

ON THE MINIMUM CORE MASS FOR GIANT PLANET FORMATION

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ABSTRACT

1. INTRODUCTION

Kelvin-Helmholz (KH) contraction considered previously (Ikoma, PapNel05). Say why useful. This paper develops a simplified model of KH contraction to both elucidate the relevant physical processes and explore a wide range of disk parameter space...

This paper is organized as follows. In section 2 we describe the assumptions of our atmosphere model, and derive the basic equations that govern the structure and evolution of the atmosphere. In section 3, we present a simplified analytic model that predicts the qualitative behavior of the numerical model. Results for atmospheric structure and evolution are presented in section 4, and implications for the critical core mass are presented in section 5. The discussion in section 6 addresses various approximations and neglected effects. We summarize our findings in section 7. The appendices contain...

2. ATMOSPHERE MODELS

To model the growth of planetary atmospheres around a solid core, we develop a simplified two layer model for time-dependent atmospheric cooling, i.e. Kelvin-Helmholtz (KH) contraction. With a convective interior and radiative exterior, this model is motivated by similar models of hot Jupiters (Arras & Bildsten 2006; Youdin & Mitchell 2010).

Our model can accurately model the growth of the atmosphere up to the crossover mass, when the atmosphere mass equals the mass of the core. Beyond the crossover mass, our approximate treatment of the radiative zone breaks down. Since subsequent growth is a rapid runaway process (*cite a numerical core accretion paper*), our model can investigate to good accuracy the timescale requirement of core accretion. Our simplified treatment is also inappropriate for hot, short-period planets, where dust sublimation gives deeper radiative zones that require more detailed models.

Our main assumptions are summarized as follows:

1. The atmosphere is spherically symmetric and remains in hydrostatic balance during its thermal evolution.
2. The core mass and radius are fixed in evolutionary calculations, neglecting ongoing planetesimal or dust accretion.
3. At the planet's Hill radius, the atmospheric temperature and pressure match the conditions of the disk midplane.

4. The only source of planetary luminosity is the gravitational contraction of the atmosphere.
5. In the radiative zone, luminosity generation is neglected, i.e. the luminosity is held constant.
6. A global cooling model connects independent static solutions into a time-dependent sequence.
7. A polytropic EOS is assumed for simplicity.
8. Dust grains provide the opacity in the radiative zone, which remains cool enough to avoid dust sublimation.

The remainder of this section develops our model in more detail.

2.1. Disk and Opacity Model

We adopt a minimum mass solar nebula (MMSN) model for a passively irradiated disk (Chiang & Youdin 2010). With the semi-major axis a normalized to the outer disk as $a_{10\oplus} = a/(10 \text{ AU})$, the gas surface density and mid-plane temperature are

$$\Sigma_d = 70 F_\Sigma a_{10\oplus}^{-3/2} \text{ g cm}^{-2} \quad (1a)$$

$$T_d = 45 F_T a_{10\oplus}^{-3/7} \text{ K}. \quad (1b)$$

The normalization factors F_Σ and F_T adjust the model relative to the fiducial MMSN. We fix $F_\Sigma = F_T = 1$ unless noted otherwise.

For a vertically isothermal disk in hydrostatic balance (with no self-gravity), the mid-plane pressure of disk gas is

$$P_d = 6.9 \times 10^{-3} F_\Sigma \sqrt{F_T} a_{10\oplus}^{-45/14} \text{ dyne cm}^2 \quad (2)$$

for a molecular weight of $\mu = 2.35$ proton masses and a Solar mass star.

The (thermodynamically isothermal) sound speed in the disk is

$$c_d = \sqrt{\mathcal{R}T_d} = 0.4 \sqrt{F_T} a_{10\oplus}^{3/14} \text{ km s}^{-1} \quad (3)$$

in terms of the specific gas constant \mathcal{R} . The disk scale-height is

$$H_d = c_d/\Omega = 0.42 \sqrt{F_T} a_{10\oplus}^{9/7} \text{ AU}. \quad (4)$$

in terms of the Keplerian frequency $\Omega = \sqrt{GM_*/a^3}$ with G the gravitational constant and M_* the stellar (in this work Solar) mass.

