SIT787: Mathematics for AI
Practical Week 2

Asef Nazari

1. For these vectors
$$\boldsymbol{u} = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 2 \end{bmatrix} \text{ and } \boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix}$$
   - Find $\boldsymbol{u} + \boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v}, 2\boldsymbol{u} + 3\boldsymbol{v}$
   - Find the cosine between these two vecors and their lengthes
   - Find the distance between them.
2. Are these vectors linearly independent?
   - $$\boldsymbol{u} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} \text{ and } \boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
   - $$\boldsymbol{u} = \begin{bmatrix} 4 \\ 2 \\ -6 \end{bmatrix} \text{ and } \boldsymbol{v} = \begin{bmatrix} 10 \\ 5 \\ -15 \end{bmatrix}$$
   - $$\boldsymbol{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } \boldsymbol{v} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$
3. For these vectors
$$\boldsymbol{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ and } \boldsymbol{v} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$
   - Find the projection of $\boldsymbol{v}$ over $\boldsymbol{u}$: $\boldsymbol{a}_1 = \text{proj}_{\boldsymbol{u}}^{\boldsymbol{v}}$
   - Find $\boldsymbol{a}_2 = \boldsymbol{v} - \boldsymbol{a}_1$ using the definition.
   - Are $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ perpendicular (orthogonal)?
   - The formulas
$$\boldsymbol{a}_1 = \left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\boldsymbol{u} \cdot \boldsymbol{u}}\right) \boldsymbol{u}$$
$$\boldsymbol{a}_2 = \boldsymbol{v} - \boldsymbol{a}_1 = \boldsymbol{v} - \left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\boldsymbol{u} \cdot \boldsymbol{u}}\right) \boldsymbol{u}$$

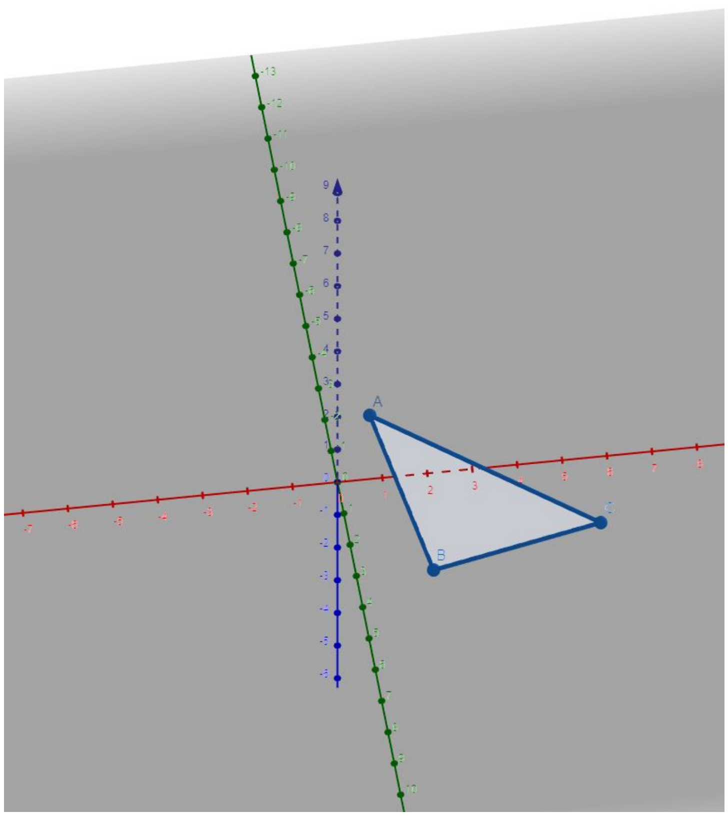
4. which two vectors are more similar considering both the distance and cosine of an angle between them?

$$\boldsymbol{u} = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 2 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix} \text{ and } \boldsymbol{w} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

5. Find all the norms for these vectors:

$$\boldsymbol{u} = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 2 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix} \text{ and } \boldsymbol{w} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

6. Which of the angles (if any) of triangle $\triangle ABC$, with $A = (1, -2, 0)$, $B = (2, 1, -2)$ and $C = (6, -1, -3)$ is a right angle?



