SIT787: Mathematics for AI
Practical Week 4

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1. Consider this matrix
$$A = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{bmatrix}$$
   - are the columns independent?
   - what is the column space of A?
   - what is the nullspace of A?
   - what is the rank of A?
   - is the matrix invertible? If so find its inverse.

2. Find the product of these two matrices using inner and outer products:
$$A = \begin{bmatrix} 1 & -4 & 0 \\ 0 & 2 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}$$

3. For matrix
$$A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$$
   - Find $A^2 = AA$ and $A^3 = AA^2$
   - find $f(A)$ for $f(x) = x^2 - x + 2$

4. Solve these systems using Gauss Elimination:
$$\begin{cases} x + 3y - 2z = 5 \\ 3x + 5y + 6z = 7 \\ 2x + 4y + 3z = 8 \end{cases}$$
$$\begin{cases} x + z = 5 \\ 2x - y + 3z = 7 \\ 4x - 2y + 6z = 8 \end{cases}$$

5. Consider
$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 3 & 0 & 4 \end{bmatrix}$$

Find matrices L and U so that $A = LU$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} x & y & z \\ 0 & t & u \\ 0 & 0 & v \end{bmatrix}$$

6. Find the inverse of these matrices

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \\ 4 & -2 & 6 \end{bmatrix}$$

1. Consider this matrix

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{bmatrix}$$

- 1 • are the columns independent?
- 2 • what is the column space of A?
- 3 • what is the nullspace of A?
- 4 • what is the rank of A?
- 5 • is the matrix invertible? If so find its inverse.

① Columns of $A = \left\{ \underbrace{\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}}_{A^1}, \underbrace{\begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}}_{A^2}, \underbrace{\begin{bmatrix} -2 \\ 6 \\ 3 \end{bmatrix}}_{A^3} \right\}$

Let a linear combination of these vectors is the zero vector

$$c_1 A^1 + c_2 A^2 + c_3 A^3 = \vec{O}$$

$$c_1 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{cases} c_1 + 3c_2 - 2c_3 = 0 & \text{eq 1} \\ 3c_1 + 5c_2 + 6c_3 = 0 & \text{eq 2} \\ 2c_2 + 4c_2 + 3c_3 = 0 & \text{eq 3} \end{cases}$$

from eq 1 : $c_1 = -3c_2 + 2c_3$

plug this in eq 2 and eq 3

$$3(-3c_2 + 2c_3) + 5c_2 + 6c_3 = 0$$
$$2(-3c_2 + 2c_3) + 4c_2 + 3c_3 = 0$$

$$\rightarrow \begin{cases} -4c_2 + 12c_3 = 0 \rightarrow 4c_2 = 12c_3 \quad \boxed{c_2 = 3c_3} \\ -2c_2 + 7c_3 = 0 \quad * \end{cases}$$

plug $c_2 = 3c_3$ in eq *

$$-2(3c_3) + 7c_3 = 0$$
$$\boxed{c_3 = 0}$$

$$-4c_2 + 12c_3 = 0$$
$$-4c_2 = 0 \rightarrow c_2 = 0$$
$$c_1 = -3c_2 + 2c_3 \longrightarrow c_1 = 0$$

From putting a linear combination of columns of A to be equal to zero vector, we got $c_1 = c_2 = c_3 = 0$. This implies that the columns of A are independent.

(2)

Column space of A is all the linear combination of columns of A:

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 6 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + 2x_2 - 2x_3 \\ 3x_1 + 5x_2 + 6x_3 \\ 2x_1 + 4x_2 + 3x_3 \end{bmatrix} \qquad x_1, x_2, x_3 \in \mathbb{R}$$

As columns A are independent, and they are vectors in $\mathbb{R}^3$, then they make a basis for $\mathbb{R}^3$.

therefore, all the combinations of columns of A is $\mathbb{R}^3$.

They can span all vector space.

Three independent vectors in $\mathbb{R}^3$ form a basis for $\mathbb{R}^3$.

(3)

Null space of A = the set of solutions for system $A\vec{x} = 0$

$$\begin{bmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

if we solve this system, the only solution is $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

So, the null space of A is the trivial vector space.

(4) rank of A is the number of linearly independent columns of A which is 3. $\text{rank}(A) = 3$.

Also, it is the number of linearly independent rows of A.

As $\operatorname{rank}(A)=3$, then there are 3 independent rows in A.

(5) The matrix A is invertible as

- the columns of A are independent.
- the system $A\vec{x}=0$ has only solution $\vec{x}=\vec{0}$.

