SIT787: Mathematics for AI
Practical Week 2

Asef Nazari

1. For these vectors

$$\boldsymbol{u} = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 2 \end{bmatrix} \text{ and } \boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix}$$

- Find $\boldsymbol{u}+\boldsymbol{v}, \boldsymbol{u}-\boldsymbol{v}, 2\boldsymbol{u}+3\boldsymbol{v}$
- Find the cosine between these two vecors and their lengthes
- Find the distance between them.

$$\vec{u}+\vec{v} = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0+1 \\ 4+0 \\ -1+3 \\ 2+(-1) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{u}-\vec{v} = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0-1 \\ 4-0 \\ -1-3 \\ 2-(-1) \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -4 \\ 3 \end{bmatrix}$$

$$2\vec{u}+3\vec{v} = 2\begin{bmatrix} 0 \\ 4 \\ -1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 2(0) \\ 2(4) \\ 2(-1) \\ 2(2) \end{bmatrix} + \begin{bmatrix} 3(1) \\ 3(0) \\ 3(3) \\ 3(-1) \end{bmatrix}$$

a linear combination

$$= \begin{bmatrix} 2 \\ 8 \\ -2 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 9 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 7 \\ 1 \end{bmatrix}$$

cosine between two vectors $\vec{u}$ and $\vec{v}$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{v}|}$$

$$\vec{u}\cdot\vec{v} = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix} = (0)(1) + (4)(0) + (-1)(3) + (2)(-1)$$

$$= 0 + 0 - 3 - 2 = -5$$

$$|\vec{u}| = \sqrt{(0)^2 + (4)^2 + (-1)^2 + (2)^2} = \sqrt{0+16+1+4} = \sqrt{21}$$

$$|\vec{v}| = \sqrt{(1)^2 + (0)^2 + (3)^2 + (-1)^2} = \sqrt{1+0+9+1} = \sqrt{11}$$

$$\cos\theta = \frac{-5}{\sqrt{21}\,\sqrt{11}} = \frac{-5}{\sqrt{231}}$$

$$\boxed{(\sqrt{a})(\sqrt{b}) = \sqrt{ab}}$$

The distance between two vectors

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \quad \text{dist}(\vec{u}, \vec{v}) = \sqrt{\sum_{i=1}^{n} (u_i - v_i)^2}$$

$$\text{dist}(\vec{u}, \vec{v}) = \sqrt{(0-1)^2 + (4-0)^2 + (-1+3)^2 + (2-(-1))^2}$$

$$= \sqrt{1 + 16 + 16 + 9} = \sqrt{42}$$

2. Are these vectors linearly independent?

① $$\boldsymbol{u} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} \text{ and } \boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

② $$\boldsymbol{u} = \begin{bmatrix} 4 \\ 2 \\ -6 \end{bmatrix} \text{ and } \boldsymbol{v} = \begin{bmatrix} 10 \\ 5 \\ -15 \end{bmatrix}$$

③ $$\boldsymbol{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } \boldsymbol{v} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Let we have a set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$. These vectors are called independent if a linear combination of vectors is zero vector, we can conclude that all the coefficients are zero.

if $\underbrace{c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k}_{\text{a linear combination}} = \vec{0}$, then $c_1 = c_2 = \dots = c_k = 0$.

① Let $c_1\vec{u} + c_2\vec{v} = \vec{0}$

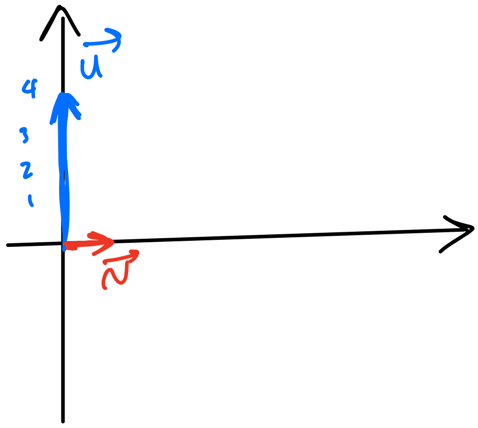
$$c_1\begin{bmatrix} 0 \\ 4 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 4c_1 \end{bmatrix} + \begin{bmatrix} c_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} c_2 \\ 4c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$c_2 = 0$

$\rightarrow 4c_1 = 0 \rightarrow c_1 = 0 \Longrightarrow c_1 = c_2 = 0$

these two vectors are independet.



