exploring higher-dimensional phase transitions

a brief odyssey

an overview

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1. the Lenz-Ising model

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2. simulations and ~computational topology~

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3. speedbumps

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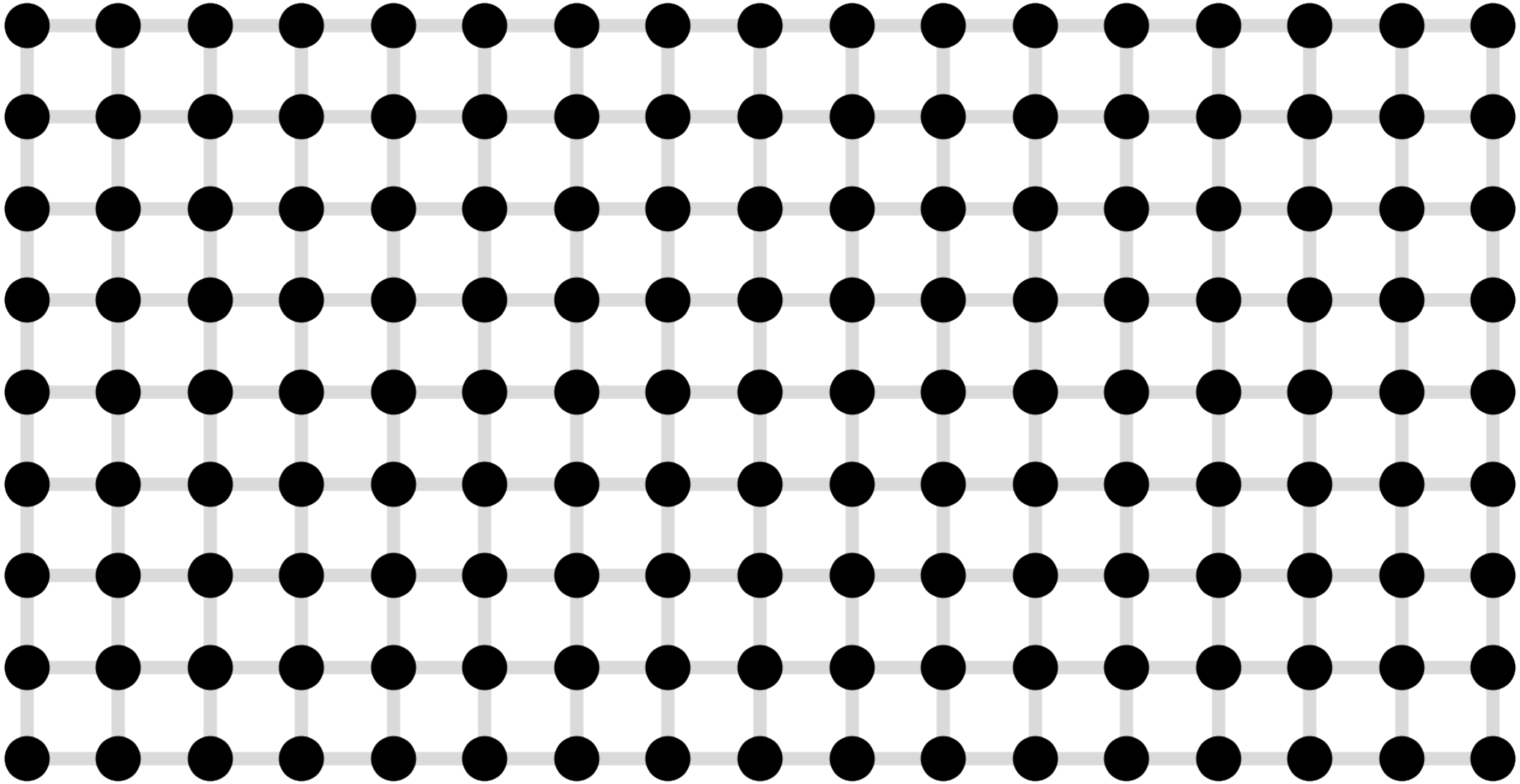
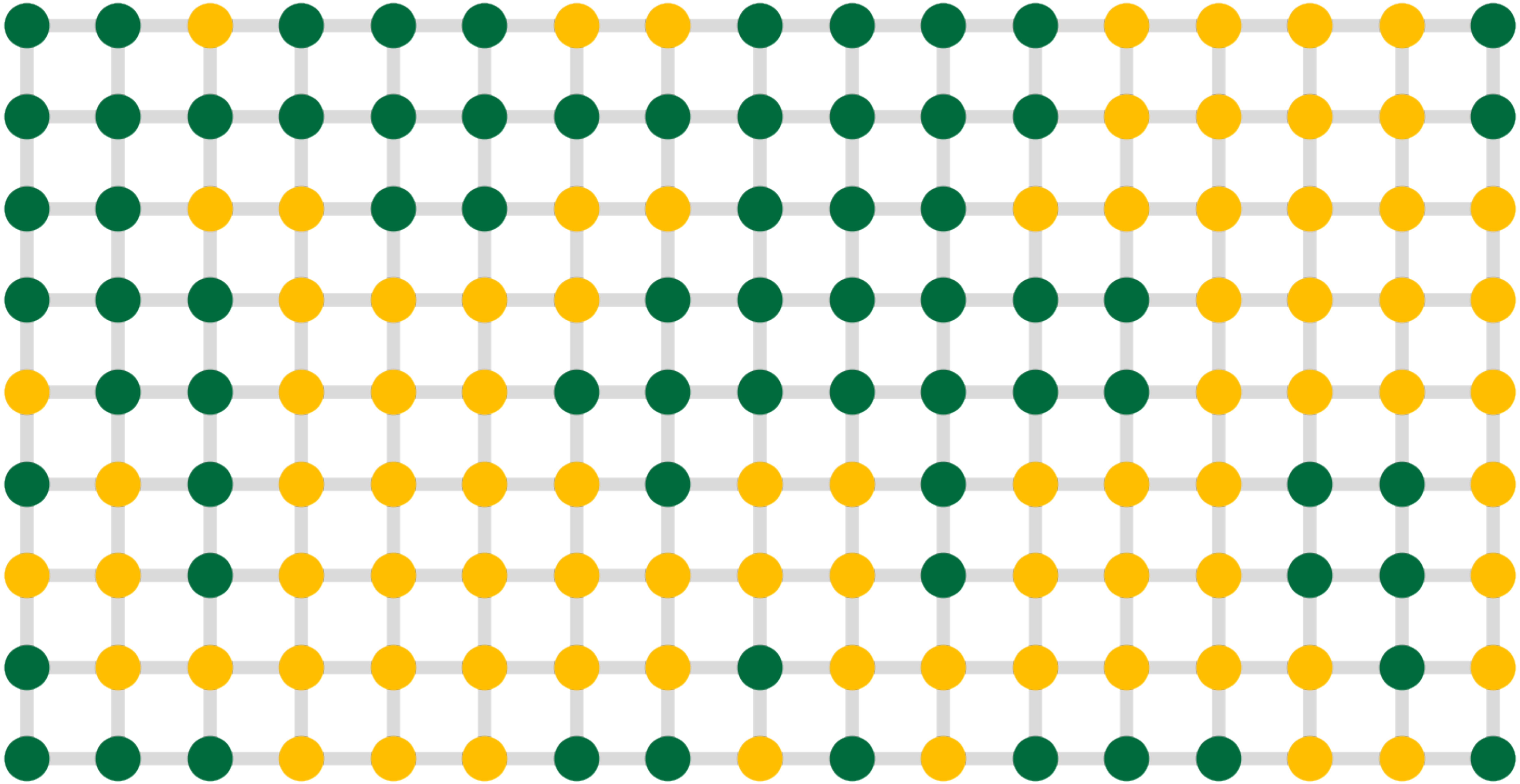
2. simulations and ~computational topology~
3. speedbumps
4. future work

1. the Lenz-Ising model

how do we simulate *magnetism*?

how do we simulate magnetism *well*?

