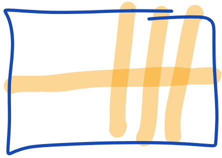
Topic 1 : preliminaries
Topic 2 : Linear Algebra

1 vectors
2
3

vector space

SIT787: Mathematics for Artificial Intelligence
Topic 2: Linear Algebra
Part 1

data table



NLP

word vector
document vector

Asef Nazari

School of Information Technology, Deakin University

vectors

Linear Algebra

$\vec{x}$ vectors

A matrices

- A good understanding of Linear Algebra
  - is essential for understanding and working with many machine learning algorithms,
  - especially deep learning algorithms
- The entities we deal with are
  - scalars $1, \; 5/2, \; \sqrt{3}, \; -\frac{\sqrt{2}}{\sqrt{2}}$
  - vectors
  - matrices
  - tensors

$3 \rightarrow \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \rightarrow$ matrix

- Scalars: a single number $c \in \mathbb{R}$ $c = 2 \in \mathbb{R}$ $c = -\sqrt{3} \in \mathbb{R}$ $c = 0 \in \mathbb{R}$
- Vectors: an array of numbers (scalars)
  - order is important, and each element of a vector has an index

ordered list

$$\underset{\sim}{x} = \underline{\vec{x}} = \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

$x_1 \in \mathbb{R}$, $x_2 \in \mathbb{R}$, $x_n \in \mathbb{R}$

n-elements of scalars

$\mathbb{R}^n$

column vectors

row vectors

$\mathbb{R}^4$

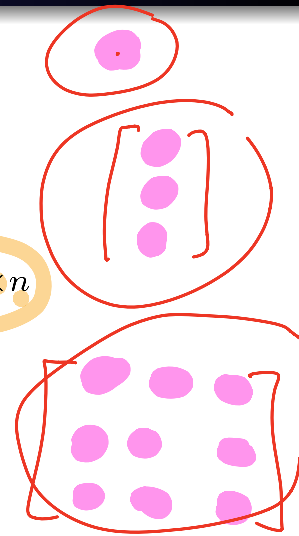
$\begin{bmatrix} 1 \\ 0 \\ -\sqrt{2} \\ 5 \end{bmatrix} \in \mathbb{R}^4$ $\begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^4$

- Matrices: a 2-D array of numbers
  - each element is identifiedby two indices

$A_{2\times3} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 7 \end{bmatrix} \in \mathbb{R}^{2\times3}$

a_{ij}

$$A_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in \mathbb{R}^{m\times n}$$



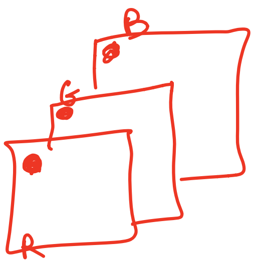
- Tensors: an array with more than two axes
  - A_{ijk}
  - an RGB color image has three axes

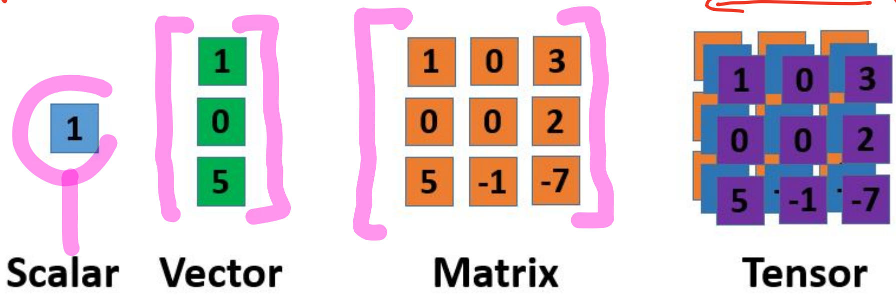
m rows n columns p layers

$A_{m\times n\times p}$

a_{ijk}

red green & blue





3x3x3

Operations

scalars → vectors → matrices → tensors

LA

- Transpose
- Addition (subtraction)
- Multiplying by a scalar
- Products
  - vector product
  - matrix product

Data tables

relational table

matrix features

cases

	A	B	C	D	E	F	G	H	I
1		Database clients of Jolly Day							
2	№	Customer	Type	Country	City	Contract Number	Date	Limitation years	Contact Manager
3	1	Intersection	com.network	USA	New York	2314589	12.12.2012	2	Aaron
4	2	Magnet	com.network	USA	New York	2432656	27.08.2014	3	Alex
5	3	Perspective korp.	warehouse	Belarus	Minsk	2456983	31.12.2014	2	Ashley
6	4	Driveway	enterprise	USA	New York	2408570	24.04.2014	5	Aaron
7	5	near	enterprise	USA	Los Angeles	2481553	06.05.2015	2	Ashley
8	6	Nori	warehouse	Japan	Tokyo	2506369	09.09.2015	2	Blake
9	7	Nevsky comp.	com.network	Russia	Moscow	2337735	15.04.2013	1	Caroline
10	8	Perspective korp.	enterprise	Belarus	Minsk	2361112	17.08.2013	2	Daniel
11	9	in touch	warehouse	USA	San Francisco	2384723	20.12.2013	2	Alex
12	10	Nardis	com.network	Japan	Tokyo	2531433	14.01.2016	3	Blake

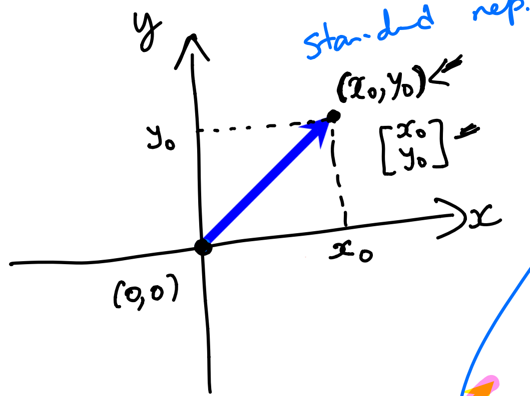
- 2D array
- each case is a vector
- each variable is a vector

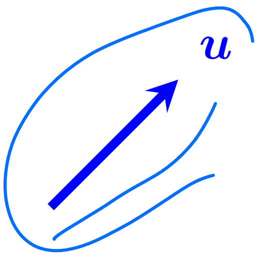
Scalars $\mathbb{N}, \mathbb{Z}, \mathbb{R}$

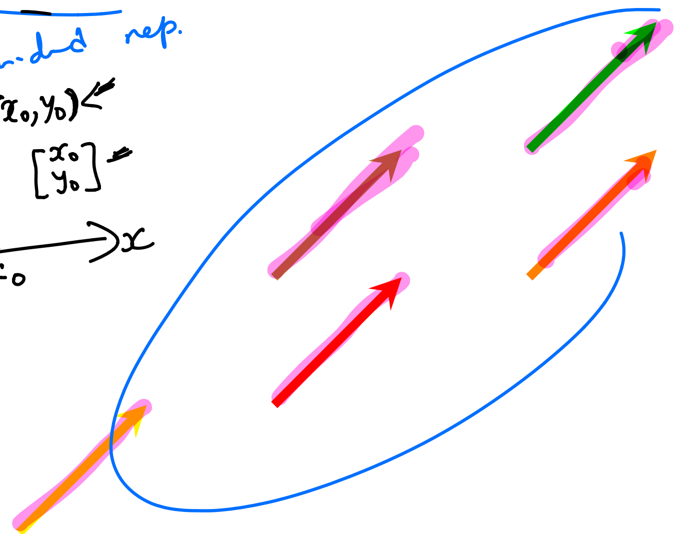
3

- a quantity with length and direction

standard rep.



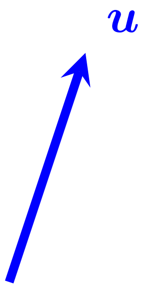




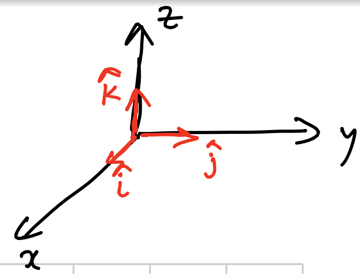
$$\frac{2}{3} = \frac{4}{6} = \frac{20}{30}$$

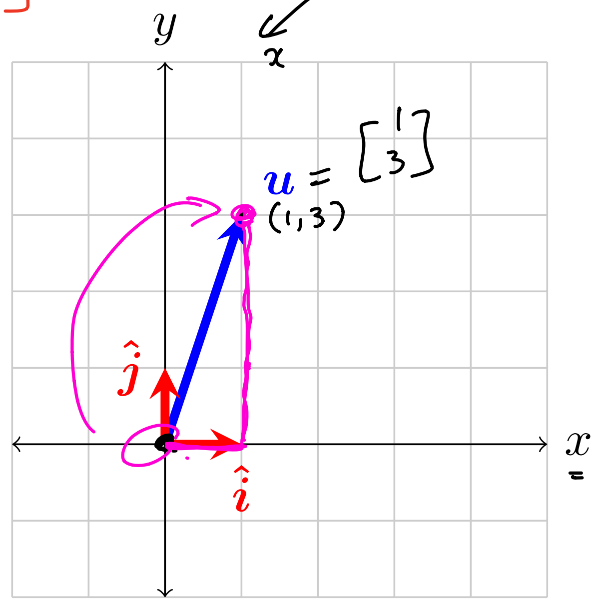
standard

Vectors in a coordinating system



$$\vec{u} = \begin{bmatrix}1\\3\end{bmatrix} = 1\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix}$$
$$= \hat{i} + 3\hat{j}$$





$$\boldsymbol{u} = \begin{bmatrix}1\\3\end{bmatrix} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$$

$$\hat{\boldsymbol{i}} = \begin{bmatrix}1\\0\end{bmatrix} \text{ and } \hat{\boldsymbol{j}} = \begin{bmatrix}0\\1\end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

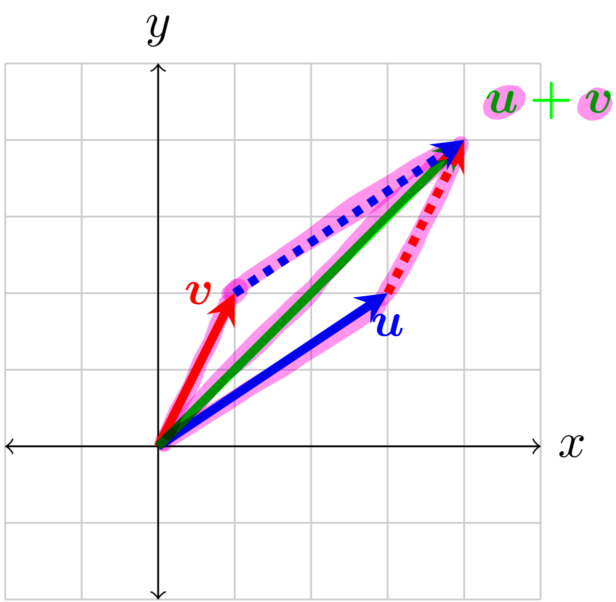
$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \text{ and } \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

$\mathbb{R}^n$

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \text{ and } \boldsymbol{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$$

$$\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix}$$



- $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$ commutative
- $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$

associative

$\vec{u} \parallel 2\vec{u}$ $2\vec{u}$ $\vec{u}$ $\vec{u} \parallel -\vec{u}$ $\vec{u}$ $-2\vec{u}$

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^n$$

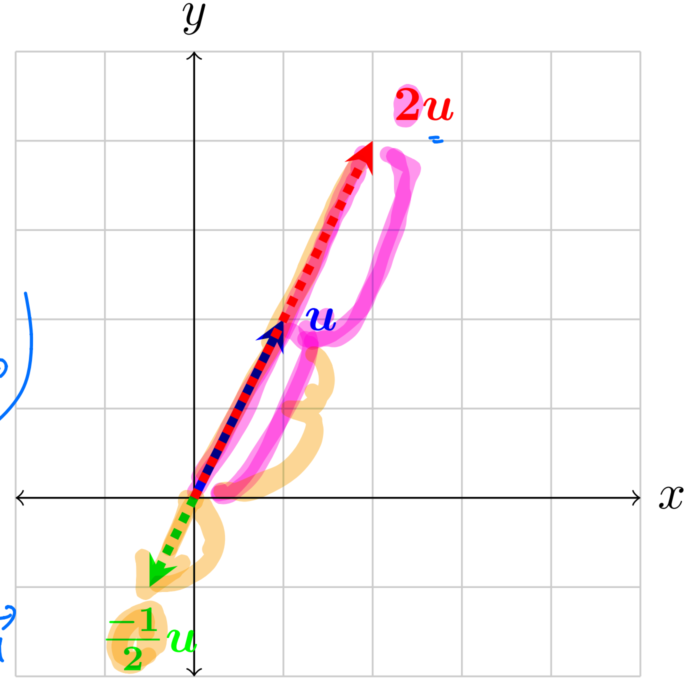
$\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}$ $2\vec{x} = \begin{bmatrix} 2 \\ -2 \\ 14 \end{bmatrix}$ $0\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$

For $c \in \mathbb{R}$

scaling a vector

$\vec{x}$ $2\vec{x}$ $-2\vec{x}$ $\vec{v}$ $\vec{u}$

$$c\boldsymbol{u} = \begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix} \in \mathbb{R}^n$$

