Homework Sheet 7

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Exercise 25

We choose the probability space

$$\Omega = \{1, 2, 3, 4, 5, 6\}^3,$$
$$\mathcal{F} = \text{Power set of } \Omega,$$
$$P(A) = U_\Omega$$

The events are

$$A_1 = \{(a, b, c) \in \Omega \mid a + b < 5\},$$
$$A_2 = \{(a, b, c) \in \Omega \mid b \cdot c = 6\},$$
$$A_3 = \{(a, b, c) \in \Omega \mid b = 2\}$$
$$A_1 \cap A_3 = \{(a, 2, c) \in \Omega \mid a < 3\},$$
$$(A_1 \cup A_2) \setminus A_3 = \{(a, b, c) \in \Omega \mid (a + b < 5) \text{ or } (b \cdot c = 6) \text{ and } b \neq 2\}$$

Since P is uniform we count the number of elements in each event.
Calculating the probabilities:

6 many (a,b) pairs times 6 choices for c

$$P(A_1) = \frac{6 \cdot 6}{216} = \frac{36}{216} = \frac{1}{6},$$

4 many (b,c) pairs times 6 choices for a

$$P(A_2) = \frac{6 \cdot 4}{216} = \frac{24}{216} = \frac{1}{9},$$

6 choices for a times 1 choice for b times 6 choices for c

$$P(A_3) = \frac{36}{216} = \frac{1}{6},$$

2 choices for a times 1 choice for b times 6 choices for c

$$P(A_1 \cap A_3) = \frac{2 \cdot 6}{216} = \frac{1}{18},$$

For the last one we need to calculate $|A_1 \cup A_2|$ first:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|,$$
$$|A_1| = 36,$$
$$|A_2| = 24,$$

Calculating $|A_1 \cap A_2|$:

$$\begin{aligned}
&\text{if } b = 1, \; c = 6: \; a + 1 < 5 \Rightarrow a < 4 \Rightarrow 3 \text{ choices} \\
&\text{if } b = 2, \; c = 3: \; a + 2 < 5 \Rightarrow a < 3 \Rightarrow 2 \text{ choices} \\
&\text{if } b = 3, \; c = 2: \; a + 3 < 5 \Rightarrow a < 2 \Rightarrow 1 \text{ choice} \\
&\text{if } b = 6, \; c = 1: \; a + 6 < 5 \Rightarrow \text{no choices.}
\end{aligned}$$

So $|A_1 \cap A_2| = 3 + 2 + 1 + 0 = 6$.

$$|A_1 \cup A_2| = 36 + 24 - 6 = 54,$$
$$P(A_1 \cup A_2) = \frac{54}{216} = \frac{1}{4},$$
$$P((A_1 \cup A_2) \setminus A_3) = P(A_1 \cup A_2) - P(A_3) + P(A_1 \cap A_3) = \frac{1}{4} - \frac{1}{6} + \frac{1}{18} = \frac{5}{36}.$$

Exercise 26

Lets define a set for students

$$S := \{s \in \mathbb{N} \mid s < 30\}$$

Each number represents a student. And note that $|S| = 30$. And the first 7 students belong to tutorial A following 12 students belong to tutorial B and the last 11 students belong to tutorial C.
We choose the probability space

$$\begin{aligned}
\Omega &= \{(a, b, c) \mid a, b, c \in S \text{ and } a \neq b \neq c\}, \\
\mathcal{F} &= \text{Power set of } \Omega, \\
P(A) &= U_\Omega
\end{aligned}$$

We can choose uniform distribution because each student has equal chance of being selected.
The events are:

$$\begin{aligned}
A_1 &= \{(a, b, c) \in \Omega \mid a \in \{7, \dots, 18\}\}, \\
A_2 &= \{(a, b, c) \in \Omega \mid \big|\{p \in \{a, b, c\} \mid p \in \{0, \dots, 6\}\}\big| = 1 \quad \wedge \\
&\quad \big|\{p \in \{a, b, c\} \mid p \in \{7, \dots, 18\}\}\big| = 1 \quad \wedge \\
&\quad \big|\{p \in \{a, b, c\} \mid p \in \{19, \dots, 29\}\}\big| = 1\}
\end{aligned}$$

Here note that $|\Omega| = 30 \cdot 29 \cdot 28$
Calculating the probabilities:

12 choices for the first paper, 29 choices for the second paper and 28 choices for the third paper

$$P(A_1) = \frac{12 \cdot 29 \cdot 28}{|\Omega|} = \frac{12 \cdot 29 \cdot 28}{30 \cdot 29 \cdot 28} = \frac{12}{30} = \frac{2}{5},$$

choose a student from tutorial A (7 choices), tutorial B (12 choices) and tutorial C (11 choices). Then permute them $(3! = 6)$ since order doesnt matter

$$P(A_2) = \frac{7 \cdot 12 \cdot 11 \cdot 6}{|\Omega|} = \frac{7 \cdot 12 \cdot 11 \cdot 6}{30 \cdot 29 \cdot 28} = \frac{5544}{24360} = \frac{231}{1015}.$$

Exercise 27

Lets create a set for the tickets

$$T := B \uplus NCP \uplus TP$$
$$B := \{b \in \mathbb{N} \mid b < 160\}$$
$$NCP := \{n \in \mathbb{N} \mid 160 \leq n < 198\}$$
$$TP := \{t \in \mathbb{N} \mid 198 \leq t < 200\}$$

B is the set of blank tickets, NCP is the set of noncash prizes and TP is the set of top prizes and they are disjoint.

We choose the probability space

$$\Omega = \{X \subseteq T \mid |X| = 10\},$$
$$\mathcal{F} = \text{Power set of } \Omega,$$
$$P(A) = U_\Omega$$

We can choose uniform distribution because each ticket has equal chance of being selected.
The events are:

$$A_1 = \{X \in \Omega \mid |X \cap TP| = 1 \text{ and } |X \cap NCP| \geq 2\},$$
$$A_2 = \{X \in \Omega \mid |X \cap B| < 6\}$$

Here note that $|\Omega| = \binom{200}{10}$
Calculating the probabilities:

choose 1 top prize, choose 2 noncash prize, choose 7 either blank or noncash prizes to make it to 10 total

$$P(A_1) = \frac{\binom{2}{1} \cdot \binom{38}{2} \cdot \binom{196}{7}}{|\Omega|}$$
$$= \frac{\binom{2}{1} \cdot \binom{38}{2} \cdot \binom{196}{7}}{\binom{200}{10}},$$
$$= 53428/431233 \quad \text{link to the calculator: } \texttt{https://www.wolframalpha.com/input?i=choose\%282\%2C1\%}$$

we separetely count when there is k blank tickets where k = 0, 1, 2, 3, 4, 5 and sum them up

$$P(A_2) = \frac{\sum_{k=0}^{5} \binom{160}{k} \cdot \binom{40}{10-k}}{|\Omega|}$$

$$= \frac{1}{\binom{200}{10}} \sum_{k=0}^{5} \binom{160}{k} \cdot \binom{40}{10-k}.$$