Introductory Mathematical Methods

STUDY GUIDE

Contents

1

Algebra

Introduction

Algebra, the written language of mathematics, is defined in terms of numbers, letters, and mathematical symbols. As with any language, a variety of rules are needed to enhance precision, and to resolve ambiguity. Ultimately, all mathematics uses algebra to describe and clarify relations between variables.

This topic covers algebraic expressions, decimal places, the rules of algebra, the algebra of fractions, inequalities, and the solution of equations in both one and two variables.

After studying this topic, you should be able to:

- evaluate algebraic expressions for given values of the variables;
- round fractions to a specified number of decimal places;
- understand and apply the rules of algebra;
- understand and apply the rules to simplify operations on fractions;
- re-arrange and solve equations in one variable;
- simplify and solve inequalities;
- solve pairs of simultaneous linear equations in two variables.

1.1 Algebraic expressions

An algebraic **expression** contains numbers, letters and mathematical symbols. The standard symbols (**operations**) used are the familiar:

+ addition;
− subtraction;
× multiplication;
÷ division;
and ( ) brackets.

e.g. $5x-3$, $2(x+9)$ and $\frac{3x+2}{x}$ are expressions.

Note that the division of one quantity by another can be denoted in three separate ways, e.g. $3 \div 5$, $\frac{3}{5}$ and $3/5$ all mean 3 divided by 5.

The letters appearing in an expression are called the **variables** (or unknowns). So x is the variable in the expression $5x-3$.

Expressions can be **evaluated** when the variable has a given value,
e.g. when $x=4$, $5x-3=5\times 4-3=20-3=17$.

Expressions may involve more than one variable, e.g. $\frac{3(x-4y)}{2x+1}$.

Examples

1. Evaluate $5x-3$ when
(i) $x=7$ (ii) $x=0$ (iii) $x=-3$.

2. Evaluate $\frac{10-2x}{x+2y}$ when

(i) $x = 2$ and $y = 0$ (ii) $x = -3$ and $y = 1$

(iii) $x = 5$ and $y = -1$ (iv) $x = 1$ and $y = 3$

Answers

1. (i) When $x = 7$, $5x - 3 = 5 \times 7 - 3 = 35 - 3 = 32$.

(ii) When $x = 0$, $5x - 3 = 5 \times 0 - 3 = 0 - 3 = -3$.

(iii) When $x = -3$, $5x - 3 = 5 \times -3 - 3 = -15 - 3 = -18$.

2. (i) When $x = 2$ and $y = 0$,

$$\frac{10-2x}{x+2y} = \frac{10-2\times2}{2+2\times0} = \frac{10-4}{2+0} = \frac{6}{2} = 3.$$

(ii) When $x = -3$ and $y = 1$,

$$\frac{10-2x}{x+2y} = \frac{10-2\times-3}{-3+2\times1} = \frac{10+6}{-3+2} = \frac{16}{-1} = -16.$$

(iii) When $x = 5$ and $y = -1$,

$$\frac{10-2x}{x+2y} = \frac{10-2\times5}{5+2\times-1} = \frac{10-10}{5-2} = \frac{0}{3} = 0.$$

(iv) When $x = 1$ and $y = 3$,

$$\frac{10-2x}{x+2y} = \frac{10-2\times1}{1+2\times3} = \frac{10-2}{1+6} = \frac{8}{7}.$$

Problems

1. Evaluate $6x - 5$ when

(i) $x = 7$ (ii) $x = 0$ (iii) $x = -3$.

2. Evaluate $\frac{16-3x}{y+4}$ when

(i) $x = 2$ and $y = 6$ (ii) $x = -3$ and $y = 1$

(iii) $x = 5$ and $y = -1$ (iv) $x = 1$ and $y = 3$

Answers

1. (i) 37 (ii) -5 (iii) -23.

2. (i) 1 (ii) 5 (iii) $\frac{1}{3}$ (iv) $\frac{13}{7}$.

1.2 Decimal places

When a fraction is converted to a decimal, the resulting answer is often an approximation. For instance, a calculator gives the answer to $\frac{254}{21}$ as 12.0952381. This answer is not exact, and is therefore an approximation.

For any fraction, the decimal will be recurring or terminating (exact). For fractions in simplest form, terminating decimals result when the denominator consists of powers of the factors 2 and/or 5 only, otherwise recurring decimals result. E.g. $\frac{3}{50}$ has an exact decimal since $50 = 2 \times 5 \times 5$ and $\frac{7}{12}$ has a recurring decimal since $12 = 2 \times 2 \times 3$.

To distinguish between an approximate and exact answer, the symbol ≈ (read as 'is approximately equal to') is used, e.g. $\frac{254}{21} \approx 12.0952381$. Other symbols, such as ≑ also denote approximations.

In approximations, decimals are usually **rounded** off to a specified number of decimal places. The number of places establishes the last digit quoted in the approximation. The procedure is as follows: Identify the specified final digit in the decimal. If the next digit is 4 or less, leave the final digit as it is. If, however, the final digit is 5 or more, add 1 to the final digit. Then drop all digits after the specified final digit.

Examples

1. Round $\frac{254}{21}$ to

(i) 6 decimal places

(ii) 5 decimal places

(iii) 3 decimal places

(iv) 2 decimal places.

2. Round $\frac{17}{29}$ to

(i) 7 decimal places

(ii) 4 decimal places

(iii) 1 decimal place.

Answers

1. A calculator gives $\frac{254}{21} \approx 12.0952381$

(i) 12.095238

(ii) 12.09524 (next digit is 8 so 3 is rounded up to 4)

(iii) 12.095

(iv) 12.10 (the 3rd decimal place is 5 so 9 is rounded up, which means changing two digits)

2. A calculator gives $\frac{17}{29} \approx 0.58620689$

(i) 0.5862069 (next digit is 9 so 8 is rounded up to 9)

(ii) 0.5862

(iii) 0.6 (the 2nd decimal place is 8 so 5 is rounded up to 6)

Problems

1. Round $\frac{34}{67}$ to

(i) 6 decimal places

(ii) 4 decimal places

(iii) 3 decimal places

(iv) 2 decimal places.

2. Round $\frac{47}{31}$ to

(i) 7 decimal places

(ii) 5 decimal places

(iii) 1 decimal place.

Answers

1. (i) 0.507463 (ii) 0.5075 (iii) 0.507 (iv) 0.51

2. (i) 1.5161290 (ii) 1.51613 (iii) 1.5

1.3 Rules of algebra

Rules of algebra are needed to avoid ambiguity when more than one of the mathematical operations is used. Without any rules, both of the following answers could be correct:

$$7 + 4 \times 6 = \begin{cases} 11 \times 6 = 66 \\ 7 + 24 = 31 \end{cases}$$

For this reason a precedence rule is required. The rule, BOMDAS (or BODMAS) specifies that operations are performed in the following order:

Brackets
Of (p**o**wers)
Multiply
Divide
Add
Subtract

Using the BODMAS rule, $7 + 4 \times 6 = 7 + 24 = 31$, the multiplication (4×6) being performed before the addition. If the addition is required to be performed before the multiplication, brackets need be inserted, i.e. $(7 + 4) \times 6 = 11 + 6 = 66$.

Whole number powers of a variable, e.g. x^2, x^3 can be considered as multiplications, as $x^2 = x \times x$, etc. Note the difference between, say, $-5^2 = -5 \times 5 = -25$, and $(-5)^2 = -5 \times -5 = 25$.

The square root, denoted $\sqrt{\ }$, can be considered to act like a bracket. Note the difference between, say, $\sqrt{9} + \sqrt{16} = 3 + 4 = 7$, and $\sqrt{9 + 16} = \sqrt{25} = 5$.

The **distributive law** is used to expand brackets, as follows:
$a(b + c) = ab + ac$, and $a(b - c) = ab - ac$.
Note that the left side (L.S.) in each case is ultimately a product of terms, and the right side (R.S.) is a sum (or difference) of terms. In general, the distributive rule is used to convert products to sums (by **expanding** brackets), or to convert sums to products (called '**factorising**'). When the distributive law is used to expand more than one bracket, all of the required terms must be multiplied, i.e.
$(a + b)(c + d) = a(c + d) + b(c + d)$ (where a multiplies all terms in the second bracket, then b multiplies all terms in the second bracket).
Continuing the expansion, $a(c + d) + b(c + d) = ac + ad + bc + bd$.
In general, if there are 2 terms in each bracket, there will be $2 \times 2 = 4$ terms in the expansion. Similarly, 3 terms in one bracket and 2 terms in the other means $3 \times 2 = 6$ terms in the expansion.

Examples

1. Evaluate

(i) $8 + 4 \times 3$ (ii) $(8 + 4) \times 3$ (iii) $6 \div 3 - 1$

(iv) $8 \div (4 - 2)$ (v) $5 \times 3 - 4$ (vi) $5 \times (3 - 4)$

(vii) $4 - 3^2$ (viii) $4 + (-3)^2$ (ix) $\sqrt{5^2 - 4^2}$.

2. Expand

(i) $2(x - 6)$ (ii) $-3(2x + 3)$ (iii) $-2(x - 7)$

(iv) $-(2x - 3y)$ (v) $x(y + 9)$ (vi) $x^2(x - 5)$.

3. Expand and simplify

(i) $(x + 2)(x + 3)$ (ii) $(2x + 1)(x - 3)$

(iii) $(3x - 2y)(x - y)$ (iv) $(x + 4)(x^2 - 3x + 7)$.

4. Factorise

(i) $5x + 10y$ (ii) $xy - 5x$ (iii) $3x^2 + 2x$.

Answers

1. (i) $8 + 4 \times 3 = 8 + 12 = 20$

(ii) $(8 + 4) \times 3 = 12 \times 3 = 36$

(iii) $6 \div 3 - 1 = 2 - 1 = 1$

(iv) $8 \div (4 - 2) = 8 \div 2 = 4$

(v) $5 \times 3 - 4 = 15 - 4 = 11$

(vi) $5 \times (3 - 4) = 5 \times -1 = -5$

(vii) $4 - 3^2 = 4 - 9 = -5$

(viii) $4 + (-3)^2 = 4 + 9 = 13$

(ix) $\sqrt{5^2 - 4^2} = \sqrt{25 - 16} = \sqrt{9} = 3$

2. (i) $2(x - 6) = 2 \times x - 2 \times 6 = 2x - 12$

(ii) $-3(2x + 3) = -3 \times 2x - 3 \times 3 = -6x - 9$

(iii) $-2(x - 7) = -2x - 2 \times -7 = -2x + 14$

(iv) $-(2x - 3y) = -2x - (-3y) = -2x + 3y$

(v) $x(y + 9) = xy + 9x$

(vi) $x^2(x - 5) = x^3 - 5x^2$

3. (i) $(x + 2)(x + 3) = x(x + 3) + 2(x + 3) = x^2 + 3x + 2x + 6$

$= x^2 + 5x + 6$ (after simplifying)

(ii) $(2x + 1)(x - 3) = 2x(x - 3) + 1(x - 3) = 2x^2 - 6x + x - 3$

$= 2x^2 - 5x - 3$ (after simplifying)

(iii) $(3x - 2y)(x - y) = 3x(x - y) - 2y(x - y) = 3x^2 - 3xy - 2xy + 2y^2$

$= 3x^2 - 5xy + 2y^2$ (after simplifying)

(iv) $(x + 4)(x^2 - 3x + 7) = x(x^2 - 3x + 7) + 4(x^2 - 3x + 7)$

$= x^3 - 3x^2 + 7x + 4x^2 - 12x + 28$ (note the 6 terms)

$= x^3 + x^2 - 5x + 28$ (after simplifying)

4. (i) $5x + 10y = 5(x + 2y)$

(ii) $xy - 5x = x(y - 5)$

(iii) $3x^2 + 2x = x(3x + 2)$

Problems

1. Evaluate

(i) $8 + 5 \times 2$ (ii) $(8 + 5) \times 2$ (iii) $4 \div 4 - 6$

(iv) $4 \div (4 - 6)$ (v) $5 \times 6 - 8$ (vi) $5 \times (6 - 8)$

(vii) $9 - 4^2$ (viii) $9 + (-4)^2$ (ix) $\sqrt{5^2 - 3^2}$.

2. Expand

(i) $3(x-3)$ (ii) $-2(3x+1)$ (iii) $-5(x-3)$

(iv) $-(4x-5y)$ (v) $2x(y+7)$ (vi) $x^2(3x-7)$.

3. Expand and simplify

(i) $(x+5)(x+3)$ (ii) $(2x+3)(x-2)$

(iii) $(3x-y)(x-2y)$ (iv) $(x+3)(x^2-3x+2)$.

4. Factorise

(i) $3x+9y$ (ii) $2xy-4x$ (iii) $5x^2+9x$.

Answers

1. (i) 18 (ii) 26 (iii) -5

(iv) -2 (v) 22 (vi) -10

(vii) -7 (viii) 25 (ix) 4

2. (i) $3x-9$ (ii) $-6x-2$ (iii) $-5x+15$

(iv) $-4x+5y$ (v) $2xy+14x$ (vi) $3x^3-7x^2$

3. (i) $x^2+8x+15$ (ii) $2x^2-x-6$

(iii) $3x^2-7xy+2y^2$ (iv) x^3-7x+6

4. (i) $3(x+3y)$ (ii) $2x(y-2)$ (iii) $x(5x+9)$

1.4 The algebra of fractions

There are 4 rules which govern the algebra of fractions. These 4 rules are as follows:

RULE I $\frac{am}{bm}=\frac{a}{b}$ (Cancellation Rule)

i.e. common factors can be cancelled.

e.g. $\frac{6}{15}=\frac{2\times 3}{5\times 3}=\frac{2}{5}$ and $\frac{7x}{4x}=\frac{7}{4}$ but $\frac{2x+1}{x+1}$ does not simplify, as there is no factor common to both the top and bottom lines.

RULE II $\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$ (Addition Rule)

i.e. fractions must have the same (common) denominator in order to be added. The same rule applies to subtraction of fractions, i.e.

$$\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$$

e.g. $\frac{5}{9}+\frac{3}{9}=\frac{5+3}{9}=\frac{8}{9}$ and $\frac{5}{9}-\frac{2}{9}=\frac{5-2}{9}=\frac{3}{9}=\frac{1}{3}$.

If fractions do not have a common denominator, the cancellation property can be used to form the common denominator before adding (or subtracting) the fractions. For instance, to evaluate $\frac{2}{3}+\frac{1}{6}$, it is necessary to write $\frac{2}{3}$ as $\frac{2\times 2}{3\times 2}=\frac{4}{6}$ before adding. Then, $\frac{2}{3}+\frac{1}{6}=\frac{4}{6}+\frac{1}{6}=\frac{5}{6}$.

Similarly, $\frac{x}{2}+\frac{3x}{4}=\frac{2x}{4}+\frac{3x}{4}=\frac{5x}{4}$ and $\frac{1}{6}+\frac{3}{8}=\frac{1\times 4}{6\times 4}+\frac{3\times 3}{8\times 3}=\frac{4}{24}+\frac{9}{24}=\frac{13}{24}$.

If a common denominator is not readily apparent, it can always be found by multiplying the denominators of the two fractions together. For instance, $\frac{2}{7}$ and $\frac{4}{5}$ have a common denominator of $7\times 5=35$. So, $\frac{2}{7}+\frac{4}{5}=\frac{2\times 5}{7\times 5}+\frac{4\times 7}{5\times 7}=\frac{10}{35}+\frac{28}{35}=\frac{38}{35}$.

Similarly, $\frac{2x}{x+1}$ and $\frac{x}{2x+1}$ have a common denominator of $(x+1)(2x+1)$.

So, $\frac{2x}{x+1}-\frac{x}{2x+1}=\frac{2x(2x+1)}{(x+1)(2x+1)}-\frac{x(x+1)}{(2x+1)(x+1)}$

$$=\frac{2x(2x+1)-x(x+1)}{(x+1)(2x+1)}$$

$$=\frac{4x^2+2x-x^2-x}{(x+1)(2x+1)}$$

$$=\frac{3x^2+x}{(x+1)(2x+1)}=\frac{x(3x+1)}{(x+1)(2x+1)}$$

RULE III $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$ (Multiplication Rule)

i.e. numerators are multiplied, and denominators are multiplied, there being no need for a common denominator.

e.g. $\frac{3}{5}\times\frac{4}{7}=\frac{3\times 4}{5\times 7}=\frac{12}{35}$ and $\frac{3}{x}\times\frac{2x+7}{3x-5}=\frac{3(2x+7)}{x(3x-5)}$.

Note also, that $4\times\frac{2}{3}=\frac{4}{1}\times\frac{2}{3}=\frac{4\times 2}{1\times 3}=\frac{8}{3}$, i.e. when multiplying by a whole number, the numerator of the fraction is multiplied by that number, and the denominator remains unchanged.

RULE IV $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}$ (Division Rule)

i.e. the second fraction is inverted, then multiplied by the first fraction.

e.g. $\frac{2}{7}\div\frac{3}{5}=\frac{2}{7}\times\frac{5}{3}=\frac{2\times 5}{7\times 3}=\frac{10}{21}$ and $\frac{x}{4}\div\frac{2}{3}=\frac{x}{4}\times\frac{3}{2}=\frac{3x}{8}$.

Similarly $\frac{3/4}{2/3}=\frac{3}{4}\div\frac{2}{3}=\frac{3}{4}\times\frac{3}{2}=\frac{9}{8}$.

Examples

1. Simplify, i.e. write as a single fraction with no common factors

(i) $\frac{84}{16}$ (ii) $\frac{7}{5}-\frac{1}{10}$ (iii) $\frac{2}{7}\left(\frac{5}{4}-\frac{3}{8}\right)$

(iv) $\frac{5/9}{5/4}$ (v) $\frac{2+3/5}{6-4/5}$ (vi) $\frac{7}{9}+\frac{8}{11}$

2. Simplify, i.e. write as a single fraction with no common factors

(i) $\frac{5x(y+3)}{15x^2y}$ (ii) $\frac{3xy^2-2y}{y-4xy}$ (iii) $\frac{x}{x+3}-\frac{3}{4}$

(iv) $\frac{2x-1}{x+3}-7$ (v) $\frac{2x}{2x+5}-\frac{4}{2x+1}$ (vi) $\frac{3x-7}{x+3}-\frac{3x-4}{x-5}$

Answers

1. (i) $\frac{84}{16}=\frac{21\times4}{4\times4}=\frac{21}{4}$

(ii) $\frac{7}{5}-\frac{1}{10}=\frac{7\times2}{5\times2}-\frac{1}{10}=\frac{14}{10}-\frac{1}{10}=\frac{13}{10}$

(iii) $\frac{2}{7}\left(\frac{5}{4}-\frac{3}{8}\right)=\frac{2}{7}\left(\frac{10}{8}-\frac{3}{8}\right)=\frac{2}{7}\times\frac{7}{8}=\frac{2}{8}=\frac{1}{4}$

(iv) $\frac{5/9}{5/4}=\frac{5}{9}\div\frac{5}{4}=\frac{5}{9}\times\frac{4}{5}=\frac{4}{9}$

(v) $\frac{2+3/5}{6-4/5}=\frac{10/5+3/5}{30/5-4/5}=\frac{13/5}{26/5}=\frac{13}{5}\times\frac{5}{26}=\frac{13}{26}=\frac{1}{2}$

(vi) $\frac{7}{9}+\frac{8}{11}=\frac{7\times11}{9\times11}+\frac{8\times9}{11\times9}=\frac{77}{99}+\frac{72}{99}=\frac{149}{99}$

2. (i) $\frac{5x(y+3)}{15x^2y}=\frac{5x(y+3)}{5x\times3xy}=\frac{y+3}{3xy}$

(ii) $\frac{3xy^2-2y}{y-4xy}=\frac{y(3xy-2)}{y(1-4x)}=\frac{3xy-2}{1-4x}$

(iii) $\frac{x}{x+3}-\frac{3}{4}=\frac{4x}{4(x+3)}-\frac{3(x+3)}{4(x+3)}=\frac{4x-3(x+3)}{4(x+3)}$

$$=\frac{4x-3x-9}{4(x+3)}=\frac{x-9}{4(x+3)}$$

(iv) $\frac{2x-1}{x+3}-7=\frac{2x-1}{x+3}-\frac{7(x+3)}{x+3}=\frac{2x-1-7(x+3)}{x+3}$

$$=\frac{2x-1-7x-21}{x+3}=\frac{-5x-22}{x+3}$$

(v) $\frac{2x}{2x+5}-\frac{4}{2x+1}=\frac{2x(2x+1)}{(2x+5)(2x+1)}-\frac{4(2x+5)}{(2x+1)(2x+5)}$

$$=\frac{2x(2x+1)-4(2x+5)}{(2x+5)(2x+1)}$$

$$= \frac{4x^2+2x-8x-20}{(2x+5)(2x+1)} = \frac{4x^2-6x-20}{(2x+5)(2x+1)}$$

(vi) $\frac{3x-7}{x+3} - \frac{3x-4}{x-5} = \frac{(3x-7)(x-5)}{(x+3)(x-5)} - \frac{(3x-4)(x+3)}{(x-5)(x+3)}$

$$= \frac{(3x-7)(x-5)-(3x-4)(x+3)}{(x+3)(x-5)}$$

$$= \frac{3x^2-15x-7x+35-(3x^2+9x-4x-12)}{(x+3)(x-5)}$$

$$= \frac{3x^2-22x+35-(3x^2+5x-12)}{(x+3)(x-5)}$$

$$= \frac{3x^2-22x+35-3x^2-5x+12}{(x+3)(x-5)}$$

$$= \frac{-27x+47}{(x+3)(x-5)}$$

Problems

1. Simplify, i.e. write as a single fraction with no common factors

(i) $\frac{15}{95}$ (ii) $\frac{7}{8} - \frac{1}{4}$ (iii) $\frac{5}{2}\left(\frac{4}{5} - \frac{2}{3}\right)$

(iv) $\frac{6/11}{3/2}$ (v) $\frac{2+5/8}{2-7/8}$ (vi) $\frac{7}{11} + \frac{3}{5}$

2. Simplify, i.e. write as a single fraction with no common factors

(i) $\frac{4x(3y+8)}{8xy^2}$ (ii) $\frac{7xy^2-2xy}{3xy-4x^2y}$ (iii) $\frac{5x}{x+1} - \frac{6}{5}$

(iv) $\frac{3x-1}{x+6} - 3$ (v) $\frac{x}{2x+5} - \frac{2}{2x+7}$ (vi) $\frac{2x-7}{x+3} - \frac{2x+5}{x-5}$

Answers

1. (i) $\frac{3}{19}$ (ii) $\frac{5}{8}$ (iii) $\frac{1}{3}$

(iv) $\frac{4}{11}$ (v) $\frac{7}{3}$ (vi) $\frac{68}{55}$

2. (i) $\frac{3y+8}{2y^2}$ (ii) $\frac{xy(7y-2)}{xy(3-4x)} = \frac{7y-2}{3-4x}$

(iii) $\frac{25x}{5(x+1)} - \frac{6(x+1)}{5(x+1)} = \frac{19x-6}{5(x+1)}$

(iv) $\frac{3x-1}{x+6} - \frac{3(x+6)}{x+6} = \frac{-19}{x+6}$

(v) $\frac{x}{2x+5} - \frac{2}{2x+7} = \frac{x(2x+7)}{(2x+5)(2x+7)} - \frac{2(2x+5)}{(2x+7)(2x+5)}$

$$= \frac{x(2x+7)-2(2x+5)}{(2x+5)(2x+7)}$$

$$= \frac{2x^2+7x-4x-10}{(2x+5)(2x+7)} = \frac{2x^2+3x-10}{(2x+5)(2x+7)}$$

(vi) $\frac{2x-7}{x+3} - \frac{2x+5}{x-5} = \frac{(2x-7)(x-5)}{(x+3)(x-5)} - \frac{(2x+5)(x+3)}{(x-5)(x+3)}$

$$= \frac{(2x-7)(x-5)-(2x+5)(x+3)}{(x+3)(x-5)}$$

$$= \frac{2x^2-10x-7x+35-\left(2x^2+6x+5x+15\right)}{(x+3)(x-5)}$$

$$= \frac{2x^2-17x+35-2x^2-11x-15}{(x+3)(x-5)}$$

$$= \frac{-28x+20}{(x+3)(x-5)}$$

1.5 Equations

When two expressions are equal for certain values of a variable (normally x), the result is an equation, e.g. $5x - 7 = 4x + 1$. This equation is true for one value of x only; $x = 8$. This can be checked by evaluating both the left side (L.S.) and the right side (R.S.) of the equation when $x = 8$, i.e.
L.S. $= 5 \times 8 - 7 = 40 - 7 = 33$, and
R.S. $= 4 \times 8 + 1 = 32 + 1 = 33$.

This means that **the equation** $5x - 7 = 4x + 1$ **has the solution** $x = 8$.

The solution to an equation can be found by performing the same operations on both the L.S. and the R.S. of the equation. The aim is to isolate the variable x on the L.S. For the above equation,

$5x - 7 = 4x + 1$

$\therefore 5x = 4x + 1 + 7$ (adding 7 to both sides)

$\therefore 5x = 4x + 8$ (simplifying)

$\therefore 5x - 4x = 8$ (subtracting $4x$ from both sides)

$\therefore x = 8$ (simplifying)

Note that all terms not involving x are removed from the L.S. (by adding 7), and then all x terms are removed from the R.S. (by subtracting $4x$).

The process of isolating x on the L.S. can be pictured as 'undoing' what has happened to x. For instance to solve the equation $\frac{4x+3}{5} = 7$, note that x has been:
multiplied by 4, 3 has been added, then the result has been divided by 5.
To isolate x, these operations are 'undone' in reverse order, i.e. first multiply by 5, then subtract 3, then divide by 4.

So, $\frac{4x+3}{5} = 7$

$\therefore 4x + 3 = 35 \qquad [\times 5]$

$\therefore 4x = 32 \qquad [-3]$

$\therefore x = \frac{32}{4} = 8 \qquad [\div 4]$

Note that 'undoing' an operation is the same as performing the inverse operation. The common operations and their inverses are shown in the following table.

OPERATION	INVERSE
Addition	Subtraction
Subtraction	Addition
Multiplication	Division
Division	Multiplication

When solving an equation involving squares and/or square roots, the following rules should be used.
For any ≥ 0 :

If $x^2 = a$, then $x = \pm\sqrt{a}$
(the inverse of the square is the $\pm$ square root);

If $\sqrt{x} = a$, then $x = a^2$
(the inverse of the square root is the square).

Note that for any $a > 0$, there are **two solutions** to the equation $x^2 = a$. For instance, if $x^2 = 9$, then $x = \pm\sqrt{9} = \pm 3$. So both $x = 3$ and $x = -3$ are solutions. This is a consequence of the fact that the square of a negative number is positive.
However, for any $a \geq 0$, there is only **one solution** to the equation $\sqrt{x} = a$. For instance if $\sqrt{x} = 4$, then $x = 4^2 = 16$. This is a consequence of the fact that x cannot be negative if $\sqrt{x}$ exists.

An equation containing two (or more) variables can also be 'solved' for one of the variables, i.e. one variable can be made **the subject** of the equation by isolating it on the L.S. For instance, the equation $\frac{x-2y}{3} = 2y + 4$ can be re-arranged to isolate x on the L.S., as follows.

$x - 2y = 3(2y + 4) \qquad [\times 3]$

$\therefore x - 2y = 6y + 12$ (simplifying)

$\therefore x = 8y + 12 \qquad [+2y]$

In any equation, 2 expressions are equal, so the L.S. and R.S. may be switched at any time (since $a = b$ and $b = a$ are regarded as identical equations). For instance, the equation $y = 3x - 1$ can be re-arranged to isolate x as follows:

$3x - 1 = y$ (switching sides)

$\therefore 3x = y + 1$ $[+1]$

$\therefore x = \frac{y+1}{3}$ $[\div 3]$

Examples

1. Solve the following equations for x

(i) $\frac{4x-1}{5} = 3$ (ii) $\frac{x}{5} + 2x - 9 = 2$ (iii) $\frac{7x-4}{2x+1} = 5$

(iv) $\sqrt{2x^2 - 17} = 9$ (v) $3x^2 + 7 = 19$ (vi) $\frac{2x-9}{x^2+3} = 0$

2. Solve the following equations for x, i.e. make x the subject of the formula

(i) $3x + 2y = 10$ (ii) $y = \sqrt{5x + 7}$ (iii) $2x^2 - 3 = 5y$

(iv) $y = \sqrt{x^2 - 8}$ (v) $y = \frac{2x+3}{x}$ (vi) $y = \frac{3x}{x-7}$

3. Given that $y = \frac{3x+5}{x}$

(i) make x the subject of the formula

(ii) find x when $y = 4$.

Answers

1. (i) $\frac{4x-1}{5} = 3$

$\therefore 4x - 1 = 15$ $[\times 5]$

$\therefore 4x = 16$ $[+1]$

$\therefore x = \frac{16}{4} = 4$ $[\div 4]$

i.e. $x = 4$

(ii) $\frac{x}{5} + 2x - 9 = 2$

$\therefore x + 5(2x - 9) = 10$ $[\times 5]$

$\therefore x + 10x - 45 = 10$ (simplifying)

$\therefore 11x - 45 = 10$ (simplifying)

$\therefore 11x = 55$ $[+45]$

$\therefore x = \frac{55}{11} = 5$ $[\div 11]$

i.e. $x = 5$

(iii) $\frac{7x-4}{2x+1} = 5$

$\therefore 7x - 4 = 5(2x + 1)$ $\quad$ $[\times (2x + 1)]$

$\therefore 7x - 4 = 10x + 5$ $\quad$ (simplifying)

$\therefore 7x = 10x + 9$ $\quad$ $[+ 4]$

$\therefore -3x = 9$ $\quad$ $[- 10x]$

$\therefore x = \frac{9}{-3} = -3$ $\quad$ $[\div (-3)]$

i.e. $x = -3$

(iv) $\sqrt{2x^2 - 17} = 9$

$\therefore 2x^2 - 17 = 9^2$ $\quad$ (squaring)

$\therefore 2x^2 - 17 = 81$ $\quad$ (simplifying)

$\therefore 2x^2 = 98$ $\quad$ $[+ 17]$

$\therefore x^2 = 49$ $\quad$ $[\div 2]$

$\therefore x = \pm\sqrt{49} = \pm 7$ $\quad$ $[\pm\sqrt{\ }]$

i.e. $x = \pm 7$ (2 solutions)

(v) $3x^2 + 7 = 19$

$\therefore 3x^2 = 12$ $\quad$ $[- 7]$

$\therefore x^2 = \frac{12}{3} = 4$ $\quad$ $[\div 3]$

$\therefore x = \pm\sqrt{4} = \pm 2$ $\quad$ $[\pm\sqrt{\ }]$

i.e. $x = \pm 2$ (2 solutions)

(vi) $\frac{2x-9}{x^2+3} = 0$

$\therefore 2x - 9 = 0(x^2 + 3)$ $\quad$ $[\times (x^2 + 3)]$

$\therefore 2x - 9 = 0$ $\quad$ (simplifying)

$\therefore 2x = 9$ $\quad$ $[+ 9]$

$\therefore x = \frac{9}{2}$ $\quad$ $[\div 2]$

i.e. $x = \frac{9}{2}$

2. (i) $3x + 2y = 10$

$\therefore 3x = 10 - 2y$ $[-\,2y]$

$\therefore x = \frac{10-2y}{3}$ $[\div 3]$

i.e. $x = \frac{10-2y}{3}$

(ii) $y = \sqrt{5x+7}$

$\therefore y^2 = 5x + 7$ (squaring)

$\therefore 5x + 7 = y^2$ (switching sides)

$\therefore 5x = y^2 - 7$ $[-\,7]$

$\therefore x = \frac{y^2-7}{5}$ $[\div 5]$

i.e. $x = \frac{y^2-7}{5}$

(iii) $2x^2 - 3 = 5y$

$\therefore 2x^2 = 5y + 3$ $[+\,3]$

$\therefore x^2 = \frac{5y+3}{2}$ $[\div 2]$

$\therefore x = \pm\sqrt{\frac{5y+3}{2}}$ $[\pm\sqrt{}]$

i.e. $x = \pm\sqrt{\frac{5y+3}{2}}$

(iv) $y = \sqrt{x^2 - 8}$

$\therefore y^2 = x^2 - 8$ (squaring)

$\therefore x^2 - 8 = y^2$ (switching sides)

$\therefore x^2 = y^2 + 8$ $[+\,8]$

$\therefore x = \pm\sqrt{y^2 + 8}$ $[\pm\sqrt{}]$

i.e. $x = \pm\sqrt{y^2 + 8}$

(v) $y = \frac{2x+3}{x}$

$\therefore xy = 2x + 3$ $[\times x]$

$\therefore xy - 2x = 3$ $[-\,2x]$

$\therefore x(y - 2) = 3$ (factorising)

$\therefore x = \frac{3}{y-2}$ $\quad [\div (y-2)]$

i.e. $x = \frac{3}{y-2}$

(vi) $y = \frac{3x}{x-7}$

$\therefore y(x-7) = 3x$ $\quad [\times (x-7)]$

$\therefore xy - 7y = 3x$ (expanding)

$\therefore xy - 3x - 7y = 0$ $\quad [-3x]$

$\therefore xy - 3x = 7y$ $\quad [+7y]$

$\therefore x(y-3) = 7y$ (factorising)

$\therefore x = \frac{7y}{y-3}$ $\quad [\div (y-3)]$

i.e. $x = \frac{7y}{y-3}$

3. (i) $y = \frac{3x+5}{x}$

$\therefore xy = 3x + 5$ $\quad [\times x]$

$\therefore xy - 3x = 5$ $\quad [-3x]$

$\therefore x(y-3) = 5$ (factorising)

$\therefore x = \frac{5}{y-3}$ $\quad [\div (y-3)]$

i.e. $x = \frac{5}{y-3}$

(ii) When $y = 4$, $x = \frac{5}{4-3} = \frac{5}{1} = 5$

Problems

1. Solve the following equations for x

(i) $\frac{7x-2}{9} = 6$ (ii) $\frac{x}{3} - 2x - 4 = 6$ (iii) $\frac{3x-1}{2x-4} = 2$

(iv) $\sqrt{5x-11} = 7$ (v) $2x^2 + 5 = 77$ (vi) $\frac{2x+5}{x^2+8} = 0$

2. Solve the following equations for x, i.e. make x the subject of the formula

(i) $8x - 3y = 10$ (ii) $y = \sqrt{9x+2}$ (iii) $5x^2 - 1 = 7y$

(iv) $y = \sqrt{x^2+4}$ (v) $y = \frac{x+4}{2x}$ (vi) $y = \frac{3x-1}{x-5}$

3. Given that $y = \frac{7x+4}{2x}$

(i) make x the subject of the formula

(ii) find x when $y = 3$.

