# Basic Equations in Integral Form for a Control Volume

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Ch - 4

N = Arbitrary extensive property of a system

$$N\Big|_{SYSEM} = \int_{Mass\,(System)} \eta\, dm = \int_{\cancel{V}\,(System)} \eta\, \rho\, d\cancel{V}$$

η = Corresponding intensive property

**Basic Equations in Integral Form for a CV**

**Conservation of Mass**

**Conservation of Momentum**

## Relation of System Derivatives to the Control Volume Formulation

$$\left.\frac{\partial N}{\partial t}\right|_{SYSEM} = \frac{\partial}{\partial t}\int_{CV} \eta \rho d\cancel{V} + \int_{CS} \eta \rho \vec{V} . d\vec{A}$$

$\vec{V}$ is measured relative to the CV

$\left.\frac{\partial N}{\partial t}\right|_{SYSEM}$ = Total rate of change of any arbitrary extensive property of the system

$\frac{\partial}{\partial t}\int_{CV} \eta \rho d\cancel{V}$ = Time rate of change of the arbitrary extensive property within the CV

$\int_{CS} \eta \rho \vec{V} . d\vec{A}$ = Net rate of efflux of the extensive property, N, through the control surface

## Conservation of Mass

$$\left.\frac{\partial N}{\partial t}\right|_{SYSEM} = \frac{\partial}{\partial t}\int_{CV} \eta \rho d\cancel{V} + \int_{CS} \eta \rho \vec{V} . d\vec{A}$$

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho d\cancel{V} + \int_{CS} \rho \vec{V} . d\vec{A}$$

Incompressible Fluid

$$0 = \int_{CS} \rho \vec{V} . d\vec{A}$$

The size of the CV is fixed

$$\int_{CS} \rho \vec{V} . d\vec{A} = \pm \left| \rho_n V_n A_n \right|$$

When uniform flow at section n is assumed

## Momentum Equation for Inertial CV

$$\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{CV} \vec{V}\rho d\cancel{V} + \int_{CS} \vec{V}\rho \vec{V}.d\vec{A}$$

$$\vec{F}_B = \int_{CV} \vec{B}\rho d\cancel{V} \qquad \vec{F}_S = \int_{A} -p\, d\vec{A}$$

Scalar Component $$F_x = F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\cancel{V} + \int_{CS} u\,\rho\vec{V}.d\vec{A}$$

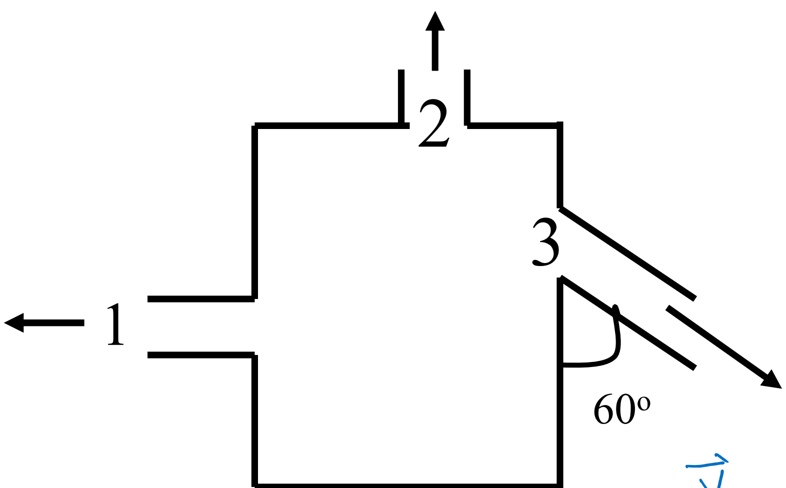
1. To determine the sign of

$$\rho\vec{V}.d\vec{A} = \pm\left|\rho V\, d A\cos\alpha\right|$$

2. To determine the sign of each velocity component

$$u\;\rho\vec{V}.d\vec{A} = u\,\left\{\pm\left|\rho V\, d A\cos\alpha\right|\right\}$$

Steady Incompressible Flow




Fluid with ρ = 1050 kg/m3 is flowing through the box,
$A_1 = 0.05\ m^2$, $A_2 = 0.01\ m^2$, $A_3 = 0.06 m^2$

$$\vec{V}_1 = 4\hat{\imath}\ m/s \quad \vec{V}_2 = -8\hat{\jmath}\ m/s$$

Find V3

$$\vec{V}_1 . \bar{A}_1 + \bar{V}_2 . \vec{A}_2 + \bar{V}_3 \bar{A}_3 = 0$$

$$\vec{V}_3 . \vec{A}_3 = -\bar{V}_1 . \bar{A}_1 - \bar{V}_2 . \bar{A}_2$$

$$= -4\hat{i} . 0.05(-\hat{i}) - (-8\hat{j}) . 0.01\hat{j}$$

$\bar{V}_3 . \bar{A}_3 = 0.28\ m^3/s$. Since $\bar{V}_3 \bar{A}_3 > 0$, flow at section 3 is out of CV. $V_3 = \frac{1}{A_3} \times 0.28 m^3/s = 4.67\ m/s$

