Homework Sheet 5

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Exercise 17

We are given the function

$$f(x) := -e^x + 2x + 1.$$

We calculte the newtons method using this formula

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

(i)

We want to find x_2 when $x_0 = -1$

Lets start by finding $f'(x)$

$$f'(x) = -e^x + 2$$

Now we can calculate x_1 and x_2

$$
\begin{aligned}
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\
&= -1 - \frac{-e^{-1} + 2(-1) + 1}{-e^{-1} + 2} \\
&= -1 - \frac{-\frac{1}{e} - 2 + 1}{-\frac{1}{e} + 2} \\
&= -1 - \frac{-\frac{1}{e} - 1}{-\frac{1}{e} + 2} \\
&= -1 + \frac{\frac{1}{e} + 1}{-\frac{1}{e} + 2} \\
&= -1 + \frac{e + 1}{-1 + 2e} \\
&= \frac{-2e + e + 1 + 1}{-1 + 2e} \\
&= \frac{-e + 2}{-1 + 2e} \\
&\approx -0.16190048965915385 \quad \text{(via using a calculator)}
\end{aligned}
$$

$$
\begin{aligned}
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\
&\approx -0.16190048965915385 - \frac{f(-0.16190048965915385)}{f'(-0.16190048965915385)} \\
&\approx -0.16190048965915385 - \frac{-0.174326815763479123074}{f'(-0.16190048965915385)} \quad \text{again using a calculator} \\
&\approx -0.16190048965915385 - \frac{-0.174326815763479123074}{1.149474163554828576926} \quad \text{again using a calculator} \\
&\approx -0.16190048965915385 - -0.151657880873425944806890147658365522997056738243886290320790176143 \\
&\approx -0.010242608785727905193109852341634477002943261756113709679209823856167266 9 \quad \text{again using a c}
\end{aligned}
$$

Links to the calculator:
`https://www.wolframalpha.com/input?i=f%28x%29+%3D+-e%5Ex+%2B2x+%2B1+at+x%3D%E2%88%920.16190048965915385&assumption=%7B%22C%22%2C+%22at%22%7D+-%3E+%7B%22EnglishWord%22%7D`
`https://www.wolframalpha.com/input?i=g%28x%29+%3D+-e%5Ex+%2B2+where+x+%3D+%E2%88%920.16190048965915385`

(ii)

We want to find x_2 when $x_0 = 1$
The derivatie

$$f'(x) = -e^x + 2$$

Now we can calculate x_1 and x_2

$$\begin{aligned}
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\
&= 1 - \frac{-e^1 + 2(1) + 1}{-e^1 + 2} \\
&= 1 - \frac{-e + 2 + 1}{-e + 2} \\
&= 1 - \frac{-e + 3}{-e + 2} \\
&= 1 + \frac{e - 3}{-e + 2} \\
&= \frac{-e + 2 + e - 3}{-e + 2} \\
&= \frac{-1}{-e + 2} \\
&\approx 1.3922111911 \quad \text{(via using a calculator)}
\end{aligned}$$

$$\begin{aligned}
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\
&= 1.3922111911 - \frac{f(1.3922111911)}{f'(1.3922111911)} \\
&\approx 1.3922111911 - \frac{-0.23931509377336}{f'(1.3922111911)} \quad \text{again using a calculator} \\
&\approx 1.3922111911 - \frac{-0.23931509377336}{-2.02373747597336} \quad \text{again using a calculator} \\
&\approx 1.3922111911 - 0.1182540208967846811355339766932549 \quad \text{again using a calculator} \\
&\approx 1.2739571702032153188644660233067451 \quad \text{again using a calculator}
\end{aligned}$$

Exercise 18

We are given the function

$$f(x_1, x_2) := \begin{pmatrix} x_1^3 + x_2 - 2 \\ x_1 + x_2^3 - 2 \end{pmatrix}$$

The newtons method for multivariable functions is given by

$$\mathbf{x_n} = \mathbf{x_{n-1}} - J_f(\mathbf{x_{n-1}})^{-1} f(\mathbf{x_{n-1}})$$

