Mathematics Homework Sheet 4

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Problem 1

We want to prove

$$(\forall n \in N) \wedge (x \in R) \wedge (x \geq -1) \qquad (1+x)^n \geq 1 + nx$$

by using mathematical induction.
Base Case: For $n = 1$, we have

$$(1+x)^1 = 1 + x \geq 1 + 1 \cdot x$$

Which is true for every $x \in R$ so it means it is also true for $x \in [-1, \infty]$

Inductive step:
We assume that the statement is true for $n = k$, i.e.

$$(1+x)^k \geq 1 + kx$$

Multiply both sides by $1 + x$, since $1 + x > 0$ because $x \in [-1, \infty]$, we have

$$(1+x)^k(1+x) \geq (1+kx)(1+x)$$
$$(1+x)^{k+1} \geq 1 + kx + x + kx^2$$
$$0 \geq -kx^2$$

Add these two together, we get

$$(1+x)^{k+1} \geq 1 + (k+1)x$$

And this completes the proof.

Give a counterexample to show that the condition x $\geq$ -1 is necessary:
Let's take $x = -2$ and $n = 2$. Then we have

$$(1-2)^2 \geq 1 + 2 \cdot (-2)$$
$$(-1)^2 \geq 1 - 4$$
$$1 \geq -3$$

Which is not true. In fact it would be false when n is even. So the condition $x \geq -1$ is necessary. Because that way $1 + x$ is never negative

Problem 2

Problem 2 (a)

$$X_1 := \{x \in R : x^2 - 2x \leq 0\}$$

What x values satisfy this condition?

$$x^2 - 2x \leq 0$$
$$x(x-2) \leq 0$$

In order this inequality to be satisfied the signs of x and $x - 2$ must be different or one of them needs to be zero, and this only happens when $0 \leq x \leq 2$.
So this means:

$$X_1 = [0, 2]$$

In this case X_1 is bounded from below and above.

$$supX_1 = 2$$
$$infX_1 = 0$$

And $supX_1 \in X_1$ which means $supX_1$ is also the maximum value.
$infX_1 \in X_1$ which means $infX_1$ is also the minimum value.

Problem 2 (b)

$$X_2 := \{x \in R \setminus \{0\} : 5 - x^2 > \frac{4}{x^2}\}$$

What x values satisfy this condition?

$$5 - x^2 > \frac{4}{x^2}$$
$$5 - x^2 - \frac{4}{x^2} > 0$$
$$\frac{5x^2 - x^4 - 4}{x^2} > 0$$

Since x^2 is always positive, we can multiply both sides by x^2.

$$\begin{aligned}5x^2 - x^4 - 4 &> 0\\ x^4 - 5x^2 + 4 &< 0\\ (x^2-4)(x^2-1) &< 0\\ (x-2)(x+2)(x-1)(x+1) &< 0\end{aligned}$$

So, this inequality is satisfied when $-2 < x < -1 \quad \vee \quad 1 < x < 2$.
So this means:

$$X_2 = (-2,-1) \cup (1,2)$$

In this case X_2 is bounded from below and above.

$$\begin{aligned}&supX_2 = 2\\ &infX_2 = -2\end{aligned}$$

And $supX_2 \notin X_2$ which means $supX_2$ is not the maximum value.
$infX_2 \notin X_2$ which means $infX_2$ is not the minimum value.
Maximum and minimum values are not in the set.

Problem 3

We say that x' is an supremum of Y if $\forall x \in Y, x' > x$
so given that $supY$ exists for set Y we can take an element $x \in Y$ such that we know that the following relation holds true for $\forall y \in Y$ because of the existence of a supremum

$$\forall y \in Y(supY > y) \tag{1}$$

Now according to the second property of ordered fields

$$\forall a,b,c \in F : (a \leq b) \wedge (c \leq 0) \implies a.c \geq b.c \tag{2}$$

let a = y, c = -1, b = supY from 1 we know that $a < b$ and we know that $-1 < 0$ thus using 2 we can conclude

$$-1.y > -1.supY$$

from the 9th property of fields we can conclude

$$-y > -supY \tag{3}$$

from 1 we know that 3 holds true for all y ∈ Y and thus by the defintion of the infimum, the infimum of the set -Y exists and it is equal to -supY

Problem 4

Problem 4 (a)

We want to prove

$$\forall x, y \in R \quad |x+y| \leq |x| + |y|$$

Let's continue with this inequality

$$\begin{aligned} &\forall x, y \in R \quad x \leq |x|,\ y \leq |y|,\ -x \leq |x|,\ -y \leq |y| \\ &\qquad x + y \leq |x| + |y| && \text{(Considering the first two inequalities above)} \\ &\qquad -x - y \leq |x| + |y| && \text{(Considering the last two inequalities above)} \end{aligned}$$

$(x+y)$ and $(-x-y)$ is nothing but two possible outcomes of $|x+y|$
So, we have

$$|x+y| \leq |x| + |y|$$

And this completes the proof.

Problem 4 (b)

Let us consider the following 4 cases:
Assume $x \geq y$:
When y is positive, the following is true:

$$|x| - |y| = x - y \tag{1}$$

When y is negative, the following is true:

$$|x| - |-y| = x - y \quad \wedge \quad x - (-y) = x + y$$

From above, we can conclude:

$$|x| - |-y| \leq x - (-y)$$

$$\begin{aligned} &\text{and both sides of the inequality are} \quad \geq 0 \quad \text{due to the assumption } x \geq y && (2) \\ &|x| - |-y| \leq x - (-y) \geq 0 && (2) \end{aligned}$$

Now let us assume that $x < y$:
when x is positive the following is true:

$$|x| - |y| = x - y \tag{3}$$

when x is negative the following is true:

$$|-x| - |y| = x - y \ \wedge\ -x - y = -(x+y)\ \wedge\ |-(x+y)| \geq |-x| - |y| \tag{4}$$

from 1, 2, 3, 4 we can conclude that

$$||x| - |y|| \leq |x - y| \quad \text{for all } x, y \in \mathbb{R}.$$