From geometry $\bar{V}_3 = V_3 \sin\theta\,\hat{i} - V_3 \cos\theta\,\hat{j} = 4.04\hat{i} - 2.34\hat{j}$

Find the net rate of efflux of momentum through the CV

The net rate of momentum efflux is given by $\int_{CS} \bar{V} \rho \bar{V} \cdot d\vec{A}$

$$= \bar{V}_1 \rho \bar{V}_1 \cdot \bar{A}_1 + \bar{V}_2 \rho \bar{V}_2 \cdot \bar{A}_2 + \bar{V}_3 \rho \bar{V}_3 \bar{A}_3$$

$$\overrightarrow{mf} = \left[u_1\{-|\rho V_1 A_1|\} + u_2\{-|\rho V_2 A_2|\} + u_3\{|\rho V_3 A_3|\}\right]\hat{i}$$

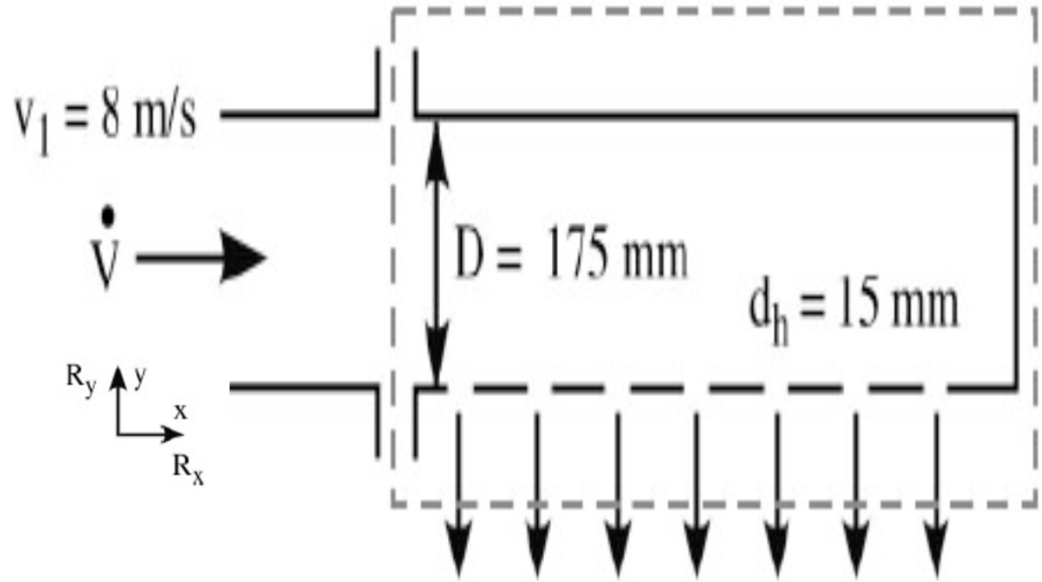
$u_1 = 4\ m/s$ $\quad u_2 = 0$ $\quad u_3 = 4.04\ m/s$

$$+ \left[v_1\{-|\rho V_1 A_1|\} + v_2\{-|\rho V_2 A_2|\} + v_3\{|\rho V_3 A_3|\}\right]\hat{j}$$

$v_1 = 0$ $\quad v_2 = -8\ m/s$ $\quad v_3 = -2.33\ m/s$

$$\overline{mf} = 349\hat{i} - 13.5\hat{j}\ \ N$$

Water exits a pipe from a series of 109 holes drilled into the side as shown in the figure, along with the coordinate systems. The press at the inlet section is 35 kPa. Calculate the forces required to hold the spray pipe in plac The pipe and the water ($\rho = 10^3$ kg/m$^3$) it contains weighs 5 kg.




$$F_x = F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho \, d\mathcal{V} + \int_{CS} u\,\rho\,\vec{V}.d\vec{A}$$

$$F_y = F_{Sy} + F_{By} = \frac{\partial}{\partial t}\int_{CV} v\rho \, d\mathcal{V} + \int_{CS} v\,\rho\,\vec{V}.d\vec{A}$$

