SIT787: Mathematics for Artificial Intelligence
Topic 2: Linear Algebra
Part 2

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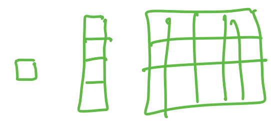
scalars $\longrightarrow$ vectors $\longrightarrow$ $\boxed{\text{matrices}}$

vector spaces
linear independence
basis, dimension
Gram-Schmidt process



Matrices

- A matrix with m rows and n columns

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n}$$

i^{th} row of A

$A \begin{cases} A_i \;,\; A_{i*} \\ A^j \;,\; A_{*j} \end{cases}$

j^{th} column of A

- rows $A_i = \left[a_{i1}, a_{i2}, \ldots, a_{in}\right]$ a row vector
- columns $A^j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$ a column vector
- entries a_{ij} you need two indices to access an entry

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix}_{2\times 3}$$

$$A_1 = \begin{bmatrix} 3 & -1 & 0 \end{bmatrix} = A_{1*}$$

$$A_2 = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} = A_{2*}$$

$$A^1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = A_{*1} \qquad A^2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = A_{*2} \qquad A^3 = \begin{bmatrix} 0 \\ 2 \end{bmatrix} = A_{*3}$$

$a_{22} = 1 \qquad a_{13} = 0$

$A_{m\times n}$
- row i
- col j
- entry a_{ij}

Special Matrices

$\vec{v} \in \mathbb{R}^n$ $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

- A column vector is a $n \times 1$ matrix

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}_{n \times 1}$$

$\vec{0}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

- A row vector is a $1 \times n$ matrix

$$[v_1, \ldots, v_n]_{1 \times n}$$

- Zero matrix: all entries are zero.

$$\mathbf{0} = \mathbf{0}_{\underbrace{m}_{\text{row}} \times \underbrace{n}_{\text{col}}} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$A_{m\times n}\begin{cases} m=n & A_{n\times n} \quad \text{square matrix} \\ m\neq n & A_{m\times n} \quad \text{rectangular matrix}\end{cases}$$

- if $m \neq n$, the matrix is rectangular.
- Square matrices: when the number of rows is the same as the number of columns.
  - they have the main diagonal

$$A_{3\times 3} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 7 & 1 & 2 \end{bmatrix}_{3\times 3}$$

$$\mathbb{R}^n \quad B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$$

- Identity Matrix

$$I_{m\times m} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I_m = I$$

$$I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \dots$$

Equal matrices

equal vectors $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$ $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}_{n \times 1}$ $\vec{x} = \vec{y}$ if $\begin{cases} x_1 = y_1 \\ x_2 = y_2 \\ \vdots \\ x_n = y_n \end{cases}$

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad B_{m\times n} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

- $A = B$ if
  - they have the same number of rows and columns
  - for every i and j, $a_{ij} = b_{ij}$
- This applies to vectors as well.

Operations in Matrices: Addition and subtraction

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad B_{m\times n} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m3} & \cdots & a_{mn} + b_{mn} \end{bmatrix}_{m\times n}$$

- If $A_{m\times n}$ and $B_{m\times n}$, then $\boxed{\underset{m\times n}{C} = A + B}$ is a $m \times n$ natrix and

$$c_{ij} = a_{ij} + b_{ij}$$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 3 \end{bmatrix}_{3\times 2} \qquad B = \begin{bmatrix} 1 & 7 & 5 \\ 2 & 1 & 0 \end{bmatrix}_{2\times 3}$$

$$C = \begin{bmatrix} 7 & -6 \\ 0 & -2 \\ 1 & 1 \end{bmatrix}_{3\times 2}$$

$A + B$ not defined.

$$A + C = \begin{bmatrix} 1+7 & 1+(-6) \\ 2+0 & 1+(-2) \\ 0+1 & 3+1 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 8 & -5 \\ 2 & -1 \\ 1 & 4 \end{bmatrix}$$

Operations in Matrices: Transpose

$[\,]^T \to [\,—\,]$

- if $A = [a]$ is a 1×1 matrix, then $A^T = A$
- The transpoose of a row vector is a column vectors:
  $A_{1\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$

$$A^T = \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \end{bmatrix}_{n \times 1}$$

row

column

- The transpoose of a column vector is a row vectors:

$$A_{m\times 1} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \qquad A^T = \underline{\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \end{bmatrix}}$$

- If A is a $m \times n$ matrix, itstranspose is a $n \times m$ matrix
- rows of A will become columns of A^T

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}_{3\times 2} \quad A^T = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \end{bmatrix}_{2\times 3} \quad (A^T)^T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$$

tall

wide

Operations in Matrices: Scalar Multiplication

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ $c \in \mathbb{R}$ $\quad c\vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}$

$$A_{\underbrace{m}_{\text{row}} \times \underbrace{n}_{\text{col}}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$cA = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix}_{m \times n}$$

- if $A_{m \times n}$, and $c = 0$, then $0.A = \mathbf{0}_{m \times n}$

$\vec{u}, \vec{v} \in \mathbb{R}^n$ $\quad$ $\vec{u} \cdot \vec{v} \in \mathbb{R}$

- Consider $A_{m\times n}$ and $B_{n\times p}$. The product between A and B is a $\boxed{m \times p}$ matrix:

$$C_{m\times p} = AB$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$c_{ij} = A_i \cdot B^j = A_i^T \cdot B^j$$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{2\times 3} \qquad B = \begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 1 & 2 & 5 \end{bmatrix}_{2\times 4}$$

$C = AB$ this is not defined.

$$D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}_{3\times 3}$$

$$AD \longrightarrow 2\times 3 \qquad \begin{bmatrix} 3 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$2\times 3 \quad 3\times 3$

① check dimension compatibility

add $\longrightarrow$ same dimens

product $\longrightarrow$ $m\times n \quad n\times p$

Week 4

vector $\rightarrow$ what we do : linear combinations

$\vec{u}, \vec{v}, \vec{w}$ $\quad c_1\vec{u} + c_2\vec{v} + c_3\vec{w}$ $\quad c_1, c_2, c_3 \in \mathbb{R}$

$2\vec{u} + (-1)\vec{v} + 3\vec{w}$ particular lin. comb

all possible linear combinations of $\vec{u}, \vec{v}, \vec{w}$

$= \{ c_1\vec{u} + c_2\vec{v} + c_3\vec{w} \mid c_1, c_2, c_3 \in \mathbb{R} \} = \text{span}(\{\vec{u}, \vec{v}, \vec{w}\})$

a vector space : a big set of vectors

V is a vector space

it has a basis B a set of linearly independent vectors

$|B| = \dim(V) = n$ $\quad B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$

$\text{span}(B) = V$

any vector in V : $\vec{y} \in V$ $\quad \vec{y} = \boxed{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n}$

many Basis for a vector space

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consider $A = \{\vec{u}, \vec{v}, \vec{w}\}$

$\boxed{\text{span}(A)} = \{ c_1\vec{u} + c_2\vec{v} + c_3\vec{w} \mid c_1, c_2, c_3 \in \mathbb{R} \}$

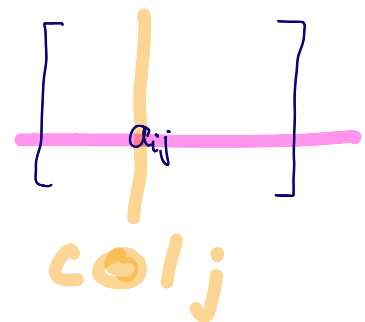
is a vector space. $\quad \vec{x}, \vec{y} \in V$

$\Rightarrow \vec{x} + \vec{y} \in V$

$\Rightarrow c\vec{x} \in V \quad c \in \mathbb{R}$

e.g. $c\vec{x} + d\vec{y} \in V \quad c, d \in \mathbb{R}$

Matrices: $\underbrace{A}_{}{}_{\underset{\text{rows}}{m} \times \underset{\text{cols}}{n}} = [a_{ij}]$



$A \in \underbrace{\mathbb{R}^{m\times n}}_{\text{the set of all } m\times n \text{ matrices.}}$