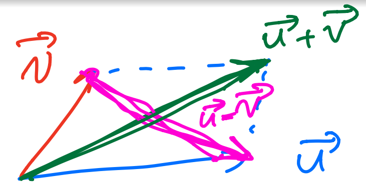


- $c\boldsymbol{u}$ depends on $\boldsymbol{u}$ so they are linearly dependent, and they are not linearly independent!
- $c\boldsymbol{u}$ and $\boldsymbol{u}$ are parallel

$\vec{u} \parallel c\vec{u}$

$\vec{u} \parallel c\vec{u}$ $\vec{u} \parallel -c\vec{u}$

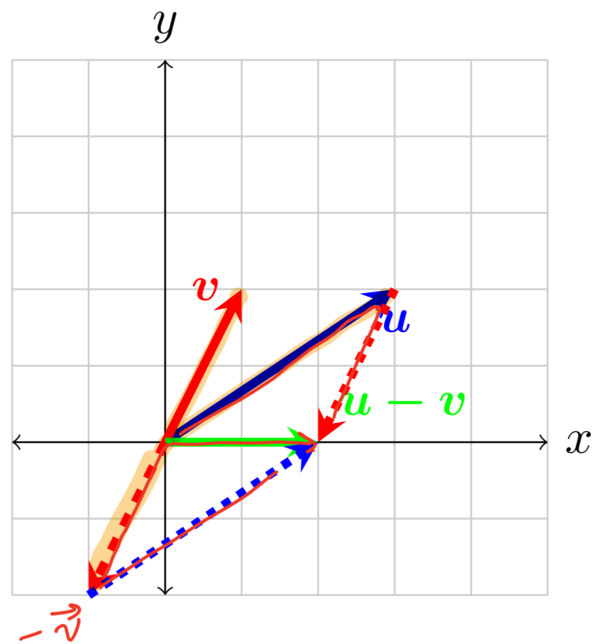
$\vec{u}$ $c\vec{u}$ $\vec{u}$ $c\vec{u}$



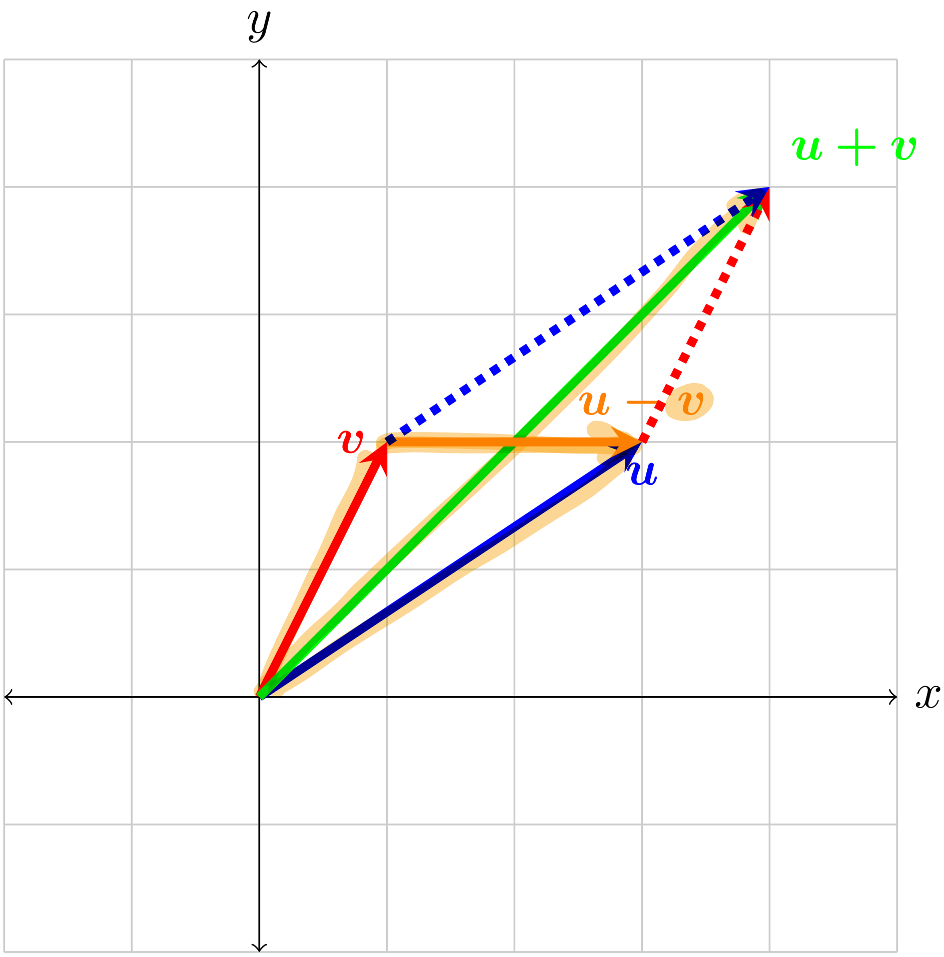
- Vectors $\boldsymbol{u} \in \mathbb{R}^n$ and $\boldsymbol{v} \in \mathbb{R}^n$ are given
- To find $\boldsymbol{u} - \boldsymbol{v}$
- find $-\boldsymbol{v}$ first
- then $\boldsymbol{u} + (-\boldsymbol{v})$

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^n \text{ and } \boldsymbol{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$$

$$\boldsymbol{u} - \boldsymbol{v} = \begin{bmatrix} u_1 - v_1 \\ \vdots \\ u_n - v_n \end{bmatrix} \in \mathbb{R}^n$$



Vector Addition and Subtraction: Paralleogram



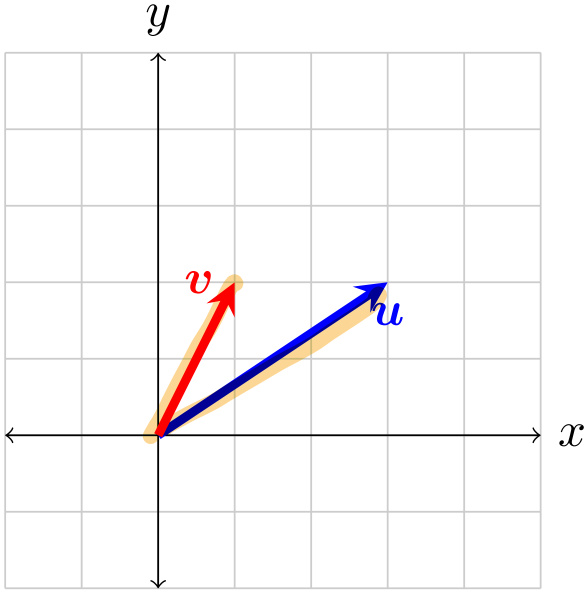
Independent vectors

linear

linear dependence $\vec{u} \parallel \vec{v}$ $\quad \vec{u} = c\vec{v}$

linear independence $\vec{u} \not\parallel \vec{v}$

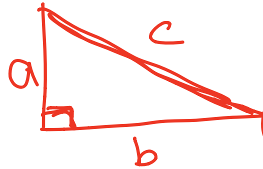
- There is no way one can express $\boldsymbol{u}$ as a scalar product of $\boldsymbol{v}$, or $\boldsymbol{v}$ as a scalar product of $\boldsymbol{u}$.
- They are independent vectors.
- But $\boldsymbol{u}$ and $c\boldsymbol{u}$ are dependent vectors. They are parallel.



Modulus, length, or magnitude of a vector

$\|u\| \geq 0$

vector quantity $\begin{cases} \text{length} \\ \text{direction} \end{cases}$



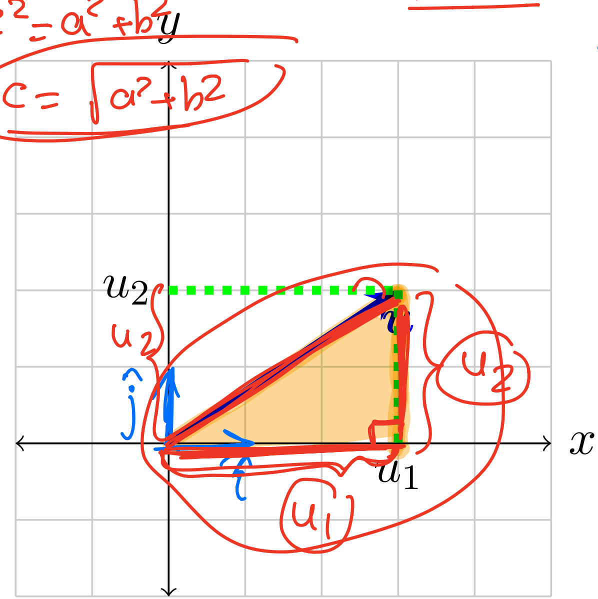
$c^2 = a^2 + b^2$

$c = \sqrt{a^2 + b^2}$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

- $\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^2$
- $\boldsymbol{u} = u_1 \hat{\boldsymbol{i}} + u_2 \hat{\boldsymbol{j}}$
- Pythagoras' theorem:

  length of $\boldsymbol{u} = \sqrt{u_1^2 + u_2^2}$
- $||\boldsymbol{u}|| = \sqrt{u_1^2 + u_2^2}$
- If $\boldsymbol{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^n$, then



$$\boxed{||\boldsymbol{u}|| = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}}$$

$$||\boldsymbol{u}|| = \sqrt{\sum_{i=1}^{n} u_i^2}$$

$= \sqrt{2}$

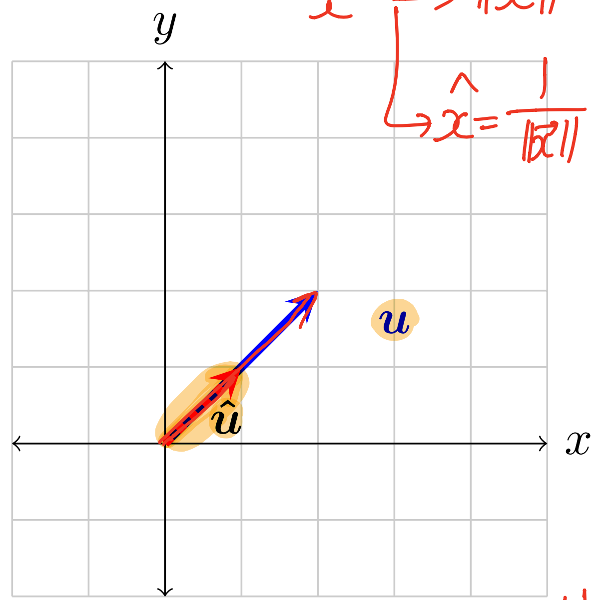
$\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\quad ||\vec{x}|| = \sqrt{(1)^2 + (-1)^2 + (0)^2} = \sqrt{}$

$\vec{x} \to \|\vec{x}\| = \sqrt{\sum_{i=1} x_i^2}$

$\to \hat{x} = \frac{1}{\|\vec{x}\|}\vec{x}$

- A vector has a length and direction
- for $\boldsymbol{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$,

$$||\boldsymbol{u}|| = \sqrt{\sum_{i=1}^{n} u_i^2}$$



- The unit vector in the doirection of $\boldsymbol{u}$ is $\hat{\boldsymbol{u}}$

$$\hat{\boldsymbol{u}} = \left(\frac{1}{\text{length } \boldsymbol{u}}\right)\boldsymbol{u} = \frac{1}{||\boldsymbol{u}||}\boldsymbol{u}$$

$$\vec{\boldsymbol{u}} = ||\boldsymbol{u}||\hat{\boldsymbol{u}}$$

$\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\quad \|\vec{x}\| = \sqrt{2}$

$\hat{x} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$

$\vec{x} \parallel \hat{x}$

linearly depent

$\vec{x} \in \mathbb{R}^n \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \vec{x} \qquad x_i \qquad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

$$\vec{x} \pm \vec{y} = \begin{bmatrix} x_1 \pm y_1 \\ x_2 \pm y_2 \\ \vdots \\ x_n \pm y_n \end{bmatrix}$$

we have m vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\} \subseteq \mathbb{R}^n$

$$\vec{v}_1 = \begin{bmatrix} v_{11} \\ v_{12} \\ \vdots \\ v_{1n} \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} v_{21} \\ v_{22} \\ \vdots \\ v_{2n} \end{bmatrix} \quad \cdots \quad \vec{v}_k = \begin{bmatrix} v_{k1} \\ v_{k2} \\ \vdots \\ v_{kn} \end{bmatrix}$$

