Fluid Mechanics

Lectures 9 – 10

Continuity Equation

Equation of Motion

Navier-Stokes Equation

Equation of Continuity

$$[\partial\rho/\partial t + (\nabla \cdot \rho\mathbf{v}) = 0]$$

Cartesian coordinates (x, y, z):

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

Cylindrical coordinates (r, θ, z):

$$\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

Spherical coordinates (r, θ, ϕ):

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 v_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial\theta}(\rho v_\theta \sin\theta) + \frac{1}{r \sin\theta}\frac{\partial}{\partial\phi}(\rho v_\phi) = 0$$

Equation of Motion for a Newtonian Fluid with Constant ρ and μ

$$[\rho D\mathbf{v}/Dt = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}]$$

Cartesian coordinates (x, y, z):

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right] + \rho g_x$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right] + \rho g_y$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$

Cylindrical coordinates (r, θ, z):

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\cancel{\frac{\partial v_\theta}{\partial \theta}} + \cancel{v_z}\frac{\partial v_\theta}{\partial z} + \cancel{\frac{v_r v_\theta}{r}}\right) = \cancel{-\frac{1}{r}\frac{\partial p}{\partial \theta}} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_\theta}{\partial z^2}} + \cancel{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}\right] + \cancel{\rho g_\theta}$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$

$$\mu\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(r v_\theta)\right) = 0$$

Spherical coordinates (r, θ, ϕ):

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right) = -\frac{\partial p}{\partial r}$$

$$+ \mu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r) + \frac{1}{r^2 \sin \theta}\frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial v_r}{\partial \theta}\right) + \frac{1}{r^2 \sin^2 \theta}\frac{\partial^2 v_r}{\partial \phi^2}\right] + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta - v_\phi^2 \cot \theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta}$$

$$+ \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin \theta}\frac{\partial}{\partial \theta}(v_\theta \sin \theta)\right) + \frac{1}{r^2 \sin^2 \theta}\frac{\partial^2 v_\theta}{\partial \phi^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{2 \cot \theta}{r^2 \sin \theta}\frac{\partial v_\phi}{\partial \phi}\right] + \rho g_\theta$$

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r + v_\theta v_\phi \cot \theta}{r}\right) = -\frac{1}{r \sin \theta}\frac{\partial p}{\partial \phi}$$

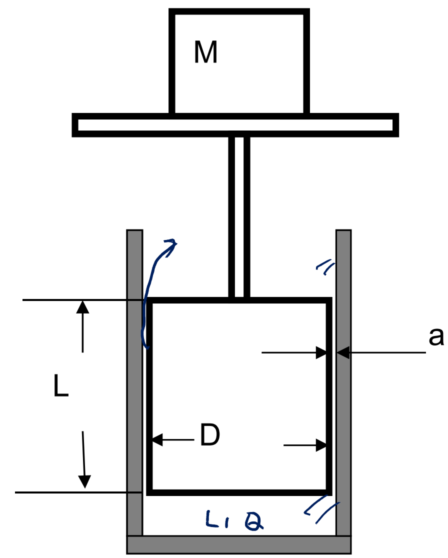
$$+ \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin \theta}\frac{\partial}{\partial \theta}(v_\phi \sin \theta)\right) + \frac{1}{r^2 \sin^2 \theta}\frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{2}{r^2 \sin \theta}\frac{\partial v_r}{\partial \phi} + \frac{2 \cot \theta}{r^2 \sin \theta}\frac{\partial v_\theta}{\partial \phi}\right] + \rho g_\theta$$

$$\rho \left(\frac{\partial\, v_z}{\partial t} + v_x \frac{\partial\, v_z}{\partial x} + v_y \frac{\partial\, v_z}{\partial y} + v_z \frac{\partial\, v_z}{\partial z} \right) = -\frac{\partial\, P}{\partial z} + \mu \left[\frac{\partial^2\, v_z}{\partial x^2} + \frac{\partial^2\, v_z}{\partial y^2} + \frac{\partial^2\, v_z}{\partial z^2} \right] + \rho\, g_z$$

$$\rho(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z}) = \rho g_z - \frac{\partial p}{\partial z} + \mu \{ \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial v_z}{\partial r}) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \}$$

$$\rho(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r\, v_\theta}{r}) = -\frac{1}{r} \frac{\partial p}{\delta \theta} + \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r\, v_\theta \right) \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} \right] + \rho g_\theta$$

