Draft Modified November 9, 2012
Preprint typeset using LaTeX style emulateapj v. 5/2/11

CORE ACCRETION OF SELF-GRAVITATING ATMOSPHERES

Andrew N. Youdin
Harvard Smithsonian Center for Astrophysics

1. INTRODUCTION

The core accretion model (Perri & Cameron 1974; Mizuno et al. 1978) proposes that giant planets form by the accretion of gas onto a solid protoplanetary core. Early versions of the theory found a "critical core mass", $M_{\rm crit}$, above which static solutions do not exist and unstable atmospheric collapse would occur (Harris 1978; Mizuno 1980). These works used hydrostatic models that include heating by planetesimal accretion, but neglect the heat generated by the gravitational, or Kelvin-Helmholtz (KH), contraction of the atmosphere.

In evolutionary models that include KH contraction (Bodenheimer & Pollack 1986), the atmosphere does not undergo a hydrodynamic collapse. Instead the quasi-static contraction of the envelope accelerates continuously. Contraction becomes particularly rapid after the the "crossover" mass, when core and atmospheric masses are equal (Pollack et al. 1996). The concepts of critical and crossover masses are similar, and even agree numerically when the planetesimal accretion luminosity exceeds the KH luminosity. However if the planetesimal accretion rate is small and KH contraction dominates, the critical core mass will be smaller than the more relevant crossover mass (Ikoma et al. 2000).

This paper will consider 1D atmospheres where KH contraction dominates over planetesimal accretion. Such solutions give realistic lower limits on the core mass required to accrete a massive gaseous atmosphere. Indeed even models that include planetesimal accretion obtain the fastest formation times by stopping planetesimal accretion after the core reaches a certain mass (Hubickyj et al. 2005).

The early phases of atmospheric growth are usually treated with 1D models where the outer boundary of the atmosphere matches onto the conditions in the disk midplane. As the atmosphere continues to grow, 1D solutions are no longer adequate. The atmosphere both notices that the disk is not spherically symmetric and at large enough masses will start to open a gap in the disk. In this case the accretion of gas is limited by the ability of the disk to supply gas to the growing protoplanet (D'Angelo & Lubow 2008). This later stage evolution is crucial for determining the final mass of the planet, but not considered in our calculations of the initial collapse.

Section 3 describes the structure of planet atmospheres embedded in a gas disk. We then derive the critical core mass to trigger runaway collapse of the atmosphere in §6.3.

2. DISK MODEL

As a fiducial disk model, we adopt the version of the minimum mass, passively irradiated model of Chiang & Youdin (2010), which gives the surface density and midplane temperature as

$$\Sigma \;=\; 70\, F_\Sigma a_{10\oplus}^{-3/2} \text{ g cm}^{-2} \tag{1a}$$

$$T \;=\; 45\, F_T\, a_{10\oplus}^{-3/7} \text{ K} \tag{1b}$$

where $a_{10\oplus} \equiv a/10$ AU scales the disk radius and the normalization factors F_Σ and F_T adjust the disk mass and temperature relative to the canonical values. The resulting disk midplane pressure

$$P_{\rm o} = 6.9\times 10^{-9} F_\Sigma \sqrt{F_T m_*}\, a_{10\oplus}^{-45/14} \text{ bar} \tag{2}$$

for a mean molecular weight of 2.35. The stellar mass in solar units is $m_* \equiv M_*/M_\odot$, which only includes the gravitational effects, though stellar mass also influences temperature and probably correlates with average disk mass. Away from the midplane the pressure drops with height z as a Gaussian with scale-height

$$H = \frac{\sqrt{\mathcal{R}T_{\rm o}}}{\Omega} = 0.42\sqrt{\frac{F_T}{m_*}}\, a_{10\oplus}^{9/7} \text{ AU}\,. \tag{3}$$

3. ATMOSPHERIC STRUCTURE

In this section we review the structure of atmospheres surrounding a solid protoplanetary core embedded in a gas disk. The core has mass $M_{\rm c}$ and radius $R_{\rm c} = [3M_{\rm c}/(4\pi\rho_{\rm c})]^{1/3}$ with internal density $\rho_{\rm c} \approx 2$ g cm^{-3}. For protoplanetary disk parameters we use the minimum mass, passively irradiated disk model presented in Chiang & Youdin (2010, hereafter CY10). The disk has a midplane gas density $\rho_{\rm o}$ and temperature $T_{\rm o}$.

The core can attract a dense atmosphere inside its Bondi radius

$$R_B = \frac{GM_{\rm c}}{\mathcal{R}T_{\rm o}} \approx 0.17 \frac{m_{{\rm c}10\oplus}\, a_{10\oplus}^{3/7}}{F_T} \text{ AU} \tag{4}$$

where $m_{{\rm c}10\oplus} \equiv M_{\rm c}/(10\ M_\oplus)$ and $\mathcal{R}$ is the specific gas constant.

Low mass planets satisfy a hierarchy of length scales, $R_{\rm c} \ll R_B < R_{\rm H} < H$, where

$$R_{\rm H} = \left(\frac{M_{\rm c}}{3M_*}\right)^{1/3} a \approx 0.22 \frac{m_{{\rm c}10\oplus}^{1/3}\, a_{10\oplus}}{m_*^{1/3}} \text{ AU} \tag{5}$$

is the Hill radius, inside which the planet's gravity competes with Keplerian shear.

As protoplanets grow in mass the length scales $R_{\rm H} \sim R_{\rm B} \sim H$ are roughly equal at a characteristic "transitional" mass

$$M_{\rm tr} > \frac{(\mathcal{R}T_{\rm o})^{3/2}}{G\Omega} \approx 25\, \frac{F_T^{3/2}}{\sqrt{m_*}}\, a_{10\oplus}^{6/7}\ M_\oplus\,. \tag{6}$$

Technically the length scales re-order in three steps of increasing mass. First for $M > M_{\rm tr}/\sqrt{3}$ the ordering becomes $R_\mathrm{H} < R_\mathrm{B} < H$ as tidal truncation becomes more important than thermal constriants. While the shear flow past the atmosphere is clearly not spherically symmetric, it is still common take the atmosphere as spherically symmetric with an outer boundary at the Hill radius. When the mass increases further to $M > M_\mathrm{tr}$ then $R_\mathrm{H} < H < R_\mathrm{B}$, which does not change the fact that tidal truncation is the stronger effect at the outer boundary.

