

# Mathematics Homework Sheet 7

Author: Abdullah Oguz Topcuoglu & Yousef Farag

## Problem 1

If  $b_n$  is convergent then

$$\liminf_{n \rightarrow \infty} b_n = \limsup_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} b_n$$

From the first inequality we have

$$\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$$

And from the second inequality we have

$$\limsup_{n \rightarrow \infty} (a_n + b_n) \geq \limsup_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$$

Combining these two inequalities we get

$$\limsup_{n \rightarrow \infty} (a_n + b_n) = \limsup_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$$

## Problem 3

$a_n$  is monotonically decreasing and  $\forall n \in N, a_n \geq 0$  and  $\lim_{n \rightarrow \infty} a_n = 0$ .  
We want to show that  $\sum_{n=1}^{\infty} (-1)^n a_n$  converges.

$$\sum_{n=1}^{\infty} (-1)^n a_n = -a_1 + a_2 - a_3 + a_4 - \dots$$

We can write this as

$$\sum_{n=1}^{\infty} (-1)^n a_n = -(a_1 - a_2) - (a_3 - a_4) - \dots$$

Since  $a_n$  is monotonically decreasing, we have

$$-(a_1 - a_2) - (a_3 - a_4) - \dots \leq 0$$

Because every paranthesis group is positive and they have minus sign before them so we are adding zero or negative terms.

Therefore,  $\sum_{n=1}^{\infty} (-1)^n a_n$  is bounded above by 0.

And we can also rewrite the sum as follows:

$$\sum_{n=1}^{\infty} (-1)^n a_n = -a_1 + (a_2 - a_3) + (a_4 - a_5) + \dots$$

Since  $a_n$  is monotonically decreasing, every paranthesis group is positive or zero. We are adding positive or zero elements to  $-a_1$  which means the sum is greater or equal to  $-a_1$

$$-a_1 \leq -a_1 + (a_2 - a_3) + (a_4 - a_5) + \dots$$

Therefore,  $\sum_{n=1}^{\infty} (-1)^n a_n$  is bounded below by  $-a_1$ .  
Let's take a look at odd and even indexed terms:

$$\sum_{n=1}^{\infty} (-1)^{2n} a_{2n} = a_2 + a_4 + a_6 + a_8 + \dots \quad (1)$$

$$\sum_{n=1}^{\infty} (-1)^{2n+1} a_{2n+1} = -a_1 - a_3 - a_5 - a_7 + \dots \quad (2)$$

Since  $a_n \geq 0$ , (2) is monotonically decreasing and similarly (1) is monotonically increasing.

$$s_n = \sum_{k=1}^n (-1)^k a_k$$

$$s_{2n} = \sum_{k=1}^{2n} (-1)^k a_k = (-a_1 + a_2) + (-a_3 + a_4) + \dots$$

$s_{2n}$  is monotonically decreasing because every paranthesis group is negative or zero. So we are adding  $\leq 0$  elements each iteration which means  $s_{2n} \geq s_{2(n+1)}$

$$s_{2n+1} = \sum_{k=1}^{2n+1} (-1)^k a_k = -a_1 + (a_2 - a_3) + (a_4 - a_5) + \dots$$

$s_{2n+1}$  is monotonically increasing because every paranthesis group is positive or zero. So we are adding  $\geq 0$  elements each iteration which means  $s_{2n+1} \leq s_{2(n+1)+1}$

And now it suffices to show that  $s_{2l-1} \leq s_{2l}, \forall l \in N$

$$\begin{aligned}
 s_{2l} &= \sum_{k=1}^{2l} (-1)^k a_k \\
 s_{2l-1} &= \sum_{k=1}^{2l-1} (-1)^k a_k \\
 s_{2l} - s_{2l-1} &= \sum_{k=1}^{2l} (-1)^k a_k - \sum_{k=1}^{2l-1} (-1)^k a_k \\
 &= (-1)^{2l} a_{2l} + \sum_{k=1}^{2l-1} (-1)^k a_k - \sum_{k=1}^{2l-1} (-1)^k a_k \\
 &= (-1)^{2l} a_{2l} \\
 &= a_{2l} \\
 &\geq 0 \quad \text{because } a_n \geq 0 \quad \forall n \in N
 \end{aligned}$$

Therefore,  $s_{2l-1} \leq s_{2l}, \forall l \in N$

Therefore,  $\sum_{n=1}^{\infty} (-1)^n a_n$  converges.

I followed the hint in the homework sheet.