Homework Sheet 8

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Let N_k denote the set of natural numbers from 1 to k.

$$N_k = \{i \in \mathbb{N} \bigm| 0 < i \leq k\}$$

So that N_k has k elements.

Exercise 29

(i)

The probability space would be

$$\begin{aligned} \Omega &= N_6 \times N_{10} \\ \mathcal{F} &= \text{Power set of } \Omega \\ P &= U_\Omega \end{aligned}$$

I chose the uniform distribution because each outcome is equally likely when rolling fair dice.
The random variable X would be

$$\begin{aligned} X &: \Omega \to \mathbb{N} \\ (d, h) &\mapsto \max(d, h) \end{aligned}$$

The range of X

$$\begin{aligned} X(\Omega) &= \{\max(d, h) \mid d \in N_6, h \in N_{10}\} \\ &= N_{10} \end{aligned}$$

(ii)

We want to calculate $P[\{X = k\}]$ for $k \in N_{10}$.
The random variable X takes the value k only in one of the following three cases:

- The decahedron shows k and the dice shows a number less than k.
- The dice shows k and the decahedron shows a number less than k.
- Both the dice and the decahedron show k.

Here note that the cases are disjoint. Lets count the number of possibilities for each case and add them up. We need to consider those cases when k is less than or equal to 6 and when k is greater than 6 separately.

- For $k \in N_6$:
  - Case 1: There are $k-1$ choices for the dice (1 to $k-1$) and 1 choice for the decahedron (k). So there are $k-1$ possibilities.
  - Case 2: There are $k-1$ choices for the decahedron (1 to $k-1$) and 1 choice for the dice (k). So there are $k-1$ possibilities.
  - Case 3: There is only 1 possibility where both show k.

  So in total there are $2(k-1)+1 = 2k-1$ possibilities.
- For $k \in \{7, 8, 9, 10\}$:
  - Case 1: There are 6 choices for the dice (1 to 6) and 1 choice for the decahedron (k). So there are 6 possibilities.
  - Case 2: Cant happen since the dice only goes up to 6.
  - Case 3: Cant happen since the dice only goes up to 6.

  So in total there are $6+0+0 = 6$ possibilities.

$$P[\{X = k\}] = \begin{cases} \frac{2k-1}{60}, & k \in N_6 \\ \frac{6}{60}, & k \in \{7, 8, 9, 10\} \end{cases}$$
$$60 = |\Omega| = |N_6| \times |N_{10}| = 6 \times 10$$

(iii)

We want to calculate $P[\{2 \leq X \leq 5\}]$.

$$\begin{aligned} P[\{2 \leq X \leq 5\}] &= \sum_{k=2}^{5} P[\{X = k\}] \\ &= \sum_{k=2}^{5} \frac{2k-1}{60} \\ &= \frac{1}{60} \sum_{k=2}^{5} (2k-1) \\ &= \frac{1}{60}(3+5+7+9) \\ &= \frac{24}{60} \\ &= \frac{2}{5} \end{aligned}$$

Exercise 30

(i)

The probability space would be

$$\begin{aligned}
\Omega &= N_7 \times N_4 \\
\mathcal{F} &= \text{Power set of } \Omega \\
P &= U_\Omega
\end{aligned}$$

I chose the uniform distribution because each outcome is equally likely when pulling marbles from urns.
The random variables D_7 and D_4 would be

$$\begin{gathered}
D_7 : \Omega \to \mathbb{N} \\
(m_7, m_4) \mapsto m_7
\end{gathered}$$

$$\begin{gathered}
D_4 : \Omega \to \mathbb{N} \\
(m_7, m_4) \mapsto m_4
\end{gathered}$$

The ranges of D_7 and D_4

$$D_7(\Omega) = \{m_7 \mid m_7 \in N_7, m_4 \in N_4\} = N_7$$

$$D_4(\Omega) = \{m_4 \mid m_7 \in N_7, m_4 \in N_4\} = N_4$$

(ii)

The range of X

$$\begin{aligned}
X(\Omega) &= \{\max(|D_7 - D_4|, 0) \mid D_7 \in N_7, D_4 \in N_4\} \\
&= N_6 \cup \{0\}
\end{aligned}$$

