International Journal of Approximate Reasoning 51 (2010) 305–334

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

journal homepage: www.elsevier.com/locate/ijar

The orness measures for two compound quasi-arithmetic mean aggregation operators ☆

Xinwang Liu *

School of Economics and Management, Southeast University, Nanjing 210096, China
Ming Hsieh Department of Electrical Engineering, University of Southern California, Los Angeles, CA 90089-2564, USA

A R T I C L E I N F O

Article history:
Received 29 October 2008
Received in revised form 13 October 2009
Accepted 16 October 2009
Available online 26 October 2009

Keywords:
Aggregation operators
Orness
Quasi-arithmetic mean
Quasi-OWA operator
Bajraktarević mean

A B S T R A C T

The paper first summarizes the orness measures and their common characteristics of some averaging operators: the quasi-arithmetic mean, the ordered weighted averaging (OWA) operator, the regular increasing monotone (RIM) quantifier and the weighted function average operator, respectively. Then it focuses on the aggregation properties and operator determination methods for two kinds of quasi-arithmetic mean-based compound aggregation operators: the quasi-OWA (ordered weighted averaging) operator and the Bajraktarević mean. The former is the combination of the quasi-arithmetic mean and the OWA operator, while the latter is the combination of the quasi-arithmetic mean and the weighted function average operator. Two quasi-OWA operator forms are given, where the OWA operator is assigned directly or generated from a RIM (regular increasing monotone) quantifier indirectly. The orness indexes to reflect the or-like level of the quasi-OWA operator and Bajraktarević mean are proposed. With generating function techniques, the properties of the quasi-OWA operator and Bajraktarević mean are discussed to show the rationality of these orness definitions. Based on these properties, two families of parameterized quasi-OWA operator and Bajraktarević mean with exponential and power function generators are proposed and compared. It shows that the method of this paper can also be applied to other function-based aggregation operators.

1. Introduction

A large range of aggregation operators have been introduced in the last few years [9,1,5,7,8,37,31,53]. These operators include the triangular norms and conorms, ordered weighted average aggregation, operators based on Choquet and Sugeno integrals and others. For the recent theoretical and application results in this domain, we recommend the overviews [2,5,31,43]. Among them, averaging aggregation is the most common way in practice, which makes the aggregation result between maximum and minimum. It is commonly used in voting, multicriteria and group decision making, statistical analysis, etc. [2,3,5]. An important index to describe the characteristic of an averaging aggregation operator is the concept of orness, which can make the aggregation value become some or-like (close to the maximum) or and-like (close to the minimum) in real application problems [2,5,53].

In this paper, we will summarize the common characteristics of some averaging operators, and then propose the orness measures for two compound quasi-arithmetic mean aggregation operators, the quasi-OWA operator and the Bajraktarević

☆ The work is supported by the National Natural Science Foundation of China (NSFC) under project 70771025, and Program for New Century Excellent Talents in University of China NCET-06-0467.

* Address: School of Economics and Management, Southeast University, Nanjing 210096, China.
E-mail address: xwliu@seu.edu.cn

0888-613X/$ - see front matter
doi:10.1016/j.ijar.2009.10.003

mean. The quasi-OWA operator is the combination of the quasi-arithmetic mean and the OWA operator or the combination of the quasi-arithmetic mean and the RIM quantifier [15,18,4]. The quasi-arithmetic mean covers a wide spectrum of means from arithmetic, quadratic, geometric and harmonic to the general root-power and exponential means, which are commonly used in both theory and applications [22,3,17,31,41]. The OWA operator was introduced by Yager [46]. It provides a general class of parameterized aggregation operators that include the *min*, *max*, *average*. And it has been applied in many areas such as decision making, expert systems, data mining, approximate reasoning, fuzzy system and control, etc. [53,21,20,42,51,27]. If the OWA operator is connected with the RIM quantifier [48], it can also be applied to the computing with words theory [50]. The properties of RIM quantifier correspond to the properties of OWA operator with dimension number approaching infinity [48,25,27]. Another compound quasi-arithmetic mean aggregation operator is the Bajraktarević mean, which is the combination of the weighted function average and the quasi-arithmetic mean operator [3,14]. Some properties and relevant issues of it were discussed [3,36,13]. The weighted function average is also called additive neat OWA (ANOWA) operator [30], which is a generation of a neat OWA operator family called BADD (BAsic Defuzzification Distribution) OWA operator [52]. It can be used to aggregate the linguistic labels represented by partially ordered fuzzy numbers in fuzzy group decision making problems where the elements of the input vector do not have to be ordered [34,35]. Pereira and Ribeiro [38] also called this weighted function average as the generalized mixture operator, where the monotonicity condition was provided.

The measure of orness was first introduced by Dujmović [10] for the power means under the name of disjunction degree. In the following works, Dujmović [11,13,12] proposed various forms of generalized conjunction/disjunction function. Yager defined the orness concept independently in the case of OWA operator [46]. It can be proved that Yager's orness measure for OWA operator coincides with Dujmović's definition as a special case [40,32,4]. Marichal [32,33] proposed the orness definition of discrete Choquet integral for multicriteria decision problems. Marichal [31] also proposed that the degrees of orness can be defined for any compensative aggregation operator. Based on the OWA operator orness, Yager [48] proposed the orness of the RIM quantifier with the OWA dimension number approaching infinity. Calvo et al. [6] suggested an extension of OWA operators by applying the concept of weighting triangles in place of the standard OWA weighting vector. Salido and Murakami [40] extended the OWA orness measure to fuzzy aggregation operators. Larsen [24] proposed an orness measure for the root-power mean in the quasi-arithmetic mean family. Yager also extended the orness concept to the generalized OWA (GOWA) operator [49]. Pereira [39] applied the OWA orness measure directly in the mixture operator and quasi-arithmetic mean with exponential function cases. Kolesárová and Mesiar [23] also proposed a general framework with the unipolar and bipolar parametric characterization of aggregation functions, where the orness/andness measure in the class of averaging aggregation functions were discussed.

As the general orness uses the multiple integral with the integral fold number being the number of the aggregated elements, the computation becomes complicated when the number of the aggregated number is large, and furthermore, the analytical formula of the orness measure often cannot be obtained. Some methods to measure the orness without the number of aggregated elements and the form of multiple integral were attempted. Larsen [24] proposed the orness of the quasi-arithmetic mean with power function. Dujmović [11] also proposed an orness formula for the quasi-arithmetic mean with arithmetic operations of the generator function. Some properties were discussed. Liu [26] proposed an orness definition by considering the formula of the quasi-arithmetic mean. Some properties and characteristics of the quasi-arithmetic mean associated with the orness measure were analyzed and proved. Liu and Lou [30] also proposed an orness definition for the weighted function average operator. The rationality of these orness definitions and the function determination methods for the quasi-arithmetic mean and the weighted function average were discussed. A common feature of these orness definitions is that they are independent on the dimension number of the aggregated input vector, where the multiple integral can be avoided. The orness value keeps as a constant for the same function definition of different dimension instantiations. The orness formula can be obtained and some properties of these aggregation operators can be revealed [26,30]. Another fact is that both the orness definitions of quasi-arithmetic mean and weighted function average operator demonstrate some common characteristics, which are also held by the orness concepts of the OWA operator and the RIM quantifier. A summary of these results are given in Section 2.

Based on the summary of the orness measures of some related aggregation operators, we will try to connect these orness measures together by considering two kinds of compound aggregation operators: the quasi-OWA operator and the Bajraktarević mean. As an extension of the quasi-arithmetic mean orness, we will propose the orness measures for the quasi-OWA operator and the Bajraktarević mean, respectively. Two extension forms of the quasi-OWA operator are proposed, where the quasi-OWA operator orness either is the combination of the quasi-arithmetic mean and the OWA operator or the combination of the quasi-arithmetic mean and the RIM quantifier, respectively. It is also a generalization of the GOWA operator orness. Similarly, the orness measure of the Bajraktarević mean [3] is also proposed, which is the combination of the quasi-arithmetic mean [26] orness and the weighted function average orness definitions [30]. Some properties associated with these orness measures are discussed with the generating function techniques. Two parameterized quasi-OWA operator and Bajraktarević mean families with exponential functions and power functions are proposed. The quasi-OWA operator and Bajraktarević mean with exponential functions are symmetrical to their orness parameters, and are also shift invariant, but those with the power functions are ratio invariant. Some properties of them are discussed and compared. With these results, we can further understand the characteristics of these aggregation operators and their orness measures. Furthermore, these orness definition methods can also be extended to other function-based operators. As the compound aggregation operators are usually more complicated than the ordinary ones [4, p.56], the results of this paper is helpful to the further study of these compound aggregation operators and other averaging operators both in theory and applications.

The remainder of this paper is organized as follows. Section 2 summarizes some common properties and characteristics of the orness measures for the OWA operator, the RIM quantifier, the quasi-arithmetic mean and the weighted function average operators, respectively. Section 3 proposes an orness measure for quasi-OWA operator. Some properties associated with this orness measure are discussed, which verify the rationality of this orness definition. Two parameterized quasi-OWA operators with exponential functions and power functions in the OWA operator and quasi-arithmetic mean generators are given, respectively. Some properties of them are discussed. Similarly, Section 4 proposes an orness measure for the Bajraktarević mean. Two parameterized Bajraktarević mean with exponential functions and power functions in the weighted function average and quasi-arithmetic means generators are also proposed, respectively. Section 5 gives some discussions on the common characteristics of these orness measures and also some comparisons on the orness properties of the quasi-OWA operator and the Bajraktarević mean. Section 6 summarizes the main results and draws conclusions.

2. A summary on the orness of some related aggregation operators

Here, we will give a summary on the orness definitions and their properties for the aggregation operators. Some common characteristics of these orness measures can be observed. The origin and proofs of these conclusions can be found in the related references.

2.1. The orness measures for quasi-arithmetic mean

Definition 1. Let f be a continuous strictly monotone function. For input vector $X = (x_1, x_2, \ldots, x_n) \in [a,b]^n$, a quasi-arithmetic mean can be defined as the aggregation operator $M_f : [a,b]^n \to [a,b]$ given by

$$M_f(x_1, x_2, \ldots, x_n) = f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right) \tag{1}$$

where f^{-1} is its inverse function. f is called a generator of the quasi-arithmetic mean M_f. The aggregation results can be denoted as $M_f(X)$.

The general orness measure for a mean aggregation operator $M(x_1, x_2, \ldots, x_n) \in [0,1]^n$ can be defined as [10]:

$$orness(M) = \frac{E(M) - E(\min)}{E(\max) - E(\min)} \tag{2}$$

where E is the expectation value on $[0,1]^n$. With $E(\max) = \frac{n}{n+1}$, $E(\min) = \frac{1}{n+1}$, (2) can be expressed more specifically:

$$orness(M) = \frac{(n+1)\int_0^1 \int_0^1 \cdots \int_0^1 M(x_1, x_2, \ldots, x_n) dx_1 dx_2 \cdots dx_n - 1}{n-1} \tag{3}$$

For quasi-arithmetic mean with generator $f(x)$, (3) can also be expressed as:

$$orness(M_f) = \frac{(n+1)\int_0^1 \int_0^1 \cdots \int_0^1 f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right) dx_1 dx_2 \cdots dx_n - 1}{n-1} \tag{4}$$

This orness definition is widely recognized and used in various applications and extensions [31,?,16,32,33,4]. However, when it is applied to the function-based aggregation operators, especially the quasi-arithmetic mean case, as the orness level associates with the dimension number of the input vector with multiple integral comoutation, we cannot obtain the orness formula in analytical form even under the most commonly used conditions except for some special cases. Some characteristics of the aggregation operators cannot be revealed.

As an example, for quasi-arithmetic mean (1) on $[0,1]^n$, it seems that the general orness formula (3) can only be analytical expressed for generator functions $f(x) = x$ and $f(x) = \ln(x)$, which correspond to the cases of arithmetic average and geometric average, respectively [4, p. 54]. It is difficult to get the orness formula even for the very simple case of $f(x) = x^2$ with $n = 3$, and it is usually computed in numerical way:

$$orness(M_f) = 2\int_0^1 \int_0^1 \int_0^1 \sqrt{\frac{1}{3}(x_1^2 + x_2^2 + x_2^2)} dx_1 dx_2 dx_3 - \frac{1}{2} = 0.6092$$

From (4), for quasi-arithmetic mean with power function generator $f(x) = x^r$, when $r = 1$, the quasi-arithmetic mean becomes the ordinary arithmetic average. The orness level is $\frac{1}{2}$, which is neither and-like nor or-like. From (1) and (4), it can also be observed that on $[0,1]^n$, when $r > 1$, the quasi-arithmetic mean is or-like, and the bigger value of r is, the more or-like the aggregation operator will be. Here, a problem arises, if we want to obtain a quasi-arithmetic mean with given orness level, say $orness(M_f) = 0.7$ and $n = 5$, how the parameter r in $f(x) = x^r$ can be determined? It seems that this general orness definition is not convenient for such computation.

On the other hand, for the aggregation operators determined by continuous functions, such as the quasi-arithmetic mean and the weighted function average, there are facts that the or-like or and-like property of the aggregation operator is just determined by the generator function itself. Such orness definition makes the orness value change with the dimension number for the same function definition. Some orness definitions for quasi-arithmetic mean which is independent on the dimension number were proposed [11,24,26].

Next, we will give some results of [11,26]. Dujmović [11,14] proposed an orness measure which is directly computed with the generator function $f(x)$ for $x_i \in [0,1]$:

$$\omega_f = \frac{f(1) - \int_0^1 f(x)dx}{f(1) - f(0)} \tag{5}$$

With (5), it is observed that:

Proposition 1 [11]. *If $f'(x)f''(x) < 0$, then $\omega_f < 0.5$, the quasi-arithmetic mean is and-like, and if $f'(x)f''(x) > 0$, then $\omega_f > 0.5$, the quasi-arithmetic mean is or-like.*

Liu [26] proposed another orness formula considering the expression of quasi-arithmetic mean, and some properties were discussed with the generating function technique. We will introduce the main results for the summary of the common properties of orness concepts and the comparison with (5).

Definition 2 [26]. The orness measure of a quasi-arithmetic mean (1) determined by generator $f(x)$ can be defined as:

$$\Omega_f = \frac{f^{-1}\left(\frac{\int_a^b f(x)dx}{b-a}\right) - a}{b - a} \tag{6}$$

Proposition 2 [26]. $0 < \Omega_f < 1$.

Proposition 3 [26]. *If $X = (x_1, x_2, \ldots, x_n)$ is evenly distributed on $[a,b]$, that is $x_i = a + \frac{n-i}{n-1}(b-a)$, then $\lim_{n\to+\infty} M_f(X) = a + \Omega_f(b-a)$.*

Proposition 4 [26]. $\Omega_{f(x)} + \Omega_{f(a+b-x)} = 1$.

Proposition 5 [26]. *For all $X = (x_1, x_2, \ldots, x_n)$, $M_{kf+c}(X) = M_f(X)$, $\Omega_{kf+c} = \Omega_f$ $(k \neq 0)$.*

Proposition 5 means that the quasi-arithmetic mean is invariant for linear transformation of the generator. For any quasi-arithmetic mean M_f on $[a,b]$, we can always choose a generator g such that g is increasing, $g(a) = 0$ and $g(b) = 1$, that makes $g = kf + c$ $(k \neq 0)$ such that $M_g = M_f$ and $\Omega_g = \Omega_f$. This continuous, strictly monotonic increasing, non-negative function family on a closed interval $[a,b]$ is denoted by $\mathscr{F}$.

To further analyze the properties of quasi-arithmetic mean, a generating function representation method of $f(x)$ was proposed.

Definition 3 [26]. For any mapping φ on $[a,b]$ and a generator of quasi-arithmetic mean $f(x) \in \mathscr{F}$, $\varphi(x)$ is called the generating function of $f(x)$, if it satisfies

$$f(x) = \int_a^x \varphi(t)dt$$

where $\varphi(t) \geqslant 0$, and $\int_a^b \varphi(t)dt = 1$.

With any $\varphi(x)$, $f(x)$ can be uniquely determined, and vice versa. If $f(x)$ is differentiable, $\varphi(x)$ can be set as its first order differential function $f'(x)$ directly. Some properties associated with the orness measure were discussed.

Proposition 6 [26]. *For quasi-arithmetic mean generator $f(x)$, $g(x)$ with generating functions $\varphi(x)$ and $\psi(x)$, respectively, if for all $s, t \in [a,b]$, $s > t$, $\frac{\varphi(s)}{\varphi(t)} \geqslant (>) \frac{\psi(s)}{\psi(t)}$, then $\Omega_f \geqslant (>) \Omega_g$, and for any $X = (x_1, x_2, \ldots, x_n) \in [a,b]^n$, $M_f(X) \geqslant M_g(X)$.*

If $g(x) = x$, then $\psi(x) = 1$, $\Omega_g = \frac{1}{2}$, and $M_g(X) = A(X) = \frac{1}{n}\sum_{i=1}^n x_i$. From Proposition 6, it can be easily obtained that:

Corollary 1 [26]. *For any quasi-arithmetic mean generator $f(x) \in \mathscr{F}$ with generating function $\varphi(t)$ on $[a,b]$ and $X = (x_1, x_2, \ldots, x_n) \in [a,b]^n$, if $f(x)$ is (strictly) convex, which means that $\varphi(t)$ is (strictly) increasing on $[a,b]$, then $\Omega_f \geqslant (>) \frac{1}{2}$, and $M_f(X) \geqslant \frac{1}{n}\sum_{i=1}^n x_i$, the operator is or-like; if $f(x)$ is (strictly) concave, which means that $\varphi(t)$ is (strictly) decreasing on $[a,b]$, then $\Omega_f \leqslant (<) \frac{1}{2}$, and $M_f(X) \leqslant \frac{1}{n}\sum_{i=1}^n x_i$, the operator is and-like.*

If the generating function is differentiable and nonzero, the premise of Proposition 6 can be replaced with a more simplified form.

Proposition 7 [26]. *For quasi-arithmetic mean generators $f(x)$, $g(x)$ with differentiable generating functions $\varphi(x)$, $\psi(x)$, respectively, if for all $t \in [a,b]$, $\frac{\varphi'(t)}{\varphi(t)} \geqslant (>) \frac{\psi'(t)}{\psi(t)}$, then $\Omega_f \geqslant (>) \Omega_g$, and for any $X = (x_1, x_2, \ldots, x_n) \in [a,b]^n$, $M_f(X) \geqslant M_g(X)$.*

Without the generating function, we usually have $\frac{\varphi'(t)}{\varphi(t)} = \frac{f''(x)}{f'(x)}$. As $\frac{f''(x)}{f'(x)}$ does not change if $f(x)$ is replaced with $-f(x)$, with Proposition 5, $f(x)$ and $-f(x)$ can be seen as the same both in data set aggregation and the orness value computation. Proposition 7 can be expressed as

Corollary 2. *For quasi-arithmetic mean generators $f(x)$ and $g(x)$, if for all $x \in [a,b]$, $\frac{f''(x)}{f'(x)} \geqslant (>) \frac{g''(x)}{g'(x)}$, then $\Omega_f \geqslant (>) \Omega_g$, and for any $X = (x_1, x_2, \ldots, x_n) \in [a,b]^n$, $M_f(X) \geqslant M_g(X)$.*

It can be derived that $\frac{f''(x)}{f'(x)}$ reflects the orness extent of a quasi-arithmetic mean. The bigger the $\frac{f''(x)}{f'(x)}$ value is, the more or-like the quasi-arithmetic mean will be. Furthermore, as when $g(x) = x$, $\Omega_g = \frac{1}{2}$, we can get that, if $\frac{f''(x)}{f'(x)} < 0$, then $\Omega_f < \frac{1}{2}$, $M_f(X) \leqslant \frac{1}{n}\sum_{i=1}^n x_i$, the quasi-arithmetic mean is and-like. If $\frac{f''(x)}{f'(x)} > 0$, then $\Omega_f > \frac{1}{2}$, $M_f(X) \geqslant \frac{1}{n}\sum_{i=1}^n x_i$, the quasi-arithmetic mean is or-like. These can be seen as an extension of Proposition 1 in [11].

2.2. The orness measure for OWA operator

An OWA operator of dimension n is a mapping $F_W : \mathbb{R}^n \to \mathbb{R}$ that has an associated weighting vector $W = (w_1, w_2, \ldots, w_n)$ having the properties [46]

$$w_1 + w_2 + \cdots + w_n = 1; \quad 0 \leqslant w_i \leqslant 1, \quad i = 1, 2, \ldots, n$$

and such that

$$F_W(X) = F_W(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} w_j y_j \tag{7}$$

with y_j being the jth largest of the x_i.