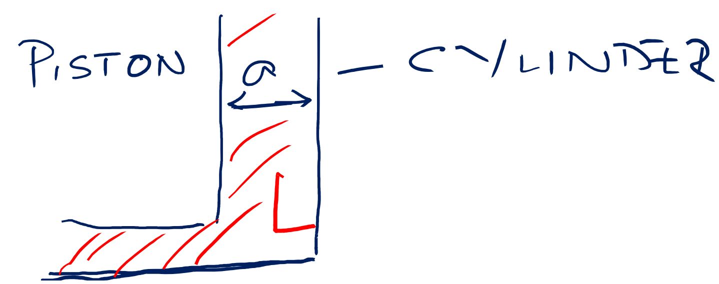
The basic component of a pressure gage tester consists of a piston-cylinder apparatus. The piston, 6 mm in diameter is loaded to develop a pressure of known magnitude. The radial clearance, a, is very small compared to the piston diameter D. The piston length, L, is 25 mm. Calculate the mass, M, required to produce 1.5 MPa (gage) in the cylinder. Determine the leakage flow rate as a function of radial clearance, a, for this load if the liquid is oil at 20°C (viscosity 0.42 N.s/m^2 and density 700 kg/m^3). Specify the maximum allowable radial clearance so that the vertical movement of the piston due to leakage will be less than 1 mm/min.



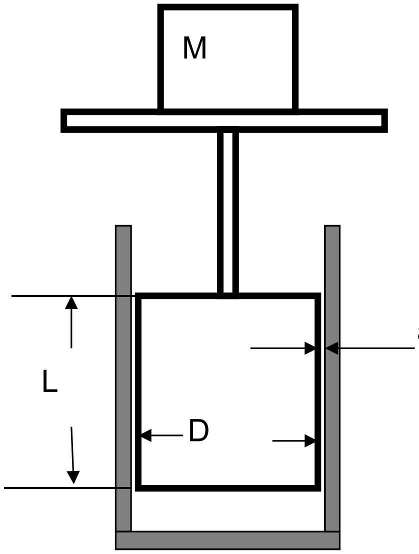
$$\frac{\pi D^2}{4}\left(p - p_{atm}\right) = Mg$$

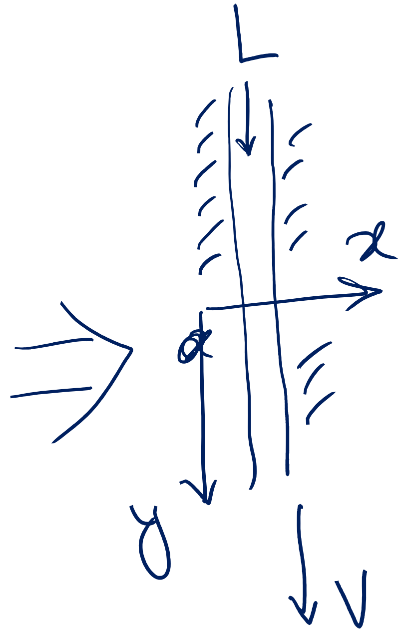
1.5×10^6

$$M = 4.32\ kg$$



$\frac{\Delta P}{L}$:





$$\mu \frac{d^2 v_y}{dx^2} = \frac{dp}{dy} - \rho g$$

$v_y = 0 \quad x = 0$

$v_y = V \, , \quad x = a$

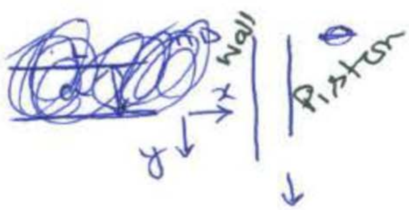
$$\rho\left(\cancel{\frac{\partial v_y}{\partial t}} + \cancel{v_x}\frac{\partial v_y}{\partial x} + v_y\cancel{\frac{\partial v_y}{\partial y}} + \cancel{v_z}\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v_y}{\partial x^2} + \cancel{\frac{\partial^2 v_y}{\partial y^2}} + \cancel{\frac{\partial^2 v_y}{\partial z^2}}\right] + \rho g_y$$

$$v_y = \frac{1}{2\mu}\frac{\Delta P}{L}\left[x^2 - \frac{x}{a}a^2\right] + v_x$$

$$\langle v_y \rangle = -\frac{1}{12\mu}\frac{\Delta P}{L}a^2 + \frac{Va}{2}$$

$$\boxed{a = 1.28 \times 10^{-5}\ m}$$

Since the gap between the cylinder and the piston is very small the flow can be treated as a flow between two parallel plates of length L and width πD, separated by 'a'.



