

# Mathematics Homework Sheet 6

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## Problem 1

### Problem 1 (a)

We want to compute the following limit:

$$\lim_{n \rightarrow \infty} \frac{(n+1)^4(1-4n^3)^2}{(1+2n^2)^5}$$

The top part look somethin like this:

$$(n+1)^4(1-4n^3)^2 = 16n^{10} + \sum_{i=0}^{i=9} k_i n^i$$

And the bottom part look like this:

$$(1+2n^2)^5 = 32n^{10} + \sum_{i=0}^{i=9} k_i n^i$$

When we substitute these into the limit, we get:

$$\lim_{n \rightarrow \infty} \frac{16n^{10} + \sum_{i=0}^{i=9} k_i n^i}{32n^{10} + \sum_{i=0}^{i=9} k_i n^i}$$

Divide the top and bottom by  $n^{10}$ :

$$\lim_{n \rightarrow \infty} \frac{16 + \sum_{i=0}^{i=9} k_i n^{i-10}}{32 + \sum_{i=0}^{i=9} k_i n^{i-10}}$$

$$\frac{\lim_{n \rightarrow \infty} (16 + \sum_{i=0}^{i=9} k_i n^{i-10})}{\lim_{n \rightarrow \infty} (32 + \sum_{i=0}^{i=9} k_i n^{i-10})}$$

$$\frac{16 + \sum_{i=0}^{i=9} k_i \lim_{n \rightarrow \infty} n^{i-10}}{32 + \sum_{i=0}^{i=9} k_i \lim_{n \rightarrow \infty} n^{i-10}}$$

And we know that  $\lim_{n \rightarrow \infty} 1/n^i$  is zero for all  $i > 0$ . Therefore, the limit is:

$$\frac{16}{32} = \frac{1}{2}$$

## Problem 1 (b)

We want to compute the following limit:

$$\lim_{n \rightarrow \infty} \sqrt{n+1} - \sqrt{n}$$

Multiply and divide by the conjugate:

$$\lim_{n \rightarrow \infty} \sqrt{n+1} - \sqrt{n} \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$$

$$\lim_{n \rightarrow \infty} \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}}$$

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

And observe that

$$\frac{1}{n^2} < \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n}}$$

when  $n > 10$  (10 is not a magic number, it is just a number that is big enough) and we only care about the tail of the sequences not the head.

And we know that:

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} = 0$$

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$$

From the sandwich theorem, we can conclude that:

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0$$

## Problem 2

$$a_n = \left(1 + \frac{1}{n}\right)^n$$

$$b_n = \left(1 + \frac{1}{n}\right)^{n+1}$$

## Problem 2 (a)

We want to prove that

$$\frac{a_{n+1}}{a_n} = \left(1 - \frac{1}{(n+1)^2}\right)^{n+1} \frac{n+1}{n}$$

Let's start

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{(1 + \frac{1}{n+1})^{n+1}}{(1 + \frac{1}{n})^n} \\
&= \frac{(n+2)^{n+1} n^n}{(n+1)^{2n+1}} \\
&= \frac{(n+2)^{n+1} n^{n+1}}{(n+1)^{2n+2}} \frac{n+1}{n} \\
&= \frac{(n+2)^{n+1} n^{n+1}}{((n+1)^2)^{n+1}} \frac{n+1}{n} \\
&= \left( \frac{(n+2)n}{(n+1)^2} \right)^{n+1} \frac{n+1}{n} \\
&= \left( \frac{n^2 + 2n}{(n+1)^2} \right)^{n+1} \frac{n+1}{n} \\
&= \left( \frac{n^2 + 2n + 1 - 1}{(n+1)^2} \right)^{n+1} \frac{n+1}{n} \\
&= \left( \frac{n^2 + 2n + 1}{(n+1)^2} - \frac{1}{((n+1)^2)} \right)^{n+1} \frac{n+1}{n} \\
&= \left( 1 - \frac{1}{(n+1)^2} \right)^{n+1} \frac{n+1}{n}
\end{aligned}$$

That's what we wanted to show.