② Let $c_1\vec{u} + c_2\vec{v} = \vec{0}$

$$c_1\begin{bmatrix}4\\2\\-6\end{bmatrix} + c_2\begin{bmatrix}10\\5\\-15\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \rightarrow \begin{bmatrix}4c_1+10c_2\\2c_1+5c_2\\-6c_1-15c_2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \begin{cases}4c_1 + 10c_2 = 0\\ 2c_1 + 5c_2 = 0\\ -6c_1 - 15c_2 = 0\end{cases}$$

from the first equation $4c_1 + 10c_2 = 0$

$$4c_1 = -10c_2$$

$$c_1 = \frac{-10c_2}{4} = -\frac{5}{2}c_2$$

I plug this value in the 2^{nd} equation

$$2c_1 + 5c_2 = 0 \rightarrow 2\left(-\frac{5}{2}c_2\right) + 5c_2 = 0$$

$$\rightarrow -5c_2 + 5c_2 = 0 \rightarrow 0 = 0$$

This is a trivial identity.

We need to try the third equation as well:

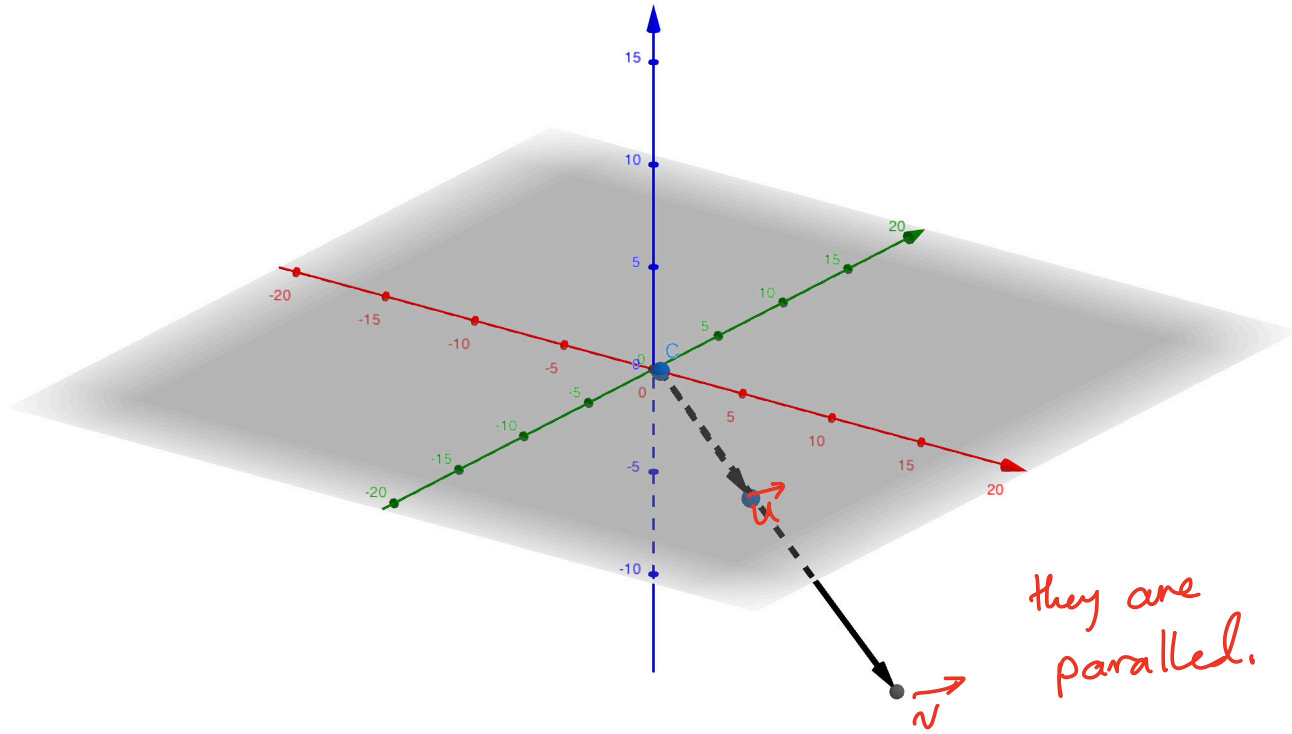
If I plug $c_1 = -\frac{5}{2} c_2$ in the third equation, I will get