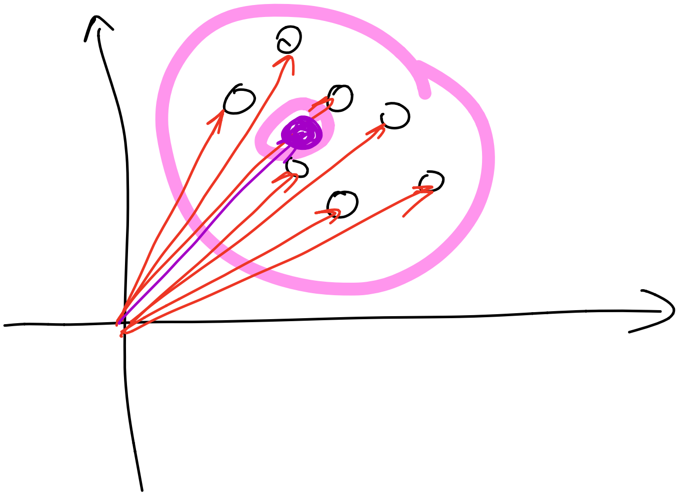
Find the center of these m vectors

center $\quad \vec{a} = \frac{1}{n}(\vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_m)$

$$= \frac{1}{n}\left(\begin{bmatrix} v_{11} \\ v_{12} \\ \vdots \\ v_{1n} \end{bmatrix} + \begin{bmatrix} v_{21} \\ v_{22} \\ \vdots \\ v_{2n} \end{bmatrix} + \cdots + \begin{bmatrix} v_{m1} \\ v_{m2} \\ \vdots \\ v_{mn} \end{bmatrix} \right)$$

$$= \frac{1}{n}\left(\begin{bmatrix} v_{11} + v_{21} + \cdots + v_{m1} \\ v_{12} + v_{22} + \cdots + v_{m2} \\ \vdots \\ v_{1n} + v_{2n} + \cdots + v_{mn} \end{bmatrix} \right)$$

$$= \begin{bmatrix} \frac{1}{n}(v_{11} + v_{21} + \cdots + v_{m1}) \\ \frac{1}{n}(v_{12} + v_{22} + \cdots + v_{m2}) \\ \vdots \\ \frac{1}{n}(v_{1n} + v_{2n} + \cdots + v_{mn}) \end{bmatrix}$$



$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

$$\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

direction of $\vec{x}$ $\qquad \hat{x} = \frac{1}{\|\vec{x}\|}\vec{x}$

$$\vec{x} \pm \vec{y}$$

product between two vectors

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \qquad \vec{x} \bullet \vec{y} = \overset{\text{number}}{\boxed{x_1y_1 + x_2y_2 + \cdots + x_ny_n}}$$

dot product

inner product

$$\underline{\boldsymbol{u}} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \text{ and } \underline{\boldsymbol{v}} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$$

$$\underline{\boldsymbol{u} \cdot \boldsymbol{v}} = \underline{u_1 v_1} + u_2 v_2$$

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \text{ and } \boldsymbol{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$$

$$||u|| = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$$

$$||u||^2 = u_1^2 + u_2^2 + \dots + u_n^2 = \sum_{i=1}^{n} u_i^2$$

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n = \underline{\sum_{i=1}^{n} u_i v_i}$$

Commutative

$$||\vec{u}||^2 = \vec{u} \cdot \vec{u}$$

- $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$
- $\boldsymbol{u} \cdot \boldsymbol{u} = \sum_{i=1}^{n} u_i u_i = \sum_{i=1}^{n} u_i^2 = ||\boldsymbol{u}||^2 \implies ||\boldsymbol{u}|| = \underline{\sqrt{\boldsymbol{u} \cdot \boldsymbol{u}}}$
- $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}$

$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

number $\cancel{\cdot}$ is not defined

number

- $\boldsymbol{u} \cdot (\boldsymbol{v} \cdot \boldsymbol{w}) = (\boldsymbol{u} \cdot \boldsymbol{v}) \cdot \boldsymbol{w}$?

scalar

- For $c_1, c_2 \in \mathbb{R}$, $(c_1\boldsymbol{u}) \cdot (c_2\boldsymbol{v}) = c_1 c_2 \boldsymbol{u} \cdot \boldsymbol{v}$?
- For $c \in \mathbb{R}$, $c(\boldsymbol{u} \cdot \boldsymbol{v}) = (c\boldsymbol{u}) \cdot \boldsymbol{v} = \boldsymbol{u} \cdot (c\boldsymbol{v})$

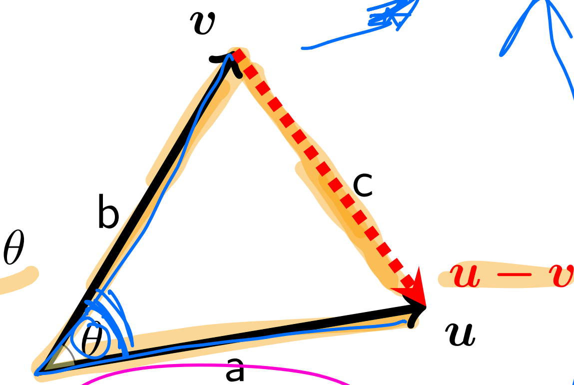
$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}$

$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ $\quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

$\boxed{\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n}$

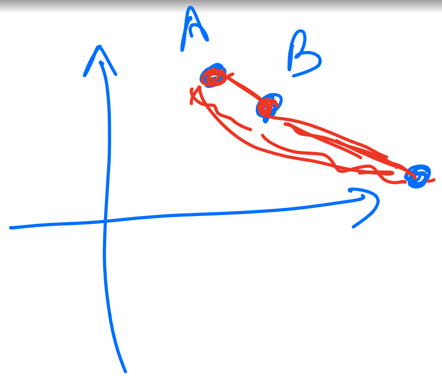
- Cosine rule in triangles
  $c^2 = a^2 + b^2 - 2ab\cos\theta$

  $||\boldsymbol{u} - \boldsymbol{v}||^2 = ||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2 - 2||\boldsymbol{u}||\,||\boldsymbol{v}||\cos\theta$



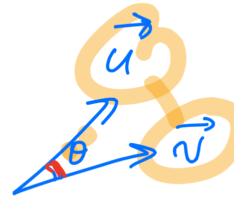
- We know that $\boldsymbol{u} \cdot \boldsymbol{u} = ||\boldsymbol{u}||^2$
  - $||\boldsymbol{u} - \boldsymbol{v}||^2 = (\boldsymbol{u} - \boldsymbol{v}) \cdot (\boldsymbol{u} - \boldsymbol{v}) =$ $\vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}$ $\quad = 2\vec{u}\cdot\vec{v}$
    $(\boldsymbol{u} \cdot \boldsymbol{u}) + (\boldsymbol{v} \cdot \boldsymbol{v}) - 2(\boldsymbol{u} \cdot \boldsymbol{v}) = ||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2 - 2(\boldsymbol{u} \cdot \boldsymbol{v})$
  - $||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2 - 2(\boldsymbol{u} \cdot \boldsymbol{v}) = ||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2 - 2||\boldsymbol{u}||\,||\boldsymbol{v}||\cos\theta$
    Then

$$\boxed{\boldsymbol{u} \cdot \boldsymbol{v} = ||\boldsymbol{u}||\,||\boldsymbol{v}||\cos\theta}$$

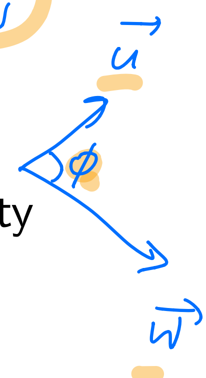


distance-based similarity

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \text{ and } \boldsymbol{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$$



$\theta < \phi$



- $\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n = \sum_{i=1}^{n} u_i v_i$
- $\boldsymbol{u} \cdot \boldsymbol{v} = ||\boldsymbol{u}||\,||\boldsymbol{v}|| \cos\theta$
- Cosine between two vectors is a measure of their similarity

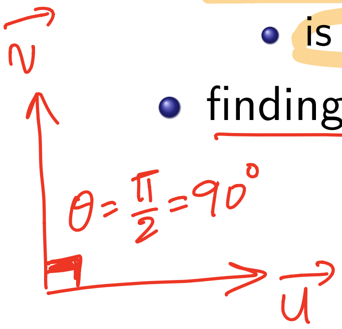
$$\cos\theta = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{||\boldsymbol{u}||\,||\boldsymbol{v}||} = \frac{\sum_{i=1}^{n} u_i v_i}{\left(\sqrt{\sum_{i=1}^{n} u_i^2}\right)\left(\sqrt{\sum_{i=1}^{n} v_i^2}\right)}$$

measure of similarity

- Finding vector length $||\boldsymbol{u}|| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}}$
  - is used to find unit vectors

$\hat{u} = \frac{1}{||\vec{u}||} \vec{u}$

- finding angle between vectors



$$\cos\theta = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{||\boldsymbol{u}||\,||\boldsymbol{v}||}$$

$\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{||\vec{u}||\,||\vec{v}||}\right)$

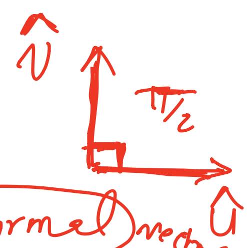
$\cos\left(\frac{\pi}{2}\right) = 0$

- To compare vectors: How similar they are!
- The less the angle between them the more similar they are!
- If two vectors are perpendicular (orthogonal), the angle between them is $\theta = \frac{\pi}{2}$, and $\boxed{\cos\frac{\pi}{2} = 0}$, then

$$\boxed{\boldsymbol{u} \cdot \boldsymbol{v}} = ||\boldsymbol{u}||\,||\boldsymbol{v}|| \underbrace{\cos\theta}_{=0} = 0$$

- Orthogonal vectors $\boldsymbol{u} \perp \boldsymbol{v} \Leftrightarrow \boldsymbol{u} \cdot \boldsymbol{v} = 0$
- Orthonormal vectors

orthogonality of unit vectors



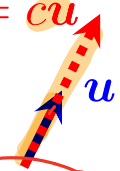
orthonormal vectors

$\boldsymbol{v} = c\boldsymbol{u}$

u//v

dependent linearly

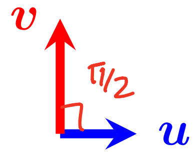
$\cos(0) = 1$



$\theta = 0$

$\boldsymbol{u} \cdot \boldsymbol{v} = ||\boldsymbol{u}|| \, ||\boldsymbol{v}|| \cdot 1$

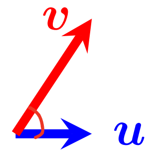
$c > 0$



$\theta = \frac{\pi}{2}$

$\boldsymbol{u} \cdot \boldsymbol{v} = 0$

orthogonal

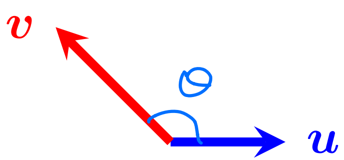


$\theta < \frac{\pi}{2}$

$\boldsymbol{u} \cdot \boldsymbol{v} > 0$

$\vec{u} \cdot \vec{v} = \underbrace{||\vec{u}||}_{\geq 0} \, \underbrace{||\vec{v}||}_{\geq 0} \, \underbrace{\cos(\theta)}_{> 0}$

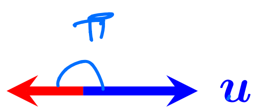
$$\vec{u} \cdot \vec{v} = \begin{cases} > 0 & \theta < \frac{\pi}{2} \\ = 0 & \theta = \pi/2 \\ < 0 & \theta > \pi/2 \end{cases}$$



$\theta > \frac{\pi}{2}$

$\boldsymbol{u} \cdot \boldsymbol{v} < 0$

$\boldsymbol{v} = c\boldsymbol{u}$



$\theta = \pi$

$\boldsymbol{u} \cdot \boldsymbol{v} = -||\boldsymbol{u}|| \, ||\boldsymbol{v}||$

$c < 0$

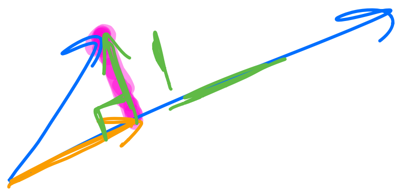
$\vec{u} \, // \, \vec{v}$

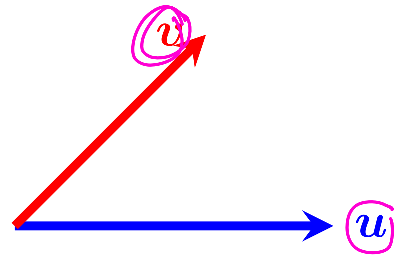
dependent

Projection

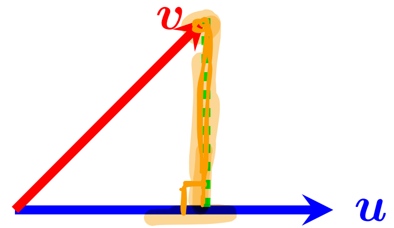
$\vec{a}_1, \vec{a}_2$ $\vec{u}, \vec{v}$

- Find the projection of $\boldsymbol{v}$ over $\boldsymbol{u}$





- draw a vertical line from the end of $\boldsymbol{v}$ to $\boldsymbol{u}$

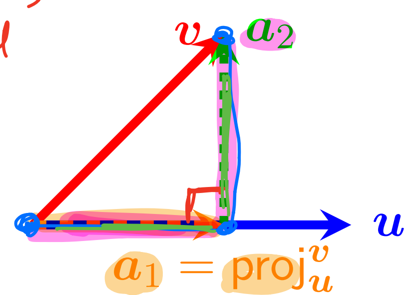


$\boxed{\vec{v} = \vec{a}_1 + \vec{a}_2}$ $\vec{a}_1 \perp \vec{a}_2$

$\vec{a}_1$ is in the direction of $\vec{u}$

$\vec{a}_2$ is perpendicular to $\vec{u}$

- aim: find $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$



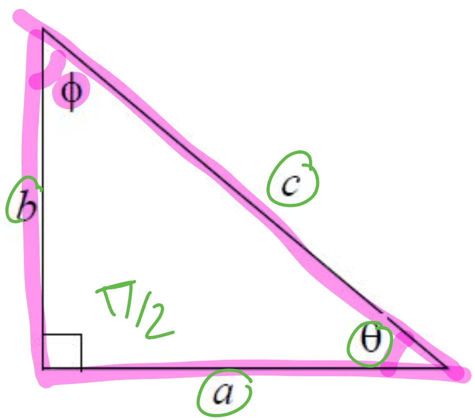
Projection

- In a right angle triangle
  - $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{c}$
  - $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{b}{c}$

$$\theta + \phi = \frac{\pi}{2}$$

$$\cos(\phi) = \frac{b}{c}$$

$$\sin(\phi) = \frac{a}{c}$$



- a_1
  - Length: $\cos\theta = \frac{||a_1||}{||v||} \to ||a_1|| = ||v||\cos\theta$
  - Direction: same as u, which is $\frac{u}{||u||}$ unit vector in the direction of $\vec{u}$
  - as $u \cdot v = ||u||||v||\cos\theta$;

