SIT787: Mathematics for Artificial Intelligence
Topic 2: Linear Algebra
Part 2

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Matrices

- A matrix with m rows and n columns

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n}$$

- rows $A_i = \begin{bmatrix} a_{i1}, a_{i2}, \ldots, a_{in} \end{bmatrix}$ a row vector
- columns $A^j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$ a column vector
- entries a_{ij} you need two indices to access an entry

Special Matrices

- A column vector is a $n \times 1$ matrix

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}_{n\times 1}$$

- A row vector is a $1 \times n$ matrix

$$\begin{bmatrix} v_1, \dots, v_n \end{bmatrix}_{1\times n}$$

- Zero matrix: all entries are zero.

$$\mathbf{0} = \mathbf{0}_{m\times n} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

- if $m \neq n$, the matrix is rectangular.
- Square matrices: when the number of rows is the same as the number of columns.
  - they have the main diagonal
- Identity Matrix

$$I_{m\times m} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I_m = I$$

Equal matrices

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad B_{m\times n} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

- $A = B$ if
  - they have the same number of rows and columns
  - for every i and j, $a_{ij} = b_{ij}$
- This applies to vectors as well.

Operations in Matrices: Addition and subtraction

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad B_{m\times n} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

$$A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m3} & \cdots & a_{mn}+b_{mn} \end{bmatrix}_{m\times n}$$

- If $A_{m\times n}$ and $B_{m\times n}$, then $C = A + B$ is a $m \times n$ natrix and

$$c_{ij} = a_{ij} + b_{ij}$$

- if $A = [a]$ is a 1×1 matrix, then $A^T = A$
- The transpoose of a row vector is a column vectors:
  $A_{1\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$

$$A^T = \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \end{bmatrix}$$

- The transpoose of a column vector is a row vectors:

$$A_{m\times 1} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \end{bmatrix}$$

Operations in Matrices: Transpose

- If A is a $m \times n$ matrix, itstranspose is a $n \times m$ matrix
- rows of A will become columns of A^T

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$cA = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix}_{m\times n}$$

- if $A_{m\times n}$, and $c = 0$, then $0.A = \mathbf{0}_{m\times n}$

- Consider $A_{m\times n}$ and $B_{n\times p}$. The product between A and B is a $m \times p$ matrix:

$$C_{m\times p} = AB$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$c_{ij} = A_i \cdot B^j = A_i^T \cdot B^j$$

Inner and outer products in matrix form

- Consider these two vectors

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \text{ and } \boldsymbol{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

- They are $n \times 1$ matrices
- $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}^T \boldsymbol{v} = \sum_{i=1}^{n} u_i v_i$, the result is a number
- Matrix representation of inner product

$$\underbrace{\boldsymbol{u} \cdot \boldsymbol{v}}_{n\times 1, n\times 1} = \underbrace{\boldsymbol{u}^T}_{1\times n} \underbrace{\boldsymbol{v}}_{n\times 1}$$

- Outer product: the result is a matrix

$$\underbrace{\boldsymbol{u}}_{n\times 1} \underbrace{\boldsymbol{v}^T}_{1\times n}$$

- Consider $A_{m\times n}$ and $B_{n\times p}$. The product between A and B is a $m \times p$ matrix:

$$C_{m\times p} = AB$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = A^1 B_1 + A^2 B_2 + A^3 B_3$$

$$= \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \end{bmatrix} + \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} \begin{bmatrix} b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \begin{bmatrix} b_{31} & b_{32} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} \\ a_{21}b_{11} & a_{21}b_{12} \end{bmatrix} + \begin{bmatrix} a_{12}b_{21} & a_{12}b_{22} \\ a_{22}b_{21} & a_{22}b_{22} \end{bmatrix} + \begin{bmatrix} a_{13}b_{31} & a_{13}b_{32} \\ a_{23}b_{31} & a_{23}b_{32} \end{bmatrix}$$

Matrix Multiplication is not Commutative

- Consider $A_{m\times n}$ and $B_{n\times p}$.
  - the product is only defined if the number of column in the first matrix is the same as the niumber of rows in the second matrix
- It is possible AB is defined but BA is not
  - $A_{2\times 3}$ and $B_{3\times 5}$
- even if AB and BA are defined, they may not be equal.
- Example:

Properties of Matrix Operations

- $A + B = B + A$
- $A + (B + C) = (A + B) + C$
- $c(A + B) = cA + cB$
- $(c_1c_2)A = c_1(c_2A)$
- $A(BC) = (AB)C$
- $(A + B)C = AC + BC$, and $A(B + C) = AB + AC$
- $c(AB) = (cA)B = A(cB)$
- $A + \mathbf{0} = A$
- $A\mathbf{0} = \mathbf{0}A = \mathbf{0}$
- $AI = IA = A$
- $cA = (cI)A$

Rules of Transposition

- $(A+B)^T = A^T + B^T$
- $(A^T)^T = A$
- $(AB)^T = B^T A^T$
  - $(ABC)^T = C^T B^T A^T$
  - $(A_1 A_2 \dots A_{n-1} A_n)^T = A_n^T A_{n-1}^T \dots A_2^T A_1^T$

Symmetric and anti-symmetric matrices

- If $A = A^T$ the matrix is called symmetric, or $a_{ij} = a_{ji}$
- if $A = -A^T$ the matrix is called anti-symmetric, or $a_{ij} = -a_{ji}$
- skew-symmetric, antisymmetric, or antimetric
- Show that the main diagonal of an anti-symmetric matrix is zero.
- example symmetric

$$S = \begin{bmatrix} 1 & 7 & 3 \\ 7 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix}, \; S^T = \begin{bmatrix} 1 & 7 & 3 \\ 7 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix} = S$$

- example anti-symmetric

$$A = \begin{bmatrix} 0 & 2 & -4 \\ -2 & 0 & -4 \\ 4 & 4 & 0 \end{bmatrix}, \; A^T = \begin{bmatrix} 0 & -2 & 4 \\ 2 & 0 & 4 \\ -4 & -4 & 0 \end{bmatrix} = -A$$

Triangular matrices

- Lower and upper triangular matrices

$$L_{n\times n} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad U_{n\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

- Diagonal

$$D_{n\times n} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

$$= \mathrm{diag}(a_{11}, a_{22}, \ldots, a_{nn})$$

Matrix and its row and column vectors

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n}$$

- Columns of $A = \{A^1, A^2, \dots, A^n\}$

$$\left\{ \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \dots, \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \right\}$$

- Rows of $A = \{A_1, A_2, \dots, A_m\}$

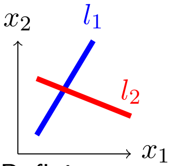
$$\{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \end{bmatrix}, \dots, \begin{bmatrix} a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}\}$$

- Having these vector independent is important.

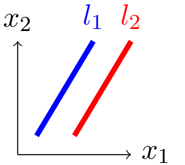
Linear System of two equations

- System of 2 equations and 2 unknowns

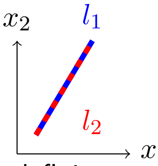
$$\begin{cases} a_1x_1 + b_1x_2 = c_1 \dashrightarrow (l_1) \\ a_2x_1 + b_2x_2 = c_2 \dashrightarrow (l_2) \end{cases}$$



Definite system
single solution



Inconsistent system
no solution



Indefinite system
many solutions

- System of 3 equations and 3 unknowns

$$\begin{cases} a_1x_1 + b_1x_2 + c_1x_3 = d_1 \dashrightarrow (\text{plane } 1) \\ a_2x_1 + b_2x_2 + c_2x_3 = d_2 \dashrightarrow (\text{plane } 2) \\ a_3x_1 + b_3x_2 + c_3x_3 = d_3 \dashrightarrow (\text{plane } 3) \end{cases}$$

Simple systems to solve: compare

$$(1)\begin{cases} x_1 + 2x_2 = 3 \\ 2x_1 - x_2 = 1 \end{cases} \qquad \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$(2)\begin{cases} x_1 + 2x_2 = 3 \\ 0x_1 - 5x_2 = -5 \end{cases} \qquad \begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

$$(3)\begin{cases} x_1 + 0x_2 = 1 \\ 0x_1 + x_2 = 1 \end{cases} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Matrix vector product

- Matrices are considered as transfomations
- A matrix A applies to a vector $\boldsymbol{x}$ and creates $A\boldsymbol{x}$
- $A\boldsymbol{x}$ is a vector. It is a linear combination of columns of matrix A
- $A\boldsymbol{x} = \boldsymbol{b}$ is a system of linear equations
    - We want to see if $\boldsymbol{b}$ can be represented as a linear combination of columns of A
    - if this is possible, we say that the system has a solution.
    - Otherwise, the system does not have solution