$$-6c_1 - 15c_2 = 0$$

$$-6\left(-\frac{5}{2} c_2\right) - 15c_2 = 0$$

$$15c_2 - 15c_2 = 0 \longrightarrow 0 = 0$$

This is a trivial identity. We did not concluded that $c_1 = c_2 = 0$. therefore, these two vectors are not independent.



The reason we need to check all the equations is seen in the following example. Imagine, the vectors $\vec{u}$ and $\vec{v}$ are defined as follows:

$$\vec{u} = \begin{bmatrix} 4 \\ 2 \\ -6 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 10 \\ 5 \\ -14 \end{bmatrix}$$

I just slightly altered the 3rd component of $\vec{v}$. to check their linear dependency, I need to solve the following system:

$$\begin{cases} 4c_1 + 10c_2 = 0 \longrightarrow c_1 = -\frac{5}{2}c_2 \\ 2c_1 + 5c_2 = 0 \quad \text{II} \\ -6c_1 - 14c_2 = 0 \quad \text{III} \end{cases}$$

If I plug $c_1 = -\frac{5}{2}c_2$ in the second equation, I get $0=0$. However, if I plug $c_1 = -\frac{5}{2}c_2$ in the third equation, I get

$$-6\left(-\frac{5}{2}c_2\right) - 14c_2 = 0 \longrightarrow 15c_2 - 14c_2 = 0$$

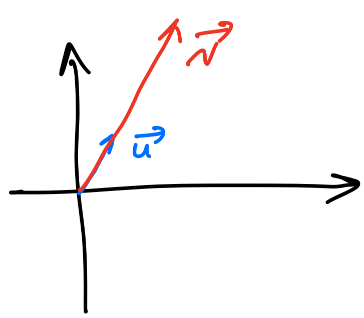
$\longrightarrow \boxed{c_2 = 0}$ if $c_2 = 0$ then $c_1 = -\frac{5}{2}c_2 = 0.$

The we get $c_1 = c_2 = 0$ and $\vec{u}, \vec{v}$ are independent. So, we need to check all the equations.

③ Let $c_1\vec{u} + c_2\vec{v} = \vec{0}$

$$c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}3\\6\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \longrightarrow \begin{bmatrix}c_1 + 3c_2\\2c_1 + 6c_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\longrightarrow \begin{cases} c_1 + 3c_2 = 0 \longrightarrow c_1 = -3c_2 \\ 2c_1 + 6c_2 = 0 \end{cases}$$ plug in the 2^{nd} eq.



$2(-3c_2) + 6c_2 = 0 \rightarrow -6c_2 + 6c_2 = 0 \rightarrow 0=0.$

we did not conclude that $c_1 = c_2 = 0 \rightarrow$ they are not indep.