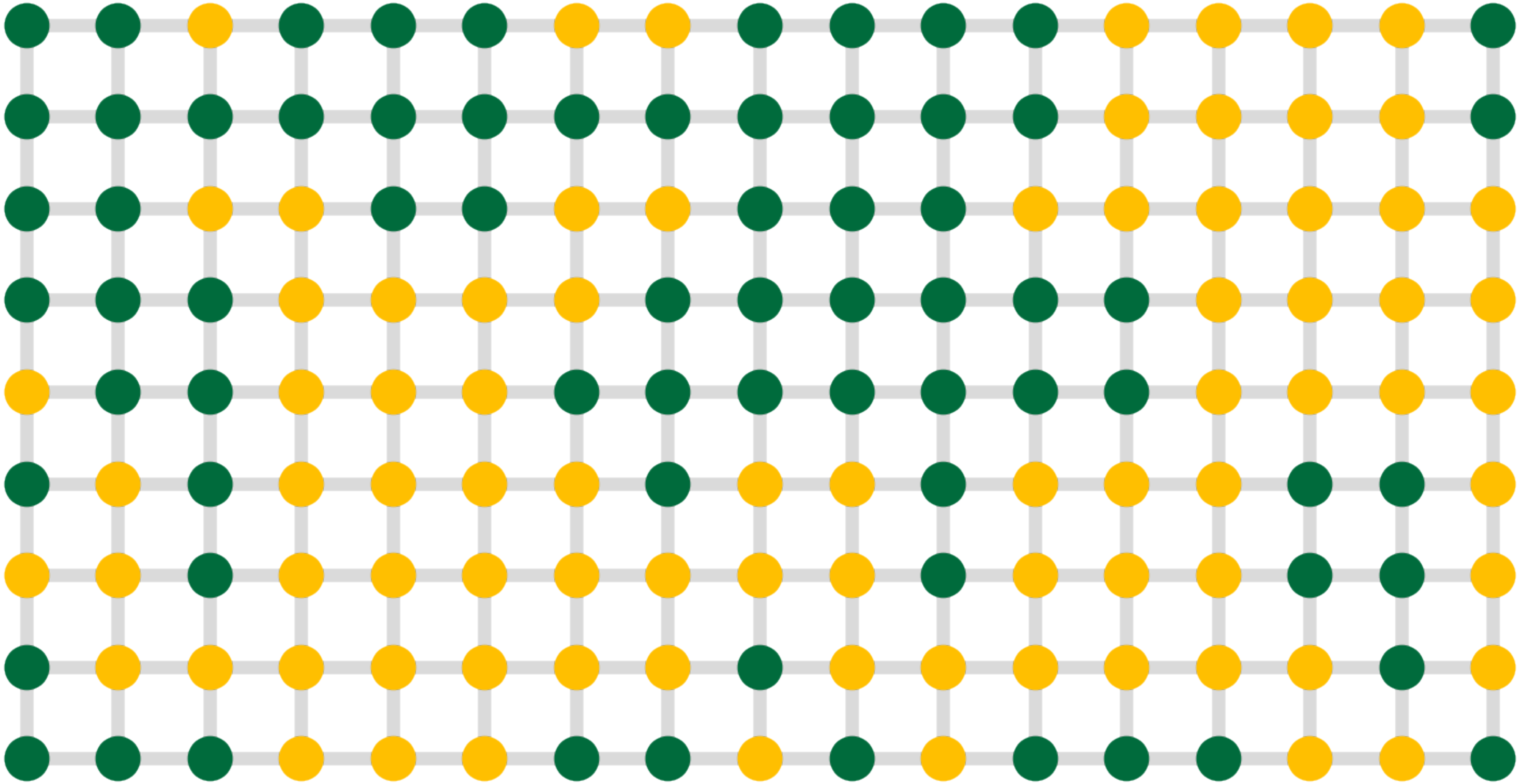


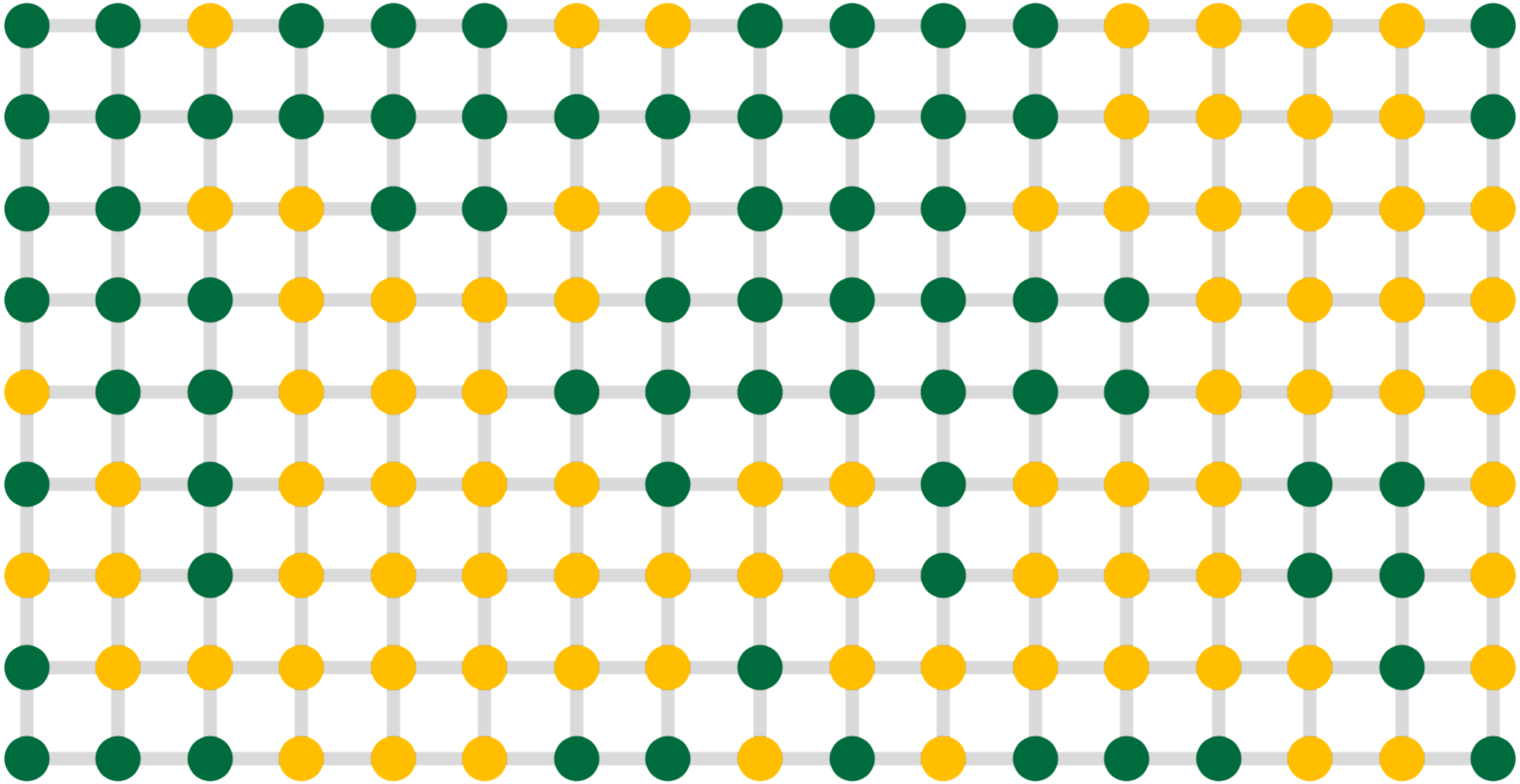


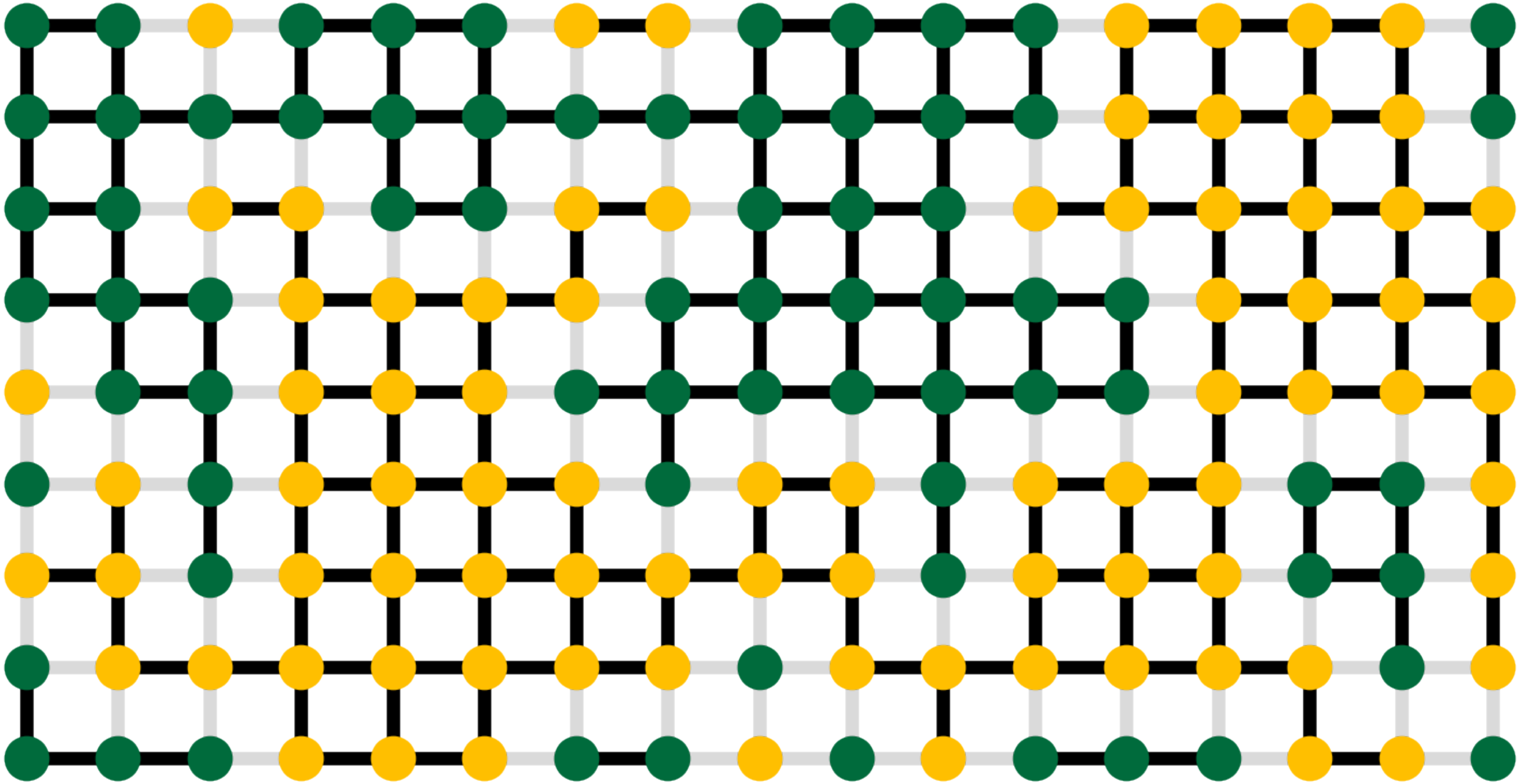
spin configuration

$$\sigma : V \to \mathbb{F}_2$$