1. For these vectors

$$\boldsymbol{u} = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 2 \end{bmatrix} \text{ and } \boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix} \in \mathbb{R}^4$$

combinations

- Find $\boldsymbol{u}+\boldsymbol{v}$, $\boldsymbol{u}-\boldsymbol{v}$, $2\boldsymbol{u}+3\boldsymbol{v}$
- Find the cosine between these two vecors and their lengthes
- Find the distance between them.

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{u}+\vec{v} = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0+1 \\ 4+0 \\ -1+3 \\ 2+(-1) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \\ 1 \end{bmatrix}$$

$c\vec{u} + d\vec{v}$

$c \in \mathbb{R} \quad d \in \mathbb{R}$

$$\vec{u}-\vec{v} = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0-1 \\ 4-0 \\ -1-3 \\ 2-(-1) \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -4 \\ 3 \end{bmatrix}$$

vectores are closed under addition/subtraction.

$$2\vec{u}+3\vec{v} = 2\begin{bmatrix} 0 \\ 4 \\ -1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ -2 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 9 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 7 \\ 1 \end{bmatrix} \checkmark$$

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\vec{u}\cdot\vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos(\theta) \rightarrow \cos(\theta) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}$$

$$\vec{u}\cdot\vec{v} = \boxed{\sum_{i=1}^{n} u_i v_i} \Leftarrow$$

$$\vec{u}\cdot\vec{v} = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix} = (0)(1) + (4)(0) + (-1)(3) + (2)(-1)$$

$$= 0 + 0 - 3 - 2 = -5$$

$$\|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}} \leftarrow L_2$$

$\mathbb{R}^n$

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2} = \sqrt{0^2 + 4^2 + (-1)^2 + (2)^2} = \sqrt{16+1+4}$$

$$= \sqrt{21}$$

V vector space $\quad V = \mathbb{R}^n$

dot product

$$\boxed{\vec{v} \cdot \vec{u} = \sum_{i=1}^{n} u_i v_i}$$

$$\boxed{\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}} = \sqrt{\sum_{i=1}^{n} v_i^2} = \underbrace{\|\vec{v}\|_2}_{L_2}$$

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{\sum_{i=1}^{4} v_i^2} = \sqrt{1^2 + 0^2 + 3^2 + (-1)^2} = \sqrt{1+9+1} = \sqrt{11}$$

$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|} = \frac{\vec{u} \cdot \vec{v}}{\left(\sqrt{\vec{u} \cdot \vec{u}}\right)\left(\sqrt{\vec{v} \cdot \vec{v}}\right)}$$

$$\boxed{\sqrt{ab} = \sqrt{a}\sqrt{b}}$$

$$= \frac{-5}{(\sqrt{21})(\sqrt{11})} = \frac{-5}{\sqrt{(21)(11)}} = \frac{-5}{\sqrt{231}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{-5}{\sqrt{(21)(11)}}\right)}$$

$(21)(11) = 231$

$\sqrt{231} = 15.2$

$$= \cos^{-1}(-0.33) = 109.2^\circ$$

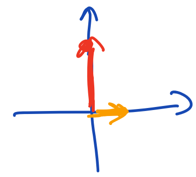
$$\text{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \left\| \begin{bmatrix} -1 \\ +4 \\ -4 \\ 3 \end{bmatrix} \right\|$$

$$= \sqrt{1 + 16 + 16 + 9} = \sqrt{46}$$

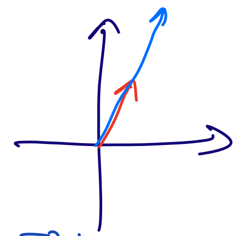
$\vec{u} \parallel \vec{v}$ $\qquad$ $\vec{u} = c\vec{v}$ $\qquad$ dependent

2. Are these vectors linearly independent?

- $\boxed{\boldsymbol{u} = \begin{bmatrix}0\\4\end{bmatrix} \text{ and } \boldsymbol{v} = \begin{bmatrix}1\\0\end{bmatrix}}$



- $\boldsymbol{u} = \begin{bmatrix}4\\2\\-6\end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix}10\\5\\-15\end{bmatrix}$



