Homework Sheet 7

Author: Abdullah Oğuz Topçuoğlu & Yousef Mostafa Farouk

Task 2

(1)

Language formulation:

$L = \{w\$q\$x \mid w \text{ is a encoding of } M, q \text{ is a state of } M, \text{ and } M \text{ on input } x \text{ reaches state } q\}$

Claim: L is **undecidable**.

Proof: We reduce the universal language to this problem.

Recall the universal language (the langauge accepted by the universal TM):

$$UNIV = \{w\$x \mid x \in L(M_w)\}$$

Given an instance $w\$x$ of the UNIV, we construct an instance of our problem as follows:

- Let q_{halt} be the halting (accepting) state of M_w.
- The instance is $M\$q_{halt}\$x$. (actually encoding of M, but i omit it for simplicity)

How M behaves is as follows:

- On input x, M simulates M_w on input x.
- If M_w accepts x, then M transitions to state q_{halt}.
- If M_w rejects x or loops indefinitely, then M never reaches q_{halt}.

So in this setting if $w\$x \in UNIV$ then $M\$q_{halt}\$x$ is in our langugae L, and if $w\$x \notin UNIV$ then $M\$q_{halt}\$x \notin L$ because we never reach the state q_{halt}. Meaning that this is a valid reduction from UNIV to our problem. And since UNIV is undecidable, our problem is also undecidable.

(2)

Language formulation:

$L = \{w\$y \mid w \text{ is a encoding of a simple TM } M, \text{ and there exists some input } x \text{ with prefix } y$
$\text{such that } M \text{ moves its head left on input } x\}$

Claim: L is **undecidable**.

Proof: We reduce the problem in (1) to this problem.

The idea is to find the rule that the TM M moves its head left. Lets say the rule is $(q, a) \rightarrow (q', b, -)$. And lets say the state q is the only state that moves the head left. (If there are multiple states that move the head left, we can modify the TM to have only one such state by adding intermediate states that do not move the head left). That means that we first need to decide whether the TM M reaches state q on some input x. But we know from (1) that this problem is undecidable.

So we reduce the problem in (1) to this problem as follows:

- Given an instance $M\$q\$x$ of the problem in (1), we construct an instance $M'\$y$ of this problem as follows:
- The TM M' behaves as follows on input z:
  - If z does not have prefix y, then M' rejects.
  - If z has prefix y, then M' simulates M on input x.
    If M reaches state q, then M' transitions to state q (the state that moves the head left).

So in this setting if $M\$q\$x$ is in the langugae of problem (1), then $M'\$y$ is in our langugae L, and if $M\$q\$x$ is not in the language of problem (1), then $M'\$y$ is not in L because we never reach the state q that moves the head left. Meaning that this is a valid reduction from problem (1) to our problem. And since problem (1) is undecidable, our problem is also undecidable.