Homework Sheet 8

Author	**Matriculation Number**	**Tutor**
Abdullah Oğuz Topçuoğlu	7063561	Maryna Dernovaia
Ahmed Waleed Ahmed Badawy Shora	7069708	Jan-Hendrik Gindorf
Yousef Mostafa Farouk Farag	7073030	Thorben Johr

Exercise 1 (Coin Flips)

a.

This is deterministic. She always stops after exactly n flips.

$$\mathbb{E}[T] = n.$$

b.

Let T be the time of the first head. This is a geometric random variable with success probability $p = \frac{1}{2}$:

$$\mathbb{E}[T] = \frac{1}{p} = 2.$$

c.

This is the negative-binomial stopping time for n successes with success probability $p = \frac{1}{2}$:

$$\mathbb{E}[T] = \frac{n}{p} = 2n.$$

d.

Let E_0, E_1, E_2 denote the expected remaining flips given that the current run of consecutive tails has length $0, 1, 2$. We stop when we reach a run of 3 tails, so $E_3 = 0$.

We set up recurrences:

$$\begin{aligned} E_0 &= 1 + \tfrac{1}{2}E_1 + \tfrac{1}{2}E_0, \\ E_1 &= 1 + \tfrac{1}{2}E_2 + \tfrac{1}{2}E_0, \\ E_2 &= 1 + \tfrac{1}{2} \cdot 0 + \tfrac{1}{2}E_0. \end{aligned}$$

Solving these:

$$E_2 = 1 + \tfrac{1}{2}E_0, \qquad E_0 = 2 + E_1.$$

Substitute into the equation for E_1:

$$E_1 = 1 + \tfrac{1}{2}(1 + \tfrac{1}{2}E_0) + \tfrac{1}{2}E_0 = \tfrac{3}{2} + \tfrac{3}{4}E_0.$$

Thus,

$$E_0 = 2 + \tfrac{3}{2} + \tfrac{3}{4}E_0 \quad \Rightarrow \quad E_0 - \tfrac{3}{4}E_0 = \tfrac{7}{4} \quad \Rightarrow \quad E_0 = 14.$$

Hence the expected number of flips until three consecutive tails appear is

$$\mathbb{E}[T] = 14.$$

Exercise 3

```
void Permute(A[1..n])
   if (n == 1) return
   int randomIndex = rand(n) // rand() function from the lecture
   swap(A[1], A[randomIndex])
   Permute(A[2..n])
```

Correctness Proof:
We prove by induction that the algorithm produces a uniform random permutation of the array $A[1..n]$.

Base Case: For $n = 1$, there is only one permutation. The algorithm returns the array unchanged.

Inductive Step: Assume the algorithm produces a uniformly random permutation for arrays of size $k-1$. For size k, the algorithm selects a random index from 1 to k, placing each element in the first position with probability $1/k$. Then it recursively permutes the remaining $k-1$ elements uniformly by the induction hypothesis. Thus the resulting permutation is uniform.

Running Time:

$$T(1) = 1, \qquad T(n) = T(n-1) + 1,$$

so the running time is

$$T(n) = O(n).$$