The degree of "orness" associated with this operator is defined as [46]:

$$orness(W) = \sum_{j=1}^{n} \frac{n-j}{n-1} w_j \tag{8}$$

The *max*, *min* and *average* correspond to W^*, W_* and W_A, respectively, where $W^* = (1, 0, \ldots, 0)$, $W_* = (0, 0, \ldots, 1)$ and $W_A = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, that is $F_{W_*}(X) = \min_{1 \leqslant i \leqslant n}\{x_i\}$, $F_{W^*}(X) = \max_{1 \leqslant i \leqslant n}\{x_i\}$ and $F_{W_A}(X) = \frac{1}{n}\sum_{i=1}^n x_i = A(X)$. Obviously, $orness(W^*) = 1$, $orness(W_*) = 0$ and $orness(W_A) = \frac{1}{2}$.

From (8), some properties about OWA operator are listed in the following.

Proposition 8. $0 \leqslant orness(W) \leqslant 1$.

Proposition 9. *If $X = (x_1, x_2, \ldots, x_n)$ is evenly distributed on $[a,b]$, that is $x_i = a + \frac{n-i}{n-1}(b-a)$, then $F_W(X) = a + orness(W) * (b-a)$.*

Proposition 10 [47, p. 127]. *For OWA operator weighting vector $W = (w_1, w_2, \ldots, w_n)$, $orness(W) = \alpha$, then for the reverse order of W, $\widetilde{W} = (w_n, w_{n-1}, \ldots, w_1)$, $orness(\widetilde{W}) = 1 - \alpha$, that is $orness(W) + orness(\widetilde{W}) = 1$.*

Proposition 11 [29, p. 167]. *For OWA weighting vector $W = (w_1, w_2, \ldots, w_n)$, $W' = (w'_1, w'_2, \ldots, w'_n)$, if $\frac{w_i}{w_{i+1}} \geqslant \frac{w'_i}{w'_{i+1}}$, $i = 1, 2, \ldots, n-1$, then $orness(W) \geqslant orness(W')$ and for all $X = (x_1, x_2, \ldots, x_n)$, $F_W(X) \geqslant F_{W'}(X)$.*

If $W' = W_A = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, $orness(W') = \frac{1}{2}$ and $F_W(X) = A(X) = \frac{1}{n}\sum_{i=1}^n x_i$. From Proposition 11, it can be easily obtained that:

Corollary 3 [29, p. 169]. *For an OWA operator $W = (w_1, w_2, \ldots, w_n)$, if $w_1 \geqslant w_2 \geqslant \cdots \geqslant w_n$, then $orness(W) \geqslant \frac{1}{2}$, and $F_W(X) \geqslant \frac{1}{n}\sum_{i=1}^n x_i$; if $w_1 \leqslant w_2 \leqslant \cdots \leqslant w_n$, then $orness(W) \leqslant \frac{1}{2}$ and $F_W(X) \leqslant \frac{1}{n}\sum_{i=1}^n x_i$.*

The former means that the OWA aggregation is or-like and the latter means the OWA aggregation is and-like.

2.3. The orness measure for RIM quantifier with quantifier guided OWA aggregation

In [48], Yager proposed a method for obtaining the OWA weighting vectors via fuzzy linguistic quantifiers, especially the RIM quantifier, which can provide information aggregation procedures guided by verbally expressed concepts and a dimension independent description of the desired aggregation.

Definition 4. A fuzzy subset Q of the real line is called a regular increasing monotone (RIM) quantifier if $Q(0) = 0$, $Q(1) = 1$, and $Q(x) \geqslant Q(y)$ for $x > y$.

With a RIM quantifier Q, the OWA weighting vector can be obtained as:

$$w_j = Q\left(\frac{j}{n}\right) - Q\left(\frac{j-1}{n}\right)$$

The quantifier guided aggregation with OWA operator is

$$F_Q(X) = F_W(X) = \sum_{i=1}^{n}\left(Q\left(\frac{j}{n}\right) - Q\left(\frac{j-1}{n}\right)\right)y_j$$

Yager also extended the orness measure of OWA operator, and defined the *orness* of a RIM quantifier Q [48]:

$$orness(Q) = \lim_{n\to\infty}\sum_{j=1}^{n}\frac{n-j}{n-1}\left(Q\left(\frac{j}{n}\right) - Q\left(\frac{j-1}{n}\right)\right) = \lim_{n\to\infty}\frac{1}{n-1}\sum_{j=1}^{n-1}Q\left(\frac{j}{n}\right) = \int_0^1 Q(x)dx$$

The orness degree of a RIM quantifier is equal to the area under its membership function.

To analyze the properties of RIM quantifier, a generating function representation method was proposed, and the relationship between OWA operator and RIM quantifier was provided, where the RIM quantifier can be seen as the continuous case of OWA operator with dimension free in mathematical view [25].

Definition 5 [25]. For a RIM quantifier $Q(x),\ q(x)$ is called the generating function of $Q(x)$, if it satisfies

$$Q(x) = \int_0^x q(t)dt \tag{9}$$

where $q(t) \geqslant 0$ and $\int_0^1 q(t)dt = 1$.

Obviously, for any RIM quantifier $Q(x)$, its generating function $q(x)$ exists. If $Q(x)$ is differentiable, $q(x)$ can be set as its first order differential function $Q'(x)$ directly.

The orness of $Q(x)$ can be represented as

$$orness(Q) = \int_0^1 Q(x)dx = \int_0^1\int_0^x q(t)dtdx = \int_0^1\int_t^1 q(t)dxdt = \int_0^1 (1-t)q(t)dt \tag{10}$$

The similarities between (8) and (10) can be observed. Furthermore, it can also be verified that $orness(Q^*) = 1,\ orness(Q_*) = 0$ and $orness(Q_A) = \frac{1}{2}$.

The properties of RIM quantifier which correspond to that of OWA operator can be obtained [28,25,27].

Proposition 12. $0 \leqslant orness(Q) \leqslant 1$.

Proposition 13. *If $X = (x_1, x_2, \ldots, x_n)$ is evenly distributed on $[a, b]$, that is $x_i = a + \frac{n-i}{n-1}(b-a)$, then $\lim_{n\to+\infty}F_Q(X) = a + orness(Q)(b-a)$.*

Proposition 14 [25, p. 584]. *For RIM quantifiers $Q(x)$ and $G(x)$ with generating functions $q(x)$ and $p(x)$, respectively, if $p(x) = q(1-x)$, then $orness(Q) + orness(G) = 1$.*

Proposition 15 [28, Theorem 1]. *For RIM quantifiers $Q(x),\ G(x)$ with their generating functions $q(x),\ p(x)$, if for all $s,t \in [0,1],\ s \leqslant t,\ q(s)p(t) \geqslant q(t)p(s)$, then $orness(Q) \geqslant orness(G)$, and for any $X,\ F_Q(X) \geqslant F_G(X)$.*

Let $p(x) = 1$ in Proposition 15, then $G(x) = x$, so

$$orness(G) = \frac{1}{2}, \quad F_G(X) = F_{Q_A}(X) = A(X) = \frac{1}{n}\sum_{i=1}^{n}x_i$$

thus

Corollary 4. *For RIM quantifier $Q(x)$ with generating function $q(x)$, if $Q(x)$ is convex, which means that $q(x)$ is increasing, then $orness(Q) \leqslant \frac{1}{2}$, and for any $X,\ F_Q(X) \leqslant \frac{1}{n}\sum_{i=1}^{n}x_i$; if $Q(x)$ is concave, which means that $q(x)$ is decreasing, then $orness(Q) \geqslant \frac{1}{2}$, and for any $X,\ F_Q(X) \geqslant \frac{1}{n}\sum_{i=1}^{n}x_i$.*

If the generating function is differentiable and positive, the premise of Proposition 15 can be replaced with a more simplified form.

Proposition 16 [28, Theorem 3]. *For RIM quantifiers $Q(x),\ G(x)$ with generating functions $q(x),\ p(x)$, respectively, if for all $x \in [0,1],\ \frac{q'(x)}{q(x)} \leqslant \frac{p'(x)}{p(x)}$, then $orness(Q) \geqslant orness(G)$, and for any X, $F_Q(X) \geqslant F_G(X)$.*

Without the generating function, Proposition 16 can be expressed as

Corollary 5. *For RIM quantifiers $Q(x),\ G(x)$, if for all $x \in [0,1],\ \frac{Q''(x)}{Q'(x)} \leqslant \frac{G''}{G'(x)}$, then $orness(Q) \geqslant orness(G)$, and for any $X,\ F_Q(X) \geqslant F_G(X)$.*

2.4. The orness measure for weighted function average

In [52,47], Yager and Filev proposed the argument dependent method to generate OWA operator weights with power function $f(x) = x^r$. For input vector $X = (x_1, x_2, \ldots, x_n)$, the aggregation weights can be generated with $w_j = x_j^\alpha / \sum_{i=1}^n x_i^\alpha$ that (7) becomes $F(X) = \sum_{i=1}^n x_j^{\alpha+1} / \sum_{i=1}^n x_i^\alpha$. Yager called it the BADD (BAsic Defuzzification Distribution) aggregation methods. He also called it "neat" as the order information of the aggregated elements is not needed.

In [30], Liu and Lou extended it to the weighted function average operator, which was once called the Additive Neat OWA (ANOWA) operator. An orness measure for the weighted function average operator was proposed.

Definition 6 [30]. For input vector $X = (x_1, x_2, \ldots, x_n)$, $x_i \in [a, b]$, and $w(x_i) \neq 0$ for at least one i, the weighted function average (ANOWA operator) is determined by weighting function $w(x)$ with weighs $W_w = (w_1, w_2, \ldots, w_n)$ defined as

$$w_i = \frac{w(x_i)}{\sum_{j=1}^n w(x_j)}$$

The aggregation result is

$$F_w(X) = \sum_{i=1}^n w_i x_i = \sum_{i=1}^n \frac{x_i w(x_i)}{\sum_{j=1}^n w(x_j)} \tag{11}$$

Definition 7 [30]. For $X = (x_1, x_2, \ldots, x_n)$, $x_i \in [a, b]$, and the weighted function average operator defined by (11) with $w(x)$, the orness measure of $w(x)$ on $[a, b]$ can be defined as:

$$\Omega_w = \frac{\int_a^b (x-a) w(x) dx}{(b-a) \int_a^b w(x) dx} \tag{12}$$

As $w(x) \geqslant 0$, $0 < \frac{x-a}{b-a} < 1$ $(x \in (a, b))$, so we have

Proposition 17. $0 < \Omega_w < 1$.

Proposition 18 [30]. *If $X = (x_1, x_2, \ldots, x_n)$ is evenly distributed on $[a, b]$, that is $x_i = a + \frac{n-i}{n-1}(b-a)$, $i = 1, 2, \ldots, n$, and $W_w = (w_1, w_2, \ldots, w_n)$ are the weights generated with weighted function average operator of $w(x)$, then $\lim_{n\to+\infty} orness(W_w) = \Omega_w$.*

From Proposition 9, when $X = (x_1, x_2, \ldots, x_n)$ is evenly distributed on $[a, b]$, that is $x_i = a + \frac{n-i}{n-1}(b-a)$, $i = 1, 2, \ldots, n$, then

$$F_w(X) = \sum_{i=1}^n \left(a + \frac{n-i}{n-1}(b-a) \right) w_i = a + (b-a) orness(W_w)$$

Therefore, an alternative form of Proposition 18 can be expressed as

Proposition 19. *If $X = (x_1, x_2, \ldots, x_n)$ is evenly distributed on $[a, b]$, that is $x_i = a + \frac{n-i}{n-1}(b-a)$, $i = 1, 2, \ldots, n$, then $\lim_{n\to+\infty} F_w(X) = a + (b-a)\Omega_w$.*

Proposition 20 [30]. $\Omega_{w(x)} + \Omega_{w(a+b-x)} = 1$.

Proposition 21 [30]. *For weighted function average operators with weighting function $w_1(x)$ and $w_2(x)$, respectively, if for all $x, y \in [a, b]$, $x \geqslant y$, $w_1(x)w_2(y) - w_2(x)w_1(y) \geqslant 0$, then $\Omega_{w_1} \geqslant \Omega_{w_2}$ on $[a, b]$, and for all $X = (x_1, x_2, \ldots, x_n)$, $x_i \in [a, b]$, $F_{w_1}(X) \geqslant F_{w_2}(X)$.*

Let $w_2(x) = 1$ in Proposition 21, then $\Omega_{w_2} = \frac{1}{2}$, $F_{w_2}(X) = A(X) = \frac{1}{n}\sum_{i=1}^n x_i$, thus

Corollary 6 [30]. *For a weighted average function $w(x)$, if $w(x)$ is monotone increasing then $\Omega_w \geqslant \frac{1}{2}$, and for any n-elements $X = (x_1, x_2, \ldots, x_n)$, $F_w(X) \geqslant \frac{1}{n}\sum_{i=1}^n x_i$, which means it is or-like; if $w(x)$ is monotone decreasing, then $\Omega_w \leqslant \frac{1}{2}$, and $F_w(X) \leqslant \frac{1}{n}\sum_{i=1}^n x_i$, which means it is and-like.*

If the weighting function is differential and positive, the condition of Proposition 21 can be replaced with a more simplified form.

Proposition 22 [30]. *For weighting function $w_1(x)$, $w_1(x)$, if for all $x \in [a, b]$, $\frac{w_1'(x)}{w_1(x)} \geqslant \frac{w_2'(x)}{w_2(x)}$, then $\Omega_{w_1} \geqslant \Omega_{w_2}$, and for all $X = (x_1, x_2, \ldots, x_n)$, $x_i \in [a, b]$, $F_{w_1}(X) \geqslant F_{w_2}(X)$.*

From the summary of these orness definitions and the relevant properties of these aggregation operators, it can be easily observed that these orness concepts hold some common characteristics despite their different forms and backgrounds. For an averaging aggregation function $\mathscr{F}$, where $\mathscr{F}$ can be one of the above four categories, its orness measure of $orness(\mathscr{F})$ should satisfy the following four properties:

1. **Standardization:** $0 \leqslant orness(\mathscr{F}) \leqslant 1$, which is shown in Propositions 2, 8, 12 and 17. In Propositions 2 and 17, " $<$" also becomes " $\leqslant$" with the limit conditions taking into consideration. We also have that if $\mathscr{F}$ becomes or approaches the maximum operator, then $orness(\mathscr{F}) = 1$, if $\mathscr{F}$ becomes or approaches the minimum operator, then $orness(\mathscr{F}) = 0$, and if $\mathscr{F}$ becomes the arithmetic average operator, then $orness(\mathscr{F}) = \frac{1}{2}$. This is shown in Corollaries 1, 3, 4 and 6.
2. **Uniformity:** If the input vector $X = (x_1, x_2, \ldots, x_n)$ distributes evenly on $[a, b]$, then $\mathscr{F}(X) = a + orness(\mathscr{F})(b - a)$ or $\lim_{n\to+\infty}\mathscr{F}(X) = a + orness(\mathscr{F})(b - a)$, which means that the aggregation value reflects the orness extent exactly if the aggregated elements distributed evenly. This is shown in Propositions 3, 9, 13 and 19.
3. **Duality:** There are always a pair of dual operators $\mathscr{F}$ and $\mathscr{F}'$, which satisfy $orness(\mathscr{F}) + orness(\mathscr{F}') = 1$. This is shown in Propositions 4, 10, 14 and 20.
4. **Consistency:** There is a condition that for any input vector $X = (x_1, x_2, \ldots, x_n),\ orness(\mathscr{F})$ and $\mathscr{F}(X)$ can always change in the same manner. This is shown in Propositions 6, 11, 16 and 22.

These properties make it possible to construct a parameterized aggregation operator family, which makes $\mathscr{F}(X)$ change monotonically with $orness(\mathscr{F})$, with $orness(\mathscr{F}) = 0, \frac{1}{2}, 1$ corresponding to $\max_{1\leqslant i\leqslant n}\{x_i\},\ \frac{1}{n}\sum_{i=1}^{n} x_i,\ \min_{1\leqslant i\leqslant n}\{x_i\}$, respectively. For more details of these parameterized aggregation operators, we recommend [30,26,25,29].

3. The orness measure for quasi-OWA operator and its properties

Next, instead of discussing the orness of aggregation operators in their simple way, we will study the orness definition and their properties with two kinds of quasi-arithmetic mean-based compound aggregation operators: the quasi-OWA operator and the Bajraktarević mean. The former is the combination of quasi-arithmetic mean and RIM quantifier (OWA operator), while the latter is the combination of quasi-arithmetic mean and the weighted function average. The orness properties such as standardization, uniformity, duality and consistency of the aggregation operators are still held but demonstrated in a compound way.

With the study of these two kinds of compound aggregation operators, we can better understand the characteristics of aggregation operators and the orness definitions about them. Furthermore, as the complicated forms of these two kinds of compound aggregation operators, the researches on their properties and characteristics are relatively sparse. The study of the orness concepts and the corresponding properties of these compound aggregation operators should be helpful for the further research of them both in theory and applications.

3.1. An orness measure for quasi-OWA operator

Instead of the quasi-OWA operator defined on $[0, 1]^n$ as Grabisch [18, p. 288] and Calvo et al. [4, p. 57], we intend to adopt the quasi-OWA operator concept of Fodor et al. [15], where the input vector $(x_1, x_2, \ldots, x_n) \in R^n$. For convenience and also as shown in Definition 1 and Remark 2 in the following, the definition domain can influence the properties of the aggregation operator. Here, a closed domain $[a, b]^n$ is considered, where $a,\ b$ can be determined from input vector data set, such as $a = \min_{1\leqslant i\leqslant n}\{x_i\},\ b = \max_{1\leqslant i\leqslant n}\{x_i\}$. It is also natural that $f(x)$ should be always meaningful on $[a, b]$.

Definition 8. Let f be a continuous strictly monotone function, $W = (w_1, w_2, \ldots, w_n)$ is an OWA operator weighting vector satisfying $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. For input vector $X = (x_1, x_2, \ldots, x_n) \in [a, b]^n$, a quasi-OWA operator can be defined as the aggregation operator $M_{f,W} : [a, b]^n \to [a, b]$ given by

$$M_{f,W}(x_1, x_2, \ldots, x_n) = f^{-1}\left(\sum_{j=1}^{n} w_j f(y_j)\right) \tag{13}$$

where f^{-1} is the inverse function of $f,\ y_j$ is the jth largest of the x_i. $f(x)$ is the quasi-arithmetic mean generator and $W = (w_1, w_2, \ldots, w_n)$ is the OWA operator weight.

For convenience, we will always assume $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n$ that $y_j = x_j$, which means that the orders of the aggregated elements do not need to be considered.

The quasi-OWA operator includes the following three aggregation operators as special cases:

1. When $f(x) = x^r,\ r \in (-\infty, +\infty)$, then
$$M_{f,W}(x_1, x_2, \ldots, x_n) = \left(\sum_{j=1}^{n} w_j x_j^r\right)^{1/r}$$
This becomes the GOWA operator which is an extension of the OWA operator [49]. The ordinary OWA operator becomes a special case for $r = 1$.
2. When $f(x) = \ln(x)$, or $r \to 0$ in the GOWA operator,
$$M_{f,W}(x_1, x_2, \ldots, x_n) = \prod_{j=1}^{n} x_j^{w_j}$$
This becomes the ordered weighted geometric operator which is used in group decision making [45,19].

3. When the OWA weighting vector $W = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$,

$$M_{f,W}(x_1, x_2, \ldots, x_n) = f^{-1}\left(\frac{1}{n}\sum_{j=1}^{n} f(x_j)\right) = f^{-1}\left(\frac{1}{n}\sum_{j=1}^{n} f(x_j)\right)$$

This becomes the quasi-arithmetic mean operator which is also commonly used in the literature [44,17,31,42,41].

If w_i is generated with the quantifier guided aggregation method $w_j = Q\left(\frac{j}{n}\right) - Q\left(\frac{j-1}{n}\right)$, the quasi-OWA operator (13) can be expressed as

$$M_{f,Q}(x_1, x_2, \ldots, x_n) = f^{-1}\left(\sum_{j=1}^{n}\left(Q\left(\frac{j}{n}\right) - Q\left(\frac{j-1}{n}\right)\right) f(x_j)\right) \tag{14}$$

The quasi-arithmetic mean (13) with $W = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ corresponds to $Q(x) = x$. One merit of this method is that the aggregation property depends on the characteristic of $Q(x)$, and it has no relation with the dimension number of W.

It is obvious that, among all the three cases of OWA operator, RIM quantifier and quasi-arithmetic mean, the aggregation results are determined by the characteristics of the operator weights or the generating functions. This characteristic is expressed with the orness concept, which reflects the and-like or or-like property of the aggregation operator. Based on the orness definitions of quasi-arithmetic mean, OWA operator and RIM quantifier, respectively, an orness concept for quasi-OWA operator which can be seen as the orness combination of the OWA operator, the RIM quantifier and the quasi-arithmetic mean together is proposed in the following. This orness concept and its properties can be seen as the extension of the known results in Section 2.