- Consider these vectors

$$\boldsymbol{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- A linear combination $x_1\boldsymbol{u} + x_2\boldsymbol{v}$

$$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ 2x_1 + x_2 \end{bmatrix}$$

- Matrix representation

$$A\boldsymbol{x} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ 2x_1 + x_2 \end{bmatrix}$$

Matrix vector product: row view and column view

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n}, \text{ and } \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$A\boldsymbol{x} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix}$$

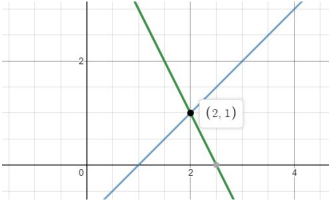
Matrix vector product: from row perspective

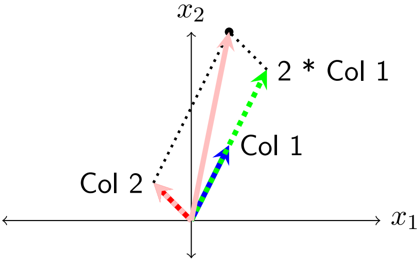
$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}_{2\times 2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}_{2\times 1}$$

$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}_{2\times 2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1} = \begin{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \\ \begin{bmatrix} 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ 2x_1 + x_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \equiv \begin{cases} x_1 - x_2 = 1 \\ 2x_1 + x_2 = 5 \end{cases} \equiv x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

- Row view: Two lines that cut each other
- Columns view: a combination of columns that gives the right hand side





A system of linear equations

- The column picture of $A\boldsymbol{x} = \boldsymbol{b}$: A combination of n columns of A produces $\boldsymbol{b}$.
- if $A = \begin{bmatrix}\boldsymbol{a}_1 \dots \boldsymbol{a}_n\end{bmatrix}$:

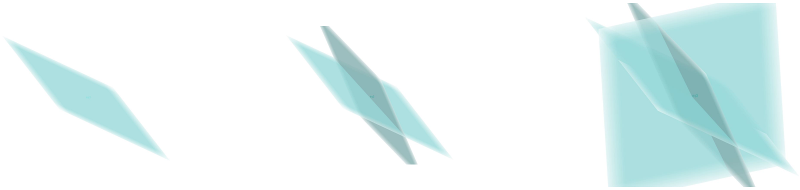
$$A\boldsymbol{x} = x_1\boldsymbol{a}_1 + \ldots + x_n\boldsymbol{a}_n = \boldsymbol{b}$$

- When $\boldsymbol{b} = \mathbf{0}$, one possibility is $\boldsymbol{x} = \begin{bmatrix}0 & 0 & \dots & 0\end{bmatrix}^T$
- The row picture: m equations from m rows give m lines, planes, or hyperlanes meeting at $\boldsymbol{x}$.
- When $\boldsymbol{b} = \mathbf{0}$, all the lines, planes, or hyperlanes go through the origin $(0, 0, \ldots, 0)$.

Three equations with three unkowns

$$A\boldsymbol{x} = \boldsymbol{b}: \qquad \begin{cases} x + 2y + 3z = 6 \\ 2x + 5y + 2z = 4 \\ 6x - 3y + z = 2 \end{cases}$$

- The row picture with solution $(x, y, z) = (0, 0, 2)$



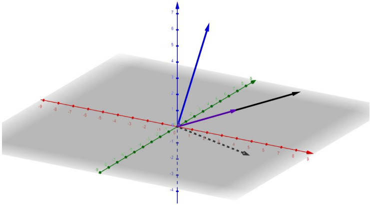
Three equations with three unkowns

$$Ax = b: \qquad \begin{cases} x + 2y + 3z = 6 \\ 2x + 5y + 2z = 4 \\ 6x - 3y + z = 2 \end{cases}$$

- The column picture with solution $(x, y, z) = (0, 0, 2)$

$$x \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + y \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} + z \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

$$0 \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$



Useful but harmless operations

- Interchange two equations
- Multiply each element in an equation by a non-zero number
- Multiply an equation by a non-zero number and add the result to another equation.