Some operations on matrices.

$$C_{m\times n} = A_{m\times n} \pm B_{m\times n}$$

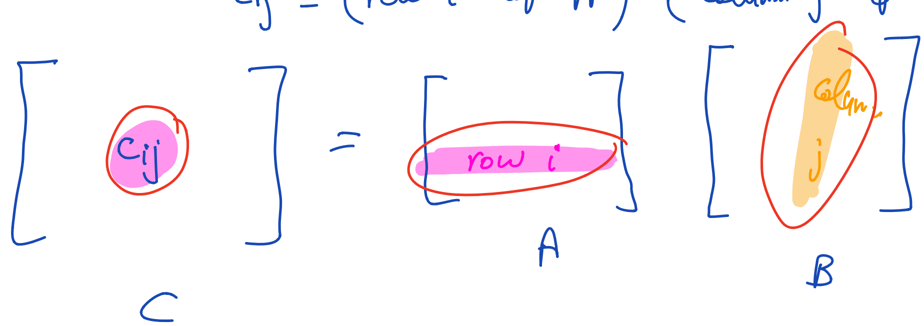
$\alpha A_{m\times n}$ $\quad \alpha \in \mathbb{R}$ $\quad$ scalar multiplication

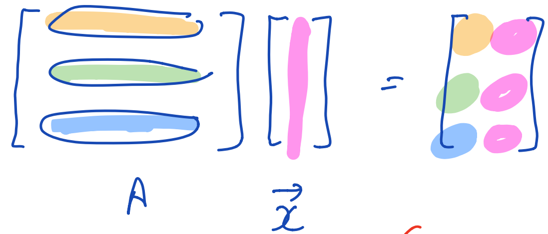
$\mathbb{R}^{m\times n}$ the set of all $m\times n$ matrices is a vector space.

product between two matrix

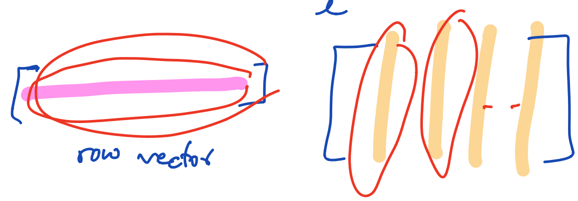
$$C_{m\times p} = A_{m\times n} B_{n\times p}$$

$$c_{ij} = (\text{row } i \text{ of } A)\cdot(\text{column } j \text{ of } B)$$



$A\vec{x} =$ 

$m \times n \rightarrow n \times 1$

$\vec{y}\,A_{m \times n} =$ 

$1 \times m$

Inner and outer products in matrix form

dot

- Consider these two vectors

$\vec{u} \bullet \vec{v} = \sum_{i=1}^{n} u_i v_i$

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}_{n\times 1} \text{ and } \boldsymbol{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}_{n\times 1} \in \mathbb{R}^n$$

$\vec{u}_{n\times 1}$

$\underset{n\times 1}{\vec{u}} \bullet \underset{n\times 1}{\vec{v}}$

- They are $n \times 1$ matrices
- $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}^T \boldsymbol{v} = \sum_{i=1}^{n} u_i v_i$, the result is a number

$\underbrace{\underset{1\times n}{\vec{u}^T}\ \underset{n\times 1}{\vec{v}}}_{1\times 1}$

- Matrix representation of inner product

$\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$

$\boxed{\vec{u}^T \vec{v} = \vec{v}^T \vec{u}} = \sum_{i=1}^{n} u_i v_i$

$$\underbrace{\boldsymbol{u} \cdot \boldsymbol{v}}_{n\times 1, n\times 1} = \underbrace{\boldsymbol{u}^T}_{1\times n} \underbrace{\boldsymbol{v}}_{n\times 1}$$

scalar

$\|\vec{u}\| = \sqrt{\vec{u} \bullet \vec{u}}$

$\|\vec{u}\|^2 = \vec{u} \bullet \vec{u}$

$= \vec{u}^T \vec{u}$

- Outer product: the result is a matrix

$(\vec{v}_{n\times 1})^T_{1\times n}$

$\vec{u}\ \vec{v}^T$

$(n\times 1\ \ 1\times n)\ \boxed{n\times n}$

$$\underbrace{\underbrace{\boldsymbol{u}}_{n\times 1} \underbrace{\boldsymbol{v}^T}_{1\times n}}$$

$\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad$ $\vec{v} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

dot (inner) product: $\vec{u}^T \vec{v} = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$= (1)(0) + (2)(3) = 0 + 6$

$= 6$

outer product

$\underbrace{\vec{u}\ \vec{v}^T}_{\substack{2\times1 \quad 1\times2 \\ 2\times2}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 0 & 3 \end{bmatrix} = \begin{bmatrix} (1)(0) & (1)(3) \\ (2)(0) & (2)(3) \end{bmatrix}_{2\times2}$

$= \begin{bmatrix} 0 & 3 \\ 0 & 6 \end{bmatrix}_{2\times2}$

- Consider $A_{m\times n}$ and $B_{n\times p}$. The product between A and B is a $m\times p$ matrix:

$$C_{m\times p} = AB$$

C A B

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

2x3 3x2

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = A^1B_1 + A^2B_2 + A^3B_3$$

2x2

$$= \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \end{bmatrix} + \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} \begin{bmatrix} b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \begin{bmatrix} b_{31} & b_{32} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} \\ a_{21}b_{11} & a_{21}b_{12} \end{bmatrix} + \begin{bmatrix} a_{12}b_{21} & a_{12}b_{22} \\ a_{22}b_{21} & a_{22}b_{22} \end{bmatrix} + \begin{bmatrix} a_{13}b_{31} & a_{13}b_{32} \\ a_{23}b_{31} & a_{23}b_{32} \end{bmatrix}$$

$C = AB$

- using inner product — high school
- using outer product → new

$A_{m \times n} = \begin{bmatrix} A^1 & A^2 & \cdots & A^n \end{bmatrix}$

$B_{n \times p} = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix}$

$$C = M_1 + M_2 + \cdots + M_n$$

$$C = A^1 B_1 + A^2 B_2 + \cdots + A^n B_n$$

Matrix Multiplication is not Commutative

$C = AB$

- Consider $A_{m\times n}$ and $B_{n\times p}$.
  - the product is only defined if the number of column in the first matrix is the same as the niumber of rows in the second matrix
- It is possible AB is defined but BA is not
  - $A_{2\times 3}$ and $B_{3\times 5}$
- even if AB and BA are defined, they may not be equal.
- Example:

$\underbrace{\underset{2\times 3}{A}\ \underset{3\times 5}{B}}_{2\times 5}$ $\quad \cancel{\underset{3\times 5}{B}\ \underset{2\times 3}{A}}$

$C = AB$ $\qquad \underset{n\times p}{B}\ \underset{m\times n}{A}$

$\vec{u} + \vec{v} = \vec{v} + \vec{u}$

$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

commutative

In general

$AB \neq BA$

Properties of Matrix Operations

$AB \neq BA$

- $A + B = B + A$
- $A + (B + C) = (A + B) + C$ $= A + B + C$
- $c(A + B) = cA + cB$
- $(c_1 c_2)A = c_1(c_2 A)$
- $A(BC) = (AB)C$ associative ABC
- $(A + B)C = AC + BC$, and $A(B + C) = AB + AC$
- $c(AB) = (cA)B = A(cB)$
- $\underset{m \times n}{A} + \underset{m \times n}{\mathbf{0}} = \underset{m \times n}{A}$
- $A\mathbf{0} = \mathbf{0}A = \mathbf{0}$
- $AI = IA = A$
- $cA = (cI)A$

$$[\;]^T \to [\quad] \qquad [\quad]^T = [\;]$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix}$$

2x3 3x2

$$\boxed{\|\vec{x}\|_2^2 = \vec{x}^T \vec{x}}$$

- $(A+B)^T = A^T + B^T$
- $(A^T)^T = A$
- $(AB)^T = B^T A^T$
  - $(ABC)^T = C^T B^T A^T$
  - $(A_1 A_2 \dots A_{n-1} A_n)^T = A_n^T A_{n-1}^T \dots A_2^T A_1^T$