Finally when $M > 3M_\mathrm{tr}$ the high mass scaling $H < R_\mathrm{H} < R_\mathrm{B}$ is attained. The assumption of spherical symmetry fails for two reasons. First the planet now notices the vertical density stratification of the disk. Second, the planet's torques on the disk can begin to open gaps, creating further density inhomogeneity. Since runaway accretion typically begins for masses well below $\sim 75 M_\oplus$, the assumption of spherical symmetry is a good, if not perfect, approximation for understanding whether cores can attrach massive atmospheres. Moreover the approximation becomes even better in the disk's outer regions.

Another relevant for atmospheric structure is the mass of disk gas that fills the Bondi radius, at the ambient density. The ratio of this mass to the core mass is

$$\theta_\mathrm{c} \equiv \frac{4\pi\rho_\mathrm{o} R_B^3}{3M_\mathrm{c}} \approx 5\times 10^{-3}\frac{m_{\mathrm{c}10\oplus}^2}{a_{10\oplus}^{3/2}}\frac{F_\Sigma\sqrt{m_*}}{F_T^{7/2}}\,. \tag{7}$$

The smaller this parameter the more gravitational compression is required for an atmosphere to become self-gravitating.

There are two limiting cases of atmospheric structure, isothermal and adiabatic. Atmospheres that are isothermal at T_o have cooled completely and give the most massive atmosphere. Convectively unstable atmospheres transport heat very efficiently and are nearly adiabatic. Adiabatic atmospheres are the opposite extreme of the isothermal case, giving the steepest increase of temperature with atmospheric dept. The least dense atmospheres are fully adiabatic with the entropy of the disk.

The structure of an embedded protoplanet consists of an inner adiabatic region that matches onto an outer radiative zone, that in some cases, can be approximated as nearly isothermal. This basic structure was demonstrated by R06 for an atmosphere supported by planetesimal accretion luminosity. PN05 showed that atmospheres powered by gas accretion alone have a similar structure. Due to their intense irradiation, "hot Jupiters" similarly have an isothermal layer that matches onto a deeper adiabatic interior (Arras & Bildsten 2006; Youdin & Mitchell 2010).

3.1. *Isothermal Atmosphere*

We now consider the structure of a low mass, i.e. non-self gravitating, isothermal atmosphere. We assume the atmosphere matches onto a constant background density, ρ_o, at a distance $r_\mathrm{fit} = n_\mathrm{fit} R_\mathrm{B}$. The resulting density profile is

$$\rho = \rho_\mathrm{o}\exp\left(\frac{R_B}{r} - \frac{1}{n_B}\right) \approx \rho_\mathrm{o}\exp\left(\frac{R_B}{r}\right) \tag{8}$$

where the approximate inequality holds deep inside the atmosphere ($r \ll R_\mathrm{B}$) for any $n_\mathrm{fit} \gtrsim 1$. However the choice of boundary condition does have an order unity effect on the density near the Bondi radius.

The mass of the atmosphere typically determined by integrating the density profile from the core to the Bondi radius. Planets can attract massive atmospheres if $\theta_\mathrm{c} \equiv R_B/R_\mathrm{c} \gg 1$. In this case

$$M_\mathrm{iso} \approx 4\pi\rho_\mathrm{o}\frac{R_\mathrm{c}^4}{R_B}e^{R_B/R_\mathrm{c}} = 4\pi\rho_\mathrm{o}\frac{R_\mathrm{c}^3}{\theta_\mathrm{c}}e^{\theta_\mathrm{c}}\,. \tag{9}$$

This result is the leading order term in a series expansion. Furthermore, because the atmospheric scale-height at R_c is $H_\rho = |dr/d\ln\rho| = R_\mathrm{c}^2/R_B$, the result is intuitively the correct order of magnitude.

From Equation (9) the ratio of isothermal atmosphere to core mass is

$$\frac{M_\mathrm{iso}}{M_\mathrm{c}} = 3\frac{\rho_\mathrm{o}}{\rho_\mathrm{c}}\frac{R_\mathrm{c}}{R_B}e^{R_B/R_\mathrm{c}} \tag{10}$$

While $\rho_\mathrm{o} \ll \rho_\mathrm{c}$, the exponential dependence allows the atmosphere to become massive. As a concrete example, consider a disk with $T = 60$ K and $\rho_\mathrm{o} = 3\times 10^{-11}$ (in cgs) and a core density $\rho_\mathrm{c} = 2$. The atmosphere self-gravitating for $\theta_\mathrm{c} = 27.1$ or $M_\mathrm{c} > 0.046 M_\oplus$. The fact that very low mass cores can have very massive isothermal atmospheres is well known (Sasaki 1989). However such solutions are unrealistic because there is inadequate time to radiate away all the atmosphere's thermal energy.

We not consider the mass exterior to the Bondi radius. For a meaningful evaluation we only include the mass coming from the overdensity relative to the background density. The resulting external mass for an isothermal atmosphere is

$$\begin{aligned} M_\mathrm{ext} &= 4\pi\int_{R_\mathrm{B}}^{r_\mathrm{fit}}(\rho - \rho_\mathrm{o})r^2 dr \qquad (11a)\\ &= M_\mathrm{c}\theta_\mathrm{c}\int_1^{n_\mathrm{fit}} 3\left[\exp\left(\frac{1}{x} - \frac{1}{n_\mathrm{fit}}\right) - 1\right]x^2 dx \\ &\equiv M_\mathrm{c}\theta_\mathrm{c} I(n_\mathrm{fit}) \qquad (11b)\end{aligned}$$

where the dimensionless integral $I(n_\mathrm{fit})$ obeys the limits $I(1) = 0$ and $I \to n_\mathrm{fit}^2/2$ as $n_\mathrm{fit}\to\infty$. Since this external mass does not converge the choice of an outer boundary does matter in principle. In practice however the assumption that $r_\mathrm{fit} = R_\mathrm{H}$ limits n_fit to modest values

$$n_\mathrm{fit} = \frac{R_\mathrm{H}}{R_\mathrm{B}} \approx 1.3\frac{a_{10\oplus}^{4/7}}{m_{\mathrm{c}10\oplus}^{2/3}}\frac{F_T}{m_*^{1/3}} \tag{12}$$

since for instance $I(2) = 1.1$ these modest n_fit values will only produce a small external mass.

However the effect is still worth including for the effect on the interior mass.