(we like this to be linear)

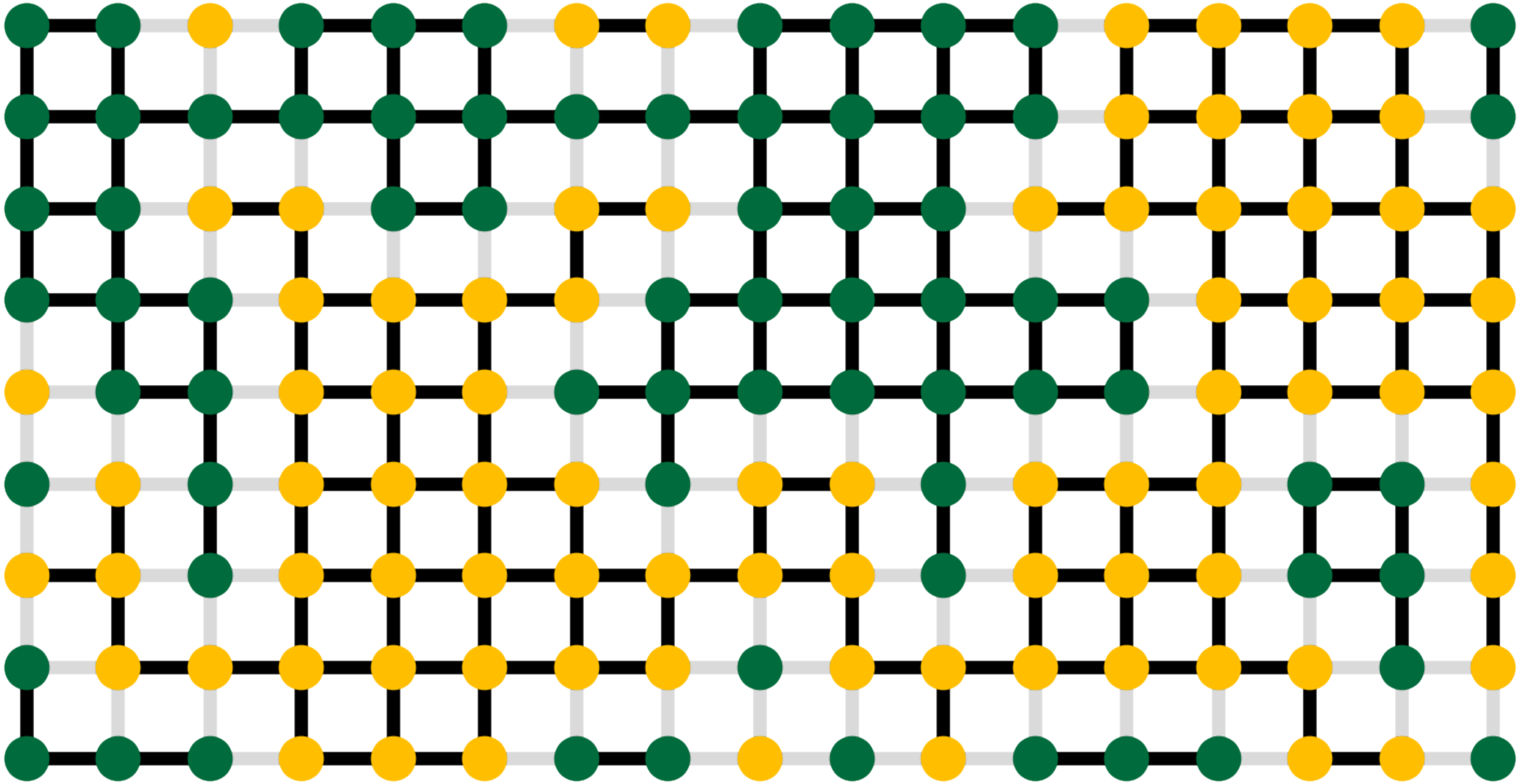






edge configuration

$\omega : E \to \{0,1\}$



we want to characterize *how likely* a given (σ, ω) pair is.

