HW9

November 13, 2020

Numerical Computation: HW9

Andrew Pickner I worked alone on this assignment.

---

[1]:
```
# global imports

import numpy as np

%matplotlib inline
import matplotlib.pyplot as plt
plt.style.use('seaborn-whitegrid')
```

[2]:
```
def get_error(obs, truth):
    diff = np.subtract(obs, truth)
    return np.sqrt(np.linalg.norm(diff, ord=2) / np.linalg.norm(truth, ord=2))

def get_size(matrix):
    return matrix.shape[0], matrix.shape[1]

def is_symmetric(matrix):
    if matrix.shape[0] == matrix.shape[1]:
        if isinstance(matrix[0][0], float):
            return np.allclose(matrix,matrix.T, atol=1e-05)
        return np.array_equal(matrix,matrix.T)
    return False

def get_rank(matrix):
    if is_symmetric(matrix):
        return np.linalg.matrix_rank(matrix, tol=None, hermitian=True)
    return np.linalg.matrix_rank(matrix, tol=None, hermitian=False)

def get_condition_number(matrix):
    return np.linalg.cond(matrix)

# reworked a function from homework 5 where we use cholesky or LU decomp.␣
↪depending on what
```

```python
def NE_solve1(A,b):
    if is_symmetric(A) and get_rank(A) == get_size(A)[1]:
        print("NE used: Cholesky")
        return cholesky_solve(A,b)
    print("NE used: LU")
    return lu_solve(A,b)

# reworked a function from homework 5 where we use cholesky or LU decomp.␣
↪depending on what
def NE_solve2(A,b):
    if is_symmetric(A) and get_rank(A) == get_size(A)[1]:
        print("NE used: Cholesky")
        return cholesky_solve2(A,b)
    print("NE used: LU")
    return lu_solve2(A,b)

def lu(A):
    #  n = number of rows
    n = A.shape[0]
    U = A.copy()
    L = np.eye(n, dtype=np.double)
    #Loop over rows
    for i in range(n):
        #Eliminate entries below i with row operations
        #on U and reverse the row operations to
        #manipulate L
        factor = U[i+1:, i] / U[i, i]
        L[i+1:, i] = factor
        U[i+1:] -= factor[:, np.newaxis] * U[i]
    return L, U

def lu_solve(A, b):
    new_A = np.matmul(A.T, A)
    new_b = np.matmul(A.T, b)
    L, U = lu(new_A)
    # solve Ly=b for y using forward sub
    y = forward_sub(L, new_b)
    # then solve Ux=y for x using back sub
    return back_sub(U, y)
    # seemed rather consistent with the scipy method so I kept the forward and␣
↪backward that I have.
    # return solve_triangular(U, b)

def lu_solve2(A, b):
    L, U = lu(A)
    # solve Ly=b for y using forward sub
    y = forward_sub(L, b)
```

```python
    # then solve Ux=y for x using back sub
    return back_sub(U, y)
    # seemed rather consistent with the scipy method so I kept the forward and␣
↪backward that I have.
    # return solve_triangular(U, b)

def cholesky(A):
    #  n = number of rows
    n = A.shape[0]
    L = np.zeros((n, n), dtype=np.double)
    for k in range(n):
        L[k, k] = np.sqrt(A[k, k] - np.sum(L[k, :] ** 2))
        L[(k+1):, k] = (A[(k+1):, k] - L[(k+1):, :] @ L[:, k]) / L[k, k]
    return L

def cholesky_solve(A, b):
    new_A = np.matmul(A.T, A)
    new_b = np.matmul(A.T, b)
    L = cholesky(new_A)
#     y = np.linalg.solve(L, new_b)
#     return np.linalg.solve(L.T, y)
    # just special case of LU decomp where U = L.T
    y = forward_sub(L, new_b)
    return back_sub(L.T, y)

def cholesky_solve2(A, b):
    L = cholesky(A)
#     y = np.linalg.solve(L, new_b)
#     return np.linalg.solve(L.T, y)
    # just special case of LU decomp where U = L.T
    y = forward_sub(L, b)
    return back_sub(L.T, y)

def forward_sub(L, b):
    #  n = number of rows
    n = L.shape[0]
    # allocating space for the solution vector
    y = np.zeros_like(b, dtype=np.double);
    #Here we perform the forward-substitution.
    #Initializing  with the first row.
    y[0] = b[0] / L[0, 0]
    #Looping over rows in reverse (from the bottom  up),
    #starting with the second to last row, because  the
    #last row solve was completed in the last step.
    for i in range(1, n):
        y[i] = (b[i] - np.dot(L[i,:i], y[:i])) / L[i,i]
    return y
```

```python
def back_sub(U, y):
    #  n = number of rows
    n = U.shape[0]
    # allocating space for the solution vector
    x = np.zeros_like(y, dtype=np.double);
    #Here we perform the back-substitution.
    #Initializing with the last row.
    x[-1] = y[-1] / U[-1, -1]
    #Looping over rows in reverse (from the bottom up),
    #starting with the second to last row, because the
    #last row solve was completed in the last step.
    for i in range(n-2, -1, -1):
        x[i] = (y[i] - np.dot(U[i,i:], x[i:])) / U[i,i]
    return x

# 1 NE
def NE_solver1(A, b):
    return NE_solve1(A,b)

def NE_solver2(A, b):
    new_A = np.matmul(A.T, A)
    new_b = np.matmul(A.T, b)
    return NE_solve1(new_A,new_b)

# 2 QR
def QR_solver(A, b):
    Q, R = np.linalg.qr(A)
    return np.linalg.solve(R, Q.T.dot(b))
```