- $\boldsymbol{u} = \begin{bmatrix}1\\2\end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix}3\\6\end{bmatrix}$

For a given set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$,

if from $\underbrace{c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \vec{0}}$, we conclude

that $c_1 = c_2 = \dots = c_k = 0 \implies$ we say that the set of vectors are linearly independent.

$$\boxed{c_1\vec{u} + c_2\vec{v} = \begin{bmatrix}0\\0\end{bmatrix} = \vec{0}} \longrightarrow \; ? \; c_1 = c_2 = 0$$

$$c_1\begin{bmatrix}0\\4\end{bmatrix} + c_2\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\underbrace{\begin{bmatrix}0\\4c_1\end{bmatrix} + \begin{bmatrix}c_2\\0\end{bmatrix}} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\underbrace{\begin{bmatrix}0+c_2\\4c_1+0\end{bmatrix}} = \underbrace{\begin{bmatrix}0\\0\end{bmatrix}} \implies \begin{cases}\boxed{c_2 = 0}\\ 4c_1 = 0 \longrightarrow \boxed{c_1 = 0}\end{cases}$$

$$\implies \boxed{c_1 = c_2 = 0} \implies \vec{u} \text{ and } \vec{v} \text{ are}$$

linearly indep.

$\boldsymbol{u} = \begin{bmatrix}1\\2\end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix}3\\6\end{bmatrix}$

$c_1\vec{u} + c_2\vec{v} = \vec{0}$

$c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}3\\6\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$\underbrace{\begin{bmatrix}c_1\\2c_1\end{bmatrix} + \begin{bmatrix}3c_2\\6c_2\end{bmatrix}} = \begin{bmatrix}0\\0\end{bmatrix}$

$\begin{bmatrix}c_1+3c_2\\2c_1+6c_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \rightarrow \begin{cases} c_1 + 3c_2 = 0 \rightarrow \boxed{c_1 = -3c_2} \\ 2c_1 + 6c_2 = 0 \end{cases}$

$2(-3c_2) + 6c_2 = 0$

$-6c_2 + 6c_2 = 0$

$\boxed{0 = 0}$

$\rightarrow \boxed{\boxed{c_1 = c_2 = 0 \ ?}} \leftarrow$

$c_1 + 3c_2 = -3+3(1) = 0$

$c_1 = -3 \quad c_2 = 1$

$2c_1 + 6c_2 = 2(-3) + 6(1) = 0$

$\vec{u}$ and $\vec{v}$ are linearly dependent.

$$\boldsymbol{u} = \begin{bmatrix}4\\2\\-6\end{bmatrix} \in \mathbb{R}^3 \text{ and } \boldsymbol{v} = \begin{bmatrix}10\\5\\-15\end{bmatrix} \in \mathbb{R}^3 \qquad \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$c_1\vec{u} + c_2\vec{v} = \vec{0} \overset{?}{\Longrightarrow} c_1 = c_2 = 0$$

$$c_1\begin{bmatrix}4\\2\\-6\end{bmatrix} + c_2\begin{bmatrix}10\\5\\-15\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}4c_1 + 10c_2\\2c_1 + 5c_2\\-6c_1 - 15c_2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \longrightarrow \begin{cases}4c_1 + 10c_2 = 0\\2c_1 + 5c_2 = 0\\-6c_1 - 15c_2 = 0\end{cases}$$

$$4c_1 = -10c_2 \longrightarrow c_1 = -\frac{10}{4}c_2 = -\frac{5}{2}c_2$$

plug in the 2^{nd} equation:

$$\cancel{2}\left(-\frac{5}{\cancel{2}}c_2\right) + 5c_2 = 0 \longrightarrow -5c_2 + 5c_2 = 0$$