The Jacobian matrix J_f is

$$\begin{aligned} J_f(x_1, x_2) &= \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} \\ &= \begin{pmatrix} 3x_1^2 & 1 \\ 1 & 3x_2^2 \end{pmatrix} \end{aligned}$$

We start with $\mathbf{x_0} = (1, 1)$

$$\begin{aligned} \mathbf{x_1} &= \mathbf{x_0} - J_f(\mathbf{x_0})^{-1} f(\mathbf{x_0}) \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 3(1)^2 & 1 \\ 1 & 3(1)^2 \end{pmatrix}^{-1} \begin{pmatrix} (1)^3 + (1) - 2 \\ (1) + (1)^3 - 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

Doing the same calculation $\mathbf{x_2}$ will also be $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. It turns out that i didnt even need to calculate the jacobian matrix :)

Exercise 19

We are given the functions

$$\begin{aligned} f(x_1, x_2) &:= \frac{1}{3}x_1^3 + \frac{1}{4}x_2^2 + x_1 + \frac{1}{2}x_2 + 2, \\ g(x_1, x_2) &:= x_1 + \frac{1}{2}x_2. \end{aligned}$$

Choose $h(x_1) = -2x_1$ so that $g(x_1, h(x_1)) = 0$.

$$\begin{aligned} f|_M(x_1) &= f(x_1, h(x_1)) \\ &= \frac{1}{3}x_1^3 + \frac{1}{4}(-2x_1)^2 + x_1 + \frac{1}{2}(-2x_1) + 2 \\ &= \frac{1}{3}x_1^3 + \frac{1}{4}(4x_1^2) + x_1 - x_1 + 2 \\ &= \frac{1}{3}x_1^3 + x_1^2 + 2 \end{aligned}$$

Now we need to find the extremum points of $f|_M$. Lets continue with derivaties

$$f|_M'(x_1) = x_1^2 + 2x_1$$
$$f|_M''(x_1) = 2x_1 + 2$$

Setting the first derivatie to zero

$$\begin{aligned} f|_M'(x_1) = 0 &\implies x_1^2 + 2x_1 = 0 \\ &\implies x_1(x_1 + 2) = 0 \\ &\implies x_1 = 0 \quad \text{or} \quad x_1 = -2 \end{aligned}$$

Now we can use the second derivatie test to classify these points

- For $x_1 = 0$:

  $$f|_M''(0) = 2(0) + 2 = 2 > 0$$

  So we have a local minimum at $x_1 = 0$. The corresponding x_2 value is

  $$x_2 = h(0) = -2(0) = 0$$

  Thus one local minimum point is $(0, 0)$.

- For $x_1 = -2$:

  $$f|_M''(-2) = 2(-2) + 2 = -2 < 0$$

  So we have a local maximum at $x_1 = -2$. The corresponding x_2 value is

  $$x_2 = h(-2) = -2(-2) = 4$$

  Thus one local maximum point is $(-2, 4)$.

Exercise 20

We are given the function

$$f(x_1, x_2, x_3) := x_1 - 2x_2 + x_3^2$$

and the set

$$M := \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : \|\mathbf{x}\|^2 = 1\}$$

(i)

The set M is closed and bounded in $\mathbb{R}^3$. So by the extreme value theorem the continuous function $f|_M : M \to \mathbb{R}$ also has a maximum and a minimum on M.

Why f is continous is obvious because its a polynomial function.

Why M is closed: intuitively the set M is just points on the surface of the unit sphere. And if you think about a random point in the complement space then we can find a ball around that point that doesnt touch the surface of the sphere. And since complement of M is open that makes the M closed.

Why M is bounded: again intuitively the set M is just points on the surface of the unit sphere and thats obviously bounded.

(ii)

Let $g(x_1, x_2, x_3) = \|\mathbf{x}\| - 1$. so that we have a name for the constraint of M. But notice here that i dropped the square because norm is always positive so its ok to do that.

