Homework Sheet 1

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Problem 1

We are given the function:

$$f : \mathbb{R}^2 \to \mathbb{R} \quad f(x_1, x_2) = x_1^2 + 2x_2^2$$

We want to show that f is continuous on $\mathbb{R}^2$. We want to find a delta that satisfies:

$$\forall \varepsilon > 0, \exists \delta > 0 : \forall x \in \mathbb{R}^2 \|x - x^0\| < \delta \implies |f(x) - f(x^0)| < \varepsilon$$

for an arbitrary but fixed x^0.
Fix $x^0 = (x_1^0, x_2^0)$. Fix $\varepsilon > 0$. We want to find a delta that makes this inequality satisfied always:

$$|f(x) - f(x^0)| < \varepsilon$$

We are gonna find a relation between delta and $|f(x) - f(x^0)|$ and from there we are gonna look for values of delta where it is less than epsilon.

$$\begin{aligned}
|f(x) - f(x^0)| &= |x_1^2 + 2x_2^2 - (x_1^0)^2 - 2(x_2^0)^2| \\
&= |(x_1^2 - (x_1^0)^2) + 2(x_2^2 - (x_2^0)^2)| \\
&\leq |x_1^2 - (x_1^0)^2| + 2|x_2^2 - (x_2^0)^2| \\
&= |(x_1 - x_1^0)(x_1 + x_1^0)| + 2|(x_2 - x_2^0)(x_2 + x_2^0)| \\
&\leq |x_1 - x_1^0||x_1 + x_1^0| + 2|x_2 - x_2^0||x_2 + x_2^0|
\end{aligned}$$

(replace invidual components of the vector with the vector itself. that makes the overall value larger)

$$\begin{aligned}
&\leq \|x - x^0\|\|x + x^0\| + 2\|x - x^0\|\|x + x^0\| \\
&= 3\|x - x^0\|\|x + x^0\| \\
&\leq 3\delta\|x + x^0\|
\end{aligned}$$

and now assume that $\delta \leq 1$, then we can say that:

$$\|x\| \leq \|x^0\| + 1$$

$$
\begin{aligned}
|f(x) - f(x^0)| &\leq 3\delta \|x + x^0\| \\
&\leq 3\delta(\|x\| + \|x^0\|) \\
&\leq 3\delta(\|x^0\| + 1 + \|x^0\|) \\
&= 3\delta(2\|x^0\| + 1)
\end{aligned}
$$

We are almost done. Now if we chose a delta that satisfies:

$$
\begin{aligned}
3\delta(2\|x^0\| + 1) &< \varepsilon \\
\delta &< \frac{\varepsilon}{3(2\|x^0\| + 1)}
\end{aligned}
$$

And at the top we assumed that $\delta \leq 1$, so we can chose:

$$\delta = \min\left(1, \frac{\varepsilon}{3(2\|x^0\| + 1)}\right)$$

So found a delta meaning that f is continuous at x^0. Since x^0 was arbitrary, f is continuous on $\mathbb{R}^2$.

Problem 2

We are given the function:

$$f : \mathbb{R}^2 \to \mathbb{R} \quad f(x_1, x_2) = \begin{cases} \frac{x_1^2 sin(x_1+x_2)}{\sqrt{x_1^4+x_2^4}} & (x_1, x_2) \neq (0, 0) \\ 0 & (x_1, x_2) = (0, 0) \end{cases}$$

We want to show that for every sequence $(x^N)_{N\in\mathbb{N}}$ that converges to $(0, 0)$, the sequence $(f(x^N))_{N\in\mathbb{N}}$ also converges to $f(0, 0) = 0$.

Fix an arbitrary sequence $(x^N)_{N\in\mathbb{N}}$ that converges to $(0, 0)$.
If $(x_1, x_2) \neq (0, 0)$

$$\sqrt{x_1^4 + x_2^4} \geq \sqrt{x_1^4} = x_1^2$$

Hence

$$\frac{x_1^2}{\sqrt{x_1^4 + x_2^4}} \leq 1$$

Also note that:

$$|sin(x)| \leq |x|$$

$$
\begin{aligned}
|f(x^{(N)})| &= \frac{(x_1^{(N)})^2 |sin(x_1^{(N)} + x_2^{(N)})|}{\sqrt{(x_1^{(N)})^4 + (x_2^{(N)})^4}} \\
&\leq \frac{(x_1^{(N)})^2 |x_1^{(N)} + x_2^{(N)}|}{\sqrt{(x_1^{(N)})^4 + (x_2^{(N)})^4}} \\
&\leq |x_1^{(N)} + x_2^{(N)}| \\
&\leq |x_1^{(N)}| + |x_2^{(N)}| \\
&\leq |x^{(N)}| + |x^{(N)}| \\
&\leq 2|x^{(N)}|
\end{aligned}
$$

Since $|f(x^{(N)})|$ is less than $2|x^{(N)}|$ it must also converge to zero. $|f(x^{(N)})|$ converging zero means that $f(x^{(N)})$ converging to $(0,0)$ which equals to $f(0,0)$. And thats what we wanted to show.