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what we *know*:

1. for each edge e in the graph,
    a. e is included in some cluster with probability p, or
    b. e is included in *no* cluster, with probability $(1-p)$;
2. there are *exactly* two states each cluster can take on.

$$\mu(\sigma, \omega) \propto \frac{1}{Z} \cdot \left(\prod_{e \in E_G} (1-p)^{1-\omega(e)} \cdot p^{\omega(e)} \right) \cdot 2^{\kappa(G)}$$

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for each edge...

chance this edge *isn't* in a cluster

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chance this edge *isn't* in a cluster

number of ways to assign spins to each cluster

for each edge...

chance this edge *is* in a cluster

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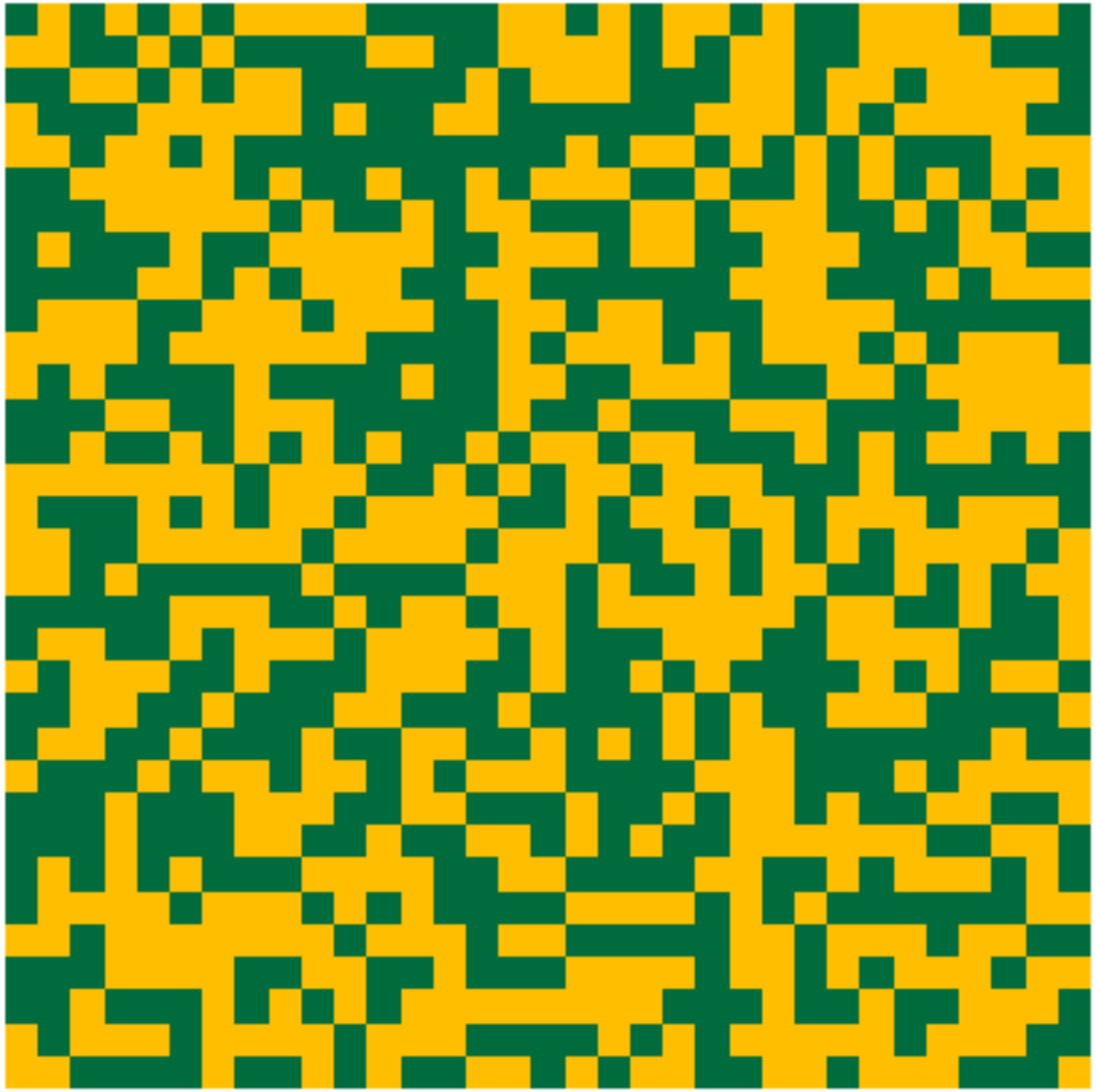
Random Cluster model

