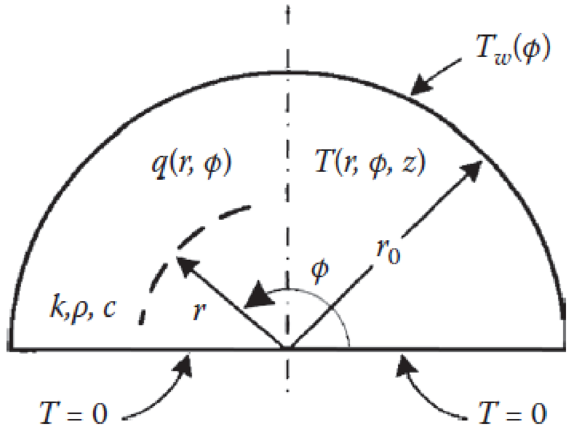
Debasis Sarkar

Advanced Heat Conduction CH61014

Indian Institute of Technology Kharagpur

Three Dimensional Problems in Cylindrical Coordinates: Integral Transforms Method

Consider a half cylinder of semi-infinite length, $0 \le r \le r0$, $0 \le \phi \le \pi$ and $0 \le z < \infty$ as illustrated in cross-section in the following Figure.



Internal energy is generated in this cylinder at a rate of $q(r, \phi)$ per unit volume. The surfaces at $\phi = 0$, $\phi = \pi$, and $z = 0$ are at zero temperature, while the surface at $r = r_0$ is kept at temperature $T_w(\phi)$. We wish to find the steady-state temperature distribution $T(r, \phi, z)$ in the cylinder. Assuming constant thermo-physical properties, the problem can be formulated as:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}(r,\phi)}{k} = 0$$

$$T(0,\phi,z) = 0 \text{ and } T(r_0,\phi,z) = T_w(\phi)$$

$$T(r,0,z) = T(r,\pi,z) = 0$$

$$T(r,\phi,0) = 0 \text{ and } T(r,\phi,\infty) = \textit{finite}$$

(Eq. A)

The partial derivative with respect to the variable ϕ can be removed by Fourier transforms. The range of ϕ is $(0, \pi)$, and in this finite interval the finite Fourier transform of $T(r, \phi, z)$ with respect to the variable ϕ can be defined as

$$\overline{T}_n(r,z) = \int_0^{\pi} T(r,\phi,z) K_n(\phi) d\phi$$

Inversion: (Eq. B)

$$T(r,\phi,z) = \sum_{n=1}^{\infty} \overline{T}_n(r,z) K_n(\phi)$$

where the kernels $K_n(\phi)$ are the normalized characteristic functions of the following characteristic-value problem:

$$\frac{d^2\psi}{d\phi^2} + n^2\psi = 0$$

$$\psi(0) = \psi(\pi) = 0$$

Kernel:

$$K_n(\phi) = \sqrt{\frac{2}{\pi}} \sin n\phi, \; n = 1, 2, 3, \ldots$$

The transform of the heat conduction equation (Eq. A) with respect to ϕ, through the use of Eq. (B) yields

$$\frac{\partial^2 \overline{T}_n}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{T}_n}{\partial r} - \frac{n^2}{r^2}\overline{T}_n(r,z) + \frac{\partial^2 \overline{T}_n}{\partial z^2} + \frac{\overline{\dot{q}}_n(r)}{k} = 0$$

(Eq. C)

where

$$\bar{\dot{q}}_n(r) = \int_0^{\pi} \dot{q}(r,\phi) K_n(\phi) d\phi$$

The above equation (Eq. C) involves the differential operator

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2}$$

In order to remove this differential operator, we define the finite Hankel transform in the finite interval (0, r_0) as

$$\bar{\bar{T}}_n(\lambda_m, z) = \int_0^{r_0} \bar{T}_n(r,z) K_n(\lambda_m, r) r\, dr$$

