# SIT787: Mathematics for AI
# Practical Week 1

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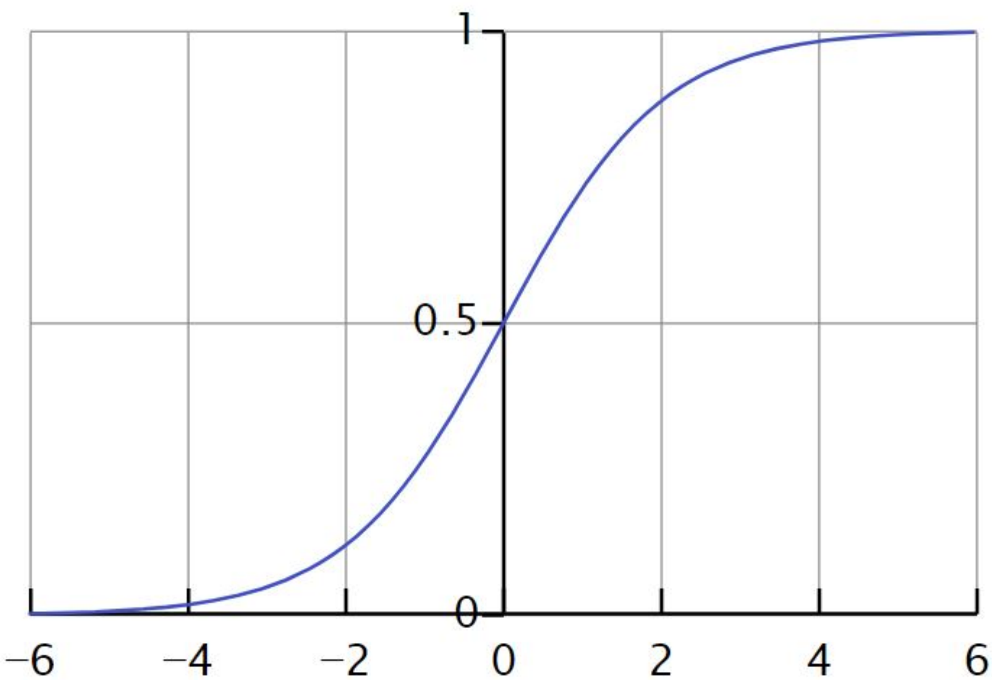
1. Consider this functions

$$f(x) = \frac{1}{1+e^{-x}}$$

- show that it can be represented as $f(x) = \frac{e^x}{1+e^x}$
- $f(0) = \frac{1}{2}$
- $f'(x) = f(x)(1 - f(x))$. [Hint: $\frac{d}{dx}(e^{kx}) = ke^{kx}$]
- A plot of the function is given as follows




  - what is the behaviour of $f(x)$ when $x$ gets too large?
  - what is the behaviour of $f(x)$ when $x$ gets too large from the negative side?
  - What are the domain and range of this function?

2. Consider the step function

$$f(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x \geq 0 \end{cases}$$

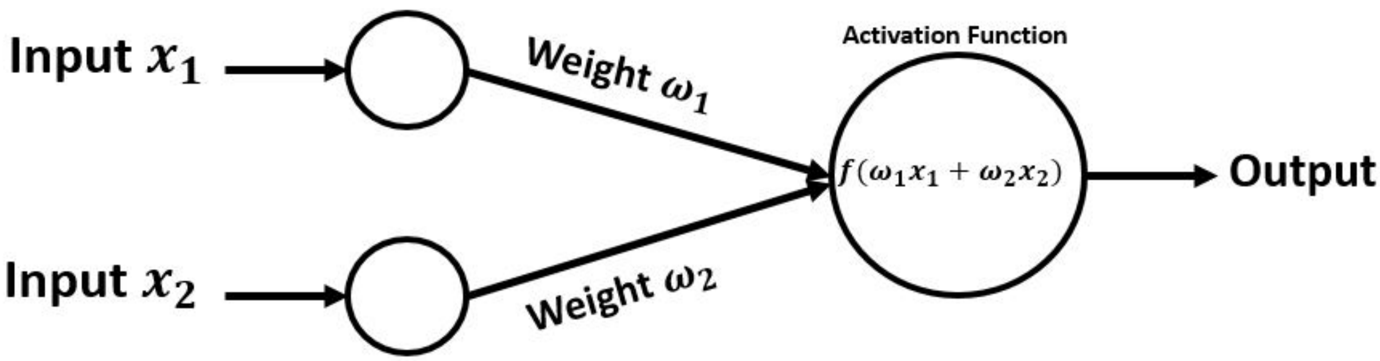
   - Draw a plot of this function.

