Homework Sheet 6

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Exercise 21

We are given the function

$$f(x_1, x_2) := \begin{pmatrix} \log(1 + x_1^2 + 2x_2^2) \\ x_1 e^{2x_2} \end{pmatrix}.$$

(i)

The jacobian matrix is invertible at the points where the determinant is not zero.
The jacobian matrix:

$$J_f(x_1, x_2) = \begin{pmatrix} \frac{2x_1}{1+x_1^2+2x_2^2} & \frac{4x_2}{1+x_1^2+2x_2^2} \\ \\ e^{2x_2} & 2x_1 e^{2x_2} \end{pmatrix}.$$

The determinant of the jacobian matrix:

$$\begin{aligned} \det(J_f(x_1, x_2)) &= \frac{2x_1 \cdot 2x_1 e^{2x_2}}{1 + x_1^2 + 2x_2^2} - \frac{4x_2 \cdot e^{2x_2}}{1 + x_1^2 + 2x_2^2} \\ &= \frac{4x_1^2 e^{2x_2} - 4x_2 e^{2x_2}}{1 + x_1^2 + 2x_2^2} \\ &= \frac{4e^{2x_2}(x_1^2 - x_2)}{1 + x_1^2 + 2x_2^2}. \end{aligned}$$

The determinant is zero when

$$4e^{2x_2}(x_1^2 - x_2) = 0 \implies x_1^2 - x_2 = 0 \implies x_2 = x_1^2.$$

The set A would be

$$A = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \neq x_1^2\}.$$

(ii)

f is locally invertible at a point if the jacobian matrix is invertible at that point and we know that the jacobian matrix is invertible at each point in A.
So f is locally invertible at each point $x \in A$.

(iii)

To find the jacobian matrix of the local inverse of f at $f(x)$, we can use the formula:

$$J_{f^{-1}}(f(x)) = (J_f(x))^{-1}.$$

We already have $J_f(x)$ so we need to find its inverse.
The jacobian matrix:

$$J_f(x_1, x_2) = \begin{pmatrix} \frac{2x_1}{1+x_1^2+2x_2^2} & \frac{4x_2}{1+x_1^2+2x_2^2} \\ e^{2x_2} & 2x_1 e^{2x_2} \end{pmatrix}.$$

And its determinant:

$$\det(J_f(x_1, x_2)) = \frac{4e^{2x_2}(x_1^2 - x_2)}{1 + x_1^2 + 2x_2^2}.$$

We calculated those in the previous parts.
The inverse of jacobian matrix:

$$\begin{aligned} (J_f(x_1, x_2))^{-1} &= \frac{1}{\det(J_f(x_1, x_2))} \begin{pmatrix} 2x_1 e^{2x_2} & -\frac{4x_2}{1+x_1^2+2x_2^2} \\ -e^{2x_2} & \frac{2x_1}{1+x_1^2+2x_2^2} \end{pmatrix} \\ &= \frac{1 + x_1^2 + 2x_2^2}{4e^{2x_2}(x_1^2 - x_2)} \begin{pmatrix} 2x_1 e^{2x_2} & -\frac{4x_2}{1+x_1^2+2x_2^2} \\ -e^{2x_2} & \frac{2x_1}{1+x_1^2+2x_2^2} \end{pmatrix} \end{aligned}$$

Exercise 22

(i)

We are given the integral

$$\int_A \frac{1}{x_1^3} + \frac{2}{x_2^2}\, d(x_1, x_2)$$

where $A = [1, 3] \times [2, 4]$.
The function $\frac{1}{x_1^3} + \frac{2}{x_2^2}$ is continuos on the set A because it is obviously a C^1 function which implies continuity. It is a C^1 function because all the partial derivatives exist and are continuous.
And continuity of the integrand implies integrability therefore the integral exists.

Since it is integrable we can use the Fubini theorem

$$\begin{aligned}
\int_A \frac{1}{x_1^3} + \frac{2}{x_2^2}\, d(x_1,x_2) &= \int_1^3 \int_2^4 \left(\frac{1}{x_1^3} + \frac{2}{x_2^2}\right) dx_2 dx_1 \\
&= \int_1^3 \left(\int_2^4 \frac{1}{x_1^3} dx_2 + \int_2^4 \frac{2}{x_2^2} dx_2\right) dx_1 \\
&= \int_1^3 \left(\frac{1}{x_1^3}(4-2) + 2\left(-\frac{1}{x_2}\Big|_2^4\right)\right) dx_1 \\
&= \int_1^3 \left(\frac{2}{x_1^3} + 2\left(-\frac{1}{4} + \frac{1}{2}\right)\right) dx_1 \\
&= \int_1^3 \left(\frac{2}{x_1^3} + 2\left(\frac{1}{4}\right)\right) dx_1 \\
&= \int_1^3 \left(\frac{2}{x_1^3} + \frac{1}{2}\right) dx_1 \\
&= 2\left(-\frac{1}{2x_1^2}\Big|_1^3\right) + \frac{1}{2}(3-1) \\
&= 2\left(-\frac{1}{18} + \frac{1}{2}\right) + 1 \\
&= 2\left(\frac{9-1}{18}\right) + 1 = 2\left(\frac{8}{18}\right) + 1 = \frac{8}{9} + 1 = \frac{17}{9}.
\end{aligned}$$

(ii)

We are given the integral

$$\int_A \cos(x_1) x_2^3 \, d(x_1,x_2)$$

where $A = \left[-\frac{\sqrt{\pi}}{2}, \frac{\sqrt{\pi}}{2}\right] \times [-1,1]$.
The function $\cos(x_1)x_2^3$ is continuos on the set A because it is obviously a C^1 function which implies continuity. It is a C^1 function because all the partial derivatives exist and are continuous.
And continuity of the integrand implies integrability therefore the integral exists.

