

Homework Sheet 8

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Let N_k denote the set of natural numbers from 1 to k.

$$N_k = \{i \in \mathbb{N} \mid 0 < i \leq k\}$$

So that N_k has k elements.

Exercise 29

(i)

The probability space would be

$$\begin{aligned}\Omega &= N_6 \times N_{10} \\ \mathcal{F} &= \text{Power set of } \Omega \\ P &= U_\Omega\end{aligned}$$

I chose the uniform distribution because each outcome is equally likely when rolling fair dice.

The random variable X would be

$$\begin{aligned}X : \Omega &\rightarrow \mathbb{N} \\ (d, h) &\mapsto \max(d, h)\end{aligned}$$

The range of X

$$\begin{aligned}X(\Omega) &= \{\max(d, h) \mid d \in N_6, h \in N_{10}\} \\ &= N_{10}\end{aligned}$$

(ii)

We want to calculate $P[\{X = k\}]$ for $k \in N_{10}$.

The random variable X takes the value k only in one of the following three cases:

- The decahedron shows k and the dice shows a number less than k .
- The dice shows k and the decahedron shows a number less than k .
- Both the dice and the decahedron show k .

Here note that the cases are disjoint. Lets count the number of possibilities for each case and add them up. We need to consider those cases when k is less than or equal to 6 and when k is greater than 6 separately.

- For $k \in N_6$:

- Case 1: There are $k - 1$ choices for the dice (1 to $k - 1$) and 1 choice for the decahedron (k). So there are $k - 1$ possibilities.
- Case 2: There are $k - 1$ choices for the decahedron (1 to $k - 1$) and 1 choice for the dice (k). So there are $k - 1$ possibilities.
- Case 3: There is only 1 possibility where both show k .

So in total there are $2(k - 1) + 1 = 2k - 1$ possibilities.

- For $k \in \{7, 8, 9, 10\}$:

- Case 1: There are 6 choices for the dice (1 to 6) and 1 choice for the decahedron (k). So there are 6 possibilities.
- Case 2: Can't happen since the dice only goes up to 6.
- Case 3: Can't happen since the dice only goes up to 6.

So in total there are $6 + 0 + 0 = 6$ possibilities.

$$P[\{X = k\}] = \begin{cases} \frac{2k-1}{60}, & k \in N_6 \\ \frac{6}{60}, & k \in \{7, 8, 9, 10\} \end{cases}$$

$$60 = |\Omega| = |N_6| \times |N_{10}| = 6 \times 10$$

(iii)

We want to calculate $P[\{2 \leq X \leq 5\}]$.

$$\begin{aligned} P[\{2 \leq X \leq 5\}] &= \sum_{k=2}^5 P[\{X = k\}] \\ &= \sum_{k=2}^5 \frac{2k-1}{60} \\ &= \frac{1}{60} \sum_{k=2}^5 (2k-1) \\ &= \frac{1}{60} (3 + 5 + 7 + 9) \\ &= \frac{24}{60} \\ &= \frac{2}{5} \end{aligned}$$

Exercise 30

(i)

The probability space would be

$$\begin{aligned}\Omega &= N_7 \times N_4 \\ \mathcal{F} &= \text{Power set of } \Omega \\ P &= U_\Omega\end{aligned}$$

I chose the uniform distribution because each outcome is equally likely when pulling marbles from urns.

The random variables D_7 and D_4 would be

$$\begin{aligned}D_7 : \Omega &\rightarrow \mathbb{N} \\ (m_7, m_4) &\mapsto m_7\end{aligned}$$

$$\begin{aligned}D_4 : \Omega &\rightarrow \mathbb{N} \\ (m_7, m_4) &\mapsto m_4\end{aligned}$$

The ranges of D_7 and D_4

$$D_7(\Omega) = \{m_7 \mid m_7 \in N_7, m_4 \in N_4\} = N_7$$

$$D_4(\Omega) = \{m_4 \mid m_7 \in N_7, m_4 \in N_4\} = N_4$$

(ii)

The range of X

$$\begin{aligned}X(\Omega) &= \{\max(|D_7 - D_4|, 0) \mid D_7 \in N_7, D_4 \in N_4\} \\ &= N_6 \cup \{0\}\end{aligned}$$

The range of Y

$$\begin{aligned}Y(\Omega) &= \{\min(|D_7 - D_4|, 5) \mid D_7 \in N_7, D_4 \in N_4\} \\ &= N_5 \cup \{0\}\end{aligned}$$

To find the distributions of X and Y , we need to calculate $P[\{X = k\}]$ and $P[\{Y = k\}]$ for k in their respective ranges.

The random variable X takes the value k in the following cases:

- When $|D_7 - D_4| = k$

The random variable Y takes the value k in the following cases:

- When $|D_7 - D_4| = k$ for $k \in N_5$
- When $|D_7 - D_4| \geq 5$ for $k = 5$

Exercise 31

Lets first create the probability space.

$$\Omega = \text{set of all light bulbs} = M_1 \uplus M_2 \uplus M_3$$

$$\mathcal{F} = \text{Power set of } \Omega$$

$$P = U_\Omega$$

where M_i is the set of all light bulbs from manufacturer i for $i \in \{1, 2, 3\}$.

Here M_i also represent the event that a randomly selected bulb is from manufacturer i .

Let D be the event that a randomly selected bulb is defective.

We are given that

$$P[M_1] = 0.3$$

$$P[M_2] = 0.5$$

$$P[M_3] = 0.2$$

$$P[D|M_1] = 0.05$$

$$P[D|M_2] = 0.02$$

$$P[D|M_3] = 0.08$$

We want to find $P[D]$.

Using the law of total probability:

$$\begin{aligned} P[D] &= \sum_{i=1}^3 P[D|M_i]P[M_i] \\ &= P[D|M_1]P[M_1] + P[D|M_2]P[M_2] + P[D|M_3]P[M_3] \\ &= 0.05 \cdot 0.3 + 0.02 \cdot 0.5 + 0.08 \cdot 0.2 \\ &= 0.015 + 0.01 + 0.016 \\ &= 0.041 \end{aligned}$$