Answers

1. (i) $\frac{7x-2}{9} = 6$

$\therefore 7x - 2 = 54$ $[\times 9]$

$\therefore 7x = 56$ $[+2]$

$\therefore x = 8$ $[\div 7]$

i.e. $x = 8$

(ii) $\frac{x}{3} - 2x - 4 = 6$

$\therefore x + 3(-2x - 4) = 18$ $[\times 3]$

$\therefore x - 6x - 12 = 18$ (simplifying)

$\therefore -5x - 12 = 18$ (simplifying)

$\therefore -5x = 30$ $[+12]$

$\therefore x = \frac{30}{-5} = -6$ $[\div (-5)]$

i.e. $x = -6$

(iii) $\frac{3x-1}{2x-4} = 2$

$\therefore 3x - 1 = 2(2x - 4)$ $[\times (2x-4)]$

$\therefore 3x - 1 = 4x - 8$ (simplifying)

$\therefore 3x = 4x - 7$ $[+1]$

$\therefore -x = -7$ $[-4x]$

$\therefore x = \frac{-7}{-1} = 7$ $[\div (-1)]$

i.e. $x = 7$

(iv) $\sqrt{5x - 11} = 7$

$\therefore 5x - 11 = 7^2$ (squaring)

$\therefore 5x - 11 = 49$ (simplifying)

$\therefore 5x = 60$ $[+11]$

$\therefore x = 12$ [$\div 5$]

i.e. $x = 12$

(v) $2x^2 + 5 = 77$

$\therefore 2x^2 = 72$ [$- 5$]

$\therefore x^2 = 36$ [$\div 2$]

$\therefore x = \pm\sqrt{36} = \pm 6$ [$\pm\sqrt{\ }$]

i.e. $x = \pm 6$ (2 solutions)

(vi) $\frac{2x+5}{x^2+8} = 0$

$\therefore 2x + 5 = 0(x^2 + 8)$ [$\times (x^2 + 8)$]

$\therefore 2x + 5 = 0$ (simplifying)

$\therefore 2x = -5$ [$- 5$]

$\therefore x = \frac{-5}{2}$ [$\div 2$]

i.e. $x = \frac{-5}{2}$

2. (i) $8x - 3y = 10$

$\therefore 8x = 10 + 3y$ [$+ 3y$]

$\therefore x = \frac{10+3y}{8}$ [$\div 8$]

i.e. $x = \frac{10+3y}{8}$

(ii) $y = \sqrt{9x + 2}$

$\therefore y^2 = 9x + 2$ (squaring)

$\therefore 9x + 2 = y^2$ (switching sides)

$\therefore 9x = y^2 - 2$ [$- 2$]

$\therefore x = \frac{y^2-2}{9}$ [$\div 9$]

i.e. $x = \frac{y^2-2}{9}$

(iii) $5x^2 - 1 = 7y$

$\therefore 5x^2 = 7y + 1$ [$+ 1$]

$\therefore x^2 = \frac{7y+1}{5}$ [$\div 5$]

$\therefore x = \pm\sqrt{\frac{7y+1}{5}}$ [$\pm\sqrt{\ }$]

i.e. $x = \pm\sqrt{\frac{7y+1}{5}}$

(iv) $y = \sqrt{x^2 + 4}$

$\therefore y^2 = x^2 + 4$ (squaring)

$\therefore x^2 + 4 = y^2$ (switching sides)

$\therefore x^2 = y^2 - 4$ [$- 4$]

$\therefore x = \pm\sqrt{y^2 - 4}$ [$\pm\sqrt{\ }$]

i.e. $x = \pm\sqrt{y^2 - 4}$

(v) $y = \frac{x+4}{2x}$

$\therefore 2xy = x + 4$ [$\times 2x$]

$\therefore 2xy - x = 4$ [$- x$]

$\therefore x(2y - 1) = 4$ (factorising)

$\therefore x = \frac{4}{2y-1}$ [$\div (2y - 1)$]

i.e. $x = \frac{4}{2y-1}$

(vi) $y = \frac{3x-1}{x-5}$

$\therefore y(x - 5) = 3x - 1$ [$\times (x - 5)$]

$\therefore xy - 5y = 3x - 1$ (expanding)

$\therefore xy - 3x - 5y = -1$ [$- 3x$]

$\therefore xy - 3x = 5y - 1$ [$+ 5y$]

$\therefore x(y - 3) = 5y - 1$ (factorising)

$\therefore x = \frac{5y-1}{y-3}$ [$\div (y - 3)$]

i.e. $x = \frac{5y-1}{y-3}$

3. (i) $y = \frac{7x+4}{2x}$

$\therefore 2xy = 7x + 4$ $[\times 2x]$

$\therefore 2xy - 7x = 4$ $[-7x]$

$\therefore x(2y - 7) = 4$ (factorising)

$\therefore x = \frac{4}{2y-7}$ $[\div (2y - 7)]$

i.e. $x = \frac{4}{2y-7}$

(ii) When $y = 3$, $x = \frac{4}{6-7} = \frac{4}{-1} = -4$

1.6 Inequalities

An inequality is a relation between the L.S. and R.S. expressions that does not use an equals sign. Instead, inequalities use the symbols $>$, $<$, $\geq$ or $\leq$. E.g. $5 > 2$ or $-1 < 7$.

Symbol	Definition	Example
$>$	greater than	$x + 3 > 5$
$<$	less than	$3x - 1 < 0$
$\geq$	greater than or equal to	$7 - x \geq 2$
$\leq$	less than or equal to	$2x - 5 \leq -1$

There is not one unique solution to an inequality, but rather a range of values. E.g. if $x + 3 > 5$, any value of x that is greater than 2 makes the statement true. So $x = 3, x = 10.1$ and $x = 103$ are all possible answers. All the possible solutions are more concisely written as $x > 2$.
To simplify and solve inequalities, the aim is the same as that for equations – isolate the variable (usually x) on the L.S. of the inequality.
Some operations can be performed to each side of the inequality as they would for equations, without the direction of the inequality being affected. For example, to solve $x + 3 > 5$, 3 is subtracted from each side of the inequality to give:
$x + 3 - 3 > 5 - 3$ which simplifies to $x > 2$.

e.g for $3x - 1 < 0$,

$3x < 1$ $[+1]$

$x < \frac{1}{3}$ $[\div 3]$

So $x < \frac{1}{3}$ is the solution.

Adding or subtracting a number on both sides, dividing both sides by a positive number and simplifying a single side are operations that do not affect the direction of the inequality.

Some operations require the inequality direction to be reversed. For example, $-5 < -1$, but $5 > 1$. When dividing or multiplying both sides by a negative number or swapping the L.S and R.S., it is necessary to reverse the direction of the inequality.
e.g for $7 - x \geq 2$,

$-x \geq -5$ [-7 and leave the inequality unchanged]

$x \leq 5$ [$\div -1$, and change the inequality direction]
So $x \leq 5$ is the solution.

e.g for $3 \geq x + 2$,
$x + 2 \leq 3$ [swap sides and change the inequality direction]
$x \leq 1$ [$-$ 2, and leave the inequality unchanged]
So $x \leq 1$ is the solution.

Examples

1. Solve the following inequalities for x

(i) $2x + 5 < 11$ (ii) $\frac{x}{5} - 9 > -2$

(iii) $\frac{4x-1}{5} \leq 3$ (iv) $1 + 3x \geq 0$

2. Solve the following inequalities for x

(i) $1 < 2x + 7$ (ii) $1 - 2x > 2$

(iii) $\frac{-x+2}{3} \leq 1$ (iv) $-1 \geq 1 - 2x$

-

Answers

1. (i) $2x + 5 < 11$

$\therefore 2x < 6$ $[-5]$

$\therefore x < 3$ $[\div 2]$

i.e. $x < 3$

(ii) $\frac{x}{5} - 9 > -2$

$\therefore \frac{x}{5} > 7$ $[+9]$

$\therefore x > 35$ $[\times 5]$

i.e. $x > 35$

(iii) $\frac{4x-1}{5} \leq 3$

$\therefore 4x - 1 \leq 15$ $[\times 5]$

$\therefore 4x \leq 16$ $[+1]$

$\therefore x \leq 4$ $[\div 4]$

i.e. $x \leq 4$

(iv) $1 + 3x \geq 0$

$\therefore 3x \geq -1$ [$-$ 1]

$\therefore x \geq \frac{-1}{3}$ [$\div$ 3]

i.e. $x \geq \frac{-1}{3}$

2. (i) $1 < 2x + 7$

$\therefore 2x + 7 > 1$ [swap sides and change the inequality direction]

$\therefore 2x > -6$ [$-$ 7]

$\therefore x > -3$ [$\div$ 2]

i.e. $x > -3$

(ii) $1 - 2x > 2$

$\therefore -2x > 1$ [$-$ 1]

$\therefore x < \frac{1}{-2}$ [$\div -2$, and change the inequality direction]

i.e. $x < \frac{-1}{2}$

(iii) $\frac{-x+2}{3} \leq 1$

$\therefore -x + 2 \leq 3$ [$\times$ 3]

$\therefore -x \leq 1$ [$-$ 2]

$\therefore x \geq -1$ [$\div -1$, and change the inequality direction]

i.e. $x \geq -1$

(iv) $-1 \geq 1 - 2x$

$\therefore 1 - 2x \leq -1$ [swap sides and change the inequality direction]

$\therefore -2x \leq -2$ [$-$ 1]

$\therefore x \geq 1$ [$\div -2$, and change the inequality direction]

i.e. $x \geq 1$

Alternatively, add $2x$ to each side for the first step:

i.e. $2x - 1 \geq 1$

$\therefore 2x \geq 2$ [+ 1]

$\therefore x \geq 1$ [$\div$ 2]

Problems

1. Solve the following inequalities for x

(i) $3x - 2 < 10$ (ii) $\frac{x}{2} + 5 > -1$

(iii) $\frac{2x+1}{3} \leq 5$ (iv) $5x - 2 \geq 0$

2. Solve the following inequalities for x

(i) $3 > 6x - 2$ (ii) $5 - x < -2$

(iii) $\frac{-2x+1}{2} \geq 0$ (iv) $3 \leq -1 + 4x$

Answers

1. (i) $3x - 2 < 10$

$\therefore 3x < 12$ [$+ 2$]

$\therefore x < 4$ [$\div 3$]

i.e. $x < 4$

(ii) $\frac{x}{2} + 5 > -1$

$\therefore \frac{x}{2} > -6$ [$- 5$]

$\therefore x > -12$ [$\times 2$]

i.e. $x > -12$

(iii) $\frac{2x+1}{3} \leq 5$

$\therefore 2x + 1 \leq 15$ [$\times 3$]

$\therefore 2x \leq 14$ [$- 1$]

$\therefore x \leq 7$ [$\div 2$]

i.e. $x \leq 7$

(iv) $5x - 2 \geq 0$

$\therefore 5x \geq 2$ [$+ 2$]

$\therefore x \geq \frac{2}{5}$ [$\div 5$]

i.e. $x \geq \frac{2}{5}$

2. (i) $3 > 6x - 2$

$\therefore 6x - 2 < 3$ [swap sides and change the inequality direction]

$\therefore 6x < 5$ [+ 2]

$\therefore x < \frac{5}{6}$ [÷ 6]

i.e. $x < \frac{5}{6}$

(ii) $5 - x < -2$

$\therefore -x < -7$ [− 5]

$\therefore x > 7$ [÷ −1, and change the inequality direction]

i.e. $x > 7$

(iii) $\frac{-2x+1}{2} \geq 0$

$\therefore -2x + 1 \geq 0$ [× 2]

$\therefore -2x \geq -1$ [− 1]

$\therefore x \leq \frac{1}{2}$ [÷ −2, and change the inequality direction]

i.e. $x \leq \frac{1}{2}$

(iv) $3 \leq -1 + 4x$

$\therefore -1 + 4x \geq 3$ [swap sides and change the inequality direction]

$\therefore 4x \geq 4$ [+ 1]

$\therefore x \geq 1$ [÷ 4]

i.e. $x \geq 1$

1.7 Simultaneous equations

A **linear equation** in 2 variables (normally x and y) has the form $Ax + By = D$, where A, B and D are constants. A linear equation is so called because its graph is a **straight line**. This can be seen mathematically by re-arranging the equation $Ax + By = D$ as follows:

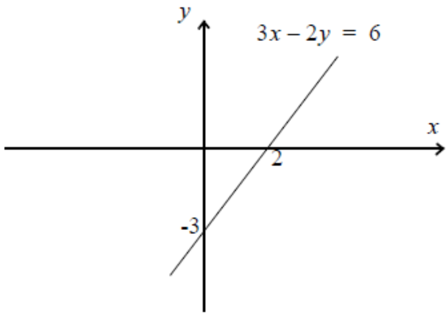
$By = -Ax + D$ [$- Ax$]

$\therefore y = \frac{-Ax+D}{B}$ [$\div B$]

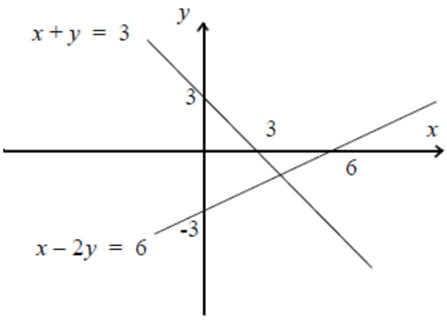
$\therefore y = \left(\frac{-A}{B}\right)x + \frac{D}{B}$, which is of the form of the straight line

$y = mx + c$, with $m = \frac{-A}{B}$ and $= \frac{D}{B}$.

A linear equation can be sketched by finding any two points on the line, and connecting them with a straight line. In general, this is done by simply finding the y value when $x = 0$ (the y-intercept), and finding the x-value when $y = 0$ (the x-intercept).
For instance, given the equation $3x - 2y = 6$:
when $x = 0, -2y = 6 \quad \therefore y = -3$ (the y-intercept);
when $y = 0, 3x = 6 \quad \therefore x = 2$ (the x-intercept).
So, the sketch is as follows:



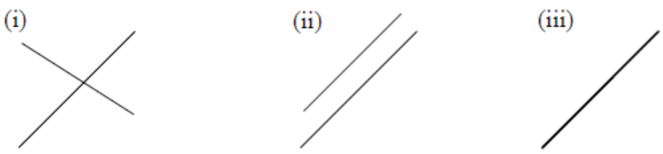
A linear equation has infinitely many solutions, and each solution is a single point on the straight line graph of the equation. Some solutions of the equation $3x - 2y = 6$ are $x = 1$ and $y = \frac{-3}{2}$, $x = 4$ and $y = 3$, $x = -2$ and $y = -6$, etc.
Consider 2 linear equations in 2 variables, e.g.
$x - 2y = 6$ and $x + y = 3$.
Each equation has infinitely many solutions, and has a straight line graph. The graphs are sketched below.



It can be seen that the graphs intersect. The point of intersection indicates the particular values of x and y which are on both straight lines simultaneously. These particular values of x and y represent the **solution to the two simultaneous equations**. From the sketch, the x co-ordinate of the solution is between 3 and 6. The corresponding y co-ordinate of the solution is between 0 and -3.

Any pair of simultaneous linear equations in 2 variables has either:
(i) exactly one solution (the 2 lines intersect at a single point);
(ii) no solution (the 2 lines are parallel, and hence don't intersect);
(iii) infinitely many solutions (the 2 lines are identical).

The 3 cases are shown in the diagrams below.



In all cases, the solution(s) can be found using either substitution or elimination, as described below. Both methods are used to find the solution to the equations sketched above, i.e.
$x - 2y = 6$ (1) and $x + y = 3$ (2)

Substitution

I **Make one variable the subject of one of the equations**
e.g. from (1), $x = 2y + 6$

II **Substitute for this variable in the other equation**
e.g. in (2), x can be replaced by $2y + 6$.
So, (2) becomes $2y + 6 + y = 3$, or $3y + 6 = 3$ when simplified.

III **Solve this equation to find the solution for one variable**
e.g. since $3y + 6 = 3$
$\therefore 3y = -3$
$\therefore y = \frac{-3}{3} = -1$ So, $y = -1$.

IV **Substitute the answer found in III into the equation obtained in I to find the solution for the remaining variable**
e.g. $y = -1$ (from III), and $x = 2y + 6$ (from I),
$x = -2 + 6 = 4$.
So, the solution is $x = 4$ and $y = -1$, i.e. the co-ordinates of the solution are $(4, -1)$.
Note that this solution is consistent with the intersection of the two lines sketched earlier.

Elimination

I **Eliminate one variable by adding (or subtracting) a multiple of one equation to (or from) the other equation**
e.g. for $x - 2y = 6$ (1) and $x + y = 3$ (2), the coefficient of x in each equation is 1.
So, x can be eliminated by subtracting one equation from the other, i.e. (2) subtract (1) gives
$x + y - (x - 2y) = 3 - 6$. Simplifying gives
$x + y - x + 2y = -3$ $\therefore 3y = -3$

II **Solve this equation to find the solution for one variable**
e.g. since $3y = -3$, $y = \frac{-3}{3} = -1$ $\therefore y = -1$

III **Substitute the answer found in II into either of the original equations to find the solution for the remaining variable**
e.g. since $y = -1$, using (1) gives $x - 2(-1) = 6$
$\therefore x + 2 = 6$ $\therefore x = 6 - 2 = 4$
So, again, the solution is $x = 4$ and $y = -1$, i.e. the co-ordinates of the solution are $(4, -1)$.

In the elimination method, the first step is the most important. **The coefficients of one variable must be 'matched' before adding or subtracting.** For instance, as the coefficients of y are -2 and 1, y could be eliminated by multiplying (2) by 2 as follows.
As $x - 2y = 6$ (1) and $x + y = 3$ (2)
(2) $\times$ 2 gives $2x + 2y = 6$ (3)
Now, adding (1) and (3) gives $x - 2y + 2x + 2y = 6 + 6$
Simplifying gives $3x = 12$ $\therefore x = \frac{12}{3} = 4$
Using (2), $4 + y = 3$ $\therefore y = 3 - 4 = -1$. So, again the solution is $x = 4$ and $y = -1$.

The solution to a pair of simultaneous equations can be checked by substituting the answers into the original equations. In the example above, when $x = 4$ and $y = -1$, the L.S. of (1) becomes
$4 - (-2) = 4 + 2 = 6, =$ R.S., as required.
Similarly, the L.S. of (2) becomes
$4 + (-1) = 4 - 1 = 3, =$ R.S., as required.

Examples

1. Solve the following equations for x and y by

(a) elimination (b) substitution

(i) $2x + 3y = -1$ (1) and $x - 4y = 16$ (2)

(ii) $4x + 3y = 8$ (1) and $5x - y = 10$ (2)

2. Use elimination to show that there is no solution to the equations

$-2x + 4y = 6$ (1) and $x - 2y = 1$ (2)

3. Use elimination to show that there are infinitely many solutions to the equations

$2x + 4y = -12$ (1) and $x + 2y = -6$ (2)

Answers

1. (i)(a) Comparing $2x + 3y = -1$ (1) and $x - 4y = 16$ (2),
it can be seen that the x coefficients can be matched by multiplying (2) by 2
i.e. $2x - 8y = 32$ (3).
Now, (1) subtract (3) gives $2x + 3y - (2x - 8y) = -1 - 32$.
Simplifying gives $2x + 3y - 2x + 8y = -33$
$\therefore 11y = -33 \ \therefore y = -3$
When $y = -3$ (2) becomes $x - 4(-3) = 16$
$\therefore x + 12 = 16 \ \therefore x = 4$.
So the solution is $x = 4$ and $y = -3$.

(b) From (2), $x = 4y + 16$. Substituting in (1) gives
$2(4y + 16) + 3y = -1$

$\therefore 8y + 32 + 3y = -1$

$\therefore 11y + 32 = -1$

$\therefore 11y = -1 - 32 = -33$
$\therefore y = \frac{-33}{11} = -3$. Substituting in $x = 4y + 16$ gives
$x = 4(-3) + 16 = -12 + 16 = 4$
So, again, the solution is $x = 4$ and $y = -3$.

(ii)(a) Comparing $4x + 3y = 8$ (1) and $5x - y = 10$ (2),
it can be seen that the y coefficients can be matched by multiplying (2) by 3
i.e. $15x - 3y = 30$ (3).
Now, adding (1) and (3) gives $4x + 3y + 15x - 3y = 8 + 30$.
Simplifying gives $19x = 38$
$\therefore x = \frac{38}{19} = 2$
When $x = 2$ (2) becomes $10 - y = 10$
$\therefore -y = 0 \ \therefore y = 0$.
So, the solution is $x = 2$ and $y = 0$.

(b) From (2), $-y = 10 - 5x \quad \therefore y = -10 + 5x$

Substituting in (1) gives
$4x + 3(-10 + 5x) = 8$

$\therefore 4x - 30 + 15x = 8$

$\therefore 19x = 8 + 30 = 38$
$\therefore x = \frac{38}{19} = 2$ Substituting in $y = -10 + 5x$ gives
$y = -10 + 10 = 0$
So, again, the solution is $x = 2$ and $y = 0$.

2. Comparing $-2x + 4y = 6$ (1) and $x - 2y = 1$ (2),
it can be seen that the x coefficients can be matched by multiplying (2) by 2
i.e. $2x - 4y = 2$ (3).
Now, adding (1) and (3) gives $-2x + 4y + 2x - 4y = 6 + 2$.
Simplifying gives $0 = 8$, which is impossible.
Hence, there is no solution (and the equations represent 2 parallel lines).

3. Comparing $2x + 4y = -12$ (1) and $x + 2y = -6$ (2),
it can be seen that the x coefficients can be matched by multiplying (2) by 2
i.e. $2x + 4y = -12$ (3).
Now, subtracting (1) from (3) gives
$2x + 4y - (2x + 4y) = -12 - (-12)$.
Simplifying gives $2x + 4y - 2x + 4y = -12 + 12$
$\therefore 0 = 0$, which is always true.
Hence, there are infinitely many solutions (and the equations represent 2 identical lines).

Problems

1. Solve the following equations for x and y by

(a) elimination (b) substitution

(i) $x + 5y = 4$ (1) and $2x + 7y = 2$ (2)

(ii) $4x + 3y = 4$ (1) and $3x - y = 16$ (2)

2. Use elimination to show that there is no solution to the equations

(i) $-3x + 6y = 7$ (1) and $x - 2y = -1$ (2)

3. Use elimination to show that there are infinitely many solutions to the equations

(i) $x + 4y = 7$ (1) and $2x + 8y = 14$ (2)

Answers

1. (i)(a) Comparing $x + 5y = 4$ (1) and $2x + 7y = 2$ (2),
it can be seen that the x coefficients can be matched by multiplying (1) by 2
i.e. $2x + 10y = 8$ (3).
Now, (3) subtract (2) gives $2x + 10y - (2x + 7y) = 8 - 2$.
Simplifying gives $2x + 10y - 2x - 7y = 6$
$\therefore 3y = 6$ $\therefore y = 2$
When $y = 2$, (1) becomes $x + 10 = 4$
$\therefore x = 4 - 10 = -6$.
So, the solution is $x = -6$ and $y = 2$.

(b) From (1), $x = 4 - 5y$. Substituting in (2) gives
$2(4 - 5y) + 7y = 2$

$\therefore 8 - 10y + 7y = 2$

$\therefore -3y + 8 = 2$

$\therefore -3y = 2 - 8 = -6$
$\therefore y = \frac{-6}{-3} = 2$. Substituting in $x = 4 - 5y$ gives
$x = 4 - 10 = -6$
So, again, the solution is $x = -6$ and $y = 2$.

(ii)(a) Comparing $4x + 3y = 4$ (1) and $3x - y = 16$ (2),
it can be seen that the y coefficients can be matched by multiplying (2) by 3
i.e. $9x - 3y = 48$ (3).
Now, adding (1) and (3) gives $4x + 3y + 9x - 3y = 4 + 48$.
Simplifying gives $13x = 52$
$\therefore x = \frac{52}{13} = 4$
When $x = 4$ (2) becomes $12 - y = 16$
$\therefore -y = 16 - 12 = 4$ $\therefore y = -4$.
So the solution is $x = 4$ and $y = -4$.

(b) From (2), $-y = 16 - 3x$ $\therefore y = -16 + 3x$
Substituting in (1) gives
$4x + 3(-16 + 3x) = 4$

$\therefore 4x - 48 + 9x = 4$

$\therefore 13x = 4 + 48 = 52$
$\therefore x = \frac{52}{13} = 4$ Substituting in $y = -16 + 3x$ gives
$y = -16 + 12 = -4$
So, again, the solution is $x = 4$ and $y = -4$.

2. Comparing $-3x + 6y = 7$ (1) and $x - 2y = -1$ (2),
it can be seen that the x coefficients can be matched by multiplying (2) by 3
i.e. $3x - 6y = -3$ (3).
Now, adding (1) and (3) gives $-3x + 6y + 3x - 6y = 7 - 3$.
Simplifying gives $0 = 4$, which is impossible.
Hence, there is no solution (and the equations represent 2 parallel lines).

3. Comparing $x + 4y = 7$ (1) and $2x + 8y = 14$ (2),
it can be seen that the x coefficients can be matched by multiplying (1) by 2
i.e. $2x + 8y = 14$ (3).
Now, subtracting (2) from (3) gives
$2x + 8y - (2x + 8y) = 14 - 14$.
Simplifying gives $2x + 8y - 2x - 8y = 0$
$\therefore 0 = 0$, which is always true.
Hence, there are infinitely many solutions (and the equations represent 2 identical lines).

2

Functions and graphs

Introduction

A function is a mathematical rule describing how one quantity depends on another. The expression of relations between physical quantities as functions provides a means of modelling the real world in mathematical terms.The concept of functions is essential to calculus - the mathematics of motion and change.

This topic covers the definition and the properties of functions, linear functions, sketches of straight lines, the solution of quadratic equations, sketches of quadratic functions, and two special types of quadratic functions.

After studying this topic, you should be able to:

- determine whether a given graph represents a function;
- find the domain of a function;
- find the equation of the straight line connecting two given points;
- sketch a straight line given its equation;
- solve a quadratic equation using factorisation or the Quadratic Formula;
- sketch a quadratic function;
- factorise perfect squares and the difference of two squares

2.1 Properties of functions

1. The formula $y = f(x)$ defines a **function**, provided that **there is exactly one *y* value for each possible *x* value**. So, $y = \frac{1}{x}$ is a function which is defined for all x values except $x = 0$.

2. Given a function $y = f(x)$, the **domain** is the set of **all possible *x* values**, and the **range** is the set of **all possible *y* values**. In general, the range is difficult to determine unless a graph of the function is given. However, the domain can often be found by examining the form of the function, without recourse to the graph.

   Many functions are defined for all values of x, e.g. $y = x^2$ (as **any** real number x can be squared). A function which has a denominator and/or a square root may, however, have a restricted domain. In summary:

   (i) $y = f(x) = \frac{1}{u}$ is not defined when $u = 0$;

   (ii) $y = f(x) = \sqrt{u}$ is defined only for $u \geq 0$
   (i.e. u is greater than or equal to 0).

   For instance, $y = f(x) = \frac{1}{x-3}$ is not defined when $x - 3 = 0$, i.e. $x = 3$. So, the domain of the function is all x except $x = 3$.

   Also, $y = f(x) = \sqrt{x+4}$ is defined only for $x + 4 \geq 0$, i.e. $x \geq -4$. So, the domain of the function is $x \geq -4$.

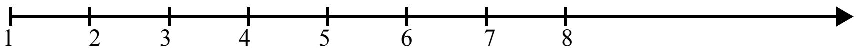
Note: Inequalities, such as $x + 4 \geq 0$ are solved in the same way as equations (in this case by subtracting 4 from both sides).

3. Apart from real numbers, there are several number systems which are frequently used in mathematics.

I Natural Numbers

The counting numbers 1,2,3,4,.... are referred to as the natural numbers. There are infinitely many natural numbers, and the set of all natural numbers is denoted by **N**. In set notation, **N** = {1,2,3,4,5,6,....}

The natural numbers can be represented by equally spaced points on the real number line.

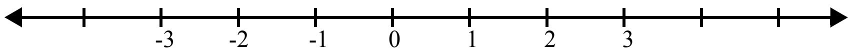


II Integers

The integers consist of the natural numbers, negative whole numbers and zero. The integers are denoted by **Z** or **J**. In set notation,

Z = {.....,-3,-2,-1,0,1,2,3,4,....}

The natural numbers can also be represented by equally spaced points on the real number line.



III Rational Numbers

All fractions can be expressed as one integer divided by another, i.e. $\frac{a}{b}$, where a and b are integers with $b \neq 0$, e.g. $\frac{2}{3}, \frac{-5}{7}, \frac{103}{12}, 4\frac{5}{9}$.

As a fraction is a ratio of integers, the set of all fractions is also known as the set of rational numbers, denoted **Q**.

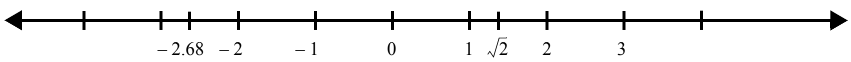
Note: Any integer (and hence any natural number) is also a rational number.

e.g. $3 = \frac{3}{1} = \frac{9}{3} = \frac{-18}{-6}$.

IV Real Numbers

An **irrational number** is a number whose decimal form has infinitely many non-zero digits after the decimal point, but with no systematic repetition. Examples are $\pi = 3.14159265$ ..... and $\sqrt{2} = 1.41421$......
The set of irrationals is denoted **Q'** (the complement of **Q**).

The real numbers, **R**, are made up of the rationals and irrationals. The reals encompass all the number systems above, and can be represented on the usual real number line.



4. The set of **Complex Numbers**, denoted **C**, extends the number system beyond real numbers. For instance, the equation $x^2 = -1$ has no solutions in the real number system (as the square of a real number

cannot be negative), but does have complex number solutions. **Only real numbers (and subsets thereof) are considered in this unit.**

5. There are several ways of specifying intervals on the real number line. These are summarised below.

   If the variable x can lie between 5 and 10 **inclusive**, this can be written as $5 \le x \le 10$, and the corresponding interval on the number line is

$$[5,10]$$

   If x lies between 5 and 10 **exclusive**, this is written as $5 < x < 10$, and the corresponding interval on the number line is

$$(5, 10)$$

   Here, $\le$ means 'is less than or equal to', while $<$ means 'is less than'.

   Similarly, if, say, x lies between 0 and 4, and can be 0 but cannot be 4, this is written as $0 \le x < 4$. The corresponding interval on the number line is [0,4). This is marked on the number line as:

6. If a graph of a function is given, any points of discontinuity can be readily seen. In general, the functions studied in this unit are continuous for all values of x in their domains.

7. If a graph of a relationship between x and y is given, the graph represents a function if and only if any vertical line cuts the graph in at most one point.This is known as the **Vertical Line Test** (VLT).

Examples

1. Evaluate

   (i) $f(-1)$ (ii) $f(0)$ (iii) $f(3)$

   for each of the following functions:

   (a) $f(x) = \dfrac{4}{3x+2}$ (b) $f(x) = \sqrt{4x+4}$

2. Find the domain for each of the following functions:

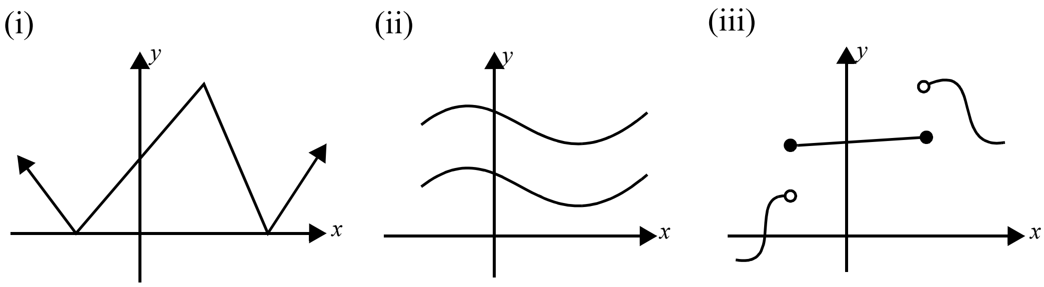
   (i) $f(x) = \dfrac{3}{2x-1}$ (ii) $f(x) = \sqrt{4x+5}$

   (iii) $f(x) = \dfrac{3}{x^2+1}$ (iv) $f(x) = \sqrt{x^2+7}$.

3. Mark each of the following intervals on a number line

   (i) $(-2, 1\,]$ (ii) $-2 \le x < 4$.

4. Determine whether each of the following graphs represents a function



1. (a) For $f(x) = \dfrac{4}{3x+2}$,

(i) $f(-1) = \dfrac{4}{-3+2} = \dfrac{4}{-1} = -4$

(ii) $f(0) = \dfrac{4}{0+2} = \dfrac{4}{2} = 2$

(iii) $f(3) = \dfrac{4}{9+2} = \dfrac{4}{11}$.

(b) For $f(x) = \sqrt{4x+4}$,

(i) $f(-1) = \sqrt{-4+4} = \sqrt{0} = 0$

(ii) $f(0) = \sqrt{0+4} = \sqrt{4} = 2$

(iii) $f(3) = \sqrt{12+4} = \sqrt{16} = 4$.

2. (i) $f(x) = \dfrac{3}{2x-1}$ is not defined when $2x-1 = 0$, i.e. $2x = 1$, i.e. $x = \dfrac{1}{2}$. So, the domain of the function is all x except $x = \dfrac{1}{2}$.

(ii) $f(x) = \sqrt{4x+5}$ is defined only for $4x+5 \geq 0$, i.e. $4x \geq -5$. i.e. $x \geq \dfrac{-5}{4}$. So, the domain of the function is $x \geq \dfrac{-5}{4}$.

(iii) $f(x) = \dfrac{3}{x^2+1}$ is defined for all values of x, as the denominator can never be 0 (as $x^2 + 1 \geq 1$). So, the domain of the function is all x.

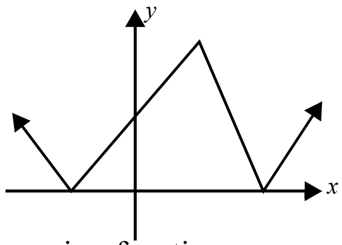
(iv) $f(x) = \sqrt{x^2+7}$ is defined for all values of x, as $x^2 + 7 \geq 7$. So, the domain of the function is all x.

3. (i) $(-2, 1]$;

(ii) $-2 \leq x < 4$.

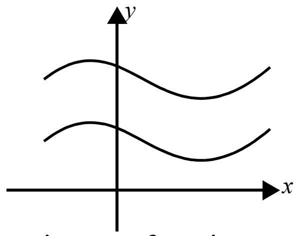
4. By the Vertical Line Test (V.L.T),

(i)



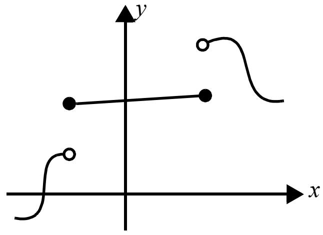
is a function.

(ii)



is not a function.

(iii)



is a function.

Problems

1. Evaluate

   (i) $f(-1)$ (ii) $f(4)$ (iii) $f(7)$

   for each of the following functions:

   (a) $f(x) = \dfrac{5}{x^2+2}$ (b) $f(x) = \sqrt{3x+4}$

2. Find the domain for each of the following functions

   (i) $f(x) = \dfrac{9}{5x-2}$ (ii) $f(x) = \sqrt{4x+1}$

   (iii) $f(x) = 3x^2 + 7$ (iv) $f(x) = \sqrt{4x^2+1}$.

Answers

1. (a) (i) $\dfrac{5}{3}$; (ii) $\dfrac{5}{18}$; (iii) $\dfrac{5}{51}$.

1. (b) (i) 1 ; (ii) 4 ; (iii) 5

2. (i) All x except $x = \dfrac{2}{5}$; (ii) $x \geq \dfrac{-1}{4}$

   (iii) All x; (iv) All x.

2.2 Linear functions

1. A **linear function** has a straight line graph. Any straight line (except a vertical line) can be represented by the equation

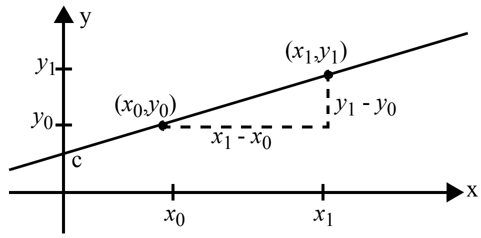
$$y = f(x) = mx + c \qquad (1)$$

where m is the **gradient (or slope)** of the line and c is the **y-intercept**.

Note that a vertical line has the equation $x = k$, where k is a constant. A vertical line does not represent a function as, for $x = k$, there are infinitely many y values on the graph.

2. The gradient m measures the steepness of the slope of the line. To calculate m, any two points on the line are required. Labelling the two points as (x_0, y_0) and (x_1, y_1) in the diagram below gives:

$$\boxed{m = \frac{rise}{run} = \frac{y_1 - y_0}{x_1 - x_0}}$$



A positive gradient indicates that, as x runs from left to right, the line tends upwards. A negative gradient means that the line tends downwards, while $m = 0$ means that the line is horizontal.

Once m has been found, the value of c can be determined by substituting the coordinates of any point on the line into (1).

3. If the linear function $y = f(x) = mx + c$ is given, the corresponding graph can be sketched by finding any two points which satisfy the equation, and connecting them with a straight line. The simplest points to use are the **y-intercept** (where $x = 0$), and the **x-intercept** (where $y = 0$).