We assume a dust opacity following Bell & Lin (1994):

$$\kappa = 2F_\kappa \left(\frac{T}{100 \text{ K}} \right)^\beta \text{ cm}^2 \text{ g}^{-1}, \quad (5)$$

with a powerlaw index $\beta = 2$ and normalization $F_\kappa = 1$ unless noted otherwise. Roughly speaking, grain growth will lower both F_κ and β , while dust abundance scales with F_κ . Section 6.2 discusses dust sublimation and more realistic opacity laws.

2.2. Length Scales

The characteristic length scales for protoplanetary atmospheres are crucial for choosing boundary conditions and for understanding the validity of spherical symmetry in a disk of scaleheight H_d . The radius of the solid core:

$$R_c \equiv \left(\frac{3M_c}{4\pi\rho_c} \right)^{1/3} \approx 10^{-4} m_{c10\oplus}^{1/3} \text{ AU}, \quad (6)$$

where the core mass, M_c , is normalized to 10 Earth masses as $m_{c10\oplus} \equiv M_c/(10 M_\oplus)$. The core density is held fixed at $\rho_c = 3.2 \text{ g cm}^{-3}$. We thus neglect the detailed equation of state of the solids core (Fortney et al. 2007).

A planet can bind a dense atmosphere if the escape speed exceeds the sound speed. This criterion is satisfied inside the Bondi radius

$$R_B \equiv \frac{GM_{\text{pl}}}{c_d^2} \approx 0.17 \frac{m_{c10\oplus} a_{10\oplus}^{3/7}}{F_T} \text{ AU} \quad (7)$$

where the planet mass, $M_{\text{pl}} = M_c + M_{\text{atm}}$, includes the mass of the core and atmosphere (defined as lying within the Bondi radius). The numerical estimates above and in Equation (8) assume the core mass dominates, $M_{\text{pl}} \sim M_c$.

Stellar tides dominate the planet's gravity beyond the Hill radius

$$R_H = \left(\frac{M_{\text{pl}}}{3M_*} \right)^{1/3} a \approx 0.22 m_{c10\oplus}^{1/3} a_{10\oplus} \text{ AU} \quad (8)$$

where hydrostatic balance clearly breaks down.

The relevant length scales of the atmosphere and disk satisfy the relation $R_B H_d^2 = 3R_H^3$. The lengthscales are roughly equal at the ‘‘thermal mass’’

$$M_{\text{th}} > \frac{c_d^3}{G\Omega} \approx 25 \frac{F_T^{3/2}}{\sqrt{m_*}} a_{10\oplus}^{6/7} M_\oplus. \quad (9)$$

In the low mass regime, $M_{\text{pl}} < M_{\text{th}}/\sqrt{3}$, the lengthscales order as $R_B < R_H < H_d$ (R06). In this regime, many studies assume the atmosphere matches the disk conditions at R_B . We however use R_H as the matching radius in both this low mass and other higher mass regimes. This choice is justified by the fact that, for hydrostatic solutions, the density at R_B exceeds the disk's background density by an order unity factor (R06). This modest density change has a similarly modest effect on our results.

For a finite range of intermediate masses, $M_{\text{th}}/\sqrt{3} < M_{\text{pl}} < 3M_{\text{th}}$, the Hill radius is the smallest scale, satisfying both $R_H < R_B$ and $R_H < H_d$. Spherical symmetry remains a good, if imperfect, approximation because

the bound atmosphere within R_H doesn't see the vertical stratification of the disk on scales $\gtrsim H_d$.

At higher planet masses where $M_{\text{pl}} > 3M_{\text{th}}$ and $H_d < R_H < R_B$, spherical symmetry is no longer a good approximation, due to both the vertical stratification of the disk and gap opening. See §6.1 for discussion of neglected non-hydrostatic effects on all mass scales.

Note that while R_H is the outer boundary of our structure calculations, we define planet masses to include only the mass inside the smaller of R_B or R_H . This conservative choice in quoting planet masses is usually a minor distinction because (when $R_B < R_H$) the gas between R_B and R_H is weakly compressed.

