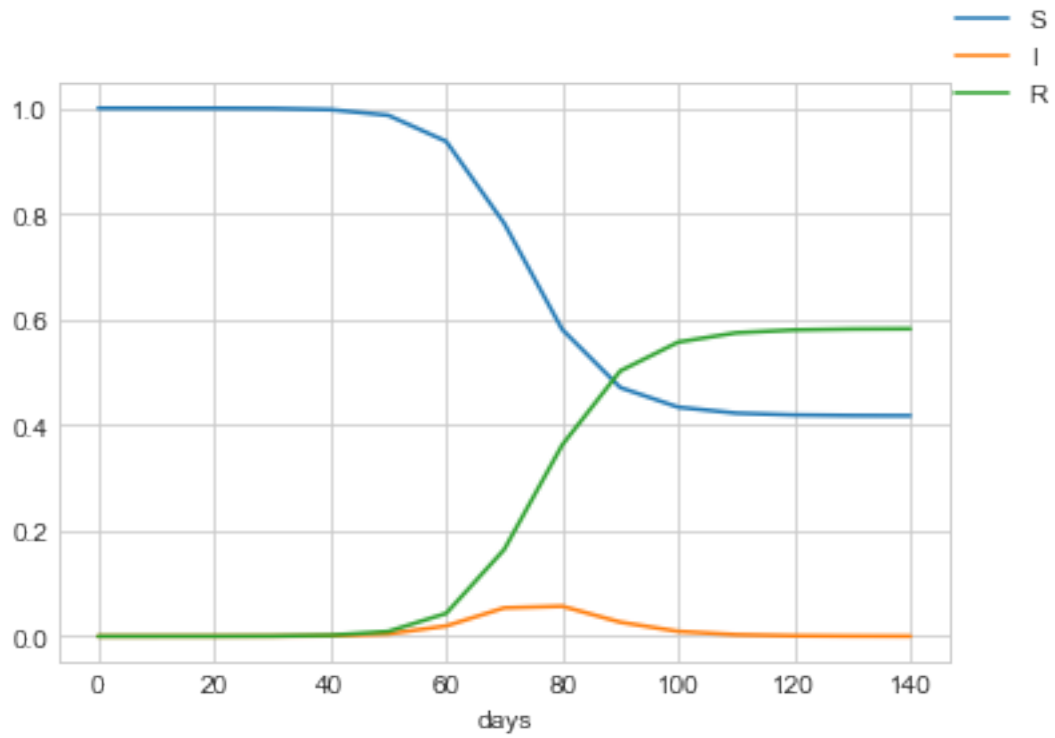
4.375



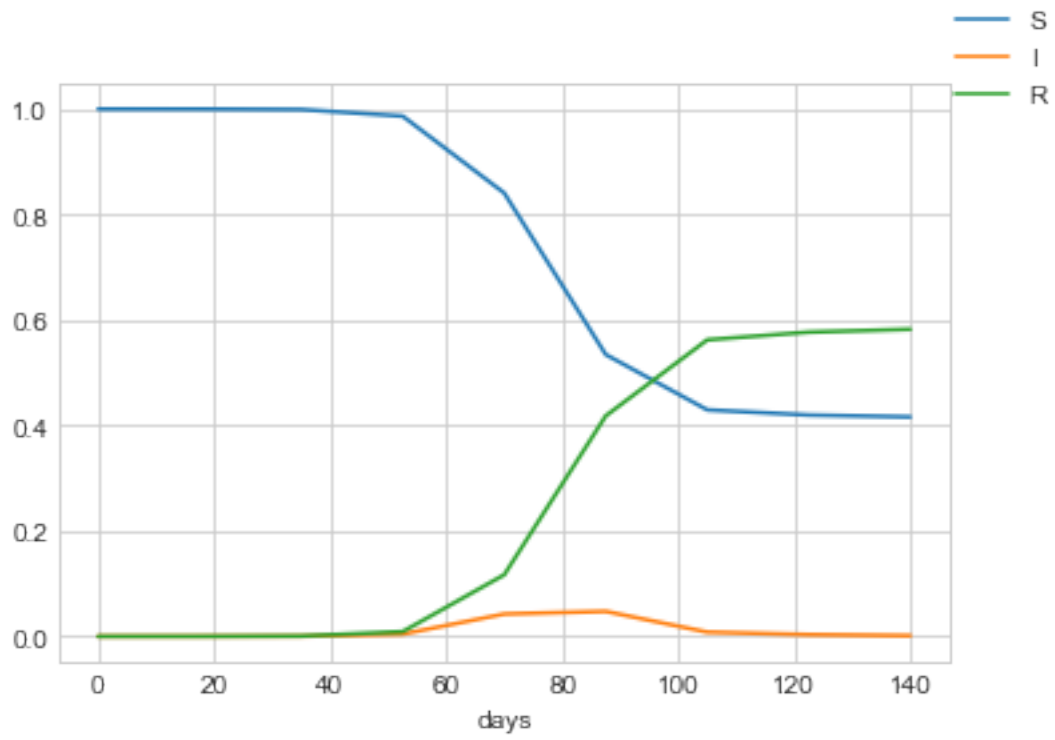
8.75



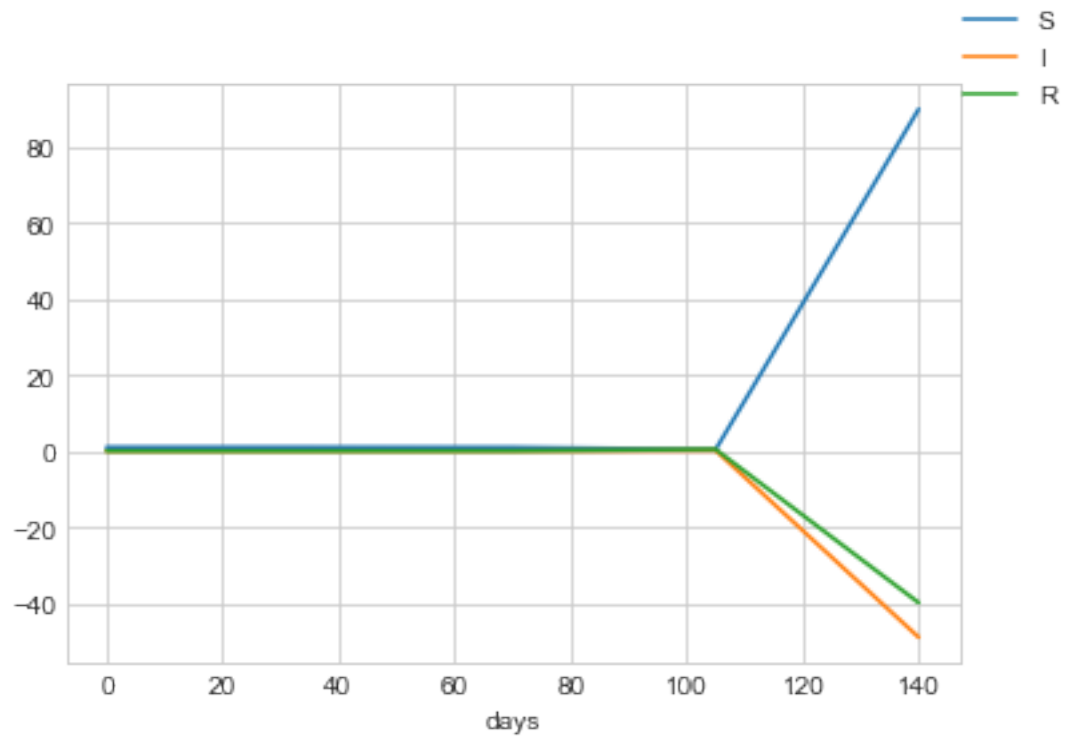
10



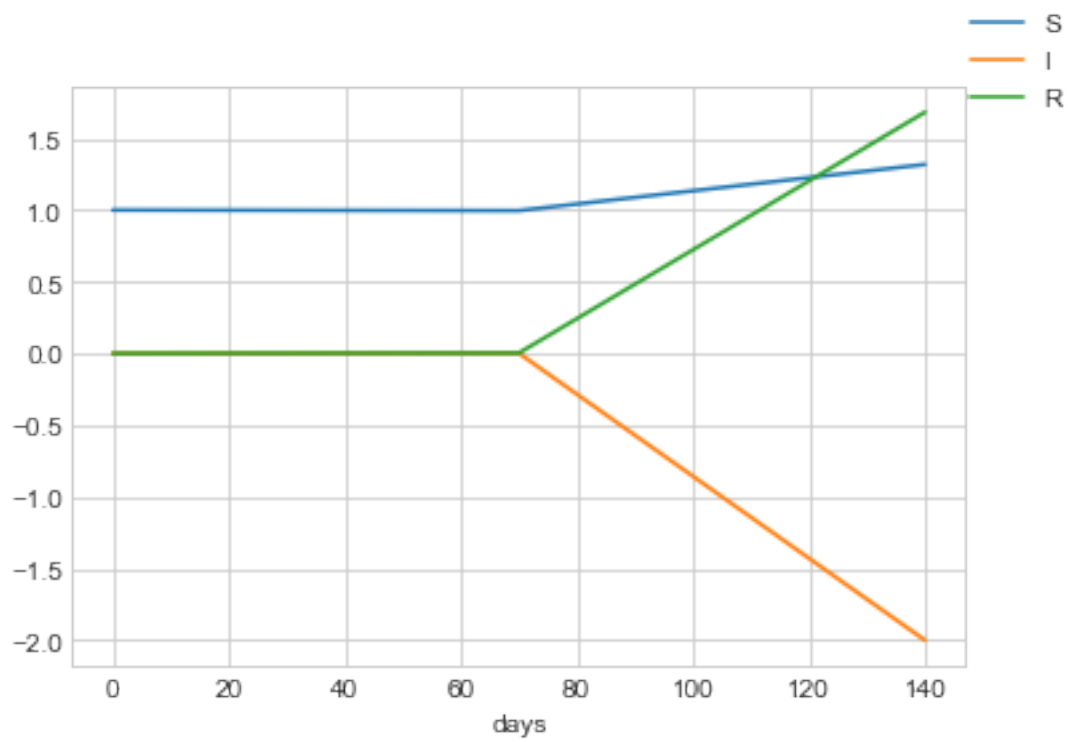
17.5

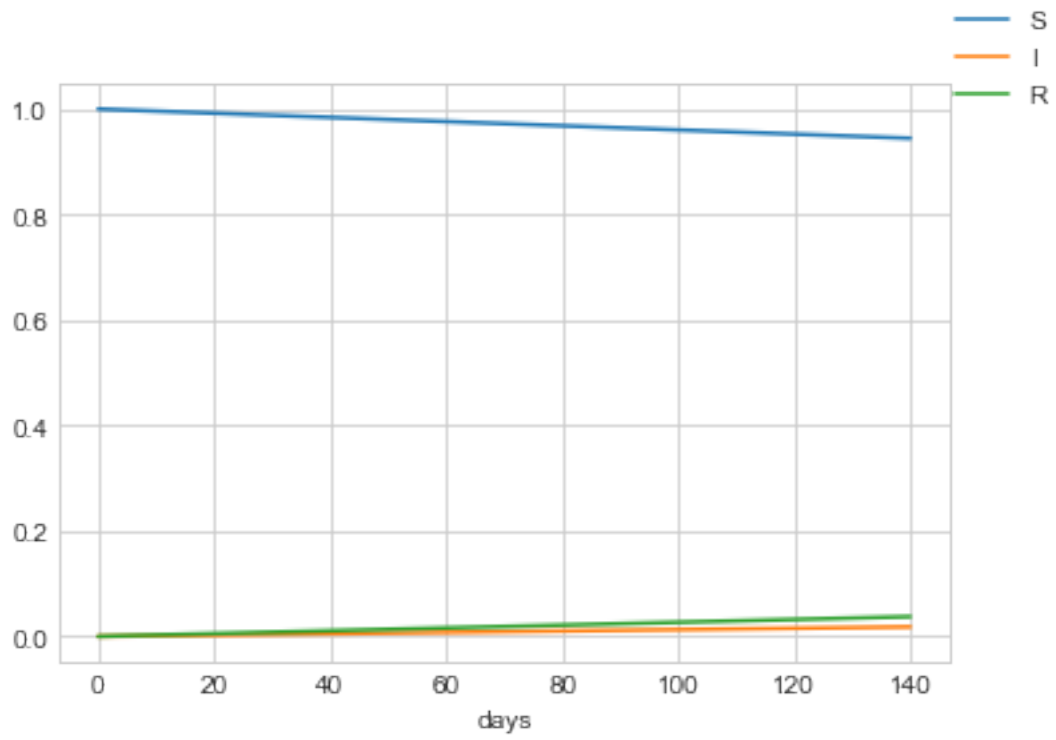


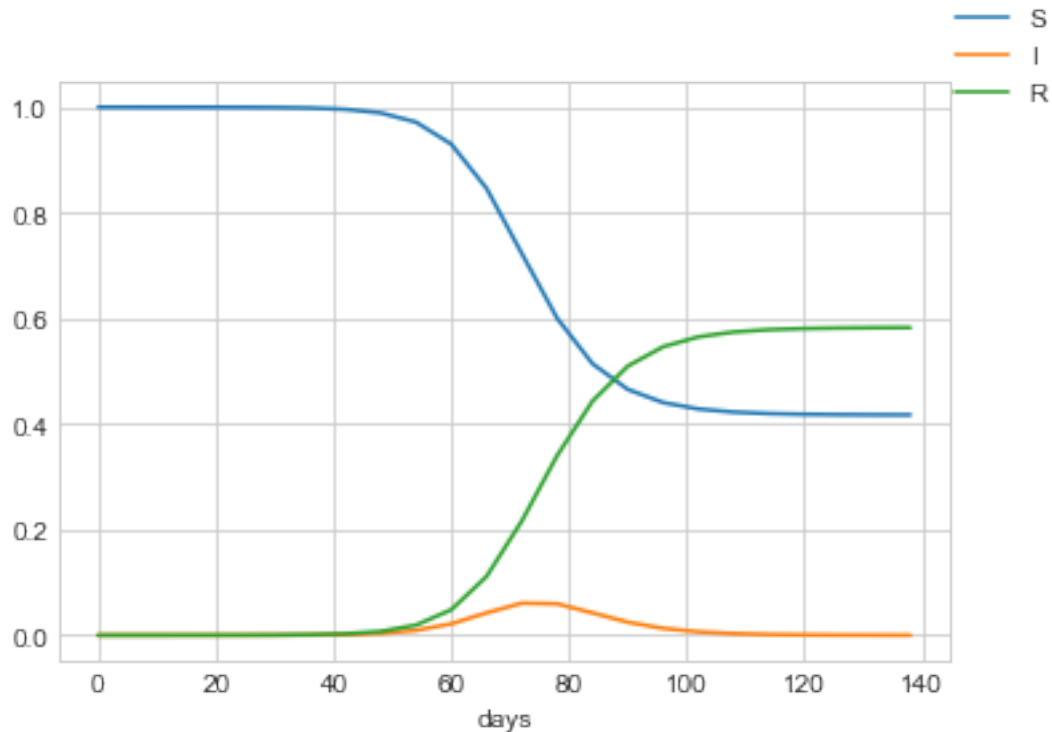
35



70







```
[16]: def f(t, u):
    # If we guess that each infected would make a possibly infecting contact
    # every two days, then  $b = 1/2$ .
    # This is simply a guess.
    b = 1/2