$A_{in} = \frac{\pi}{4}(0.173)^2 = 2.404\times10^{-2}\,m^2$

$V_{in} = 8\,m/s$

$Q = 1.92\times10^{-1}\,m^3/s$

$A_{OUT}/\text{hole} = \frac{\pi}{4}(0.015)^2 = 1.766\times10^{-4}\,m^2$

$A_{OUT}(\text{Total}) = 109 \times 1.766\times10^{-4}\,m^2$

$= 1.925\times10^{-2}\,m^2$

$\therefore\quad 109 \times V_{OUT} \times 1.766\times10^{-4} = 1.9\times10^{-1}$

$V_{OUT} = 9.98\,m/s$

$x$ comp

$$F_{Sx} + \underbrace{\cancel{F_{Bx}}}_{=0} = \underbrace{\cancel{\frac{\partial}{\partial t}}}_{=0}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho\, \vec{V}\cdot d\vec{A}$$

$$R_x + P_{1g}A_{IN} = -\rho Q u_1 + \rho Q \underbrace{\cancel{u_2}}_{=0} = -\rho Q u_1$$

$$R_x = -1000 \times 1.092\times10^{-1} \times 8 - 35\times10^{3} \times 2.404\times10^{-2}$$

$$\underline{R_x = -2377\ N}$$

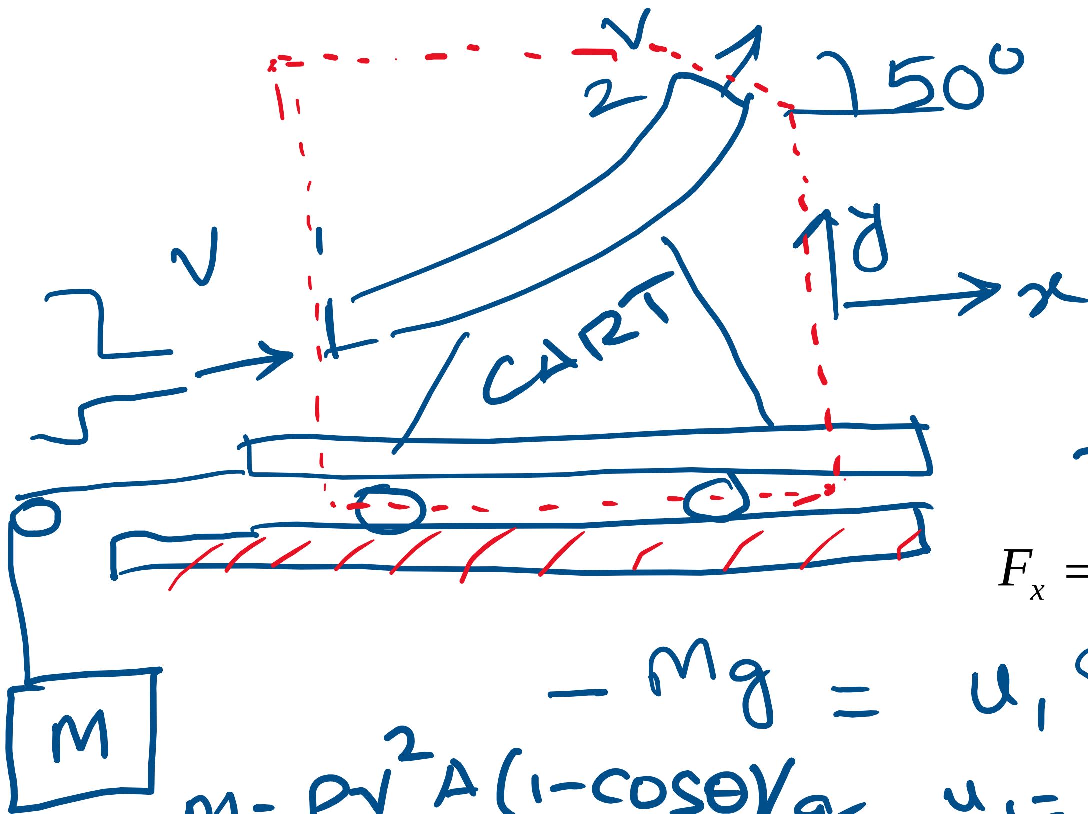
For $y$ direct$^n$

$$R_y + \underbrace{\cancel{P_2}}_{=0} g A_{out} - \rho g \forall = -\rho Q \underbrace{\cancel{v_1}}_{=0} + \rho Q v_{2y}$$

$$R_y = \rho Q v_2 + \rho g \forall$$

$$\underline{R_y = -1867\ N}$$

Vane with a block of mass attached to it




$V = 15\,m/s$

$A = 0.05\,m^2$

$\theta = 50^\circ$

FIND THE MASS, M, NEEDED TO HOLD THE CART STATIONARY.

