Mathematics Homework Sheet 4

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Problem 1

We want to prove

$$(\forall n \in N) \wedge (x \in R) \wedge (x \geq -1) \qquad (1+x)^n \geq 1 + nx$$

by using mathematical induction.
Base Case: For $n = 1$, we have

$$(1+x)^1 = 1 + x \geq 1 + 1 \cdot x$$

Which is true for every $x \in R$ so it means it is also true for $x \in [-1, \infty]$

Inductive Step: Assume that the statement is true for $n = k$, i.e.

$$(1+x)^k \geq 1 + kx$$

is true for every $x \in [-1, \infty]$.
We want to prove that the statement is also true for $n = k + 1$, i.e.

$$(1+x)^{k+1} \geq 1 + (k+1)x$$

for every $x \in [-1, \infty]$.

$$\begin{aligned}
(1+x)^{k+1} &= (1+x)^k \cdot (1+x) && (1)\\
&\geq (1+kx) \cdot (1+x) && \text{(Using inductive step)} \quad (2)\\
&= 1 + kx + x + kx^2 && (3)\\
&= 1 + (k+1)x + kx^2 && (4)\\
&\geq 1 + (k+1)x && (\text{Since } kx^2 \geq 0) \quad (5)
\end{aligned}$$

And this completes the proof.

Inductive step alternative solution:
We assume that the statement is true for $n = k$, i.e.

$$(1+x)^k \geq 1 + kx$$

Multiply both sides by $1 + x$, since $1 + x > 0$ because $x \in [-1, \infty]$, we have

$$(1+x)^k(1+x) \geq (1+kx)(1+x)$$

$$(1+x)^{k+1} \geq 1 + kx + x + kx^2$$
$$0 \geq -kx^2$$

Add these two together, we get

$$(1+x)^{k+1} \geq 1 + (k+1)x$$

And this completes the proof.

Give a counterexample to show that the condition x $\geq$ -1 is necessary:
Let's take $x = -2$ and $n = 2$. Then we have

$$(1-2)^2 \geq 1 + 2 \cdot (-2)$$
$$(-1)^2 \geq 1 - 4$$
$$1 \geq -3$$

Which is not true. In fact it would be false when n is even. So the condition $x \geq -1$ is necessary. Because that way $1 + x$ is never negative