Mathematics Homework Sheet 5

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Problem 1

1. $-(x+y) = (-x) + (-y)$:
   By the definition of the additive inverse, we have:
$$-(x+y) + (x+y) = 0$$
   Use distributivity property:
$$\begin{aligned}(-x) + (-y) + (x+y) &= 0 \\ ((-x) + (-y)) + (x+y) &= 0\end{aligned}$$
   So, since adding $((-x) + (-y))$ to $(x+y)$ gives 0, we can conclude that it is the additive inverse of $(x+y)$. And that's what we are trying to prove.

2. $-(x-y) = (-x) + y$:
   Apply the rule above.
$$\begin{aligned}-(x + (-y)) &= (-x) + (-(-y)) \\ &= (-x) + y\end{aligned}$$

3. $x \cdot 0 = 0 \cdot x = 0$:
$$x \cdot 0 + x \cdot 0 = x \cdot (0+0) = x \cdot 0$$
$$\begin{aligned}x \cdot 0 + x \cdot 0 &= x \cdot 0 \qquad \text{(add additive inverse of } x \cdot 0) \\ x \cdot 0 + (x \cdot 0 + -(x \cdot 0)) &= x \cdot 0 + -(x \cdot 0) \\ x \cdot 0 + 0 &= 0 \\ x \cdot 0 &= 0\end{aligned}$$
   And by commutativity we have $0 \cdot x = 0$.

4. $(-x) \cdot y = -(x \cdot y)$:
$$(x \cdot y) + ((-x) \cdot y) = (x + (-x)) \cdot y = 0 \cdot y = 0$$
   So, $(-x) \cdot y$ is additive inverse of $(x \cdot y)$.

5. $x \cdot (-y) = -(x \cdot y)$:

$$\begin{aligned} x \cdot (-y) &= x \cdot (-y) && \text{(commutativity)} \\ &= (-y) \cdot x && \text{(insert this into original equation)} \\ (-y) \cdot x &= -(x.y) && \text{(true by the previous rule)} \end{aligned}$$

6. $(-x) \cdot (-y) = x \cdot y$:
   Use rule (4) to get:
   $$(-x) \cdot (-y) = -(x \cdot (-y))$$
   Now use rule (5) to get:
   $$-(x \cdot (-y)) = -(-(x \cdot y))$$
   And by the definition of additive inverse, we have:
   $$-(-(x \cdot y)) = x \cdot y$$

7. $x + y = z$ if and only if $x = z - y$:
   By the definition of addition, we have:
   $$x + y = z \implies x = z - y$$
   and
   $$x = z - y \implies x + y = z$$

Problem 3

We want to show that only solution to the equation

$$a_1 \begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} = 0$$

is $a_1 = 0$ and $a_2 = 0$ where $a_1, a_2 \in \mathbb{C}$. We can write this as a system of equations:

$$\begin{aligned} a_1 + 0 \cdot a_2 &= 0 \\ a_1 i + a_2 &= 0 \\ a_1(1+i) + a_2 i &= 0 \end{aligned}$$

The first equation gives us $a_1 = 0$. Substituting this into the second equation gives us $a_2 = 0$. Thus, the only solution is $a_1 = 0$ and $a_2 = 0$.

Now we want to show that the set

$$S = \left\{ \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} \right\}$$

is a spanning set for $\mathbb{C}^3$. This means that any vector in $\mathbb{C}^3$ can be written as a linear combination of the vectors in S. We can write this as:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = a_1 \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$$

This gives us the system of equations:

$$\begin{aligned} x_1 &= a_1 i \\ x_2 &= a_2 + a_3 i \\ x_3 &= a_3 \end{aligned}$$

We can solve this system of equations for a_1, a_2, a_3 in terms of x_1, x_2, x_3:

$$\begin{aligned} a_1 &= \frac{x_1}{i} \\ a_3 &= x_3 \\ a_2 &= x_2 - a_3 i = x_2 - x_3 i \end{aligned}$$

Thus, any vector in $\mathbb{C}^3$ can be written as a linear combination of the vectors in S, so S is a spanning set for $\mathbb{C}^3$.

$$S = \left\{ \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} \right\}$$

Create a ne set by adding a vector from T to S:

$$S_1 = \left\{ \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix} \right\}$$

We can write the vector $\begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix}$, as a linear combination of the vectors in S_1:

$$\begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix} = a_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} + a_3 \begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix}$$

With the coeeficients:

$$
\begin{aligned}
a_1 &= -i \\
a_2 &= 1 - i \\
a_3 &= i
\end{aligned}
$$

Thus we can get rid of the vector $\begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix}$ and replace it with the vector $\begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix}$ from T.

$$
S_2 = \left\{ \begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} \right\}
$$

Let's add the other vector from T:

$$
S_3 = \left\{ \begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} \right\}
$$

We can write the vector $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, as a linear combination of the vectors in S_3:

$$
\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}
$$

With the coeeficients:

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -i/2 \\
a_3 &= 1/2
\end{aligned}
$$

Thus we can get rid of the vector $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and replace it with the vector $\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}$ from T.

$$
S_4 = \left\{ \begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}, \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} \right\}
$$

S_4 still spans the same space as S.

Problem 4

(a)

We can use Steinitz theorem recursively to show that $\langle v_1, ..., v_n \rangle = \langle v_1 - v_2, ..., v_n \rangle$.
We can replace v_1 with $v_1 - v_2$ and get:

$$\langle v_1 - v_2, v_2 - v_3, ..., v_n \rangle = \langle v_1, v_2 - v_3, ..., v_n \rangle$$

by choosing every coefficient to be 1.
We can repeat this for v_2. Replace $v_2 - v_3$ with v_2 to get:

$$\langle v_1, v_2 - v_3, ..., v_n \rangle = \langle v_1, v_2, ..., v_n \rangle$$

Again by choosing every coefficient to be 1.
We can repeat this for every v_i. And at the end we will have:

$$\langle v_1 - v_2, v_2 - v_3, ..., v_n \rangle = \langle v_1, v_2, ..., v_n \rangle$$

(b)

If $v_1, v_2, ..., v_n$ are linearly independent means that the only solution to the equation

$$a_1 v_1 + a_2 v_2 + ... + a_n v_n = 0$$

is $a_1 = a_2 = ... = a_n = 0$.

We want to show that the only solution to the equation

$$b_1(v_1 - v_2) + b_2(v_2 - v_3) + ... + b_n(v_n) = 0$$

is $b_1 = b_2 = ... = b_n = 0$.
We can write this as:

$$b_1 v_1 + (b_2 - b_1)v_2 + (b_3 - b_2)v_3 + ... + b_n v_n = 0$$

and since $v_1, v_2, ..., v_n$ are linearly independent, we have:

$$b_1 = 0, b_2 - b_1 = 0, b_3 - b_2 = 0, ..., b_n = 0$$

This means that $b_1 = b_2 = ... = b_n = 0$.