$$F_x = F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, d\forall + \int_{CS} u\,\rho\,\vec{V}.d\vec{A}$$

$$-Mg = u_1\{-|\rho V_1 A_1|\} + u_2\{+|\rho V_2 A_2|\}$$

$u_1 = V$ $\quad u_2 = V\cos\theta$

$M = \rho V^2 A(1-\cos\theta)/g$

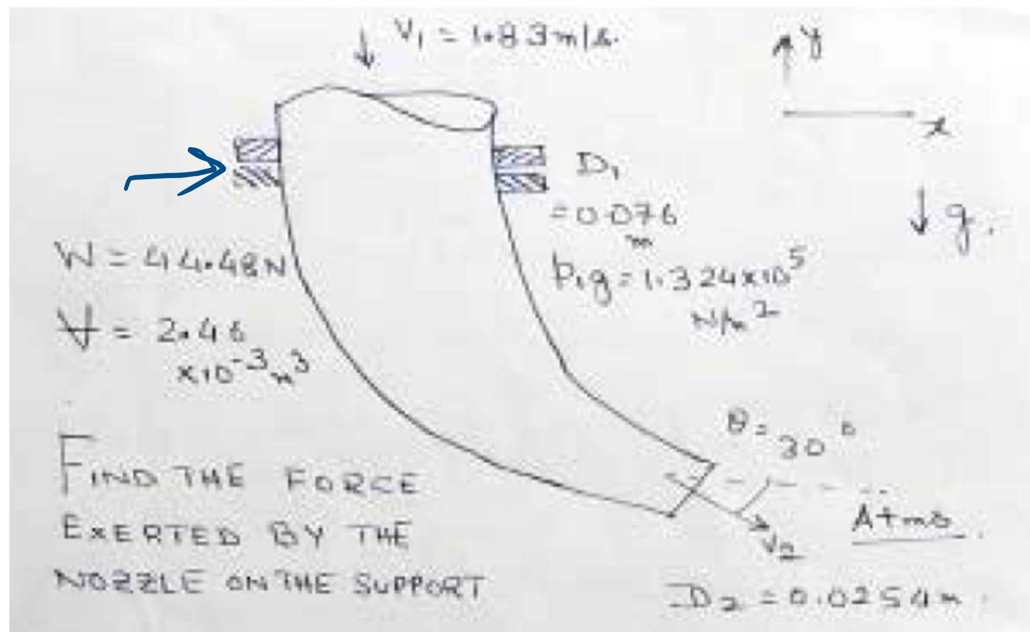
Curved nozzle Assembly, discharging water to the atmosphere, as shown in the figure

Given:

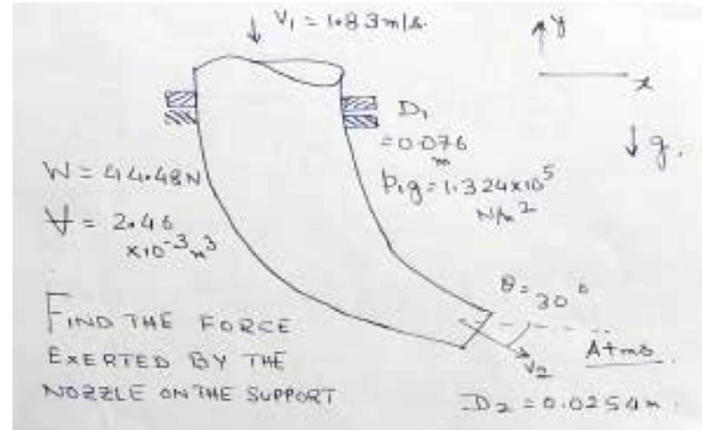
$$W = 44.48N,\ p_{1g} = 1.324x10^5\ N/m^2, V = 2.46x10^{-3}\,m^3, V_1 = 1.83\,m/s,\ D_1 = 0.076\ m,\ D_2 = 0.0254m$$

The angle at the exit is 30º with the horizontal.

Calculate the reaction force exerted by the nozzle on the coupling of the inlet pipe.



$$F_x = F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, d\mathcal{V} + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$$