$$\mu \frac{d v_y^2}{dx^2} + \cancel{\rho g} - \frac{dp}{dx} = 0.$$

≈ 0, quite small compared to $\frac{dp}{dx}$

$\rho \to O(10^3)$ $g \to O(10)$ $\rho g \to O(10^4)$

$$\frac{dp}{dx} = \frac{1.5\times10^6}{2.5\times10^{-3}} \approx O(10^8)$$

$$\mu \frac{d v_y^2}{dx^2} = \frac{dp}{dx} = \frac{\Delta p}{L}$$

$$v_y = \frac{1}{2\mu} \frac{\Delta p}{L} x^2 + c_1 x + c_2$$

at $x=0$, $v_y = 0 \Rightarrow c_2 = 0$.

at $x=a$, $v_y = V \Rightarrow c_1 = \frac{1}{a}\left[V - \frac{1}{2\mu}\frac{\Delta p}{L} a^2\right]$

$$v_y = \frac{1}{2\mu}\frac{\Delta p}{L}\left[x^2 - \frac{x}{a}a^2\right] + V\frac{x}{a}$$

$$\langle v_y \rangle = \frac{1}{a}\int_0^a v_y\, dx = -\frac{1}{12\mu}\frac{\Delta p}{L}a^2 + \frac{Va}{2}$$

... the is quite small

$$\langle v_y \rangle = \frac{1}{a}\int_0^a v_y\,dx = -\frac{1}{12\mu}\frac{\Delta P}{L}a^2 + \frac{Va}{2}$$

Again $V = 1\,mm/min$ $\therefore$ 2nd term on rhs is quite small.

$$\therefore \langle v_y \rangle = -\frac{1}{12\mu}\frac{\Delta P}{L}a^2$$

$$Q = \langle v_y \rangle a\pi D = -\frac{1}{12\mu}\frac{\Delta P}{L}a^3\pi D,$$ The flow is in the -ve y directn as it should be.

For downward movement (V m/s) the vol. displaced is

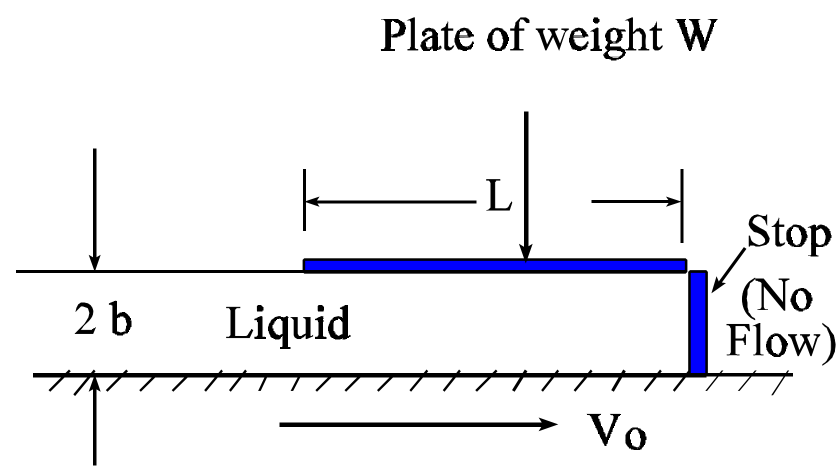
$$Q = \frac{\pi D^2}{4}V = \frac{\pi}{4}(0.006)^2 \times \frac{0.001}{60}\frac{m^3}{s} = 4.71\times10^{-10}\,m^3/s.$$

