Homework Sheet 2

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Problem 5

We are given the function

$$f(x_1, x_2) := |x_1| + |x_2|.$$

We are looking for points $x^0 \in \mathbb{R}^2$ where $\frac{\partial f}{\partial x_1^0}$ and $\frac{\partial f}{\partial x_2^0}$ exist. First things first that we realize the function is symmetric in both variables, so we can just focus on one of them and the other will follow the same logic.
Let's consider the partial derivative with respect to x_1^0:

$$\begin{aligned}
\frac{\partial f}{\partial x_1^0} &= \lim_{h \to 0} \frac{f(x_1^0 + h, x_2^0) - f(x_1^0, x_2^0)}{h}. \\
&= \lim_{h \to 0} \frac{|x_1^0 + h| + |x_2^0| - (|x_1^0| + |x_2^0|)}{h}. \\
&= \lim_{h \to 0} \frac{|x_1^0 + h| - |x_1^0|}{h}.
\end{aligned}$$

Now we have to consider different cases for x_1^0:

- If $x_1^0 > 0$:

$$\frac{\partial f}{\partial x_1^0} = \lim_{h \to 0} \frac{(x_1^0 + h) - x_1^0}{h} = \lim_{h \to 0} \frac{h}{h} = 1.$$

- If $x_1^0 < 0$:

$$\frac{\partial f}{\partial x_1^0} = \lim_{h \to 0} \frac{-(x_1^0 + h) + x_1^0}{h} = \lim_{h \to 0} \frac{-h}{h} = -1.$$

- If $x_1^0 = 0$:

$$\begin{aligned}
\frac{\partial f}{\partial x_1^0} &= \lim_{h \to 0^+} \frac{|h| - 0}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1, \\
\frac{\partial f}{\partial x_1^0} &= \lim_{h \to 0^-} \frac{|-h| - 0}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1.
\end{aligned}$$

Since the left-hand limit and right-hand limit are not equal, the partial derivative does not exist at this point.

So partial derivates exist for all points where $x_1^0 \neq 0$. By symmetry, the same applies for x_2^0.
Partial Derivaties:

$$\frac{\partial f}{\partial x_1^0} = \begin{cases} 1, & x_1^0 > 0 \\ -1, & x_1^0 < 0 \end{cases},$$

$$\frac{\partial f}{\partial x_2^0} = \begin{cases} 1, & x_2^0 > 0 \\ -1, & x_2^0 < 0 \end{cases}$$

Problem 6

We are given the set

$$D := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \neq x_2^2, x_2 > 0\}$$

and the function

$$f : D \to \mathbb{R}, \quad f(x_1, x_2) := \frac{x_1 \ln(x_2)}{(x_1 - x_2^2)x_2}.$$

(i)

We need to show that for any point $x \in D$ we can find an open ball around it.

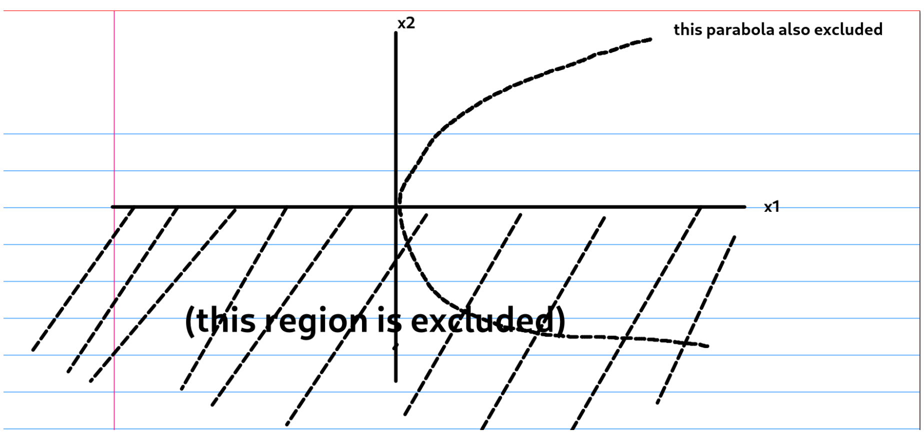

Figure 1: The set D

So if we pick a poitn in D we can get the distance to x_1 axis and distance to the parabola in the image and if we choose the radius of the open ball to be the smaller of both then every point in that open ball will be in D because we are guaranteed to not touch the x_1 axis and the parabola.
Thats why D is open.

