Mathematics Homework Sheet 4

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Problem 1

We want to prove

$$(\forall n \in N) \wedge (x \in R) \wedge (x \geq -1) \qquad (1+x)^n \geq 1+nx$$

by using mathematical induction.
Base Case: For $n = 1$, we have

$$(1+x)^1 = 1 + x \geq 1 + 1 \cdot x$$

Which is true for every $x \in R$ so it means it is also true for $x \in [-1, \infty]$

Inductive Step: Assume that the statement is true for $n = k$, i.e.

$$(1+x)^k \geq 1 + kx$$

is true for every $x \in [-1, \infty]$.
We want to prove that the statement is also true for $n = k + 1$, i.e.

$$(1+x)^{k+1} \geq 1 + (k+1)x$$

for every $x \in [-1, \infty]$.

$$\begin{aligned}
(1+x)^{k+1} &= (1+x)^k \cdot (1+x) && (1)\\
&\geq (1+kx) \cdot (1+x) && \text{(Using inductive step)} \quad (2)\\
&= 1 + kx + x + kx^2 && (3)\\
&= 1 + (k+1)x + kx^2 && (4)\\
&\geq 1 + (k+1)x && \text{(Since } kx^2 \geq 0) \quad (5)
\end{aligned}$$

And this completes the proof.

Inductive step alternative solution:
We assume that the statement is true for $n = k$, i.e.

$$(1+x)^k \geq 1 + kx$$

Multiply both sides by $1 + x$, since $1 + x > 0$ because $x \in [-1, \infty]$, we have

$$(1+x)^k(1+x) \geq (1+kx)(1+x)$$

$$
\begin{aligned}
(1+x)^{k+1} &\geq 1+kx+x+kx^2 \\
0 &\geq -kx^2
\end{aligned}
$$

Add these two together, we get

$$(1+x)^{k+1} \geq 1+(k+1)x$$

And this completes the proof.

Give a counterexample to show that the condition x $\geq$ -1 is necessary:
Let's take $x=-2$ and $n=2$. Then we have

$$
\begin{aligned}
(1-2)^2 &\geq 1+2\cdot(-2) \\
(-1)^2 &\geq 1-4 \\
1 &\geq -3
\end{aligned}
$$

Which is not true. In fact it would be false when n is even. So the condition $x \geq -1$ is necessary. Because that way $1+x$ is never negative

Problem 2

Problem 2 (a)

$$X_1 := \{x \in R : x^2 - 2x \leq 0\}$$

What x values satisfy this condition?

$$
\begin{aligned}
x^2 - 2x &\leq 0 \\
x(x-2) &\leq 0
\end{aligned}
$$

In order this inequality to be satisfied the signs of x and $x-2$ must be different or one of them needs to be zero, and this only happens when $0 \leq x \leq 2$.
So this means:

$$X_1 = [0,2]$$

In this case X_1 is bounded from below and above.

$$
\begin{aligned}
supX_1 &= 2 \\
infX_1 &= 0
\end{aligned}
$$

And $supX_1 \in X_1$ which means $supX_1$ is also the maximum value.
$infX_1 \in X_1$ which means $infX_1$ is also the minimum value.

Problem 2 (b)

$$X_2 := \{x \in R \setminus \{0\} : 5 - x^2 > \frac{4}{x^2}\}$$

What x values satisfy this condition?

$$\begin{aligned} 5 - x^2 &> \frac{4}{x^2} \\ 5 - x^2 - \frac{4}{x^2} &> 0 \\ \frac{5x^2 - x^4 - 4}{x^2} &> 0 \end{aligned}$$

Since x^2 is always positive, we can multiply both sides by x^2.

$$\begin{aligned} 5x^2 - x^4 - 4 &> 0 \\ x^4 - 5x^2 + 4 &< 0 \\ (x^2 - 4)(x^2 - 1) &< 0 \\ (x - 2)(x + 2)(x - 1)(x + 1) &< 0 \end{aligned}$$

So, this inequality is satisfied when $-2 < x < -1 \quad \vee \quad 1 < x < 2$.
So this means:

$$X_2 = (-2, -1) \cup (1, 2)$$

In this case X_2 is bounded from below and above.

$$\begin{aligned} supX_2 &= 2 \\ infX_2 &= -2 \end{aligned}$$

And $supX_2 \notin X_2$ which means $supX_2$ is not the maximum value.
$infX_2 \notin X_2$ which means $infX_2$ is not the minimum value.
Maximum and minimum values are not in the set.

Problem 4

Problem 4 (a)

We want to prove

$$\forall x, y \in R \quad |x + y| \leq |x| + |y|$$

Let's continue with this inequality

$$\begin{aligned}
\forall x, y \in R \quad x \leq |x|,\ y \leq |y|,\ -x \leq |x|,\ -y \leq |y| \\
x + y \leq |x| + |y| \qquad & \text{(Considering the first two inequalities above)} \\
-x - y \leq |x| + |y| \qquad & \text{(Considering the last two inequalities above)}
\end{aligned}$$

$(x + y)$ and $(-x - y)$ is nothing but two possible outcomes of $|x + y|$
So, we have

$$|x + y| \leq |x| + |y|$$

And this completes the proof.