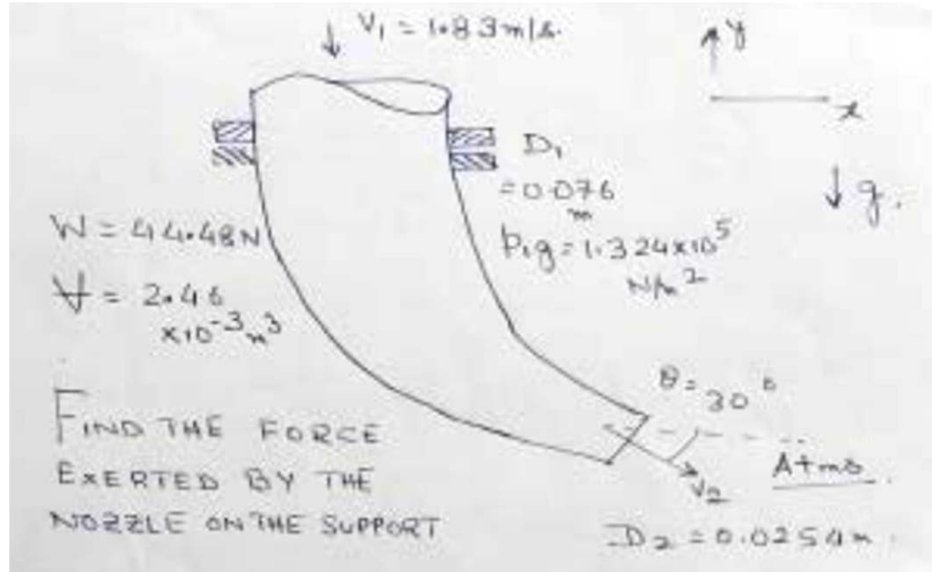

$$R_x = \underbrace{\cancel{u_1}}_{=0}\{-|\rho V_1 A_1|\} + u_2\{+|\rho V_2 A_2|\}$$

$$\boxed{R_x = \rho V_2^2 A_2 \cos\theta} = 117.3\,N$$

$$R_y - p_{1g}A_1 - W - \rho g \forall = v_1\{-|\rho V_1 A_1|\} + v_2\{+|\rho V_2 A_2|\}$$

$$\boxed{= 616\,N}$$

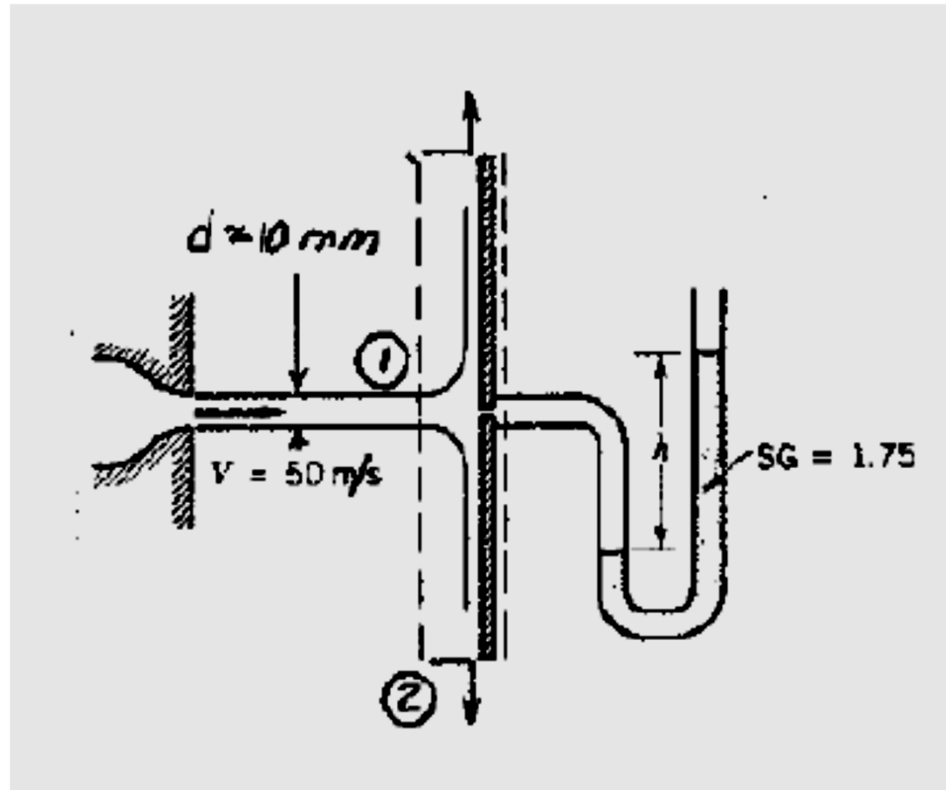
$$v_1 = -V_1, \quad v_2 = -V_2 \sin\theta$$


V1 = 1.83 m/s
y
x
D1
= 0.076 m
g
W = 44.48 N
p1g = 1.324x10^5 N/m^2
V = 2.46 x10^-3 m^3
θ = 30°
FIND THE FORCE
EXERTED BY THE
NOZZLE ON THE SUPPORT
V2
Atmo.
D2 = 0.0254 m

A horizontal, axi-symmetric jet of air ($\rho = 1.23$ $Kg/m^3$) with a diameter of 10 mm strikes the centre of a vertical disk of 200 mm diameter. The jet speed is 50 m/s at the nozzle exit.