2.3. Structure Equations and Boundary Conditions

Our atmosphere calculations use the standard structure equations of mass conservation, hydrostatic balance, thermal gradients, and energy conservation:

$$\frac{dm}{dr} = 4\pi r^2 \rho \quad (10a)$$

$$\frac{dP}{dr} = -\frac{Gm}{r^2} \rho \quad (10b)$$

$$\frac{dT}{dr} = \nabla \frac{T}{P} \frac{dP}{dr} \quad (10c)$$

$$\frac{dL}{dr} = 4\pi r^2 \rho \left(\epsilon - T \frac{\partial S}{\partial t} \Big|_m \right), \quad (10d)$$

where r is the radial coordinate, P , T , ρ and L are the gas pressure, temperature, density and luminosity, respectively. The enclosed mass at radius r is m . Equation (10c) simply defines the temperature gradient $\nabla \equiv d \ln T / d \ln P$. In radiative zones radiative diffusion gives a temperature gradient

$$\nabla_{\text{rad}} \equiv \frac{3\kappa P}{64\pi G m \sigma T^4} L, \quad (11)$$

where σ is the Stefan-Boltzmann constant. In convectively unstable regions, efficient convection gives an isentropic temperature gradient with $\nabla = \nabla_{\text{ad}}$, the adiabatic gradient. According to the Schwarzschild criterion, convective instability occurs when $\nabla_{\text{rad}} > \nabla_{\text{ad}}$. Thus $\nabla = \min(\nabla_{\text{rad}}, \nabla_{\text{ad}})$ set the temperature gradient.

In the energy equation (10d), ϵ represents all local sources of heat input, which excludes the motion of the atmosphere itself. In stars, nuclear burning contributes to ϵ . In a protoplanetary atmosphere, dissipative drag on planetesimals contributes to ϵ . Our simplified models set $\epsilon = 0$, consistent with our neglect of planetesimal accretion luminosity at the base of the atmosphere. The $\epsilon_g = -T \partial S / \partial t$ term gives the energy input from gravitational contraction.¹ The partial time derivative would normally require our radial derivatives should be partial derivatives. However our subsequent developments will replace the local energy equation (10d) with global energy balance, reverting the structure equations to time-independent ordinary differential equations (ODEs).

To solve the equation set (10) an equation of state (EOS) is required for closure. In our study, we adopt an

¹ In general, any motion in a non-stationary atmosphere is accounted for by this term. The partial time derivative is performed on shells of fixed mass.

ideal gas law with a polytropic EOS

$$P = \rho RT, \quad (12a)$$

$$P = K\rho^\gamma, \quad (12b)$$

where K is the adiabatic constant. The adiabatic index $\gamma = 1/(1 - \nabla_{\text{ad}})$. An ideal monatomic gas has $\nabla_{\text{ad}} = 2/5$. This work uses $\nabla_{\text{ad}} = 2/7$ for an ideal diatomic gas. While our reference mean molecular weight ($\mu = 2.35$ proton masses) includes Helium, we ignore Helium's effect on the EOS, which is already greatly simplified. The second law of thermodynamics gives the relative entropy as

$$S = \mathcal{R} \ln \left(\frac{T^{1/\nabla_{\text{ad}}}}{P} \right) \quad (13)$$

eliminating the need for K .

Boundary conditions must be satisfied at both the base and the top of the atmosphere with $m(R_c) = M_c$, $T(R_H) = T_d$ and $P(R_H) = P_d$. In principle our solutions describe atmospheres with $L(R_c) = 0$. In practice, since we do not directly integrate Equation (10d) we need not directly impose this boundary condition, as described in §2.5.

2.4. Global Cooling of an Embedded Planet

This section describes the global energy balance of a planet embedded in a gas disk, or more generally any spherical, hydrostatic object in pressure equilibrium with a background medium. The total atmospheric energy includes gravitational and internal energies, $E = E_G + U$:

$$E_G = - \int_{M_c}^M \frac{Gm}{r} dm, \quad (14a)$$

$$U = \int_{M_c}^M u dm. \quad (14b)$$

The specific internal energy $u = C_V T = \mathcal{R}(\nabla_{\text{ad}}^{-1} - 1)T$ for a polytropic EOS. For a star or coreless planet, $L_c = M_c = 0$.

We start with the global energy balance for an isolated planet with a free surface:

$$L_M = L_c + \Gamma - \dot{E}. \quad (15)$$

The surface luminosity, L_M , includes contributions from the core luminosity L_c – e.g. planetesimal accretion or radioactive decay – from the total heat generation Γ – given by the integral of ϵ over the object – and from the rate of change of atmospheric energy \dot{E} , a loss term.