Now  $b_n$ . We want to prove

$$\frac{b_n}{b_{n+1}} = \left( 1 + \frac{1}{n(n+2)} \right)^{n+2} \frac{n}{n+1}$$

Let's start

$$\begin{aligned}
\frac{b_n}{b_{n+1}} &= \frac{(1 + \frac{1}{n})^{n+1}}{(1 + \frac{1}{n+1})^{n+2}} \\
&= \frac{(n+1)^{2n+3}}{n^{n+1}(n+2)^{n+2}} \\
&= \frac{(n+1)^{2n+4}}{n^{n+2}(n+2)^{n+2}} \frac{n}{n+1} \\
&= \frac{((n+1)^2)^{n+2}}{n^{n+2}(n+2)^{n+2}} \frac{n}{n+1} \\
&= \left(\frac{(n+1)^2}{n(n+2)}\right)^{n+2} \frac{n}{n+1} \\
&= \left(\frac{n^2 + 2n + 1}{n(n+2)}\right)^{n+2} \frac{n}{n+1} \\
&= \left(\frac{n(n+2) + 1}{n(n+2)}\right)^{n+2} \frac{n}{n+1} \\
&= \left(1 + \frac{1}{n(n+2)}\right)^{n+2} \frac{n}{n+1}
\end{aligned}$$

That's what we wanted to show.

## Problem 2 (b)

We want to show that

$$a_{n+1} \geq a_n \quad \forall n \in \mathbb{N}$$

Let's start  $a_{n+1} \geq a_n$  means that  $\frac{a_{n+1}}{a_n} \geq 1$ . Because  $a_n > 0 \quad \forall n \in \mathbb{N}$ . And we computed what  $\frac{a_{n+1}}{a_n}$  is in the previous part. It is  $\left(1 - \frac{1}{(n+1)^2}\right)^{n+1} \frac{n+1}{n}$ . So we want to show  $\left(1 - \frac{1}{(n+1)^2}\right)^{n+1} \frac{n+1}{n} \geq 1$ .  
From Bernoulli's inequality we have

$$(1+x)^n \geq 1+nx$$

Choose  $x = -\frac{1}{n^2}$ . Then we have

$$\left(1 - \frac{1}{n^2}\right)^n \geq 1 - \frac{n}{n^2} = 1 - \frac{1}{n}$$

We can rewrite this by substituting  $n$  with  $n+1$ :

$$\left(1 - \frac{1}{(n+1)^2}\right)^{n+1} \geq 1 - \frac{1}{n+1}$$

$$\begin{aligned}
\left(1 - \frac{1}{(n+1)^2}\right)^{n+1} \frac{n+1}{n} &= \left(1 - \frac{1}{(n+1)^2}\right)^{n+1} \frac{n+1}{n} \\
&= \left(1 - \frac{1}{(n+1)^2}\right)^{n+1} \frac{n+1}{n} \geq \left(1 - \frac{1}{n+1}\right) \frac{n+1}{n} \\
&= \frac{n}{n+1} \frac{n+1}{n} = 1
\end{aligned}$$

And we showed that  $a_{n+1} \geq a_n \quad \forall n \in N$ .

We want to show that  $b_n$  is monotonically decreasing.

$$b_{n+1} \leq b_n \quad \forall n \in N$$

Let's start  $b_{n+1} \leq b_n$  means that  $\frac{b_n}{b_{n+1}} \geq 1$ . Because  $b_{n+1} > 0 \quad \forall n \in N$ . And we know what  $\frac{b_n}{b_{n+1}}$  is from previous part. It is  $\left(1 + \frac{1}{n(n+2)}\right)^{n+2} \frac{n}{n+1}$ . So we want to show that

$$\left(1 + \frac{1}{n(n+2)}\right)^{n+2} \frac{n}{n+1} \geq 1$$

In the Berboulli's inequality choose  $x = \frac{1}{n(n+2)}$ . Then we have

$$\left(1 + \frac{1}{n(n+2)}\right)^n \geq 1 + \frac{n}{n(n+2)} = 1 + \frac{1}{n+2}$$

We can rewrite this by substituting  $n$  with  $n+2$ :

$$\left(1 + \frac{1}{(n+2)n}\right)^{n+2} \geq 1 + \frac{1}{n}$$

$$\begin{aligned}
\left(1 + \frac{1}{n(n+2)}\right)^{n+2} \frac{n}{n+1} &= \left(1 + \frac{1}{n(n+2)}\right)^{n+2} \frac{n}{n+1} \\
&= \left(1 + \frac{1}{n(n+2)}\right)^{n+2} \frac{n}{n+1} \geq \left(1 + \frac{1}{n}\right) \frac{n}{n+1} \\
&= \frac{n+1}{n} \frac{n}{n+1} = 1
\end{aligned}$$

And we showed that  $b_{n+1} \leq b_n \quad \forall n \in N$ .

