Haskell

Livro: Learning Hakell

Functional Programming

Programs in C or Java are usually imperative: sequences of commands that modify variables in memory.

In a purely functional program, we **never modify variables**: we only apply functions.

Example of a Functional Program in Haskell

```haskell
main = print (sum (map (^2) [1..10]))
```

- [1..10] is the sequence of integers from 1 to 10
- map (^2) calculates the square of each value
- sum sums the sequence
- print prints the result

Advantages of Functional Programming

- More concise programs
- Closer to a mathematical specification
- More focus on problem analysis and less on "debugging"
- Helps improve programming skills in any language!
- Greater modularity: breaks problems into small, reusable components
- Correction guarantees: allows for proof-based correctness and simplifies automated testing
- Concurrency/parallelism: the execution order does not affect the results

Disadvantages of Functional Programming

- Compilers and interpreters are more complex
- Hard to predict execution costs (time/space)
- Some low-level programs require precise control over time/space
- Certain algorithms are more efficient when implemented imperatively

Interpreter Commands

Comand	Abrev.	Meaning
`:load ficheiro`	`:l`	load a file
`:reload`	`:r`	reload modifications
`:edit`		edit the current file
`:set editor prog`		set the editor

Comand	Abrev.	Meaning
`:type expr`	`:t`	show the type of an expression
`:help`		get help
`:quit`	`:q`	end the session

1. Introduction

1.2. Simple Expressions

Operator	Comment
+, -	Addition, subtraction
*, /	Multiplication, division (floating-point)
^, ^^, **	Exponentiation (non-negative integer, integer, floating-point)
div	Integer division quotient
mod	Integer division remainder
==	Equality
/=	Inequality
<, <=, >, >=	Comparison
not	Negation
&&, ||	Logical conjunction and disjunction

Table 1.1 Some basic numeric and logical operators in Haskell

> **EA-1:** IN-3, IN-4

Syntax

- Function arguments are separated by spaces.
- Function application has higher precedence than any operator.

Haskell	Usual Notation
`f x`	`f(x)`
`f (g x)`	`f(g(x))`
`f (g x) (h x)`	`f(g(x), h(x))`
`f x y + 1`	`f(x, y) + 1`
`f x (y+1)`	`f(x, y + 1)`

Haskell	Usual Notation
sqrt x + 1	√x + 1
sqrt (x + 1)	√(x + 1)

- An operator can be used as a function by enclosing it in parentheses.
- Conversely, a function can be used as an operator by enclosing it in backticks.

Haskell	Equivalent
(+) x y	x + y
(*) y 2	y * 2
x `mod` 2`	mod x 2
f x `div` n	div (f x) n

1.3. Simple functions

In Haskell, since functions behave like any other entity, they are defined using the same syntax as variables. In fact, variables (in the sense as those used in imperative languages) are a particular case of a function with no arguments.

The general syntax to define a function is: `<function name > <argument 1> <argument 2> ... = <expression >`

Valid function and argument names begin with a lowercase letter, followed by letters, numbers, underscores (example: func_2) and apostrophes (func'). The names used cannot be any of the following reserved keywords: case class data default deriving do else if import in infix infixl infixr instance let module newtype of then type where .

Notes

- Function and variable names must start with lowercase letters and can include letters, digits, underscores, and apostrophes.
- Indentation indicates the scope of declarations

```haskell
a = b+c
  where b = 1
        c = 2
d = a*2
```

corresponds to:

```haskell
a = b+c
  where {b = 1;
```

```
        c = 2}
d = a*2
```

- All definitions in the same scope must start in the same column.

```
a = 1          a = 1      a = 1
 b = 2       b   = 2      b = 2
c = 3          c = 3      c = 3
ERRADO       ERRADO       OK
```

> **EA-1:** IN-6

1.4. Conditional structures

There are various ways of writing conditional structures in Haskell, which, given a condition, determine which expression will be computed. The four main conditional structures are:

- if-then-else expressions.

```
absoluto :: Float -> Float
absoluto x = if x>=0 then x else -x

-- Note1: in haskell, 'if...then...else' is an expression not a command
-- Note2: 'else' is mandatory
```

- guards.

```
absoluto :: Float -> Float
absoluto | x>=0 = x
         | otherwise = -x

-- Note: In Haskell, local definitions cover all alternatives if the where clause
is aligned with the guards. This allows you to define auxiliary functions or
variables .

raizes :: Float -> Float -> Float -> [Float]
raizes a b c
    | delta>0 = [(-b+sqrt delta)/(2*a),
                 (-b-sqrt delta)/(2*a)]
    | delta==0 = [-b/(2*a)]
    | otherwise = []
    where delta = b^2 - 4*a*c
```

- pattern matching.

```
(&&) :: Bool -> Bool -> Bool
True && True = True
True && False = False
False && True = False
False && False = False

-- Note:
x && x = x -- ERROR, we cannot repeat variables in patterns
x && y | x==y = x -- OK
```

- case expressions.

```
null :: [a] -> Bool
null xs = case xs of
            [] -> True
            (_:_) -> False

equivalent to

null :: [a] -> Bool
null xs = case xs of
            [] -> True
            _ -> False
```

> **EA-1:** IN-13

1.5. Recursion

In Haskell, there is not iteration, namely `for` and `while` cycles. To execute a fragment of code a certain number of times until a condition is met, one must use recursion, where a function's expression contains a call to itself.

> **EA-2:** IN-17, IN-18

Lambda Expressions

In Haskell, we can define an anonymous function (i.e., a function without a name) using a lambda expression. Lambda expressions:

- Allow the definition of functions whose results are other functions.
- Help avoid naming short functions.

Here's a simple example of a lambda function:

```
\x -> 2*x + 1  -- A function that corresponds to 2x + 1 for each x.
```

You can use this lambda expression like this:

```
> (\x -> 2*x+1) 1
3
> (\x -> 2*x+1) 3
7
```

You can also define named functions using lambda expressions. For example:

```
soma x y = x + y

-- This is equivalent to:

soma = \x -> (\y -> x + y)
```

Lambda expressions are particularly useful when working with higher-order functions. For instance, instead of defining a separate function to square numbers, you can use a lambda expression directly:

```
quadrados = map f [1..10]
    where f x = x^2

-- We can write this as:

quadrados = map (\x -> x^2) [1..10]
```

Operators and Sections

Expressions of the form `(x ⊗)` and `(⊗ x)` are called **sections**. They define the resulting function by applying one of the operator's arguments.

Here are a couple of examples using sections:

```
> (+1) 2
3
> (/2) 1
0.5
```

In these examples:

- (+ 1) is a section that takes a number and adds 1 to it.

- (/ 2) is a section that takes a number and divides it by 2.

For instance, consider the following function definition:

```
quadrados = map (\x -> x^2) [1..10]
```

This can be expressed more succinctly using a section:

```
quadrados = map (^2) [1..10]
```

---

2. Fundamentals on types

2.1. Elementary types

The type of any expression or function can be checked in GHCI using the `:type` command (or `:t` , for short). Examples:

```
Prelude > :type True
True :: Bool
```

```
Prelude > :type 'a'
'a' :: Char
```

Type	Comment
Bool	Boolean. Two values: True and False.
Char	Character. Denoted by single quotes.
Int	Integer. Fixed-precision (i.e. has a maximum size) depending on the system's architecture.
Integer	Integer. Arbitrary precision.
Float	Floating-point. Single precision (32 bits).
Double	Floating-point. Double precision (64 bits).

Table 2.1 Most common elementary types in Haskell

The Int type has the advantage over Integral of being more efficient in terms of time and space, since the variable's size is fixed. However, if not handled properly the former may lead to silent overflow issues.

2.2. Tuples

A tuple is a sequence of elements with a fixed size. The elements do not have to be all of the same type. They are denoted by parenthesis.

Tuples with one element do not exist as they have the type of the actual element.There is a single type for a tuple with zero elements, the unit type ().

```
Prelude > :type (True ,'a')

(True ,'a') :: (Bool , Char)

Prelude > :type ('x',(True ,False))

('x',(True ,False)) :: (Char , (Bool , Bool))
```

Function	Comment	Example
fst	Returns the first element of a pair (binary tuple).	`fst (3, 8) -> 3`
snd	Returns the second element of a pair (binary tuple).	`snd (3, 8) -> 8`

Table 2.2 Some Prelude functions for tuples

> **EA-1:** FT-3, FT-4

2.3. Lists

A list is a variable-sized sequence of elements of the same type. It has several key differences from tuples:

- Lists have a variable size, which means that a function that receives a list as input can handle lists with 2 or 1000 elements.
- They are homogeneous: all of the elements in a list must have the same type.
- A list can have one element (singleton list)

Empty lists are represented as [] . A String is a particular case of a list. It corresponds to an array of characters, [Char] .

Lists can be defined in various ways:

- Using square-brackets. This is used to represent lists of fixed size.

  `[1,4,7,10,13]`

- Using the "cons" operator (:).

  `1:4:7:10:13:[]`

- Using ranges.

  `[1,4..13]`

- Using list comprehensions

```
[x | x <- [1..15], mod x 3 == 1]
```

Function	Comment	Example
(++)	Appends two lists.	`[1,2] ++ [3,4] -> [1,2,3,4]`
head	Extracts the first element of non-empty lists.	`head [1,2,3] -> 1`
tail	Removes the first element of non-empty lists (output can be [] if input is a singleton).	`tail [1,2,3] -> [2,3]`
last	Extracts the last element of non-empty lists.	`last [1,2,3] -> 3`
init	Removes the last element of non-empty lists.	`init [1,2,3] -> [1,2]`
elem	Checks if a value is contained in a list.	`elem 3 [1,2,3] -> True`
(!!)	Returns the n-th element of a list (with indices starting at 0).	`[1,2,3] !! 1 -> 2`
length	Returns the number of elements in a list.	`length [1,2,3] -> 3`
reverse	Inverts the order of the elements in a list.	`reverse [1,2,3] -> [3,2,1]`
take	Extracts the first n elements of a list.	`take 2 [1,2,3] -> [1,2]`
drop	Removes the first n elements of a list.	`drop 2 [1,2,3] -> [3]`
repeat	Creates an infinite list with x as the value of all elements.	`repeat 1 -> [1,1,1,1,1,1...]`
cycle	Creates an infinite repetition of a list.	`cycle [1,2,3] -> [1,2,3,1,2,3...]`
zip	Creates a list with the corresponding pairs of two lists.	`zip [1,2,3] "abc" -> [(1,'a'),(2,'b'),(3,'c')]`
sum	Adds all of the numbers in the list.	`sum [1,2,3,4] -> 10`
product	Multiplies all of the numbers in the list.	`product [1,2,3,4] -> 24`

Table 2.3 Some Prelude functions for lists

Notes:

```
> head []
ERRO
```

```
head x:_ = x     -- ERRO
head (x:_) = x   -- OK
```

```
> init [1]
[]
> init []
*** Exception: Prelude.init: empty list
```

There is also a module with more useful functions to work with lists, Data.List. `import Data.List`

