Homework Sheet 4

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Exercise 13

We are given the function:

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x_1, x_2) := |x_1 - x_2|$$

and the points:

$$a := (0, 1), \quad b := (1, 2)$$

(i)

Choose the G:

$$G := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 - x_2 < 0\}$$

It is obvious that $a, b \in G$.
Now we need to show that the line segment connecting a and b lies in G.

$$\begin{aligned} \text{line segment} &= \{a + t(b - a) \mid t \in [0, 1]\} \\ &= \{(0, 1) + t(1 - 0, 2 - 1) \mid t \in [0, 1]\} \\ &= \{(t, 1 + t) \mid t \in [0, 1]\} \end{aligned}$$

which obviously is in G.

(ii)

When we consider the function f in the domain G, f is equal to

$$f(x_1, x_2) = x_2 - x_1$$

Lets calculate the gradient of f

$$\nabla f(x_1, x_2) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Now we need to find $\theta \in (0, 1)$ such that

$$f(b) - f(a) = \langle \nabla f(\xi), b - a \rangle$$

where $\xi := a + \theta(b - a)$.
Calculating the left hand side:

$$\begin{aligned} f(b) - f(a) &= f(1, 2) - f(0, 1) \\ &= 1 - 1 = 0 \end{aligned}$$

Calculating the right hand side:

$$\begin{aligned}\langle \nabla f(\xi), b-a\rangle &= \langle \nabla f(a+\theta(b-a)), b-a\rangle \\ &= \langle \nabla f((0,1)+\theta(1,1)), (1,1)\rangle \\ &= \langle \nabla f(\theta, 1+\theta), (1,1)\rangle \\ &= \langle \begin{pmatrix}-1\\1\end{pmatrix}, (1,1)\rangle \\ &= 0\end{aligned}$$

We get $0 = 0$ which is true always and doesnt depend on what θ is. So any $\theta \in (0,1)$ satisfies the equation.

(iii)

We are given the points:

$$\tilde{a} := (-1,-1), \quad \tilde{b} := (1,1)$$

No, we cant directly use the part (ii) here because simply the points $\tilde{a}$ and $\tilde{b}$ are not in G we chose. And actually there is no G that contains a point in the diagonal line (where $x_1 = x_2$) because f is not differentiable on that line.

Exercise 14

We are given the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x_1, x_2) := e^{1+x_1} + x_1 \sin(\pi x_2)$$

Taylor polynomial of order 1 and 2 is given by

$$\begin{aligned}T_1^f(x^0; y) &= f(x^0) + \nabla f(x^0) \cdot (y - x^0) \\ T_2^f(x^0; y) &= T_1(x^0; y) + \frac{1}{2}(y-x^0)^T H_f(x^0)(y-x^0)\end{aligned}$$

First and second order ferivaties of f exist and given by

$$\begin{aligned}\frac{\partial f}{\partial x_1} &= e^{1+x_1} + \sin(\pi x_2) \\ \frac{\partial f}{\partial x_2} &= \pi x_1 \cos(\pi x_2) \\ \frac{\partial^2 f}{\partial x_1^2} &= e^{1+x_1} \\ \frac{\partial^2 f}{\partial x_2^2} &= -\pi^2 x_1 \sin(\pi x_2) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} = \frac{\partial^2 f}{\partial x_1 \partial x_2} &= \pi \cos(\pi x_2)\end{aligned}$$

Since all the partial derivaites are addition and multiplication of other continuous functions, they are continous as well.

$T_1^f(x^0; y)$ at $(0, \frac{1}{2})$:

$$f(0, \tfrac{1}{2}) = e^{1+0} + 0 \cdot \sin(\pi \cdot \tfrac{1}{2}) = e$$
$$\nabla f(0, \tfrac{1}{2}) = \begin{pmatrix} e^{1+0} + \sin(\pi \cdot \frac{1}{2}) \\ \pi \cdot 0 \cdot \cos(\pi \cdot \frac{1}{2}) \end{pmatrix} = \begin{pmatrix} e+1 \\ 0 \end{pmatrix}$$
$$T_1^f((0, \tfrac{1}{2}); (y_1, y_2)) = e + \begin{pmatrix} e+1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} y_1 - 0 \\ y_2 - \frac{1}{2} \end{pmatrix} = e + (e+1)y_1$$

$T_2^f(x^0; y)$ at $(0, \frac{1}{2})$:

$$H_f(0, \tfrac{1}{2}) = \begin{pmatrix} e^{1+0} & \pi \cos(\pi \cdot \frac{1}{2}) \\ \pi \cos(\pi \cdot \frac{1}{2}) & -\pi^2 \cdot 0 \cdot \sin(\pi \cdot \frac{1}{2}) \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$$
$$\begin{aligned} T_2^f((0, \tfrac{1}{2}); (y_1, y_2)) &= T_1^f((0, \tfrac{1}{2}); (y_1, y_2)) + \frac{1}{2} \begin{pmatrix} y_1 - 0 & y_2 - \frac{1}{2} \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 - 0 \\ y_2 - \frac{1}{2} \end{pmatrix} \\ &= e + (e+1)y_1 + \frac{1}{2} e (y_1)^2 \end{aligned}$$