

Mathematics Homework Sheet 2

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Problem 2

$$\sum_{k=0}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

is what we want to prove using mathematical induction
We need to do two things:

- Prove the base case
- Prove the inductive step

Let's start with the base case:

Insert $n = 0$ into the equation:

$$\sum_{k=0}^0 k^2 = 0^2 = 0 = \frac{0(0+1)(2 \cdot 0 + 1)}{6} = 0$$

Now the inductive step:

Assume that the equation holds for $n = m$, that is:

$$\sum_{k=0}^m k^2 = \frac{m(m+1)(2m+1)}{6}$$

Now we need to prove that the equation holds for $n = m + 1$:

$$\sum_{k=0}^{m+1} k^2 = \frac{(m+1)(m+2)(2m+3)}{6}$$

$$\begin{aligned}
\sum_{k=0}^{m+1} k^2 &= \sum_{k=0}^m k^2 + (m+1)^2 \\
&= \frac{m(m+1)(2m+1)}{6} + (m+1)^2 && \text{(This step uses the induction assumption)} \\
&= \frac{m(m+1)(2m+1) + 6(m+1)^2}{6} \\
&= \frac{(m+1)(m(2m+1) + 6(m+1))}{6} \\
&= \frac{(m+1)(2m^2 + m + 6m + 6)}{6} \\
&= \frac{(m+1)(2m^2 + 7m + 6)}{6} \\
&= \frac{(m+1)((m+2)(2m+3))}{6}
\end{aligned}$$

Problem 3

Problem 3 (a)

Symmetric relation means that

$$\forall x, y \in X, xRy \iff yRx$$

R_1 and R_2 are symmetric.

$$\forall x, y \in X, xR_1y \iff yR_1x$$

$$\forall x, y \in X, xR_2y \iff yR_2x$$

Let's pick an element from $R_1 \cup R_2$. $(x, y) \in R_1 \cup R_2$

Which means that $(x, y) \in R_1$ or $(x, y) \in R_2$

If it is in R_1 , then $(y, x) \in R_1$

If it is in R_2 , then $(y, x) \in R_2$

In both cases $(y, x) \in R_1 \cup R_2$

Which means that $R_1 \cup R_2$ is symmetric.

Problem 3 (b)

Reflexive relation means that

$$\forall x \in X, xRx$$

R_1 is reflexive.

$$\forall x \in X, xR_1x$$

R_2 is arbitrary.
 Let R_3 be $R_1 \cup R_2$.
 Let $x \in X$
 Is xR_3x ?
 If xR_3x , then $(x, x) \in R_3$
 Which means that $(x, x) \in R_1$ or $(x, x) \in R_2$
 And we know that xR_1x is true because R_1 is reflexive.
 Thus xR_3x is true which means R_3 is reflexive

Problem 3 (c)

Antisymmetric relation means that

$$\forall x, y \in X, xRy \wedge yRx \implies x = y$$

R_1 and R_2 is antisymmetric.
 Let's take a look at this example:

$$X = \{1, 2\}$$

$$R_1 = \{(1, 2)\}$$

$$R_2 = \{(2, 1)\}$$

R_1 and R_2 are antisymmetric. But $R_1 \cup R_2$ is not antisymmetric because it contains $(1, 2)$ and $(2, 1)$ and $1 \neq 2$.

Problem 4

$$X = Z \times N$$

$$(a, b) \sim (c, d) \iff ad = bc$$

Problem 4 (a)

Relation \sim being an equivalence relation means that it is reflexive, symmetric and transitive.
 Let's start with reflexivity:

$$(a, b) \sim (a, b) \implies ab = ba$$

Above statement is true for all $a \in Z, b \in N$.
 Thus \sim is reflexive.

Now let's check symmetry:

$$\forall a, c \in Z \quad \forall b, d \in N \quad (a, b) \sim (c, d) \iff ad = bc \iff cb = da \iff (c, d) \sim (a, b)$$

Thus \sim is symmetric.

Now let's check transitivity:

$$\forall a, b, c, d, e, f \in Z \quad \forall x, y, z \in N \quad (a, b) \sim (c, d) \wedge (c, d) \sim (e, f) \implies (a, b) \sim (e, f)$$

$$(a, b) \sim (c, d) \iff ad = bc$$

$$(c, d) \sim (e, f) \iff cf = de$$

Multiply these two equations together, we get:

$$adc f = bcde$$

$$af = be$$

$$af = be \iff (a, b) \sim (e, f)$$

Thus \sim is transitive.

Problem 4 (b)

$(a, b) \sim (a', b') \wedge (c, d) \sim (c', d')$ means that

$$ab' = a'b \wedge cd' = c'd$$

We want prove $(ad + cb, bd) \sim (a'd' + c'b', b'd')$

$(ad + cb, bd) \sim (a'd' + c'b', b'd')$ means that

$$(ad + cb)b'd' = (a'd' + c'b')bd$$

$$adb'd' + cbb'd' = a'd'bd + c'b'bd$$

$$adb'd' - a'd'bd = c'b'bd - cbb'd'$$

$$dd'(ab' - a'b) = bb'(c'd - cd')$$

$$dd'(0) = bb'(0) \quad (\text{Since } ab' = a'b \text{ and } cd' = c'd)$$

$0 = 0$ which is true for every $a, b, c, d, a', b', c', d'$. Thus $(ad + cb, bd) \sim (a'd' + c'b', b'd')$