The lagrange function is

$$L : \mathbb{R}^4 \to \mathbb{R}, \qquad L(x_1, x_2, x_3, \lambda) = f(x_1, x_2, x_3) + \lambda g(x_1, x_2, x_3)$$

$$L(x_1, x_2, x_3, \lambda) = x_1 - 2x_2 + x_3^2 + \lambda(\sqrt{x_1^2 + x_2^2 + x_3^2} - 1)$$

The gradient of L

$$\nabla L(x_1, x_2, x_3, \lambda) = \begin{pmatrix} 1 + \lambda \frac{x_1}{\sqrt{x_1^2+x_2^2+x_3^2}} \\ -2 + \lambda \frac{x_2}{\sqrt{x_1^2+x_2^2+x_3^2}} \\ 2x_3 + \lambda \frac{x_3}{\sqrt{x_1^2+x_2^2+x_3^2}} \\ \sqrt{x_1^2 + x_2^2 + x_3^2 - 1} \end{pmatrix}$$

Setting the gradient to zero

$$\nabla L(x_1,x_2,x_3,\lambda)=0 \implies \begin{pmatrix} 1+\lambda\frac{x_1}{\sqrt{x_1^2+x_2^2+x_3^2}} \\ -2+\lambda\frac{x_2}{\sqrt{x_1^2+x_2^2+x_3^2}} \\ 2x_3+\lambda\frac{x_3}{\sqrt{x_1^2+x_2^2+x_3^2}} \\ \sqrt{x_1^2+x_2^2+x_3^2}-1 \end{pmatrix}=0$$

$$\implies \begin{cases} 1+\lambda\frac{x_1}{\sqrt{x_1^2+x_2^2+x_3^2}}=0 \\ -2+\lambda\frac{x_2}{\sqrt{x_1^2+x_2^2+x_3^2}}=0 \\ 2x_3+\lambda\frac{x_3}{\sqrt{x_1^2+x_2^2+x_3^2}}=0 \\ \sqrt{x_1^2+x_2^2+x_3^2}-1=0 \end{cases}$$

Subsitute the last equationinto first three

$$\implies \begin{cases} 1+\lambda x_1=0 \\ -2+\lambda x_2=0 \\ 2x_3+\lambda x_3=0 \\ x_1^2+x_2^2+x_3^2=1 \end{cases}$$

$$\implies \begin{cases} x_1=-\frac{1}{\lambda} \\ x_2=\frac{2}{\lambda} \\ x_3(2+\lambda)=0 \\ x_1^2+x_2^2+x_3^2=1 \end{cases}$$

From the third equation we have two cases:

- If $x_3=0$:

$$\begin{aligned} x_1^2+x_2^2+x_3^2=1 &\implies \left(-\frac{1}{\lambda}\right)^2+\left(\frac{2}{\lambda}\right)^2+0^2=1 \\ &\implies \frac{1}{\lambda^2}+\frac{4}{\lambda^2}=1 \\ &\implies \frac{5}{\lambda^2}=1 \\ &\implies \lambda^2=5 \\ &\implies \lambda=\sqrt{5} \quad \text{or} \quad \lambda=-\sqrt{5} \end{aligned}$$

  For $\lambda=\sqrt{5}$:

$$x_1=-\frac{1}{\sqrt{5}}, \quad x_2=\frac{2}{\sqrt{5}}, \quad x_3=0$$

  For $\lambda=-\sqrt{5}$:

$$x_1=\frac{1}{\sqrt{5}}, \quad x_2=-\frac{2}{\sqrt{5}}, \quad x_3=0$$

- If $\lambda = -2$:

$$x_1 = -\frac{1}{-2} = \frac{1}{2}, \quad x_2 = \frac{2}{-2} = -1, \quad x_3 \text{ is free}$$

Using the constraint equation:

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 = 1 &\implies \left(\frac{1}{2}\right)^2 + (-1)^2 + x_3^2 = 1 \\ &\implies \frac{1}{4} + 1 + x_3^2 = 1 \\ &\implies x_3^2 = 1 - \frac{5}{4} = -\frac{1}{4} \end{aligned}$$

which has no real solution.

Long story short the extremum points are

$$\left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right) \quad \text{and} \quad \left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}, 0\right)$$