Definition 9. For quasi-OWA operator (13) that is determined by the quasi-arithmetic mean generator $f(x)$, and the OWA weights W, respectively, the orness measure of it can be defined as:

$$\Omega_{f,W} = \frac{f^{-1}\left(\sum_{j=1}^{n} f\left(a + \frac{n-j}{n-1}(b-a)\right) w_j\right) - a}{b-a} \tag{15}$$

Otherwise, if the quasi-OWA operator is determined by the quasi-arithmetic mean generator $f(x)$ and the RIM quantifier $Q(x)$, respectively, as (14), the orness measure can be defined as:

$$\Omega_{f,Q} = \frac{f^{-1}\left(\int_0^1 f(ax + (1-x)b)dQ(x)\right) - a}{b-a} \tag{16}$$

With the generating function of the RIM quantifier (9), (16) can be expressed as

$$\Omega_{f,q} = \frac{f^{-1}\left(\int_0^1 f(ax + (1-x)b)q(x)dx\right) - a}{b-a} \tag{17}$$

(15), (16) (or (17)) can be seen as the discrete and continuous cases of the same formula with fixed dimension and the dimension number approaching infinity, respectively.

When $f(x) = x$,

$$\Omega_{f,W} = \frac{\sum_{j=1}^{n}\left(a + \frac{n-j}{n-1}(b-a)\right) w_j - a}{b-a} = \sum_{j=1}^{n} \frac{n-j}{n-1} w_j = orness(W)$$

$$\begin{aligned}\Omega_{f,Q} &= \frac{f^{-1}\left(\int_0^1 f(ax + (1-x)b)dQ(x)\right) - a}{b-a} = \frac{\left((ax + (1-x)b)Q(x)|_0^1 - \int_0^1 (a-b)Q(x)dx\right) - a}{b-a} \\ &= \frac{\left(a + (b-a)\int_0^1 Q(x)dx\right) - a}{b-a} = \int_0^1 Q(x)dx = orness(Q)\end{aligned}$$

When $Q(x) = x$,

$$\Omega_{f,Q} = \frac{f^{-1}\left(\int_0^1 f(ax + (1-x)b)dx\right) - a}{b-a} = \frac{f^{-1}\left(\frac{\int_a^b f(y)dy}{b-a}\right) - a}{b-a} = \Omega_f$$

All these indicate that the orness concepts of the OWA operator, the RIM quantifier and the quasi-arithmetic mean are the special cases of quasi-OWA operator orness definition, respectively.

Remark 1. The orness measures of the quasi-OWA operator and the quasi-arithmetic mean are determined both by $f(x)$ and the distribution interval of $X = (x_1, x_2, \ldots, x_n)$, $[a, b]$ (we can set $a = \min_{1\leqslant i\leqslant n}\{x_i\}$, $b = \max_{1\leqslant i\leqslant n}\{x_i\}$). It is natural that a prerequisite condition is that $f(x)$ should be meaningful on $[a,b]$. As an example, for the special case of harmonic mean in the quasi-arithmetic mean with $f(x)=\frac{1}{x}$, the orness level does not exist on $[0,1]$ as $f(x)=\frac{1}{x}$ is meaningless at $x=0$.

If $f(x)=x^r$ and $a=0,\ b=1$ in (15), then

$$\Omega_{f,W}=\left(\sum_{j=1}^{n}\left(\frac{n-j}{n-1}\right)^r w_j\right)^{\frac{1}{r}} \tag{18}$$

This becomes the orness (attitude character) of Yager's GOWA operator [49]. Especially when $r=1$, it becomes the orness concept of the ordinary OWA operator.

Remark 2. In the case of GOWA operator, the main difference between the orness measure (15) and that of Yager (18) is that the former depends on the interval where the aggregated elements are distributed in. This context dependent orness definition does not hinder us from getting the same or-like (orness value is greater than $\frac{1}{2}$) or and-like (orness value is smaller than $\frac{1}{2}$) conclusion for specific r. But their orness values may be different. Consider an example with $w_1=w_2=0.5$ and $f(x)=x^2$, the GOWA operator orness value with Yager's method is 0.707, which is the same value of $\Omega_{f,W}$ for $a=0,\ b=1$. But for $a=100,\ b=101,\ \Omega_{f,W}=0.501$. Both of them are and-like, which means the aggregation values are more than the average of the aggregated elements, but the orness values show that $f(x)=x^2$ is more and-like on $[0,1]$ than it does on $[100,101]$. Let us illustrate this with examples. For $X=(x_1,x_2)=(0,1),\ F_{f,W}(X)=0.707$, but for $X'=(x_1',x_2')=(100,101)$, $F_{f,W}(X')=100.501,\ F_{f,W}(X')=100.501$ is closer to the average value 100.5 than $F_{f,W}(X)=0.707$ does to the average value 0.5.

The orness of quasi-OWA operator can be seen as the orness measures combination of the OWA operator (or the RIM quantifier) and the quasi-arithmetic mean together. The main difference between the OWA operator (or the RIM quantifier) and the quasi-arithmetic mean is that the former is determined by the order relation of the aggregated elements but the latter associates with the direct computation of the aggregated elements. This makes that the orness of OWA operator (or RIM quantifier) has no relation with the specific values of aggregated elements, but for the orness of quasi-arithmetic mean, the interval in which the aggregated elements distributed must be considered.

As the orness of the quasi-OWA operator is determined by these two kinds of operators: the quasi-arithmetic mean and the OWA operator (or the RIM quantifier). As shown in Section 3.2, under some function formulation cases, they play different roles in the orness determination of a quasi-OWA operator, which reflect the overall aggregation effect on the aggregated elements. The OWA operator (RIM quantifier) orness can be seen as a basis of the quasi-OWA orness level value, and the orness of quasi-arithmetic mean can be seen as a fine tuning method of the quasi-OWA orness measure.

The following theorems verify the rationality of the orness measure for quasi-OWA operator. As shown in Section 2, similar results can also be found in the OWA operator, the RIM quantifier and the quasi-arithmetic mean, respectively [47,29,26].

Theorem 1. $0\leqslant \Omega_{f,W}\leqslant 1,\ 0\leqslant \Omega_{f,Q}\leqslant 1.$

Proof. As $f(x)$ is strictly monotone, it can be assumed that $f(x)$ is strictly increasing, with $\sum_{j=1}^n w_j=1$ and $\int_0^1 Q(x)dx=1$, then $f(a)\leqslant \sum_{j=1}^n f\left(a+\frac{n-j}{n-1}(b-a)\right)w_j\leqslant f(b),\ f(a)\leqslant \int_0^1 f(at+(1-t)b)dQ(t)\leqslant f(b)$, from (15) and (16), $0\leqslant \Omega_{f,W}\leqslant 1,\ 0\leqslant \Omega_{f,Q}\leqslant 1.$ □

Theorem 2. *If $X=(x_1,x_2,\ldots,x_n)$ is evenly distributed in $[a,b]$, that is $x_j=a+\frac{n-j}{n-1}(b-a),\ j=1,2,\ldots,n$, then $M_{f,W}(X)=a+\Omega_{f,W}(b-a)$ and $\lim_{n\to+\infty}M_{f,Q}(X)=a+\Omega_{f,Q}(b-a)$.*

Proof. From (13) and (15),

$$M_{f,W}(X)=f^{-1}\left(\sum_{j=1}^n f\left(a+\frac{n-j}{n-1}(b-a)\right)w_j\right)=a+\Omega_{f,W}(b-a)$$

As $f^{-1}(x)$ is continuous, from (14),

$$\begin{aligned}\lim_{n\to\infty}M_{f,Q}(x_1,x_2,\ldots,x_n)&=\lim_{n\to\infty}f^{-1}\left(\sum_{j=1}^n\left(Q\left(\frac{j}{n}\right)-Q\left(\frac{j-1}{n}\right)\right)f\left(a+\frac{n-j}{n-1}(b-a)\right)\right)\\&=f^{-1}\left(\lim_{n\to+\infty}\sum_{j=1}^n\left(Q\left(\frac{j}{n}\right)-Q\left(\frac{j-1}{n}\right)\right)f\left(a+\frac{n-j}{n-1}(b-a)\right)\right)\end{aligned}$$

Consider an even division of $[0,1]$ with n intervals $[d_{j-1},d_j]\ (j=1,2,\ldots,n)$, where $d_j=\frac{j}{n}\ (j=0,1,2,\ldots,n)$. Select $d_j'=\frac{j-1}{n-1}\ (j=1,2,\ldots,n)$, then $d_j'\in[d_{j-1},d_j]$, therefore

$$\begin{aligned}\lim_{n\to+\infty}\sum_{j=1}^n\left(Q\left(\frac{j}{n}\right)-Q\left(\frac{j-1}{n}\right)\right)f\left(a+\frac{n-j}{n-1}(b-a)\right)&=\lim_{n\to+\infty}\sum_{j=1}^n\left(Q\left(\frac{j}{n}\right)-Q\left(\frac{j-1}{n}\right)\right)f\left(a+\left(1-\frac{j-1}{n-1}\right)(b-a)\right)\\&=\int_0^1 f(a+(1-x)(b-a))dQ(x)=\int_0^1 f(ax+(1-x)b)dQ(x)\end{aligned}$$

So, with (16),

$$\lim_{n\to+\infty} M_{f,Q}(x_1,x_2,\ldots,x_n) = f^{-1}\left(\int_0^1 f(ax+(1-x)b)dQ(x)\right) = a + \Omega_{f,Q}(b-a) \qquad \square$$

Theorem 3. *For all* $X=(x_1,x_2,\ldots,x_n),\ M_{kf+c,W}(X) = M_{f,W}(X),\ \Omega_{kf+c,W} = \Omega_{f,W},\ \Omega_{kf+c,Q} = \Omega_{f,Q}\ (k \neq 0)$.

Proof. Let $g(x) = kf(x)+c$, and $g^{-1}(x) = y$, then $x = g(y) = kf(y)+c,\ y = f^{-1}\left(\frac{x-c}{k}\right)$, that is $g^{-1}(x) = f^{-1}\left(\frac{x-c}{k}\right)$.

$$M_{g,W}(X) = g^{-1}\left(w_i\sum_{i=1}^n g(x_i)\right) = g^{-1}\left(w_i\sum_{i=1}^n kf(x_i)+c\right) = f^{-1}\left(w_i\sum_{i=1}^n f(x_i)\right) = M_{f,W}(X)$$

As a special case, let $x_j = a + \frac{n-j}{n-1}(b-a)$, from Theorem 2, it can be obtained that $\Omega_{g,W} = \Omega_{f,W}$:

$$\begin{aligned}\Omega_{g,Q} &= \frac{g^{-1}\left(\int_0^1 g(ax+(1-x)b)dQ(x)\right)-a}{b-a} = \frac{f^{-1}\left(\int_0^1 (kf(ax+(1-x)b)+c)dQ(x)/k-c\right)-a}{b-a}\\ &= \frac{f^{-1}(f(ax+(1-x)b)dQ(x))-a}{b-a} = \Omega_{f,Q} \qquad \square\end{aligned}$$

As (16) and (17) are the different expressions of the same formula, for simplification, we will mainly use (17) instead of (16), due to (16) involves more complicated Riemann–Stieltjes integral computation.

The orness measure also fulfills the duality property.

Theorem 4. *For a quasi-OWA operator determined by the quasi-arithmetic mean generator $f(x)$ and the OWA weighting vector $W=(w_1,w_2,\ldots,w_n)$ (or a RIM quantifier with generating function $q(x)$), $\Omega_{f(a+b-x),\widetilde{W}} = 1-\Omega_{f(x),W},\ \Omega_{f(a+b-x),q(1-x)} = 1-\Omega_{f(x),q(x)}$, where $\widetilde{W} = (w_n,w_{n-1},\ldots,w_1)$ is the reverse order of W.*

Proof. Let $f(a+b-x) = g(x),\ g^{-1}(x) = y$, then $x = g(y) = f(a+b-y),\ y = a+b-f^{-1}(x)$, that is $g^{-1}(x) = a+b-f^{-1}(x)$:

$$\begin{aligned}\Omega_{f(a+b-x),\widetilde{W}} &= \frac{g^{-1}\left(\sum_{i=1}^n g\left(a+\frac{n-i}{n-1}(b-a)\right)w_{n-i+1}\right)-a}{b-a} = \frac{b-f^{-1}\left(\sum_{i=1}^n f\left(b-\frac{n-i}{n-1}(b-a)\right)w_{n-i+1}\right)}{b-a}\\ &= \frac{b-f^{-1}\left(\sum_{j=1}^n f\left(a+\frac{n-j}{n-1}(b-a)\right)w_j\right)}{b-a} = 1-\Omega_{f(x),W}\end{aligned}$$

$$\begin{aligned}\Omega_{f(a+b-x),q(1-x)} &= \frac{g^{-1}\left(\int_0^1 g(ax+(1-x)b)q(1-x)dx\right)-a}{b-a} = \frac{b-f^{-1}\left(\int_0^1 f(bx+(1-x)a)q(1-x)dx\right)}{b-a}\\ &= \frac{b-f^{-1}\left(\int_0^1 f(ay+(1-y)b)q(y)dy\right)}{b-a} = 1-\Omega_{f(x),q(x)} \qquad \square\end{aligned}$$

From (17), the orness measure of quasi-OWA operator $\Omega_{f,q} = \alpha$ is the solution of equation

$$f(a+(b-a)\alpha) = \int_0^1 f(ax+(1-x)b)q(x)dx$$

With the generating function method of quasi-arithmetic mean $f(x) = \int_a^x \varphi(t)dt$,

$$\int_a^{a+(b-a)\alpha} \varphi(t)dt = \int_0^1 \int_a^{ax+(1-x)b} \varphi(t)q(x)dtdx$$

that is

$$\begin{aligned}\int_0^\alpha \varphi(a+(b-a)t)dt &= \int_0^1\int_0^{1-t} \varphi(a+(b-a)x)q(t)dxdt = \int_0^1\int_0^{1-x}\varphi(a+(b-a)x)q(t)dtdx\\ &= \int_0^1\int_0^{1-t}\varphi(a+(b-a)t)q(x)dxdt \end{aligned} \tag{19}$$

Similarly, from (14), it can be obtained that the quasi-OWA operator
$M_{f,Q}(x_1,x_2,\ldots,x_n) = y$ is the solution of equation

$$f(y) = \sum_{j=1}^n \left(Q\left(\frac{j}{n}\right) - Q\left(\frac{j-1}{n}\right)\right)f(x_j)$$

that is

$$\int_a^y \varphi(t)dt = \sum_{j=1}^n \left(Q\left(\frac{j}{n}\right) - Q\left(\frac{j-1}{n}\right)\right)\int_a^{x_j}\varphi(t)dt = \sum_{j=1}^n \left(\int_0^{\frac{j}{n}} q(x)dx - \int_0^{\frac{j-1}{n}} q(x)dx\right)\int_a^{x_j}\varphi(t)dt$$
$$= \sum_{j=1}^n \int_{\frac{j-1}{n}}^{\frac{j}{n}} q(x)dx \int_a^{x_j}\varphi(t)dt \tag{20}$$

or

$$\int_a^y \varphi(t)dt = \sum_{i=1}^n w_j \int_a^{x_j}\varphi(t)dt$$

As a special case, $\Omega_{f,W} = \alpha$ is the solution of:

$$\int_a^\alpha \varphi(x)dx = \sum_{j=1}^n w_j \int_a^{a+\frac{n-j}{n-1}(b-a)} \varphi(t)dt$$

Lemma 1. *For nonzero OWA weighting vector $W = (w_1, w_2, \ldots, w_n)$, $W' = (w'_1, w'_2, \ldots, w'_n)$, if for all $i \leqslant j$ $(i,j = 1,2,\ldots,n)$, $\frac{w_i}{w_j} \geqslant \frac{w'_i}{w'_j}$, then for all $k = 1,2,\ldots,n$, $\sum_{k=1}^i w_i \geqslant \sum_{k=1}^i w'_i$. Correspondingly, for nonzero RIM quantifiers $Q(x)$, $P(x)$ with generating functions $q(x)$, $p(x)$, respectively, if for all $u, v \in [0,1]$, $u \leqslant v$, $\frac{q(u)}{q(v)} \geqslant \frac{p(u)}{p(v)}$, then for all $t \in [0,1]$, $\int_0^t q(x)dx \geqslant \int_0^t p(x)dx$, that is for all $x \in [0,1]$, $Q(x) \geqslant P(x)$.*

Proof. As $\sum_{i=1}^n w_i = \sum_{i=1}^n w'_i = 1$,

$$\sum_{i=1}^k w_i - \sum_{i=1}^k w'_i = \sum_{i=1}^k w_i \sum_{j=1}^n w'_j - \sum_{i=1}^k w'_i \sum_{j=1}^n w_j = \sum_{i=1}^k w_i \sum_{j=1}^k w'_j + \sum_{i=1}^k w_i \sum_{j=k+1}^n w'_j - \sum_{i=1}^k w'_i \sum_{j=1}^k w_j - \sum_{i=1}^k w'_i \sum_{j=k+1}^n w_j$$
$$= \sum_{i=1}^k w_i \sum_{j=k+1}^n w'_j - \sum_{i=1}^k w'_i \sum_{j=k+1}^n w_j = \sum_{i=1}^k \sum_{j=k+1}^n \left(w_i w'_j - w'_i w_j\right)$$

If for all $i \leqslant j$ $(i,j = 1,2,\ldots,n)$, $\frac{w_i}{w_j} \geqslant \frac{w'_i}{w'_j}$, then $w_i w'_j - w'_i w_j \geqslant 0$, that is $\sum_{i=1}^k w_i \geqslant \sum_{i=1}^k w'_i$.

Similarly, as $\int_0^1 q(x)dx = \int_0^1 p(x)dx = 1$,

$$\int_0^t q(x)dx - \int_0^t p(x)dx = \int_0^t q(x)dx \int_0^1 p(x)dx - \int_0^t p(x)dx \int_0^1 q(x)dx = \int_0^t q(x)dx \int_t^1 p(x)dx - \int_0^t p(x)dx \int_t^1 q(x)dx$$
$$= \iint_{\substack{0 \leqslant x \leqslant t \\ t \leqslant y \leqslant 1}} q(x)p(y)dxdy - \iint_{\substack{0 \leqslant x \leqslant t \\ t \leqslant y \leqslant 1}} p(x)q(y)dx = \iint_{\substack{0 \leqslant x \leqslant t \\ t \leqslant y \leqslant 1}} (q(x)p(y) - p(x)q(y))dxdy$$

If for all $u, v \in [0,1]$, $u \leqslant v$, $\frac{q(u)}{q(v)} \geqslant \frac{p(u)}{p(v)}$, then $q(u)p(v) - p(u)q(v) \geqslant 0$, so for $0 \leqslant x \leqslant t$, $t \leqslant y \leqslant 1$, $q(x)p(y) - p(x)q(y) \geqslant 0$ stands, thus for all $t \in [0,1]$, $\int_0^t q(x)dx \geqslant \int_0^t p(x)dx$. $\square$

Theorem 5. *For quasi-OWA operators determined by the quasi-arithmetic mean generator and the RIM quantifier pairs $f(x)$, $Q(x)$ and $g(x)$, $P(x)$ with generating functions $\varphi(x)$, $q(x)$ and $\psi(x)$, $p(x)$, respectively, if for all $s, t \in [a,b]$, $s \geqslant t$, $\frac{\varphi(s)}{\varphi(t)} \geqslant \frac{\psi(s)}{\psi(t)}$, and for all $u, v \in [0,1]$, $u \leqslant v$, $\frac{q(u)}{q(v)} \geqslant \frac{p(u)}{p(v)}$, then $\Omega_{f,Q} \geqslant \Omega_{g,P}$, and for any $X = (x_1, x_2, \ldots, x_n)$ with $\min_{1 \leqslant i \leqslant n}\{x_i\} = a$, $\max_{1 \leqslant i \leqslant n}\{x_i\} = b$, $M_{f,Q}(X) \geqslant M_{g,P}(X)$. Replace the RIM quantifiers $Q(x)$ and $P(x)$ with OWA operator weighting vectors $W = (w_1, w_2, \ldots, w_n)$ and $W' = (w'_1, w'_2, \ldots, w'_n)$, if for all $i \leqslant j$ $(i,j = 1,2,\ldots,n)$, $\frac{w_i}{w_j} \geqslant \frac{w'_i}{w'_j}$, then $\Omega_{f,W} \geqslant \Omega_{g,W'}$ and $M_{f,W}(X) \geqslant M_{g,W'}(X)$.*

Proof. Let $\Omega_{f,Q} = \alpha$, $\Omega_{g,P} = \beta$, from (19),

$$\int_0^\alpha \varphi(a + (b-a)t)dt = \int_0^1 \int_0^{1-t} \varphi(a + (b-a)t)q(x)dxdt$$
$$\int_0^\beta \psi(a + (b-a)t)dt = \int_0^1 \int_0^{1-t} \psi(a + (b-a)t)p(x)dxdt$$