3. For these vectors

$$\boldsymbol{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and } \boldsymbol{v} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

- Find the projection of $\boldsymbol{v}$ over $\boldsymbol{u}$: $\boldsymbol{a}_1 = \text{proj}_{\boldsymbol{u}}^{\boldsymbol{v}}$
- Find $\boldsymbol{a}_2 = \boldsymbol{v} - \boldsymbol{a}_1$ using the definition.
- Are $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ perpendicular (orthogonal)?
- The formulas

$$\boldsymbol{a}_1 = \left(\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\boldsymbol{u}\cdot\boldsymbol{u}}\right)\boldsymbol{u}$$

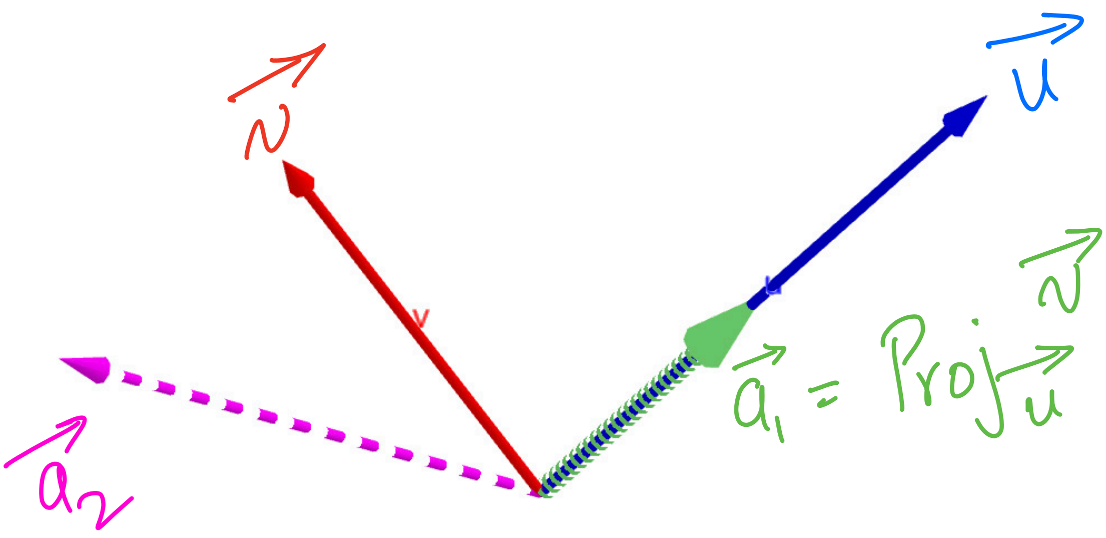
$$\boldsymbol{a}_2 = \boldsymbol{v} - \boldsymbol{a}_1 = \boldsymbol{v} - \left(\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\boldsymbol{u}\cdot\boldsymbol{u}}\right)\boldsymbol{u}$$

$$\vec{u}\cdot\vec{v} = (1)(0) + (0)(-1) + (1)(1) = 0 + 0 + 1 = 1$$

$$\vec{u}\cdot\vec{u} = (1)(1) + (0)(0) + (1)(1) = 1 + 0 + 1 = 2$$

$$\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}} = \frac{1}{2} \qquad a_1 = \left(\tfrac{1}{2}\right)\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

$$a_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1 \\ 1/2 \end{bmatrix}$$



To show that $\vec{a}_1$ and $\vec{a}_2$ are perpendicular, I need to show that their dot product is zero. let's check it:

$$\vec{a}_1 \cdot \vec{a}_2 = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} -1/2 \\ -1 \\ 1/2 \end{bmatrix} = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) + (0)(-1) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$= -\frac{1}{4} + 0 + \frac{1}{4} = 0$$

$$\Longrightarrow \vec{a}_1 \perp \vec{a}_2$$

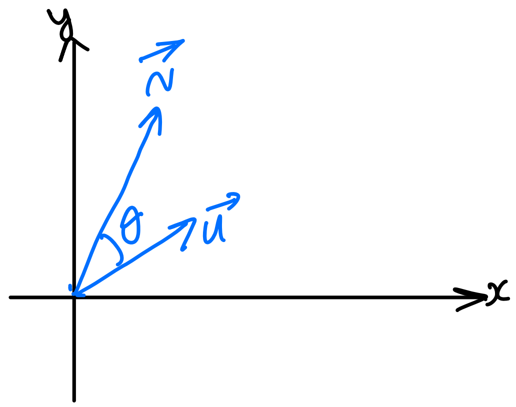
4. which two vectors are more similar considering both the distance and cosine of an angle between them?