## Problem 2 (c)

We want to show

$$a_n \leq b_n \quad \forall n \in N$$

Let's start

$$a_n \leq b_n \quad \forall n \in N \Rightarrow \frac{a_n}{b_n} \leq 1$$

From definition of  $a_n$  and  $b_n$ , we have

$$\frac{a_n}{b_n} = \frac{(1 + \frac{1}{n})^n}{(1 + \frac{1}{n})^{n+1}} = \frac{1}{1 + \frac{1}{n}} = \frac{n}{n+1} \leq 1$$

And we showed that  $a_n \leq b_n \quad \forall n \in N$ .

Now we want to show why  $a_n$  and  $b_n$  are convergent.

We know that  $b_n$  is monotonically decreasing. This means that  $\sup b_n$  exists. And we also know that  $a_n \leq b_n \quad \forall n \in N$ . This means that  $a_n \leq \sup b_n$ . And we also know that  $a_n$  is monotonically increasing. This means that  $a_n$  is convergent because it is monotonically increasing and bounded above.

We know that  $a_n$  is monotonically increasing that means that  $\inf a_n$  exists. And we also know that  $a_n \leq b_n \quad \forall n \in N$ . This means that  $b_n \geq \inf a_n$ . And we also know that  $b_n$  is monotonically decreasing. This means that  $b_n$  is convergent.

We consider the limit of  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$ .

$$\begin{aligned} \frac{a_n}{b_n} &= \frac{(1 + \frac{1}{n})^n}{(1 + \frac{1}{n})^{n+1}} = \frac{1}{1 + \frac{1}{n}} = \frac{n}{n+1} \\ \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \\ &= \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n} \\ \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} b_n \end{aligned}$$

That's what we wanted to show.

## Problem 3

### Problem 3(a)

$$\begin{aligned} a_1 &= 1 \\ a_{n+1} &= \frac{1}{1 + a_n} \quad \forall n \in N \end{aligned}$$

The limit: Whatever the limit of  $a_n$  is, it is also the limit of  $a_{n+1}$ . Let's call the limit  $l$ .

$$l = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_{n+1}$$

Using the recursion definition, we have

$$\begin{aligned}\lim_{n \rightarrow \infty} a_{n+1} &= \lim_{n \rightarrow \infty} \frac{1}{1 + a_n} \\ \lim_{n \rightarrow \infty} a_{n+1} &= \frac{1}{1 + \lim_{n \rightarrow \infty} a_n} \\ l &= \frac{1}{1 + l} \\ l + l^2 &= 1 \\ l_1 &= \frac{1 + \sqrt{5}}{2}, \quad l_2 = \frac{1 - \sqrt{5}}{2}\end{aligned}$$

And we know that  $\forall n \in \mathbb{N} \quad a_n > 0$  so the limit is  $l = l_1 = \frac{1 + \sqrt{5}}{2}$

### Problem 3(b)

We know that  $f_{n+2} = f_{n+1} + f_n$  if we then divide both sides by  $f_{n+1}$  then we get  $\frac{f_{n+2}}{f_{n+1}} = 1 + \frac{f_n}{f_{n+1}}$   
we let  $x_n = \frac{f_{n+1}}{f_n}$  if we substitute in the statment above then we get:

$$x_{n+1} = 1 + \frac{1}{x_n}$$

Let the limit of  $x_n = L$  by the properties of limits we know that the limit of  $x_{(n+1)}$  is also equal to  $L$  By substituting in the statment above we get :

$$L = 1 + \frac{1}{L}$$

Multiply both sides by  $L$ :

$$\begin{aligned}L^2 &= L + 1 \\ L^2 - L - 1 &= 0\end{aligned}$$

By solving this equation we get two numbers:

$$\frac{1}{2} + \frac{\sqrt{5}}{2}, \frac{1}{2} - \frac{\sqrt{5}}{2}$$

Since this sequence doesn't produce any negative numbers then we can say that this sequence converges to  $\frac{1}{2} + \frac{\sqrt{5}}{2}$  because it is monotonically increasing and bounded from below by 1 ( $x_1 = 1$ ) and bounded from above by  $\frac{1}{2} + \frac{\sqrt{5}}{2}$