Homework Sheet 3

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Exercise 9

We are given the function:

$$f(x_1, x_2) := \begin{cases} \frac{x_1^2 x_2}{x_1^2 + x_2^2}, & (x_1, x_2) \neq (0,0) \\ 0, & (x_1, x_2) = (0,0) \end{cases}$$

(i)

To show that f is continuous at the point $(0,0)$, we need to verify that every sequence $(x_1^{(n)}, x_2^{(n)})$ converging to $(0,0)$, $f(x_1^{(n)}, x_2^{(n)})$ also converges to $f(0,0) = (0,0)$.
Fix a sequence $(x_1^{(n)}, x_2^{(n)})$ such that $(x_1^{(n)}, x_2^{(n)}) \to (0,0)$.
Then, we have:
If $(x_1^{(n)}, x_2^{(n)}) \neq (0,0)$

$$\begin{aligned} |f(x_1^{(n)}, x_2^{(n)}) - f(0,0)| &= \left| \frac{(x_1^{(n)})^2 x_2^{(n)}}{(x_1^{(n)})^2 + (x_2^{(n)})^2} - 0 \right| \\ &= \left| \frac{(x_1^{(n)})^2 x_2^{(n)}}{(x_1^{(n)})^2 + (x_2^{(n)})^2} \right| \\ &\leq \left| \frac{(x_1^{(n)})^2 x_2^{(n)}}{(x_1^{(n)})^2} \right| \quad (\text{since } (x_1^{(n)})^2 + (x_2^{(n)})^2 \geq (x_1^{(n)})^2) \\ &= |x_2^{(n)}| \end{aligned}$$

$|x_2^{(n)}| \to 0$ since $(x_1^{(n)}, x_2^{(n)}) \to (0,0)$. Therefore $f(x_1^{(n)}, x_2^{(n)}) \to (0,0)$.

If $(x_1^{(n)}, x_2^{(n)}) = (0,0)$

$$|f(x_1^{(n)}, x_2^{(n)}) - f(0,0)| = |0 - 0| = 0$$

which also converges to 0.
Thats what we wanted to show.

(ii)