Mathematics Homework Sheet 7

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Problem 1

A transformation T is linear if and only if it satisfies the following two properties for all vectors u, v and scalar c:

1. $T(u + v) = T(u) + T(v)$
2. $T(cu) = cT(u)$

1. $T : \mathbb{R}^2 \to \mathbb{R}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto x + 2y$ is linear.
2. $T : \mathbb{R}^2 \to \mathbb{R}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto x + y^2$ is not linear.
   Because rule (1) is not satisfied.
3. $T : \mathbb{R}^2 \to \mathbb{R}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto xy$ is not linear.
   Because rule (2) is not satisfied.
4. $T : \mathbb{C} \to \mathbb{C}, z \mapsto \overline{z}$ is linear.
5. $T : \mathbb{R}^2 \to \mathbb{R}^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+1 \\ y-1 \end{pmatrix}$ is not linear.
   Because rule (1) is not satisfied.
6. $T : \mathbb{R}^2 \to \mathbb{R}^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x-y \\ x+2y \end{pmatrix})$ is linear.
7. $T : \mathbb{R}^n[x] \to \mathbb{R}, p(x) \mapsto p(1)$ is linear.
8. $T : \mathbb{R}^n[x] \to \mathbb{R}^{n+2}[x], p(x) \mapsto x^2 p(x)$ is linear.

Problem 2

(a)

The way we construct the matrix given a linear transformation is that we take the each basis vector and apply the transformation to it and write the result to corresponding column of the matrix.

$$
\begin{aligned}
T(e_1) &= T(1,0,0) = (1-0, 1+2\cdot 0-0, 2\cdot 1+0+0) = (1,1,2) \\
T(e_2) &= T(0,1,0) = (0-1, 0+2\cdot 1-0, 2\cdot 0+1+0) = (-1,2,1) \\
T(e_3) &= T(0,0,1) = (0-0, 0+2\cdot 0-1, 2\cdot 0+0+1) = (0,-1,1)
\end{aligned}
$$

Put each result as a column in the matrix A:

$$
A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 2 & 1 & 1 \end{pmatrix}
$$

(b)

The standard basis looks like this: $\{1, x, ..., x^n\}$

$$
\begin{aligned}
T(1) &= 1 \\
T(x) &= x+1 \\
T(x^2) &= (x+1)^2 = x^2+2x+1 \\
&\vdots \\
T(x^k) &= (x+1)^k = \sum_{i=0}^{k} \binom{k}{i} x^i \\
&\vdots \\
T(x^n) &= (x+1)^n
\end{aligned}
$$

$$
m_{i,j} = \begin{cases} \binom{j}{i} & \text{if } 0 \le i \le j \le n \\ 0 & \text{if } i > j \end{cases}
$$

Each column of the matrix has the coefficients of a binomial expansion of $(x+1)^k$ for $k = 0, 1, \ldots, n$. For example the first column looks like this:

$$
\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}
$$

The second column looks like this:

$$
\begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}
$$

(c)

The standard basis of $\mathbb{R}^{2\times 2}$ is:

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

We compute $T(E_{ij})$ for each basis element:

$$\begin{aligned}
T(E_{11}) &= T\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = E_{11} \\
T(E_{12}) &= T\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 2E_{12} \\
T(E_{21}) &= T\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix} = 3E_{21} \\
T(E_{22}) &= T\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} = 4E_{22}
\end{aligned}$$

Thus, the matrix of T with respect to the basis $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ is:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

Problem 3

$$M_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad (3\times 1), \qquad M_2 = \begin{pmatrix} 2 & -1 & 3 \end{pmatrix} \quad (1\times 3), \qquad M_3 = \begin{pmatrix} 4 & 1 \\ 1 & 3 \\ -1 & 1 \end{pmatrix} \quad (3\times 2),$$

$$M_4 = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} \quad (2\times 2), \qquad M_5 = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 1 & -1 \end{pmatrix} \quad (2\times 3), \qquad M_6 = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 4 & 3 \end{pmatrix} \quad (3\times 3).$$

We can compute M_iM_j only when the number of columns of M_i equals the number of rows of M_j.