Inversion: (Eq. D)

$$\bar{T}_n(r,z) = \sum_{m=1}^{\infty} \bar{\bar{T}}_n(\lambda_m, z) K_n(\lambda_m, r)$$

where the kernels $K_n(\lambda_m, r)$ are the normalized characteristic functions of the following characteristic value problem:

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + (\lambda^2 r^2 - n^2)R = 0$$

$$R(0) = \textit{finite} \text{ and } R(r_0) = 0$$

The kernels K_n (λ_m, r) are given by

$$K_n(\lambda_m, r) = \frac{\sqrt{2}}{r_0}\frac{J_n(\lambda_m r)}{J_{n+1}(\lambda_m r_0)}$$

where the characteristic values λ_m are positive roots of

$$J_n(\lambda r_0) = 0$$

Now, the transform of Eq. (C) with respect to r, through the use of transform (Eq. D), yields

$$\frac{d^2\bar{\bar{T}}_n}{dz^2} - \lambda_m^2 \bar{\bar{T}}_n(\lambda_m, z) = r_0 \frac{dK_n(\lambda_m, r_0)}{dr} \bar{T}_{wn} - \frac{1}{k} \bar{\bar{\dot{q}}}_n(\lambda_m)$$

(Eq. E)

where we have defined

$$\bar{T}_{wn} = \int_0^{\pi} K_n(\phi) T_w(\phi) d\phi$$

and

$$\bar{\bar{\dot{q}}}_n(\lambda_m) = \int_0^{r_0} \bar{\dot{q}}_n(r) K_n(\lambda_m, r) r\, dr$$

Equation (Eq. E) can further be transformed with respect to z in the semi-infinite interval (0, ∞) to reduce it to an algebraic equation. However, here we prefer to solve this ordinary differential equation. The solution can be written as

$$\bar{\bar{T}}_n(\lambda_m, z) = A_n^m e^{-\lambda_m z} + B_n^m e^{\lambda_m z} + \bar{\bar{T}}_{pn}(\lambda_m)$$

where the particular solution $\bar{\bar{T}}_{pn}(\lambda_m)$ is given by

$$\bar{\bar{T}}_{pn}(\lambda_m) = -\frac{1}{\lambda_m^2}\left[r_0 \frac{dK_n(\lambda_m, r_0)}{dr} \bar{T}_{wn} - \frac{1}{k} \bar{\bar{\dot{q}}}_n(\lambda_m) \right]$$

Since the temperature distribution $T(r, \phi, z)$ is to be finite as $z \to \infty$, we have

$$\lim_{z\to\infty} \bar{\bar{T}}_n(\lambda_m, z) = finite$$

which yields $B_n^m = 0$. On the other hand, since $T(r, \phi, 0) = 0$,

$$\bar{\bar{T}}_n(\lambda_m, z) = 0$$

which gives

$$A_n^m = -\bar{\bar{T}}_p(\lambda_m, n)$$

The solution for $\bar{\bar{T}}_n(\lambda_m, z)$ now becomes

$$\bar{\bar{T}}_n(\lambda_m, z) = (1 - e^{-\lambda_m z})\bar{\bar{T}}_{pn}(\lambda_m)$$

When this double transform is inverted successively through the use of inversion relations (Eq. B) and (Eq. D), we obtain the temperature distribution as

$$T(r, \phi, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(1 - e^{-\lambda_m z})\bar{\bar{T}}_{pn}(\lambda_m)K_n(\lambda_m, r)K_n(\phi)$$

which can also be written as

$$T(r, \phi, z) = \frac{4}{\pi r_0}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1 - e^{-\lambda_m z}}{\lambda_m^2}\sin n\phi \frac{J_n(\lambda_m r)}{J_{n+1}(\lambda_m r_0)}\left[\lambda_m \int_0^{\pi} \sin n\phi' T_w(\phi')\, d\phi' \right.$$
$$\left. + \frac{1}{k r_0 J_{n+1}(\lambda_m r_0)}\int_0^{r_0}\int_0^{\pi} J_n(\lambda_m r')\sin n\phi' \dot{q}(r', \phi')r'\, dr' d\phi'\right]$$