Mathematics Homework Sheet 3

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Problem 1

(a)

$$3z^2 + z = 1$$

solve the quadratic equations and we get

$$z = \frac{-1 \pm \sqrt{1^2 - 4.3.(-1)}}{(3.2)}$$

$$z = \frac{-1 \pm \sqrt{1+12}}{6}$$

$$z = \frac{-1 \pm \sqrt{13}}{6}$$

$$z_0 = \frac{-1+\sqrt{13}}{6} + 0i, \quad z_1 = \frac{-1-\sqrt{13}}{6} + 0i$$

(b)

$$3z^2 + z = 0$$

$$z(3z+1) = 0$$

$$z = 0 \quad \text{or} \quad 3z + 1 = 0$$

$$z = 0 \quad \text{or} \quad z = -\frac{1}{3}$$

(c)

$$z^2 - (3+i)z + 4 + 3i = 0$$

$$z = \frac{(3+i) \pm \sqrt{(3+i)^2 - 4(4+3i)}}{2}$$

$$z = \frac{(3+i) \pm \sqrt{(9+6i-1-16-12i)}}{2}$$

$$z = \frac{(3+i) \pm \sqrt{-8-6i}}{2}$$

$$z = \frac{(3+i) \pm (1-3i)}{2}$$

$$z_0 = \frac{4-2i}{2} = 2 - i, \quad z_1 = \frac{2+4i}{2} = 1 + 2i$$

$\sqrt{-8-6i} = \pm(1-3i)$ because $\sqrt{-8-6i} = \pm(\sqrt{\frac{r+(-8)}{2}} + sign(-6)\sqrt{\frac{r-(-8)}{2}}i)$ where $r = |-8-6i|$

(d)

$$\sinh z = i$$

By definiton of sinh

$$\frac{e^z - e^{-z}}{2} = i$$

Multiplying both sides by $2e^z$, we get:

$$e^{2z} - 1 = 2ie^z$$

$$e^{2z} - 2ie^z - 1 = 0$$

Solving for e^z with $a = 1, b = -2i, c = -1$:

$$e^z = \frac{2i}{2} \pm \sqrt{\frac{-4+4}{4}} = i = 1 \cdot e^{i\frac{\pi}{2}}$$

$$z = i(\frac{\pi}{2} + 2n\pi), n \in \mathbb{Z}$$

(e)

$$tan(z) = 1$$

$$\frac{sin(z)}{cos(z)} = 1$$

$$sin(z) = cos(z)$$

By definition of sin(z) and cos(z)

$$\frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2}$$

$$e^{iz} - e^{-iz} = i(e^{iz} + e^{-iz})$$

$$e^{2iz} - 1 = i(e^{2iz} + 1)$$

$$e^{2iz}(1 - i) = 1 + i$$

$$e^{2iz} = \frac{1+i}{1-i} = \frac{\sqrt{2}e^{i\frac{\pi}{4}}}{\sqrt{2}e^{-i\frac{\pi}{4}}} = e^{i\frac{\pi}{2}}$$

$$2z = \frac{\pi}{2} + 2n\pi, n \in \mathbb{Z}$$

$$z = \frac{\pi}{4} + n\pi, n \in \mathbb{Z}$$

(f)

$$cos(z) = -\frac{5}{4}$$

By definition of cos(z)

$$\frac{e^{iz} + e^{-iz}}{2} = -\frac{5}{4}$$

$$e^{iz} + e^{-iz} = -\frac{5}{2}$$

Let $q = e^{iz}$

$$q + \frac{1}{q} = -\frac{5}{2}$$

$$q^2 + \frac{5}{2}q + 1 = 0$$
$$q = \frac{-\frac{5}{2} \pm \sqrt{(\frac{5}{2})^2 - 4}}{2}$$
$$q = \frac{-\frac{5}{2} \pm \frac{3}{2}}{2}$$
$$q_0 = \frac{-1}{2}, \qquad q_1 = -2$$
$$e^{iz_0} = -\frac{1}{2} = \frac{1}{2}e^{i\pi}$$
$$e^{iz_0 - i\pi} = \frac{1}{2}$$
$$iz_0 - i\pi = ln(\frac{1}{2})$$
$$z_0 = \frac{ln(\frac{1}{2}) + i\pi}{i}$$

$$e^{iz_1} = -2 = 2e^{i\pi}$$
$$e^{iz_1 - i\pi} = 2$$
$$iz_1 - i\pi = ln(2)$$
$$z_1 = \frac{ln(2) + i\pi}{i}$$

