Mathematics Homework Sheet 7

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Problem 1

A transformation T is linear if and only if it satisfies the following two properties for all vectors u, v and scalar c:

1. $T(u+v) = T(u) + T(v)$
2. $T(cu) = cT(u)$

1. $T : \mathbb{R}^2 \to \mathbb{R}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto x + 2y$ is linear.
2. $T : \mathbb{R}^2 \to \mathbb{R}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto x + y^2$ is not linear.
   Because rule (1) is not satisfied.
3. $T : \mathbb{R}^2 \to \mathbb{R}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto xy$ is not linear.
   Because rule (2) is not satisfied.
4. $T : \mathbb{C} \to \mathbb{C}, z \mapsto \overline{z}$ is linear.
5. $T : \mathbb{R}^2 \to \mathbb{R}^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+1 \\ y-1 \end{pmatrix}$ is not linear.
   Because rule (1) is not satisfied.
6. $T : \mathbb{R}^2 \to \mathbb{R}^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x-y \\ x+2y \end{pmatrix})$ is linear.
7. $T : \mathbb{R}^n[x] \to \mathbb{R}, p(x) \mapsto p(1)$ is linear.
8. $T : \mathbb{R}^n[x] \to \mathbb{R}^{n+2}[x], p(x) \mapsto x^2 p(x)$ is linear.

Problem 2

(a)

The way we construct the matrix given a linear transformation is that we take the each basis vector and apply the transformation to it and write the result to corresponding column of the matrix.

$$
\begin{aligned}
T(e_1) &= T(1,0,0) = (1-0, 1+2\cdot 0-0, 2\cdot 1+0+0) = (1,1,2)\\
T(e_2) &= T(0,1,0) = (0-1, 0+2\cdot 1-0, 2\cdot 0+1+0) = (-1,2,1)\\
T(e_3) &= T(0,0,1) = (0-0, 0+2\cdot 0-1, 2\cdot 0+0+1) = (0,-1,1)
\end{aligned}
$$

Put each result as a column in the matrix A:

$$
A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 2 & 1 & 1 \end{pmatrix}
$$

(b)

The standard basis looks like this: $\{1, x, ..., x^n\}$

$$
\begin{aligned}
T(1) &= 1\\
T(x) &= x+1\\
T(x^2) &= (x+1)^2 = x^2+2x+1\\
&\vdots\\
T(x^k) &= (x+1)^k = \sum_{i=0}^{k} \binom{k}{i} x^i\\
&\vdots\\
T(x^n) &= (x+1)^n
\end{aligned}
$$

$$
m_{i,j} = \begin{cases} \binom{j}{i} & \text{if } 0 \le i \le j \le n \\ 0 & \text{if } i > j \end{cases}
$$

Each column of the matrix has the coefficients of a binomial expansion of $(x+1)^k$ for $k = 0, 1, \ldots, n$. For example the first column looks like this:

$$
\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}
$$

The second column looks like this:

$$
\begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}
$$

(c)

The standard basis of $\mathbb{R}^{2\times 2}$ is:

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

We compute $T(E_{ij})$ for each basis element:

$$\begin{aligned}
T(E_{11}) &= T\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = E_{11} \\
T(E_{12}) &= T\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 2E_{12} \\
T(E_{21}) &= T\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix} = 3E_{21} \\
T(E_{22}) &= T\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} = 4E_{22}
\end{aligned}$$

Thus, the matrix of T with respect to the basis $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ is:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$