For an object with no core luminosity (or no core) and no internal heat sources, the energy equation $L_M = -\dot{E}$ describes KH contraction in its simplest form. For a main sequence star $L_M = \Gamma$ as the internal heat of nuclear burning supplies the total luminosity.

For a protoplanetary atmosphere embedded in a gas disk, the full energy equation,

$$L_M = L_c + \Gamma - \dot{E} + e_M \dot{M} - P_M \left. \frac{\partial V_M}{\partial t} \right|_M, \quad (16)$$

acquires surface terms as derived in Appendix A. The energy accreted across the surface is given by the specific energy $e_M = u_M - GM/R$ and the mass accretion rate

\dot{M} . The work done by the surface is $P_M \partial V_M / \partial t$, with the partial derivative performed at fixed mass. This generalized energy equation applies on any spherical shell where hydrostatic balance holds.² Thus M no longer refers to a uniquely defined total mass, but to the chosen mass level, where the instantaneous radius is R . Values on this shell are labelled by M subscripts.

For static solutions, which are not the focus of this Paper, the surface terms (and also \dot{E}) vanish. Static solutions are valid when imposed heat sources, i.e. L_c and Γ exceed the atmospheric losses. Quantitatively, static solutions apply when the evolutionary timescale exceeds the KH timescale,

$$\tau_{\text{KH}} \sim \frac{|E|}{L_M}, \quad (17)$$

where surface terms are assumed to be subdominant. Thus evolutionary calculations, including the quasistatic calculations of this Paper, are needed to consider the fastest possible evolution that occurs on τ_{KH} .

2.5. The Two-Layer Model

To simplify our calculations of atmospheric contraction we use a two layer model with a bottom convective region and an upper radiative layer. The existence of such a structure is well known from previous studies (R06) and can be readily understood. Before the protoplanetary atmosphere can cool, it has the entropy of the disk. As the atmosphere cools the deep interior remains convective. Convective interiors are a common feature of low mass cool objects (brown dwarfs and planets) that results from the behavior of ∇_{rad} for realistic opacity laws. However the entropy of the deep interior decreases as the atmosphere cools. A region of outwardly increasing entropy, i.e. a radiative layer, is required to connect the convective interior to the disk. A more complicated structure, with radiative windows in the convection zone, is possible as discussed in §6.2.

In convective regions, the adiabatic structure is independent of luminosity and can be calculated without local energy balance, Equation (10d). Thus for fully convective objects, a cooling sequence can be established by connecting a series of adiabatic solutions using a global energy equation, $L_M = -\dot{E}$ or Equation (15). Such methods are commonly used for their computational efficiency and are sometime referred to as “following the adiabats,” since the steady state solutions evolve in order of decreasing entropy (Marleau & Cumming 2013).

In the radiative zone, local energy balance, Equation (10d), does affect the atmospheric structure. We proceed by assuming that the majority of energy is lost from the convective interior, and thus the luminosity can be treated as constant in the outer radiative zone. With this approximation we can construct solutions from equations (10a – c) that “follow the mass,” i.e. gradually increase the atmospheric mass. We then use the global energy balance, Equation (16) to connect these solution

² Equation (16) also applies in the interior of objects with a free surface. The work term, $P_M \partial V_M / \partial t$, vanishes at the free surface. The accretion energy, $e_M \dot{M}$, vanishes for a truly isolated object, but would be included to account for accretion onto (or mass loss from) an otherwise free surface.

in a cooling sequence. The validity of neglecting luminosity generation in the radiative zone can be checked *a posteriori*.

To make a single atmosphere model (indexed by i) we choose a planet mass M_i . At the outer boundary, at $R_H(M_i)$, the temperature and pressure are set to the disk values. To integrate equations (10a–c), the luminosity is required to compute ∇_{rad} . The correct value of the luminosity is the eigenvalue of the problem, which we find by the shooting method. Only for the eigenvalue of luminosity does the integrated value of mass at the core, $m(R_c)$, match the actual core mass, M_c .