> **EA-1:** FT-9, FT-10, FT-11

> **EA-2:** FT-14, FT-18, FT-19

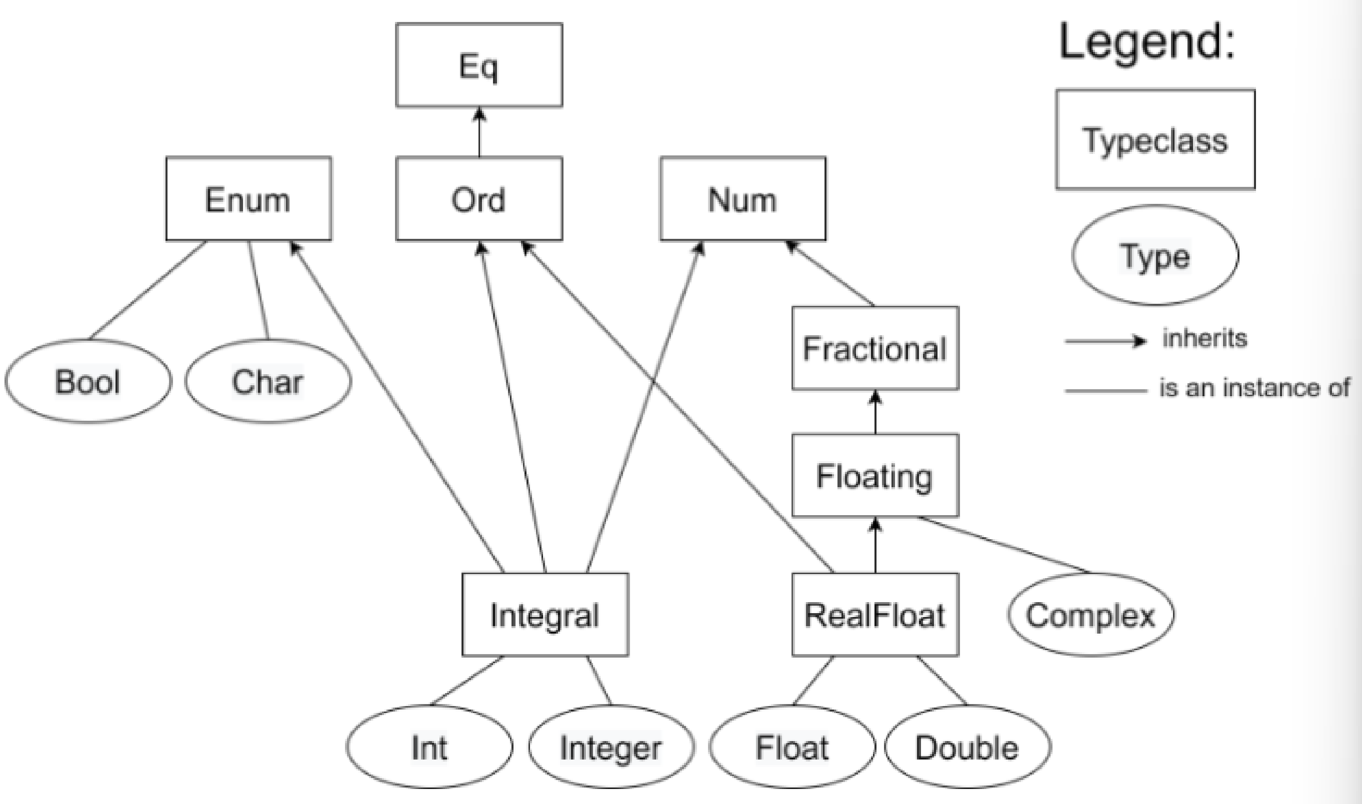
2.4. Typeclasses

Certain functions operate over a certain group of types but not over every single type. To allow this behavior, Haskell defines typeclasses which group a set of types by a common property. Types are instances of typeclasses, just like classes are implementations of interfaces in object-oriented programming languages (like Java).

Typeclass	Comment
Num	Numeric types.
Integral	Integer types.
Fractional	Floating-point types. Supports real number division with `(/)`.
Floating	Floating-point types. Includes a type for complex numbers, `Complex`. Defines certain functions with irrational numbers, such as `sqrt`, `log`, `sin`, `asin` and `sinh`.
RealFloat	Another floating-point typeclass that does not include complex numbers.
Eq	Types for which the equality and inequality operators (`==`, `/=`) are defined.
Ord	Types for which the comparison operators (`>`, `<`, `>=`, `<=`) are defined.
Enum	Types that can be enumerated.

Table 2.4 Some relevant typeclasses

Simplified hierarchy of the **most common** typeclasses and types in Haskell:




```
1 :: Int
1 :: Float
1 :: Num a => a -- tipo mais geral

3.0 :: Float
3.0 :: Double
3.0 :: Fractional a => a -- tipo mais geral

1/3 :: Float
(1 + 1.5 + 2) :: Float
```

Mixing Numeric Types

```
media xs = sum xs / length xs
```

If you try to compile this code, you might get an error like:

```
Could not deduce (Fractional Int) ...
```

Problem:

```
(/) :: Fractional a => a -> a -> a -- fractional division
length xs :: Int -- returns an Int, which is not a fractional type
```

Solution:

```
media xs = sum xs / fromIntegral (length xs)   -- use an explicit conversion
```

`fromIntegral` converts any integer type to any other numeric type, allowing length xs to be used in fractional operations.

Type Annotations

In Haskell, we can write function definitions and let the interpreter infer their types. However, it's generally recommended to always annotate function definitions with their types.

Benefits of Type Annotations:

- Documentation: Makes the purpose and expected inputs/outputs of the function clearer.
- Guidance: Helps guide the process of writing function definitions by clarifying expected types.
- Error Clarity: Can sometimes make error messages more understandable by providing context on expected types.

In Haskell, it's often easier to start with a concrete type and then generalize as you refine the function. This approach allows you to test the function with a specific type to ensure it works correctly before adapting it to operate over a broader range of types.

2.5. Type variables

When documenting the type of a variable, instead of making the commitment of assigning a variable to a certain type, one could instead associate a variable to a typeclass (a more general type declaration). This can be achieved using the notation: `e :: TC a =>a` . This line denotes that e is of type a, which is an instance of typeclass TC. a is a type variable: e belongs to any data type a that is an instance of typeclass TC. a is intentionally left undefined. The arrow `=>` denotes a class constraint.

For example:

```
zip [1,2] "abc" :: Num a => [(a, Char)]
```

When asked about the type of numbers (using the `:type` command), GHCI typically responds with a type variable to link it to a type class (usually Num) rather than a specific type.

> **EA-1:** FT-21

2.6. Functional Types

- **Type Declarations**: In Haskell, it's good practice to declare a function's type above its definition. This has several advantages:
  - **Error Reduction**: Helps programmers reason about their functions, reducing programming errors.
  - **Documentation**: Provides useful documentation for users to understand the function better.
  - **Clear Error Messages**: Assists in generating clearer error messages when functions are used with incorrect argument types.
- **Type Inference**: Haskell compilers can infer types for most expressions and functions, which means explicit type declarations are not always necessary.
- **Function Type Structure**: A function `f` with `n` arguments of types `t1, t2, ..., tn` produces an output of type `T`, represented as:

```
f :: t1 -> t2 -> ... -> tn -> T
```

If there are class constraints, they appear before the function name.

- **Polymorphic Functions**: Functions that have type declarations containing type variables are known as polymorphic functions.

> **EA-1:** FT-23, FT-24

---

3. Lists

3.1. Lists by range

Lists in Haskell can be defined using various range formats:

- **Basic Range with Step**:
  - Format: `[<value1>, <value2> .. <valueN>]`
  - Produces a list starting from `<value1>`, with each element incrementing by `<value2> - <value1>`, until exceeding `<valueN>`.
  - If the step is negative, `<value1>` must be greater than `<valueN>`. If the signs conflict, an empty list is returned.
- **Simple Incrementing Range**:
  - Format: `[<value1> .. <final value>]`
  - Assumes a step of `1` (i.e., `<value2> = <value1> + 1`).
- **Infinite List with Specified Step**:
  - Format: `[<value1>, <value2> ..]`
  - Creates an infinite list starting at `<value1>` with a step of `<value2> - <value1>`.
- **Infinite Incrementing List**:
  - Format: `[<value1>, ..]`

- Assumes a step of `1`.

Examples:

```
Prelude > [1..]
[1,2,3,4,5,6,7,8,9,10...]
Prelude > [1..5]
[1,2,3,4,5]
Prelude > [1,3..10]
[1,3,5,7,9]
Prelude > [1,0..]
[1,0,-1,-2,-3,-4,-5,-6,-7,-8...]
Prelude > [1,3..]
[1,3,5,7,9,11,13,15,17,19...]
Prelude > ['a'..]
"abcdefghijklmnopqrstuvwxyz{|}~\DEL\128\129..."
Prelude > [0.1,0.2..1]
[0.1, 0.2, 0.30000000000000004, 0.4, 0.5, 0.6, 0.7000000000000001, 0.8, 0.9, 1.0]
```

Important Notes:

- When using ranges with floating-point values, be cautious of numeric imprecision.
- Elements of a list range must belong to a type that is an instance of the Enum typeclass:

```
Prelude > f a b = [a .. b]
Prelude > :t f
f :: Enum a => a -> a -> [a]
```

Lazy Evaluation Haskell handles infinite lists through lazy evaluation, computing values only when needed. For example, using `take` to retrieve the first N elements from an infinite list results in finite computation:

```
Prelude > take 5 [1..]
[1,2,3,4,5]
```

Working with infinite lists separates the logic of generating a list from processing it, making certain functions easier to implement and more readable.

> **EA-4:** LI-2

3.2. Lists by recursion

The previous chapter introduced some examples of recursive functions with lists. This section contains exercises to implement recursive functions with lists that return new lists.

> **EA-2:** LI-13, LI-14, LI-15, LI-16, LI-17, LI-18, LI-20

> **EA-4:** LI-10

3.3. Lists by comprehension

List comprehensions are a concise way to build lists using other lists. The general structure is:

`[<pattern> | <generator 1>, <generator 2>, ..., <guard 1>, <guard 2> ...]`

- **Generators**: Each generator has the format `<pattern> <- <list>`. They iterate through their respective lists and produce values for each element visited.

```haskell
-- Ordem entre geradores
-- x primeiro, y depois
> [(x,y) | x<-[1,2,3], y<-[4,5]]
[(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)]
-- y primeiro, x depois
> [(x,y) | y<-[4,5], x<-[1,2,3]]
[(1,4),(2,4),(3,4),(1,5),(2,5),(3,5)]

> [(x,y) | x<-[1..3], y<-[x..3]]
[(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)]

> [(x,y) | y<-[x..3], x<-[1..3]]
error: Variable not in scope: x
```

- **Guards**: Conditions that must be met for an instance of the pattern to be included in the output list. They function as filters.

```haskell
-- os inteiros x tal que x está entre 1 e 10 e x é par.
> [x | x<-[1..10], x /`mod/` 2==0]
[2,4,6,8,10]

-- divisores de um numero inteiro positivo
divisores :: Int -> [Int]
divisores n = [x | x<-[1..n], n /`mod/` x==0]
```

Examples:

1. **Basic List Comprehension**:

   ```haskell
   Prelude > [x^2 | x <- [1..10]]
   [1, 4, 9, 16, 25, 36, 49, 64, 81, 100]
   ```

2. **Using Guards**:

```
Prelude > [x^2 | x <- [1..10], odd x]
[1, 9, 25, 49, 81]

Prelude > [x^2 | x <- [1..10], odd x, mod x 3 == 0]
[9, 81]
```

3. **Patterns in Generators**:
   - determina quais os valores das variáveis no padrão
   - e a ordem pela qual os valores são gerados

```
Prelude > [x | (x:_) <- [[1, 2], [3, 4]]]
[1, 3]

Prelude > [(a, b) | (a, b) <- zip [1..3] [1..]]
[(1, 1), (2, 2), (3, 3)]
```

Multiple Generators:

Using multiple generators behaves like nested loops: for each value of the leftmost generator, all combinations of values from the generators to the right are produced. Changing the order of generators affects the resulting list.

Examples:

1. **Order Matters**:

```
Prelude > [(x, y) | x <- [1, 2], y <- "ab"]
[(1, 'a'), (1, 'b'), (2, 'a'), (2, 'b')]

Prelude > [(x, y) | y <- "ab", x <- [1, 2]]
[(1, 'a'), (2, 'a'), (1, 'b'), (2, 'b')]
```

2. **Nested List Comprehension**:

```
Prelude > [[x, y] | x <- "ab", y <- x:"ab"]
["aa", "aa", "ab", "bb", "ba", "bb"]
```

3. **Dependencies** The values used in generators can depend on previous values but not on subsequent ones.

```
> [(x,y) | x<-[1..3], y<-[x..3]]
[(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)]
```

```
> [(x,y) | y<-[x..3], x<-[1..3]]
error: Variable not in scope: x
```

Conclusion

List comprehensions provide a powerful and expressive way to create and manipulate lists in Haskell, allowing for concise code that can replace more verbose looping constructs.

> **EA-2:** LI-29, LI-31, LI-32, LI-33, LI-35, LI-36, LI-39

> **EA-4:** LI-34, LI-37, LI-38, LI-40, LI-41, LI-42

`zip` function

The `zip` function in Haskell combines two lists into a list of pairs containing corresponding elements.

```
zip :: [a] -> [b] -> [(a,b)]
```

Example:

```
> zip ['a','b','c'] [1,2,3,4]
[('a',1), ('b',2), ('c',3)]
```

If the lists have different lengths, the result will have the length of the shorter list.

Using `zip` to Find Indices

You can also use `zip` to combine elements of a list with their indices. For instance, if you want to find the indices of occurrences of a value in a list:

```
indices :: Eq a => a -> [a] -> [Int]
indices x ys = [i | (y,i)<-zip ys [0..n], x==y]
where n = length ys - 1
```

```
> indices 'a' ['b','a','n','a','n','a']
[1,3,5]
```

This function generates a list of indices where the value x occurs in the list ys.

Using `zip` and `tail` for Consecutive Pairs

You can also use `zip` together with `tail` to create a list of consecutive pairs from a list:

```
pares :: [a] -> [(a,a)]
pares xs = zip xs (tail xs)

xs = [x1, x2, ..., x_n-1, x_n]
tail xs = [x2, x3, ..., x_n]
zip xs (tail xs) = [(x1,x2), (x2,x3), ..., (x_n-1, x_n)]
```

In this case:

- xs could be [x1, x2, ..., x_n]
- tail xs would be [x2, x3, ..., x_n]
- The zip function combines these two lists into pairs: [(x1, x2), (x2, x3), ..., (x_n-1, x_n)]

```
> pares [1,2,3,4]
[(1,2),(2,3),(3,4)]
> pares ['a','b','b','a']
[('a','b'),('b','b'),('b','a')]
> pares [1,2]
[(1,2)]
> pares [1]
[]
```

Using `zip` to Count Consecutive Equal Elements

You can also define a function to count the number of consecutive elements that are equal in a list:

```
paresIguais :: Eq a => [a] -> Int
paresIguais xs = length [(x, x') | (x, x') <- zip xs (tail xs), x == x']
```

```
> paresIguais [1, 1, 2, 2, 3]
2

> paresIguais ['a', 'b', 'b', 'a']
1
```

This function effectively counts how many times adjacent elements in the list are equal by leveraging the `zip` function.

String

In Haskell, the String type is predefined as a synonym for a list of characters:

```
type String = [Char] -- definido no prelúdio-padrão
```

For example, the string `"abba"` is equivalent to the list of characters `['a', 'b', 'b', 'a']`.

Since strings are essentially lists of characters, you can use list functions directly on them.

Exemplos:

```
> length "abcde"
5

> take 3 "abcde"
"abc"

> zip "abc" [1,2,3,4]
[('a',1),('b',2),('c',3)]
```

You can also use list comprehensions with strings. For example, to count the number of uppercase letters in a string:

```
contarLetras :: String -> Int
contarLetras txt = length [c | c<-txt, c>='A' && c<='Z']
```

Many specialized functions related to characters are defined in modules, which need to be imported to use their functions. For example, the `Data.Char` module includes several useful functions:

```
isUpper :: Char -> Bool
  -- testar se é letra maiúscula
isLower :: Char -> Bool
  -- testar se é letra minúscula
isLetter :: Char -> Bool
  -- testar se é letra (qualquer)
toUpper :: Char -> Char
  -- converter para maiúscula (ou for letra)
toLower :: Char -> Char
  -- converter para minúscula (se for letra)
```

```
import Data.Char

countLetters :: String -> Int
countLetters xs = length [x | x<-xs, isLetter x]

stringToUpper :: String -> String
stringToUpper xs = [toUpper x | x<-xs]
```

```
> countLetters "Abba123"
4
> stringToUpper "Abba123"
"ABBA123"
```

List Comprehensions vs. Recursive Functions

Any definition using list comprehensions can be translated into recursive functions.

However, recursive definitions are often more general than those using list comprehensions.

Example 1: Listing Squares

```
-- versão com lista em compreensão
listarQuadrados n = [i^2 | i<-[1..n]]
-- versão recursiva
listarQuadrados’ n = quadrados 1
where
quadrados i
| i<=n = i^2 : quadrados (i+1)
| otherwise = []
```

Example 2: Summing Squares

```
-- versão com lista em compreensão
somarQuadrados n = sum [i^2 | i<-[1..n]]
-- versão recursiva sem listas
somarQuadrados’ n = quadrados 1
where
quadrados i
| i<=n = i^2 + quadrados (i+1)
| otherwise = 0
```

This shows how list comprehensions can sometimes be transformed into more efficient recursive functions that do not require intermediate lists.

---

4. Higher-order functions

This chapter covers **higher-order functions**, central to functional programming for enabling greater abstraction and flexibility by allowing functions to take or return other functions. It introduces *lambdas* (anonymous functions), *currying* (transforming multi-argument functions into single-argument ones), and *key Prelude functions* like function *composition* (.), *function application* ($), and *folds* (foldl, foldr). The chapter concludes with *point-free style*, a concise way of writing functions without explicit arguments for cleaner, more readable code.

4.1. Fundamentals on higher-order functions

In functional type declarations, the `->` symbol is right-associative, meaning `a -> b -> c` is equivalent to `a -> (b -> c)`. Parentheses are used to clarify when an argument is a function. For example, a function f with a functional argument and returning another function would be declared as `f :: (a -> b) -> c -> (d -> e)`. To use such a function, the usual prefix notation can be applied: `f xyz`, where `x`, `y`, and `z` are the function's arguments.

> **EA-3:** HO-3, HO-4, HO-7

4.2. Lambdas

Lambdas, or anonymous functions, are a convenient way to define functions on the fly without giving them a name. They are particularly useful when the function is only needed once or in a small scope. The general form of a lambda in functional programming languages, such as Haskell, is:

```
\x1 x2 ... xN -> f x1 x2 ... xN
```

Here, the backslash (`\`) indicates the start of the lambda function, followed by the arguments, and then the function's expression after the `->`.

Anonymous: Lambdas are functions without names.

On-the-fly: They are typically written where needed, such as within an expression.

Single clause: Unlike named functions, lambdas must be defined in a single clause, which can limit their use for more complex recursive cases.

Pattern matching: You can use pattern matching in lambdas, but if the pattern fails, a runtime error will occur.

Examples:

```
(\x -> x + 1) 2 evaluates to 3 (adds 1 to the argument).
(\(x:xs) -> x) [1..10] extracts the first element of a list (1 from [1..10]).
```

Lambdas are commonly used in places where defining a full function is unnecessary.

> **EA-3:** HO-8

4.3. Currying

In Haskell, all functions actually only accept one argument. Functions with multiple arguments can be considered as a series of functions which receive an argument and return a function which receives the second argument, and so on. This is known as currying, a reference to the mathematician Haskell Curry, who shares his first name with the programming language.

- **Aplication:** Arguments are passed to functions, one-by-one, by putting spaces between the function's name and the name of each argument.
- **Partially applied**: If fewer arguments are passed than required, the function doesn't immediately compute a result; instead, it returns another function that waits for the remaining arguments. This can help reduce the need for defining new functions and makes the code more concise.

```
map (drop 2) [[1,2,3],[4,5,6]]
-- In this case, drop 2 is partially applied, resulting in a new function that removes the first two elements of any list passed to it.
```

- **Sections:** Infix operators can also be partially applied by enclosing them in parentheses, which is known as sections. This feature allows for concise function definitions using operators.

```
map (*2) [1..5]     -- Multiplies each element by 2
map (2*) [1..5]     -- Also multiplies each element by 2
(`elem` [1..3]) 3  -- Checks if 3 is in the list [1..3]
```

- **Curried vs. Tupled Arguments:** While functions can alternatively take a tuple as an argument (unary function with a tuple), this is discouraged. Curried functions offer more flexibility than tuple arguments. Tuples should only be used when the argument itself is inherently a tuple.

```
-- Example of a curried function:
add :: Int -> Int -> Int
add x y = x + y

-- Example of a function with a tuple argument:
add' :: (Int, Int) -> Int
add' (x, y) = x + y
```

> **EA-3:** HO-10

4.4. Common higher-order functions

Function	Comment	Example
map	Applies a function to each element of a list.	`map succ [1,2,3] -> [2,3,4]`
filter	Returns a sublist with the elements that satisfy a predicate (i.e. a function that returns a boolean).	`filter odd [1..5] -> [1,3,5]`
any	Checks if at least one element of a list satisfies a predicate.	`any even [1,1,1,3,1] -> False`

Function	Comment	Example
all	Checks if all the elements of a list satisfy a predicate.	`all odd [1,1,1,3,1] -> True`
takeWhile	Returns the longest prefix of a list that satisfies a predicate.	`takeWhile odd [1,1,1,2,3] -> [1,1,1]`
dropWhile	Returns the remainder of a list after calling `takeWhile`.	`dropWhile odd [1,1,1,2,3] -> [2,3]`
iterate	Returns an infinite list where the i-th element is the application of a function `f` on a value `x` i times (with indices starting at 0).	`iterate succ 0 -> [0,1,2,3,4,5,6,7,8,9,...]`
zipWith	Zips two lists, then combines each pair using a binary function.	`zipWith (+) [1,2,3,4] [3,2,4,1] -> [4,4,7,5]`
flip	Swaps the order of the arguments in a binary function.	`flip (/) 2 0 -> 0.0`
(.)	Function composition.	(Check section 4.5)
($)	Function application.	(Check section 4.5)
foldr	Right-associative fold of a structure.	(Check section 4.6)
foldl	Left-associative fold of a structure.	(Check section 4.6)

`Filter`

```haskell
filter :: (a -> Bool) -> [a] -> [a]

> filter (\n->n‘mod‘2==0) [1..10]
[2,4,6,8,10]
> filter isLower "Hello, world!"
"elloworld"

-- can be defined as
filter p xs = [x | x<-xs, p x]
-- or
filter p [] = []
filter p (x:xs)
    | p x = x : filter p xs
    | otherwise = filter p xs
```

`takeWhile` and `dropWhile`

```haskell
takeWhile, dropWhile :: (a -> Bool) -> [a] -> [a]

> takeWhile isLetter "Hello, world!"
"Hello"
> dropWhile isLetter "Hello, world!"
```

```
", world!"

-- can be defined as
takeWhile :: (a -> Bool) -> [a] -> [a]
takeWhile p [] = []
takeWhile p (x:xs)
    | p x = x : takeWhile p xs
    | otherwise = []
dropWhile :: (a -> Bool) -> [a] -> [a]
dropWhile p [] = []
dropWhile p (x:xs)
    | p x = dropWhile p xs
    | otherwise = x:xs
```

`all` and `any`

```
all, any :: (a -> Bool) -> [a] -> Bool

> all (\n -> n‘mod‘2==0) [2,4,6,8]
True
> any (\n -> n‘mod‘2/=0) [2,4,6,8]
False
> all isLower "Hello, world!"
False
> any isLower "Hello, world!"
True

-- can be defined as
all p xs = and (map p xs)
any p xs = or (map p xs)
-- or
all p [] = True
all p (x:xs) = p x && all p xs
any p [] = False
any p (x:xs) = p x || any p xs
```

`foldr`

```
sum [] = 0                          z = 0
sum (x:xs) = x + sum xs            ⊕ = +

sum = foldr (+) 0

product [] = 1                      z = 1
product (x:xs) = x * product xs    ⊕ = *

product = foldr (*) 1

and [] = True                       z = True
```

```
and (x:xs) = x && and xs            ⊕ = &&

and = foldr (&&) True

or [] = False                        z = False
or (x:xs) = x || or xs              ⊕ = ||

or = foldr (||) False

length [] = 0                        z = 0
length (x:xs)= 1 + length xs        ⊕ = \_ n → 1 + n

length = foldr (\_ n->n+1) 0

-- can be defined as
foldr :: (a -> b -> b) -> b -> [a] -> b
foldr f z [] = z
foldr f z (x:xs) = f x (foldr f z xs)

foldr f z [1,2,3,4,5] = foldr f z (1:2:3:4:5:[]) = f 1 (f 2 (f 3 (f 4 (f 5 z))))
```

`foldl`

`foldr` vs `foldl`:

```
foldl (+) 0 [1,2,3,4] = (((0+1)+2)+3)+4 = 10
foldr (+) 0 [1,2,3,4] = 1+(2+(3+(4+0))) = 10
```

```
-- can be defined as
foldl :: (a -> b -> a) -> a -> [b] -> a
foldl f z [] = z
foldl f z (x:xs) = foldl f (f z x) xs

foldl f z [1,2,3,4,5] = foldl f z (1:2:3:4:5:[]) = f (f (f (f (f z 1) 2) 3) 4) 5
```

Composição (.)

```
(.) :: (b -> c) -> (a -> b) -> a -> c
f . g = \x -> f (g x)

f xs = sum (map (^2) (filter even xs))
-- is equivalent to
f = sum . map (^2) . filter even
```

> **EA-3:** HO-13, HO-14, HO-15, HO-16, HO-17, HO-18, HO-22, HO-23

> **EA-4:** HO-19, HO-24

4.5 Application and composition

In Haskell, **function application** is done by placing the function name followed by its arguments, separated by spaces. `f x y` is interpreted as `f x y = (f x) y`. By default, function application is *left-associative*, meaning it is processed from left to right. Application has a very high precedence, so it occurs before other operations.

- **Dollar Sign ($):** This operator is used to apply a function with lower precedence than regular application. This allows for reduced parentheses in expressions.

```
succ (succ 1)          -- 3
succ succ 1            -- error
succ $ succ 1          -- 3
(*3) $ succ $ (*8) 2  -- 51
```

Function composition allows the output of one function to be used as the input for another, creating a new function. It is denoted by `(g . f)`, where `g` is applied to the result of `f`. Composition is *right-associative*, meaning `h . g . f` is evaluated as `h . (g . f)`.

```
(head . tail) [1..5] -- 2
(succ . (*8)) 4      -- 33
```

> **EA-3:** HO-27, HO-29

4.6 Folds

Folds are higher-order functions that process data structures (typically lists) in a specified order and return a value. Folds have usually **two ingredients**:

- **Combining Function:** A binary function that takes two inputs(an accumulator and an element from the data structure).
- **Accumulator:** A value that accumulates results as the fold processes the list.

Folding functions receive three arguments (in order): the *combining function*, the *initial value of the accumulator* and the *list*:

- *Accumulator:* must be the same type as the return value type of the fold.
- *Initial value:* usually the identity/neutral element of the combining function.

The **two main folding functions** are:

- `foldr` (Right Fold):
  - it recursively combines the result of the list's head and accumulator with the result combining with the tail.

- `foldl` (Left Fold):
  - it recursively combines the result of combining all but the list's last element and the accumulator with the last element.

Using left or right folds gives the same result if the operation of the combining f function is *associative*: f(fab)c = fa(fbc).

Difference between left and right folds:

```
foldr (-) 0 [1..5] -- = (1 - (2 - (3 - (4 - (5 - 0))))) = 3
foldl (-) 0 [1..5] -- = (((((0 - 1) - 2) - 3) - 4) - 5) = -15
```

Folds have the *advantage* of allowing for more compact code, relative to recursive functions.

`scanr` and `scanl`: work mostly like `foldr`/`foldl` but instead return a list with all the intermidiate values of the computations.

```
scanr (-) 0 [1..5] -- = [3,-2,4,-1,5,0]
scanl (-) 0 [1..5] -- = [0,-1,-3,-6,-10,-15]
```

> **EA-3:** HO-32, HO-33, HO-35, HO-37, HO-40, HO-42, HO-43

4.7. Point-free style

Point-free style: the arguments of the function are omitted from its definition.

The main tools to program in point-free style are using composition and other higher-order functions (namely maps, filters and folds), rather than application.

Advantage: function definitions are more readable, elegant and concise.

Example:

```
import Data.Char
capitalize :: [Char] -> [Char]
capitalize = map toUpper

{- map is used to apply toUpper to each element of the string. The function is
defined in point-free style by leaving map partially applied: only the functional
argument is provided, while the list is left to be applied by those who call
capitalize .
This solution shows an example of how to write an unary function in point-
free style. -}
```

> **EA-3:** O-47, HO-48, HO-49, HO-50, HO-51, HO-52, HO-53

Infinite Lists

- Because of lazy evaluation, lists are calculated as needed and only as far as necessary.

```
uns :: [Integer]
uns = 1 : uns

head uns = head (1:uns) = 1
```

- A computation that needs to traverse an entire infinite list does not finish.

```
length uns = length (1:uns) = 1 + length uns = 1 + length (1:uns) = 1 + (1 +
length uns) = ... não termina
```

---

5. User-defined types

5.1. Creating type synonyms with the type keyword

The `type` keywork can be used to define **type synonyms**.

- *Advantage:* increase the readability of Haskell code by providing syntatic sugar.

```
type <synonym name > <type variable 1> <type variable 2> ... = <expression >
```

- Synonym's name must start with an uppercase.
- Synonyms cannot have recursive definitions.

```
type Pos = (Int,Int)
type Cells = [Pos]             -- OK
type Tree = (Int,[Tree])       -- ERROR
```

```
type String = [Char]   -- from Prelude
type Pair a = (a,a)
type HashMap k v = [(k,v)]
```

> **EA-4:** UT-3, UT-4

5.2. Creating algebraic data types with the data keyword

If one wants to define a structure for a person with two strings: one with their name and another with their email:

```
type Person = (String,String)
```

However, the declaration above does not allow one to distinguish a person from any other pair composed of two strings.

If one deines a function that receives as input a pair representing a country and its capital (`type CountryCapital = (String,String)`) one could also pass a Person as input, which is semantically wrong, even though it is syntactically correct.

The `data` keyword circumvents this problem by defining new algebric data types.

- The data statement lists the alternative values of the new type.
- `True` and `False` are the constructors of the `Bool type`

```
data Bool = False | True    -- Prelude
```

- Constructors must be unique (cannot be used in different types)
- The names of types and constructors must begin with a capital letter
- *Advantages:* better structured code, readability and improvestype safety.

```
data <type name > <type variable 1> <type variable 2> ... =
<value constructor 1> <type 1> <type 2> ... |
<value constructor 2> <type 1> <type 2> ... |
...
```

Examples:

```
data Bool = False | True              -- Prelude
data Maybe a = Just a | Nothing       -- Prelude
data Shape = Circle Double Double Double | Rectangle Double Double Double Double
```

- **Maybe:** can be used as an alternative to errors in functions that may fail (e. g. `head` with an empty list)
  - *Just:* has a variable type
  - *Nothing:* has no arguments/fields

Each value constructor must only be used once in a data declaration. They can be used in two different ways: as functions or in patterns. Examples:

```
Prelude > data Shape = Circle Double Double Double | Rectangle Double Double
Double Double
```

```
Prelude > :type Circle
  Circle :: Double -> Double -> Double -> Shape
Prelude > :type ( Circle 2.0)
  ( Circle 2.0) :: Double -> Double -> Shape
Prelude > let area (Circle _ _ r) = pi*r^2
Prelude > :type area
  area :: Shape -> Double
Prelude > area( Circle 1.0 2.0 1.0)
  3.141592653589793
Prelude > map (area .( Circle 1.0 2.0)) [1..5]
  [3.141592653589793 ,12.566370614359172 ,28.274333882308138 ,50.26548245743669
,78.53981633974483]
```

The type of the area function is Shape -> Double rather than Circle -> Double , since Shape is the actual type's name, while Circle is the name of one of its value constructors.

- Unlike type synonyms, type definitions can be recursive.

```
Prelude > data MyList a = List a ( MyList a) | EmptyList
*Main > :type (List 4 (List 6 EmptyList ))
  (List 4 (List 6 EmptyList )) :: Num a => MyList a
```

> **EA-4:** UT-6, UT-7, UT-8

5.3. Derived types

Consider the `Shape` type defined in the previous section. To print a shape or compare two shapes one must define that it derives the typeclass Eq, using the `deriving`keyword.

```
Prelude > data Shape = Circle Double Double Double | Rectangle Double Double
Double Double
Prelude > Rectangle 1.0 2.0 3.0 4.0
<interactive >:2:1: error:
* No instance for (Show Shape) arising from a use of 'print'
* In a stmt of an interactive GHCi command : print it
Prelude > Circle 3 4 5 == Circle 3 5 5
<interactive >:3:1: error:
* No instance for (Eq Shape) arising from a use of '=='
* In the expression : Circle 3 4 5 == Circle 3 5 5
In an equation for 'it': it = Circle 3 4 5 == Circle 3 5 5
```

```
Prelude > data Shape = Circle Double Double Double | Rectangle Double Double
Double Double deriving (Show ,Eq)
Prelude > Rectangle 1.0 2.0 3.0 4.0
Rectangle 1.0 2.0 3.0 4.0
Prelude > Circle 3 4 5 == Circle 3 5 5
```

```
False
Prelude > Circle 3 4 5 == Circle 3 4 5
True
```

Other examples:

```
Prelude > data Tempo = Adagio | Andante | Moderato | Allegro | Presto deriving (Eq
,Ord ,Show ,Enum)

-- Ord
Prelude > Andante < Allegro
True
Prelude > Adagio >= Moderato
False

-- Enum
Prelude > [Adagio ..]
[Adagio ,Andante ,Moderato ,Allegro , Presto]
Prelude > [Adagio , Moderato ..]
[Adagio ,Moderato , Presto ]
Prelude > [Presto , Allegro ..]
[Presto ,Allegro ,Moderato ,Andante , Adagio]
```

For types T with parameters that derive Ord , a value A is less than a value B if the value constructor of A comes before the one for B in the definition of T.

```
Prelude > data Shape = Circle Double Double Double | Rectangle Double Double
Double Double deriving (Eq ,Ord)
Prelude > Circle 1 2 3 < Rectangle 1 2 3 4
True
Prelude > Circle 1 2 3 < Circle 1 2 3
False
Prelude > Circle 1 50 3 < Circle 1 2 3
False
Prelude > Circle 1 2 3 < Circle 1 3 3
True
```

5.4. Named fields

When defining a new type using data , the fields of a value can be given names using the record syntax. If a class with named fields is an instance of Show, then they are printed in a different manner.

```
Prelude > data Date = Date { day :: Int , month :: Int , year :: Int} deriving
(Show)
Prelude > Date 18 6 2006
Date {day = 18, month = 6, year = 2006}
```

```
Prelude > Date {day = 18, year = 2006 , month = 6}
Date {day = 18, month = 6, year = 2006}
```

- *Advantage:* allowing one to forget their order in the definition of a value constructor.

It also avoids the need to write "getter" functions, which retrieve a field of a value.

```
Prelude > let d = Date {day = 18, year = 2006 , month = 6}
Prelude > month(d)
6
```

> **EA-4:** UT-9

Polimorphismo e sobrecarga

In functional programming, a definition that works with multiple types is said to have a **polymorphic type**. For example, the function `length` is polymorphic, as it can operate on a list of any type: Exemplo: a função length.

```
length :: [a] -> Int
length [] = 0
length (x:xs) = 1 + length xs
```

In object-oriented programming, this is similar to having a **generic type**.

Type classes allow for **operator overloading**, enabling the same operator to have different implementations for different types. For instance, the equality operator (==) can be defined for multiple types:

```
(==) :: Int -> Int -> Bool -- tipo Int
(==) :: Float -> Float -> Bool -- tipo Float
(==) :: String -> String -> Bool -- tipo String
...
(==) :: Eq a => a -> a -> Bool -- tipo mais geral
```

Instances Declarations

An **instance** declaration defines how a type conforms to a type class. For example, here is the equality instance for Bool (as defined in the standard Prelude):

```
instance Eq Bool where
True == True = True
False == False = True
_ == _ = False
```

You can also create instances for custom types. For instance, defining equality for the recursive type `Nat`:

```
data Nat = Zero | Succ Nat
instance Eq Nat where
Zero == Zero = True -- base case
Succ x == Succ y = x==y -- recursive case
_ == _ = False
```

Here, the recursive nature of `Nat` requires a recursive definition for equality as well.

Classes Restrictions

Type class constraints enable additional requirements for types within a class. For example, the class `Ord` (total ordering) requires that any type in it also has equality (`Eq`):

```
class (Eq a) => Ord a where
(<), (<=), (>), (>=) :: a -> a -> Bool
min, max :: a -> a -> a
```

You can also apply constraints in instance declarations, such as defining equality for lists based on equality of their elements:

```
instance (Eq a) => Eq [a] where
[] == [] = True
(x:xs) == (y:ys) = x==y && xs==ys
_ == _ = False -- diferentes
```

Example: `Num` and `Fractional` Classes

The `Num` type class requires `Eq` and `Show` instances and includes basic arithmetic:

```
class (Eq a, Show a) => Num a where
(+), (-), (*) :: a -> a -> a
negate :: a -> a
abs, signum :: a -> a
fromInteger :: Integer -> a
class (Num a) => Fractional a where
(/) :: a -> a -> a
recip :: a -> a
```

Example: Implementing Rational Numbers

Using type classes, we can create a data type for rational numbers, or fractions. This includes:

- Show instance for converting to text
- Eq and Ord instances for comparisons
- Num and Fractional instances for arithmetic

```
data Fraction = Frac Integer Integer

num, denom :: Fraction -> Integer
num (Frac p q) = p
denom (Frac p q) = q
```

Define an infix operator % to construct rational numbers, allowing for easier representation and automatic normalization:

```
(%) :: Integer -> Integer -> Fraction
p % q
| q==0 = error "%: division by zero"
| q<0 = (-p) % (-q)
| otherwise = Frac (p`div`d) (q`div`d)
where d = mdc p q
mdc :: Integer -> Integer -> Integer
mdc a 0 = a
mdc a b = mdc b (a`mod`b)
```

Advantages:

- **Readability** (e.g., we write 1 % 2 instead of Frac 1 2);
- **Encapsulation** of the representation;
- **Automatic normalization** of the representation.

```
-- pré-condição: as frações são normalizadas
instance Eq Fraction where
(Frac p q) == (Frac r s) = p==r && q==s
instance Show Fraction where
show (Frac p q) = show p ++ (’%’: show q)

instance Num Fraction where
(Frac p q) + (Frac r s) = (p*s+q*r) % (q*s)
(Frac p q) * (Frac r s) = (p*r) % (q*s)
negate (Frac p q) = Frac (-p) q
fromInteger n = Frac n 1
```

Comparison of Fractions: We define one comparison operator (e.g., <=), and Haskell will derive the rest:

```
instance Ord Fraction where
(Frac p q) <= (Frac r s) = p*s-q*r<=0
```

Os operadores `<`, `>`, `>=` ficam definidos apartir de `<=`, `==` e `/=`.

5.5. Modules

A module is a collection of related definitions, such as functions, types, and typeclasses. Modules help organize code, making it easier to reuse in different projects. Some standard modules in Haskell include `Data.List` and `Data.Char`.

Benefits of Using Modules:

- Code Reusability: Avoids duplication by allowing commonly used functions or definitions to be reused in multiple projects.

Basic Syntax for Importing a Module:

```
import <module name>

import Data.Char
```

The import statement is used to bring a module's definitions into scope, making them available in your source code or in the interactive console.

Selective Importing: You can also import only specific functions, types, or typeclasses from a module to keep the namespace clean.

```
import <module name> (<definition 1>, <definition 2>)

import Data.List (nub , sort)
```

Defining Custom Modules: Users can create their own modules by defining them at the beginning of a source code file. You can specify what to export by listing the functions, types, or constructors you want to make available to other files.

```
module <module name> (<definition 1, definition 2,...) where

module Shape (
Shape , -- export the data type
Circle , Rectangle -- export the value constructors
area -- export the function
) where

module Fraction (Fraction, (%)) where
```

> **EA-5:** UT-11

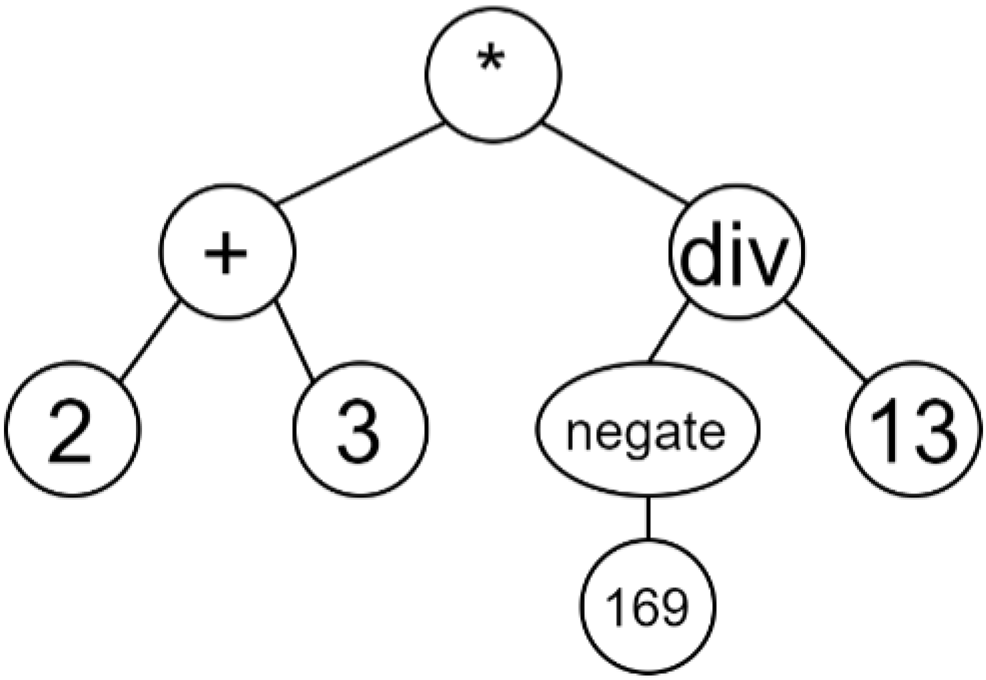
5.6. Case study 1: Syntax trees

Syntax trees represent expressions in a structured, hierarchical form. Each node in a syntax tree represents an operator and has at least one child, while leaves represent constant values and have no children.

`(+) 2 3 * (negate 169 `div` 13)`

In a syntax tree, operators like `+`, `*`, and `div` are represented as nodes, and values like `2`, `3`, `13` and `169` are the leaves of the tree.

Here's a visualization of the expression as a syntax tree:

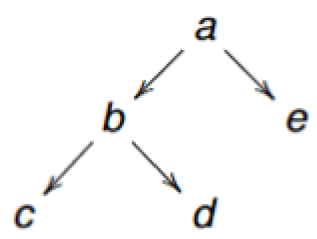


> **EA-5:** UT-12, UT-13, UT-14

Binary Trees

A BST can be represented in haskell as:

```haskell
data Tree = Node Tree Tree | Leaf
```



$$\underbrace{\texttt{No}\ \underbrace{(\texttt{No}\ \overbrace{\texttt{Folha}}^{c}\ \overbrace{\texttt{Folha}}^{d})}_{b}\ \overbrace{\texttt{Folha}}^{e}}_{a}$$

We can associate additional information to tree structures by adding annotations to nodes, leaves, or both. Here are a few examples:

- Nodes annotated with integers: `data Tree = Node Int Tree Tree | Leaf`
- Leaves annotated with integers: `data Tree = Node Tree Tree | Leaf Int`
- Nodes annotated with integers, leaves with booleans: `data Tree = Node Int Tree Tree | Leaf Bool`

Instead of using specific types for annotations, we can generalize the tree structure by parameterizing the annotation types. This approach enables greater flexibility by allowing annotations of any type.

- Nodes annotated with a generic type `a`: `data Tree a = Node a (Tree a) (Tree a) | Leaf`
- Leaves annotated with a generic type `a`: `data Tree a = Node (Tree a) (Tree a) | Leaf a`
- Nodes annotated with type `a`, leaves with type : `data Tree a b = Node a (Tree a b) (Tree a b) | Leaf b`

This generalized approach lets you create trees with any kind of annotation on nodes and leaves, enhancing reusability and adaptability for various data types.

To **list all the values** in a tree in in-order traversal (infix order), we need to follow these steps:

- Traverse the left subtree.
- Visit the node itself.
- Traverse the right subtree.

In this recursive function, the base case is an empty tree, which returns an empty list.

Here's how the function is defined in Haskell:

```haskell
data Tree a = Node a (Tree a) (Tree a) | Empty

listar :: Tree a -> [a]
listar Empty = []
listar (Node x left right) = listar left ++ [x] ++ listar right
```

If a binary tree is sorted (meaning it's a **binary search tree**, or BST), then `listar` will produce the values in ascending order. We can use this property to check if a tree is ordered by verifying that the list produced by `listar` is strictly increasing.

Here's how the `ordenada` function works:

- It lists all values in the tree using listar.
- It checks if this list is in ascending order by using a helper function ascendente.

The function in Haskell is defined as follows:

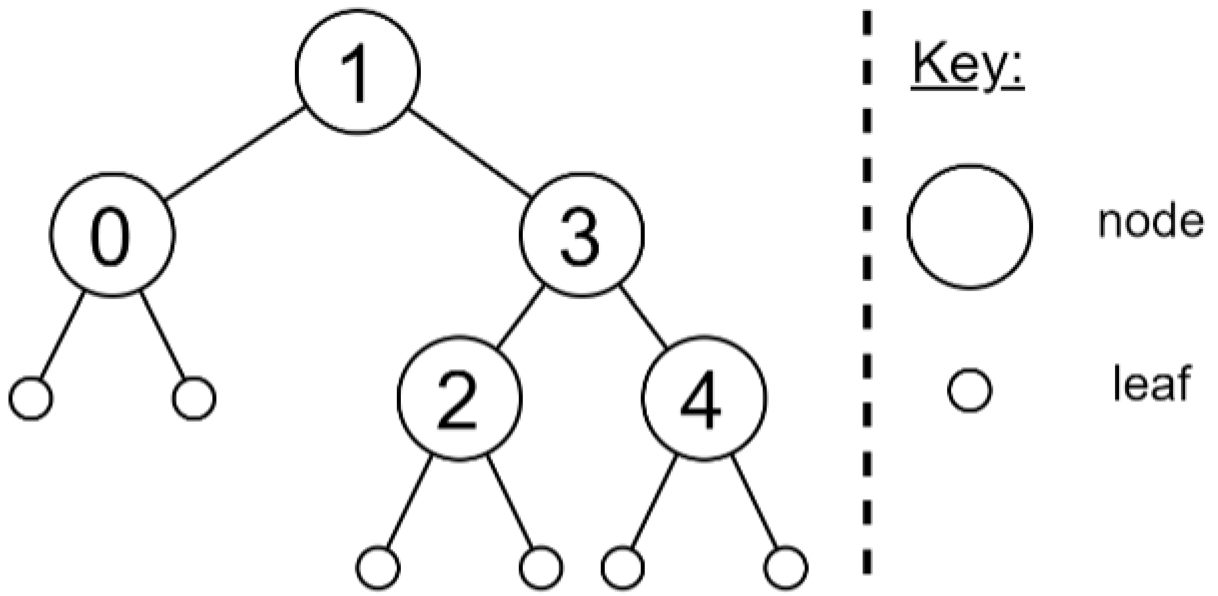
```haskell
ordenada :: Ord a => Tree a -> Bool
ordenada tree = ascendente (listar tree)
  where
    ascendente [] = True
    ascendente [_] = True
    ascendente (x:y:xs) = x < y && ascendente (y:xs)
```

5.7. Case study 2: Binary search trees

A **Binary Search Tree** (**BST**) is a tree data structure where each node has up to two children and contains a key from a type that supports the `<` operator (i.e., a type that is an instance of Ord). Each node has:

- **Left Child**: Contains only keys smaller than the node's key.
- **Right Child**: Contains only keys greater than the node's key.

In a BST, keys are stored in an ordered way, allowing efficient searching, insertion, and deletion. Each subtree of a node is itself a BST, and it is assumed that all keys in the tree are unique. Leaves (nodes without children) do not contain any data.



Searching for a Value in a Sorted Tree

To search for a value in a **BST**:

1. Compare the target value with the node's value.
2. Recursively search in the left or right subtree depending on the comparison result.

Here is the function procurar in Haskell:

```
procurar :: Ord a => a -> Tree a -> Bool
procurar x Vazia = False -- Base case: value not found
procurar x (No y left right)
  | x == y    = True       -- Found the value
  | x < y     = procurar x left -- Search left subtree
  | otherwise = procurar x right -- Search right subtree
```

- `Ord a =>` indicates we need the `Ord` class because we're using comparison operators (`<`, `==`).
- The base case `procurar x Vazia = False` returns `False` when the tree is empty, meaning the value is not present.

Inserting a Value into a Sorted Tree To insert a value into a BST:

1. Compare the value with each node to decide which subtree to insert into.
2. Recursively insert into the left or right subtree as appropriate.
3. If the value already exists in the tree, we do not insert it again.

Here's the function `inserir` in Haskell:

```
inserir :: Ord a => a -> Arv a -> Arv a
inserir x Vazia = No x Vazia Vazia  -- Insert into an empty tree
inserir x (No y esq dir)
  | x == y    = No y esq dir        -- Value already exists; no insertion
```

```
    | x < y     = No y (inserir x esq) dir  -- Insert into left subtree
    | otherwise = No y esq (inserir x dir)  -- Insert into right subtree
```

- This function returns a new tree with the value inserted.
- The tree remains sorted after insertion.

Building a Tree from a List

We can build a **BST** from a list of values by inserting each element into an initially empty tree.

1. Recursively insert elements from the list.
2. Use an empty tree as the starting point.

Here's how to build a tree from a list using recursion:

```
construir :: Ord a => [a] -> Tree a
construir [] = Empty
construir (x:xs) = inserir x (construir xs)
```

Alternatively, we can use `foldr` to make the code shorter:

```
construir :: Ord a => [a] -> Tree a
construir xs = foldr inserir Empty xs
```

- `foldr inserir Vazia xs` starts with an empty tree and inserts each element from the list xs.
- *Note*: While `construir` keeps the tree sorted, it doesn't balance it. This can lead to unbalanced trees, especially if the list is sorted in ascending or descending order.

Example of Building a Tree

For example, calling `construir [3,1,2]` produces the tree:

```
     2
    / \
   1   3
```

Calling `construir [4,3,2,1]` produces a skewed tree:

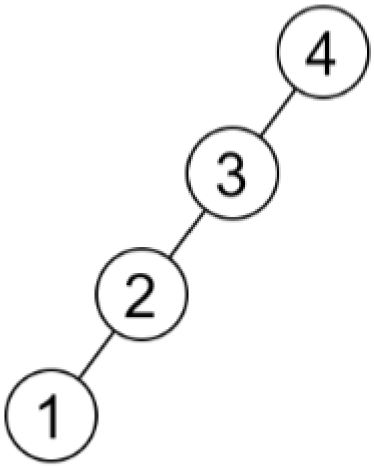
```
4
 \
  3
   \
    2
     \
      1
```

This tree is **sorted but unbalanced**, which can **slow down operations like search**.

