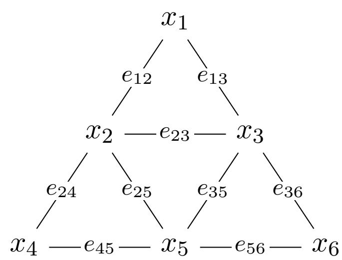
Beginnings

Consider the following graph G:



Definitions (Chains, boundary, coboundary). Recall that C_0 is the free abelian group generated by the set of vertices of G; each of these elements is a 0-*dimensional chain* or 0-*chain*. C_1 is the free abelian group generated by the set of (possibly directed) edges; each of these elements is called a 1-*dimensional chain* or 1-*chain*.

The *boundary operator* ∂ sends 1-chains to 0-chains, and the *coboundary operator* δ sends 0-chains to 1-chains: that is,

$$\partial : C_1 \to C_0 \text{ and } \delta : C_0 \to C_1.$$

Each is linear; for $x = uv$ an edge of G,

$$\partial(uv) = u + v,$$

which is a 0-chain; for u a vertex of G,

$$\delta(u) = \sum \varepsilon_i x_i,$$

where ε_i is 1 whenever (the edge) x_i is incident to u. A *coboundary* is the coboundary of some 0-chain in G.

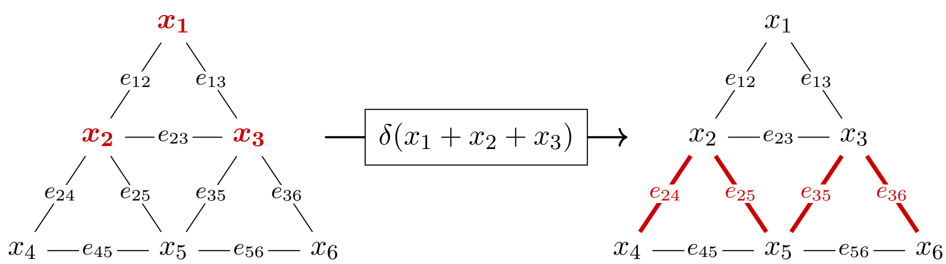
A 0-chain σ in G is

$$\sigma = x_1 + x_2 + x_3$$

and has as its coboundary

$$\begin{aligned} \delta(\sigma) &= (e_{12} + e_{13}) + (e_{12} + e_{23} + e_{24} + e_{25}) + (e_{13} + e_{23} + e_{35} + e_{36}) \\ &= 2e_{12} + 2e_{13} + 2e_{23} + e_{24} + e_{25} + e_{35} + e_{36} \\ &= e_{24} + e_{25} + e_{35} + e_{36}. \end{aligned}$$

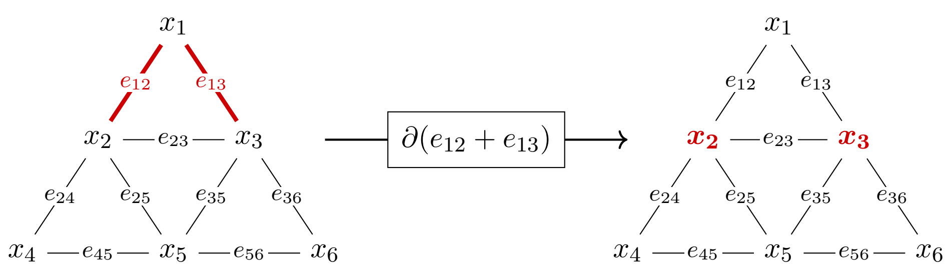
Graphically, we have



For a 1-chain like $\omega = e_{12} + e_{13}$, we have

$$\begin{aligned} \partial(\omega) &= \partial(e_{12} + e_{13}) \\ &= \partial(e_{12}) + \partial(e_{13}) \\ &= (x_1 + x_2) + (x_1 + x_3) \\ &= 2x_1 + x_2 + x_3 \\ &= x_2 + x_3. \end{aligned}$$

Graphically, we get



Basically, ***the coboundary operator sends sets of vertices V_σ to the set of edges E_σ incident to exactly one of the vertices in V_σ***, while ***the boundary operator sends sets of edges E_ω to the set of vertices V_ω incident to exactly one of the edges in E_ω***. Note that our addition here is over $\mathbb{F}_2$ ($\cong \mathbb{Z}_2 \cong \mathbb{Z}/2\mathbb{Z}$).

