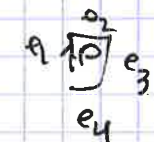
HIGHER-DIMENSIONAL STUFF!

Potts Lattice-gauge thm. (one-dimensional) (q-state)

a random 1-cochain in a 2-d cell complex (i.e. assigning spins to edges) → $f \in C^1(X, \mathbb{Z}_q)$ induced

induced by the Hamiltonian

~~δ(f)p~~ $\delta f(p) = f(\partial p)$



$$H = -\underbrace{\sum_{f \in X} K(\delta f(p), 0)}_{\text{2d plaquettes, with vanishing coboundary!}}$$

$$\mathbb{P}(f) \propto = e^{-\beta H(f)}$$

random variables of interest:

wilson loop variable: given a cycle, we can evaluate f on a cycle, associated to

$$\gamma \in Z_1(X; \mathbb{Z}_q)$$

is $f(\gamma) \in \mathbb{C} \longrightarrow$ roots of unity!

$= W_\gamma$

conjecture: ~~there exists~~ let $\gamma_{n,m} = \partial \underbrace{[0,n] \times [0,m]}_{\in \mathbb{Z}^d} \times 0^{d-2}$

we want to study the asymptotics of the wilson loop variable: there exists a temp $\beta_c > 0$ such that

$$\mathbb{E}[W_\gamma] \sim \begin{cases} e^{-c(\beta) \cdot \text{Perimeter}(\gamma)}, & \beta > \beta_c \\ e^{-c(\beta) \cdot \text{Area}(\gamma)}, & \beta < \beta_c \end{cases}$$

For some ~~\$~~ $c(\beta) \geq 0$.

the i=0 case is the sharpness? for q-state potts model recently proven in 2018

the i=1, d=3, ~~case~~ q=2, (or i=d-1 case) proven soon!

spins cover $i=1$, $d=4$ (which has definite, closed-form β!)

for $i=d-1$, $q=2$ case, we do the same clustering thing!

given a temp β and a cochain $f \in C^1(X, \mathbb{Z}_q)$, sample a random 2-complex by starting with $X^{(1)}$ (2d skeleton) and including each plaquette σ of X so that

$$\underbrace{\delta(f\sigma) = 0}_{\text{spins add up to } 0}$$

one step is computing this P

with probability $1-e^{-\beta}$

q prime (for field reasons)

if we sample P' from q-state Potts Lattice Gauge theory with inverse temp β, then sample P,

Hiraoka, Shirai

$$\mathbb{P}(P) = \frac{1}{\hat{Z}} \, p^{(\#\text{ of plaquettes})} \cdot (1-p)^{(\#\text{ non-plaquettes})}$$

$$\mathbb{P}(P) = \frac{1}{\hat{Z}} \cdot p^{\#\text{ of plaquettes}} \cdot (1-p)^{\#\text{ of non-plaquettes}} \cdot \operatorname{rank}(H_1(P, \mathbb{Z}_q))$$

Thm. Let V_γ (a cycle) be the event that $[\gamma]=0$ in $H_1(P, \mathbb{Z}_q)$ (boundary in 2-complex).
Then,

$$\boxed{\mathbb{E}(W_\gamma) = \mathbb{P}(V_\gamma)}$$

Prop. The conditional dist'ns are:

$(P \mid f)$ — indep. percol'n w/ prob $p = 1-e^{-\beta}$ on plaquettes in $\{\sigma : \delta f\sigma = 0\}$

$(f \mid P)$ — uniform distr. on $Z^1(P, \mathbb{Z}_q)$

need to find uniformly random element of null space!

Pf. We want to condition on P.
Then,

$$W_\gamma = \begin{cases} 0 & \text{if } V_\gamma \text{ happens} \\ \text{uniform on } \mathbb{Z}_q & \text{if } \neg V_\gamma \end{cases}$$

if V_γ occurs, we can write $\gamma = \partial(a_1\sigma_1 + \dots + a_k\sigma_k)$, where $\delta f(\sigma_1) = \dots = \delta f(\sigma_k) = 0$. Thus, we get

$$W_\gamma = \delta f(\gamma) = f(\partial\gamma).$$

if $\neg V_\gamma$, then $[\gamma] \neq 0$ in $H_1(P, \mathbb{Z}_q)$. As

$$H_1(P, \mathbb{Z}_q) \cong \operatorname{Hom}(H_1(P, \mathbb{Z}_q), \mathbb{Z}_q) \cong H^1(P, \mathbb{Z}_q)$$

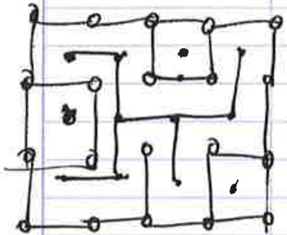
so we can find $f \in Z^1(P, \mathbb{Z}_q)$, so $\delta f(\gamma) \neq 0$.

f is uniformly distributed on a nonzero subgroup of $\mathbb{Z}_q$, but q prime, so this subgroup is $\mathbb{Z}_q$

$$\Longrightarrow \mathbb{E}(V_\gamma \mid \neg P) = 0$$

as $\mathbb{E}[W_\gamma] = \mathbb{E}[W_\gamma \mid V_\gamma]\, \mathbb{P}[V_\gamma] + \mathbb{E}[W_\gamma \mid \neg V_\gamma]\, \mathbb{P}[\neg V_\gamma]$

first dual of this!



DUALITY! (in the plane first)

Let $L = \mathbb{Z}_2$ and the dual $L^\circ = \mathbb{Z}^2 + (1/2, 1/2)$ — here, unique edges intersect unique edges! Let P be a subgraph of L; define P° to be the subgraph of L° which contains each edge of L for which L° for which the corresponding edge is <u>not</u> in P.

Let ~~Z~~ Λ (capital lambda?!) $\Lambda = [0, n^2]$ and $\Lambda^\circ = [-1/2, n+1/2]^2$

Prop. $\mathbb{R}^2 - \Lambda^\circ - P$ (blue graph) is a deformation retract to P°.
By Alexander duality (Betti numbers!

$$\beta_1(P^\circ) = 4$$

$$\beta_0(P^\circ) = \beta_1(P) + 1$$

no cycles

betti

Euler–Poincaré (special case):

$$\#V - \#E = \beta_0 + \beta_1 \quad \checkmark$$

free bdy cond's

Now, let P be the random subgraph of $[0, N^2]$ with ~~$\mathbb{P}(P)$~~

p,q coordinated

$$\mu^{free}_{p,q} \longrightarrow \mathbb{P}(P) \propto p^{\#\text{edges}} (1-p)^{\#\text{non-edges}}\, q^{\beta_0}$$

and P° the random subgraph of ~~Z~~ $[1/2, N+1/2]^2$ containing all edges ~~inside~~ on the boundary of the box (wired cond's) with prob ~~Thm.~~

$$\mu^{wired}_{p,q} \longrightarrow \mathbb{P}(P^\circ) \propto p^{\circ\,\#\text{edges}} (1-p^\circ)^{\#\text{non-edges}}\, q^{\beta_0^\circ}$$

self dual!

Thm.

$$\mu^{free}_{p,q}(P) = \mu^{wired}_{p^*(p,q),q}(P^\circ)$$

loc'n in plane

where

$p^*(p,q)$ satisfies $\dfrac{p\,p^*}{(1-p)(1-p^*)} = q$

$$p_c = p^*(p_c) = p_c$$