$\vec{u}, \vec{v}$

$$\|\vec{u}-\vec{v}\|^2 = \sum_{i=1}^{n} (u_i - v_i)^2 = (\vec{u}-\vec{v})^T (\vec{u}-\vec{v})$$

$$= (\vec{u}^T - \vec{v}^T)(\vec{u}-\vec{v})$$

$$= \vec{u}^T\vec{u} - \vec{u}^T\vec{v} - \vec{v}^T\vec{u} + \vec{v}^T\vec{v}$$

$$= \|\vec{u}\|^2 - 2\vec{u}^T\vec{v} + \|\vec{v}\|^2$$

$$\boxed{\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\vec{u}^T\vec{v}}$$

square
$n \times n$

- If $A = A^T$ the matrix is called symmetric, or $a_{ij} = a_{ji}$
- if $A = -A^T$ the matrix is called anti-symmetric, or $a_{ij} = -a_{ji}$
- skew-symmetric, antisymmetric, or antimetric
- Show that the main diagonal of an anti-symmetric matrix is zero.
- example symmetric

$$S = \begin{bmatrix} 1 & 7 & 3 \\ 7 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix}, \quad S^T = \begin{bmatrix} 1 & 7 & 3 \\ 7 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix} = S$$

3×3

- example anti-symmetric

$$A = \begin{bmatrix} 0 & 2 & -4 \\ -2 & 0 & -4 \\ 4 & 4 & 0 \end{bmatrix}, \quad A^T = \begin{bmatrix} 0 & -2 & 4 \\ 2 & 0 & 4 \\ -4 & -4 & 0 \end{bmatrix} = -A$$

Triangular matrices

square

- Lower and upper triangular matrices

$$L_{n\times n} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad U_{n\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

- Diagonal

$$D_{n\times n} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

$$= \mathrm{diag}(a_{11}, a_{22}, \ldots, a_{nn})$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 2 & 3 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$L^T \rightsquigarrow$ upper tri

Matrix and its row and column vectors

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in \underline{\mathbb{R}^{m\times n}}$$

rows cols

- Columns of $A = \{A^1, A^2, \ldots, A^n\}$

$$\left\{ \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \right\}$$

- Rows of $A = \{A_1, A_2, \ldots, A_m\}$

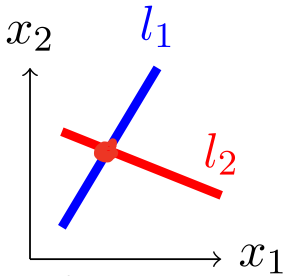
$$\{\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \end{bmatrix}, \ldots, \begin{bmatrix} a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}\}$$

- Having these vector independent is important.

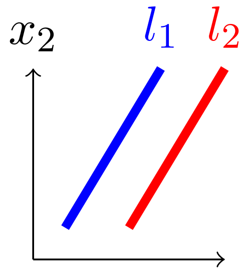
Linear System of two equations

- System of 2 equations and 2 unknowns

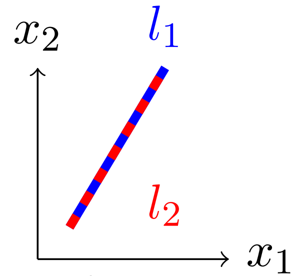
$$\begin{cases} a_1x_1 + b_1x_2 = c_1 \dashrightarrow (l_1) \\ a_2x_1 + b_2x_2 = c_2 \dashrightarrow (l_2) \end{cases}$$



Definite system
single solution



Inconsistent system
no solution



Indefinite system
many solutions

- System of 3 equations and 3 unknowns

$$\begin{cases} a_1x_1 + b_1x_2 + c_1x_3 = d_1 \dashrightarrow (\text{plane } 1) \\ a_2x_1 + b_2x_2 + c_2x_3 = d_2 \dashrightarrow (\text{plane } 2) \\ a_3x_1 + b_3x_2 + c_3x_3 = d_3 \dashrightarrow (\text{plane } 3) \end{cases}$$

$$(1)\begin{cases} x_1 + 2x_2 = 3 \\ 2x_1 - x_2 = 1 \end{cases} \qquad \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$(2)\begin{cases} x_1 + 2x_2 = 3 \\ 0x_1 - 5x_2 = -5 \end{cases} \qquad \begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

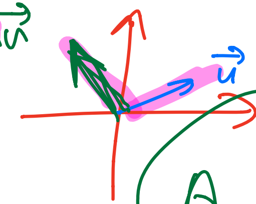
$$(3)\begin{cases} x_1 + \overset{0}{0x_2} = 1 \\ 0x_1 + x_2 = 1 \end{cases} \quad \begin{cases} x_1 = 1 \\ x_2 = 1 \end{cases} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

AB $m \times n$ $n \times p$ $A\vec{x}$ $m \times n$ $n \times 1$ col

$A\vec{x} \to$ new vector

$A_{2\times3}\vec{x}_{3\times1}$ $_{2\times1}\begin{bmatrix}0\\0\end{bmatrix}$ $A_{3\times3}\vec{x}_{3\times1}$ $\begin{bmatrix}0\\0\\0\end{bmatrix}_{3\times1}$ $A\vec{u}$

$A\vec{u}$



$A_{2\times2}[\ ]_{2\times1}$ 2×1

- Matrices are considered as transfomations
- A matrix A applies to a vector x and creates Ax vector
- Ax is a vector. It is a linear combination of columns of matrix A Ax $A\vec{x}$
- $Ax = b$ is a system of linear equations $A\vec{x} = \vec{b}$
  system
  - We want to see if b can be represented as a linear combination of columns of A
  - if this is possible, we say that the system has a solution.
  - Otherwise, the system does not have solution

$A_{m\times n} = [A^1\ A^2 \cdots A^n]$ $\vec{x}_{n\times1} = \begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}$

$A\vec{x} = [A^1\ A^2 \cdots A^n]\vec{x} = x_1A^1 + x_2A^2 + \cdots + x_nA^n$

$$\begin{cases} x + y = 3 \\ x - 3y = 1 \end{cases}$$

$$\underbrace{\underset{A}{\begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix}}_{2\times 2} \underset{\vec{v}}{\begin{bmatrix} x \\ y \end{bmatrix}}_{2\times 1} = \underset{\vec{b}}{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{2\times 1}} \equiv \underbrace{\boxed{A\vec{v}} = \vec{b}}$$

$$\rightarrow \begin{bmatrix} x + 2y \\ x - 3y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \rightarrow \begin{cases} x + 2y = 3 \\ x - 3y = 1 \end{cases}$$

$$A\vec{v} = \vec{b}$$

$$x\begin{bmatrix} 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ x \end{bmatrix} + \begin{bmatrix} 2y \\ -3y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x + 2y \\ x - 3y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \rightarrow \begin{cases} x + 2y = 3 \\ x - 3y = 1 \end{cases}$$

$$A\vec{x} = \vec{b}$$

column view: by solving this system, can we write $\vec{b}$ as a linear combination of the columns of A.

