
How to Write Mathematics

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CHAPTER 0

Preface

*Question: How many months have 28 days?
Mathematician's answer: All of them.*

The concept

This booklet is about writing mathematics at university. At pre-university level a lot of mathematics involves writing down a sequence of equations, a number or function appears at the bottom of the page and you get a tick or a cross depending on whether you are right or wrong. This is not the way mathematics is written at university. Writing mathematics involves putting together a coherent argument.

I'm not saying you have write an essay, rather that you write mathematics so someone else can understand it. You will get more marks that way and when you go out into the big bad world and get a job you will have a useful skill – the ability to write well.

How to Think Like a Mathematician - The book

The material in this booklet forms chapters 3 and 4 of my book *How to Think Like a Mathematician*. I want students like you to be able to think like a mathematician and the book jam-packed with practical advice and helpful hints on how to acquire specific skills to do this. Some points are subtle, others appear obvious when you have been told them. For example, when trying to show an equation holds you should take the most complicated side and reduce it until you get to the other side.

How to Think Like a Mathematician - The Movie

There are various videos to accompany this booklet on my YouTube channel:

<http://www.youtube.com/user/DrKevinHouston>

There is also a video on mathematics and playing cards. It's not a magic trick with cards but a property of what is called a perfect shuffle.

Some friendly advice

And now for some friendly advice that you have probably heard before – but is worth repeating.

- It's up to you* – Your actions are likely to be the greatest determiner of the outcome of your studies at university. Consider the ancient proverb: The teacher can open the door, but you must enter by yourself.
- Be active* – Read the lecture notes. Do the exercises set and hand them in.
- Think for yourself* – Always good advice.
- Question everything* – Be sceptical of all results presented to you. Don't accept them until you are sure you believe them.
- Observe* – The power of Sherlock Holmes came not from his deductions but his observations.
- Prepare to be wrong* – You will often be told you are wrong when doing mathematics. Don't despair, mathematics is hard, but the rewards are great. Use it to spur yourself on.
- Don't memorize – Seek to understand* – It is easy to remember what you truly understand.
- Develop your intuition* – But don't trust it completely.
- Collaborate* – Work with others, if you can, to understand the mathematics. This isn't a competition. Don't merely copy from them though!
- Reflect* – Look back and see what you have learned. Ask yourself how you could have done better.

Comments and suggestions please

If you have any comments, criticisms, suggestions or spotted any mistakes, then email me: k.houston@leeds.ac.uk. A list of corrections and solutions to exercises can be found at

<http://www.maths.leeds.ac.uk/~khouston/httlam.html>

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CHAPTER 3

Writing mathematics I

We have a habit in writing articles published in scientific journals to make the work as finished as possible, to cover up all the tracks, to not worry about the blind alleys or describe how you had the wrong idea first, and so on.

Richard Feynman, Nobel Lecture, 1966.

As a lecturer my toughest initial task in turning enthusiastic students into able mathematicians is to force them (yes, force them) to write mathematics correctly. Their first submitted assessments tend to be incomprehensible collections of symbols, with no sentences or punctuation. ‘What’s the point of writing sentences?’ they ask, ‘I’ve got the correct answer. There it is – see, underlined – at the bottom of the page.’ I can sympathise but in mathematics we have to get to the right answer in a rigorous way and we have to be able to show to others that our method is rigorous.

A common response when I indicate a non-sensical statement in a student’s work is ‘But you are a lecturer, you know what I meant’. I have sympathy with this view too, but there are two problems with it.

- (i) If the reader has to use their intelligence to work out what was intended, then the student is getting marks because of the reader’s intelligence, not their own intelligence.¹
- (ii) The second point is perhaps the most important for students. Sorting through a jumble of symbols and half-baked poorly expressed ideas is likely to frustrate and annoy any assessor – not a good recipe for obtaining good marks.

My students performed well at school and are frustrated at losing marks over what seems to them unimportant details. However, by the end of the year they generally accept that writing well has improved their performance. You have to trust me that this works! Besides, writing well in any subject is a useful skill to possess.

¹To be honest, students don’t mind this!

Writing well is good for you

Writing well

There are many reasons for writing – you might be making notes for future use or wish to communicate an idea to another person. Whatever the reason, writing mathematics is a difficult art and requires practice to produce clear and effective work.

Good writing is clearly important if you wish to be understood, but it has a bonus: it clarifies for you the material being communicated and thus adds to your understanding. In fact, I believe that if I can't explain an idea in writing, then I don't understand it. This is one reason why writing well helps you to think like a mathematician.

Generally, we write to explain to another person, so have this person in mind. Two points to remember:

- Have mercy on the reader. Do not make it difficult for them – particularly someone marking your work.
- The responsibility of communication lies with you. If someone at your level can't understand it, then the problem is with your writing!

What follows is a collection of ideas on how to improve your writing. The ideas presented have been tried and tested over many years with students and are not merely theoretical ideas. They may seem troublesome and pedantic, but if you follow them, you will produce clearer explanations, and hence gain more marks in assessments.