Examples

1. Find the equation of the straight line
   (i) passing through the points $(-2, 4)$ and $(4, 1)$.
   (ii) passing through the point $(-2, -3)$ with gradient 2.
2. Sketch $y = 3x - 6$.

1.(i) $m = \frac{\text{rise}}{\text{run}} = \frac{1-4}{4-(-2)} = \frac{1-4}{4+2} = \frac{-3}{6} = \frac{-1}{2}$

So, the equation has the form $y = \frac{-1}{2}x + c$ (1)

Since (4,1) is a point on the line, when $x = 4$, $y = 1$, substituting in (1) gives

$1 = \frac{-1}{2} \times 4 + c$

$\therefore 1 = -2 + c \qquad \therefore c = 3$

Hence, the required equation is $y = \frac{-1}{2}x + 3$, i.e. $y = \frac{-x}{2} + 3$.

Note: The answer here can be verified by checking that the point $(-2, 4)$ satisfies the equation, i.e. when $x = -2$, $y = \frac{-(-2)}{2} + 3 = 1 + 3 = 4$, as required.

1. (ii) Given $m = 2$, the equation has the form $y = 2x + c$ (2)

Since $(-2, -3)$ is a point on the line, when $x = -2$, $y = -3$, substituting in (2) gives

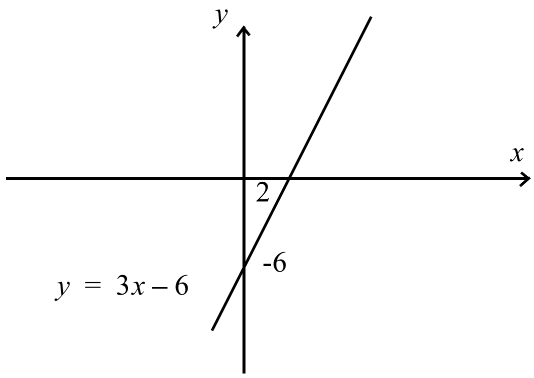
$-3 = 2 \times -2 + c \qquad \therefore -3 = -4 + c \qquad \therefore c = 1$

Hence, the required equation is $y = 2x + 1$

2. For $y = 3x - 6$, when $x = 0, y = -6$ (y-intercept).

Also, when $y = 0$, $3x - 6 = 0$

$\therefore 3x = 6 \qquad \therefore x = 2$ (x-intercept). Hence the sketch



Problems

1. Find the equation of the straight line
   - (i) passing through the points $(-2, 4)$ and $(-1, 1)$.
   - (ii) passing through the point $(-2, 4)$ with gradient -5 .
2. Sketch $y = 4 - x$.

Answers

1.(i) $m = \frac{\text{rise}}{\text{run}} = \frac{1-4}{-1-(-2)} = \frac{1-4}{-1+2} = \frac{-3}{1} = -3$

So, the equation has the form $y = -3x + c$ (1)

Since is $(-1, 1)$ a point on the line, when $x = -1$, $y = 1$, substituting in (1) gives

$1 = (-3) \times (-1) + c$

$\therefore 1 = 3 + c$ $\therefore c = -2$

Hence, the required equation is $y = -3x - 2$.

1. (ii) Given $m = -5$, the equation has the form $y = -5x + c$ (2)

Since $(-2, 4)$ is a point on the line, when $x = -2$, $y = 4$, substituting in (2) gives

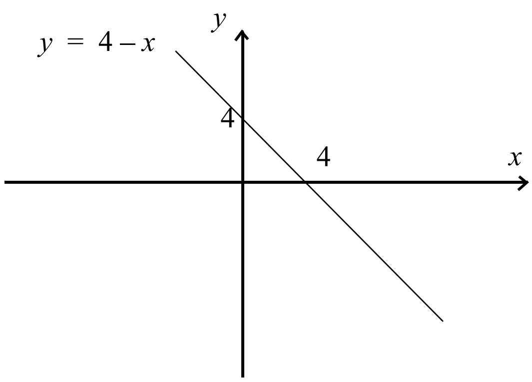
$4 = (-5) \times (-2) + c$ $\therefore 4 = 10 + c$

$\therefore c = -6$

Hence, the required equation is $y = -5x - 6$.

2. For $y = 4 - x$, when $x = 0$, $y = 4$, and when $y = 0$, $x = 4$.

Hence the sketch



2.3 Quadratic functions

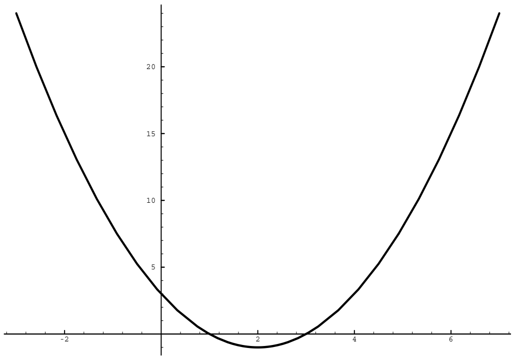
1. A **quadratic function** can be written in the form

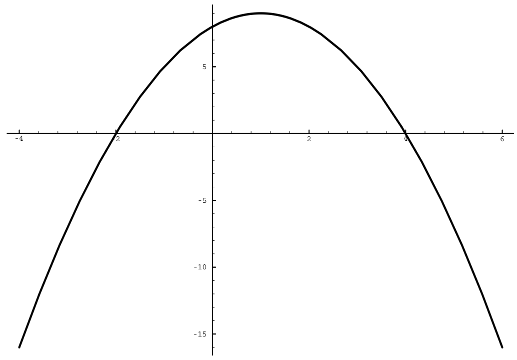
$y = f(x) = ax^2 + bx + c$, where a, b, and c are constants with $a \neq 0$

Note that, if $a = 0$, the equation becomes $y = bx + c$, and is then a linear function.

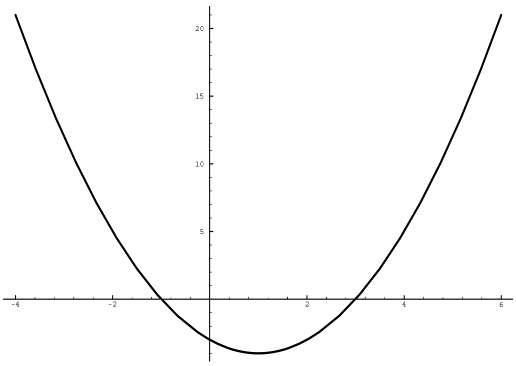
2. The graphs of these functions are **parabolas**, and have a 'cup' shape or a 'frown' shape , as can be seen in the diagrams below.

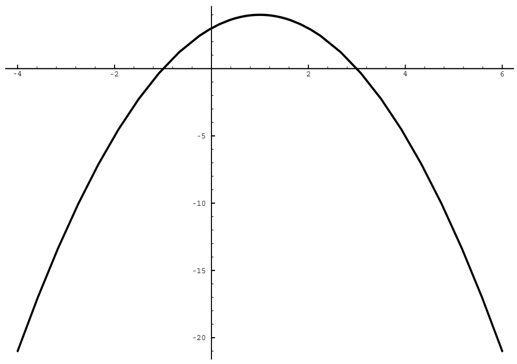
$y = x^2 - 4x + 3$ $y = -x^2 + 2x + 8$



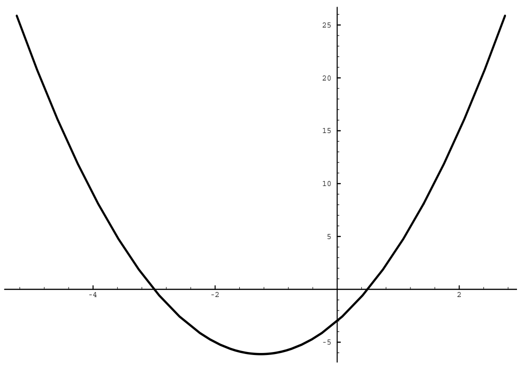


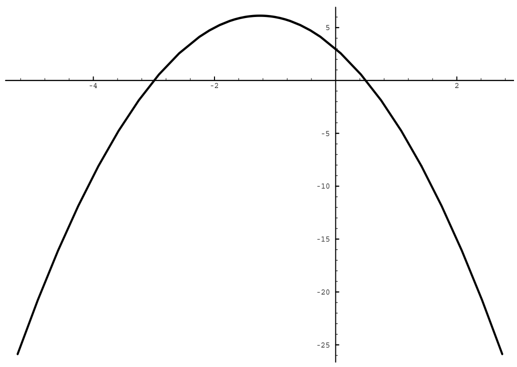
$y = x^2 - 2x - 3$ $\qquad\qquad$ $y = -x^2 + 2x + 3$





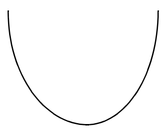
$y = 2x^2 + 5x - 3$ $\qquad\qquad$ $y = -2x^2 - 5x + 3$



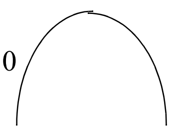


3. The sketches above indicate that the quadratic function $y = ax^2 + bx + c$ is:

   'cup' shaped if $a > 0$; $\qquad$ 'frown' shaped if $a < 0$



$a > 0$ $\qquad$ $a < 0$



Clearly, the graph has a lowest point (minimum) if $a > 0$, and a highest point (maximum) if $a < 0$. The graph is then symmetric about a vertical line drawn through this lowest (or highest) point.

4. The y-intercept for a quadratic function is found by setting $x = 0$ in the equation, i.e. when $x = 0$, $y = c$.

   The x-intercepts for a quadratic function are found by setting $y = 0$ in the equation, i.e. when $y = 0$, $ax^2 + bx + c = 0$.

   Solving this quadratic equation allows any quadratic function to be sketched without relying on plotting points. The 2 methods of solution are covered in Chapter 2.4 below.

Examples

1. For each of the following quadratic functions:

   (i) identify a, b, and c

   (ii) find the y-intercept

   (iii) determine whether the graph is 'cup' shaped or 'frown' shaped.

   (a) $y = 7x^2 - 3x + 8$ (b) $y = 4x - 3x^2$

   (c) $y = 3 + 5x^2$ (d) $y = -6x^2 + 3 - x$

1. (a) (i) $a = 7, b = -3, c = 8$.

   (ii) When $x = 0$, $y = 8$

   (iii) As $a = 7 > 0$, curve is 'cup' shaped.

   (b) (i) $a = -3, b = 4, c = 0$.

   (ii) When $x = 0$, $y = 0$

   (iii) As $a = -3 < 0$, curve is 'frown' shaped.

   (c) (i) $a = 5, b = 0, c = 3$.

   (ii) When $x = 0$, $y = 3$

   (iii) As $a = 5 > 0$, curve is 'cup' shaped.

   (d) (i) $a = -6, b = -1, c = 3$.

   (ii) When $x = 0$, $y = 3$

   (iii) As $a = -6 < 0$, curve is 'frown' shaped.

Problems

1. For each of the following quadratic functions

   (i) identify a, b, and c

   (ii) find the y-intercept

(iii) determine whether the graph is 'cup' shaped or 'frown' shaped.

(a) $y = 9x^2 + 8$ (b) $y = 4 - 3x^2$

(c) $y = 3x + 5x^2$ (d) $y = -6x^2 - 1 - x$

Answers

1. (a) (i) $a = 9, b = 0, c = 8$.
   (ii) When $x = 0$, $y = 8$
   (iii) As $a = 9 > 0$, curve is 'cup' shaped.

   (b) (i) $a = -3, b = 0, c = 4$.
   (ii) When $x = 0$, $y = 4$
   (iii) As $a = -3 < 0$, curve is 'frown' shaped.

   (c) (i) $a = 5, b = 3, c = 0$.
   (ii) When $x = 0$, $y = 0$
   (iii) As $a = 5 > 0$, curve is 'cup' shaped.

   (d) (i) $a = -6, b = -1, c = -1$.
   (ii) When $x = 0$, $y = -1$
   (iii) As $a = -6 < 0$, curve is 'frown' shaped.

2.4 Quadratic equations

1. There are 2 methods of solving the **quadratic equation**

$$ax^2 + bx + c = 0$$

The first method, **factorising**, should only be used when $a = 1$, i.e. when the quadratic equation has the form $x^2 + bx + c = 0$.

The factorisation method relies on writing the quadratic expression

$x^2 + bx + c$ (1) as the product of 2 bracketed terms, i.e.

$$x^2 + bx + c = (x + m)(x + n) \ ,$$

where the constants *m* and *n* are to be found.

Since $(x + m)(x + n) = x(x + n) + m(x + n)$

$$= x^2 + nx + mx + mn$$

$$= x^2 + (n + m)x + mn \qquad (2)$$

Comparing (1) and (2), it can be seen that the numbers *m* and *n* must satisfy both of the equations:

$mn = c$ and $n + m = b$.

In summary, $x^2 + bx + c = (x+m)(x+n)$, where the numbers **$m$ and n have a product equal to c, and a sum equal to b**.

2. Once the quadratic expression $x^2 + bx + c$ has been factorised, the quadratic equation $x^2 + bx + c = 0$ can be solved immediately. The factorised form gives

$$(x+m)(x+n) = 0$$

So, either $x + m = 0$, or $x + n = 0$

Hence, the solutions are $x = -m$, and $x = -n$

3. Before factorising and solving, the quadratic equation must have the form $x^2 + bx + c = 0$,

i.e. **the quadratic expression must be on the L.S, and the R.S. must be zero**.

For instance, the solutions to $x^2 + 8x + 8 = 1$ are found by first subtracting 1 to give $x^2 + 8x + 7 = 0$, then factorising and solving.

4. The **Quadratic Formula** is the second method of solving the quadratic equation $ax^2 + bx + c = 0$.

The Quadratic Formula provides the solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and **should be remembered**. It is used (in most cases) when $a \neq 1$.

As is the case with the factorising method, **the quadratic expression must be on the L.S, and the R.S. must be zero**.

5. The expression $b^2 - 4ac$ completely determines the number of solutions to the equation $ax^2 + bx + c = 0$.

Such an equation has either 2 solutions, 1 solution, or no solution.

If $b^2 - 4ac > 0$, there are 2 real solutions.

If $b^2 - 4ac = 0$, there is 1 real solution.

If $b^2 - 4ac < 0$, there are no real solutions.

Examples

1. Factorise, and solve the following quadratic equations

(i) $x^2 + 8x + 12 = 0$ (ii) $x^2 - 8x + 7 = 0$

(iii) $x^2 - 3x - 10 = 0$ (iv) $x^2 + 9x - 10 = 0$

(v) $x^2 + 8x + 8 = 1$ (vi) $x^2 = 4x - 4$.

2. Use the Quadratic Formula to solve the following quadratic equations

(i) $4x^2 - 8x + 3 = 0$ (ii) $6x^2 + x - 2 = 0$

(iii) $3x^2 - 12x + 9 = 0$ (iv) $2x^2 - x + 6 = 1$.

1. (i) $x^2 + 8x + 12 = 0$

[Here, $mn = 12$ (so m and n have the same sign)

and $n + m = 8$ (so m and n are both positive)

Possibilities for m and n are: 1 and 12; 2 and 6; 3 and 4.

Of these, 2 and 6 satisfy both equations.]

$\therefore x^2 + 8x + 12 = (x + 2)(x + 6) = 0$

So, there are 2 solutions: $x = -2$, and $x = -6$.

(ii) $x^2 - 8x + 7 = 0$

[Here, $mn = 7$ (so m and n have the same sign)

and $n + m = -8$ (so m and n are both negative)

Possibilities for m and n are: -1 and -7 only.]

$\therefore x^2 - 8x + 7 = (x - 1)(x - 7) = 0$

So, there are 2 solutions: $x = 1$, and $x = 7$.

(iii) $x^2 - 3x - 10 = 0$

[Here, $mn = -10$ (so m and n are of opposite sign)

and $n + m = -3$ (so m and n have a negative sum)

Possibilities for m and n are: 1 and -10; 2 and -5.

Of these, 2 and -5 satisfy both equations.]

$\therefore x^2 - 3x - 10 = (x + 2)(x - 5) = 0$

So, there are 2 solutions: $x = -2$, and $x = 5$.

(iv) $x^2 + 9x - 10 = 0$

[Here, $mn = -10$ (so m and n are of opposite sign)

and $n + m = 9$ (so m and n have a positive sum)

Possibilities for m and n are: -1 and 10; -2 and 5.

Of these, -1 and 10 satisfy both equations.]

$\therefore x^2 + 9x - 10 = (x-1)(x+10) = 0$

So, there are 2 solutions: $x = 1$, and $x = -10$.

(v) $x^2 + 8x + 8 = 1$

$\therefore x^2 + 8x + 7 = 0$ (correct form)

[Here, $mn = 7$ (so m and n have the same sign)

and $n + m = 8$ (so m and n are both positive)

Possibilities for m and n are: 1 and 7 only]

$\therefore x^2 + 8x + 7 = (x+1)(x+7) = 0$

So, there are 2 solutions: $x = -1$, and $x = -7$.

(vi) $x^2 = 4x - 4$

$\therefore x^2 - 4x + 4 = 0$ (correct form)

[Here, $mn = 4$ (so m and n have the same sign)

and $n + m = -4$ (so m and n are both negative)

Possibilities for m and n are: -1 and -4; -2 and -2.

Of these, -2 and -2 satisfy both equations.]

$\therefore x^2 - 4x + 4 = (x-2)(x-2) = 0$

So, there is one solution: $x = 2$.

2. (i) $4x^2 - 8x + 3 = 0$ ($a = 4$, $b = -8$, $c = 3$)

$$\therefore x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4 \times 4 \times 3}}{2 \times 4}$$

$$\therefore x = \frac{8 \pm \sqrt{64 - 48}}{8} = \frac{8 \pm \sqrt{16}}{8} = \frac{8 \pm 4}{8}$$

$$\therefore x = \frac{12}{8} = \frac{3}{2}\text{, and } x = \frac{4}{8} = \frac{1}{2}.$$

So, there are 2 solutions: $x = \frac{3}{2}$, and $x = \frac{1}{2}$.

(ii) $6x^2 + x - 2 = 0$; ($a = 6$, $b = 1$, $c = -2$)

$$\therefore x = \frac{-1 \pm \sqrt{1^2 - 4 \times 6 \times (-2)}}{2 \times 6}$$

$$\therefore x = \frac{-1 \pm \sqrt{1+48}}{12} = \frac{-1 \pm \sqrt{49}}{12} = \frac{-1 \pm 7}{12}$$

$$\therefore x = \frac{6}{12} = \frac{1}{2}, \text{ and } x = \frac{-8}{12} = \frac{-2}{3}.$$

So, there are 2 solutions: $x = \frac{1}{2}$, and $x = \frac{-2}{3}$.

(iii) $3x^2 - 12x + 9 = 0$ $(a = 3, b = -12, c = 9)$

$$\therefore x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4 \times 3 \times 9}}{2 \times 3}$$

$$\therefore x = \frac{12 \pm \sqrt{144 - 108}}{6} = \frac{12 \pm \sqrt{36}}{6} = \frac{12 \pm 6}{6}$$

$$\therefore x = \frac{18}{6} = 3, \text{ and } x = \frac{6}{6} = 1.$$

So, there are 2 solutions: $x = 3$ and $x = 1$.

NOTE: The original equation can be divided by 3 to give $x^2 - 4x + 3 = 0$.

As the coefficient of x is now 1, the quadratic can be factorised, i.e.

$$x^2 - 4x + 3 = (x-3)(x-1) = 0$$

Hence, again the solutions are $x = 3$ and $x = 1$.

(iv) $2x^2 - x + 6 = 1$

$\therefore 2x^2 - x + 5 = 0$ (correct form)

$(a = 2, b = -1, c = 5)$

$$\therefore x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 2 \times 5}}{2 \times 2}$$

$$\therefore x = \frac{1 \pm \sqrt{1-40}}{4} = \frac{1 \pm \sqrt{-39}}{4}$$

Since $\sqrt{-39}$ is not a real number, there is no solution.

Problems

1. Factorise, and solve the following quadratic equations

(i) $x^2 + 13x + 12 = 0$ (ii) $x^2 - 15x + 26 = 0$

(iii) $x^2 - 3x - 4 = 0$ (iv) $x^2 + 13x - 14 = 0$

(v) $x^2 + 6x + 10 = 1$ (vi) $x^2 = 6x - 5$.

2. Use the Quadratic Formula to solve the following quadratic equations

(i) $4x^2 + 4x + 1 = 0$ (ii) $5x^2 + 4x - 1 = 0$

(iii) $3x^2 - 2x - 1 = 0$ (iv) $2x^2 + 3 = 3x$.

Answers

1. (i) $x = -1$ and $x = -12$. (ii) $x = 2$ and $x = 13$.

(iii) $x = -1$ and $x = 4$. (iv) $x = -14$ and $x = 1$.

(v) $x = -3$. (vi) $x = 1$, and $x = 5$.

2. (i) $x = \frac{-1}{2}$. (ii) $x = \frac{1}{5}$, and $x = -1$.

(iii) $x = \frac{-1}{3}$ and $x = 1$. (iv) No real solutions.

2.5 Sketching quadratic functions

1. Whether obtained by factorising or the Quadratic Formula, the solutions of the quadratic equation $ax^2 + bx + c = 0$ completely determine the x-intercepts of the quadratic function $y = ax^2 + bx + c$.

   In conjunction with the y-intercepts, the 'cup' or 'frown' shape, and the axis of symmetry, the x-intercepts can be used to sketch any quadratic function quickly. **All quadratics can be sketched without either plotting points or using a graphics package or calculator.**

2. The x value of the axis of symmetry of the parabola

$$y = ax^2 + bx + c$$

   is the **average value of the x-intercepts**. In general, since the x-intercepts are given by the Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

   these intercepts can be labelled as

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The average of x_1 and x_2 is thus

$$\frac{1}{2}(x_1 + x_2) = \frac{1}{2}\left(\frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

$$= \frac{1}{2}\left(\frac{-b-b}{2a}\right) = \frac{1}{2}\left(\frac{-2b}{2a}\right) = \frac{-b}{2a}.$$

Hence, the axis of symmetry occurs at $x = \dfrac{-b}{2a}$.

3. To sketch the quadratic function $y = ax^2 + bx + c$:

I **Find the y-intercept** (occurs when $x = 0$)

II **Find the x-intercept** (occurs when $y = 0$)

III **Note the sign of *a*** ($a > 0$: 'cup', and $a < 0$: 'frown')

IV **Find the co-ordinates of the highest or lowest point**

(occurs when $x = \dfrac{-b}{2a}$)

Examples

1. For each of the following quadratic functions

(i) find the y-intercept

(ii) find the x-intercepts

(iii) determine whether the curve has a highest or lowest point

(iv) find the co-ordinates of the highest or lowest point

(v) sketch the curve.

(a) $y = x^2 - 2x - 15$ (b) $y = -x^2 + 8x - 16$

(a) $y = x^2 - 2x - 15$

(i) When $x = 0$, $y = -15$

(ii) When $y = 0$, $x^2 - 2x - 15 = 0$

Factorising gives $x^2 - 2x - 15 = (x+3)(x-5) = 0$

$\therefore x = -3$, and $x = 5$. (two x-intercepts)

(iii) As $a = 1$ ($a > 0$), the curve is 'cup' shaped, and has a lowest point.

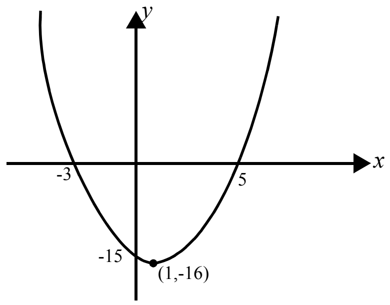
(iv) As $a = 1$, $b = -2$, and $c = -15$, for the axis of symmetry, $x = \frac{-b}{2a} = \frac{-(-2)}{2 \times 1} = \frac{2}{2} = 1$.

(alternatively, the average of the x-intercepts is $\frac{-3+5}{2} = \frac{2}{2} = 1$)

When $x = 1$, $y = 1 - 2 - 15 = -16$.

So, the lowest point is $(1, -16)$.

(v)



NOTE: The only points required to be shown on the graph are the x and y-intercepts, and the lowest (or highest) point.

(b) $y = -x^2 + 8x - 16$

(i) When $x = 0, y = -16$

(ii) When $y = 0, -x^2 + 8x - 16 = 0$

Multiplying by -1 gives $x^2 - 8x + 16 = 0$

Factorising gives $x^2 - 8x + 16 = (x-4)(x-4) = 0$

$\therefore x = 4$. (one x-intercept)

(iii) As $a = -1$ ($a < 0$), the curve is 'frown' shaped, and has a highest point.

(iv) As $a = -1$, $b = 8$, and $c = -16$,

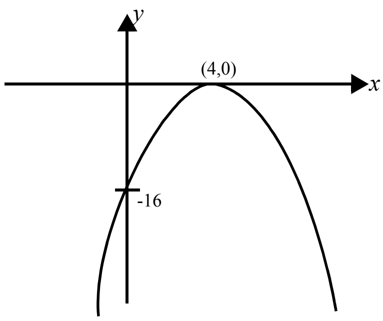
for the axis of symmetry, $x = \frac{-b}{2a} = \frac{-8}{2 \times (-1)} = \frac{-8}{-2} = 4$.

(alternatively, the average of the x-intercepts is $\frac{4+4}{2} = \frac{8}{2} = 4$)

When $x = 4$, $y = 0$ (x-intercept).

So, the highest point is $(4, 0)$.

(v)



Problems

1. For each of the following quadratic functions
   - (i) find the y-intercept
   - (ii) find the x-intercepts
   - (iii) determine whether the curve has a highest or lowest point
   - (iv) find the co-ordinates of the highest or lowest point
   - (v) sketch the curve.

   (a) $y = x^2 - 3x - 4$ (b) $y = -x^2 + 8x - 12$

Answers

1. (a) (i) $y = -4$.

(ii) $x = -1$, and $x = 4$. (two x-intercepts)

(iii) As $a = 1 > 0$, curve is 'cup' shaped, and has a lowest point

(iv) The lowest point is $(3/2, -25/4)$.

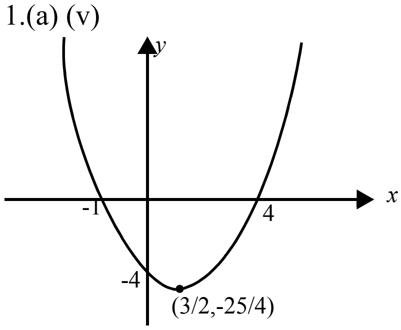
1. (b) (i) $y = -12$

(ii) $x = 2$, and $x = 6$ (two x-intercepts)

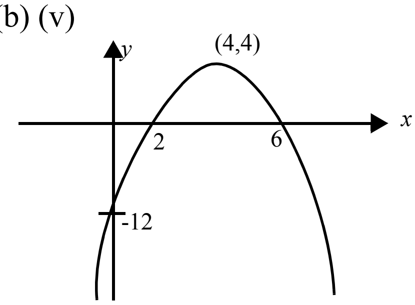
(iii) As $a = -1 < 0$, curve is 'frown' shaped, and has a highest point

(iv) The highest point is $(4, 4)$.

1.(a) (v)



1.(b) (v)



2.6 Special quadratic forms

1. For quadratics, factorisation can be simplified in 2 special cases; **perfect squares** and **the difference of two squares**.

   I Perfect squares

   Since $(a+b)^2 = (a+b)(a+b) = a(a+b)+b(a+b)$

   $= a^2+ab+ab+b^2 \qquad = a^2+2ab+b^2$,

   the perfect square expansion is

   $$(a+b)^2 = a^2+2ab+b^2 .$$

   Similarly, $\qquad (a-b)^2 = a^2-2ab+b^2 .$

   Note that there are 3 terms in the perfect square expansion. The first and third terms are simply the squares of each term in the original brackets, while the second term is twice the product of the terms in the original brackets.

   For instance, $(x+3)^2 = x^2+2\times x\times 3+3^2 = x^2+6x+9$,

   and $\quad (x-5)^2 = x^2+2\times x\times(-5)+(-5)^2 = x^2-10x+25$.

   II Difference of two squares

   Since $(a+b)(a-b) = a(a-b)+b(a-b)$

   $= a^2-ab+ab-b^2 \qquad = a^2-b^2$,

   the difference of two squares expansion is

   $$(a+b)(a-b) = a^2-b^2 .$$

   Note that there are 2 terms in the difference of two squares expansion. The R.S. is the difference of the square of the first term in each of the original brackets and the square of the second term in each of the original brackets.

   For instance, $(x+4)(x-4) = x^2-4^2 = x^2-16$,

   and $\quad (6+x)(6-x) = 6^2-x^2 = 36-x^2$.

   A difference of two squares is the simplest quadratic form to factorise.

   For instance, $x^2-64 = x^2-8^2 = (x+8)(x-8)$.

   Note, however, that the sum of two squares a^2+b^2 does not factorise.

2. Many quadratics can be written as a difference of two squares using the technique known as **completing the square**. The technique relies on the following result:

   $$\left(x+\frac{b}{2a}\right)^2 = x^2+2\times x\times\frac{b}{2a}+\left(\frac{b}{2a}\right)^2 \qquad \text{(perfect square)}$$

$$\therefore\left(x+\frac{b}{2a}\right)^2 = x^2+\frac{bx}{a}+\left(\frac{b}{2a}\right)^2$$

$$\therefore\left(x+\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2 = x^2+\frac{bx}{a} \qquad \left[\ -\left(\frac{b}{2a}\right)^2\right]$$

Switching the L.S. and the R.S. gives

$$x^2+\frac{bx}{a} = \left(x+\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2 \qquad \text{(a difference of two squares)}$$

The above result can be used to complete the square for the quadratic expression ax^2+bx+c as follows:

COMPLETING THE SQUARE

STEP I Take out the co-efficient a as a factor

i.e. $ax^2+bx+c = a\left[x^2+\frac{bx}{a}+\frac{c}{a}\right]$.

STEP II Replace $x^2+\frac{bx}{a}$ by $\left(x+\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2$

i.e. $ax^2+bx+c = a\left[x^2+\frac{bx}{a}+\frac{c}{a}\right] = a\left[\left(x+\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right]$

STEP III Simplify the terms $-\left(\frac{b}{2a}\right)^2+\frac{c}{a}$

i.e. $-\left(\frac{b}{2a}\right)^2+\frac{c}{a} = \frac{-b^2}{4a^2}+\frac{c}{a} \qquad = \frac{-b^2+4ac}{4a^2}$

i.e. $ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2+\frac{-b^2+4ac}{4a^2}\right]$

STEP IV Multiply through by a

i.e. $ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2+\frac{-b^2+4ac}{4a}$

Note: The important part of completing the square occurs in Step II, and if $a = 1$, the process is much simpler, as shown in the following example.

For x^2+6x-1, Step I is unnecessary, and in Step II the first two terms, i.e. x^2+6x are written as $(x+3)^2-3^2$.

Note that the term with *x* in the brackets is half the co-efficient of *x* in the original quadratic (here, half of +6 is +3). **This term is then squared, and subtracted** (here, -3^2).

Hence, $x^2+6x-1 = (x+3)^2-3^2-1$

$= (x+3)^2-9-1 \qquad = (x+3)^2-10$,

and the process is complete.

3. The **Quadratic Formula** introduced in Chapter 2.4 is derived using the technique for completing the square. The equation obtained in Step IV above is

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{-b^2 + 4ac}{4a}.$$

As L.S. = R.S. in the above equation, the solutions to the quadratic equation

$$ax^2 + bx + c = 0$$

are the same as the solutions to the equation

$$a\left(x + \frac{b}{2a}\right)^2 + \frac{-b^2 + 4ac}{4a} = 0.$$

Solving for x gives

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a} \qquad \left[\ + \frac{b^2 - 4ac}{4a}\right]$$

$$\therefore \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \qquad [\ \div a]$$

$$\therefore x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \qquad [\pm\sqrt{\ }\ ]$$

$$\therefore x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \qquad \text{(simplifying)}$$

$$\therefore x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \qquad \left[\ -\frac{b}{2a}\right]$$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{(simplifying),}$$

which is the required Quadratic Formula.

Examples

1. Expand, and simplify

   (i) $(x-9)^2$ (ii) $(x+2)^2$

   (iii) $(x+10)(x-10)$ (iv) $(2x+1)(2x-1)$

2. Factorise

   (i) $x^2 - 49$ (ii) $4x^2 - 81$ (iii) $121 - x^2$

3. Complete the square for the following quadratic expressions

(i) $x^2 + 10x$ (ii) $x^2 - 14x$ (iii) $x^2 + 8x - 3$

(iv) $x^2 - 5x + 2$ (v) $2x^2 + 6x - 1$.

1. (i) $(x-9)^2 = x^2 - 2 \times 9 \times x + (-9)^2$

$= x^2 - 18x + 81$.

(ii) $(x+2)^2 = x^2 + 2 \times 2 \times x + 2^2$

$= x^2 + 4x + 4$

(iii) $(x+10)(x-10) = x^2 - 10^2 = x^2 - 100$

(iv) $(2x+1)(2x-1) = (2x)^2 - 1^2 = 4x^2 - 1$.

2. (i) $x^2 - 49 = x^2 - 7^2 = (x+7)(x-7)$

(ii) $4x^2 - 81 = (2x)^2 - 9^2 = (2x+9)(2x-9)$

(iii) $121 - x^2 = 11^2 - x^2 = (11+x)(11-x)$.

3. (i) For $x^2 + 10x$, since $\dfrac{10}{2} = 5$,

$x^2 + 10x = (x+5)^2 - 5^2 \qquad = (x+5)^2 - 25$

(ii) For $x^2 - 14x$, since $\dfrac{-14}{2} = -7$,

$x^2 - 14x = (x-7)^2 - 7^2 \qquad = (x-7)^2 - 49$

(iii) For $x^2 + 8x - 3$, since $\dfrac{8}{2} = 4$,

$x^2 + 8x - 3 = (x+4)^2 - 4^2 - 3 \qquad = (x+4)^2 - 16 - 3$

$x^2 + 8x - 3 = (x+4)^2 - 19$

(iv) For $x^2 - 5x + 2$, since $\dfrac{-5}{2}$ doesn't simplify,

$$x^2 - 5x + 2 = \left(x - \frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 2 \qquad = \left(x - \frac{5}{2}\right)^2 - \frac{25}{4} + 2$$

$$= \left(x - \frac{5}{2}\right)^2 + \frac{8-25}{4} \qquad = \left(x - \frac{5}{2}\right)^2 - \frac{17}{4}$$

(v) For $2x^2 + 6x - 1$, first taking out a factor of 2 gives

$$2x^2 + 6x - 1 = 2\left[x^2 + 3x - \frac{1}{2}\right]$$

Now, since $\frac{3}{2}$ doesn't simplify,

$$2\left[x^2 + 3x - \frac{1}{2}\right] = 2\left[\left(x + \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 - \frac{1}{2}\right]$$

$$= 2\left[\left(x + \frac{3}{2}\right)^2 - \frac{9}{4} - \frac{1}{2}\right] \qquad = 2\left[\left(x + \frac{3}{2}\right)^2 - \frac{9}{4} - \frac{2}{4}\right]$$

$$= 2\left[\left(x + \frac{3}{2}\right)^2 - \frac{11}{4}\right] \qquad = 2\left(x + \frac{3}{2}\right)^2 - \frac{11}{2}$$

Problems

1. Expand, and simplify

   (i) $(x - 12)^2$ (ii) $(2x + 3)^2$

   (iii) $(x + 12)(x - 12)$ (iv) $(3x + 1)(3x - 1)$

2. Factorise

   (i) $x^2 - 1$ (ii) $9x^2 - 16$ (iii) $169 - x^2$

3. Complete the square for the following quadratic expressions

   (i) $x^2 + 12x$ (ii) $x^2 - 16x$ (iii) $x^2 + 4x - 7$

   (iv) $x^2 - 7x + 12$ (v) $2x^2 + 8x - 1$.

Answers

1. (i) $x^2 - 24x + 144$ (ii) $4x^2 + 12x + 9$

   (iii) $x^2 - 144$ (iv) $9x^2 - 1$.

2. (i) $(x + 1)(x - 1)$ (ii) $(3x + 4)(3x - 4)$

   (iii) $(13 + x)(13 - x)$.