$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\,||\vec{v}||}$

$$||a_1|| = \frac{u \cdot v}{||u||}$$

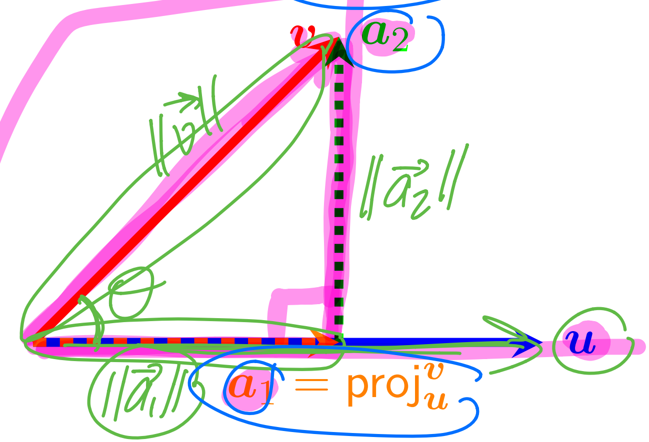
$||\vec{a}_1|| = ||\vec{v}||.\cos\theta = \cancel{||\vec{v}||}.\frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\,\cancel{||\vec{v}||}}$

number

$$a_1 = (\text{length})(\text{direction}) = \left(\frac{u\cdot v}{||u||}\right)\left(\frac{u}{||u||}\right) = \left(\frac{u\cdot v}{u\cdot u}\right)u$$

$\boxed{\vec{v} = ||\vec{v}||.\hat{v}}$

$\boxed{\vec{a}_1 = \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\vec{u}}$



- $a_2 = v - a_1 = v - \left(\frac{u\cdot v}{u\cdot u}\right)u$

$\vec{v} = \vec{a}_1 + \vec{a}_2$

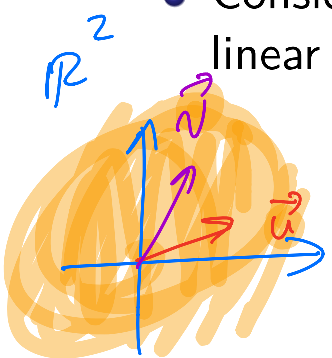
$\vec{a}_2 = \vec{v} - \vec{a}_1 = \vec{v} - \left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)\vec{u}$

Linear Combination of vectors

- consider $\{\boldsymbol{v}\}$. For different values of $c \in \mathbb{R}$, $c\boldsymbol{v}$ produces a line.
- consider $\{\boldsymbol{u}, \boldsymbol{v}\}$. For $c_1 \in \mathbb{R}$ and $c_2 \in \mathbb{R}$, a linear combination of those two vectors is $\boxed{c_1\boldsymbol{u} + c_2\boldsymbol{v}}$
- Note that he linear combination of any number of vectors is a vector.
- In general, for $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$, a linear combination of vectors of this set is

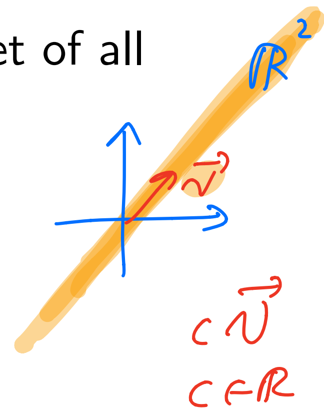
$$c_1\boldsymbol{v}_1 + \ldots + c_k\boldsymbol{v}_k$$

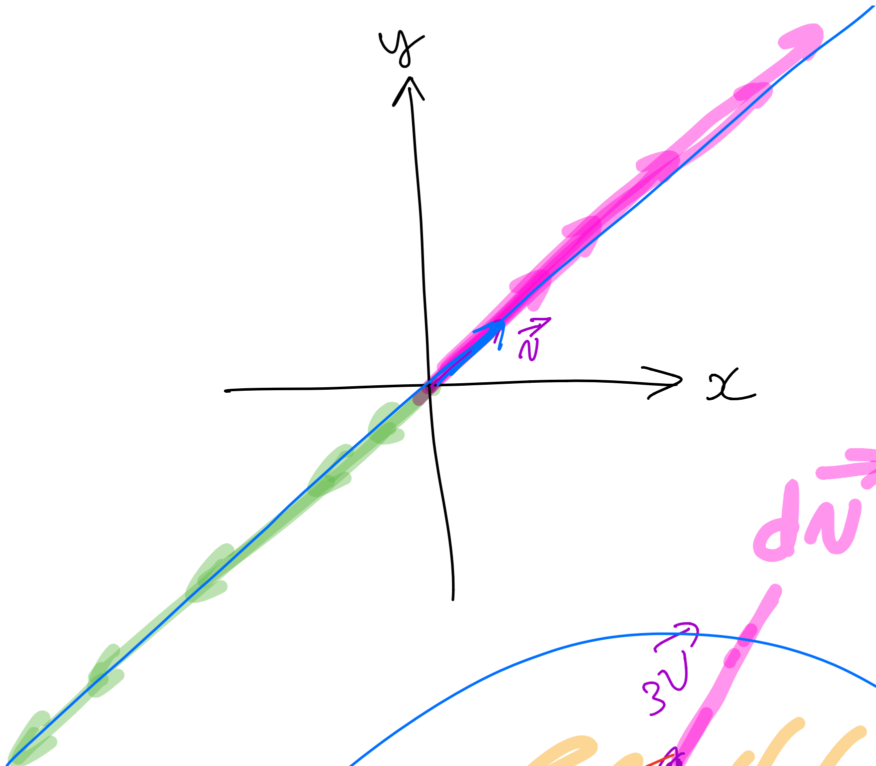
- Consider $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$. The span of S is the set of all linear combinations of its vectors



$$\text{span}(S) = \left\{ \sum_{i=1}^{k} c_i\boldsymbol{v}_i \,\middle|\, \text{for all } c_i \in \mathbb{R} \right\}$$