Solving a system: the idea of Gaussian elimination

- Consider $m = n = 3$

$$A\boldsymbol{x} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \text{ or } \begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

- aim: eliminate x_1 from the second equation
  - Multiply the first equation by $\frac{a_{21}}{a_{11}}$ and subtract it from the second
  - then x_1 eliminated from the second equation.
- The entry a_{11} is called the first **pivot** and the ratio $\frac{a_{21}}{a_{11}}$ is called the first **multiplier**.
- aim: eliminate x_1 from the i^{th} equation
  - Multiply the first equation by $\frac{a_{i1}}{a_{11}}$ and subtract it from the i^{th}
  - then x_1 eliminated from the i^{th} equation.

Solving a system: the idea of Gaussian elimination

- Before

$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases}$$

- multiply the first equation by $\frac{3}{1}$ and subtract it from the second equation
- After

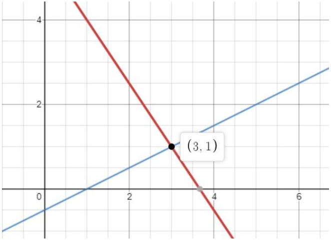
$$\begin{cases} x - 2y = 1 \\ \quad\;\; 8y = 8 \end{cases}$$

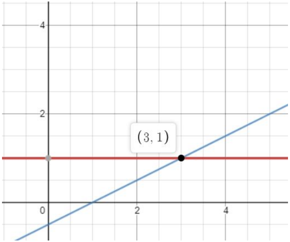
- the first pivot is 1 and the first multiplier is 3.
- matrix representation of changes

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 8 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 11 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

- matrix representation of changes

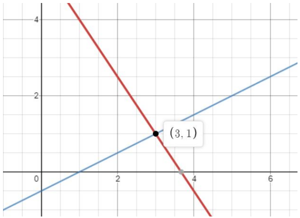
$$\begin{bmatrix} {\color{blue}*} & * \\ * & * \end{bmatrix} \to \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$$

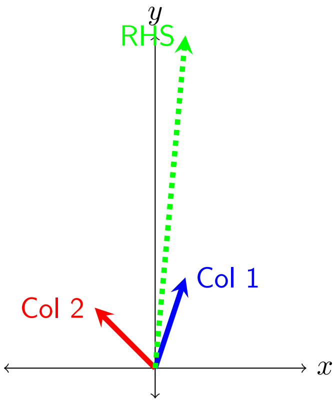




- only one soluton
- lines are cutting eachother at a single point
- Columns are independent

$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases} \rightarrow \begin{cases} x - 2y = 1 \\ \quad\quad 8y = 8 \end{cases}$$

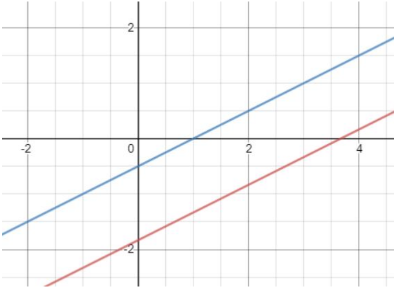


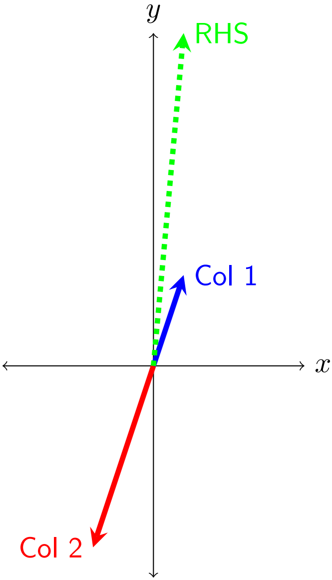


Elimination failures

- No soluton
- lines are parallel
- Columns arn't independent

$$\begin{cases} x - 2y = 1 \\ 3x - 6y = 11 \end{cases} \rightarrow \begin{cases} x - 2y = 1 \\ \qquad 0y = 8 \end{cases}$$

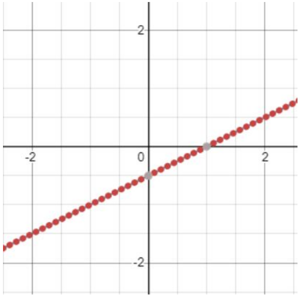


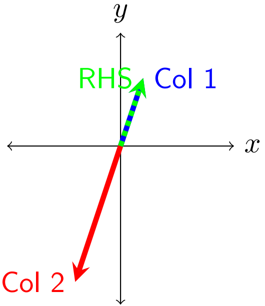


Elimination failures

- many solutons
- lines are overlapping
- Columns arn't independent, but the right-hand side is in the direction of one of the columns

$$\begin{cases} x - 2y = 1 \\ 3x - 6y = 3 \end{cases} \rightarrow \begin{cases} x - 2y = 1 \\ 0y = 0 \end{cases}$$