As $\varphi(t), \psi(t) \geqslant 0$, to prove $\alpha \geqslant \beta$, we only need to prove that

$$\int_0^\alpha \varphi(a + (b-a)t)dt \geqslant \int_0^\beta \varphi(a + (b-a)t)dt$$

Consider that $\int_0^1 q(x)dx = \int_0^1 p(x)dx = 1$,

$$\begin{aligned}
&\int_0^{\alpha} \varphi(a+(b-a)t)dt - \int_0^{\beta} \varphi(a+(b-a)t)dt \\
&= \int_0^1 \int_0^{1-t} \varphi(a+(b-a)t)q(x)dxdt - \int_0^{\beta} \varphi(a+(b-a)t)dt \\
&= \int_0^{\beta} \left(\int_0^{1-t} q(x)dx - 1\right)\varphi(a+(b-a)t)dt + \int_{\beta}^1 \int_0^{1-t} \varphi(a+(b-a)t)q(x)dxdt \\
&= \int_0^{\beta} \left(\int_0^{1-t} q(x)dx - \int_0^1 q(x)dx\right)\varphi(a+(b-a)t)dt + \int_{\beta}^1 \int_0^{1-t} \varphi(a+(b-a)t)q(x)dxdt \\
&= \int_{\beta}^1 \int_0^{1-t} \varphi(a+(b-a)t)q(x)dxdt - \int_0^{\beta} \int_{1-t}^1 \varphi(a+(b-a)t)q(x)dxdt
\end{aligned}$$

For all $s,t \in [a,b]$, $s \geqslant t$, $\frac{\varphi(s)}{\varphi(t)} \geqslant \frac{\psi(s)}{\psi(t)}$, it can be obtained that $\frac{\varphi(s)}{\psi(s)} \geqslant \frac{\varphi(t)}{\psi(t)}$. Let $\mu_1(x) = \frac{\varphi(x)}{\psi(x)}$, then $\mu_1(x)$ is increasing, and $\mu_1(x) \geqslant 0$. $\varphi(x) = \mu_1(x)\psi(x)$, thus

$$\begin{aligned}
&\int_0^{\alpha} \varphi(a+(b-a)t)dt - \int_0^{\beta} \varphi(a+(b-a)t)dt \\
&= \int_{\beta}^1 \int_0^{1-t} \mu_1(a+(b-a)t)\psi(a+(b-a)t)q(x)dxdt - \int_0^{\beta} \int_{1-t}^1 \mu_1(a+(b-a)t)\psi(a+(b-a)t)q(x)dxdt \\
&\geqslant \int_{\beta}^1 \int_0^{1-t} \mu_1(a+(b-a)\beta)\psi(a+(b-a)t)q(x)dxdt - \int_0^{\beta} \int_{1-t}^1 \mu_1(a+(b-a)\beta)\psi(a+(b-a)t)q(x)dxdt \\
&= \mu_1(a+(b-a)\beta)\left(\int_{\beta}^1 \int_0^{1-t} \psi(a+(b-a)t)q(x)dxdt - \int_0^{\beta} \int_{1-t}^1 \psi(a+(b-a)t)q(x)dxdt\right) \\
&= \mu_1(a+(b-a)\beta)\left(\int_{\beta}^1 \int_0^{1-t} \psi(a+(b-a)t)q(x)dxdt - \int_0^{\beta} \left(1 - \int_0^{1-t} q(x)dx\right)\psi(a+(b-a)t)dt\right) \\
&= \mu_1(a+(b-a)\beta)\left(\int_0^1 \int_0^{1-t} \psi(a+(b-a)t)q(x)dxdt - \int_0^{\beta} \psi(a+(b-a)t)dxdt\right) \\
&= \mu_1(a+(b-a)\beta)\left(\int_0^1 \int_0^{1-t} \psi(a+(b-a)t)q(x)dxdt - \int_0^1 \int_0^{1-t} \psi(a+(b-a)t)p(x)dxdt\right) \\
&= \mu_1(a+(b-a)\beta)\int_0^1 \psi(a+(b-a)t)\left(\int_0^{1-t}(q(x)-p(x))dx\right)dt
\end{aligned}$$

As for all $u,v \in [0,1]$, $u \leqslant v$, $\frac{q(u)}{q(v)} \leqslant \frac{p(u)}{p(v)}$, from Lemma 1, $\int_0^{1-t}(q(x)-p(x))dx \geqslant 0$, so $\int_0^{\alpha} \varphi(a+(b-a)t)dt - \int_0^{\beta} \varphi(a+(b-a)t)dt \geqslant 0$, thus $\alpha \geqslant \beta$, that is $\Omega_{f,Q} \geqslant \Omega_{g,P}$.

Let $M_{f,Q}(X) = \gamma$, $M_{g,P}(X) = \delta$, from (20),

$$\int_a^{\gamma} \varphi(t)dt = \sum_{j=1}^n \int_{\frac{j-1}{n}}^{\frac{j}{n}} q(x)dx \int_a^{x_j} \varphi(t)dt, \quad \int_a^{\delta} \psi(t)dt = \sum_{j=1}^n \int_{\frac{j-1}{n}}^{\frac{j}{n}} q(x)dx \int_a^{x_j} \psi(t)dt \tag{21}$$

To prove $\gamma \geqslant \delta$, we only need to prove that $\int_a^{\gamma} \varphi(t)dt \geqslant \int_a^{\delta} \varphi(t)dt$.

With $\int_0^1 q(x)dx = \int_0^1 p(x)dx = 1$, $\mu_1(x)$ is increasing, $\mu_1(x) \geqslant 0$, $\varphi(x) = \mu_1(x)\psi(x)$, and (21),

$$\begin{aligned}
\int_a^{\gamma} \varphi(t)dt - \int_a^{\delta} \varphi(t)dt &= \sum_{j=1}^n \int_{\frac{j-1}{n}}^{\frac{j}{n}} q(x)dx \int_a^{x_j} \varphi(t)dt - \int_0^1 q(x)dx \int_a^{\delta} \varphi(t)dt \\
&= \sum_{j=1}^n \int_{\frac{j-1}{n}}^{\frac{j}{n}} q(x)dx\left(\int_a^{x_j} \varphi(t)dt - \int_a^{\delta} \varphi(t)dt\right) \\
&= \sum_{j=1}^n \int_{\frac{j-1}{n}}^{\frac{j}{n}} q(x)dx\left(\int_a^{x_j} \mu_1(t)\psi(t)dt - \int_a^{\delta} \mu_1(t)\psi(t)dt\right) \\
&= \sum_{\substack{x_j>\delta \\ 1\leqslant j\leqslant n}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} q(x)dx \int_{\delta}^{x_j} \mu_1(t)\psi(t)dt - \sum_{\substack{x_j<\delta \\ 1\leqslant j\leqslant n}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} q(x)dx \int_{x_j}^{\delta} \mu_1(t)\psi(t)dt \\
&\geqslant \sum_{\substack{x_j>\delta \\ 1\leqslant j\leqslant n}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} q(x)dx \int_{\delta}^{x_j} \mu_1(\delta)\psi(t)dt - \sum_{\substack{x_j<\delta \\ 1\leqslant j\leqslant n}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} q(x)dx \int_{x_j}^{\delta} \mu_1(\delta)\psi(t)dt
\end{aligned}$$

$$
\begin{aligned}
&= \mu_1(\delta)\sum_{j=1}^{n}\int_{\frac{j-1}{n}}^{\frac{j}{n}} q(x)dx\left(\int^{x_j}\psi(t)dt - \int_a^{\delta}\psi(t)dt\right)\\
&= \mu_1(\delta)\sum_{j=1}^{n}\int_{\frac{j-1}{n}}^{\frac{j}{n}} q(x)dx\int_a^{x_j}\psi(t)dt - \mu_1(\delta)\int_0^1 q(x)dx\int_a^{\delta}\psi(t)dt\\
&= \mu_1(\delta)\sum_{j=1}^{n}\int_{\frac{j-1}{n}}^{\frac{j}{n}} q(x)dx\int_a^{x_j}\psi(t)dt - \mu_1(\delta)\int_a^{\delta}\psi(t)dt\\
&= \mu_1(\delta)\sum_{j=1}^{n}\int_{\frac{j-1}{n}}^{\frac{j}{n}} q(x)dx\int_a^{x_j}\psi(t)dt - \mu_1(\delta)\int_0^1 p(x)dx\int_a^{\delta}\psi(t)dt\\
&= \mu_1(\delta)\sum_{j=1}^{n}\int_{\frac{j-1}{n}}^{\frac{j}{n}} q(x)dx\int_a^{x_j}\psi(t)dt - \mu_1(\delta)\sum_{j=1}^{n}\int_{\frac{j-1}{n}}^{\frac{j}{n}} p(x)dx\int_a^{x_j}\psi(t)dt\\
&= \mu_1(\delta)\sum_{j=1}^{n}\int_{\frac{j-1}{n}}^{\frac{j}{n}} (q(x)-p(x))dx\int_a^{x_j}\psi(t)dt\\
&= \mu_1(\delta)\sum_{j=1}^{n}\left(\int_0^{\frac{j}{n}}(q(x)-p(x))dx - \int_0^{\frac{j-1}{n}}(q(x)-p(x))dx\right)\int_a^{x_j}\psi(t)dt\\
&= \mu_1(\delta)\int_0^1(q(x)-p(x))dx\int_a^{x_n}\psi(t)dt + \mu_1(\delta)\sum_{j=1}^{n-1}\int_0^{\frac{j}{n}}(q(x)-p(x))dx\left(\int_a^{x_j}\psi(t)dt - \int_a^{x_{j+1}}\psi(t)dt\right)\\
&= \mu_1(\delta)\sum_{j=1}^{n-1}\int_0^{\frac{j}{n}}(q(x)-p(x))dx\int_{x_{j+1}}^{x_j}\psi(t)dt
\end{aligned}
$$

As for all $u, v \in [0,1],\ u \leqslant v,\ \frac{q(u)}{q(v)} \geqslant \frac{p(u)}{p(v)}$, from Lemma 1, $\int_0^{\frac{j}{n}}(q(x)-p(x))dx \geqslant 0$, so $\int_0^{\gamma}\varphi(a+(b-a)t)dt - \int_0^{\delta}\varphi(a+(b-a)t)dt \geqslant 0$, thus $\gamma \geqslant \delta$, that is $M_{f,Q}(X) \geqslant M_{g,P}(X)$.

Similarly, it can also be proved that $M_{f,W}(X) \geqslant M_{g,W'}(X)$. From Theorem 2, as a special case of $x_i = a + \frac{n-i}{n-1}(b-a)$, it has $\Omega_{f,W} \geqslant \Omega_{g,W'}$. □

Remark 3. The conditions of Lemma 1 and Theorem 5 can be relaxed to the case of OWA weighting vectors or RIM quantifier generating functions with zero, that is the condition if for all $i \leqslant j\ (i,j = 1,2,\ldots,n),\ \frac{w_i}{w_j} \geqslant \frac{w'_i}{w'_j}$ can be replaced with if for all $i \leqslant j\ (i,j = 1,2,\ldots,n),\ w_i w'_j - w'_i w_j \geqslant 0$, and for all $u, v \in [0,1],\ u \leqslant v,\ \frac{q(u)}{q(v)} \geqslant \frac{p(u)}{p(v)}$ with for all $u, v \in [0,1],\ u \leqslant v$, $q(u)p(v) - p(u)q(v) \geqslant 0$. These fractional expressions can make the similarities among the OWA weighting vector, the RIM quantifier generating function and the quasi-arithmetic mean generating function be more clearly observed.

Remark 4. For nonzero OWA weighting vector $W = (w_1, w_2, \ldots, w_n)$ and $W' = (w'_1, w'_2, \ldots, w'_n)$, the condition for all $i \leqslant j\ (i,j = 1,2,\ldots,n),\ \frac{w_i}{w_j} \geqslant \frac{w'_i}{w'_j}$ in Lemma 1 and Theorem 5 can also be replaced with a more simplified and seemingly weaker condition that $\frac{w_i}{w_{i+1}} \geqslant \frac{w'_i}{w'_{i+1}},\ i = 1,2,\ldots,n-1$, as $\frac{w_i}{w_{i+1}} \geqslant \frac{w'_i}{w'_{i+1}}$ implies $\frac{w_i}{w'_i} \geqslant \frac{w_{i+1}}{w'_{i+1}}$, so $\frac{w_1}{w'_1} \geqslant \frac{w_2}{w'_2} \geqslant \cdots \geqslant \frac{w_n}{w'_n}$, that is for $1 \leqslant i \leqslant j \leqslant n$, $w_i w'_j \geqslant w_j w'_i$, so $\frac{w_i}{w_j} \geqslant \frac{w'_i}{w'_j}$.

As the quasi-OWA operator is an extension of the quasi-arithmetic mean and the OWA operator, it can be easily verified that the quasi-OWA operator is commutative and idempotent, monotonic, bounded. The satisfaction of these properties implies that the quasi-OWA operator is a mean operator for any choice of $f(x)$ and $Q(x)$.

If $\psi(t) = 1,\ p(x) = 1$ in Theorem 5, then $g(x) = x,\ P(x) = x = Q_A(x)$, so

$$\Omega_{g,P} = \frac{1}{2}, \quad M_{g,P}(X) = A(X) = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Replace the RIM quantifier $P(x) = Q_A(x)$ with $W' = W_A = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$, then

$$\Omega_{g,W'} = \frac{1}{2}, \quad M_{g,W'}(X) = A(X)$$

It can be obtained that

Corollary 7. *For quasi-OWA operator determined by quasi-arithmetic mean generator $f(x)$ and RIM quantifier $Q(x)$, with generating functions $\varphi(x)$ and $q(x)$, respectively, if $\varphi(x)$ is increasing and $q(x)$ is decreasing, then $\Omega_{f,Q} \geqslant \frac{1}{2}$, and $M_{f,Q}(X) \geqslant \frac{1}{n}\sum_{i=1}^{n} x_i$, alternatively, if $W = (w_1, w_2, \ldots, w_n)$ with $w_1 \geqslant w_2 \geqslant \cdots \geqslant w_n$, then $\Omega_{f,W} \geqslant \frac{1}{2}$, and $M_{f,W}(X) \geqslant \frac{1}{n}\sum_{i=1}^{n} x_i$, which means it is or-like; if $\varphi(x)$ is decreasing and $q(x)$ is increasing, then $\Omega_{f,Q} \leqslant \frac{1}{2}$, and $M_{f,Q}(X) \leqslant \frac{1}{n}\sum_{i=1}^{n} x_i$, alternatively, if $W = (w_1, w_2, \ldots, w_n)$ with $w_1 \leqslant w_2 \leqslant \cdots \leqslant w_n$, then $\Omega_{f,W} \leqslant \frac{1}{2}$, and $M_{f,W}(X) \leqslant \frac{1}{n}\sum_{i=1}^{n} x_i$, which means it is and-like.*

If the generating function is differential and positive, the premise of Theorem 5 can be replaced with a more simplified form.

Theorem 6. *For quasi-OWA operators determined by the quasi-arithmetic mean generator and the RIM quantifier pairs* $f(x),\ Q(x)$ *and* $g(x),\ P(x)$ *with generating functions* $\varphi(x),\ q(x)$ *and* $\psi(x),\ p(x)$*, respectively, if for all* $t \in [a,b],\ \frac{\varphi'(t)}{\varphi(t)} \geqslant \frac{\psi'(t)}{\psi(t)}$ *and for all* $u \in [0,1],\ \frac{q'(u)}{q(u)} \leqslant \frac{p'(u)}{p(u)}$*, then* $\Omega_{f,Q} \geqslant \Omega_{g,P}$*, and for any* $X,\ M_{f,Q}(X) \geqslant M_{g,P}(X)$.

Proof. Let $\mu(t) = \frac{\varphi(t)}{\psi(t)}$, then $\mu'(t) = \frac{\varphi'(t)\psi(t)-\varphi(t)\psi'(t)}{\psi^2(t)}$. As $\varphi(x),\ \psi(x) > 0$, if for all $t \in [a,b],\ \frac{\varphi'(t)}{\varphi(t)} \geqslant \frac{\psi'(t)}{\psi(t)}$, then $\varphi'(t)\psi(t)-\varphi(t)\psi'(t) \geqslant 0$, it has $\mu'(t) \geqslant 0$, which means $\mu(t)$ is increasing on $[a,b]$, so for all $s,t \in [a,b],\ s \geqslant t,\ \frac{\varphi(s)}{\psi(s)} \geqslant \frac{\varphi(t)}{\psi(t)}$, that is $\frac{\varphi(s)}{\varphi(t)} \geqslant \frac{\psi(s)}{\psi(t)}$. Similarly, with the same method. As for all $u \in [0,1],\ \frac{q'(u)}{q(u)} \leqslant \frac{p'(u)}{p(u)}$, it can be obtained that for all $u,v \in [0,1],\ u \leqslant v,\ \frac{q(u)}{q(v)} \geqslant \frac{p(u)}{p(v)}$. From Theorem 5, if for all $t \in [a,b],\ \frac{\varphi'(t)}{\varphi(t)} \geqslant \frac{\psi'(t)}{\psi(t)}$ and for all $v \in [0,1],\ \frac{q'(v)}{q(v)} \leqslant \frac{p'(v)}{p(v)}$, then $\Omega_{f,Q} > \Omega_{g,P}$, and for any $X,\ M_{f,Q}(X) \geqslant M_{g,P}(X)$. □

Without involving the generating function, Theorem 6 can also be expressed as:

Corollary 8. *For quasi-OWA operators determined by the quasi-arithmetic mean generator and the RIM quantifier pairs* $f(x),\ Q(x)$ *and* $g(x),\ P(x)$*, if for all* $t \in [a,b],\ \frac{f''(t)}{f'(t)} \geqslant \frac{g''(t)}{g'(t)}$ *and for all* $u \in [0,1],\ \frac{Q''(u)}{Q'(u)} \leqslant \frac{P''(u)}{P'(u)}$*, then* $\Omega_{f,Q} \geqslant \Omega_{g,P}$*, and for any* $X,\ M_{f,Q}(X) \geqslant M_{g,P}(X)$.

With the convex concept between two functions [3, pp. 49–50], the expression for all $t \in [a,b],\ \frac{f''(t)}{f'(t)} \geqslant \frac{g''(t)}{g'(t)}$ actually means that $f \circ g^{-1}$ is convex, and for all $u \in [0,1],\ \frac{Q''(u)}{Q'(u)} \leqslant \frac{P''(u)}{P'(u)}$ means $Q \circ P^{-1}$ is concave. That is the more convex of the quasi-arithmetic generator $f(x)$ and the more concave of the RIM quantifier $Q(x)$ is, the bigger the orness level of the quasi-OWA operator and the aggregation value will be.

3.2. Two families of parameterized quasi-OWA operators

From Corollary 8, the orness value of a quasi-OWA operator and the aggregation value change with the quasi-arithmetic mean function and the RIM quantifier function (or the OWA weighting vector). With Definition 9, the orness measure of quasi-OWA operator has two forms with the OWA operator weighting vector and the RIM quantifier, which correspond to the discrete (with fixed dimension n) and continuous cases (with $n \to \infty$) of OWA aggregation, respectively. In this section, two parameterized quasi-OWA operator families with exponential functions and power functions are proposed. For simplification, only the RIM quantifier based quasi-OWA operator is considered, which corresponds to the continuous case of the orness measure. The conclusions also stand if the RIM quantifier is replaced with the corresponding discrete OWA weight elements.

These can also be seen as the direct extensions of the corresponding conclusions in the quasi-arithmetic mean operator [26].