3.2. *Temperature Contrast at Convective Boundary*

We now show that the temperature contrast between the convective boundary, T_CB, and the ambient disk T_o, is modest. We express the radiative lapse rate

$$\nabla_\mathrm{rad} \equiv \frac{d\ln T}{d\ln P} = \frac{3\kappa P}{64\pi G M\sigma T^4}L = \nabla_\mathrm{o}\frac{(P/P_\mathrm{o})^{1+\alpha}}{(T/T_\mathrm{o})^{4-\beta}} \tag{13}$$

where M is the sum of the core mass and the atmosphere mass below the pressure level P. If the mass in the radiative zone is small, then we can hold $M = M_\mathrm{c} + M_\mathrm{conv}$ fixed at the sum of core and convective zone masses.

With this assumption and powerlaw opacity, we get a constant value for $\nabla_{\rm o}$. The temperature profile then integrates to

$$\left(\frac{T}{T_{\rm o}}\right)^{4-\beta} - 1 = \frac{\nabla_{\rm o}}{\nabla_\infty}\left[\left(\frac{P}{P_{\rm o}}\right)^{1-\alpha} - 1\right], \qquad (14)$$

where $\nabla_\infty = (1+\alpha)/(4-\beta)$ is $\nabla_{\rm rad}$ for $T, P \to \infty$.[1] We apply Equations (13) and (14) at the convective boundary $\nabla_{\rm rad} = \nabla_{\rm ad}$ under the assumption that the pressure there is $P_{\rm CB} \gg P_{\rm o}$. The resulting temperature contrast at the convective boundary is

$$\chi \equiv \frac{T_{\rm CB}}{T_{\rm o}} \simeq \left(1 - \frac{\nabla_{\rm ad}}{\nabla_\infty}\right)^{-\frac{1}{4-\beta}}. \qquad (15)$$

For the low temperatures in protoplanetary disks, opacity is dominated by dust with $\alpha = 0$ and $\beta \approx 2$ or 1, giving $\chi \approx 1.5$ or 1.9, respectively, assuming $\nabla_{\rm ad} = 2/7$.

3.3. *Location of Convective Boundary*

The pressure at the convective boundary follows from Equations (13) and (15) as

$$\frac{P_{\rm CB}}{P_{\rm o}} \simeq \left(\frac{\nabla_{\rm ad}/\nabla_{\rm o}}{1 - \nabla_{\rm ad}/\nabla_\infty}\right)^{\frac{1}{1+\alpha}} \qquad (16)$$

This pressure contrast can be quite large due to the smallness of $\nabla_{\rm o}$ in low luminosity atmospheres.

It is also useful to obtain a relation between T and P that eliminates $\nabla_{\rm o}$ in favor of $P_{\rm CB}$:

$$\frac{T}{T_{\rm o}} = \left\{1 + \frac{1}{\frac{\nabla_\infty}{\nabla_{\rm ad}} - 1}\left[\left(\frac{P}{P_{\rm CB}}\right)^{1-\alpha} - \left(\frac{P_{\rm o}}{P_{\rm CB}}\right)^{1-\alpha}\right]\right\}^{\frac{1}{4-\beta}}. \qquad (17)$$

We can determine the radius of the convective boundary $r_{\rm CB}$ from the hydrostatic balance equation as

$$\frac{R_B}{r_{\rm CB}} = \int_{P_{\rm o}}^{P_{\rm CB}} \frac{T}{T_{\rm o}}\frac{dP}{P}. \qquad (18)$$

An isothermal atmosphere gives a simple logarithmic dependence on $P_{\rm CB}$. However using Equation (17) in the integral gives

$$\frac{R_B}{r_{\rm CB}} = \ln\left(\frac{P_{\rm CB}}{P_{\rm o}}\right) - \ln\theta, \qquad (19)$$

with an extra correction term, $\theta < 1$. In the $P_{\rm CB} \gg P_{\rm o}$ limit the correction term is a constant that depends on α, β and $\nabla_{\rm ad}$. The form of θ was chosen to allow us to express

$$P_{\rm CB} = \theta P_{\rm o} e^{R_B/r_{\rm CB}}. \qquad (20)$$

A simple analytic expression for θ is not possible.

The T-P profile can also be combined with the equation of hydrostatic balance to relate the radius and pressure of the convective boundary. If the convective boundary is well inside the Bondi radius, i.e. $r_{\rm CB} \ll R_B$ and $P_{\rm CB} \gg P_{\rm o}$ then

$$P_{\rm CB} = \theta P_{\rm o} e^{R_B/r_{\rm CB}}. \qquad (21)$$

[1] Our definitions of $\nabla_{\rm o}$ and ∇_∞ are precisely opposite to R06, but consistent with other works and the general convention of labeling a quantity f in the disk as $f_{\rm o}$.

TABLE 1
Parameters Describing Structure of Radiative Zone.

$\gamma = 7/5$ ($\nabla_{\rm ad} = 2/7$), $\alpha = 0$					
β	1/2	3/4	1	3/2	2
∇_∞	2/7 [a]	4/13	1/3	2/5	1/2
χ	···	2.25245	1.91293	1.65054	1.52753
θ	···	0.145032	0.285824	0.456333	0.556069

[a] Since $\nabla_{\rm ad} = \nabla_\infty$ there is no convective transition at depth for this case.

The order unity constant $\theta < 1$ accounts for the fact that the radiative layer is not perfectly isothermal, as θ would be exactly one in that case. The value of theta depends on the equation of state and the pressure and temperature dependence of the opacity, i.e. the $\alpha(=0)$ and β values. We find $\theta(\beta = 2) \approx 0.556$ and $\theta(\beta = 1) = 0.286$. Closed form expressions for these integrals are too cumbersome to be useful.

4. VIRIAL THEOREM

The virial theorem for self-gravitating atmospheres can be applied to protoplanetary atmospheres. The virial theorem is derived as usual by integrating the equation of hydrostatic balance to give

$$E_G = -3\int_{M_{\rm c}}^{M} \frac{P}{\rho}dm + 4\pi\left(R^3 P_M - r_{\rm c}^3 P_{\rm c}\right) \qquad (22)$$

where the gravitational energy

$$E_G = -\int_{M_{\rm c}}^{M} \frac{Gm}{r}dm. \qquad (23)$$

At the outer surface $m = M$, $r = R$ and other values are given a subscripted M. This outer surface can be evaluated anywhere (e.g. Bondi radius, Hill radius or RCB) that hydrostatic balance holds. The boundary terms include the effects of finite surface pressure and finite radius of the solid core.

