Mathematics Homework Sheet 6

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Problem 1

Problem 1 (a)

We want to compute the following limit:

$$\lim_{n\to\infty} \frac{(n+1)^4(1-4n^3)^2}{(1+2n^2)^5}$$

The top part look somethin like this:

$$(n+1)^4(1-4n^3)^2 = 16n^{10} + \sum_{i=0}^{i=9} k_i n^i$$

And the bottom part look like this:

$$(1+2n^2)^5 = 32n^{10} + \sum_{i=0}^{i=9} k_i n^i$$

When we substitute these into the limit, we get:

$$\lim_{n\to\infty} \frac{16n^{10} + \sum_{i=0}^{i=9} k_i n^i}{32n^{10} + \sum_{i=0}^{i=9} k_i n^i}$$

Divide the top and bottom by n^{10}:

$$\lim_{n\to\infty} \frac{16 + \sum_{i=0}^{i=9} k_i n^{i-10}}{32 + \sum_{i=0}^{i=9} k_i n^{i-10}}$$

$$\frac{\lim_{n\to\infty}(16 + \sum_{i=0}^{i=9} k_i n^{i-10})}{\lim_{n\to\infty}(32 + \sum_{i=0}^{i=9} k_i n^{i-10})}$$

$$\frac{16 + \sum_{i=0}^{i=9} k_i \lim_{n\to\infty} n^{i-10}}{32 + \sum_{i=0}^{i=9} k_i \lim_{n\to\infty} n^{i-10}}$$

And we know that $\lim_{n\to\infty} 1/n^i$ is zero for all $i > 0$. Therefore, the limit is:

$$\frac{16}{32} = \frac{1}{2}$$

Problem 1 (b)

We want to compute the following limit:

$$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n}$$

Multiply and divide by the conjugate:

$$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$$

$$\lim_{n\to\infty} \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

And observe that

$$\frac{1}{n^2} < \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n}}$$

when $n > 10$ (10 is not a magic number, it is just a number that is big enough) and we only care about the tail of the sequences not the head.
And we know that:

$$\lim_{n\to\infty} \frac{1}{n^2} = 0$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

From the sandwich theorem, we can conclude that:

$$\lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0$$

Problem 2

$$a_n = (1 + \frac{1}{n})^n$$

$$b_n = (1 + \frac{1}{n})^{n+1}$$

Problem 2 (a)

We want to prove that

$$\frac{a_{n+1}}{a_n} = \left(1 - \frac{1}{(n+1)^2}\right)^{n+1} \frac{n+1}{n}$$

Let's start

$$
\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{(1+\frac{1}{n+1})^{n+1}}{(1+\frac{1}{n})^n} \\
&= \frac{(n+2)^{n+1}n^n}{(n+1)^{2n+1}} \\
&= \frac{(n+2)^{n+1}n^{n+1}}{(n+1)^{2n+2}}\frac{n+1}{n} \\
&= \frac{(n+2)^{n+1}n^{n+1}}{((n+1)^2)^{n+1}}\frac{n+1}{n} \\
&= (\frac{(n+2)n}{(n+1)^2})^{n+1}\frac{n+1}{n} \\
&= (\frac{n^2+2n}{(n+1)^2})^{n+1}\frac{n+1}{n} \\
&= (\frac{n^2+2n+1-1}{(n+1)^2})^{n+1}\frac{n+1}{n} \\
&= (\frac{n^2+2n+1}{(n+1)^2} - \frac{1}{((n+1)^2)})^{n+1}\frac{n+1}{n} \\
&= (1 - \frac{1}{(n+1)^2})^{n+1}\frac{n+1}{n}
\end{aligned}
$$

That's what we wanted to show.

Now b_n. We want to prove

$$
\frac{b_n}{b_{n+1}} = \left(1+\frac{1}{n(n+2)}\right)^{n+2}\frac{n}{n+1}
$$

Let's start

$$\begin{aligned}
\frac{b_n}{b_{n+1}} &= \frac{(1+\frac{1}{n})^{n+1}}{(1+\frac{1}{n+1})^{n+2}} \\
&= \frac{(n+1)^{2n+3}}{n^{n+1}(n+2)^{n+2}} \\
&= \frac{(n+1)^{2n+4}}{n^{n+2}(n+2)^{n+2}}\frac{n}{n+1} \\
&= \frac{((n+1)^2)^{n+2}}{n^{n+2}(n+2)^{n+2}}\frac{n}{n+1} \\
&= (\frac{(n+1)^2}{n(n+2)})^{n+2}\frac{n}{n+1} \\
&= (\frac{n^2+2n+1}{n(n+2)})^{n+2}\frac{n}{n+1} \\
&= (\frac{n(n+2)+1}{n(n+2)})^{n+2}\frac{n}{n+1} \\
&= (1+\frac{1}{n(n+2)})^{n+2}\frac{n}{n+1}
\end{aligned}$$

That's what we wanted to show.