[3]:
```python
f = lambda x: np.sin(2 * np.pi * x) + np.cos(3 * np.pi * x)
```

[4]:
```python
num_points = 33
interval = -1, 1

xs = np.linspace(interval[0], interval[1], num=num_points)
```

[5]:
```python
# degree of p(x)
d = 7

# vandermonde matrix to represent our over-determined system
A = np.vander(xs, N=d+1, increasing=True)

# actual values of f(x) that were trying to approximate
ys = np.array([f(x) for x in xs])

# our coefficients for the polynomial of
```

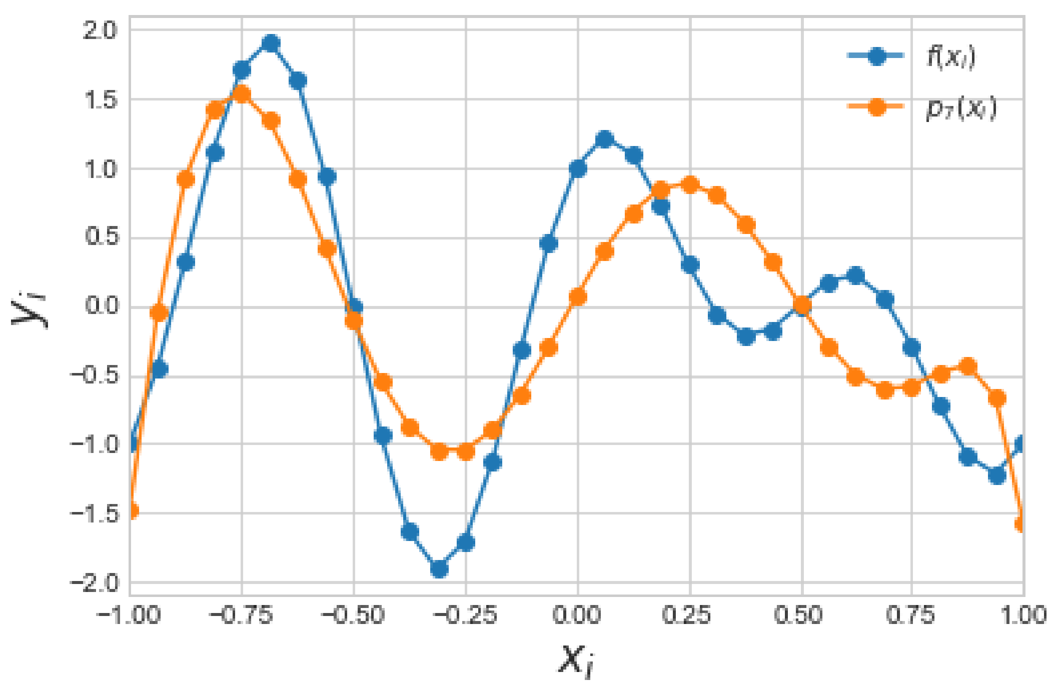
```
cs = np.linalg.lstsq(A, ys, rcond=None)[0]
```