3. swish function

$$f(x) = \frac{x}{1 + e^{-x}}$$

   - Considering $\sigma(x) = \frac{1}{1+e^{-x}}$, show that $f(x) = x\sigma(x)$.
   - show that $f'(x) = f(x) + \sigma(x)(1 - f(x))$

4. The concept of perceptron:The functions we considered so far are types are activation functions, which are used in designing neural networks. The simplest possible neural network is a perceptron. This is called a single-layer Perceptron.




Classifying using Perceptron:

- input: $x_1, x_2, x_3, \ldots, x_n$
- weights: $\omega_1, \omega_2, \omega_3, \ldots, \omega_n$
- using a step function as an activation function

$$\text{output} = \begin{cases} 1 & \text{if } \sum_{j=1}^{n} \omega_j x_j > \text{threshold} \\ 0 & \text{if } \sum_{j=1}^{n} \omega_j x_j \leq \text{threshold} \end{cases}$$

Consider after training your model (means finding the weights) you know that you weights are $\omega_1 = 2$ and $\omega_2 = 5$, and the threshold is 4. Classify these new observation whether they belong to class 0 or class 1.

- $x_1 = 3, x_2 = -1$
- $x_1 = -2, x_2 = 7$
- $x_1 = 0, x_2 = 0$
- $x_1 = 9, x_2 = 120$

① $f(x) = \frac{1}{1+e^{-x}}$

show that $f(x) = \frac{e^x}{1+e^x}$.

$f(x) = \frac{1}{1+e^{-x}}$ multiply both the numerator and denominator by $e^x$. this possible as we know $e^x \neq 0$ for all $x$.

$$= \frac{1}{1+e^{-x}} \times \frac{e^x}{e^x} = \frac{e^x}{(1+e^{-x})(e^x)}$$

in the denominator, do the product

$$= \frac{e^x}{(1)(e^x) + (e^{-x})(e^x)}$$

we know that $(a^m)(a^n) = a^{m+n}$

$$= \frac{e^x}{e^x + e^{-x+x}} = \frac{e^x}{e^x + e^0} = \frac{e^x}{e^x+1}.$$

To find $f(0)$, we plug 0 instead of $x$ in the function:

$$f(0) = \frac{1}{1+e^{-(0)}} = \frac{1}{1+e^0} = \frac{1}{1+1} = \frac{1}{2}.$$

Show that $f'(x) = f(x)(1 - f(x))$

I will find $f'(x)$ using the qustient rule, and then will find $f(x)(1-f(x))$ and show that they are the same.

quotien rule; if $y = \frac{f(x)}{g(x)}$ then $y' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

$$f'(x) = \frac{(0)(1+e^{-x}) - (1)(-e^{-x})}{(1+e^{-x})^2} = \frac{e^{-x}}{(1+e^{-x})^2} \quad \text{I}$$

Now, I find the right-hand side:

$$f(x)(1-f(x)) = \left(\frac{1}{1+e^{-x}}\right)\left(1 - \frac{1}{1+e^{-x}}\right)$$

$$= \left(\frac{1}{1+e^{-x}}\right)\left(\frac{1+e^{-x}-1}{1+e^{-x}}\right)$$

$$= \left(\frac{1}{1+e^{-x}}\right)\left(\frac{e^{-x}}{1+e^{-x}}\right)$$

$$= \frac{e^{-x}}{(1+e^{-x})^2} \quad \text{II}$$

As I = II then $f'(x) = f(x)(1-f(x))$.

Based on the plot:

- as $x$ grows from the positive side, the graph of the function approaches to 1.