Definitions (Cycle vector, cycle space, cutset, cocycle, cocycle space). A 1-chain with boundary 0 is called a *cycle vector* and is a collection of edge-disjoint cycles; each cycle vector belongs to the *cycle space* of G, a vector space over $\mathbb{F}_2$. The *cycle basis* for the cycle space of G is the set of cycles $Z(T)$: $Z(T)$ is the set of minimal cycles induced by adjoining, one-by-one, each edge in the cotree T^*. These cycles are independent, as each of them differs by at least one edge. Any cycle Z in the cycle space of T can be written as

$$\sum_{i=1}^{k} \varepsilon_i C_i,$$

where $\varepsilon_i = 1$ if Z shares boundary ***(or point? not sure)*** with C_i.

A *coboundary* of G is the coboundary of some 0-chain in G. A *cutset* of a connected graph G is a collection of edges that, when deleted, disconnects G. Every coboundary is a cutset, as the coboundary of any vertex x_i is simply the edges to all its neighbors; when the coboundary is deleted, x_i is an isolated point. A *cocycle* is, equivalently: a minimal cutset of G; a minimal nonzero coboundary. The collection of cocycles is called the *cocycle space*; given a spanning tree T, the set of cocycles constructed by adding a single edge *not* in T^* serves as a basis for this space.

Definitions (The above, more generally). We can even couch the above definitions in more topological language. Let an *n-simplex* be the n-dimensional analogue of a triangle: for example, the *standard n-simplex* is defined by

$$\Delta^n = \left\{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} : \sum t_i = 1 \quad \text{and} \quad t_i \geq 0\right\}.$$

The group $\Delta_n(X)$ is then the free abelian group generated by the ("open," i.e. with faces deleted) n-simplices.

A Δ-*complex* is just a bunch of simplices glued together; more precisely, it's the quotient space of simplices we get by identifying faces. Note that vertices (0-simplices) are ordered by the orientations of their edges (1-simplices); higher-dimensional simplices can be ordered as well. An *n-chain* is a linear combination of n-simplices in a complex, and the *boundary homomorphism* $\partial_n : \Delta_n(X) \to \Delta_{n+1}(X)$ sends n-chains to $(n-1)$-chains as given by

$$\partial_n(\sigma) = \sum_k (-1)^k (v_0, \ldots, \hat{v}_k, \ldots, v_n),$$

where "$\hat{v}_k$" indicates that the k^{th} vertex was deleted from the sequence of vertices $v_0, \ldots, v_n$. Note that the composition $\partial_n \circ \partial_{n+1}$ (often written using juxtaposition, as $\partial_n \partial_{n+1}$) is always zero by the linearity of the ∂ operator, as the signs of each face are swapped.

Using these operators, we get a *sequence of homomorphisms of abelian groups*

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0,$$

called a *chain complex* (where C_n is a free abelian group on some basis of n-simplices; it's possible that $C_n = \Delta_n(X)$). Recalling that $\partial_n \partial_{n+1} = 0$, we know that ∂_n sends everything in $\text{Im } \partial_{n+1}$ to 0, so $\text{Im } \partial_{n+1} \subseteq \text{Ker } \partial_n$. The *$n^{th}$ homology group* is then $H_n = \text{Ker } \partial_n / \text{Im } \partial_{n+1}$, which gives us a way to classify *cycles* (elements of $\text{Ker } \partial_n$) by the *boundaries* (elements of $\text{Im } \partial_{n+1}$) to which they belong; elements of H_n are called *homology classes*, and two cycles representing the same boundary are *homologous*.

If we want to go in the reverse direction (i.e. get all the possible boundaries from a given cycle), we define the *coboundary operator* $\delta_n = \partial_n^* : C_{n-1}^* \to C_n^*$, where $C_n^* = C^n = \mathbf{Hom}(C_n, G)$, the group of homomorphisms from C_n (as a module over and) into some fixed abelian group G; alternatively, $C_n^* = C^n$ is the *dual* of C_n over G. Given that $\delta_n = \partial_n^*$, $\partial\partial = 0$ implies that $\partial^*\partial^* = \delta\delta = 0$, we can define $H^n(C_n; G) = \text{Ker } \delta_{n+1} / \text{Im } \delta_n$ to be the *n^{th} cohomology group* of X with respect to G. These groups are determined only by the n^{th} homology groups $H_n(C_n)$ and G.