It should be noted that there is a huge difference between finding the answer to a problem and presenting it. These rules apply to the final polished product. When trying to solve a problem or do an exercise it is acceptable to break all these rules. What is important is that they are followed when writing up the solution for someone else to read.

An example

In a geometry course I stated the cosine rule.

Cosine rule: Suppose that a triangle has edges of length a , b and c with the angle opposite a equal to θ . Then,

$$a^2 = b^2 + c^2 - 2bc \cos \theta.$$

If you have not met this before, then this is a good chance to ‘Check the text’. (This a reference to Chapter 2 of the book: have pen and paper with you as you read and check the text by choosing examples to play with). Try drawing some pictures and trying some examples. More techniques for investigating such a statement is to be found in Chapter 16 of the full book.

The cosine rule is a useful result which can be regarded as a generalization of Pythagoras’ Theorem when we take $\theta = \pi/2$. (Check the text!) During the geometry course I proved this formula in the case that θ was an acute angle and left the case of an obtuse angle as an exercise. Figure 3.1 shows one solution I

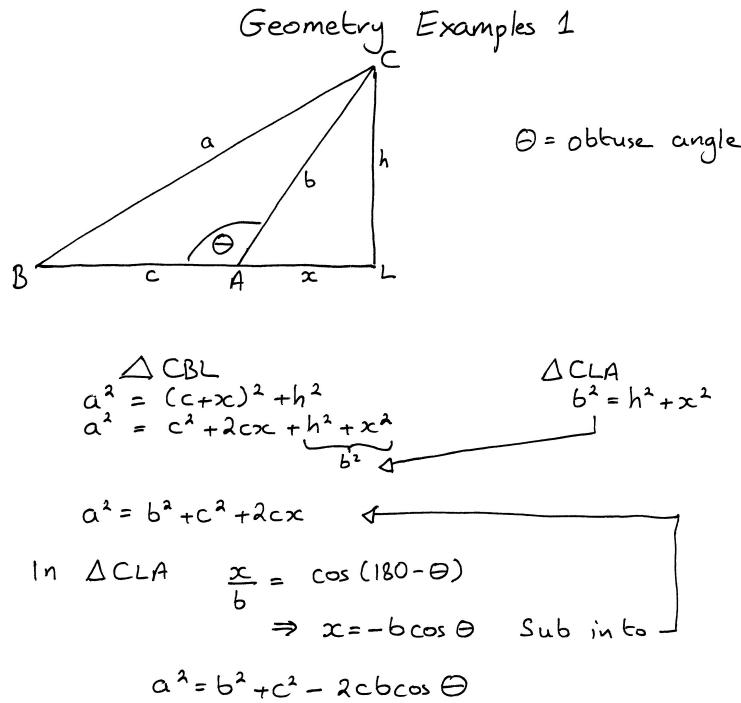


Figure 3.1: Student's proof of cosine formula

received. We will refer to this as we proceed. As an exercise take a look at it and try to spot as many errors as possible. Does it make sense? Is it easy to read? Most importantly, is it right?

Basic rules

The primary rule is that you should write in simple, correctly punctuated sentences. Let's put some more detail on this.

Write in sentences

Write in sentences. Write in sentences. And once more to really hammer it home: Write in sentences.

This advice has precedence over all others and is the one that can really change the way you present your work.

One of most common erroneous beliefs of the novice mathematician is that because mathematics is a highly symbolic language we need only provide a list of symbols to answer a problem. This is wrong, symbols are merely shorthand for certain concepts; they need to be incorporated into sentences for there to be any meaning.

Consider this student's answer to an exercise on finding the solution of a set of equations:

' $0=1$, \therefore no solutions, empty set (\emptyset).'

It is obvious what the student meant: 'Since the equations reduce to the equation ' $0=1$ ', which doesn't have any solutions, the solution set is empty.' This

vital fact - that no solutions exist - is certainly included. He also showed that he knows that the empty set is denoted by \emptyset . However, the inclusion of this symbol is unnecessary, it serves no purpose.

But what he wrote is not a sentence – it is a string of symbols and conveys no meaning in itself.

The answer could be better expressed as

‘Since the equation $0 = 1$ is present, the system of equations is inconsistent and so no solutions exist.’

We could add ‘That is, the solution set is empty’, but it is not necessary. Understanding is clearly shown in this answer, and so more marks will be forthcoming.

All the other usual rules of written English apply, for example the use of paragraphs and punctuation. The rules of grammar are just as important: every sentence should have a verb, subjects should agree with verbs, and so on.

Let us look at the example in Figure 3.1 of the proof of the cosine formula. Examine the first two lines below the student’s diagram.

$$\Delta CBL \quad a^2 = (c+x)^2 + h^2$$

$$\Delta CLA \quad b^2 = h^2 + x^2$$

If I read from left to right in the standard fashion, I read

$$\Delta CBL \Delta CLA \quad a^2 = (c+x)^2 + h^2 \quad b^2 = h^2 + x^2.$$

Now what does that mean? It is obvious what is intended. But why should we have to work out what was intended? It would be better to say what was meant from the start:

In triangle ΔCBL we have $a^2 = (c+x)^2 + h^2$ and in ΔCLA we have $b^2 = h^2 + x^2$.