3.2.1. Quasi-OWA operator with exponential functions

From Theorem 6, for a quasi-OWA operator with quasi-arithmetic mean generator $f(x)$. To make the quasi-OWA operator more or-like or and-like, we can increase or decrease the value of expression $\frac{f''(x)}{f'(x)}$. The simplest case is to set $\frac{f''(x)}{f'(x)} = \lambda$, so $f(x) = C_1 + C_2 e^{\lambda x}\ (\lambda \neq 0)$ or $f(x) = C_1 + C_2 x\ (\lambda = 0)$. Considering Theorem 3, only $f(x)$ with the following form needs to be considered:

$$f(x) = \begin{cases} e^{\lambda x} & \text{if } \lambda \neq 0 \\ x & \text{otherwise} \end{cases} \tag{22}$$

With the same idea, for the generating function of the RIM quantifier in the quasi-OWA operator, let $\frac{Q''(x)}{Q'(x)} = \mu$, consider that $Q(0) = 0$ and $Q(1) = 1$, then

$$Q(x) = \begin{cases} \frac{1-e^{\mu x}}{1-e^{\mu}} & \text{if } \mu \neq 0 \\ x & \text{otherwise} \end{cases} \tag{23}$$

For simplification, we will always assume that the aggregated elements are listed in descending order, that is $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n$. The quasi-OWA operator (14), becomes

$$M_{f,Q}(x_1,x_2,\ldots,x_n) = \begin{cases} \frac{1}{\lambda}\ln\left(\frac{1-e^{\frac{\mu}{n}}}{1-e^{\mu}}\sum_{i=1}^{n} e^{\frac{\mu(i-1)}{n}+\lambda x_i}\right) & \text{if } \lambda \neq 0,\ \mu \neq 0 \\ \frac{1-e^{\frac{\mu}{n}}}{1-e^{\mu}}\sum_{i=1}^{n} e^{\frac{\mu(i-1)}{n}} x_i & \text{if } \lambda = 0,\ \mu \neq 0 \\ \frac{1}{\lambda}\ln\left(\frac{1}{n}\sum_{i=1}^{n} e^{\lambda x_i}\right) & \text{if } \lambda \neq 0,\ \mu = 0 \\ \frac{1}{n}\sum_{i=1}^{n} x_i & \text{if } \lambda = \mu = 0 \end{cases} \tag{24}$$

This is called the exponential function quasi-OWA operator. From (16) in Definition 9, the orness value of (24) can be obtained:

$$\Omega_{f,Q} = \begin{cases} \dfrac{\ln\left(\frac{\mu(e^{\mu}-e^{\lambda(b-a)})}{(e^{\mu}-1)(\mu-(b-a)\lambda)}\right)}{\lambda(b-a)} & \text{if } \lambda \neq 0,\ \mu \neq 0 \\ \frac{1}{\lambda(b-a)} \ln\left(\frac{e^{\lambda(b-a)}-1}{\lambda(b-a)}\right) & \text{if } \lambda \neq 0,\ \mu = 0 \\ \frac{e^{\mu}-\mu-1}{\mu e^{\mu}-\mu} & \text{if } \lambda = 0,\ \mu \neq 0 \\ \frac{1}{2} & \text{if } \lambda = \mu = 0 \end{cases} \tag{25}$$

Remark 5. It can be verified that in (25), $\Omega_{f,Q}$ is continuous for λ and μ.

Property 1. *For quasi-OWA operator with exponential function (22) and (23), $\Omega_{f,Q}$ and $M_{f,Q}(X)$ are monotonically increasing for λ and monotonically decreasing for μ.*

Proof. This can be verified with Corollary 8 and the above deduction directly. □

From (25), the orness measure can be completely determined by the generating function changing slope parameters $\lambda,\ \mu$, and the interval length $b-a$ is concerned. Furthermore, $\Omega_{f,Q}$ is symmetrical for λ and μ, respectively, that is

Property 2. *For quasi-OWA operator with exponential functions, the orness level is only related with the aggregated elements distribution interval length $b-a$, but have no relationship with its start or end point. And $\Omega_{f,Q}$ is a symmetrical function for its parameters $\lambda,\ \mu$, that is $\Omega_{f,Q}(\lambda,\mu) + \Omega_{f,Q}(-\lambda,-\mu) = 1$.*

Proof. This can be verified from (25) directly. □

The symmetrical property makes that with a given or-like or and-like quasi-OWA operator, it is easy to get its complementary operator with the same level of and-like or or-like. For a quasi-OWA operator of exponential functions (22) and (23), with given orness level α, replacing $\lambda,\ \mu$ with $-\lambda,\ -\mu$, its dual part with complimentary orness level $1-\alpha$ can be obtained.

The relationships between $\Omega_{f,Q}$ and $\lambda,\ b-a$ for different μ are shown in Fig. 1.

From these figures, it can be observed that $\lambda,\ \mu$ and $b-a$ play different roles in the determination of orness value $\Omega_{f,Q}$, respectively. The parameter μ in the RIM quantifier function determines the basic orness aggregation level. When μ is fixed, the orness values of the quasi-OWA operator usually have a common intersection point for the distribution interval $[a,b]$ or

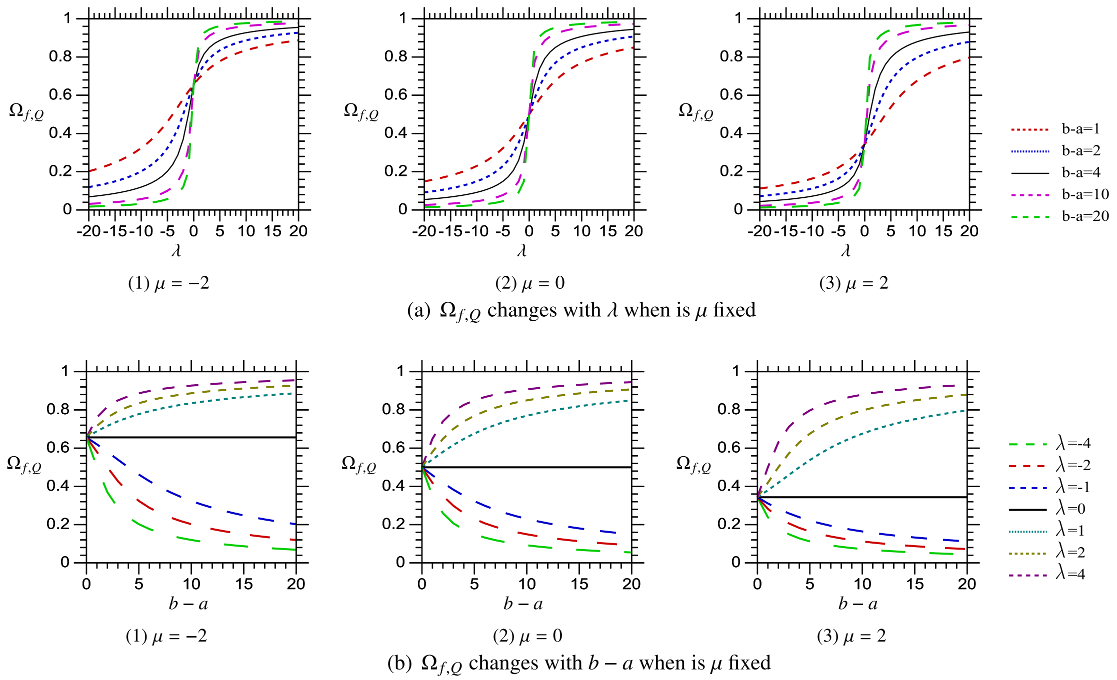


Fig. 1. The orness of exponential function quasi-OWA operator $\Omega_{f,Q}$ changes when is μ fixed.

the parameter of quasi-arithmetic mean λ. Furthermore, the orness value usually changes more flatly for fixed λ than the case for fixed μ. So the orness value is mainly determined by the RIM quantifier parameter μ, and the quasi-arithmetic mean generator parameter λ can be seen as a fine tuning of the orness level. If μ is not selected appropriately, the absolute value of λ must be very large to get the desired orness level.

As the OWA operator or RIM quantifier does not change the aggregation form of quasi-arithmetic mean, associating the relationship between $\Omega_{f,Q}$ and $b-a$ in Theorem 2, the shift invariant of the ordinary quasi-arithmetic mean with exponential function can be extended to the quasi-OWA operator case directly.

Property 3. *The quasi-OWA operator is shift invariant if and only if the quasi-arithmetic mean generator has the form of an exponential function, that is, for any* $X=(x_1,\ldots,x_n)$ *and c,*

$$M_{f,W}(c+x_1,c+x_2,\ldots,c+x_n)=c+M_{f,W}(x_1,x_2,\ldots,x_n)$$

or

$$M_{f,Q}(c+x_1,c+x_2,\ldots,c+x_n)=c+M_{f,Q}(x_1,x_2,\ldots,x_n)$$

if and only if $f(x)=e^{\lambda x}$ $(\lambda\neq 0)$ *or* $f(x)=x$.

Proof. Omitted. □

Remark 6. According to (22), the quasi-arithmetic mean generator $f(x)=x$ can be seen as the complementary case of the exponential function $f(x)=e^{\lambda x}$ $(\lambda\neq 0)$ with $\lambda=0$. And with Theorem 3, $f(x)$ can also be replaced with $kf(x)+c$ $(k\neq 0)$.

3.2.2. Quasi-OWA operator with power functions

The quasi-OWA operator with power functions is another family of parameterized quasi-OWA operator changing consistently with its orness level parameters. But the aggregated elements should be the domain $\mathbb{R}^+$. If the RIM quantifier $Q(x)=x$ (or $q(x)=1$), this becomes the well known root-power mean operator. Some of its properties have been well known [22,31,24], and are widely used in many areas. Here, only the properties associated with the orness measure are discussed.

Like the quasi-OWA operator with exponential functions, from Corollary 8, if $\frac{f''(x)}{f'(x)}=\frac{r-1}{x}$, then $f(x)=C_1+C_2x^r$ $(r\neq 0)$ or $f(x)=C_1+C_2\ln x$ $(r=0)$. Considering Theorem 3, only the $f(x)$ in the following form needs to be considered:

$$f(x)=\begin{cases} x^r & \text{if } r\neq 0\\ \ln(x) & \text{otherwise}\end{cases}\tag{26}$$

To avoid $f(x)$ being meaningless, only the domain $[a,b]$ with $b>a>0$ can be considered. That is, the aggregated elements must be positive real numbers of $\mathbb{R}^+$ if the quasi-OWA operator with power functions can be applied. This is different from the quasi-OWA operator with exponential functions, where the quasi-arithmetic mean generator is always meaningful in $\mathbb{R}$. The aggregated elements can be any real numbers without limitation.

With the same idea, let $\frac{Q''(x)}{Q'(x)}=\frac{s-1}{x}$ then $Q(x)=C_1+C_2x^s$ $(s\neq 0)$ or $Q(x)=C_1+C_2\ln x$ $(s=0)$. With $Q(0)=0,\ Q(1)=1$, only the case of $s\neq 0$ can be considered, so

$$Q(x)=x^s,\quad s\neq 0\tag{27}$$

Obviously, with the RIM quantifier concept of Definition 4, it should have $s>0$.

The quasi-OWA operator becomes

$$M_{f,Q}(x_1,x_2,\ldots,x_n)=\begin{cases}\left(\sum_{i=1}^{n}\left(\left(\frac{i}{n}\right)^s-\left(\frac{i-1}{n}\right)^s\right)x_i^r\right)^{\frac{1}{r}} & \text{if } r\neq 0\\ \prod_{i=1}^{n}x_i^{\left(\frac{i}{n}\right)^s-\left(\frac{i-1}{n}\right)^s} & \text{otherwise}\end{cases}$$

where $x_i>0$.

The orness value of this kind of quasi-OWA operator is

$$\Omega_{f,Q}=\begin{cases}\dfrac{\left(s\int_0^1(at+(1-t)b)^r t^{s-1}dt\right)^{\frac{1}{r}}-a}{b-a} & \text{if } r\neq 0\\ \dfrac{e^{s\int_0^1\ln(at+(1-t)b)t^{s-1}dt}-a}{b-a} & \text{otherwise}\end{cases}\tag{28}$$

If $a\neq 0$, it can also be expressed as

$$\Omega_{f,Q}=\begin{cases}\dfrac{\left(s\int_0^1\left(t+(1-t)\frac{b}{a}\right)^r t^{s-1}dt\right)^{\frac{1}{r}}-1}{\frac{b}{a}-1} & \text{if } r\neq 0\\ \dfrac{e^{s\int_0^1\ln\left(t+(1-t)\frac{b}{a}\right)t^{s-1}dt}-1}{\frac{b}{a}-1} & \text{otherwise}\end{cases}\tag{29}$$

As mentioned before, in order to keep quasi-arithmetic mean generator (26) always meaningful on $[a,b]$, it should have $b > a > 0$. In some areas, especially the fuzzy logic, we often need to consider the arguments in the domain $[0,1]$. For the quasi-OWA operator with exponential function quasi-arithmetic mean generator (22), a quasi-arithmetic mean that makes $\Omega_{f,Q} \in (0,1)$ can always be obtained. But for the quasi-OWA operator with power function quasi-arithmetic mean generator (26), $f(x)$ must be continuous and monotone on $[0,1]$. As the generator $f(x)$ is meaningless at 0 for $r \leqslant 0$, only the case of $r > 0$ can be considered. With $a = 0,\ b = 1$, (28) becomes

$$\Omega_{f,Q} = \left(\frac{s\Gamma(s)\Gamma(r+1)}{\Gamma(r+s+1)}\right)^{\frac{1}{r}}, \quad r,s > 0 \tag{30}$$

where Γ is gamma function.

From (30), for $r \in (0,+\infty)$, $\Omega_{f,Q}$ can only ranges in $\left(e^{-\frac{\gamma s+\Psi(s)s+1}{s}}, 1\right)$, where Ψ is the digamma function, the logarithmic derivative of the gamma function. $\gamma = 0.5772156649\ldots$ is Euler's constant. The feasible value of $\Omega_{f,Q}$ is shown in Fig. 2 (shaded area). As $\lim_{s\to 0} e^{-\frac{\gamma s+\Psi(s)s+1}{s}} = 1$ and $\lim_{s\to+\infty} e^{-\frac{\gamma s+\Psi(s)s+1}{s}} = 0$. This means that for the very "and-like" ($\Omega_{f,Q} = \delta$ is very small) quasi-OWA operator in the power function form, s must be set big enough to make $e^{-\frac{\gamma s+\Psi(s)s+1}{s}} \leqslant \alpha$. To avoid such condition, we can use the dual property of Theorem 4 for arguments on $[0,1]$, to get a quasi-OWA operator with $\Omega_{f,Q} = \delta < e^{-\frac{\gamma s+\Psi(s)s+1}{s}}$ for some specific s, first, get a function $g(x) = x^r$ and $P(x) = x^s$ with $\Omega_{g,P} = 1-\delta$, then $f(x) = g(1-x) = (1-x)^r,\ Q(x) = 1-(1-x)^s$ will be the needed "and-like" quasi-OWA operator functions. This technique was also used in [24, p. 71] for the quasi-arithmetic mean with $Q(x) = x$ as the special case.

An alternative way is that, if these arguments are nonzero and a lower bound $a > 0$ of them can be obtained, then by considering the interval $[a,1]$ rather than $[0,1]$, (26) will be meaningful for any $r \in \mathbb{R}$, (28) becomes

$$\Omega_{f,Q} = \begin{cases} \dfrac{\left(s\int_0^1 (at+(1-t))^r t^{s-1}\,dt\right)^{\frac{1}{r}} - a}{1-a} & \text{if } r \neq 0 \\[2ex] \dfrac{e^{s\int_0^1 \ln(at+(1-t))t^{s-1}\,dt} - 1}{1-a} & \text{otherwise} \end{cases}$$

For any $\delta \in (0,1)$, with fixed s, the value of r for (26) can always be obtained on the input arguments interval $[a,1]$, that makes $\Omega_{f,Q} = \delta$.

Similar to the properties of quasi-OWA operator with Corollary 8 and (29), we can get that

Property 4. *For quasi-OWA operator with power functions (26) and (27), $\Omega_{f,Q}$ and $M_{f,Q}(X)$ are monotonically increasing for r and monotonically decreasing for s.*

Property 5. *For quasi-OWA operator mean with power function, the orness level is only related with the start and end points ratio b/a, but has no relationship with its start or end point.*

The relationships between $\Omega_{f,Q}$ and $r,\ b/a$ for different s are shown in Fig. 3.

It can be observed that, unlike the quasi-OWA operator with exponential functions the quasi-OWA operator with power functions is not symmetrical for its parameter r and s.

In the same way, the scale invariant property of the quasi-OWA operator with exponential quasi-arithmetic generator can also be extended to the quasi-OWA operator case.

Property 6. *The quasi-OWA operator is scale invariant if and only if the quasi-arithmetic mean generator has the form of a power function, that is, for any $X = (x_1,\ldots,x_n)$ $(x_i > 0,\ i = 1,2,\ldots,n)$ and $c > 0$,*

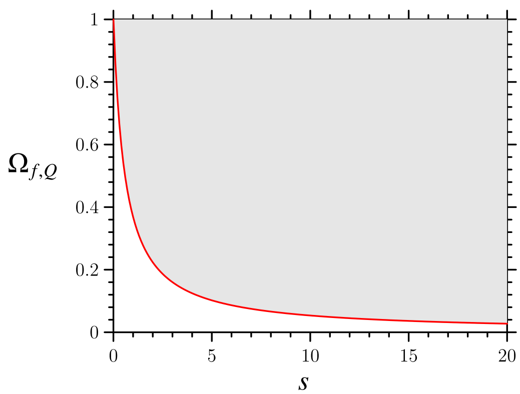


Fig. 2. The feasible value of $\Omega_{f,Q}$ for $a = 0,\ b = 1$.

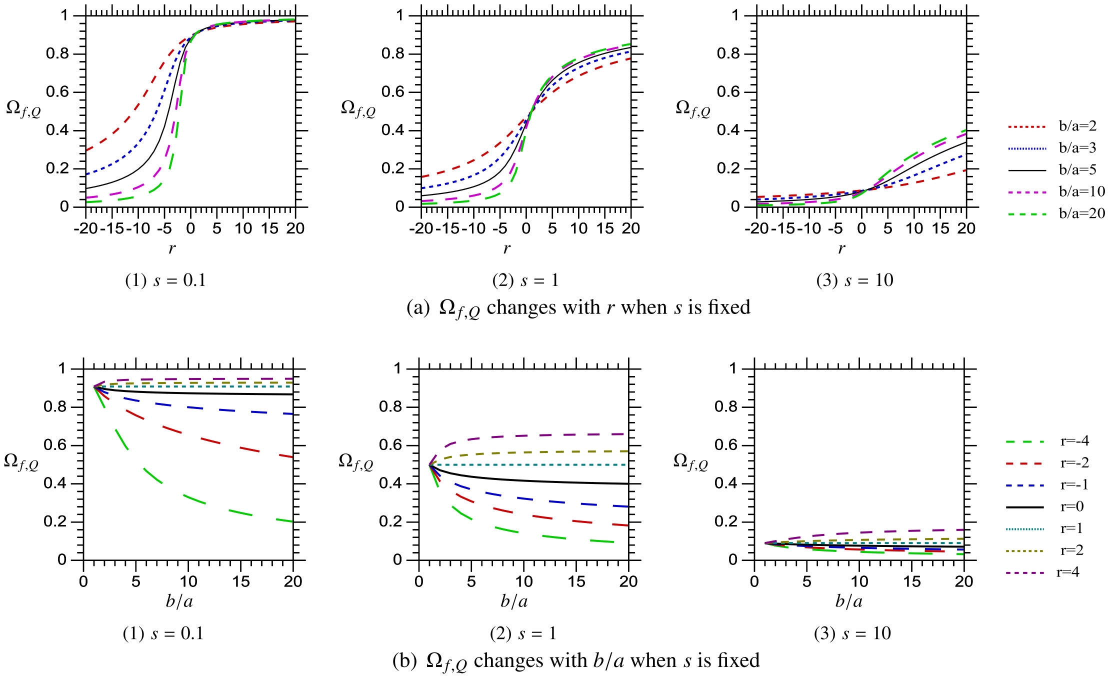


Fig. 3. The orness of power function quasi-OWA operator $\Omega_{f,Q}$ changes when is s fixed.

$$M_{f,W}(cx_1, cx_2, \ldots, cx_n) = cM_{f,W}(x_1, x_2, \ldots, x_n)$$

or

$$M_{f,Q}(cx_1, cx_2, \ldots, cx_n) = cM_{f,Q}(x_1, x_2, \ldots, x_n)$$

if and only if $f(x) = x^r \ (r \neq 0)$ *or* $f(x) = \ln x$.

Remark 7. According to (26), the quasi-arithmetic mean generator $f(x) = \ln(x)$ can be seen as the complementary case of the power function $f(x) = x^r \ (r \neq 0)$ with $r = 0$. And with Theorem 3, $f(x)$ can also be replaced with $kf(x) + c \ (k \neq 0)$.

By comparing these two kinds of quasi-OWA operators, we can see that both of them have some interesting properties in common and of their own. The exponential function quasi-OWA operator is shift invariant and the power function quasi-OWA operator is ratio invariant for their own quasi-arithmetic mean generators. Another one is that the exponential function quasi-OWA operator is symmetrical for its parameters λ and μ, and a pair of operators which have complementary orness measure can be easily obtained, while the power function quasi-OWA operator is not. The last difference between them is that the exponential function quasi-OWA operator can be applied to the values of any real number in $\mathbb{R}$, but the power function-based quasi-OWA operator can only be used for the positive real numbers in $\mathbb{R}^+$ (the feasible domain can be extended to $\mathbb{R}^+ \cup \{0\}$ with the dual property).

