# Mathematics Homework Sheet 6

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## Problem 2

### (a)

$U_1$:
$U_1$ is subspace of $R[x]$.
**Not empty:**
$U_1$ is not empty because $0 \in U_1$ (zero polynomial).
**Closed under addition:**
Let $p(x), q(x) \in U_1$. Then $p(0) = 0$ and $q(0) = 0$.
Then, $(p+q)(0) = p(0) + q(0) = 0 + 0 = 0$.
**Closed under scalar multiplication:**
Let $p(x) \in U_1$ and $c \in R$. Then, $(cp)(0) = c(p(0)) = c(0) = 0$.
Thus, $U_1$ is closed under scalar multiplication.

$U_2$:
$U_2$ is not a subspace of $R[x]$. Because $U_2$ doesn't contain the zero polynomial. (every vector space has to contain the zero vector which is the zero polynomial in this case)

$U_3$:
$U_3$ is subspace of $R[x]$.
**Not empty:**
$U_3$ is not empty because $0 \in U_3$ (zero polynomial).
**Closed under addition:**
Let $p(x), q(x) \in U_3$. Then $p(1) = 0$ and $q(1) = 0$.
Then, $(p+q)(1) = p(1) + q(1) = 0 + 0 = 0$.
**Closed under scalar multiplication:**
Let $p(x) \in U_3$ and $c \in R$. Then, $(cp)(1) = c(p(1)) = c(0) = 0$.
Thus, $U_3$ is closed under scalar multiplication.

$U_4$:
$U_4$ is subspace of $R[x]$.
**Not empty:**
$U_4$ is not empty because $0 \in U_4$ (zero polynomial).
**Closed under addition:**
Let $p(x), q(x) \in U_4$. Then $\int_0^1 p(x)dx = 0$ and $\int_0^1 q(x)dx = 0$.
Then, $\int_0^1 (p+q)(x)dx = \int_0^1 p(x)dx + \int_0^1 q(x)dx = 0 + 0 = 0$.
**Closed under scalar multiplication:**

Let $p(x) \in U_4$ and $c \in R$. Then, $\int_0^1 (cp)(x)dx = c\int_0^1 p(x)dx = c(0) = 0$.
Thus, $U_4$ is closed under scalar multiplication.

$U_5$:
$U_5$ is subspace of $R[x]$.
**Not empty:**
$U_5$ is not empty because $0 \in U_5$ (zero polynomial).
**Closed under addition:**
Let $p(x), q(x) \in U_5$. Then $p'(0) + p''(0) = 0$ and $q'(0) + q''(0) = 0$.
Then, $(p+q)'(0) + (p+q)''(0) = p'(0) + q'(0) + p''(0) + q''(0) = 0 + 0 = 0$.
**Closed under scalar multiplication:**
Let $p(x) \in U_5$ and $c \in R$. Then, $(cp)'(0) + (cp)''(0) = c(p'(0)) + c(p''(0)) = c(p'(0) + p''(0)) = c(0) = 0$.
Thus, $U_5$ is closed under scalar multiplication.

$U_6$:
$U_6$ is not a subspace of $R[x]$. Becuase it is not closed under addition
Let $p(x), q(x) \in U_6$. Then $p'(0)p''(0) = 0$ and $q'(0)q''(0) = 0$.
Then, $(p+q)'(0)(p+q)''(0) = (p'(0) + q'(0))(p''(0) + q''(0)) = p'(0)p''(0) + p'(0)q''(0) + q'(0)p''(0) + q'(0)q''(0) = p'(0)q''(0) + q'(0)p''(0)$
Which is not necessarily equal to 0. Thus $U_6$ is not closed under addition.

## (b)

$S_1$:
$S_1$ is a subspace of $R^{2\times 2}$.
**Not empty:**
$S_1$ is not empty because $0 \in S_1$ (2 by 2 zero matrix).
**Closed under addition:**
Let $A, B \in S_1$. Then $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ where $a = b$ and $e = f$.
Then, $A + B = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$ where $a + e = b + f$.
**Closed under scalar multiplication:**
Let $A \in S_1$ and $c \in R$. Then, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a = b$.
Then, $cA = \begin{pmatrix} ca & cb \\ cc & cd \end{pmatrix}$ where $ca = cb$.