The range of Y

$$\begin{aligned}
Y(\Omega) &= \{\min(|D_7 - D_4|, 5) \mid D_7 \in N_7, D_4 \in N_4\} \\
&= N_5 \cup \{0\}
\end{aligned}$$

To find the distributions of X and Y, we need to calculate $P[\{X = k\}]$ and $P[\{Y = k\}]$ for k in their respective ranges.
The random varaible X takes the value k in the following cases:

- When $|D_7 - D_4| = k$

The random varaible Y takes the value k in the following cases:

- When $|D_7 - D_4| = k$ for $k \in N_5$
- When $|D_7 - D_4| \geq 5$ for $k = 5$

Exercise 31

Lets first create the probability space.

$$\begin{aligned}\Omega &= \text{set of all light bulbs } = M_1 \uplus M_2 \uplus M_3 \\ \mathcal{F} &= \text{Power set of } \Omega \\ P &= U_\Omega\end{aligned}$$

where M_i is the set of all light bulbs from manufacturer i for $i \in \{1, 2, 3\}$.
Here M_i also represent the event that a randomly selected bulb is from manufacturer i.
Let D be the event that a randomly selected bulb is defective.
We are given that

$$\begin{aligned}P[M_1] &= 0.3 \\ P[M_2] &= 0.5 \\ P[M_3] &= 0.2 \\ P[D|M_1] &= 0.05 \\ P[D|M_2] &= 0.02 \\ P[D|M_3] &= 0.08\end{aligned}$$

We want to find $P[D]$.
Using the law of total probability:

$$\begin{aligned}P[D] &= \sum_{i=1}^{3} P[D|M_i]P[M_i] \\ &= P[D|M_1]P[M_1] + P[D|M_2]P[M_2] + P[D|M_3]P[M_3] \\ &= 0.05 \cdot 0.3 + 0.02 \cdot 0.5 + 0.08 \cdot 0.2 \\ &= 0.015 + 0.01 + 0.016 \\ &= 0.041\end{aligned}$$

Exercise 32

Lets first create the probability space.

$$\begin{aligned}\Omega &= \text{set of all inspected cars } = A \uplus A^c \\ \mathcal{F} &= \text{Power set of } \Omega \\ P &= U_\Omega\end{aligned}$$

where A is the set of all inspected cars that receive a TÜV sticker and A^c is the set of all inspected cars that do not receive a TÜV sticker.
Let B be the set of cars older than ten years.

We are given that

$$\begin{aligned} P[A^c] &= 0.08 \\ P[B|A^c] &= 0.7 \\ P[B^c \cap A] &= 0.25 \end{aligned}$$

(i)

We want to find $P[B]$.
Using the law of total probability:

$$\begin{aligned} P[B] &= P[B|A]P[A] + P[B|A^c]P[A^c] \\ &= P[B|A](1 - P[A^c]) + P[B|A^c]P[A^c] \\ &= P[B|A](1 - 0.08) + 0.7 \cdot 0.08 \end{aligned}$$

To find $P[B|A]$, we can use the complement of the event $B^c \cap A$:

$$\begin{aligned} P[B|A] &= 1 - P[B^c|A] \\ &= 1 - \frac{P[B^c \cap A]}{P[A]} \\ &= 1 - \frac{0.25}{1 - 0.08} \\ &= 1 - \frac{0.25}{0.92} \\ &= 1 - \frac{25}{92} \\ &= \frac{67}{92} \end{aligned}$$

Substituting back:

$$\begin{aligned} P[B] &= \frac{67}{92} \cdot 0.92 + 0.7 \cdot 0.08 \\ &= 0.67 + 0.056 \\ &= 0.726 \end{aligned}$$

(ii)

We want to find $P[A^c|B]$.
Using Bayes theorem:

$$\begin{aligned} P[A^c|B] &= \frac{P[B|A^c]P[A^c]}{P[B]} \\ &= \frac{0.7 \cdot 0.08}{0.726} \\ &= \frac{0.056}{0.726} \\ &= 0.07713498622589532 \end{aligned}$$

(iii)

We want to find $P[B|A]$.
We have already calculated this in part (i):

$$P[B|A] = \frac{67}{92}$$