with $\mu = 0.42\ \frac{Ns}{m^2}$

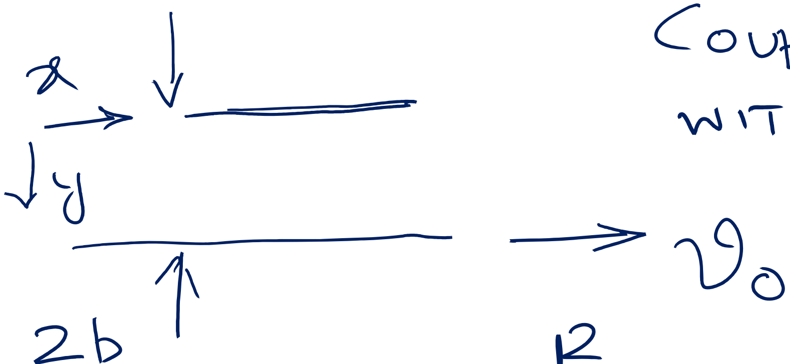
$$a = \left[\frac{12\mu Q L}{\pi D \Delta P}\right]^{1/3} = \left[\frac{12}{\pi}\times\frac{0.42\times 4.71\times10^{-10}\times0.025}{0.006\times1.5\times10^6}\right]^{1/3}$$

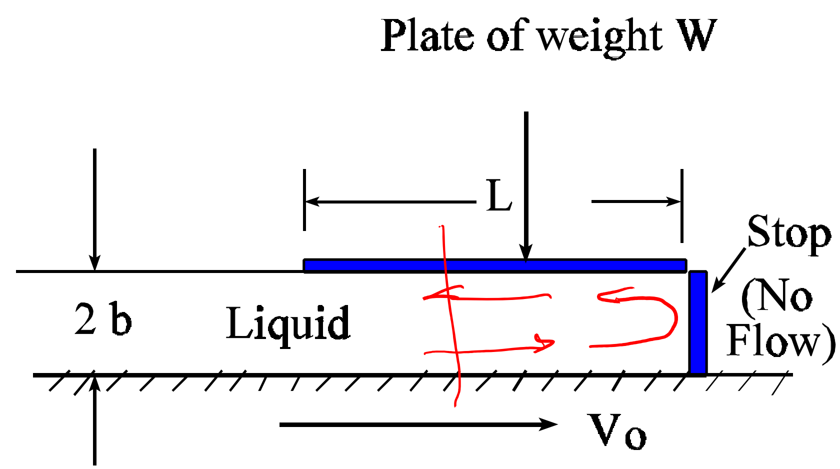
$$a = 1.28\times10^{-5}\,m$$

The lower plate of a lubricated thrust bearing moves to the right at velocity Vo . The stop at the right prevents any liquid flow beyond that point. Find the weight W that can be supported by the fluid (of viscosity μ and of density ρ) Assume the plate to be wide so that the end effects can be neglected. It can be assumed further that even if two unequal pressures act at the two ends (x=0 and x=L) of the plate it will not topple and the whole plate can be assumed to be acted on by an average of the two pressures at the two ends.



COUETTE FLOW
WITH A -Ve Pr. GRAD.





$$0 = \mu \frac{d^2 u}{dy^2} - \frac{dp}{dx}$$

$u = 0 \quad y = 0, \quad u = v_0, \; y = 2b$

$$u = \frac{1}{2\mu}\left(\frac{dp}{dx}\right)\left[y^2 - 2by\right] + \frac{v_0}{2b}\, y$$

$$\langle u \rangle = \frac{1}{2b}\int_0^{2b} u\, dy \Rightarrow \langle u \rangle = \frac{1}{2b}\left[\frac{1}{2\mu}\left(\frac{dp}{dx}\right)\left(\frac{8b^3}{3} - 4b^3\right) + v_0 b\right]$$