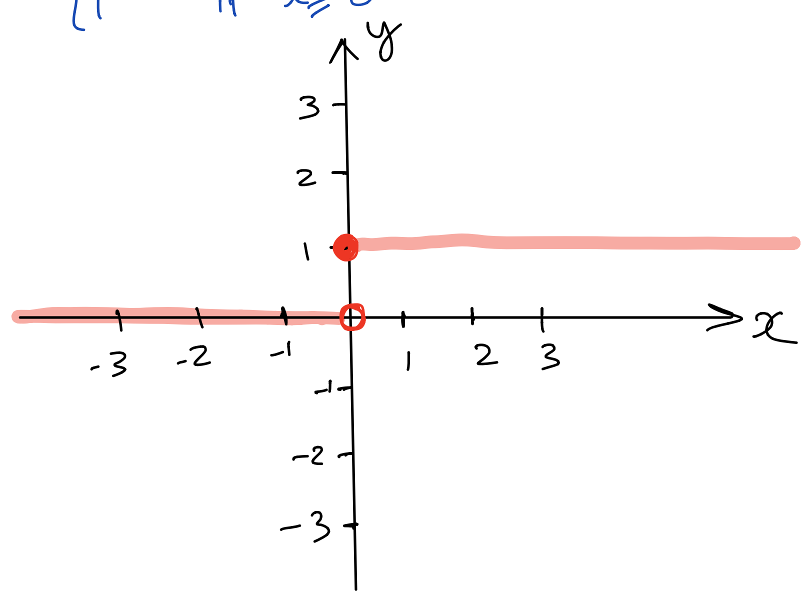
$$\lim_{x \to +\infty} f(x) = 1$$

- as $x$ grows from the negative side, the graph of the function approaches to 0.

$$\lim_{x \to -\infty} f(x) = 0$$

- The domain of the function is $\mathbb{R}$. There is no restriction.
- The range of the function is $(0, 1)$.

② $$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$



③ $f(x) = \dfrac{x}{1+e^{-x}}$

considering $\sigma(x) = \dfrac{1}{1+e^{-x}}$ show that $f(x) = x\sigma(x)$.

$$f(x) = \frac{x}{1+e^{-x}} = (x)\left(\frac{1}{1+e^{-x}}\right) = x\sigma(x).$$

show that $f'(x) = f(x) + \sigma(x)(1-f(x))$

I will find $f'(x)$ using the quotient rule, and the compute the right-hand-side.

$$f'(x) = \frac{(1)(1+e^{-x}) - (x)(-e^{-x})}{(1+e^{-x})^2}$$

$$= \frac{1+e^{-x} + xe^{-x}}{(1+e^{-x})^2} \quad \text{I}$$

The right-hand-side:

$$f(x) + \sigma(x)(1-f(x)) = \frac{x}{1+e^{-x}} + \left(\frac{1}{1+e^{-x}}\right)\left(1 - \frac{x}{1+e^{-x}}\right)$$

$$= \frac{x}{1+e^{-x}} + \left(\frac{1}{1+e^{-x}}\right)\left(\frac{1+e^{-x}-x}{1+e^{-x}}\right)$$

$$= \frac{x}{1+e^{-x}} + \left(\frac{1}{1+e^{-x}}\right)\left(\frac{1-x+e^{-x}}{1+e^{-x}}\right)$$

$$= \frac{x}{1+e^{-x}} + \frac{1-x+e^{-x}}{(1+e^{-x})^2}$$

create the common denominator in both fractions:

$$= \frac{x}{1+e^{-x}} \times \frac{1+e^{-x}}{1+e^{-x}} + \frac{1-x+e^{-x}}{(1+e^{-x})^2}$$

$$= \frac{x+xe^{-x}}{(1+e^{-x})^2} + \frac{1-x+e^{-x}}{(1+e^{-x})^2}$$

$$= \frac{\cancel{x}+xe^{-x}+1-\cancel{x}+e^{-x}}{(1+e^{-x})^2} = \frac{1+e^{-x}+xe^{-x}}{(1+e^{-x})^2} \quad \text{II}$$

I = II the $f'(x) = f(x) + \sigma(x)\,(1-f(x))$

4) $w_1 = 2$ $w_2$ $5$ threshould $= 4$

$x_1 = 3$ $x_2 = -1$

$w_1 x_1 + w_2 x_2 = (2)(3) + (5)(-1) = 6 - 5 = 1$

because $1 \leq 4$ then output is $0$.

$x_1 = -2$, $x_2 = 7$

$w_1 x_1 + w_2 x_2 = (2)(-2) + (5)(7) = -4 + 35 = 31$

because $31 > 4$, then output is $1$.

$x_1 = 0$ $x_2 = 0$

$w_1 x_1 + w_2 x_2 = (2)(0) + (5)(0) = 0$

because $0 \leq 4$ then output is $0$.

$x_1 = 9$, $x_2 = 120$

$w_1 x_1 + w_2 x_2 = (2)(9) + (5)(120) = 618$

because $618 > 4$, then output $= 1$.