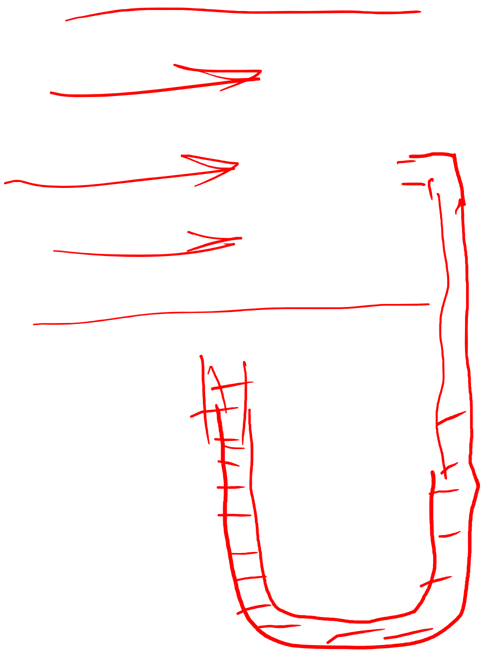
There is a small hole at the centre of the disk, where the air jet strikes and a manometer with a manometric liquid of specific gravity equal to 1.75.

Calculate (i) the deflection, h, of the manometer and (ii) the force exerted by the jet on the disk.

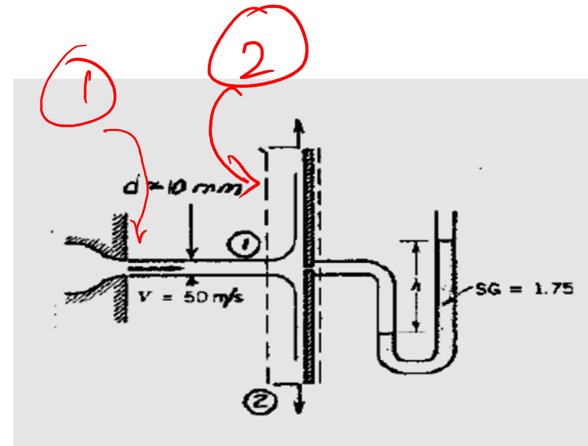



Pitot Tube

# Pitot Tube



$$\frac{p_1}{\rho} + \frac{V_1^2}{2} = \frac{p_2}{\rho} + \frac{\cancel{V_2^2}}{2} \quad = 0$$




$$\boxed{\frac{\rho V_1^2}{2} = \rho_m g h}$$

$$F_x = F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, d\forall + \int_{CS} u\,\rho\,\vec{V}.d\vec{A}$$

$$h = 0.089 \text{ m}$$

$$R_x = \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$$

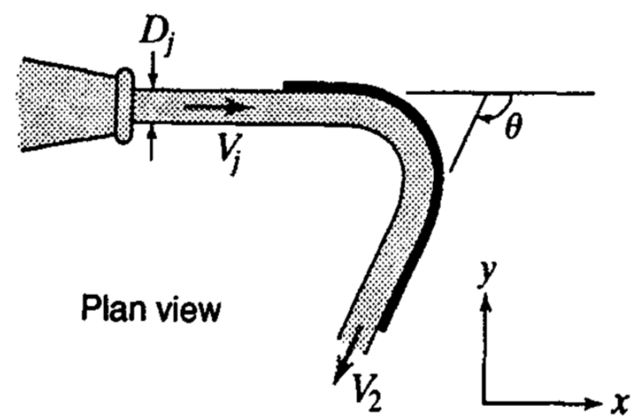
$$R_x = u_1\{-\rho V A\} + \underset{=0}{\cancel{u_2}}\{+\rho V A\}$$

✓

$$R_x = -\rho V^2 A = -0.24\,\text{N}$$

FORCE BY THE JET $= +0.24$ N

Refer to the given figure where a water jet is forced to change its direction because of the presence of a blade in its path. Assume that friction is negligible, that $\theta = 115^{o}$, and that the water jet has a velocity of 25 m/s and a diameter of 40 mm. Find (a) the component of the force acting on the blade (the portion shown by the dark line ~~at~~ the bend) in the direction of the jet; (b) the force component normal to the jet; and (c) the magnitude and direction of the resultant force exerted on the blade.