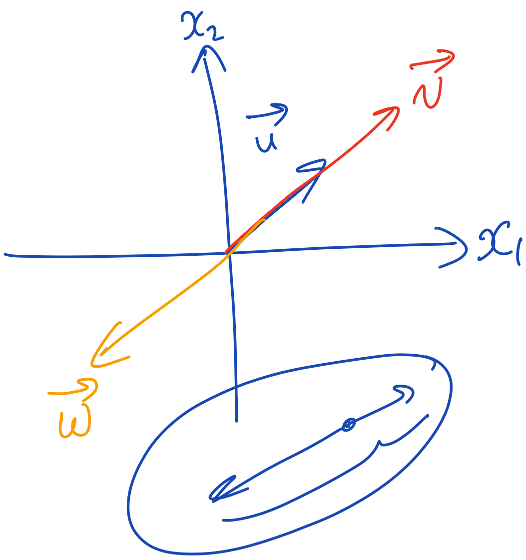
$$\longrightarrow 0 = 0$$

plug in the 3^{rd} equation

$$-\overset{3}{\cancel{6}}\left(-\frac{5}{\cancel{2}}c_2\right) - 15c_2 = 0$$

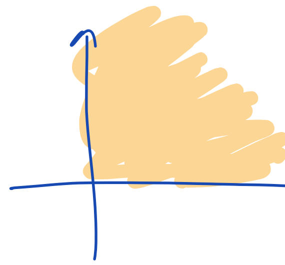
$$15c_2 - 15c_2 = 0 \longrightarrow 0 = 0$$

I did not conclude that $c_1 = c_2 = 0$ is the only solution $\Longrightarrow$ $\vec{u}$ and $\vec{v}$ are linearly dep.



$\vec{u} = c\vec{v}$ $\quad c > 0$

$\vec{w} = d\vec{u}$ $\quad d < 0$



✓

3. For these vectors

$$\boldsymbol{u} = \begin{bmatrix}1\\0\\1\end{bmatrix} \text{ and } \boldsymbol{v} = \begin{bmatrix}0\\-1\\1\end{bmatrix}$$

$\text{proj}_{\vec{u}} \vec{v}$

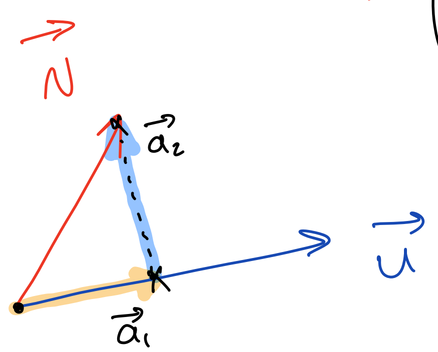
- Find the projection of $\boldsymbol{v}$ over $\boldsymbol{u}$: $\boldsymbol{a}_1 = \text{proj}_{\boldsymbol{u}}^{\boldsymbol{v}}$
- Find $\boldsymbol{a}_2 = \boldsymbol{v} - \boldsymbol{a}_1$ using the definition.
- Are $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ perpendicular (orthogonal)?
- The formulas

$\vec{u} \circ \vec{v}$

$\vec{v} \circ \vec{u}$

$$\boxed{\boldsymbol{a}_1 = \left(\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\boldsymbol{u}\cdot\boldsymbol{u}}\right)\boldsymbol{u}}$$

$$\boxed{\boldsymbol{a}_2 = \boldsymbol{v} - \boldsymbol{a}_1 = \boldsymbol{v} - \left(\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\boldsymbol{u}\cdot\boldsymbol{u}}\right)\boldsymbol{u}}$$



$$\boxed{\begin{aligned} \vec{v} &= \vec{a}_1 + \vec{a}_2 \\ \vec{a}_1 &\perp \vec{a}_2 \\ \vec{a}_1 &= c\,\vec{u} \end{aligned}}$$

$$\vec{u} \circ \vec{v} = \begin{bmatrix}1\\0\\1\end{bmatrix}\begin{bmatrix}0\\-1\\1\end{bmatrix} = \underbrace{(1)(0)}_{0} + \underbrace{(0)(-1)}_{0} + \underbrace{(1)(1)}_{1} = 1$$

$$\vec{u} \circ \vec{u} = \begin{bmatrix}1\\0\\1\end{bmatrix} \cdot \begin{bmatrix}1\\0\\1\end{bmatrix} = (1)(1) + (0)(0) + (1)(1) = 2 \leftarrow$$

$$\vec{a}_1 = \left(\frac{1}{2}\right)\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}(\frac{1}{2})(1)\\(\frac{1}{2})(0)\\(\frac{1}{2})(1)\end{bmatrix} = \boxed{\begin{bmatrix}1/2\\0\\1/2\end{bmatrix}}$$

$$\vec{a}_2 = \vec{v} - \vec{a}_1 = \begin{bmatrix}0\\-1\\1\end{bmatrix} - \begin{bmatrix}1/2\\0\\1/2\end{bmatrix} = \boxed{\begin{bmatrix}-1/2\\-1\\1/2\end{bmatrix}}$$