Matrix vector product Examples

- Consider these vectors

$$A = [\vec{u} \;\; \vec{v}] = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \qquad \boldsymbol{u} = \underset{2\times 1}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}, \boldsymbol{v} = \underset{2\times 1}{\begin{bmatrix} -1 \\ 1 \end{bmatrix}} \in \mathbb{R}^2$$

- A linear combination $\overset{\in\mathbb{R}}{x_1}\boldsymbol{u} + \overset{\in\mathbb{R}}{x_2}\boldsymbol{v}$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \boxed{x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}} = \begin{bmatrix} x_1 - x_2 \\ 2x_1 + x_2 \end{bmatrix}$$

- Matrix representation

$$A\boldsymbol{x} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ 2x_1 + x_2 \end{bmatrix}$$

Matrix vector product: row view and column view

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n}, \text{ and } \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} n\times 1$$

$$A\boldsymbol{x} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix}$$

Matrix vector product: from row perspective

$$\overset{A}{\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}_{2\times 2}} \overset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1}} = \overset{\vec{b}}{\begin{bmatrix} 1 \\ 5 \end{bmatrix}_{2\times 1}}$$

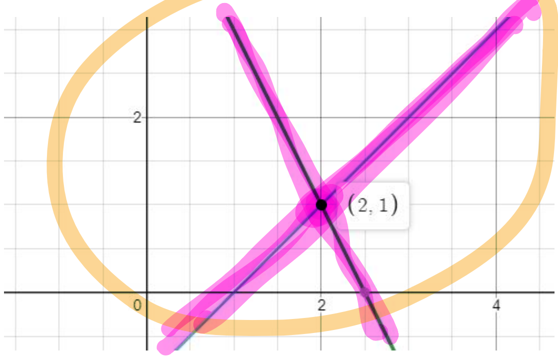
$$\underbrace{\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}_{2\times 2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1}}_{A\vec{x}} = \begin{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \\ \begin{bmatrix} 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \end{bmatrix} = \boxed{\begin{bmatrix} x_1 - x_2 \\ 2x_1 + x_2 \end{bmatrix}}$$

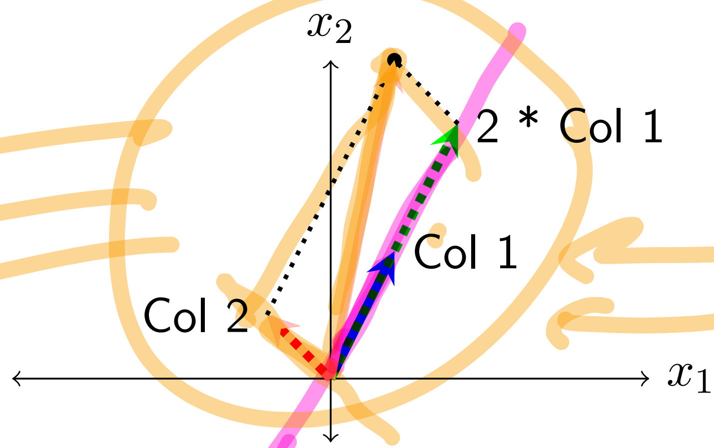
a linear combination of cols of A

Matrix vector product: row view and column view

$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \equiv \begin{cases} x_1 - x_2 = 1 \\ 2x_1 + x_2 = 5 \end{cases} \equiv x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

- Row view: Two lines that cut each other
- Columns view: a combination of columns that gives the right hand side





A system of linear equations

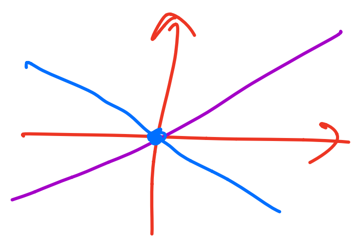
- The column picture of $A\boldsymbol{x} = \boldsymbol{b}$: A combination of n columns of A produces $\boldsymbol{b}$.
- if $A = \begin{bmatrix}\boldsymbol{a}_1 \ldots \boldsymbol{a}_n\end{bmatrix}$:

$$A\boldsymbol{x} = x_1\boldsymbol{a}_1 + \ldots + x_n\boldsymbol{a}_n = \boldsymbol{b}$$

- When $\boldsymbol{b} = \boldsymbol{0}$, one possibility is $\boldsymbol{x} = \begin{bmatrix}0 & 0 & \ldots & 0\end{bmatrix}^T$
- The row picture: m equations from m rows give m lines, planes, or hyperlanes meeting at $\boldsymbol{x}$.
- When $\boldsymbol{b} = \boldsymbol{0}$, all the lines, planes, or hyperlanes go through the origin $(0, 0, \ldots, 0)$.

$A\vec{x} = \begin{bmatrix}0\\0\end{bmatrix}$

2x2

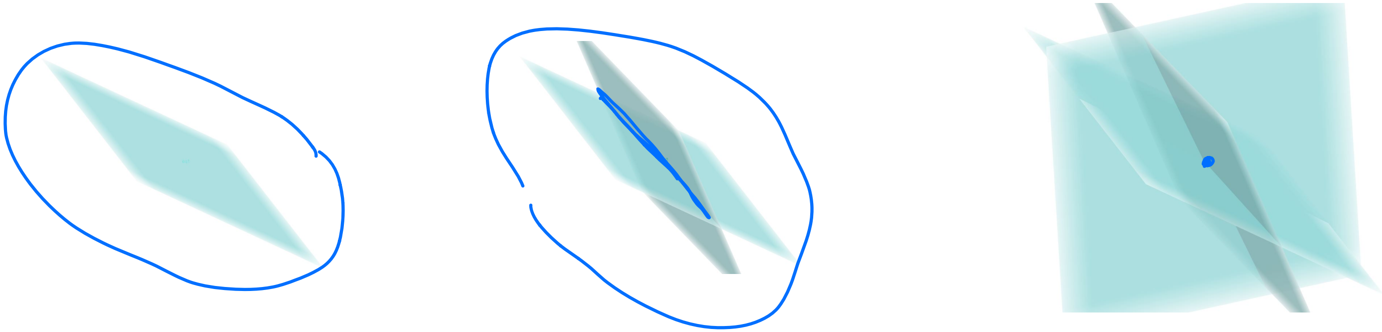


row picture

$$\boxed{A\boldsymbol{x} = \boldsymbol{b}}: \quad \begin{cases} x + 2y + 3z = 6 \\ 2x + 5y + 2z = 4 \\ 6x - 3y + z = 2 \end{cases}$$

plane

- The row picture with solution $(x, y, z) = (0, 0, 2)$



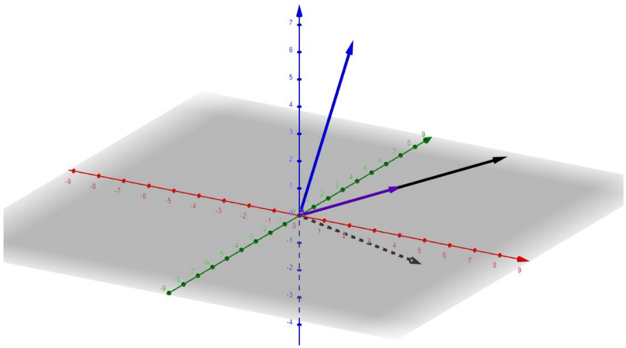
Three equations with three unkowns

$$A\boldsymbol{x} = \boldsymbol{b}: \qquad \begin{cases} x + 2y + 3z = 6 \\ 2x + 5y + 2z = 4 \\ 6x - 3y + z = 2 \end{cases}$$

- The column picture with solution $(x, y, z) = (0, 0, 2)$

$$x \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + y \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} + z \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

$$0 \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$



Useful but harmless operations

elementary row operations

$$\begin{cases} 2x + 5y = 3 \\ x - y = 2 \end{cases}$$

- Interchange two equations
- Multiply each element in an equation by a non-zero number
- Multiply an equation by a non-zero number and add the result to another equation.

Solving a system: the idea of Gaussian elimination

- Consider $m = n = 3$

$$A\boldsymbol{x} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \text{ or } \begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

- aim: eliminate x_1 from the second equation
  - Multiply the first equation by $\frac{a_{21}}{a_{11}}$ and subtract it from the second
  - then x_1 eliminated from the second equation.
- The entry a_{11} is called the first **pivot** and the ratio $\frac{a_{21}}{a_{11}}$ is called the first **multiplier**.