Problem 3

We are given the function:

$$
f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \quad f(x_1, x_2) = \begin{cases} \frac{\sqrt{x_1 x_2^2 + x_1 x_2^2}}{x_1 + 2x_2^2} & ,(x_1, x_2) \neq (0,0) \\ 0 & ,(x_1, x_2) = (0,0) \end{cases}
$$

(i)

$f_{x_1^0}$:

Fix a sequence $x^{(N)} = (x_1^0, x_2^{(N)})$ that converges to (x_1^0, x_2^0). We want to show that $f_{x_1^0}(x_2^{(N)})$ converges to $f_{x_1^0}(x_2^0)$.

$$
|f_{x_1^0}(x_2^{(N)}) - f_{x_1^0}(x_2^0)| = \left| \frac{\sqrt{x_1^0 (x_2^{(N)})^2} + x_1^0 (x_2^{(N)})^2}{x_1^0 + 2(x_2^{(N)})^2} - \frac{\sqrt{x_1^0 (x_2^0)^2} + x_1^0 (x_2^0)^2}{x_1^0 + 2(x_2^0)^2} \right|
$$

take the squares out of the square roots

$$
= \left| \frac{|x_2^{(N)}| \sqrt{x_1^0} + x_1^0 (x_2^{(N)})^2}{x_1^0 + 2(x_2^{(N)})^2} - \frac{|x_2^0| \sqrt{x_1^0} + x_1^0 (x_2^0)^2}{x_1^0 + 2(x_2^0)^2} \right|
$$

$\sqrt{x_1}^2 = x_1$ since x_1 is never negative.

$$= \left| \frac{|x_2^{(N)}|\sqrt{x_1^0}(1+|x_2^{(N)}|\sqrt{x_1^0})}{x_1^0+2(x_2^{(N)})^2} - \frac{|x_2^0|\sqrt{x_1^0}(1+|x_2^0|\sqrt{x_1^0})}{x_1^0+2(x_2^0)^2} \right|$$

now lets take the limit as N goes to infinity: We know that x_2^N converges to x_2^0

$$= \left| \frac{|x_2^0|\sqrt{x_1^0}(1+|x_2^0|\sqrt{x_1^0})}{x_1^0+2(x_2^0)^2} - \frac{|x_2^0|\sqrt{x_1^0}(1+|x_2^0|\sqrt{x_1^0})}{x_1^0+2(x_2^0)^2} \right|$$
$$= 0$$

So $f_{x_1^0}(x_2^{(N)})$ converges to $f_{x_1^0}(x_2^0)$.

$f_{x_2^0}$:
Fix a sequence $x^{(N)} = (x_1^{(N)}, x_2^0)$ that converges to (x_1^0, x_2^0). We want to show that $f_{x_2^0}(x_1^{(N)})$ converges to $f_{x_2^0}(x_1^0)$.

$$|f_{x_2^0}(x_1^{(N)}) - f_{x_2^0}(x_1^0)| = \left| \frac{\sqrt{x_1^{(N)}(x_2^0)^2} + x_1^{(N)}(x_2^0)^2}{x_1^{(N)}+2(x_2^0)^2} - \frac{\sqrt{x_1^0(x_2^0)^2} + x_1^0(x_2^0)^2}{x_1^0+2(x_2^0)^2} \right|$$

take the squares out of the square roots

$$= \left| \frac{|x_2^0|\sqrt{x_1^{(N)}} + x_1^{(N)}(x_2^0)^2}{x_1^{(N)}+2(x_2^0)^2} - \frac{|x_2^0|\sqrt{x_1^0} + x_1^0(x_2^0)^2}{x_1^0+2(x_2^0)^2} \right|$$

$\sqrt{x_1}^2 = x_1$ since x_1 is never negative.

$$= \left| \frac{|x_2^0|\sqrt{x_1^{(N)}}(1+|x_2^0|\sqrt{x_1^{(N)}})}{x_1^{(N)}+2(x_2^0)^2} - \frac{|x_2^0|\sqrt{x_1^0}(1+|x_2^0|\sqrt{x_1^0})}{x_1^0+2(x_2^0)^2} \right|$$

now lets take the limit as N goes to infinity: We know that x_1^N converges to x_1^0

$$= \left| \frac{|x_2^0|\sqrt{x_1^0}(1+|x_2^0|\sqrt{x_1^0})}{x_1^0+2(x_2^0)^2} - \frac{|x_2^0|\sqrt{x_1^0}(1+|x_2^0|\sqrt{x_1^0})}{x_1^0+2(x_2^0)^2} \right|$$
$$= 0$$

So $f_{x_2^0}(x_1^{(N)})$ converges to $f_{x_2^0}(x_1^0)$.

(ii)

We want to show that f is not continuous at $(0,0)$.
We are gonna show that there exists a sequence $(x^{(N)})_{N\in\mathbb{N}}$ that converges to $(0,0)$ but $(f(x^{(N)}))_{N\in\mathbb{N}}$ does not converge to $f(0,0)=0$.
Fix the sequence:

$$x^{(N)} = \left(\frac{1}{N^2}, \frac{1}{N}\right)$$

We can see that this sequence converges to $(0,0)$ as N goes to infinity.

$$\begin{aligned} f(x^{(N)}) &= \frac{\sqrt{\frac{1}{N^2}\cdot\frac{1}{N^2}} + \frac{1}{N^2}\cdot\frac{1}{N^2}}{\frac{1}{N^2} + 2\cdot\frac{1}{N^2}} \\ &= \frac{\frac{1}{N^2} + \frac{1}{N^4}}{\frac{3}{N^2}} \\ &= \frac{1 + \frac{1}{N^2}}{3} \end{aligned}$$

As N goes to infinity, $f(x^{(N)})$ converges to $\frac{1}{3}$ which is not equal to $f(0,0)=0$.
So we found a sequence that converges to $(0,0)$ but f of that sequence does not converge to $f(0,0)$.
Hence f is not continuous at $(0,0)$.