To find the inverse, we use Gauss-Jordan elimination.

$$[A \,\vdots\, I] \longrightarrow [I \,\vdots\, A^{-1}]$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 2 & 5 & 6 & 0 & 1 & 0 \\ 3 & 4 & 3 & 0 & 0 & 1 \end{array}\right]$$

First step is to make 2 and 3 zero by adding appropriate multiple of row1 to rows 2 and 3.

row2 $\leftarrow$ row2 $-$ 2 row1

row3 $\leftarrow$ row3 $-$ 3 row1

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & -4 & 12 & -3 & 1 & 0 \\ 0 & -2 & 7 & -2 & 0 & 1 \end{array}\right]$$

multiply row 2 by $-\frac{1}{4}$, to make this element 1.

row2 $\leftarrow -\frac{1}{4}$ row2

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 3/4 & -1/4 & 0 \\ 0 & -2 & 7 & -2 & 0 & 1 \end{array}\right]$$

next we want to make 3 and -4 zero.

row1 = row1 − 3 row2

row3 = row3 + 2 row2

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 7 & -5/4 & 3/4 & 0 \\ 0 & 1 & -5 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & -1/2 & -1/2 & 1 \end{array}\right]$$

row1 ← row1 − (7) row3

row2 ← row2 − (−5) row3

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9/4 & 17/4 & -7 \\ 0 & 1 & 0 & -3/4 & -7/4 & 3 \\ 0 & 0 & 1 & -1/2 & -1/2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 9/4 & 17/4 & -7 \\ -3/4 & -7/4 & 3 \\ -1/2 & -1/2 & 1 \end{bmatrix}$$

- Is the matrix invertible? If so find its inverse.

2. Find the product of these two matrices using inner and outer products:

$$A = \begin{bmatrix} 1 & -4 & 0 \\ 0 & 2 & -1 \end{bmatrix}_{2\times 3}, \qquad B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}_{3\times 2}$$

$$(AB)_{2\times 2}$$

using inner product

$$AB = \begin{bmatrix} \begin{bmatrix} 1 & -4 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 & -4 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -4 \\ -2 & 1 \end{bmatrix}$$

using outer product

$$AB = \underset{2\times 1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}\underset{1\times 2}{\begin{bmatrix} 1 & 0 \end{bmatrix}} + \underset{2\times 1}{\begin{bmatrix} -4 \\ 2 \end{bmatrix}}\underset{1\times 2}{\begin{bmatrix} -1 & 1 \end{bmatrix}} + \underset{2\times 1}{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}\underset{1\times 2}{\begin{bmatrix} 0 & 1 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2} + \begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix}_{2\times 2} + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}_{2\times 2}$$

$$= \begin{bmatrix} 1+4+0 & 0-4+0 \\ 0-2+0 & 0+2-1 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -2 & 1 \end{bmatrix}_{2\times 2}$$

3. For matrix

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$$

- Find $A^2 = AA$ and $A^3 = AA^2$
- find $f(A)$ for $f(x) = x^2 - x + 2$

$$A^2 = AA = \underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}}_{2\times 2} = \begin{bmatrix} 1 & -3 \\ 0 & 4 \end{bmatrix}$$

$$A^3 = AA^2 = \underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} 1 & -3 \\ 0 & 4 \end{bmatrix}}_{2\times 2} = \begin{bmatrix} 1 & -7 \\ 0 & 8 \end{bmatrix}$$

$$f(A) = A^2 - A + 2I$$

$$= \begin{bmatrix} 1 & -3 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - 1 + 2 & -3 + 1 + 0 \\ 0 - 0 + 0 & 4 - 2 + 2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 0 & 4 \end{bmatrix}$$

4. Solve these systems using Gauss Elimination:

$$\begin{cases} x + 3y - 2z = 5 \\ 3x + 5y + 6z = 7 \\ 2x + 4y + 3z = 8 \end{cases}$$

$$\begin{cases} x + z = 5 \\ 2x - y + 3z = 7 \\ 4x - 2y + 6z = 8 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 3 & 5 & 6 & 7 \\ 2 & 4 & 3 & 8 \end{array}\right]$$

row2 $\leftarrow$ row2 $-$ 3 row1

row3 $\leftarrow$ row3 $-$ 2 row1

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & -4 & 12 & -8 \\ 0 & -2 & 7 & -2 \end{array}\right]$$