(g)

$$z + \bar{z} = 1$$

Let $z = x + iy$

$$x + iy + x - iy = 1$$
$$2x = 1$$
$$x = \frac{1}{2}$$
$$z = \frac{1}{2} + iy, \qquad y \in R$$

(h)

$$z^2 + 2\bar{z}^2 + z - \bar{z} + 9 = 0$$

Let $z = x + iy$

$$(x+iy)^2 + 2(x-iy)^2 + (x+iy) - (x-iy) + 9 = 0$$
$$9 - 2iy + x^2 + 2xyi - y^2 + 2x^2 - 4xyi - 2y^2 = 0$$
$$9 - 2iy + 3x^2 - 2xyi - 3y^2 = 0$$
$$9 + 3x^2 - 3y^2 = 0, \quad -2y - 2xy = 0$$
$$x = 1$$

Insert this in the first equation

$$9 + 3 - 3y^2 = 0$$
$$y^2 = 4$$
$$y = \pm 2$$

$$z = 1 \pm 2i$$

(i)

$$(1-i)z^2 = 1+7i$$
$$z^2 = \frac{1+7i}{1-i} = \frac{(1+7i)(1+i)}{2} = \frac{1+i+7i-7}{2} = -3+4i$$
$$z = \pm\sqrt{-3+4i}$$

(j)

$$(1-i)z^2 = (1+i)z$$
$$(1-i)z^2 - (1+i)z = 0$$
$$z\left[(1-i)z - (1+i)\right] = 0$$
$$z = 0 \quad \text{or} \quad (1-i)z = 1+i$$

$$z = \frac{1+i}{1-i} = \frac{(1+i)(1+i)}{(1-i)(1+i)} = \frac{(1+2i+i^2)}{1+1} = \frac{1+2i-1}{2} = \frac{2i}{2} = i$$

$$\Rightarrow z = 0 \quad \text{or} \quad z = i$$

(k)

$$z^4 - 4z^2 + 16 = 0$$

Solving for z^2 using a = 1, b = -4 and c = 16, we get:

$$z^2 = \frac{4 \pm \sqrt{16 - 4 \cdot 16}}{2} = 2 \pm \frac{\sqrt{-48}}{2} = 2 \pm 2\sqrt{3}i = 4e^{\pm i\frac{\pi}{3}}$$
$$z = \pm 2e^{\pm i\frac{\pi}{6}}$$

(l)

$$z^3 = 1$$

We know that $z = 1$ is one of the solutions. We divide $z^3 - 1$ by $z - 1$ using polynomial long division:

$\underline{z^3 - 1 \div z - 1}$

Step 1: $z^3 \div z = z^2$

$\Rightarrow z^2(z-1) = z^3 - z^2$

Subtract: $(z^3 - 1) - (z^3 - z^2) = z^2 - 1$

Step 2: $z^2 \div z = z$

$\Rightarrow z(z-1) = z^2 - z$

Subtract: $(z^2 - 1) - (z^2 - z) = z - 1$

Step 3: $z \div z = 1$

$\Rightarrow 1(z-1) = z - 1$

Subtract: $(z-1) - (z-1) = 0$

$$\therefore \quad \frac{z^3-1}{z-1} = z^2 + z + 1$$

$$z^3 - 1 = (z-1)(z^2+z+1) = 0$$
$$z = -\frac{1}{2} \pm \sqrt{-\frac{3}{4}} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$
$$z = 1 \quad \text{or} \quad z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

(m)

$$(z^2-1)^3 = (2z)^3 \implies z^2 - 1 = 2z, \quad z^2 - 1 = 2z\left(-\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}\right), \quad z^2 - 1 = 2z\left(-\tfrac{1}{2} - i\tfrac{\sqrt{3}}{2}\right).$$

Case 1: $z^2 - 1 = 2z$

$$z^2 - 2z - 1 = 0 \quad \implies \quad z = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}.$$