3. (i) $(x + 6)^2 - 36$ (ii) $(x - 8)^2 - 64$

   (iii) $(x + 2)^2 - 11$ (iv) $\left(x - \frac{7}{2}\right)^2 - \frac{1}{4}$

   (v) $2(x + 2)^2 - 9$.

3

Exponentials and logarithms

Introduction

Two of the most widely applicable functions in mathematics are the exponential function and the logarithmic function. Exponential growth is common in biology, and exponential decay is prevalent in both physics and chemistry. Logarithmic functions occur in chemistry and geology, and are used to solve exponential equations.

This topic covers the index laws, the use of the base *e* for both exponentials and logarithms, the rules governing exponentials and logarithms, the inverse relationship of the two functions, and solutions of both exponential and logarithmic equations. After studying this topic, you should be able to:

- understand and apply the index laws;
- understand and use the base *e*;
- use the rules for exponentials to simplify expressions;
- use the rules for logarithms to simplify expressions;
- solve exponential equations;
- solve logarithmic equations.

3.1 Index laws

1. An **exponential** is formed when a number (the base) is raised to a power (also called the exponent or **index**). For instance, in 4^2, the base is 4 and the index is 2. Similarly, in x^3, the base is x and the index is 3.
2. The rules governing exponentials are called the **index laws**.There are 7 index laws, as follows.

I $a^m \times a^n = a^{m+n}$ **(to multiply, add indices)**

e.g. $4^2 \times 4^3 = 4^5$, and $x^2 \times x^4 = x^6$.

II $\dfrac{a^m}{a^n} = a^{m-n}$ **(to divide, subtract indices)**

e.g. $\dfrac{2^5}{2^2} = 2^3$, and $\dfrac{x^8}{x^3} = x^5$.

III $(ab)^n = a^n b^n$ **(all terms in brackets raised to the power *n*)**

e.g. $(2 \times 5)^2 = 2^2 \times 5^2$, and $(xy)^4 = x^4 y^4$.

IV $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$ **(all terms in brackets raised to the power *n*)**

e.g. $\left(\dfrac{3}{5}\right)^2 = \dfrac{3^2}{5^2}$, and $\left(\dfrac{x}{y}\right)^3 = \dfrac{x^3}{y^3}$.

V $(a^m)^n = a^{mn}$ **(to find a power of a power, multiply powers)**

e.g. $(2^3)^4 = 2^{12}$, and $(x^2)^3 = x^6$.

VI $a^0 = 1$ and $a^1 = a$ **(by definition)**

e.g. $x^0 = 1$, $3x^0 = 3 \times 1 = 3$, and $4x^1 = 4x$.

VII $a^{-n} = \dfrac{1}{a^n}$ **(switching a power from top to bottom)**

e.g. $4^{-2} = \dfrac{1}{4^2}$, and $x^{-3} = \dfrac{1}{x^3}$

NOTE: The index laws apply for both $a \neq 0$ and $b \neq 0$.

3. Law VII is used to convert negative powers of a variable to positive powers of that variable. In general, a power in the top line changes sign when switched to the bottom line. Similarly, a power in the bottom line changes sign when switched to the top line. For instance,

   $\dfrac{1}{x^2} = x^{-2}$, and $\dfrac{1}{x^{-2}} = x^2$.

4. When **sums** of variables appear in an expression, law VII cannot be used, e.g. $\dfrac{x^{-1}}{4+x^{-3}} \neq \dfrac{4+x^3}{x^1}$, i.e. the sum $4 + x^{-3}$ cannot be switched to the top line. Such expressions can be simplified (and written in terms of non-negative indices) as follows:

   first locate the most negative power of *x*, say x^{-n};

   then multiply by $\dfrac{x^n}{x^n}$.

   For instance, in $\dfrac{x^{-1}}{4+x^{-3}}$, the most negative power of *x* is x^{-3}.

   Multiplying by $\dfrac{x^3}{x^3}$ gives $\dfrac{x^{-1}}{4+x^{-3}} = \dfrac{x^3x^{-1}}{x^3(4+x^{-3})} = \dfrac{x^2}{4x^3+x^3x^{-3}}$

   $$= \frac{x^2}{4x^3+x^0} \qquad = \frac{x^2}{4x^3+1}.$$

Examples

1. Simplify, and express all answers with non-negative indices

(i) $\dfrac{x^4x^3}{x^2x^8}$	(ii) $\dfrac{5x^3y^{-1}}{x^{-2}y^4}$	(iii) $\left(\dfrac{3x^{-2}}{y}\right)(2x^3y^{-4})$
(iv) $\left(\dfrac{x^2}{xy^{-2}}\right)^3$	(v) $\dfrac{x^{-2}+2x^{-4}}{x-3x^{-3}}$	(vi) $\dfrac{7x^2}{x^3+4x^{-1}}$.

1. (i) $\dfrac{x^4x^3}{x^2x^8} = \dfrac{x^7}{x^{10}} = x^{7-10} = x^{-3} = \dfrac{1}{x^3}$

(ii) $\dfrac{5x^3y^{-1}}{x^{-2}y^4} = \dfrac{5x^3x^2}{y^1y^4}$ (switching lines for y^{-1} and x^{-2})

$= \dfrac{5x^5}{y^5}$

(iii) $\left(\dfrac{3x^{-2}}{y}\right)(2x^3y^{-4}) = \dfrac{3 \times 2x^{-2}x^3y^{-4}}{y} = \dfrac{6x^1y^{-4}}{y} = \dfrac{6x}{yy^4}$

$= \dfrac{6x}{y^5}$

(iv) $\left(\dfrac{x^2}{xy^{-2}}\right)^3 = \dfrac{x^{2\times 3}}{x^3y^{-2\times 3}} = \dfrac{x^6}{x^3y^{-6}} = \dfrac{x^6y^6}{x^3} = x^{6-3}y^6$

$= x^3y^6$

(v) For $\dfrac{x^{-2}+2x^{-4}}{x-3x^{-3}}$, a sum is involved. The most negative power of x is x^{-4}. Multiplying by $\dfrac{x^4}{x^4}$ gives

$$\frac{x^{-2}+2x^{-4}}{x-3x^{-3}} = \frac{x^4(x^{-2}+2x^{-4})}{x^4(x-3x^{-3})} = \frac{x^4x^{-2}+2x^4x^{-4}}{x^4x+-3x^4x^{-3}}$$

$$= \frac{x^2+2x^0}{x^5-3x^1} \qquad = \frac{x^2+2}{x^5-3x}$$

(vi) For $\dfrac{7x^2}{x^3+4x^{-1}}$, a sum is involved. The most negative power of x is x^{-1}. Multiplying by $\dfrac{x^1}{x^1}$, i.e. $\dfrac{x}{x}$ gives

$$\frac{7x^2}{x^3+4x^{-1}} = \frac{x \times 7x^2}{x(x^3+4x^{-1})} = \frac{7x^3}{xx^3+4xx^{-1}} = \frac{7x^3}{x^4+4x^0}$$

$$= \frac{7x^3}{x^4+4}.$$

Problems

1. Simplify, and express all answers with non-negative indices

(i) $\dfrac{x^6x^3}{x^2x^9}$ (ii) $\dfrac{6x^{-4}y}{x^{-2}y^{-3}}$ (iii) $\left(\dfrac{x^{-2}}{4y^2}\right)(2x^5y^{-2})$

(iv) $\left(\dfrac{3x^{-1}}{xy^{-3}}\right)^2$ (v) $\dfrac{x^2 + 4x^{-3}}{x^3 - 3}$ (vi) $\dfrac{7x^{-2}}{5x^{-1} + x^{-3}}$.

Answers

1. (i) $\dfrac{1}{x^2}$ (ii) $\dfrac{6y^4}{x^2}$ (iii) $\dfrac{x^3}{2y^4}$

(iv) $\dfrac{9y^6}{x^4}$ (v) $\dfrac{x^5 + 4}{x^6 - 3x^3}$ (vi) $\dfrac{7x}{5x^2 + 1}$.

3.2 Fractional indices

1. Fractional indices are equivalent to n$^{\text{th}}$ roots:

e.g. $a^{1/2} = \sqrt{a}$ (square root of a);

$a^{1/3} = \sqrt[3]{a}$ (cube root of a);

$a^{1/4} = \sqrt[4]{a}$ (fourth root of *a*), etc.

In general $a^{\frac{1}{n}} = \sqrt[n]{a}$ (n$^{\text{th}}$ root of *a*).

Two very useful consequences of this are:

$\sqrt{x} = x^{1/2}$ and $\dfrac{1}{\sqrt{x}} = \dfrac{1}{x^{1/2}} = x^{-1/2}$.

2. The 7 index laws apply to fractional indices, and can be used to simplify expressions containing square roots.

To achieve such simplifications the following rules are required.

For both $a \geq 0$ and $b \geq 0$,

$\sqrt{a^2} = a$ (as $\sqrt{a^2} = (a^2)^{1/2} = a^{2 \times 1/2} = a^1 = a$)

$\sqrt{ab} = \sqrt{a}\sqrt{b}$ (as $\sqrt{ab} = (ab)^{1/2} = a^{1/2}b^{1/2} = \sqrt{a}\sqrt{b}$)

$\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$ (as $\sqrt{\dfrac{a}{b}} = \left(\dfrac{a}{b}\right)^{1/2} = \dfrac{a^{1/2}}{b^{1/2}} = \dfrac{\sqrt{a}}{\sqrt{b}}$) (here $b > 0$).

Examples

1. Simplify, and express all answers with non-negative indices

(i) $\dfrac{x^{4/3}x}{x^{-2/3}}$ (ii) $\dfrac{(x^{2/3})^{3/4}}{x^{-5/2}}$.

2. Simplify

(i) $(5 + 2\sqrt{6})(5 - 2\sqrt{6})$ (ii) $\sqrt{x}(3\sqrt{x} - \sqrt{4x})$.

1. (i) $\dfrac{x^{4/3}x}{x^{-2/3}} = x^{2/3}x^{4/3}x$ (switching $x^{-2/3}$)

$= x^{6/3}x \qquad = x^2x = x^3$

(ii) $\dfrac{(x^{2/3})^{3/4}}{x^{-5/2}} = \dfrac{x^{2/3 \times 3/4}}{x^{-5/2}} = \dfrac{x^{1/2}}{x^{-5/2}}$ (as $\dfrac{2}{3} \times \dfrac{3}{4} = \dfrac{2}{4} = \dfrac{1}{2}$)

$= x^{1/2}x^{5/2}$ (switching $x^{-5/2}$)

$= x^3$ (as $\dfrac{1}{2} + \dfrac{5}{2} = \dfrac{6}{2} = 3$).

2. (i) $(5 + 2\sqrt{6})(5 - 2\sqrt{6}) = 5(5 - 2\sqrt{6}) + 2\sqrt{6}(5 - 2\sqrt{6})$

$= 25 - 10\sqrt{6} + 10\sqrt{6} - 4\sqrt{6}\sqrt{6}$

$= 25 - 4 \times 6 \qquad = 25 - 24 = 1$

(ii) $\sqrt{x}(3\sqrt{x} - \sqrt{4x}) = 3\sqrt{x}\sqrt{x} - \sqrt{x}\sqrt{4x}$

$= 3x - 2\sqrt{x}\sqrt{x}$ (as $\sqrt{4x} = \sqrt{4}\sqrt{x} = 2\sqrt{x}$)

$= 3x - 2x = x$.

Problems

1. Simplify, and express all answers with non-negative indices

(i) $\dfrac{x^{7/4}x^{-1}}{x^{-1/4}}$ (ii) $\dfrac{(x^{5/3})^{3/4}}{x^{-3/4}}$.

2. Simplify

(i) $(8 + 5\sqrt{11})(8 - 5\sqrt{11})$ (ii) $3\sqrt{x}(6\sqrt{x} - \sqrt{9x})$.

Answers

1. (i) $\dfrac{x^{7/4}x^{-1}}{x^{-1/4}} = x^{7/4}x^{1/4}x^{-1}$ (switching $x^{-1/4}$)

$= x^2 x^{-1} \qquad (\text{as } \frac{7}{4} + \frac{1}{4} = \frac{8}{4} = 2)$

$= x^1 = x$

(ii) $\dfrac{(x^{5/3})^{3/4}}{x^{-3/4}} = \dfrac{x^{5/3 \times 3/4}}{x^{-3/4}} = \dfrac{x^{5/4}}{x^{-3/4}} \qquad (\text{as } \frac{5}{3} \times \frac{3}{4} = \frac{5}{4})$

$= x^{5/4} x^{3/4} \qquad (\text{switching } x^{-3/4})$

$= x^2 \qquad (\text{as } \frac{5}{4} + \frac{3}{4} = \frac{8}{4} = 2).$

2. (i) $(8 + 5\sqrt{11})(8 - 5\sqrt{11}) = 8(8 - 5\sqrt{11}) + 5\sqrt{11}(8 - 5\sqrt{11})$

$= 64 - 40\sqrt{11} + 40\sqrt{11} - 25\sqrt{11}\sqrt{11}$

$= 64 - 25 \times 11 \qquad = 64 - 275 = -211$

(ii) $3\sqrt{x}(6\sqrt{x} - \sqrt{9x}) = 18\sqrt{x}\sqrt{x} - 3\sqrt{x}\sqrt{9x}$

$= 18x - (3\sqrt{x})(3\sqrt{x}) \qquad (\text{as } \sqrt{9x} = \sqrt{9}\sqrt{x} = 3\sqrt{x})$

$= 18x - 9x = 9x$.

3.3 Exponential functions

1. A simple exponential function has the form $y = a^x$, where the base a is greater than zero, and the variable x is the power or exponent. Common bases are $a = 2$ (in computing) and $a = 10$ (in the physical sciences).
2. The number ***e*** is used as the base for exponentials in almost all mathematics. To 3 decimal places, $e \approx 2.718$. This choice of base is important in the calculus of exponentials.
3. The standard exponential function can be written in one of the two forms:

   $y = ke^x$ (exponential growth); or

   $y = ke^{-x}$ (exponential decay), where k is a constant.

   The fundamentals of exponential growth and decay can be observed when the graphs of the functions $y = e^x$ and $y = e^{-x}$ are sketched on the same set of axes.

   Both graphs have:

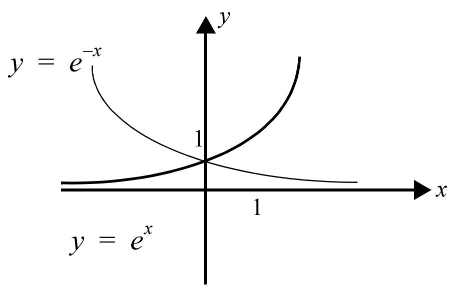
   I **no x-intercept;**

   II **a y-intercept of 1.**

For $y = e^x$, y increases as x increases (growth),

whereas for $y = e^{-x}$, y decreases as x increases (decay).

More generally, the same properties are observed for the graphs of both $y = ke^x$ and $y = ke^{-x}$, except that the y-intercept is then k. This follows from the fact that, when $x = 0$, $y = ke^x = ke^0 = k \times 1 = k$. Similarly, when $x = 0$, $y = ke^{-x} = ke^0 = k \times 1 = k$.



4. Expressions involving exponentials can be simplified using the following three rules, all of which follow from the index laws:

$e^a \times e^b = e^{a+b}$ (Product of powers)

$\dfrac{e^a}{e^b} = e^{a-b}$ (Quotient of powers)

$(e^a)^b = e^{ab}$ (Power of a power)

$\dfrac{1}{e^a} = e^{-a}$ (Negative powers)

Examples

1. Simplify (by writing as a single exponential)

(i) $\dfrac{e^x}{e^{x-2}}$ (ii) $\dfrac{(e^{x+3})^4}{e^{x+2}}$ (iii) $e^{2x-5}(e^{3-x})^2$.

2. Expand and simplify

(i) $(e^x + e^{-x})^2$ (ii) $(e^{2x} + e^{-2x})(e^{2x} - e^{-2x})$.

1. (i) $\dfrac{e^x}{e^{x-2}} = e^{x-(x-2)} = e^{x-x+2} = e^2$

(ii) $\dfrac{(e^{x+3})^4}{e^{x+2}} = \dfrac{e^{4x+12}}{e^{x+2}} = e^{4x+12-x-2} = e^{3x+10}$

(iii) $e^{2x-5}(e^{3-x})^2 = e^{2x-5} \times e^{6-2x}$

$= e^{2x-5+6-2x} = e^{-5+6} = e\,.$

2. (i) $(e^x + e^{-x})^2 = (e^x)^2 + 2e^x e^{-x} + (e^{-x})^2$

$= e^{2x} + 2e^0 + e^{-2x} = e^{2x} + 2 + e^{-2x}$

(ii) $(e^{2x} + e^{-2x})(e^{2x} - e^{-2x}) = (e^{2x})^2 - (e^{-2x})^2$

$= e^{4x} - e^{-4x}\,.$

Problems

1. Simplify (by writing as a single exponential)

(i) $\dfrac{e^{3x}}{e^{2x-3}}$ (ii) $\dfrac{(e^{x+1})^3}{e^{5-3x}}$ (iii) $e^{6x-5}(e^{3-2x})^3\,.$

2. Expand and simplify

(i) $(e^x - e^{-x})^2$ (ii) $e^{3x}(e^{2x} + e^{-2x})\,.$

Answers

1. (i) e^{x+3} (ii) e^{6x-2} (iii) $e^4\,.$

2. (i) $e^{2x} - 2 + e^{-2x}$ (ii) $e^{5x} + e^x\,.$

3.4 Logarithmic functions

1. If $y = a^x$, then ***x* is called the logarithm of *y* to base *a***, and denoted

   $x = \log_a y\,.$

   The expression $\log_a y$ is thus the power of *a* needed to obtain *y*.

   For instance, $\log_2 8$ is equal to 3, because $2^3 = 8\,.$

   Similarly, $\log_2\left(\dfrac{1}{4}\right) = -2$, because $2^{-2} = \dfrac{1}{4}\,.$

2. As is the case with exponentials, the base *e* is the usual base for logarithms in mathematics.Using base *e*, the natural logarithm of *x* is always written as $\ln x\,.$

3. If $y = \ln x$, then *y* is the power to which *e* must be raised to give *x*.

So, $e^y = x$ and $y = \ln x$ mean the same thing.

More importantly, $e^{\ln x} = x$, and $\ln(e^x) = x$.

Thus, the operations e and ln are inverse operations, in the same way as multiplying and dividing by 3 are inverse operations.

4. The graph of the function $y = \ln x$ can be sketched by analogy with the exponential function. Since $e^y = x$ and $y = \ln x$ mean the same thing, the required sketch is the sketch of the equation $e^y = x$, i.e. $x = e^y$.

   Now, when $x = 0$, $e^y = 0$, which has no solution.

   So, the graph has no y-intercept.

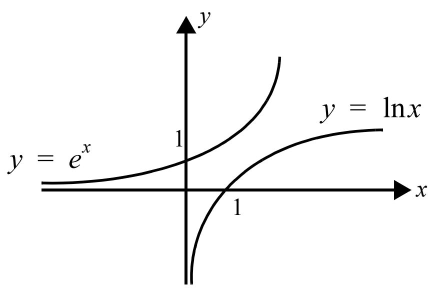
   When $y = 0$, $x = e^0 = 1$.

   So, the graph has an x-intercept at $x = 1$.

   Also, as x increases, y increases. This can be checked by plotting selected points on the curve $y = \ln x$,

   e.g. when $x = -1$, $y = e^{-1} \approx 0.3$, and when $x = 1$, $y = e^1 \approx 2.718$.

   The graphs of both $y = e^x$ and $y = \ln x$ are sketched on the same set of axes in the diagram below.



5. The properties of logarithms are very important. Stated in base e they are as follows;

Law 1:	$\ln x + \ln y = \ln(xy)$	(Sum of logs)
Law 2:	$\ln x - \ln y = \ln\left(\frac{x}{y}\right)$	(Difference of logs)
Law 3:	$r\ln x = \ln(x^r)$	(Multiple of a log)

Examples

1 Simplify (by expressing as the natural logarithm of a single number)

(i) $4\ln 2 + 2\ln 5 - \ln 10$ (ii) $\ln 12 - 2\ln 3 + \ln 6$.

(i) $4\ln 2 + 2\ln 5 - \ln 10 \quad = \ln(2^4) + \ln(5^2) - \ln 10$ (Law 3)

$= \ln 16 + \ln 25 - \ln 10 \quad = \ln(16 \times 25) - \ln 10$ (Law 1)

$= \ln\left[\frac{16 \times 25}{10}\right]$ (Law 2)

$= \ln\left[\frac{16 \times 5}{2}\right]$ $= \ln 40$

ii) $\ln 12 - 2\ln 3 + \ln 6 = \ln 12 - \ln(3^2) + \ln 6$ (Law 3)

$= \ln 12 - \ln 9 + \ln 6$ $= \ln\left[\frac{12}{9}\right] + \ln 6$ (Law 2)

$= \ln\left[\frac{12 \times 6}{9}\right]$ (Law 1)

$= \ln\left[\frac{72}{9}\right]$ $= \ln 8$.

2. Simplify (by expressing as a single natural logarithm)

(i) $\ln(5x) - 4\ln(2x) + 2\ln(4x^2)$

(ii) $2\ln(x^2 y) + 3\ln x - \ln(x^3 y)$.

(i) $\ln(5x) - 4\ln(2x) + 2\ln(4x^2)$

$= \ln(5x) - \ln[(2x)^4] + \ln[(4x^2)^2]$ (Law 3)

$= \ln(5x) - \ln[16x^4] + \ln[16x^4]$

$= \ln(5x)$

(ii) $2\ln(x^2 y) + 3\ln x - \ln(x^3 y)$

$= \ln[(x^2 y)^2] + \ln(x^3) - \ln(x^3 y)$ (Law 3)

$= \ln[x^4 y^2] + \ln(x^3) - \ln(x^3 y)$

$= \ln(x^4 y^2 x^3) - \ln(x^3 y)$ (Law 1)

$= \ln(x^7 y^2) - \ln(x^3 y)$

$= \ln\left[\frac{x^7 y^2}{x^3 y}\right]$ (Law 2)

$= \ln(x^4 y)$.

Problems

1. Simplify (by expressing as the natural logarithm of a single number)

(i) $2\ln 3 + 2\ln 5 - \ln 15$ (ii) $5\ln 2 - 2\ln 4 + \ln 7$.

2. Simplify (by expressing as a single natural logarithm)

   (i) $\ln(8x) - 2\ln(2x) + 3\ln(2x^2)$

   (ii) $3\ln(xy) + 2\ln x - \ln(x^3y^2)$.

Answers

1. (i) $\ln 15$ (ii) $\ln 14$.

2. (i) $\ln(16x^5)$ (ii) $\ln(x^2y)$.

3.5 Logarithmic and exponential equations

1. When solving any equation for *x*, the operations performed on *x* and the order of those operations should be noted. These operations should then be 'undone' in reverse order to find *x* (see Chapter 1 of this Study Guide).'Undoing' an operation means performing the corresponding inverse operation (on both sides of the equation). Example 1.(i) below gives an elementary illustration of this, and indicates that the techniques of solving equations covered in Chapter 1 still apply.

2. As exponentials and logarithms are inverse operations, an exponential can be 'undone' by taking logarithms of both sides. Similarly, a logarithm can be 'undone' by equating exponentials of both sides (see Examples below).

Examples

1. Solve the following equations for *x:*

   (i) $\dfrac{3x-2}{5} + 3 = -1$ (ii) $3 + 2\ln(x-5) = 7$

   (iii) $\dfrac{e^{3x-4} + 2}{6} = 1$.

(i) $\dfrac{3x-2}{5} + 3 = -1$

[Note that the order of operations performed on *x* to form the left side of the equation is as follows: multiply by 3; subtract 2; divide by 5; add 3. So, solving for *x* requires the following operations to be performed to both sides of the equation **in this order**: subtract 3; multiply by 5; add 2; divide by 3]. So,

$$\frac{3x-2}{5} = -4 \qquad \therefore 3x - 2 = -20$$

$$\therefore 3x = -18 \qquad \therefore x = \frac{-18}{3}$$

Hence, $x = -6$.

(ii) $3 + 2\ln(x-5) = 7$

(subtract 3) $\therefore 2\ln(x-5) = 4$

(divide by 2) $\therefore \ln(x-5) = 2$

(exponentials of both sides) $\therefore e^{\ln(x-5)} = e^2$

So, $x - 5 = e^2$

(add 5) Hence, $x = e^2 + 5$.

(iii) $\dfrac{e^{3x-4}+2}{6} = 1$

(multiply by 6) $\therefore e^{3x-4} + 2 = 6$

(subtract 2) $\therefore e^{3x-4} = 4$

(ln of both sides) $\therefore \ln(e^{3x-4}) = \ln 4$.

So, $3x - 4 = \ln 4$

(add 4) $\therefore 3x = 4 + \ln 4$

(divide by 3) Hence, $x = \dfrac{4+\ln 4}{3}$.

2. Solve the following equations for x

(i) $7 + \frac{1}{2}\ln(3x-2) = y$ (ii) $y\ln(5\sqrt{x}) = \dfrac{y+3}{2}$

(iii) $\dfrac{3e^{5x}+2}{6} = y - 1$.

(i) $7 + \frac{1}{2}\ln(3x-2) = y$

$\therefore \frac{1}{2}\ln(3x-2) = y - 7$

$\therefore \ln(3x-2) = 2(y-7)$

$\therefore 3x - 2 = e^{2(y-7)}$

$\therefore 3x = e^{2(y-7)} + 2$

$x = \dfrac{e^{2(y-7)}+2}{3}$

(ii) $y\ln(5\sqrt{x}) = \dfrac{y+3}{2}$

$$\therefore \ln(5\sqrt{x}) = \frac{y+3}{2y}$$

$$\therefore 5\sqrt{x} = e^{(y+3)/2y}$$

$$\therefore \sqrt{x} = \frac{e^{(y+3)/2y}}{5}$$

$$\therefore x = \left[\frac{e^{(y+3)/2y}}{5}\right]^2$$

$$\therefore x = \frac{e^{(y+3)/y}}{25}$$

(iii) $$\frac{3e^{5x}+2}{6} = y-1$$

$$\therefore 3e^{5x}+2 = 6(y-1)$$

$$\therefore 3e^{5x}+2 = 6y-6$$

$$\therefore 3e^{5x} = 6y-8$$

$$\therefore e^{5x} = \frac{6y-8}{3}$$

$$\therefore 5x = \ln\left[\frac{6y-8}{3}\right]$$

$$\therefore x = \frac{1}{5}\ln\left[\frac{6y-8}{3}\right].$$

Problems

1. Solve the following equations for x

   (i) $4-2\ln(2x-1) = 0$ (ii) $3+6\ln(x-1) = 0$

   (iii) $\dfrac{e^{3x-2}-7}{4} = -1$.

2. Solve the following equations for x

   (i) $7-3\ln(9x-1) = y$ (ii) $\dfrac{e^{3x-1}-11}{2} = 3y-7$

   (iii) $5y\ln(1+\sqrt{x}) = y-4$.

Answers

1. (i) $x = \dfrac{1 + e^2}{2}$ (ii) $x = 1 + e^{-1/2}$ (iii) $x = \dfrac{2 + \ln 3}{3}$.

2. (i) $x = \dfrac{1 + e^{(7-y)/3}}{9}$ (ii) $x = \dfrac{1 + \ln(6y - 3)}{3}$.

   (iii) $x = [e^{(y-4)/5y} - 1]^2$

4

Trigonometric functions

Introduction

Trigonometry is a branch of mathematics which has been studied since ancient times. Its earliest uses were confined to the exact measurement of lengths and angles. Such applications are still used in surveying, navigation and astronomy. In more recent times its applications include several types of periodic phenomena, including sound waves and electronics.

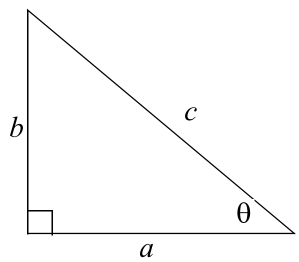
This topic covers trigonometric ratios, right-angled triangles, trigonometric identities, radian measure of angles, and the graphs of the three major trigonometric functions.After studying this topic, you should be able to:

- understand and use the definitions of the three major trigonometric ratios;
- find lengths and angles in right-angled triangles;
- understand the basic trigonometric identity;
- convert angles from degrees to radians and vice versa;
- sketch graphs of the three major trigonometric functions.

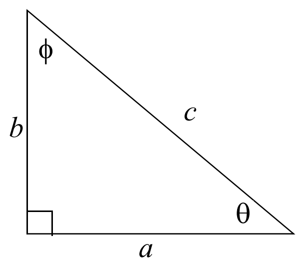
4.1 Trigonometric ratios

1. By definition, in the right-angled triangle shown:

$$\sin\theta = \frac{b}{c}; \qquad \cos\theta = \frac{a}{c}; \qquad \tan\theta = \frac{b}{a}.$$



2. Pythagoras's Theorem ($a^2 + b^2 = c^2$), and the definitions of $\sin\theta$, $\cos\theta$, and $\tan\theta$ can be used to find unknown sides and angles in any **right-angled** triangle. Also, the sum of all angles in **any triangle** (whether right-angled or not) must be $180°$.



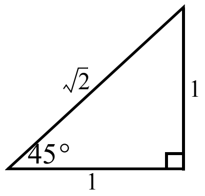
In the right-angled triangle above, this means that

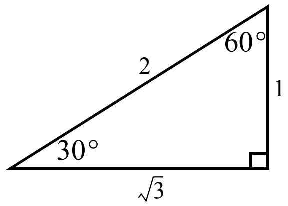
$$\theta + \phi + 90° = 180°$$

$$\therefore \theta + \phi = 90°.$$

3. If a trigonometric ratio is known, the angle can be found using the **inverse trigonometric function**. For instance, if it is known that $\sin\theta = k$, then $\theta = \sin^{-1}k$, read as 'the **inverse sine** of *k*'). Most calculators use the notation $\sin^{-1}k$, though some calculators require using the inverse instruction (normally **inv**), followed by the trigonometric function, e.g. inv sin *k*. (See Example 1(iv)).

4. There are two special right-angled triangles which occur regularly in mathematics. These triangles are called **Standard Triangles**, and should be remembered. The two standard triangles are:



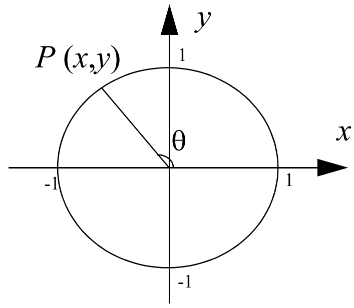


Using these two triangles, all trigonometric ratios of the angles $45°, 30°$, and $60°$ can be found exactly. The relevant trigonometric ratios are shown in the table below:

θ	$30°$	$45°$	$60°$
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\tan\theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

5. Note that, in any right-angled triangle, the angle θ cannot be greater than 90^o. So the above definitions of $\sin\theta$, $\cos\theta$, and $\tan\theta$ apply only for angles between 0^o and 90^o. For angles beyond this interval, further definitions are required.

6. The trigonometric ratios can be defined for angles greater than $90°$. This is done by considering points $P(x, y)$ on the unit circle $x^2 + y^2 = 1$, and measuring the angle θ **anti-clockwise from the positive *x* axis** (as shown in the diagram below).



By definition, $\cos\theta = x$ and $\sin\theta = y$. Hence, $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{y}{x}$.

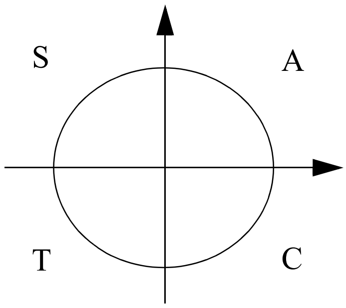
As x and y are the co-ordinates of points on the circle $x^2 + y^2 = 1$,

i.e. $-1 \le x \le 1$ and $-1 \le y \le 1$, and the inequalities
$-1 \le \cos\theta \le 1$ and $-1 \le \sin\theta \le 1$ are always true.

Whenever x is positive, $\cos\theta$ is positive, and whenever y is positive, $\sin\theta$ is positive. Similarly, whenever x is negative, $\cos\theta$ is negative, and whenever y is negative, $\sin\theta$ is negative.

7. There are several ways to remember the signs of the trigonometric ratios in each quadrant.

   One such method is to start in the 1st Quadrant (where both x and y are positive, and move around the circle anti-clockwise. The initial letters of the phrase '**A**ll **S**tations **T**o **C**entral' give the signs of the ratios which are positive in a quadrant.
   Here, **A** means that **a**ll of the ratios are positive (in the 1st quadrant);
   S means that only the **s**ine is positive (in the 2nd quadrant);
   T means that only the **t**angent is positive (in the 3rd quadrant);
   C means that only the **c**osine is positive (in the 4th quadrant);
   Another similar method starts in the 4th quadrant, and moves around the circle anti-clockwise. The letters of the word '**CAST**' then give the correct signs, as shown in the diagram below.



8. **An angle measured clockwise is considered to be a negative angle.** For instance, an angle of $-90°$ represents the same point on the unit circle as an angle of $270°$. In general, adding or subtracting $360°$ to an angle does not change the trigonometric ratio.

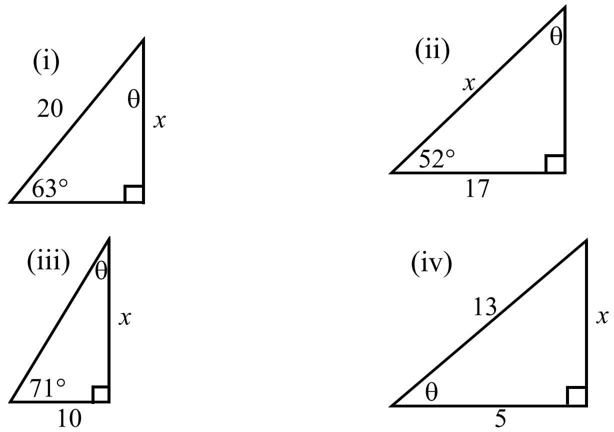
Examples

1. Use a calculator to evaluate (to 4 decimal places)

   (i) $\cos 132°$ (ii) $\sin 204°$

   (iii) $\tan 263°$ (iv) $\cos(-78°)$.

2. In each of the following triangles, find the unknown length, x and angle, θ. Note: round all answers to 1 decimal place.



1. (i) $\cos 132° \approx -0.6691$ (ii) $\sin 204° \approx -0.4067$

   (iii) $\tan 263° \approx 8.1443$ (iv) $\cos(-78°) \approx 0.2079$.

2. (i) As $\theta + 63° = 90°$, $\theta = 90° - 63° = 27°$

   $\therefore \theta = 27°$

   From the diagram, $\sin 63° = \dfrac{x}{20}$.

   $\therefore x = 20 \sin 63° \qquad \approx 20 \times 0.8910$

   $\therefore x \approx 17.820 \qquad$ i.e. $x \approx 17.8$.

   (ii) As $\theta + 52° = 90°$, $\theta = 90° - 52° = 38°$

   $\therefore \theta = 38°$

   From the diagram, $\cos 52° = \dfrac{17}{x}$.

   $\therefore x \cos 52° = 17 \qquad \therefore x = \dfrac{17}{\cos 52°}$

   $\therefore x \approx \dfrac{17}{0.6157} \qquad \approx 27.6126$

   $\therefore x \approx 27.6$.

   (iii) As $\theta + 71° = 90°$, $\theta = 90° - 71° = 19°$

   From the diagram, $\tan 71° = \dfrac{x}{10}$.

   $\therefore x = 10 \tan 71° \qquad \approx 10 \times 2.9042$

   $\therefore x \approx 29.042 \qquad$ i.e. $x \approx 29.0$.

   (iv) By Pythagoras, $x^2 + 5^2 = 13^2$

   $\therefore x^2 = 13^2 - 5^2 \qquad = 169 - 25 = 144$

$\therefore x = \sqrt{144} = 12$

(the negative square root is not needed, as x is a length).

From the diagram, $\cos\theta = \dfrac{5}{13} \approx 0.3846$, and θ is the inverse cosine of 0.3846, i.e. $\theta = \cos^{-1} 0.3846 \approx 67.4°$.