(ii)

We want to show that for any point in the set D the partial derivatives exist.
Lets start with $\frac{\partial f}{\partial x_1}$:
Define $f_{x_2}(x_1) = f(x_1, x_2)$. Now we want to show that $\frac{df_{x_2}}{dx_1}$ exists.
And yes it does because $f_{x_2}(x_1)$ is a rational function where the numerator and denominator are polynomials in x_1 (with x_2 being constant) and demominator is never zero.

Now we want to show that $\frac{\partial f}{\partial x_2}$ exists:
Define $f_{x_1}(x_2) = f(x_1, x_2)$. Now we want to show that $\frac{df_{x_1}}{dx_2}$ exists.
And yes it does again because the numerator is differentiable and the denomainator is also differentiable and never zero.

Partial derivaties:

$$\frac{\partial f}{\partial x_1} = \frac{df_{x_2}}{dx_1}$$

apply quotient rule

$$\begin{aligned}
&= \frac{ln(x_2)((x_1 - x_2^2)x_2) - x_1 ln(x_2)x_2}{(x_1 - x_2^2)^2 x_2^2}, \\
&= \frac{ln(x_2)x_2(x_1 - x_2^2 - x_1)}{(x_1 - x_2^2)^2 x_2^2}, \\
&= \frac{ln(x_2)x_2(-x_2^2)}{(x_1 - x_2^2)^2 x_2^2}, \\
&= \frac{-ln(x_2)x_2^3}{(x_1 - x_2^2)^2 x_2^2}, \\
&= \frac{-ln(x_2)x_2}{(x_1 - x_2^2)^2},
\end{aligned}$$

$$\frac{\partial f}{\partial x_2} = \frac{df_{x_1}}{dx_2}$$

apply quotient rule

$$
\begin{aligned}
&= \frac{x_1/x_2((x_1 - x_2^2)x_2) - (x_1(ln(x_2)(x_1 - 3x_2^2)))}{(x_1 - x_2^2)^2 x_2^2} \\
&= \frac{x_1(x_1 - x_2^2) - (x_1 ln(x_2)(x_1 - 3x_2^2))}{(x_1 - x_2^2)^2 x_2^2} \\
&= \frac{x_1(x_1 - x_2^2 - ln(x_2)(x_1 - 3x_2^2))}{(x_1 - x_2^2)^2 x_2^2}
\end{aligned}
$$

This is the furthest i could simplify :)

(iii)

The fradient is simply the vector of partial derivatives which we just calculated in (ii):

$$
\begin{aligned}
\nabla f(x_1, x_2) &= \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \\ \frac{\partial f}{\partial x_2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{-ln(x_2)x_2}{(x_1 - x_2^2)^2} \\ \\ \frac{x_1(x_1 - x_2^2 - ln(x_2)(x_1 - 3x_2^2))}{(x_1 - x_2^2)^2 x_2^2} \end{pmatrix}
\end{aligned}
$$

Problem 7

We are given the function

$$
f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x_1, x_2) := \begin{cases} (x_1^2 + x_2^2) \sin\left(\frac{1}{x_1^2 + x_2^2}\right), & (x_1, x_2) \neq (0,0) \\ 0, & (x_1, x_2) = (0,0) \end{cases}
$$

(i)

We want to show that f is partially differentiable.
Lets start with $\frac{\partial f}{\partial x_1}$:

$$
\frac{\partial f}{\partial x_1} = \frac{df_{x_2}}{dx_1}
$$

where

$$
f_{x_2}(x_1) = f(x_1, x_2) = (x_1^2 + x_2^2) \sin\left(\frac{1}{x_1^2 + x_2^2}\right)
$$

And f_{x_2} is differentiable for all $x_1 \in \mathbb{R}$ because it is a product of differentiable functions.

