Homework Sheet 1

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Problem 1

(1.)

The given definition of antisymmetry is not sensible because it implies that for any elements $a, a' \in A$, if $(a, a') \in R$, then $(a', a) \notin R$. When I hear "antisymmetric" i think of something that is the opposite of symmetric and symmetric means if $(a, a') \in R$ then $(a', a) \in R$. And opposite of that would be if $(a, a') \in R$, then $(a', a) \notin R$ as in the question but with a difference that it only applies when $a \neq a'$. Because otherwise in the current definition if we plug in $a = a'$, we get $(a, a) \in R \iff (a, a) \notin R$ which is a contradiction.

(2.)

(a) True.
If $f : A \to B$ and $g : B \to C$ are bijective functions.
To prove that $g \circ f$ is bijective, we need to show that it is both injective and surjective.
Since f and g are injective their composition is also injective. And since f and g are surjective their composition is also surjective.
Therefore, $g \circ f$ is bijective.

(b) False.
Let $A = \{1, 2\}, B = \{1, 2\}, C = \{1\}$.
Define $f : A \to B$ by $f(1) = 1$, $f(2) = 2$ (which is injective) and $g : B \to C$ by $g(1) = 1, g(2) = 1$ (which is surjective).
Then, $g \circ f(1) = g(f(1)) = g(1) = 1$.
Also, $g \circ f(2) = g(f(2)) = g(2) = 1$.
Therefore $g \circ f$ is not injective since $g \circ f(1) = g \circ f(2)$.

(c) False.
Consider this example: Let $A = \{1\}, B = \{1, 2\}, C = \{1, 2\}$.
Define $f : A \to B$ by $f(1) = 1$ (which is injective) and $g : B \to C$ by $g(1) = 1, g(2) = 2$ (which is surjective).
In this configuration there is no element in A that maps to 2 in C through $g \circ f$.
Thus, $g \circ f$ is not surjective.

Problem 2

(1.)

B^A is set of all functions from A to B. And a function is a relation on $A \times B$ such that for every $a \in A$ there is exactly one $b \in B$ such that (a, b) is in the relation. So $|B^A|$ is just how many different ways to find such a relation. For every element in A we have $|B|$ choices to map it to an element in B. Which is $|B| \times |B| \times ... \times |B|$ ($|A|$ times) $= |B|^{|A|}$.
Thats what we wanted to show.

(2.)

Fix a set A and $a \in A$ and $k \in \mathbb{N}$. We need to find a bijection between:

$$\binom{A}{k} \leftrightarrow \binom{A \setminus \{a\}}{k} \cup \binom{A \setminus \{a\}}{k-1}$$

Define a function $f : \binom{A}{k} \to \binom{A \setminus \{a\}}{k} \cup \binom{A \setminus \{a\}}{k-1}$ as follows:

$$f(S) = \begin{cases} S & \text{if } a \notin S \\ S \setminus \{a\} & \text{if } a \in S \end{cases}$$

In the first case $S \in \binom{A \setminus \{a\}}{k}$ and in the second case $S \setminus \{a\} \in \binom{A \setminus \{a\}}{k-1}$. Now we need to show that this function is bijective by showing that it is both injective and surjective.
Injective: Assume $f(S_1) = f(S_2)$ for some $S_1, S_2 \in \binom{A}{k}$. We need to show that $S_1 = S_2$.
If $a \notin S_1$ and $a \notin S_2$, then $f(S_1) = S_1$ and $f(S_2) = S_2$. Thus, $S_1 = S_2$.
If $a \in S_1$ and $a \in S_2$, then $f(S_1) = S_1 \setminus \{a\}$ and $f(S_2) = S_2 \setminus \{a\}$. Thus, $S_1 \setminus \{a\} = S_2 \setminus \{a\}$ which implies $S_1 = S_2$.
If $a \in S_1$ and $a \notin S_2$, then $f(S_1) = S_1 \setminus \{a\}$ and $f(S_2) = S_2$. This leads to a contradiction since $S_1 \setminus \{a\}$ has size $k - 1$ while S_2 has size k.
Similarly, if $a \notin S_1$ and $a \in S_2$, we reach a contradiction.
Thus, f is injective.

Surjective: Let $T \in \binom{A \setminus \{a\}}{k} \cup \binom{A \setminus \{a\}}{k-1}$. We need to find $S \in \binom{A}{k}$ such that $f(S) = T$.
If $T \in \binom{A \setminus \{a\}}{k}$, then let $S = T$. Then, $f(S) = S = T$.
If $T \in \binom{A \setminus \{a\}}{k-1}$, then let $S = T \cup \{a\}$. Then, $f(S) = S \setminus \{a\} = T$.
Thus, f is surjective.
Since f is both injective and surjective, it is bijective.
Thats what we wanted to show.

Problem 3

We want to show that $|B^A| > |A|$.
If A and B are finite sets then we can use the coclusion from Problem 2.1 that $|B^A| = |B|^{|A|}$. And since $|B| > 1$ we have $|B|^{|A|} > |A|$.
In other cases,
Assume for the sake of contradiction that there exists a surjective function $f : A \to B^A$. This means that for every function $g : A \to B$, there exists an element $a \in A$ such that $f(a) = g$.
Now, we can construct a function $h : A \to B$ such that for each $a \in A$, $h(a)$ is different from $f(a)(a)$. This is possible since B has more than one element.
However, by construction, h cannot be equal to $f(a)$ for any $a \in A$, which is a contradiction.
Therefore, there is no surjective mapping from $A \to B^A$, and thus $|B^A| > |A|$.

Problem 4

Let $k \in \mathbb{N}$. We want to show that the set of monotonic functions from $\mathbb{N}$ to $\{0, \ldots, k\}$ is countable.
By countable we mean that there exists a bijection between the set of monotonic functions $(\{f | f : \mathbb{N} \to \{0, \cdots, k\}\})$ and $\mathbb{N}$.
I am gonna map each monotonic function into a $\mathbb{N}^{k+1}$ tuple where the ith element in the tuple represents the first index n where the function reaches value i or ith element is -1 indicating that the value never appears in the function. (for example constant zero function would be mapped to $(0, -1, -1, ..., -1)$).
So lets say f is a monotonic function and i mapped it to the tuple a. $a[5] = 44$ means that $f(44) = 5$ and every value that comes before index 44 is less than 5. This monotonic function to tuple mapping uniquely identifies the monotonic function since there is only one way to fill the values in between the indices in the tuple to make it monotonic. Also we can reconstruct the monotonic fuction from the given the tuple.
So now question becomes whether or not the set $\mathbb{N}^{k+1}$ is countable. And as we seen in the lecture $\mathbb{N}^m$ is countable for any $m \in \mathbb{N}$. (using the diagonal technic)