Since it is integrable we can use the Fubini theorem

$$\begin{aligned}
\int_A \cos(x_1)x_2^3 \, d(x_1,x_2) &= \int_{-\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{\pi}}{2}} \int_{-1}^{1} \cos(x_1)x_2^3 \, dx_2 \, dx_1 \\
&= \int_{-\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{\pi}}{2}} \cos(x_1) \left(\int_{-1}^{1} x_2^3 \, dx_2 \right) dx_1 \\
&= \int_{-\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{\pi}}{2}} \cos(x_1) \left(\frac{x_2^4}{4} \Big|_{-1}^{1} \right) dx_1 \\
&= \int_{-\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{\pi}}{2}} \cos(x_1) \left(\frac{1}{4} - \frac{1}{4} \right) dx_1 \\
&= \int_{-\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{\pi}}{2}} \cos(x_1) \cdot 0 \, dx_1 = 0.
\end{aligned}$$

Exercise 23

We are given the set

$$A_{a,b} := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \leq 0, a^2 \leq x_1^2 + x_2^2 \leq b^2\}.$$

and the integral

$$\int_{A_{a,b}} \sqrt{x_1^2 + x_2^2} \, d(x_1, x_2).$$

The set A is intuitively the area between two circles with radius a and b and we only take the half where $x_2 \leq 0$.
The integral exists because the integrand $\sqrt{x_1^2 + x_2^2}$ is continuous on the set $A_{a,b}$ which implies integrability.
We can use the transformation theorem with the polar coordinates transformation

$$g(r, \varphi) = (r\cos(\varphi), r\sin(\varphi))$$

where $r \in [a, b]$ and $\varphi \in [\pi, 2\pi]$ to cover the area where $x_2 \leq 0$.
The jacobian determinant of the transformation is

$$\det(J_g(r, \varphi)) = r.$$

Using the transformation theorem we have

$$\begin{aligned}
\int_{A_{a,b}} \sqrt{x_1^2 + x_2^2}\, d(x_1, x_2) &= \int_a^b \int_\pi^{2\pi} \sqrt{r^2} \cdot r \, d\varphi \, dr \\
&= \int_a^b \int_\pi^{2\pi} r^2 \, d\varphi \, dr \\
&= \int_a^b r^2 \left(\varphi \Big|_\pi^{2\pi} \right) dr \\
&= \int_a^b r^2 (\pi) \, dr \\
&= \pi \left(\frac{r^3}{3} \Big|_a^b \right) \\
&= \pi \left(\frac{b^3 - a^3}{3} \right).
\end{aligned}$$

Exercise 24

We are given the set

$$A_{(c_1,c_2),g} := \{(x_1, x_2) \in \mathbb{R}^2 : (x_1 - c_1)^2 + (x_2 - c_2)^2 \le g\}.$$

and the integral

$$\int_{A_{(c_1,c_2),g}} h \, d(x_1, x_2).$$

where $g, h \in \mathbb{R}^+$.

The set A is just a circle with center (c_1, c_2) and radius $\sqrt{g}$.
The integral exists because the integrand h is continuous on the set $A_{(c_1,c_2),g}$ which implies integrability.
We can use the transformation theorem with the polar coordinates transformation

$$g(r, \varphi) = (r \cos(\varphi) + c_1, r \sin(\varphi) + c_2)$$

where $r \in [0, \sqrt{g}]$ and $\varphi \in [0, 2\pi]$ to cover the area of the circle and shifted to the center (c_1, c_2).
The general formula is

$$\int_{g(A)} f(x) \, d(x) = \int_A f(g(y)) |\det(J_g(y))| \, d(y)$$

The jacobian determinant of the transformation is

$$\det(J_g(r, \varphi)) = r.$$

Using the transformation theorem we have

$$\begin{aligned}
\int_{A_{(c_1,c_2),g}} h\, d(x_1,x_2) &= \int_0^{\sqrt{g}} \int_0^{2\pi} h \cdot r\, d\varphi\, dr \\
&= \int_0^{\sqrt{g}} hr \left(\varphi \Big|_0^{2\pi} \right) dr \\
&= \int_0^{\sqrt{g}} hr(2\pi)\, dr \\
&= 2\pi h \left(\frac{r^2}{2} \Big|_0^{\sqrt{g}} \right) \\
&= 2\pi h \left(\frac{g}{2} - 0 \right) = \pi h g.
\end{aligned}$$