Mathematics Homework Sheet 8

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Problem 2

Let's do elementary row operations.

$$
\begin{aligned}
A_\lambda &= \begin{pmatrix} 1 & \lambda & 0 & 0 \\ \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \end{pmatrix} \\
&= \left(\begin{array}{cccc|cccc} 1 & \lambda & 0 & 0 & 1 & 0 & 0 & 0 \\ \lambda & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \lambda & 1 & 0 & 0 & 0 & 1 \end{array}\right) \\
&\left(\begin{array}{cccc|cccc} 1 & \lambda & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1-\lambda^2 & 0 & 0 & -\lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \lambda & 1 & 0 & 0 & 0 & 1 \end{array}\right) \qquad r_2 = r_2 - \lambda r_1
\end{aligned}
$$

if $\lambda^2 \neq 1$, then we can divide by $1-\lambda^2$ and get the following:

$$A_\lambda = \left(\begin{array}{cccc|cccc} 1 & \lambda & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{\lambda}{1-\lambda^2} & \frac{1}{1-\lambda^2} & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \lambda & 1 & 0 & 0 & 0 & 1 \end{array}\right) \qquad r_2 = r_2/(1-\lambda^2)$$

$$\left(\begin{array}{cccc|cccc} 1 & \lambda & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{\lambda}{1-\lambda^2} & \frac{1}{1-\lambda^2} & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{\lambda^2}{1-\lambda^2} & -\frac{\lambda}{1-\lambda^2} & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 & 0 & 0 & 1 \end{array}\right) \qquad r_3 = r_3 - \lambda r_2$$

$$\left(\begin{array}{cccc|cccc} 1 & \lambda & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{\lambda}{1-\lambda^2} & \frac{1}{1-\lambda^2} & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{\lambda^2}{1-\lambda^2} & -\frac{\lambda}{1-\lambda^2} & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{\lambda^3}{1-\lambda^2} & \frac{\lambda^2}{1-\lambda^2} & 0 & 1 \end{array}\right) \qquad r_4 = r_4 - \lambda r_3$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1+\frac{\lambda^2}{1-\lambda^2} & -\frac{\lambda}{1-\lambda^2} & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{\lambda}{1-\lambda^2} & \frac{1}{1-\lambda^2} & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{\lambda^2}{1-\lambda^2} & -\frac{\lambda}{1-\lambda^2} & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{\lambda^3}{1-\lambda^2} & \frac{\lambda^2}{1-\lambda^2} & 0 & 1 \end{array}\right) \qquad r_1 = r_1 - \lambda r_2$$

So if $\lambda^2 \neq 1$, then we can write the inverse matrix as follows:

$$A_\lambda^{-1} = \begin{pmatrix} 1+\frac{\lambda^2}{1-\lambda^2} & -\frac{\lambda}{1-\lambda^2} & 0 & 0 \\ -\frac{\lambda}{1-\lambda^2} & \frac{1}{1-\lambda^2} & 0 & 0 \\ \frac{\lambda^2}{1-\lambda^2} & -\frac{\lambda}{1-\lambda^2} & 0 & 0 \\ -\frac{\lambda^3}{1-\lambda^2} & \frac{\lambda^2}{1-\lambda^2} & 0 & 1 \end{pmatrix}$$

If $\lambda^2 = 1$, then we have two cases:

- If $\lambda = 1$, then the matrix becomes:

$$A_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

  which is not invertible because the first row and the second row are the same.

- If $\lambda = -1$, then the matrix becomes:

$$A_{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

which is also not invertible because $r_1 = -r_2$.