    # We have already estimated the average period of infectiousness at three
    # days, so that would suggest  $k = 1/3$ 
    # This is simply a guess.
    k = 1/3
    S, I, R = u
    #  $ds/dt = -b * s(t) * i(t)$ 
    #  $di/dt = b * s(t) * i(t) - k * i(t)$ 
    #  $dr/dt = k * i(t)$ 
    return np.array([-b*S*I, b*S*I - k*I, k*I])

# num days
T = 140
#  $s(t) = S(t)/N$  : the susceptible fraction of the population,
s_0 = 1
#  $i(t) = I(t)/N$  : the infected fraction of the population,
i_0 = 1.27e-6
#  $r(t) = R(t)/N$  : the recovered fraction of the population,
r_0 = 0
```

```
# initial condition vector
U_0 = [s_0, i_0, r_0]
```

```
[17]: n = 5

hs = [(2**-i) for i in range(n)]
truth = [scipy.integrate.solve_ivp(f, (0.0, T), U_0, max_step=h).y for h in hs]
RK2 = [rungeKutta2(f, U_0, h, T) for h in hs]
RK4 = [rungeKutta4(f, U_0, h, T) for h in hs]

print(hs)
print("truth: ", len(truth))
print("RK2: ", len(RK2))
print("RK4: ", len(RK4))

# print("shape truth: ", np.array(truth).shape)
print("shape RK2: ", np.array(RK2).shape)
print("shape RK4: ", np.array(RK4).shape)
```

```
[1, 0.5, 0.25, 0.125, 0.0625]
truth: 5
RK2: 5
RK4: 5
shape RK2: (5, 2)
shape RK4: (5, 2)
```

I really struggled at getting the scipy method to give me output that I knew how to work with. I would have used that as my truth and found the error that way. Unfortunately, I couldn't do this.

0.0.2 COVID BONUS?

I thought I'd fool around and see how this looked with some COVID numbers!

