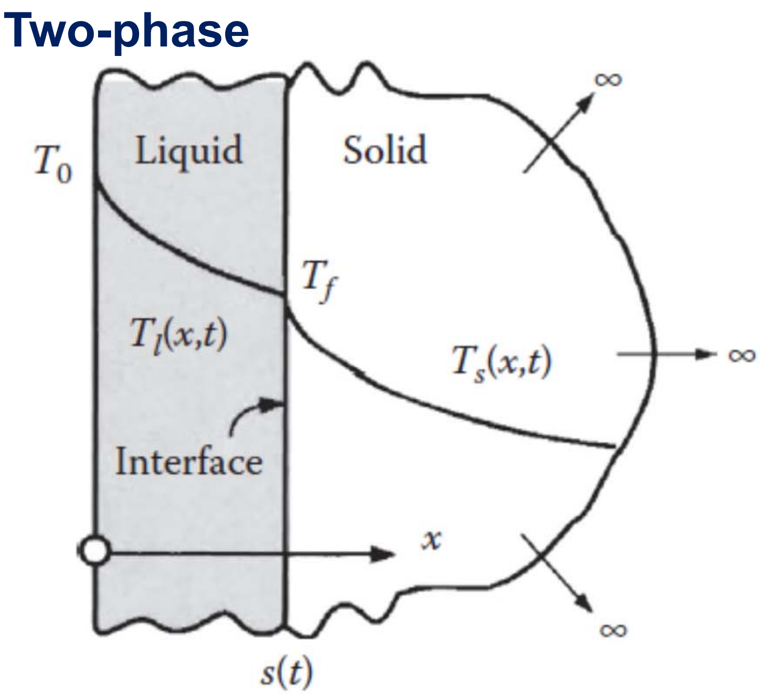
CH61014: Advanced Heat Transfer

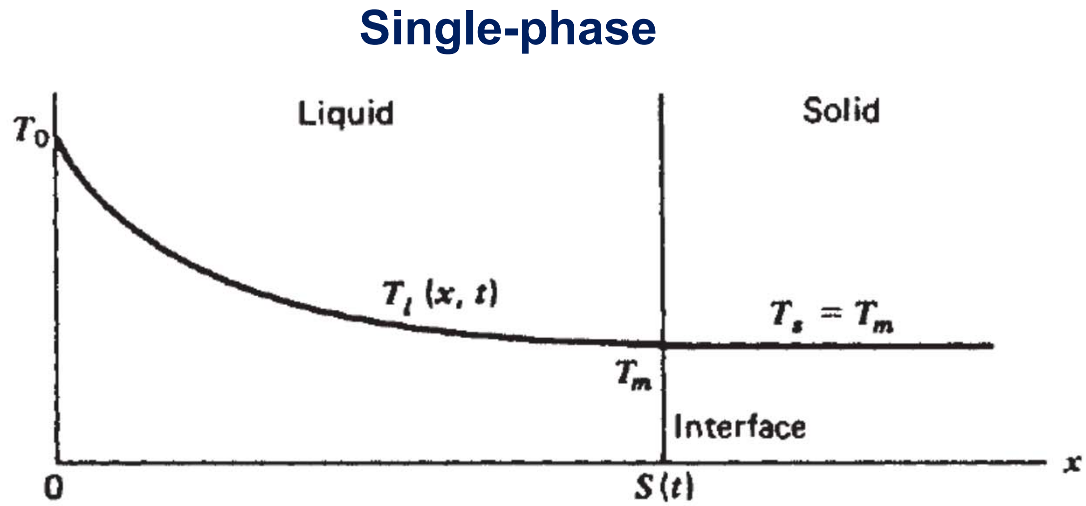
Debasis Sarkar

Solution of Phase Change Problems: Integral Method

The integral method provides a relatively simple and straightforward approach for the solution of one-dimensional transient phase-change problems.

Melting of Solid: Single Phase: Integral Method





Consider the melting of a solid confined in a semi-infinite region. The solid is initially at the melting temperature T_m.

For times $t > 0$, the boundary surface at $x = 0$ is kept at a constant temperature T_0, which is greater than the solid's melting temperature T_m.

