

Homework Sheet 5

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Exercise 17

We are given the function

$$f(x) := -e^x + 2x + 1.$$

We calculate the newtons method using this formula

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

(i)

We want to find x_2 when $x_0 = -1$

Lets start by finding $f'(x)$

$$f'(x) = -e^x + 2$$

Now we can calculate x_1 and x_2

$$\begin{aligned}
 x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\
 &= -1 - \frac{-e^{-1} + 2(-1) + 1}{-e^{-1} + 2} \\
 &= -1 - \frac{-\frac{1}{e} - 2 + 1}{-\frac{1}{e} + 2} \\
 &= -1 - \frac{-\frac{1}{e} - 1}{-\frac{1}{e} + 2} \\
 &= -1 + \frac{\frac{1}{e} + 1}{-\frac{1}{e} + 2} \\
 &= -1 + \frac{e + 1}{-1 + 2e} \\
 &= \frac{-2e + e + 1 + 1}{-1 + 2e} \\
 &= \frac{-e + 2}{-1 + 2e} \\
 &\approx -0.16190048965915385 \quad (\text{via using a calculator})
 \end{aligned}$$

$$\begin{aligned}
 x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\
 &\approx -0.16190048965915385 - \frac{f(-0.16190048965915385)}{f'(-0.16190048965915385)} \\
 &\approx -0.16190048965915385 - \frac{-0.174326815763479123074}{f'(-0.16190048965915385)} \quad \text{again using a calculator} \\
 &\approx -0.16190048965915385 - \frac{-0.174326815763479123074}{1.149474163554828576926} \quad \text{again using a calculator} \\
 &\approx -0.16190048965915385 - -0.151657880873425944806890147658365522997056738243886290320790176143 \\
 &\approx -0.0102426087857279051931098523416344770029432617561137096792098238561672669 \quad \text{again using a calculator}
 \end{aligned}$$

Links to the calculator:

<https://www.wolframalpha.com/input?i=f%28x%29+%3D+-e%5Ex+%2B2x+%2B1+at+x%3D%E2%88%920.16190048965915385&assumption=%7B%22C%22%2C+%22at%22%7D+-%3E+%7B%22EnglishWord%22%7D>
<https://www.wolframalpha.com/input?i=g%28x%29+%3D+-e%5Ex+%2B2+where+x+%3D+%E2%88%920.16190048965915385>

(ii)

We want to find x_2 when $x_0 = 1$

The derivative

$$f'(x) = -e^x + 2$$

Now we can calculate x_1 and x_2

$$\begin{aligned}x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\&= 1 - \frac{-e^1 + 2(1) + 1}{-e^1 + 2} \\&= 1 - \frac{-e + 2 + 1}{-e + 2} \\&= 1 - \frac{-e + 3}{-e + 2} \\&= 1 + \frac{e - 3}{-e + 2} \\&= \frac{-e + 2 + e - 3}{-e + 2} \\&= \frac{-1}{-e + 2} \\&\approx 1.3922111911 \quad (\text{via using a calculator})\end{aligned}$$

$$\begin{aligned}x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\&= 1.3922111911 - \frac{f(1.3922111911)}{f'(1.3922111911)} \\&\approx 1.3922111911 - \frac{-0.23931509377336}{f'(1.3922111911)} \quad \text{again using a calculator} \\&\approx 1.3922111911 - \frac{-0.23931509377336}{-2.02373747597336} \quad \text{again using a calculator} \\&\approx 1.3922111911 - 0.1182540208967846811355339766932549 \quad \text{again using a calculator} \\&\approx 1.2739571702032153188644660233067451 \quad \text{again using a calculator}\end{aligned}$$

Exercise 18

We are given the function

$$f(x_1, x_2) := \begin{pmatrix} x_1^3 + x_2 - 2 \\ x_1 + x_2^3 - 2 \end{pmatrix}$$

The newtons method for multivariable functions is given by

$$\mathbf{x}_n = \mathbf{x}_{n-1} - J_f(\mathbf{x}_{n-1})^{-1}f(\mathbf{x}_{n-1})$$

The Jacobian matrix J_f is

$$\begin{aligned} J_f(x_1, x_2) &= \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} \\ &= \begin{pmatrix} 3x_1^2 & 1 \\ 1 & 3x_2^2 \end{pmatrix} \end{aligned}$$

We start with $\mathbf{x}_0 = (1, 1)$

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{x}_0 - J_f(\mathbf{x}_0)^{-1}f(\mathbf{x}_0) \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 3(1)^2 & 1 \\ 1 & 3(1)^2 \end{pmatrix}^{-1} \begin{pmatrix} (1)^3 + (1) - 2 \\ (1) + (1)^3 - 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

Doing the same calculation \mathbf{x}_2 will also be $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. It turns out that i didnt even need to calculate the jacobian matrix :)

Exercise 19

We are given the functions

$$f(x_1, x_2) := \frac{1}{3}x_1^3 + \frac{1}{4}x_2^2 + x_1 + \frac{1}{2}x_2 + 2,$$

$$g(x_1, x_2) := x_1 + \frac{1}{2}x_2.$$

Choose $h(x_1) = -2x_1$ so that $g(x_1, h(x_1)) = 0$.

$$\begin{aligned} f|_M(x_1) &= f(x_1, h(x_1)) \\ &= \frac{1}{3}x_1^3 + \frac{1}{4}(-2x_1)^2 + x_1 + \frac{1}{2}(-2x_1) + 2 \\ &= \frac{1}{3}x_1^3 + \frac{1}{4}(4x_1^2) + x_1 - x_1 + 2 \\ &= \frac{1}{3}x_1^3 + x_1^2 + 2 \end{aligned}$$

Now we need to find the extremum points of $f|_M$. Lets continue with derivaties

$$\begin{aligned}f|'_M(x_1) &= x_1^2 + 2x_1 \\f|''_M(x_1) &= 2x_1 + 2\end{aligned}$$

Setting the first derivatie to zero

$$\begin{aligned}f|'_M(x_1) = 0 &\implies x_1^2 + 2x_1 = 0 \\&\implies x_1(x_1 + 2) = 0 \\&\implies x_1 = 0 \quad \text{or} \quad x_1 = -2\end{aligned}$$

Now we can use the second derivatie test to classify these points

- For $x_1 = 0$:

$$f|''_M(0) = 2(0) + 2 = 2 > 0$$

So we have a local minimum at $x_1 = 0$. The corresponding x_2 value is

$$x_2 = h(0) = -2(0) = 0$$

Thus one local minimum point is $(0, 0)$.

- For $x_1 = -2$:

$$f|''_M(-2) = 2(-2) + 2 = -2 < 0$$

So we have a local maximum at $x_1 = -2$. The corresponding x_2 value is

$$x_2 = h(-2) = -2(-2) = 4$$

Thus one local maximum point is $(-2, 4)$.