This is now a proper sentence. As an aside, notice how I explained my notation Δ by using the word ‘triangle.’

Now look at the words after the \Rightarrow sign:

$$\begin{aligned} \frac{x}{b} &= \cos(180 - \theta) \\ \Rightarrow x &= -b \cos \theta \quad \text{Sub into } \end{aligned}$$

This is a perfect example of where we can understand what the student had intended but it is not well written. It is much clearer as

... $x = -b \cos \theta$. Substituting this into ...

Use punctuation

The purpose of punctuation is to make the sentence clear. Punctuation should be used in accordance with standard practice. In particular, all sentences begin with a capital letter and end with a full stop. The latter holds even if the sentence ends in a mathematical expression. For example,

'Let $x = y^4 + 2y^2$ Then x is positive.'

needs a full stop after the expression $y^4 + 2y^2$ as it is obvious that the second part is a new sentence – it begins with a capital letter. This is true for a list of equal expressions:

$$\begin{aligned} x &= y^2 + 2y \\ &= y(y + 2) \end{aligned}$$

This should end with a full stop. Note that some authors do not adhere to this rule of punctuation. They are wrong.²

Mathematical expressions need to be punctuated. For example,

'Let $x = 4a + 3b$ where $a \in \mathbb{R}$ $b \in \mathbb{Z}$,'

should have commas like so

'Let $x = 4a + 3b$, where $a \in \mathbb{R}$, $b \in \mathbb{Z}$.'

Notice the three commas and the final full stop in the following example.

$$\text{Let } f(x) = \begin{cases} x^2, & \text{for } x \geq 0, \\ 0, & \text{for } x < 0. \end{cases}$$

Look at the example of the proof of the cosine formula. As you can see there is no punctuation! Presumably a sentence starts at 'In $\Delta CLA\dots$ ' but it is not proceeded by a full stop so who knows?

Keep it simple

Mathematics is written in a very economical way. To achieve this use short words and sentences. Short sentences are easy to read. To eliminate ambiguities avoid complicated sentences with lots of negations.

Consider the following hard-to-read example:

'The functions f and g are defined to be equal to the function defined on the set of non-positive integers given by x maps to its square and x maps to the negative of its square respectively.'

This would be better as:

'Let $\mathbb{Z}^{\leq 0} = \{\dots, -5, -4, -3, -2, -1, 0\}$ be the set of non-positive integers. Let $f : \mathbb{Z}^{\leq 0} \rightarrow \mathbb{R}$ be given by $f(x) = x^2$ and $g : \mathbb{Z}^{\leq 0} \rightarrow \mathbb{R}$ be given by $g(x) = -x^2$.'

Note that we separated the definition of the domains of the maps into a separate sentence. Also we defined the set in words and clarified by writing it in a different way. The definitions of f and g are mixed together in the first sentence due to the use of 'respectively', while in the second sentence they are separated and defined using symbols. Sometimes using symbols is clearer, sometimes not, see page 10.

²A number of people think this is a controversial statement. 'What does it matter, as long as you are consistent?' Well, we could apply that argument to any sentence and we can get rid of all full stops! The majority opinion is that sentences end with a full stop – go with that.

Expressing yourself clearly

The purpose of writing is communication – you are supposed to be transferring a thought to someone else (or yourself at a later date). Unfortunately – and I have lots of experience of this – it is easy to communicate an incorrect or unintended idea. The following advice is offered to prevent this from happening.

Explain what you are doing – keeping the reader informed

Readers are not psychic. It is crucial to explain what you are doing. To do this imagine that you are giving a running commentary. As stated earlier, it is not sufficient to produce a list of symbols, formulas, or unconnected statements. A good explanation will help gain marks as it demonstrates understanding.

You can introduce an argument by saying what you are about to do, e.g.,

‘We now show that X is a finite set’,

‘We shall prove that ...’.

Similarly you can end by

‘This concludes the proof that X is a finite set’, or

‘We have proved ...’.

Make clear, bold assertions. Avoid phrases like ‘it should be possible’; either it is possible or it isn’t, so claim ‘it is possible’. Be positive.

Of course, avoid going to the extreme of explaining every last detail. A balance, which will come from practice and having your written work criticized, needs to be struck.

If we look at the end of the example in Figure 3.1, then we see the following.

$$\Rightarrow x = -b \cos \theta \quad \text{Sub into } \square$$

$$a^2 = b^2 + c^2 - 2cb \cos \theta$$

This ending would be better as

‘... $x = -b \cos \theta$. Substituting this into the above we deduce that
 $a^2 = b^2 + c^2 - 2cb \cos \theta$.’

This is certainly much better as it implicitly makes the claim that what we had to prove has been proved. Otherwise it may look like we wrote the cosine formula at the end to fool the marker into thinking that the solution had been given. Also, using the word ‘deduce’ in the final sentence explains where the result came from.

Explain your assertions

Rather than merely make an assertion, say where it comes from. That is, use sentences containing

‘as, because, since, due to, in view of, from, using, we have,’ and so on.