The integral on the RHS Equation (22) is related to the internal energy. For an ideal gas, $P/\rho = \mathcal{R}T = (\gamma-1)u$, where u is the internal energy per mass, and $\gamma = C_P/C_V$. For a polytrope with constant γ we can express the virial theorem in terms of internal energy $E_i = \int u dm$ as

$$E_G = -\zeta E_i + 4\pi\left(R^3 P_M - r_{\rm c}^3 P_{\rm c}\right) \qquad (24)$$

where $\zeta \equiv 3(\gamma - 1)$. In this case the total energy $W = E_G + E_i$ becomes

$$W = (1-\zeta)E_i + 4\pi\left(R^3 P_M - r_{\rm c}^3 P_{\rm c}\right) \qquad (25a)$$

$$= \frac{\zeta - 1}{\zeta}E_G + \frac{4\pi}{\zeta}\left(R^3 P_M - r_{\rm c}^3 P_{\rm c}\right). \qquad (25b)$$

The surface pressure acts to unbind the atmosphere by increasing W. While a realistic EOS is neither ideal nor polytropic, Equation (22) is general.

5. COOLING MODELS

An isolated sphere satisfies a simple global energy equation

$$L = \Gamma - \dot{E} \qquad (26)$$

where the luminosity is balanced by the rate of heat generation, Γ, and the rate at which total (gravitational plus thermal, at least) energy is lost.

For a protoplanetary atmosphere embedded in a gas disk, the cooling equation is more complicated:

$$L = L_{\rm c} + \Gamma - \dot{E} + e_{\rm acc}\dot{M} - P_M \frac{\partial V_M}{\partial t} \, . \tag{27}$$

We explain these extra terms and then derive the result. Equation (27) assumes a well (but arbitrarily) defined surface at radius R and mass M, which can evolve in time. Examples include the Bondi radius or the radiative-convective boundary. The luminosity L represents the luminosity from that surface. The luminosity from the solid core is given by $L_{\rm c}$, and includes planetesimal accretion energy and radioactive decay. The core is assumed to have a fixed mass, $M_{\rm c}$, and radius, $R_{\rm c}$, to simplify the atmospheric calculation. The Γ and $\dot{E}$ integrals are identical to Equation (26), and include the atmosphere from the core to the top boundary. Sources of direct heating that contribute to Γ include atmospheric drag on sedimenting planetesimals and the dissipation of any atmospheric turbulence.

Mass accretion at the rate $\dot{M}$ brings specific energy $e_{\rm acc} = u - GM/R$, the sum of internal and gravitational energies. The energy of accreted matter, $e_{\rm acc}$, is zero at a modified Bondi radius, $R_{\rm out} = GM/u \approx (\mathcal{R}/C_V)R_B$. For $R < R_{\rm out}$ ($R > R_{\rm out}$, respectively) the accreted mass has negative (positive) energy and is (un)bound. The final term in Equation (27) gives the pressure work done on a surface mass element, so the partial time derivative of volume holds the mass fixed. Our quasistatic models we neglect bulk kinetic energy.

5.1. *Derivation of Global Energy Equation*

We now derive the global energy equation (27). We follow the simpler example in §4.3 of Kippenhahn & Weigert (1994), adding the effects of finite core radius, surface pressure and mass accretion. We assume that hydrostatic balance holds. Integrating the local energy equation from core to surface gives:

$$L - L_{\rm c} = \int_{M_{\rm c}}^{M} \frac{\partial L}{\partial m} dm \tag{28a}$$

$$= \int_{M_{\rm c}}^{M} \left(\epsilon - T\frac{\partial S}{\partial t}\right) dm \tag{28b}$$

$$= \Gamma - \int_{M_{\rm c}}^{M} \frac{\partial u}{\partial t} dm + \int_{M_{\rm c}}^{M} \frac{P}{\rho^2}\frac{\partial \rho}{\partial t} dm \, . \tag{28c}$$

with $\Gamma = \int \epsilon dm$ the integral of the direct heating rate.

In what follows, we must carefully distinguish between partial time derivatives, $\partial/\partial t$, (performed at fixed mass) and total time derivatives, denoted with overdots (which include the effect of mass accreted through the outer boundary). For instance the evolution of surface radius, R, evolves as

$$\dot{R} = \frac{\partial R}{\partial t} + \frac{\dot{M}}{4\pi R^2 \rho_M} \tag{29}$$

where $\partial R/\partial t$ gives the Lagrangian contraction of surface mass elements, and $\dot{M}$ denotes mass accretion rate through the upper boundary. The subscript M denotes quantities at the upper boundary of total mass M (though it is omitted from M and R). Similarly the volume, $V = (4\pi/3)r^3$ and pressure at the surface evolve as

$$\dot{V}_M = \frac{\partial V_{\rm M}}{\partial t} + \frac{\dot{M}}{\rho_{\rm M}} \tag{30a}$$

$$\dot{P}_M = \frac{\partial P_{\rm M}}{\partial t} + \frac{\partial P_M}{\partial m}\dot{M} \tag{30b}$$

$$= \frac{\partial P_{\rm M}}{\partial t} - \frac{GM}{4\pi R^4}\dot{M} \, . \tag{30c}$$

For the purpose of this derivation we will hold the core mass fixed $\dot{M}_{\rm c} = 0$ which further gives $\dot{P}_{\rm c} = \partial P_{\rm c}/\partial t$.

To derive the global energy equation we must move the (partial) time derivatives in Equation (28c) outside their integrals. The internal energy integral follows simply from Leibniz's rule as

$$\int_{M_{\rm c}}^{M(t)} \frac{\partial u}{\partial t} dm = \dot{E}_i - \dot{M} u_M \, . \tag{31}$$

To evaluate the work integral, we derive a pair of expressions for the rate of change of gravitational energy. The time derivative of Equation (22) gives

$$\dot{E}_G = 3\int_{M_{\rm c}}^{M} \frac{P}{\rho^2}\frac{\partial \rho}{\partial t} dm - 3\int_{M_{\rm c}}^{M} \frac{\partial P}{\partial t}\frac{dm}{\rho} - 3\frac{P_M}{\rho_M}\dot{M} + 3\dot{P}_M V_M - 3\dot{P}_{\rm c} V_{\rm c} + 3 P_M \dot{V}_M \, . \tag{32}$$

The first integral in Equation (32) is the one we want, but the next one must be eliminated. The time derivative of Equation (23) (times four) gives

$$4\dot{E}_G = -4\frac{GM\dot{M}}{R} + 4\int_{M_{\rm c}}^{M} \frac{Gm}{r^2}\frac{\partial r}{\partial t} dm \tag{33a}$$

$$= -4\frac{GM\dot{M}}{R} + 4\pi\int_{M_{\rm c}}^{M} r^3 \frac{\partial}{\partial m}\frac{\partial P}{\partial t} dm \tag{33b}$$

$$= -4\frac{GM\dot{M}}{R} - 3\int_{M_{\rm c}}^{M} \frac{\partial P}{\partial t}\frac{dm}{\rho} + 3V_M\frac{\partial P_M}{\partial t} - 3V_{\rm c}\frac{\partial P_{\rm c}}{\partial t} \tag{33c}$$

where Equations (33b) and (33c) use hydrostatic balance and integration by parts.