Problem 2 (b)

We want to show that

$$a_{n+1} \geq a_n \quad \forall n \in N$$

Let's start $a_{n+1} \geq a_n$ means that $\frac{a_{n+1}}{a_n} \geq 1$. Because $a_n > 0 \quad \forall n \in N$. And we computed what $\frac{a_{n+1}}{a_n}$ is in the previous part. It is $\left(1 - \frac{1}{(n+1)^2}\right)^{n+1} \frac{n+1}{n}$. So we want to show $\left(1 - \frac{1}{(n+1)^2}\right)^{n+1} \frac{n+1}{n} \geq 1$

From Bernoulli's inequality we have

$$(1+x)^n \geq 1 + nx$$

Choose $x = -\frac{1}{n^2}$. Then we have

$$\left(1 - \frac{1}{n^2}\right)^n \geq 1 - \frac{n}{n^2} = 1 - \frac{1}{n}$$

We can rewrite this by substituting n with $n+1$:

$$\left(1 - \frac{1}{(n+1)^2}\right)^{n+1} \geq 1 - \frac{1}{n+1}$$

$$\begin{aligned}
\left(1-\frac{1}{(n+1)^2}\right)^{n+1}\frac{n+1}{n} &= \left(1-\frac{1}{(n+1)^2}\right)^{n+1}\frac{n+1}{n} \\
&= \left(1-\frac{1}{(n+1)^2}\right)^{n+1}\frac{n+1}{n} \geq (1-\frac{1}{n+1})\frac{n+1}{n} \\
&= \frac{n}{n+1}\frac{n+1}{n} = 1
\end{aligned}$$

And we showed that $a_{n+1} \geq a_n \quad \forall n \in N$.

We want to show that b_n is monotonically decreasing.

$$b_{n+1} \leq b_n \quad \forall n \in N$$

Let's start $b_{n+1} \leq b_n$ means that $\frac{b_n}{b_{n+1}} \geq 1$. Because $b_{n+1} > 0 \quad \forall n \in N$. And we know what $\frac{b_n}{b_{n+1}}$ is from previous part. It is $\left(1+\frac{1}{n(n+2)}\right)^{n+2}\frac{n}{n+1}$. So we want to show that

$$\left(1+\frac{1}{n(n+2)}\right)^{n+2}\frac{n}{n+1} \geq 1$$

In the Berboulli's ineqaulity choose $x = \frac{1}{n(n-2)}$. Then we have

$$\left(1+\frac{1}{n(n-2)}\right)^{n} \geq 1+\frac{n}{n(n-2)} = 1+\frac{1}{n-2}$$

We can rewrite this by substituting n with $n+2$:

$$\left(1+\frac{1}{(n+2)n}\right)^{n+2} \geq 1+\frac{1}{n}$$

$$\begin{aligned}
\left(1+\frac{1}{n(n+2)}\right)^{n+2}\frac{n}{n+1} &= \left(1+\frac{1}{n(n+2)}\right)^{n+2}\frac{n}{n+1} \\
&= \left(1+\frac{1}{n(n+2)}\right)^{n+2}\frac{n}{n+1} \geq (1+\frac{1}{n})\frac{n}{n+1} \\
&= \frac{n+1}{n}\frac{n}{n+1} = 1
\end{aligned}$$

And we showed that $b_{n+1} \leq b_n \quad \forall n \in N$.

Problem 2 (c)

We want to show

$$a_n \leq b_n \quad \forall n \in N$$

Let's start

$$a_n \leq b_n \quad \forall n \in N \Rightarrow \frac{a_n}{b_n} \leq 1$$

From definiton of a_n and b_n, we have

$$\frac{a_n}{b_n} = \frac{(1+\frac{1}{n})^n}{(1+\frac{1}{n})^{n+1}} = \frac{1}{1+\frac{1}{n}} = \frac{n}{n+1} \leq 1$$

And we showed that $a_n \leq b_n \quad \forall n \in N$.

Now we want to show why a_n and b_n are convergent.
We know that b_n is monotonically decreasing. This means that $\sup b_n$ exists. And we also know that $a_n \leq b_n \quad \forall n \in N$. This means that $a_n \leq \sup b_n$. And we also know that a_n is monotonically increasing. This means that a_n is convergent because it is monotonically increasing and bounded above.
We know that a_n is monotonically increasing that means that $\inf a_n$ exists. And we also know that $a_n \leq b_n \quad \forall n \in N$. This means that $b_n \geq \inf a_n$. And we also know that b_n is monotonically decreasing. This means that b_n is convergent.
We consider the limit of $\lim_{n\to\infty} \frac{a_n}{b_n}$.

$$\begin{aligned}
&\frac{a_n}{b_n} = \frac{(1+\frac{1}{n})^n}{(1+\frac{1}{n})^{n+1}} = \frac{1}{1+\frac{1}{n}} = \frac{n}{n+1} \\
&\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n+1} = 1 \\
&= \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \\
&\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n
\end{aligned}$$

That's what we wanted to show.