To establish the time difference between neighboring solutions, we apply Equation (16) at the radiative-convective boundary (RCB) of the solutions. Using exact solutions, energy balance could be evaluated at any level. The approximation of constant luminosity in the radiative zone makes the RCB the self-consistent choice. After setting $\Gamma = L_c = 0$, we solve for the elapsed time Δt between states i and $i+1$ by finite differencing to give

$$\Delta t = \frac{-\Delta E + \langle e \rangle \Delta M - \langle P \rangle \Delta V_{\langle M \rangle}}{\langle L \rangle}. \quad (18)$$

Brackets indicate an average of, and Delta (Δ) indicates a difference between, neighboring solutions. All values are evaluated at the RCB. Due to the partial derivative in Equation (16), the volume difference $\Delta V_{\langle M \rangle}$ is performed at fixed mass, here the average of the masses at the RCB.

3. ANALYTIC COOLING MODEL

4. QUASI-STATIC KELVIN-HELMHOLTZ CONTRACTION

5. CRITICAL CORE MASS

6. NEGLECTED EFFECTS

6.1. Hydrodynamic Effects

The neglect of hydrodynamical effects in our model is best discussed in terms of the thermal mass, M_{th} , and the lengthscales introduced in §2.2. In the low mass regime, $M_{\text{pl}} < M_{\text{th}}/\sqrt{3}$, where $R_B < R_H$ we assume that hydrostatic balance holds out to the outer boundary at R_H . In this low mass regime, Ormel (2013) calculated the

flow patterns driven by stellar tides and disk headwinds. On scales $\gtrsim R_B$ the flows no longer circulate the planet: they belong to the disk. Nevertheless, the density structure outside R_B remains spherically symmetric and hydrostatic. Even if these flows do not destroy hydrostatic balance, they could affect the planet’s cooling. We expect such effects to be weak, as heat losses at greater depths dominate planetary cooling, but more study is needed.

At higher masses, non-hydrostatic effects become more severe. At $M_{\text{pl}} \gtrsim M_{\text{th}}$ planets can open significant gaps (Zhu et al. 2013). At yet higher masses accretion instabilities could occur (Ayliffe & Bate 2012). However in this high mass regime we already know that the spherically symmetric approximation breaks down, as does our approximate cooling model (*discuss elsewhere and cite*).

Thus by restricting our attention to low masses, neglected hydrodynamic effects should be minor. Moreover since $M_{\text{th}} \propto a^{6/7}$ increases with disk radius, spherical hydrostatic models like ours have a greater range of applicability in the outer regions of disks.

6.2. Realistic Opacities and EOS

some comments on sublimation and radiative windows here (current text rough). upshot is the outer disk is good.

Real dust opacities exhibit a more complicated behavior that depends on grain composition. Opacities drop by order unity when ice grains sublimate for $T \gtrsim 150$ K and they drop by orders of magnitude when silicate grains evaporate above $T \gtrsim 1500$ K (Semenov et al. 2003; Ferguson et al. 2005). While protoplanetary atmospheres get significantly hotter than 1500 K, our model depends on opacity only in the upper radiative zones, which are cool enough to remain dusty. For our opacity approximations to be valid, we must restrict our model to the lower temperatures of the outer disk.

A second radiative zone inside the convection zone, a “radiative window,” is possible. Dust sublimation at shallow depths – giving a low κ and P in Equation (11) – could cause such a window. Our two layer model ignores such complications.

7. CONCLUSIONS

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APPENDIX

A. DERIVATION OF THE GLOBAL ENERGY EQUATION

To derive the global energy equation (16) for an embedded protoplanets, we generalize the analogous calculations in stellar structure theory, e.g. in §4.3 of Kippenhahn & Weigert (1990). For our problem we add the effects of finite core radius, surface pressure and mass accretion. We start with the local energy equation (10d) whose more natural form

in Lagrangian (mass) coordinates is $\partial L/\partial m = \epsilon - T\partial S/\partial t$. Integrating from the core to a higher shell with enclosed mass M gives:

$$L - L_c = \int_{M_c}^M \frac{\partial L}{\partial m} dm \quad (\text{A1a})$$

$$= \int_{M_c}^M \left(\epsilon - T \frac{\partial S}{\partial t} \right) dm \quad (\text{A1b})$$

$$= \Gamma - \int_{M_c}^M \frac{\partial u}{\partial t} dm + \int_{M_c}^M \frac{P}{\rho^2} \frac{\partial \rho}{\partial t} dm. \quad (\text{A1c})$$

with $\Gamma = \int \epsilon dm$ the integral of the direct heating rate and applying the law of thermodynamics in the final step.