As shown Theorems 1 and 4, both of these quasi-OWA operators give their aggregation value for any element set consistently changing with their parameters. In fact, both the exponential function and power function-based quasi-OWA operators can be generated from the same expression $\frac{f''(x)}{f'(x)} = K_1(x)$ by setting $K_1(x) = \lambda$ or $K_1(x) = \frac{r-1}{x}$ in the quasi-arithmetic mean, and $\frac{Q''(x)}{Q'(x)} = K_2(x)$ by setting $K_2(x) = \mu$ or $K_2(x) = \frac{s-1}{x}$ in the RIM quantifier, respectively. Many other parameterized quasi-OWA operators can be obtained that make the orness value and the aggregation value for any element set change consistently with $K_1(x)$ and $K_2(x)$ as their parameterized functions. Unfortunately, the solutions of $\frac{f''(x)}{f'(x)} = K_1(x)$ and $\frac{Q''(x)}{Q'(x)} = K_2(x)$ cannot be expressed in an analytical formula in many cases. Trying to find out consistent quasi-OWA operators with other form should be an interesting topic of the future research. These conclusions can also be extended widely if the quasi-arithmetic mean function and the RIM quantifier function (or OWA weighting vector) are replaced with other appropriate forms. The unique prerequisite condition is that the premise of Corollary 8 or Theorem 5 can be satisfied.

4. The orness measure for Bajraktarević mean and its properties

In this section, we will discuss another compound quasi-arithmetic mean aggregation operator – the Bajraktarević mean, which is the combination of the quasi-arithmetic mean and the weighted function average method. Similar results to the quasi-OWA operator can be obtained.

4.1. An orness measure for Bajraktarević mean

The Bajraktarević mean is also called Losonczi mean. It is the combination of the weighted function average operator and quasi-arithmetic mean [3]. Some properties and related issues were discussed [3,36,13]. Here, the context of $[a,b]^n$ instead of $[0,1]^n$ is considered, where $[a,b]$ is the interval the aggregated elements distributed in. It is also natural that $f(x)$ should be always meaningful on $[a,b]$.

Definition 10. Let f be a continuous strictly monotone function for the quasi-arithmetic mean generator and $w(x)$ be the weighted function average operator. For input vector $X=(x_1,x_2,\dots,x_n)\in[a,b]^n$, a Bajraktarević mean can be defined as the aggregation operator $M_{f,w}:[a,b]^n\to[a,b]$ given by

$$M_{f,w}(x_1,x_2,\dots,x_n)=f^{-1}\left(\frac{\sum_{i=1}^n w(x_i)f(x_i)}{\sum_{i=1}^n w(x_i)}\right) \tag{31}$$

where f^{-1} is the inverse function of f.

When $f(x)=x$, then

$$M_{f,w}(x_1,x_2,\dots,x_n)=\frac{\sum_{i=1}^n w(x_i)x_i}{\sum_{i=1}^n w(x_i)}$$

This becomes the weighted function average operator.

When $w(x)=1$, then

$$M_{f,w}(x_1,x_2,\dots,x_n)=f^{-1}\left(\frac{1}{n}\sum_{i=1}^n f(x_i)\right)$$

This becomes the commonly used quasi-arithmetic mean operator.

Remark 8. A general form of the Bajraktarević mean is $M_{f,\mathbf{w}}(x_1,x_2,\dots,x_n)=f^{-1}\left(\frac{\sum_{i=1}^n w_i(x_i)f(x_i)}{\sum_{i=1}^n w_i(x_i)}\right)$. For simplification, we assume that all the weighting functions have the same form.

Similar to the quasi-OWA operator, an orness concept for Bajraktarević mean that can combine the orness measures of the weighted function average operator and the quasi-arithmetic mean together is proposed in the following.

Definition 11. For Bajraktarević mean (31) which is determined by the quasi-arithmetic mean generator $f(x)$ and weighting function $w(x)$, respectively, the orness measure can be defined as:

$$\Omega_{f,w}=\frac{f^{-1}\left(\frac{\int_a^b w(x)f(x)dx}{\int_a^b w(x)dx}\right)-a}{b-a} \tag{32}$$

When $f(x)=x$,

$$\Omega_{f,w}=\frac{\frac{\int_a^b xw(x)dx}{\int_a^b w(x)dx}-a}{b-a}=\frac{\int_a^b (x-a)w(x)dx}{(b-a)\int_a^b w(x)dx}$$

These are the orness concepts of the weighted function average [30].

When $w(x)=x$,

$$\Omega_{f,w}=\frac{f^{-1}\left(\int_0^1 f(ax+(1-x)b)dx\right)-a}{b-a}=\frac{f^{-1}\left(\frac{\int_a^b f(y)dy}{b-a}\right)-a}{b-a}$$

This becomes the orness definition of the quasi-arithmetic mean [26]. So the orness of Bajraktarević mean can be seen as a combination of the orness measures of the weighted function average and the quasi-arithmetic mean together.

The following theorems show the rationality of the orness measure for Bajraktarević mean. As shown in Section 2, similar results can also be found in the weighted function average operator and the quasi-arithmetic mean, respectively [47,29,26].

Theorem 7. $0\leqslant\Omega_{f,w}\leqslant 1$.

Proof. As $f(x)$ is strictly monotone, it can be assumed that $f(x)$ is strictly increasing, $f(a)\leqslant\frac{\int_a^b w(x)f(x)dx}{\int_a^b w(x)dx}\leqslant f(b)$, so $0\leqslant\Omega_{f,w}\leqslant 1$. □

Theorem 8. *If* $X=(x_1,x_2,\dots,x_n)$ *is evenly distributed in* $[a,b]$, *that is* $x_j=a+\frac{j-1}{n-1}(b-a)$, $j=1,2,\dots,n$, *then* $\lim_{n\to+\infty}M_{f,w}(X)=a+\Omega_{f,w}(b-a)$.

Proof. As $f^{-1}(x)$ is continuous, from (31),

$$\lim_{n\to\infty} M_{f,w}(x_1,x_2,\ldots,x_n) = \lim_{n\to\infty} f^{-1}\left(\frac{\sum_{j=1}^n f\left(a+\frac{j-1}{n-1}(b-a)\right)w\left(a+\frac{j-1}{n-1}(b-a)\right)}{\sum_{j=1}^n w\left(a+\frac{j-1}{n-1}(b-a)\right)}\right)$$
$$= f^{-1}\left(\lim_{n\to\infty}\frac{\sum_{j=1}^n f\left(a+\frac{j-1}{n-1}(b-a)\right)w\left(a+\frac{j-1}{n-1}(b-a)\right)}{\sum_{j=1}^n w\left(a+\frac{j-1}{n-1}(b-a)\right)}\right)$$
$$= f^{-1}\left(\lim_{n\to\infty}\frac{\frac{b-a}{n}\sum_{j=1}^n f\left(a+\frac{j-1}{n-1}(b-a)\right)w\left(a+\frac{j-1}{n-1}(b-a)\right)}{\frac{b-a}{n}\sum_{j=1}^n w\left(a+\frac{j-1}{n-1}(b-a)\right)}\right)$$

Consider an even division of $[a,b]$ with n intervals $[d_{j-1},d_j]$ $(j=1,2,\ldots,n)$, where $d_j=\frac{j}{n}$ $(j=0,1,2,\ldots,n)$. Select $d'_j=\frac{j-1}{n-1}$ $(j=1,2,\ldots,n)$, then $d'_j\in[d_{j-1},d_j]$. Therefore

$$\lim_{n\to\infty}\frac{\frac{b-a}{n}\sum_{j=1}^n f\left(a+\frac{j-1}{n-1}(b-a)\right)w\left(a+\frac{j-1}{n-1}(b-a)\right)}{\frac{b-a}{n}\sum_{j=1}^n w\left(a+\frac{j-1}{n-1}(b-a)\right)} = \frac{\int_a^b f(x)w(x)dx}{\int_a^b w(x)dx}$$

So, with (32), $\lim_{n\to+\infty} M_{f,w}(x_1,x_2,\ldots,x_n) = f^{-1}\left(\frac{\int_a^b w(x)f(x)dx}{\int_a^b w(x)dx}\right) = a+\Omega_{f,w}(b-a)$. $\square$

Theorem 9. *For all* $X=(x_1,x_2,\ldots,x_n)$, $M_{k_1f+c,k_2w}(X)=M_{f,w}(X)$ $(k_1,k_2\neq 0)$.

Proof. Let $h(x)=k_1f(x)+c$, and $h^{-1}(x)=y$, then $x=h(y)=k_1f(y)+c$, $y=f^{-1}\left(\frac{x-c}{k_1}\right)$, that is $h^{-1}(x)=f^{-1}\left(\frac{x-c}{k_1}\right)$

$$M_{k_1f+c,k_2w}(X)=M_{h,k_2w}(X)=h^{-1}\left(\frac{\sum_{i=1}^n h(x_i)k_2w(x_i)}{\sum_{i=1}^n k_2w(x_i)}\right)$$
$$= h^{-1}\left(\frac{k_1\sum_{i=1}^n f(x_i)w(x_i)}{\sum_{i=1}^n w(x_i)}+c\right) = f^{-1}\left(\frac{\sum_{i=1}^n f(x_i)w(x_i)}{\sum_{i=1}^n w(x_i)}\right) = M_{f,w}(X) \qquad \square$$

From Theorem 9, the linear transformation invariance in the quasi-arithmetic mean is still kept for the Bajraktarević mean. Similar to the quasi-arithmetic mean, only the case that f is in the set $\mathscr{F}$ needs to be considered, where $f(x)$ is continuous, strictly monotone increasing, non-negative, with $f(a)=0$ and $f(b)=1$.

Theorem 10. *For a Bajraktarević mean determined by the quasi-arithmetic mean generator* $f(x)$ *and the weighting function* $w(x)$, $\Omega_{f(a+b-x),w(a+b-x)}=1-\Omega_{f(x),w(x)}$.

Proof. Let $f(a+b-x)=h(x)$, $h^{-1}(x)=y$, then $x=h(y)=f(a+b-y)$, $y=a+b-f^{-1}(x)$, that is $h^{-1}(x)=a+b-f^{-1}(x)$

$$\Omega_{f(a+b-x),w(a+b-x)} = \frac{h^{-1}\left(\int_a^b h(x)w(a+b-x)dx\right)-a}{b-a} = \frac{b-f^{-1}\left(\int_a^b f(a+b-x)w(a+b-x)dx\right)}{b-a} = \frac{b-f^{-1}\left(\int_a^b f(y)w(y)dy\right)}{b-a}$$
$$= 1-\Omega_{f(x),w(a+b-x)} \qquad \square$$

From (32), the orness measure of Bajraktarević mean $\Omega_{f,w}=\alpha$ is the solution of equation

$$f(a+(b-a)\alpha)=\frac{\int_a^b f(x)w(x)dx}{\int_a^b w(x)dx}$$

With the generating function method of quasi-arithmetic mean $f(x)=\int_a^x \varphi(t)dt$,

$$\int_a^{a+(b-a)\alpha}\varphi(t)dt = \frac{\int_a^b\int_a^x \varphi(t)dtw(x)dx}{\int_a^b w(x)dx} \tag{33}$$

Similarly, from (31), it can be obtained that the Bajraktarević mean $M_{f,w}(x_1,x_2,\ldots,x_n)=y$ is the solution of equation

$$f(y)=\frac{\sum_{i=1}^n w(x_i)f(x_i)}{\sum_{i=1}^n w(x_i)}$$

that is

$$\int_a^y \varphi(t)dt = \frac{\sum_{i=1}^n w(x_i)\int_a^{x_i}\varphi(t)dt}{\sum_{i=1}^n w(x_i)} \tag{34}$$

Similar to Theorem 5 in the quasi-OWA case, we also have

Theorem 11. *For Bajraktarević means determined by the quasi-arithmetic mean generator and the weighting function pairs $f(x)$, $w_1(x)$ and $g(x)$, $w_2(x)$, the quasi-arithmetic mean generator $f(x)$, $g(x)$ with generating functions $\varphi(x)$, and $\psi(x)$, respectively, if for all $s,t \in [a,b]$, $s \geqslant t$, $\frac{\varphi(s)}{\varphi(t)} \geqslant \frac{\psi(s)}{\psi(t)}$, and $\frac{w_1(s)}{w_1(t)} \geqslant \frac{w_2(s)}{w_2(t)}$, then $\Omega_{f,w_1} \geqslant \Omega_{g,w_2}$, and for any $X = (x_1, x_2, \ldots, x_n)$ with $\min_{1\leqslant i\leqslant n}\{x_i\} = a$, $\max_{1\leqslant i\leqslant n}\{x_i\} = b$, $M_{f,w_1}(X) \geqslant M_{g,w_2}(X)$.*

Proof. Let $\Omega_{f,w_1} = \alpha$, $\Omega_{g,w_2} = \beta$, from (33),

$$\int_a^{a+\alpha(b-a)} \varphi(t)dt = \frac{\int_a^b \int_a^x \varphi(t)dt w_1(x)dx}{\int_a^b w_1(x)dx}, \quad \int_a^{a+\beta(b-a)} \psi(t)dt = \frac{\int_a^b \int_a^x \psi(t)dt w_2(x)dx}{\int_a^b w_2(x)dx} \tag{35}$$

As $\varphi(t)$, $\psi(t) \geqslant 0$, to prove $\alpha \geqslant \beta$, it only needs to prove that

$$\int_a^{a+\alpha(b-a)} \varphi(t)dt \geqslant \int_a^{a+\beta(b-a)} \varphi(t)dt$$

$$\begin{aligned}
\int_a^{a+\alpha(b-a)} \varphi(t)dt - \int_a^{a+\beta(b-a)} \varphi(t)dt &= \frac{\int_a^b \int_a^x \varphi(t)dt w_1(x)dx}{\int_a^b w_1(x)dx} - \int_a^{a+\beta(b-a)} \varphi(t)dt \\
&= \frac{\int_a^b \int_a^x \varphi(t)dt w_1(x)dx - \int_a^b \int_a^{a+\beta(b-a)} \varphi(t)dt w_1(x)dx}{\int_a^b w_1(x)dx} \\
&= \frac{\int_{a+\beta(b-a)}^b \int_{a+\beta(b-a)}^x \varphi(t)dt w_1(x)dx - \int_a^{a+\beta(b-a)} \int_x^{a+\beta(b-a)} \varphi(t)dt w_1(x)dx}{\int_a^b w_1(x)dx}
\end{aligned}$$

For all $s,t \in [a,b]$, $s \geqslant t$, $\frac{\varphi(s)}{\varphi(t)} \geqslant \frac{\psi(s)}{\psi(t)}$, $\frac{w_1(s)}{w_1(t)} \geqslant \frac{w_2(s)}{w_2(t)}$, it can be obtained that $\frac{\varphi(s)}{\psi(s)} \geqslant \frac{\varphi(t)}{\psi(t)}$, $\frac{w_1(s)}{w_2(s)} \geqslant \frac{w_1(t)}{w_2(t)}$. Let $\mu_1(x) = \frac{\varphi(x)}{\psi(x)}$, $\mu_2(x) = \frac{w_1(x)}{w_2(x)}$, then $\mu_1(x)$, $\mu_2(x)$ are increasing, and $\mu_1(x)$, $\mu_2(x) \geqslant 0$. $\varphi(x) = \mu_1(x)\psi(x)$, $w_1(x) = \mu_2(x)w_2(x)$, thus

$$\begin{aligned}
&\int_{a+\beta(b-a)}^b \int_{a+\beta(b-a)}^x \varphi(t)dt w_1(x)dx - \int_a^{a+\beta(b-a)} \int_x^{a+\beta(b-a)} \varphi(t)dt w_1(x)dx \\
&= \int_{a+\beta(b-a)}^b \int_{a+\beta(b-a)}^x \mu_1(t)\psi(t)dt \mu_2(x)w_2(x)dx - \int_a^{a+\beta(b-a)} \int_x^{a+\beta(b-a)} \mu_1(t)\psi(t)dt \mu_2(x)w_2(x)dx \\
&\geqslant \mu_1(a+\beta(b-a))\mu_2(a+\beta(b-a))\left(\int_{a+\beta(b-a)}^b \int_{a+\beta(b-a)}^x \psi(t)dt w_2(x)dx - \int_a^{a+\beta(b-a)} \int_x^{a+\beta(b-a)} \psi(t)dt w_2(x)dx\right)
\end{aligned}$$

From (35),

$$\int_a^b \int_a^x \psi(t)dt w_2(x)dx - \int_a^b w_2(x)dx \int_a^{a+\beta(b-a)} \psi(t)dt = 0$$

then

$$\int_{a+\beta(b-a)}^b \int_{a+\beta(b-a)}^x \psi(t)dt w_2(x)dx - \int_a^{a+\beta(b-a)} \int_x^{a+\beta(b-a)} \psi(t)dt w_2(x)dx = 0$$

So $\int_a^{a+\alpha(b-a)} \varphi(t)dt \geqslant \int_a^{a+\beta(b-a)} \varphi(t)dt \geqslant 0$, thus $\alpha \geqslant \beta$, that is $\Omega_{f,w_1} \geqslant \Omega_{g,w_2}$.
Let $M_{f,w}(X) = \gamma$, $M_{g,w_2}(X) = \delta$, from (34),

$$\int_a^{\gamma} \varphi(t)dt = \frac{\sum_{i=1}^n w_1(x_i)\int_a^{x_i}\varphi(t)dt}{\sum_{i=1}^n w_1(x_i)}, \quad \int_a^{\delta} \psi(t)dt = \frac{\sum_{i=1}^n w_2(x_i)\int_a^{x_i}\psi(t)dt}{\sum_{i=1}^n w_2(x_i)} \tag{36}$$

To prove $\gamma \geqslant \delta$, it only needs to prove that $\int_a^{\gamma} \varphi(t)dt \geqslant \int_a^{\delta} \varphi(t)dt$.

$$\int_a^{\gamma}\varphi(t)dt-\int_a^{\delta}\varphi(t)dt=\frac{\sum_{i=1}^n w_1(x_i)\int_a^{x_i}\varphi(t)dt}{\sum_{i=1}^n w_1(x_i)}-\int_a^{\delta}\varphi(t)dt=\frac{\sum_{i=1}^n w_1(x_i)\left(\int_a^{x_i}\varphi(t)dt-\int_a^{\delta}\varphi(t)dt\right)}{\sum_{i=1}^n w_1(x_i)}$$
$$=\frac{\sum_{\substack{1\leqslant i\leqslant n\\ x_i>\delta}} w_1(x_i)\int_{\delta}^{x_i}\varphi(t)dt-\sum_{\substack{1\leqslant i\leqslant n\\ x_i<\delta}} w_1(x_i)\int_{x_i}^{\delta}\varphi(t)dt}{\sum_{i=1}^n w_1(x_i)}$$
$$=\frac{\sum_{\substack{1\leqslant i\leqslant n\\ x_i>\delta}}\mu_2(x_i)w_2(x_i)\int_{\delta}^{x_i}\mu_1(t)\psi(t)dt-\sum_{\substack{1\leqslant i\leqslant n\\ x_i<\delta}}\mu_2(x_i)w_2(x_i)\int_{x_i}^{\delta}\mu_1(t)\psi(t)dt}{\sum_{i=1}^n w_1(x_i)}$$
$$\geqslant\frac{\mu_1(\delta)\mu_2(\delta)\left(\sum_{\substack{1\leqslant i\leqslant n\\ x_i>\delta}} w_2(x_i)\int_{\delta}^{x_i}\psi(t)dt-\sum_{\substack{1\leqslant i\leqslant n\\ x_i<\delta}} w_2(x_i)\int_{x_i}^{\delta}\psi(t)dt\right)}{\sum_{i=1}^n w(x_i)}$$

From (36),

$$\sum_{i=1}^n w_2(x_i)\int_a^{\delta}\psi(t)dt-\sum_{i=1}^n w_2(x_i)\int_a^{x_i}\psi(t)dt=0$$

that is

$$\sum_{\substack{1\leqslant i\leqslant n\\ x_i>\delta}} w_2(x_i)\int_{\delta}^{x_i}\psi(t)dt-\sum_{\substack{1\leqslant i\leqslant n\\ x_i<\delta}} w_2(x_i)\int_{x_i}^{\delta}\psi(t)dt=0$$

So $\int_a^{\gamma}\varphi(t)dt-\int_a^{\delta}\varphi(t)dt\geqslant 0$, thus $\gamma\geqslant\delta$, that is $M_{f,g}(X)\geqslant M_{p,q}(X)$. □

If $g(x)=x,\ w_2(x)=1$ in Theorem 11, then

$$\Omega_{g,w_2}=\frac{1}{2},\quad M_{g,w_2}(X)=A(X)=\frac{1}{n}\sum_{i=1}^n x_i$$

It can be obtained that

Corollary 9. *For Bajraktarević mean determined by quasi-arithmetic mean generator $f(x)$ and weighting function $w(x)$, respectively, if $f(x)$ is convex and $w(x)$ is decreasing, then $\Omega_{f,w}\geqslant\frac{1}{2}$, and $M_{f,w}(X)\geqslant\frac{1}{n}\sum_{i=1}^n x_i$, the operator is or-like; if $f(x)$ is concave and $w(x)$ is increasing, then $\Omega_{f,w}\leqslant\frac{1}{2}$, and $M_{f,w}(X)\leqslant\frac{1}{n}\sum_{i=1}^n x_i$, the operator is and-like.*

If the generating function is differential and positive, the premise of Theorem 11 can be replaced with a more simplified form.