[6]:
```
def f_tilde(cs, x):
    return sum([cs[i] * (x ** i) for i in range(len(cs))])
```

[7]:
```
approx = [f_tilde(cs, x) for x in xs]
```

[8]:
```
plt.plot(xs, ys, '-o', label='$f(x_{i})$')
plt.plot(xs, approx, '-o', label='$p_{7}(x_{i})$')
plt.xlim([interval[0], interval[1]])
plt.xlabel("$x_{i}$", fontsize='xx-large')
plt.ylabel("$y_{i}$", fontsize='xx-large')
plt.legend()
```

[8]:
```
<matplotlib.legend.Legend at 0x10b186710>
```



[9]:
```
def generate_random_xs(low=-1, high=1, n=100):
    l1 = np.append(np.array([low]), low * np.random.rand(n//2 - 1))
    l2 = np.append(np.array([high]), high * np.random.rand(n//2 - 1))
    ret = np.append(l1, l2)
    return np.sort(ret)
```

[10]:
```
new_size = 100
```

```python
# i
new_xs = generate_random_xs()

# ii
new_ys = np.array([f(x) for x in new_xs])
```

[11]:

```python
d_upper = 31

qr_error = []
ne_error = []

for i in range(1, d_upper + 1):
    print(i)
    ai = np.vander(new_xs, N=i+1, increasing=True)

    qr_cs   = QR_solver(ai, new_ys)
    ne_cs   = NE_solver1(ai, new_ys)

    qr_approx = [f_tilde(qr_cs, x) for x in new_xs]
    ne_approx = [f_tilde(ne_cs, x) for x in new_xs]

    qr_error.append(get_error(qr_approx, new_ys))
    ne_error.append(get_error(ne_approx, new_ys))
```

```
1
NE used: LU
2
NE used: LU
3
NE used: LU
4
NE used: LU
5
NE used: LU
6
NE used: LU
7
NE used: LU
8
NE used: LU
9
NE used: LU
10
NE used: LU
11
NE used: LU
12
```

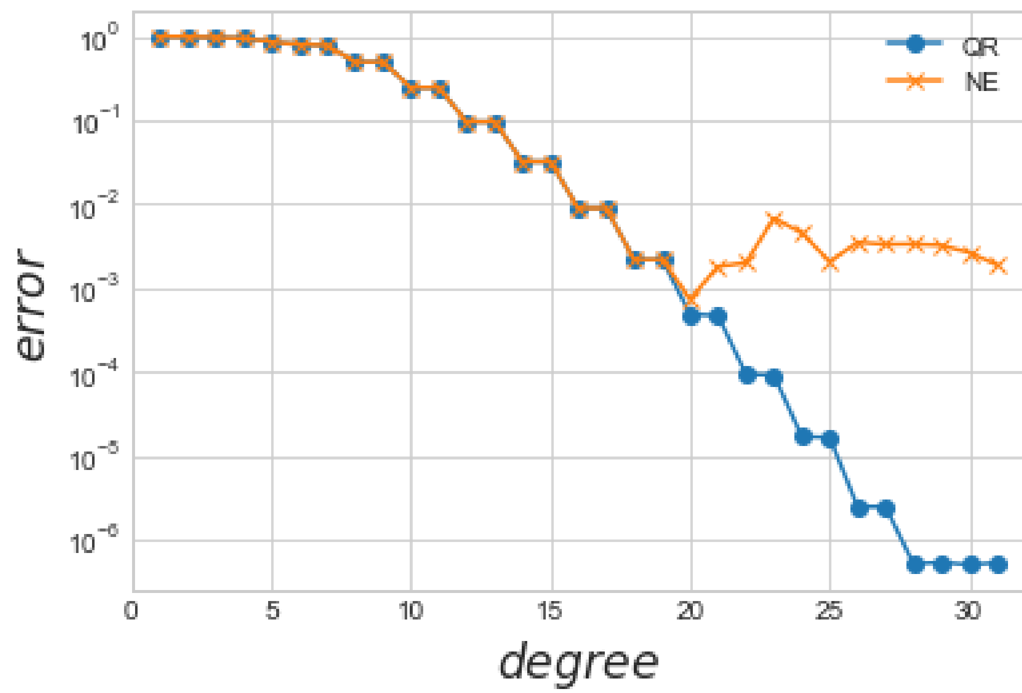
```
NE used: LU
13
NE used: LU
14
NE used: LU
15
NE used: LU
16
NE used: LU
17
NE used: LU
18
NE used: LU
19
NE used: LU
20
NE used: LU
21
NE used: LU
22
NE used: LU
23
NE used: LU
24
NE used: LU
25
NE used: LU
26
NE used: LU
27
NE used: LU
28
NE used: LU
29
NE used: LU
30
NE used: LU
31
NE used: LU
```