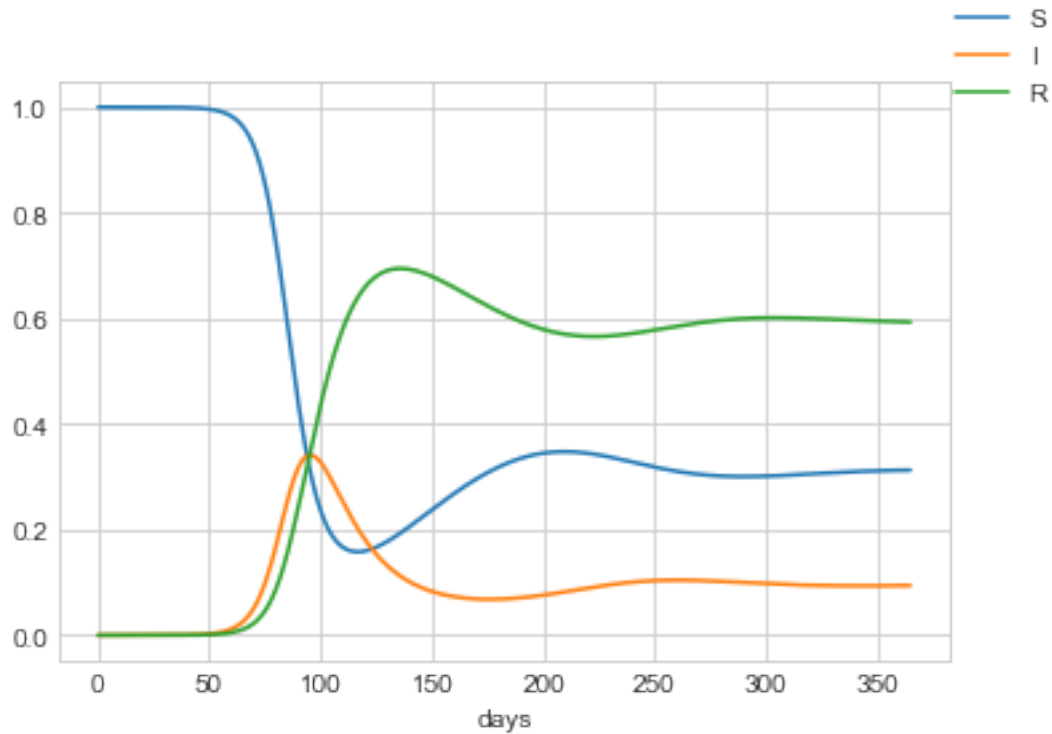
I was going to plot actual data to

```
[18]: # takes the avg. person 4-5 days to start showing symptoms
# infectiousness goes away 10 days after symptoms show up
# k = 1 / 14.5
k=1/14.5

# maybe because people dont develop symptoms for 4.5 days so they can transmit
# → the disease then?
# b = 1/5
b=1/4.5
```

```
# i couldn't find conclusive data on immunity...
# moderna says their vaccine will provide 3 months,
# i saw various studies that say natural immunity is in the 3-12 month range.
# who knows...
v=1/90
```

```
SIR_Demo((1/24), rungeKutta4, num_days=365, b=b, k=k, v=v)
```



```
[19]: df = pd.read_csv('/Users/AndrewMacbook/Downloads/
↳United_States_COVID-19_Cases_and_Deaths_by_State_over_Time.csv')
df.head()
```

```
[19]: submission_date state tot_cases conf_cases prob_cases new_case \
0 01/22/2020 CO 0 NaN NaN 0
1 01/23/2020 CO 0 NaN NaN 0
2 01/24/2020 CO 0 NaN NaN 0
3 01/25/2020 CO 0 NaN NaN 0
4 01/26/2020 CO 0 NaN NaN 0

pnew_case tot_death conf_death prob_death new_death pnew_death \
0 NaN 0 NaN NaN 0 NaN
1 NaN 0 NaN NaN 0 NaN
```

2	NaN	0	NaN	NaN	0	NaN
3	NaN	0	NaN	NaN	0	NaN
4	NaN	0	NaN	NaN	0	NaN

	created_at	consent_cases	consent_deaths
0	03/26/2020 04:22:39 PM	Agree	Agree
1	03/26/2020 04:22:39 PM	Agree	Agree
2	03/26/2020 04:22:39 PM	Agree	Agree
3	03/26/2020 04:22:39 PM	Agree	Agree
4	03/26/2020 04:22:39 PM	Agree	Agree

[]: