Discrete Differential Geometry, A.I. Bobenko, P. Schröder, J.M. Sullivan and G.M. Ziegler, eds.
Oberwolfach Seminars, Vol. 38, 275–286

Convergence of the Cotangent Formula: An Overview

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Abstract. The cotangent formula constitutes an intrinsic discretization of the Laplace–Beltrami operator on polyhedral surfaces in a finite-element sense. This note gives an overview of approximation and convergence properties of discrete Laplacians and mean curvature vectors for polyhedral surfaces located in the vicinity of a smooth surface in euclidean 3-space. In particular, we show that mean curvature vectors converge in the sense of distributions, but fail to converge in L^2.

Keywords. Cotangent formula, discrete Laplacian, Laplace–Beltrami operator, convergence, discrete mean curvature.

1. Introduction

There are various approaches toward a purely discrete theory of surfaces for which classical differential geometry, and in particular the notion of *curvature*, appears as the limit case. Examples include the theory of *spaces of bounded curvature* [1, 24], *Lipschitz–Killing curvatures* [5, 12, 13], *normal cycles* [6, 7, 30, 31], *circle patterns* and *discrete conformal structures* [2, 17, 26, 28], and geometric *finite elements* [10, 11, 15, 20, 29]. In this note we take a finite-element viewpoint, or, more precisely, a functional-analytic one, and give an overview over convergence properties of weak versions of the Laplace–Beltrami operator and the mean curvature vector for embedded polyhedral surfaces.

Convergence. Consider a sequence of polyhedral surfaces $\{M_n\}$, embedded into euclidean 3-space, which converges (in an appropriate sense) to a smooth embedded surface M. One may ask: What are the measures and conditions such that metric and geometric objects on M_n—like intrinsic distance, area, mean curvature, Gauss curvature, geodesics and the Laplace–Beltrami operator—converge to the corresponding objects on M? To date no complete answer has been given to this question in its full generality. For example, the approach of *normal cycles* [6, 7], while well-suited for treating convergence of curvatures of embedded polyhedra in the sense of measures, cannot deal with convergence of elliptic operators such as the Laplacian. The *finite-element approach*, on the other hand, while well-suited for treating convergence of elliptic operators (cf. [10, 11]) and mean curvature vectors, has its difficulties with Gauss curvature.

Despite the differences between these approaches, there is a remarkable similarity: The famous *lantern of Schwarz* [27] constitutes a quite general example of what can go wrong—pointwise convergence of surfaces without convergence of their normal fields. Indeed, while one cannot expect convergence of metric and geometric properties of embedded surfaces from pointwise convergence alone, it often suffices to additionally require *convergence of normals*. The main technical step, to show that this is so, is the construction of a bi-Lipschitz map between a smooth surface M, embedded into euclidean 3-space, and a polyhedral surface M_h nearby, such that the metric distortion induced by this map is bounded in terms of the Hausdorff distance between M and M_h, the deviation of normals, and the shape operator of M. (See Theorem 3.3 and compare [19] for a similar result.) This map then allows for *explicit error estimates* for the distortion of area and length, and—when combined with a functional-analytic viewpoint—error estimates for the Laplace–Beltrami operator and the mean curvature vector.

We treat convergence of Laplace–Beltrami operators in *operator norm*, and we discuss two distinct concepts of mean curvature: a *functional* representation (in the sense of distributions) as well as a representation as a *piecewise linear function*. We observe that one concept (the functional) converges whereas the other (the function) in general does not. This is in accordance with what has been observed in geometric measure theory [5, 6, 7]: for polyhedral surfaces approximating smooth surfaces, in general, one cannot expect pointwise convergence of curvatures, but only convergence in an integrated sense.

A brief history of the cotangent formula. The cotangent representation for the Dirichlet energy of piecewise linear functions on triangular nets seems to have first appeared in Duffin's work [9] in 1959. In 1988 Dziuk [10, 11] studied linear finite elements on polyhedral surfaces—without explicit reference to the cotangent formula. In 1993 Pinkall and Polthier [20] employed the cotangent formula for a *functional representation* of the discrete mean curvature vector, leading to explicitly computable discrete minimal surfaces [16, 21, 22, 23, 25]. Later, Desbrun et al. [8, 18] used the cotangent formula for expressing the area gradient of piecewise linear surfaces. Their approach rescales with an area factor, which effectively means dealing with *functions* (pointwise quantities) instead of *functionals* (integrated quantities). Based on intrinsic Delaunay triangulations, Bobenko and Springborn [3] recently derived an intrinsic version of the cotangent formula on polyhedral surfaces that obeys the discrete maximum principle.

2. Polyhedral surfaces

By a *polyhedral surface* M_h, we mean a metric space obtained by gluing finitely many flat euclidean triangles isometrically along their edges, such that the result is homeomorphic to a 2-dimensional manifold. Metrically, polyhedral surfaces are length spaces in the sense of Gromov [14]. Two triangles which share an edge can always be unfolded such that they become coplanar, so that no intrinsic curvature occurs across edges. All intrinsic

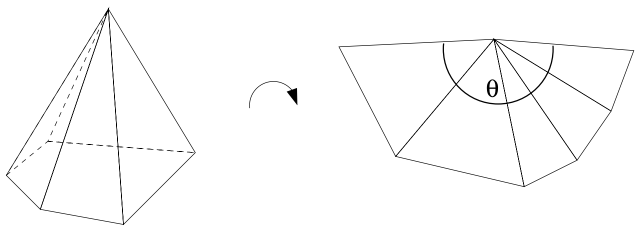

FIGURE 1. A neighborhood of an inner vertex of M_h with angle defect $2\pi - \theta$ is isometric to a metric cone with cone angle θ.

curvatures are concentrated at vertices which we treat as euclidean cone points. A *metric cone* C_θ with *cone angle* θ is the set $\{(r,\varphi)|0 \leq r;\ \varphi \in \mathbb{R}/\theta\mathbb{Z}\}/_{(0,\varphi_1)\sim(0,\varphi_2)}$ together with the (infinitesimal) metric $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2$, see Figure 1.

2.1. Finite elements on polyhedra

Polyhedral surfaces are piecewise linear; hence, they can be treated naturally by the finite-element method, in particular, by studying finite-dimensional subspaces of the Sobolev space $\mathcal{H}^1(M_h)$. First, we review the finite-element setting, then we discuss Sobolev spaces on polyhedral surfaces.

Given $f \in \mathcal{L}^2(M_h)$, the *Dirichlet problem* on M_h is to find $u \in \mathcal{H}^1_0(M_h)$ such that

$$\int_{M_h} g_h(\nabla_h u, \nabla_h \varphi)\,\mathrm{d}\,\mathrm{vol}_h = \int_{M_h} f\cdot\varphi\,\mathrm{d}\,\mathrm{vol}_h \quad \forall\,\varphi \in \mathcal{H}^1_0(M_h)\,,$$

where g_h is the euclidean cone metric on M_h and ∇_h is the associated gradient. As in the planar case, an *abstract Galerkin scheme* is defined by restricting the space of test functions and the space of solutions to a *finite-dimensional* subspace $V_0 \subset \mathcal{H}^1_0(M_h)$. As usual, the subscript 0 denotes zero boundary conditions (we assume $\partial M_h \neq \emptyset$; the case $\partial M_h = \emptyset$ is treated similarly by setting $\mathcal{H}^1_0(M_h) = \{u \in \mathcal{H}^1(M_h) \mid \int u = 0\}$).

Definition 2.1 (Finite-element space). For vertices $p \in M_h \setminus \partial M_h$ and $q \in M_h$, let

$$\phi_p(q) := \begin{cases} 1 & \text{if} \quad q = p \\ 0 & \qquad \text{else}\,, \end{cases}$$

and extend ϕ_p to all of M_h by linearly interpolating on triangles. Then $\{\phi_p\}$ is a *nodal basis* for the finite-dimensional space $S_{h,0}$.

Every $u_h \in S_{h,0}$ can be written as $u_h = \sum_q u_h^q \phi_q$ with coefficients $u_h^q \in \mathbb{R}$. Let

$$-\Delta_{pq} := \int_{M_h} g_h(\nabla_h\phi_p, \nabla_h\phi_q)\,\mathrm{d}\,\mathrm{vol}_h$$
$$\mathrm{b}_p := \int_{M_h} f\cdot\phi_p\,\mathrm{d}\,\mathrm{vol}_h\,.$$

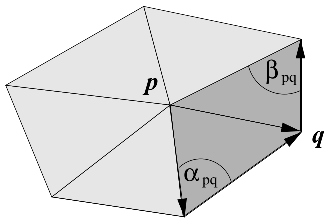

FIGURE 2. Only the angles α_{pq} and β_{pq} enter into the expression for Δ_{pq}.

Then the *discrete Dirichlet problem* amounts to a finite-dimensional linear solve: find u_h^q such that

$$-\sum_q \Delta_{pq} u_h^q = \mathrm{b}_p \ .$$

With this notation one readily verifies the *cotangent* representation of Δ_{pq}, see [20].

Lemma 2.2 (Cotangent formula). *The nonzero entries of the discrete cotan-Laplacian on a polyhedral surface are given by*

$$\Delta_{pq} = \frac{1}{2}(\cot\alpha_{pq} + \cot\beta_{pq}) \quad \textit{and} \quad \Delta_{pp} = -\sum_{q_i \in link(p)} \Delta_{pq_i} \ ,$$

if p and q share an edge, and where α_{pq} and β_{pq} denote the angles opposite to the edge $\overline{pq}$ in the two triangles adjacent to $\overline{pq}$.

2.2. Sobolev spaces on polyhedra

On a smooth manifold M the definition of $\mathcal{H}^1(M)$—the space of square-integrable functions with square-integrable weak derivatives—can be based on a locally finite partition of M into smooth charts. The requirement on these charts is that the difference between the metric tensor on M and the flat metric tensor on $\mathbb{E}^n$ is uniformly bounded. (Such charts exist, for example, under the assumption of uniform curvature bounds on M.) In the polyhedral case, a difficulty arises from the fact that M_h is only of class $C^{0,1}$. A very general framework for defining $\mathcal{H}^1(M_h)$ on compact Lipschitz manifolds (of which finite polyhedra are just a special case) via local charts is provided by a consequence of Rademacher's theorem: weak differentiability is preserved under bi-Lipschitz maps. (See, e.g., Cheeger [4] and Ziemer [32].) In the following we will base our definition of $\mathcal{H}^1(M_h)$ on the assumption that there is a smooth surface M in the vicinity of M_h and a bi-Lipschitz map between M and M_h with uniformly bounded Lipschitz constant (for the existence of such a map, see Theorem 3.3), so that we can identify $\mathcal{H}^1(M_h)$ with $\mathcal{H}^1(M)$.

3. Convergence and approximation

3.1. Comparing two surfaces

If $M \subset \mathbb{E}^3$ is a compact smooth surface, and M_h is a polyhedral surface close to it, we need a map in order to compare the two surfaces. One way to define such a map is to

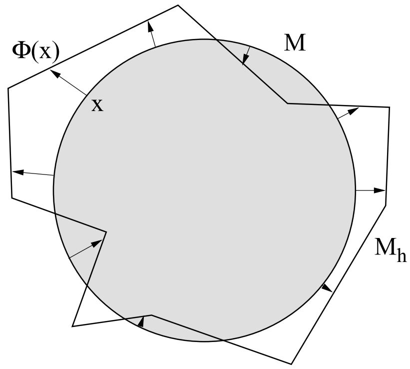

FIGURE 3. If M_h is a normal graph over M, then Φ takes a point x on M to the intersection of M_h with the normal line through x. The inverse, $\Psi = \Phi^{-1}$, thus realizes the pointwise distance from M_h to M.

map each point on M_h to its closest point on M. This is well defined for points within the *reach* of M. The reach of M is the distance of M to its *medial axis*. The medial axis of M is the set of those points in $\mathbb{E}^3$ which do not have a unique nearest point in M. The reach is related to local curvature properties of M, more precisely,

$$\operatorname{reach}(M) \leq \inf_{x \in M} \frac{1}{|\kappa|_{\max}(x)} , \tag{3.1}$$

where $|\kappa|_{\max}(x)$ denotes the maximal absolute value of the normal curvatures at $x \in M$.

Definition 3.1 (Normal graph). A compact polyhedral surface M_h is a *normal graph* over a compact smooth surface M if it is strictly within the reach of M and the map $\Psi : M_h \to M$ which takes every point on M_h to its closest point on M is a homeomorphism.

The inverse of this map, $\Phi = \Psi^{-1} : M \to M_h$, satisfies

$$\operatorname{dist}_{\mathbb{E}^3}(\Phi(x), M) = \|\Phi(x) - x\|_{\mathbb{E}^3} ,$$

see Figure 3. Φ is called the *shortest distance map*.

Definition 3.2 (Normal convergence). We say that a sequence of polyhedra $\{M_n\}$, with normal fields N_n, converges *normally* to a smooth surface M, with normal field N, if each M_n is a normal graph over M and the sequence of normal fields converges in $L^\infty(M)$ under the shortest distance maps,

$$\|N_n \circ \Phi_n - N\|_\infty \to 0 .$$

The sequence is said to converge *totally normally* if it converges normally and the Hausdorff distances $\mathrm{d}_H(M_n, M)$ tend to zero as well.

3.2. Measuring metric distortion

We use the shortest distance map to pull back the length metric on the polyhedral surface M_h to a metric g_h defined on the smooth surface M. This amounts to thinking of M as being equipped with two metric structures—the smooth Riemannian metric g and the polyhedral metric g_h. The *metric distortion tensor* A measures the distortion between g and g_h. It is defined by

$$g(A(X), Y) := g_h(X, Y) := g_{\mathbb{E}^3}(\mathrm{d}\Phi(X), \mathrm{d}\Phi(Y)) \quad \text{a.e.}\,, \tag{3.2}$$

where X and Y are smooth vector fields on M. The matrix field A is symmetric and positive definite outside a measure zero set. The next theorem shows that A only depends on the distance between M and M_h, the angle between their normals, and the shape operator of M (for a proof see [15, 29]; see also Morvan and Thibert [19] for a similar result).

Theorem 3.3 (Metric distortion tensor splitting). *Let M_h be a polyhedral surface with normal field N_h. Assume M_h is a normal graph over an embedded smooth surface M with normal field N. Then the metric distortion tensor satisfies*

$$A = P \circ Q^{-1} \circ P \quad \textit{a.e.}\,, \tag{3.3}$$

a decomposition into symmetric positive definite matrices P and Q which can pointwise be diagonalized (possibly in different orthonormal frames) by

$$P = \begin{pmatrix} 1 - d \cdot \kappa_1 & 0 \\ 0 & 1 - d \cdot \kappa_2 \end{pmatrix} \tag{3.4}$$

$$Q = \begin{pmatrix} \langle N, N_h \circ \Phi \rangle^2 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.5}$$

Here κ_1 and κ_2 denote the principal curvatures of the smooth manifold M, and $d(x)$ is the signed distance function, defined by $\Phi(x) = x + d(x) \cdot N(x)$.

3.3. Convergence of Laplace–Beltrami operators

The domain $\mathcal{H}_0^1(M)$ of the Laplace–Beltrami operator on M is the subspace of functions in $\mathcal{H}^1(M)$ with zero boundary condition; its range $\mathcal{H}^{-1}(M)$ is the space of bounded linear functionals on $\mathcal{H}_0^1(M)$. Here, $\mathcal{H}_0^1(M)$ will always be equipped with the norm

$$\|u\|^2_{\mathcal{H}_0^1(M)} = \int_M g(\nabla u, \nabla u)\, \mathrm{d}\,\mathrm{vol}\,.$$

Using the shortest distance map to pull back the Laplace–Beltrami operator on M_h to the smooth surface M, we think of M as being equipped with *two* elliptic operators:

$$\Delta, \Delta_h : \mathcal{H}_0^1(M) \to \mathcal{H}^{-1}(M)\,. \tag{3.6}$$

Let $\langle \cdot | \cdot \rangle$ denote the dual pairing between $\mathcal{H}^{-1}(M)$ and $\mathcal{H}_0^1(M)$. Then the weak definition of these operators is given by

$$\langle \Delta u | v \rangle = -\int_M g(\nabla u, \nabla v)\, \mathrm{d}\,\mathrm{vol} \tag{3.7}$$

$$\langle \Delta_h u | v \rangle = -\int_M g(A^{-1}\nabla u, \nabla v)(\det A)^{1/2}\, \mathrm{d}\,\mathrm{vol}\ . \tag{3.8}$$

Convergence of Laplace–Beltrami operators is understood in the operator norm of bounded linear operators between the spaces $\mathcal{H}_0^1(M)$ and $\mathcal{H}^{-1}(M)$.

Remark. The last formula is justified by the fact that the metric distortion tensor induces the following transformations for gradients and volume forms:

$$\mathrm{d}\,\mathrm{vol}_h = (\det A)^{1/2}\, \mathrm{d}\,\mathrm{vol}\ ,$$
$$\nabla_h = A^{-1}\nabla\ .$$

For a proof of the next theorem, see [15, 29].

Theorem 3.4 (Convergence of Laplacians). *Let $M_h \subset \mathbb{E}^3$ be an embedded compact polyhedral surface which is a normal graph over a smooth surface M with corresponding distortion tensor A. Define $\bar{A} := (\det A)^{1/2} A^{-1}$. Then*

$$\frac{1}{2}\|\mathrm{tr}(\bar{A} - \mathrm{Id})\|_\infty \le \|\Delta_h - \Delta\|_{\mathrm{op}} \le \|\bar{A} - \mathrm{Id}\|_\infty\ . \tag{3.9}$$

Hence, if a sequence of polyhedral surfaces converges to M totally normally, then the corresponding Laplace–Beltrami operators converge in norm.

From here on, in order to show convergence for the discrete cotan-Laplacian, one proceeds similarly to Dziuk [10] who studies linear finite elements for *interpolating* polyhedra. For details about the extension to our case—*approximating* polyhedra—see [29].

3.4. Convergence of mean curvature

Analogously to the smooth case, we define the *mean curvature vector* as the result of applying the Laplace–Beltrami operator to the embedding function of a surface. In the polyhedral case this yields a *functional* (a distribution) rather than a function.

Definition 3.5 (Mean curvature functional). Let $\vec{I}_M : M \to \mathbb{E}^3$ and $\vec{I}_{M_h} : M_h \to \mathbb{E}^3$ denote the embeddings of M and M_h, respectively, and let $\vec{I}_h = \vec{I}_{M_h} \circ \Phi : M \to \mathbb{E}^3$. Then the mean curvature vectors are *functionals* on M defined by

$$\vec{H}_M := \Delta \vec{I}_M \in (\mathcal{H}^{-1}(M))^3\ ,$$
$$\vec{H}_h := \Delta_h \vec{I}_h \in (\mathcal{H}^{-1}(M))^3\ ,$$

defining one equation for each of the three components of these embeddings.

Lemma 3.6 (Connection with cotangent formula). *The mean curvature functional, when restricted to the subspace spanned by nodal basis functions, can be expressed using the cotangent formula:*

$$\langle \vec{H}_h | \phi_p \rangle = \frac{1}{2} \sum_{q \in link(p)} (\cot \alpha_{pq} + \cot \beta_{pq}) \cdot (q - p) \,. \tag{3.10}$$

The mean curvature functional is $\mathbb{R}^3$-valued. We need to say what we mean by the norm of such a functional. Let $\vec{F}$ be an $\mathbb{R}^n$-valued bounded linear operator on $\mathcal{H}_0^1$. We define

$$\|\vec{F}\|_{\mathcal{H}^{-1}} = \sup_{0 \neq u \in \mathcal{H}_0^1} \frac{\|\langle \vec{F} | u \rangle\|_{\mathbb{R}^n}}{\|u\|_{\mathcal{H}_0^1}} \,,$$

where $\langle \cdot | \cdot \rangle$ denotes the dual pairing between $\mathcal{H}^{-1}$ and $\mathcal{H}_0^1$. The following result gives an a-priori error bound for the mean curvature functionals.

Theorem 3.7 (Convergence of mean curvature functionals). *Let M_h be a normal graph over a smooth surface M with associated shortest distance map Φ and metric distortion tensor A. Then*

$$\|\vec{H}_M - \vec{H}_h\|_{\mathcal{H}^{-1}} \leq \sqrt{|M|}\,(C_A - 1 + C_A \|\mathrm{Id} - \mathrm{d}\Phi\|_\infty) \,, \tag{3.11}$$

where $C_A = \|(\det A)^{1/2} A^{-1}\|_\infty$, $|M|$ is the total area of M, and $\|\mathrm{Id} - \mathrm{d}\Phi\|_\infty$ denotes the essential supremum over the pointwise operator norm of the operator $(\mathrm{Id} - \mathrm{d}\Phi)(x) : \mathrm{T}_x M \to \mathbb{R}^3$. *Hence, if a sequence of polyhedral surfaces converges to M totally normally, then the mean curvature functionals converge in norm.*

Sketch of proof. Inequality (3.11) is a consequence of the triangle inequality applied to

$$(\Delta \vec{I}_M - \Delta_h \vec{I}_h) = (\Delta \vec{I}_M - \Delta_h \vec{I}_M) + (\Delta_h \vec{I}_M - \Delta_h \vec{I}_h) \,.$$

The convergence statement follows from an application of Theorem 3.3 to estimate the two terms $(C_A - 1)$ and $\|\mathrm{Id} - \mathrm{d}\Phi\|_\infty$, respectively. □

A counterexample to the convergence of discrete mean curvature. The reason to care about functions instead of functionals (distributions) is *scaling*. The mean curvature functional scales differently from the (classical) mean curvature function: If a surface is uniformly scaled by a factor λ, then the mean curvature functional also scales with λ, whereas the mean curvature function scales with $1/\lambda$.

Definition 3.8 (Discrete mean curvature vector). The *discrete mean curvature vector* is the unique $\mathbb{R}^3$-valued piecewise linear function $\vec{H}_{\mathrm{dis}} \in (S_{h,0})^3$, corresponding to the mean curvature functional $\vec{H}_h$, evaluated on $S_{h,0}$. It is defined by

$$(\vec{H}_{\mathrm{dis}}, u_h)_{\mathcal{L}^2(M_h)} = \langle \vec{H}_h | u_h \rangle \quad \forall\, u_h \in S_{h,0} \,, \tag{3.12}$$

where $(\cdot,\cdot)_{\mathcal{L}^2(M_h)}$ denotes the $\mathcal{L}^2$ inner product on M_h, and $\langle \vec{H}_h | u_h \rangle$ denotes the evaluation of the mean curvature functional on u_h.

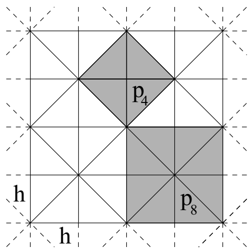

FIGURE 4. Discrete mean curvature does not converge in $\mathcal{L}^2$ for a 4–8 tessellation of a regular quad-grid, because the ratio between the areas of the stencils of p_4 and p_8 does not converge to 1.

Note that it is possible to associate a discrete function to the mean curvature functional only because the dimension of $S_{h,0}$ is finite. There is no infinite-dimensional analogue of this construction. The mean curvature function can be computed explicitly:

$$\vec{H}_{\text{dis}} = \sum_{p,q \in M_h \setminus \partial M_h} \langle \vec{H}_h | \phi_p \rangle \mathcal{M}^{pq} \phi_q \ , \tag{3.13}$$

where $\mathcal{M}^{pq}$ denotes the inverse of the *mass matrix*, $\mathcal{M}_{pq}$, which is given by

$$\mathcal{M}_{pq} = \int_{M_h} \phi_p \phi_q \, \mathrm{d}\,\mathrm{vol}_h \ .$$

Remark. Instead of using the full mass matrix, it is common to use a *lumped* version (such as obtained by forming a diagonal matrix with entries equal to the row sums of $\mathcal{M}$).

Counterexample. This example shows that, in general, discrete mean curvature fails to converge in $\mathcal{L}^2$ (and in fact, in any pointwise sense). Let M be a smooth cylinder of height 2π and radius 1. We construct a sequence $\{M_n\}$ of polyhedral cylinders whose vertices lie on M and which converges to M totally normally, but for which the mean curvature *functions* fail to converge. Let M be parameterized as follows:

$$x = \cos u, \quad y = \sin u, \quad z = v \ .$$

Let the vertices of M_n be given by

$$\begin{aligned} u &= \frac{i\pi}{n} & i &= 0, \dots, 2n-1, \\ v &= \begin{cases} 2j \sin \frac{\pi}{2n} & j = 0, \dots, 2n-1, \\ 2\pi & j = 2n. \end{cases} \end{aligned}$$

This corresponds (up to the upper-most layer) to folding along vertical lines a regular planar quad-grid of edge length $h_n = 2\sin(\pi/2n)$, see Figure 4. In other words, all faces of M_n are square (except for the upper-most layer). It will now depend on the *tessellation pattern* (i.e., the choice of diagonals) of this quad-grid whether there is $\mathcal{L}^2$-convergence of the mean curvature function or not. Indeed, consider the regular 4–8 tessellation scheme depicted in Figure 4. There are two kinds of vertices: those of valence 4 and those of valence 8. Call them p_4 and p_8, respectively, and let ϕ_{p_4} and ϕ_{p_8} denote the corresponding nodal basis functions. By the symmetry of the problem there exist constants $a_n, b_n \in \mathbb{R}$ such that

$$\vec{H}_{\mathrm{dis},n} = \sum_{p_4} a_n \cdot \phi_{p_4} \cdot \partial_r + \sum_{p_8} b_n \cdot \phi_{p_8} \cdot \partial_r + \text{boundary contributions} ,$$

where ∂r is the smooth cylinder's outer normal and the contributions from the boundary include all vertices one layer away from the upper boundary. Then, as the edge lengths h_n tend to zero, it turns out that $b_n \to 0$ and $a_n \to -3$. Hence, $\vec{H}_{\mathrm{dis},n}$ is a family of continuous functions oscillating between $a_n \approx -3$ (at the vertices of valence 4) and $\mathrm{b}_n \approx 0$ (at the vertices of valence 8) with ever growing frequencies. Such a family, although it does converge in $\mathcal{H}^{-1}(M)$, cannot converge in $\mathcal{L}^2(M)$.

3.5. A general convergence result

We have chosen to elaborate on the concept of mean curvature here. In order to complete the picture, we conclude this overview with a summary of related results. For a more complete story, see [15, 29].

Theorem 3.9 (Geometric conditions for normal convergence). *Let $\{M_n\}$ be a sequence of compact polyhedral surfaces which are normal graphs over a compact smooth surface M, converging to M in Hausdorff distance. Then the following conditions are* equivalent*:*

1. *convergence of normals,*
2. *convergence of metric tensors ($A_n \to \mathrm{Id}$),*
3. *convergence of area measure,*
4. *convergence of Laplace–Beltrami operators in norm.*

The fact that totally normal convergence implies convergence of metric tensors has several other consequences: convergence of shortest geodesics on M_n to geodesics on M, convergence of solutions to the Dirichlet problem, convergence of Hodge decompositions (for appropriate discrete notions of gradient, curl, and divergence), and convergence of the spectrum of the Laplacian. For details we refer to [29].

Acknowledgments

This work grew out of many stimulating discussions with Michael Herrmann, Klaus Hildebrandt, and Konrad Polthier. I wish to thank Jean-Marie Morvan for bringing to my attention the relation between the gradient of the shortest distance map and convergence of normals. This work was supported by the DFG Research Center MATHEON "Mathematics for key technologies" in Berlin.

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