Mathematics Homework Sheet 5

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Problem 1

Problem 1(a)

We want to prove

$$X \setminus \bigcap_{i \in I} Y_i = \bigcup_{i \in I} (X \setminus Y_i), \qquad \forall i \in I \quad Y_i \subseteq X$$

and

$$X \setminus \bigcup_{i \in I} Y_i = \bigcap_{i \in I} (X \setminus Y_i), \qquad \forall i \in I \quad Y_i \subseteq X$$

Let's start with the first one.

$$X \setminus \bigcap_{i \in I} Y_i = \bigcup_{i \in I} (X \setminus Y_i)$$

We are going to prove this by proving both sides are subsets of each other.

First, $X \setminus \bigcap_{i \in I} Y_i \subseteq \bigcup_{i \in I} (X \setminus Y_i)$:
Let $x \in X \setminus \bigcap_{i \in I} Y_i$. This means that $x \in X$ and $x \notin \bigcap_{i \in I} Y_i$. This means that there is at least one Y_i such that $x \notin Y_i$. So, $x \in X \setminus Y_i$ for that Y_i. Thus, $x \in \bigcup_{i \in I} (X \setminus Y_i)$.

Second, $\bigcup_{i \in I} (X \setminus Y_i) \subseteq X \setminus \bigcap_{i \in I} Y_i$:
Let $x \in \bigcup_{i \in I} (X \setminus Y_i)$. This means that there is at least one Y_i such that $x \in X \setminus Y_i$. This means that $x \in X$ and $x \notin Y_i$. This means that $x \notin \bigcap_{i \in I} Y_i$. Thus, $x \in X \setminus \bigcap_{i \in I} Y_i$.
And this completes the proof for the first one.

Now, let's prove the second one.

$$X \setminus \bigcup_{i \in I} Y_i = \bigcap_{i \in I} (X \setminus Y_i)$$

We are going to prove this by proving both sides are subsets of each other.

First, $X \setminus \bigcup_{i \in I} Y_i \subseteq \bigcap_{i \in I} (X \setminus Y_i)$:
Let $x \in X \setminus \bigcup_{i \in I} Y_i$. This means that $x \in X$ and $x \notin \bigcup_{i \in I} Y_i$. This means that $x \notin Y_i$ for all $i \in I$. This means that $x \in X \setminus Y_i$ for all $i \in I$. Thus, $x \in \bigcap_{i \in I} (X \setminus Y_i)$.

Second, $\bigcap_{i\in I}(X \setminus Y_i) \subseteq X \setminus \bigcup_{i\in I} Y_i$:
Let $x \in \bigcap_{i\in I}(X \setminus Y_i)$. This means that $x \in X \setminus Y_i$ for all $i \in I$. This means that $x \in X$ and $x \notin Y_i$ for all $i \in I$. This means that $x \notin \bigcup_{i\in I} Y_i$. Thus, $x \in X \setminus \bigcup_{i\in I} Y_i$.
And this completes the proof for the second one.

Problem 1(b)

Problem 1(b)(i)

We want to prove $\bigcap_{i\in I} U_i$ is closed.
We are given $(\forall i \in I \quad U_i \subseteq R)$ is closed.
A set being closed means that its complement is open. So we want to prove that $\bigcup_{i\in I} U_i^c$ is open.
Since each U_i is closed, we know that U_i^c is open.
And from the lecture we know that union or intersection of open sets is open.
Thus $\bigcup_{i\in I} U_i^c$ is open. Which means that $\bigcap_{i\in I} U_i$ is closed.
And this completes the proof.

Problem 1(b)(ii)

We want to prove $\bigcup_{i=1}^{n} U_i$ is closed.
We are given $(U_1, ..., U_n \subseteq R)$ are closed.
A set being closed means that its complement is open. So we want to prove that $\bigcap_{i=1}^{n} U_i^c$ is open.
Since each U_i is closed, we know that U_i^c is open.
And from the lecture we know that union or intersection of open sets is open.
Thus $\bigcap_{i=1}^{n} U_i^c$ is open. Which means that $\bigcup_{i=1}^{n} U_i$ is closed.
And this completes the proof.

Problem 3

Problem 3(a)

$$a_n := (-1)^n$$

a_n is not convergent. Because, for example, if we take $\epsilon = 1/10$ then there is no N that satisfies

$$\forall n \geq N \quad |a_n - a| < 1/10$$

a_n alternates between -1 and 1. So we can't find a value a that stays in the neighborhood of both -1 and 1. For example, when $\epsilon = 1/10$ there is no $a \in R$ that satisfies

$$|1 - a| < 1/10 \quad \text{and} \quad |-1 - a| < 1/10$$

No matter what you choose N to be you will always get for some $j \in N$ $a_j = 1$ and $a_j = -1$.
Thus a_n is divergent.

Problem 3(b)

$$b_n := \frac{(-1)^n}{n}$$

b_n is convergent.
Because, we can find an N that satisfies

$$\forall \epsilon > 0 \quad \forall n \geq N \quad |b_n - 0| < \epsilon$$

We are trying to find an N such that this inequality holds for any choice of ϵ.

$$|\frac{(-1)^n}{n}| < \epsilon \qquad \text{when } n \geq N$$

When n is even, we have

$$|\frac{1}{n}| < \epsilon \qquad \text{when } n \geq N$$
$$\frac{1}{n} < \epsilon \qquad \text{when } n \geq N$$
$$n > \frac{1}{\epsilon} \qquad \text{when } n \geq N$$

So if we choose N to be the any integer greater than $\frac{1}{\epsilon}$ then the inequality holds for even n. So, such N exist when n is even.
When n is odd, we have

$$|\frac{-1}{n}| < \epsilon \qquad \text{when } n \geq N$$
$$\frac{1}{n} < \epsilon \qquad \text{when } n \geq N$$
$$n > \frac{1}{\epsilon} \qquad \text{when } n \geq N$$

Basically, we have the same thing for odd n. So, such N exist when n is odd too.
Thus, we can find an N that satisfies the inequality for any choice of ϵ if we choose a to be 0.

Since $a = 0$, the limit is zero.

$$\lim_{n \to \infty} b_n = \frac{(-1)^n}{n} = 0$$