Mathematics Homework Sheet 9

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Problem 1

From the given hint i observe that the columns of the matrix is just the images of the standard basis vectors. And since e_2 and e_3 are in the kernel, second and third columns of D are all zeros.

$$D = \begin{pmatrix} x & 0 & 0 & x \\ x & 0 & 0 & x \\ x & 0 & 0 & x \\ x & 0 & 0 & x \end{pmatrix}$$

And also from the hint i observe that the image is just span of the column vectors of the matrix. So i will just plug the given vectors in the image of D into the columns of D

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Problem 2

We can write the equations in matrix form as follows:

$$\begin{pmatrix} 2 & -3 & -1 & 1 \\ 3 & 4 & -4 & -3 \\ 0 & 17 & -5 & -9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

To find the nontrivial solutions, we can row reduce the matrix to echelon form.

$$\begin{pmatrix} 2 & -3 & -1 & 1 \\ 3 & 4 & -4 & -3 \\ 0 & 17 & -5 & -9 \end{pmatrix} r_2 = 2r_2 - 3r_1$$
$$\begin{pmatrix} 2 & -3 & -1 & 1 \\ 0 & 17 & -5 & -9 \\ 0 & 17 & -5 & -9 \end{pmatrix} r_3 = r_3 - r_2$$
$$\begin{pmatrix} 2 & -3 & -1 & 1 \\ 0 & 17 & -5 & -9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Which gives us the following system of equations:

$$\begin{aligned} 2x_1 - 3x_2 - x_3 + x_4 &= 0 \\ 17x_2 - 5x_3 - 9x_4 &= 0 \end{aligned}$$

Let $x_3 = \lambda$ and $x_4 = \mu$ then we can express x_1 and x_2 in terms of λ and μ:

$$x_2 = \frac{5\lambda + 9\mu}{17}$$

$$x_1 = \frac{3x_2 + x_3 - x_4}{2} = \frac{3\left(\frac{5\lambda+9\mu}{17}\right) + \lambda - \mu}{2}$$

Thus, the general solution can be expressed as:

$$\begin{pmatrix} \frac{3\left(\frac{5\lambda+9\mu}{17}\right)+\lambda-\mu}{2} \\ \frac{5\lambda+9\mu}{17} \\ \lambda \\ \mu \end{pmatrix} \lambda, \mu \in \mathbb{R}$$

Problem 3

$$\det \begin{pmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{pmatrix}$$

Expand along the first row:

$$= (a^2+1)\det\begin{pmatrix} b^2+1 & bc \\ bc & c^2+1 \end{pmatrix} - ab\det\begin{pmatrix} ab & ac \\ ac & c^2+1 \end{pmatrix} + ac\det\begin{pmatrix} ab & b^2+1 \\ ac & bc \end{pmatrix}$$
$$= (a^2+1)((b^2+1)(c^2+1) - b^2c^2) - ab(ab(c^2+1) - a^2c^2) + ac(ab^2c - ac(b^2+1))$$
$$= (a^2+1)(b^2+1)(c^2+1) - (a^2+1)b^2c^2 - a^2b^2(c^2+1) + a^3bc^2 + a^2b^2c^2 - a^2c^2(b^2+1)$$
$$= (a^2b^2 + a^2 + b^2 + 1)(c^2+1) - (a^2+1)b^2c^2 - a^2b^2(c^2+1) + a^3bc^2 + a^2b^2c^2 - a^2c^2(b^2+1)$$
$$= a^2b^2c^2 + a^2b^2 + a^2c^2 + a^2 + b^2c^2 + b^2 + c^2 + 1 - (a^2+1)b^2c^2 - a^2b^2(c^2+1) + a^3bc^2 + a^2b^2c^2 - a^2c^2(b^2+1)$$
$$= a^2 + b^2 + c^2 + 1$$

Problem 5

If we can find a basis consistiong of eigenvectors for the matrix, then the matrix is diagonalizable. Lets first calculate the eigenvalues:

$$\det\begin{pmatrix} -3-\lambda & 4 \\ -1 & 1-\lambda \end{pmatrix} = (-3-\lambda)(1-\lambda) + 4 = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2$$

$$E_{-1} = \ker \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix} = \ker \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$$

Which is not enough to form a basis because the dimension of the eigenspace is 1, but the matrix is 2×2. So the first matrix is not diagonalizable over $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.

The second matrix:

$$\det \begin{pmatrix} -3-\lambda & 1 \\ -2 & -1-\lambda \end{pmatrix} = (-3-\lambda)(-1-\lambda) + 2 = \lambda^2 + 4\lambda + 5$$

The discriminant of the characteristic polynomial is -4, which means the eigenvalues are complex. And since the eigenvalues are dsitinct, and we get at least one eigenvector per eigenvalue, we can find two linearly independent eigenvectors. Which means that the second matrix is diagonalizable over $\mathbb{C}$ but not over $\mathbb{R}$ or $\mathbb{Q}$.

The third matrix:

$$\det \begin{pmatrix} -3-\lambda & 3 \\ 2 & -2-\lambda \end{pmatrix} = (-3-\lambda)(-2-\lambda) - 6 = \lambda^2 + 5\lambda + 0 = \lambda(\lambda+5)$$

We get two distinct eigenvalues, $\lambda_1 = 0$ and $\lambda_2 = -5$. The matrix is diagonalizable over $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$.

The fourth matrix:

$$\det \begin{pmatrix} 2-\lambda & -1 \\ -1 & 3-\lambda \end{pmatrix} = (2-\lambda)(3-\lambda) - 1 = \lambda^2 - 5\lambda + 5$$

Solving the quadratic equation yields to $\lambda = \frac{5 \pm \sqrt{5}}{2}$, which are distinct real eigenvalues. The matrix is diagonalizable over $\mathbb{R}$ and $\mathbb{C}$ but not over $\mathbb{Q}$ since the eigenvalues are not rational.