Homework Sheet 7

Author: Abdullah Oğuz Topçuoğlu

Exercise 25

We choose the probability space

$$\begin{aligned}\Omega &= \{1,2,3,4,5,6\}^3, \\ \mathcal{F} &= \text{Power set of } \Omega, \\ P(A) &= U_\Omega\end{aligned}$$

The events are

$$\begin{aligned}A_1 &= \{(a,b,c) \in \Omega \mid a+b<5\}, \\ A_2 &= \{(a,b,c) \in \Omega \mid b \cdot c = 6\}, \\ A_3 &= \{(a,b,c) \in \Omega \mid b = 2\} \\ A_1 \cap A_3 &= \{(a,2,c) \in \Omega \mid a < 3\}, \\ (A_1 \cup A_2) \setminus A_3 &= \{(a,b,c) \in \Omega \mid (a+b<5) \text{ or } (b \cdot c = 6) \text{ and } b \neq 2\}\end{aligned}$$

Since P is uniform we count the number of elements in each event.
Calculating the probabilities:

6 many (a,b) pairs times 6 choices for c

$$P(A_1) = \frac{6 \cdot 6}{216} = \frac{36}{216} = \frac{1}{6},$$

4 many (b,c) pairs times 6 choices for a

$$P(A_2) = \frac{6 \cdot 4}{216} = \frac{24}{216} = \frac{1}{9},$$

6 choices for a times 1 choice for b times 6 choices for c

$$P(A_3) = \frac{36}{216} = \frac{1}{6},$$

2 choices for a times 1 choice for b times 6 choices for c

$$P(A_1 \cap A_3) = \frac{2 \cdot 6}{216} = \frac{1}{18},$$

For the last one we need to calculate $|A_1 \cup A_2|$ first:

$$\begin{aligned}|A_1 \cup A_2| &= |A_1| + |A_2| - |A_1 \cap A_2|, \\ |A_1| &= 36, \\ |A_2| &= 24,\end{aligned}$$

Calculating $|A_1 \cap A_2|$:

$$\begin{aligned}
&\text{if } b=1,\ c=6:\ a+1<5 \Rightarrow a<4 \Rightarrow 3 \text{ choices}\\
&\text{if } b=2,\ c=3:\ a+2<5 \Rightarrow a<3 \Rightarrow 2 \text{ choices}\\
&\text{if } b=3,\ c=2:\ a+3<5 \Rightarrow a<2 \Rightarrow 1 \text{ choice}\\
&\text{if } b=6,\ c=1:\ a+6<5 \Rightarrow \text{no choices.}
\end{aligned}$$

So $|A_1 \cap A_2| = 3+2+1+0 = 6$.

$$\begin{aligned}
|A_1 \cup A_2| &= 36+24-6 = 54,\\
P(A_1 \cup A_2) &= \frac{54}{216} = \frac{1}{4},\\
P((A_1 \cup A_2) \setminus A_3) &= P(A_1 \cup A_2) - P(A_3) + P(A_1 \cap A_3) = \frac{1}{4} - \frac{1}{6} + \frac{1}{18} = \frac{5}{36}.
\end{aligned}$$

Exercise 26

Lets define a set for students

$$S := \{s \in \mathbb{N} \mid s < 30\}$$

Each number represents a student. And note that $|S| = 30$. And the first 7 students belong to tutorial A following 12 students belong to tutorial B and the last 11 students belong to tutorial C.

We choose the probability space

$$\begin{aligned}
\Omega &= \{(a,b,c) \mid a,b,c \in S \text{ and } a \neq b \neq c\},\\
\mathcal{F} &= \text{Power set of } \Omega,\\
P(A) &= U_\Omega
\end{aligned}$$

We can choose uniform distribution because each student has equal chance of being selected.

The events are:

$$\begin{aligned}
A_1 &= \{(a,b,c) \in \Omega \mid a \in \{7,\ldots,18\}\},\\
A_2 &= \{(a,b,c) \in \Omega \mid \big|\{p \in \{a,b,c\} \mid p \in \{0,\ldots,6\}\}\big| = 1 \quad \wedge\\
&\quad \big|\{p \in \{a,b,c\} \mid p \in \{7,\ldots,18\}\}\big| = 1 \quad \wedge\\
&\quad \big|\{p \in \{a,b,c\} \mid p \in \{19,\ldots,29\}\}\big| = 1\}
\end{aligned}$$

Here note that $|\Omega| = 30 \cdot 29 \cdot 28$

Calculating the probabilities:

12 choices for the first paper, 29 choices for the second paper and 28 choices for the third paper

$$P(A_1) = \frac{12 \cdot 29 \cdot 28}{|\Omega|} = \frac{12 \cdot 29 \cdot 28}{30 \cdot 29 \cdot 28} = \frac{12}{30} = \frac{2}{5},$$

choose a student from tutorial A (7 choices), tutorial B (12 choices) and tutorial C (11 choices). Then permute them ($3! = 6$) since order doesnt matter

$$P(A_2) = \frac{7 \cdot 12 \cdot 11 \cdot 6}{|\Omega|} = \frac{7 \cdot 12 \cdot 11 \cdot 6}{30 \cdot 29 \cdot 28} = \frac{5544}{24360} = \frac{231}{1015}.$$

Exercise 27

Lets create a set for the tickets

$$\begin{aligned} T &:= B \uplus NCP \uplus TP \\ B &:= \{b \in \mathbb{N} \mid b < 160\} \\ NCP &:= \{n \in \mathbb{N} \mid 160 \leq n < 198\} \\ TP &:= \{t \in \mathbb{N} \mid 198 \leq t < 200\} \end{aligned}$$

B is the set of blank tickets, NCP is the set of noncash prizes and TP is the set of top prizes and they are disjoint.