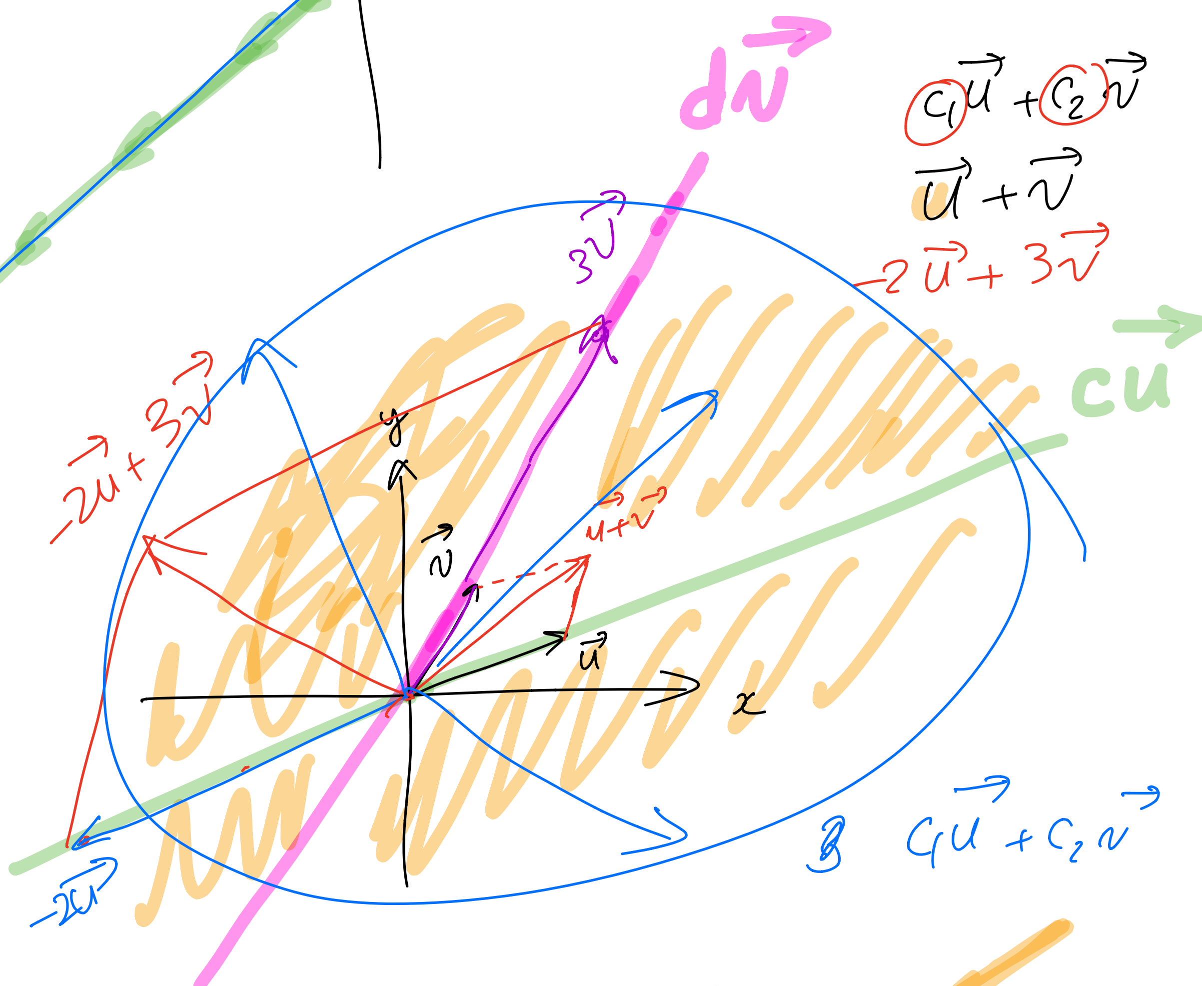
Space

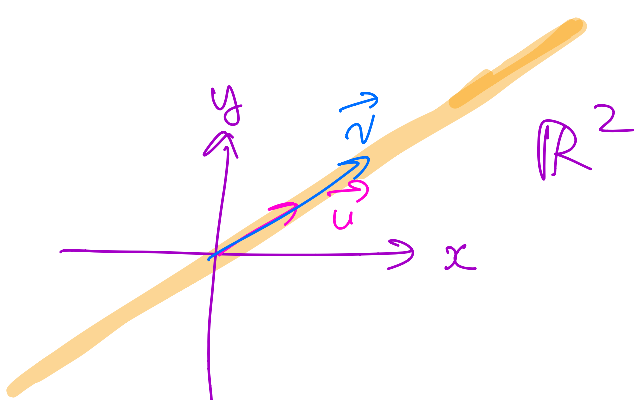




$\vec{v}$

$c\vec{v}$ $c \in \mathbb{R}$

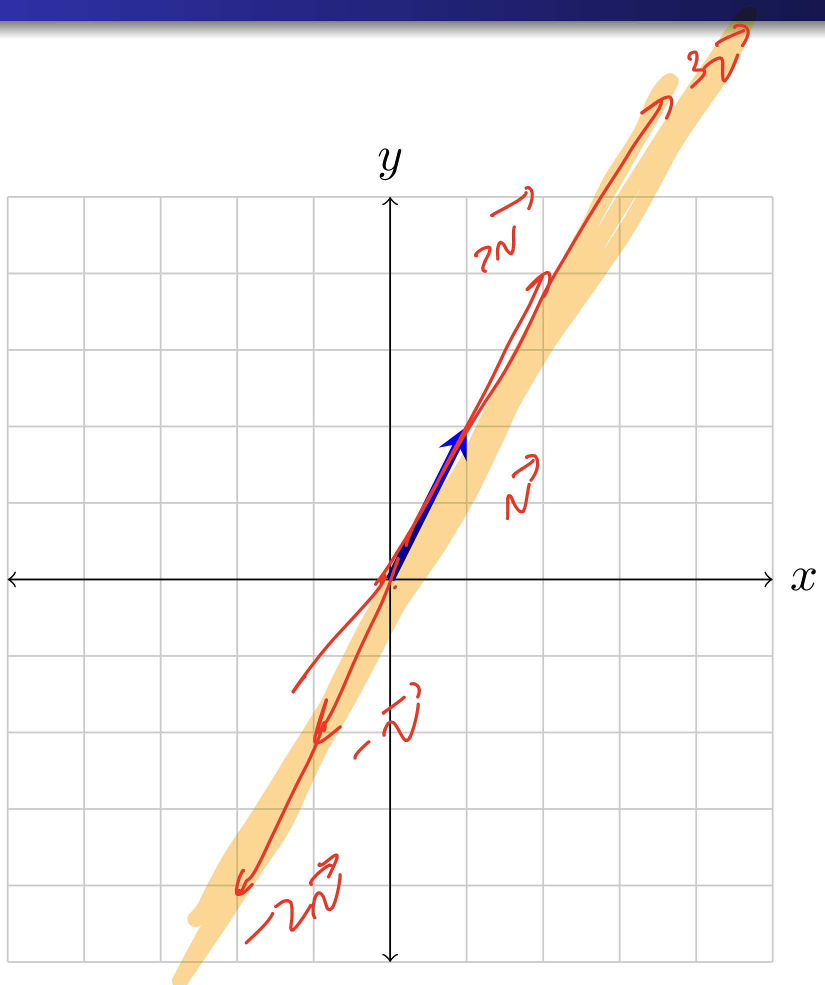




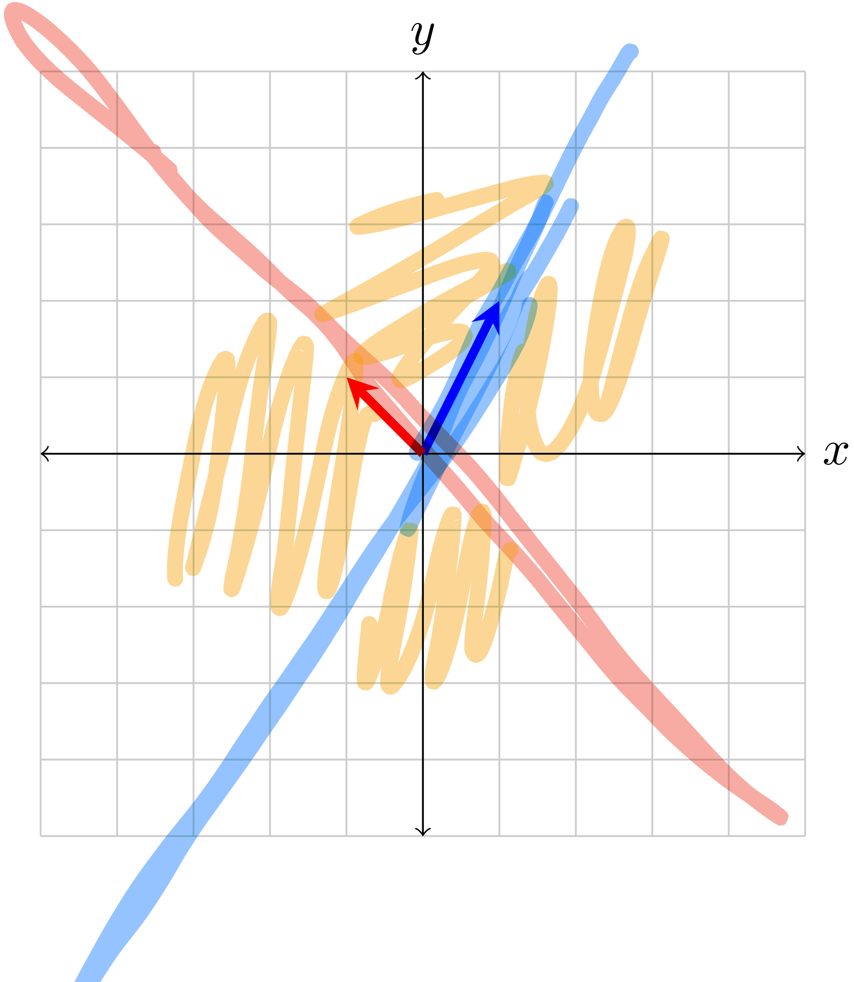
- For $S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ find span(S). $\mathbb{R}^2$
- For $S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ find span(S). $\mathbb{R}^2$
- For $S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ find span(S).

$\mathbb{R}^2$

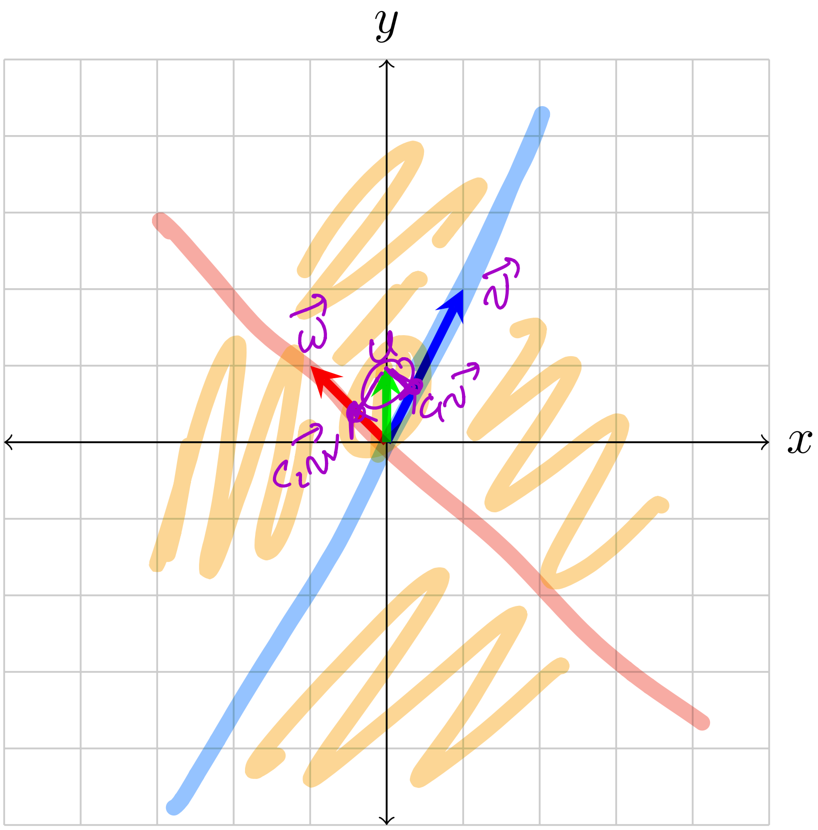
Linear Combination of vectors: Examples



Linear Combination of vectors: Examples



Linear Combination of vectors: Examples



$\vec{u} = \underline{c_1\vec{v}} + \underline{c_2\vec{w}}$

$\mathbb{R}^2$

Week 03

how add/subtract ①

vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ $\qquad \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^n$

$$\vec{u} \pm \vec{v} = \begin{bmatrix} u_1 \pm v_1 \\ u_2 \pm v_2 \\ \vdots \\ u_n \pm v_n \end{bmatrix}$$

scalar multiplication $c \in \mathbb{R}$ $\qquad c\vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}$ ②

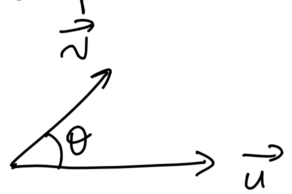
①, ② $\rightarrow$ $\boxed{\text{linear combinations}}$

$\vec{v}, \vec{u}$ $\qquad c\vec{v} + d\vec{u}$ $\qquad c, d \in \mathbb{R}$

$c=0,\ d=0 \rightarrow 0\vec{v} + 0\vec{u} = \vec{0}$

$c=1,\ d=-1 \rightarrow (1)\vec{v} + (-1)\vec{u} = \vec{v} - \vec{u}$

dot product $\vec{u} \cdot \vec{v} = \underbrace{\sum_{i=1}^{n} u_i v_i} = \underbrace{\|\vec{u}\| \, \|\vec{v}\| \cos(\theta)}$



$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}$$

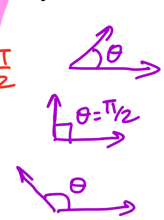
vectors $\begin{cases} \text{length} \\ \text{direction} \end{cases}$

$$\|\vec{v}\| = \sqrt{\sum_{i=1}^{n} v_i^2} = \sqrt{\vec{v} \cdot \vec{v}}$$

unit vector in the direction of $\vec{v}$

$$\hat{v} = \frac{1}{\|\vec{v}\|} \vec{v} \qquad \|\hat{v}\| = 1$$

$\vec{u} \cdot \vec{v} \in \mathbb{R}$ $\begin{cases} > 0 & 0 < \theta < \frac{\pi}{2} \\ = 0 & \theta = \frac{\pi}{2} \\ < 0 & \theta > \frac{\pi}{2} \end{cases}$

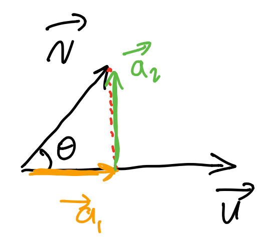


2 measures of similarities between $\vec{u}$ and $\vec{v}$

① $dist(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\|$

② $\cos(\theta)$ closer to 1 or θ closer to 0 they are more similar.

Projection



$$\{\vec{u}, \vec{v}\} \rightsquigarrow \{\vec{a_1}, \vec{a_2}\} \rightsquigarrow \{\hat{a}_1, \hat{a}_2\}$$

not orthogonal — orthogonal — orthonormal

$$\vec{a_1} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)\vec{u}$$

$$\vec{a_2} = \vec{v} - \vec{a_1}$$

$\vec{a_1} \perp \vec{a_2}$ $\quad (\vec{a_1}\cdot\vec{a_2} = 0)$

$\vec{v} = \vec{a_1} + \vec{a_2}$

→ the component of $\vec{v}$ in the direction of $\vec{u}$

→ $\vec{a_2}$ is the component of $\vec{v}$ in the direction which is perpendicular to $\vec{u}$.

later $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_k}\} \rightsquigarrow \{\vec{u_1}, \vec{u_2}, \dots, \vec{u_k}\}$

linearly indep. — all are mutually orthonormal

Vectors

$\mathbb{R}^n$

- A n-vectors: a list (tuple) of n numbers.
- the set of all n-vectors is $\mathbb{R}^n$

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$$

$$\cancel{\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}}$$

broadcasting
Python

- 2-vectors,

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^2 \qquad \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \in \mathbb{R}^3$$

- 1-vectors,

$$\boldsymbol{v} = \begin{bmatrix} v_1 \end{bmatrix} \in \mathbb{R}$$

$3 \in \mathbb{R}$

$[3] \in \mathbb{R}^1$

- A column vector and a row vector

$\vec{v} \in \mathbb{R}^n$

$v \in \mathbb{R}$

$\vec{u}\ \vec{v} \quad \vec{u} \pm \vec{v} \quad c\vec{v}$

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$$

$\vec{v} \in \mathbb{R}^n \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

Column vector

$$\boldsymbol{v}^T = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \end{bmatrix}$

$\begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

- $(\boldsymbol{v}^T)^T = \boldsymbol{v}$
- Zero vector

$$\mathbf{0}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$$

the origin

- Vector of ones

$$\mathbf{1}_n^T = \begin{bmatrix} 1 & \dots & 1 \end{bmatrix} \in \mathbb{R}^n$$

$$\vec{1}_n = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^n \qquad \vec{0}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$$

$$\boxed{\vec{1}_n \bullet \vec{u}} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \bullet \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = (1)(u_1) + (1)(u_2) + \cdots + (1)(u_n)$$

$$= \boxed{u_1 + u_2 + \cdots + u_n}$$

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \sum_{i=1}^{n} u_i = \vec{1}_n \bullet \vec{u} = \vec{u} \bullet \vec{1}_n$$

$c \in \mathbb{R}$

$$c\vec{u} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix} \qquad c = 0 \qquad (0)\vec{u} = \begin{bmatrix} (0)u_1 \\ (0)u_2 \\ \vdots \\ (0)u_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{0}_n$$

$$\vec{1}_n = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^n \qquad \boxed{\vec{1}_n^{\,T} = [1 \ 1 \ \cdots \ 1]}$$

vectors → big set of vectors / space of vectors } V vector space

$\vec{x}, \vec{y} \in V$

$\vec{x} + \vec{y} \in V$

V is closed under addition

$\vec{x} \in V \quad c \in \mathbb{R}$

$c\vec{x} \in V$

V is closed under scalar multiplication

- A set of elements (which are called vectors), with two properties
  - The elements can be added to each other

$$\mathbf{x}, \mathbf{y} \in V, \rightarrow \mathbf{x} + \mathbf{y} \in V$$

  - an element can be multiplied by a scaler

$$\mathbf{x} \in V \text{ and } c \in \mathbb{R}, \rightarrow c\mathbf{x} \in V$$

V is called a vector space

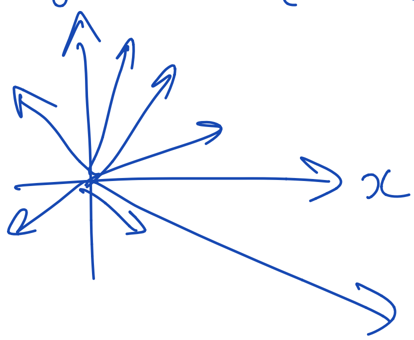
- sometimes it is called $\boxed{\text{linear space}}$
- Every vector space has a $\boxed{\text{null element}}$ $\mathbf{0} \in V$,

$$\boxed{\mathbf{x} + \mathbf{0}} = \mathbf{0} + \mathbf{x} = \mathbf{x}$$

- All elements has an $\boxed{\text{additive inverse}}$ $-\mathbf{x}$,

$$\mathbf{x} + (-\mathbf{x}) = \vec{\mathbf{0}}$$

$$\mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; x, y \in \mathbb{R} \right\}$$