Subtracting Equations (31) and (33c) and rearranging for the desired integral gives

$$\int_{M_{\rm c}}^{M} \frac{P}{\rho^2}\frac{\partial \rho}{\partial t} dm = -\dot{E}_G - \frac{GM\dot{M}}{R} - P_M\frac{\partial \dot{V}_M}{\partial t} \tag{34}$$

with Equation (30) used to combine terms. Combining Equations (28c), (31) and (34), we reproduce Equation (27) with the accreted energy $e_{\rm acc} \equiv u_M - GM/R$.

6. SIMPLIFIED COOLING MODEL

We consider an analytic cooling model that assumes an ideal gas polytrope for the convective region and neglects self-gravity.

6.1. *Luminosity, Energy & Mass*

The luminosity that emerges at the radiative convective boundary is

$$L_{\rm CB} = \frac{64\pi G M_{\rm CB}\sigma T_{\rm CB}^4}{3\kappa P_{\rm CB}}\nabla_{\rm ad} \approx L_{\rm o}\frac{P_{\rm o}}{P_{\rm CB}} \tag{35}$$

for a dust opacity with $\kappa \propto T^\beta$ and

$$L_{\rm o} \equiv \frac{64\pi G M_{\rm CB}\sigma T_{\rm o}^4}{3\kappa(T_{\rm o})P_{\rm o}}\nabla_{\rm ad}\chi^{4-\beta}\,. \tag{36}$$

normalized to disk conditions, with $T = \chi T_{\rm o}$ at the RCB.

We adopt an opacity law

$$\kappa = 2F_\kappa\left(\frac{T}{100\ {\rm K}}\right)^\beta\ {\rm cm^2/g} \tag{37}$$

independent of pressure as appropriate for dust opacities. For $\beta = 2$ and $F_\kappa = 1$, this gives the Bell & Lin (1994) opacity for icy grains. By varying F_κ, dust depletion or enhancement can be considered. Grain properties affect both F_κ and β which generally satisfies $1/2 \lesssim \beta \lesssim 2$ (aside from discontinuities across sublimation regions), see Semenov et al. (2003).

To make further analytic progress, we ignore the self-gravity in the convective zone, holding the mass fixed at $M_{\rm c}$. The density profile of an adiabatic atmosphere follows from hydrostatic balance as

$$\rho = \rho_{\rm CB}\left[1 + \frac{R_B'}{r} - \frac{R_B'}{r_{\rm CB}}\right]^{1/(\gamma-1)}\,. \tag{38}$$

where we define an effective Bondi radius,

$$R_B' \equiv \frac{GM_{\rm c}}{C_P T_{\rm CB}} = \frac{\nabla_{\rm ad}}{\chi}R_B \tag{39}$$

to simplify expressions.

Deep in the atmosphere, where $r \ll r_{\rm CB} \lesssim R_b'$ the density profile is $\rho \propto r^{-1/(\gamma-1)}$. Since energy scale as $\rho r^2 \propto r^{(2\gamma-3)/(\gamma-1)}$ only polytropes with $\gamma < 3/2$ (i.e. $\gamma = 7/5$, but not $\gamma = 5/3$) have the bulk of energy at the bottom of the atmospheres. We will thus focus on the $\gamma = 7/5$ case, even though dissociation occurs deep in real protoplanetary atmospheres.

The total (thermal and gravitational) energy in an adiabatic atmosphere could be evaluated from the virial theorem. More simply, we use the result for temperature profiles in deep (but non-self-gravitating) convective regions:

$$T \approx \frac{GM_{\rm c}}{C_P r} = T_{\rm CB}\frac{R_B'}{r}\,. \tag{40}$$

The internal energy per unit mass of an ideal gas is thus $u = C_V T = (1 - \nabla_{\rm ad})GM_{\rm c}/r$ and the specific energy deep in the atmosphere is

$$e = e_g + u = -\nabla_{\rm ad}\frac{GM_{\rm c}}{r}\,. \tag{41}$$

Thus the total energy for $\gamma < 3/2$ is thus

$$E = -4\pi\nabla_{\rm ad}GM_{\rm c}\int_{R_{\rm c}}^{r_{\rm CB}}\rho r dr \tag{42}$$

$$\approx -4\pi P_{\rm CB}R_B'^{\frac{1}{\nabla_{\rm ad}}}\left(\frac{\gamma-1}{3-2\gamma}\right)R_{\rm c}^{\frac{2\gamma-3}{\gamma-1}} \tag{43}$$

$$\approx -8\pi P_{\rm CB}\frac{R_B'^{7/2}}{\sqrt{R_{\rm c}}} \tag{44}$$

where the final expression takes $\gamma = 7/5$.

The mass of the adiabatic atmosphere is given by

$$M_{\rm atm} = 4\pi\int_{R_{\rm c}}^{r_{\rm CB}}\rho r^2 dr \tag{45}$$

$$= \frac{5\pi^2}{4}\rho_{\rm CB}{R_B'}^{5/2}\sqrt{r_{\rm CB}} \tag{46}$$

in the limit $R_{\rm c} \ll r_{\rm CB} \ll R_B'$. The mass in the isothermal region is $M_{\rm iso} \sim 4\pi\rho_{\rm CB}r_{\rm CB}^4/R_B \ll M_{\rm atm}$, and can be neglected.