The melting starts at the surface $x = 0$ and the solid–liquid interface moves in the positive x-direction. Solid phase remains at T_m. It is a Single phase problem.

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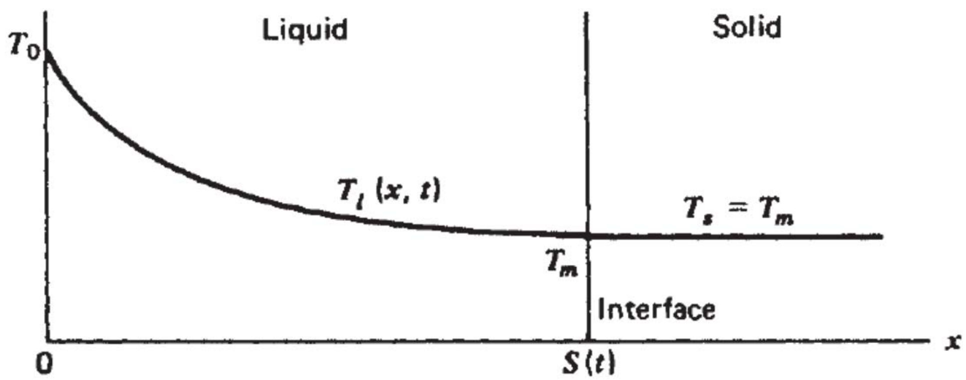

Single-phase

$$\frac{\partial^2 T_l}{\partial x^2} = \frac{1}{\alpha_l}\frac{\partial T_l(x,t)}{\partial t} \qquad \text{in} \qquad 0 < x < s(t), \qquad t > 0$$

$$T_s = T_m \qquad \text{in} \qquad x > s(t), \qquad t > 0$$

Boundary condition: $T_l(x = 0, t) = T_0$

At the interface: $T_l(x = s, t) = T_m \qquad \text{at} \qquad x = s(t)$

$$-k_l \left.\frac{\partial T_l}{\partial x}\right|_{x=s(t)} = \rho L \frac{ds(t)}{dt} \qquad \text{at} \qquad x = s(t)$$

$s(0) = 0$

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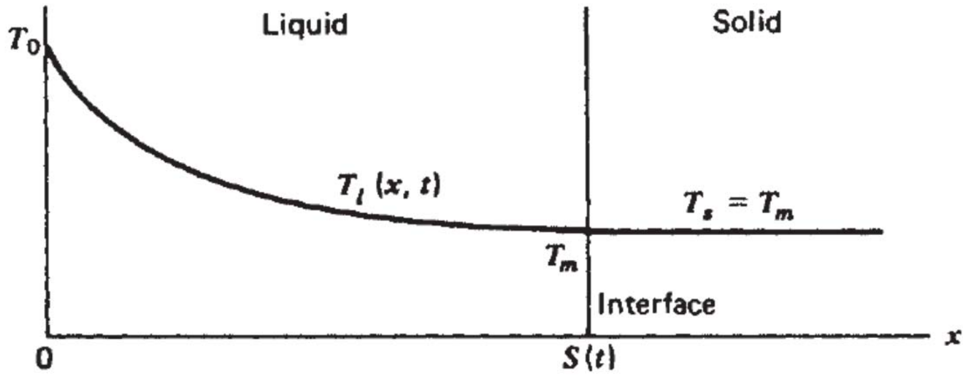

First, define a *thermal layer thickness:*

We choose the region $0 \le x \le s(t)$ as the thermal layer

Next, integrate the heat conduction equation from $x = 0$ to $x = s(t)$,

$$\int_{x=0}^{s(t)} \frac{\partial^2 T}{\partial x^2} dx = \frac{1}{\alpha} \int_{x=0}^{s(t)} \frac{\partial T}{\partial t} dx$$

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$$\int_{x=0}^{s(t)} \frac{\partial^2 T}{\partial x^2} dx = \frac{1}{\alpha} \int_{x=0}^{s(t)} \frac{\partial T}{\partial t} dx$$

LHS: $$\int_{x=0}^{s(t)} \frac{\partial^2 T}{\partial x^2} dx = \left. \frac{\partial T}{\partial x} \right|_0^{s(t)}$$