[12]:

```
k = [n for n in range(1, d_upper + 1)]
plt.semilogy(k, qr_error, '-o', label='QR')
plt.semilogy(k, ne_error, '-x', label='NE')

plt.xlim([0, d_upper + 1])
plt.xlabel("$degree$", fontsize='xx-large')
plt.ylabel("$error$", fontsize='xx-large')
```

```
plt.legend()
```

[12]: <matplotlib.legend.Legend at 0x10d350650>



[13]:
```
d_upper = 31

qr_error = []
ne_error = []

for i in range(1, d_upper + 1):
    print(i)
    ai = np.vander(new_xs, N=i+1, increasing=True)

    qr_cs   = QR_solver(ai, new_ys)
    ne_cs   = NE_solver2(ai, new_ys)

    qr_approx = [f_tilde(qr_cs, x) for x in new_xs]
    ne_approx = [f_tilde(ne_cs, x) for x in new_xs]

    qr_error.append(get_error(qr_approx, new_ys))
    ne_error.append(get_error(ne_approx, new_ys))
```

```
1
NE used: Cholesky
```

```
2
NE used: Cholesky
3
NE used: Cholesky
4
NE used: Cholesky
5
NE used: Cholesky
6
NE used: Cholesky
7
NE used: Cholesky
8
NE used: Cholesky
9
NE used: Cholesky
10
NE used: Cholesky
11
NE used: Cholesky
12
NE used: Cholesky
13
NE used: Cholesky
14
NE used: Cholesky
15
NE used: Cholesky
16
NE used: Cholesky
17
NE used: Cholesky
18
NE used: Cholesky
19
NE used: Cholesky
20
NE used: LU
21
NE used: LU
22
NE used: LU
23
NE used: LU
24
NE used: LU
25
NE used: LU
```

```
26
NE used: LU
27
NE used: LU
28
NE used: LU
29
NE used: LU
30
NE used: LU
31
NE used: LU
```