Theorem 12. *For two Bajraktarević means determined by the function pairs $f(x),\ w_1(x)$ and $g(x),\ w_2(x)$, the quasi-arithmetic mean operators $f(x),\ g(x)$ having generating functions $\varphi(x)$ and $\psi(x)$, respectively, if for all $t\in[a,b]$, $\frac{\varphi'(t)}{\varphi(t)}\geqslant\frac{\psi'(t)}{\psi(t)}$ and for all $u\in[a,b]$, $\frac{w_1'(u)}{w_1(u)}\geqslant\frac{w_2'(u)}{w_2(u)}$, then $\Omega_{f,w_1}\geqslant\Omega_{g,w_2}$, and for any X, $M_{f,w}(X)\geqslant M_{g,w_2}(X)$.*

Proof. Similar to Theorem 6, omitted. □

Theorem 12 can be expressed without the generating function:

Corollary 10. *For a pair of Bajraktarević means determined by the quasi-arithmetic mean generator and the weighting function pairs $f(x),\ w_1(x)$ and $g(x),\ w_2(x)$, respectively, if for all $t\in[a,b]$, $\frac{f''(t)}{f'(t)}\geqslant\frac{g''(t)}{g'(t)}$ and for all $u\in[a,b]$, $\frac{w_1'(u)}{w_1(u)}\geqslant\frac{w_2'(u)}{w_2(u)}$, then $\Omega_{f,w_1}\geqslant\Omega_{g,w_2}$, and for any X, $M_{f,w_1}(X)\geqslant M_{g,w_2}(X)$.*

Similar to the quasi-OWA operator, with the convex relation between functions [3, p.49-50], Corollary 10 means that the more convex the quasi-arithmetic mean generator is and the more increasing the weighting function is, the bigger the orness level of the Bajraktarević mean and its aggregation value will be.

4.2. Two families of parameterized Bajraktarević means

From Corollary 10, the orness value of a Bajraktarević mean and the aggregation value are determined by the forms of the quasi-arithmetic mean function and the weighting function, respectively. With these properties, two parameterized Bajraktarević mean families with exponential functions and power functions are proposed, which can be regarded as another extension of the ordinary quasi-arithmetic mean case [26], and can also be compared with the quasi-OWA case in parallel. The contents will be introduced in a brief way.

4.2.1. Bajraktarević mean with exponential functions

From Corollary 10, for a Bajraktarević mean with quasi-arithmetic mean generator $f(x)$, the generating function of $f(x)$ is $\varphi(x)$, we can set $\frac{f''(x)}{f'(x)}$ and $\frac{w'(x)}{w(x)} = \mu$ as the parameter to make a Bajraktarević mean become and-like or or-like. Let $\frac{f''(x)}{f'(x)} = \lambda,\ \frac{w'(x)}{w(x)} = \mu$, and considering Theorem 9, we can get

$$f(x) = \begin{cases} e^{\lambda x} & \text{if } \lambda \neq 0 \\ x & \text{otherwise} \end{cases} \tag{37}$$

$$w(x) = e^{\mu x} \tag{38}$$

The Bajraktarević mean becomes

$$M_{f,w}(x_1, x_2, \ldots, x_n) = \begin{cases} \frac{1}{\lambda} \ln\left(\frac{\sum_{i=1}^n e^{(\lambda+\mu)x_i}}{\sum_{i=1}^n e^{\mu x_i}}\right) & \text{if } \lambda \neq 0 \\ \frac{\sum_{i=1}^n x_i e^{\mu x_i}}{\sum_{i=1}^n e^{\mu x_i}} & \text{if } \lambda = 0 \end{cases} \tag{39}$$

This is called the exponential function Bajraktarević mean. From (32) in Definition 11, the orness level with exponential functions (37) and (38) can be obtained:

$$\Omega_{f,w} = \begin{cases} \frac{1}{2} & \text{if } \lambda = \mu = 0 \\ \frac{1}{\lambda(b-a)} \ln\left(\frac{e^{\lambda(b-a)}-1}{\lambda(b-a)}\right) & \text{if } \mu = 0,\ \lambda \neq 0 \\ \frac{\mu(b-a)e^{\mu(b-a)} - e^{\mu(b-a)} + 1}{\mu(b-a)e^{\mu(b-a)} - \mu(b-a)} & \text{if } \mu \neq 0,\ \lambda = 0 \\ -\frac{1}{\mu(b-a)} \ln\left(\frac{\mu(b-a)}{e^{\mu(b-a)}-1}\right) & \text{if } \lambda = -\mu \neq 0 \\ \frac{1}{\lambda(b-a)} \ln\left(\frac{\mu(e^{(\lambda+\mu)(b-a)}-1)}{(\lambda+\mu)(e^{\mu(b-a)}-1)}\right) & \text{if } \lambda \neq 0,\ \mu \neq 0,\ \lambda+\mu \neq 0 \end{cases} \tag{40}$$

Remark 9. It can be verified that in (40), $\Omega_{f,w}$ is continuous for λ and μ.

Property 7. *For Bajraktarević mean with exponential function (37) and (38), $\Omega_{f,w}$ and $M_{f,w}(X)$ are monotonically increasing with both λ and μ.*

Proof. This can be obtained with Corollary 10 and the above deduction directly. □

From (40), the orness measure can be completely determined by $\lambda,\ \mu$ and $b-a$, which are the generating function changing slope parameters and the interval length, respectively. Furthermore, $\Omega_{f,w}$ is symmetrical for λ and μ, respectively, that is

Property 8. *For Bajraktarević mean with exponential functions, the orness level is only related with the aggregated elements distribution interval length $b-a$, but has no relationship with its start or end point. And $\Omega_{f,w}$ is a symmetrical function for its parameters $\lambda,\ \mu$, that is $\Omega_{f,w}(\lambda,\mu) + \Omega_{f,w}(-\lambda,-\mu) = 1$.*

Proof. This can be verified from (40) directly. □

Similarly, the symmetrical property means that for a Bajraktarević mean of exponential functions (37) and (38), the operator with $\lambda,\ \mu$ and $-\lambda,\ -\mu$ have complementary orness level α and $1-\alpha$, respectively.

The relationships between $\Omega_{f,w}$ and $\lambda,\ b-a$ for different μ are shown in Fig. 4.

From these figures, it can be observed that the orness level $\Omega_{f,w}$ increases with λ and μ, respectively, and the orness level $\Omega_{f,w}$ usually changes more flatly when μ is fixed than when λ is fixed.

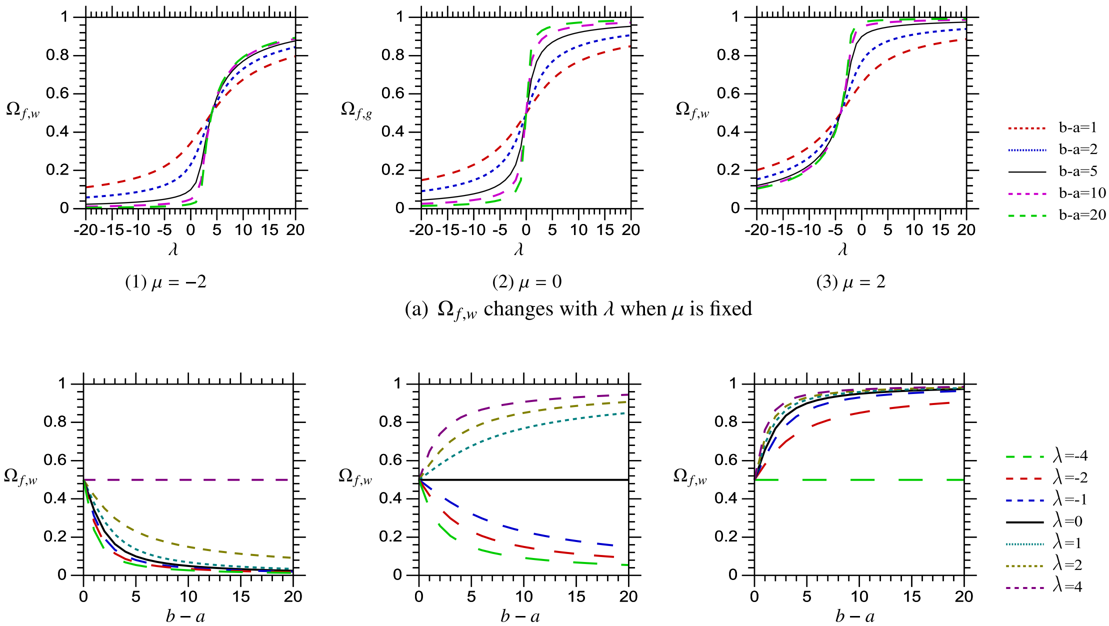
From Fig. 4 and (40), it can be observed that when $\lambda + 2\mu = 0,\ \Omega_{f,w} = \frac{1}{2}$, (39) becomes

$$M(X,\mu) = \begin{cases} -\frac{1}{2\mu} \ln\left(\frac{\sum_{i=1}^n e^{-\mu x_i}}{\sum_{i=1}^n e^{\mu x_i}}\right) & \text{if } \mu \neq 0 \\ \frac{\sum_{i=1}^n x_i}{n} & \text{if } \mu = 0 \end{cases} \tag{41}$$

We will call this the *generalized arithmetic mean* which always keeps orness level $\frac{1}{2}$. The function of parameter μ in (41) is shown in the following theorem.

Property 9. *$M(X,\mu)$ is symmetrical for μ, that is $M(X,\mu) = M(X,-\mu)$, and furthermore* $\lim_{\mu\to\pm\infty} M(X,\mu) = (\max_{1\leqslant i\leqslant n}\{x_i\} + \min_{1\leqslant i\leqslant n}\{x_i\})/2$.

Proof. With Theorem 8, we only need to prove the case of $\mu \to +\infty$. From (41), with the L'Hospital rule,



(1) $\mu = -2$ (2) $\mu = 0$ (3) $\mu = 2$

(a) $\Omega_{f,w}$ changes with λ when μ is fixed

(1) $\mu = -2$ (2) $\mu = 0$ (3) $\mu = 2$

(b) $\Omega_{f,w}$ changes with $b - a$ when μ is fixed

Fig. 4. The orness of exponential function Bajraktarević mean $\Omega_{f,w}$ changes when is μ fixed.

$$\lim_{\mu\to+\infty} M(X,\mu) = \lim_{\mu\to+\infty} -\frac{1}{2\mu}\ln\left(\frac{\sum_{i=1}^n e^{-\mu x_i}}{\sum_{i=1}^n e^{\mu x_i}}\right) = \lim_{\mu\to+\infty} -\frac{\ln\left(\sum_{i=1}^n e^{-\mu x_i}\right)}{2\mu} + \frac{\ln\left(\sum_{i=1}^n e^{\mu x_i}\right)}{2\mu} = \lim_{\mu\to+\infty} \frac{\sum_{i=1}^n x_i e^{-\mu x_i}}{2\sum_{i=1}^n e^{-\mu x_i}} + \frac{\sum_{i=1}^n x_i e^{\mu x_i}}{2\sum_{i=1}^n e^{\mu x_i}}$$
$$= \lim_{\mu\to+\infty} \frac{\sum_{i=1}^n x_i e^{-\mu(x_i - \min_{1\leqslant i\leqslant n}\{x_i\})}}{2\sum_{i=1}^n e^{-\mu(x_i - \min_{1\leqslant i\leqslant n}\{x_i\})}} + \frac{\sum_{i=1}^n x_i e^{\mu(x_i - \max_{1\leqslant i\leqslant n}\{x_i\})}}{2\sum_{i=1}^n e^{\mu(x_i - \min_{1\leqslant i\leqslant n}\{x_i\})}} = \frac{\max_{1\leqslant i\leqslant n}\{x_i\} + \min_{1\leqslant i\leqslant n}\{x_i\}}{2} \qquad \square$$

From Theorem 9, we can see that $|\mu|$ in (41) can be seen as an index of the emphasis on the uneven condition of $X = (x_1, x_2, \ldots, x_n)$, when $\mu = 0, M(X,\mu)$ becomes the arithmetic average of all the elements $\sum_{i=1}^n x_i/n$. And when $|\mu| \to +\infty$, it becomes the arithmetic average of two extreme points $(\max_{1\leqslant i\leqslant n}\{x_i\} + \min_{1\leqslant i\leqslant n}\{x_i\})/2$. If all the elements are distributed evenly, $M(x_1, x_2, \ldots, x_n, \mu)$ becomes a constant for all μ.

As an example, for $X = (2, 3, 7, 16)$, $\sum_{i=1}^4 x_i/4 = 7$, $(\max_{1\leqslant i\leqslant 4}\{x_i\} + \min_{1\leqslant i\leqslant 4}\{x_i\})/2 = 9$. Replacing μ with $-\mu$ in (41) for simplification, the aggregation value becomes

$$M(X,\mu) = \begin{cases} \frac{1}{2\mu}\ln\left(\frac{e^{2\mu}+e^{3\mu}+e^{7\mu}+e^{16\mu}}{e^{-2\mu}+e^{-3\mu}+e^{-7\mu}+e^{-16\mu}}\right) & \mu \neq 0 \\ 7 & \mu = 0 \end{cases}$$

The plot of $M(X,\mu)$ changing with μ is shown in Fig. 5.

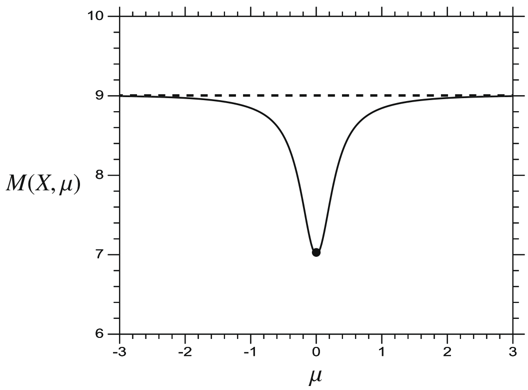


Fig. 5. $M(X,\mu)$ changes with μ of the generalized arithmetic mean example.

From (39), the shift invariant property of the quasi-arithmetic mean with exponential function can also be extended to the Bajraktarević mean, but the uniqueness cannot be proved in a simple way.

Property 10. *The Bajraktarević mean with exponential function forms is shift invariant, that is, if both* $f(x) = e^{\lambda x}$ $(\lambda \neq 0)$ *or* $f(x) = x$ *and* $w(x) = e^{\mu x}$*, then for any* $X = (x_1, \ldots, x_n)$ *and* c,

$$M_{f,w}(c + x_1, c + x_2, \ldots, c + x_n) = c + M_{f,w}(x_1, x_2, \ldots, x_n)$$

Remark 10. Similar to Property 3 in quasi-OWA operator, $f(x) = x$ can be seen as the complementary case of $f(x) = e^{\lambda x}$ $(\lambda \neq 0)$. According to Theorem 9, f and w can be replaced with $k_1 f + c$ and $k_2 w$, respectively $(k_1, k_2 \neq 0)$.

4.2.2. Bajraktarević mean with power functions

The Bajraktarević mean with power functions is another family of parameterized Bajraktarević mean changing consistently with its orness level parameters. But the aggregated elements should be the domain $\mathbb{R}^+$. If the weighting function $w(x) = 1$, this becomes the well known root-power mean operator.

Similar to the previous method, let $\frac{f''(x)}{f'(x)} = \frac{r-1}{x}$ and $\frac{w'(x)}{w(x)} = \frac{s}{x}$, and consider Theorem 9, then

$$f(x) = \begin{cases} x^r & \text{if } r \neq 0 \\ \ln(x) & \text{otherwise} \end{cases} \tag{42}$$

$$w(x) = x^s \tag{43}$$

To avoid $f(x)$ being meaningless, only the domain $[a, b]$ with $b > a > 0$ can be considered. That is the aggregated elements must be positive real numbers of $\mathbb{R}^+$ if the Bajraktarević mean with power function generators is applied.

The Bajraktarević mean becomes

$$M_{f,w}(x_1, x_2, \ldots, x_n) = \begin{cases} \left(\frac{\sum_{i=1}^{n} x_i^{r+s}}{\sum_{i=1}^{n} x_i^{s}} \right)^{\frac{1}{r}} & \text{if } r \neq 0 \\ \exp\left(\frac{\sum_{i=1}^{n} \ln(x_i) x_i^{s}}{\sum_{i=1}^{n} x_i^{s}} \right) & \text{otherwise} \end{cases} \tag{44}$$

where $x_i > 0$.

In this case, the orness value of this kind of Bajraktarević mean is

$$\Omega_{f,w} = \begin{cases} \frac{\left(\frac{\int_a^b x^{r+s} dx}{\int_a^b x^s dx} \right)^{\frac{1}{r}} - a}{b - a} & \text{if } r \neq 0 \\ \frac{e^{\frac{\int_a^b \ln(x) x^s dx}{\int_a^b x^s dx}} - a}{b - a} & \text{otherwise} \end{cases} \tag{45}$$

that is

$$\Omega_{f,w} = \begin{cases} \frac{\left(\frac{(b^{r+s+1} - a^{r+s+1})(s+1)}{(b^{s+1} - a^{s+1})(r+s+1)} \right)^{\frac{1}{r}} - a}{b - a} & \text{if } r \neq 0 \\ \frac{e^{\frac{b^{s+1}(\ln(b)s + \ln(b) - 1) - a^{s+1}(\ln(a)s + \ln(a) - 1)}{(s+1)(b^{s+1} - a^{s+1})}} - a}{b - a} & \text{otherwise} \end{cases}$$

If $a \neq 0$, it can be expressed as

$$\Omega_{f,w} = \begin{cases} \frac{\left(\frac{\left(\left(\frac{b}{a}\right)^{r+s+1} - 1 \right)(s+1)}{\left(\left(\frac{b}{a}\right)^{s+1} - 1 \right)(r+s+1)} \right)^{\frac{1}{r}} - 1}{\frac{b}{a} - 1} & \text{if } r \neq 0 \\ \frac{e^{\frac{\left(\frac{b}{a}\right)^{s+1} \left(\ln(\frac{b}{a}) s + \ln(\frac{b}{a}) - 1 \right) + 1}{(s+1)\left(\left(\frac{b}{a}\right)^{s+1} - 1 \right)}} - 1}{\frac{b}{a} - 1} & \text{otherwise} \end{cases} \tag{46}$$

In order to keep (42) be always meaningful on $[a, b]$, it should have $b > a > 0$. Similar to the case of quasi-OWA operator, we will consider the case of arguments in the domain $[0, 1]$ as in fuzzy logic. For the Bajraktarević mean with exponential function generator (37), a quasi-arithmetic mean which makes $\Omega_{f,w} \in (0, 1)$ can always be obtained. But for the Bajraktarević mean with power function generators (42) and (43), the generators $f(x)$ or $w(x)$ are meaningless at 0 for $r \leqslant 0$ or $s < 0$, only the case of $r > 0$ and $s \geqslant 0$ can be considered. With $a = 0,\ b = 1$, (45) becomes

$$\Omega_{f,w} = \left(\frac{s + 1}{r + s + 1} \right)^{\frac{1}{r}}, \quad r > 0,\ s \geqslant 0 \tag{47}$$

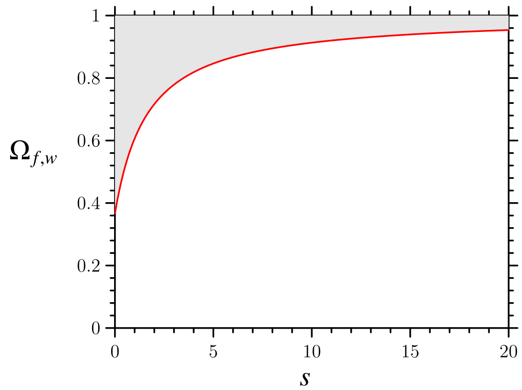


Fig. 6. The feasible value of $\Omega_{f,w}$ for $[0,1]$.

From (47), for $r \in (0,+\infty)$, $\Omega_{f,w}$ can only range in $(e^{-1/(1+s)},1)$. The feasible area of $\Omega_{f,w}$ is shown in Fig. 6. This means there does not exist Bajraktarević mean with power functions (42) and (43) of orness level $\Omega_{f,w} < e^{-1/(1+s)}$. Suppose that we want to get Bajraktarević mean with $\Omega_{f,w} = \delta < e^{-1/(1+s)}$ for certain s. With the dual property of Theorem 10, as the technique used in [24, p. 71], we can first get a function $u(x) = x^r$, $v(x) = x^s$ with $\Omega_{u,v} = 1-\delta$, then $f(x) = u(1-x)$, $w(x) = v(1-x)$ will be the needed function generators.

