Homework Sheet 3

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Exercise 9

We are given the function:

$$f(x_1, x_2) := \begin{cases} \frac{x_1^2 x_2}{x_1^2 + x_2^2}, & (x_1, x_2) \neq (0,0) \\ 0, & (x_1, x_2) = (0,0) \end{cases}$$

(i)

To show that f is continuous at the point $(0,0)$, we need to verify that every sequence $(x_1^{(n)}, x_2^{(n)})$ converging to $(0,0)$, $f(x_1^{(n)}, x_2^{(n)})$ also converges to $f(0,0) = (0,0)$.
Fix a sequence $(x_1^{(n)}, x_2^{(n)})$ such that $(x_1^{(n)}, x_2^{(n)}) \to (0,0)$.
Then, we have:
If $(x_1^{(n)}, x_2^{(n)}) \neq (0,0)$

$$\begin{aligned} |f(x_1^{(n)}, x_2^{(n)}) - f(0,0)| &= \left| \frac{(x_1^{(n)})^2 x_2^{(n)}}{(x_1^{(n)})^2 + (x_2^{(n)})^2} - 0 \right| \\ &= \left| \frac{(x_1^{(n)})^2 x_2^{(n)}}{(x_1^{(n)})^2 + (x_2^{(n)})^2} \right| \\ &\leq \left| \frac{(x_1^{(n)})^2 x_2^{(n)}}{(x_1^{(n)})^2} \right| \quad \text{(since } (x_1^{(n)})^2 + (x_2^{(n)})^2 \geq (x_1^{(n)})^2\text{)} \\ &= |x_2^{(n)}| \end{aligned}$$

$|x_2^{(n)}| \to 0$ since $(x_1^{(n)}, x_2^{(n)}) \to (0,0)$. Therefore $f(x_1^{(n)}, x_2^{(n)}) \to (0,0)$.

If $(x_1^{(n)}, x_2^{(n)}) = (0,0)$

$$|f(x_1^{(n)}, x_2^{(n)}) - f(0,0)| = |0 - 0| = 0$$

which also converges to 0.
Thats what we wanted to show.

(ii)

We need to show that the following limit doesnt converge to zero

$$\lim_{x \to 0} \frac{f(x) - f(0) - Jf(0)x}{\|x\|}$$

The J is the Jacobian matrix.

$$J = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix}$$

Lets compute the partials at (0, 0)

$$\frac{\partial f}{\partial x_1}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$
$$\frac{\partial f}{\partial x_2}(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

Thus, the Jacobian matrix at (0, 0) is:

$$Jf(0,0) = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Substituting this into our limit we get

$$\lim_{x\to 0} \frac{f(x) - f(0)}{\|x\|}$$

We also know that $f(0) = 0$, so we can simplify this to:

$$\lim_{x\to 0} \frac{f(x)}{\|x\|}$$

So it is enough if i can show a vector x where this limit is not zero.
Lets choose $x = (x_1, x_1)$. Then we have:

$$\begin{aligned} \lim_{x_1\to 0} \frac{f(x_1,x_1)}{\|(x_1,x_1)\|} &= \lim_{x_1\to 0} \frac{\frac{x_1^2 x_1}{x_1^2+x_1^2}}{\sqrt{x_1^2+x_1^2}} \\ &= \lim_{x_1\to 0} \frac{\frac{x_1^3}{2x_1^2}}{\sqrt{2}|x_1|} \\ &= \lim_{x_1\to 0} \frac{x_1}{2\sqrt{2}|x_1|} \\ &= \lim_{x_1\to 0} \frac{1}{2\sqrt{2}} \cdot \frac{x_1}{|x_1|} \end{aligned}$$

This limit doesnt even exist so it cant be equal to zero.
Thats what we wanted to show.

Exercise 10

We are given the function:

$$f(x_1,x_2,x_3) := \begin{bmatrix} x_1^4 \ln(3+2x_2^2) \\ x_1 \sin(x_2 x_3) e^{x_1} \end{bmatrix}$$

(i)

To show that f is a C^1 function we need to verify that all partial derivatives of f exist and are continuous.
Lets compute the partial derivatives of each component of f which corresponds to the entries of Jacobian matrix.

$$Jf = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \end{bmatrix}$$

Calculating the partial derivatives:

$$\frac{\partial f_1}{\partial x_1} = 4x_1^3 \ln(3 + 2x_2^2)$$
$$\frac{\partial f_1}{\partial x_2} = x_1^4 \cdot \frac{4x_2}{3 + 2x_2^2}$$
$$\frac{\partial f_1}{\partial x_3} = 0$$
$$\frac{\partial f_2}{\partial x_1} = \sin(x_2 x_3) e^{x_1} + x_1 \sin(x_2 x_3) e^{x_1}$$
$$\frac{\partial f_2}{\partial x_2} = x_1 x_3 \cos(x_2 x_3) e^{x_1}$$
$$\frac{\partial f_2}{\partial x_3} = x_1 x_2 \cos(x_2 x_3) e^{x_1}$$

All these partial derivatives are polynomials, logarithmic, exponential, sine and cosine functions which are continuous everywhere in $\mathbb{R}^3$ Thus, all partial derivatives of f exist and are continuous.
(we can explicitly use the remark 1.3.13 in lecture notes to prove continuity but thats trivial so i cut it short with one sentence. Can i cut it short like this in the exam?)
Thats what we wanted to show.

(ii)

Yes it is because we know that continuously partially differentiability implies total differentiability. In (i) we showed that f is continuously partially differentiable function.