For example,

‘Using Theorem 4(i), we see that the solution set is non-empty’

is obviously preferable to

‘The solution set is non-empty,’

and

‘ $x^3 > 0$ because x is positive’

is better than the bare

‘ $x^3 > 0$ ’

since, for a general $x \in \mathbb{R}$, we don’t have $x^3 > 0$. The point is that the reader may be misled into thinking the statement is ‘obviously false’ if they had forgotten that x was positive. It doesn’t hurt to include such helpful comments.

Another example is to say when a rule has been used:

‘ $f'(x) = 2x \cos(x^2)$ by the Chain Rule.’

In this way, you demonstrate your understanding.

Returning to the first few lines of the student’s proof of the cosine formula in Figure 3.1

$$\triangle CBL \quad a^2 = (c+x)^2 + h^2$$

$$\triangle CLA \quad b^2 = h^2 + x^2$$

we have already seen that it would be better to have said

‘In triangle $\triangle CBL$ we have $a^2 = (c+x)^2 + h^2$ and in $\triangle CLA$ we have $b^2 = h^2 + x^2$.’

But what about the next line? It says simply

$$a^2 = c^2 + 2cx + h^2 + x^2$$

Is this a deduction from the diagram? Certainly the first two equalities were, i.e., $a^2 = (c+x)^2 + h^2$ and $b^2 = h^2 + x^2$. In this case the line is not deduced from the diagram but from the first equation by expanding the bracket. So we should say so.

Expanding the brackets we get $a^2 = c^2 + 2cx + h^2 + x^2$

We’ll see that it is not necessary to phrase it this way when we look at the next line:

$$a^2 = b^2 + c^2 + 2cx$$

This comes from substituting the second equation, $b^2 = h^2 + x^2$, into the expanded version of the first, $a^2 = c^2 + 2cx + h^2 + x^2$. Let’s say so.

‘In triangle $\triangle CBL$ we have $a^2 = (c+x)^2 + h^2$ and in $\triangle CLA$ we have $b^2 = h^2 + x^2$. Expanding this first equation and substituting in b^2 from the second we get $a^2 = b^2 + c^2 + 2cx$.’

Note that we have left out the expansion of the brackets. You can include it if you wish but the calculation is so trivial that it is not worth the ink. The reader can check it themselves if they don’t believe us.

Say what you mean

In any writing, saying what you mean is important – and difficult. Precise use of grammar can help in this task.

The first rule is that the reader should not have to deduce what you mean from context, all the necessary information should be there. Nothing should be ambiguous.

The true mathematician is pedantic, and requires that mathematics is precise. Without precision mathematics is nothing. Without it we cannot build with one concept placed on top of another. If one of the ideas is vague or open to different interpretations by different parties, then errors can creep in and the endeavour is unsound. So, be precise!

As an example, use the quantifiers ‘some’ and ‘all’. Rather than say

$$'f(x) = 5,'$$

which is ambiguous – the reader may ask ‘Is it for one x ? At least one x ? All x ? – say

$$'f(x) = 5 \text{ for some } x \in \mathbb{R},' \text{ or } 'f(x) = 5 \text{ for all } x \in \mathbb{R},'$$

depending on the situation.

More will be said in Chapter 10 of the full book on quantifiers to explain the importance of precision in this area.

Using symbols

We now come to tips concerning symbols. There is no escaping that mathematics is highly symbolic, but using lots of mathematical symbols does not make an argument a mathematical one.

Words or symbols?

Symbols are shorthand. For example, a famous theorem by Euler in the theory of complex numbers,

$$e^{2\pi\sqrt{-1}} = 1,$$

is concisely expressed in symbols.³ The equivalent statement written out in words is less impressive:

‘The exponential of two times the circumference of a circle divided by its diameter times the square root of minus one is equal to one.’

However, a good general rule of thumb is to use words. For example, use ‘therefore’ rather than the \therefore symbol. Very few books use it. Similarly,

$$'x \text{ is a rational number} \Rightarrow x^2 \text{ is real},'$$

can be written as

³This is a great theorem – it relates many great numbers, e , π , the square root of -1 and of course two important natural numbers: 1 and the only even prime, 2. In a poll of mathematicians, (Mathematical Intelligencer, Vol 12 no 3, 1990, p37-41), this theorem was voted the most beautiful theorem in mathematics.

‘ x is a rational number implies that x^2 is real.’

In some sentences it is best to avoid mixing symbols and words. For example,

‘The answer=1’

should be written as

‘The answer equals 1.’

Otherwise we produce sentences like

‘The number of people aged over 40 = 5,’

which reads all right, but the eye is drawn to the (erroneous) expression $40 = 5$.

Small numbers used as adjectives should be spelled out. For example,

‘the two sets.’

They should be in numerals when used as names or numbers, as in

‘Lemma 3’ and ‘...has mean equal to 23.’

Another example:

‘One of the roots is 3.’

An exception is the number 1, which traditionally can be either.

Note that symbols which are similar can cause confusion: Clearly differentiate between \in and ε . The former usually denotes membership of a set and the latter is the Greek letter epsilon, but be aware that other writers use them the other way round.

