Mathematics Homework Sheet 3

Author: Abdullah Oguz Topcuoglu & Yousef Farag

Problem 1

Problem 1 (a)

For i=0:

$$f_1(-1) = -1$$
$$f_1(0) = 0$$
$$f_1(1) = 0$$

Meaning that:

$$f_1(\{-1, 0, 1\}) = \{-1, 0, 0\} = \{-1, 0\}$$

Now lets take a look at the inverse:

$$f_1(-1) = -1$$
$$f_1(0) = 0$$
$$f_1(1) = 0$$
$$f_1(2) = 1$$

These are the only values we can get -1, 0, 1. Meaning that:

$$f_1^{-1}(\{-1, 0, 1\}) = \{-1, 0, 1, 2\}$$

Now, for i=2 (the second function):

$$f_2(-1) = -4$$
$$f_2(0) = -3$$
$$f_2(1) = -2$$

Meaning that:

$$f_2(\{-1, 0, 1\}) = \{-4, -3, -2\}$$

Now lets take a look at the inverse:

$$f_2(2) = -1$$
$$f_2(3) = 0$$
$$f_2(4) = 1$$

These are the only values we can get -1, 0, 1. Meaning that:

$$f_2^{-1}(\{-1,0,1\}) = \{2,3,4\}$$

Now, for i=3 (the third function):

$$\begin{aligned} f_3(-1) &= -2 \\ f_3(0) &= 0 \\ f_3(1) &= 2 \end{aligned}$$

Meaning that:

$$f_3(\{-1,0,1\}) = \{-2,0,2\}$$

Now lets take a look at the inverse:

$$\begin{aligned} &f_3(x) = -1, \text{ No such } x \text{ exists in the domain of } f_3 \\ &\qquad \text{In other words, } f_3^{-1}(\{-1\}) = \emptyset \\ &f_3(0) = 0 \\ &f_3(x) = 1, \text{ No such } x \text{ exists in the domain of } f_3 \\ &\qquad \text{In other words, } f_3^{-1}(\{1\}) = \emptyset \end{aligned}$$

These are the only values we can get -1, 0, 1. Meaning that:

$$f_3^{-1}(\{-1,0,1\}) = \{0\}$$

Problem 1 (b)

Lets start with f_1:
f_1 is not injective because it maps 0 and 1 to the same value, that is:

$$f_1(0) = f_1(1) = 0$$

f_1 is surjective because it maps to all the values in the codomain.
Lets take an element from the codomain $z \in Z$.
If $z \leq 0$ then we can find an element in the domain of f_3, $x \in Z$ that maps to z. Simply $x = z$ works.
If $z > 0$ then we can find an element in the domain of f_3, $x \in Z$ that maps to z. Simply $x = z + 1$ works.

Now, lets take a look at f_2:

f_2 is injective because it maps each element in the domain to a unique element in the codomain, that is:

$$f_2(x) = f_2(y) \Rightarrow x = y$$

$$f_2(x) = x - 3, \quad f_2(y) = y - 3$$
$$x - 3 = y - 3 \Rightarrow x = y$$

f_2 is surjective because it maps to all the values in the codomain.
Lets take an element from the codomain of f_2, $z \in Z$.
We cand find an element in the domain of f_2, $x \in Z$ that maps to z. Simply $x = z + 3$ works.

Now, lets take a look at f_3:
f_3 is injective because it maps each element in the domain to a unique element in the codomain, that is:

$$f_3(x) = f_3(y) \Rightarrow x = y$$

$$f_3(x) = 2x, \quad f_3(y) = 2y$$
$$2x = 2y \Rightarrow x = y$$

f_3 is not surjective because it does not map to all the values in the codomain. For example there is no element in the domain of f_3 that maps to 1. Generally all the odd numbers are not in the range of f_3.

Problem 1 (c)

By looking at Problem 1 (b),
We see that f_1 is not injective thus f_1 is not bijective.
We see that f_2 is injective and surjective thus f_2 is bijective.
We see that f_3 is not surjective thus f_3 is not bijective.

Inverse of f_2:

$$f_2 : Z \to Z, \quad x \rightsquigarrow x - 3$$
$$f_2^{-1} : Z \to Z, \quad x \rightsquigarrow x + 3$$

Problem 2

Proof of Problem 2(a)
We are given that $f_s : A \to B$ is a surjective function, and we want to show that if $g \circ f_s = h \circ f_s$, then $g = h$.

1. **Assume** $g \circ f_s = h \circ f_s$, such that for every $x \in A$ we have:
$$g(f_s(x)) = h(f_s(x)).$$
2. Since f_s is surjective, for every $y \in B$, there exists some $x \in A$ such that $f_s(x) = y$.
3. By substituting $f_s(x) = y$ into the equation $g(f_s(x)) = h(f_s(x))$, we get:
$$g(y) = h(y).$$
This equality is true for all $y \in B$, since f_s is surjective and thus covers all elements of B.

Thus, the implication $(g \circ f_s = h \circ f_s) \Rightarrow g = h$ holds true when f_s is surjective.

Problem 2 (b)

We are given that $f_i : C \to D$ is an injective function, and we want to show that if $f_i \circ g = f_i \circ h$, then $g = h$.

1. **Assume** $f_i \circ g = f_i \circ h$ such that for every $x \in B$ we have:
$$f_i(g(x)) = f_i(h(x))$$
2. Since f_i is injective thus for each $y1, y2 \in C$, $f_i(y1) = f_i(y2) \Rightarrow\ y1 = y2$
3. By substituting $g(x), h(x)$ into 2 we get
$$f_i(g(x)) = f_i(h(x)) \Rightarrow\ g(x) = h(x)$$
from 1 and 2 we get that $g = h$ for all $x \in B$

Problem 3

Problem 3 (a)

Lets assume that opposite of the statement is true, that is there are more than one identity element. We are going to try to find a contradiction.
Lets name them e_1 and e_2.
We know that both identity elements are in the set:

$$e_1 \in F$$
$$e_2 \in F$$

$$
\begin{aligned}
e_1 &= e_1 \cdot e_2 && \text{(By definition of identity element } e_2\text{)} && (1)\\
&= e_2 \cdot e_1 && \text{(By commutative property of multiplication)} && (2)\\
&= e_2 && \text{(By definition of identity element } e_1\text{)} && (3)
\end{aligned}
$$

So, this is a contradiction, we assumed there are two different identity elements but they turned out to be the same element. Thus, there can be only one identity element in a field.

Problem 3 (b)

$$\forall a, b, c \in F \quad a + c = b + c \implies a = b$$

Lets go step by step:

$$
\begin{aligned}
a &= a + 0 && \text{Identity element of addtiion} && (4)\\
&= a + (c - c) && \text{Inverse element of addition} && (5)\\
&= (a + c) - c && \text{Associative property of addition} && (6)\\
&= (b + c) - c && \text{Given (a+c = b+c)} && (7)\\
&= b + (c - c) && \text{Associative property of addition} && (8)\\
&= b + 0 && \text{Inverse element of addition} && (9)\\
&= b && \text{Identity element of addition} && (10)
\end{aligned}
$$

So, we have proven that if $a + c = b + c$ then $a = b$.