RHS: $$\frac{1}{\alpha} \int_{x=0}^{s(t)} \frac{\partial T}{\partial t} dx = \frac{1}{\alpha} \left[\frac{d}{dt} \left(\int_{x=0}^{s(t)} T \, dx \right) - \frac{ds(t)}{dt} \left. T \right|_{x=s(t)} \right]$$

$$\left. \frac{\partial T}{\partial x} \right|_{x=s(t)} - \left. \frac{\partial T}{\partial x} \right|_{x=0} = \frac{1}{\alpha} \left[\frac{d}{dt} \left(\int_{x=0}^{s(t)} T \, dx \right) - \frac{ds(t)}{dt} \left. T \right|_{x=s(t)} \right]$$

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$$\left.\frac{\partial T}{\partial x}\right|_{x=s(t)} - \left.\frac{\partial T}{\partial x}\right|_{x=0} = \frac{1}{\alpha}\left[\frac{d}{dt}\left(\int_{x=0}^{s(t)} T\,dx\right) - \frac{ds(t)}{dt}\left.T\right|_{x=s(t)}\right]$$

$$\Rightarrow \quad -\frac{\rho L}{k}\frac{ds(t)}{dt} - \left.\frac{\partial T}{\partial x}\right|_{x=0} = \frac{1}{\alpha}\frac{d}{dt}\left[\theta - s(t)T_m\right]$$

$$\theta(x) \equiv \int_{x=0}^{s(t)} T(x,t)\,dx$$

This equation is the *energy integral equation* for this problem.

We choose a second-degree polynomial approximation for the temperature in the form

$$T(x,t) = a + b(x-s) + c(x-s)^2$$

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$$T(x,t) = a + b(x-s) + c(x-s)^2$$

Three conditions are needed to determine these three coefficients.

$$T(x=0,t) = T_0$$

$$T(x=s(t),t) = T_m$$

First and Second conditions

Derive Third Condition:

Use of Stefan Condition is inconvenient

Differentiate BC: $T_l(x=s,t) = T_m$ at $x = s(t)$

$$dT \equiv \left[\frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial t}dt\right]_{x=s(t)} = 0 \quad \Rightarrow \quad \frac{\partial T}{\partial x}\frac{ds(t)}{dt} + \frac{\partial T}{\partial t} = 0$$

Eliminate ds(t)/dt using Stefan Condition

$$\left(\frac{\partial T}{\partial x}\right)^2 = \frac{\rho L}{k}\frac{\partial T}{\partial t} \quad \text{at} \quad x = s(t)$$

Eliminate $\frac{\partial T}{\partial t}$ using Heat Eq.

$$\left(\frac{\partial T}{\partial x}\right)^2 = \frac{\alpha\rho L}{k}\frac{\partial^2 T}{\partial x^2} \quad \text{at} \quad x = s(t)$$

Third Condition

Melting of Solid: Single Phase: Integral Method

The resulting temperature profile becomes

$$T(x,t) = T_m + b(x-s) + c(x-s)^2$$

where

$$b = \frac{\alpha\rho L}{ks}[1-(1+\mu)^{1/2}]$$

$$c = \frac{bs + (T_0 - T_m)}{s^2}$$

$$\mu = \frac{2k}{\alpha\rho L}(T_0 - T_m) = \frac{2C(T_0 - T_m)}{L}$$

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Substituting the temperature profile into the energy integral equation, we obtain the following ordinary differential equation for the determination of the location of the solid–liquid interface *s*(*t*), that is,

$$s\frac{ds}{dt} = 6\alpha\frac{1-(1+\mu)^{1/2}+\mu}{5+(1+\mu)^{1/2}+\mu}$$

with $s(t=0)=0$

The solution of the above ODE:

$$s(t) = 2\lambda\sqrt{\alpha t}$$

where

$$\lambda \equiv \left[3\frac{1-(1+\mu)^{1/2}+\mu}{5+(1+\mu)^{1/2}+\mu}\right]^{1/2}$$

Exact Solution (Similarity Solution) for the Same Problem:

$$s(t) = 2\lambda(\alpha_l t)^{1/2}$$

where λ the root of

$$\lambda e^{\lambda^2}\operatorname{erf}(\lambda) = \frac{C(T_0 - T_m)}{L\sqrt{\pi}}$$

NOTE: The approximate solution for *s*(*t*) is of the same form as the exact solution of the same problem. However, definition of λ is different.