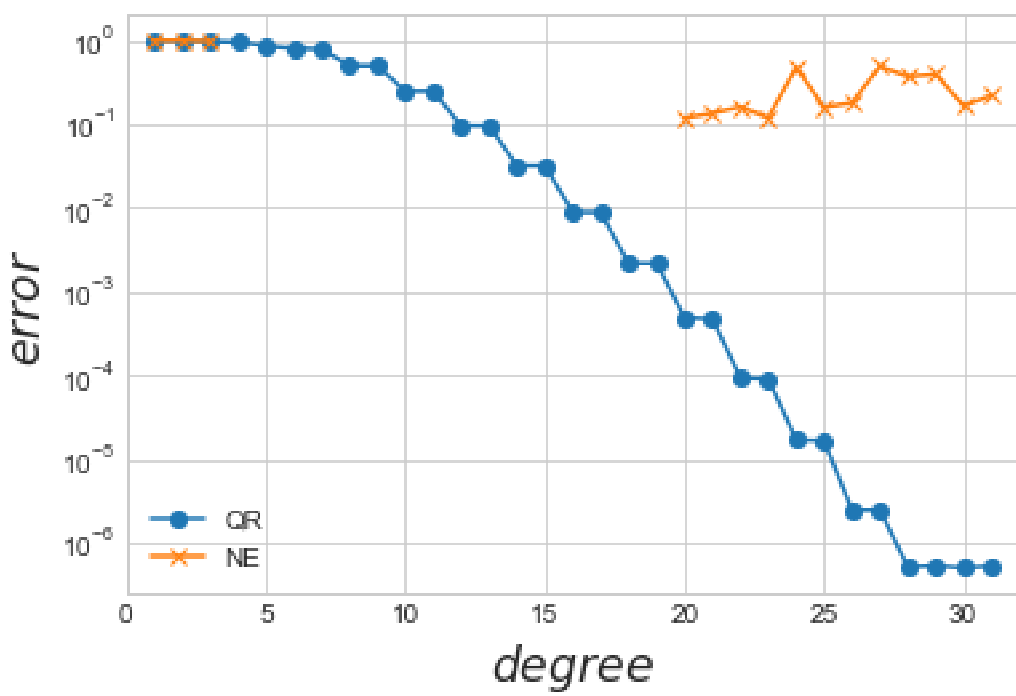
```
/usr/local/lib/python3.7/site-packages/ipykernel_launcher.py:79: RuntimeWarning:
invalid value encountered in sqrt
```

[14]:

```
k = [n for n in range(1, d_upper + 1)]
plt.semilogy(k, qr_error, '-o', label='QR')
plt.semilogy(k, ne_error, '-x', label='NE')

plt.xlim([0, d_upper + 1])
plt.xlabel("$degree$", fontsize='xx-large')
plt.ylabel("$error$", fontsize='xx-large')
plt.legend()
```

[14]:

```
<matplotlib.legend.Legend at 0x10d519f50>
```



0.0.1 BONUS

Here some US COVID case counts from back in March:

Days since Feb 29	Case count
1	89
2	105
3	125
4	159
5	227
6	331
7	444
8	564
9	728
10	1000
11	1267
12	1645
13	2204
14	2826
15	3485
17	7038

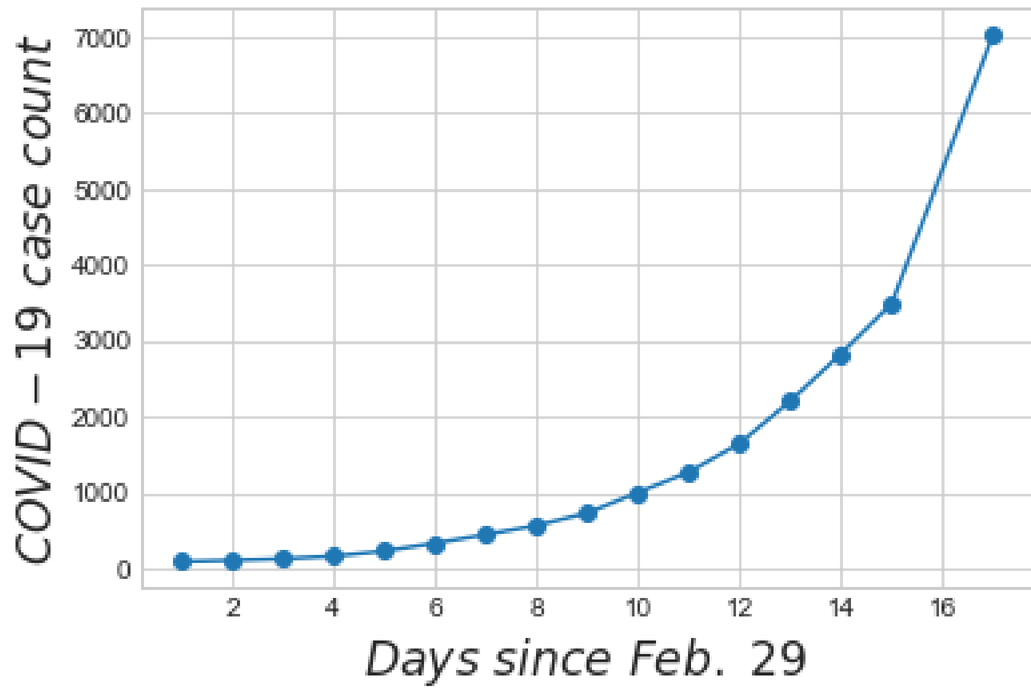
Derive the linear least-squares system whose solution contains the coefficients of a log-linear model for case count over time. Plot the data on top of the model on a log scale. How well does the log-linear model (which represents exponential growth) appear to model the growth in the case count? What was the case doubling time over this roughly two week period?

