Mathematics Homework Sheet 1

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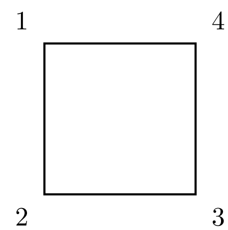
Problem 1

Symmetry group S will consist of rotations and reflections.

- Rotations: R_{90}, R_{180}, R_{270}
- Reflections: T_x, T_y, T_d, $T_{d'}$
- Identity: I



R_i rotates i degrees clockwise.
T_x reflects over the x-axis, T_y reflects over the y-axis, T_d reflects diagonally, and $T_{d'}$ reflects over the other diagonal.
When we take a look at S_4, S_4 has $4! = 24$ elements.
Our group S has 8 elements.
Lets start with identity I.

- $()$

Rotations:

- $R_{90} = (1, 2, 3, 4)$
- $R_{180} = (1, 3)(2, 4)$
- $R_{270} = (1, 4, 3, 2)$

Reflections:

- $T_x = (1, 2)(3, 4)$
- $T_y = (1, 4)(2, 3)$
- $T_d = (1, 3)$
- $T_{d'} = (2, 4)$

So, when combined, S can be identified with this subset of S_4:

$$\{(), (1, 2, 3, 4), (1, 3)(2, 4), (1, 4, 3, 2), (1, 2)(3, 4), (1, 4)(2, 3), (1, 3), (2, 4)\}$$

Problem 2

Problem 2(i)

$$f_{a,b}(x) = ax + b$$

$$(G, \diamond) = \{f_{a,b} : a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R}\}, f_{a,b} \diamond f_{c,d} = f_{ac,ad+b}$$

We want to show $(G, \diamond)$ is a group. To do that, we need to show that $(G, \diamond)$ satisfies the properties of group.
Associativity:

$$f_{a,b} \diamond (f_{c,d} \diamond f_{e,f}) = f_{a,b} \diamond f_{ce,cf+d} = f_{ace,acf+ad+b}$$

$$(f_{a,b} \diamond f_{c,d}) \diamond f_{e,f} = f_{ac,ad+b} \diamond f_{e,f} = f_{ace,acf+ad+b}$$

Thus $f_{a,b} \diamond (f_{c,d} \diamond f_{e,f}) = (f_{a,b} \diamond f_{c,d}) \diamond f_{e,f}$.
Existence of a neutral elemenet:

$$f_{1,0} \diamond f_{a,b} = f_{1,0} \diamond f_{a,b} = f_{a,b}$$

$f_{1,0}$ is the neutral element.
Existence of inverses:

$$f_{a,b} \diamond f_{1/a,-b/a} = f_{a*(1/a),(-ab/a)+b} = f_{1,0}$$

Thus, $f_{1/a,-b/a}$ is the inverse of $f_{a,b}$.
Therefore, $(G, \diamond)$ is a group.

Problem 2(ii)

$$H = f_{1,b} : b \in \mathbb{R}$$

We want to show $(H, \diamond)$ is a subgroup of $(G, \diamond)$ which is isomorphic to $(\mathbb{R}, +)$.
We need to show identity element of $(G, \diamond)$ is in H:

$$f_{1,0} \in H$$

We need to show H is closed under $\diamond$ that is $x_1, x_2 \in H \implies x_1.x_2 \in H$:

$$f_{1,b_1} \diamond f_{1,b_2} = f_{1,b_1+b_2}$$

Thus, $f_{1,b_1} \diamond f_{1,b_2} \in H$.
We need to show H is closed under inverses that is $x \in H \implies x^{-1} \in H$:

$$f_{1,b} \diamond f_{1,-b} = f_{1,0}$$

Thus, $f_{1,-b} \in H$.

Problem 3

We are given $(X, .)$ is a group. We are also given that
(i) $e \in X$ satisfies $e.x = x$ for all $x \in X$
(ii) for each $x \in X$, there exists $x^{-1} \in X$ such that $x^{-1}.x = e$

We want to show $x.e = x$ and $x.x^{-1} = e$ for all $x \in X$