SIT787: Mathematics for AI
Practical Week 3

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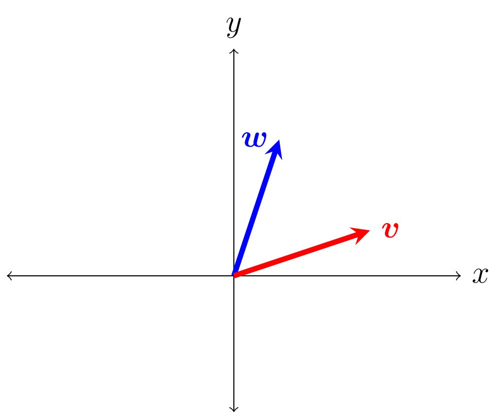
1. For $\boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, describe all points $c\boldsymbol{v}$ with
   - $c \in \mathbb{I}$ an integer number
   - nonnegative real number $c \geq 0$

2. Find scalars c and d so that the linear combination $c\boldsymbol{v} + d\boldsymbol{w}$ equals $\boldsymbol{b}$:

$$\boldsymbol{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

3. draw all combinations of $c\boldsymbol{v} + d\boldsymbol{w}$
   - Restricted by $c \geq 0$ and $d \geq 0$
   - Restricted by $0 \leq c \leq 1$ and $0 \leq d \leq 1$



4. For the vectors

$$\boldsymbol{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

   - Test the Schwarz inequality: $|\boldsymbol{v} \cdot \boldsymbol{w}| \leq ||\boldsymbol{v}|| ||\boldsymbol{w}||$
   - Test the Triangle inequality: $||\boldsymbol{v} + \boldsymbol{w}|| \leq ||\boldsymbol{v}|| + ||\boldsymbol{w}||$
   - Find the cosine of angle between them: $\cos\theta = \frac{\boldsymbol{v} \cdot \boldsymbol{w}}{||\boldsymbol{v}|| ||\boldsymbol{w}||}$
   - Knowing that for every angle θ, $|\cos\theta| \leq 1$, can you show the Schwarz inequality?

5. Describe the vector space spanned by these three vectors and state a subspace of this vector space.

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad \boldsymbol{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

6. Show that

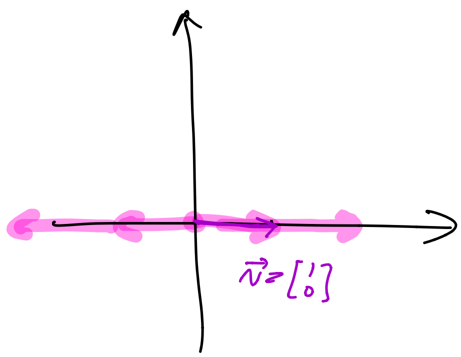
$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}$$

- are linearly independent, hence, form a basis for $\mathbb{R}^3$.
- Are the vectors mutually orthogonal?
- Use Gram-Schmidt process and orthognalise them.

1. For $\boldsymbol{v} = \begin{bmatrix}1\\0\end{bmatrix} \in \mathbb{R}^2$, describe all points $c\boldsymbol{v}$ with

- $c \in \mathbb{Z}$ an integer number
- nonnegative real number $c \geq 0$

$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$



$0\vec{v} = 0\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$1\vec{v} = 1\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$

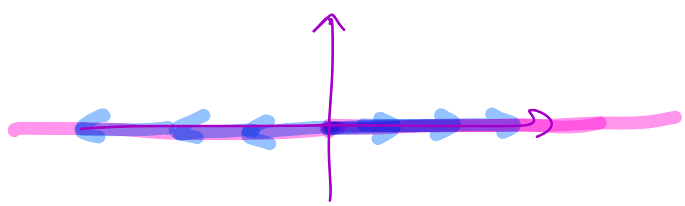
$(-1)\vec{v} = (-1)\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$

$(2)\vec{v} = \begin{bmatrix}2\\0\end{bmatrix}$ $\quad (-2)\vec{v} = \begin{bmatrix}-2\\0\end{bmatrix}$