$\neq 0$

- aim: eliminate x_1 from the i^{th} equation
  - Multiply the first equation by $\frac{a_{i1}}{a_{11}}$ and subtract it from the i^{th}
  - then x_1 eliminated from the i^{th} equation.

row 2 $\leftarrow$ (row 2) $-$ $\left(\frac{3}{1}\right)$ (row 1)

- Before

$3(x - 2y = 1)$

$3x - 6y = 3$

$$\begin{cases} 1x - 2y = 1 \\ 3x + 2y = 11 \end{cases}$$

- multiply the first equation by $\frac{3}{1}$ and subtract it from the second equation
- After

$$\begin{cases} x - 2y = 1 \\ \quad\; 8y = 8 \end{cases}$$

- the first pivot is 1 and the first multiplier is 3. $\frac{3}{1}$
- matrix representation of changes

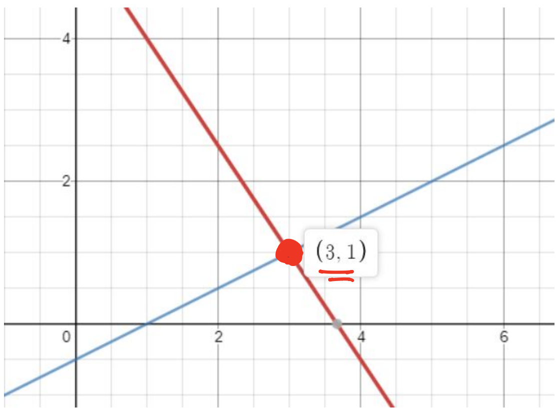
$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 8 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 11 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

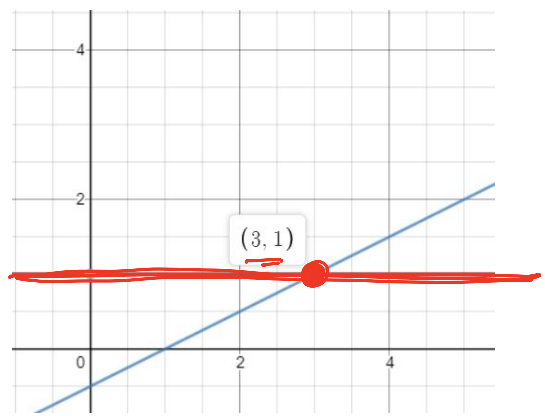
Solving a system: the idea of elimination

- matrix representation of changes

$$\begin{bmatrix} * & * \\ \cancel{*} & * \end{bmatrix} \to \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$$

original syst

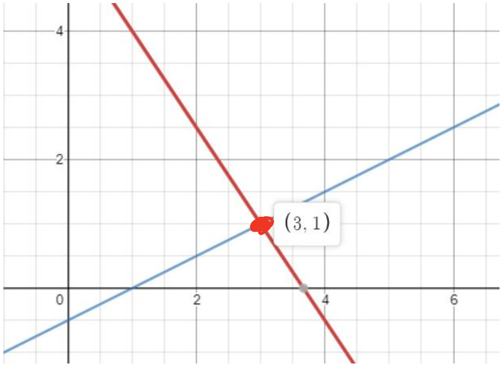


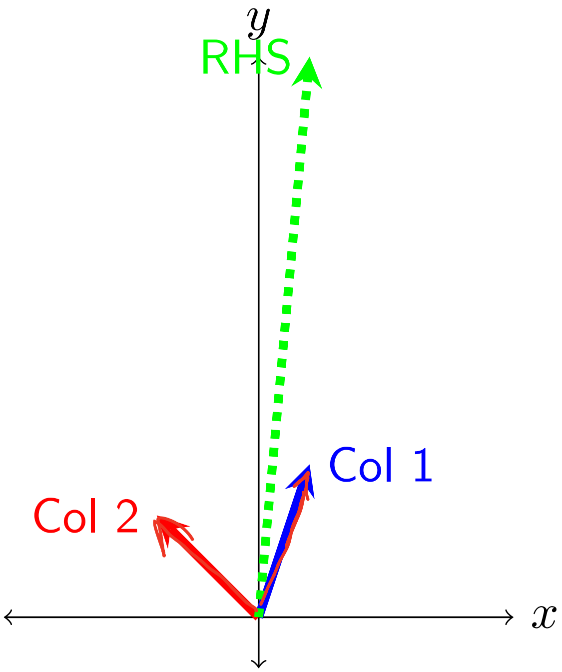


Elimination

- only one soluton
- lines are cutting eachother at a single point
- Columns are independent

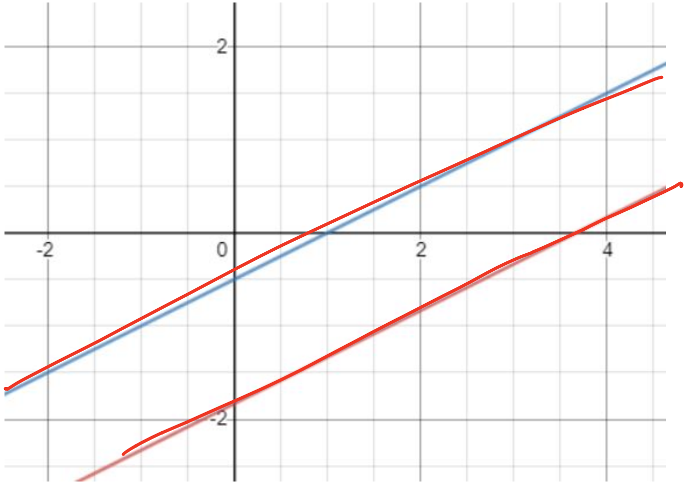
$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases} \to \begin{cases} x - 2y = 1 \\ \quad\;\; 8y = 8 \end{cases}$$

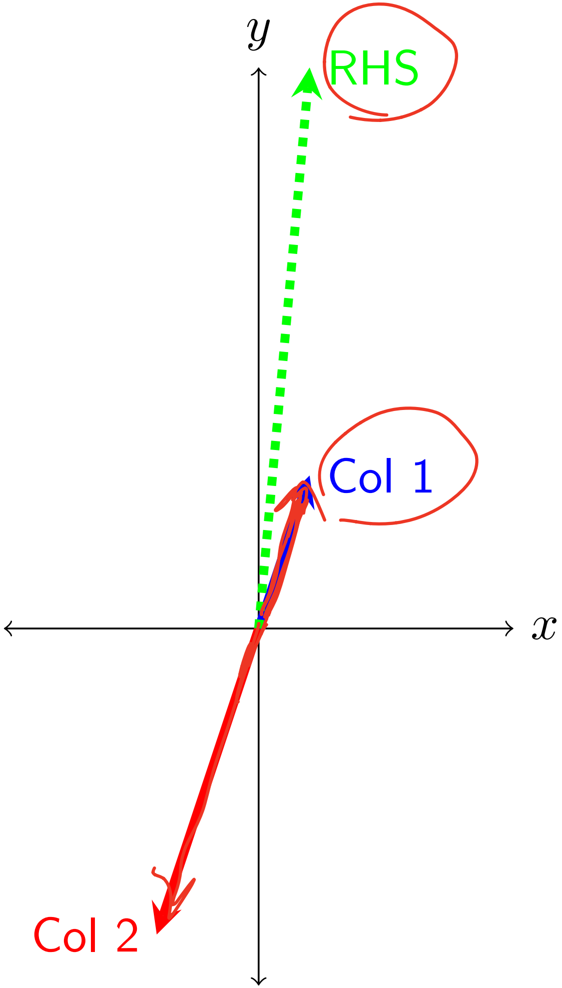




- No soluton
- lines are parallel
- Columns arn't independent

$$\begin{cases} x - 2y = 1 \\ 3x - 6y = 11 \end{cases} \rightarrow \begin{cases} x - 2y = 1 \\ 0y = 8 \end{cases}$$