NO FLOW $\langle u \rangle = 0$

$$\Rightarrow \frac{dp}{dx} = \frac{3\mu v_0}{2b^2}$$

$$\text{AV. } P_r = \frac{3\mu v_0}{4b^2} L$$

$$\text{LOAD} = P_{av} \times L \times 1$$

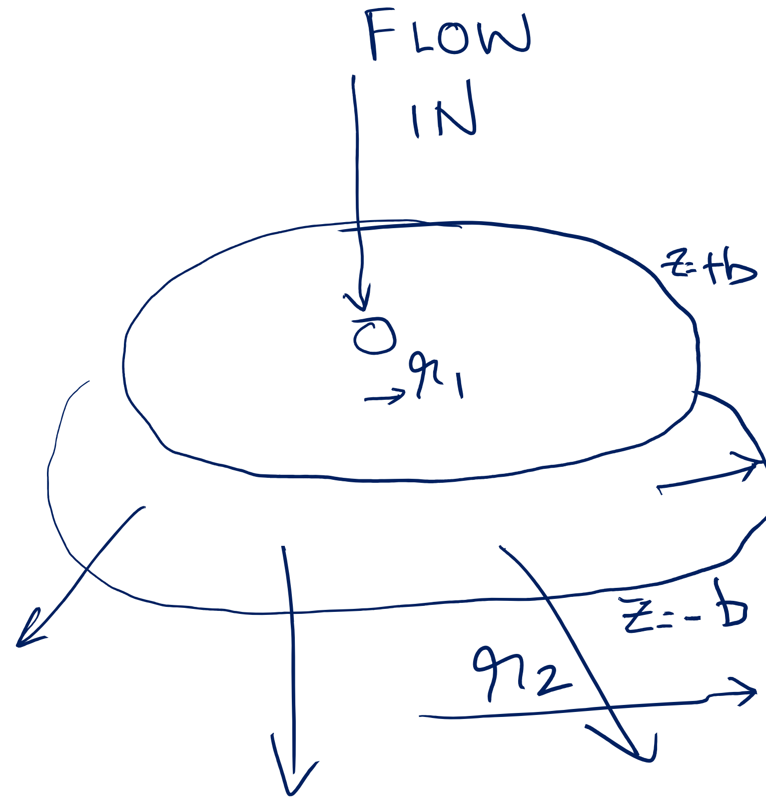
Radial Flow between Two Parallel Disks

Equation of Continuity

Cylindrical coordinates (r, θ, z):

$$\cancel{\frac{\partial \rho}{\partial t}} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \cancel{\frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta)} + \cancel{\frac{\partial}{\partial z}(\rho v_z)} = 0$$

$$v_r = \frac{\phi}{r} \leftarrow f(z)$$



$$v_r = f(r, z)$$

✓

Cylindrical coordinates (r, θ, z):

$$\rho\left(\cancel{\frac{\partial v_r}{\partial t}} + v_r\frac{\partial v_r}{\partial r} + \cancel{\frac{v_\theta}{r}}\frac{\partial v_r}{\partial \theta} + \cancel{v_z}\frac{\partial v_r}{\partial z} - \cancel{\frac{v_\theta^2}{r}}\right) = -\frac{\partial p}{\partial r} + \mu\left[\cancel{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)} + \cancel{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}} + \frac{\partial^2 v_r}{\partial z^2} - \cancel{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}\right] + \cancel{\rho g_r}$$

FROM CONTINUITY

$$v_r(r, z) = \frac{\phi(z)}{r}$$

$$\cancel{-\rho\frac{\phi^2}{r^3}} = -\frac{dp}{dr} + \frac{1}{r}\frac{d^2\phi}{dz^2}$$

≈ 0, CREEPING FLOW (NEGLIGIBLE CONVECTION)

$$\frac{\mu}{r}\frac{d^2\phi}{dz^2} = \frac{dp}{dr} \Rightarrow \mu\frac{d^2\phi}{dz^2}\int_{r_1}^{r_2}\frac{dr}{r} = \Delta p$$

$$\left(\mu \ln \frac{r_2}{r_1}\right) \frac{d^2\phi}{dz^2} + \Delta p = 0,$$

$$\phi = \frac{-\Delta p\, z^2}{2\mu \ln \frac{r_2}{r_1}} + c_1 z + c_2$$

$$v_r = \frac{\phi}{r} = \frac{-\Delta p\, z^2}{2\mu r \ln \frac{r_2}{r_1}} + \frac{c_1}{r} z + \frac{c_2}{r}$$

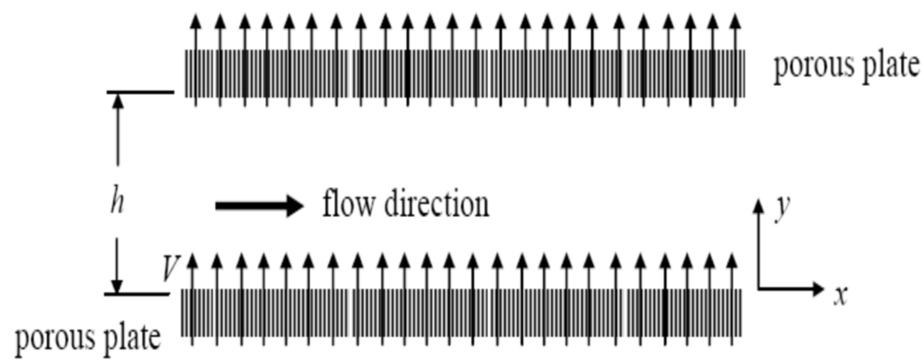
$$z = \pm b, \quad v_r = 0 \implies c_1, \; c_2$$