As noted earlier we rewrote the first few lines and to include the standard notation ΔCBL :

‘In triangle ΔCBL we have $a^2 = (c + x)^2 + h^2$ and in ΔCLA we have $b^2 = h^2 + x^2$.’

Equals means equals

The equals sign, $=$, is one of the most common in mathematics, and one of the earliest learned by children. Despite this, or maybe because of it, it is still badly abused.

Let’s go back to the beginning and note that, in using the equals sign, we are asserting that the two objects on either side are *exactly the same* – being almost the same is not enough, being close is not enough, being similar in a poor light and from a distance is not enough! For numbers this idea of equality should be second nature. But it holds for other objects. Thus remember:

Equals means equals.

One consequence is that if on one side of the sign there is a function, then on the other side there must be a function. If on one side there is a set, then on the other there must be a set.

In answer to a question on factorising numbers into primes, one of my students wrote:

Factors: $6 = 2$ and 3 .

Leaving aside the observations that this is a poor sentence and ‘ $6=2$ ’ is not good on the eye, the idea expressed is false. True, 6 is *equal to the product* of 2 and 3 , and so has 2 and 3 as factors, but it is not *equal to* ‘ 2 and 3 ’. A better answer is:

The prime factors of 6 are 2 and 3 .

Similarly, consider the exercise, ‘Find the derivative of x^3 ’, the answer is not

$$x^3 = 3x^2.$$

Now, this does give a mathematical expression, in fact an equation, but it is not what the student wanted to assert. One correct way to write it is

$$\frac{d}{dx}(x^3) = 3x^2.$$

A very common mistake is to use the equals as a link from one line to the next, almost like a sign saying this is the next part of the process. The correct way of displaying results is given next.

Displaying results with the equals sign

If an expression is short, we show working by writing across the page. For example, $(x+3)^2 = (x+3)(x+3) = x^2 + 6x + 9$.

For a longer calculation it is traditional to write down the page like so:⁴

$$\begin{aligned} (x+3)^2 + x^2 &= (x+3)(x+3) + x^2 \\ &= x^2 + 6x + 9 + x^2 \\ &= 2x^2 + 6x + 9. \end{aligned}$$

Sometimes we need to indicate where a particular result came from. Avoid interrupting the flow of the argument like so:

$$\begin{aligned} &= x^2 + 5y \\ \text{by theorem 6 } &= x^2 + 25 \dots \end{aligned}$$

If the details of why a particular step is true need to be included, then do the following. For the sake of argument suppose that $y = 3$ by Theorem 4.6. Then we write

$$\begin{aligned} x^2 + 4x + y &= x^2 + 4x + 3, \text{ by Theorem 4.6,} \\ &= (x+1)(x+3) \dots \end{aligned}$$

Note the punctuation after the symbolic expression on the first line and after the mention of the theorem. It doesn’t read well, but is clear on the page.

⁴Unfortunately, this violates our rule on punctuation, but we do it anyway as it is practical and traditional.

Don't draw arrows everywhere

If a result requires an earlier one, it is tempting to draw a long arrow to point to it. Don't do this on aesthetic grounds. Instead, give the required result a name, number or symbol, so you can refer to it.

Our example in Figure 3.1 uses arrows.

$$\begin{aligned}
 & +h^2 \\
 & + h^2 + x^2 \\
 & \quad \underbrace{\qquad\qquad}_{b^2} \quad \text{---} \quad \Delta \text{CLA} \\
 & \quad \quad \quad b^2 = h^2 + x^2 \\
 & -x \quad \leftarrow \\
 & = \cos(180 - \theta) \\
 & \Rightarrow x = -b \cos \theta \quad \text{Sub into } \quad \boxed{\qquad}
 \end{aligned}$$

We can change this to

'Expanding this first equation and substituting in b^2 from the second we get $a^2 = b^2 + c^2 + 2cx$. (*)

⋮

$\dots \Rightarrow x = -b \cos \theta$. Substituting this into (*) we deduce that $a^2 = b^2 + c^2 - 2cb \cos \theta$.

Exercise 3.1

Re-write the proof of the Cosine Rule so that it follows the suggestions given.

Finishing off

Proof read

Always proofread your work. That is, read through it looking for errors. These could be typographical errors, (also known as typos), where the wrong character is used, e.g., cay instead of cat, or spelling mistakes, e.g., parrallel instead of parallel, grammatical mistakes, e.g., 'A herd of cows are in the field', or even mathematical errors.

Read your work slowly. Reading aloud can help catch many errors as it stops you skimming. Get someone else to read your work as you will often read what you think is there, rather than what actually is there. If your checker misses mistakes, then you are not allowed to blame them. The final responsibility always rests with the writer!

A useful proofreading method is to concentrate on one aspect of proofreading at a time. That is, read through first for accuracy, i.e., is it true? Next, check for spelling, typos, are all the brackets closed?, etc. After that check that the order of the material is correct and that it flows as you read it.

