Mathematics Homework Sheet 6

Authors: Abdullah Oguz Topcuoglu & Ahmed Waleed Ahmed Badawy Shora

Problem 2

(a)

U_1:
U_1 is subspace of $R[x]$.
Not empty:
U_1 is not empty because $0 \in U_1$ (zero polynomial).
Closed under addition:
Let $p(x), q(x) \in U_1$. Then $p(0) = 0$ and $q(0) = 0$.
Then, $(p + q)(0) = p(0) + q(0) = 0 + 0 = 0$.
Closed under scalar multiplication:
Let $p(x) \in U_1$ and $c \in R$. Then, $(cp)(0) = c(p(0)) = c(0) = 0$.
Thus, U_1 is closed under scalar multiplication.

U_2:
U_2 is not a subspace of $R[x]$. Because U_2 doesn't contain the zero polynomial. (every vector space has to contain the zero vector which is the zero polynomial in this case)

U_3:
U_3 is subspace of $R[x]$.
Not empty:
U_3 is not empty because $0 \in U_3$ (zero polynomial).
Closed under addition:
Let $p(x), q(x) \in U_3$. Then $p(1) = 0$ and $q(1) = 0$.
Then, $(p + q)(1) = p(1) + q(1) = 0 + 0 = 0$.
Closed under scalar multiplication:
Let $p(x) \in U_3$ and $c \in R$. Then, $(cp)(1) = c(p(1)) = c(0) = 0$.
Thus, U_3 is closed under scalar multiplication.

U_4:
U_4 is subspace of $R[x]$.
Not empty:
U_4 is not empty because $0 \in U_4$ (zero polynomial).
Closed under addition:
Let $p(x), q(x) \in U_4$. Then $\int_0^1 p(x)dx = 0$ and $\int_0^1 q(x)dx = 0$.
Then, $\int_0^1 (p + q)(x)dx = \int_0^1 p(x)dx + \int_0^1 q(x)dx = 0 + 0 = 0$.
Closed under scalar multiplication:

Let $p(x) \in U_4$ and $c \in R$. Then, $\int_0^1 (cp)(x)dx = c \int_0^1 p(x)dx = c(0) = 0$.
Thus, U_4 is closed under scalar multiplication.

U_5:
U_5 is subspace of $R[x]$.
Not empty:
U_5 is not empty because $0 \in U_5$ (zero polynomial).
Closed under addition:
Let $p(x), q(x) \in U_5$. Then $p'(0) + p''(0) = 0$ and $q'(0) + q''(0) = 0$.
Then, $(p+q)'(0) + (p+q)''(0) = p'(0) + q'(0) + p''(0) + q''(0) = 0 + 0 = 0$.
Closed under scalar multiplication:
Let $p(x) \in U_5$ and $c \in R$. Then, $(cp)'(0) + (cp)''(0) = c(p'(0)) + c(p''(0)) = c(p'(0) + p''(0)) = c(0) = 0$.
Thus, U_5 is closed under scalar multiplication.

U_6:
U_6 is not a subspace of $R[x]$. Becuase it is not closed under addition
Let $p(x), q(x) \in U_6$. Then $p'(0)p''(0) = 0$ and $q'(0)q''(0) = 0$.
Then, $(p+q)'(0)(p+q)''(0) = (p'(0) + q'(0))(p''(0) + q''(0)) = p'(0)p''(0) + p'(0)q''(0) + q'(0)p''(0) + q'(0)q''(0) = p'(0)q''(0) + q'(0)p''(0)$
Which is not necessarily equal to 0. Thus U_6 is not closed under addition.

(b)

S_1:
S_1 is a subspace of $R^{2\times 2}$.
Not empty:
S_1 is not empty because $0 \in S_1$ (2 by 2 zero matrix).
Closed under addition:
Let $A, B \in S_1$. Then $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ where $a = b$ and $e = f$.
Then, $A + B = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$ where $a + e = b + f$.
Closed under scalar multiplication:
Let $A \in S_1$ and $c \in R$. Then, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a = b$.
Then, $cA = \begin{pmatrix} ca & cb \\ cc & cd \end{pmatrix}$ where $ca = cb$.

S_2:
S_2 is not a subspace of $R^{2\times 2}$. Because S_2 doesn't contain the zero matrix. (every vector space has to contain the zero vector which is the zero matrix in this case)

S_3:
S_3 is not a subspace of $R^{2\times 2}$. Because S_3 is not closed under addition.

Let $A, B \in S_3$. Then $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ where $a^2 = b^2$ and $e^2 = f^2$.
Then, $A + B = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$ where $(a+e)^2 = (b+f)^2$ is not necessarily true.

$$\begin{aligned} (a+e)^2 &= (b+f)^2 \\ a^2 + 2ae + e^2 &= b^2 + 2bf + f^2 \qquad \text{use } a^2 = b^2 \text{ and } e^2 = f^2 \\ 2ae &= 2bf \\ ae &= bf \end{aligned}$$

Which is not always true. Thus S_3 is not closed under addition.

Problem 3

We need to show two things: vectors are linearly independent and they span the subspace W.
Vectors are linearly independent:
Let $c_1(x^3 - x^2) + c_2(x^3 - x) = 0$.
Then, $c_1x^3 - c_1x^2 + c_2x^3 - c_2x = 0$.
Then, $(c_1 + c_2)x^3 - c_1x^2 - c_2x = 0$.
The coefficients of x^3, x^2 and x must be equal to 0.
Thus, c_1 and c_2 must be equal to 0.
Vectors span the subspace W:
Let $p(x) \in W$. Then, $p(0) = 0$ and $p(1) = 0$.
Then, $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$.
Then, $p(0) = a_0 = 0$.
Then, $p(1) = a_3 + a_2 + a_1 = 0$.
Thus, $p(x) = a_3x^3 + a_2x^2 + (-a_3 - a_2)x$.
Then, $p(x) = a_3(x^3 - x^2) + a_2(x^3 - x)$.
Thus, $p(x)$ can be written as a linear combination of the vectors $x^3 - x^2$ and $x^3 - x$.

Extend the basis to a basis for $R^3[x]$:
We want to be able write any polynomial in $R^3[x]$ as a linear combination of the basis vectors. We know that $dimR^3[x] = 4$. So we need 2 more linearly independent vectors.
From the basis extension theorem we know that if we add two more linearly independent vectors to our original set of vectors, we will have a basis for $R^3[x]$. So let's peak two vectors outside of the subspace W which are linearly independent.
$q_1(x) = 1$ and $q_2(x) = x$ are linearly independent and not in the subspace W.

They are not in the subspace W because $q_1(1) \neq 0$ and $q_2(1) \neq 0$.
Thus, a basis for $R^3[x]$ is $\{x^3 - x^2, x^3 - x, 1, x\}$.