$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix} \in \mathbb{R}^2$$

$$c \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix} \in \mathbb{R}^2$$

$\mathbb{R}^2$ is a vector space

$\mathbb{R}^n$ is a vector space

Eclidean vector space

vector addition
scalar multiplication } → linear combinations

$\{\vec{u}, \vec{v}\} \rightarrow c\vec{u} + d\vec{v}$

$c=0$ and $d=0$ $\quad 0\vec{u} + 0\vec{v} = \vec{0}$

$c=(-1)$ and $d=0$ $\quad -\vec{u}$

additive inverse of $\vec{u}$

Vector space operations

- Addition:

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n, \boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{R}^n \rightarrow \boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{bmatrix}$$

- Scaling

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \text{ and } c \in \mathbb{R} \rightarrow c\boldsymbol{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}$$

- Linear combination

$$c\boldsymbol{v} + d\boldsymbol{w} = \begin{bmatrix} cv_1 + dw_1 \\ cv_2 + dw_2 \\ \vdots \\ cv_n + dw_n \end{bmatrix}$$

$c, d \in \mathbb{R}$

Vector space examples

- Euclidean space
  - 1-Dimensional space $\mathbb{R}$
  - 2-Dimensional space $\mathbb{R}^2$
  - 3-Dimensional space $\mathbb{R}^3$
  - n-Dimensional space $\mathbb{R}^n$

Properties

- Addition: $\boldsymbol{v}, \boldsymbol{w} \in V \to \boldsymbol{v} + \boldsymbol{w} \in V$
- Commutativity: $\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v}$
- Zero vector: $\mathbf{0}$
- Identity element: $v + \mathbf{0} = \mathbf{0} + v = \boldsymbol{v}$
- additive Inverses: $\boldsymbol{v} + (-\boldsymbol{v}) = (-v) + v = \mathbf{0}$
- Associativity: $\boldsymbol{v} + (\boldsymbol{w} + \boldsymbol{z}) = (\boldsymbol{v} + \boldsymbol{w}) + \boldsymbol{z}$ $= \vec{v} + \vec{w} + \vec{z}$

$$\vec{v} + (\vec{v} + \vec{v}) = 3\vec{v}$$

$$\vec{v} + \vec{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{bmatrix} = \begin{bmatrix} w_1 + v_1 \\ w_2 + v_2 \\ \vdots \\ w_n + v_n \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \vec{w} + \vec{v}$$

$V = \mathbb{R}^n$ Euclidean space (vector space) $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

- $V = \{$all real polynomials of degree 3 or less$\}$
- $ax^3 + bx^2 + cx + d, \quad a, b, c, d \in \mathbb{R}$
- addition
- scaling
- zero vector $p(x) = 0$
- inverse (additive inverse)

$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

$p(x) = x^2 + 2x + 1$ 2nd degree polynomial

$p(x) = x - 1$ 1st degree pol

$p(x) = 3$ polynomial of degree 0

$p_1(x) = x^3 + 2 \in V$

$p_2(x) = x^2 - 2x + 1 \in V$

$p_1(x) + p_2(x) = x^3 + x^2 - 2x + 3 \in V$

$3p_1(x) = 3x^3 + 6 \in V$

$p_1(x) = x^3 + 2$

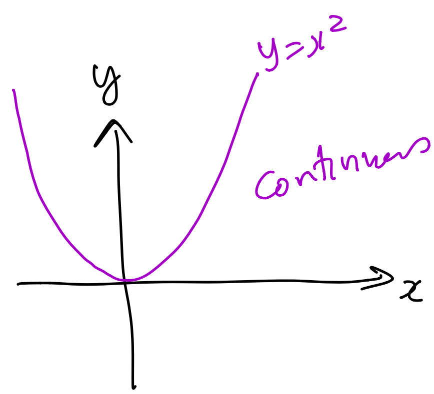
$-p_1(x) = -x^3 - 2$

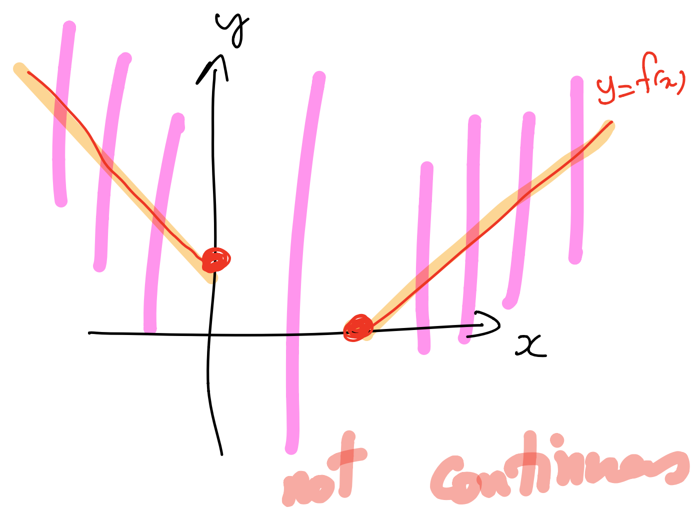
$(x^3 + 2) + (-x^3 - 3) = 0$

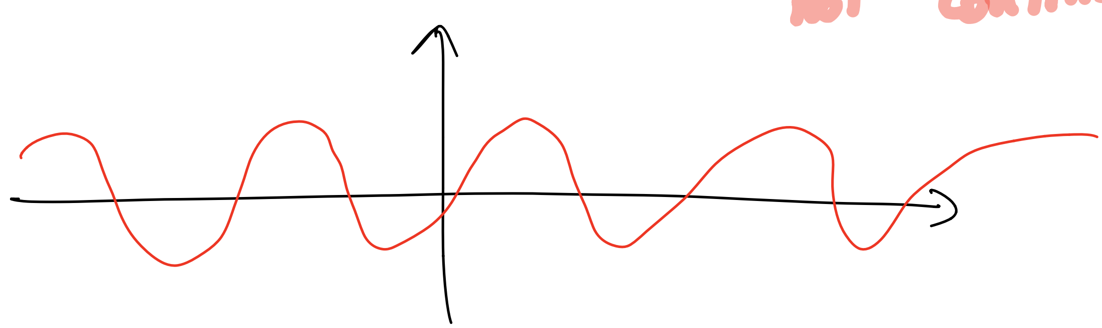
$$V = \mathbb{R}^4 = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \;\middle|\; a, b, c, d \in \mathbb{R} \right\}$$

$$V = \{\text{all polynomials of degree at most } 3\}$$

$$= \{ ax^3 + bx^2 + cx + d \mid a, b, c, d \in \mathbb{R} \}$$







$y = x^2 \quad x^2 \in V$

$\sin(x) \in V$

$\cos(x) \in V \ldots$

- $V = \{f(x) \mid f(x)$ is continuous on $\mathbb{R}\}$
- addition
- scaling
- zero vector
- inverse

$$\begin{cases} f_1(x) + f_2(x) & \text{continues} \\ c f_1(x) & \text{conti} \end{cases}$$

$0 \in V$

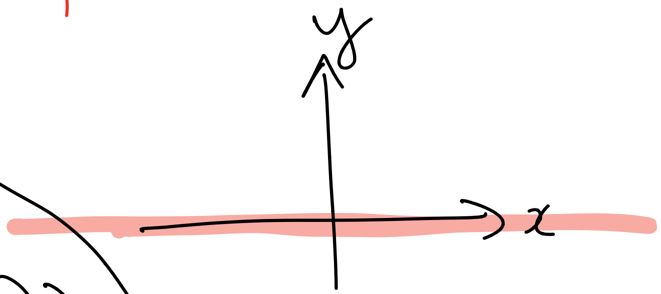
$y = 0$

additive

$f_1(x) \in V$

$-f_1(x) \in V$

$f_1(x) + (-f_1(x)) = 0$



Subspace of a vector space

vector spaces $V \to \begin{cases} \vec{x}+\vec{y} \in V \\ c\vec{x} \in V \end{cases}$

$L \subset V$

- Vector spaces can contain other vector spaces.
- $L \subset V$
  - $\mathbf{0} \in L$
  - $\mathbf{x}, \mathbf{y} \in L, \to \mathbf{x} + \mathbf{y} \in L$
  - $\mathbf{x} \in L$ and $c \in \mathbb{R}, \to c\mathbf{x} \in L$
- $V \subset V$
- $\{\mathbf{0}\} \subset V$ trivial subspace
- a line passing through the origin is a subspace of Euclidean space

$\vec{u}, \vec{w} \in L \to \vec{u}+\vec{w} \in L$

$c \in \mathbb{R} \quad c\vec{u} \in L$

L vector space

a subspace of V

$\boxed{L = \{\vec{0}\}}$

$\vec{x}, \vec{y} \in L \quad \vec{x} = \vec{0} \quad \vec{y} = \vec{0}$

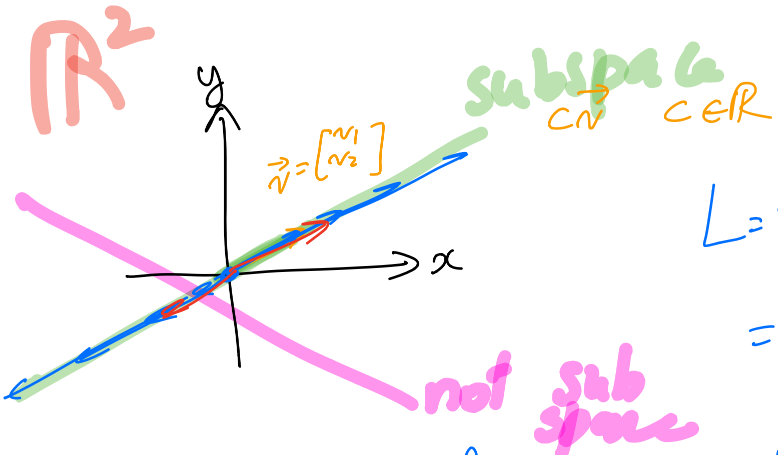
$\vec{x} \in L \;, \; 0 \in \mathbb{R}$

$\boxed{\begin{aligned} \vec{x}+\vec{y} &= \vec{0}+\vec{0} = \vec{0} \in L \\ 0 \cdot \vec{x} &= \vec{0} \in L \end{aligned}}$

$L \subset V$

subspace

trivial

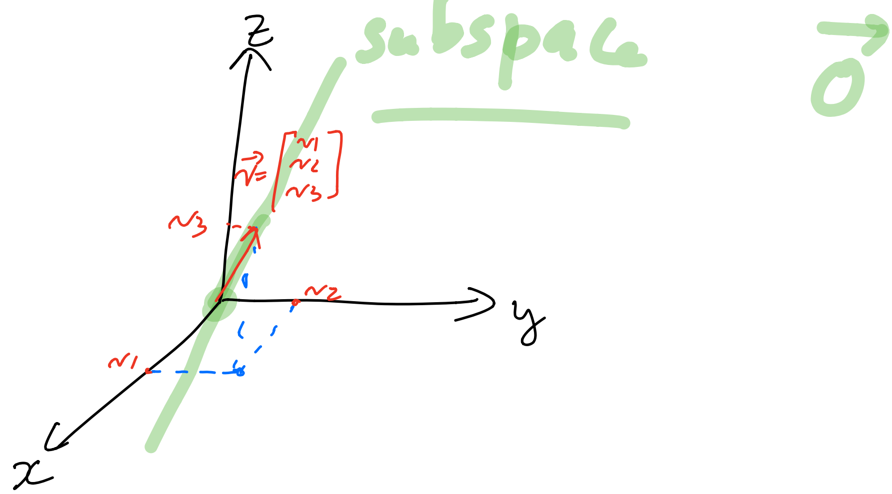


$$L = \{\text{line through } \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\}$$

$$= \left\{ c \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix} \right\}$$

if $c = 0 \rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in L$

line passing throug origin is a subspace of $\mathbb{R}^2$



Linear combinations

- Consider a vector space V
- $\mathbf{x}_1, \ldots, \mathbf{x}_n \in V$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$
- A linear combination

$$\mathbf{y} = \boxed{\alpha_1 \mathbf{x}_1 + \ldots + \alpha_n \mathbf{x}_n} \in V$$

$$\vec{y} = \boxed{\sum_{i=1}^{n} \alpha_i \vec{x}_i} \in V$$

Span

- Consider $\boxed{X = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}}$
- The set of all linear combinations of members of X,

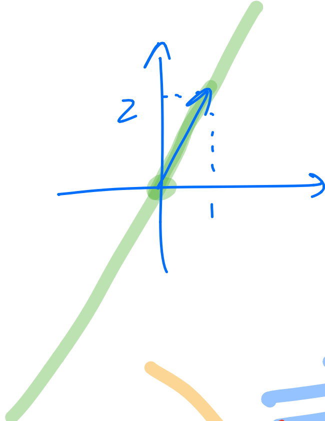
$$\boxed{\text{span}(X)} = \text{span}(\mathbf{x}_1, \ldots, \mathbf{x}_n) =$$

$$\{\alpha_1\mathbf{x}_1 + \ldots + \alpha_n\mathbf{x}_n | \alpha_1, \ldots, \alpha_n \in \mathbb{R}\} = \left\{ \sum_{i=1}^{n} \alpha_i \vec{x}_i \,\middle|\, \begin{array}{l} \vec{x}_i \in V \\ \alpha_i \in \mathbb{R} \end{array} \right\}$$