$\begin{bmatrix}1/2\\0\end{bmatrix}$ $\quad \begin{bmatrix}3/2\\0\end{bmatrix}$

$\begin{bmatrix}c\\0\end{bmatrix}$ $\quad c \in \mathbb{Z}$

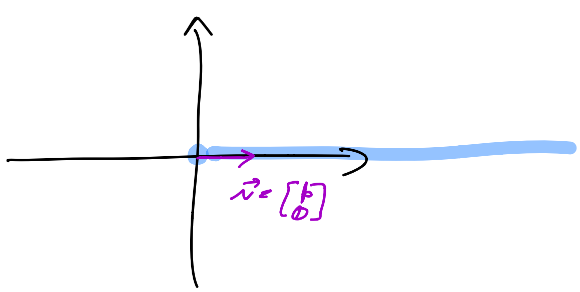
$\begin{bmatrix}c\\0\end{bmatrix}$ $\quad c \in \mathbb{R}$



$\vec{v} = \begin{bmatrix}1\\0\end{bmatrix}$ $\quad c \geq 0$

$c = 0$ $\quad \begin{bmatrix}0\\0\end{bmatrix}$

$c\vec{v} = \begin{bmatrix}c\\0\end{bmatrix}$ $\quad c \geq 0$ $\quad \begin{bmatrix}c\\0\end{bmatrix}$



2. Find scalars c and d so that the linear combination $c\boldsymbol{v} + d\boldsymbol{w}$ equals $\boldsymbol{b}$:

$$\boldsymbol{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$c\vec{v} + d\vec{w} = \vec{b}$$

$$\begin{bmatrix} 2c \\ -c \end{bmatrix} + \begin{bmatrix} -d \\ 2d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2c - d \\ -c + 2d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow \begin{cases} 2c - d = 1 \\ -c + 2d = 0 \end{cases} \longrightarrow \boxed{c = 2d}$$

plug $c = 2d$ in the 1st equation

$$2(2d) - d = 1 \longrightarrow 4d - d = 1 \longrightarrow 3d = 1 \longrightarrow \boxed{d = 1/3}$$

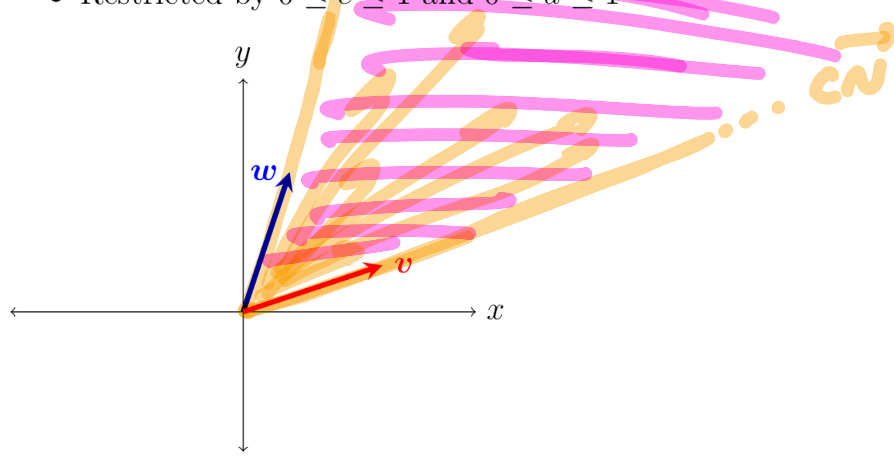
$$c = 2d = c = 2(1/3) = 2/3$$

$$\underbrace{\left(\frac{2}{3}\right)\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \left(\frac{1}{3}\right)\begin{bmatrix} -1 \\ 2 \end{bmatrix}} = \begin{bmatrix} 4/3 \\ -2/3 \end{bmatrix} + \begin{bmatrix} -1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 4/3 - 1/3 \\ -2/3 + 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 3/3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

3. draw all combinations of $c\boldsymbol{v} + d\boldsymbol{w}$

- Restricted by $c \geq 0$ and $d \geq 0$
- Restricted by $0 \leq c \leq 1$ and $0 \leq d \leq 1$



3. draw all combinations of $c\boldsymbol{v} + d\boldsymbol{w}$

- Restricted by $c \geq 0$ and $d \geq 0$
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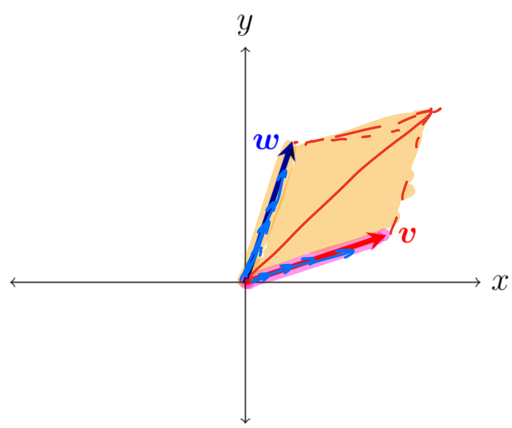