We can eliminate $r_{\rm CB}$ with Equation (21). The ratio of atmosphere to core mass is then

$$\frac{M_{\rm atm}}{M_{\rm c}} \approx \frac{P_{\rm CB}/P_M}{\sqrt{\ln[P_{\rm CB}/(\theta P_{\rm o})]}} \tag{47}$$

where we introduce a characteristic pressure

$$P_M \equiv \frac{4\nabla_{\rm ad}^{3/2}}{5\pi^2\sqrt{\chi}}\frac{GM_{\rm c}^2}{{R_B'}^4}\,. \tag{48}$$

For the atmosphere to become self-gravitating, with $M_{\rm atm} = M_{\rm c}$, we thus require

$$P_{\rm CB} = \xi P_M \tag{49}$$

where the logarithmic factor

$$\xi \equiv \sqrt{\ln[P_{\rm CB}/(\theta P_{\rm o})]} = \sqrt{\ln[\xi P_M/(\theta P_{\rm o})]} \tag{50}$$

is found by numerically solving the above transcendental equation. The physical solution has $\xi > 1$, but typically order unity. Numerical solutions exit for $\xi \ll 1$, but are unphysical as they imply $P_{\rm CB} < P_{\rm o}$.

6.2. *Cooling Time*

Neglecting surface terms, the total time to cool the atmosphere from an initially fully adiabatic state is

$$t_{\rm cool} = -\int\frac{dE}{L} = -\int_{P_{\rm o}}^{P_{\rm CB}}\frac{dE/dP_{\rm CB}}{L}dP_{\rm CB} \tag{51}$$

$$\approx 4\pi\frac{P_{\rm CB}^2}{P_{\rm o}}\frac{R_B'^{7/2}}{L_{\rm o}\sqrt{R_{\rm c}}}\,, \tag{52}$$

where we neglect self-gravity and take $M_{\rm CB} = M_{\rm c}$ in $L_{\rm o}$.

The cooling timescale for an atmosphere to become self-gravitating is found from Equations (52) and (49) as

$$t_{\rm cool} \approx 2\times 10^8\frac{F_T^{5/2}F_\kappa\left(\frac{\xi}{3.4}\right)^2}{\left(\frac{m_{\rm c\oplus}}{10}\right)^{5/3}\left(\frac{a_\oplus}{10}\right)^{\frac{15}{14}}}\ {\rm yr} \tag{53}$$

for $\beta = 2$. Clearly this timescale is too long, and this is likely due to missing physics of self-gravity and EOS. Nevertheless the trends are informative. A reduction in opacity naturally gives faster cooling, as long as the optically thick assumption holds. Lower disk temperatures also give faster cooling. Even though higher temperatures give higher luminosities, they also give higher dust opacities and lower the Bondi radius and gas density.

In balance higher temperatures suppress cooling in this model. For different β values, the temperature dependence is $t_{\rm cool} \propto F_T^{1/2+\beta}$. The cooling timescale only weakly depends on disk mass or pressure, via the logarithmic factor ξ. This weak dependence is a consequence of cooling coming from the convective boundary. Note that the scaling laws do not reflect the changes to ξ which remains order unity and is here given the appropriate value for the nominal parameters.

6.3. *Critical Core Mass*

We can define a critical more mass as that which gives a self-gravitating atmospheres within a typical gas disk lifetime:

$$t_{\rm cool} = 3 \times 10^6 \tau_{\rm cool} \ {\rm yr}\,, \tag{54}$$

with $\tau_{\rm cool}$ a scaling factor. The resulting critical core mass is

$$M_{\rm crit} \approx 100 \frac{F_T^{3/2} F_\kappa^{3/5} \left(\frac{\xi}{2.6}\right)^{6/5}}{\left(\frac{a_\oplus}{10}\right)^{\frac{9}{14}}} M_\oplus \tag{55}$$

as with the cooling time, the critical mass is too large due to missing physics. The scaling with disk properties is similar, in fact all quantities are simply raised to the 3/5 power. The ξ value is changed to match the value for the nominal solution. The lower value reflects the fact that an atmosphere around such a massive core does not need to contract as much to be self-gravitating.

The numerical values for a different opacity law do not change the numerical answers above very much. That is mostly because we have chosen a disk location where T_o is not far from 100 K, where our opacity law is normalized. More generally a higher β value is more favorable for core accretion at large distances due to the sharper drop in opacity.

6.4. *The Opacity Effect*

A lower opacity could lower the core mass. Reducing the opacity by a factor of one hundred cuts the core mass by more than a factor of 10, specifically to 9 $M_\oplus$ for the parameters in Equation (55). The reduction is not as strong as the nominal scaling would imply, $0.01^{3/5} \approx 0.06$, because ξ increases.

Even with significantly lower opacities, radiative diffusion remains a good approximation at the convective boundary. We estimate the optical depth as (for $\beta = 2$):

$$\tau_{\rm CB} \sim \frac{\kappa P_{\rm CB}}{g} \sim 7 \times 10^4 \frac{F_T^4 F_\kappa}{\left(\frac{m_{c\oplus}}{10}\right)\left(\frac{a_\oplus}{10}\right)^{\frac{12}{7}}} \tag{56}$$

where $P_{\rm CB} \sim P_M$ for a self-gravitating atmosphere and $g \sim GM_{\rm c}/R_B^2$, with both approximation good to within the the order unity factor ξ. Clearly $\tau_{\rm CB} \gg 1$ even for $F_\kappa \lesssim 0.01$ out to very wide separations.

A hotter disk would increase core masses. Instead of our passive disk model, adopting the standard Hayashi temperature profile would increase core masses by $\sim$ 50%. A hotter accretion phase would further increase core masses, but such phases are presumably short lived.

6.5. *Surface Terms*

We now check the relevance of the neglected surface terms in Equation (27). We already showed that accretion energy can be exactly eliminated by choosing an outer boundary near the Bondi radius. We now show that accretion energy is also a small correction at the RCB. A rough comparison, (ignoring terms of order ξ) of accretion luminosity vs. $\dot{E}$ gives

$$\frac{GM\dot{M}}{R\dot{E}} = \frac{GM}{R}\frac{dM}{dE} \sim \frac{GM_{\rm c}^2}{R_B E}\frac{P_{\rm CB}}{P_M} \sim \sqrt{\frac{R_{\rm c}}{R_B}} \ll 1 \tag{57}$$

, where we assume $P_{\rm CB} \sim P_M$ for a massive atmosphere. Accretion energy at the protoplanetary surface is thus very weak for marginally self-gravitating atmospheres, and even weaker for lower mass atmospheres. A similar scaling analysis shows that the work term, $P_M \partial V_M/\partial t$ is similarly weak. Nevertheless numerical calculations should include, or check, the importance of these surface terms in more realistic self-gravitating atmospheres.