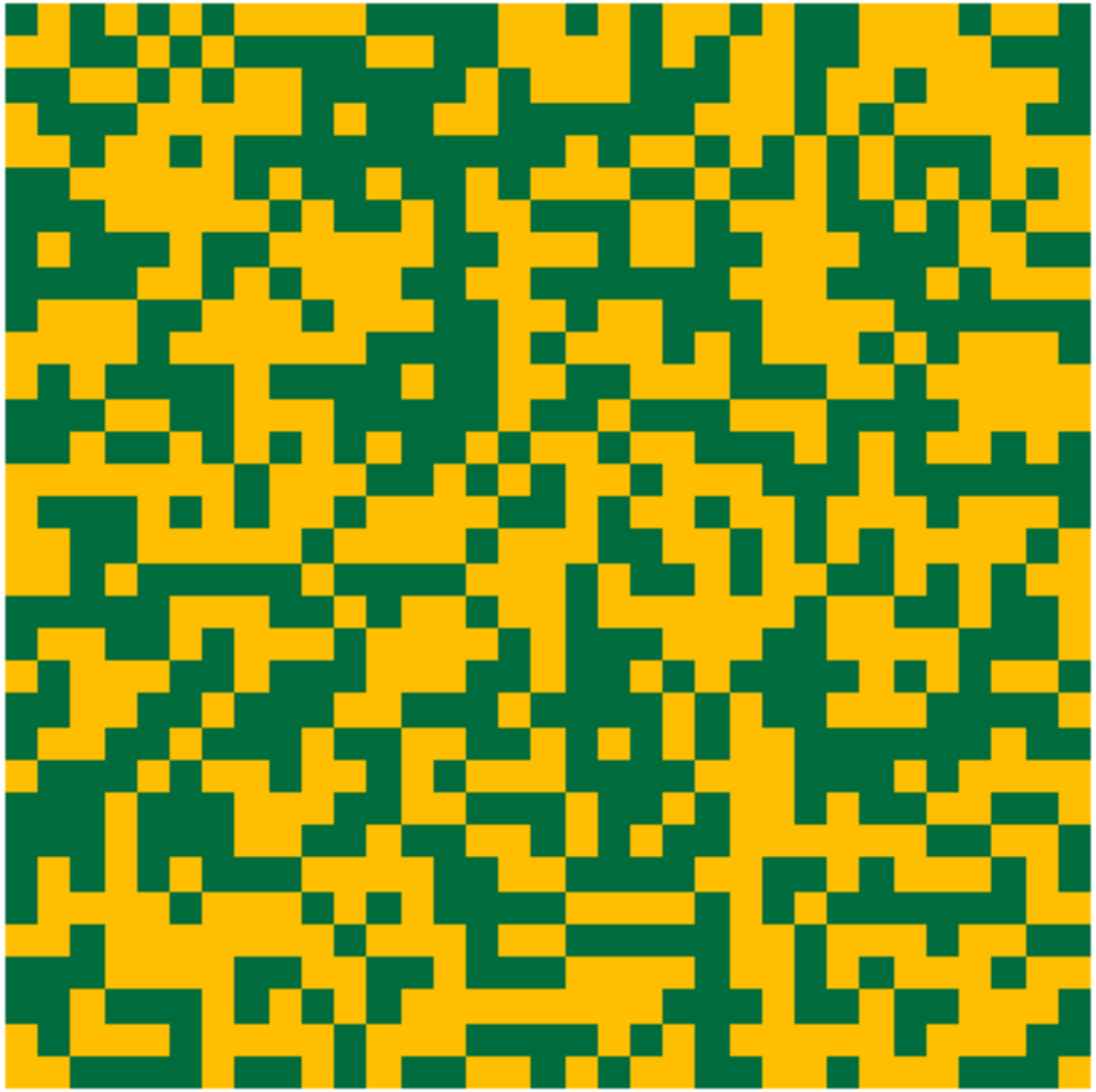
most general

2. simulations and ~computational topology~

a *random process* is a series of draws from a random variable.

a *Markov chain* is a random process where the outcome of the next draw depends *only* on the value of the previous draw traditional Metropolis-Hastings algorithm.





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    c. get the *induced subgraph* S from G by ignoring all edges not in C.