Now we want to show that $\frac{\partial f}{\partial x_2}$:

$$\frac{\partial f}{\partial x_2} = \frac{df_{x_1}}{dx_2}$$

where

$$f_{x_1}(x_2) = f(x_1, x_2) = (x_1^2 + x_2^2)\sin\left(\frac{1}{x_1^2 + x_2^2}\right)$$

And the same thing as above applies here because of the symmetry of the function.
Now we need to check the point $(0, 0)$:

$$\begin{aligned}\frac{\partial f}{\partial x_1} &= \lim_{h\to 0}\frac{f(0+h,0) - f(0,0)}{h}\\ &= \lim_{h\to 0}\frac{h^2\sin\left(\frac{1}{h^2}\right) - 0}{h}\\ &= \lim_{h\to 0} h\sin\left(\frac{1}{h^2}\right)\\ &= 0\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial x_2} &= \lim_{h\to 0}\frac{f(0,0+h) - f(0,0)}{h}\\ &= \lim_{h\to 0}\frac{h^2\sin\left(\frac{1}{h^2}\right) - 0}{h}\\ &= \lim_{h\to 0} h\sin\left(\frac{1}{h^2}\right)\\ &= 0\end{aligned}$$

So we have shown that f is partially differentiable everywhere.

(ii)

We want to show that f is totally differentiable at point $(0, 0)$.
So we are going to show that jacobian matrix exists at point $(0, 0)$ and the limit condition holds.
Limit condition is:

$$\lim_{(h_1,h_2)\to(0,0)}\frac{f(0+h_1, 0+h_2) - f(0,0) - J_f(0,0)\cdot(h_1,h_2)^T}{\|(h_1,h_2)\|} = 0$$

Jacobian matrix is:

$$J_f = \begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{pmatrix}$$

we are interested in $J_f(0,0)$:

$$J_f(0,0) = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

Values we calculated in (i).
Now we can calculate the limit:
Case: $(h_1, h_2) \neq (0,0)$

$$\begin{aligned}
&\lim_{(h_1,h_2)\to(0,0)} \frac{f(0+h_1, 0+h_2) - f(0,0) - J_f(0,0)\cdot(h_1,h_2)^T}{\|(h_1,h_2)\|} \\
&= \lim_{(h_1,h_2)\to(0,0)} \frac{(h_1^2+h_2^2)\sin\left(\frac{1}{h_1^2+h_2^2}\right) - 0 - 0}{\sqrt{h_1^2+h_2^2}} \\
&= \lim_{(h_1,h_2)\to(0,0)} \sqrt{h_1^2+h_2^2}\sin\left(\frac{1}{h_1^2+h_2^2}\right) \\
&= 0
\end{aligned}$$

Case: $(h_1, h_2) = (0,0)$

$$\begin{aligned}
&\lim_{(h_1,h_2)\to(0,0)} \frac{f(0+h_1, 0+h_2) - f(0,0) - J_f(0,0)\cdot(h_1,h_2)^T}{\|(h_1,h_2)\|} \\
&= \lim_{(h_1,h_2)\to(0,0)} \frac{0-0-0}{0} \\
&= 0
\end{aligned}$$

So the limit condition holds and f is totally differentiable at point $(0,0)$. Thats what we wanted to show.

