Mathematics Homework Sheet 3

Authors: Abdullah Oguz Topcuoglu & Ahmed Waleed Ahmed Badawy Shora

Problem 1

(a)

$$3z^2 + z = 1$$

solve the quadratic equations and we get

$$z = \frac{-1 \pm \sqrt{1^2 - 4.3.(-1)}}{(3.2)}$$

$$z = \frac{-1 \pm \sqrt{1 + 12}}{6}$$

$$z = \frac{-1 \pm \sqrt{13}}{6}$$

$$z_0 = \frac{-1 + \sqrt{13}}{6} + 0i, \quad z_1 = \frac{-1 - \sqrt{13}}{6} + 0i$$

(b)

$$3z^2 + z = 0$$

$$z(3z + 1) = 0$$

$$z = 0 \quad \text{or} \quad 3z + 1 = 0$$

$$z = 0 \quad \text{or} \quad z = -\frac{1}{3}$$

(c)

$$z^2 - (3 + i)z + 4 + 3i = 0$$

$$z = \frac{(3 + i) \pm \sqrt{(3 + i)^2 - 4(4 + 3i)}}{2}$$

$$z = \frac{(3 + i) \pm \sqrt{(9 + 6i - 1 - 16 - 12i)}}{2}$$

$$z = \frac{(3 + i) \pm \sqrt{-8 - 6i}}{2}$$

$$z = \frac{(3 + i) \pm (1 - 3i)}{2}$$

$$z_0 = \frac{4-2i}{2} = 2-i, \quad z_1 = \frac{2+4i}{2} = 1+2i$$

$\sqrt{-8-6i} = \pm(1-3i)$ because $\sqrt{-8-6i} = \pm(\sqrt{\frac{r+(-8)}{2}} + sign(-6)\sqrt{\frac{r-(-8)}{2}}i)$
where $r = |-8-6i|$

(d)

$$\sinh z = i$$

By definiton of sinh

$$\frac{e^z - e^{-z}}{2} = i$$

By definition of e^z

$$\frac{e^x(cos(y) + sin(y)i) + e^{-x}(cos(-y) + sin(-y)i))}{2} = i$$

where $z = x + iy$.

$$e^x(cos(y) + sin(y)i) + e^{-x}(cos(-y) + sin(-y)i)) = 2i$$

$$e^x(cos(y) + sin(y)i) + e^{-x}(cos(y) - sin(y)i)) = 2i$$

$$e^x cos(y) + e^x sin(y)i + e^{-x} cos(y) - e^{-x} sin(y)i = 2i$$

$$(e^x cos(y) + e^{-x} cos(y)) + (e^x sin(y) - e^{-x} sin(y))i = 0 + 2i$$

$$e^x cos(y) + e^{-x} cos(y) = 0, \qquad e^x sin(y) - e^{-x} sin(y) = 2$$

$$e^x cos(y) + e^{-x} cos(y) = 0 \implies cos(y) = 0$$

$$y = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$$

$$e^x sin(y) - e^{-x} sin(y) = 2$$

$$sin(y)(e^x - e^{-x}) = 2$$

$$sin(\frac{\pi}{2} + k\pi)(e^x - e^{-x}) = 2 \implies e^x - e^{-x} = \pm 2$$

Let $r = e^x$

$$r - \frac{1}{r} = 2 \implies r^2 - 2r - 1 = 0$$

or

$$r - \frac{1}{r} = -2 \implies r^2 + 2r - 1 = 0$$

$$r = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$$

$$r = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$$

$$e^x = \sqrt{2} \pm 1 \implies x = ln(\sqrt{2} \pm 1), \qquad \sqrt{2} \pm 1 > 0$$

$$z = x + yi \text{ where } x = ln(\sqrt{2}+1) \text{ or } x = ln(\sqrt{2}-1) \text{ and } y = \tfrac{\pi}{2} + k\pi$$

