Mathematics Homework Sheet 3

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Problem 1

(a)

$$3z^2 + z = 1$$

solve the quadratic equations and we get

$$z = \frac{-1 \pm \sqrt{1^2 - 4.3.(-1)}}{(3.2)}$$

$$z = \frac{-1 \pm \sqrt{1 + 12}}{6}$$

$$z = \frac{-1 \pm \sqrt{13}}{6}$$

$$z_0 = \frac{-1 + \sqrt{13}}{6} + 0i, \quad z_1 = \frac{-1 - \sqrt{13}}{6} + 0i$$

(b)

$$3z^2 + z = 0$$

$$z(3z + 1) = 0$$

$$z = 0 \quad \text{or} \quad 3z + 1 = 0$$

$$z = 0 \quad \text{or} \quad z = -\frac{1}{3}$$

(c)

$$z^2 - (3 + i)z + 4 + 3i = 0$$

$$z = \frac{(3 + i) \pm \sqrt{(3 + i)^2 - 4(4 + 3i)}}{2}$$

$$z = \frac{(3 + i) \pm \sqrt{(9 + 6i - 1 - 16 - 12i)}}{2}$$

$$z = \frac{(3 + i) \pm \sqrt{-8 - 6i}}{2}$$

$$z = \frac{(3 + i) \pm (1 - 3i)}{2}$$

$$z_0 = \frac{4-2i}{2} = 2-i, \quad z_1 = \frac{2+4i}{2} = 1+2i$$

$\sqrt{-8-6i} = \pm(1-3i)$ because $\sqrt{-8-6i} = \pm(\sqrt{\frac{r+(-8)}{2}}+sign(-6)\sqrt{\frac{r-(-8)}{2}}i)$
where $r = |-8-6i|$

(d)

$$\sinh z = i$$

By definiton of sinh

$$\frac{e^z - e^{-z}}{2} = i$$

By definition of e^z

$$\frac{e^x(cos(y) + sin(y)i) + e^{-x}(cos(-y) + sin(-y)i))}{2} = i$$

where $z = x + iy$.

$$e^x(cos(y) + sin(y)i) + e^{-x}(cos(-y) + sin(-y)i)) = 2i$$

$$e^x(cos(y) + sin(y)i) + e^{-x}(cos(y) - sin(y)i)) = 2i$$

$$e^x cos(y) + e^x sin(y)i + e^{-x}cos(y) - e^{-x}sin(y)i = 2i$$

$$(e^x cos(y) + e^{-x}cos(y)) + (e^x sin(y) - e^{-x}sin(y))i = 0 + 2i$$

$$e^x cos(y) + e^{-x}cos(y) = 0, \qquad e^x sin(y) - e^{-x}sin(y) = 2$$

$$e^x cos(y) + e^{-x}cos(y) = 0 \implies cos(y) = 0$$

$$y = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$$

$$e^x sin(y) - e^{-x}sin(y) = 2$$

$$sin(y)(e^x - e^{-x}) = 2$$

$$sin(\frac{\pi}{2} + k\pi)(e^x - e^{-x}) = 2 \implies e^x - e^{-x} = \pm 2$$

Let $r = e^x$

$$r - \frac{1}{r} = 2 \implies r^2 - 2r - 1 = 0$$

or

$$r - \frac{1}{r} = -2 \implies r^2 + 2r - 1 = 0$$

$$r = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$$

$$r = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$$

$$e^x = \sqrt{2} \pm 1 \implies x = ln(\sqrt{2} \pm 1), \qquad \sqrt{2} \pm 1 > 0$$

$z = x + yi$ where $x = ln(\sqrt{2}+1)$ or $x = ln(\sqrt{2}-1)$ and $y = \frac{\pi}{2} + k\pi$

(e)

$$tan(z) = 1$$
$$\frac{sin(z)}{cos(z)} = 1$$
$$sin(z) = cos(z)$$

By definition of sin(z) and cos(z)

$$\frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2}$$
$$e^{iz} - e^{-iz} = i(e^{iz} + e^{-iz})$$

Let $q = e^{iz}$

$$q - \frac{1}{q} = i(q + \frac{1}{q})$$
$$q^2 - 1 = iq^2 + 1$$
$$q^2(1 - i) - 2 = 0$$
$$q = \sqrt{\frac{2}{1 - i}} = \sqrt{\frac{2(1 + i)}{2}} = \sqrt{1 + i}$$

(f)

$$cos(z) = -\frac{5}{4}$$

By definition of cos(z)

$$\frac{e^{iz} + e^{-iz}}{2} = -\frac{5}{4}$$
$$e^{iz} + e^{-iz} = -\frac{5}{2}$$

Let $q = e^{iz}$

$$q + \frac{1}{q} = -\frac{5}{2}$$
$$q^2 + \frac{5}{2}q + 1 = 0$$
$$q = \frac{-\frac{5}{2} \pm \sqrt{(\frac{5}{2})^2 - 4}}{2}$$
$$q = \frac{-\frac{5}{2} \pm \frac{3}{2}}{2}$$
$$q_0 = \frac{-1}{2}, \qquad q_1 = -2$$

$$e^{iz_0} = -\frac{1}{2} = \frac{1}{2}e^{i\pi}$$
$$e^{iz_0 - i\pi} = \frac{1}{2}$$
$$iz_0 - i\pi = ln(\frac{1}{2})$$
$$z_0 = \frac{ln(\frac{1}{2}) + i\pi}{i}$$

$$e^{iz_1} = -2 = 2e^{i\pi}$$
$$e^{iz_1 - i\pi} = 2$$
$$iz_1 - i\pi = ln(2)$$
$$z_1 = \frac{ln(2) + i\pi}{i}$$