Homework Sheet 3

Author: Abdullah Oğuz Topçuoğlu

Exercise 9

We are given the function:

$$f(x_1, x_2) := \begin{cases} \frac{x_1^2 x_2}{x_1^2 + x_2^2}, & (x_1, x_2) \neq (0,0) \\ 0, & (x_1, x_2) = (0,0) \end{cases}$$

(i)

To show that f is continuous at the point $(0,0)$, we need to verify that every sequence $(x_1^{(n)}, x_2^{(n)})$ converging to $(0,0)$, $f(x_1^{(n)}, x_2^{(n)})$ also converges to $f(0,0) = (0,0)$.
Fix a sequence $(x_1^{(n)}, x_2^{(n)})$ such that $(x_1^{(n)}, x_2^{(n)}) \to (0,0)$.
Then, we have:
If $(x_1^{(n)}, x_2^{(n)}) \neq (0,0)$

$$\begin{aligned} |f(x_1^{(n)}, x_2^{(n)}) - f(0,0)| &= \left| \frac{(x_1^{(n)})^2 x_2^{(n)}}{(x_1^{(n)})^2 + (x_2^{(n)})^2} - 0 \right| \\ &= \left| \frac{(x_1^{(n)})^2 x_2^{(n)}}{(x_1^{(n)})^2 + (x_2^{(n)})^2} \right| \\ &\leq \left| \frac{(x_1^{(n)})^2 x_2^{(n)}}{(x_1^{(n)})^2} \right| \quad (\text{since } (x_1^{(n)})^2 + (x_2^{(n)})^2 \geq (x_1^{(n)})^2) \\ &= |x_2^{(n)}| \end{aligned}$$

$|x_2^{(n)}| \to 0$ since $(x_1^{(n)}, x_2^{(n)}) \to (0,0)$. Therefore $f(x_1^{(n)}, x_2^{(n)}) \to (0,0)$.

If $(x_1^{(n)}, x_2^{(n)}) = (0,0)$

$$|f(x_1^{(n)}, x_2^{(n)}) - f(0,0)| = |0 - 0| = 0$$

which also converges to 0.
Thats what we wanted to show.

(ii)

We need to show that the following limit doesnt converge to zero

$$\lim_{x \to 0} \frac{f(x) - f(0) - Jf(0)x}{\|x\|}$$

The J is the Jacobian matrix.

$$J = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix}$$

Lets compute the partials at (0, 0)

$$\frac{\partial f}{\partial x_1}(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$
$$\frac{\partial f}{\partial x_2}(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$

Thus, the Jacobian matrix at (0, 0) is:

$$Jf(0,0) = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Substituting this into our limit we get

$$\lim_{x \to 0} \frac{f(x) - f(0)}{\|x\|}$$

We also know that $f(0) = 0$, so we can simplify this to:

$$\lim_{x \to 0} \frac{f(x)}{\|x\|}$$

So it is enough if i can show a vector x where this limit is not zero.
Lets choose $x = (x_1, x_1)$. Then we have:

$$\begin{aligned} \lim_{x_1 \to 0} \frac{f(x_1, x_1)}{\|(x_1, x_1)\|} &= \lim_{x_1 \to 0} \frac{\frac{x_1^2 x_1}{x_1^2 + x_1^2}}{\sqrt{x_1^2 + x_1^2}} \\ &= \lim_{x_1 \to 0} \frac{\frac{x_1^3}{2x_1^2}}{\sqrt{2}|x_1|} \\ &= \lim_{x_1 \to 0} \frac{x_1}{2\sqrt{2}|x_1|} \\ &= \lim_{x_1 \to 0} \frac{1}{2\sqrt{2}} \cdot \frac{x_1}{|x_1|} \end{aligned}$$

This limit doesnt even exist so it cant be equal to zero.
Thats what we wanted to show.

Exercise 10

We are given the function:

$$f(x_1, x_2, x_3) := \begin{bmatrix} x_1^4 \ln(3 + 2x_2^2) \\ x_1 \sin(x_2 x_3) e^{x_1} \end{bmatrix}$$

(i)

To show that f is a C^1 function we need to verify that all partial derivatives of f exist and are continuous.
Lets compute the partial derivatives of each component of f which corresponds to the entries of Jacobian matrix.

$$Jf = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \end{bmatrix}$$

Calculating the partial derivatives:

$$\frac{\partial f_1}{\partial x_1} = 4x_1^3 \ln(3 + 2x_2^2)$$
$$\frac{\partial f_1}{\partial x_2} = x_1^4 \cdot \frac{4x_2}{3 + 2x_2^2}$$
$$\frac{\partial f_1}{\partial x_3} = 0$$
$$\frac{\partial f_2}{\partial x_1} = \sin(x_2 x_3) e^{x_1} + x_1 \sin(x_2 x_3) e^{x_1}$$
$$\frac{\partial f_2}{\partial x_2} = x_1 x_3 \cos(x_2 x_3) e^{x_1}$$
$$\frac{\partial f_2}{\partial x_3} = x_1 x_2 \cos(x_2 x_3) e^{x_1}$$

All these partial derivatives are polynomials, logarithmic, exponential, sine and cosine functions which are continuous everywhere in $\mathbb{R}^3$ Thus, all partial derivatives of f exist and are continuous.
(we can explicitly use the remark 1.3.13 in lecture notes to prove continuity but thats trivial so i cut it short with one sentence. Can i cut it short like this in the exam?)
Thats what we wanted to show.