$$v_r = \frac{BSL}{}$$

$$Q = 2\pi \int_{-b}^{+b} r\, v_r \, dr$$

$$Q = \frac{4\pi \Delta p\, b^3}{3\mu \ln r_2/r_1}$$

An incompressible fluid flows between two porous, parallel flat plates as shown in the figure. An identical fluid is injected at a constant speed V through the bottom plate and simultaneously extracted from the upper plate at the same velocity. Assume the flow to be steady, fully-developed, the pressure gradient in the x-direction is a constant, and neglect body forces. Determine appropriate expressions for the y component of velocity. Show that the x component of velocity can be expressed as



$$u_x = \frac{h}{\rho V}\left[\frac{\partial p}{\partial x}\right]\left[\left\{\frac{1-\exp\left(\frac{\rho\, V\, y}{\mu}\right)}{1-\exp\left(\frac{\rho\, V\, h}{\mu}\right)}\right\} - \frac{y}{h}\right]$$

Steady, fully developed flow and therefore no change in time and in the flow direction. Channel is not bounded in the z-direction and therefore nothing happens in the z-direction.

$$\left.\begin{aligned} x-\text{direction}: &\quad u = function(y) \\ y-\text{direction}: &\quad v = function(y) \\ z-\text{direction}: &\quad w = 0 \end{aligned}\right\} (1)$$

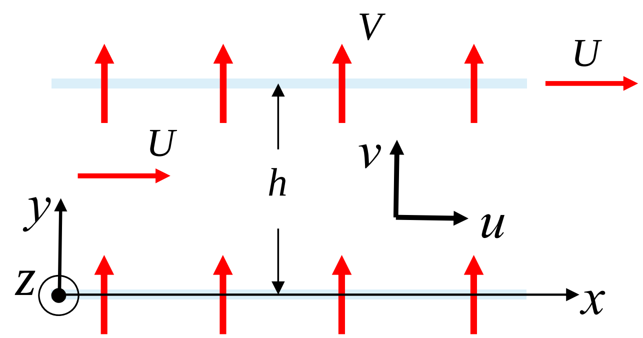
Use the continuity equation in Cartesian coordinates

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial v}{\partial y} = 0$$

$$v = \text{constant} \quad \text{or} \quad v = 0$$

$$\Downarrow$$

$$v = V$$



The functional dependence of the velocity components therefore reduces to

$$\left.\begin{array}{ll} \textit{\textbf{x}}\ \text{direction:} & \boldsymbol{u = \textbf{function of } (y)} \\ \textit{\textbf{y}}\ \text{direction:} & \boldsymbol{v = V} \\ \textit{\textbf{z}}\ \text{direction:} & \boldsymbol{w = 0} \end{array}\right\} \quad (2)$$

Step 5: Using the N-S equation, we get

x - component:

$$\rho\left(\cancel{\frac{\partial u}{\partial t}}+\cancel{u\frac{\partial u}{\partial x}}+v\frac{\partial u}{\partial y}+\cancel{w\frac{\partial u}{\partial z}}\right)=-\frac{\partial p}{\partial x}+\mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}}+\frac{\partial^2 u}{\partial y^2}+\cancel{\frac{\partial^2 u}{\partial z^2}}\right)+\cancel{\rho g_x}$$

y - component:

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\rho g_y$$

z - component:

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\rho g_z\,.$$

$v = V \neq f(y)$

$$\rho V \frac{du}{dy} = -\frac{\partial p}{\partial x} + \mu \frac{d^2 u}{dy^2} \quad \Bigg| \quad \begin{array}{ll} \text{BC} & \\ u = 0 & y = 0 \\ u = 0 & y = h \end{array}$$

$\curvearrowright A$

$$\Downarrow$$

$$u_x = \frac{h}{\rho V}\left[\frac{\partial p}{\partial x}\right]\left[\left\{\frac{1 - \exp\left(\frac{\rho\, V\, y}{\mu}\right)}{1 - \exp\left(\frac{\rho\, V\, h}{\mu}\right)}\right\} - \frac{y}{h}\right]$$