An alternative way is that, if these arguments are nonzero and a lower bound $a > 0$ of them can be obtained, then by considering the interval $[a,1]$ rather than $[0,1]$, (42) will be meaningful for any $r \in \mathbb{R}$, (45) becomes

$$\Omega_{f,w} = \begin{cases} \dfrac{\left(\frac{(1-a^{r+s+1})(s+1)}{(1-a^{s+1})(r+s+1)}\right)^{\frac{1}{r}} - a}{1-a} & \text{if } r \neq 0 \\ \dfrac{e^{\frac{(-1-a^{s+1})(\ln(a)s+\ln(a)-1)}{(s+1)(1-a^{s+1})}} - a}{1-a} & \text{otherwise} \end{cases}$$

From the discussions above, for any $\delta \in (0,1)$, with fixed s, the value of r for (42) can always be obtained on the aggregated elements distribution interval $[a,1]$, which makes $\Omega_{f,w} = \delta$.

Property 11. *For Bajraktarević mean with power functions, the orness level is only related with the start and end points ratio b/a, but has no relationship with its start or end point.*

Proof. This can be seen from (46) directly. □

The relationships between the orness value $\Omega_{f,w}$ and r, b/a for different s are plotted in Fig. 7.

It can be observed that, unlike the Bajraktarević mean with exponential functions, the orness level of the Bajraktarević mean with power functions is not symmetrical for its parameter r and s.

Similar to the properties of Bajraktarević mean with exponential functions, it also has

Property 12. *For Bajraktarević mean with power functions (42) and (43), $\Omega_{f,w}$ and $M_{f,w}(X)$ are monotonically increasing for both r and s.*

Proof. Similar to Theorem 7, this can be obtained with Corollary 10 directly. □

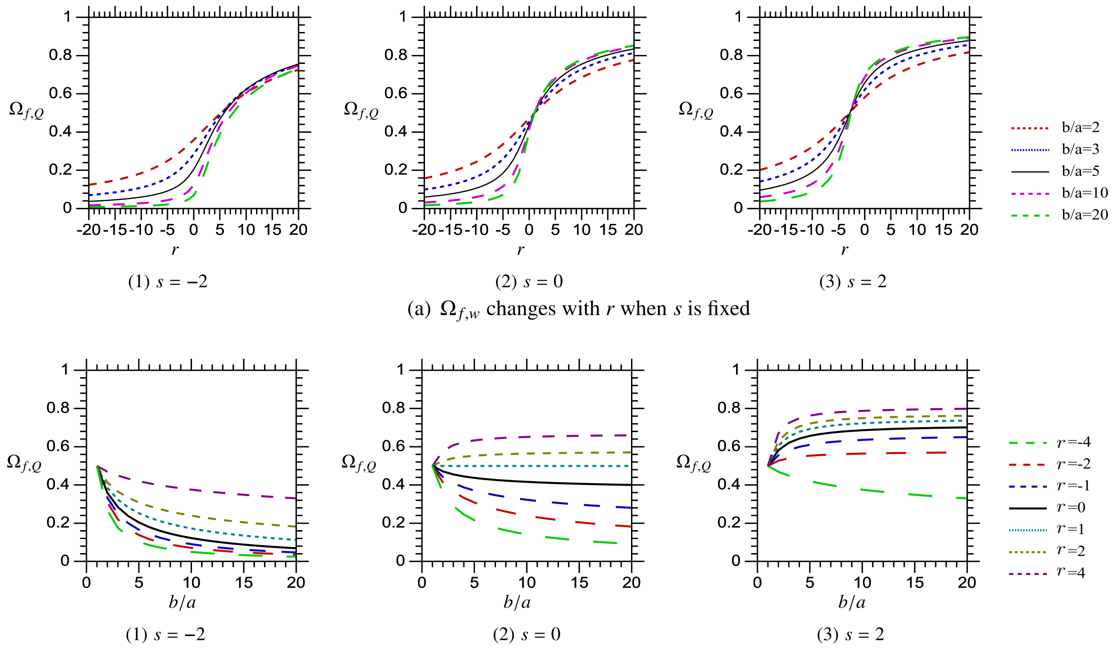
From (44), the Bajraktarević mean with power function quasi-arithmetic mean generator is a parameterized generalized mean of scale invariance.

Property 13. *The Bajraktarević mean with power function forms is scale invariant, that is if $f(x) = x^r$ $(r \neq 0)$ or $f(x) = \ln x$ and $w(x) = x^s$, then for any $X = (x_1,\ldots,x_n)$ $(x_i > 0,\ i = 1,2,\ldots,n)$ and $c > 0$,*

$$M_{f,w}(cx_1, cx_2, \ldots, cx_n) = cM_{f,w}(x_1, x_2, \ldots, x_n)$$

Remark 11. Similar to Property 6 in quasi-OWA operator, $f(x) = \ln x$ can be seen as the complementary case of $f(x) = x^r$ $(r \neq 0)$. According to Theorem 9, f and w can be replaced with $k_1 f + c$ and $k_2 w$, respectively $(k_1, k_2 \neq 0)$.

The comparisons of the exponential function and power function Bajraktarević means are very similar to that of the quasi-OWA operator. A very interesting point is that for exponential function Bajraktarević mean, there is a very simple expression of the extended arithmetic mean (41) to keep the orness level 0.5, but such principle is not found in the power function Bajraktarević mean. Furthermore, from Theorems 7 and 12, these Bajraktarević mean operator determination methods can also be extended to some more general forms such as $\frac{f''(x)}{f'(x)} = K_1(x)$ and $\frac{w'(x)}{w(x)} = K_2(x)$ instead of the two special cases with $K_1(x) = \lambda$ and $K_2(x) = \mu$ or $K_1(x) = \frac{r-1}{x}$ and $K_2(x) = \frac{s-1}{x}$.



(1) $s = -2$ (2) $s = 0$ (3) $s = 2$

(a) $\Omega_{f,w}$ changes with r when s is fixed

(1) $s = -2$ (2) $s = 0$ (3) $s = 2$

(b) $\Omega_{f,w}$ changes with $b - a$ when s is fixed

Fig. 7. The orness of exponential function Bajraktarević mean $\Omega_{f,w}$ changes when s is fixed.

5. Comparisons and some discussions

By considering the conclusions of the quasi-OWA operator and the Bajraktarević mean together, as two extension forms of the quasi-arithmetic mean, both of them have some interesting properties. Here, we will try to summarize and give a comparison of them:

1. The orness measure is an important property of the aggregation operator. For aggregation operator determination problems, we often need to derive the parameter of aggregation operator yielding a desired orness. The paper gives the definitions of two compound quasi-arithmetic mean operators, where the analytical expression can always be obtained. They can avoid the multiple integral computation of the general orness concept, which makes it possible to obtain the orness formula for some commonly used function expressions. By deriving the parameter of aggregation operator from such analytical expressions, we can determine the aggregation operator if the orness value is given.
2. Both of these two kinds of orness definitions satisfy the common properties which other aggregation operator orness have: standardization, uniformity, duality and consistency. The orness level of these aggregation operators is determined by both the function forms and the interval in which aggregation elements distributed. In a general sense, for a continuous function-based average operator $\mathscr{F}$ on $[a,b]$, with the uniformity property, the orness of a general average aggregation operator $\mathscr{F}$ can be defined as:
$$orness(\mathscr{F}) = \frac{\lim_{n\to+\infty}\mathscr{F}(X) - a}{b-a} \tag{48}$$
where $X = (x_1, x_2, \ldots, x_n)$ distributes evenly on $[a,b]$ that $x_i = a + \frac{n-i}{n-1}(b-a)$.The orness definitions of the RIM quantifier, the quasi-arithmetic mean, the weighted function average operator, the quasi-OWA operator with RIM quantifier and the Bajraktarević mean can all be derived from this expression. The standardization property can be partially derived from the bound characteristic of the average operator. This is also the reason that why (6) and (12) are very different at first glance, but they obey the same principle. Here, we just give the orness definitions for the two forms of quasi-arithmetic extension. With (48), the method in this paper can be extended to other function-based simple or compound aggregation operators.
3. The common characteristics of the orness measure still hold in these two types of compound aggregation operators. Both of these aggregation operators can be seen as the extensions of the quasi-arithmetic mean. For quasi-OWA operator, the more convex the quasi-arithmetic mean generator is and the more concave the RIM quantifier membership function is, the bigger the orness level of the quasi-OWA operator and its aggregation value will be. But for Bajraktarević mean, the more convex the quasi-arithmetic mean generator is and the more increasing the weighting function is, the bigger the orness level of the quasi-OWA operator and its aggregation value will be. These characteristics not only verify the rationality of these orness definitions, but also make it possible to be extended to other aggregation operators. With these

characteristics, and the general expression (48), we can also consider the axiomatic systems for the orness concept in some general way.

4. Both the function generators in the quasi-OWA operator and the Bajraktarević mean can be set in the exponential and power function forms, respectively. With these aggregation operators, the aggregation value for any set changes consistently with the orness level. The aggregation operators with exponential function are shift invariant and their orness values are symmetrical to their parameters. The aggregation operators with power function are ratio invariant, but usually are not symmetrical to their parameters despite that they are commonly used in applications. For quasi-OWA operator, the uniqueness of shift invariance and ratio invariance for the exponential function and power function operator are verified. But the uniqueness property is not proved in the Bajraktarević mean.
5. From these discussions, we can also see that the quasi-arithmetic mean generator, the RIM quantifier, and the weighting function play different roles in the aggregation process. For quasi-OWA operator, the RIM quantifier (OWA operator) determines the basic orness level and the quasi-arithmetic mean can be seen as a fine tuning tool. The orness levels usually have a common intersection point if the RIM quantifier is fixed. Such phenomenon does not hold for the Bajraktarević mean, although the parameter in weighting function has more influence than that in the quasi-arithmetic mean, and they usually have a common intersection point with orness level 0.5.
6. The common properties of the quasi-OWA operator and the Bajraktarević mean make us consider the aggregation operators in a more generic form

$$M_{f,\mathbf{g}}(x_1, x_2, \ldots, x_n) = f^{-1}(\mathbf{g}(f(x_1), f(x_2), \ldots, f(x_n)))$$

where $\mathbf{g}$ can be some average operator or the ordinary aggregation operator which can include the OWA operator or the weighted function average as special cases.

6. Conclusions

The paper first summarizes the orness measures of the four types of averaging operators: the quasi-arithmetic mean, the OWA operator, the RIM quantifier, and the weighted function average operator, respectively. They all can be seen as the parameterized extension of the minimum, the arithmetic average, and maximum. Some common characteristics of these orness measures can be observed. They include the standardization, uniformity, duality and consistency properties.

Then, the paper proposes the orness measures for two compound forms of the quasi-arithmetic mean: the quasi-OWA operator and the Bajraktarević mean. With the generating function technique, some properties of these orness measures are discussed. Two kinds of parameterized quasi-OWA operators and Bajraktarević means with exponential functions and power functions are proposed, respectively. The exponential function quasi-OWA operator and Bajraktarević mean are symmetrical for their parameters. They are shift invariant. One can easily get its complementary part with given orness level. The power function quasi-OWA operator can be seen as the extension of the generalized OWA operator, while the power function Bajraktarević mean can be seen as the extension of the commonly used power root means and the weighted function average method. They are ratio invariant. The relationships and some extensions on the orness expressions of these aggregation operators are also discussed.

As the researches on the compound operators are relatively sparse due to their complicated forms, the results of this paper should be useful for the further studies on these compound operators and other averaging aggregation operators both in theories and applications.

Acknowledgements

The author is very grateful to the Editor and the anonymous reviewers for their valuable comments and suggestions that improve the previous versions of this paper.

The author is also especially grateful to Professor Jozo J. Dujmović, Professor JanaŠpirková, Professor Ricardo Alberto Marques Pereira, Professor Rita Almeida Ribeiro and Professor Henrik Legind Larsen for their valuable help of giving some of their research papers.

The author is thankful to Miss Yuwen Pan and Mr. Ming Liu for their many valuable comments in the writing of this paper.

References

[1] G. Beliakov, How to build aggregation operators from data, International Journal of Intelligent Systems 18 (8) (2003) 903–923.
[2] G. Beliakov, A. Pradera, T. Calvo, Aggregation Functions: A Guide for Practitioners, Springer, Berlin/Heidelberg, 2007.
[3] P.S. Bullen, Handbook of Means and Their Inequalities, Kluwer Academic Publishers, 2003.
[4] T. Calvo, A. Kolesárová, M. Komorníková, R. Mesiar, Aggregation operators: properties, classes and construction methods, in: T. Calvo, G. Mayor, R. Mesiar (Eds.), Aggregation Operators: New Trends and Applications, Physica-Verlag, Heilderberg/New York, 2002.
[5] T. Calvo, G. Mayor, R. Mesiar (Eds.), Aggregation Operators: New Trends and Applications, Physica-Verlag, Heilderberg/New York, 2002.
[6] T. Calvo, G. Mayor, J. Torrens, J. Suner, M. Mas, M. Carbonell, Generation of weighting triangles associated with aggregation functions, International Journal of Uncertainty Fuzziness and Knowledge-Based Systems 8 (4) (2000) 417–451.
[7] T. Calvo, R. Mesiar, Aggregation operators: ordering and bounds, Fuzzy Sets and Systems 139 (3) (2003) 685–697.
[8] J. Chiang, Aggregating membership values by a Choquet-fuzzy-integral based operator, Fuzzy Sets and Systems 114 (3) (2000) 367–375.

[9] D. Dubois, H. Prade, A review of fuzzy set aggregation connectives, Information Sciences 36 (1) (1985) 85–121.
[10] J.J. Dujmović, Weighted conjunctive and disjunctive means and their application in system evaluation, Journal of the University of Belgrade, EE Department, Series Mathematics and Physics 483 (1974) 147–158.
[11] J.J. Dujmović, Seven flavors of andness/orness, in: B.D. Baets, J. Fodor, D. Radojević (Eds.), Proceedings of Eurofuse 2005, Belgrade, 2005.
[12] J.J. Dujmović, A comparison of andness/orness indicators, in: Proceedings of the 11th Information Processing and Management of Uncertainty international conference (IPMU), Paris, 2006.
[13] J.J. Dujmović, H.L. Larsen, Properties and modeling of partial conjunction/disjunction, in: B.D. Baets et al. (Eds.), Current Issues in Data and Knowledge Engineering, Warsaw, 2004.
[14] J.J. Dujmović, H.L. Larsen, Generalized conjunction/disjunction, International Journal of Approximate Reasoning 46 (3) (2007) 423–446.
[15] J. Fodor, J.L. Marichal, M. Roubens, Characterization of the ordered weighted averaging operators, IEEE Transactions on Fuzzy Systems 3 (2) (1995) 236–240.
[16] J.C. Fodor, M. Roubens, Fuzzy Preference Modelling and Multicriteria Decision Support, Kluwer, Dordrecht, 1994.
[17] L. Godo, V. Torra, On aggregation operators for ordinal qualitative information, IEEE Transactions on Fuzzy Systems 8 (2) (2000) 143–154.
[18] M. Grabisch, Fuzzy integral in multicriteria decision making, Fuzzy Sets and Systems 69 (3) (1995) 279–298.
[19] F. Herrera, E. Herrera-Viedma, F. Chiclana, A study of the origin and uses of the ordered weighted geometric operator in multicriteria decision making, International Journal of Intelligent Systems 18 (6) (2003) 689–707.
[20] E. Herrera-Viedma, O. Cordón, M. Luque, A.G. Lopez, A.M. Muñoz, A model of fuzzy linguistic IRS based on multi-granular linguistic information, International Journal of Approximate Reasoning 34 (2–3) (2003) 221–239.
[21] J. Kacprzyk, S. Zadrozny, Computing with words in intelligent database querying: standalone and internet-based applications, Information Sciences 134 (1) (2001) 71–109.
[22] A. Kolesárová, Limit properties of quasi-arithmetic means, Fuzzy Sets and Systems 124 (1) (2001) 65–71.
[23] A. Kolesárová, R. Mesiar, Parametric characterization of aggregation functions, Fuzzy Sets and Systems 160 (6) (2009) 816–831.
[24] H.L. Larsen, Efficient andness-directed importance weighted averaging operators, International Journal of Uncertainty Fuzziness and Knowledge-Based Systems 11 (Suppl.) (2003) 67–82.
[25] X. Liu, On the properties of equidifferent RIM quantifier with generating function, International Journal of General Systems 34 (5) (2005) 579–594.
[26] X. Liu, An orness measure for quasi-arithmetic means, IEEE Transactions on Fuzzy Systems 14 (6) (2006) 837–848.
[27] X. Liu, Some properties of the weighted OWA operator, IEEE Transactions on Systems, Man and Cybernetics, Part B 36 (1) (2006) 118–127.
[28] X. Liu, On the properties of regular increasing monotone RIM quantifiers with maximum entropy, International Journal of General Systems 37 (2) (2008) 167–179.
[29] X. Liu, L. Chen, On the properties of parametric geometric OWA operator, International Journal of Approximate Reasoning 35 (2) (2004) 163–178.
[30] X. Liu, H. Lou, Parameterized additive neat OWA operators with different orness levels, International Journal of Intelligent Systems 21 (10) (2006) 1045–1072.
[31] J.L. Marichal, Aggregation operators for multicriteria decision aid, Ph.D. Thesis, Institute of Mathematics, University of Liège, Liège, Belgium, 1998.
[32] J.L. Marichal, Behavioral analysis of aggregation in multicriteria decision aid, in: J. Fodor, B. de Baets, P. Perny (Eds.), Preferences and Decisions under Incomplete Knowledge, Studies in Fuzziness and Soft Computing, vol. 51, Physica Verlag, Heidelberg, 2000.
[33] J.L. Marichal, Tolerant or intolerant character of interacting criteria in aggregation by the Choquet integral, European Journal of Operational Research 155 (3) (2004) 771–791.
[34] M. Marimin, M. Umano, I. Hatono, H. Tamura, Linguistic labels for expressing fuzzy preference relations in fuzzy group decision making, IEEE Transactions on Systems, Man and Cybernetics, Part B 28 (2) (1998) 205–218.
[35] M. Marimin, M. Umano, I. Hatono, H. Tamura, Hierarchical semi-numeric method for pairwise fuzzy group decision making, IEEE Transactions on Systems, Man and Cybernetics, Part B 32 (5) (2002) 691–700.
[36] R. Mesiar, J. Spirkova, Weighted means and weighting functions, Kybernetika 42 (2) (2006) 151–160.
[37] V. Peneva, I. Popchev, Properties of the aggregation operators related with fuzzy relations, Fuzzy Sets and Systems 139 (3) (2003) 615–633.
[38] R. Pereira, R. Ribeiro, Aggregation with generalized mixture operators using weighting functions, Fuzzy Sets and Systems 137 (1) (2003) 43–58.
[39] R.A.M. Pereira, The orness of mixture operators: the exponential case, in: Proceedings of the Eighth International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU2000), vol. 2, Madrid, Spain, 2000.
[40] J.M.F. Salido, S. Murakami, Extending Yager's orness concept for the OWA aggregators to other mean operators, Fuzzy Sets and Systems 139 (3) (2003) 515–542.
[41] V. Torra, Learning weights for the quasi-weighted means, IEEE Transactions on Fuzzy Systems 10 (5) (2002) 653–666.
[42] V. Torra, OWA operators in data modeling and reidentification, IEEE Transactions on Fuzzy Systems 12 (5) (2004) 652–660.
[43] V. Torra, Y. Narukawa, Modeling Decisions: Information Fusion and Aggregation Operators, Springer, Berlin/Heidelberg, 2007.
[44] J.S. Ume, Y.H. Kim, Some mean values related to the quasi-arithmetic mean, Journal of Mathematical Analysis and Applications 252 (1) (2000) 167–176.
[45] Z.S. Xu, Q.L. Da, The ordered weighted geometric averaging operators, International Journal of Intelligent Systems 17 (7) (2002) 709–716.
[46] R.R. Yager, On ordered weighted averaging aggregation operators in multicriteria decision making, IEEE Transactions on Systems, Man and Cybernetics 18 (1) (1988) 183–190.
[47] R.R. Yager, Families of OWA operators, Fuzzy Sets and Systems 59 (2) (1993) 125–143.
[48] R.R. Yager, Quantifier guided aggregation using OWA operators, International Journal of Intelligent Systems 11 (1) (1996) 49–73.
[49] R.R. Yager, Generalized OWA aggregation operators, Fuzzy Optimization and Decision Making 3 (1) (2004) 93–107.
[50] R.R. Yager, On the retranslation process in Zadeh's paradigm of computing with words, IEEE Transactions on Systems, Man and Cybernetics, Part B 34 (2) (2004) 1184–1195.
[51] R.R. Yager, OWA aggregation over a continuous interval argument with applications to decision making, IEEE Transactions on Systems, Man and Cybernetics, Part B 34 (5) (2004) 1952–1963.
[52] R.R. Yager, D.P. Filev, Parameterized and-like and or-like OWA operators, International Journal of General Systems 22 (3) (1994) 297–316.
[53] R.R. Yager, J. Kacprzyk, The Ordered Weighted Averaging Operators—Theory and Applications, Kluwer Academic Publishers, 1997.