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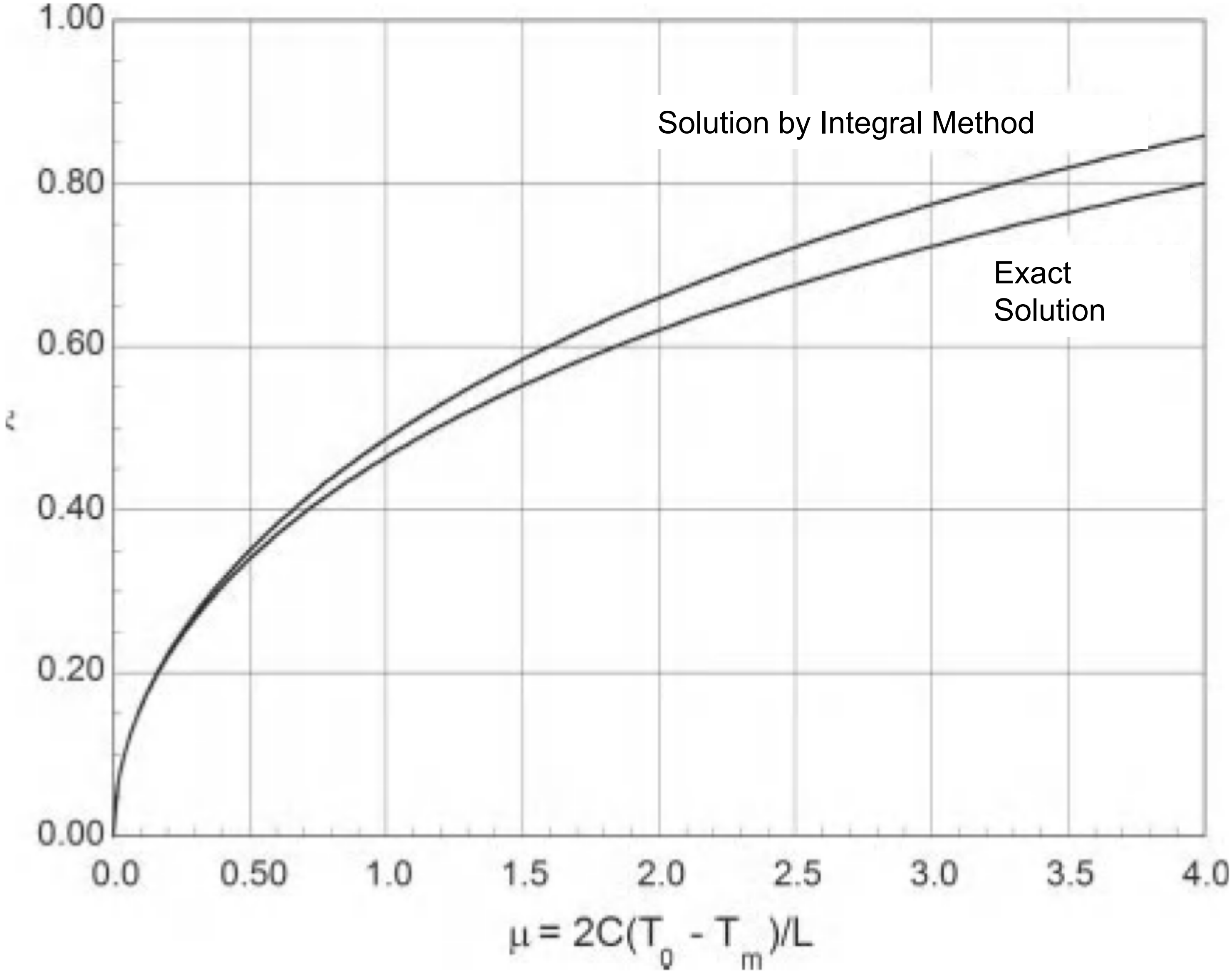

Comparison of exact and approximate solutions

Thank You