$\vec{a}_1 \perp \vec{a}_2$

$$\vec{a}_1 \cdot \vec{a}_2 = (1/2)(-1/2) + (0)(-1) + (1/2)(1/2)$$

$$= -\frac{1}{4} + 0 + \frac{1}{4} = 0.$$

$\vec{u} \cdot \vec{v}$

$$\text{proj}_{\vec{u}}\vec{v} = \left(\frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\right)\vec{u} = \left(\frac{1}{2}\right)\vec{u}$$

4. which two vectors are more similar considering both the distance and cosine of an angle between them?

$$\boldsymbol{u} = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 2 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix} \text{ and } \boldsymbol{w} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{dist}(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\| = \left\| \begin{bmatrix} -1 \\ 4 \\ -4 \\ 3 \end{bmatrix} \right\| = \sqrt{(-1)^2+(4)^2+(-4)^2+(3)^2}$$

$$= \sqrt{1+16+16+9} = \sqrt{42}$$

$$\text{dist}(\vec{u},\vec{w}) = \|\vec{u}-\vec{w}\|$$

$$= \left\| \begin{bmatrix} -1 \\ 4 \\ -2 \\ 1 \end{bmatrix} \right\| = \sqrt{(-1)^2+(4)^2+(-2)^2+(1)^2}$$

$$= \sqrt{1+16+4+1} = \sqrt{22}$$

$$\text{dist}(\vec{v},\vec{w}) = \|\vec{v}-\vec{w}\| = \left\| \begin{bmatrix} 0 \\ 0 \\ 2 \\ -2 \end{bmatrix} \right\| = \sqrt{(0)^2+(0)^2+(2)^2+(-2)^2}$$

$$= \sqrt{4+4} = \sqrt{8}$$

$$\sqrt{8} < \sqrt{22} < \sqrt{42}$$

$\vec{v}$ and $\vec{w}$ are more similar based on distance.

$$\vec{u}\cdot\vec{v} = 0+0-3-2 = -5$$

$$\vec{u}\cdot\vec{w} = 0+0-1+2 = 1$$

$$\vec{v}\cdot\vec{w} = 1+0+3-1 = 3$$

$$\cos(\theta_{\vec{u},\vec{v}}) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

$$\|\vec{u}\| = \sqrt{0+16+1+4} = \sqrt{21} \qquad \|\vec{v}\| = \sqrt{1+0+9+1} = \sqrt{11}$$

$$\sqrt{\vec{w}} = \sqrt{1+0+1+1} = \sqrt{3}$$

$$\cos(\theta_{\vec{u},\vec{v}}) = \frac{-5}{(\sqrt{21})(\sqrt{11})} \simeq -0.3 \qquad \cos(\theta_{\vec{u},\vec{w}}) = \frac{1}{(\sqrt{21})(\sqrt{3})} \simeq \boxed{0.12}$$

$$\cos(\theta_{\vec{v},\vec{w}}) = \frac{3}{(\sqrt{11})(\sqrt{3})} \simeq 0.5$$

$\vec{v}$ and $\vec{w}$ are more similar based on cosine similarity.

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\vec{u} \perp \vec{v} \qquad \underbrace{\vec{u} \cdot \vec{v}}_{\sum_{i=1}^{n} u_i v_i = 0} = 0$$