Essentially, we are going to need to 'linearize' a exponential model: $y = c_1 e^{c_2 \cdot t}$.

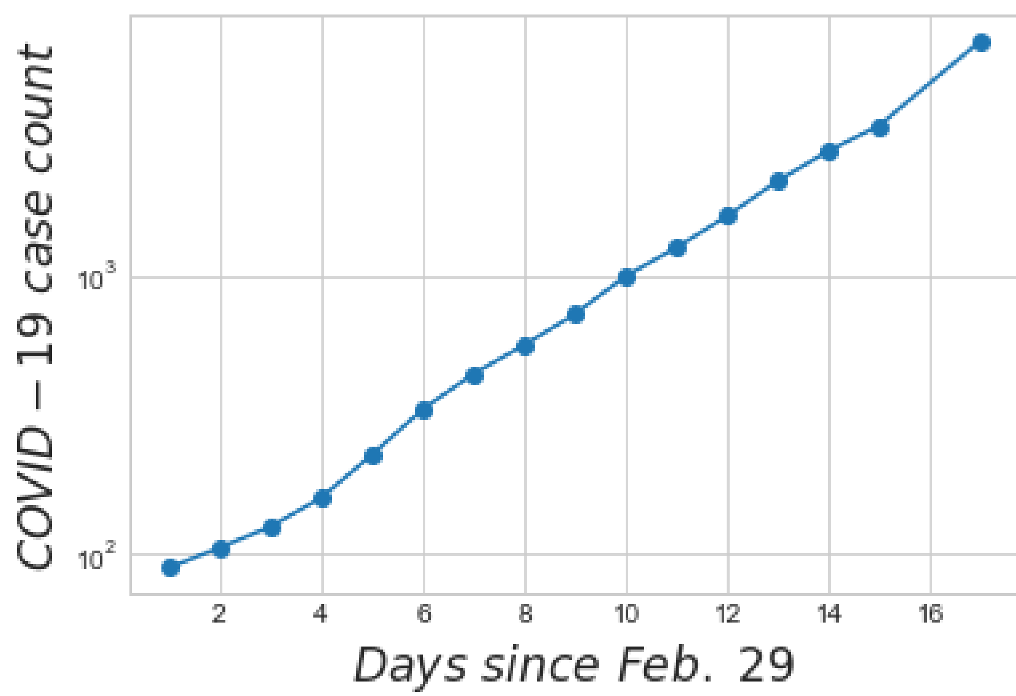
We can do this like so: $ln(y) = ln(c_1) + c_2 \cdot t = k + c_2 \cdot t$

[15]:
```
days_since_feb_29 = [day for day in range(1, 16)] + [17]
case_count        = [89, 105, 125, 159, 227, 331, 444, 564, 728, 1000, 1267,␣
 ↪1645, 2204, 2826, 3485, 7038]

plt.plot(days_since_feb_29, case_count,  '-o')
plt.xlabel("$Days\ since\ Feb.\ 29$", fontsize='xx-large')
plt.ylabel("$COVID-19\ case\ count$", fontsize='xx-large')
plt.show()
plt.semilogy(days_since_feb_29, case_count,  '-o')
plt.xlabel("$Days\ since\ Feb.\ 29$", fontsize='xx-large')
plt.ylabel("$COVID-19\ case\ count$", fontsize='xx-large')
```

```
plt.show()
```



