Mathematics Homework Sheet 9

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Problem 1

From the given hint i observe that the columns of the matrix is just the images of the standard basis vectors. And since e_2 and e_3 are in the kernel, second and third columns of D are all zeros.

$$D = \begin{pmatrix} x & 0 & 0 & x \\ x & 0 & 0 & x \\ x & 0 & 0 & x \\ x & 0 & 0 & x \end{pmatrix}$$

And also from the hint i observe that the image is just span of the column vectors of the matrix. So i will just plug the given vectors in the image of D into the columns of D

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Problem 2

We can write the equations in matrix form as follows:

$$\begin{pmatrix} 2 & -3 & -1 & 1 \\ 3 & 4 & -4 & -3 \\ 0 & 17 & -5 & -9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

To find the nontrivial solutions, we can row reduce the matrix to echelon form.

$$\begin{pmatrix} 2 & -3 & -1 & 1 \\ 3 & 4 & -4 & -3 \\ 0 & 17 & -5 & -9 \end{pmatrix} r_2 = 2r_2 - 3r_1$$

$$\begin{pmatrix} 2 & -3 & -1 & 1 \\ 0 & 17 & -5 & -9 \\ 0 & 17 & -5 & -9 \end{pmatrix} r_3 = r_3 - r_2$$

$$\begin{pmatrix} 2 & -3 & -1 & 1 \\ 0 & 17 & -5 & -9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Which gives us the following system of equations:

$$\begin{aligned} 2x_1 - 3x_2 - x_3 + x_4 &= 0 \\ 17x_2 - 5x_3 - 9x_4 &= 0 \end{aligned}$$

Let $x_3 = \lambda$ and $x_4 = \mu$ then we can express x_1 and x_2 in terms of λ and μ:

$$\begin{aligned} x_2 &= \frac{5\lambda + 9\mu}{17} \\ x_1 &= \frac{3x_2 + x_3 - x_4}{2} = \frac{3\left(\frac{5\lambda+9\mu}{17}\right) + \lambda - \mu}{2} \end{aligned}$$

Thus, the general solution can be expressed as:

$$\begin{pmatrix} \frac{3\left(\frac{5\lambda+9\mu}{17}\right)+\lambda-\mu}{2} \\ \frac{5\lambda+9\mu}{17} \\ \lambda \\ \mu \end{pmatrix} \lambda, \mu \in \mathbb{R}$$

Problem 3

$$\det \begin{pmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{pmatrix}$$

Expand along the first row:

$$\begin{aligned}
&= (a^2+1)\det\begin{pmatrix} b^2+1 & bc \\ bc & c^2+1 \end{pmatrix} - ab\det\begin{pmatrix} ab & ac \\ ac & c^2+1 \end{pmatrix} + ac\det\begin{pmatrix} ab & b^2+1 \\ ac & bc \end{pmatrix} \\
&= (a^2+1)((b^2+1)(c^2+1) - b^2c^2) - ab(ab(c^2+1) - a^2c^2) + ac(ab^2c - ac(b^2+1)) \\
&= (a^2+1)(b^2+1)(c^2+1) - (a^2+1)b^2c^2 - a^2b^2(c^2+1) + a^3bc^2 + a^2b^2c^2 - a^2c^2(b^2+1) \\
&= (a^2b^2 + a^2 + b^2 + 1)(c^2+1) - (a^2+1)b^2c^2 - a^2b^2(c^2+1) + a^3bc^2 + a^2b^2c^2 - a^2c^2(b^2+1) \\
&= a^2b^2c^2 + a^2b^2 + a^2c^2 + a^2 + b^2c^2 + b^2 + c^2 + 1 - (a^2+1)b^2c^2 - a^2b^2(c^2+1) + a^3bc^2 + a^2b^2c^2 - a^2c^2(b^2+1) \\
&= a^2 + b^2 + c^2 + 1
\end{aligned}$$