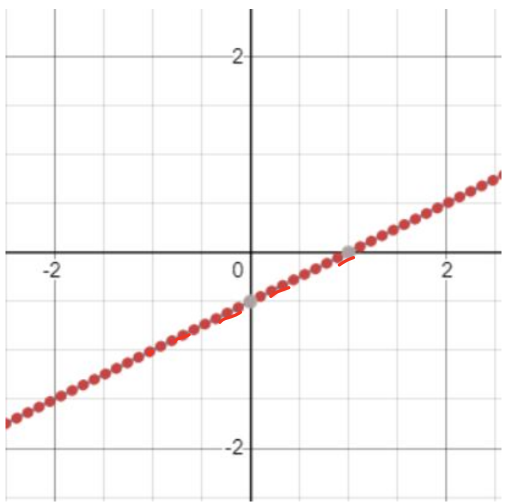
Elimination failures

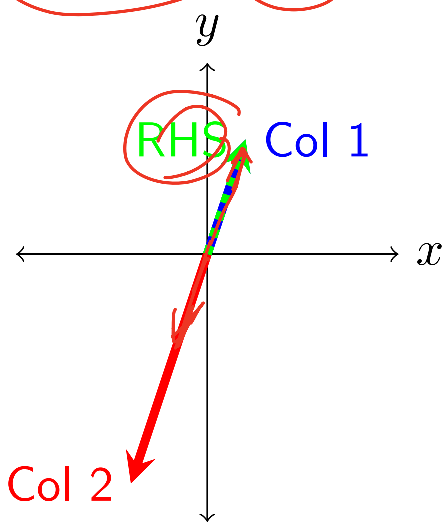
- many solutons
- lines are overlapping
- Columns arn't independent, but the right-hand side is in the direction of one of the columns

$$\begin{cases} x - 2y = 1 \\ 3x - 6y = 3 \end{cases} \rightarrow \begin{cases} x - 2y = 1 \\ \quad 0y = 0 \end{cases}$$

$0(1) = 0$

$0(2) = 0$





Elimination: Failures

- No solution
  - $0y = 8$
  - no value of y satusfy here.
  - We expect two pivots, but there are only one.
- Many solutions
  - $0y = 0$
  - every y satisfies here
  - the unknown y is free
  - $x = 1, y = 0, x = 0, y = -0.5,$ etc.
  - We expect two pivots, but there are only one.
- For n equations, we don't get n pivots and elimination leads to
  - $0 \neq 0$ (no solution)
  - $0 = 0$ (many solutions)

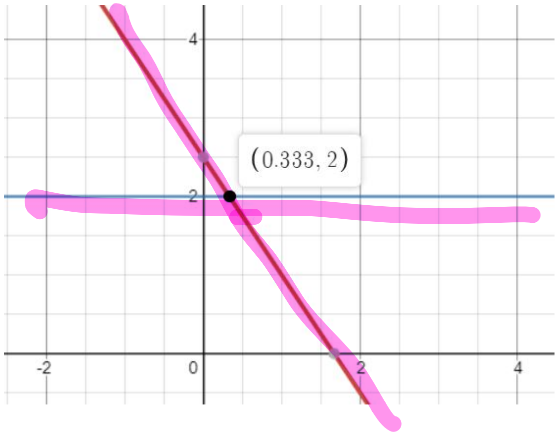
- Zero in pivot

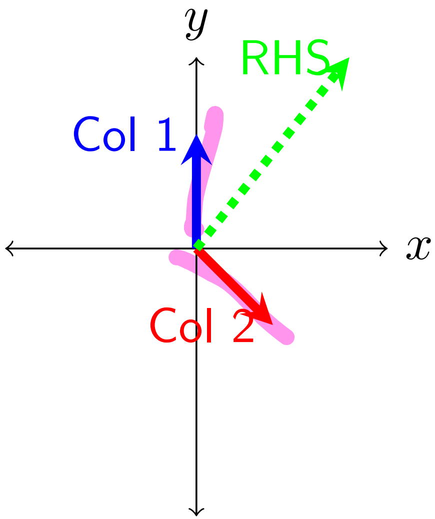
$\begin{cases} 2y = 4 \\ 3x - 2y = 5 \end{cases}$

$$\begin{cases} 0x + 2y = 4 \\ 3x - 2y = 5 \end{cases} \rightarrow \begin{cases} 3x - 2y = 5 \\ \quad\;\; 2y = 4 \end{cases}$$

pivot

- a row exchange solves the problem.





Augmented matrix representation $Ax = b$ as $[A{:}b]$

$\{$ $Ax=b$ $\rightarrow$

$m \times n$

$[A \vdots b]$

$m \times (n+1)$

- A unique solution

$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases} \rightarrow \begin{cases} x - 2y = 1 \\ \quad\;\; 8y = 8 \end{cases}$$

$$\begin{array}{c} A \qquad\qquad b \\ \begin{bmatrix} 1 & -2 & \vdots & 1 \\ 3 & 2 & \vdots & 11 \end{bmatrix} \end{array} \rightarrow \begin{bmatrix} 1 & -2 & \vdots & 1 \\ 0 & 8 & \vdots & 8 \end{bmatrix}$$

- No solution

$$\begin{cases} x - 2y = 1 \\ 3x - 6y = 11 \end{cases} \rightarrow \begin{cases} x - 2y = 1 \\ \quad\;\; 0y = 8 \end{cases}$$

$$\begin{array}{c} \begin{bmatrix} 1 & -2 & \vdots & 1 \\ 3 & -6 & \vdots & 11 \end{bmatrix} \\ A \qquad\qquad b \end{array} \rightarrow \begin{bmatrix} 1 & -2 & \vdots & 1 \\ 0 & 0 & \vdots & 8 \end{bmatrix}$$

Augmented matrix representation

- many solutions

$$\begin{cases} x - 2y = 1 \\ 3x - 6y = 3 \end{cases} \rightarrow \begin{cases} x - 2y = 1 \\ \quad\ \ 0y = 0 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & \vdots & 1 \\ 3 & -6 & \vdots & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & \vdots & 1 \\ 0 & 0 & \vdots & 0 \end{bmatrix}$$

- zero pivot

$$\begin{cases} 0x + 2y = 1 \\ 3x - 2y = 5 \end{cases} \rightarrow \begin{cases} 3x - 2y = 5 \\ 0x + 2y = 1 \end{cases} \rightarrow \begin{cases} x - 2y = 1 \\ \quad\ \ 2y = 4 \end{cases}$$

$$\begin{bmatrix} 0 & 2 & \vdots & 1 \\ 3 & -2 & \vdots & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & -2 & \vdots & 5 \\ 0 & 2 & \vdots & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & -2 & \vdots & 5 \\ 0 & 2 & \vdots & 1 \end{bmatrix}$$

Three equations in three unknowns: The procedure

$A\vec{x} = \vec{b} \to 3\times 1$

- Use the first equation (column) to create zeros below the first pivot.

3×3 $\quad$ 3×1

- Use the second equation (column) to create zeros below the second pivot.
- keep going until you have an upper triangular matrix

A $\quad$ b

$$\left[\begin{array}{ccc|c} \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast \end{array}\right] \to \left[\begin{array}{ccc|c} \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast \end{array}\right] \to \left[\begin{array}{ccc|c} \ast & \ast & \ast & \ast \\ 0 & \star & \star & \star \\ 0 & \star & \star & \star \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} \ast & \ast & \ast & \ast \\ 0 & \star & \star & \star \\ 0 & \star & \star & \star \end{array}\right] \to \left[\begin{array}{ccc|c} \ast & \ast & \ast & \ast \\ 0 & \star & \star & \star \\ 0 & 0 & \bullet & \bullet \end{array}\right] \to \left[\begin{array}{ccc|c} \ast & \ast & \ast & \ast \\ 0 & \star & \star & \star \\ 0 & 0 & \bullet & \bullet \end{array}\right]$$

Upper

Three equations in three unknowns

$$\begin{cases} 2x + 4y - 2z = 2 \\ 4x + 9y - 3z = 8 \\ -2x - 3y + 7z = 10 \end{cases} \rightarrow \begin{cases} 2x + 4y - 2z = 2 \\ \quad\quad y + z = 4 \\ \quad\quad\quad -4z = 8 \end{cases}$$

$y = 6$

$\to z = \frac{8}{-4} = -2$

- $A\boldsymbol{x} = \boldsymbol{b}$ becomes $U\boldsymbol{x} = \boldsymbol{c}$

$$\begin{bmatrix} 2 & 4 & -2 & \vdots & 2 \\ 4 & 9 & -3 & \vdots & 8 \\ -2 & -3 & 7 & \vdots & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & -2 & \vdots & 2 \\ 0 & 1 & 1 & \vdots & 4 \\ 0 & 0 & -4 & \vdots & 8 \end{bmatrix}$$