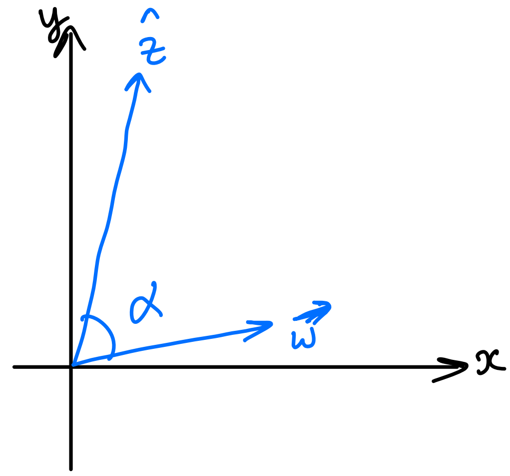
$$\boldsymbol{u} = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 2 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix} \text{ and } \boldsymbol{w} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Table of cos(angle)

Angle	cos(a)	Angle	cos(a)	Angle	cos(a)	Angle	cos(a)
0.0	1.00	25.0	.9063	46.0	.6947	71.0	.3256
1.0	.9998	26.0	.8988	47.0	.6820	72.0	.3090
2.0	.9994	27.0	.8910	48.0	.6691	73.0	.2924
3.0	.9986	28.0	.8829	49.0	.6561	74.0	.2756
4.0	.9976	29.0	.8746	50.0	.6428	75.0	.2588
5.0	.9962	30.0	.8660	51.0	.6293	76.0	.2419
6.0	.9945	31.0	.8571	52.0	.6157	77.0	.2249
7.0	.9926	32.0	.8480	53.0	.6018	78.0	.2079
8.0	.9903	33.0	.8387	54.0	.5878	79.0	.1908
9.0	.9877	34.0	.8290	55.0	.5736	80.0	.1736
10.0	.9848	35.0	.8191	56.0	.5592	81.0	.1564
11.0	.9816	36.0	.8090	57.0	.5446	82.0	.1392
12.0	.9781	37.0	.7986	58.0	.5299	83.0	.1219
13.0	.9744	38.0	.7880	59.0	.5150	84.0	.1045
14.0	.9703	39.0	.7772	60.0	.5000	85.0	.0872
15.0	.9659	40.0	.7660	61.0	.4848	86.0	.0698
16.0	.9613	41.0	.7547	62.0	.4695	87.0	.0523
17.0	.9563	42.0	.7431	63.0	.4540	88.0	.0349
18.0	.9511	43.0	.7314	64.0	.4384	89.0	.0174
19.0	.9455	44.0	.7193	65.0	.4226	90.0	0.0
20.0	.9397	45.0	.7071	66.0	.4067		
21.0	.9336			67.0	.3907		
22.0	.9272			68.0	.3746		
23.0	.9205			69.0	.3584		
24.0	.9135			70.0	.3420		

As the angle getting closer and closer to zero, the cosine is getting closer and closer to 1.





as $\theta < \alpha$, then $\vec{u}$ and $\vec{v}$ are more similar than $\vec{w}$ and $\vec{z}$.

$1 > \cos(\theta) > \cos(\alpha)$.

To solve this problem, we find cos of angle between each pair of vectors, and observe which one is closer to 1.

