Mathematics Homework Sheet 1

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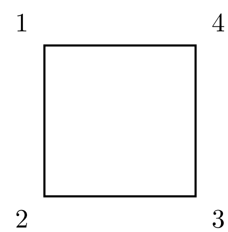
Problem 1

Symmetry group S will consist of rotations and reflections.

- Rotations: R_{90}, R_{180}, R_{270}
- Reflections: T_x, T_y, T_d, $T_{d'}$
- Identity: I



R_i rotates i degrees clockwise.
T_x reflects over the x-axis, T_y reflects over the y-axis, T_d reflects diagonally, and $T_{d'}$ reflects over the other diagonal.
When we take a look at S_4, S_4 has $4! = 24$ elements.
Our group S has 8 elements.
Lets start with identity I.

- ()

Rotations:

- $R_{90} = (1, 2, 3, 4)$
- $R_{180} = (1, 3)(2, 4)$
- $R_{270} = (1, 4, 3, 2)$

Reflections:

- $T_x = (1, 2)(3, 4)$
- $T_y = (1, 4)(2, 3)$
- $T_d = (1, 3)$
- $T_{d'} = (2, 4)$

So, when combined, S can be identified with this subset of S_4:

$$\{(), (1, 2, 3, 4), (1, 3)(2, 4), (1, 4, 3, 2), (1, 2)(3, 4), (1, 4)(2, 3), (1, 3), (2, 4)\}$$

Problem 2

Problem 2(i)

$$f_{a,b}(x) = ax + b$$

$$(G, \diamond) = \{f_{a,b} : a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R}\}, f_{a,b} \diamond f_{c,d} = f_{ac,ad+b}$$

We want to show $(G, \diamond)$ is a group. To do that, we need to show that $(G, \diamond)$ satisfies the properties of group.
Associativity:

$$f_{a,b} \diamond (f_{c,d} \diamond f_{e,f}) = f_{a,b} \diamond f_{ce,cf+d} = f_{ace,acf+ad+b}$$

$$(f_{a,b} \diamond f_{c,d}) \diamond f_{e,f} = f_{ac,ad+b} \diamond f_{e,f} = f_{ace,acf+ad+b}$$

Thus $f_{a,b} \diamond (f_{c,d} \diamond f_{e,f}) = (f_{a,b} \diamond f_{c,d}) \diamond f_{e,f}$.
Existence of a neutral elemenet:

$$f_{1,0} \diamond f_{a,b} = f_{1,0} \diamond f_{a,b} = f_{a,b}$$

$f_{1,0}$ is the neutral element.
Existence of inverses:

$$f_{a,b} \diamond f_{1/a,-b/a} = f_{a*(1/a),(-ab/a)+b} = f_{1,0}$$

Thus, $f_{1/a,-b/a}$ is the inverse of $f_{a,b}$.
Therefore, $(G, \diamond)$ is a group.

Problem 2(ii)

$$H = f_{1,b} : b \in \mathbb{R}$$

We want to show $(H, \diamond)$ is a subgroup of $(G, \diamond)$ which is isomorphic to $(\mathbb{R}, +)$.
We need to show identity element of $(G, \diamond)$ is in H:

$$f_{1,0} \in H$$

We need to show H is closed under $\diamond$ that is $x_1, x_2 \in H \implies x_1.x_2 \in H$:

$$f_{1,b_1} \diamond f_{1,b_2} = f_{1,b_1+b_2}$$

Thus, $f_{1,b_1} \diamond f_{1,b_2} \in H$.
We need to show H is closed under inverses that is $x \in H \implies x^{-1} \in H$:

$$f_{1,b} \diamond f_{1,-b} = f_{1,0}$$

Thus, $f_{1,-b} \in H$.