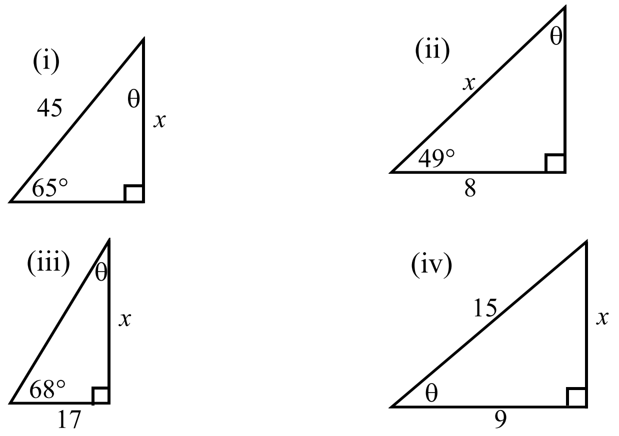
Problems

1. Use a calculator to evaluate (to 4 decimal places)

   (i) $\cos 341°$ (ii) $\sin 111°$

   (iii) $\tan 202°$ (iv) $\sin(-78°)$.

2. In each of the following triangles, find the unknown length, x and angle, θ. Note: round all answers to 1 decimal place.



Answers

1. (i) 0.9455 (ii) 0.9336 (iii) 0.4040 (iv) -0.9781.

2. (i) $\theta = 25°$ and $x = 45\sin 65° \approx 40.8$

   (ii) $\theta = 41°$ and $x = \dfrac{8}{\cos 49°} \approx 12.2$

   (iii) $\theta = 22°$ and $x = 17\tan 68° \approx 42.1$

   (iv) $x^2 = 15^2 - 9^2 = 144 \quad \therefore x = 12$

   and $\cos\theta = \dfrac{9}{15} = 0.6 \quad \therefore \theta \approx 53.1°$.

4.2 Trigonometric identities

1. Many trigonometric identities are derived using the symmetry of the unit circle. Of the symmetry identities, the most important are the following.

   For **any** angle θ,

   $\sin(-\theta) = -\sin\theta$; $\cos(-\theta) = \cos\theta$; $\tan(-\theta) = -\tan\theta$.

2. The **basic trigonometric identity** is

   $$\cos^2\theta + \sin^2\theta = 1$$

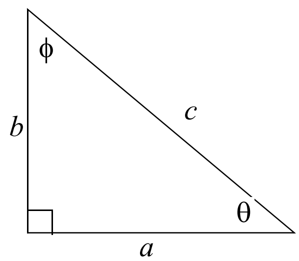
   Note that the value of $\cos^2\theta$ is found by first finding $\cos\theta$, then squaring, i.e. $\cos^2\theta$, pronounced 'cos squared theta', is defined as $(\cos\theta)^2$.

   For instance, $\cos^2 48° = (\cos 48°)^2 \approx (0.6691)^2 \approx 0.4477$.

   Similarly, $\sin^2 48° = (\sin 48°)^2 \approx (0.7431)^2 \approx 0.5523$.

   So, $\cos^2 48° + \sin^2 48° = 0.4477 + 0.5523 = 1$.

3. Another useful set of trigonometric identities can be observed from the right-angled triangle below.



   Since $\theta + \phi = 90°$, the angle ϕ is given by $\phi = 90° - \theta$.

   Using the definitions of $\sin\phi$, $\cos\phi$, and $\tan\phi$,

   $$\sin\phi = \frac{a}{c} = \cos\theta \qquad \therefore \sin(90° - \theta) = \cos\theta$$

   $$\cos\phi = \frac{b}{c} = \sin\theta \qquad \therefore \cos(90° - \theta) = \sin\theta$$

   $$\tan\phi = \frac{a}{b} = \frac{1}{\tan\theta} \qquad \therefore \tan(90° - \theta) = \frac{1}{\tan\theta}$$

Examples

1. Use a calculator to verify that $\cos^2\theta + \sin^2\theta = 1$ for

   (i) $\theta = 289°$ (ii) $\theta = -146°$.

1. (i) For $\theta = 289°$, $\cos^2 289° = (\cos 289°)^2 \approx (0.3256)^2 \approx 0.1060$.

   Similarly, $\sin^2 289° = (\sin 289°)^2 \approx (-0.9455)^2 \approx 0.8940$.

   So, $\cos^2 289° + \sin^2 289° = 0.1060 + 0.8940 = 1$.

   (ii) For $\theta = -146°$,

   $\cos^2(-146°) = [\cos(-146°)]^2 \approx (-0.8290)^2 \approx 0.6873$.

   Similarly,

   $\sin^2(-146°) = [\sin(-146°)]^2 \approx (-0.5592)^2 \approx 0.3127$.

   So, $\cos^2(-146°) + \sin^2(-146°) = 0.6873 + 0.3127 = 1$.

Problems

1. Use a calculator to verify that $\cos^2\theta + \sin^2\theta = 1$ for $\theta = 190°$.

Answers

1. As $\cos 190° \approx -0.9844$ and $\sin 190° \approx -0.1736$,
   $(-0.9844)^2 + (-0.1736)^2 = 0.96985 + 0.03015 = 1$.

4.3 Radian measure

1. Almost all mathematical applications use angles measured in radians, rather than degrees.**In particular, the calculus of trigonometric functions is always performed in radians.**

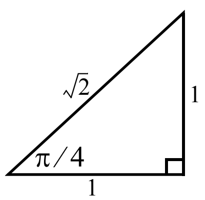
2. To convert angles from degrees to radians, multiply by $\frac{\pi}{180}$.

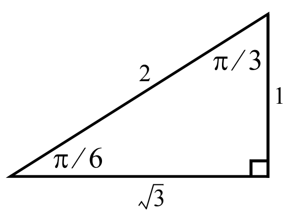
   To convert angles from radians to degrees, multiply by $\frac{180}{\pi}$.

3. If the units of an angle are not stated, the angle is, by convention, measured in radians. For instance, $\cos 2$ means $\cos 2^R$. Hence

   $\cos 2 = -0.4161$. Note, however, that $\cos 2° = 0.9994$.

4. When angles are measured in radians, the two standard triangles become:





The relevant trigonometric ratios in radians are shown in the table below:

θ	$\frac{\pi}{6}^R$	$\frac{\pi}{4}^R$	$\frac{\pi}{3}^R$
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\tan\theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

Examples

1. Express in radians in terms of π

   (i) $18°$ (ii) $20°$ (iii) $35°$.

2. Express in degrees

   (i) $\frac{7\pi}{10}$ (ii) $\frac{7\pi}{4}$ (iii) $\frac{2\pi}{15}$.

1. (i) $18° = \frac{18 \times \pi}{180} = \frac{\pi}{10}$ (ii) $20° = \frac{20 \times \pi}{180} = \frac{\pi}{9}$

   (iii) $35° = \frac{35 \times \pi}{180} = \frac{5 \times 7\pi}{5 \times 36} = \frac{7\pi}{36}$.

2. (i) $\frac{7\pi}{10} = \frac{7\pi \times 180°}{10\pi} = 7 \times 18° = 126°$

   (ii) $\frac{7\pi}{4} = \frac{7\pi \times 180°}{4\pi} = 7 \times 45° = 315°$

   (iii) $\frac{2\pi}{15} = \frac{2\pi \times 180°}{15\pi} = 2 \times 12° = 24°$

Problems

1. Express in radians in terms of π

   (i) $54°$ (ii) $100°$ (iii) $135°$.

2. Express in degrees

   (i) $\frac{\pi}{5}$ (ii) $\frac{2\pi}{3}$ (iii) $\frac{7\pi}{9}$.

Answers

1. (i) $\dfrac{3\pi}{10}$ (ii) $\dfrac{5\pi}{9}$ (iii) $\dfrac{3\pi}{4}$.

2. (i) $36°$ (ii) $120°$ (iii) $140°$.

4.4 Trigonometric functions and graphs

1. The graphs of the three major trigonometric functions:

   $y = \sin x$; $y = \cos x$; and $y = \tan x$ are sketched in note 3 below.

2. The graphs of $y = \sin x$ and $y = \cos x$ are similar in many ways.

   Both

   (a) have a wave shape,

   (b) are defined and continuous for all values of x,

   (c) repeat at intervals of 2π,

   (d) are restricted to the interval $-1 \le y \le 1$.

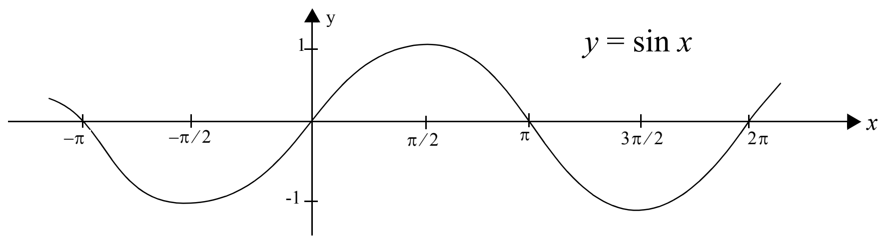
3. The graph of $y = \tan x$

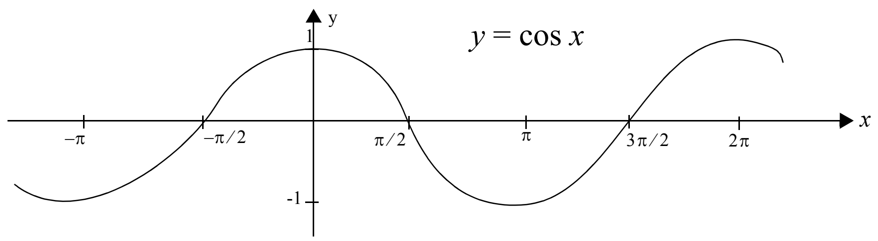
   (a) does not have a wave shape,

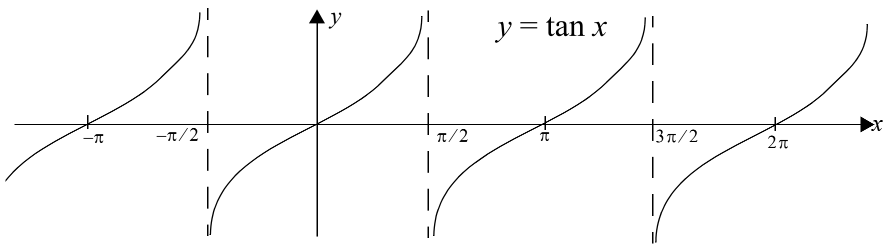
   (b) is defined for all values of x except when x is an odd multiple of $\dfrac{\pi}{2}$,

   (c) repeats at intervals of π,

   (d) can take on all possible y values, i.e. $-\infty < y < \infty$







Examples

1. Verify that $\sin 2x \neq 2\sin x$ for

   (i) $x = 1.2$ (ii) $x = -0.4$.

1. (i) For $x = 1.2$, $2x = 2.4$

   $\therefore \sin 2x = \sin 2.4 \approx 0.6755$

   Similarly, $2\sin x = 2\sin 1.2 = 2 \times 0.9320 \approx 1.8641$

   (ii) For $x = -0.4$, $2x = -0.8$

   $\therefore \sin 2x = \sin(-0.8) \approx -0.7174$

   Similarly, $2\sin x = 2\sin(-0.4) = 2 \times (-0.3894) \approx -0.7788$.

Problems

1. Verify that $\cos 2x \neq 2\cos x$ for

   (i) $x = 1$ (ii) $x = -0.6$.

Answers

1. (i) For $x = 1$, $2x = 2$

   $\therefore \cos 2x = \cos 2 \approx -0.4161$,

   whereas $2\cos x = 2\cos 1 = 2 \times 0.5403 \approx 1.0806$.

   (ii) For $x = -0.6$, $2x = -1.2$

   $\therefore \cos 2x = \cos(-1.2) \approx 0.3624$,

   whereas $2\cos x = 2\cos(-0.6) = 2 \times 0.8253 \approx 1.6507$.

5

The derivative

Introduction

Calculus, the mathematics of change and motion, relies on the fundamental concept of the derivative. Derivatives arise in each of the sciences, and in many other fields of study. Velocity and acceleration in physics, currents in electronics, rates of reaction in chemistry, growth of populations in biology and marginal revenue in economics are all direct applications of the derivative.

This topic covers average rates of change, the definition of the derivative as an instantaneous rate of change, the gradient of a function represented as a derivative, and the simple rules needed to differentiate polynomials, exponentials, logarithms, and trigonometric functions.

After studying this topic, you should be able to:

- find the average rate of change of a function;
- understand and use the rules to find the derivatives of polynomial, trigonometric, exponential and logarithmic functions;
- find and classify stationary points;
- sketch polynomials using the derivative.

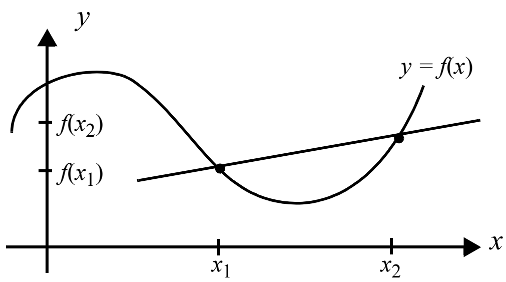
5.1 Rates of change

1. Given the function $y = f(x)$, the **average rate of change of *f* is the change in *f* divided by the change in *x***.

   In function notation, for $x_1 \le x \le x_2$,

   $$\text{average rate of change} = \frac{\text{change in } f(x)}{\text{change in } x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

   Pictorially, the average rate of change is given by the slope of the straight line joining two points on the curve, as shown in the diagram .

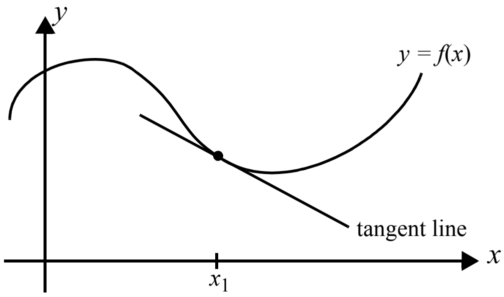


   For the straight line shown, the slope is given by the usual formula,

   i.e. $m = \dfrac{rise}{run} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

2. If *x* represents time, and $y = f(x)$ represents displacement, then the average rate of change is the average speed.

3. Given the function $y = f(x)$, the **instantaneous rate of change of *f*** when $x = x_1$ **is the slope of the tangent line to the curve** $y = f(x)$ at the point where $x = x_1$.



The formula for the instantaneous rate of change of *f* when $x = x_1$ is obtained from the average rate of change by first writing $x_2 = x_1 + h$.

The instantaneous rate of change of f is then given by

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{x_1 + h - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}$$

The smaller the distance between x_1 and x_2, i.e. the smaller the distance between *h* and 0, the closer the value of the average rate of change to the instantaneous rate of change of *f* when $x = x_1$.

So, as *h* approaches 0, $\dfrac{f(x_1 + h) - f(x_1)}{h}$ approaches the instantaneous rate of change of $f(x)$ at x_1.

Mathematically, the formula for the instantaneous rate of change of *f* when $x = x_1$ is given by

$$\text{instantaneous rate of change} = \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}$$

The process of letting *h* approach 0 is called a limit, and is called ‘the limit as *h* tends to 0’.

Note: When $h = 0$, $\dfrac{f(x_1 + h) - f(x_1)}{h} = \dfrac{f(x_1) - f(x_1)}{0} = \dfrac{0}{0}$, which is undefined.

4. The formula for the instantaneous rate of change of *f* at **any** value of *x* is given by replacing x_1 by *x* in the limit used above. This gives the **gradient function** : $\displaystyle\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$.

Examples

1. The following table shows the distance covered by a swimmer at various times during a 30 second swim

Distance (m.)	0	12	23	32	42	51	60
Time (s.)	0	5	10	15	20	25	30

Find the average speed

(i) over the first 5 seconds

(ii) between 10 and 20 seconds

(iii) over the last 5 seconds

(iv) over the entire 30 seconds.

2. The temperatures at various times on a certain day are shown in the following table.

Temperature	9° C	27° C	29° C	x	21° C
Time	7 a.m.	11 a.m.	3 p.m.	7 p.m.	11 p.m

(i) Find the average rate of change of temperature between 7 a.m. and 3 p.m.

(ii) Find the average rate of change of temperature between 11 a.m. and 11 p.m.

(iii) Given that the average rate of change of temperature between 7 p.m. and 11 p.m. is $-1.5°$ C per hour, find the temperature, x, at 7 p.m.

1. The required average speed:

(i) over the first 5 seconds is $\dfrac{12-0}{5-0} = \dfrac{12}{5} = 2.4$ (m/sec);

(ii) between 10 and 20 seconds is $\dfrac{42-23}{20-10} = \dfrac{19}{10} = 1.9$ (m/sec);

(iii) over the last 5 seconds is $\dfrac{60-51}{30-25} = \dfrac{9}{5} = 1.8$ (m/sec);

(iv) over the entire 30 seconds is $\dfrac{60-0}{30-0} = \dfrac{60}{30} = 2.0$ (m/sec).

Note: The units of the rate of change here are the units for distance (m.) divided by the units for time (sec.).

2. The required rate of change of temperature:

(i) between 7 a.m. and 3 p.m. is $\dfrac{29-9}{15-7} = \dfrac{20}{8} = 2.5$ (° C per hour)

(ii) between 11 a.m. and 11 p.m.is

$$\frac{21-27}{23-11} = \frac{-6}{12} = -0.5 \text{ (° C per hour)}$$

(iii) Since the average rate of change of temperature between 7 p.m. and 11 p.m. is $-1.5°$ C per hour,

$$\frac{21-x}{11-7} = -1.5 \qquad \therefore \frac{21-x}{4} = -1.5$$

$$\therefore 21-x = -1.5\times 4 \qquad \therefore 21-x = -6$$

$$\therefore -x = -27 \qquad \therefore x = 27$$

Hence, the temperature at 7 p.m.is $27°$ C.

Problems

1. The following table shows the total distance covered by a walker at various times during a hike

Distance (km)	0	2.4	3.9	4.8	6.0	7.1	8.4
Time (hours.)	0	0.5	1	1.5	2	2.5	3

Find the average speed

(i) over the first hour

(ii) between 1.5 and 3 hours

(iii) over the last half hour

(iv) over the entire 3 hours.

2. The temperatures at various times on a certain day are shown in the following table.

Temperature	$2.8°$ C	$11.6°$ C	$14.8°$ C	x	$8.4°$ C
Time	7 a.m.	11 a.m.	3 p.m.	7 p.m.	11 p.m

(i) Find the average rate of change of temperature between 11a.m. and 3 p.m.

(ii) Find the average rate of change of temperature between 7 a.m. and 3 p.m.

(iii) Given that the average rate of change of temperature between7 p.m. and 11 p.m. is $-1.1°$ C per hour, find the temperature, x, at 7 p.m.

Answers

1. (i) 3.9 (km/h) (ii) 2.4 (km/h)

(iii) 2.6 (km/h) (iv) 2.8 (km/h).

2. (i) 0.8 ($°$C per hour) (ii) 1.5 ($°$C per hour) (iii) 12.8 ($°$C).

5.2 The derivative

1. Given the function $y = f(x)$, the instantaneous rate of change of f at the general point x is called **the derivative** of $y = f(x)$.

   The derivative is most commonly denoted by $\frac{dy}{dx}$, $f'(x)$, or y'.

   By definition,

   $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

   (the first-principles formula for the derivative).

2. **All derivatives can be calculated without using the first-principles formula.** Example 1 below indicates that using the formula can be quite tedious.

3. The **derivative represents the slope** (gradient) of a curve at any point on that curve.

4. The rule for finding the derivative of a power of x is sometimes called **the power rule**, i.e. if $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$, where n is a constant.

   The power rule applies for **any** constant *n,* including fractions and decimals.

   For instance, if $y = x^{7/5}$, then $\frac{dy}{dx} = \frac{7}{5}x^{7/5-1} = \frac{7}{5}x^{2/5}$.

   Note that, before differentiating, y must be in the form $y = x^n$, e.g.

   $y = \frac{1}{x^6}$ must be re-written as $y = x^{-6}$, before differentiating. Then

   $\frac{dy}{dx} = -6\,x^{-6-1} = -6\,x^{-7}$

5. The rule for finding the derivative of a constant multiple of a function is intuitively obvious:

   if $y = kf(x)$, then $\frac{dy}{dx} = k\frac{df}{dx}$, where k is a constant.

   For instance, if $y = 6x^{5/3}$, then $\frac{dy}{dx} = 6 \times \frac{5}{3}x^{5/3-1} = 10x^{2/3}$.

6. The rule for finding the derivative of a sum or difference of functions is also intuitively obvious:

   if $y = f(x) \pm g(x)$, then $\frac{dy}{dx} = \frac{df}{dx} \pm \frac{dg}{dx}$.

For instance, if $y = x^{5/3} - x^{1/4}$, then

$$\frac{dy}{dx} = \frac{5}{3}x^{5/3-1} - \frac{1}{4}x^{1/4-1} = \frac{5}{3}x^{2/3} - \frac{1}{4}x^{-3/4}.$$

7. The derivative of any constant function is 0, i.e. if $y = c$, then $\frac{dy}{dx} = 0$.

   Similarly, the derivative of any constant multiple of x is a constant, i.e. if $y = mx = mx^1$, then $\frac{dy}{dx} = m \times 1x^{1-1} = mx^0 = m$.

   More generally, for the straight line $y = mx + c$, these 2 results give $\frac{dy}{dx} = m + 0 = m$ (as expected, since m is the slope of $y = mx + c$).

8. The above rules can be used to differentiate any polynomial, e.g. for the most general quadratic $y = ax^2 + bx + c$,

$$\frac{dy}{dx} = a \times 2x^{2-1} + b + 0$$

$$\therefore \frac{dy}{dx} = 2ax + b.$$

9. An important application of rate of change is that of the **motion of an object in a straight line**. The conventional notation is as follows:

   ***s* is the displacement of the object;**

   ***t* is the time;**

   ***v* is the velocity;**

   ***a* is the acceleration.**

   Using this notation, $v = \frac{ds}{dt}$, and $a = \frac{dv}{dt}$.

   Hence, if the displacement is given, the velocity can be found by differentiation. Then, the acceleration can be found by differentiating the velocity.

Examples

1. Use the first-principles formula $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ to find the derivative of the following functions

   (i) $y = f(x) = 4x - 7$ (ii) $y = f(x) = x^2$.

2. Find $\frac{dy}{dx}$ for

   (i) $y = 3\sqrt{x}$ (ii) $y = \sqrt{3x}$ (iii) $y = \frac{3}{\sqrt{x}}$

   (iv) $y = \frac{1}{3\sqrt{x}}$ (v) $y = x^4 - \frac{1}{x^4}$ (vi) $y = 4x^{1/6} + \pi^3$

(vii) $y = 4x^2 - 5x + 6$ (viii) $y = 2x^3 + 7x^2 + 9x - 1$.

3. Find the slope of the curve $y = 5x^2 + x - 3$ at:

(i) the point $(-1, 1)$ (ii) the point $(2, 19)$.

4. It takes 12 minutes to fill an oil tank. The volume of oil V litres, after time t minutes, is given by

$$V = 24t - t^2, \quad \text{for } 0 \le t \le 12 \; .$$

(i) Find the instantaneous rate of change of volume when $t = 10$.

(ii) Find the volume of oil in the tank when the instantaneous rate of change of volume is 23 litres/minute.

(iii) Find the instantaneous rate of change of volume when the volume of oil in the tank is 44 litres.

5. The displacement s (metres) of an object at time t (seconds) is given by $s = -t^3 + 3t^2 + 6t$, for $t \ge 0$. Find

(i) the velocity and acceleration

(ii) the velocity after 2 seconds

(iii) the acceleration after 7 seconds

(iv) the velocity when the acceleration is 0

(v) the acceleration when the velocity is –3 (metres/second).

1. (i) For $y = f(x) = 4x - 7$,

$$f(x+h) = 4(x+h) - 7 = 4x + 4h - 7 .$$

$$\therefore f(x+h) - f(x) = 4x + 4h - 7 - (4x - 7)$$

$$\therefore f(x+h) - f(x) = 4x + 4h - 7 - 4x + 7 = 4h$$

$$\therefore \frac{f(x+h) - f(x)}{h} = \frac{4h}{h} = 4$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 4 = 4 .$$

i.e. $f'(x) = 4$.

Note: this agrees with the fact that the slope of the straight line r $y = f(x) = 4x - 7$ is 4 (constant slope for all values of x).

(ii) For $y = f(x) = x^2$,

$$f(x+h) = (x+h)^2 = x^2 + 2xh + h^2 .$$

$$\therefore f(x+h) - f(x) = x^2 + 2xh + h^2 - x^2$$

$$\therefore f(x+h) - f(x) = 2xh + h^2$$

$$\therefore \frac{f(x+h)-f(x)}{h} = \frac{2xh+h^2}{h} = \frac{h(2x+h)}{h} = 2x+h$$

$$f\,'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}(2x+h) = 2x+0 = 2x$$

i.e. $f\,'(x) = 2x$.

Note that the limit is obtained by cancelling as far as is possible, then replacing h by 0.

2. (i) $y = 3\sqrt{x} \qquad \therefore y = 3x^{1/2}$

$$\therefore \frac{dy}{dx} = 3\times\frac{1}{2}x^{1/2-1} = \frac{3}{2}\,x^{-1/2}$$

(ii) $y = \sqrt{3x} \qquad \therefore y = \sqrt{3}\sqrt{x} = \sqrt{3}x^{1/2}$

$$\therefore \frac{dy}{dx} = \sqrt{3}\times\frac{1}{2}x^{1/2-1} = \frac{\sqrt{3}}{2}\,x^{-1/2}$$

(iii) $y = \dfrac{3}{\sqrt{x}} \qquad \therefore y = \dfrac{3}{x^{1/2}} = 3x^{-1/2}$

$$\therefore \frac{dy}{dx} = 3\times\frac{-1}{2}x^{-1/2-1} = \frac{-3}{2}\,x^{-3/2}$$

(iv) $y = \dfrac{1}{3\sqrt{x}} \qquad \therefore y = \dfrac{1}{3x^{1/2}} = \dfrac{1}{3}\,x^{-1/2}$

$$\therefore \frac{dy}{dx} = \frac{1}{3}\times\left(\frac{-1}{2}\right)x^{-1/2-1} = \frac{-1}{6}\,x^{-3/2}$$

(v) $y = x^4 - \dfrac{1}{x^4} \qquad \therefore y = x^4 - x^{-4}$

$$\therefore \frac{dy}{dx} = 4x^{4-1} - (-4)x^{-4-1} = 4x^3 + 4x^{-5}$$

(vi) $y = 4x^{1/6} + \pi^3$

$$\therefore \frac{dy}{dx} = \left(4\times\frac{1}{6}\right)x^{1/6-1} + 0 = \frac{2}{3}x^{-5/6} \quad \text{(Note: } \pi^3 \text{ is a constant)}$$

(vii) $y = 4x^2 - 5x + 6$

$$\therefore \frac{dy}{dx} = 4\times 2x^{2-1} - 5 + 0 = 8x - 5$$

(viii) $y = 2x^3 + 7x^2 + 9x - 1$

$$\therefore \frac{dy}{dx} = 2\times 3x^{3-1} + 7\times 2x^{2-1} + 9 + 0 = 6x^2 + 14x + 9\,.$$

3. For $y = 5x^2 + x - 3$, $\qquad \frac{dy}{dx} = 5 \times 2x^{2-1} + 1 + 0 = 10x + 1$

Hence, the slope at a point is $10x + 1$.

(i) at the point $(-1, 1)$, i.e. when $x = -1$,

$\frac{dy}{dx} = (10)(-1) + 1 = -10 + 1 = -9$

(ii) at the point $(2, 19)$, i.e. when $x = 2$,

$\frac{dy}{dx} = 10 \times 2 + 1 = 20 + 1 = 21$.

Note that when $x = -1$, $y = 5x^2 + x - 3 = 5 - 1 - 3 = 1$, i.e. the point $(-1, 1)$ **is** a point on the curve.

Similarly, when $x = 2$, $y = 5 \times 4 + 2 - 3 = 20 + 2 - 3 = 19$, i.e. the point $(-2, 19)$ **is also** a point on the curve.

4. Given $V = 24t - t^2$, for $0 \le t \le 12$, the instantaneous rate of change of volume is given by $\frac{dV}{dt} = 24 - 2t$

(i) When $t = 10$, $\frac{dV}{dt} = 24 - 20 = 4$ (litres per minute).

(ii) The question is to find V when $\frac{dV}{dt} = 23$, i.e. when

$24 - 2t = 23 \qquad \therefore -2t = 23 - 24 = -1$

$\therefore t = \frac{-1}{-2} = \frac{1}{2}$.

When $t = \frac{1}{2}$, $V = 24t - t^2 = 24 \times \frac{1}{2} - \left(\frac{1}{2}\right)^2 = 12 - \frac{1}{4} = \frac{47}{4}$,

i.e. $V = \frac{47}{4} = 11.75$ (litres).

(iii) The question is to find $\frac{dV}{dt}$ when $V = 44$, i.e. when

$24t - t^2 = 44 \qquad \therefore 0 = t^2 - 24t + 44 \qquad$ (a quadratic)

Factorising gives $t^2 - 24t + 44 = (t - 2)(t - 22) = 0$, with solutions $t = 2$ and $t = 22$.

Since $0 \le t \le 12$, the only permissible solution is $t = 2$, i.e. 2 (minutes).

When $t = 2$, $\frac{dV}{dt} = 24 - 4 = 20$ (litres per minute).

5. Given $s = -t^3 + 3t^2 + 6t$, for $t \geq 0$,

   (i) the velocity is given by $v = \frac{ds}{dt} = -3t^2 + 6t + 6$ (m/sec), and

   the acceleration by $a = \frac{dv}{dt} = -6t + 6$ (m/sec/sec).

   (ii) When $t = 2$, $v = -3 \times 4 + 6 \times 2 + 6 = -12 + 12 + 6 = 6$, i.e. the velocity is 6 (m/sec).

   (iii) When $t = 7$, $a = -6 \times 7 + 6 = -42 + 6 = -36$, i.e. the acceleration is -36 (m/sec/sec).

   (iv) When the acceleration is 0, $-6t + 6 = 0$

   $6t = 6 \qquad \therefore t = 1$.

   When $t = 1$, $v = -3 + 6 + 6 = 9$ (m/sec).

   (v) When the velocity is -3, $-3t^2 + 6t + 6 = -3$

   $\therefore -3t^2 + 6t + 9 = 0$. Dividing by -3 gives

   $t^2 - 2t - 3 = 0$ Factorising gives

   $t^2 - 2t - 3 = (t-3)(t+1) = 0$, which has solutions

   $t = 3$ and $t = -1$.

   Since $t \geq 0$, the only permissible solution is $t = 3$.

   When $t = 3$, $a = -6 \times 3 + 6 = -18 + 6 = -12$ (m/sec/sec).

Problems

1. Find $\frac{dy}{dx}$ for

   (i) $y = 2\sqrt{x}$ (ii) $y = \sqrt{2x}$ (iii) $y = \frac{2}{\sqrt{x}}$

   (iv) $y = \frac{1}{2\sqrt{x}}$ (v) $y = x^3 - \frac{1}{x^2}$ (vi) $y = 6x^{2/3} + \sqrt{3}$

   (vii) $y = 7x^2 - 9x + 8$ (viii) $y = 3x^3 - 7x^2 + 10x - 1$.

2. Find the slope of the curve $y = 7x^2 - 4x - 6$ at

   (i) the point $(-1, 5)$ (ii) the point $(2, 14)$.

3. It takes 10 minutes to fill a water tank. The volume of water V litres, after time t minutes, is given by

$$V = 20t - t^2, \quad \text{for } 0 \leq t \leq 10.$$

(i) Find the instantaneous rate of change of volume when $t = 9$.

(ii) Find the volume of water in the tank when the instantaneous rate of change of volume is 14 litres/minute.

(iii) Find the instantaneous rate of change of volume when the volume of water in the tank is 84 litres.

4. The displacement s (metres) of an object at time t (seconds) is given by $s = -2t^3 + 6t^2 + 18t$, for $t \geq 0$. Find

(i) the velocity and acceleration

(ii) the velocity after 2 seconds

(iii) the acceleration after 3 seconds

(iv) the velocity when the acceleration is 0

(v) the acceleration when the velocity is -30 (metres/second).

Answers

1. (i) $y = 2\sqrt{x} \qquad \therefore y = 2x^{1/2}$

$$\therefore \frac{dy}{dx} = 2 \times \frac{1}{2}x^{1/2-1} = x^{-1/2}$$

(ii) $y = \sqrt{2x} \qquad \therefore y = \sqrt{2}\sqrt{x} = \sqrt{2}x^{1/2}$

$$\therefore \frac{dy}{dx} = \sqrt{2} \times \frac{1}{2}x^{1/2-1} = \frac{\sqrt{2}}{2}x^{-1/2} \; (= \frac{1}{\sqrt{2}}x^{1/2})$$

(iii) $y = \frac{2}{\sqrt{x}} \qquad \therefore y = \frac{2}{x^{1/2}} = 2x^{-1/2}$

$$\therefore \frac{dy}{dx} = 2 \times \frac{-1}{2}x^{-1/2-1} = -x^{-3/2}$$

(iv) $y = \frac{1}{2\sqrt{x}} \qquad \therefore y = \frac{1}{2x^{1/2}} = \frac{1}{2}x^{-1/2}$

$$\therefore \frac{dy}{dx} = \frac{1}{2} \times \left(\frac{-1}{2}\right)x^{-1/2-1} = \frac{-1}{4}x^{-3/2}$$

(v) $y = x^3 - \frac{1}{x^2} \qquad \therefore y = x^3 - x^{-2}$

$$\therefore \frac{dy}{dx} = 3x^{3-1} - (-2)x^{-2-1} = 3x^2 + 2x^{-3}$$

(vi) $y = 6x^{2/3} + \sqrt{3}$

$$\therefore \frac{dy}{dx} = \left(6 \times \frac{2}{3}\right)x^{2/3-1} + 0 = 4x^{-1/3}$$

(vii) $y = 7x^2 - 9x + 8$

$$\therefore \frac{dy}{dx} = 2 \times 7x^{2-1} - 9 + 0 = 14x - 9$$

(viii) $y = 3x^3 - 7x^2 + 10x - 1$

$$\therefore \frac{dy}{dx} = 3 \times 3x^{3-1} - 7 \times 2x^{2-1} + 10 - 0 = 9x^2 - 14x + 10\,.$$

2. As $y = 7x^2 - 4x - 6$, $\frac{dy}{dx} = 14x - 4$

(i) At the point $(-1, 5)$, $\frac{dy}{dx} = -14 - 4 = -18$.

(ii) At the point $(2, 14)$, $\frac{dy}{dx} = 28 - 4 = 24$.

3. As $V = 20t - t^2$, $\frac{dV}{dt} = 20 - 2t$

(i) When $t = 9$, $\frac{dV}{dt} = 20 - 18 = 2$ (litres per minute).

(ii) When $\frac{dV}{dt} = 14$, i.e. when $20 - 2t = 14$

$\therefore -2t = 14 - 20 = -6$ $\qquad \therefore t = 3$.

When $t = 3$, $V = 20t - t^2 = 20 \times 3 - 3^2 = 60 - 9 = 51$ (litres)

(iii) When $V = 84$, i.e. when

$20t - t^2 = 84$ $\qquad \therefore 0 = t^2 - 20t + 84$

As $t^2 - 20t + 84 = (t - 6)(t - 14) = 0$, the solutions are $t = 6$ and $t = 14$.

Since $0 \le t \le 10$, the only permissible solution is $t = 6$, i.e. 6 (minutes).

When $t = 6$, $\frac{dV}{dt} = 20 - 12 = 8$ (litres per minute).

4. As $s = -2t^3 + 6t^2 + 18t$,

(i) the velocity $v = \frac{ds}{dt} = -6t^2 + 12t + 18$ (m/sec), and

the acceleration $a = \frac{dv}{dt} = -12t + 12$ (m/sec/sec).

(ii) When $t = 2$, the velocity is

$v = -6 \times 4 + 12 \times 2 + 18 = -24 + 24 + 18 = 18$ (m/sec).