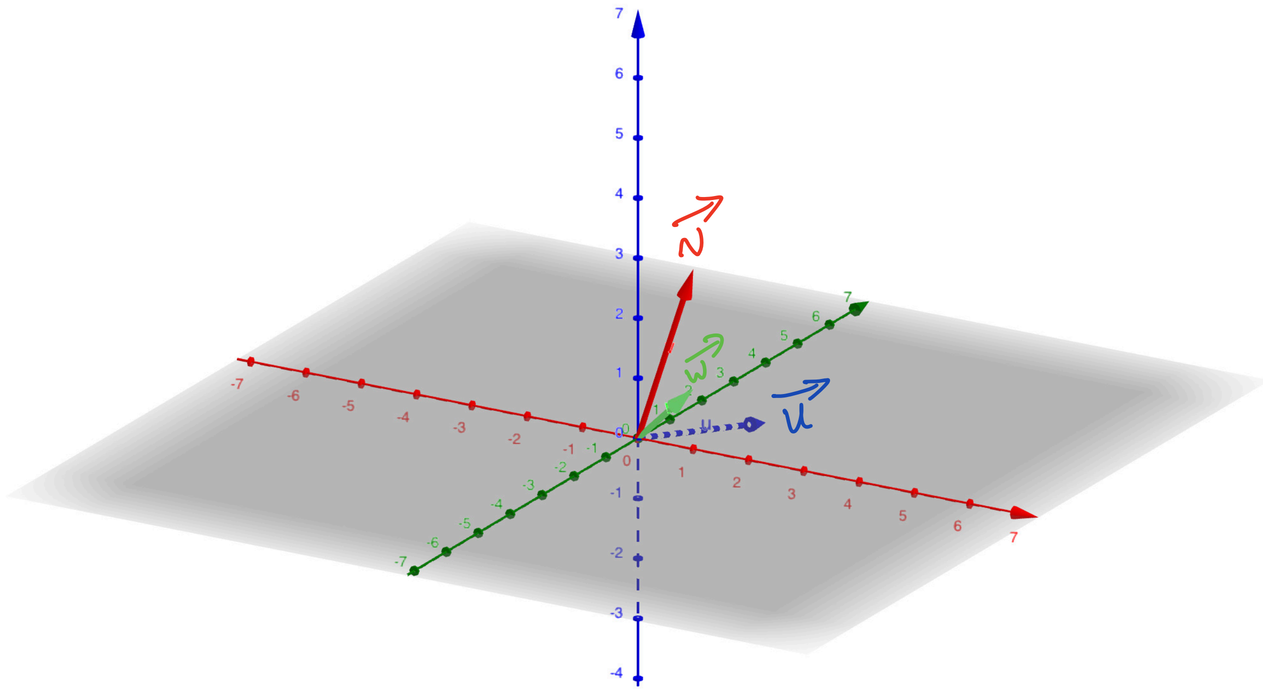
$$\cos(\theta_{u,v}) = \frac{\vec{u}.\vec{v}}{|\vec{u}|\,|\vec{v}|} = \frac{-5}{\sqrt{21}\,\sqrt{11}} = \frac{-5}{\sqrt{231}} \simeq -0.3 \qquad \theta_{u,v} \simeq 109.2^{0}$$

$$\cos(\theta_{u,w}) = \frac{1}{\sqrt{21}\,\sqrt{3}} = \frac{1}{\sqrt{63}} \simeq 0.12 \qquad \theta_{u,w} \simeq 82.7^{0}$$

$$\cos(\theta_{v,w}) = \frac{3}{\sqrt{11}\,\sqrt{3}} = \frac{3}{\sqrt{33}} \simeq 0.5 \qquad \theta_{v,w} \simeq 58.5^{0}$$

vectors $\vec{v}$ and $\vec{w}$ are more similar

as the angle with them is smaller than others. Equivalently, the cosine of angle between $\vec{v}$ and $\vec{w}$ is closer to 1 regarding others.