Case 2: $z^2 - 1 = 2z\left(-\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}\right)$

$$z^2 + \left(1 - i\sqrt{3}\right)z - 1 = 0 \quad \implies \quad z = \frac{-\left(1 - i\sqrt{3}\right) \pm \sqrt{(1 - i\sqrt{3})^2 + 4}}{2} = \frac{-1 + \sqrt{3}}{2} \pm i\,\frac{\sqrt{3} - 1}{2}.$$

Case 3: $z^2 - 1 = 2z\left(-\tfrac{1}{2} - i\tfrac{\sqrt{3}}{2}\right)$

$$z^2 + \left(1 + i\sqrt{3}\right)z - 1 = 0 \quad \implies \quad z = \frac{-\left(1 + i\sqrt{3}\right) \pm \sqrt{(1 + i\sqrt{3})^2 + 4}}{2} = \frac{-1 - \sqrt{3}}{2} \pm i\,\frac{\sqrt{3} + 1}{2}.$$

$$\boxed{\begin{aligned} &z = 1 \pm \sqrt{2}, \\ &z = \frac{-1+\sqrt{3}}{2} \pm i\,\frac{\sqrt{3}-1}{2}, \\ &z = \frac{-1-\sqrt{3}}{2} \pm i\,\frac{\sqrt{3}+1}{2}. \end{aligned}}$$

(n)

$$z^4 + 1 = 0 \implies z^4 = -1 = e^{i(\pi + 2\pi k)}, \quad k \in \mathbb{Z}$$
$$z = e^{i\left(\frac{\pi + 2\pi k}{4}\right)} = e^{i\left(\frac{\pi}{4} + \frac{\pi k}{2}\right)}, \quad k \in \mathbb{Z}$$

Taking k s that will get us an angle in $(-\pi, \pi]$,

$$z = e^{i\pi/4},\ e^{i3\pi/4},\ e^{i5\pi/4},\ e^{i7\pi/4},$$

(o)

$$z^6 - 3i\,z^3 - 2 = 0.$$

Set $w = z^3$. Then

$$w^2 - 3i\,w - 2 = 0 \implies w = \frac{3i \pm \sqrt{(-3i)^2 + 8}}{2} = \frac{3i \pm i}{2} = \begin{cases} 2i, \\ i. \end{cases}$$

Hence we must solve

$$z^3 = 2i \quad \text{and} \quad z^3 = i.$$

By taking cube-roots (as in your solution of $z^3 = 1$), we get
1. For $z^3 = 2i = 2e^{i\pi/2}$:

$$z = (2)^{1/3} \exp\Big(i\frac{\pi/2 + 2\pi k}{3}\Big) = 2^{1/3} e^{i(\pi/6 + 2\pi k/3)}, \quad k = 0, 1, 2.$$

2. For $z^3 = i = e^{i\pi/2}$:

$$z = \exp\Big(i\frac{\pi/2 + 2\pi k}{3}\Big) = e^{i(\pi/6 + 2\pi k/3)}, \quad k = 0, 1, 2.$$

Thus the six solutions are

$$\boxed{z = 2^{1/3} e^{i(\pi/6 + 2\pi k/3)}, \quad z = e^{i(\pi/6 + 2\pi k/3)}, \quad k = 0, 1, 2.}$$

(p)

We factor first:

$$z^3 + 2z^2 + 2z = z(z^2 + 2z + 2) = 0.$$

One solution is:

$$z = 0.$$

For the quadratic $z^2 + 2z + 2 = 0$, use the quadratic formula with $a = 1$, $b = 2$, $c = 2$:

$$z = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i.$$

$$\boxed{z = 0, \quad z = -1 + i, \quad z = -1 - i}$$

(q)

We write $1 = e^{2\pi ik}$, so:

$$e^z = e^{2\pi ik} \Rightarrow z = 2\pi ik, \quad k \in \mathbb{Z}.$$

$$\boxed{z = 2\pi ik}$$

(r)

We solve:

$$e^z = e^{iz}.$$

Equating exponents up to a multiple of $2\pi i$, we write:

$$z = iz + 2\pi ik, \quad k \in \mathbb{Z}.$$

Solving:

$$z - iz = 2\pi ik \Rightarrow z(1 - i) = 2\pi ik \Rightarrow z = \frac{2\pi ik}{1 - i}.$$