5.8. Case study 3: AVL trees

Problem with Unbalanced Binary Search Trees (BSTs)

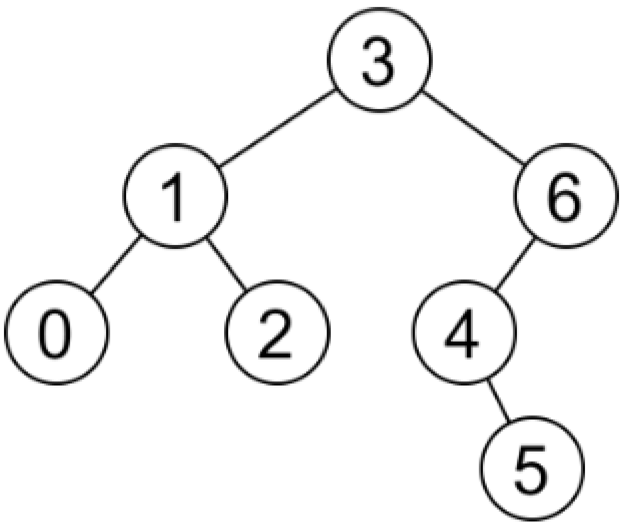
Searching in a standard BST can have a worst-case complexity of O(n). It occurs if the height of the tree is equal to n, as shown below:



Introduction to AVL Trees

AVL trees, named after inventors Adelson-Velsky and Landis, are **self-balancing BSTs** that automatically maintain balance after insertions and deletions. An AVL tree ensures that for every node, the height difference between its left and right subtrees is at most 1 (known as the height invariant).

An example of an unbalanced node:



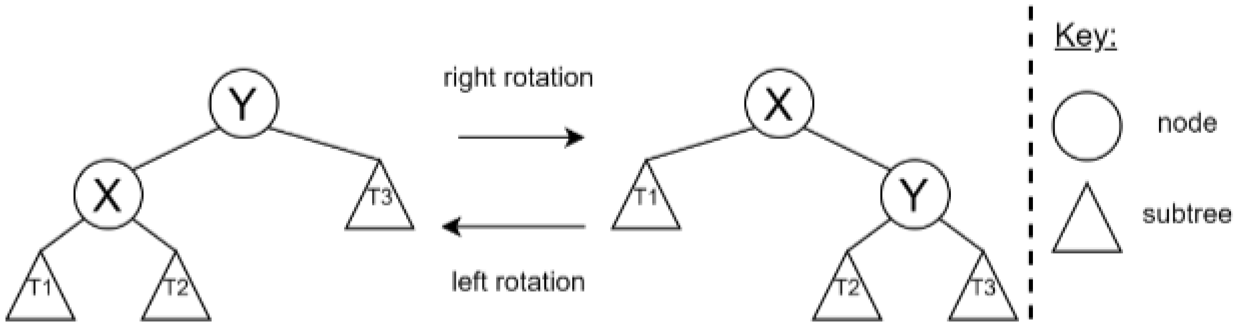
Here, the node with key 6 has a left subtree height of 2 and a right subtree height of 0, violating the AVL balance condition.

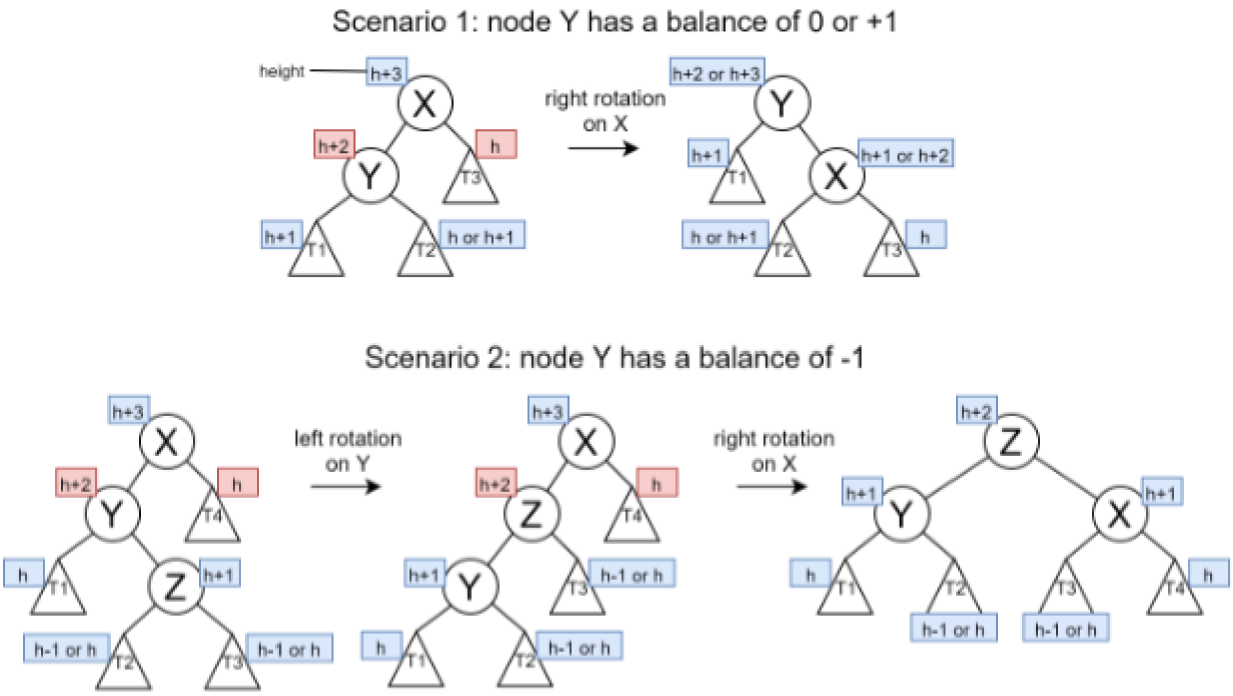
Rebalancing with Rotations

To maintain balance, AVL trees perform rotations on nodes when they become unbalanced. There are four types of rotations:

1. Left Rotation
2. Right Rotation
3. Left-Right Rotation
4. Right-Left Rotation

The appropriate rotation is selected based on the subtree causing the imbalance, as illustrated below:





These rotations ensure that an AVL tree with n nodes has a height of log n, providing efficient O(log n) complexity for lookup, insertion, and deletion.

Haskell AVL Tree Definition

The AVL tree type in Haskell is defined similarly to a regular BST. The `contains` and `smallest` functions are also identical to those for a regular BST:

```haskell
data AVLTree a = Empty
                | Node ( AVLTree a) a ( AVLTree a)
                deriving (Show ,Eq)

contains :: (Eq a, Ord a) => a -> AVLTree a -> Bool
contains a Empty = False
contains a (Node t1 v t2)
    | (a == v) = True
    | (a < v) = contains a t1
    | otherwise = contains a t2

smallest :: AVLTree a -> Maybe a
smallest Empty = Nothing
smallest (Node Empty v t2) = Just v
smallest (Node t1 v t2) = smallest t1
```

Building a Balanced Tree from a Sorted List

Using a sorted list, we can build a balanced binary tree by repeatedly dividing the list in half, where the middle element becomes the root of the subtree. This approach ensures that the tree remains balanced.

```haskell
-- Assumes input list is sorted
construir :: [a] -> Arv a
construir [] = Vazia
construir xs = No x (construir xs') (construir xs'')
    where
        n = length xs `div` 2
        xs' = take n xs
        x:xs'' = drop n xs
```

Creating a balanced tree from a sorted list `[1, 2, 3, 4, 5, 6, 7]`:

```
   4
 /   \
2     6
```

/ \ /
1 3 5 7

Removing a Value

Removing a value from an AVL tree requires careful handling to maintain balance. The removal operation involves:

1. **Searching for the node**: If `x < y`, search the left subtree; if `x > y`, search the right subtree.
2. **Removing the node**:
    - If the node has a single child, replace the node with its child.

    - If the node has two children, replace it with the smallest node in the right subtree (or the largest in the left subtree).
3. **Rebalancing** after removal if necessary.

Example:

```
1. Removing a node with a single child:

    5                 5
  / \      →        /
  3   7            3

2. Removing a node with two children by replacing it with the smallest value in
the right subtree.
```

Here's the function to find the **smallest value**:

```haskell
maisEsq :: Arv a -> a
maisEsq (No x Vazia _) = x
maisEsq (No _ esq _) = maisEsq esq
```

And the function to **remove a node**, handling all cases:

```haskell
remover :: Ord a => a -> Arv a -> Arv a
remover x Vazia = Vazia -- Value not found
remover x (No y Vazia dir) | x == y = dir -- One child
remover x (No y esq Vazia) | x == y = esq -- One child
remover x (No y esq dir)
    | x < y = No y (remover x esq) dir
    | x > y = No y esq (remover x dir)
    | x == y = let z = maisEsq dir in No z esq (remover z dir)
```

This function allows safe deletion while ensuring the tree remains ordered.

> **EA-5:** UT-23, UT-24, UT-25, UT-26, UT-27, UT-28

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6. Interactive programs

6.1. Standard I/O

In Haskell, standard input/output (I/O) operations allow programs to interact with the console using `stdin` for input and `stdout` for output. Here's a breakdown of essential functions and concepts for Haskell I/O, with practical examples and an introduction to the `do` notation. Relevant Prelude functions for input/output on the console:

Function	Comment
`putChar`	Writes a character on the stdout.
`putStr`	Writes a string on the stdout.
`putStrLn`	Writes a string followed by a newline (`'\n'`) on the stdout.
`print`	Prints a value (such as an `Int`) on the stdout.
`getChar`	Reads a character from the stdin. Only returns after a newline is read.
`getLine`	Reads text from the console until a newline is read from the stdin.
`getContents`	Reads all text from the console until an end-of-file (EOF) character is read from the stdin.
`return`	Returns an empty action with type `IO ()`.

The **EOF** (**End-of-File**) character can be simulated in the console using:

- `CTRL+D` on Unix-based systems.
- `CTRL+Z` on Windows.

A sample module `IOUtils` provides useful functions for command-line applications, including functions to:

- **Clear the screen**: Erase console output.
- **Draw at specific positions**: Position text output at given coordinates.
- **Change text color**: Adjust foreground and background text colors.
- **Pause execution**: Wait for a specified time.

I/O Actions

An I/O action in Haskell, of type `IO t`, is a computation that, when executed, interacts with the outside world and may return a value of type `t`. For instance:

- `putChar` has the type `String -> IO ()` . It receives a Char and returns an action responsible for printing it on the console, so this action has no need to hold a value.
- `getChar`, on the other hand, has the type `IO Char` . It takes no arguments and returns an action responsible for asking the user for a character which it can then return.

Writing and Running a Basic Program

Example code (`printHi.hs`):

```
printHi :: IO ()
printHi = putStrLn "Hi!"
main = putStrLn "Hello, world!"
```

One can call main using the GHCi or compile the file and then run it:

```
$ ghc printHi.hs
[1 of 1] Compiling Main      ( printHi.hs, printHi.o )
```

```
Linking printHi.exe ...
$ ./printHi
Hello, world!
```

By convention, main is the entry point for the program, with type IO ().

Combining two comands

```
(>>) :: IO () -> IO () -> IO ()
```

You can sequence I/O actions with the `(>>)` operator, which combines two I/O actions. For example:

```
putChar '?' >> putChar '!'
```

This command prints a question mark followed by an exclamation mark.

Doing Nothing: To perform a “no-operation” action, Haskell provides `done :: IO ()`. This action represents an action that does nothing, though it technically denotes a command that, when performed, will result in no visible effect. Compare thinking about doing nothing with actually doing nothing — they’re not the same thing!

Printing a String

```
putStr :: String -> IO ()
putStr [] = done
putStr (x:xs) = putChar x >> putStr xs
```

For example, `putStr "?!"` is equivalent to `putChar '?' >> (putChar '!' >> done)`.

Using **higher-order functions**:

```
putStr "?!"
= foldr (>>) done (map putChar ['?','!'])
= foldr (>>) done [putChar '?', putChar '!']
= putChar '?' >> (putChar '!' >> done)
```

Doing nothing and returning a value

The command `return :: a -> IO a` does nothing and but returns the given value. `return 42 :: IO Int` yields the value 42 and leaves the input unchanged.

Combining Commands with Values

The operator `>>=` (pronounced **"bind"**) is crucial for chaining I/O actions together while passing values from one action to the next. It allows you to sequence operations in a way that the result of one operation can directly influence the next.

```
(>>=) :: IO a -> (a -> IO b) -> IO b
```

For example, performing the command

```
getChar >>= \x -> putChar (toUpper x)
```

when the input is "abc" produces the output "A" and the remaning input is "bc".