(iii) When $t = 3$, the acceleration is

$a = -12 \times 3 + 12 = -36 + 12 = -24$ (m/sec/sec).

(iv) When $a = 0$, $-12t + 12 = 0$

$12t = 12 \qquad \therefore t = 1$.

When $t = 1$, $v = -6 + 12 + 18 = 24$ (m/sec).

(v) When $v = -30$, $-6t^2 + 12t + 18 = -30$, i.e.

$$-6t^2 + 12t + 48 = 0$$

$\therefore t^2 - 2t - 8 = 0 \qquad \therefore (t-4)(t+2) = 0$

$t = 4$ and $t = -2$.

Since $t \geq 0$, the only permissible solution is $t = 4$.

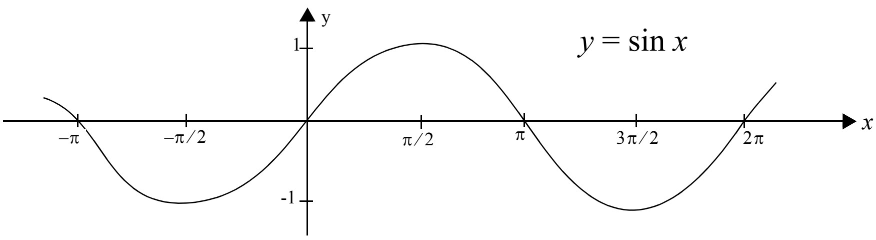
When $t = 4$, $a = -12 \times 4 + 12 = -48 + 12 = -36$ (m/sec/sec).

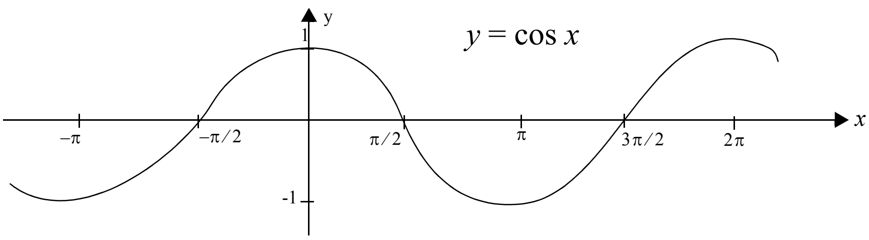
5.3 Trigonometric, exponential and logarithmic functions

1. Given the graph of a function $y = f(x)$, the graph of its derivative $\frac{dy}{dx}$ can be drawn (roughly). Although this is not a rigorous method of finding the derivative, it does give an illustration of the possible prescription for $\frac{dy}{dx}$.

2. The graphs of the two trigonometric functions $y = \sin x$ and $y = \cos x$ are reproduced below. From the graphs, two results for derivatives follow: If $y = \sin x$, $\quad \frac{dy}{dx} = \cos x$.

   If $y = \cos x$, $\quad \frac{dy}{dx} = -\sin x$.

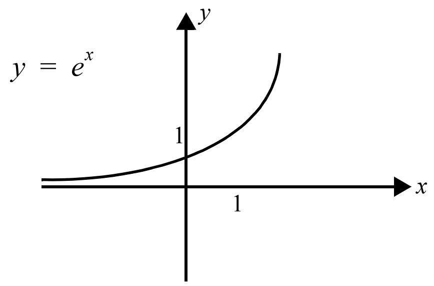


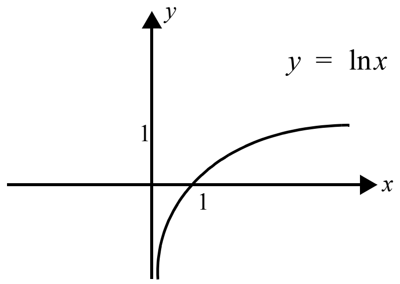


3. The graphs of the two functions $y = e^x$ and $y = \ln x$ are reproduced below. From the graphs, two results for derivatives follow:

$$\text{If } y = e^x, \quad \frac{dy}{dx} = e^x;$$

$$\text{If } y = \ln x, \quad \frac{dy}{dx} = \frac{1}{x}.$$





4. The results for the derivatives of trigonometric, exponential and logarithmic functions can be generalised as follows:
 For any constant k,

$$\text{if } y = \sin kx, \qquad \frac{dy}{dx} = k\cos kx;$$

$$\text{if } y = \cos kx, \qquad \frac{dy}{dx} = -k\sin kx;$$

$$\text{if } y = e^{kx}, \qquad \frac{dy}{dx} = ke^{kx};$$

$$\text{if } y = \ln x, \qquad \frac{dy}{dx} = \frac{1}{x}.$$

Note that the angle (kx)appearing in a trigonometric function is preserved after differentiation. Similarly, the power (kx) appearing in an exponential function is also preserved.

5. In summary, the 5 major results for the derivatives of the commonly used functions are shown in the table below.

Function: y	**Derivative:** $\frac{dy}{dx}$
x^n	nx^{n-1}
$\sin kx$	$k\cos kx$
$\cos kx$	$-k\sin kx$
e^{kx}	ke^{kx}
$\ln x$	$\frac{1}{x}$

Examples

1. Find $\frac{dy}{dx}$ for

(i) $y = \sin(3x)$ (ii) $y = \cos(5x)$

(iii) $y = 4e^{2x}$ (iv) $y = 8\sin\left(\frac{x}{2}\right)$

(v) $y = 4\ln x + 7e^{-2x}$ (vi) $y = \ln(5x)$

(vii) $y = \ln(x^4)$ (viii) $y = e^{3x} + 3e^x - \frac{4}{e^{2x}}$

(ix) $y = \frac{1}{4}\cos(8x) - 3\sin\left(\frac{x}{2}\right)$ (x) $y = x^3 + 3e^x$.

1. (i) $y = \sin(3x)$

$$\therefore \frac{dy}{dx} = 3\cos(3x) \qquad (k = 3)$$

(ii) $y = \cos(5x)$

$$\therefore \frac{dy}{dx} = -5\sin(5x) \qquad (k = 5)$$

(iii) $y = 4e^{2x}$

$$\therefore \frac{dy}{dx} = 4 \times 2e^{2x} = 8e^{2x} \qquad (k = 2)$$

(iv) $y = 8\sin\left(\frac{x}{2}\right)$ (Note that $\frac{x}{2} = \frac{1}{2}x$)

$\therefore \frac{dy}{dx} = 8 \times \frac{1}{2}\cos\left(\frac{x}{2}\right) = 4\cos\left(\frac{x}{2}\right)$ ($k = \frac{1}{2}$)

(v) $y = 4\ln x + 7e^{-2x}$

$\therefore \frac{dy}{dx} = 4 \times \frac{1}{x} + 7 \times (-2e^{-2x})$ ($k = -2$)

$\therefore \frac{dy}{dx} = \frac{4}{x} - 14e^{-2x}$

(vi) $y = \ln(5x)$ Using the log rules, $y = \ln 5 + \ln x$.

$\therefore \frac{dy}{dx} = 0 + \frac{1}{x} = \frac{1}{x}$ (Note that $\ln 5$ is a constant)

(vii) $y = \ln(x^4)$ Using the log rules, $y = 4\ln x$.

$\therefore \frac{dy}{dx} = 4 \times \frac{1}{x} = \frac{4}{x}$.

(viii) $y = e^{3x} + 3e^{x} - \frac{4}{e^{2x}}$ $\therefore y = e^{3x} + 3e^{x} - 4e^{-2x}$

$\therefore \frac{dy}{dx} = 3e^{3x} + 3e^{x} - 4 \times (-2e^{-2x}) = 3e^{3x} + 3e^{x} + 8e^{-2x}$

($k = 3$, $k = 1$,and $k = -2$, respectively)

(ix) $y = \frac{1}{4}\cos(8x) - 3\sin\left(\frac{x}{2}\right)$ (Note that $\frac{x}{2} = \frac{1}{2}x$)

$y = \frac{1}{4}\cos(8x) - 3\sin\left(\frac{1}{2}x\right)$

$\therefore \frac{dy}{dx} = \frac{1}{4} \times (-8)\sin(8x) - 3 \times \frac{1}{2}\cos\left(\frac{1}{2}x\right)$

($k = 8$,and $k = \frac{1}{2}$, respectively)

$\therefore \frac{dy}{dx} = -2\sin(8x) - \frac{3}{2}\cos\left(\frac{x}{2}\right)$

(x) $y = x^3 + 3e^{x}$

$\therefore \frac{dy}{dx} = 3x^2 + 3e^{x}$ ($k = 1$)

Problems

1. Find $\frac{dy}{dx}$ for:

(i) $y = \sin(6x)$ (ii) $y = \cos(2x)$

(iii) $y = 5e^{3x}$ (iv) $y = 6\sin\left(\frac{x}{3}\right)$

(v) $y = 5\ln x - 3e^{-4x}$ (vi) $y = \ln(2x)$

(vii) $y = \ln(x^2)$ (viii) $y = 2e^{5x} + 3e^{4x} - \frac{7}{e^x}$

(ix) $y = \frac{2}{3}\cos(9x) - 5\sin\left(\frac{x}{4}\right)$ (x) $y = x^4 + 3e^{4x}$.

Answers

1. (i) $y = \sin(6x)$ $\therefore \frac{dy}{dx} = 6\cos(6x)$ $(k = 6)$

(ii) $y = \cos(2x)$ $\therefore \frac{dy}{dx} = -2\sin(2x)$ $(k = 2)$

(iii) $y = 5e^{3x}$ $\therefore \frac{dy}{dx} = 15e^{3x}$ $(k = 3)$

(iv) $y = 6\sin\left(\frac{x}{3}\right)$ $\therefore \frac{dy}{dx} = 2\cos\left(\frac{x}{3}\right)$ $(k = \frac{1}{3})$

(v) $y = 5\ln x - 3e^{-4x}$ $\therefore \frac{dy}{dx} = \frac{5}{x} + 12e^{-4x}$ $(k = -4)$

(vi) $y = \ln(2x)$ $\therefore y = \ln 2 + \ln x$ $\therefore \frac{dy}{dx} = \frac{1}{x}$

(vii) $y = \ln(x^2)$ $\therefore y = 2\ln x$ $\therefore \frac{dy}{dx} = \frac{2}{x}$

(viii) $y = 2e^{5x} + 3e^{4x} - \frac{7}{e^x}$ $\therefore y = 2e^{5x} + 3e^{4x} - 7e^{-x}$

$\therefore \frac{dy}{dx} = 10e^{5x} + 12e^{4x} + 7e^{-x}$

($k = 5$, $k = 4$, and $k = -1$, respectively)

(ix) $y = \frac{2}{3}\cos(9x) - 5\sin\left(\frac{x}{4}\right)$ $\therefore \frac{dy}{dx} = -6\sin(9x) - \frac{5}{4}\cos\left(\frac{x}{4}\right)$

($k = 9$, and $k = \frac{1}{4}$, respectively)

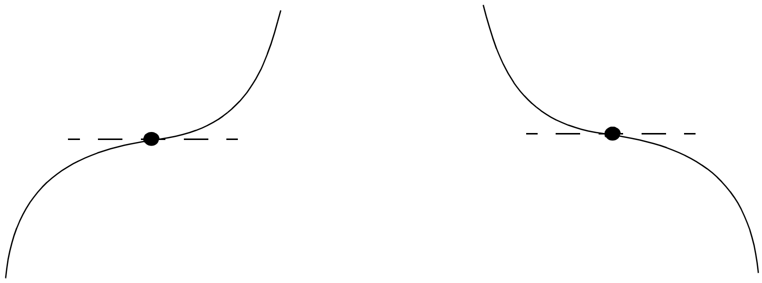
(x) $y = x^4 + 3e^{4x}$ $\therefore \frac{dy}{dx} = 4x^3 + 12e^{4x}$ $(k = 4)$

5.4 Maxima and minima

1. The points on a curve $y = f(x)$ at which the slope (gradient) is 0 are called **stationary points (or critical points)**. In mathematical terms, the definition of a stationary point is as follows:

   For the function $y = f(x)$, if $\frac{dy}{dx} = 0$ when $x = c$, then $x = c$ is a stationary point.

2. A stationary point can be either a **local maximum**, a **local minimum**, or a **horizontal point of inflection**. The two types of horizontal point of inflection are reproduced below.



   Note: 'horizontal' means the slope is 0, and 'inflection' means 'kink'.

3. To find all stationary points for a given function $y = f(x)$:

   I Find $\frac{dy}{dx}$;

   II Solve the equation $\frac{dy}{dx} = 0$ to obtain the x co-ordinates of each stationary point;

   III Substitute these x co-ordinates into $y = f(x)$ to find the corresponding y co-ordinates of each stationary point.

   For instance, if $y = x^2 - 6x$, then $\frac{dy}{dx} = 2x - 6$.

   Hence, for stationary points: $2x - 6 = 0$, $\quad \therefore 2x = 6$, so $x = 3$.

   Now, when $x = 3$, $y = 3^2 - 6 \times 3 = 9 - 18 = -9$,

   i.e. $(3, -9)$ is the (only) stationary point.

4. A stationary point can be classified as a local maximum, a local minimum, or a horizontal point of inflection using the following **First Derivative Test**.

   The stationary point at $x = c$ is:

   (a) a local maximum if, for $\begin{cases} x < c, & f'(x) > 0 \\ x > c, & f'(x) < 0 \end{cases}$;

(b) a local minimum if, for $\begin{cases} x < c, & f'(x) < 0 \\ x > c, & f'(x) > 0 \end{cases}$;

(c) a horizontal point of inflection otherwise.

Using the sign pattern notation for $f\ '(x)$ (not $f(x)$)

a local maximum has the pattern +, 0, -

a local minimum has the pattern -, 0, +

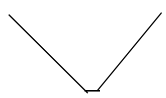
a horizontal point of inflection has the pattern -, 0, - or +, 0, +

For instance, the function $y = x^2 - 6x$ has $\frac{dy}{dx} = 2x - 6$. There is only one stationary point, at $(3, -9)$ (see note 3 above). This stationary point can be classified by investigating the sign of $\frac{dy}{dx}$ (i.e. $f\ '(x)$) on either side of $x = 3$.

Choosing, say, $x = 2$, $\frac{dy}{dx} = 4 - 6 = -2 < 0$.

Choosing, say, $x = 4$, $\frac{dy}{dx} = 8 - 6 = 2 > 0$.

So, the sign pattern of $f\ '(x)$ near $x = 3$ is -, 0, +, and the point $(3, -9)$ is a local minimum. Alternatively, from the above analysis of the sign of $f\ '(x)$, the following (rough) diagram can be drawn.



This also shows that $(3, -9)$ is a local minimum.

Examples

1. For each of the following functions find the x and y co-ordinates of the stationary points. Then classify each of the points as either a local maximum, a local minimum, or a horizontal point of inflection.

   (i) $y = -3x^2 + 12x + 2$

   (ii) $y = x^3 - 6x^2 + 12x + 9$

   (iii) $y = 3x^3 - 9x^2 + 1$

1. (i) $y = -3x^2 + 12x + 2$

$$\therefore \frac{dy}{dx} = -6x + 12.$$

For stationary points, $-6x + 12 = 0$ $\therefore -6x = -12$

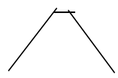
$\therefore x = \dfrac{-12}{-6} = 2$.

When $x = 2$, $y = -3 \times 4 + 12 \times 2 + 2 = -12 + 24 + 2 = 14$.

So, the one stationary point is $(2, 14)$.

Now, for $x < 2$, e.g. $x = 0$, $\dfrac{dy}{dx} = 0 + 12 = 12 > 0$.

Also, for $x > 2$, e.g. $x = 3$, $\dfrac{dy}{dx} = -18 + 12 = -6 < 0$.

So, near $x = 2$, the curve looks like: 

$\therefore (2, 14)$ is a local maximum.

(ii) $y = x^3 - 6x^2 + 12x + 9$

$\therefore \dfrac{dy}{dx} = 3x^2 - 12x + 12$

For stationary points, $3x^2 - 12x + 12 = 0$. Dividing by 3,

$x^2 - 4x + 4 = 0$. Factorising gives

$x^2 - 4x + 4 = (x - 2)(x - 2) = 0$

$\therefore x = 2$

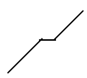
When $x = 2$, $y = 8 - 6 \times 4 + 12 \times 2 + 9 = 8 - 24 + 24 + 9 = 17$

So, the one stationary point is $(2, 17)$.

Now, for $x < 2$, e.g. $x = 0$, $\dfrac{dy}{dx} = 0 - 0 + 12 = 12 > 0$.

Also, for $x > 2$, e.g. $x = 3$,

$\dfrac{dy}{dx} = 3 \times 9 - 12 \times 3 + 12 = 27 - 36 + 12 = 3 > 0$.

So, near $x = 2$, the curve looks like: 

$\therefore (2, 17)$ is a horizontal point of inflection.

Note: as $\dfrac{dy}{dx} = 3x^2 - 12x + 12 = 3(x^2 - 4x + 4) = 3(x - 2)^2$

(a perfect square), the slope cannot be negative, so $x = 2$ has to be a horizontal point of inflection.

(iii) $y = 3x^3 - 9x^2 + 1$

$\therefore \dfrac{dy}{dx} = 9x^2 - 18x$

For stationary points, $9x^2 - 18x = 0$. Dividing by 9,

$x^2 - 2x = 0$. Factorising gives

$x^2 - 2x = x(x-2) = 0$

$\therefore x = 0$, and $x = 2$ (2 stationary points)

When $x = 0$, $y = 0 - 0 + 1 = 1$

So, one stationary point is $(0, 1)$.

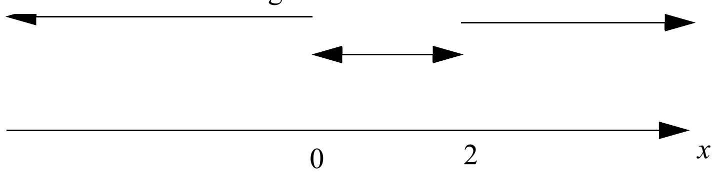
When $x = 2$, $y = 3 \times 8 - 9 \times 4 + 1 = 24 - 36 + 1 = -11$

So, the other stationary point is $(2, -11)$.

It is necessary to find the sign of $f'(x)$ in 3 separate intervals, i.e.

$x < 0$; $0 < x < 2$; $x > 2$,

as shown in the diagram below:



So, for $x < 0$, e.g. $x = -1$,

$$\frac{dy}{dx} = 9 \times 1 - 18 \times (-1) = 9 + 18 = 27 > 0$$

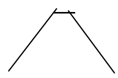
For $0 < x < 2$, e.g. $x = 1$,

$$\frac{dy}{dx} = 9 \times 1 - 18 \times 1 = 9 - 18 = -9 < 0$$

Also, for $x > 2$, e.g. $x = 3$,

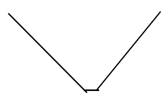
$$\frac{dy}{dx} = 9 \times 9 - 18 \times 3 = 81 - 54 = 27 > 0.$$

So, near $x = 0$, the curve looks like:



$\therefore (0, 1)$ is a local maximum.

Also, near $x = 2$, the curve looks like:



$\therefore (2, -11)$ is a local minimum.

Problems

1. For each of the following functions find the x and y co-ordinates of the stationary points. Then classify each of the points as either a local maximum, a local minimum, or a horizontal point of inflection.

   (i) $y = 5x^2 - 20x + 9$

   (ii) $y = 2x^3 - 9x^2 + 12x + 1$

   (iii) $y = x^3 + 3x^2 + 3x + 1$

Answers

1. (i) $y = 5x^2 - 20x + 9 \quad \therefore \frac{dy}{dx} = 10x - 20 = 10(x-2)$

   The only stationary point is $(2, -11)$, which is a local minimum.

   (ii) $y = 2x^3 - 9x^2 + 12x + 1 \quad \therefore \frac{dy}{dx} = 6x^2 - 18x + 12$

   $$\therefore \frac{dy}{dx} = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$$

   There are two stationary points; $(1, 6)$ is a local maximum and $(2, 5)$ is a local minimum.

   (iii) $y = x^3 + 3x^2 + 3x + 1 \quad \therefore \frac{dy}{dx} = 3x^2 + 6x + 3$

$$\therefore \frac{dy}{dx} = 3(x^2 + 2x + 1) = 3(x+1)^2$$

The only stationary point is $(-1, 0)$, which is a horizontal point of inflection.

5.5 Graph sketching

1. The graphs of straight lines ($y = mx + c$), and parabolas ($y = ax^2 + bx + c$) can be sketched using simple methods. For polynomials of degree greater than 2, stationary points can be used as a valuable aid in sketching.
2. **To sketch a polynomial of degree greater than 2**, e.g. the cubic $y = ax^3 + bx^2 + cx + d$, the following three criteria should be examined:

   I Find the x and y-intercepts;

   II Find and classify the stationary points;

   III Sketch, labelling all intercepts, and stationary points.

Examples

1. For $y = 3x^3 - 9x^2$
   (i) find the x and y-intercepts
   (ii) find and classify the stationary points
   (iii) sketch, labelling all intercepts, and stationary points.

1. $y = 3x^3 - 9x^2$

   (i) When $x = 0$, $y = 0 - 0 = 0$.

   When $y = 0$, $3x^3 - 9x^2 = 0$

   Factorising gives $3x^3 - 9x^2 = 3x^2(x-3) = 0$,

   with solutions $x = 0$ and $x = 3$.

   So, the intercepts are $(0, 0)$, and $(3, 0)$.

   (ii) Now, $\dfrac{dy}{dx} = 9x^2 - 18x$

   For stationary points, $9x^2 - 18x = 0$. Dividing by 9,

   $x^2 - 2x = 0$. Factorising gives

   $x^2 - 2x = x(x-2) = 0$

   $\therefore x = 0$, and $x = 2$ (2 stationary points)

   When $x = 0$, $y = 0$ (an intercept, from (i))

   So, one stationary point is $(0, 0)$.

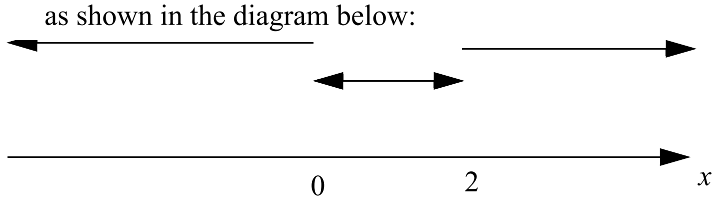
   When $x = 2$, $y = 3 \times 8 - 9 \times 4 = 24 - 36 = -12$

   So, the other stationary point is $(2, -12)$.

   It is necessary to find the sign of $f\,'(x)$ in 3 separate intervals, i.e.

   $x < 0$; $0 < x < 2$; $x > 2$,

   as shown in the diagram below:



.

   So, for $x < 0$, e.g. $x = -1$,

   $$\frac{dy}{dx} = 9 \times 1 - 18 \times (-1) = 9 + 18 = 27 > 0$$

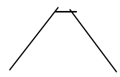
   For $0 < x < 2$, e.g. $x = 1$,

$$\frac{dy}{dx} = 9 \times 1 - 18 \times 1 = 9 - 18 = -9 < 0$$

Also, for $x > 2$, e.g. $x = 3$,

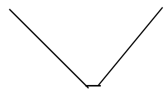
$$\frac{dy}{dx} = 9 \times 9 - 18 \times 3 = 81 - 54 = 27 > 0.$$

So, near $x = 0$, the curve looks like:



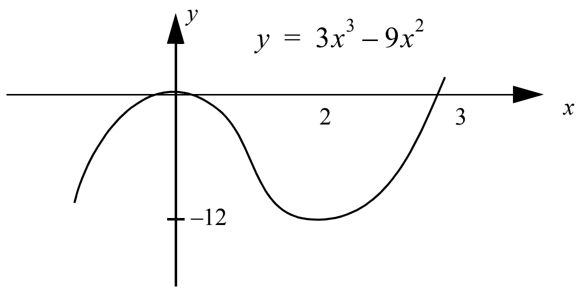
$\therefore (0, 0)$ is a local maximum.

Also, near $x = 2$, the curve looks like:



$\therefore (2, -12)$ is a local minimum.

(iii) The curve is sketched below.



Problems

1. For $y = (x-1)^3$
   (i) find the x and y-intercepts
   (ii) find and classify the stationary points
   (iii) sketch, labelling all intercepts, and stationary points.

$$y = (x-1)^3$$

Answers

1. $y = (x-1)^3$

(i) When $x = 0$, $y = (-1)^3 = -1$.

When $y = 0$, $(x-1)^3 = 0$

with one solution $x = 1$.

So, the intercepts are $(0, -1)$, and $(1, 0)$.

(ii) Now, $y = (x-1)^3 = (x-1)(x-1)^2 = (x-1)(x^2-2x+1)$

$\therefore y = x(x^2-2x+1)-(x^2-2x+1)$

$\therefore y = x^3-2x^2+x-x^2+2x-1$

$\therefore y = x^3-3x^2+3x-1$

$\therefore \dfrac{dy}{dx} = 3x^2-6x+3$

For stationary points, $3x^2-6x+3 = 0$. Dividing by 3,

$x^2-2x+1 = 0$. Factorising gives

$x^2-2x+1 = (x-1)(x-1) = 0$

$\therefore x = 1$. (1 stationary point)

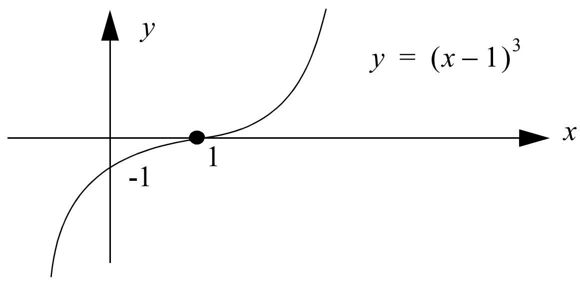
When $x = 1$, $y = 0$ (an intercept, from (i))

So, the only stationary point is $(1, 0)$.

Now, since $\dfrac{dy}{dx} = 3x^2-6x+3 = 3(x^2-2x+1) = 3(x-1)^2$ can never be negative, $(1, 0)$ is a horizontal point of inflection.

The curve has positive slope (except when $x = 1$, where the slope is zero).

(iii) The curve is sketched below.



6

Further rules for derivatives

Introduction

In many applications involving rates of change and the derivative, the relevant mathematical functions are combinations of polynomials, exponentials, logarithms, and trigonometric functions. Further rules are needed to find the derivatives of such combinations of functions. The second derivative of a function leads to a simple and elegant method of classifying stationary points.

This topic covers the 3 rules needed to find the derivative of any product, quotient or chain of functions, the definition and meaning of the second derivative, higher derivatives, and the use of the Second Derivative Test to classify stationary points.After studying this topic, you should be able to:

- distinguish between products of functions and chains of functions;
- understand and use the product rule to find derivatives;
- understand and use the quotient rule to find derivatives;
- understand and use the chain rule to find derivatives;
- find the second derivative of a given function;
- use the Second Derivative Test to classify stationary points.

6.1 Combinations of functions

1. Sums, differences and quotients of functions are easily recognisable, but care must be taken to distinguish between **products of functions** and **functions of functions**. A function of a function is also known as a **chain of functions**.
2. Any trigonometric, exponential or logarithmic function of a function of x is a chain of functions. Some examples are:

   $\sin(2x-1)$; $\cos(x^3)$; e^{4x+3}; $\ln(2x^2+7x-1)$.

   In each case, several operations have been performed on x, and then the trigonometric, exponential or logarithmic function has been calculated. The order of the operations performed on x is important.

   For instance, in $\sin(2x-1)$, x has been multiplied by 2, then 1 has been subtracted. This can be considered as one set of operations on x. After these operations, the sine has been found, and the result is a **chain of functions**.
   The function $(2x-1)\sin x$, however, is a **product** of $(2x-1)$ and $\sin x$.

Examples

1. Identify the form of the following functions

   (i) $\dfrac{\sin x}{x+1}$ (ii) xe^x (iii) $\ln(3x^2+5x-8)$

   (iv) $(x^3-4)^5$ (v) $\cos(x^2-4x+5)$.

1. (i) $\dfrac{\sin x}{x+1}$ is clearly a quotient (the function $\sin x$ being divided by the function $(x+1)$)

(ii) xe^x is a product (the function x being multiplied by the function e^x)

(iii) $\ln(3x^2+5x-8)$ is a function of a function (the first set of operations on x being the function $(3x^2+5x-8)$, followed by the ln function)

(iv) $(x^3-4)^5$ is a function of a function (the first set of operations on x being the function (x^3-4), followed by raising to the power 5)

(v) $\cos(x^2-4x+5)$ is a function of a function (the first set of operations on x being the function (x^2-4x+5), followed by the cosine function).

Problems

1. Identify the form of the following functions:

(i) $x^3\cos x$ (ii) e^{4x^2-7} (iii) $\dfrac{\ln(x+3)}{x^2+1}$

(iv) $(5x^2-4)^3$ (v) $\ln(x^3-4x+8)$.

Answers

1. (i) Product (ii) Function of a function
(iii) Quotient (iv) Function of a function
(v) Function of a function.

6.2 The product rule

1. The **product rule** can be stated as:

if $y = uv$, then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$.

Note that, as $\dfrac{dy}{dx}$ is a sum, the order of the two terms involved in the sum can be changed.

2. Before using the product rule, it is necessary to ensure that the function has the form of a product. If so, the two functions multiplied together can be labelled as u and v, respectively. After some practice, the product rule can be used without recourse to labelling u and v.

Examples

1. Find $\dfrac{dy}{dx}$ for each of the following

(i) $y = (2x-1)\sin x$ (ii) $y = (3x^2-6x+1)e^x$

(iii) $y = (6x+5)\ln x$ (iv) $y = x^2\cos x$

(v) $y = x\sin 2x$ (vi) $y = e^{3x}\cos x$.

1. (i) For $y = (2x-1)\sin x$,

$u = 2x-1$ and $v = \sin x$

$\therefore \dfrac{du}{dx} = 2$ and $\dfrac{dv}{dx} = \cos x$.

By the product rule, $\dfrac{dy}{dx} = (2x-1)\cos x + 2\sin x$.

(ii) For $y = (3x^2-6x+1)e^x$,

$u = 3x^2-6x+1$ and $v = e^x$

$\therefore \dfrac{du}{dx} = 6x-6$ and $\dfrac{dv}{dx} = e^x$.

By the product rule, $\dfrac{dy}{dx} = (3x^2-6x+1)e^x + (6x-6)e^x$

Taking out the common factor of e^x, and factorising gives

$$\frac{dy}{dx} = (3x^2-6x+1+6x-6)e^x$$

$$\therefore \frac{dy}{dx} = (3x^2-5)e^x.$$

(iii) For $y = (6x+5)\ln x$,

$u = 6x+5$ and $v = \ln x$

$\therefore \dfrac{du}{dx} = 6$ and $\dfrac{dv}{dx} = \dfrac{1}{x}$.

By the product rule, $\dfrac{dy}{dx} = (6x+5)\times\dfrac{1}{x} + 6\ln x$.

$$\therefore \frac{dy}{dx} = \frac{6x+5}{x} + 6\ln x.$$

(iv) For $y = x^2\cos x$,

$u = x^2$ and $v = \cos x$

$\therefore \dfrac{du}{dx} = 2x$ and $\dfrac{dv}{dx} = -\sin x$.

By the product rule, $\dfrac{dy}{dx} = x^2 \times (-\sin x) + 2x\cos x$.

$$\therefore \frac{dy}{dx} = -x^2 \sin x + 2x\cos x.$$

(v) For $y = x\sin 2x$,

$u = x$ and $v = \sin 2x$

$\therefore \dfrac{du}{dx} = 1$ and $\dfrac{dv}{dx} = 2\cos 2x \quad (k = 2)$

By the product rule, $\dfrac{dy}{dx} = x \times 2\cos 2x + 1\sin 2x$.

$$\therefore \frac{dy}{dx} = 2x\cos 2x + \sin 2x.$$

(vi) For $y = e^{3x}\cos x$,

$u = e^{3x}$ and $v = \cos x$

$\therefore \dfrac{du}{dx} = 3e^{3x} \quad (k = 3)$ and $\dfrac{dv}{dx} = -\sin x$.

By the product rule, $\dfrac{dy}{dx} = e^{3x} \times (-\sin x) + 3e^{3x}\cos x$.

$$\therefore \frac{dy}{dx} = e^{3x}(-\sin x + 3\cos x).$$

Problems

1. Find $\dfrac{dy}{dx}$ for each of the following

(i) $y = x^3 \sin x$ (ii) $y = (6x^2 - 12x + 5)e^x$

(iii) $y = (7x - 4)\ln x$ (iv) $y = (2x^2 + 3)\cos x$

(v) $y = x\cos 4x$ (vi) $y = e^{-x}\sin x$.

Answers

1. (i) $y = x^3 \sin x$ $\therefore \dfrac{dy}{dx} = x^3\cos x + 3x^2 \sin x$

(ii) $y = (6x^2 - 12x + 5)e^x$ $\therefore \dfrac{dy}{dx} = (6x^2 - 7)e^x$

(iii) $y = (7x - 4)\ln x$ $\therefore \dfrac{dy}{dx} = \dfrac{7x - 4}{x} + 7\ln x$

(iv) $y = (2x^2+3)\cos x \quad \therefore \dfrac{dy}{dx} = -(2x^2+3)\sin x + 4x\cos x$

(v) $y = x\cos 4x \quad \therefore \dfrac{dy}{dx} = -4x\sin 4x + \cos 4x$

(vi) $y = e^{-x}\sin x \quad \therefore \dfrac{dy}{dx} = e^{-x}(\cos x - \sin x)$.

6.3 The quotient rule

1. The **quotient rule** can be stated as:

   if $y = \dfrac{u}{v}$, then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$.

   Note that the order of the two terms involved in the top line cannot be changed.

2. When using the quotient rule, the functions on the top and bottom lines can be labelled as u and v, respectively. After some practice, the quotient rule can be used without recourse to labelling u and v.

Examples

1. Find $\dfrac{dy}{dx}$ for each of the following

   (i) $y = \dfrac{2x-3}{2x+7}$ (ii) $y = \dfrac{e^{4x}}{x+6}$

   (iii) $y = \dfrac{x^2-4}{2x^2+1}$ (iv) $y = \dfrac{\ln x}{x^2}$

   (v) $y = \dfrac{\sin 2x}{x}$ (vi) $y = \dfrac{\sin x}{\cos x}$.

1. (i) For $y = \dfrac{2x-3}{2x+7}$,

   $u = 2x-3$ and $v = 2x+7$

   $\therefore \dfrac{du}{dx} = 2$ and $\dfrac{dv}{dx} = 2$.

   By the quotient rule, $\dfrac{dy}{dx} = \dfrac{2(2x+7) - 2(2x-3)}{(2x+7)^2}$.

   $$\therefore \frac{dy}{dx} = \frac{4x+14-4x+6}{(2x+7)^2}$$

$$\therefore \frac{dy}{dx} = \frac{20}{(2x+7)^2}.$$

(ii) For $y = \dfrac{e^{4x}}{x+6}$,

$u = e^{4x}$ and $v = x + 6$

$\therefore \dfrac{du}{dx} = 4e^{4x} \quad (k = 4)$ and $\dfrac{dv}{dx} = 1$.

By the quotient rule, $\dfrac{dy}{dx} = \dfrac{(x+6)(4e^{4x}) - 1(e^{4x})}{(x+6)^2}$.

$$\therefore \frac{dy}{dx} = \frac{[4(x+6)-1]e^{4x}}{(x+6)^2} =$$

$$\therefore \frac{dy}{dx} = \frac{[4x+24-1]e^{4x}}{(x+6)^2}$$

$$\therefore \frac{dy}{dx} = \frac{(4x+23)e^{4x}}{(x+6)^2}.$$

(iii) For $y = \dfrac{x^2-4}{2x^2+1}$,

$u = x^2 - 4$ and $v = 2x^2 + 1$

$\therefore \dfrac{du}{dx} = 2x$ and $\dfrac{dv}{dx} = 4x$.