7. HYDROGEN DISSOCIATION

The dissociation of molecular Hydrogen deep in the atmospheres of accreting protoplanets plays a significant role in the energetics of core accretion. In the high density regions $r \ll r_{\rm CB}$ of a convective atmosphere, the thermal plus gravitational energy scales as

$$dE = -4\pi \nabla_{\rm ad}^{1/\nabla_{\rm ad}} \rho_{\rm CB} R_B'^{1/(\gamma-1)} r^{\frac{2\gamma-3}{\gamma-1}} \frac{dr}{r} \tag{58}$$

If $\gamma < 3/2$ then the main contribution to the energy is at the bottom of the atmosphere, i.e. the core. This is the case for a diatomic ideal gas ($\gamma = 1.4$) or a solar mixture of hydrogen and helium ($\gamma \approx 1.43$). However a monatomic gas has $\gamma = 5/3 > 3/2$. In this case, the atmosphere's energy is no longer concentrated near the bottom, but will be concentrated near the top of the convective zone.

A likely structure is an atmosphere that is dissociated near the base, but becomes molecular near the top of the convective zone. In this case the atmosphere's energy budget would be concentrated near the atomic to molecular transition.

The energy required to dissociate a hydrogen molecule, $I = 4.467$ eV can be significant to the overall energy budget. Since

$$\frac{I}{\nabla_{\rm ad} GM_{\rm c}(2m_{\rm H})/r} \approx 3\left(\frac{M_{\rm c}}{10M_\oplus}\right)^{-2/3}\frac{r}{R_{\rm c}} \tag{59}$$

we see that this energy is always relevant.

We can use the Saha equation to determine where dissociation is significant,

$$\frac{n_{\rm H}^2}{n_{\rm H_2}} = \left(\frac{\pi m_{\rm H} kT}{h}\right)^{3/2} e^{-I/(kT)} \tag{60}$$

We introduce a reaction coordinate δ so that $n_{\rm H} = 2\delta n_o$ and $n_{\rm H_2} = (1-\delta)n_o$ with $n_o = \rho X/(2m_{\rm H})$ the number density when all hydrogen is molecular. We express equilibrium as

$$\frac{\delta^2}{1-\delta} = f_\mu \frac{P_{\rm diss}(T)}{P} \tag{61}$$

with the characteristic pressure below which dissociation occurs is

$$P_{\rm diss} = \frac{(kT)^{5/2}}{4}\left(\frac{\pi m_{\rm H}}{h^2}\right)^{3/2} e^{-I/(kT)} \tag{62}$$

and the order unity factor

$$f_\mu = 2\delta + (1-\delta) + Y/2 + Z/\mu_Z \tag{63}$$

accounts for variations in the mean molecular weight with dissociation. (Take $\mu_Z = 31/2$, but not too significant.)

Thus dissociation occurs where $P \lesssim P_{\rm diss}(T)$. At disk temperatures (say 150 K) the dissociation pressure is negligibly small ($\sim 10^{-141}$ dyne cm^{-2}) and no dissociation occurs. However at core temperatures the dissociation pressure is quite large especially for massive cores. Dissociation is guaranteed.

8. STRUCTURE MODELS

The atmospheric structure equations in pressure units are

$$\frac{\partial m}{\partial P} = -\frac{4\pi r^4}{Gm} \tag{64a}$$

$$\frac{\partial r}{\partial P} = -\frac{r^2}{Gm\rho} \tag{64b}$$

$$\frac{\partial T}{\partial P} = \nabla\frac{T}{P} \tag{64c}$$

$$\frac{\partial L}{\partial P} = \frac{\partial m}{\partial P}\epsilon\,. \tag{64d}$$

When the atmospheric luminosity is carried by radiative diffusion, the lapse rate ∇ is

$$\nabla_{\rm rad} = \frac{3\kappa P}{64\pi G m \sigma T^4} L\,. \tag{65}$$

We assume efficient convection when $\nabla_{\rm rad} > \nabla_{\rm ad}$ so that $\nabla = \min(\nabla_{\rm ad}, \nabla_{\rm rad})$. The local the local energy generation, ϵ, includes heating from gravitational contraction

$$\epsilon_g = -T\frac{\partial S}{\partial t}\,. \tag{66}$$

Other sources of local heating, such as drag on infalling planetesimals could be specified but are not considered here. To solve these equations we must supply opacity law for κ and an equation of state (EOS) to give ρ and $\nabla_{\rm ad}$ as functions of T and P and the assumed composition. Note that the luminosity and opacity do not affect the structure of convection zones, but do affect their boundaries. If we can assume that the atmospheric luminosity is primarily generated in the convective zone, then the time dependence in the structure equations can be neglected.

Boundary conditions for the problem exist both at the core and at the outer accretion radius. At the core, $m = M_{\rm c}$, $r = R_{\rm c}$, $L = L_{\rm c}$. At the top boundary $T = T_{\rm o}$ and $P = P_{\rm o}$ set by conditions in the disk midplane. In addition to these four boundary conditions, an initial solution is required for the time dependence that arises when Equation (66) is included and allowed to affect the radiative zone structure.

8.1. *Shooting Method (1a)*

This is the first (and currently only) method we have numerically developed. It is not described yet! It is similar to method 1b below, but holds entropy values, S_i fixed instead of mass (or luminosity). In brief, we obtain solutions by varying the free thermodynamic variable at the core until the radius at the top of the convective boundary matches the value in Equation (19).

8.2. *Shooting Method (1b)*

We here describe an algorithm to obtain static solutions by integrating downward from the top of the convective zone. As in §8.1 this method uses simplified solutions for the radiative zone that neglect self gravity and luminosity generation. With this method we can obtain an evolutionary sequence at fixed values of either mass or luminosity, with the other quantity optimized to get a converged solution. Fixing the mass intervals has the advantage that total mass increases monotonically, while the luminosity is known to decrease to a minimum value and then increase (as with entropy in §8.1).

The algorithm proceeds as follows:

1. Guess a value for either the luminosity, $L_{\rm guess}$, or mass, $M_{\rm guess}$, at the top of the convective zone (whichever is not being held fixed). From this guess get estimates of $P_{\rm CB}$ from Equation (35), $T_{\rm CB}$ from Equation (15) and $r_{\rm CB}$ from Equation (19).

2. Integrate the structure equations from the convective boundary to the core, thereby obtaining a model value of the core mass.

