Mathematics Homework Sheet 2

Author: Abdullah Oguz Topcuoglu & Yousef Farag

Problem 1

Inductive set $M \subseteq R$ means that

$$1 \in M$$

$$\forall n \in M \implies n + 1 \in M$$

We want to show that the set $\bigcap_{i \in I} M_i$ is inductive. We know that $\forall i \in I, M_i$ is inductive.
We need to show two things:

- $1 \in \bigcap_{i \in I} M_i$
- If $n \in \bigcap_{i \in I} M_i$, then $n + 1 \in \bigcap_{i \in I} M_i$

Let's start with the first one:
Is $1 \in \bigcap_{i \in I} M_i$?
Yes because $\forall i \in I, 1 \in M_i$.

Now let's move on to the second one:
Pick an element $n \in \bigcap_{i \in I} M_i$.
This means that $\forall i \in I, n \in M_i$.
By definition of inductive set, $\forall i \in I, n + 1 \in M_i$.
Which means that $n + 1 \in \bigcap_{i \in I} M_i$.

Thus, $\bigcap_{i \in I} M_i$ is inductive.

Problem 2

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

is what we want to prove using mathematical induction
We need to do two things:

- Prove the base case
- Prove the inductive step

Let's start with the base case:
Insert n = 0 into the equation:

$$\sum_{k=0}^{0} k^2 = 0^2 = 0 = \frac{0(0+1)(2*0+1)}{6} = 0$$

Now the inductive step:
Assume that the equation holds for n = m, that is:

$$\sum_{k=0}^{m} k^2 = \frac{m(m+1)(2m+1)}{6}$$

Now we need to prove that the equation holds for n = m + 1:

$$\sum_{k=0}^{m+1} k^2 = \frac{(m+1)(m+2)(2m+3)}{6}$$

$$\begin{aligned}
\sum_{k=0}^{m+1} k^2 &= \sum_{k=0}^{m} k^2 + (m+1)^2 \\
&= \frac{m(m+1)(2m+1)}{6} + (m+1)^2 \qquad \text{(This step uses the induction assumption)} \\
&= \frac{m(m+1)(2m+1) + 6(m+1)^2}{6} \\
&= \frac{(m+1)(m(2m+1) + 6(m+1))}{6} \\
&= \frac{(m+1)(2m^2 + m + 6m + 6)}{6} \\
&= \frac{(m+1)(2m^2 + 7m + 6)}{6} \\
&= \frac{(m+1)((m+2)(2m+3))}{6}
\end{aligned}$$

Problem 3

Problem 3 (a)

Symmetric relation means that

$$\forall x, y \in X, xRy \iff yRx$$

R_1 and R_2 are symmetric.

$$\forall x, y \in X, xR_1y \iff yR_1x$$

$$\forall x, y \in X, xR_2y \iff yR_2x$$

Let's pick an element from $R_1 \cup R_2$. $(x, y) \in R_1 \cup R_2$
Which means that $(x, y) \in R_1$ or $(x, y) \in R_2$
If it is in R_1, then $(y, x) \in R_1$
If it is in R_2, then $(y, x) \in R_2$
In both cases $(y, x) \in R_1 \cup R_2$
Which means that $R_1 \cup R_2$ is symmetric.

Problem 3 (b)

Reflexive relation means that

$$\forall x \in X, xRx$$

R_1 is reflexive.

$$\forall x \in X, xR_1x$$

R_2 is arbitrary.
Let R_3 be $R_1 \cup R_2$.
Let $x \in X$
Is xR_3x?
If xR_3x, then $(x, x) \in R_3$
Which means that $(x, x) \in R_1$ or $(x, x) \in R_2$
And we know that xR_1x is true because R_1 is reflexive.
Thus xR_3x is true which means R_3 is reflexive

Problem 3 (c)

Antisymmetric relation means that

$$\forall x, y \in X, xRy \land yRx \implies x = y$$

R_1 and R_2 is antisymmetric.
Let's take a look at this example:

$$X = \{1, 2\}$$
$$R_1 = \{(1, 2)\}$$
$$R_2 = \{(2, 1)\}$$

R_1 and R_2 are antisymmetric. But $R_1 \cup R_2$ is not antisymmetric because it contains $(1, 2)$ and $(2, 1)$ and $1 \neq 2$.

Problem 4

$$X = Z \times N$$
$$(a,b) \sim (c,d) \iff ad = bc$$

Problem 4 (a)

Relation $\sim$ being an equivalence relation means that it is reflexive, symmetric and transitive.
Let's start with reflexivity:

$$(a,b) \sim (a,b) \implies ab = ba$$

Above statement is true for all $a \in Z$, $b \in N$.
Thus $\sim$ is reflexive.

Now let's check symmetry:

$$\forall a,c \in Z \ \ \forall b,d \in N \ \ (a,b) \sim (c,d) \iff ad = bc \iff cb = da \iff (c,d) \sim (a,b)$$

Thus $\sim$ is symmetric.

Now let's check transitivity:

$$\forall a,b,c,d,e,f \in Z \ \ \forall x,y,z \in N \ \ (a,b) \sim (c,d) \wedge (c,d) \sim (e,f) \implies (a,b) \sim (e,f)$$
$$(a,b) \sim (c,d) \iff ad = bc$$
$$(c,d) \sim (e,f) \iff cf = de$$

Multiply these two equations together, we get:

$$adcf = bcde$$

$$af = be$$
$$af = be \iff (a,b) \sim (e,f)$$

Thus $\sim$ is transitive.

Problem 4 (b)

$(a,b) \sim (a',b') \wedge (c,d) \sim (c',d')$ means that

$$ab' = a'b \wedge cd' = c'd$$

We want prove $(ad + cb, bd) \sim (a'd' + c'b', b'd')$
$(ad + cb, bd) \sim (a'd' + c'b', b'd')$ means that

$$(ad + cb)b'd' = (a'd' + c'b')bd$$

$$adb'd' + cbb'd' = a'd'bd + c'b'bd$$

$$adb'd' - a'd'bd = c'b'bd - cbb'd'$$
$$dd'(ab' - a'b) = bb'(c'd - cd')$$
$$dd'(0) = bb'(0) \quad (\text{Since} \quad ab' = a'b \; and \; cd' = c'd)$$

$0 = 0$ which is true for every a b c d a' b' c' d'. Thus $(ad + cb, bd) \sim (a'd' + c'b', b'd')$