- $M_1(3\times 1) \times M_2(1\times 3) \to (3\times 3)$:

$$M_1M_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 0 & 0 \\ 2 & -1 & 3 \end{pmatrix}$$

- $M_2(1\times 3) \times M_1(3\times 1) \to (1\times 1)$:

$$M_2M_1 = 2\cdot 1 + (-1)\cdot 0 + 3\cdot 1 = 5$$

- $M_2(1 \times 3) \times M_3(3 \times 2) \to (1 \times 2)$:

$$M_2M_3 = \begin{pmatrix} 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 1 & 3 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 \cdot 4 + (-1) \cdot 1 + 3 \cdot (-1) & 2 \cdot 1 + (-1) \cdot 3 + 3 \cdot 1 \end{pmatrix} = \begin{pmatrix} 8 - 1 - 3 & 2 \end{pmatrix}$$

- $M_2(1 \times 3) \times M_6(3 \times 3) \to (1 \times 3)$:

$$M_2M_6 = \begin{pmatrix} 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 + (-1) \cdot 2 + 3 \cdot 3, & 2 \cdot 2 + (-1) \cdot 3 + 3 \cdot 4, & 2 \cdot 1 + (-1) \cdot 2 + \end{pmatrix}$$

- $M_3(3 \times 2) \times M_4(2 \times 2) \to (3 \times 2)$:

$$M_3M_4 = \begin{pmatrix} 4 & 1 \\ 1 & 3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 4 \cdot 4 + 1 \cdot 2 & 4 \cdot 1 + 1 \cdot 3 \\ 1 \cdot 4 + 3 \cdot 2 & 1 \cdot 1 + 3 \cdot 3 \\ -1 \cdot 4 + 1 \cdot 2 & -1 \cdot 1 + 1 \cdot 3 \end{pmatrix} = \begin{pmatrix} 16 + 2 & 4 + 3 \\ 4 + 6 & 1 + 9 \\ -4 + 2 & -1 + 3 \end{pmatrix} = \begin{pmatrix} 18 & 7 \\ 10 & 10 \\ -2 & 2 \end{pmatrix}$$

- $M_3(3 \times 2) \times M_5(2 \times 3) \to (3 \times 3)$:

$$M_3M_5 = \begin{pmatrix} 4 & 1 \\ 1 & 3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ 3 & 1 & -1 \end{pmatrix}$$

  Compute each entry:

$$\begin{pmatrix} 4 \cdot 2 + 1 \cdot 3 & 4 \cdot (-1) + 1 \cdot 1 & 4 \cdot 1 + 1 \cdot (-1) \\ 1 \cdot 2 + 3 \cdot 3 & 1 \cdot (-1) + 3 \cdot 1 & 1 \cdot 1 + 3 \cdot (-1) \\ -1 \cdot 2 + 1 \cdot 3 & -1 \cdot (-1) + 1 \cdot 1 & -1 \cdot 1 + 1 \cdot (-1) \end{pmatrix} = \begin{pmatrix} 8 + 3 & -4 + 1 & 4 - 1 \\ 2 + 9 & -1 + 3 & 1 - 3 \\ -2 + 3 & 1 + 1 & -1 - 1 \end{pmatrix} = \begin{pmatrix} 11 & -3 & 3 \\ 11 & 2 & - \\ 1 & 2 & - \end{pmatrix}$$

- $M_4(2 \times 2) \times M_5(2 \times 3) \to (2 \times 3)$:

$$M_4M_5 = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ 3 & 1 & -1 \end{pmatrix}$$

  Compute each entry:

$$\begin{pmatrix} 4 \cdot 2 + 1 \cdot 3 & 4 \cdot (-1) + 1 \cdot 1 & 4 \cdot 1 + 1 \cdot (-1) \\ 2 \cdot 2 + 3 \cdot 3 & 2 \cdot (-1) + 3 \cdot 1 & 2 \cdot 1 + 3 \cdot (-1) \end{pmatrix} = \begin{pmatrix} 8 + 3 & -4 + 1 & 4 - 1 \\ 4 + 9 & -2 + 3 & 2 - 3 \end{pmatrix} = \begin{pmatrix} 11 & -3 & 3 \\ 13 & 1 & -1 \end{pmatrix}$$