Multiply numerator and denominator by $1 + i$:

$$z = \frac{2\pi ik(1 + i)}{2} = \pi ik(1 + i).$$

Now write:

$$z = \pi ki(1 + i) = \pi k(-1 + i), \quad \text{since } i^2 = -1.$$

So:

$$\boxed{z = \pi k(-1 + i), \quad k \in \mathbb{Z}}$$

Restricting to angles in $(-\pi, \pi]$, we consider only values of k such that:

$$\arg(e^z) = \arg(e^{iz}) \in (-\pi, \pi] \Rightarrow \text{we need } \operatorname{Im}(z) = \pi k \in (-\pi, \pi].$$

Thus $k = -1, 0, 1$. Therefore:

$$\boxed{\begin{array}{l} k = -1 \Rightarrow z = \pi(1-i) \\ \quad k = 0 \Rightarrow z = 0 \\ \quad k = 1 \Rightarrow z = \pi(-1+i) \end{array}}$$

Final boxed form:

$$\boxed{z \in \{\pi(1-i),\ 0,\ \pi(-1+i)\}}$$

(s)

Given:

$$e^{iz} + 4e^{-iz} = 4$$

Multiply both sides by e^{iz}:

$$e^{2iz} + 4 = 4e^{iz} \Rightarrow e^{2iz} - 4e^{iz} + 4 = 0$$

Let $w = e^{iz}$, then:

$$w^2 - 4w + 4 = 0 \Rightarrow (w-2)^2 = 0 \Rightarrow w = 2$$

Now solve:

$$e^{iz} = 2 \Rightarrow iz = \ln 2 + 2\pi ik \Rightarrow z = -i\ln 2 + 2\pi k, \quad k \in \mathbb{Z}$$

$$\boxed{z = 2\pi k - i\ln 2}$$

(t)

Given:

$$e^{2z} + ie^z + 1 = 0$$

Let $w = e^z$, then:

$$w^2 + iw + 1 = 0 \Rightarrow w = \frac{-i \pm \sqrt{i^2 - 4}}{2} = \frac{-i \pm \sqrt{-1-4}}{2} = \frac{-i \pm \sqrt{-5}}{2}$$

$$w = \frac{-i \pm i\sqrt{5}}{2} = i \cdot \frac{-1 \pm \sqrt{5}}{2}$$

Now solve $e^z = w$:

$$z = \ln\left(i \cdot \frac{-1 \pm \sqrt{5}}{2}\right) + 2\pi ik$$

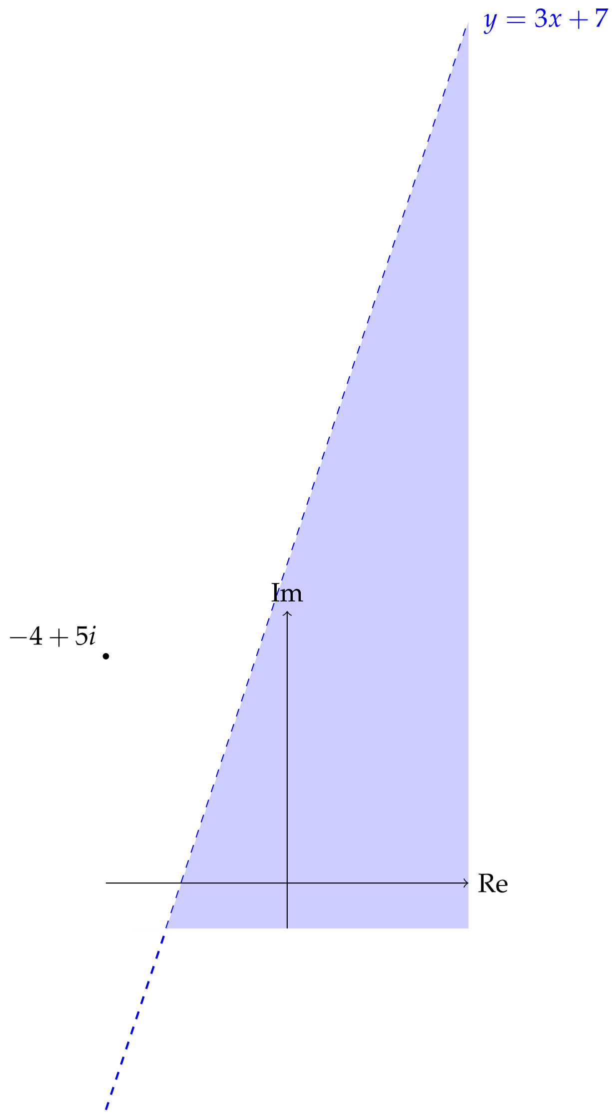
$$\boxed{z = \ln\left(i \cdot \tfrac{-1 \pm \sqrt{5}}{2}\right) + 2\pi ik}$$