5. Find all the norms for these vectors:

$$\boldsymbol{u} = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 2 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix} \text{ and } \boldsymbol{w} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

norms that we learnt:

For $\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

- 1-norm

$$||\boldsymbol{x}||_1 = \sum_{i=1}^{n} |x_i|$$

- 2-norm (Euclidean norm)

$$||\boldsymbol{x}||_2 = \sqrt{\sum_{i=1}^{n} x_i^2} = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$$

- p-norm

$$||\boldsymbol{x}||_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \qquad p \geq 1$$

- ∞-norm or max norm

$$||\boldsymbol{x}||_\infty = \max_{i=1,\ldots,n} |x_i|$$

$$||\vec{u}||_1 = \left(|0| + |4| + |-1| + |2|\right) = (0+4+1+2) = 7$$

$$||\vec{u}||_2 = \sqrt{(0)^2 + (4)^2 + (-1)^2 + (2)^2} = \sqrt{0+16+1+4} = \sqrt{21}$$

$$||\vec{u}||_p = \sqrt[p]{|0|^p + |4|^p + |-1|^p + |2|^p} = \sqrt[p]{4^p + 2^p + 1}$$

$$||\vec{u}||_\infty = \max\{|0|, |4|, |-1|, |2|\} = \max\{0, 4, 1, 2\} = 4$$

$$\|\vec{v}\|_1 = |1| + |0| + |3| + |-1| = 1+0+3+1 = 5$$

$$\|\vec{v}\|_2 = \sqrt{(1)^2+(0)^2+(3)^2+(-1)^2} = \sqrt{1+0+9+1} = \sqrt{11}$$

$$\|\vec{v}\|_p = \sqrt[p]{|1|^p+|0|^p+|3|^p+|-1|^p} = \sqrt[p]{1+0+3^p+1} = \sqrt[p]{3^p+2}$$

$$\|\vec{v}\|_\infty = \max\{|1|, |0|, |3|, |-1|\} = \{1, 0, 3, 1\} = 3$$

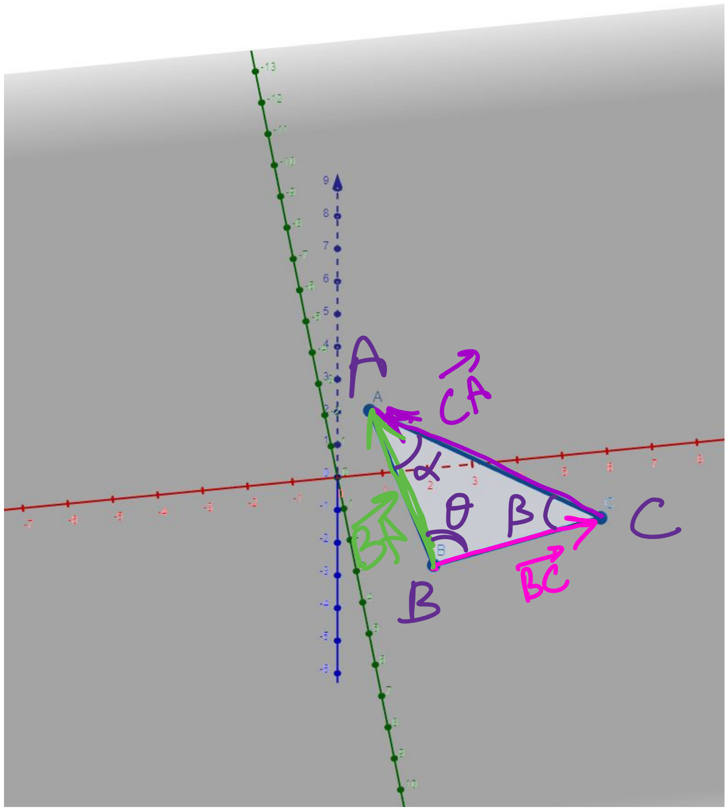
$$\|\vec{w}\|_1 = |1| + |0| + |1| + |1| = 3$$

