Mathematics Homework Sheet 7

Author: Abdullah Oguz Topcuoglu & Yousef Farag

Problem 1

If b_n is convergent then

$$\liminf_{n\to\infty} b_n = \limsup_{n\to\infty} b_n = \lim_{n\to\infty} b_n$$

From the first inequality we have

$$\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

And from the second inequality we have

$$\limsup_{n\to\infty}(a_n + b_n) \ge \limsup_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

Combining these two inequalities we get

$$\limsup_{n\to\infty}(a_n + b_n) = \limsup_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

Problem 2

We want to show that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \quad \text{converges}$$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n(n+1)} &= \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} \\
s_n &= \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1} \\
s_n &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \ldots + \frac{1}{n} - \frac{1}{n+1} \\
s_n &= 1 - \frac{1}{n+1} \\
\sum_{n=1}^{\infty} \frac{1}{n(n+1)} &= \lim_{n\to\infty} s_n = 1 - \lim_{n\to\infty} \frac{1}{n+1} = 1 - 0 = 1
\end{aligned}$$

Therefore, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges and its value is 1.

Problem 3

a_n is monotonically decreasing and $\forall n \in N, a_n \geq 0$ and $\lim_{n\to\infty} a_n = 0$.
We want to show that $\sum_{n=1}^{\infty}(-1)^n a_n$ converges.

$$\sum_{n=1}^{\infty}(-1)^n a_n = -a_1 + a_2 - a_3 + a_4 - \dots$$

We can write this as

$$\sum_{n=1}^{\infty}(-1)^n a_n = -(a_1 - a_2) - (a_3 - a_4) - \dots$$

Since a_n is monotonically decreasing, we have

$$-(a_1 - a_2) - (a_3 - a_4) - \dots \leq 0$$

Because every paranthesis group is positive and they have minus sign before them so we are adding zero or negative terms.
Therefore, $\sum_{n=1}^{\infty}(-1)^n a_n$ is bounded above by 0.
And we can also rewrite the sum as follows:

$$\sum_{n=1}^{\infty}(-1)^n a_n = -a_1 + (a_2 - a_3) + (a_4 - a_5) + \dots$$

Since a_n is monotonically decreasing, every paranthesis group is positive or zero. We are adding positive or zero elements to $-a_1$ which means the sum is greater or equal to $-a_1$

$$-a_1 \leq -a_1 + (a_2 - a_3) + (a_4 - a_5) + \dots$$

Therefore, $\sum_{n=1}^{\infty}(-1)^n a_n$ is bounded below by $-a_1$.
Let's take a look at odd and even indexed terms:

$$\sum_{n=1}^{\infty}(-1)^{2n} a_{2n} = a_2 + a_4 + a_6 + a_8 + \dots \tag{1}$$

$$\sum_{n=1}^{\infty}(-1)^{2n+1} a_{2n+1} = -a_1 - a_3 - a_5 - a_7 + \dots \tag{2}$$

Since $a_n \geq 0$, (2) is monotonically decreasing and similarly (1) is monotonically increasing.

$$s_n = \sum_{k=1}^{n}(-1)^k a_k$$

$$s_{2n} = \sum_{k=1}^{2n} (-1)^k a_k = (-a_1 + a_2) + (-a_3 + a_4) + \dots$$

s_{2n} is monotonically decreasing because every paranthesis group is negative or zero. So we are adding ≤ 0 elements each iteration which means $s_{2n} \geq s_{2(n+1)}$

$$s_{2n+1} = \sum_{k=1}^{2n+1} (-1)^k a_k = -a_1 + (a_2 - a_3) + (a_4 - a_5) + \dots$$

s_{2n+1} is monotonically increasing because every paranthesis group is positive or zero. So we are adding ≥ 0 elements each iteration which means $s_{2n+1} \leq s_{2(n+1)+1}$

And now it suffices to show that $s_{2l-1} \leq s_{2l}, \forall l \in N$

$$\begin{aligned}
s_{2l} &= \sum_{k=1}^{2l} (-1)^k a_k \\
s_{2l-1} &= \sum_{k=1}^{2l-1} (-1)^k a_k \\
s_{2l} - s_{2l-1} &= \sum_{k=1}^{2l} (-1)^k a_k - \sum_{k=1}^{2l-1} (-1)^k a_k \\
&= (-1)^{2l} a_{2l} + \sum_{k=1}^{2l-1} (-1)^k a_k - \sum_{k=1}^{2l-1} (-1)^k a_k \\
&= (-1)^{2l} a_{2l} \\
&= a_{2l} \\
&\geq 0 \quad \text{because} \quad a_n \geq 0 \quad \forall n \in N
\end{aligned}$$

Therefore, $s_{2l-1} \leq s_{2l}, \forall l \in N$
Therefore, $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.
I followed the hint in the homework sheet.