Problem 8

We are given the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x_1,x_2) := \begin{cases} \frac{x_1x_2^3 - x_1^3x_2}{x_1^2+x_2^2}, & (x_1,x_2) \neq (0,0) \\ 0, & (x_1,x_2) = (0,0) \end{cases}$$

(ii)

We need to show that the PD condition holds for f. $PD_{(0,0)(i,j)}$ for $i,j \in \{1,2\}$.
Lets start with $PD_{(0,0)(1,1)}$:

- f is partially differentiable at $(0,0)$ with respect to x_1:

$$\begin{aligned}\frac{\partial f}{\partial x_1} &= \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0}\frac{0-0}{h}=0\end{aligned}$$

- $\frac{\partial f}{\partial x_1}$ is partially differentiable at $(0,0)$ with respect to x_1:

$$\frac{\partial f}{\partial x_1} = \frac{df_{x_2}}{dx_1}$$

apply quotient rule

$$= \frac{x_2^3 - 3x_1^2 x_2}{x_1^2+x_2^2} - \frac{(x_1x_2^3 - x_1^3x_2)2x_1}{(x_1^2+x_2^2)^2}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial x_1^2} &= \lim_{h\to 0}\frac{\frac{\partial f}{\partial x_1}(0+h,0)-\frac{\partial f}{\partial x_1}(0,0)}{h}\\ &= \lim_{h\to 0}\frac{0-0}{h}=0\end{aligned}$$

Now lets do $PD_{(0,0)(1,2)}$:

- f is partially differentiable at $(0,0)$ with respect to x_1: (Already shown above)
- $\frac{\partial f}{\partial x_1}$ is partially differentiable at $(0,0)$ with respect to x_2:

$$\begin{aligned}\frac{\partial^2 f}{\partial x_2 \partial x_1} &= \lim_{h\to 0}\frac{\frac{\partial f}{\partial x_1}(0,0+h)-\frac{\partial f}{\partial x_1}(0,0)}{h}\\ &= \lim_{h\to 0}\frac{h-0}{h}=1\end{aligned}$$

Now lets do $PD_{(0,0)(2,1)}$:

- f is partially differentiable at $(0,0)$ with respect to x_2:

$$\begin{aligned}\frac{\partial f}{\partial x_2} &= \lim_{h\to 0}\frac{f(0,0+h)-f(0,0)}{h}\\ &= \lim_{h\to 0}\frac{0-0}{h}=0\end{aligned}$$

- $\frac{\partial f}{\partial x_2}$ is partially differentiable at $(0,0)$ with respect to x_1:

$$\frac{\partial f}{\partial x_2} = \frac{df_{x_1}}{dx_2}$$

apply quotient rule

$$= \frac{3x_1x_2^2 - x_1^3}{x_1^2 + x_2^2} - \frac{(x_1x_2^3 - x_1^3x_2)2x_2}{(x_1^2 + x_2^2)^2}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial x_1 \partial x_2} &= \lim_{h \to 0} \frac{\frac{\partial f}{\partial x_2}(0+h,0) - \frac{\partial f}{\partial x_2}(0,0)}{h} \\ &= \lim_{h \to 0} \frac{-h-0}{h} = -1\end{aligned}$$

Now lets do $PD_{(0,0)(2,2)}$:

- f is partially differentiable at $(0,0)$ with respect to x_2: (Already shown above)
- $\frac{\partial f}{\partial x_2}$ is partially differentiable at $(0,0)$ with respect to x_2:

$$\begin{aligned}\frac{\partial^2 f}{\partial x_2^2} &= \lim_{h \to 0} \frac{\frac{\partial f}{\partial x_2}(0,0+h) - \frac{\partial f}{\partial x_2}(0,0)}{h} \\ &= \lim_{h \to 0} \frac{0-0}{h} = 0\end{aligned}$$

So we have shown that f is twice partially differentiable at point $(0,0)$ and the Hessian matrix at this point is:

$$H_f(0,0) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Schwarz theorem reqiures the function to be twice continuously partially differentiable at a point but our function f is only twice partially differentiable.