(e)

$$tan(z) = 1$$
$$\frac{sin(z)}{cos(z)} = 1$$
$$sin(z) = cos(z)$$

By definition of sin(z) and cos(z)

$$\frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2}$$
$$e^{iz} - e^{-iz} = i(e^{iz} + e^{-iz})$$

Let $q = e^{iz}$

$$q - \frac{1}{q} = i(q + \frac{1}{q})$$
$$q^2 - 1 = iq^2 + 1$$
$$q^2(1 - i) - 2 = 0$$
$$q = \sqrt{\frac{2}{1-i}} = \sqrt{\frac{2(1+i)}{2}} = \sqrt{1+i}$$

(f)

$$cos(z) = -\frac{5}{4}$$

By definition of cos(z)

$$\frac{e^{iz} + e^{-iz}}{2} = -\frac{5}{4}$$
$$e^{iz} + e^{-iz} = -\frac{5}{2}$$

Let $q = e^{iz}$

$$q + \frac{1}{q} = -\frac{5}{2}$$
$$q^2 + \frac{5}{2}q + 1 = 0$$
$$q = \frac{-\frac{5}{2} \pm \sqrt{(\frac{5}{2})^2 - 4}}{2}$$
$$q = \frac{-\frac{5}{2} \pm \frac{3}{2}}{2}$$
$$q_0 = \frac{-1}{2}, \qquad q_1 = -2$$

$$e^{iz_0} = -\frac{1}{2} = \frac{1}{2}e^{i\pi}$$
$$e^{iz_0 - i\pi} = \frac{1}{2}$$
$$iz_0 - i\pi = ln(\frac{1}{2})$$
$$z_0 = \frac{ln(\frac{1}{2}) + i\pi}{i}$$

$$e^{iz_1} = -2 = 2e^{i\pi}$$
$$e^{iz_1 - i\pi} = 2$$
$$iz_1 - i\pi = ln(2)$$
$$z_1 = \frac{ln(2) + i\pi}{i}$$

(g)

$$z + \bar{z} = 1$$

Let $z = x + iy$

$$x + iy + x - iy = 1$$
$$2x = 1$$
$$x = \frac{1}{2}$$
$$z = \frac{1}{2} + iy, \qquad y \in R$$

(h)

$$z^2 + 2\bar{z}^2 + z - \bar{z} + 9 = 0$$

Let $z = x + iy$

$$(x + iy)^2 + 2(x - iy)^2 + (x + iy) - (x - iy) + 9 = 0$$
$$9 - 2iy + x^2 + 2xyi - y^2 + 2x^2 - 4xyi - y^2 = 0$$
$$9 - 2iy + 3x^2 - 2xyi - 2y^2 = 0$$
$$9 + 3x^2 - 2y^2 = 0, \quad -2 - 2xy = 0$$
$$x = -\frac{1}{y}$$

Insert this in the first equation

$$9 - \frac{3}{y^2} - 2y^2 = 0$$

$$4y^4 - 9y^2 + 3 = 0$$

$$y^2 = \frac{9 \pm \sqrt{81 - 48}}{8} = \frac{9 \pm 3\sqrt{3}}{8}$$

$$y = \pm\sqrt{\frac{9 \pm 3\sqrt{3}}{8}}$$

$$x = -\frac{1}{y}$$

$$x = \pm\sqrt{\frac{8}{9 \pm 3\sqrt{3}}}$$

$$z = \pm\sqrt{\frac{8}{9 \pm 3\sqrt{3}}} \pm i\sqrt{\frac{9 \pm 3\sqrt{3}}{8}}$$