Reflection

Reflection is an important part of the writing process. Put it away for some time and come back to it with a fresh eye. Obviously, this is not possible for work with tight deadlines, but can be done with project work.

When reading through again, ask ‘What can I take away?’ (aim for economy of words) and ‘What can I add?’ (more examples might clarify). For the former remove unnecessary words and sentences. Also ask: ‘Are all the symbols explained and are they necessary? Does it say what I mean and is it simple? Is it more than just a collection of symbols?’ And of course, most importantly, ‘Did I write in sentences?’

Exercises

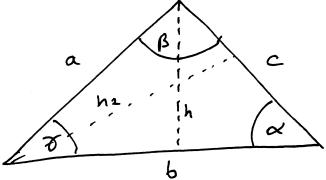
Exercises 3.2

- (i) **The Sine Rule:** Suppose that we have a triangle with sides of length a , b and c with the angles opposite these sides labelled α , β and γ respectively.

Then

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$$

In an exam a student answered the question ‘State and prove the Sine Rule’ with the following:

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$$


$$\begin{aligned} \sin \alpha &= \frac{h}{c} & h &= \sin \alpha c \\ \sin \gamma &= \frac{h}{a} & h &= \sin \gamma a \end{aligned} \quad \text{and} \quad \begin{aligned} \sin \beta &= \frac{h_2}{a} & \sin \alpha &= \frac{h_2}{b} \\ \frac{\sin \alpha}{a} &= \frac{\sin \beta}{b} & a \sin \beta &= h_2 \\ \frac{\sin \beta}{b} &= \frac{\sin \gamma}{c} & a \sin \beta &= b \sin \gamma \\ \frac{\sin \alpha}{a} &= \frac{\sin \beta}{b} & \frac{\sin \beta}{b} &= \frac{\sin \gamma}{c} \end{aligned}$$

Re-write this answer so that it is correctly written and easily comprehended.

- (ii) If you know how to find maxima and minima as well as curve sketching you should rewrite the answer to the exercise ‘Find the maximum and minimum values of the function $f(x) = 2x^3 - 12x^2 + 18x$ and sketch its graph.’

$$\begin{aligned}
 f &= 2x^3 - 12x^2 + 18x \\
 &= 6x^2 - 24x + 18 \Rightarrow x = \frac{24 \pm \sqrt{24^2 - 4 \times 18 \times 6}}{2 \times 6} \\
 &\quad \frac{24 \pm \sqrt{144}}{12} \\
 &\quad 2 \pm 1 \\
 &\quad 1, 3. \\
 \frac{d^2y}{dx^2} &= 12x - 24 \Rightarrow \frac{d^2y}{dx^2} = 12x - 24 = -12 < 0 \quad \text{max} \\
 \frac{d^2y}{dx^2} &= 12x^2 - 24 = 12 > 0 \quad \text{min} \\
 y &= 2 - 12 + 18 = 8 \\
 y &= 2 \times 27 - 12 \times 9 + 18 \times 3 \\
 &= 0.
 \end{aligned}$$

- (iii) Find some of your old mathematics exercises and re-write them so that the exposition is crystal clear. You can also take examples from friends.

Summary

- Write in simple, punctuated sentences.
- Keep it simple.
- Explain what you are doing.
- Explain your assertions.
- Say what you mean.
- In general, use words rather than symbols.
- Use equals properly – equals means equals.
- Don't draw arrows everywhere – use symbols or numbers to identify equations.
- Proof read.
- Reflect.

CHAPTER 4

Writing mathematics II

Learn as much by writing as by reading.
Lord Acton, Lectures on Modern History (1906)

In the previous chapter we were concerned with the basic principles of writing mathematics. Here we shall become more specific and give particular examples of ways to improve the presentation of mathematics.

Expressing yourself clearly

If you use ‘if’, then use ‘then’

If you use the word ‘if’, then use the word ‘then’ as well. (It is traditional to punctuate with a comma before the word ‘then’ but this seems to be dying out.) That is,

‘If x is odd, then x^2 is odd’

is preferable to

‘If x is odd, x^2 is odd.’

Statements can be written to include ‘if’ but for which ‘then’ is not needed:

‘ x^2 is odd if x is odd.’

But, in general, use ‘then’ as confusion may result when it is not employed. For example, what is the meaning of

‘If $a > 0, b > 0, a + b > 0$?’

It could mean

‘if $a > 0$, then $b > 0$ and $a + b > 0$.’

This is because maybe a is positive and so forces b to be positive, for example, $b = 5a$. Alternatively, it could mean

'If $a > 0$ and $b > 0$, then $a + b > 0$ '

which is always true. The point is that omission of 'then' can lead to ambiguity. In many cases the reader can deduce the meaning of a statement with the 'then' omitted, but it is safer to include it. The reader should not have to deduce the meaning from other clues. Remember, the responsibility of communication rests with the writer.

Not everything is a 'formula': Call things by their correct name

A lot of attention is paid to formulas in mathematics and so many beginners call any collection of symbols a formula or an equation. Generally, such a collection is called an **expression** or **term**, e.g., $3x^2 - 7x$ is an expression.

