SIT787 -Mathematics for AI
Assignment 2
Trimester 2, 2023
Due: no later than the end of <mark>Week 7, Sunday 3 September 2023, 11:59 pm AEST</mark>

Important notes:

- Your submission can be handwritten but it must be legible. If your submission is not legible, it will not be marked and will result in a zero mark. A proper way of presenting your solutions is part of the assessment.

- Please follow the order of questions in your submission.

- All steps (workings) to arrive at the answer must be clearly shown. No marks will be awarded for answers without workings.

- Generally, you need to keep your answers in the form of quotients and surds (e.g. $\frac{2}{3}$ and $\sqrt{3}$). Rarely, you may convert your solutions into decimals for plotting or comparing purposes. However, you need to show the final answer in terms of quotients and surds before converting them into decimals.

- Only (scanned) electronic submission would be accepted via the unit site (DeakinSync).

- Your submission must be in ONE pdf file. Multiple files and/or in different file format, e.g. .jpg, will NOT be accepted. It is your responsibility to ensure your file is not corrupted and can be read by a standard pdf viewer. Failure to comply will result in a zero mark.

Question 1) Consider these vectors:

$$\boldsymbol{u} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 0 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \boldsymbol{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

(i) Determine which two vectors are most similar to each other and which two vectors are least similar to each other based on the following norms:

  (a) ℓ_2 norm:

$$\text{dist}(\boldsymbol{x}, \boldsymbol{y}) = ||\boldsymbol{x} - \boldsymbol{y}||_2 = ||\boldsymbol{y} - \boldsymbol{x}||_2 = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$$

  (b) ℓ_1 norm:

$$\text{dist}(\boldsymbol{x}, \boldsymbol{y}) = ||\boldsymbol{x} - \boldsymbol{y}||_1 = ||\boldsymbol{y} - \boldsymbol{x}||_1 = \sum_{i=1}^{n}|x_i - y_i|$$

  (c) ℓ_∞ norm:

$$\text{dist}(\boldsymbol{x}, \boldsymbol{y}) = ||\boldsymbol{x} - \boldsymbol{y}||_\infty = ||\boldsymbol{y} - \boldsymbol{x}||_\infty = \max_{i=1,\dots,n}|x_i - y_i|$$

(ii) Determine which two vectors are most similar to each other and which two vectors are least similar to each other based on the cosine similarity measure:

$$\cos(\theta) = \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{||\boldsymbol{x}||||\boldsymbol{y}||}$$

  [You can use decimals here. However, you need to show the final answer in terms of quotients and surds before converting them into decimals.]

(iii) Explain the reason behind the difference in result between (i) and (ii), if you observe any.

(iv) Show that given any two data vectors, the ℓ_2 norm and cosine similarity measure produce consistent results after vector normalisation.

[5+5+5+10= 25 marks]

Question 2) Given the matrix $\boldsymbol{A}$ with an unknown a as follows:

$$\boldsymbol{A} = \begin{bmatrix} a & 3a & 1 & 2 \\ -1 & 2 & 0 & 0 \\ -3 & 0 & a & 0 \\ 0 & -a & 1 & 0 \end{bmatrix}$$

(i) Determine the values of a such that $\boldsymbol{A}$ has a full rank.

(ii) Suppose that $\boldsymbol{A}$ is not invertible. Calculate the sum of the eigenvalues of $\boldsymbol{A}$.

[10 + 10 = 20 marks]

Question 3) Consider the following matrix

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}.$$

(i) Construct $B = AA^\top$, and find the characteristic polynomial of B.

(ii) Find the eigenvalues and eigenvectors of B. Let λ_1 and λ_2 denote the eigenvalues with $\lambda_1 \geq \lambda_2$.

(iii) Convert eigenvectors into orthonormal vectors u_1 and u_2 associated with eigenvalues λ_1 and λ_2, respectively. Let $U = [u_1\ u_2]$.

(iv) Construct matrix D as follows:

$$D = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix}$$

where $\sigma_i = \sqrt{\lambda_i}$.

(v) Find three orthonormal vectors v_1, v_2, and v_3 such that

$$v_i = \frac{1}{\sigma_i} A^\top u_i, i = 1, 2.$$

Let $V = [v_1\ v_2\ v_3]$.

(vi) Compute the product $UDV^\top$.

[10+15+5+5+10+10= 55 marks]