- a linear span of X
- Span of a set of vectors: a set obtained by a linear combination of those vectors

span (X) is a vector space

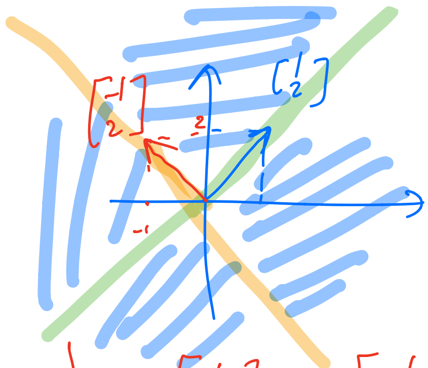
$\left\{ \vec{v} = \begin{bmatrix}1\\2\end{bmatrix} \right\}$



$\text{span}\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \left\{ c\begin{bmatrix}1\\2\end{bmatrix} \; c\in\mathbb{R} \right\}$

independent

$\left\{ \underline{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix}} \right\}$



$$\text{span}\left(\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix}\right) = \left\{ c\begin{bmatrix}1\\2\end{bmatrix} + d\begin{bmatrix}-1\\2\end{bmatrix} \,\middle|\, c,d\in\mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix}c-d\\2c+2d\end{bmatrix} \,\middle|\, c,d\in\mathbb{R} \right\}$$

$c=0,\ d=0 \qquad \begin{bmatrix}0\\0\end{bmatrix}$

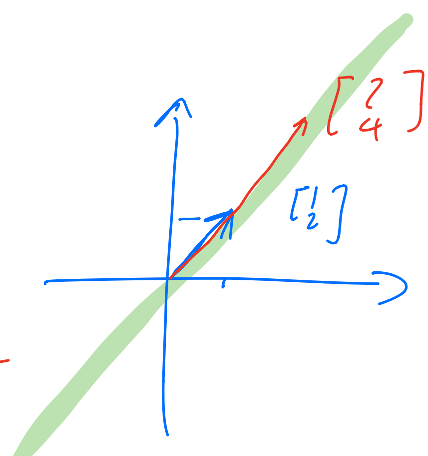
$c=1,\ d=0 \qquad \begin{bmatrix}1\\2\end{bmatrix}$

$c=-1,\ d=0 \qquad \begin{bmatrix}-1\\-2\end{bmatrix}$

$$\text{span}\left(\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix}\right) = \underline{\underline{\mathbb{R}^2}}$$

$\left\{ \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix} \right\}$

dependent



$$\text{span}\left(\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}\right) \neq \mathbb{R}^2$$

is a vector space

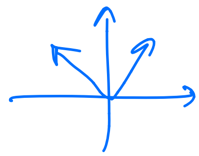
Linear dependence

- $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$
- If $\alpha_1\mathbf{x}_1 + \ldots + \alpha_n\mathbf{x}_n = \mathbf{0}$ implies that all the scalars are zero, we say vectors in X are independent.

system $\xrightarrow{\text{solve}}$ $\alpha_1, \alpha_2, -, \alpha_n$

- $\alpha_1\mathbf{x}_1 + \ldots + \alpha_n\mathbf{x}_n = \mathbf{0}$ and at least one of the scalars is not zero, we say vectors in X are dependent.

$$\left\{ \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}} \right\}$$



$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$\begin{cases} x_1 = x_2 \\ y_1 = y_2 \end{cases}$$

$$\ast \quad \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha - \beta \\ 2\alpha + 2\beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} \alpha - \beta = 0 \rightarrow \alpha = \beta. \\ 2\alpha + 2\beta = 0 \end{cases}$$

$$2(\beta) + 2\beta = 0 \rightarrow 4\beta = 0 \rightarrow \beta = 0 \Rightarrow \boxed{\alpha = 0}$$

$$\alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow \alpha = \beta = 0$$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ad $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ are indep.

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$$

$$\alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha + 2\beta \\ 2\alpha + 4\beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{cases} \alpha + 2\beta = 0 \rightarrow \boxed{\alpha = -2\beta} \\ 2\alpha + 4\beta = 0 \end{cases}$$

$$2(-2\beta) + 4\beta = 0$$

$$-4\beta + 4\beta = 0$$

$$0 = 0$$

$$\not\Longrightarrow \underline{\alpha = \beta = 0}$$

$\cancel{\beta = 0}$?

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ad $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ are dep. +

Vector spaces , spans

$\text{span}\left(\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix}\right) = \mathbb{R}^2$

$\text{span}\left(\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}\right) \neq \mathbb{R}^2$

- $B = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \subset V$ is a basis for V if and only if
  - B is linearly independent
  - $V = \text{span}(B)$

$B = \left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix}\right\}$ indep

$\text{span}(B) = \mathbb{R}^2$

a basis for $\mathbb{R}^2$

- Example $V = \mathbb{R}^n$

$B = \left\{\begin{bmatrix}0\\3\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\right\}$

standard basis for $\mathbb{R}^n$

$$B = \left\{ \mathbf{e}_1 = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}, \ldots, \mathbf{e}_n = \begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix} \right\}$$

$B = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$

indep

$\text{span}(B) = \mathbb{R}^n$

$B = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

- a basis is not unique.
- If a vector space is spanned by a finite number of vectors, it is said to be finite-dimensional. Otherwise it is in infinite-dimensional.

$\mathbb{R}^3$

$\mathbb{R}^2$ $\quad \dim(\mathbb{R}^2) = 2$ $\quad \dim(\mathbb{R}^3) = 3$ $\quad \dim(\mathbb{R}^n) = n$

- The number of vectors in a basis for a finite-dimensional vector space V is called the dimension of V and denoted $\dim(V)$.
- $\dim(V) = |B|$
- $\dim(\mathbb{R}^n) = n$
- $\dim(\mathbb{R}^2) = 2$
- $\dim(\{\mathbf{0}\}) = 0$

Finite-dimensional space

$V = \{ f(x) \mid f(x) \text{ is continuous} \}$

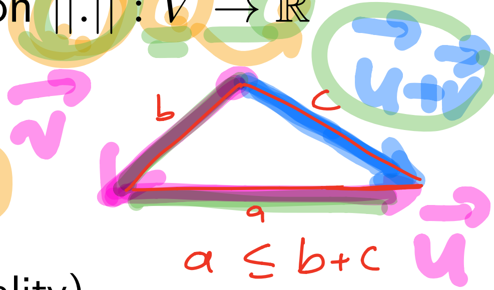
infinite-dimensional space

Normed spaces

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ $\|\vec{v}\| = \sqrt{\sum_{i=1}^{n} v_i^2}$

vector space + norm normed space

- Norms generalise the notion of length from Euclidean space
- A norm on a real vector space V is a function $||.|| : V \to \mathbb{R}$ that satisfies
  for all $\boldsymbol{x}, \boldsymbol{y} \in V$ and $\alpha \in \mathbb{R}$
  - $||\boldsymbol{x}|| \geq 0$ with equality if and only if $\boldsymbol{x} = \boldsymbol{0}$
  - $||\alpha \boldsymbol{x}|| = |\alpha| ||\boldsymbol{x}||$
  - $||\boldsymbol{x} + \boldsymbol{y}|| \leq ||\boldsymbol{x}|| + ||\boldsymbol{y}||$ (the triangle inequality)



$a \leq b + c$

- A vector space provided with a norm is called a normed vector space, or simply a normed space.
- Any norm on V induces a distance metric on V

$$\text{dist}(\boldsymbol{x}, \boldsymbol{y}) = ||\boldsymbol{x} - \boldsymbol{y}||$$

Frequent norms on $\mathbb{R}^n$

For $\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

- 1-norm

$$||\boldsymbol{x}||_1 = \sum_{i=1}^{n} |x_i|$$

- 2-norm (Euclidean norm)

$$||\boldsymbol{x}||_2 = \sqrt{\sum_{i=1}^{n} x_i^2} = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$$

- p-norm

$$||\boldsymbol{x}||_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \quad p \geq 1$$

- ∞-norm or max norm

$$||\boldsymbol{x}||_\infty = \max_{i=1,\dots,n} |x_i|$$

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} \in \mathbb{R}^4$$

$$||\vec{x}||_2 = \sqrt{(1)^2 + (0)^2 + (-1)^2 + (2)^2} = \sqrt{1+0+1+4} = \sqrt{6}$$

$$||\vec{x}||_1 = |1| + |0| + |-1| + |2| = 1 + 0 + 1 + 2 = 4$$

$$||\vec{x}||_\infty = \max\{|1|, |0|, |-1|, |2|\} = \max\{1, 0, \cancel{1}, 2\} = \max\{1, 0, 2\} = 2$$

$p=3$ 3-norm

$$\|\vec{x}\|_3 = \left(\sum_{i=1}^{n} |x_i|^3 \right)^{1/3}$$

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} \qquad \|\vec{x}\|_3 = \left(|1|^3 + |0|^3 + |-1|^3 + |2|^3 \right)^{1/3}$$

$$= (1 + 0 + 1 + 8)^{1/3}$$

$$= (10)^{1/3} = \sqrt[3]{10}$$

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \|\vec{v}\|_p = \left(|v_1|^p + |v_2|^p\right)^{1/p}$

$p=2 \quad \|\vec{v}\|_2 = \sqrt{v_1^2 + v_2^2} = 1$

$\longrightarrow p=1 \quad \|\vec{v}\|_1 = |v_1| + |v_2|$

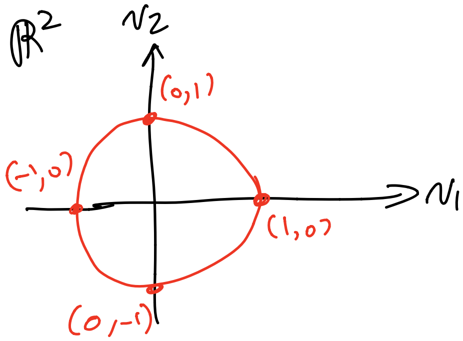
$p=\infty \quad \|\vec{v}\|_\infty = \max\{|v_1|, |v_2|\}$

- Let's see $\boxed{||\boldsymbol{v}||_p = 1}$ for $\boldsymbol{v} = \left[v_1, v_2\right]^T \in \mathbb{R}^2$
  - $p = 2$, $v_1^2 + v_2^2 = 1$
  - $p = 1$, $|v_1| + |v_2| = 1$
  - $p = \infty$, $\max\{|v_1|, |v_2|\} = 1$
- Note that $||\boldsymbol{x}|| = ||\boldsymbol{x}||_2$ and generally lenth of a vector is shown as $||\boldsymbol{x}||$.
- Cauchy-Schwarz inequality

$$\boxed{\text{for } \boldsymbol{x}, \boldsymbol{y} \in V, \quad |\boldsymbol{x} \cdot \boldsymbol{y}| \leq ||\boldsymbol{x}||\,||\boldsymbol{y}||}$$

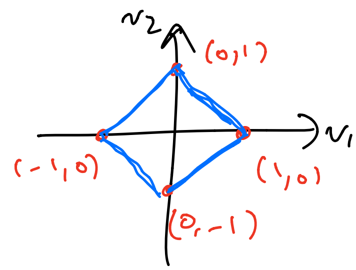
$\vec{x} \cdot \vec{y} = \|\vec{x}\| \, \|\vec{y}\| \cos(\theta) \qquad -1 \leq \cos(\theta) \leq 1 \qquad 0 \leq |\cos(\theta)| \leq 1$

$|\vec{x} \cdot \vec{y}| = \big|\|\vec{x}\|\big| \cdot \big|\|\vec{y}\|\big| \, |\cos(\theta)| \leq \|\vec{x}\| \, \|\vec{y}\|$

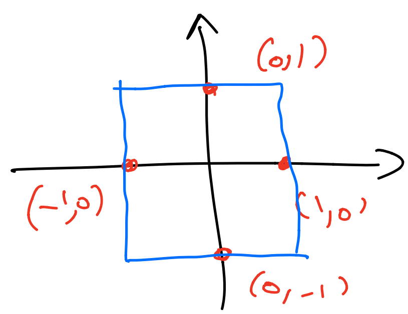


$v_1^2+v_2^2=1$

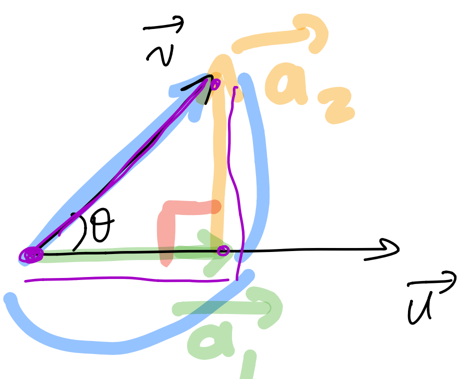
$|v_1|+|v_2|=1$



$\max\{|v_1|,|v_2|\}=1$



Gram-Schmidt process.