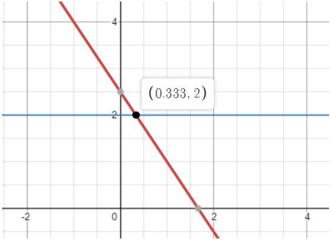
Elimination: Failures

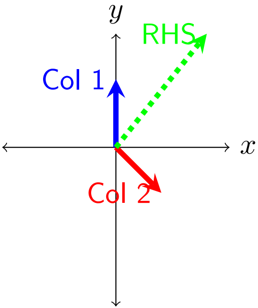
- No solution
  - $0y = 8$
  - no value of y satusfy here.
  - We expect two pivots, but there are only one.
- Many solutions
  - $0y = 0$
  - every y satisfies here
  - the unknown y is free
  - $x = 1, y = 0$, $x = 0, y = -0.5$, etc.
  - We expect two pivots, but there are only one.
- For n equations, we don't get n pivots and elimination leads to
  - $0 \neq 0$ (no solution)
  - $0 = 0$ (many solutions)

- Zero in pivot

$$\begin{cases} {\color{red}0}x + 2y = 4 \\ 3x - 2y = 5 \end{cases} \rightarrow \begin{cases} {\color{blue}3}x - 2y = 5 \\ \qquad {\color{blue}2}y = 4 \end{cases}$$

- a row exchange solves the problem.





Augmented matrix representation $Ax = b$ as $[A \vdots b]$

- A unique solution

$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases} \rightarrow \begin{cases} x - 2y = 1 \\ \quad\;\; 8y = 8 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & \vdots & 1 \\ 3 & 2 & \vdots & 11 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & \vdots & 1 \\ 0 & 8 & \vdots & 8 \end{bmatrix}$$

- No solution

$$\begin{cases} x - 2y = 1 \\ 3x - 6y = 11 \end{cases} \rightarrow \begin{cases} x - 2y = 1 \\ \quad\;\; 0y = 8 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & \vdots & 1 \\ 3 & -6 & \vdots & 11 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & \vdots & 1 \\ 0 & 0 & \vdots & 8 \end{bmatrix}$$

Augmented matrix representation

- many solutions

$$\begin{cases} x - 2y = 1 \\ 3x - 6y = 3 \end{cases} \rightarrow \begin{cases} x - 2y = 1 \\ \qquad 0y = 0 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & \vdots & 1 \\ 3 & -6 & \vdots & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & \vdots & 1 \\ 0 & 0 & \vdots & 0 \end{bmatrix}$$

- zero pivot

$$\begin{cases} 0x + 2y = 1 \\ 3x - 2y = 5 \end{cases} \rightarrow \begin{cases} 3x - 2y = 5 \\ 0x + 2y = 1 \end{cases} \rightarrow \begin{cases} x - 2y = 1 \\ \qquad 2y = 4 \end{cases}$$

$$\begin{bmatrix} 0 & 2 & \vdots & 1 \\ 3 & -2 & \vdots & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & -2 & \vdots & 5 \\ 0 & 2 & \vdots & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & -2 & \vdots & 5 \\ 0 & 2 & \vdots & 1 \end{bmatrix}$$

Three equations in three unknowns: The procedure

- Use the first equation (column) to create zeros below the first pivot.
- Use the second equation (column) to create zeros below the second pivot.
- keep going until you have an upper triangular matrix

$$\begin{bmatrix} * & * & * & \vdots & * \\ * & * & * & \vdots & * \\ * & * & * & \vdots & * \end{bmatrix} \to \begin{bmatrix} \color{blue}{*} & * & * & \vdots & * \\ \color{red}{*} & * & * & \vdots & * \\ \color{red}{*} & * & * & \vdots & * \end{bmatrix} \to \begin{bmatrix} \color{blue}{*} & * & * & \vdots & * \\ \color{red}{0} & \star & \star & \vdots & \star \\ \color{red}{0} & \star & \star & \vdots & \star \end{bmatrix}$$