Problem 3

We are given $(X, .)$ is a group. We are also given that
(i) $e \in X$ satisfies $e.x = x$ for all $x \in X$
(ii) for each $x \in X$, there exists $x^{-1} \in X$ such that $x^{-1}.x = e$

We want to show $x.e = x$ and $x.x^{-1} = e$ for all $x \in X$

Proof of $x.x^{-1} = e$:

$$
\begin{aligned}
x^{-1}.x &= e && \text{given by (ii), multiply by e from right} \\
(x^{-1}.x).e &= e.e && \text{associativity} \\
x^{-1}.(x.e) &= e.e && \text{use (i)} \\
x^{-1}.(x.e) &= e && \text{multiply by x from left} \\
x.(x^{-1}.(x.e)) &= x.e && \text{associativity} \\
(x.x^{-1}).(x.e) &= x.e && \text{associativity} \\
e.(x.e) &= x.e \implies x.x^{-1} = e && \text{use (i)}
\end{aligned}
$$

Problem 4

Definiton of subtraction:

$$a - b = a + (-b) \qquad \forall a, b \in K$$

Thus

$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \left(-\frac{c}{d}\right)$$

Definition of division:

$$\frac{a}{b} = a \cdot b^{-1} \qquad \forall a, b \in K$$

$$\frac{a}{b} = \frac{a \cdot d}{b \cdot d}, \quad \frac{c}{d} = \frac{c \cdot b}{d \cdot b}$$

Multiplying by a value and its inverse does not change the value of the fraction.

$$\frac{a}{b} - \frac{c}{d} = \frac{a \cdot d}{b \cdot d} + \left(-\frac{c \cdot b}{d \cdot b}\right) \qquad \text{(defition of subtraction)}$$

$$= \frac{a \cdot d - c \cdot b}{b \cdot d}$$

We want to show that

$$\frac{\frac{a}{b}}{\frac{d}{c}} = \frac{a.c}{b.d}$$

$$\begin{aligned}
\frac{\frac{a}{b}}{\frac{d}{c}} &= \frac{a}{b} \cdot \left(\frac{d}{c}\right)^{-1} && \text{(definition of division)} \\
&= a.b^{-1}.(d.c^{-1})^{-1} && \text{(defitiniton of division)} \\
&= a.b^{-1}.c.d^{-1} && \text{(inverse of . operation from field axioms)} \\
&= a.c.b^{-1}.d^{-1} && \text{(commutativity)} \\
&= (a.c).(b^{-1}.d^{-1}) && \text{(associativity)} \\
&= \frac{ac}{b.d} && \text{(definition of division)}
\end{aligned}$$

Thus, we have shown that

$$\frac{a}{b} - \frac{c}{d} = \frac{a.d - b.c}{b.d}$$

$$\frac{\frac{a}{b}}{\frac{d}{c}} = \frac{a.c}{b.d}$$

Problem 5

We want to show that

$$S := \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$$

is a subfield of $(\mathbb{R}, +, \cdot)$.

S is closed with respect to + and .:

$$\begin{aligned}
(a + b\sqrt{2}) + (c + d\sqrt{2}) &= (a + c) + (b + d)\sqrt{2} \\
(a + b\sqrt{2})(c + d\sqrt{2}) &= ac + ad\sqrt{2} + bc\sqrt{2} + bd(2) \\
&= (ac + 2bd) + (ad + bc)\sqrt{2}
\end{aligned}$$

S contains the identity elements of + and .:

$$0 = 0 + 0\sqrt{2}$$
$$1 = 1 + 0\sqrt{2}$$

S contains the inverses of + and .:

$$\begin{aligned}
-(a+b\sqrt{2}) &= -a-b\sqrt{2} \\
(a+b\sqrt{2})^{-1} &= \frac{1}{a+b\sqrt{2}} \cdot \frac{a-b\sqrt{2}}{a-b\sqrt{2}} \\
&= \frac{a-b\sqrt{2}}{a^2-2b^2} \\
&= \frac{a}{a^2-2b^2} - \frac{b}{a^2-2b^2}\sqrt{2}
\end{aligned}$$

We have shown that S is closed with respect to + and ., contains the identity elements of + and ., and contains the inverses of + and ..
Therefore, $(S, +, .)$ is a subfield of $(\mathbb{R}, +, \cdot)$.