By the quotient rule, $\dfrac{dy}{dx} = \dfrac{2x(2x^2+1) - 4x(x^2-4)}{(2x^2+1)^2}$

$$\therefore \frac{dy}{dx} = \frac{4x^3 + 2x - 4x^3 + 16x}{(2x^2+1)^2}$$

$$\therefore \frac{dy}{dx} = \frac{18x}{(2x^2+1)^2}.$$

(iv) For $y = \dfrac{\ln x}{x^2}$,

$u = \ln x$ and $v = x^2$

$\therefore \dfrac{du}{dx} = \dfrac{1}{x}$ and $\dfrac{dv}{dx} = 2x$.

By the quotient rule, $\dfrac{dy}{dx} = \dfrac{x^2 \times \dfrac{1}{x} - 2x\ln x}{(x^2)^2}$

$$\therefore \frac{dy}{dx} = \frac{x - 2x\ln x}{x^4} = \frac{x(1 - 2\ln x)}{x^4}$$

$$\therefore \frac{dy}{dx} = \frac{1 - 2\ln x}{x^3}.$$

(v) For $y = \dfrac{\sin 2x}{x}$,

$u = \sin 2x$ and $v = x$

$\therefore \dfrac{du}{dx} = 2\cos 2x \quad (k = 2)$ and $\dfrac{dv}{dx} = 1$.

By the quotient rule, $\dfrac{dy}{dx} = \dfrac{x(2\cos 2x) - 1\sin 2x}{x^2}$.

$$\therefore \frac{dy}{dx} = \frac{2x\cos 2x - \sin 2x}{x^2}.$$

(vi) For $y = \dfrac{\sin x}{\cos x}$,

$u = \sin x$ and $v = \cos x$

$\therefore \dfrac{du}{dx} = \cos x$ and $\dfrac{dv}{dx} = -\sin x$.

By the quotient rule, $\dfrac{dy}{dx} = \dfrac{\cos x \times \cos x - \sin x \times (-\sin x)}{(\cos x)^2}$.

$$\therefore \frac{dy}{dx} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}.$$

By the basic trigonometric identity, $\therefore \dfrac{dy}{dx} = \dfrac{1}{\cos^2 x}$.

Problems

1. Find $\dfrac{dy}{dx}$ for each of the following

(i) $y = \dfrac{3x - 7}{7x - 2}$ (ii) $y = \dfrac{e^{2x}}{4x + 1}$

(iii) $y = \dfrac{2x^2 + 3}{x^2 + 1}$ (iv) $y = \dfrac{\ln x}{2x^3}$

(v) $y = \dfrac{\sin 5x}{x}$ (vi) $y = \dfrac{\cos x}{\sin x}$.

Answers

1. (i) $y = \dfrac{3x-7}{7x-2}$ $\quad \therefore \dfrac{dy}{dx} = \dfrac{43}{(7x-2)^2}$

(ii) $y = \dfrac{e^{2x}}{4x+1}$ $\quad \therefore \dfrac{dy}{dx} = \dfrac{(8x-2)e^{2x}}{(4x+1)^2}$

(iii) $y = \dfrac{2x^2+3}{x^2+1}$ $\quad \therefore \dfrac{dy}{dx} = \dfrac{-2x}{(x^2+1)^2}$

(iv) $y = \dfrac{\ln x}{2x^3}$ $\quad \therefore \dfrac{dy}{dx} = \dfrac{1-3\ln x}{2x^4}$

(v) $y = \dfrac{\sin 5x}{x}$ $\quad \therefore \dfrac{dy}{dx} = \dfrac{5x\cos 5x - \sin 5x}{x^2}$

(vi) $y = \dfrac{\cos x}{\sin x}$ $\quad \therefore \dfrac{dy}{dx} = \dfrac{-1}{\sin^2 x}$.

6.4 The chain rule

1. The **chain rule** (or function of a function rule) is usually stated as:

   if $y = f(u)$, and $u = g(x)$, then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$.

   Note that the function *u* is not normally given, but is recognisable as the first set of operations performed on *x*.
2. In most cases, u is either a power of e, or appears in brackets, or appears under a square root sign. For instance, $u = 3x^2 - 1$ in each of:

   $y = e^{3x^2-1}$; $y = \cos(3x^2-1)$; $y = \ln(3x^2-1)$; $y = \sqrt{3x^2-1}$.
3. Note that it is not necessary to use the chain rule to find $\dfrac{dy}{dx}$ when

   $y = \sin 3x$. Using $k = 3$, the derivative is given by $\dfrac{dy}{dx} = 3\cos 3x$.
4. Combinations of the product, quotient and chain rules are needed to find the derivatives of some functions (see Example 2 below).

Examples

1. Find $\dfrac{dy}{dx}$ for each of the following

(i) $y = (3x^2-8)^4$ $\qquad$ (ii) $y = e^{4x^5}$

(iii) $y = \ln(5x^4 - 3x^2 - 1)$ $\qquad$ (iv) $y = \sqrt{x^2+7}$

(v) $y = \sin(8x^3 - 5)$ $\qquad$ (vi) $y = \cos^3 x$.

2. Find $\dfrac{dy}{dx}$ for each of the following

   (i) $y = x\ln(x^2+1)$ (ii) $y = \sin(xe^x)$.

1. (i) For $y = (3x^2-8)^4$,

   let $u = 3x^2 - 8$. $\therefore y = u^4$

   $\therefore \dfrac{du}{dx} = 6x$, and $\dfrac{dy}{du} = 4u^3$.

   By the chain rule, $\dfrac{dy}{dx} = (6x)\times(4u^3) = 24xu^3$.

   As $u = 3x^2-8$, expressing the final answer in terms of x only,

   $$\frac{dy}{dx} = 24x(3x^2-8)^3.$$

   (ii) For $y = e^{4x^5}$,

   let $u = 4x^5$. $\therefore y = e^u$

   $\therefore \dfrac{du}{dx} = 20x^4$, and $\dfrac{dy}{du} = e^u$.

   By the chain rule, $\dfrac{dy}{dx} = 20x^4 \times e^u = 20x^4e^u$.

   As $u = 4x^5$, expressing the final answer in terms of x only,

   $$\frac{dy}{dx} = 20x^4e^{4x^5}.$$

   (iii) For $y = \ln(5x^4-3x^2-1)$

   let $u = 5x^4-3x^2-1$. $\therefore y = \ln u$

   $\therefore \dfrac{du}{dx} = 20x^3-6x$, and $\dfrac{dy}{du} = \dfrac{1}{u}$.

   By the chain rule, $\dfrac{dy}{dx} = (20x^3-6x)\times\dfrac{1}{u} = \dfrac{20x^3-6x}{u}$.

   As $u = 5x^4-3x^2-1$, expressing the final answer in terms of x only,

   $$\frac{dy}{dx} = \frac{20x^3-6x}{5x^4-3x^2-1}.$$

   (iv) For $y = \sqrt{x^2+7}$, i.e. $y = (x^2+7)^{1/2}$

   let $u = x^2+7$. $\therefore y = u^{1/2}$

   $\therefore \dfrac{du}{dx} = 2x$, and $\dfrac{dy}{du} = \dfrac{1}{2}u^{-1/2}$.

   By the chain rule, $\dfrac{dy}{dx} = 2x\times\dfrac{1}{2}u^{-1/2} = xu^{-1/2} = \dfrac{x}{u^{1/2}}$.

As $u = x^2 + 7$, expressing the final answer in terms of x only,

$$\frac{dy}{dx} = \frac{x}{(x^2+7)^{1/2}} = \frac{x}{\sqrt{x^2+7}}.$$

(v) For $y = \sin(8x^3 - 5)$,

let $u = 8x^3 - 5$. $\quad \therefore y = \sin u$

$\therefore \dfrac{du}{dx} = 24x^2$, and $\dfrac{dy}{du} = \cos u$.

By the chain rule, $\dfrac{dy}{dx} = 24x^2 \cos u$.

As $u = 8x^3 - 5$, expressing the final answer in terms of x only,

$$\frac{dy}{dx} = 24x^2 \cos(8x^3 - 5).$$

(vi) For $y = \cos^3 x$, i.e. $y = (\cos x)^3$,

let $u = \cos x$. $\quad \therefore y = u^3$

$\therefore \dfrac{du}{dx} = -\sin x$, and $\dfrac{dy}{du} = 3u^2$

By the chain rule, $\dfrac{dy}{dx} = -3u^2 \sin x$.

As $u = \cos x$, expressing the final answer in terms of x only,

$$\frac{dy}{dx} = -3\cos^2 x \, \sin x.$$

2. (i) $y = x\ln(x^2 + 1)$ is a product, so the product rule must be used.

$\therefore u = x$ and $v = \ln(x^2 + 1)$

$\therefore \dfrac{du}{dx} = 1$ and the chain rule is needed to find $\dfrac{dv}{dx}$.

Let $w = x^2 + 1$ (as u has been used earlier) $\quad \therefore v = \ln w$

$\therefore \dfrac{dw}{dx} = 2x$ and $\dfrac{dv}{dw} = \dfrac{1}{w}$

By the chain rule $\dfrac{dv}{dx} = \dfrac{dv}{dw} \times \dfrac{dw}{dx} = (2x) \times \dfrac{1}{w} = \dfrac{2x}{w} = \dfrac{2x}{x^2+1}$

i.e. $u = x$ and $v = \ln(x^2 + 1)$

$$\therefore \frac{du}{dx} = 1 \quad \text{and} \quad \frac{dv}{dx} = \frac{2x}{x^2+1}$$

By the product rule, $\dfrac{dy}{dx} = x \times \dfrac{2x}{x^2+1} + 1\ln(x^2+1)$.

$$\therefore \frac{dy}{dx} = \frac{2x^2}{x^2+1} + \ln(x^2+1).$$

(ii) $y = \sin(xe^x)$ is a function of a function, so the chain rule must be used.

let $u = xe^x$ $\qquad \therefore y = \sin u$

$\therefore \dfrac{du}{dx}$ is found using the product rule, and $\dfrac{dy}{du} = \cos u$.

To find $\dfrac{du}{dx}$,

let $w = x$ (as u has been used), and $v = e^x$.

$$\therefore \frac{dw}{dx} = 1 \quad \text{and} \quad \frac{dv}{dx} = e^x.$$

Now, $\dfrac{du}{dx} = w\dfrac{dv}{dx} + v\dfrac{dw}{dx}$

$$\therefore \frac{du}{dx} = xe^x + e^x = (x+1)e^x.$$

i.e. $\quad u = xe^x \qquad y = \sin u$

$$\therefore \frac{du}{dx} = (x+1)e^x \quad \text{and} \quad \frac{dy}{du} = \cos u.$$

By the chain rule, $\dfrac{dy}{dx} = (x+1)e^x \cos u$.

As $u = xe^x$, expressing the final answer in terms of x only,

$$\frac{dy}{dx} = (x+1)e^x\cos(xe^x).$$

Problems

1. Find $\dfrac{dy}{dx}$ for each of the following

(i) $y = (2x^3 - 5)^8$ $\qquad$ (ii) $y = e^{4\sin x}$

(iii) $y = \ln(3x^3 - 5x^2 - 1)$ $\qquad$ (iv) $y = \sqrt{2x^3 + 9}$

(v) $y = \cos(4x^2 - x + 3)$ $\qquad$ (vi) $y = \sin^3 x$.

2. Find $\dfrac{dy}{dx}$ for each of the following

(i) $y = xe^{x^2+3}$ (ii) $y = (4 + x\ln x)^5$.

Answers

1. (i) $y = (2x^3 - 5)^8$ $\quad \therefore \dfrac{dy}{dx} = 48x^2(2x^3 - 5)^7$

(ii) $y = e^{4\sin x}$ $\quad \therefore \dfrac{dy}{dx} = 4\cos x \, e^{4\sin x}$

(iii) $y = \ln(3x^3 - 5x^2 - 1)$ $\quad \therefore \dfrac{dy}{dx} = \dfrac{9x^2 - 10x}{3x^3 - 5x^2 - 1}$

(iv) $y = \sqrt{2x^3 + 9}$ $\quad \therefore \dfrac{dy}{dx} = \dfrac{3x^2}{\sqrt{2x^3 + 9}}$

(v) $y = \cos(4x^2 - x + 3)$ $\quad \therefore \dfrac{dy}{dx} = -(8x - 1)\sin(4x^2 - x + 3)$

(vi) $y = \sin^3 x$ $\quad \therefore \dfrac{dy}{dx} = 3\sin^2 x \cos x$.

2. (i) $y = xe^{x^2+3}$ $\quad \therefore \dfrac{dy}{dx} = (2x^2 + 1)e^{x^2+3}$

(ii) $y = (4 + x\ln x)^5$ $\quad \therefore \dfrac{dy}{dx} = 5(1 + \ln x)(4 + x\ln x)^4$.

6.5 Higher derivatives

1. Given the function $y = f(x)$, the second derivative is the derivative of the derivative, and is generally denoted by $\dfrac{d^2y}{dx^2}$ or $f''(x)$ or y''.

   Successive differentiations produce the third, and fourth derivatives, and the process can be continued indefinitely. All derivatives beyond the first derivative are called higher derivatives.

2. For motion in a straight line, the first derivative represents velocity, and the second derivative represents acceleration. Physical meanings can be attached to other derivatives. In particular, the fourth derivative is used in the equations describing the bending of beams.

3. The Second Derivative Test can be used to determine local maxima and minima, and is stated below.

   Given the function $y = f(x)$,

   if $\dfrac{dy}{dx} = 0$ when $x = c$, then $x = c$ provides:

(i) a local maximum if $\frac{d^2y}{dx^2} < 0$;

(ii) a local minimum if $\frac{d^2y}{dx^2} > 0$;

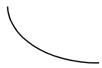
(iii) no conclusion if $\frac{d^2y}{dx^2} = 0$ (and the First Derivative test must be used instead).

4. The sign (positive or negative) of the second derivative $\frac{d^2y}{dx^2}$ determines the concavity (curvature) of the curve $y = f(x)$, as follows:

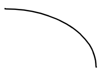
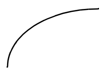
   if $\frac{d^2y}{dx^2} > 0$ at a point, the curve is concave up at that point;

   if $\frac{d^2y}{dx^2} < 0$ at a point, the curve is concave down at that point

   A **concave up** curve is **cup-shaped**

   i.e.  and  are both concave up curves.

   A **concave down** curve is **frown-shaped**

   i.e.  and  are both concave down curves.

   For the parabola $y = ax^2 + bx + c$,

   $$\frac{dy}{dx} = 2ax + b \qquad \text{and} \qquad \frac{d^2y}{dx^2} = 2a.$$

   So, if $a > 0$, $\frac{d^2y}{dx^2} = 2a > 0$, and the parabola is concave up.

   Also, if $a < 0$, $\frac{d^2y}{dx^2} = 2a < 0$, and the parabola is concave down.

   These results agree with those used in sketching parabolas in Chapter 2 of this Study Guide.

5. A point of inflection occurs when the concavity changes from up to down (or down to up). Hence, a point of inflection occurs at $x = c$ if $\frac{d^2y}{dx^2} = 0$ at $x = c$, and $\frac{d^2y}{dx^2}$ has different signs either side of $x = c$.

Examples

1. Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ for each of the following:

(i) $y = 2x^3 - 4x^2 - 5x + 9$ (ii) $y = (x-2)e^{2x}$

(iii) $y = x\ln x$ (iv) $y = 8\sqrt{x} - \cos 3x$.

2. For each of the following functions find the x and y co-ordinates of the stationary points. Then classify each of the points as either a local maximum, or a local minimum using the Second Derivative Test.

(i) $y = 3x^3 - 9x^2 + 1$; (ii) $y = x^4 - 8x^2 + 10$

1. (i) $y = 2x^3 - 4x^2 - 5x + 9$

$$\therefore \frac{dy}{dx} = 6x^2 - 8x - 5$$

$$\therefore \frac{d^2y}{dx^2} = 12x - 8\,.$$

(ii) $y = (x-2)e^{2x}$ (a product)

let $u = x - 2$ and $v = e^{2x}$

$\therefore \dfrac{du}{dx} = 1$ and $\dfrac{dv}{dx} = 2e^{2x}$.

By the product rule, $\dfrac{dy}{dx} = (x-2)\times(2e^{2x}) + 1e^{2x} = (2x - 4 + 1)e^{2x}$.

$\therefore \dfrac{dy}{dx} = (2x-3)e^{2x}$ (a product)

let $u = 2x - 3$ and $v = e^{2x}$

$\therefore \dfrac{du}{dx} = 2$ and $\dfrac{dv}{dx} = 2e^{2x}$.

By the product rule,

$$\frac{d^2y}{dx^2} = (2x-3)\times(2e^{2x}) + 2e^{2x} = (4x - 6 + 2)e^{2x}.$$

$$\therefore \frac{d^2y}{dx^2} = (4x-4)e^{2x}.$$

(iii) $y = x\ln x$ (a product)

let $u = x$ and $v = \ln x$

$\therefore \dfrac{du}{dx} = 1$ and $\dfrac{dv}{dx} = \dfrac{1}{x}$.

By the product rule, $\dfrac{dy}{dx} = x \times \dfrac{1}{x} + 1\ln x$.

$$\therefore \frac{dy}{dx} = 1 + \ln x$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{x}.$$

(iv) $y = 8\sqrt{x} - \cos 3x$ $\therefore y = 8x^{1/2} - \cos 3x$

$$\therefore \frac{dy}{dx} = 8 \times \frac{1}{2}x^{-1/2} - (-3\sin 3x)$$

$$\therefore \frac{dy}{dx} = 4x^{-1/2} + 3\sin 3x$$

$$\therefore \frac{d^2y}{dx^2} = 4 \times \left(\frac{-1}{2}x^{-3/2}\right) + 3(3\cos 3x)$$

$$\therefore \frac{d^2y}{dx^2} = -2x^{-3/2} + 9\cos 3x$$

2. (i) $y = 3x^3 - 9x^2 + 1$

$$\therefore \frac{dy}{dx} = 9x^2 - 18x$$

For stationary points, $9x^2 - 18x = 0$. Dividing by 9,

$x^2 - 2x = 0$. Factorising gives

$x^2 - 2x = x(x-2) = 0$

$\therefore x = 0$, and $x = 2$ (2 stationary points)

When $x = 0$, $y = 0 - 0 + 1 = 1$

So, one stationary point is $(0, 1)$.

When $x = 2$, $y = 3 \times 8 - 9 \times 4 + 1 = 24 - 36 + 1 = -11$

So, the other stationary point is $(2, -11)$.

Now, $\dfrac{d^2y}{dx^2} = 18x - 18 = 18(x-1)$.

For the stationary point $(0, 1)$, i.e. when $x = 0$,

$\dfrac{d^2y}{dx^2} = 0 - 18 = -18 < 0$, so $(0, 1)$ is a local maximum.

For the stationary point $(2, -11)$, i.e. when $x = 2$,

$\frac{d^2y}{dx^2} = 36 - 18 = 18 > 0$, so $(2, -11)$ is a local minimum.

(ii) $y = x^4 - 8x^2 + 10$

$\therefore \frac{dy}{dx} = 4x^3 - 16x$

For stationary points, $4x^3 - 16x = 0$.

Factorising gives $4x^3 - 16x = 4x(x^2 - 4) = 4x(x + 2)(x - 2) = 0$.

$\therefore x = 0$, $x = -2$, and $x = 2$ (3 stationary points)

When $x = 0$, $y = 0 - 0 + 10 = 10$

So, one stationary point is $(0, 10)$.

When $x = -2$, $y = (-2)^4 - 8 \times (-2)^2 + 10 = 16 - 32 + 10 = -6$

So, the second stationary point is $(-2, -6)$.

When $x = 2$, $y = 2^4 - 8 \times 2^2 + 10 = 16 - 32 + 10 = -6$.

So, the third stationary point is $(2, -6)$.

Now, $\frac{d^2y}{dx^2} = 12x^2 - 16$.

For the stationary point $(0, 10)$, i.e. when $x = 0$,

$\frac{d^2y}{dx^2} = 0 - 16 = -16 < 0$,

so $(0, 10)$ is a local maximum.

For the stationary point $(-2, -6)$, i.e. when $x = -2$,

$\frac{d^2y}{dx^2} = 12 \times (-2)^2 - 16 = 48 - 16 = 32 > 0$,

so $(2, -11)$ is a local minimum.

For the stationary point $(2, -6)$, i.e. when $x = 2$,

$\frac{d^2y}{dx^2} = 12 \times 2^2 - 16 = 48 - 16 = 32 > 0$,

so $(2, -11)$ is a local minimum.

Problems

1. Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ for each of the following

(i) $y = x^4 - 4x^3 - 7x + 7$ (ii) $y = (4x-3)e^x$

(iii) $y = x\sin x$ (iv) $y = 16x^{3/4} - 4\ln x$.

2. For each of the following functions find the x and y co-ordinates of the stationary points. Then classify each of the points as either a local maximum, or a local minimum using the Second Derivative Test.

(i) $y = 2x^3 + 9x^2 - 24x$ (ii) $y = x^3 + 3x^2 - 4$.

Answers

1. (i) $\dfrac{dy}{dx} = 4x^3 - 12x^2 - 7$ and $\dfrac{d^2y}{dx^2} = 12x^2 - 24x$

(ii) $\dfrac{dy}{dx} = (4x+1)e^x$ and $\dfrac{d^2y}{dx^2} = (4x+5)e^x$

(iii) $\dfrac{dy}{dx} = x\cos x + \sin x$ and $\dfrac{d^2y}{dx^2} = 2\cos x - x\sin x$

(iv) $\dfrac{dy}{dx} = 12x^{-1/4} - \dfrac{4}{x}$ and $\dfrac{d^2y}{dx^2} = -3x^{-5/4} + \dfrac{4}{x^2}$

2. (i) For $y = 2x^3 + 9x^2 - 24x$,

$\dfrac{dy}{dx} = 6x^2 + 18x - 24$, and $\dfrac{d^2y}{dx^2} = 12x + 18$.

The stationary points are: $(1, -13)$ (a local minimum), and $(-4, 112)$ (a local maximum).

(ii) For $y = x^3 + 3x^2 - 4$,

$\dfrac{dy}{dx} = 3x^2 + 6x$, and $\dfrac{d^2y}{dx^2} = 6x + 6$.

The stationary points are: $(0, -4)$ (a local minimum), and $(-2, 0)$ (a local maximum).

7

Integration

Introduction

This topic covers the fundamental relationship between the concepts of differentiation and integration. This fundamental relationship permits the evaluation of both indefinite and definite integrals, applications of which abound in engineering, technology and the sciences. Indefinite integrals are used to solve problems concerning the motion of an object in a straight line, and definite integrals are used to find areas of regions in the plane. After studying this topic you should be able to:

- understand the relationship between integration and differentiation;
- integrate polynomial, trigonometric, and exponential functions;
- evaluate definite integrals of polynomial, trigonometric, and exponential functions;
- find areas contained between curves;
- understand the distinction between definite integrals and areas.

7.1 Anti-derivatives

1. If the derivative of a function is known, the original function can be found (almost) by **anti-differentiating (integrating)**. The anti-derivative **always** contains an arbitrary constant C, which cannot be determined unless extra information is given.
2. When applied to sketches of graphs, anti-differentiation is the process of finding the equation to a curve from the slope of that curve. In the simple case of a linear function, if the slope m is known, e.g. $m = 3$, then the equation of the straight line is $y = 3x + C$ .The same arbitrary constant C appears in the anti-derivative of any function.
   In general, given the slope of a curve, the equation of that curve is known, except for its y-intercept. If, however, a particular point on the curve is specified, the equation is completely determined.
3. The anti-derivative is more usually called the **indefinite integral**, where the word 'indefinite' indicates the presence of the constant C in the answer.
4. The notation $\int f(x)dx$ must be strictly observed. In particular, the dx cannot be omitted.
5. After the answer to an indefinite integral has been found, it can always be checked by differentiation. For instance, the answer to $\int 3x^2dx$ is $x^3 + C$ because $\frac{d}{dx}(x^3 + C) = 3x^2$.

Examples

1. Write the following statements about derivatives as statements about indefinite integrals

   (i) $\frac{d}{dx}(x^5) = 5x^4$ (ii) $\frac{d}{dx}(e^{2x}) = 2e^{2x}$

(iii) $\frac{d}{dx}(\sin 2x) = 2\cos 2x$ (iv) $\frac{d}{dx}(\cos 3x) = -3\sin 3x$.

1. (i) $\int 5x^4 dx = x^5 + C$ (ii) $\int 2e^{2x} dx = e^{2x} + C$

(iii) $\int 2\cos 2x dx = \sin 2x + C$ (iii) $\int -3\sin 3x dx = \cos 3x + C$.

Problems

1. Write the following statements about derivatives as statements about indefinite integrals:

(i) $\frac{d}{dx}(x^7) = 7x^6$ (ii) $\frac{d}{dx}(e^{5x}) = 5e^{5x}$

(iii) $\frac{d}{dx}(\sin 6x) = 6\cos 6x$ (iv) $\frac{d}{dx}(\cos 2x) = -2\sin 2x$.

Answers

1. (i) $\int 7x^6 dx = x^7 + C$ (ii) $\int 5e^{5x} dx = e^{5x} + C$

(iii) $\int 6\cos 6x dx = \sin 6x + C$ (iii) $\int -2\sin 2x dx = \cos 2x + C$.

7.2 Rules for integration

1. Since differentiation and integration are inverse operations, every rule for derivatives can be converted into a corresponding rule for integrals. In particular, since,

   if $y = \frac{x^{n+1}}{n+1}$, then $\frac{dy}{dx} = (n+1) \times \frac{x^n}{n+1} = x^n$, it follows that:

   $$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad \text{(provided } n+1 \neq 0\text{, i.e. } n \neq -1\text{).}$$

2. Note that, in the answer above, the power of x is the same as the denominator, and is found by adding 1 to the original power of x. In terms of the power of x, **differentiation** takes the power **down**, whereas **integration** takes the power **up**.

3. The results in the table of integrals (below) can all be proved by converting a rule for differentiation into a corresponding rule for integration. For instance, the result:

   if $y = \ln x$, then $\frac{dy}{dx} = \frac{1}{x}$ leads to $\int \frac{1}{x} dx = \ln|x| + C$.

   Note that the **modulus (absolute value)**, $|x|$ ensures that the logarithm is defined for both positive and negative values of x, as, by definition $|x|$

is the distance of x from the origin, and is never negative. As examples, $|-3| = 3$; $|-8| = 8$; and $|6| = 6$.

4. More general results for integrals of trigonometric, exponential and logarithmic functions can be derived, as follows.

   Since, if $y = \frac{-1}{k}\cos kx$, then $\frac{dy}{dx} = \frac{-1}{k} \times (-k\sin kx) = \sin kx$

   $$\int \sin kx dx = \frac{-1}{k}\cos kx + C.$$

   Similarly, if $y = \frac{1}{k}\sin kx$, then $\frac{dy}{dx} = \frac{1}{k} \times (k\cos kx) = \cos kx$

   $$\therefore \int \cos kx dx = \frac{1}{k}\sin kx + C.$$

   Also, if $y = \frac{1}{k}e^{kx}$, then $\frac{dy}{dx} = \frac{1}{k} \times (ke^{kx}) = e^{kx}$

   $$\therefore \int e^{kx} dx = \frac{1}{k}e^{kx} + C.$$

5. In summary, the 5 major results for the indefinite integrals of the commonly used functions are shown in the table below (k is a constant).

$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1} + C \ (n \neq -1)$
$\sin kx$	$\frac{-1}{k}\cos kx + C$
$\cos kx$	$\frac{1}{k}\sin kx + C$
e^{kx}	$\frac{1}{k}e^{kx} + C$
$\frac{1}{x}$	$\ln\lvert x\rvert + C$

Examples

1. Find:

   (i) $I = \int x^6 dx$ (ii) $I = \int x^{-6} dx$

   (iii) $I = \int \sin 6x dx$ (iv) $I = \int \cos 6x dx$

(v) $I = \int e^{6x} dx$ (vi) $I = \int e^{-6x} dx$.

2. Find:

(i) $I = \int\left(2x + \frac{5}{x^2} - 7\right)dx$ (ii) $I = \int\left(4\sqrt{x} - \frac{3}{\sqrt{x}} + 2\right)dx$

(iii) $I = \int(2x^{-1/3} + 5x^{2/3})dx$ (iv) $I = \int 3x(x-8)dx$

(v) $I = \int(x+3)^2 dx$ (vi) $I = \int\left(\frac{4}{x} - 5\right)dx$.

3. Find:

(i) $I = \int(3e^{-9x} + 8e^{4x})dx$ (ii) $I = \int 5\cos\left(\frac{x}{2}\right)dx$

(iii) $I = \int(5\sin 2x + 3\cos 8x)dx$ (iv) $I = \int \frac{2}{e^{3x}}\,dx$.

1. (i) $I = \int x^6 dx = \frac{x^7}{7} + C$ $(n = 6)$.

(ii) $I = \int x^{-6} dx = \frac{x^{-5}}{-5} + C = \frac{-x^{-5}}{5} + C = \frac{-1}{5x^5} + C$ $(n = -6)$.

(iii) $I = \int \sin 6x dx = \frac{-1}{6}\cos 6x + C$ $(k = 6)$.

(iv) $I = \int \cos 6x dx = \frac{1}{6}\sin 6x + C$ $(k = 6)$.

(v) $I = \int e^{6x} dx = \frac{1}{6}e^{6x} + C$ $(k = 6)$.

(vi) $I = \int e^{-6x} dx = \frac{1}{-6}e^{-6x} + C = \frac{-1}{6}e^{-6x} + C$ $(k = -6)$

2. (i) $I = \int\left(2x + \frac{5}{x^2} - 7\right)dx = \int(2x^1 + 5x^{-2} - 7x^0)dx$

$(n = 2, -1, 1$ respectively$)$

$= \frac{2x^2}{2} + \frac{5x^{-1}}{-1} - \frac{7x^1}{1} + C$ $(n = 2, -1, 1$ respectively$)$

$\therefore I = x^2 - 5x^{-1} - 7x + C$.

(ii) $I = \int\left(4\sqrt{x} - \frac{3}{\sqrt{x}} + 2\right)dx = \int(4x^{1/2} - 3x^{-1/2} + 2x^0)dx$

$\left(n = \frac{1}{2}, \frac{-1}{2}, 0\right.$ respectively$\left.\right)$

$$= \frac{4x^{3/2}}{3/2} - \frac{3x^{1/2}}{1/2} + \frac{2x^1}{1} + C = 4 \times \frac{2}{3}x^{3/2} - 3 \times 2x^{1/2} + 2x + C$$

$$\therefore I = \frac{8x^{3/2}}{3} - 6x^{1/2} + 2x + C.$$

(iii) $I = \int (2x^{-1/3} + 5x^{2/3})dx \qquad (n = \frac{-1}{3}, \frac{2}{3} \text{ respectively})$

$$= \frac{2x^{2/3}}{2/3} + \frac{5x^{5/3}}{5/3} + C = 2 \times \frac{3}{2}x^{2/3} + 5 \times \frac{3}{5}x^{5/3} + C$$

$$\therefore I = 3x^{2/3} + 3x^{5/3} + C.$$

(iv) $I = \int 3x(x-8)dx = \int (3x^2 - 24x^1)dx \quad (n = 2, 1 \text{ respectively})$

$$= \frac{3x^3}{3} - \frac{24x^2}{2} + C$$

$$\therefore I = x^3 - 12x^2 + C$$

(v) $I = \int (x+3)^2 dx = \int (x^2 + 6x^1 + 9x^0)dx$

$(n = 2, 1, 0 \text{ respectively})$

$$= \frac{x^3}{3} + \frac{6x^2}{2} + \frac{9x^1}{1} + C$$

$$\therefore I = \frac{x^3}{3} + 3x^2 + 9x + C$$

(vi) $I = \int \left(\frac{4}{x} - 5\right)dx = \int (4x^{-1} - 5x^0)dx \ (n = -1, 0 \text{ respectively})$

$$= 4\ln|x| - \frac{5x^1}{1} + C$$

$$\therefore I = 4\ln|x| - 5x + C.$$

Note: Since the constant k can always be written as $k = kx^0$,

$$\int k dx = \int kx^0 dx = \frac{kx^1}{1} + C = kx + C$$

3. (i) $I = \int (3e^{-9x} + 8e^{4x})dx \qquad (k = -9, 4 \text{ respectively})$

$$= \frac{3e^{-9x}}{-9} + \frac{8e^{4x}}{4} + C$$

$$\therefore I = \frac{-e^{-9x}}{3} + 2e^{4x} + C$$

(ii) $I = \int 5\cos\left(\frac{x}{2}\right)dx \qquad (k = \frac{1}{2})$

$$= \frac{5}{1/2}\sin\left(\frac{x}{2}\right) + C = 5 \times 2\sin\left(\frac{x}{2}\right) + C$$

$$\therefore I = 10\sin\left(\frac{x}{2}\right) + C$$

(iii) $I = \int (5\sin 2x + 3\cos 8x)dx \qquad (k = 2, 8 \text{ respectively})$

$$\therefore I = \frac{-5}{2}\cos 2x + \frac{3}{8}\sin 8x + C$$

(iv) $I = \int \frac{2}{e^{3x}}\,dx = \int 2e^{-3x}dx \qquad (k = -3)$

$$= \frac{2e^{-3x}}{-3} + C$$

$$\therefore I = \frac{-2e^{-3x}}{3} + C.$$

Problems

1. Find

(i) $I = \int x^4 dx$ (ii) $I = \int x^{-4} dx$

(iii) $I = \int \sin 4x dx$ (iv) $I = \int \cos 4x dx$

(v) $I = \int e^{4x} dx$ (vi) $I = \int e^{-4x} dx$.

2. Find

(i) $I = \int \left(4x + \frac{6}{x^3} - 1\right) dx$ (ii) $I = \int \left(6\sqrt{x} - \frac{1}{\sqrt{x}} + 3\right) dx$

(iii) $I = \int (3x^{-1/4} + 5x^{1/4})dx$ (iv) $I = \int 2x(3x-2)dx$

(v) $I = \int (x-2)^2 dx$ (vi) $I = \int \left(\frac{7}{x} - 2\right) dx$.

3. Find

(i) $I = \int (5e^{-5x} + 8e^{2x})dx$ (ii) $I = \int 2\cos\left(\frac{x}{3}\right) dx$

(iii) $I = \int (7\sin 3x + 3\cos 5x)dx$ (iv) $I = \int \frac{3}{e^{2x}}\, dx$.

Answers

1. (i) $I = \int x^4 dx = \dfrac{x^5}{5} + C$

   (ii) $I = \int x^{-4} dx = \dfrac{x^{-3}}{-3} + C = \dfrac{-x^{-3}}{3} + C = \dfrac{-1}{3x^3} + C$

   (iii) $I = \int \sin 4x dx = \dfrac{-1}{4}\cos 4x + C$

   (iv) $I = \int \cos 4x dx = \dfrac{1}{4}\sin 4x + C$

   (v) $I = \int e^{4x} dx = \dfrac{1}{4}e^{4x} + C$

   (vi) $I = \int e^{-4x} dx = \dfrac{1}{-4}e^{-4x} + C = \dfrac{-1}{4}e^{-4x} + C$.

2. (i) $I = \int\left(4x + \dfrac{6}{x^3} - 1\right)dx = 2x^2 - \dfrac{3}{x^2} - x + C$

   (ii) $I = \int\left(6\sqrt{x} - \dfrac{1}{\sqrt{x}} + 3\right)dx = 4x^{3/2} - 2x^{1/2} + 3x + C$

   (iii) $I = \int(3x^{-1/4} + 5x^{1/4})dx = 4x^{3/4} + 4x^{5/4} + C$

   (iv) $I = \int 2x(3x-2)dx = 2x^3 - 2x^2 + C$

   (v) $I = \int(x-2)^2 dx = \dfrac{x^3}{3} - 2x^2 + 4x + C$

   (vi) $I = \int\left(\dfrac{7}{x} - 2\right)dx = 7\ln|x| - 2x + C$.

3. (i) $I = \int(5e^{-5x} + 8e^{2x})dx = -e^{-5x} + 4e^{2x} + C$

   (ii) $I = \int 2\cos\left(\dfrac{x}{3}\right)dx = 6\sin\left(\dfrac{x}{3}\right) + C$

   (iii) $I = \int(7\sin 3x + 3\cos 5x)dx = \dfrac{-7}{3}\cos 3x + \dfrac{3}{5}\sin 5x + C$

   (iv) $I = \int \dfrac{3}{e^{2x}}dx = \dfrac{-3e^{-2x}}{2} + C$.

