Homework Sheet 2

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Problem 5

We are given the function

$$f(x_1, x_2) := |x_1| + |x_2|.$$

We are looking for points $x^0 \in \mathbb{R}^2$ where $\frac{\partial f}{\partial x_1^0}$ and $\frac{\partial f}{\partial x_2^0}$ exist. First things first that we realize the function is symmetric in both variables, so we can just focus on one of them and the other will follow the same logic.
Let's consider the partial derivative with respect to x_1^0:

$$\begin{aligned}\frac{\partial f}{\partial x_1^0} &= \lim_{h\to 0} \frac{f(x_1^0 + h, x_2^0) - f(x_1^0, x_2^0)}{h}. \\ &= \lim_{h\to 0} \frac{|x_1^0 + h| + |x_2^0| - (|x_1^0| + |x_2^0|)}{h}. \\ &= \lim_{h\to 0} \frac{|x_1^0 + h| - |x_1^0|}{h}.\end{aligned}$$

Now we have to consider different cases for x_1^0:

- If $x_1^0 > 0$:

$$\frac{\partial f}{\partial x_1^0} = \lim_{h\to 0} \frac{(x_1^0 + h) - x_1^0}{h} = \lim_{h\to 0} \frac{h}{h} = 1.$$

- If $x_1^0 < 0$:

$$\frac{\partial f}{\partial x_1^0} = \lim_{h\to 0} \frac{-(x_1^0 + h) + x_1^0}{h} = \lim_{h\to 0} \frac{-h}{h} = -1.$$

- If $x_1^0 = 0$:

$$\begin{aligned}\frac{\partial f}{\partial x_1^0} &= \lim_{h\to 0^+} \frac{|h| - 0}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1, \\ \frac{\partial f}{\partial x_1^0} &= \lim_{h\to 0^-} \frac{|-h| - 0}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1.\end{aligned}$$

  Since the left-hand limit and right-hand limit are not equal, the partial derivative does not exist at this point.

So partial derivates exist for all points where $x_1^0 \neq 0$. By symmetry, the same applies for x_2^0.
Partial Derivaties:

$$\frac{\partial f}{\partial x_1^0} = \begin{cases} 1, & x_1^0 > 0 \\ -1, & x_1^0 < 0 \end{cases},$$

$$\frac{\partial f}{\partial x_2^0} = \begin{cases} 1, & x_2^0 > 0 \\ -1, & x_2^0 < 0 \end{cases}$$