Mathematics Homework Sheet 8

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Problem 1

$$a_n := \sqrt[n]{n} - 1$$

We want to prove

$$a_n \leq \sqrt{\frac{2}{n}} \qquad \forall n \geq 2$$

and we want to prove

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

Apply the binomial formula to $(1 + a_n)^n$

$$\begin{aligned}(1 + a_n)^n &= n \\ &= \sum_{k=0}^{n} \binom{n}{k} a_n^k \\ &= \sum_{k=0}^{n} \binom{n}{k} (\sqrt[n]{n} - 1)^k \\ &= 1 + \binom{n}{1}(\sqrt[n]{n} - 1) + \binom{n}{2}(\sqrt[n]{n} - 1)^2 + \ldots + (\sqrt[n]{n} - 1)^n\end{aligned}$$

Using the fact that $a_n \leq \sqrt{\frac{2}{n}}$, we want to show

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

We know that

$$1 - \frac{1}{n} \leq \sqrt[n]{n}$$

because $1 - \frac{1}{n}$ is less than 1 and $\sqrt[n]{n}$ is greater or equal to 1 for all $n \geq 1$. Also from the previous inequality we have

$$\begin{aligned}\sqrt[n]{n} - 1 &\leq \sqrt{\frac{2}{n}} \\ \sqrt[n]{n} &\leq 1 + \sqrt{\frac{2}{n}}\end{aligned}$$

When combined we have

$$1 - \frac{1}{n} \leq \sqrt[n]{n} \leq 1 + \sqrt{\frac{2}{n}}$$

$$\lim_{n\to\infty} 1 - \frac{1}{n} = 1$$

$$\lim_{n\to\infty} 1 + \sqrt{\frac{2}{n}} = 1$$

Therefore, by the sandwich theorem, we have

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

Problem 2

Problem 2(a)

$$a_n := \frac{4 + 3n^2}{n(2n+1)^2}$$

We want to prove

$$(\frac{1}{n})_{n\in N} \in O((a_n)_{n\in N})$$

Which means $\frac{\frac{1}{n}}{a_n}$ is bounded.
Let's start

$$\begin{aligned}
\frac{1}{na_n} &= \frac{n(2n+1)^2}{n(4+3n^2)} \\
&= \frac{(2n+1)^2}{4+3n^2} \\
&= \frac{4n^2+4n+1}{4+3n^2} \\
&= \frac{4+\frac{4}{n}+\frac{1}{n^2}}{3+\frac{4}{n^2}} \\
lim_{n\to\infty} \frac{4+\frac{4}{n}+\frac{1}{n^2}}{3+\frac{4}{n^2}} &= \frac{4}{3}
\end{aligned}$$

Existence of limit implies that $\frac{1}{na_n}$ is bounded. Therefore, $(\frac{1}{n})_{n\in N} \in O((a_n)_{n\in N})$
The series $\sum_{n=1}^{\infty} a_n$ is divergent. Since $lim_{n\to\infty} \frac{\frac{1}{n}}{a_n} = \frac{4}{3}$, the limit of $lim_{n\to\infty} \frac{\frac{1}{2n}}{a_n} = \frac{2}{3} \leq 1$ which means for almost all $n \in N$ $\frac{1}{2n} \leq a_n$. And from the therom 3.44(ii) in the lecture notes, a_n is divergent. because $\sum_{n=1}^{\infty} \frac{1}{2n}$ is divergent and $a_n \geq \frac{1}{2n} \geq 0$ for almost all $n \in N$.

Problem 2(b)

Problem 2(b)(i)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

We notice that the series is alternating series. Let's check for Leibniz criterion. When is x^n/n a monotonous null sequence? The answer is when x is in $[-1, 1]$. So we know the series converges for $x \in [-1, 1]$.
For absolute convergence we consider the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

Let's do ratio test

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} \\
&= \lim_{n\to\infty} \frac{xn}{n+1} \\
&= x \lim_{n\to\infty} \frac{n}{n+1} \\
&= x
\end{aligned}$$

By the ratio test we see that if $x \in (-1, 1)$ then the series converges absolutely. And if $x > 1, x < -1$ then the series diverges.

Problem 2(b)(ii)

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

We notice that the series is alternating series. Let's check for Leibniz criterion. When is $x^{2n+1}/(2n+1)!$ a monotonous null sequence? The answer is for all x. Because in the lecture we learnt that factorial grows faster than exponential.
For absolute convergence we consider the series

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

Let's do ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} = \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+3)} = 0 \quad \forall x \in R$$

Therefore, the series converges absolutely for all x.