$$R_x = \bar{V}_1 \left\{ |\rho \vartheta_1 A_1| \right\} + V_2 \left\{ |\rho \vartheta_1 A_1| \right\}$$

$$R_x = u_1\{-|\rho v_1 A_1|\} + u_2\{+|\rho v_1 A_1|\}$$

$$u_1 = 25\,m/s, \quad u_2 = 25\cos 115$$

$$\Rightarrow \quad \underline{\underline{R_x = -1117\,N}}$$

$$R_y = v_1\{-|\rho v_1 A_1|\} + v_2\{+|\rho V_2 A_2|\}$$

$$v_1 = 0, \quad v_2 = -V_2 \sin 115$$

$$\underline{\underline{R_y = -712\,N}}$$

FORCE ON THE BLADE

$\underline{+1117\,N \;\&\; +712\,N}$

## Control Volume Moving with Constant Velocity

The previous equations (e.g., momentum) are valid for inertial control volumes. A control volume moving with constant velocity also inertial, since it has no acceleration with respect o the inertial reference frame XYZ.

$$\left.\frac{\partial N}{\partial t}\right|_{SYSEM} = \frac{\partial}{\partial t}\int_{CV} \eta \rho d\cancel{V} + \int_{CS} \eta \rho \vec{V}.d\vec{A}$$

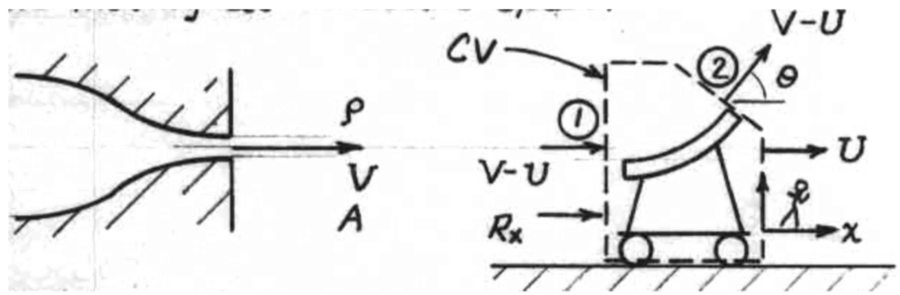
The above equation is valid for any constant velocity motion of coordinate system XYZ, fixed to the control volume if

1. All velocities are measured relative to the control volume
2. All time derivatives are measured relative to the control volume

$$F_x = F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\, \rho\, d\cancel{V} + \int_{CS} u_{xyz}\, \rho\, \overrightarrow{V_{xyz}}.d\vec{A}$$

The subscript xyz indicates that all velocities are measured relative to the control volume. Velocities are those that would be seen by an observer moving at constant speed with the control volume.

The adjoining figure shows water from a jet (of velocity V) striking a vane of mass M moving with a constant speed (U).



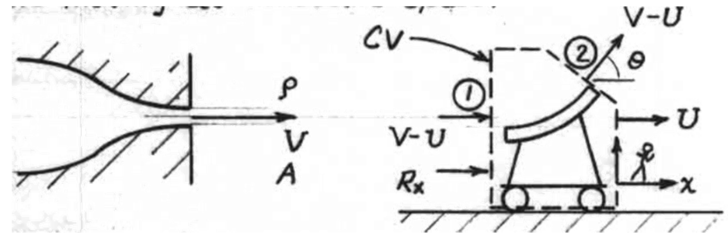

Find expressions for the

i) forces, $R_x$ and $R_y$ exerted by the vane;
ii) power produced by the vane;
iii) deduce a relation between V and U to maximize the power.

You may assume that there is no net pressure force and no change in jet speed.

$$F_x = F_{Sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\, \rho \, d\forall + \int_{CS} u_{xyz}\, \rho \, \overrightarrow{V_{xyz}} \cdot d\vec{A}$$

$$F_x = F_{Sx} + F_{Bx} = \cancel{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\,\rho\, d\forall} + \int_{CS} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$




$$R_x = \int_{cs} u_{xyz}\,\rho\,\vec{V}_{xyz}\cdot d\vec{A}$$

$$= u_{xyz_1}\left\{-\left|\rho V_{xyz} A_1\right|\right\} + u_{xyz_2}\left\{+\left|\rho V_{xyz} A_1\right|\right\}$$

$$V_{xyz_1} = V - u \qquad V_{xyz_2} = V - u$$

$$u_{xyz_1} = V - u \qquad u_{xyz_2} = (V - u)\cos\theta$$

$\cancel{R_y}$ $R_x = \rho (V-u)^2 (\cos\theta - 1)$

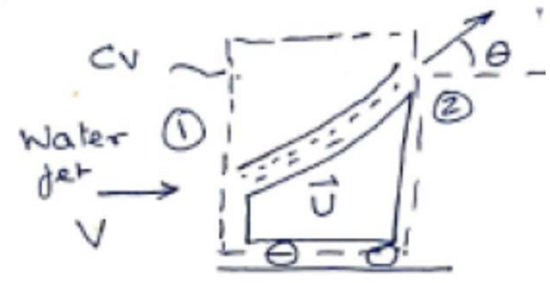
$K_x = -R_x = \rho (V-u)^2 (1 - \cos\theta)$.