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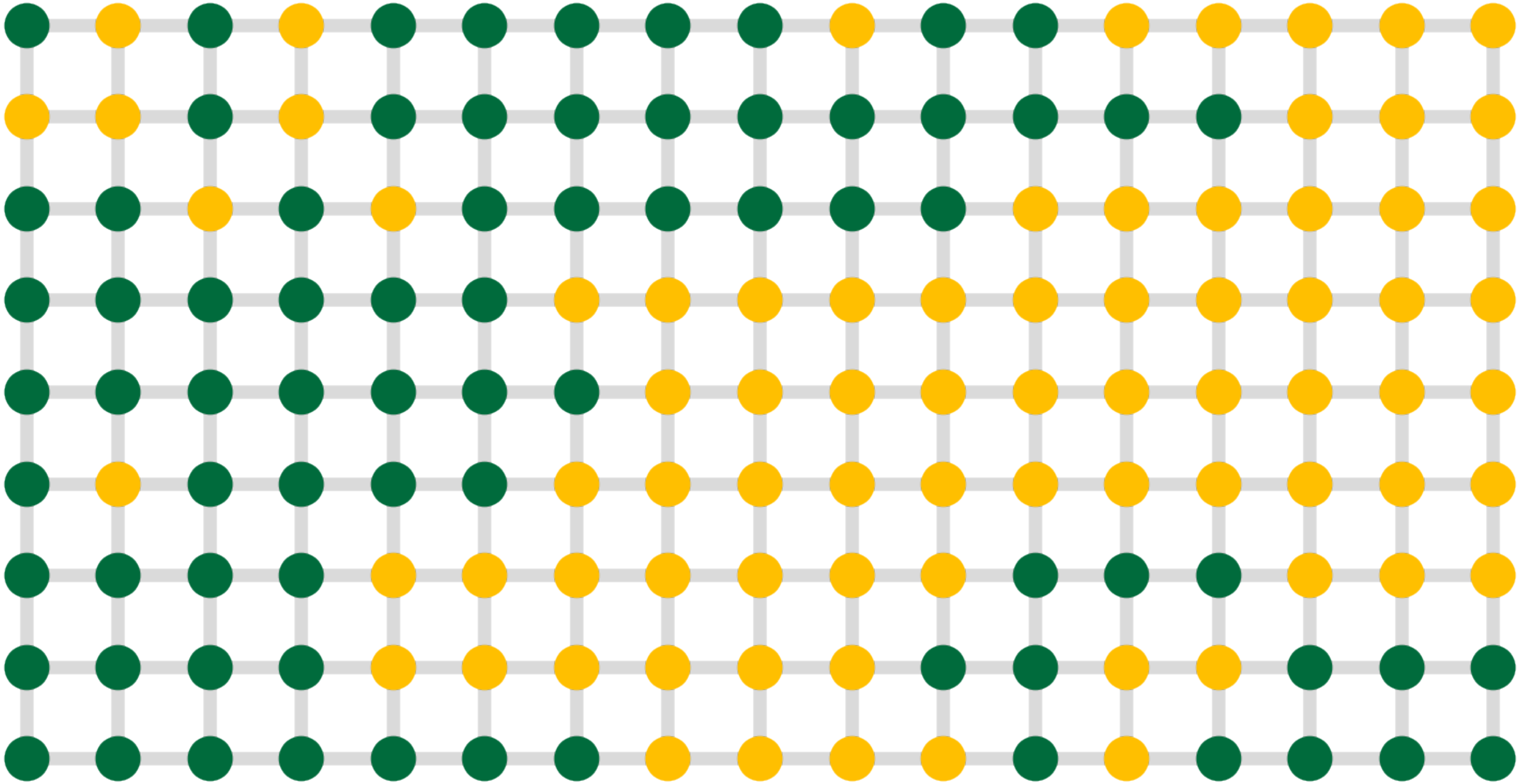
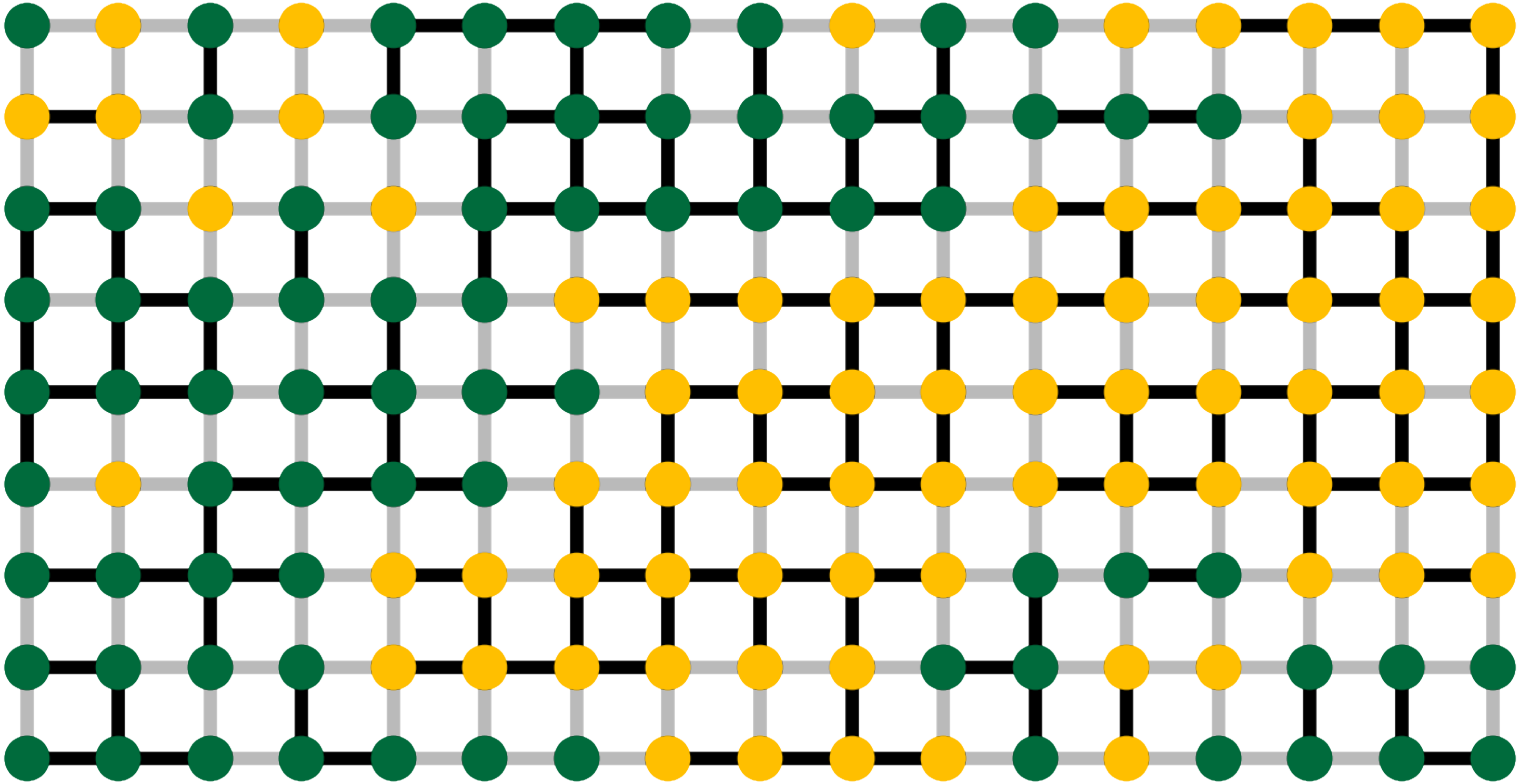
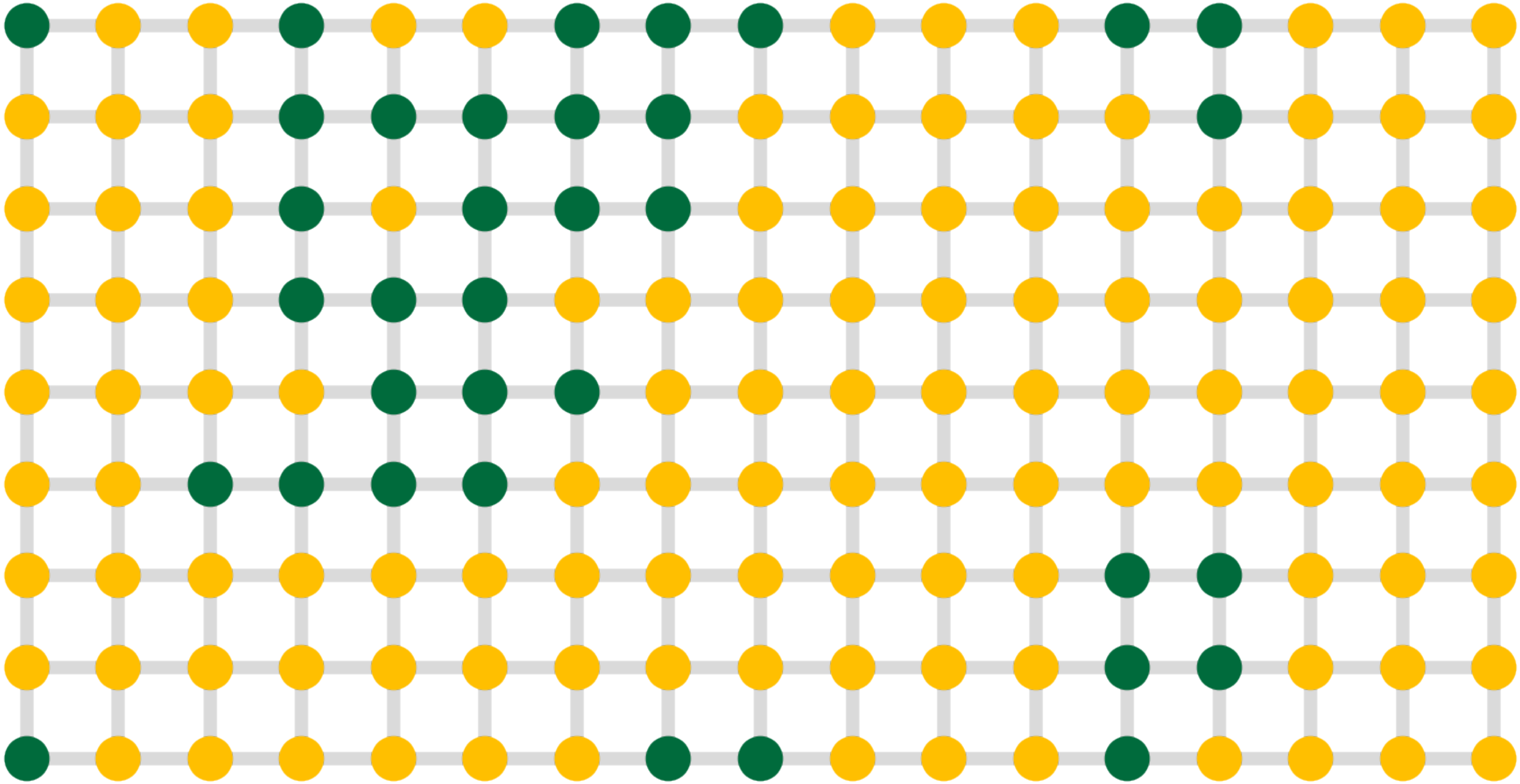
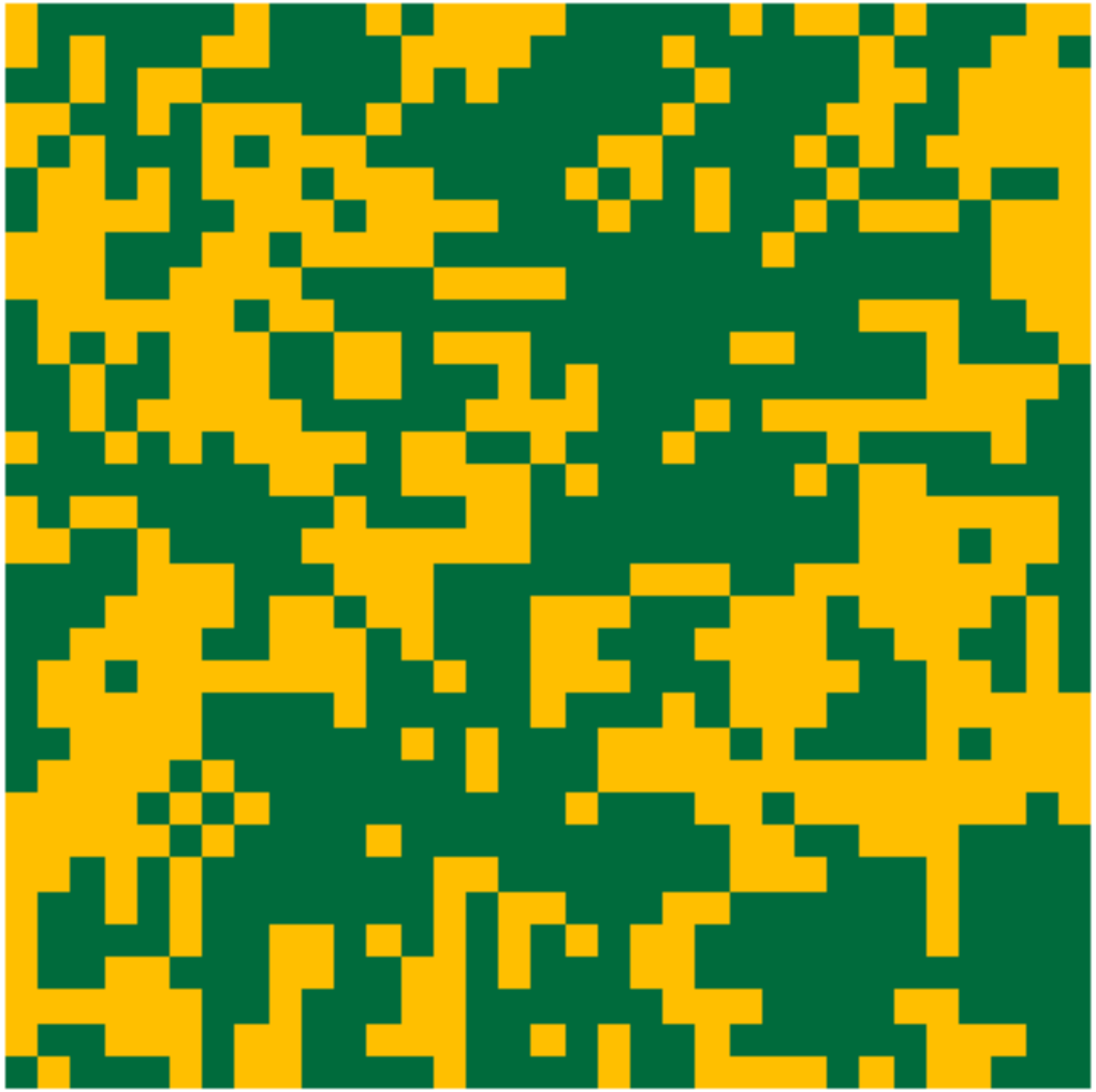
