

# Homework Sheet 1

Author: Abdullah Oğuz Topçuoğlu

## Problem 1

We are given the function:

$$f : \mathbb{R}^2 \rightarrow \mathbb{R} \quad f(x_1, x_2) = x_1^2 + 2x_2^2$$

We want to show that  $f$  is continuous on  $\mathbb{R}^2$ . We want to find a delta that satisfies:

$$\forall \varepsilon > 0, \exists \delta > 0 : \forall x \in \mathbb{R}^2 \|x - x^0\| < \delta \implies |f(x) - f(x^0)| < \varepsilon$$

for an arbitrary but fixed  $x^0$ .

Fix  $x^0 = (x_1^0, x_2^0)$ . Fix  $\varepsilon > 0$ . We want to find a delta that makes this inequality satisfied always:

$$|f(x) - f(x^0)| < \varepsilon$$

We are gonna find a relation between delta and  $|f(x) - f(x^0)|$  and from there we are look for values of delta where it is less than epsilon.

$$\begin{aligned} |f(x) - f(x^0)| &= |x_1^2 + 2x_2^2 - (x_1^0)^2 - 2(x_2^0)^2| \\ &= |(x_1^2 - (x_1^0)^2) + 2(x_2^2 - (x_2^0)^2)| \\ &\leq |x_1^2 - (x_1^0)^2| + 2|x_2^2 - (x_2^0)^2| \\ &= |(x_1 - x_1^0)(x_1 + x_1^0)| + 2|(x_2 - x_2^0)(x_2 + x_2^0)| \\ &\leq |x_1 - x_1^0||x_1 + x_1^0| + 2|x_2 - x_2^0||x_2 + x_2^0| \end{aligned}$$

(replace individual components of the vector with the vector itself. that makes the overall value larger)

$$\begin{aligned} &\leq \|x - x^0\| \|x + x^0\| + 2\|x - x^0\| \|x + x^0\| \\ &= 3\|x - x^0\| \|x + x^0\| \\ &\leq 3\delta \|x + x^0\| \end{aligned}$$

and now assume that  $\delta \leq 1$ , then we can say that:

$$\|x\| \leq \|x^0\| + 1$$

$$\begin{aligned}
|f(x) - f(x^0)| &\leq 3\delta\|x + x^0\| \\
&\leq 3\delta(\|x\| + \|x^0\|) \\
&\leq 3\delta(\|x^0\| + 1 + \|x^0\|) \\
&= 3\delta(2\|x^0\| + 1)
\end{aligned}$$

We are almost done. Now if we chose a delta that satisfies:

$$\begin{aligned}
3\delta(2\|x^0\| + 1) &< \varepsilon \\
\delta &< \frac{\varepsilon}{3(2\|x^0\| + 1)}
\end{aligned}$$

And at the top we assumed that  $\delta \leq 1$ , so we can chose:

$$\delta = \min\left(1, \frac{\varepsilon}{3(2\|x^0\| + 1)}\right)$$

So found a delta meaning that  $f$  is continuous at  $x^0$ . Since  $x^0$  was arbitrary,  $f$  is continuous on  $\mathbb{R}^2$ .

## Problem 2

We are given the function:

$$f : \mathbb{R}^2 \rightarrow \mathbb{R} \quad f(x_1, x_2) = \begin{cases} \frac{x_1^2 \sin(x_1 + x_2)}{\sqrt{x_1^4 + x_2^4}} & (x_1, x_2) \neq (0, 0) \\ 0 & (x_1, x_2) = (0, 0) \end{cases}$$

We want to show that for every sequence  $(x^N)_{N \in \mathbb{N}}$  that converges to  $(0, 0)$ , the sequence  $(f(x^N))_{N \in \mathbb{N}}$  also converges to  $f(0, 0) = 0$ .

Fix an arbitrary sequence  $(x^N)_{N \in \mathbb{N}}$  that converges to  $(0, 0)$ .  
If  $(x_1, x_2) \neq (0, 0)$

$$\sqrt{x_1^4 + x_2^4} \geq \sqrt{x_1^4} = x_1^2$$

Hence

$$\frac{x_1^2}{\sqrt{x_1^4 + x_2^4}} \leq 1$$

Also note that:

$$|\sin(x)| \leq |x|$$

$$\begin{aligned}
|f(x^{(N)})| &= \frac{(x_1^{(N)})^2 |\sin(x_1^{(N)} + x_2^{(N)})|}{\sqrt{(x_1^{(N)})^4 + (x_2^{(N)})^4}} \\
&\leq \frac{(x_1^{(N)})^2 |x_1^{(N)} + x_2^{(N)}|}{\sqrt{(x_1^{(N)})^4 + (x_2^{(N)})^4}} \\
&\leq |x_1^{(N)} + x_2^{(N)}| \\
&\leq |x_1^{(N)}| + |x_2^{(N)}| \\
&\leq |x^{(N)}| + |x^{(N)}| \\
&\leq 2|x^{(N)}|
\end{aligned}$$

Since  $|f(x^{(N)})|$  is less than  $2|x^{(N)}|$  it must also converge to zero.  $|f(x^{(N)})|$  converging zero means that  $f(x^{(N)})$  converging to  $(0, 0)$  which equals to  $f(0, 0)$ . And thats what we wanted to show.