Mathematics Homework Sheet 8

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Problem 2

Let's do elementary row operations.

$$
\begin{aligned}
A_\lambda &= \begin{pmatrix} 1 & \lambda & 0 & 0 \\ \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \end{pmatrix} \\
&= \left(\begin{array}{cccc|cccc} 1 & \lambda & 0 & 0 & 1 & 0 & 0 & 0 \\ \lambda & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \lambda & 1 & 0 & 0 & 0 & 1 \end{array}\right) \\
&\left(\begin{array}{cccc|cccc} 1 & \lambda & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1-\lambda^2 & 0 & 0 & -\lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \lambda & 1 & 0 & 0 & 0 & 1 \end{array}\right) \qquad r_2 = r_2 - \lambda r_1
\end{aligned}
$$

if $\lambda^2 \neq 1$, then we can divide by $1-\lambda^2$ and get the following:

$$A_\lambda = \left(\begin{array}{cccc|cccc} 1 & \lambda & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{\lambda}{1-\lambda^2} & \frac{1}{1-\lambda^2} & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \lambda & 1 & 0 & 0 & 0 & 1 \end{array}\right) \qquad r_2 = r_2/(1-\lambda^2)$$

$$\left(\begin{array}{cccc|cccc} 1 & \lambda & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{\lambda}{1-\lambda^2} & \frac{1}{1-\lambda^2} & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{\lambda^2}{1-\lambda^2} & -\frac{\lambda}{1-\lambda^2} & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 & 0 & 0 & 1 \end{array}\right) \qquad r_3 = r_3 - \lambda r_2$$

$$\left(\begin{array}{cccc|cccc} 1 & \lambda & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{\lambda}{1-\lambda^2} & \frac{1}{1-\lambda^2} & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{\lambda^2}{1-\lambda^2} & -\frac{\lambda}{1-\lambda^2} & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{\lambda^3}{1-\lambda^2} & \frac{\lambda^2}{1-\lambda^2} & 0 & 1 \end{array}\right) \qquad r_4 = r_4 - \lambda r_3$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1+\frac{\lambda^2}{1-\lambda^2} & -\frac{\lambda}{1-\lambda^2} & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{\lambda}{1-\lambda^2} & \frac{1}{1-\lambda^2} & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{\lambda^2}{1-\lambda^2} & -\frac{\lambda}{1-\lambda^2} & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{\lambda^3}{1-\lambda^2} & \frac{\lambda^2}{1-\lambda^2} & 0 & 1 \end{array}\right) \qquad r_1 = r_1 - \lambda r_2$$

So if $\lambda^2 \neq 1$, then we can write the inverse matrix as follows:

$$A_\lambda^{-1} = \begin{pmatrix} 1+\frac{\lambda^2}{1-\lambda^2} & -\frac{\lambda}{1-\lambda^2} & 0 & 0 \\ -\frac{\lambda}{1-\lambda^2} & \frac{1}{1-\lambda^2} & 0 & 0 \\ \frac{\lambda^2}{1-\lambda^2} & -\frac{\lambda}{1-\lambda^2} & 0 & 0 \\ -\frac{\lambda^3}{1-\lambda^2} & \frac{\lambda^2}{1-\lambda^2} & 0 & 1 \end{pmatrix}$$

If $\lambda^2 = 1$, then we have two cases:

- If $\lambda = 1$, then the matrix becomes:

$$A_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

  which is not invertible because the first row and the second row are the same.

- If $\lambda = -1$, then the matrix becomes:

$$A_{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

which is also not invertible because $r_1 = -r_2$.

Problem 3

We know that if determinant is zero if and only if the matrix is not invertible. So lets calculate the determinant.

$$\begin{aligned}
C &= \begin{pmatrix} 13 & 7 & 6 \\ -7 & 1 & 1 \\ 3 & 8 & 7 \end{pmatrix} \\
\det(C) &= 13\det\begin{pmatrix} 1 & 1 \\ 8 & 7 \end{pmatrix} - 7\det\begin{pmatrix} -7 & 1 \\ 3 & 7 \end{pmatrix} + 6\det\begin{pmatrix} -7 & 1 \\ 3 & 8 \end{pmatrix} \\
&= 13(1\cdot 7 - 1\cdot 8) - 7(-7\cdot 7 - 1\cdot 3) + 6(-7\cdot 8 - 1\cdot 3) \\
&= 13(-1) - 7(-49-3) + 6(-56-3) \\
&= 13(-1) - 7(-52) + 6(-59) \\
&= -13 + 364 - 354 \\
&= -13 + 10 = -3
\end{aligned}$$

$\det(C) = -3 \equiv 1 \mod 2$ so if p = 2, then the matrix is invertible.
$\det(C) = -3 \equiv 0 \mod 3$ so if p = 3, then the matrix is not invertible.
$\det(C) = -3 \equiv 2 \mod 5$ so if p = 5, then the matrix is invertible.

Lets calculate the inverse using elementary row operations:

$$\left(\begin{array}{ccc|ccc} 13 & 7 & 6 & 1 & 0 & 0 \\ -7 & 1 & 1 & 0 & 1 & 0 \\ 3 & 8 & 7 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 13 & 7 & 6 & 1 & 0 & 0 \\ 0 & 13/7+7 & 13/7+6 & 1 & 13/7 & 0 \\ 0 & 8.(-13/3)+7 & 7.(-13/3)+6 & 1 & 0 & -13/3 \end{array}\right) r_2 = 13/7r_2 + r_1, r_3 = -13/3r_3 + r_1$$