Homework Sheet 1

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Problem 1

(1.)

The given definition of antisymmetry is not sensible because it implies that for any elements $a, a' \in A$, if $(a, a') \in R$, then $(a', a) \notin R$. When I hear "antisymmetric" i think of something that is the opposite of symmetric and symmetric means if $(a, a') \in R$ then $(a', a) \in R$. And opposite of that would be if $(a, a') \in R$, then $(a', a) \notin R$ as in the question but with a difference that it only applies when $a \neq a'$. Because otherwise in the current definition if we plug in $a = a'$, we get $(a, a) \in R \iff (a, a) \notin R$ which is a contradiction.

(2.)

(a) True.
If $f : A \to B$ and $g : B \to C$ are bijective functions.
To prove that $g \circ f$ is bijective, we need to show that it is both injective and surjective.
Since f and g are injective their composition is also injective. And since f and g are surjective their composition is also surjective.
Therefore, $g \circ f$ is bijective.

(b) False.
Let $A = \{1, 2\}, B = \{1, 2\}, C = \{1\}$.
Define $f : A \to B$ by $f(1) = 1$, $f(2) = 2$ (which is injective) and $g : B \to C$ by $g(1) = 1, g(2) = 1$ (which is surjective).
Then, $g \circ f(1) = g(f(1)) = g(1) = 1$.
Also, $g \circ f(2) = g(f(2)) = g(2) = 1$.
Therefore $g \circ f$ is not injective since $g \circ f(1) = g \circ f(2)$.

(c) False.
Consider this example: Let $A = \{1\}, B = \{1, 2\}, C = \{1, 2\}$.
Define $f : A \to B$ by $f(1) = 1$ (which is injective) and $g : B \to C$ by $g(1) = 1, g(2) = 2$ (which is surjective).
In this configuration there is no element in A that maps to 2 in C through $g \circ f$.
Thus, $g \circ f$ is not surjective.

Problem 2

(1.)

B^A is set of all functions from A to B. And a function is a relation on $A \times B$ such that for every $a \in A$ there is exactly one $b \in B$ such that (a, b) is in the relation. So $|B^A|$ is just how many different ways to find such a relation. For every element in A we have $|B|$ choices to map it to an element in B. Which is $|B| \times |B| \times ... \times |B|$ ($|A|$ times) $= |B|^{|A|}$.
Thats what we wanted to show.

(2.)

Fix a set A and $a \in A$ and $k \in \mathbb{N}$. We need to find a bijection between:

$$\binom{A}{k} \leftrightarrow \binom{A \setminus \{a\}}{k} \cup \binom{A \setminus \{a\}}{k-1}$$

Define a function $f : \binom{A}{k} \to \binom{A\setminus\{a\}}{k} \cup \binom{A\setminus\{a\}}{k-1}$ as follows:

$$f(S) = \begin{cases} S & \text{if } a \notin S \\ S \setminus \{a\} & \text{if } a \in S \end{cases}$$

In the first case $S \in \binom{A\setminus\{a\}}{k}$ and in the second case $S \setminus \{a\} \in \binom{A\setminus\{a\}}{k-1}$. Now we need to show that this function is bijective by showing that it is both injective and surjective.
Injective: Assume $f(S_1) = f(S_2)$ for some $S_1, S_2 \in \binom{A}{k}$. We need to show that $S_1 = S_2$.
If $a \notin S_1$ and $a \notin S_2$, then $f(S_1) = S_1$ and $f(S_2) = S_2$. Thus, $S_1 = S_2$.
If $a \in S_1$ and $a \in S_2$, then $f(S_1) = S_1 \setminus \{a\}$ and $f(S_2) = S_2 \setminus \{a\}$. Thus, $S_1 \setminus \{a\} = S_2 \setminus \{a\}$ which implies $S_1 = S_2$.
If $a \in S_1$ and $a \notin S_2$, then $f(S_1) = S_1 \setminus \{a\}$ and $f(S_2) = S_2$. This leads to a contradiction since $S_1 \setminus \{a\}$ has size $k - 1$ while S_2 has size k.
Similarly, if $a \notin S_1$ and $a \in S_2$, we reach a contradiction.
Thus, f is injective.

Surjective: Let $T \in \binom{A\setminus\{a\}}{k} \cup \binom{A\setminus\{a\}}{k-1}$. We need to find $S \in \binom{A}{k}$ such that $f(S) = T$.
If $T \in \binom{A\setminus\{a\}}{k}$, then let $S = T$. Then, $f(S) = S = T$.
If $T \in \binom{A\setminus\{a\}}{k-1}$, then let $S = T \cup \{a\}$. Then, $f(S) = S \setminus \{a\} = T$.
Thus, f is surjective.
Since f is both injective and surjective, it is bijective.
Thats what we wanted to show.

Problem 3

We want to show that $|B^A| > |A|$.
If A and B are finite sets then we can use the coclusion from Problem 2.1 that $|B^A| = |B|^{|A|}$. And since $|B| > 1$ we have $|B|^{|A|} > |A|$.
In other cases,
Assume for the sake of contradiction that there exists a surjective function $f : A \to B^A$. This means that for every function $g : A \to B$, there exists an element $a \in A$ such that $f(a) = g$.
Now, we can construct a function $h : A \to B$ such that for each $a \in A$, $h(a)$ is different from $f(a)(a)$. This is possible since B has more than one element.
However, by construction, h cannot be equal to $f(a)$ for any $a \in A$, which is a contradiction.
Therefore, there is no surjective mapping from $A \to B^A$, and thus $|B^A| > |A|$.