An **equation** involves stating that two expressions are equal, for example, $3x^2 - 7x = 4x$. Note that an **inequality**, such as $x \leq 5$, is not an equation as an *equation* should be an *equality*.

A **formula** expresses some relationship or rule. It is often used when a method of calculating something from another expression is given, e.g., for studying $ax^2 + bx + c = 0$ we look at $D = \sqrt{b^2 - 4ac}$. This is the formula for D (sometimes known as the discriminant.)

So, ensure that you call an object by its correct name. Care should also be taken to distinguish between concepts, for example, between being a set and being an element of a set.

Another example is that you should distinguish between a function and its value at a point, that is, between f and $f(x)$. The first represents a function which we have called f , while $f(x)$ is the *value* of the function f at x (see Chapter 1 of full book). Strictly speaking, you should not call $f(x)$ a function, but this distinction is often ignored by mathematicians who should know better (and that includes me).

In summary, from the above, $f(x) = 3x^2$ is an equation (or expression or formula) which gives the values for a particular function f .

Avoid 'it'

Be explicit about what you are talking about. The word 'it' is a very useful word in the English language since it allows us to talk about something without really saying what that something is. Unfortunately, 'it' can be ambiguous. When writing mathematics, the word can be used as a way of disguising that we don't really know what we are talking about. Mostly this is self-deception - we do it subconsciously rather than deliberately.

If you find that you have used the word 'it', then replace it with the proper word or phrase. Does the sentence still make sense? If not, change it.

Decimal approximations

We tend not to use decimal approximations of numbers in pure mathematics but they are common in applied mathematics or statistics. This does not mean that they are never used in pure mathematics. If you are drawing figures, then it is permissible to use decimal approximations. For example, in finding where a quadratic crosses the axes.

So in pure mathematics if the final answer is $\sin 7$, then leave it as that rather than say 0.656986598. If you have to use approximations, then use the

symbols \simeq or \approx . So $\pi \approx 3.14$ and $\sqrt{2} \simeq 1.41$ are both acceptable. The point is that writing $\pi = 3.14$ is actually wrong – recall that equals means equals, page 11.

For expressions like $\sin(\pi/6)$ use $1/2$ or 0.5 . Note that the use of 0.5 is all right as this is not an approximation - we are against approximations, not decimals.

Words or symbols?

Don't begin sentences with a symbol

Do not begin sentences with symbols. Thus, the next sentence should not be used.

‘ f is a function with domain \mathbb{R} .’

This avoids violating the rule that every sentence begins with a capital letter, but it applies even if the symbol is a capital, for example, ‘ X is a finite set’ is bad. We could get a sentence like

‘Suppose that x is an element of X . x is not in Y .’

Given that, in mathematics, the full stop functions as a multiplication symbol, then the sentence may be read as having X times x in the middle.

To avoid the problem use a description of the object the symbol represents. For instance, we can rewrite the earlier incorrect sentences as

‘The function f has domain \mathbb{R} .’

‘The set X is finite’

and

‘Suppose that x is an element of X and x is not in Y .’

Another approach is to employ phrases like ‘We have’ to begin sentences:

‘We have $g(x) = 2x^4 - 5x - 3$.’

The curse of the implication symbol

A symbol commonly abused and overused by the novice mathematician is the symbol of implication: \Rightarrow . First, this should be read as ‘implies’ or ‘implies that’, and should not be used as an equals sign, so do not write $5 - 3 \Rightarrow 2$.

As we will see in later chapters (and there is no harm in stating it now) the correct usage is

‘statement \Rightarrow statement.’

For example,

‘ x is odd $\Rightarrow x^2$ is odd.’

In much the same way no sentence would begin with an equals sign, no sentence would begin ‘implies that’ so never begin one with \Rightarrow . If you feel the need to begin with an implication, then it is probably better to write ‘This implies that ...’. Similarly many students use it as a method of connecting one line to the next, something like ‘this is what we do next.’

If you are unsure where to use it, then say the sentence out loud with the symbol \Rightarrow replaced by ‘implies’ or ‘implies that’ as often this will help.

If you are unsure how to use \Rightarrow or \Leftrightarrow , then consult Chapters 7 and 9 of the full book.

Use common symbols and notation

Some symbols are given a common fixed meaning. The prime example of this is π which represents the circumference of a circle divided by its diameter. It would be unusual to denote this ratio with anything other than π , but it is technically correct to start an argument with ‘Let α be the circumference of a circle divided by its diameter’.

However, π can mean other things as well, e.g., it is often used for a projection map (since π represents p , and p is the first letter of projection). It is also used for the fundamental group of a space in algebraic topology.

Other symbols, such as ε usually mean a small positive number, n is a natural number and so on. Get to know these conventions and use them as they make your work easier to read.

Define your symbols and notation

As we have seen, many symbols get used for certain objects, such as f regularly denotes a function, and you may have noticed that I have been using capital letters such as X and Y for sets and lower case letters, such as x , to denote the elements of those sets.