4. For the vectors

$$\boldsymbol{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

- Test the Schwarz inequality: $|\boldsymbol{v} \cdot \boldsymbol{w}| \leq ||\boldsymbol{v}||||\boldsymbol{w}||$ ✓
- Test the Triangle inequality: $||\boldsymbol{v} + \boldsymbol{w}|| \leq ||\boldsymbol{v}|| + ||\boldsymbol{w}||$ ✓
- Find the cosine of angle between them: $\cos\theta = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{||\boldsymbol{v}||||\boldsymbol{w}||}$
- Knowing that for every angle θ, $|\cos\theta| \leq 1$, can you show the Schwarz inequality?

$$|\vec{v} \cdot \vec{w}| \leq ||\vec{v}|| \, ||\vec{w}||$$

$$\vec{v} \cdot \vec{w} = (3)(4) + (4)(3) = 12 + 12 = 24$$

$$||\vec{v}|| = \sqrt{(3)^2 + (4)^2} = \sqrt{9+16} = \sqrt{25} = 5$$

$$||\vec{w}|| = \sqrt{(4)^2 + (3)^2} = \sqrt{16+9} = 5$$

$$|24| \leq (5)(5)$$

$$24 \leq 25 \quad ✓$$

$$\vec{v} + \vec{w} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

$$||\vec{v} + \vec{w}|| = \sqrt{(7)^2 + (7)^2} = \sqrt{49+49} = \sqrt{98} \simeq 9.8$$

$$||\vec{v} + \vec{w}|| \leq ||\vec{v}|| + ||\vec{w}||$$

$$9.8 \leq 5 + 5$$

$$9.8 \leq 10 \quad ✓$$

$$\cos(\theta) = \frac{\vec{v} \cdot \vec{w}}{||\vec{v}|| \, ||\vec{w}||} = \frac{24}{(5)(5)} = \frac{24}{25}$$

$$|\vec{v} \cdot \vec{w}| \leq ||\vec{v}|| \, ||\vec{w}||$$

check the lecture notes.

5. Describe the vector space spanned by these three vectors and state a subspace of this vector space.

$$\boldsymbol{v} = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \qquad \boldsymbol{u} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \quad \in \mathbb{R}^4$$

$$\text{span}(\vec{v}, \vec{w}, \vec{u}) = \{ c_1\vec{v} + c_2\vec{w} + c_3\vec{u} \mid c_1, c_2, c_3 \in \mathbb{R} \}$$

$$c_1\vec{v} + c_2\vec{w} + c_3\vec{u} = c_1\begin{bmatrix}1\\1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}1\\1\\1\\0\end{bmatrix} + c_3\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}c_1+c_2+c_3\\c_1+c_2+c_3\\c_2+c_3\\c_3\end{bmatrix}$$

$$\text{span}(\vec{v}, \vec{w}, \vec{u}) = \left\{ \begin{bmatrix}c_1+c_2+c_3\\c_1+c_2+c_3\\c_2+c_3\\c_3\end{bmatrix} \middle| \; c_1, c_2, c_3 \in \mathbb{R} \right\}$$

$$c_1 = 0, \; c_2 = 0, \; c_3 = 0 \longrightarrow \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \in \text{Span}(\vec{u}, \vec{v}, \vec{w})$$

6. Show that

$$\begin{array}{ccc} \vec{v_1} & \vec{v_2} & \vec{v_3} \end{array}$$

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}$$

- A. are linearly independent, hence, form a basis for $\mathbb{R}^3$.
- B. Are the vectors mutually orthogonal?
- C. Use Gram-Schmidt process and orthognalise them.

(A) To show that a set of vectors $\{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$ is linearly independent, we should show that if

$$c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_k\vec{v_k} = \vec{0},$$

then this implies that $c_1 = c_2 = \cdots = c_k = 0$. Otherwise, the set is linearly dependent.

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 + c_3 \\ c_1 + 2c_2 \\ c_1 + 3c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{cases} c_1 + c_3 = 0 & (1) \\ c_1 + 2c_2 = 0 & (2) \\ c_1 + 3c_3 = 0 & (3) \end{cases}$$

from equation (1): $c_1 = -c_3$

plug this in equation (3): $(-c_3) + 3c_3 = 0 \longrightarrow 2c_3 = 0 \longrightarrow c_3 = 0$

if $c_3 = 0$, then $c_1 = 0$. Plugging these in equation (2) we have $c_2 = 0$. We conclude that $c_1 = c_2 = c_3 = 0$.