$$\|\vec{w}\|_2 = \sqrt{(1)^2+(0)^2+(1)^2+(1)^2} = \sqrt{3}$$

$$\|\vec{w}\|_p = \sqrt[p]{|1|^p+|0|^p+|1|^p+|1|^p} = \sqrt[p]{1+0+1+1} = \sqrt[p]{3}$$

$$\|\vec{w}\|_\infty = \max\{|1|, |0|, |1|, |1|\} = 1$$

6. Which of the angles (if any) of triangle $\triangle ABC$, with $A = (1, -2, 0)$, $B = (2, 1, -2)$ and $C = (6, -1, -3)$ is a right angle?



the vector from B to A $\quad \vec{BA} = \begin{bmatrix} x_A - x_B \\ y_A - y_B \\ z_A - z_B \end{bmatrix} = \begin{bmatrix} 1-2 \\ -2-1 \\ 0-(-2) \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$

the vector from B to C $\quad \vec{BC} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$

the vector from C to A $\quad \vec{CA} = \begin{bmatrix} -5 \\ -1 \\ 3 \end{bmatrix}$

we need to see any pair of these vectors are perpendicular. In other words, is the dot product between them is zero.

$$\vec{BA} \cdot \vec{BC} = (-1)(-4) + (-3)(2) + (2)(1) = 4 - 6 + 2 = 0$$

$$\vec{BA} \cdot \vec{CA} = (-1)(-5) + (-3)(-1) + (2)(3) = 5 + 3 + 6 = 14$$

$$\vec{BC} \cdot \vec{CA} = (-4)(-5) + (2)(-1) + (1)(3) = 20 - 2 + 3 = 21$$

$$\vec{BA} \cdot \vec{BC} = 0 \implies \vec{BA} \perp \vec{BC}$$

the angle $\hat{B}$ is a right angle.

Let's find the angles between the vectors:

$$\cos(\theta) = \frac{\vec{BA} \cdot \vec{BC}}{\|\vec{BA}\| \, \|\vec{BC}\|} = \frac{0}{\|\vec{BA}\| \, \|\vec{BC}\|} = 0$$

$$\implies \theta = 90^\circ$$

angle between $\vec{BA}$ and $\vec{CA}$

$$\cos(\alpha) = \frac{\vec{BA} \cdot \vec{CA}}{\|\vec{BA}\| \, \|\vec{CA}\|} = \frac{14}{\sqrt{14}\,\sqrt{35}} = \frac{14}{22.13}$$

$$\simeq 0.63$$

$$\|\vec{BA}\| = \sqrt{(-1)^2+(-3)^2+(2)^2}$$

$$= \sqrt{1+9+4} = \sqrt{14}$$

$$\alpha = \arccos(0.63)$$

$$\alpha \simeq 51^0$$

$$\|\vec{CA}\| = \sqrt{(-5)^2+(-1)^2+(3)^2}$$

$$= \sqrt{25+1+9} = \sqrt{35}$$

Angle between $\vec{BC}$ and $\vec{CA}$

$$\cos(\beta) = \frac{\vec{BC} \cdot \vec{CA}}{\|\vec{BC}\| \, \|\vec{CA}\|} = \frac{21}{}$$

$$\|\vec{BC}\| = \sqrt{(-4)^2+(2)^2+(1)^2} = \sqrt{16+4+1} = \sqrt{21}$$

$$\|\vec{CA}\| = \sqrt{(-5)^2+(-1)^2+(3)^2} = \sqrt{35}$$

$$\cos(\beta) = \frac{21}{\sqrt{21}\,\sqrt{35}} \simeq \frac{21}{27.11} \simeq 0.77$$

$\beta \simeq \arccos(0.77) = 39°$

check point: $\alpha + \beta + \theta =$

$51° + 39° + 90° = 180°$ ✓