

Mathematics Homework Sheet 3

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Problem 1

Problem 1 (a)

For $i=0$:

$$f_1(-1) = -1$$

$$f_1(0) = 0$$

$$f_1(1) = 0$$

Meaning that:

$$f_1(\{-1, 0, 1\}) = \{-1, 0, 0\} = \{-1, 0\}$$

Now lets take a look at the inverse:

$$f_1(-1) = -1$$

$$f_1(0) = 0$$

$$f_1(1) = 0$$

$$f_1(2) = 1$$

These are the only values we can get -1, 0, 1. Meaning that:

$$f_1^{-1}(\{-1, 0, 1\}) = \{-1, 0, 1, 2\}$$

Now, for $i=2$ (the second function):

$$f_2(-1) = -4$$

$$f_2(0) = -3$$

$$f_2(1) = -2$$

Meaning that:

$$f_2(\{-1, 0, 1\}) = \{-4, -3, -2\}$$

Now lets take a look at the inverse:

$$f_2(2) = -1$$

$$f_2(3) = 0$$

$$f_2(4) = 1$$

These are the only values we can get -1, 0, 1. Meaning that:

$$f_2^{-1}(\{-1, 0, 1\}) = \{2, 3, 4\}$$

Now, for $i=3$ (the third function):

$$f_3(-1) = -2$$

$$f_3(0) = 0$$

$$f_3(1) = 2$$

Meaning that:

$$f_3(\{-1, 0, 1\}) = \{-2, 0, 2\}$$

Now lets take a look at the inverse:

$$f_3(x) = -1, \text{ No such } x \text{ exists in the domain of } f_3$$

$$\text{In other words, } f_3^{-1}(\{-1\}) = \emptyset$$

$$f_3(0) = 0$$

$$f_3(x) = 1, \text{ No such } x \text{ exists in the domain of } f_3$$

$$\text{In other words, } f_3^{-1}(\{1\}) = \emptyset$$

These are the only values we can get -1, 0, 1. Meaning that:

$$f_3^{-1}(\{-1, 0, 1\}) = \{0\}$$

Problem 1 (b)

Lets start with f_1 :

f_1 is not injective because it maps 0 and 1 to the same value, that is:

$$f_1(0) = f_1(1) = 0$$

f_1 is surjective because it maps to all the values in the codomain.

Lets take an element from the codomain $z \in Z$.

If $z \leq 0$ then we can find an element in the domain of f_3 , $x \in Z$ that maps to z . Simply $x = z$ works.

If $z > 0$ then we can find an element in the domain of f_3 , $x \in Z$ that maps to z . Simply $x = z + 1$ works.

Now, lets take a look at f_2 :

f_2 is injective because it maps each element in the domain to a unique element in the codomain, that is:

$$f_2(x) = f_2(y) \Rightarrow x = y$$

$$\begin{aligned} f_2(x) &= x - 3, & f_2(y) &= y - 3 \\ x - 3 &= y - 3 \Rightarrow x = y \end{aligned}$$

f_2 is surjective because it maps to all the values in the codomain.

Lets take an element from the codomain of f_2 , $z \in Z$.

We can find an element in the domain of f_2 , $x \in Z$ that maps to z . Simply $x = z + 3$ works.

Now, let's take a look at f_3 :

f_3 is injective because it maps each element in the domain to a unique element in the codomain, that is:

$$f_3(x) = f_3(y) \Rightarrow x = y$$

$$\begin{aligned} f_3(x) &= 2x, & f_3(y) &= 2y \\ 2x &= 2y \Rightarrow x = y \end{aligned}$$

f_3 is not surjective because it does not map to all the values in the codomain.

For example there is no element in the domain of f_3 that maps to 1. Generally all the odd numbers are not in the range of f_3 .

Problem 1 (c)

By looking at Problem 1 (b),

We see that f_1 is not injective thus f_1 is not bijective.

We see that f_2 is injective and surjective thus f_2 is bijective.

We see that f_3 is not surjective thus f_3 is not bijective.

Inverse of f_2 :

$$\begin{aligned} f_2 : Z &\rightarrow Z, & x &\mapsto x - 3 \\ f_2^{-1} : Z &\rightarrow Z, & x &\mapsto x + 3 \end{aligned}$$

Problem 2

Problem 2 (b)

$$(f_i \circ g = f_i \circ h) \implies g = h$$

Problem 3

Problem 3 (a)

Lets assume that opposite of the statement is true, that is there are more than one identity element. We are going to try to find a contradiction.

Lets name them e_1 and e_2 .

We know that both identity elements are in the set:

$$e_1 \in F$$

$$e_2 \in F$$

$$e_1 = e_1 \cdot e_2 \quad (\text{By definition of identity element } e_2) \quad (1)$$

$$= e_2 \cdot e_1 \quad (\text{By commutative property of multiplication}) \quad (2)$$

$$= e_2 \quad (\text{By definition of identity element } e_1) \quad (3)$$

So, this is a contradiction, we assumed there are two different identity elements but they turned out to be the same element. Thus, there can be only one identity element in a field.

Problem 3 (b)

$$\forall a, b, c \in F \quad a + c = b + c \implies a = b$$

Lets go step by step:

$$a = a + 0 \quad \text{Identity element of additiion} \quad (4)$$

$$= a + (c - c) \quad \text{Inverse element of addition} \quad (5)$$

$$= (a + c) - c \quad \text{Associative property of addition} \quad (6)$$

$$= (b + c) - c \quad \text{Given (a+c = b+c)} \quad (7)$$

$$= b + (c - c) \quad \text{Associative property of addition} \quad (8)$$

$$= b + 0 \quad \text{Inverse element of addition} \quad (9)$$

$$= b \quad \text{Identity element of addition} \quad (10)$$

So, we have proven that if $a + c = b + c$ then $a = b$.