- $M_5(2 \times 3) \times M_1(3 \times 1) \to (2 \times 1)$:

$$M_5M_1 = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 + (-1) \cdot 0 + 1 \cdot 1 \\ 3 \cdot 1 + 1 \cdot 0 + (-1) \cdot 1 \end{pmatrix} = \begin{pmatrix} 2 + 0 + 1 \\ 3 + 0 - 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

- $M_5(2 \times 3) \times M_3(3 \times 2) \to (2 \times 2)$:

$$M_5M_3 = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 1 & 3 \\ -1 & 1 \end{pmatrix}$$

  Compute each entry:

$$\begin{pmatrix} 2 \cdot 4 + (-1) \cdot 1 + 1 \cdot (-1) & 2 \cdot 1 + (-1) \cdot 3 + 1 \cdot 1 \\ 3 \cdot 4 + 1 \cdot 1 + (-1) \cdot (-1) & 3 \cdot 1 + 1 \cdot 3 + (-1) \cdot 1 \end{pmatrix} = \begin{pmatrix} 8 - 1 - 1 & 2 - 3 + 1 \\ 12 + 1 + 1 & 3 + 3 - 1 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 14 & 5 \end{pmatrix}$$

- $M_5(2 \times 3) \times M_6(3 \times 3) \to (2 \times 3)$:

$$M_5M_6 = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 4 & 3 \end{pmatrix}$$

  Compute each entry:

$$\begin{pmatrix} 2 \cdot 1 + (-1) \cdot 2 + 1 \cdot 3 & 2 \cdot 2 + (-1) \cdot 3 + 1 \cdot 4 & 2 \cdot 1 + (-1) \cdot 2 + 1 \cdot 3 \\ 3 \cdot 1 + 1 \cdot 2 + (-1) \cdot 3 & 3 \cdot 2 + 1 \cdot 3 + (-1) \cdot 4 & 3 \cdot 1 + 1 \cdot 2 + (-1) \cdot 3 \end{pmatrix} = \begin{pmatrix} 2 - 2 + 3 & 4 - 3 + 4 & 2 - 2 \\ 3 + 2 - 3 & 6 + 3 - 4 & 3 + 2 \end{pmatrix}$$

- $M_6(3 \times 3) \times M_1(3 \times 1) \to (3 \times 1)$:

$$M_6M_1 = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 2 \cdot 0 + 1 \cdot 1 \\ 2 \cdot 1 + 3 \cdot 0 + 2 \cdot 1 \\ 3 \cdot 1 + 4 \cdot 0 + 3 \cdot 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$$

- $M_6(3 \times 3) \times M_3(3 \times 2) \to (3 \times 2)$:

$$M_6M_3 = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 4 & 3 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 1 & 3 \\ -1 & 1 \end{pmatrix}$$

  Compute each entry:

$$\begin{pmatrix} 1 \cdot 4 + 2 \cdot 1 + 1 \cdot (-1) & 1 \cdot 1 + 2 \cdot 3 + 1 \cdot 1 \\ 2 \cdot 4 + 3 \cdot 1 + 2 \cdot (-1) & 2 \cdot 1 + 3 \cdot 3 + 2 \cdot 1 \\ 3 \cdot 4 + 4 \cdot 1 + 3 \cdot (-1) & 3 \cdot 1 + 4 \cdot 3 + 3 \cdot 1 \end{pmatrix} = \begin{pmatrix} 4 + 2 - 1 & 1 + 6 + 1 \\ 8 + 3 - 2 & 2 + 9 + 2 \\ 12 + 4 - 3 & 3 + 12 + 3 \end{pmatrix} = \begin{pmatrix} 5 & 8 \\ 9 & 13 \\ 13 & 18 \end{pmatrix}$$

All other products M_iM_j are undefined (inner dimensions mismatch).