Then, the vectors in the set are linearly independent. We know that every basis for $\mathbb{R}^3$, should have three independent vectors. We have three independent vectors, so the set is a basis for $\mathbb{R}^3$.

(B)

Now, we need to check whether the vectors are mutually orthogonal. The set of vectors

$\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_k}\}$ is called mutually orthogonal

if $\quad \vec{v_i} \cdot \vec{v_j} = 0 \quad$ for all $i \neq j$.

For our set of vectors, we have

$$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{v_2} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \quad \vec{v_3} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\vec{v_1} \cdot \vec{v_2} = (1)(0) + (1)(2) + (1)(0) = 2 \neq 0$$

$$\vec{v_1} \cdot \vec{v_3} = (1)(1) + (1)(0) + (1)(3) = 4 \neq 0$$

$$\vec{v_2} \cdot \vec{v_3} = (0)(1) + (2)(0) + (0)(3) = 0$$

only $\vec{v_2} \perp \vec{v_3}$. So this set is not mutually orthogonal.

(c) Now we want to convert these vectors into a new set of vectors $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ which are mutually orthogonal. This will happen through the Gram-Schmidt process. Then we normalise these vectors to get $\{\hat{u}_1, \hat{u}_2, \hat{u}_3\}$.

$$\vec{u}_1 = \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{u}_2 = \vec{v}_2 - \text{proj}_{\vec{u}_1}^{\vec{v}_2}$$

$$\text{proj}_{\vec{u}_1}^{\vec{v}_2} = \left(\frac{\vec{u}_1 \cdot \vec{v}_2}{\vec{u}_1 \cdot \vec{u}_1}\right)\vec{u}_1 = \left(\frac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}\right)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \vec{u}_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 4/3 \\ -2/3 \end{bmatrix}$$

$$\vec{u}_3 = \vec{v}_3 - \text{proj}_{\vec{u}_1}^{\vec{v}_3} - \text{proj}_{\vec{u}_2}^{\vec{v}_3}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - \left(\frac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}\right)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{\begin{bmatrix} -2/3 \\ 4/3 \\ -2/3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}}{\begin{bmatrix} -2/3 \\ 4/3 \\ -2/3 \end{bmatrix} \cdot \begin{bmatrix} -2/3 \\ 4/3 \\ -2/3 \end{bmatrix}}\right)\begin{bmatrix} -2/3 \\ 4/3 \\ -2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - \left(\frac{4}{3}\right)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{-8/3}{8/3}\right)\begin{bmatrix} -2/3 \\ 4/3 \\ -2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 4/3 \\ 4/3 \\ 4/3 \end{bmatrix} + \begin{bmatrix} 2/3 \\ 4/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 1 - 4/3 - 2/3 \\ -4/3 + 4/3 \\ 3 - 4/3 - 2/3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

The new set of vectors

$$\left\{ \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ \vec{u}_2 = \begin{bmatrix} -2/3 \\ 4/3 \\ -2/3 \end{bmatrix}, \ \vec{u}_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

These are mutually orthogonal.

$$\vec{u}_1 \bullet \vec{u}_2 = 0 \qquad \vec{u}_1 \bullet \vec{u}_3 = 0 \qquad \vec{u}_2 \bullet \vec{u}_3 = 0$$

Now, we convert each vector into a unit vector:

$$\hat{u}_1 = \frac{1}{\|\vec{u}_1\|} \vec{u}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \qquad \|\vec{u}_1\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

$$\hat{u}_2 = \frac{1}{\|\vec{u}_2\|} \vec{u}_2 = \frac{1}{\sqrt{24}/3} \begin{bmatrix} -2/3 \\ 4/3 \\ -2/3 \end{bmatrix} = \frac{3}{\sqrt{24}} \begin{bmatrix} -2/3 \\ 4/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} -2/\sqrt{24} \\ 4/\sqrt{24} \\ -2/\sqrt{24} \end{bmatrix}$$

$$\hat{u}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

Now the set of vectors

$$\left\{ \hat{u}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \ \hat{u}_2 = \begin{bmatrix} -2/\sqrt{24} \\ 4/\sqrt{24} \\ -2/\sqrt{24} \end{bmatrix}, \ \vec{u}_3 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right\}$$

is a orthonormal set.

This is a very particular basis for $\mathbb{R}^3$

1. They are independent

$$\left.\begin{array}{l}\text{(2) They are orthogonal} \\ \text{(3) They are unit vectors}\end{array}\right\} \text{orthonormal.}$$