Homework Sheet 2

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Problem 5

We are given the function

$$f(x_1, x_2) := |x_1| + |x_2|.$$

We are looking for points $x^0 \in \mathbb{R}^2$ where $\frac{\partial f}{\partial x_1^0}$ and $\frac{\partial f}{\partial x_2^0}$ exist. First things first that we realize the function is symmetric in both variables, so we can just focus on one of them and the other will follow the same logic.
Let's consider the partial derivative with respect to x_1^0:

$$\begin{aligned}\frac{\partial f}{\partial x_1^0} &= \lim_{h\to 0} \frac{f(x_1^0 + h, x_2^0) - f(x_1^0, x_2^0)}{h}. \\ &= \lim_{h\to 0} \frac{|x_1^0 + h| + |x_2^0| - (|x_1^0| + |x_2^0|)}{h}. \\ &= \lim_{h\to 0} \frac{|x_1^0 + h| - |x_1^0|}{h}.\end{aligned}$$

Now we have to consider different cases for x_1^0:

- If $x_1^0 > 0$:

$$\frac{\partial f}{\partial x_1^0} = \lim_{h\to 0} \frac{(x_1^0 + h) - x_1^0}{h} = \lim_{h\to 0} \frac{h}{h} = 1.$$

- If $x_1^0 < 0$:

$$\frac{\partial f}{\partial x_1^0} = \lim_{h\to 0} \frac{-(x_1^0 + h) + x_1^0}{h} = \lim_{h\to 0} \frac{-h}{h} = -1.$$

- If $x_1^0 = 0$:

$$\begin{aligned}\frac{\partial f}{\partial x_1^0} &= \lim_{h\to 0^+} \frac{|h| - 0}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1, \\ \frac{\partial f}{\partial x_1^0} &= \lim_{h\to 0^-} \frac{|-h| - 0}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1.\end{aligned}$$

  Since the left-hand limit and right-hand limit are not equal, the partial derivative does not exist at this point.

So partial derivates exist for all points where $x_1^0 \neq 0$. By symmetry, the same applies for x_2^0.
Partial Derivaties:

$$\frac{\partial f}{\partial x_1^0} = \begin{cases} 1, & x_1^0 > 0 \\ -1, & x_1^0 < 0 \end{cases},$$
$$\frac{\partial f}{\partial x_2^0} = \begin{cases} 1, & x_2^0 > 0 \\ -1, & x_2^0 < 0 \end{cases}$$

Problem 6

We are given the set

$$D := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \neq x_2^2, x_2 > 0\}$$

and the function

$$f : D \to \mathbb{R}, \quad f(x_1, x_2) := \frac{x_1 \ln(x_2)}{(x_1 - x_2^2)x_2}.$$

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We need to show that for any point $x \in D$ we can find an open ball around it.

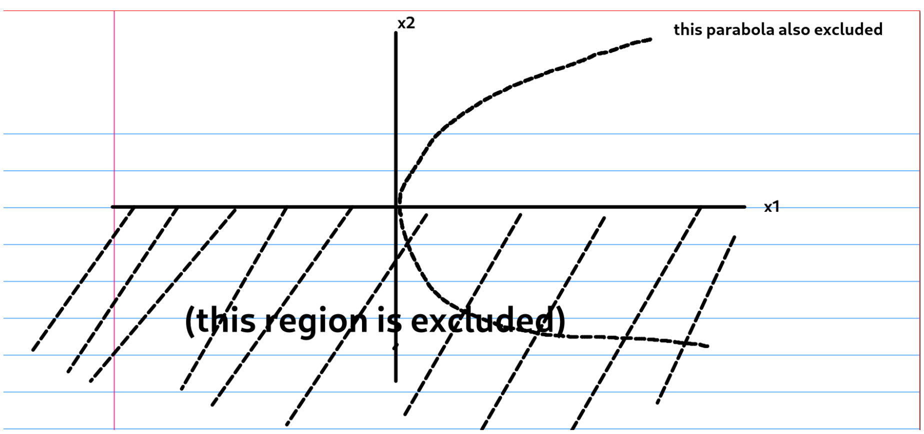

Figure 1: The set D

So if we pick a poitn in D we can get the distance to x_1 axis and distance to the parabola in the image and if we choose the radius of the open ball to be the smaller of both then every point in that open ball will be in D because we are guaranteed to not touch the x_1 axis and the parabola.
Thats why D is open.