Problem 2

Set A

$$A = \{z \in \mathbb{C} : |z - (2+3i)| < |z - (-4+5i)|\},$$

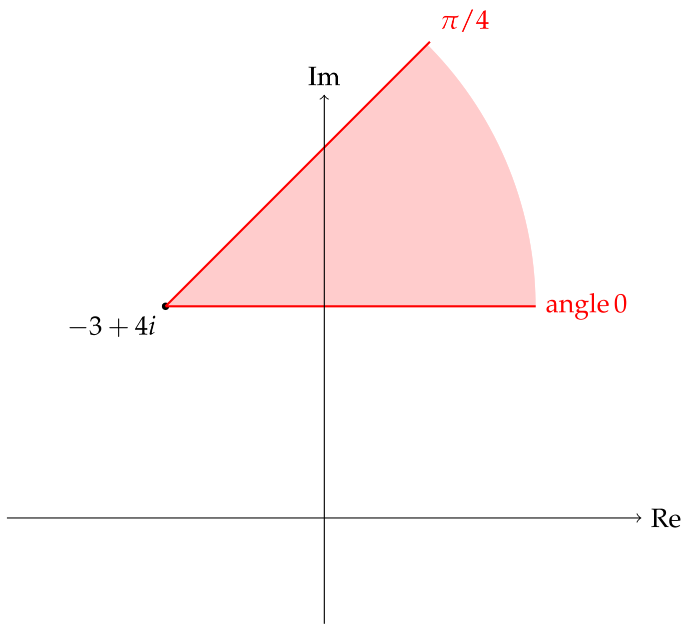
i.e. the half-plane on the side of the perpendicular bisector of $(2,3)$–$(-4,5)$ containing $(2,3)$.



Set B

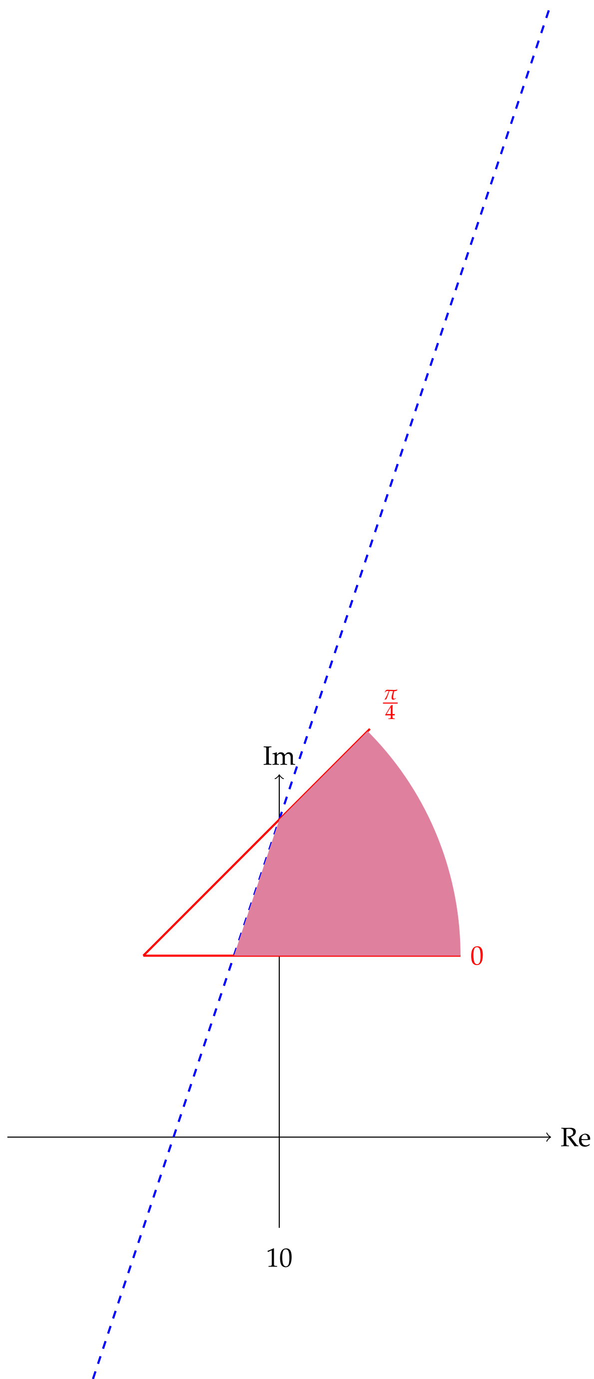
$$B = \{z \in \mathbb{C} : 0 \le \arg(z + 3 - 4i) < \tfrac{\pi}{4}\},$$

