Homework Sheet 7

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Task 3

(1) L r.e. $\Rightarrow \overline{L}$ r.e.

FALSE.

Consider the language SAM. In the lecture we saw that SAM is r.e. but its complement $\overline{SAM}$ is not r.e.

(2) L decidable $\Rightarrow \overline{L}$ decidable

TRUE.

Using the definition of decidability, L is decidable means that there is a TM that accepts L and there is another TM that accepts $\overline{L}$.
$\overline{L}$ is decidable means the same thing but in reverse order which doesnt matter.

(3) L decidable and $\overline{L}$ not r.e. $\Rightarrow L$ regular

TRUE.

From false we can imply anything. The premise is false because from (2) we know that if L is deciadable then is $\overline{L}$. $\overline{L}$ being decidable implies $\overline{L}$ is r.e..

(4) $L \preceq L'$ and L r.e. $\Rightarrow L'$ r.e.

FALSE.

Counterexample: Let $L = \emptyset$ (the empty language) and L' a language that is not r.e.

- $L = \emptyset$ is r.e. (trivially, the TM that rejects everything recognizes it).
- $L \preceq L'$: The reduction $f(x)$ goes from L to L' but since L is empty, the condition $x \in L \Leftrightarrow f(x) \in L'$ is trivially true.

So in this example the premises hold, but the conclusion does not.

(5) $L \preceq L'$ and L' r.e. $\Rightarrow$ L r.e.

TRUE.

Proof: Since $L \preceq L'$, there exists a computable function f such that $x \in L \Leftrightarrow f(x) \in L'$.

Since L' is r.e., there is a TM M' that recognizes L'.

To recognize L: on input x, compute $f(x)$ and then run M' on $f(x)$. Accept if M' accepts.

This machine accepts x iff $f(x) \in L'$ iff $x \in L$. So L is r.e.

(6) $L \preceq L'$ and L not r.e. $\Rightarrow$ L' not r.e.

TRUE.

If L' were r.e. then by (5) L would be r.e., contradiction. So L' cannot be r.e.

(7) L regular $\Rightarrow$ L decidable

TRUE.

If L is regular then its complement $\overline{L}$ is also regular. If a DFA accepts L then there is a TM that accepts L by simulating the DFA.
So there is a TM that accepts L and there is another TM that accepts $\overline{L}$. Meaning that L is decidable.

(8) L context free $\Rightarrow$ L r.e.

TRUE.

If L is context free then there is a NPDA that accepts it. Since TMs are more powerful than NPDAs, there is a TM that accepts L which means that L is r.e..

(9) L context free $\Rightarrow$ L decidable

TRUE.

In the lecture we saw that there is an algorithm that decides whether a string can be derived from a context free grammar.
We can construct a TM that simulates this algorithm to decide L. So L is decidable.

(10) $L = \{w \in \{0,1\}^* \mid L(M_w)$ is context free$\} \Rightarrow L$ is decidable

FALSE.

From Rice's theroem we know that a language with the following form is undecidable:

$$L = \{w \mid L(M_w) \text{ satisfies a property}\}$$

In this case the property is being context free but that doesnt matter. L is undecidable.