COVID − 19 case count
7000
6000
5000
4000
3000
2000
1000
0
2
4
6
8
10
12
14
16
Days since Feb. 29



COVID − 19 case count
10^3
10^2
2
4
6
8
10
12
14
16
Days since Feb. 29

[16]:
```
b = np.array([np.log(y) for y in case_count])
print(b)
A = np.array([[float(1), float(x)] for x in days_since_feb_29])
print(A)
```

```
[4.48863637 4.65396035 4.82831374 5.0689042  5.42495002 5.80211838
 6.09582456 6.33505425 6.59030105 6.90775528 7.14440718 7.40549566
 7.69802917 7.94661756 8.15622332 8.85907932]
[[ 1.  1.]
 [ 1.  2.]
 [ 1.  3.]
 [ 1.  4.]
 [ 1.  5.]
 [ 1.  6.]
 [ 1.  7.]
 [ 1.  8.]
 [ 1.  9.]
 [ 1. 10.]
 [ 1. 11.]
 [ 1. 12.]
 [ 1. 13.]
 [ 1. 14.]
 [ 1. 15.]
 [ 1. 17.]]
```

[17]:
```
print(NE_solver1(A, b))
```

```
NE used: LU
[4.0920173  0.27688609]
```

[18]:
```
print(QR_solver(A, b))
```

```
[4.0920173  0.27688609]
```

[19]:
```
print(np.linalg.lstsq(A, b, rcond=None)[0])
```

```
[4.0920173  0.27688609]
```

[20]:
```
print(np.exp(4.0920173))
```

```
59.860526625182786
```

All three solvers produce the same output, thus we can proceed and solve k for c_1:

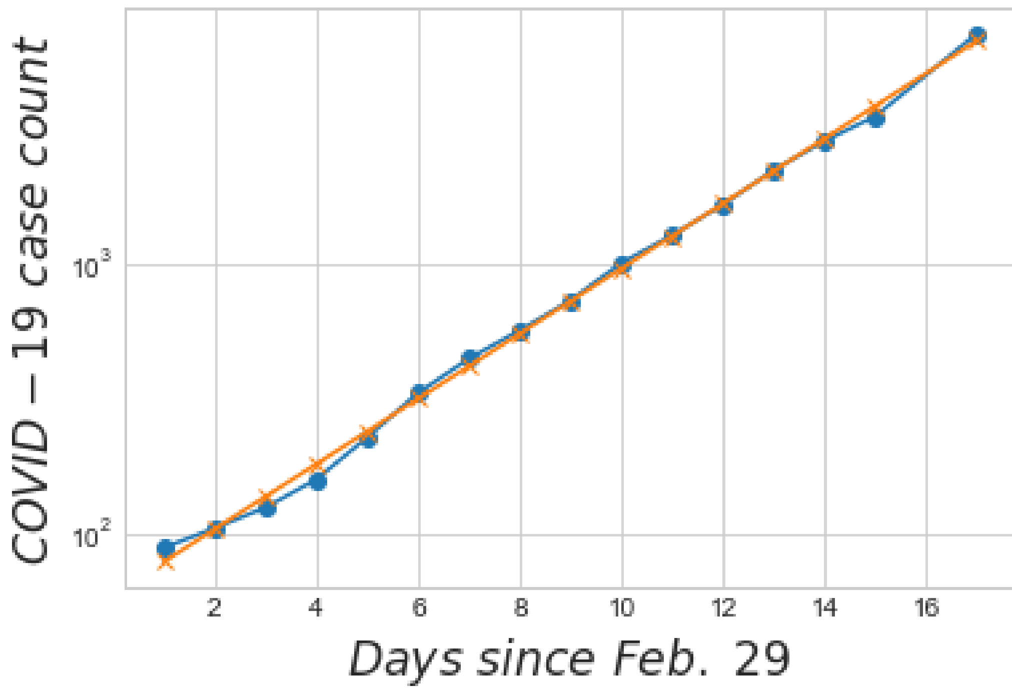
$c_1 \approx e^k$

$c_1 \approx e^{4.0920173}$

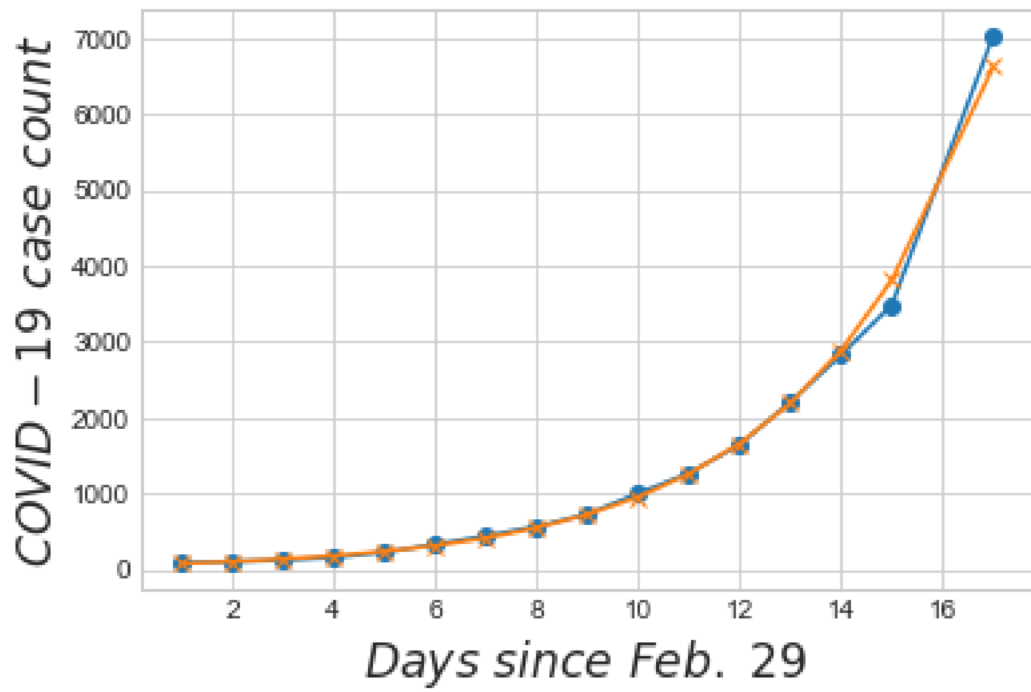
$c_1 \approx 59.8605266$

[21]:
```
approx_f = lambda t: 59.8605266 * np.exp(0.27688609 * t)
```

[22]:
```
approx = [approx_f(x) for x in days_since_feb_29]

plt.semilogy(days_since_feb_29, case_count,  '-o')
plt.semilogy(days_since_feb_29, approx,  '-x')

plt.xlabel("$Days\ since\ Feb.\ 29$", fontsize='xx-large')
plt.ylabel("$COVID-19\ case\ count$", fontsize='xx-large')
plt.show()
```



[23]:
```
approx = [approx_f(x) for x in days_since_feb_29]

plt.plot(days_since_feb_29, case_count,  '-o')
plt.plot(days_since_feb_29, approx,  '-x')

plt.xlabel("$Days\ since\ Feb.\ 29$", fontsize='xx-large')
plt.ylabel("$COVID-19\ case\ count$", fontsize='xx-large')
plt.show()
```



For starters, I always like to see the data on a log and linear scale. The log-linear model appears to model the growth of COVID-19 cases extremely well for the first two weeks and we find a tiny bit of error as we extend into the last few data points, but overall, this seems like a great model.

The simple way I remember calculating half life is by setting the equation we have above: $y = 59.8605266 \cdot e^{0.27688609 \cdot t}$ into $2 = e^{0.27688609 \cdot t}$. Which, can be further simplified into: $\frac{ln(2)}{0.27688609} = t$.

So the doubling time turns out to around 2 and a half days, and this seems to check out when looking at the data: it appears to double every 2 or 3 days.

---

[24]:
```
print(np.log(2) / 0.27688609)
```

```
2.503365844632879
```

[ ]:
```
```