Despite these conventions you should define your notation so that it is totally unambiguous since a reader may use a different notation to you. Thus, write ‘Let X be a set’ rather than just use X without explanation. Another example of this can be seen in Exercise 3.2 where the student has given a statement of the Sine Rule where the letters a , b , c , α , β , γ have not been defined. Did you include a definition of the letters in *your* answer to that question?

Some notation does not need introduction; most mathematicians will understand what π stands for, provided it is in context, and will know that Σ refers to summation, \int to integration and so on.

Making improvements

Use connecting phrases

Another common problem in writing is that assumptions and deductions in an argument are not clearly distinguished. To avoid this problem you should, in constructing an explanation, use connecting words and phrases, such as

‘hence, as, therefore, since, and so,’

to indicate that implications and deductions are being made. If you look at a mathematics book, then you will see that there are plenty of uses of since, hence and therefore.

As an example, consider explaining that the order of brackets is important when dividing numbers. It is not sufficient to write

$$\text{‘}(8/2)/4 = 1, 8/(2/4) = 16.\text{’}$$

We should say

$$\text{‘We have } (8/2)/4 = 1, \text{ but } 8/(2/4) = 16.\text{’}$$

Notice that the ‘but’ brings attention to the important idea that there is a difference between the calculations. Using the word ‘and’ would not make the statement incorrect, but attention would not have been drawn.

Use Synonyms

Repetition of words can lead to boredom for the reader. The use of synonyms helps to make the material more interesting.

Some synonyms for deduction are:

‘hence, so, it follows, it follows that, as a result, consequently, therefore, thus, accordingly, then.’

Like most synonyms there can be slight differences in usages, one cannot always replace ‘hence’ by ‘then’.

Synonyms for explanations are:

‘as, because, since, due to, in view of, owing to.’

We can use a construction involving ‘let’ in place of one involving ‘suppose’. For example,

‘Let X be a set’

in place of

‘Suppose that X is a set.’

Note that we say ‘suppose that’, i.e., ‘suppose’ is followed by ‘that.’

Exercises

Exercise 4.1

- (i) Can you improve on your rewriting of the mathematics in the exercises of the previous chapter?
- (ii) Now more generally, find some of your previous answers to exercises and rewrite them using the guidance from this chapter.

Summary

- ❑ If you use if, then use then.
- ❑ Not everything is a formula. Call things by their correct name.
- ❑ Don't use the word 'it'.
- ❑ Avoid decimal approximations in pure mathematics.
- ❑ Don't begin sentences with a symbol.
- ❑ Use the implication symbol, \implies , correctly.
- ❑ Use common symbols and notation and define them first.
- ❑ Use connecting phrases and synonyms.

APPENDIX A

Greek alphabet

In the pronunciation guide the letter i should be pronounced as in big. The letter y should be pronounced as in the word my

Name	Upper	Lower	Pronunciation
Alpha	A	α	al-fa
Beta	B	β	bee-tah
Gamma	Γ	γ	gam-ah
Delta	Δ	δ	del-tah
Epsilon	E	ϵ or ε	ep-sigh-lon / ep-sil-on
Zeta	Z	ζ	zee-tah
Eta	H	η	ee-tah
Theta	Θ	θ	thee-tah
Iota	I	ι	y-oh-tah
Kappa	K	κ	cap-ah
Lambda	Λ	λ	lam-dah
Mu	M	μ	mew
Nu	N	ν	new
Xi	Ξ	ξ	ksy
Omicron	O	\circ	oh-mi-kron
Pi	Π	π	py
Rho	P	ρ	row (as in propelling a boat)
Sigma	Σ	σ	sig-mah
Tau	T	τ	taw
Upsilon	Υ	υ	up-sigh-lon or up-sil-on
Phi	Φ	ϕ or φ	fy
Chi	X	χ	ky
Psi	Ψ	ψ	psy
Omega	Ω	ω	oh-meg-ah

APPENDIX B

Commonly used symbols

e	base of natural logarithms
i	square root of -1
∞	infinity
\forall	for all
\exists	there exists
\square	end of proof marker
∇	nabla
\aleph	aleph
\emptyset	empty set
\sum	sum
\prod	product
\in	is an element of / is in
\subseteq	is a subset of
\subset	is a subset of (but is not equal to)
\cap	intersection
\cup	union
\Rightarrow	implies that (often incorrectly used)
\Leftrightarrow	equivalent to, (also known as ‘if and only if’)
$'$	prime
\mapsto	maps to
\twoheadrightarrow	surjection
\hookrightarrow	injection
\propto	proportional to
\equiv	equivalent/congruent to
\approx	approximately equal
\perp	perpendicular to
\neg	negation
\sim	tilde
$\hat{}$	hat
\therefore	therefore
\because	because
\mathbb{N}	natural numbers, defined as the set $\{1, 2, 3, 4, \dots\}$ (but some lectures include 0).
\mathbb{Z}	integers, defined as $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
\mathbb{Q}	rational numbers, defined as numbers of the form p/q where $p, q \in \mathbb{Z}$ and $q \neq 0$
\mathbb{R}	real numbers, erm ... you'll have to wait until you're older for a definition!
\mathbb{C}	complex numbers, numbers of the form $a + ib$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$

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