# Homework Sheet 6

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## Exercise 21

We are given the function

$$f(x_1, x_2) := \begin{pmatrix} \log(1 + x_1^2 + 2x_2^2) \\ x_1 e^{2x_2} \end{pmatrix}.$$

### (i)

The jacobian matrix is invertible at the points where the determinant is not zero.
The jacobian matrix:

$$J_f(x_1, x_2) = \begin{pmatrix} \frac{2x_1}{1+x_1^2+2x_2^2} & \frac{4x_2}{1+x_1^2+2x_2^2} \\ \\ e^{2x_2} & 2x_1 e^{2x_2} \end{pmatrix}.$$

The determinant of the jacobian matrix:

$$\begin{aligned} \det(J_f(x_1, x_2)) &= \frac{2x_1 \cdot 2x_1 e^{2x_2}}{1 + x_1^2 + 2x_2^2} - \frac{4x_2 \cdot e^{2x_2}}{1 + x_1^2 + 2x_2^2} \\ &= \frac{4x_1^2 e^{2x_2} - 4x_2 e^{2x_2}}{1 + x_1^2 + 2x_2^2} \\ &= \frac{4e^{2x_2}(x_1^2 - x_2)}{1 + x_1^2 + 2x_2^2}. \end{aligned}$$

The determinant is zero when

$$4e^{2x_2}(x_1^2 - x_2) = 0 \implies x_1^2 - x_2 = 0 \implies x_2 = x_1^2.$$

The set $A$ would be

$$A = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \neq x_1^2\}.$$

### (ii)

$f$ is locally invertible at a point if the jacobian matrix is invertible at that point and we know that the jacobian matrix is invertible at each point in $A$.
So $f$ is locally invertible at each point $x \in A$.

## (iii)

To find the jacobian matrix of the local inverse of $f$ at $f(x)$, we can use the formula:

$$J_{f^{-1}}(f(x)) = (J_f(x))^{-1}.$$

We already have $J_f(x)$ so we need to find its inverse.
The jacobian matrix:

$$J_f(x_1, x_2) = \begin{pmatrix} \frac{2x_1}{1+x_1^2+2x_2^2} & \frac{4x_2}{1+x_1^2+2x_2^2} \\ \\ e^{2x_2} & 2x_1 e^{2x_2} \end{pmatrix}.$$

And its determinant:

$$\det(J_f(x_1, x_2)) = \frac{4e^{2x_2}(x_1^2 - x_2)}{1 + x_1^2 + 2x_2^2}.$$

We calculated those in the previous parts.
The inverse of jacobian matrix:

$$\begin{aligned}(J_f(x_1, x_2))^{-1} &= \frac{1}{\det(J_f(x_1, x_2))} \begin{pmatrix} 2x_1 e^{2x_2} & -\frac{4x_2}{1+x_1^2+2x_2^2} \\ \\ -e^{2x_2} & \frac{2x_1}{1+x_1^2+2x_2^2} \end{pmatrix} \\ &= \frac{1 + x_1^2 + 2x_2^2}{4e^{2x_2}(x_1^2 - x_2)} \begin{pmatrix} 2x_1 e^{2x_2} & -\frac{4x_2}{1+x_1^2+2x_2^2} \\ \\ -e^{2x_2} & \frac{2x_1}{1+x_1^2+2x_2^2} \end{pmatrix}\end{aligned}$$