i.e. the sector with vertex $-3+4i$, between ray at angle 0 and $\pi/4$.



Intersection $A \cap B$

Shade only the points satisfying both conditions.


$\frac{\pi}{4}$
Im
0
Re

Problem 3

(a)

$$(4\sqrt{3} - 4i)^{88}$$

Let's convert this to polar form

$$r = \sqrt{(4\sqrt{3})^2 + (-4)^2} = \sqrt{48 + 16} = \sqrt{64} = 8$$

$$\theta = tan^{-1}(\frac{-4}{4\sqrt{3}}) = tan^{-1}(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$$

So we want to compute

$$(8e^{-\frac{\pi}{6}i})^{88}$$

$$(8e^{-\frac{\pi}{6}i})^{88} = 8^{88}e^{-\frac{88\pi}{6}i} = 8^{88}e^{-\frac{44\pi}{3}i}$$

(b)

$$z = \left(1 + i\tan\left(\frac{(4m+1)\pi}{4n}\right)\right)^n$$

$$z = \left(1 + i\frac{\sin\left(\frac{(4m+1)\pi}{4n}\right)}{\cos\left(\frac{(4m+1)\pi}{4n}\right)}\right)^n$$

$$z = \left(\frac{\cos\left(\frac{(4m+1)\pi}{4n}\right) + i\sin\left(\frac{(4m+1)\pi}{4n}\right)}{\cos\left(\frac{(4m+1)\pi}{4n}\right)}\right)^n$$

$$z = \left(\frac{1}{\cos\left(\frac{(4m+1)\pi}{4n}\right)}\right)^n \left(\cos\left(\frac{(4m+1)\pi}{4n}\right) + i\sin\left(\frac{(4m+1)\pi}{4n}\right)\right)^n$$

Using de Moivre's theorem

$$z = \frac{1}{\cos\left(\frac{(4m+1)\pi}{4n}\right)^n}\left(\cos\left(\frac{(4m+1)\pi}{4}\right) + i\sin\left(\frac{(4m+1)\pi}{4}\right)\right)$$

$$Re(z) = \frac{1}{\cos\left(\frac{(4m+1)\pi}{4n}\right)^n}\cos\left(\frac{(4m+1)\pi}{4}\right)$$

$$Im(z) = \frac{1}{\cos\left(\frac{(4m+1)\pi}{4n}\right)^n}\sin\left(\frac{(4m+1)\pi}{4}\right)$$

Problem 4

In an ordered field the following must hold $0 < 1$
Assume $i > 0$

$$
\begin{aligned}
0 &< i && \text{(multiply by i)} \\
0 &< i^2 && \\
0 &< -1 && \text{(add 1)} \\
1 &< 0 && \text{this is a contradiction}
\end{aligned}
$$

Assume $i < 0$

$$
\begin{aligned}
0 &< -i && \text{(multiply by -i)} \\
0 &< i^2 && \\
0 &< -1 && \text{(add 1)} \\
1 &< 0 && \text{this is a contradiction}
\end{aligned}
$$

Thus, we have shown that both assumptions lead to a contradiction. Therefore, $(C, +, \cdot)$ is not an ordered field.