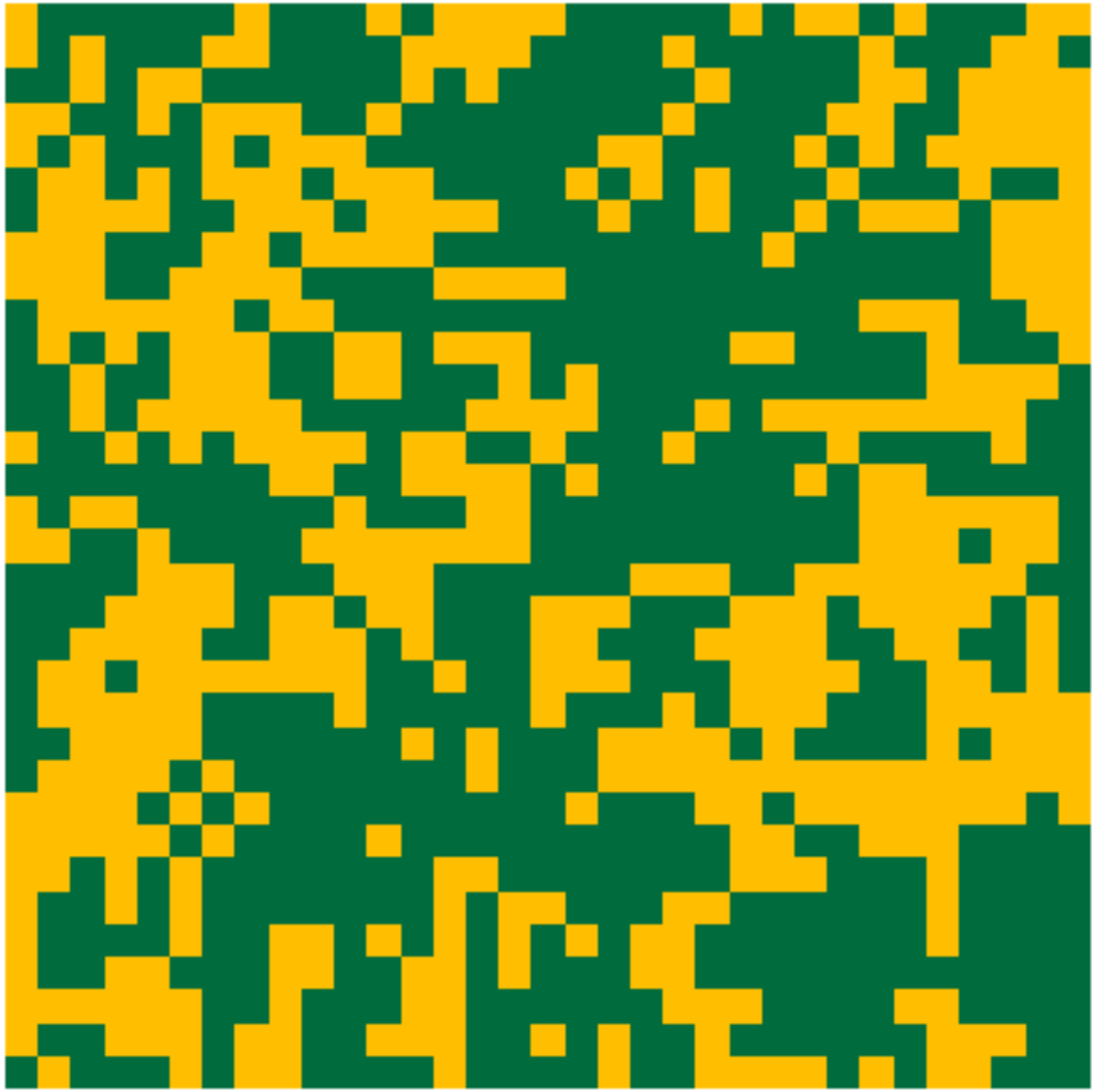
      i. uniformly randomly choose a spin value m;
      ii. set $\sigma_t(v) = m$ for all vertices v in k; σ_t is the new spin configuration.