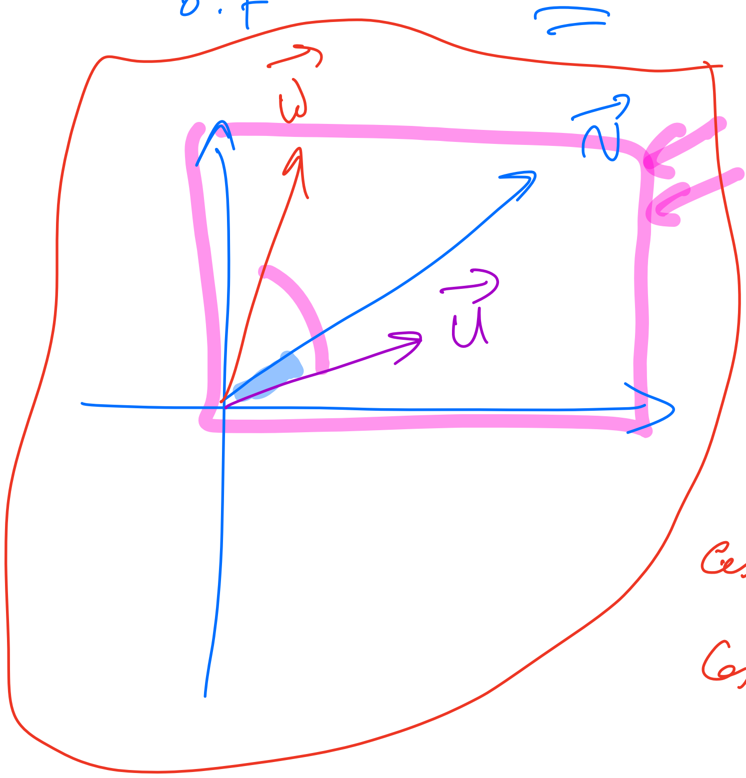
$$\underbrace{\cos(\theta_1)}_{0.7} \qquad \underbrace{\cos(\theta_2)}_{\underline{0.2}}$$

$\theta = 0^\circ$

$\cos(\theta) = 1$

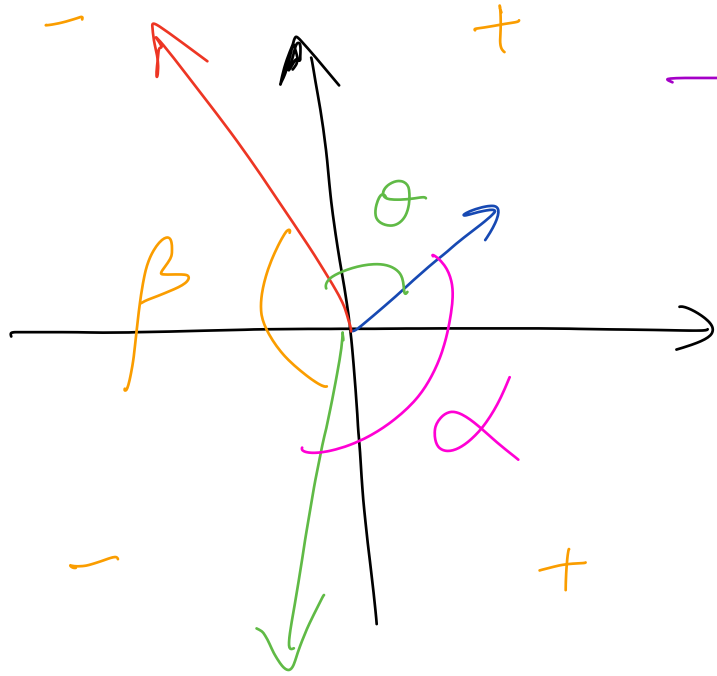
$\theta = \frac{\pi}{2}$

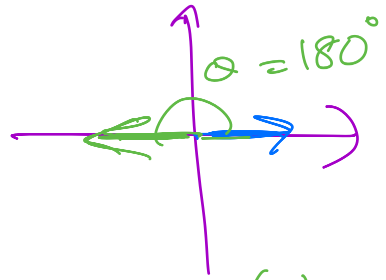
$\cos(\theta) = 0$



$\cos\theta \longrightarrow 0$

$\cos\theta \longrightarrow 1$





$\cos(\theta) = -1$

similarity

5. Find all the norms for these vectors:

$$\boldsymbol{u} = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 2 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix} \text{ and } \boldsymbol{w} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

6. Which of the angles (if any) of triangle $\triangle ABC$, with $A = (1, -2, 0)$, $B = (2, 1, -2)$ and $C = (6, -1, -3)$ is a right angle?