The global energy equation is derived by eliminating the partial time derivatives in Equation (A1c), which are performed at a fixed mass, in favor of total time derivatives, denoted with overdots. For instance the surface radius, R , evolves as

$$\dot{R} = \frac{\partial R}{\partial t} + \frac{\dot{M}}{4\pi R^2 \rho_M} \quad (\text{A2})$$

where $\partial R/\partial t$ gives the Lagrangian contraction of the “original” shell, and mass accretion through the upper boundary at rate \dot{M} also changes the shell location. Similarly the volume, $V = (4\pi/3)r^3$, and pressure at the outer shell evolve as

$$\dot{V}_M = \frac{\partial V_M}{\partial t} + \frac{\dot{M}}{\rho_M} \quad (\text{A3a})$$

$$\dot{P}_M = \frac{\partial P_M}{\partial t} + \frac{\partial P_M}{\partial m} \dot{M} = \frac{\partial P_M}{\partial t} - \frac{GM}{4\pi R^4} \dot{M}. \quad (\text{A3b})$$

This derivation holds the core mass and radius fixed, $\dot{M}_c = \dot{R}_c = 0$. Therefore the core pressure satisfies

$$\dot{P}_c = \partial P_c / \partial t. \quad (\text{A4})$$

The internal energy integral follows simply from Leibniz’s rule as

$$\int_{M_c}^{M(t)} \frac{\partial u}{\partial t} dm = \dot{U} - \dot{M} u_M. \quad (\text{A5})$$

To make further progress we require the virial theorem:

$$E_G = -3 \int_{M_c}^M \frac{P}{\rho} dm + 4\pi(R^3 P_M - R_c^3 P_c) \quad (\text{A6})$$

which follows from Equations (10a), (10b) and (14) by integrating hydrostatic balance in Lagrangian coordinates. As an aside, the integral in equation (A6) can be evaluated for a polytropic EOS to give simple expressions for the total energy:

$$E = (1 - \zeta)U + 4\pi(R^3 P_M - R_c^3 P_c) \quad (\text{A7a})$$

$$= \frac{\zeta - 1}{\zeta} E_G + \frac{4\pi}{\zeta} (R^3 P_M - R_c^3 P_c) \quad (\text{A7b})$$

where $\zeta \equiv 3(\gamma - 1)$. We will not make this assumption and will keep the EOS general.

To express the work integral, the final term in Equation (A1c), in terms of changes to gravitational energy we first take the time derivative of Equation (A6):

$$\dot{E}_G = 3 \int_{M_c}^M \frac{P}{\rho^2} \frac{\partial \rho}{\partial t} dm - 3 \int_{M_c}^M \frac{\partial P}{\partial t} \frac{dm}{\rho} - 3 \frac{P_M}{\rho_M} \dot{M} + 3 \dot{P}_M V_M - 3 \dot{P}_c V_c + 3 P_M \dot{V}_M. \quad (\text{A8})$$

The first integral in Equation (A8) is the one we want, but the next one must be eliminated. The time derivative of Equation (14) (times four) gives

$$4\dot{E}_G = -4 \frac{GM\dot{M}}{R} + 4 \int_{M_c}^M \frac{Gm}{r^2} \frac{\partial r}{\partial t} dm \quad (\text{A9a})$$

$$= -4 \frac{GM\dot{M}}{R} + 4\pi \int_{M_c}^M r^3 \frac{\partial}{\partial m} \frac{\partial P}{\partial t} dm \quad (\text{A9b})$$

$$= -4 \frac{GM\dot{M}}{R} - 3 \int_{M_c}^M \frac{\partial P}{\partial t} \frac{dm}{\rho} + 3 V_M \frac{\partial P_M}{\partial t} - 3 V_c \frac{\partial P_c}{\partial t} \quad (\text{A9c})$$

where Equations (A9b) and (A9c) use hydrostatic balance and integration by parts.

Subtracting Equations (A5) and (A9c) and rearranging terms with the help of Equations (A2), (A3) and (A4) gives

$$\int_{M_c}^M \frac{P}{\rho^2} \frac{\partial \rho}{\partial t} dm = -\dot{E}_G - \frac{GM\dot{M}}{R} - P_M \frac{\partial V_M}{\partial t}. \quad (\text{A10})$$

Combining Equations (A1c), (A5) and (A10), we reproduce Equation (16) with the accreted specific energy $e_M \equiv u_M - GM/R$.