Mathematics Homework Sheet 2

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Problem 1

(a)

Z_5:

$\oplus$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\times$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Z_7:

$\oplus$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

$\times$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

(b)

Z_4:

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\times$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

$(Z_4, \oplus, \times)$ is not a field because [2] does not have a multiplicative inverse.

Problem 2

(a)

Let $f : Z_n \to Z_n$ be:

$$f([i]) = [i] + [m], \qquad [m] \in Z_n$$

We want to show that f is a bijection.

- **Injective:** Suppose $f([i_1]) = f([i_2])$:

  $$[i_1] + [m] = [i_2] + [m]$$

  $$[i_1 + m] = [i_2 + m]$$

  Thus, we have that n divides $(i_1 + m) - (i_2 + m)$.
  Equivalently, n divides $(i_1 - i_2)$

  $$\therefore [i_1] = [i_2]$$

  Thus, f is injective.

- **Surjective:** Let $[j] \in Z_n$. We want to show that there exists an $[x]$ such that $f([x]) = [j]$:

  $$f([j - m]) = [j - m] + [m] = [j]$$

  Thus, f is surjective.

Thus, f is a bijection.

(b)

$[i] \to [i] + [1]$: (01234567)
$[i] \to [i] + [2]$: (0246)(1357)
$[i] \to [i] + [3]$: (03614725)
$[i] \to [i] + [4]$: (04)(15)(26)(37)

Problem 3

(a)

Let $f : Z_p \setminus \{[0]\} \to Z_p \setminus \{[0]\}$ be:

$$f([i]) = [m].[i], \qquad [m] \in Z_p \setminus \{[0]\}$$

We want to show that f is a bijection.

- **Injective:** Suppose $f([i_1]) = f([i_2])$:

  $$[m] \cdot [i_1] = [m] \cdot [i_2], [m \cdot i_1] = [m \cdot i_2]$$

  Thus, P divides $(m \cdot i_1 - m \cdot i_2)$.
  Equivalently, P divides $(m \cdot (i_1 - i_2))$.
  Since P is a prime number, P divides either m or $(i_1 - i_2)$.
  Since $m \in \mathbb{Z} \setminus \{[0]\}$, P divides $(i_1 - i_2)$.

  $$\therefore [i_1] = [i_2]$$

  Thus, f is injective.

- **Surjective:** Let $[j] \in Z_p \setminus \{[0]\}$. We want to show that there exists an $[x]$ such that $f([x]) = [j]$:

  $$f([j.m^{-1}]) = [j.m^{-1}].[m] = [j]$$

  Thus, f is surjective.

Thus, f is a bijection.

(b)

$[i] \to [6].[i]$: (0)(16)(25)(34)
$[i] \to [2].[i]$: (0)(124)(365)

Problem 4

Let I be an arbitrary interval. We will show that I and $\mathbb{R}$ have the same cardinality by constructing a bijection between them.

Problem 5

(a)

Let $(p, n), (q, m) \in N_0 \times N_0$. We want to show that $\sim$ is an equivalence relation on $N_0 \times N_0$:

- **Reflexive:** We want to show that $(p, n) \sim (p, n)$:

  $$p + n = p + n$$

  Thus, $\sim$ is reflexive.

- **Symmetric:** We want to show that if $(p, n) \sim (q, m)$, then $(q, m) \sim (p, n)$:

  $$p + m = q + n$$

  Thus,

  $$q + n = p + m$$

  Thus, $\sim$ is symmetric.

- **Transitive:** We want to show that if $(p, n) \sim (q, m)$ and $(q, m) \sim (r, s)$, then $(p, n) \sim (r, s)$:

  $$p + m = q + n$$

  and

  $$q + s = r + m$$

  Thus,

  $$p + s = r + n$$

  Thus, $\sim$ is transitive.

Thus, $\sim$ is an equivalence relation on $N_0 \times N_0$.

(b)

Let $k \in N_0$. We want to show that $(p, n) \sim (k + p, k + n)$:

$$p + (k + n) = (k + p) + n$$

Thus, $(p, n) \sim (k + p, k + n)$.

(c)

Let $[(k, 0)]$ be the equivalence class of $(k, 0)$. We want to find an equivalence class $-[(k, 0)]$ such that

$$-[(k, 0)] + [(k, 0)] = [(0, 0)]$$

Define $-[(k,0)]$ as $[(0,k)]$. Then,

$$[(0,k)] + [(k,0)] = [(0+k, k+0)] = [(k,k)] = [(0,0)]$$

Thus, we have

$$-[(k,0)] + [(k,0)] = [(0,0)]$$

Thus, $-[(k,0)]$ is the equivalence class with the property that $-[(k,0)]+[(k,0)] = [(0,0)]$.