3. Repeat, varying either $L_{\rm guess}$ or $M_{\rm guess}$ in step 1 until the core mass in the atmospheric solution matches the assumed value.

Solutions for a range of masses (and luminosities) can be connected in an evolutionary sequence using the global energy equation that we derive from the virial theorem.

For simplified radiative zone solutions, shooting from the top has the advantage that a monotonic sequence of mass can easily be contructed. On the other hand, integrating from the bottom can be more numerically efficient because the tabulated values of thermodynamic variables along a single adiabat can be used for each solution. When shooting from the top, the variation of the luminosity (or mass) requires integration along different adiabats, with greater interpolation costs. (But computational costs aren't really a big limitation.?)

8.3. *Shooting Method (2)*

Here we describe a method that allows us to add the self-gravity of the radiative zone. While a time dependent model could be developed (see §8.4), we first describe the method for constant radiative zone luminosity. As in §8.2 we integrate inwards, this time starting from the disk. The outer boundary conditions are $T = T_{\rm o}$, $P = P_{\rm o}$, $M = M_i$ where i is simply a label for each evolutionary stage,

$$r = r_{\rm fit} = n_{\rm fit}\frac{GM_i}{\mathcal{R}T_{\rm o}}\,, \tag{67}$$

which places the outer radius at a finite number, $n_{\rm fit} > 1$, of Bondi radii. Analytic solutions like Equation (8) show that the precise choice of fit radius has negligible effect on the atmospheric solution (R06).

To find a solution we must vary $L = L_{\rm guess}$, though as in §8.2 we could fix L and vary M. An inward integration for a trial $L_{\rm guess}$ gives the value of the core radius $R_{\rm c,guess}$ implied by the trial solution. We adjust $L_{\rm guess}$ until $R_{\rm c,guess}$ converges to the actual $R_{\rm c}$.

With atmospheric self-gravity, shooting outwards is more difficult because two parameters must be varied. It is impossible to fix the total mass shooting from the bottom, but we can try to construct models for choices of L_i. In this case, both $P_{\rm c}$ and $T_{\rm c}$ must be varied to match $T = T_{\rm o}$ and $r = r_{\rm fit}$ (using the total mass M from the trial integration itself) at the top, which is defined by $P = P_{\rm o}$. Alternatively one could try to find solutions with fixed values of entropy $S = S_i$ for the interior adiabat as in §8.1. This gives only one thermodynamic variable to specify at the core, since the trial choice of $T_{\rm c}$ gives $P_{\rm c}$ at fixed entropy. However trial values of $L_{\rm CB}$ must also be chosen, and there remain two quantities to vary ($L_{\rm CB}$ and the core thermodynamic variable) to match the two conditions at the upper boundary. With only a single variable to optimize, downward integration is more efficient.

8.4. *Time dependent shooting*

We now consider how to add time dependence to the method of §8.3. *For now this is just a sketchy description.* First a static solution must be obtained for some small mass. Then the next, slightly larger mass should be chosen. A solution now requires variation of two quantities, the luminosity and the assumed time difference between the two states. The time difference is used to calculate the heating in the convective zone using Equation (66). Convergence occurs when the atmosphere solution produces both the correct total mass and the timestep matched the global energy change between the two states.

APPENDIX

A. SELF-GRAVITATING ISOTHERMAL ATMOSPHERE

For an isothermal atmosphere, hydrostatic equilibrium gives:

$$r^2 \frac{d\ln\rho}{dr} = -\frac{G}{\mathcal{R}T}\left(M_{\rm c} + 4\pi \int_{R_{\rm c}}^{r} \rho(r') r'^2 dr'\right) \tag{A1}$$

This integro-differential equation is subject to the boundary condition that the density is $\rho_{\rm o}$ at large distances. This boundary condition can be applied at $n_B R_B$, where the Bondi radius $R_B = GM_{\rm c}/\mathcal{R}T$. The number $n_B > 1$ does not affect the solution since the gravitational perturbation from the planet is weak outside the Bondi radius.

Radial differentiation of Equation (A1) gives:

$$x^2 \frac{d^2S}{dx^2} + 2x\frac{dS}{dx} = -\theta_o e^S x^2 \tag{A2}$$

where $S = \ln(\rho/\rho_o)$, $x = r/R_{\rm c}$ and $\theta_o = 4\pi G\rho_{\rm o} R_{\rm c}^2/(\mathcal{R}T)$. Since θ_o is the squared ratio of the core radius to the disk's Jeans length, $\theta_o \lll 1$. The outer boundary condition now reads $S(n_B\theta_{\rm c}) = 0$ where $\theta_{\rm c} = R_B/R_{\rm c} > 1$. An inner boundary condition

$$\frac{dS}{dx}(x = 1) = -\theta_{\rm c} \tag{A3}$$

must also be satisfied.

REFERENCES

Arras, P., & Bildsten, L. 2006, ApJ, 650, 394
Bell, K. R., & Lin, D. N. C. 1994, ApJ, 427, 987
Bodenheimer, P., & Pollack, J. B. 1986, Icarus, 67, 391
Chiang, E., & Youdin, A. N. 2010, Annual Review of Earth and Planetary Sciences, 38, 493
D'Angelo, G., & Lubow, S. H. 2008, ApJ, 685, 560
Harris, A. W. 1978, in Lunar and Planetary Institute Science Conference Abstracts, Vol. 9, Lunar and Planetary Institute Science Conference Abstracts, 459–461
Hubickyj, O., Bodenheimer, P., & Lissauer, J. J. 2005, Icarus, 179, 415
Ikoma, M., Nakazawa, K., & Emori, H. 2000, ApJ, 537, 1013
Kippenhahn, R., & Weigert, A. 1994, Stellar Structure and Evolution
Mizuno, H. 1980, Progress of Theoretical Physics, 64, 544
Mizuno, H., Nakazawa, K., & Hayashi, C. 1978, Progress of Theoretical Physics, 60, 699
Perri, F., & Cameron, A. G. W. 1974, Icarus, 22, 416
Pollack, J. B., Hubickyj, O., Bodenheimer, P., Lissauer, J. J., Podolak, M., & Greenzweig, Y. 1996, Icarus, 124, 62
Sasaki, S. 1989, A&A, 215, 177
Semenov, D., Henning, T., Helling, C., Ilgner, M., & Sedlmayr, E. 2003, A&A, 410, 611
Youdin, A. N., & Mitchell, J. L. 2010, ApJ, 721, 1113