0 1 −3 2

row 3 $\leftarrow$ row3 $-$ $\left(\frac{-2}{-4}\right)$ row 2

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & -4 & 12 & -8 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

we have a unique solution

$$\begin{cases} x + 3y - 2z = 5 \\ -4y + 12z = -8 \\ \boxed{z = 2} \end{cases}$$

plug in eq 2: $-4y + 12(2) = -8$

$-4y + 24 = -8$

$-4y = -32 \rightarrow \boxed{y = 8}$

plug these 2 in eq 1

$x + 3(8) - 2(2) = 5$

$x + 24 - 4 = 5$

$x + 20 = 5$

$\boxed{x = -15}$

solution $\vec{x} = \begin{bmatrix} -15 \\ 8 \\ 2 \end{bmatrix}$

System 2

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 2 & -1 & 3 & 7 \\ 4 & -2 & 6 & 8 \end{array}\right]$$

row2 $\leftarrow$ row2 -2 row1

row3 $\leftarrow$ row3 -4 row1

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & -1 & 1 & -3 \\ 0 & -2 & 2 & 12 \end{array}\right]$$

row 3 $\leftarrow$ row3 $- \left(\frac{-2}{-1}\right)$ row 1

row 3 $\leftarrow$ row3 $-$ 2 row 1

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & -1 & 1 & -3 \\ 0 & 0 & 0 & 18 \end{array}\right]$$

left hand side is all zeros, but right hand side is 18. There is no solution to this system.

This system is inconsistent.

5. Consider

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 3 & 0 & 4 \end{bmatrix}$$

Find matrices L and U so that $A = LU$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} x & y & z \\ 0 & t & u \\ 0 & 0 & v \end{bmatrix}$$

lower upper

Here we want to write A is a product of a lower triangular and upper triangular matrices.

The lower triangular matrix has all 1 on its main diagonal.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 3 & 0 & 4 \end{bmatrix} = \underset{3\times3}{\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}} \underset{3\times3}{\begin{bmatrix} x & y & z \\ 0 & t & u \\ 0 & 0 & v \end{bmatrix}}$$

we need to find $a, b, c,\ x, y, z, t, u, v$.

if we do the product of the right hand side

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 3 & 0 & 4 \end{bmatrix} = \begin{bmatrix} x & y & z \\ ax & ay+t & az+u \\ bx & by+ct & bz+cu+v \end{bmatrix}$$

$x = 1, \quad y = 0, \quad z = 1$

$ax = 0 \longrightarrow a(1) = 0 \longrightarrow a = 0$

$ay + t = 2 \longrightarrow (0)(0) + t = 2 \longrightarrow \boxed{t = 2}$

$az + u = 2 \longrightarrow (0)(1) + u = 2 \longrightarrow \boxed{u = 2}$

$bx = 3 \rightarrow b(1) = 3 \rightarrow b = 3$

$by + ct = 0 \rightarrow (3)(0) + c(2) = 0 \rightarrow 2c = 0 \rightarrow \boxed{c = 0}$

$bz + cu + v = 4$

$(3)(1) + (0)(2) + v = 4$

$3 + v = 4 \rightarrow v = 1$

$\Rightarrow x = 1,\quad y = 0,\quad z = 1,\quad u = 2,\ t = 2,\ v = 1\quad a = 0,\ b = 3,\ c = 0$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 3 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

6. Find the inverse of these matrices

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \\ 4 & -2 & 6 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right]$$

row 2 ← row2 − 2 row1

row 3 ← row3 − 4 row1

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right]$$

row 2 ← (−1) row 2

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right]$$

row 3 ← row 3 − (1) row 2

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right]$$

row 3 ← (−1) row 3

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & +1 & +6 & -1 & -1 \end{array}\right]$$

row 1 $\leftarrow$ row 1 -2 row 3

row 2 $\leftarrow$ row 2 $-$ row 3

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \\ 4 & -2 & 6 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & -2 & 6 & 0 & 0 & 1 \end{array}\right]$$

row 2 $\leftarrow$ row 2 -2 row 1

row 3 $\leftarrow$ row 3 -4 row 1

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 1 & 0 \\ 0 & -2 & 2 & -4 & 0 & 1 \end{array}\right]$$

row 2 $\leftarrow$ $-$ row 2

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & +2 & -1 & 0 \\ 0 & -2 & 2 & -4 & 0 & 1 \end{array}\right]$$

row 3 $\leftarrow$ row 3 $-(-2)$ row 2

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & +2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 \end{array}\right]$$

we have a row of zeros here.

We never can make identity matrix on the left hand side.

B is not invertible.