Mathematics Homework Sheet 9

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Problem 1

Problem 1(a)

$$f : [0, \infty) \to R, \quad x \mapsto \sqrt{x}$$

We want to prove that f is continous on $(0, \infty)$.
So we want to show that $\forall x_0 \in (0, \infty)$, $\lim_{x \to x_0} f(x) = f(x_0)$
So we want to show

$$\lim_{x \to x_0} \sqrt{x} = \sqrt{x_0}, \quad \forall x_0 \in (0, \infty)$$

Let $x_0 \in (0, \infty)$ be given.
Let $\epsilon > 0$ be given.
We want to find $\delta > 0$ such that

$$0 < |x - x_0| < \delta \implies |\sqrt{x} - \sqrt{x_0}| < \epsilon$$

$$|\sqrt{x} - \sqrt{x_0}| = \frac{|x - x_0|}{|\sqrt{x} + \sqrt{x_0}|} < \epsilon$$

And since $|\sqrt{x} - \sqrt{x_0}| < \epsilon$ the minimum value of $\sqrt{x}$ can be $\sqrt{x_0} - \epsilon$. Inserting this information into the above inequality we get

$$\frac{|x - x_0|}{|\sqrt{x} + \sqrt{x_0}|} < \frac{|x - x_0|}{|2\sqrt{x_0} - \epsilon|} < \epsilon$$
$$|x - x_0| < |2\sqrt{x_0} - \epsilon|\epsilon$$

So, if I choose $\delta = |2\sqrt{x_0} - \epsilon|\epsilon$, then the condition is satisfied.

Problem 1(b)

Yes, the function f is also continous at the point 0. Because the the proof at 1(a) is also valid for $x_0 = 0$. That is

$$\lim_{x \to 0} \sqrt{x} = \sqrt{0}$$

Problem 2

We are given that f is continous at the point 0 and f(3x) = f(x) for all x.
We want to show that f is constant on R.
Let's call the the limit and the value of the function at 0 as L.

$$\lim_{x \to 0} f(x) = f(0) = L$$

So from the definiton of the continuity at 0 we have

$$\forall \epsilon > 0,\ \exists \delta > 0,\ 0 < |x| < \delta \implies |f(x) - L| < \epsilon$$

So let's focus on $x_0 = \delta/2$. From the continuity at 0 we know that $|f(x_0) - L| < \epsilon$. And from the definiton of f, the above inequality is satisfied for every $x = 3^k x_0$, where $k \in Z$. This means that the above inequality is satisfied for every $x = 3^k \delta/2$. In this case I chose x_0 to be $\delta/2$ but in fact, x_0 can be chosen to be any number in the interval $(0, \delta)$. And if we multiply this number by 3^k we can get any real number. This tells us that every real number in the codomain of f stays in the ϵ neighborhood of L. So every real number in the codomain of f is equal to L. Therefore, f is constant on R.