$S_2$:
$S_2$ is not a subspace of $R^{2\times 2}$. Because $S_2$ doesn't contain the zero matrix. (every vector space has to contain the zero vector which is the zero matrix in this case)

$S_3$:
$S_3$ is not a subspace of $R^{2\times 2}$. Because $S_3$ is not closed under addition.

Let $A, B \in S_3$. Then $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ where $a^2 = b^2$ and $e^2 = f^2$.
Then, $A + B = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$ where $(a+e)^2 = (b+f)^2$ is not necessarily true.

$$\begin{aligned} (a+e)^2 &= (b+f)^2 \\ a^2 + 2ae + e^2 &= b^2 + 2bf + f^2 \qquad \text{use } a^2 = b^2 \text{ and } e^2 = f^2 \\ 2ae &= 2bf \\ ae &= bf \end{aligned}$$

Which is not always true. Thus $S_3$ is not closed under addition.

## Problem 3

We need to show two things: vectors are linearly independent and they span the subspace $W$.
**Vectors are linearly independent:**
Let $c_1(x^3 - x^2) + c_2(x^3 - x) = 0$.
Then, $c_1x^3 - c_1x^2 + c_2x^3 - c_2x = 0$.
Then, $(c_1 + c_2)x^3 - c_1x^2 - c_2x = 0$.
The coefficients of $x^3$, $x^2$ and $x$ must be equal to 0.
Thus, $c_1$ and $c_2$ must be equal to 0.
**Vectors span the subspace $W$:**
Let $p(x) \in W$. Then, $p(0) = 0$ and $p(1) = 0$.
Then, $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$.
Then, $p(0) = a_0 = 0$.
Then, $p(1) = a_3 + a_2 + a_1 = 0$.
Thus, $p(x) = a_3x^3 + a_2x^2 + (-a_3 - a_2)x$.
Then, $p(x) = a_3(x^3 - x^2) + a_2(x^3 - x)$.
Thus, $p(x)$ can be written as a linear combination of the vectors $x^3 - x^2$ and $x^3 - x$.

**Extend the basis to a basis for $R^3[x]$:**
We want to be able write any polynomial in $R^3[x]$ as a linear combination of the basis vectors. We know that $dim R^3[x] = 4$. So we need 2 more linearly independent vectors.
From the basis extension theorem we know that if we add two more linearly independent vectors to our original set of vectors, we will have a basis for $R^3[x]$. So let's peak two vectors outside of the subspace $W$ which are linearly independent.
$q_1(x) = 1$ and $q_2(x) = x$ are linearly independent and not in the subspace $W$.

They are not in the subspace $W$ because $q_1(1) \neq 0$ and $q_2(1) \neq 0$.
Thus, a basis for $R^3[x]$ is $\{x^3 - x^2, x^3 - x, 1, x\}$.

## Problem 4

We are interested in the matrices in the folowing form:

$$A = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

So, if we determine the upper part (or lower part) of the matrix we can determine the whole matrix.
We can write the matrix $A$ as a linear combination of the following matrices:

$$A = a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + e\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + f\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

These matrices are trivially linearly independent.
Since there are 6 linearly independent matrices, the dimension of $S_3$ is 6.

**Extend this basis to a basis for $R^{3\times 3}$:**
We want to be able write any matrix in $R^{3\times 3}$ as a linear combination of the basis vectors. We know that $dim R^{3\times 3} = 9$. So we need 3 more linearly independent vectors.
Intiutively we want to be able to control the lower part of the matrix. Following matrices are linearly independent and not in the subspace $S_3$:

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus, a basis for $R^{3\times 3}$ is $\{A_1, A_2, A_3\} \cup S_3$.