POWER PRODUCED BY THE VANE

$K_x \times (VEL) = \rho (V-u)^2 u A (1-\cos\theta)$

$\cancel{d}$ $\frac{dW_{OUT}}{du} = 0 \Rightarrow \boxed{u = \frac{V}{3}}$

Since the area of the jet is constant, the velocity of the jet leaving will also be equal to $\vec{V}$ the velocity of the CV to the right is $\vec{U}$



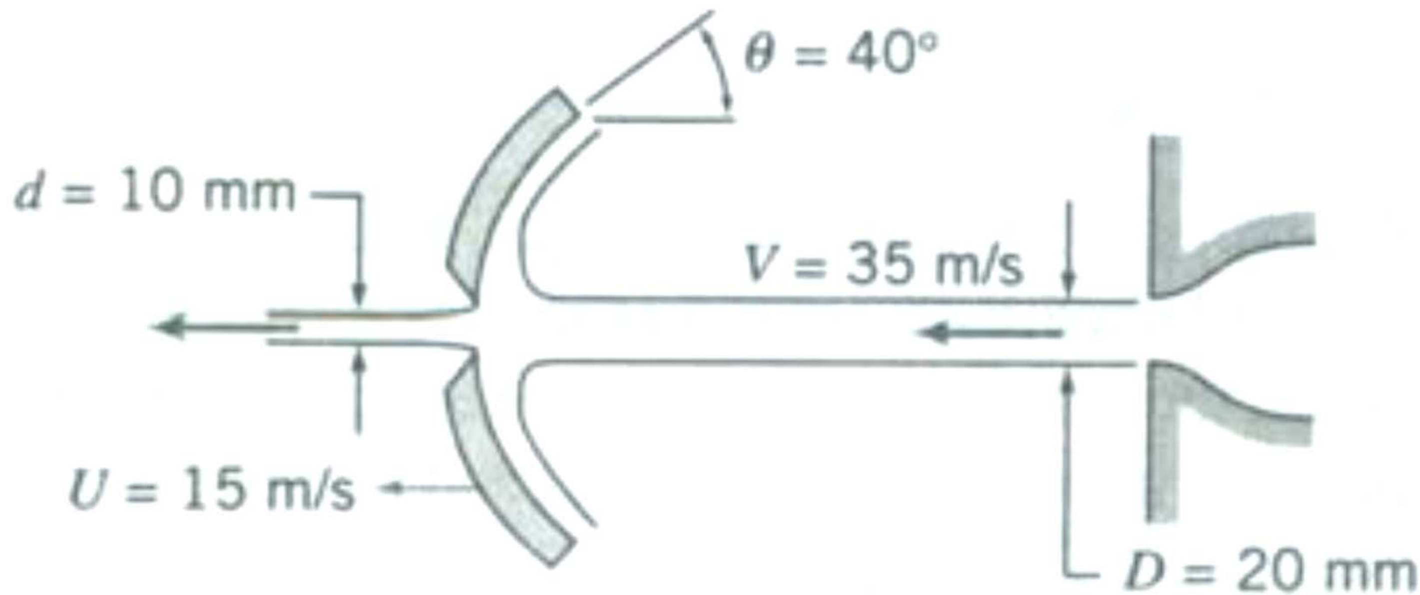

The time derivatives in the governing equations are to be measured relative to the CV. The CV is moving to the right

In order to use the equations (mass and momentum) in the integral form for **an inertial CV**, we may assume that the entire CV may be moving to the left with a velocity $\vec{U}$ such that the coordinate system fixed on the CV would appear to be stationery.

Th velocity of water entering and leaving the jet in that case would have to be taken as $\left(\vec{V} - \vec{U}\right)$ at both entry and exit, to be used in the continuity and momentum equations, i.e., $\vec{V}_{xyz} = \vec{V} - \vec{U}$

To use the 2nd tern of the momentum equation, $\int u\, \rho \vec{V}_{xyz} \cdot \overrightarrow{dA}$ u is he x component of $\vec{V}_{xyz}$

Thus u at 1 is $\vec{V}_{xyz} \cos 0^{o} = V - U$ and u at 2 is $\vec{V}_{xyz} \cos\theta = \left(V - U\right)\cos\theta$




The circular dish in the figure has an outside diameter of 0.2 m. A water jet with speed of 35 m/s strikes the dish concentrically. The dish moves to the left at 15 m/s. The jet diameter is 20 mm. The dish has a hole at its centre that allows a stream of water, 10 mm in diameter to pass through without resistance. The remainder of the jet is deflected and flows along the dish. Calculate the force required to maintain the dish motion.

Answer: $R_x = -167N$