… but can we *generalize* this?

yes — we'll use *topology* to do it.

homology lets us characterize *holes*.

sequences of edges

$$\text{sequences of edges} \xrightarrow[\textit{boundary operator}]{\partial} \text{sequences of vertices}$$

*co*homology lets us characterize *co*holes.

labelings on vertices

$$\text{labelings on vertices} \xrightarrow[\textit{coboundary operator}]{\delta} \textit{sums} \text{ of labelings on edges}$$

labelings on
n-dimensional things

$$\xrightarrow[\text{coboundary operator}]{\delta}$$

sums of labelings on
$(n+1)$-dimensional things

can be written as a matrix!!!!!!!!!!!!!!!!!!!

labelings on
n-dimensional things

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sums of labelings on
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   b. for each plaquette $\square$:

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      i. include $\square$ in C with probability p if $\delta(\sigma_{t-1})(\square) = (\sigma_{t-1} \circ \partial)(\square) = 0$

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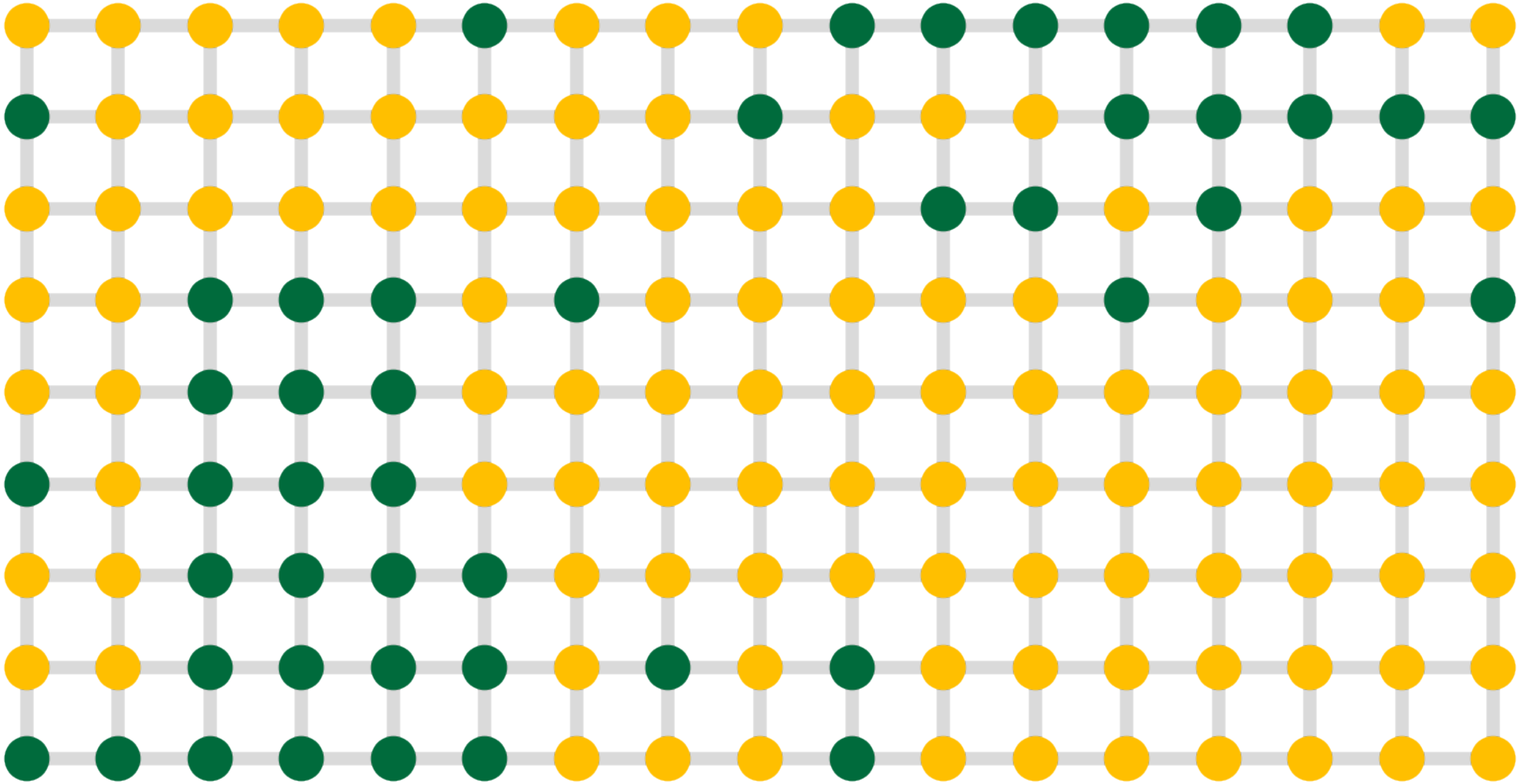
   d. construct the *coboundary matrix* M_δ on S.
   e. uniformly randomly sample a new labeling σ_t from the *kernel* of M_δ.

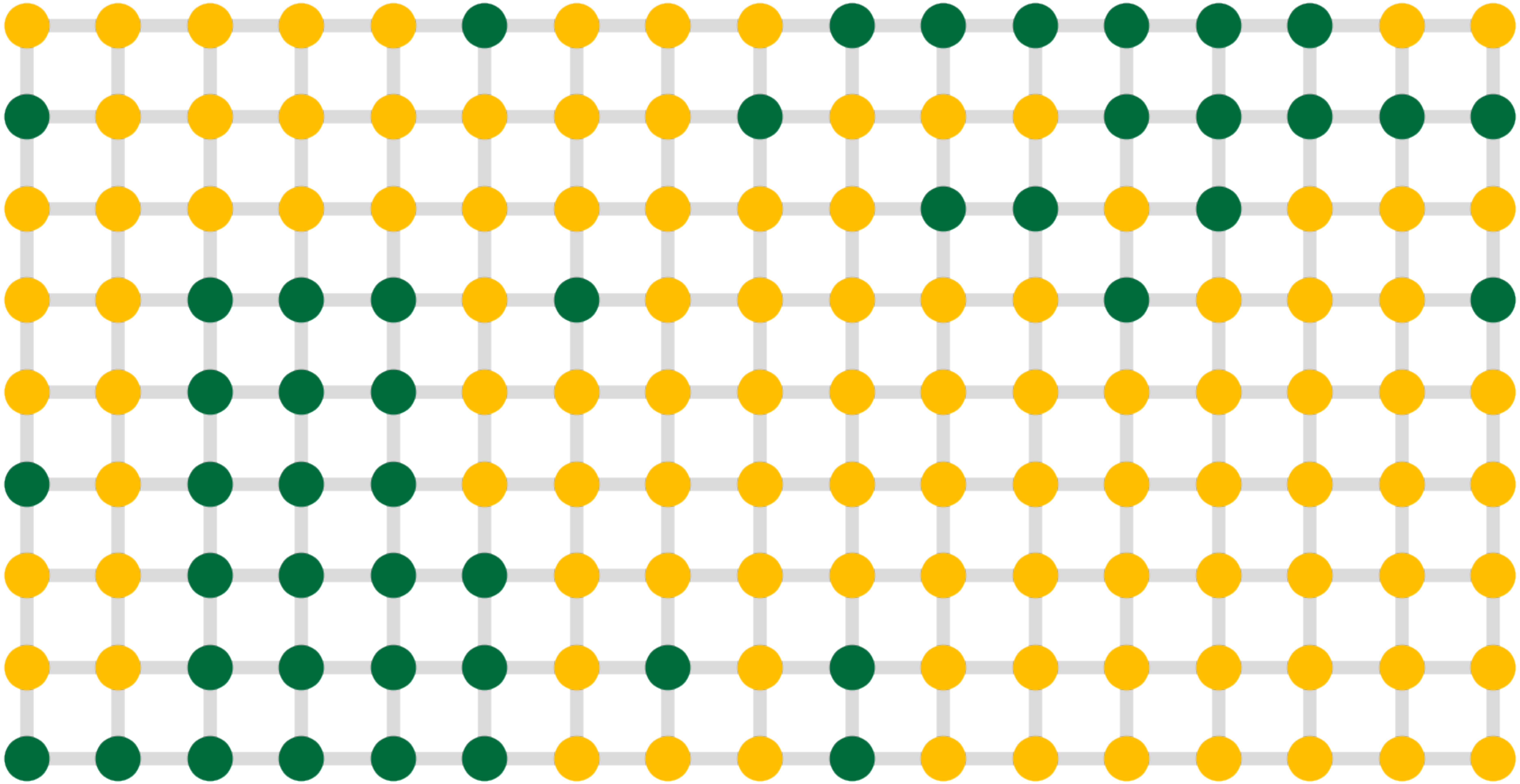
3. speedbumps

M_δ is *very, very* large.

M_δ is *very, very* large… so sampling from its null space is *intense.*



$M_\delta \in \mathrm{Mat}_{153\times 380}(\mathbb{Z})$



M_δ is *very, very* large, and *very very* sparse.

M_δ is *very, very* large, and *very very* sparse... so *storage* is an issue too.

4. future work

thank you!