Mathematics Homework Sheet 8

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Problem 1

$$a_n := \sqrt[n]{n} - 1$$

We want to prove

$$a_n \leq \sqrt{\frac{2}{n}} \qquad \forall n \geq 2$$

and we want to prove

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

Apply the binomial formula to $(1 + a_n)^n$

$$\begin{aligned}
(1 + a_n)^n &= n \\
&= \sum_{k=0}^{n} \binom{n}{k} a_n^k \\
&= \sum_{k=0}^{n} \binom{n}{k} (\sqrt[n]{n} - 1)^k \\
&= 1 + \binom{n}{1}(\sqrt[n]{n} - 1) + \binom{n}{2}(\sqrt[n]{n} - 1)^2 + \ldots + (\sqrt[n]{n} - 1)^n
\end{aligned}$$

Using the fact that $a_n \leq \sqrt{\frac{2}{n}}$, we want to show

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

We know that

$$1 - \frac{1}{n} \leq \sqrt[n]{n}$$

because $1 - \frac{1}{n}$ is less than 1 and $\sqrt[n]{n}$ is greater or equal to 1 for all $n \geq 1$. Also from the previous inequality we have

$$\sqrt[n]{n} - 1 \leq \sqrt{\frac{2}{n}}$$

$$\sqrt[n]{n} \leq 1 + \sqrt{\frac{2}{n}}$$

When combined we have

$$1 - \frac{1}{n} \leq \sqrt[n]{n} \leq 1 + \sqrt{\frac{2}{n}}$$

$$\lim_{n\to\infty} 1 - \frac{1}{n} = 1$$

$$\lim_{n\to\infty} 1 + \sqrt{\frac{2}{n}} = 1$$

Therefore, by the sandwich theorem, we have

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

Problem 2

Problem 2(a)

$$a_n := \frac{4 + 3n^2}{n(2n+1)^2}$$

We want to prove

$$(\frac{1}{n})_{n\in N} \in O((a_n)_{n\in N})$$

Which means $\frac{\frac{1}{n}}{a_n}$ is bounded.
Let's start

$$\begin{aligned}
\frac{1}{na_n} &= \frac{n(2n+1)^2}{n(4+3n^2)} \\
&= \frac{(2n+1)^2}{4+3n^2} \\
&= \frac{4n^2+4n+1}{4+3n^2} \\
&= \frac{4+\frac{4}{n}+\frac{1}{n^2}}{3+\frac{4}{n^2}}
\end{aligned}$$

$$lim_{n\to\infty} \frac{4+\frac{4}{n}+\frac{1}{n^2}}{3+\frac{4}{n^2}} = \frac{4}{3}$$

Existence of limit implies that $\frac{1}{na_n}$ is bounded. Therefore, $(\frac{1}{n})_{n\in N} \in O((a_n)_{n\in N})$
The series $\sum_{n=1}^{\infty} a_n$ is divergent. Since $lim_{n\to\infty} \frac{\frac{1}{n}}{a_n} = \frac{4}{3}$, the limit of $lim_{n\to\infty} \frac{\frac{1}{2n}}{a_n} = \frac{2}{3} \leq 1$ which means for almost all $n \in N$ $\frac{1}{2n} \leq a_n$. And from the therom 3.44(ii) in the lecture notes, a_n is divergent. because $\sum_{n=1}^{\infty} \frac{1}{2n}$ is divergent and $a_n \geq \frac{1}{2n} \geq 0$ for almost all $n \in N$.