Mathematics Homework Sheet 1

Problem 1

a)

A	B	$A \implies B$	$\neg B$	$A \wedge \neg B$	$\neg(A \wedge \neg B)$	(a)
1	1	1	0	0	1	1
1	0	0	1	1	0	1
0	1	1	0	0	1	1
0	0	1	1	0	1	1

(a) is all true

b)

A	B	$A \wedge B$	$\neg(A \wedge B)$	$\neg A$	$\neg A \vee \neg B$	(b)
1	1	1	0	0	0	1
1	0	0	1	0	1	1
0	1	0	1	1	1	1
0	0	0	1	1	1	1

(b) is all true

c)

A	B	$A \vee B$	$\neg(A \vee B)$	$\neg A \wedge \neg B$	(c)
1	1	1	0	0	1
1	0	1	0	0	1
0	1	1	0	0	1
0	0	0	1	1	1

(c) is all true

Problem 2

(a)

Let P be set of all people.

$$P := \{p : person(p)\}$$

Let D be set of all decisions.

$$D := \{d : decision(d)\}$$

$$
\begin{aligned}
(i) \qquad & (\exists d \in D) \quad (\forall p \in P) \quad content(p,d) \\
& \text{Negation (i):} \quad (\forall d \in D) \quad (\exists p \in P) \quad \neg content(p,d) \\
(ii) \qquad & (\forall p \in P) \quad (\forall d \in D) \quad content(p,d) \\
& \text{Negation (ii):} \quad (\exists p \in P) \quad (\exists d \in D) \quad \neg content(p,d)
\end{aligned}
$$

(b)

Every decision results in discontent people

$$(\forall d \in D) \quad (\exists p \in P) \quad \neg content(p,d)$$

and if we negate this, we get

$$
\begin{aligned}
\neg((\forall d \in D) \quad (\exists p \in P) \quad \neg content(p,d)) \qquad (1) \\
(\exists d \in D) \quad (\forall p \in P) \quad \neg\neg content(p,d) \qquad (2) \\
(\exists d \in D) \quad (\forall p \in P) \quad content(p,d) \qquad (3)
\end{aligned}
$$

and this is the same as (i). When we negate a statement, $\forall$ turns into $\exists$ and vice versa.

Problem 3

A $\subseteq \Omega$, B $\subseteq \Omega$

(a)

$$
\begin{aligned}
A \setminus B &:= \{a : (a \in A) \wedge (a \notin B)\} \\
A \cap B^C &:= \{a : (a \in A) \wedge (a \in B^C)\}
\end{aligned}
$$

To show that $A \setminus B = A \cap B^C$, we need to show that $A \setminus B \subseteq A \cap B^C$ and $A \cap B^C \subseteq A \setminus B$. Let's start with $A \setminus B \subseteq A \cap B^C$

Pick an element from $A \setminus B$ and call it x which means that

$$x \in A \wedge x \notin B \qquad (4)$$

We want to show that such x also exists in $A \cap B^C$.
An element in $A \cap B^C$, let's call it y, needs to satisfy this condition:

$$y \in A \wedge y \in B^C \qquad (5)$$

From condition (5) and from the definition, if $y \in B^C$ then $y \notin B$, we can get the following:

$$y \in A \wedge y \in B^C \tag{5}$$
$$y \in A \wedge y \notin B \tag{6}$$

(6) is exactly what the element x satisfies. So, any element in $A \setminus B$ is also in $A \cap B^C$. $A \setminus B \subseteq A \cap B^C$

Now the second part $A \cap B^C \subseteq A \setminus B$
Pick an element from $A \cap B^C$ and call it x which means that

$$x \in A \wedge x \in B^C \tag{6}$$

We want to show that such x also exists in $A \setminus B$.
An element in $A \setminus B$, let's call it y, needs to satisfy this condition:

$$y \in A \wedge y \notin B \tag{7}$$

From condition (7) and from the definition, if $y \in B^C$ then $y \notin B$, we can get the following:

$$y \in A \wedge y \notin B \tag{7}$$
$$y \in A \wedge y \in B^C \tag{8}$$

(8) is exactly what the element x satisfies. So, any element in $A \cap B^C$ is also in $A \setminus B$. $A \cap B^C \subseteq A \setminus B$

(b)

To show

$$P(A \cap B) = P(A) \cap P(B)$$

We need to show

$$P(A \cap B) \subseteq P(A) \cap P(B) \quad \wedge \quad P(A) \cap P(B) \subseteq P(A \cap B)$$

$$P(A \cap B) = \{X : X \subseteq A \cap B\} \tag{8}$$
$$P(A) \cap P(B) = \{X : X \subseteq A\} \cap \{X : X \subseteq B\} \tag{9}$$

Lets start with the first one $P(A \cap B) \subseteq P(A) \cap P(B)$
Lets pick an element X from (8). Is this arbitrary element X also in the set (9)?
Yes, because $A \cap B \subseteq A$ and $A \cap B \subseteq B$. So, $P(A \cap B) \subseteq P(A) \cap P(B)$.

Now the second one $P(A) \cap P(B) \subseteq P(A \cap B)$
Lets pick an element X from (9). Is this arbitrary element X also in the set (8)?
Yes, X is subset of A which means X only contains elements that are in A. And X is subset of B which means X only contains elements that are in B. When we consider above two statements, X only contains elements from A and B which corresponds to $A \cap B$.

(c)

$A \subseteq B$ means that

$$\forall x \in A \quad x \in B \tag{c1}$$

Now, lets take a look at this $B^C \subseteq A^C$

$$\forall x \in B^C \quad x \in A^C \tag{c2}$$

We try to show $(c1) \implies (c2)$. Lets try to prove it using proof by contradiction. Assume $(c1) \land \neg(c2)$ is true. Lets write down $\neg(c2)$.

$$\begin{aligned} &\neg(\forall x \in B^C \quad x \in A^C) \\ &\exists x \in B^C \quad x \notin A^C \\ &\exists x \in B^C \quad x \in A \end{aligned} \tag{c3}$$

(c3) states that there is at least one element that is not in B but in A. This contradicts with (c1) because (c1) states that all elements in A are also in B.