Prof. Dr. Henryk Zähle
Prof. Dr. Frederik Herzberg, M.Sc. Olaf Kubinkiewicz
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# Mathematics for Computer Scientists III

## Exercise Sheet 1

### Exercise 1 (3 Points)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$$f(x) := x_1^2 + 2x_2^2.$$

Show that $f$ is continuous by showing that the $\varepsilon$-$\delta$ criterion in Definition 1.3.1 is satisfied for every point $(x_1^0, x_2^0) \in \mathbb{R}^2$.

### Exercise 2 (4 Points)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$$f(x_1, x_2) := \begin{cases} \frac{x_1^2 \sin(x_1+x_2)}{\sqrt{x_1^4+x_2^4}} & , \quad (x_1, x_2) \neq (0,0) \\ 0 & , \quad (x_1, x_2) = (0,0) \end{cases}.$$

Show that $f$ is continuous.

### Exercise 3 (5 Points)

Let $\mathbb{R}_+ := [0, \infty)$. Let $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x_1, x_2) := \begin{cases} \frac{\sqrt{x_1\, x_2^2 + x_1 x_2^2}}{x_1+2x_2^2} & , \quad (x_1, x_2) \neq (0,0) \\ 0 & , \quad (x_1, x_2) = (0,0) \end{cases}.$$

(i) Let $x^0 = (x_1^0, x_2^0) \in \mathbb{R}^2$ be arbitrary, but fixed. Let the functions $f_{x_1^0} : \mathbb{R} \to \mathbb{R}$ and $f_{x_2^0} : \mathbb{R}_+ \to \mathbb{R}$ be defined by $f_{x_2^0}(x_1) := f(x_1, x_2^0)$ and $f_{x_1^0}(x_2) := f(x_1^0, x_2)$. Show that both $f_{x_1^0}$ and $f_{x_2^0}$ are continuous.

(ii) Show that $f$ is *not* continuous at the point $(0, 0)$.

### Exercise 4* (4 Bonus Points)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$$f(x_1, x_2) := \begin{cases} \frac{1}{x_1^2+x_2^2} & , \quad (x_1, x_2) \in A \\ x_1 x_2 & , \quad (x_1, x_2) \notin A \end{cases},$$

where $A := \left\{ \left( \frac{1}{x} \cos(\frac{2\pi}{x}), \frac{1}{x} \sin(\frac{2\pi}{x}) \right) : x \in (0, \infty) \right\} \subseteq \mathbb{R}^2$.

(i) Plot (or sketch) the set $A$.

(ii) Show that $f$ is *not* continuous at the point $(0, 0)$.

(iii) Show that $f(h_N v) \to f(0,0)$ (as $N \to \infty$) for *every* $v \in \mathbb{R}^2$ and every decreasing sequence $(h_N)_{N \in \mathbb{N}}$ in $\mathbb{R}$ with $h_N \to 0$.

*Hint for (iii):* Consider how "often" the sequence $(h_N v)_{N\in\mathbb{N}}$ can hit the set $A$. To do this, look at the plot (or sketch) of the set $A$.