projection $\text{Proj}_{\vec{u}}\vec{v}$

$$\vec{a_1} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)\vec{u}$$

$$\vec{a_2} = \vec{v} - \vec{a_1}$$

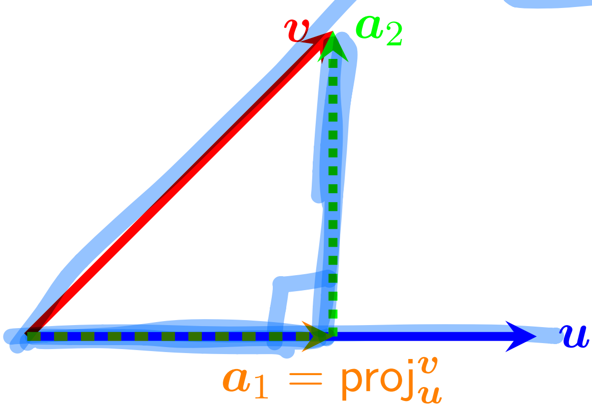
$a_1 \perp a_2$ $\qquad \vec{v} = \vec{a_1} + \vec{a_2}$

$$\{\vec{u},\vec{v}\} \longrightarrow \{\vec{a_1},\vec{a_2}\} \longrightarrow \{\hat{a}_1,\hat{a}_2\}$$

orthogonal $\qquad$ orthonormal

Gram-Schmidt Process

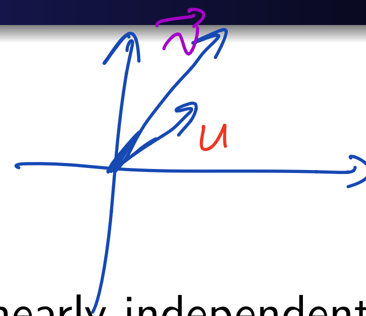
- You remember from the projection that
  - For given vectors $\boldsymbol{u}$ and $\boldsymbol{v}$
  - we can find the projection of $\boldsymbol{v}$ over $\boldsymbol{u}$
  - this gives us two vectors
    - $\boldsymbol{a}_1 = \text{proj}_{\boldsymbol{u}}^{\boldsymbol{v}} = \left(\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\boldsymbol{u}\cdot\boldsymbol{u}}\right)\boldsymbol{u}$
    - $\boldsymbol{a}_2 = \boldsymbol{v} - \boldsymbol{a}_1 = \boldsymbol{v} - \left(\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\boldsymbol{u}\cdot\boldsymbol{u}}\right)\boldsymbol{u}$
    - $\boldsymbol{a}_1$ is the component of $\boldsymbol{v}$ in the direction of $\boldsymbol{u}$
    - $\boldsymbol{a}_2$ is the component of $\boldsymbol{v}$ in the direction operpendicular to $\boldsymbol{u}$
    - $\boldsymbol{a}_1 \perp \boldsymbol{a}_2$ or $\boldsymbol{a}_1 \cdot \boldsymbol{a}_2 = 0$



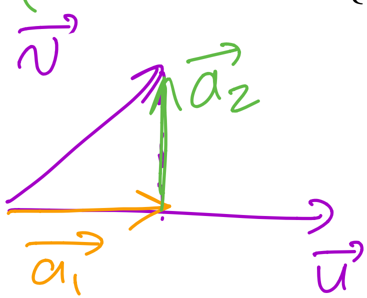
$\{\hat{a}_1, \hat{a}_2\}$

orthonormal vectors

Example



- Vectors $\boldsymbol{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\boldsymbol{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ are linearly independent
- Two independent vectors in $\mathbb{R}^2$ form a basis.
- but not orthogonal. $\vec{u} \cdot \vec{v} = 4 + 5 = 9 \neq 0$
- But $\boldsymbol{v}$ and $\boldsymbol{a}_2 = \boldsymbol{v} - \text{proj}_{\boldsymbol{u}}^{\boldsymbol{v}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ are independent and orthogonal.
- Therefore, $\{\vec{u}, \boldsymbol{a}_2\}$ form an orthogonal basis for $\mathbb{R}^2$.



$\{\vec{u}, \vec{v}\}$ are not perpendicular

$\{\vec{u}, \vec{a}_2\}$ are. $\rightarrow \{\vec{a}_1, \vec{a}_2\}$

vector space

- Vectors in $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\} \subset V$ are called mutually orthogonal when any two different members are orthogonal

$\vec{v_1} \circ \vec{v_1}$ $\vec{v_2} \circ \vec{v_2}$ $\vec{v_3} \circ \vec{v_3}$

$$\boldsymbol{v}_i \cdot \boldsymbol{v}_j = 0 \text{ for } i \neq j$$

$\{\vec{v_1}, \vec{v_2}, \vec{v_3}\} \subset V$

if $\vec{v_1} \circ \vec{v_2} = 0$ $\vec{v_1} \circ \vec{v_3} = 0$ $\vec{v_2} \circ \vec{v_3} = 0$

- **Theorem:** If the vectors in a set are mutually orthogonal and nonzero then that set is linearly independent.
- Using Gram-Schmidt Process we orthognolise this set.
- we make $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ so that they are mutually orthogonal
  1. $\boldsymbol{u}_1 = \boldsymbol{v}_1$
  2. $\boldsymbol{u}_2 = \boldsymbol{v}_2 - \text{proj}_{\boldsymbol{u}_1}^{\boldsymbol{v}_2}$
  3. $\boldsymbol{u}_3 = \boldsymbol{v}_3 - \text{proj}_{\boldsymbol{u}_1}^{\boldsymbol{v}_3} - \text{proj}_{\boldsymbol{u}_2}^{\boldsymbol{v}_3}$
  4. $\ldots$
  5. $\boldsymbol{u}_k = \boldsymbol{v}_k - \text{proj}_{\boldsymbol{u}_1}^{\boldsymbol{v}_k} - \ldots - \text{proj}_{\boldsymbol{u}_{k-1}}^{\boldsymbol{v}_k}$
  6. Finally, normalise each vector $\boldsymbol{u}_k$ by dividing it by its length to get an orthonormal set.

$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \subseteq V$

linearly indep.

we make $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\} \subseteq V$

orthogonal

$\{\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_k\} \subset V$

orthonormal

$$\left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}_{\vec{v}_2}, \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\vec{v}_3} \right\} \subset \mathbb{R}^3$$

linearly indep.

Exercise check it

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = 1 \neq 0 \qquad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 2 \neq 0$$

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 3 \neq 0$$

$$\vec{u}_1 = \vec{v}_1 \rightarrow \boxed{\vec{u}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}}$$

$$\vec{u}_2 = \vec{v}_2 - \text{proj}_{\vec{u}_1}^{\vec{v}_2} = \begin{bmatrix}0\\1\\2\end{bmatrix} - \text{proj}_{\vec{u}_1}^{\vec{v}_2}$$

$$\text{proj}_{\vec{y}}^{\vec{x}} = \left(\frac{\vec{x}\cdot\vec{y}}{\vec{y}\cdot\vec{y}}\right)\vec{y}$$

$$\cdot\ \text{proj}_{\vec{u}_1}^{\vec{v}_2}$$

$$\vec{v}_2 \cdot \vec{u}_1 = \begin{bmatrix}0\\1\\2\end{bmatrix} \cdot \begin{bmatrix}1\\1\\0\end{bmatrix} = 1$$

$$\vec{u}_1 \cdot \vec{u}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix} \begin{bmatrix}1\\1\\0\end{bmatrix} = 2$$

$$= \left(\frac{1}{2}\right)\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1/2\\1/2\\0\end{bmatrix}$$

$$\vec{u}_2 = \begin{bmatrix}0\\1\\2\end{bmatrix} - \begin{bmatrix}1/2\\1/2\\0\end{bmatrix} = \begin{bmatrix}-1/2\\1/2\\2\end{bmatrix}$$

$$\vec{u}_3 = \vec{v}_3 - \text{proj}_{\vec{u}_1}^{\vec{v}_3} - \text{proj}_{\vec{u}_2}^{\vec{v}_3} \ \cdot$$

$A=\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ check when they are dependent or independent.

$$\boxed{c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \vec{0}} \Longrightarrow \begin{cases} c_1 = c_2 = \dots = c_k = 0 \\ \quad \text{these are linearly independent} \\ \text{otherwise,} \\ \quad \text{they are linearly dependent.} \end{cases}$$

system

① if $\vec{0} \in A$, then these vectors are linearly dependent.

② $\mathbb{R}^2$ is a vector space

$$\dim(\mathbb{R}^2) = 2$$

$$A = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\} \subset \mathbb{R}^2$$

we know that these vectors are linearly dependent.

$$|A| > \dim(\mathbb{R}^2) \rightarrow$$

$A = \{ \vec{v}_1, \vec{v}_2, \dots, \vec{v}_k \} \subseteq V$

linearly independent. $\qquad k \leqslant \dim(V)$

$\rightarrow$ orthonormalise this set

① make a set of k vectors out of A so that they are mutually orthogonal

② make them unit vectors

$$A \xrightarrow{G.S} \{ \vec{u}_1, \vec{u}_2, \dots, \vec{u}_k \} \xrightarrow{\text{normalize}} \{ \hat{u}_1, \hat{u}_2, \dots, \hat{u}_k \}$$

orthogonal

$$\Big\{ \underbrace{\begin{bmatrix}1\\1\\0\end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix}0\\1\\2\end{bmatrix}}_{\vec{v}_2}, \underbrace{\begin{bmatrix}1\\1\\1\end{bmatrix}}_{\vec{v}_3} \Big\} \subset \mathbb{R}^3$$

$\boxed{\text{exercise:}}$ prove that they are linearly independent

$$\vec{u}_1 = \vec{v}_1 \qquad \vec{u}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\vec{u}_2 = \vec{v}_2 - \operatorname{proj}_{\vec{u}_1}\vec{v}_2 \qquad \vec{u}_2 = \begin{bmatrix}-1/2\\1/2\\2\end{bmatrix}$$

$$\vec{u}_3 = \vec{v}_3 - \underbrace{\operatorname{proj}_{\vec{u}_1}\vec{v}_3} - \underbrace{\operatorname{proj}_{\vec{u}_2}\vec{v}_3}$$

$$\underline{\operatorname{proj}_{\vec{u}_1}\vec{v}_3} = \left(\frac{\vec{v}_3 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\right)\vec{u}_1 = \left(\frac{\begin{bmatrix}1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\1\\0\end{bmatrix}}{\begin{bmatrix}1\\1\\0\end{bmatrix} \cdot \begin{bmatrix}1\\1\\0\end{bmatrix}}\right)\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$$

$$= \left(\frac{2}{2}\right)\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix} \checkmark$$

$$\frac{\frac{2}{1}}{\frac{9}{2}} = \frac{4}{9}$$

$$\operatorname{proj}_{\vec{u}_2}\vec{v}_3 = \left(\frac{\begin{bmatrix}1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}-1/2\\1/2\\2\end{bmatrix}}{\begin{bmatrix}-1/2\\1/2\\2\end{bmatrix} \cdot \begin{bmatrix}-1/2\\1/2\\2\end{bmatrix}}\right)\begin{bmatrix}-1/2\\1/2\\2\end{bmatrix} = \left(\frac{2}{\frac{1}{4}+\frac{1}{4}+4}\right)\begin{bmatrix}-1/2\\1/2\\2\end{bmatrix}$$

$4/9$

$$\frac{1+1+16}{4} = \frac{18}{4} = \frac{9}{2}$$

$$= \frac{9}{2}\begin{bmatrix}-1/2\\1/2\\2\end{bmatrix} = \begin{bmatrix}-9/4\\9/4\\9\end{bmatrix}$$

$$\vec{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -2/9 \\ 2/9 \\ 8/9 \end{bmatrix} = \begin{bmatrix} 2/9 \\ -2/9 \\ 1/9 \end{bmatrix} \qquad 1 - \frac{8}{9} = \frac{1}{9}$$

$$\left\{ \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ \vec{u}_2 = \begin{bmatrix} -1/2 \\ 1/2 \\ 2 \end{bmatrix}, \ \vec{u}_3 = \begin{bmatrix} 2/9 \\ -2/9 \\ 1/9 \end{bmatrix} \right\}$$

These vectors are orthogonal:

$$\vec{u}_1 \cdot \vec{u}_2 = (1)(-1/2) + (1)(1/2) + (0)(2) = 0 \quad \checkmark$$

$$\vec{u}_1 \cdot \vec{u}_3 = (1)(2/9) + (1)(-2/9) + (0)(1/9) = \frac{2}{9} - \frac{2}{9} = 0 \checkmark$$

$$\vec{u}_2 \cdot \vec{u}_3 = (-1/2)(2/9) + (1/2)(-2/9) + (2)(1/9)$$

$$= -\frac{1}{9} - \frac{1}{9} + \frac{2}{9} = 0$$

These are mutually orthogonal vectors.

$$\|\vec{u}_1\|, \quad \|\vec{u}_2\|, \quad \|\vec{u}_3\|$$

$$\hat{u}_1 = \frac{1}{\|\vec{u}_1\|}\vec{u}_1, \quad \hat{u}_2 = \frac{1}{\|\vec{u}_2\|}\vec{u}_2, \quad \hat{u}_3 = \frac{1}{\|\vec{u}_3\|}\vec{u}_3.$$

$$\|u_1\| = \sqrt{2} \qquad \hat{u}_1 = \frac{1}{\sqrt{2}}\vec{u}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \text{ unit}$$

$\{\hat{u}_1, \hat{u}_2, \hat{u}_3\}$ mutually orthonormal.

$A = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\} \subset V$

linearly independent

$\longrightarrow \{\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_k\} \subset V$

orthonormal

---

assignment, or exa

① check if they are already orthogonal.

② if orthogonal $\rightarrow$ just make them unit vectors.