Indian Institute of Technology Kharagpur
Department of Chemical Engineering
End Sem Examination, Autumn 24-25

18 Nov 2024
Time: 3 Hrs

Sub: **Adv. Mathematical Techniques in Chem Engg.** Full Marks: 50

Subject Number: CH61015 No. of Students: 140

Instructions: Answer all questions. Closed book, closed notes examination. All symbols carry their usual meaning. Assume missing data suitably. Follow the five step problem solving methodology. Data required are included in the question paper.

1. (10 marks) Consider the case of growth of two species under limited resources:

$$\frac{dx}{dt} = x(\alpha_1 - \beta_1 x - \gamma_1 y)$$
$$\frac{dy}{dt} = y(\alpha_2 - \beta_2 y - \gamma_2 x)$$

where x and y are the population of the two species, α is the growth constant, β is the self inhibition constant and γ is the constant corresponding to the inhibition by the other species.

(a) (2 marks) Obtain all the possible steady states of the system in terms of the parameters.

(b) (2 marks) Obtain the Jacobian matrix for linear stability analysis.

(c) (6 marks) Analyze the phase plane picture of each of the steady states for stability and remark under which condition of parameters they are stable/unstable focus or saddle or node.

2. Discuss the following concepts.

(a) (1 mark) Basin of attraction

(b) (1 mark) Fixed point of a map

(c) (1 mark) Contraction mapping

(d) (1 mark) Rayleigh quotient

(e) (1 mark) Hopf bifurcation

3. (10 marks) Consider the following chemical kinetic equations for two reacting species:

$$\frac{dx}{dt} = \frac{1}{\tau}(1 - x) - xy^2$$
$$\frac{dy}{dt} = \frac{1}{\tau}(y_0 - y) + xy^2 - ky$$

We fix the parameters $y_0 = 0.25$ and $k = 0.05$ and would like to employ homotopy continuation method to explore the dependence of the steady state value of x on the parameter τ.

(a) (3 marks) Obtain the set of algebraic equations defining the steady state by following these steps: i) manipulate the steady state equations to obtain a linear relation between y and x. ii) Use this relation to eliminate y and obtain the steady state relation for x.

(b) (2 marks) Use physical argument to set the initial value of the parameter τ and the corresponding steady state variables.

(c) (5 marks) Use homotopy continuation method to obtain the guess value x for $\tau = 1$

4. (5 marks) Solve completely:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0$$

The boundary condition at $r = 1, u = f(\theta, \phi)$ Use the suitable physical boundary conditions on rest of the boundaries.

5. (5 marks) The Bessel equation is given as:

$$\frac{d}{dr}\left(r\frac{dy}{dr}\right) + \lambda^2 r y = 0$$

It has solution of zeroth order Bessel function, $J_0(\lambda r)$. The relevant boundary condition is at $r = 1, y = 0$. Prove that the zeroth order Bessel functions are orthogonal functions with respect to the weight function, r.

6. (5 marks) In a rectangular channel, the wall at $y = 0$ is coated with a solute that is gradually dissolved in water with the fixed solute concentration at the wall is c_w. Pure water is entering in the channel and is flowing under laminar condition inside the channel. Because of the dissolution of the solute, there exists a thin concentration boundary layer of solute near the wall. At the steady state, the governing equation of the solute concentration within the mass transfer boundary layer in terms of non-dimensional variables is given as

$$Ay\frac{\partial c}{\partial x} = \frac{\partial^2 c}{\partial y^2}$$

The relevant boundary conditions are: at $x = 0, c = 0$; at $y = 0, c = 1$ and at $y = \infty, c = 0$. Find out concentration profile in the mass transfer boundary layer using **similarity method**.

7. (10 marks) Solve completely using **Green's function method**:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + t$$

At $t = 0, u = 0$; at $x = 0, \partial u/\partial x = 0$ and at $x = 1$, $u = t$.

End of Question Paper