- Through back substitution: $z = -2, y = 2, x = -1$

$$A\boldsymbol{x} = A\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = (-1)\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 9 \\ -3 \end{bmatrix} + 2\begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \boldsymbol{b}$$

Elimination: key ideas

$[A \vdots \vec{b}]$ $\qquad$ $[U \vdots \vec{c}]$

- A linear system $\boldsymbol{Ax} = \boldsymbol{b}$ becomes upper triangular $U\boldsymbol{x} = \boldsymbol{c}$ after elimination
- multipliers
  we subtract ℓ_{ij} times equation j from equation i to make the (i, j) entry zero
- The multiplier $\ell_{ij} = \dfrac{\text{entry to eliminate in row } i}{\text{pivot in row } j}$.
- pivots cannot be zero
- When zero in the pivot position, exchange rows if there is a nonzero row below it.
- The upper triangular system $\boxed{U\boldsymbol{x} = \boldsymbol{c}}$ is solved by back substitution
- the system may have a unique solution, no solution, or many solutions

$$\begin{cases} x+y+z=3 \\ x-y+2z=2 \\ x-2y+3z=2 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$$

$$\underset{3\times 3}{A}\ \underset{3\times 1}{\vec{x}} = \underset{3\times 1}{\vec{b}}$$

1st

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc:c} \boxed{1} & 1 & 1 & 3 \\ 1 & -1 & 2 & 2 \\ 1 & -2 & 3 & 2 \end{array}\right] \quad 3\times 4$$

$(-1)R_1$ $\quad -1 \quad -1 \quad -1 \quad -3$

$$\text{multiplier} = \frac{1}{1_{\text{pivot}}}$$

$$\boxed{R_2 \leftarrow R_2 - (1)R_1}$$

2nd

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & \boxed{-2} & 1 & -1 \\ 0 & -3 & 2 & -1 \end{array}\right]$$

$$\boxed{R_3 \leftarrow R_3 - (1)R_1}$$

$$R_3 \leftarrow R_3 - \frac{(-3)}{(-2)} R_2$$

$$R_3 \leftarrow R_3 - \frac{3}{2} R_2$$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & \boxed{1/2} & 1/2 \end{array}\right]$$

$(-1) - \frac{3}{2}(-1) = -1 + \frac{3}{2} =$

$(-3) - \frac{3}{2}(-2) =$

$-3 + 3 = 0$

$2 - \frac{3}{2}(1) = 2 - \frac{3}{2} =$

Linear Equation in n Variables

$3x_1 + 2x_2 - 7x_3 + 8x_4 = 10$

$$\boxed{a_1x_1 + a_2x_2 + \ldots + a_nx_n = b}$$

- Another representation

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \boldsymbol{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$\boxed{\vec{\boldsymbol{a}}^T \vec{\boldsymbol{x}} = b}$$

- $x_1, \ldots, x_n$ are variables (unknowns)
- $a_1, \ldots, a_n, b \in \mathbb{R}$
- The set of points satisfying in this equation (set of solutions) is called a hyperplane in $\mathbb{R}^n$
- Examples
  - in $\mathbb{R}^2$, the hyperplane $a_1x_1 + a_2x_2 = b$ is a line.
  - in $\mathbb{R}^3$, the hyperplane $a_1x_1 + a_2x_2 + a_3x_3 = b$ is a plane.

$2x_1 + 3x_2 = 1$

line 2D

$2x + 3y = 1$

$3y = 1 - 2x$

$y = \frac{1}{3} - \frac{2}{3}x$ line

$2x_1 + 3x_2 - x_3 = 5$

plane 3D

$2x_1 - x_2 - x_3 + x_4 = 7$

4D

hyper plane

$$\begin{bmatrix} 2 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 7$$

$\vec{a}^T \vec{x} = 7$

$A\vec{x} = \vec{b}$

augmented matrix $[A \vdots \vec{b}]$

we try to convert A into an upper triangular matrix U

$$[A \vdots \vec{b}] \longrightarrow [U \vdots \vec{c}]$$

solving this system is easier.

$$\begin{cases} x + y = 3 \\ x - 2y = 0 \end{cases} \qquad \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\begin{array}{l} R_1 \leftarrow \\ R_2 \leftarrow \end{array} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -2 & 0 \end{array}\right]$$

$$R_2 \leftarrow R_2 - (\text{coefficient}) R_1$$

$\text{coefficient} = \dfrac{a_{21}}{a_{11}}$; $a_{21} \rightarrow$ the entry we want to disappear; $a_{11} \leftarrow$ pivot $(\neq 0)$

$a_{11} = 1$

$a_{21} = 1$

$\text{coefficient} = \frac{1}{1} = 1$

$$R_2 \leftarrow R_2 - R_1$$

$$\begin{array}{rrrr} & 1 & -2 & 0 \\ - & 1 & 1 & 3 \\ \hline & 0 & -3 & -3 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -3 & -3 \end{array}\right] \rightarrow \begin{cases} x + y = 3 \\ -3y = -3 \rightarrow \boxed{y = 1} \quad \boxed{x = 2} \end{cases}$$

Gaussian Elimination

$$\begin{cases} x - 2y + z = 0 \\ x + 2z = 3 \\ x + 2y = 3 \end{cases}$$

pivot

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 1 & 0 & 2 & 3 \\ 1 & 2 & 0 & 3 \end{array}\right] \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3 \end{array}$$

pivot

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & 1 & 3 \\ 0 & 4 & -1 & 3 \end{array}\right]$$

$R_2 \leftarrow R_2 - R_1$

$R_3 \leftarrow R_3 - R_1$

$$R_3 \leftarrow R_3 - \left(\frac{4}{2}\right) R_2$$

2

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & -3 & -3 \end{array}\right] \to \begin{cases} x - 2y + z = 0 \\ 2y + z = 3 \\ -3z = -3 \end{cases}$$

$\boxed{z = 1}$

$2y + 1 = 3 \to \boxed{y = 1}$

$x - 2(1) + (1) = 0 \to \boxed{x = 1}$

$$\text{pivots} = \{1, 2, -3\}$$

System of m linear Equations with n Unknowns

$A\vec{x} = \vec{b}$

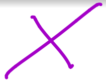
$A_{2\times 2}$
$A_{3\times 3}$

$$* \begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{cases}$$

- In matrix format $A\boldsymbol{x} = \boldsymbol{b}$

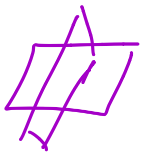
$$A_{m\times n} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{m\times n} \qquad \boldsymbol{x} = \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{= n\times 1} \in \mathbb{R}^n, \boldsymbol{b} = \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}_{= m\times 1} \in \mathbb{R}^m$$

$\mathbb{R}^2$



$$
\begin{cases}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\
\cdots \qquad \cdots \qquad \cdots \qquad \cdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m
\end{cases}
$$

$\mathbb{R}^3$



$\mathbb{R}^4$ hyperplane

- Row view: consider m hyperplanes and see whether they cut each other ina single point
- Column view: can we express the right hand side as a linear combination of the columns of A?

$A\vec{x}$

$$
x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \boldsymbol{b}
$$

Augmented Representation of a linear system of equations

$A_{n \times n}$

$A_{m \times n}$

- $A\boldsymbol{x} = \boldsymbol{b}$ equivalent to $[A|b]$

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n, \boldsymbol{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^m$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

$A\vec{x} = \vec{b}$ $\quad A^{-1}$ $\quad AA^{-1} = A^{-1}A = I$

$\underbrace{A^{-1}A}_{n\times n}\vec{x} = A^{-1}\vec{b} \rightarrow I\vec{x} = A^{-1}\vec{b} \rightarrow \boxed{\vec{x} = A^{-1}\vec{b}}$

- A single equation: $ax = b$, then $x = \frac{b}{a}$ or $\boxed{x = a^{-1}b}$
- Consider a square matrix $A_{n\times n}$. A square $n \times n$ matrix A^{-1} is called its inverse if

A $\quad A^{-1}$ non-singular – invertible

$$AA^{-1} = A^{-1}A = I_n$$

I_n or $I_{n\times n}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ I_3

- A square matrix is called singular it it does not have an inverse. Otrherwise it is called nonsingular or invertible.

