

Mathematics Homework Sheet 4

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Problem 1

We want to prove

$$(\forall n \in \mathbb{N}) \wedge (x \in \mathbb{R}) \wedge (x \geq -1) \quad (1+x)^n \geq 1+nx$$

by using mathematical induction.

Base Case: For $n = 1$, we have

$$(1+x)^1 = 1+x \geq 1+1 \cdot x$$

Which is true for every $x \in \mathbb{R}$ so it means it is also true for $x \in [-1, \infty]$

Inductive Step: Assume that the statement is true for $n = k$, i.e.

$$(1+x)^k \geq 1+kx$$

is true for every $x \in [-1, \infty]$.

We want to prove that the statement is also true for $n = k+1$, i.e.

$$(1+x)^{k+1} \geq 1+(k+1)x$$

for every $x \in [-1, \infty]$.

$$(1+x)^{k+1} = (1+x)^k \cdot (1+x) \tag{1}$$

$$\geq (1+kx) \cdot (1+x) \quad \text{(Using inductive step)} \tag{2}$$

$$= 1+kx+x+kx^2 \tag{3}$$

$$= 1+(k+1)x+kx^2 \tag{4}$$

$$\geq 1+(k+1)x \quad \text{(Since } kx^2 \geq 0 \text{)} \tag{5}$$

And this completes the proof.

Inductive step alternative solution:

We assume that the statement is true for $n = k$, i.e.

$$(1+x)^k \geq 1+kx$$

Multiply both sides by $1+x$, since $1+x > 0$ because $x \in [-1, \infty]$, we have

$$(1+x)^k(1+x) \geq (1+kx)(1+x)$$

$$\begin{aligned}(1+x)^{k+1} &\geq 1+kx+x+kx^2 \\ 0 &\geq -kx^2\end{aligned}$$

Add these two together, we get

$$(1+x)^{k+1} \geq 1+(k+1)x$$

And this completes the proof.

Give a counterexample to show that the condition $x \geq -1$ is necessary:

Let's take $x = -2$ and $n = 2$. Then we have

$$\begin{aligned}(1-2)^2 &\geq 1+2 \cdot (-2) \\ (-1)^2 &\geq 1-4 \\ 1 &\geq -3\end{aligned}$$

Which is not true. In fact it would be false when n is even. So the condition $x \geq -1$ is necessary. Because that way $1+x$ is never negative

Problem 2

Problem 2 (a)

$$X_1 := \{x \in \mathbb{R} : x^2 - 2x \leq 0\}$$

What x values satisfy this condition?

$$\begin{aligned}x^2 - 2x &\leq 0 \\ x(x-2) &\leq 0\end{aligned}$$

In order this inequality to be satisfied the signs of x and $x-2$ must be different or one of them needs to be zero, and this only happens when $0 \leq x \leq 2$. So this means:

$$X_1 = [0, 2]$$

In this case X_1 is bounded from below and above.

$$\begin{aligned}\sup X_1 &= 2 \\ \inf X_1 &= 0\end{aligned}$$

And $\sup X_1 \in X_1$ which means $\sup X_1$ is also the maximum value.
 $\inf X_1 \in X_1$ which means $\inf X_1$ is also the minimum value.

Problem 2 (b)

$$X_2 := \{x \in \mathbb{R} \setminus \{0\} : 5 - x^2 > \frac{4}{x^2}\}$$

What x values satisfy this condition?

$$\begin{aligned} 5 - x^2 &> \frac{4}{x^2} \\ 5 - x^2 - \frac{4}{x^2} &> 0 \\ \frac{5x^2 - x^4 - 4}{x^2} &> 0 \end{aligned}$$

Since x^2 is always positive, we can multiply both sides by x^2 .

$$\begin{aligned} 5x^2 - x^4 - 4 &> 0 \\ x^4 - 5x^2 + 4 &< 0 \\ (x^2 - 4)(x^2 - 1) &< 0 \\ (x - 2)(x + 2)(x - 1)(x + 1) &< 0 \end{aligned}$$

So, this inequality is satisfied when $-2 < x < -1 \quad \vee \quad 1 < x < 2$.
So this means:

$$X_2 = (-2, -1) \cup (1, 2)$$

In this case X_2 is bounded from below and above.

$$\begin{aligned} \sup X_2 &= 2 \\ \inf X_2 &= -2 \end{aligned}$$

And $\sup X_2 \notin X_2$ which means $\sup X_2$ is not the maximum value.
 $\inf X_2 \notin X_2$ which means $\inf X_2$ is not the minimum value.
Maximum and minimum values are not in the set.