More generally, given a space X, the *group of singular n-cochains with coefficients in (a fixed abelian) group G*, denoted by $C^n(X;G)$, is the dual $\mathbf{Hom}(C_n(X), G)$ of homomorphisms from the n^{th} homology group of X into G. We can then construct the cochain complex

$$\cdots \longleftarrow C^{n+1}(X;G) \xleftarrow{\delta_{n+1}} C^n(X;G) \xleftarrow{\delta_n} C^{n-1}(X;G) \longleftarrow \cdots \xleftarrow{\delta_1} C^0(X;G) \xrightarrow{\delta_0} 0,$$

so the *n^{th} cohomology group with coefficients in G* is $H^n(X;G) = \text{Ker } \delta_{n+1}/\text{Im } \delta_n$. Elements of Ker δ_{n+1} are called *cocycles* and elements of Im δ_n are called *coboundaries*. Notice that cocycles f vanish on boundaries: that is, elements of $C_i(X,G)$ (i.e. linear combinations φ of i-simplices resulting from the application of the boundary operator, or $\varphi \in \text{Ker } \partial_{n+1}$) have $f(\varphi) = 0$.

Definitions (Betti number, Euler characteristic). Using these homology groups, we can classify spaces' connectedness based on the dimension of their homology groups: given a space X and its n^{th} homology group H_n, the *Betti number* $\mathbf{b}_n(X) = \dim H_n$. This corresponds to the number of "cuts" we can make until we separate the space into two pieces; equivalently, it captures the number of "n-dimensional holes" in the space, where a "hole" is some n-dimensional cycle that is *not* a boundary of some $(n+1)$-dimensional object. Informally,

$\mathbf{b}_0$ connected components;

$\mathbf{b}_1$ "circular" (or one-dimensional) holes;

$\mathbf{b}_2$ "spherical" (or two-dimensional) holes, also called *voids* or *cavities*.

For example, the torus $T = S^1 \times S^1$ has:

$\mathbf{b}_0(T) = 1,$ as there is 1 connected component;

$\mathbf{b}_1(T) = 2,$ as one copy of S^1 bounds the hole cutting *through* the torus, and the other copy bounds the donut hole.

$\mathbf{b}_2(T) = 1,$ because T is hollow and thus traps a "void."

These can be computed using triangulations of the objects (simplicial approximations) and some really diligent fucking work. Now, we can connect the ranks of the homology groups of some space X to their *Euler characteristic* $\chi(X)$, defined as

$$\chi(X) = \sum_{i=0}^{\dim X} (-1)^i \cdot |X^i|,$$

where X^i is the set of i-simplices. The *Euler-Poincaré theorem* tells us that the Euler characteristic can be determined by the Betti numbers:

$$\chi(X) = \sum_{i=0}^{\dim X} (-1)^i \cdot |X^i| = \sum_{i=0}^{\dim X} (-1)^i \cdot \mathbf{b}_i(X),$$

which basically just says that we can tell the "shape" of a given shape based on its homology groups. Classically, the Euler characteristic of a polyhedron P is

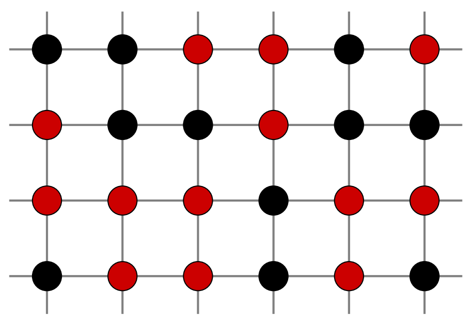
$$\chi(P) = V - E + F,$$

and any convex polyhedron has Euler characteristic 2.

Definition (Plaquette).

We can tie together these general definitions with those given for graphs: cycles in graphs are automatically sent to 0 (in addition over $\mathbb{Z}_2$), so they're in the kernel of the boundary operator $\partial_1 : C_1 \to C_0$; cycles are things with no boundary. Cocycles in graphs are automatically sent to 0 (again in addition over $\mathbb{Z}_2$) so they're in the kernel of the coboundary operator $\delta_1 : C^0 \to C^1$; cocycles are sets of vertices which admit cycles.

Given a (possibly finite) graph G, we can assign "spins," or weights $w_i \in \{0, 1\}$, to each vertex or edge in G. If we consider G as a sub-lattice of $\mathbb{Z}^2$ (or $\mathbb{Z}^n$ for some n), we put a spin on each of the pq vertices in the rectangular $p \times q$ subgraph of $\mathbb{Z}^2$:



Vertices with spin 1 are colored red and vertices with spin 0 are colored black.