$$\to \begin{bmatrix} \color{blue}{*} & * & * & \vdots & * \\ 0 & \color{blue}{\star} & \star & \vdots & \star \\ 0 & \color{red}{\star} & \star & \vdots & \star \end{bmatrix} \to \begin{bmatrix} \color{blue}{*} & * & * & \vdots & * \\ 0 & \color{blue}{\star} & \star & \vdots & \star \\ 0 & \color{red}{0} & \bullet & \vdots & \bullet \end{bmatrix} \to \begin{bmatrix} \color{blue}{*} & * & * & \vdots & * \\ 0 & \color{blue}{\star} & \star & \vdots & \star \\ 0 & 0 & \color{blue}{\bullet} & \vdots & \bullet \end{bmatrix}$$

Three equations in three unknowns

$$\begin{cases} 2x + 4y - 2z = 2 \\ 4x + 9y - 3z = 8 \\ -2x - 3y + 7z = 10 \end{cases} \rightarrow \begin{cases} 2x + 4y - 2z = 2 \\ \qquad\quad y + z = 4 \\ \qquad\qquad -4z = 8 \end{cases}$$

- $A\boldsymbol{x} = \boldsymbol{b}$ becomes $U\boldsymbol{x} = \boldsymbol{c}$

$$\begin{bmatrix} 2 & 4 & -2 & \vdots & 2 \\ 4 & 9 & -3 & \vdots & 8 \\ -2 & -3 & 7 & \vdots & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & -2 & \vdots & 2 \\ 0 & 1 & 1 & \vdots & 4 \\ 0 & 0 & -4 & \vdots & 8 \end{bmatrix}$$

- Through back substitution: $z = 2, y = 2, x = -1$

$$A\boldsymbol{x} = A\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = (-1)\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 9 \\ -3 \end{bmatrix} + 2\begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \boldsymbol{b}$$

Elimination: key ideas

- A linear system $A\boldsymbol{x} = \boldsymbol{b}$ becomes upper triangular $U\boldsymbol{x} = \boldsymbol{c}$ after elimination
- we subtract ℓ_{ij} times equation j from equation i to make the (i, j) entry zero
- The multiplier $\ell_{ij} = \frac{\text{entry to eliminate in row } i}{\text{pivot in row } j}$
- pivots cannot be zero
- When zero in the pivot position, exchange rows if there is a nonzero row below it.
- The upper triangular system $U\boldsymbol{x} = \boldsymbol{c}$ is solved by back substitution
- the system may have a unique solution, no solution, or many solutions

Linear Equation in n Variables

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

- Another representation

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \boldsymbol{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \qquad \boldsymbol{a}^T\boldsymbol{x} = b$$

- $x_1, \ldots, x_n$ are variables (unknowns)
- $a_1, \ldots, a_n, b \in \mathbb{R}$
- The set of points satisfying in this equation (set of solutions) is called a hyperplane in $\mathbb{R}^n$
- Examples
  - in $\mathbb{R}^2$, the hyperplane $a_1x_1 + a_2x_2 = b$ is a line.
  - in $\mathbb{R}^3$, the hyperplane $a_1x_1 + a_2x_2 + a_3x_3 = b$ is a plane.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots \qquad \ldots \qquad \ldots \qquad \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{cases}$$

- In matrix format $A\boldsymbol{x} = \boldsymbol{b}$

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n, \boldsymbol{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^m$$

Row and column views

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \cdots \qquad \cdots \qquad \cdots \qquad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{cases}$$

- Row view: consider m hyperplanes and see whether they cut each other ina single point
- Column view: can we express the right hand side as a linear combination of the columns of A?

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \boldsymbol{b}$$

- $A\boldsymbol{x} = \boldsymbol{b}$ equivalent to $(A|b)$

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n, \boldsymbol{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^m$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & \vdots & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & \vdots & b_m \end{bmatrix}$$

Inverse of a Matrix: Intution

- A single equation: $ax = b$, then $x = \frac{b}{a}$ or $x = a^{-1}b$
- Consider a square matrix $A_{n\times n}$. A square $n \times n$ matrix A^{-1} is called its inverse if

$$AA^{-1} = A^{-1}A = I_n$$

- A square matrix is called singular it it does not have an inverse. Otrherwise it is called nonsingular or invertible.