- The operator `>>=` behaves similarly to `let` when the continuation is a lambda expression.

Reading a line

```
getLine :: IO [Char]
getLine = getChar >>= \x ->
          if x == '\n' then
            return []
          else
            getLine >>= \xs ->
            return (x:xs)
```

Commands as special cases The general combinators for commands are:

```
return :: a -> IO a
(>>=) :: IO a -> (a -> IO b) -> IO b
```

The command done is a special case of return and >> is a special case of >>=:

```
done :: IO ()
done = return ()

(>>) :: IO () -> IO () -> IO ()
m >> n = m >>= \_ -> n
```

Using `do` Notation

The `do` notation simplifies chaining I/O actions in sequence, allowing you to write multiple actions within a single block. Here's an example function that reads input and prints a greeting:

A program that echoes each input line in upper-case:

```
echo :: IO ()
echo = getLine >>= \line ->
       if line == "" then
        return ()
       else
        putStrLn (map toUpper line) >>
        echo
```

Here's the same program using `do` notation:

```
echo :: IO ()
echo = do {
        line <- getLine;
        if line == "" then
          return ()
        else do {
          putStrLn (map toUpper line);
          echo
          }
      }
```

To translate `do` notation into lambda notation:

- Replace each line of `x <- e` with `e >>= \x ->`.
- Replace each line of `e` with `e >>`.

Inside a `do`-block:

- `x <- action` runs the action and binds its result to `x`.
- `return` creates an `IO` action with a specified value but doesn't end execution as in other languages.

```
funcX :: IO String
funcX = do
  return ()
  return "hello"
  return "123"
```

Only the last `return` ("123") is evaluated since return does not end the function execution like in other languages.

`return` is like the opposite of `<-`: `return` receives a value x and creates IO action that holds x, while `x <- <action>` extracts the x value of an action.

```
x <- return y  <=> x = y
```

You can declare local variables within a `do` block using `let`, without needing in. Example:

```
testLet :: IO ()
testLet = do
  x <- getLine
  let str = "A":x
  printStrLn str
```

Type Conversion: `show` and `read`

In Haskell, values can be converted between types using the `show` and `read` functions:

- `show`: Converts values (e.g., integers or lists) to strings. Only types in the `Show` typeclass support this.
- `read`: Parses a string into a specific value type, as long as it belongs to the `Read` typeclass.

```
Prelude > show 3
"3"
Prelude > show [1,2,3]
"[1,2,3]"
Prelude > read "3" :: Int
3
Prelude > read "[True ,False]" :: [Bool]
[True ,False]
Prelude > read "[1,2,3.5]" :: [Double]
[1.0,2.0,3.5]
```

- When using `read`, a type annotation is required if Haskell cannot infer the type. For example, `"3"` could be interpreted as an `Int`, `Integer`, or `Double`, so specifying the type ensures correct interpretation.

> **EA-5:** IP-4, IP-6, IP-7, IP-8

Monads

Monoids

A monoid is a pair (?, u) of an associative *operator* ? with an *identity value* u that satisfy the following laws:

Left-identity u ? x = x

Right-identity x ? u = x

Associativity (x ? y) ? z = x ? (y ? z)

Examples: (+) and 0; (*) and 1; (||) and False; (&&) and True; (++) and []; (>>) and done

---

Monads

A *monad* is a pair of functions (>>=, return) that satisfy the following laws:

Left-identity return a >>= f = f a

Right-identity m >>= return = m

Associativity (m >>= f) >>= g = m >>= (\x -> f x >>= g)

Monad laws in 'do' notation

```
-- (1) Left identity
do { x'<-return x ; f x' } = do { f x }
-- (2) Right identity
do { x <- m; return x } = do { m }
-- (3) Associativity
do { y <- do { x <- m; f x }
     g y
   }
  =
do { x <- m;
      do { y <- f x; g y }
   }
  =
do { x <- m;
     y <- f x;
     g y
   }
```

The monad type class

Monad operations in Haskell are overloaded in a type class.

```
-- in the Prelude
class Monad m where
return :: a -> m a
(>>=) :: m a -> (a -> m b) -> m b
instance Monad IO where
return = ... -- primitive ops
(>>=) = ...
-- other Monad instances
```

The partially monad

The `Maybe` type

`data Maybe a = Nothing | Just a` A value of type `Maybe` a is either: `Nothing` representing the absence of further information; `Just` x with a further value x :: a

```
Just 42 :: Maybe Int
Nothing :: Maybe Int

Just "hello" :: Maybe String
Nothing :: Maybe String

Just (42, "hello") :: Maybe (Int,String)
Nothing :: Maybe (Int,String)
```

Using `Maybe` to Represent Failure

The `Maybe` monad is often used for partial functions, where the function may not be able to produce a result for every possible input. A common example is the `lookup` function, which searches for a key in a list of key-value pairs:

```
lookup :: Eq a => a -> [(a, b)] -> Maybe b
lookup k ((x, v):assocs)
  | k == x     = Just v        -- Key found
  | otherwise = lookup k assocs
lookup _ []     = Nothing      -- Key not found
```

- If the key `k` is found in the list, `lookup` returns `Just v` where `v` is the associated value.
- If the key is not found, `lookup` returns `Nothing`, indicating a failure to find the value.

Example:

```
phonebook :: [(String, String)]
phonebook = [ ("Bob", "01788 665242"),
              ("Fred", "01624 556442"),
              ("Alice", "01889 985333"),
              ("Jane", "01732 187565") ]
```

```
> lookup "Bob" phonebook
Just "01788 665242"

> lookup "Alice" phonebook
Just "01889 985333"

> lookup "Zoe" phonebook
Nothing
```

Combining lookups Return the pair of phone, email and fail if either lookup fails.

```haskell
getPhoneEmail :: String -> Maybe (String,String)
getPhoneEmail name =
  case lookup name phonebook of
    Nothing -> Nothing
    Just phone -> case lookup name emails of
      Nothing -> Nothing
      Just email -> Just (phone,email)
```

Using nested case expressions for handling Maybe values can quickly become complex because each level requires checking for Nothing explicitly. To solve this:

Monad Instance for `Maybe`

The `Maybe` type has a `Monad` instance, which enables chaining of computations that may fail. Here's how the `Monad` instance for `Maybe` is implemented:

```haskell
instance Monad Maybe where
  return = Just
  Nothing >>= _ = Nothing
  Just x >>= f = f x
```

Specific types of the monad operations:

```haskell
return :: a -> Maybe a
(>>=) :: Maybe a -> (a -> Maybe b) -> Maybe b
```

- `return` wraps a value in the `Just` constructor.
- `>>=` (bind) will propagate `Nothing` values (like an exception) through the computation chain.
- When applied to `Just x`, it will apply the function `f` to `x`, continuing the computation.

Example: The code gets much shorter with >>= handling the failure cases.

```haskell
getPhoneEmail :: String -> Maybe (String,String)
getPhoneEmail name =
  lookup name phonebook >>= \phone ->
  lookup name emails >>= \email ->
  return (phone,email)
```

Gets even simpler by using `do` notation.

```haskell
getPhoneEmail :: String -> Maybe (String,String)
getPhoneEmail name
  = do phone <- lookup name phonebook
```

```
        email <- lookup name emails
        return (phone,email)
```

The error monad

Representing Errors with `Either`

The `Either a b` type is used to represent computations that may either produce a result (`Right b`) or fail with an error (`Left a`). We use `Left` to indicate an error, and `Right` to indicate a valid result.

```haskell
-- from the Prelude
data Either a b = Left a | Right b
```

We can use:

- Left to tag errors (the *exceptions*);
- Right to tag valid results.

Example: **Integer Division with Either**

We define a function `myDiv` that performs integer division with error handling:

```haskell
myDiv :: Int -> Int -> Either String Int
myDiv x y
  | y == 0            = Left "zero division"
  | x `mod` y /= 0    = Left "not exact"
  | otherwise         = Right (x `div` y)
```

Usage:

```
> myDiv 42 2
Right 21

> myDiv 42 0
Left "zero division"

> myDiv 42 5
Left "not exact"
```

Monad Instance for `Either`

The `Either` type has a `Monad` instance in the Haskell Prelude, specifically `Either e`. Here's how it's defined:

```haskell
instance Monad (Either e) where
    return x        = Right x
    Left e >>= k    = Left e
    Right x >>= k   = k x
```

This means:

- `return x` produces a `Right` value, representing a successful result.
- `>>=` (bind) applies a function to the `Right` value, while `Left` values propagate through the computation like *exceptions*.

Examples:

```haskell
> Right 41 >>= \x -> return (x + 1)
Right 42

> Left "boom" >>= \x -> return (x + 1)
Left "boom"

> Right 100 >>= \x -> Left "no way!"
Left "no way!"
```

Proving the Monad Laws for Either

Note that `Either e` is a **monad** but `Either` itself is not a **monad**. To verify that `Either e` is indeed a monad, we need to show it satisfies the *three monad laws*:

```
**Left Identity**: return a >>= k should be the same as k a.

    Proof:

```haskell
return a >>= k
-- by the monad definition of return for Either
= Right a >>= k
-- applying >>= for Right
= k a
```
```

Right Identity: m >>= return should be the same as m.

Proof: If `m = Left e`:

```
    ```haskell

    Left e >>= return
```
```

````
    -- by the monad definition of >>= for Left
    = Left e
    ```
````

If `m = Right x`:

````
    ```haskell
    Right x >>= return
    -- by the monad definition of >>= for Right
    = return x
    -- by the monad definition of return
    = Right x
    ```
````

Associativity: (m >>= k) >>= h should be the same as m >>= (\x -> k x >>= h).

Proof: If `m = Left e`:

```haskell
(Left e >>= k) >>= h
-- by the monad definition of >>= for Left
= Left e >>= h
-- by the monad definition of >>= for Left again
= Left e
```

```haskell
Left e >>= (\x -> k x >>= h)
-- by the monad definition of >>= for Left
= Left e
```

If `m = Right x`:

```haskell
(Right x >>= k) >>= h
-- by the monad definition of >>= for Right
= k x >>= h
```

```haskell
Right x >>= (\x -> k x >>= h)
-- by the monad definition of >>= for Right
= k x >>= h
```

By satisfying these laws, `Either e` is confirmed to be a monad. This lets us chain computations that may fail, with errors propagating through `Left` values.
```

The state monad

The `State` monad is used to represent stateful computations in Haskell. Here's a breakdown of how it works:

Representing Stateful Computations

Stateful computations can be seen as functions that take a state and return a result along with a new state:

```
state -> (result, new state)
```

This means each computation both consumes and produces a state, allowing the state to be threaded through a series of computations.

Definition of the State Monad

The `State` monad is defined in `Control.Monad.State`.

```haskell
newtype State s a = State (s -> (a, s))
-- type for state computations

run :: State s a -> s -> (a, s)
run (State f) s = f s
instance Monad (State s) where
  return a = State (\s -> (a, s))
  m >>= k = State (\s ->
             let (x, s') = run m s
             in run (k x) s')
```

- Note: For something to be a monad it should also satisfy the three monad laws.

Modules and Abstract Types

Concrete Types

These are defined explicitly by listing their constructors. For example:

```haskell
data Bool = False | True
data Nat = Zero | Succ Nat
```

This approach is considered **concrete** because it defines the actual data representation, without specifying any operations or behaviors.

Abstract Types

Instead of detailing the internal representation, abstract types focus on specifying the operations a type should support.

This is considered an **abstract** specification because it omits the concrete details of the data representation, focusing instead on what the type can do rather than how it is implemented.

Stack

A **stack** is a data structure that follows the *LIFO* (Last In, First Out) principle, meaning that the last value added is the first one to be removed.

Stack Operations

The fundamental operations associated with a stack are:

- **push**: Adds a value to the top of the stack.
- **pop**: Removes the value from the top of the stack.
- **top**: Retrieves the value at the top of the stack without removing it.
- **empty**: Creates an empty stack.
- **isEmpty**: Checks if the stack is empty.

Stack Specification

We can specify a stack in Haskell using a type Stack along with the functions for each operation:

```haskell
data Stack a -- A stack that holds values of type 'a'

push :: a -> Stack a -> Stack a   -- Adds a value to the top of the stack
pop :: Stack a -> Stack a          -- Removes the value from the top of the stack
top :: Stack a -> a                -- Gets the value at the top of the stack
empty :: Stack a                   -- Creates an empty stack
isEmpty :: Stack a -> Bool         -- Checks if the stack is empty
```

This specification provides a clear interface for using stacks while abstracting away the details of their implementation.