Problem 2

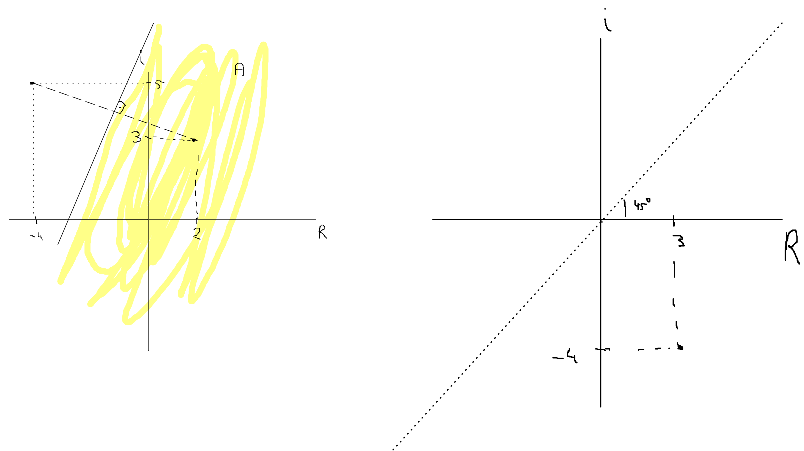

Figure 1: Visualization for Problem 2

Problem 3

(a)

$$(4\sqrt{3} - 4i)^{88}$$

Let's convert this to polar form

$$r = \sqrt{(4\sqrt{3})^2 + (-4)^2} = \sqrt{48 + 16} = \sqrt{64} = 8$$

$$\theta = tan^{-1}(\frac{-4}{4\sqrt{3}}) = tan^{-1}(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$$

So we want to compute

$$(8e^{-\frac{\pi}{6}i})^{88}$$

$$(8e^{-\frac{\pi}{6}i})^{88} = 8^{88}e^{-\frac{88\pi}{6}i} = 8^{88}e^{-\frac{44\pi}{3}i}$$

(b)

$$z = \left(1 + i\tan\left(\frac{(4m+1)\pi}{4n}\right)\right)^n$$

$$z = \left(1 + i\frac{\sin\left(\frac{(4m+1)\pi}{4n}\right)}{\cos\left(\frac{(4m+1)\pi}{4n}\right)}\right)^n$$

$$z = \left(\frac{\cos\left(\frac{(4m+1)\pi}{4n}\right) + i\sin\left(\frac{(4m+1)\pi}{4n}\right)}{\cos\left(\frac{(4m+1)\pi}{4n}\right)}\right)^n$$

$$z = \left(\frac{1}{\cos\left(\frac{(4m+1)\pi}{4n}\right)}\right)^n \left(\cos\left(\frac{(4m+1)\pi}{4n}\right) + i\sin\left(\frac{(4m+1)\pi}{4n}\right)\right)^n$$

Using de Moivre's theorem

$$z = \frac{1}{\cos\left(\frac{(4m+1)\pi}{4n}\right)^n}\left(\cos\left(\frac{(4m+1)\pi}{4}\right) + i\sin\left(\frac{(4m+1)\pi}{4}\right)\right)$$

$$Re(z) = \frac{1}{\cos\left(\frac{(4m+1)\pi}{4n}\right)^n}\cos\left(\frac{(4m+1)\pi}{4}\right)$$

$$Im(z) = \frac{1}{\cos\left(\frac{(4m+1)\pi}{4n}\right)^n}\sin\left(\frac{(4m+1)\pi}{4}\right)$$

Problem 4

In an ordered field the following must hold $0 < 1$
Assume $i > 0$

$$\begin{aligned} 0 &< i && \text{(multiply by i)} \\ 0 &< i^2 && \\ 0 &< -1 && \text{(add 1)} \\ 1 &< 0 && \text{this is a contradiction} \end{aligned}$$

Assume $i < 0$

$$\begin{aligned} 0 &< -i && \text{(multiply by -i)} \\ 0 &< i^2 && \\ 0 &< -1 && \text{(add 1)} \\ 1 &< 0 && \text{this is a contradiction} \end{aligned}$$

Thus, we have shown that both assumptions lead to a contradiction. Therefore, $(C, +, \cdot)$ is not an ordered field.