Inverse Matrix Properties

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$ given that both are nonsingular
- For a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then } A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Examples:
  - $3 \times (\frac{1}{3}) = 1$
  - Find the inverse of $\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}^{-1} = \frac{1}{4 \times 6 - 7 \times 2}\begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}, \quad \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Identity and Inverse Matrices

- $I_n \in \mathbb{R}^{n \times n}$

$$\text{for all } \boldsymbol{x} \in \mathbb{R}^n, \quad I_n \boldsymbol{x} = \boldsymbol{x}$$

- $A \in \mathbb{R}^{n \times n}$, its inverse A^{-1} if exists

$$A^{-1} A = I_n$$

- Solving a system $A\boldsymbol{x} = \boldsymbol{b}$ when A has an inverse

$$A\boldsymbol{x} = \boldsymbol{b} \rightarrow A^{-1} A \boldsymbol{x} = A^{-1} \boldsymbol{b} \rightarrow I_n \boldsymbol{x} = A^{-1} \boldsymbol{b}$$

$$\boldsymbol{x} = A^{-1} \boldsymbol{b}$$

Finding the inverse of a larger matrix using elimination

- Consider a square matrix $A_{n\times n}$
- To have an inverse, a matrix should have n nonzero pivots.
- The system $A\boldsymbol{x} = \mathbf{0}$ must have only solution $\boldsymbol{x} = \mathbf{0}$.
- Gauss-Jordan elimination

$$[A \vdots I] \rightarrow [I \vdots A^{-1}]$$

  - Using Gauss elimination, convert the left hand side to I
  - the right hand side will be A^{-1}

Gauss-Jordan elimination for A^{-1}

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

- Make the Augmented matrix $[A \vdots I]$,

$$[A \vdots I] = \begin{bmatrix} 2 & -1 & 0 & \vdots & 1 & 0 & 0 \\ -1 & 2 & -1 & \vdots & 0 & 1 & 0 \\ 0 & -1 & 2 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

- row2 $\leftarrow$ ($\frac{1}{2}$row1+row2),

$$\begin{bmatrix} 2 & -1 & 0 & \vdots & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \vdots & \frac{1}{2} & 1 & 0 \\ 0 & -1 & 2 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

- row3 $\leftarrow$ ($\frac{2}{3}$row2+row3),

$$\left[\begin{array}{ccc:ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right]$$

- row2 $\leftarrow$ ($\frac{3}{4}$row3+row2),

$$\left[\begin{array}{ccc:ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right]$$

Gauss-Jordan elimination for A^{-1}

- row1 $\leftarrow$ ($\frac{2}{3}$row2+row1),

$$\begin{bmatrix} 2 & 0 & 0 & \vdots & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & \frac{3}{2} & 0 & \vdots & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \vdots & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix}$$

- divide row1 by 2, row2 by $\frac{3}{2}$, and row3 by $\frac{4}{3}$

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \vdots & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \vdots & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix} = [I \vdots A^{-1}]$$

Gauss-Jordan elimination for A^{-1}

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

- Check the correctness

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix} = I_3$$

- Find A^{-1} by Gauss-Jordan elimination strating from

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$$

$$[A \vdots I] = \begin{bmatrix} 2 & 3 & \vdots & 1 & 0 \\ 4 & 7 & \vdots & 0 & 1 \end{bmatrix} \xrightarrow{\text{row2=row2-2row1}} \begin{bmatrix} 2 & 3 & \vdots & 1 & 0 \\ 0 & 1 & \vdots & -2 & 1 \end{bmatrix}$$

$$\xrightarrow{\text{row1=row1-3row2}} \begin{bmatrix} 2 & 0 & \vdots & 7 & -3 \\ 0 & 1 & \vdots & -2 & 1 \end{bmatrix}$$

$$\xrightarrow{\text{row1=0.5row1}} \begin{bmatrix} 1 & 0 & \vdots & \frac{7}{2} & \frac{-3}{2} \\ 0 & 1 & \vdots & -2 & 1 \end{bmatrix} = [I \vdots A^{-1}]$$

$$A^{-1} = \begin{bmatrix} \frac{7}{2} & \frac{-3}{2} \\ -2 & 1 \end{bmatrix}$$

Reduced row echelon form R

- When A is rectangular , elimination will not stop at the upper triangular matrix U
- To make this matrix simpler we can take two actions
  - roduce zeros above the pivots
  - produce ones in the pivots

$$U = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \rightarrow U = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$