Implementation of an Abstract Type in Haskell

To implement an abstract type in Haskell, we follow these steps:

1. **Choose a Concrete Representation**: We decide on how the data will be represented internally.
2. **Implement Operations**: We define the operations that can be performed on this type.
3. **Hide the Representation**: We ensure that users of the type only have access to the type itself and its operations, without exposing the underlying representation.

Using Modules in Haskell

Modules in Haskell allow us to group related definitions, such as types, constants, and functions, while controlling what is accessible from outside the module.

- **Defining a Module**: We define a module Foo in a file named Foo.hs with the declaration:

  ```
  module Foo where
  ```

- **Importing a Module**: To use the Foo module in another module, we add an import statement:

  ```
  import Foo
  ```

- **Exporting Specific Entities**: We can specify which types and functions are exported by the module to control visibility:

  ```
  module Foo (T1, T2, f1, f2, ...) where
  ```

Stack implementation

Here's how we can implement the stack abstract type using a module:

```
module Stack (Stack, push, pop, top, empty, isEmpty) where

-- Define the Stack type using a concrete representation (list)
data Stack a = Stk [a]  -- Stack implemented with lists

-- Push a value onto the stack
push :: a -> Stack a -> Stack a
push x (Stk xs) = Stk (x:xs)

-- Pop a value from the stack
pop :: Stack a -> Stack a
pop (Stk (_:xs)) = Stk xs
pop _ = error "Stack.pop: empty stack"

-- Get the value at the top of the stack
top :: Stack a -> a
top (Stk (x:_)) = x
top _ = error "Stack.top: empty stack"

-- Create an empty stack
empty :: Stack a
empty = Stk []

-- Check if the stack is empty
isEmpty :: Stack a -> Bool
```

```haskell
isEmpty (Stk []) = True
isEmpty (Stk _)  = False
```

Using the Abstract Type

To determine if a string of characters has correctly matched parentheses, we can utilize an auxiliary stack data structure. Here's how we can implement this logic in Haskell using the Stack module we defined earlier.

Algorithm

1. Push '(' onto the stack when encountered.
2. Pop from the stack when encountering ')', ensuring that the top of the stack is the matching '('.
3. After processing the entire string, if the stack is empty, then the parentheses are correctly matched.

Here's how the implementation looks in Haskell:

```haskell
import Stack

-- Main function to check if parentheses are correctly matched
parent :: String -> Bool
parent str = parentAux str empty

-- Auxiliary function that takes a string and a stack of characters
parentAux :: String -> Stack Char -> Bool
parentAux [] stk = isEmpty stk  -- If no characters left, check if the stack is
empty
parentAux (x:xs) stk
  | x == '(' = parentAux xs (push '(' stk)  -- Push '(' onto the stack
  | x == ')' = not (isEmpty stk) &&          -- Check for matching ')'
               top stk == '(' &&
               parentAux xs (pop stk)          -- Pop if there is a match
  | otherwise = parentAux xs stk  -- Ignore other characters
```

Explanation: Function `parent`: This is the main function that initializes the check by calling `parentAux` with an empty stack. Function `parentAux`:

- If the string is empty ([]), it checks if the stack is empty, indicating all parentheses were matched.
- If the character is '(', it pushes it onto the stack.
- If the character is ')', it checks if the stack is not empty and the top of the stack is '('. If so, it pops the stack and continues.
- If the character is neither (nor), it simply continues processing the rest of the string.

Properties of Stacks

We can specify the behavior of stack operations using algebraic equations:

Pop and Push: `pop (push x s) = s` This states that pushing an element x onto stack s and then popping it will return the original stack s.

Top Operation: `top (push x s) = x` This indicates that after pushing x onto the stack s, the top element will be x.

Empty Stack: `isEmpty empty = True` An empty stack is indeed empty. `isEmpty (push x s) = False` A stack with any elements is not empty.

These properties are intuitively valid and can be formally proven based on the definitions of the stack operations.

Set Operations

- **member**: Tests if an element belongs to a set.
- **insert**: Adds an element to a set.
- **delete**: Removes an element from a set.
- **union**: Combines two sets into one.
- **intersection**: Gets the common elements between two sets.
- **empty**: Creates an empty set.
- **isEmpty**: Checks if a set is empty.
- **fromList**: Constructs a set from a list of elements.
- **toList**: Lists the elements in a set.

Here are some examples illustrating the usage of set operations:

```
> insert 1 (insert 2 (insert 1 empty))
fromList [1,2]

> delete 2 (fromList [1,2,3])
fromList [1,3]

> member 2 (fromList [1,3,4,6])
False

> intersection (fromList [1..4]) (fromList [3..6])
fromList [3,4]
```

Implementation of Sets using Binary Search Trees

In this implementation, we will represent sets using binary search trees (BST), which allow for more efficient insertion and searching than a list. For this approach, we require a total order among the elements in the set.

We will define the `Set` type as follows:

```
data Set a = Empty
           | Node a (Set a) (Set a)
```

- `Empty`: Represents an empty set.

- `Node a (Set a) (Set a)`: Represents a node containing an element `a`, along with two subtrees: one for elements less than `a` (left subtree) and one for elements greater than `a` (right subtree).

We will place all definitions into a module that exports the `Set` type and its associated operations, while hiding the constructors `Empty` and `Node`.

```haskell
module Set
  ( Set,
    empty, insert, delete, member,
    union, intersection, fromList, toList) where
```

Function Implementations

1. **Member Function**: Checks if an element is in the set.

```haskell
member :: Ord a => a -> Set a -> Bool
member x Empty = False
member x (Node y left right)
  | x == y = True
  | x > y  = member x right
  | x < y  = member x left
```

2. **Insert Function**: Inserts an element into the set.

```haskell
insert :: Ord a => a -> Set a -> Set a
insert x Empty = Node x Empty Empty
insert x (Node y left right)
  | x == y = Node y left right  -- No duplicates allowed
  | x > y  = Node y left (insert x right)
  | x < y  = Node y (insert x left) right
```

3. **Delete Function**: Removes an element from the set. (Implementation will follow later.)

```haskell
delete :: Ord a => a -> Set a -> Set a
delete x Empty = Empty  -- Nothing to delete
delete x (Node y left right)
  | x < y   = Node y (delete x left) right
  | x > y   = Node y left (delete x right)
  | otherwise = case (left, right) of
      (Empty, Empty) -> Empty  -- Leaf node case
      (Empty, _)     -> right  -- Only right subtree
      (_, Empty)     -> left   -- Only left subtree
      _              -> let (minRight, newRight) = removeMin right
                        in Node minRight left newRight
  where
    removeMin (Node z Empty r) = (z, r)
```

```haskell
    removeMin (Node z l r) = let (minLeft, newLeft) = removeMin l
                             in (minLeft, Node z newLeft r)
```

4. **Empty Set**: Returns an empty set.

```haskell
empty :: Set a
empty = Empty
```

5. **Check if Set is Empty**:

```haskell
isEmpty :: Set a -> Bool
isEmpty Empty = True
isEmpty _ = False
```

6. **Convert Set to List**: Lists the elements in the set in sorted order.

```haskell
toList :: Set a -> [a]
toList Empty = []
toList (Node x l r) = toList l ++ [x] ++ toList r
```

7. **Create Set from List**: Constructs a set from a list of elements.

```haskell
fromList :: Ord a => [a] -> Set a
fromList = foldr insert Empty
```

8. **Union Function**: Combines two sets.

```haskell
union :: Ord a => Set a -> Set a -> Set a
union s1 Empty = s1
union Empty s2 = s2
union (Node x left right) s2 = union left (union right (insert x s2))
```

9. **Intersection Function**: Gets common elements between two sets.

```haskell
intersection :: Ord a => Set a -> Set a -> Set a
intersection Empty _ = Empty
intersection _ Empty = Empty
intersection (Node x left right) s2
  | member x s2 = Node x (intersection left s2) (intersection right s2)
  | otherwise = union (intersection left s2) (intersection right s2)
```

Properties of Set Operations

We can specify certain properties that should hold for the operations:

1. **Membership**:
   - `member x empty = False`
   - `member x (insert x a) = True`
   - `member x (insert y a) = member x a` if `x /= y`
2. **Insertion**:
   - `insert x (insert x a) = insert x a`
   - `insert x (insert y a) = insert y (insert x a)`

Association Tables

In addition to sets, we can define tables (also known as **maps** or dictionaries) where each key is associated with a single value. The operations typically include:

- **Insert**: Adds a new key-value pair.
- **Delete**: Removes a key (if it exists).
- **Lookup**: Retrieves a value by key.
- **Empty**: Creates an empty table.
- **FromList**: Constructs a table from a list of pairs.
- **ToList**: Lists the pairs in the table.

```
> let tbl = insert 'a' 1 (insert 'b' 2 empty)
fromList [('a',1), ('b',2)]

> lookup 'b' tbl
Just 2

> lookup 'c' tbl
Nothing

> insert 'b' 3 tbl
fromList [('a',1), ('b',3)]
```

These examples illustrate how to use the defined operations for managing both sets and association tables in Haskell. This structure allows for efficient data management and retrieval based on the underlying tree implementations.

Implementation of Association Tables using Binary Search Trees

In this implementation, we will represent association tables (also known as maps or dictionaries) using binary search trees (BST), where each key is associated with a value. We only require a total order for the keys, allowing for efficient insertion and lookup.

We will define the `Map` type as follows:

```
data Map k v = Empty
             | Node k v (Map k v) (Map k v)
```

- `Empty`: Represents an empty map.
- `Node k v (Map k v) (Map k v)`: Represents a node containing a key `k`, a value `v`, and two subtrees: one for keys less than `k` (left subtree) and one for keys greater than `k` (right subtree).

We will place all definitions into a module that exports the `Map` type and its associated operations, while hiding the constructors `Empty` and `Node`.

```
module Map
  ( Map,
    empty, insert, delete,
    lookup, fromList, toList) where
```

Function Implementations

1. **Insert Function**: Inserts a key-value pair into the map.

```
insert :: Ord k => k -> v -> Map k v -> Map k v
insert x v Empty = Node x v Empty Empty
insert x v (Node y u l r)
  | x == y = Node x v l r  -- Update value if key already exists
  | x > y  = Node y u l (insert x v r)
  | x < y  = Node y u (insert x v l) r
```

2. **Delete Function**: Removes a key from the map. The implementation of the delete function will follow, usually involving finding the minimum value from the right subtree to replace the deleted node's key.

```
delete :: Ord k => k -> Map k v -> Map k v
delete x Empty = Empty  -- Nothing to delete
delete x (Node y v l r)
  | x < y    = Node y v (delete x l) r
  | x > y    = Node y v l (delete x r)
  | otherwise = case (l, r) of
      (Empty, Empty) -> Empty  -- Leaf node case
      (Empty, _)     -> r       -- Only right subtree
      (_, Empty)     -> l       -- Only left subtree
```

```haskell
      _              -> let (minRight, newRight) = removeMin r
                          in Node minRight v l newRight
  where
    removeMin (Node z Empty r) = (z, r)
    removeMin (Node z l r) = let (minLeft, newLeft) = removeMin l
                                in (minLeft, Node z newLeft r)
```

3. **Lookup Function**: Finds a value associated with a given key. If the key is not found, it will return `Nothing`. We will use the `Maybe` type for this purpose.

```haskell
data Maybe a = Nothing | Just a

lookup :: Ord k => k -> Map k v -> Maybe v
lookup x Empty = Nothing  -- Not found
lookup x (Node y v l r)
  | x == y = Just v         -- Found
  | x > y  = lookup x r
  | x < y  = lookup x l
```

Some Properties of the Operations

The following properties hold for the map operations:

1. **Lookup on Empty Map**:
   - `lookup x empty = Nothing`
2. **Lookup on Insert**:
   - `lookup x (insert x v a) = Just v` (if the key was inserted)
   - `lookup x (insert y v a) = lookup x a` if `x /= y` (lookup remains unchanged)
3. **Insertion**:
   - `insert x v (insert x u a) = insert x v a` (inserting the same key does not change the map)
   - `insert x v (insert y u a) = insert y u (insert x v a)` if `x /= y` (order of insertion doesn't matter for different keys)

Handling Namespaces with Qualified Imports

In Haskell, if you want to use operations from different modules that share the same name (like `insert`), you can import them qualified. This is useful to avoid name clashes, especially with Prelude functions.

```haskell
import qualified Set
import qualified Map

Set.empty      -- Accessing the empty set
```

```
Map.empty      -- Accessing the empty map
Set.insert     -- Inserting into a set
Map.insert     -- Inserting into a map
Map.lookup     -- Looking up a value in a map
lookup         -- Looking up a value in a list (from Prelude)
```

This implementation provides a clear structure for managing associations between keys and values in a way that allows for efficient operations based on binary search trees.

THE END 😃