We choose the probability space

$$\begin{aligned} \Omega &= \{X \subseteq T \mid |X| = 10\}, \\ \mathcal{F} &= \text{Power set of } \Omega, \\ P(A) &= U_\Omega \end{aligned}$$

We can choose uniform distribution because each ticket has equal chance of being selected.
The events are:

$$\begin{aligned} A_1 &= \{X \in \Omega \mid |X \cap TP| = 1 \text{ and } |X \cap NCP| \geq 2\}, \\ A_2 &= \{X \in \Omega \mid |X \cap B| < 6\} \end{aligned}$$

Here note that $|\Omega| = \binom{200}{10}$
Calculating the probabilities:

choose 1 top prize, choose 2 noncash prize, choose 7 either blank or noncash prizes to make it to 10 total

$$\begin{aligned} P(A_1) &= \frac{\binom{2}{1} \cdot \binom{38}{2} \cdot \binom{196}{7}}{|\Omega|} \\ &= \frac{\binom{2}{1} \cdot \binom{38}{2} \cdot \binom{196}{7}}{\binom{200}{10}}, \\ &= 53428/431233 \quad \text{link to the calculator: } \texttt{https://www.wolframalpha.com/input?i=choose\%282\%2C1\%} \end{aligned}$$

we separetely count when there is k blank tickets where k = 0, 1, 2, 3, 4, 5 and sum them up

$$
\begin{aligned}
P(A_2) &= \frac{\sum_{k=0}^{5} \binom{160}{k} \cdot \binom{40}{10-k}}{|\Omega|} \\
&= \frac{1}{\binom{200}{10}} \sum_{k=0}^{5} \binom{160}{k} \cdot \binom{40}{10-k}.
\end{aligned}
$$

Exercise 28

We are given the functions

$$
\begin{aligned}
p_1(k) &= \begin{cases} -\frac{1}{3}, & k \in \{1,3\} \\ -\frac{1}{6}, & k \in \{2,5\} \\ 1, & k \in \{4,6\} \end{cases}, \\
p_2(k) &= \begin{cases} \frac{1}{3}, & k \in \{1,3\} \\ \frac{1}{6}, & k \in \{2,5\} \\ \frac{1}{2}, & k \in \{4,6\} \end{cases}, \\
p_q(k) &= q^k, \quad q \in (0,1).
\end{aligned}
$$

(i)

A probability counting density must satisfy two properties:

- $p(k) \geq 0$ for all $k \in \Omega$
- $\sum_{k \in \Omega} p(k) = 1$

p_1 is not a probability counting density because it assigns negative probabilities to some outcomes.
p_2 is not a probability counting density function because the sum exceeds 1.

(ii)

We need to find $\alpha, \beta \in \mathbb{R}$ such that $\alpha p_1 + \beta p_2$ is a probability counting density. So we need to satisfy the two properties:

- $\alpha p_1(k) + \beta p_2(k) \geq 0$ for all $k \in \Omega$
- $\sum_{k \in \Omega} \alpha p_1(k) + \beta p_2(k) = 1$

From the second property we have:

$$
\begin{aligned}
\sum_{k \in \Omega} \alpha p_1(k) + \beta p_2(k) &= \alpha \sum_{k \in \Omega} p_1(k) + \beta \sum_{k \in \Omega} p_2(k) \\
&= \alpha \cdot 1 + \beta \cdot 2 = \alpha + 2\beta = 1
\end{aligned}
$$

We are gonna replace α with $1-2\beta$ in the first property and write down all the inequalities and find one pair that satisfies all of them.

$$(1-2\beta)p_1(k)+\beta p_2(k)\geq 0$$

for $k=1$ or $k=3$

$$\begin{aligned}(1-2\beta)\left(-\frac{1}{3}\right)+\beta\left(\frac{1}{3}\right)&\geq 0\\ -\frac{1}{3}+\frac{2\beta}{3}+\frac{\beta}{3}&\geq 0\\ -1+3\beta&\geq 0\\ \beta&\geq\frac{1}{3}\end{aligned}$$

for $k=2$ or $k=5$

$$\begin{aligned}(1-2\beta)\left(-\frac{1}{6}\right)+\beta\left(\frac{1}{6}\right)&\geq 0\\ -\frac{1}{6}+\frac{2\beta}{6}+\frac{\beta}{6}&\geq 0\\ -1+3\beta&\geq 0\\ \beta&\geq\frac{1}{3}\end{aligned}$$

for $k=4$ or $k=6$

$$\begin{aligned}(1-2\beta)(1)+\beta\left(\frac{1}{2}\right)&\geq 0\\ 1-2\beta+\frac{\beta}{2}&\geq 0\\ 2-4\beta+\beta&\geq 0\\ 2-3\beta&\geq 0\\ \beta&\leq\frac{2}{3}\end{aligned}$$

Choose $\beta=\frac{1}{2}$ and $\alpha=1-2\beta=0$.
So one possible linear combination is:

$$0\cdot p_1+\frac{1}{2}\cdot p_2$$

(iii)

We need to find $q\in(0,1)$ such that p_q is a probability counting density.
We need to satisfy the two properties:

- $p_q(k) = q^k \geq 0$ for all $k \in \Omega$
- $\sum_{k\in\Omega} p_q(k) = \sum_{k\in\mathbb{N}} q^k = 1$

The first property is satisfied for all $q \in (0,1)$ since exponentials with positive bases are always positive.
For the second property we use the formula for infinite geometric series:

$$\begin{aligned}\sum_{k=1}^{\infty} q^k = \frac{q}{1-q} &= 1 \\ q &= 1-q \\ 2q &= 1 \\ q &= \frac{1}{2}\end{aligned}$$

So the only q that makes p_q a probability counting density is $q = \frac{1}{2}$.