7.3 Finding the constant *C*

1. If $\dfrac{dy}{dx} = f(x)$, then $y = \int f(x)dx$.

   The solution for *y* contains an unknown arbitrary constant *C*, which can only be determined if extra information is given. Such extra information is usually given in the form of one point on the solution curve, e.g. when $x = 0$, $y = 2$. The given point is sometimes written in function notation, e.g. $y(0) = 2$, where the x-value appears in the brackets.

2. For many applications of integration, the variable t (time) is used, rather than x. In such cases, a given point on the solution curve is called an **initial condition**.

3. Equations of the form $\frac{dy}{dx} = f(x)$, or $\frac{dy}{dt} = f(t)$ are called differential equations.

4. For motion in a straight line, with the usual notation, i.e.

   ***s* is the displacement of the object;**

   ***t* is the time;**

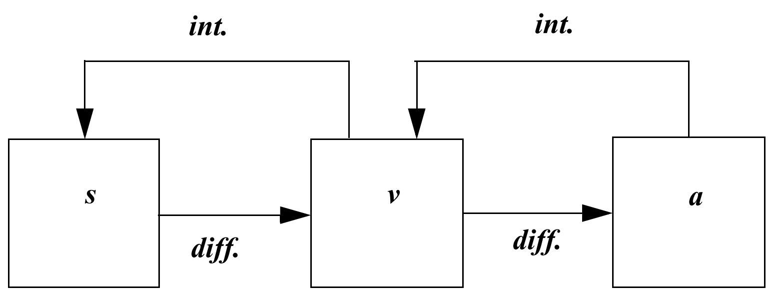
   ***v* is the velocity;**

   ***a* is the acceleration.**

   the relations $v = \frac{ds}{dt}$, and $a = \frac{dv}{dt}$ can be re-written as:

   $s = \int v dt$ and $v = \int a dt$.

   If any one of the displacement, velocity or acceleration is known, the other two can be found using differentiation and integration, as shown in the diagram below.



Examples

1. Find y given that:

   (i) $\frac{dy}{dx} = 4x - 5$ and $y(2) = 3$

   (ii) $\frac{dy}{dx} = 6x^2 + 8x - 5$ and $y(0) = -1$

   (iii) $\frac{dy}{dx} = 2e^{-6x}$ and $y(0) = 0$.

2. The acceleration of an object is given by:
   $a = 12t + 2$ (m/sec/sec.), for $0 \le t \le 8$.
   If the object is initially at rest, find the following:
   (i) the velocity after 5 seconds
   (ii) the distance travelled in the first 2 seconds
   (iii) the acceleration when the velocity is 8 (m/sec).

1. (i) $\frac{dy}{dx} = 4x - 5$ and $y(2) = 3$

$$\therefore y = \int (4x-5)dx = \frac{4x^2}{2} - 5x + C$$

$$\therefore y = 2x^2 - 5x + C$$

Since $y = 3$ when $x = 2$,

$$3 = 2 \times 2^2 - 5 \times 2 + C = 8 - 10 + C = -2 + C$$

$$\therefore C = 3 + 2 = 5$$

$$\therefore y = 2x^2 - 5x + 5$$

(ii) $\frac{dy}{dx} = 6x^2 + 8x - 5$ and $y(0) = -1$

$$\therefore y = \int (6x^2 + 8x - 5)dx = \frac{6x^3}{3} + \frac{8x^2}{2} - 5x + C$$

$$\therefore y = 2x^3 + 4x^2 - 5x + C$$

Since $y = -1$ when $x = 0$,

$$-1 = 0 + 0 - 0 + C$$

$$\therefore C = -1$$

$$\therefore y = 2x^3 + 4x^2 - 5x - 1$$

(iii) $\frac{dy}{dx} = 2e^{-6x}$ and $y(0) = 0$

$$\therefore y = \int 2e^{-6x}dx = \frac{2e^{-6x}}{-6} + C$$

$$\therefore y = \frac{-e^{-6x}}{3} + C$$

Since $y = 0$ when $x = 0$,

$$0 = \frac{-e^0}{3} + C = \frac{-1}{3} + C$$

$$\therefore C = \frac{1}{3}$$

$$\therefore y = \frac{-e^{-6x}}{3} + \frac{1}{3} = \frac{1 - e^{-6x}}{3}.$$

2. $a = 12t + 2$, for $0 \le t \le 8$.

As the object is initially at rest, $v(0) = 0$

(i) Since $v = \int a\,dt = \int (12t+2)\,dt = \frac{12t^2}{2} + 2t + C$,

$$v = 6t^2 + 2t + C.$$

Since $v = 0$ when $t = 0$,

$$0 = 0 + 0 + C \qquad \therefore C = 0.$$

$$\therefore v = 6t^2 + 2t.$$

When $t = 5$, $v = 6 \times 5^2 + 2 \times 5 = 150 + 10 = 160$.

So, the velocity after 5 seconds is 160 (m/sec).

(ii) Since $s = \int v\,dt = \int (6t^2 + 2t)\,dt = \frac{6t^3}{3} + \frac{2t^2}{2} + C$,

$$s = 2t^3 + t^2 + C$$

Since $s = 0$ when $t = 0$,

$$0 = 0 + 0 + C \qquad \therefore C = 0.$$

$$s = 2t^3 + t^2.$$

When $t = 2$, $s = 2 \times 2^3 + 2^2 = 16 + 4 = 20$

So, the distance travelled in the first 2 seconds is 20 (m).

(iii) When $v = 8$, $6t^2 + 2t = 8$. Dividing by 2 gives

$3t^2 + t = 4$, i.e. $3t^2 + t - 4 = 0$

Using the Quadratic Formula (with $a = 3$, $b = 1$, $c = -4$),

$$t = \frac{-1 \pm \sqrt{1^2 - 4 \times 3 \times (-4)}}{2 \times 3}$$

$$\therefore t = \frac{-1 \pm \sqrt{1 + 48}}{6} = \frac{-1 \pm \sqrt{49}}{6}$$

$$\therefore t = \frac{-1 \pm 7}{6}. \qquad \text{So, } t = \frac{-8}{6} = \frac{-4}{3}, \text{ and } t = \frac{6}{6} = 1.$$

As $0 \le t \le 8$, the only permissible value of t is $t = 1$.

Since $a = 12t + 2$, when $t = 1$, $a = 12 + 2 = 14$

So, the acceleration when the velocity is 8 (m/sec), i.e. after 1 sec, is 14 (m/sec/sec).

Problems

1. Find y given that

   (i) $\dfrac{dy}{dx} = 8x - 1$ and $y(2) = 5$

   (ii) $\dfrac{dy}{dx} = 9x^2 + 12x - 4$ and $y(1) = 8$

   (iii) $\dfrac{dy}{dx} = 2e^{8x}$ and $y(0) = 1$.

2. The acceleration of an object is given by:
   $a = 72t - 12t^2 + 1$ (m/sec/sec.), for $0 \le t \le 6$.
   If the object is initially at rest, find the following

   (i) the velocity after 5 seconds

   (ii) the distance travelled in the first 2 seconds

   (iii) the velocity when the acceleration is 109 (m/sec/sec).

Answers

1. (i) $y = 4x^2 - x - 9$ (ii) $y = 3x^3 + 6x^2 - 4x + 3$

   (iii) $y = \dfrac{3 + e^{8x}}{4}$.

2. Velocity $v = 36t^2 - 4t^3 + t$;displacement $s = 12t^3 - t^4 + \dfrac{t^2}{2}$.

   (i) 405 (m/sec) (ii) 82 (m)

   (iii) When $a = 109$, $72t - 12t^2 + 1 = 109$

   $\therefore 12t^2 - 72t + 108 = 0$ Dividing by 12 gives

   $t^2 - 6t + 9 = 0$, i.e. $(t - 3)^2 = 0$ $\therefore t = 3$.

   When $t = 3$,

   $v = 36 \times 3^2 - 4 \times 3^3 + 3 = 324 - 108 + 3 = 219$.

   $\therefore v = 219$ (m/sec).

7.4 The definite integral

1. $\int_a^b f(x)dx$ is called the **definite integral** of f for $a \le x \le b$. The constant ***a* and *b* are called the terminals of integration**. By definition,

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a),$$

where $F(x)$ is any anti-derivative of $f(x)$.

Note that **the definite integral is a number** (not a function), whose value is obtained by evaluating $F(x)$ at the top terminal *b*, and the bottom terminal *a,* before subtracting in the correct order, $F(b) - F(a)$.

2. For definite integrals, the constant *C* is not needed, as it cancels after subtraction, i.e.

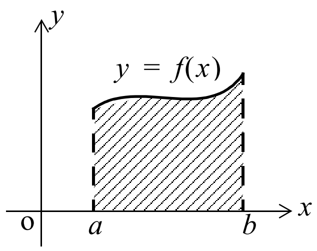
$$\int_a^b f(x)dx = [F(x) + C]_a^b = \{F(b) + C\} - \{F(a) + C\}$$

$$= F(b) + C - F(a) - C = F(b) - F(a).$$

3. It is important to note that the definite integral $\int_a^b f(x)dx$ represents an **area only if the curve** $y = f(x)$ **is non-negative**, i.e.

$$f(x) \geq 0 \quad \text{for } a \leq x \leq b.$$

The relevant area is shaded in the diagram below.



4. In general, **the definite integral may be positive, negative or zero**, and evaluating a definite integral should not be confused with finding an area.

5. Since the definite integral is a number, the variable of integration is a 'dummy' variable. So,

$\int_a^b f(x)dx$, $\int_a^b f(t)dt$, and $\int_a^b f(z)dz$ all represent the same number.

Examples

1. Evaluate the following definite integrals

(i) $I = \int_0^6 x^2 dx$ (ii) $I = \int_1^9 \frac{1}{\sqrt{x}} dx$

(iii) $I = \int_0^3 (x^2 - 4)dx$ (iv) $I = \int_1^3 (2 - x)dx$

(v) $I = \int_0^1 (2-3x^2)dx$ (vi) $I = \int_0^6 3\,dx$

(vii) $I = \int_1^9 \left(\frac{3}{\sqrt{x}} - 2\right)dx$ (viii) $I = \int_1^3 \frac{1}{x^4}dx$.

2. Evaluate the following definite integrals:

(i) $I = \int_0^{\pi/2} \sin x\,dx$ (ii) $I = \int_0^{\ln 2} 8e^{4x}\,dx$

(iii) $I = \int_0^{\pi} (2\cos x - \sin 2x)dx$ (iv) $I = \int_2^8 \frac{1}{4x}dx$.

1. (i) $I = \int_0^6 x^2 dx \quad = \left[\frac{x^3}{3}\right]_0^6 \quad = \frac{1}{3}[x^3]_0^6$

$\therefore I = \frac{1}{3}[6^3 - 0^3] \quad = \frac{1}{3}(216-0) \quad = 72$

$\therefore I = 72$

(ii) $I = \int_1^9 \frac{1}{\sqrt{x}}dx \quad = \int_1^9 x^{-1/2}dx$

$\therefore I = \left[\frac{x^{1/2}}{1/2}\right]_1^9 \quad = 2[x^{1/2}]_1^9$

$\therefore I = 2[9^{1/2} - 1^{1/2}] \quad = 2(3-1) \quad = 2 \times 2 = 4$

$\therefore I = 4$

(iii) $I = \int_0^3 (x^2-4)dx \quad = \left[\frac{x^3}{3} - 4x\right]_0^3$

$\therefore I = \left(\frac{3^3}{3} - 4 \times 3\right) - (0-0) \quad = \frac{27}{3} - 12 - 0$

$\therefore I = 9 - 12$

$\therefore I = -3$.

(iv) $I = \int_1^3 (2-x)dx \quad = \left[2x - \frac{x^2}{2}\right]_1^3$

$$\therefore I = \left(2 \times 3 - \frac{3^2}{2}\right) - \left(2 \times 1 - \frac{1^2}{2}\right) \qquad = 6 - \frac{9}{2} - \left(2 - \frac{1}{2}\right)$$

$$\therefore I = 6 - \frac{9}{2} - 2 + \frac{1}{2} \qquad = 4 + \frac{1}{2} - \frac{9}{2}$$

$$\therefore I = \frac{9}{2} - \frac{9}{2} = 0$$

$$\therefore I = 0.$$

(v) $$I = \int_0^1 (2 - 3x^2)dx \qquad = \left[2x - \frac{3x^3}{3}\right]_0^1$$

$$\therefore I = [2x - x^3]_0^1 \qquad = (2 \times 1 - 1^3) - (0 - 0)$$

$$\therefore I = 2 - 1 - 0 = 1$$

$$\therefore I = 1.$$

(vi) $$I = \int_0^6 3dx \qquad = [3x]_0^6 \qquad = (3 \times 6) - (3 \times 0)$$

$$\therefore I = 18 - 0 = 18$$

$$\therefore I = 18.$$

(vii) $$I = \int_1^9 \left(\frac{3}{\sqrt{x}} - 2\right)dx \qquad = \int_1^9 (3x^{-1/2} - 2)dx$$

$$\therefore I = \left[\frac{3x^{1/2}}{1/2} - 2x\right]_1^9$$

$$\therefore I = [6x^{1/2} - 2x]_1^9 = (6 \times 9^{1/2} - 2 \times 9) - (6 \times 1^{1/2} - 2 \times 1)$$

$$\therefore I = 6 \times 3 - 2 \times 9 - 6 \times 1 + 2 \times 1$$

$$\therefore I = 18 - 18 - 6 + 2$$

$$\therefore I = -4.$$

(viii) $$I = \int_1^3 \frac{1}{x^4}dx \qquad = \int_1^3 x^{-4}dx$$

$$\therefore I = \left[\frac{x^{-3}}{-3}\right]_1^3 = -\frac{1}{3}\left[\frac{1}{x^3}\right]_1^3$$

$$\therefore I = -\frac{1}{3}\left(\frac{1}{3^3} - \frac{1}{1^3}\right) = -\frac{1}{3}\left(\frac{1}{27} - \frac{1}{1}\right) = -\frac{1}{3} \times -\frac{26}{27}$$

$$\therefore I = \frac{26}{81}.$$

2. (i) $I = \int_0^{\pi/2} \sin x\,dx \qquad = [-\cos x]_0^{\pi/2}$

$$\therefore I = -[\cos x]_0^{\pi/2} \qquad = -\left(\cos\frac{\pi}{2} - \cos 0\right)$$

$$\therefore I = -(0-1) = 0+1 = 1$$

$$\therefore I = 1.$$

(ii) $I = \int_0^{\ln 2} 8e^{4x}dx \qquad = \left[\frac{8e^{4x}}{4}\right]_0^{\ln 2}$

$$\therefore I = [2e^{4x}]_0^{\ln 2} \qquad = 2[e^{4x}]_0^{\ln 2}$$

$$\therefore I = 2(e^{4\ln 2} - e^0)$$

Using log. rules, $\quad 4\ln 2 = \ln(2^4) = \ln 16$

$$\therefore e^{4\ln 2} = e^{\ln 16} = 16.$$

$$\therefore I = 2(16-1) = 2\times 15 = 30$$

$$\therefore I = 30.$$

(iii) $I = \int_0^{\pi} (2\cos x - \sin 2x)dx$

$$\therefore I = \left[2\sin x - \left(\frac{-1}{2}\cos 2x\right)\right]_0^{\pi} \qquad = \left[2\sin x + \frac{1}{2}\cos 2x\right]_0^{\pi}$$

$$\therefore I = \left(2\sin\pi + \frac{1}{2}\cos 2\pi\right) - \left(2\sin 0 + \frac{1}{2}\cos 0\right)$$

$$\therefore I = \left(2\times 0 + \frac{1}{2}\times 1\right) - \left(2\times 0 + \frac{1}{2}\times 1\right)$$

$$\therefore I = 0 + \frac{1}{2} - 0 - \frac{1}{2} = 0$$

$$\therefore I = 0.$$

(iv) $I = \int_2^8 \frac{1}{4x}dx \qquad = \frac{1}{4}\int_2^8 \frac{1}{x}dx$

$$\therefore I = \frac{1}{4}[\ln|x|]_2^8 \qquad = \frac{1}{4}(\ln|8| - \ln|2|)$$

$$\therefore I = \frac{1}{4}(\ln 8 - \ln 2) \qquad = \frac{1}{4}\ln\left(\frac{8}{2}\right) = \frac{1}{4}\ln 4$$

$$\therefore I = \frac{1}{4}\ln 4.$$

Problems

1. Evaluate the following definite integrals

(i) $I = \int_0^4 x^3 dx$ (ii) $I = \int_4^{16} \frac{2}{\sqrt{x}} dx$

(iii) $I = \int_3^6 (2x^2 - 9) dx$ (iv) $I = \int_1^2 (3 - 2x) dx$

(v) $I = \int_0^2 (1 - 6x^2) dx$ (vi) $I = \int_1^2 \frac{1}{x^3} dx$.

2. Evaluate the following definite integrals:

(i) $I = \int_0^{\pi/2} \cos x dx$ (ii) $I = \int_0^{\ln 2} 4e^{2x} dx$

(iii) $I = \int_0^{\pi} (3\cos 3x - \sin x) dx$ (iv) $I = \int_3^9 \frac{1}{5x} dx$.

Answers

1. (i) $I = \int_0^4 x^3 dx = \quad = \left[\frac{x^4}{4}\right]_0^4 = 64$

(ii) $I = \int_4^{16} \frac{2}{\sqrt{x}} dx = 4[x^{1/2}]_1^{16} = 12$

(iii) $I = \int_3^6 (2x^2 - 9) dx = \left[\frac{2x^3}{3} - 9x\right]_3^6 = 99$

(iv) $I = \int_1^2 (3 - 2x) dx = [3x - x^2]_1^2 = 0$

(v) $I = \int_0^2 (1 - 6x^2) dx = [x - 2x^3]_0^2 = -14$

(vi) $I = \int_1^2 \frac{1}{x^3} dx = \left[\frac{x^{-2}}{-2}\right]_1^2 = \frac{3}{8}$.

2. (i) $I = \int_0^{\pi/2} \cos x dx = \quad = [\sin x]_0^{\pi/2} = 1$

(ii) $I = \int_0^{\ln 2} 4e^{2x} dx = \left[\frac{4e^{2x}}{2}\right]_0^{\ln 2} = 6$

(iii) $I = \int_0^{\pi} (3\cos 3x - \sin x) dx = [\sin 3x + \cos x]_0^{\pi} = -2$

(iv) $I = \int_3^9 \frac{1}{5x} dx = \frac{1}{5}[\ln|x|]_3^9 = \frac{1}{5}\ln 3$.

7.5 Area between curves

1. Given the curve $y = f(x)$, the definite integral $\int_a^b f(x) dx$ represents an area only if $f(x) \ge 0$ for $a \le x \le b$.

   If, however, $f(x) \le 0$ for $a \le x \le b$, the definite integral $\int_a^b f(x) dx$ is negative, and does not represent an area.

   In general, evaluating $\int_a^b f(x) dx$ is not the same as finding the area contained between the curve $y = f(x)$ and the x-axis, for $a \le x \le b$.

   To find a required area, the curve (or curves) must be sketched first, before forming the definite integral corresponding to the area.

2. The area contained between two curves can be found using the following rule.

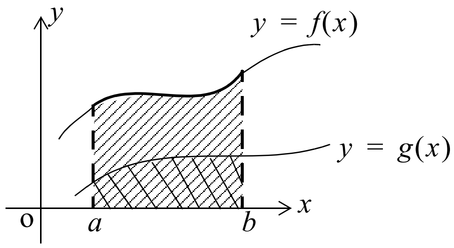
AREAS

The area enclosed between $y = f(x)$ and $y = g(x)$

(where $f(x) \ge g(x)$ for $a \le x \le b$) is given by:

$$A = \int_a^b [f(x) - g(x)] dx.$$

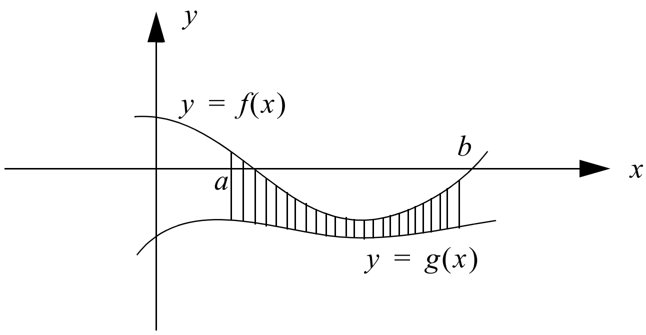
The formula is derived by subtracting the area underneath the curve $y = g(x)$ from the area underneath the curve $y = f(x)$, as shown in the following diagram.



i.e. $A = \int_a^b f(x)dx - \int_a^b g(x)dx$

$\therefore A = \int_a^b [f(x) - g(x)]dx.$

3. The above integral determines the required area in all cases where $f(x) \geq g(x)$ for $a \leq x \leq b$, irrespective of whether $f(x)$ and $g(x)$ themselves take negative values. Provided $f(x) \geq g(x)$, the difference $f(x)-g(x)$ is positive, and the area is given by $A = \int_a^b [f(x) - g(x)]dx$, as shown below.



4. Once the curves $y = f(x)$ and $y = g(x)$ have been sketched, **the function to be integrated is simply the difference between the y value on the top curve and the y value on the bottom curve.**

5. When sketching the curves $y = f(x)$ and $y = g(x)$, **all intersections should be found**. Often, the terminals of integration are determined by the intersections of the curves.

Examples

1. Find the area bounded by the curve $y = x^2 - 4$ and the x-axis for $0 \leq x \leq 2$.

2. (i) Evaluate $I = \int_0^2 (1 - x^2)dx$

(ii) Find the area bounded by the curve $y = 1 - x^2$ and the x-axis for $0 \leq x \leq 2$.

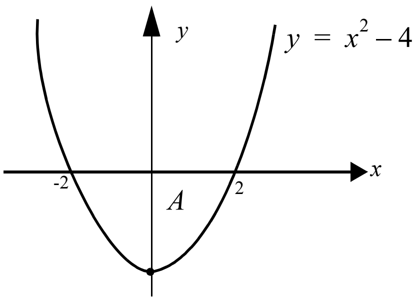
3. Find the area bounded by the curve $y = 6x^2 + 7$ and the x-axis for $1 \leq x \leq 3$

4. Find the area bounded by the curve $y = 3x^2$ and the straight line $y = 6x$

5. Find the area bounded by the curve $y = x^2 + 2x + 1$ and the straight line $y = 5x + 1$.

1. The curve $y = x^2 - 4$ is a cup shaped parabola with a y-intercept of -4.

For the x-intercepts, $x^2 - 4 = 0 \qquad \therefore x^2 = 4 \qquad \therefore x = \pm 2$.



The top 'curve' is $y = 0$, and the bottom curve is $y = x^2 - 4$.

So, $f(x) - g(x) = 0 - (x^2 - 4) = -x^2 + 4$.

$$\therefore A = \int_a^b [f(x) - g(x)]dx = \int_0^2 (-x^2 + 4)dx$$

$$\therefore A = \left[\frac{-x^3}{3} + 4x\right]_0^2 \qquad = \left(\frac{-2^3}{3} + 4 \times 2\right) - (0 + 0)$$

$$\therefore A = \frac{-8}{3} + 8 = \frac{-8 + 24}{3} = \frac{16}{3}$$

$$\therefore A = \frac{16}{3}$$

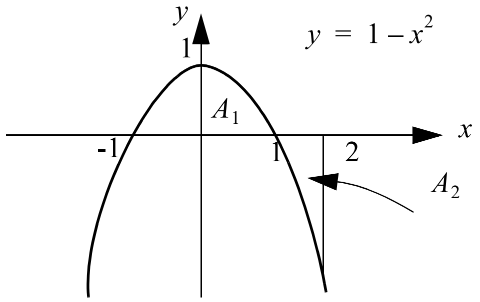
2. (i) $I = \int_0^2 (1 - x^2)dx \qquad = \left[x - \frac{x^3}{3}\right]_0^2$

$$\therefore I = \left(2 - \frac{2^3}{3}\right) - (0 - 0) \qquad = 2 - \frac{8}{3} = \frac{6 - 8}{3}$$

$$\therefore I = \frac{-2}{3}$$

(ii) The curve $y = 1 - x^2$ is a frown shaped parabola with a y-intercept of 1 .

For the x-intercepts, $1 - x^2 = 0 \qquad \therefore x^2 = 1 \qquad \therefore x = \pm 1$.



The area consists of 2 distinct areas, A_1 (for $0 \le x \le 1$), and A_2 (for $1 \le x \le 2$).

For $0 \le x \le 1$, the top curve is $y = 1 - x^2$, and the bottom 'curve' is $y = 0$.

So, $f(x) - g(x) = 1 - x^2 - 0 = 1 - x^2$.

$$\therefore A_1 = \int_0^1 (1 - x^2)dx \qquad = \left[x - \frac{x^3}{3}\right]_0^1$$

$$\therefore A_1 = \left(1 - \frac{1^3}{3}\right) - (0 - 0) \qquad = 1 - \frac{1}{3} = \frac{2}{3}$$

For $1 \le x \le 2$, the top 'curve' is $y = 0$, and the bottom curve is $y = 1 - x^2$.

So, $f(x) - g(x) = 0 - (1 - x^2) = -1 + x^2$.

$$\therefore A_2 = \int_1^2 (-1 + x^2)dx \qquad = \left[-x + \frac{x^3}{3}\right]_1^2$$

$$\therefore A_2 = \left(-2 + \frac{2^3}{3}\right) - \left(-1 + \frac{1^3}{3}\right) \qquad = -2 + \frac{8}{3} + 1 - \frac{1}{3}$$

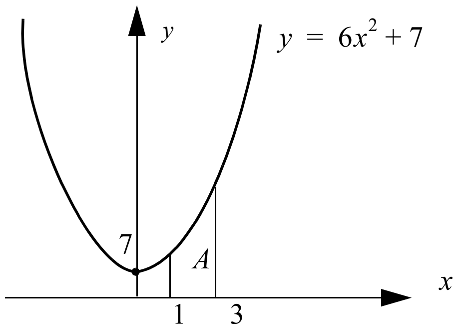
$$\therefore A_2 = -1 + \frac{7}{3} = \frac{-3 + 7}{3} = \frac{4}{3}.$$

$$\therefore A = A_1 + A_2 = \frac{2}{3} + \frac{4}{3} = \frac{6}{3} = 2$$

$$\therefore A = 2.$$

3. The curve $y = 6x^2 + 7$ is a cup shaped parabola with a y-intercept of 7 .

As $y = 6x^2 + 7 \geq 0$ for all values of x, there are no x-intercepts.



The top curve is $y = 6x^2 + 7$, and the bottom 'curve' is $y = 0$.

So, $f(x)–g(x) = 6x^2 + 7 - 0 = 6x^2 + 7$.

$$\therefore A = \int_1^3 (6x^2 + 7)dx \qquad = \left[\frac{6x^3}{3} + 7x\right]_1^3$$

$$\therefore A = [2x^3 + 7x]_1^3 \qquad = (2 \times 3^3 + 7 \times 3) - (2 \times 1^3 + 7 \times 1)$$

$$\therefore A = (54 + 21) - (2 + 7) \qquad = 75 - 9 = 66$$

$$\therefore A = 66.$$

4. The curve $y = 3x^2$ is a cup shaped parabola with a y-intercept of 0.

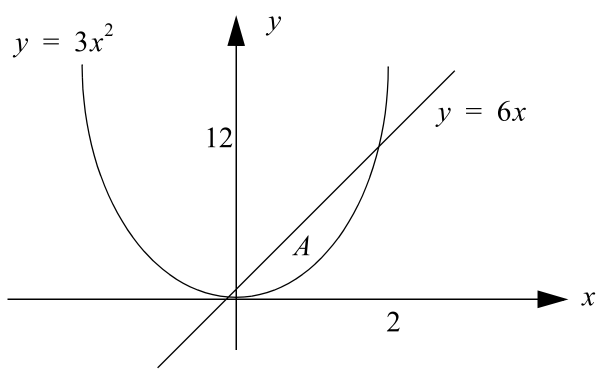
   For the x-intercepts, $3x^2 = 0 \qquad \therefore x^2 = 0 \qquad \therefore x = 0$.

   The straight line $y = 6x$ has slope 6, and passes through the origin.

   The curve and the straight line intersect when $3x^2 = 6x$, i.e. when

   $3x^2 - 6x = 0 \qquad \therefore 3x(x-2) = 0 \qquad \therefore x = 0$ and $x = 2$.

   The graphs can be sketched without finding the corresponding y-values at the points of intersection, but can be found easily using either equation. When $x = 0$, $y = 6 \times 0 = 0$, and, when $x = 2$, $y = 6 \times 2 = 12$.



From the diagram, the terminals of integration are $x = 0$ and $x = 2$. The top 'curve' is $y = 6x$, and the bottom curve is $y = 3x^2$.

So, $f(x)-g(x) = 6x - 3x^2$.

$$\therefore A = \int_0^2 (6x - 3x^2)dx \qquad = \left[\frac{6x^2}{2} - \frac{3x^3}{3}\right]_0^2$$

$$\therefore A = [3x^2 - x^3]_0^2 \qquad = (3 \times 2^2 - 2^3) - (0 - 0)$$

$$\therefore A = 12 - 8 = 4$$

$$\therefore A = 4.$$

5. The curve $y = x^2 + 2x + 1$ is a cup shaped parabola with a y-intercept of 1.

For the x-intercepts,

$$x^2 + 2x + 1 = 0 \qquad \therefore (x+1)^2 = 0 \qquad \therefore x = -1.$$

The straight line $y = 5x + 1$ has slope 5, and a y-intercept of 1.

When $y = 0$, $5x + 1 = 0$ $\qquad \therefore 5x = -1 \qquad \therefore x = \frac{-1}{5}$.

The curve and the straight line intersect when $x^2 + 2x + 1 = 5x + 1$,

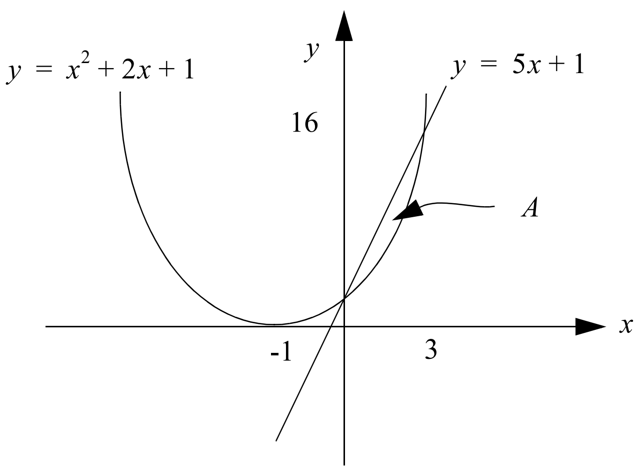
i.e. when $\qquad x^2 + 2x = 5x \qquad \therefore x^2 - 3x = 0 \qquad \therefore x(x-3) = 0$,

i.e. $x = 0$ and $x = 3$.

When $x = 0$, $y = 0 + 1 = 1$.

When $x = 3$, $y = 5 \times 3 + 1 = 15 + 1 = 16$.

So, the intersections are $(0, 1)$, and $(3, 16)$.



From the diagram, the terminals of integration are $x = 0$ and $x = 3$.

The top 'curve' is $y = 5x + 1$, and the bottom curve is $y = x^2 + 2x + 1$.

So,

$$f(x) - g(x) = 5x + 1 - (x^2 + 2x + 1) = 5x + 1 - x^2 - 2x - 1.$$

$$f(x) - g(x) = 3x - x^2$$

$$\therefore A = \int_0^3 (3x - x^2)dx \qquad = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3$$

$$\therefore A = \left(\frac{3 \times 3^2}{2} - \frac{3^3}{3}\right) - (0 - 0) \qquad = \frac{27}{2} - 9 = \frac{27 - 18}{2}$$

$$\therefore A = \frac{9}{2}.$$

Problems

1. Find the area bounded by the curve $y = 3x^2 + 1$ and the x-axis for $1 \le x \le 3$.

2. (i) Evaluate $I = \int_0^3 (4 - x^2)dx$

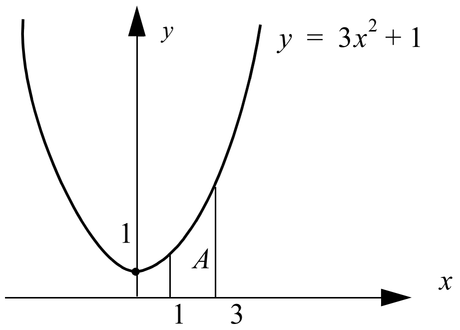
   (ii) Find the area bounded by the curve $y = 4 - x^2$ and the x-axis for $0 \le x \le 3$.

3. Find the area bounded by the curve $y = 12x^2 + 5$ and the x-axis for $1 \le x \le 2$

4. Find the area bounded by the curve $y = x^2$ and the straight line $y = 4x - 3$

Answers

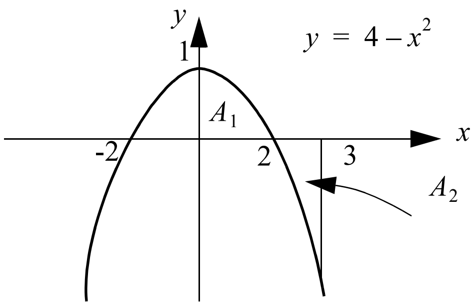
1.



$$A = \int_1^3 (3x^2 + 1)dx \qquad = [x^3 + x]_1^3 \qquad \therefore A = 28.$$

2. (i) $$I = \int_0^3 (4 - x^2)dx \qquad = \left[4x - \frac{x^3}{3}\right]_0^3 \qquad \therefore I = 3.$$

(ii)



$$A_1 = \int_0^2 (4 - x^2)dx \qquad = \left[4x - \frac{x^3}{3}\right]_0^2$$

$$\therefore A_1 = \frac{16}{3}.$$

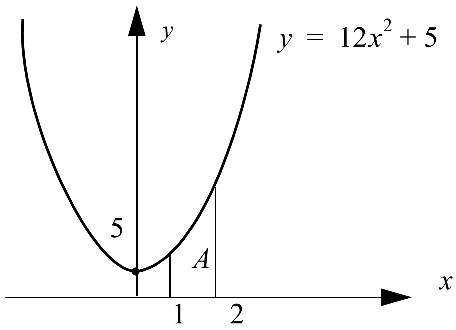
$$A_2 = \int_2^3 (-4 + x^2)dx \qquad = \left[-4x + \frac{x^3}{3}\right]_2^3$$

$$\therefore A_2 = \frac{7}{3}.$$

$$\therefore A = A_1 + A_2 = \frac{16}{3} + \frac{7}{3} = \frac{23}{3}$$

$$\therefore A = \frac{23}{3}.$$

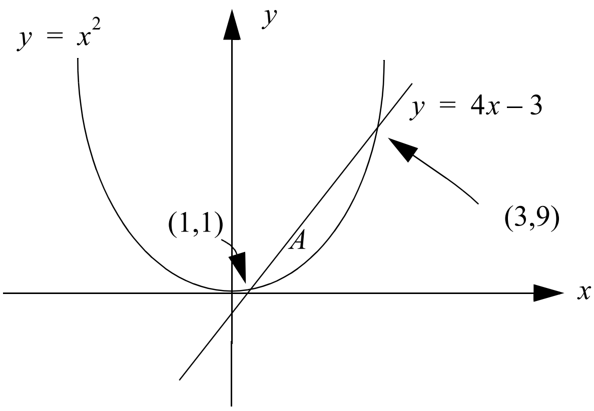
3.



$$A = \int_1^2 (12x^2 + 5)dx \qquad = [4x^3 + 5x]_1^2$$

$$\therefore A = 33\,.$$

4.



$$A = \int_1^3 (4x - 3 - x^2)dx \qquad = \left[2x^2 - 3x - \frac{x^3}{3}\right]_1^3$$

$$\therefore A = \frac{4}{3}\,.$$