We can also assign spins to *edges* via cochains — that is, elements f of C^i which send linear combinations σ of edges into some abelian group (generally the field $\mathbb{Z}_2$). We can use a so-far-unnamed score (which I'm calling the *H-score*) which counts the number of linear combinations of edges, or 1-*plaquettes*, which have vanishing boundary, as defined by

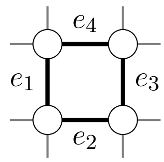
$$\begin{aligned} H &= -\sum_{\sigma \in C_0} K\Big((\delta_1(f))(\sigma), 0\Big) \\ &= -\sum_{\sigma \in C_0} K\Big(f(\partial_1(\alpha)), 0\Big) \end{aligned}$$

where K is the Kronecker delta and

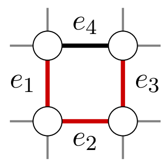
- $\delta_1 : C^0(G, \mathbb{Z}_2) \to C^1(G, \mathbb{Z}_2)$ is the coboundary operator on G with respect to the field $\mathbb{Z}_2$, which sends functionals (or *cochains*) in the dual of C_0 to functionals in the dual of C_1;
- f is an element of $C^0(G, \mathbb{Z}_2)$, i.e. a homomorphism from the 0-chains (linear combinations of vertices with coefficients in $\mathbb{Z}_2$) to $\mathbb{Z}_2$;

- ▷ $\partial_1 : C_1(G) \to C_0(G)$ is the boundary operator taking linear combinations of edges with coefficients in $\mathbb{Z}_2$ to a linear combination of vertices with coefficients in $\mathbb{Z}_2$;
- ▷ σ is a linear combination of edges in G;
- ▷ K is the Kronecker delta.

Consider the following pq-sublattice of $\mathbb{Z}^2$, where $p = 1 = q$:



We want to sample ***FIX THIS SHIT!***



Edges with spin $w_i = 1$ are in red, while edges with spin $w_i = 0$ are in black.

We can explicitly compute $K((\delta f)(\sigma), 0)$:

$$\begin{aligned}(\delta f)(\sigma) &= g(\sigma)\\ &= g(e_1 + e_2 + e_3 + e_4)\\ &= g(e_1) + g(e_2) + g(e_3) + g(e_4)\\ &= 1 + 1 + 1 + 0\\ &= 3\\ &\equiv 1 \pmod 2,\end{aligned}$$

so σ is a 1-plaquette with non-vanishing boundary over $\mathbb{Z}_2$. ***(How many homomorphisms f are there which, given a plaquette σ, assign it a vanishing boundary? It should be dependent only on the number of edges in σ, but can we calculate them using something like Stirling numbers? I think we can!)***

Probability

We want to conduct experiments on these cell complexes because reasons (magnetism, physics stuff, etc.), so we need a way to sample from them in a precise way.

Definitions (Random cluster model, Potts lattice gauge theory). Now, let X be a finite d-dimensional cell complex (first, we'll address $d = 2$ and $d = 3$), and let $0 < i < d$. Fix some field $\mathbb{F}$, and choose parameters $p \in [0, 1]$ and $q \in (0, \infty)$. The *i-plaquette random cluster model* *on* X is the random i-complex P containing the full $(i-1)$-skeleton of X according to

$$\begin{aligned}\mu_X(P) &= \frac{1}{\mathcal{Z}} p^{|P|} \cdot (1-p)^{|X^i|-|P|} \cdot q^{\mathbf{b}(P;\mathbb{F})_{i-1}} \\ &= p^{(\text{number of plaquettes in } P)} \cdot (1-p)^{(\text{number of plaquettes } \textit{not} \text{ in } P)} \cdot q^{\mathbf{b}_{i-1}},\end{aligned}$$

where $\mathcal{Z} = \mathcal{Z}(X, p, q, i, \mathbb{F})$ is a normalizing constant (to make our distribution "normal") and $|X^i|$ and $|P|$ denote the number of i-cells of X and P, respectively.

Now, letting G be an arbitrary finite abelian group and X a *cubical complex* (i.e. a sublattice of $\mathbb{Z}^d$ for $1 < d$ an integer), the $(i-1)$ *Potts lattice gauge theory on X with coefficients in G* is the measure

$$\nu(f) = \frac{1}{\mathcal{Z}} e^{-\beta \cdot H(f)} = \mu_\beta(f),$$

where $\mathcal{Z}$ is again a normalizing constant and β is an inverse temperature parameter. We'll be focusing specifically on the case where $\mathbb{F} \cong \mathbb{Z}_q$, where q is prime (which corresponds to the multiplicative group $\mathbb{Z}(q)$ of q^{th} roots of unity in $\mathbb{C}$); when we set $\mathbb{F} = \mathbb{Z}_q$, we call this the *$q$-state Potts lattice gauge theory*. This gives rise to the probability of selecting such a cocycle f:

$$\mathbf{P}[f] = e^{-p \cdot H(f)}.$$

Special cases of 2- and 3-dimensional Potts lattice gauge theory are the Ising model (i.e. $\mathbb{Z}(2)$) and the "clock" (i.e. $\mathbb{Z}(3)$) lattice gauge theory.

