Mathematics Homework Sheet 8

Author: Abdullah Oguz Topcuoglu & Yousef Farag

Problem 1

$$a_n := \sqrt[n]{n} - 1$$

We want to prove

$$a_n \leq \sqrt{\frac{2}{n}} \qquad \forall n \geq 2$$

and we want to prove

$$\lim_{n \to \infty} \sqrt[n]{n} = 1$$

Apply the binomial formula to $(1 + a_n)^n$

$$\begin{aligned}
(1 + a_n)^n &= n \\
&= \sum_{k=0}^{n} \binom{n}{k} a_n^k \\
&= \sum_{k=0}^{n} \binom{n}{k} (\sqrt[n]{n} - 1)^k \\
&= 1 + \binom{n}{1} (\sqrt[n]{n} - 1) + \binom{n}{2} (\sqrt[n]{n} - 1)^2 + \ldots + (\sqrt[n]{n} - 1)^n
\end{aligned}$$

Using the fact that $a_n \leq \sqrt{\frac{2}{n}}$, we want to show

$$\lim_{n \to \infty} \sqrt[n]{n} = 1$$

We know that

$$1 - \frac{1}{n} \leq \sqrt[n]{n}$$

because $1 - \frac{1}{n}$ is less than 1 and $\sqrt[n]{n}$ is greater or equal to 1 for all $n \geq 1$. Also from the previous inequality we have

$$\begin{aligned}
\sqrt[n]{n} - 1 &\leq \sqrt{\frac{2}{n}} \\
\sqrt[n]{n} &\leq 1 + \sqrt{\frac{2}{n}}
\end{aligned}$$

When combined we have

$$1 - \frac{1}{n} \leq \sqrt[n]{n} \leq 1 + \sqrt{\frac{2}{n}}$$

$$\lim_{n \to \infty} 1 - \frac{1}{n} = 1$$

$$\lim_{n \to \infty} 1 + \sqrt{\frac{2}{n}} = 1$$

Therefore, by the sandwich theorem, we have

$$\lim_{n \to \infty} \sqrt[n]{n} = 1$$