$A_{n\times n}$ $\begin{cases} \text{non-singular } A^{-1} \\ \text{singular} \end{cases}$

Inverse Matrix Properties

$(A^T)^T = A$ $\quad$ $(AB)^T = B^T A^T$

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$ given that both are nonsingular
- For a 2×2 matrix

scalar

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then } A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

2x2

3x3

- Examples:
  - $3 \times (\frac{1}{3}) = 1$
  - Find the inverse of $\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$

$(4)(6) - (7)(2) = 10$

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}^{-1} = \frac{1}{4 \times 6 - 7 \times 2}\begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$AA^{-1} = A^{-1}A = I$

$$= \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}, \quad \underbrace{\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}}_{A^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I_2}$$

$A\vec{x}$ transformatio

$1(3) = 3$

square

- $I_n \in \mathbb{R}^{n\times n}$

$$\text{for all } \boldsymbol{x} \in \mathbb{R}^n, \quad I_n\boldsymbol{x} = \boldsymbol{x}$$

- $A \in \mathbb{R}^{n\times n}$, its inverse A^{-1} if exists

$$A^{-1}A = I_n$$

- Solving a system $A\boldsymbol{x} = \boldsymbol{b}$ when A has an inverse

$$A\boldsymbol{x} = \boldsymbol{b} \rightarrow A^{-1}A\boldsymbol{x} = A^{-1}\boldsymbol{b} \rightarrow I_n\boldsymbol{x} = A^{-1}\boldsymbol{b}$$

$$\boxed{\boldsymbol{x} = A^{-1}\boldsymbol{b}}$$

$A\vec{x} = \vec{0}$ homogenous system

$\vec{x} = \vec{0}$ $\quad A\vec{0} = \vec{0}$

Gaussian elimination

n

- Consider a square matrix $\boxed{A_{n\times n}}$
- To have an inverse, a matrix should have n nonzero pivots.
- The system $\boxed{A\boldsymbol{x} = \mathbf{0}}$ must have only solution $\boldsymbol{x} = \mathbf{0}$.
- Gauss-Jordan elimination

homogenous system $A\vec{x} = \vec{0}$: $\vec{x} = \vec{0}$ is the only solution → A is invertible; other nonzero solution → A is singular

$$[A \vdots I] \rightarrow [I \vdots A^{-1}]$$

- Using Gauss elimination, convert the left hand side to I
- the right hand side will be A^{-1}

$$A_{n\times n} \qquad [A \vdots I_{n\times n}] \xrightarrow{\text{eliminate}} [I \vdots A^{-1}]$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I}$$

Gauss-Jordan elimination for A^{-1}

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}_{3\times 3}$$

- Make the Augmented matrix $[A \vdots I]$,

$$[A \vdots I] = \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

- row2 $\leftarrow$ ($\frac{1}{2}$row1+row2),

$-1/2$

row2 $\Leftarrow$ row2 $-$ $(-\frac{1}{2})$row1

row2 $\Leftarrow$ row2 $+ \frac{1}{2}$ row1

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

Gauss-Jordan elimination for A^{-1}

- row3 $\leftarrow$ ($\frac{2}{3}$row2+row3),

$$\left[\begin{array}{ccc:ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ {\color{blue}0} & \frac{3}{2} & {\color{red}-1} & \frac{1}{2} & 1 & 0 \\ {\color{blue}0} & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right]$$

- row2 $\leftarrow$ ($\frac{3}{4}$row3+row2),

$$\left[\begin{array}{ccc:ccc} 2 & {\color{red}-1} & {\color{blue}0} & 1 & 0 & 0 \\ {\color{blue}0} & \frac{3}{2} & {\color{blue}0} & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ {\color{blue}0} & {\color{blue}0} & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right]$$

Gauss-Jordan elimination for A^{-1}

- row1 $\leftarrow$ ($\frac{2}{3}$row2+row1),

$$\begin{bmatrix} 2 & 0 & 0 & \vdots & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & \frac{3}{2} & 0 & \vdots & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \vdots & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix}$$

- divide row1 by 2, row2 by $\frac{3}{2}$, and row3 by $\frac{4}{3}$

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \vdots & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \vdots & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix} = [I \vdots A^{-1}]$$

Gauss-Jordan elimination for A^{-1}

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

- Check the correctness

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix} = I_3$$

Gauss-Jordan elimination for A^{-1}

- Find A^{-1} by Gauss-Jordan elimination strating from

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{14-12}\begin{bmatrix} 7 & -3 \\ -4 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 7 & -3 \\ -4 & 2 \end{bmatrix}$$

$$[A \vdots I] = \left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{array}\right] \xrightarrow{\text{row2=row2-2row1}} \left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{\text{row1=row1-3row2}} \left[\begin{array}{cc|cc} 2 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{\text{row1=0.5row1}} \left[\begin{array}{cc|cc} 1 & 0 & \frac{7}{2} & \frac{-3}{2} \\ 0 & 1 & -2 & 1 \end{array}\right] = [I \vdots A^{-1}]$$

$$A^{-1} = \begin{bmatrix} \frac{7}{2} & \frac{-3}{2} \\ -2 & 1 \end{bmatrix}$$

$$\boxed{AA^{-1} = A^{-1}A = I}$$

Reduced row echelon form R

$A_{n\times n}$ $\quad A\vec{x}=\vec{b} \quad [A \vdots \vec{b}] \to [U \vdots \vec{c}]$

upper triangular matrix

$A_{m\times n}$ $\quad [A \vdots \vec{b}] \to [R \vdots \vec{c}]$

reduced row echelon form (rref)

- When A is rectangular , elimination will not stop at the upper triangular matrix U
- To make this matrix simpler we can take two actions
  - roduce zeros above the pivots
  - produce ones in the pivots

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \to U = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

rref

$$\begin{cases} x + 2y + 2z = 4 \\ 0 + 2y + 0 = 4 \end{cases}$$

2 eq $\quad$ 3 unknowns

$$\begin{cases} x + y - z + w = 3 \\ x - 2z - w = -1 \\ y + z - w = 1 \end{cases}$$

$$A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 0 & -2 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix}_{3\times 4}$$

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

$[A \vdots b]$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 3 \\ 1 & 0 & -2 & -1 & -1 \\ 0 & 1 & 1 & -1 & 1 \end{array}\right] \quad R_2 \leftarrow R_2 - (1)\, R_1$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 3 \\ 0 & -1 & -1 & -2 & -4 \\ 0 & 1 & 1 & -1 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \leftarrow R_1 - (-1)\, R_2 \\ R_3 \leftarrow R_3 - (-1)\, R_2 \end{array}$$

$R_1 \leftarrow R_1 + R_2$

$R_3 \leftarrow R_3 + R_2$

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & -1 \\ 0 & \boxed{-1} & -1 & -2 & -4 \\ 0 & 0 & 0 & -3 & -3 \end{array}\right] \quad R_2 \leftarrow (-1)\, R_2$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & -1 \\ 0 & +1 & +1 & +2 & +4 \\ 0 & 0 & 0 & \cancel{-3}\ 1 & \cancel{-3}\ 1 \end{array}\right] \quad R_3 \leftarrow \left(-\tfrac{1}{3}\right) R_3$$

$R_2 \leftarrow R_2 - 2\boxed{R_3}$

$R_1 \leftarrow R_1 + \boxed{R_3}$

free variable

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & -2 & 0 & 0 \\ 0 & \boxed{+1} & +1 & 0 & 2 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{array}\right]$$

rref

$$\begin{cases} x - 2z = 0 \\ y + z = 2 \\ \boxed{w = 1} \end{cases}$$

$z = t \in \mathbb{R}$

$y = 2 - z = 2 - t$

$x = 2t$

$$\vec{x} = \begin{bmatrix} 2t \\ 2 - t \\ t \\ 1 \end{bmatrix} \quad t \in \mathbb{R}$$

$t = 0 \quad \vec{x} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

$t = 1 \quad \vec{x} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

many solutions here.