Definition (Wilson loop variable). Let f be an $(i-1)$ cocycle. The *generalized Wilson loop variable* associated to the cycle $\gamma \in C_{i-1}(G)$ is the random variable $W_\gamma : C^{i-1}(G; \mathbb{F}_q) \to \mathbb{C}$ is given by

$$W_\gamma(f) = (f(\gamma))^{\mathbb{C}},$$

where the superscript denotes the "evaluation" of f at γ in the complex numbers. That is, if $f(\gamma) = g \in \mathbb{F}_q$, $g^{\mathbb{C}}$ is the corresponding q^{th} root of unity in $\mathbb{C}$.

We want to investigate the asymptotics of the Wilson loop variable; to do so, we have a nifty conjecture:

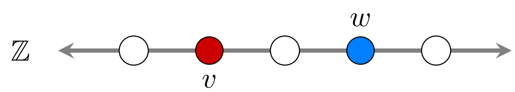
Conjecture (Area and perimeter laws). Given an inverse temperature β, a critical inverse temperature β_c, and a cycle γ,

$$\mathbb{E}[W_\gamma] \propto \begin{cases} e^{-c(\beta) \cdot \mathrm{Perim}(\gamma)} & \beta > \beta_c \\ e^{-c(\beta) \cdot \mathrm{Area}(\gamma)} & \beta < \beta_c \end{cases}$$

where $\mathrm{Perim}(\gamma)$ is the number of plaquettes in its support (i.e. the plaquettes in the linear combination) and $\mathrm{Area}(\gamma)$ is the number of plaquettes in the support of its *minimal bounding chain* (i.e. the smallest $(i-1)$-boundary which "surrounds" γ).

An example of the "perimeter" and "area" concepts are if $i = 1$, then γ consists of exactly two vertices

$\{v, w\}$, its perimeter is 2, and its area is the distance between v and w:



$$\gamma = v + w, \qquad \begin{array}{l} \text{Perim}\,(\gamma) = 2, \\ \text{Area}\,(\gamma) = |v - w| = 2, \end{array}$$

Now, suppose we have a sublattice G of $\mathbb{Z}^d$ for some $d > 1$, an inverse temperature β, and a cochain $f \in C^1(G; \mathbb{Z}_q)$. Then, we sample a random 2-complex P starting with the skeleton (open faces) and including plaquettes $\sigma \in C_2$ such that $(\delta f)(\sigma) = 0$ (i.e. so the spins add up to 0) with probability $1 - e^{-\beta}$. We get the following theorem (stated above, but more nicely here):

Theorem (Hiraoka, Shirai). Sampling P from a q-state Potts lattice gauge theory gives

$$\mathbf{P}\,[P] \propto p^{(\text{number of plaquettes in } P)} \cdot (1 - p)^{(\text{number of plaquettes } \textit{not} \text{ in } P)} \cdot |H_1(P; \mathbb{Z}_q)|.$$

Moreover,

Theorem. Let $\gamma \in C_{i-1}$, and V_γ the event that $\partial(\gamma) = 0$ (i.e. γ is a cycle) in $H_1(P; \mathbb{Z}_q)$. Then,

$$\mathbb{E}\,[W_\gamma] = \mathbf{P}\,[V_\gamma]\,.$$

Equivalently, the expectation of the Wilson loop variable with respect to γ is the same as the probability that γ has boundary 0 over $\mathbb{Z}_q$.

Algorithms

The following algorithm, which starts out in $\mathbb{Z}^3$ (i.e. the clock model), tests out these hypotheses: given a cocycle $f \in C^0(G, \mathbb{Z}_2)$, an inverse temperature parameter $-\beta$, and G a sublattice of $\mathbb{Z}_3$, we want to update f in the following way:

1. compute $G(f)$, the random graph obtained from evaluating f on G, only including plaquettes σ such that $(\delta f)(\sigma) = 0$ with probability $p = 1 - e^{-\beta}$;
2. find the *components* of $G(f)$;
3. uniformly randomly assign spins to each of the components of $G(f)$, and call these new spins (i.e. the new cocycle) f_{new}.

This samples uniformly from μ_β. This is called the *Swendson-Wang algorithm*, and is typically used in $\mathbb{Z}^2$ and $\mathbb{Z}^3$; we want to generalize this to higher dimensions.