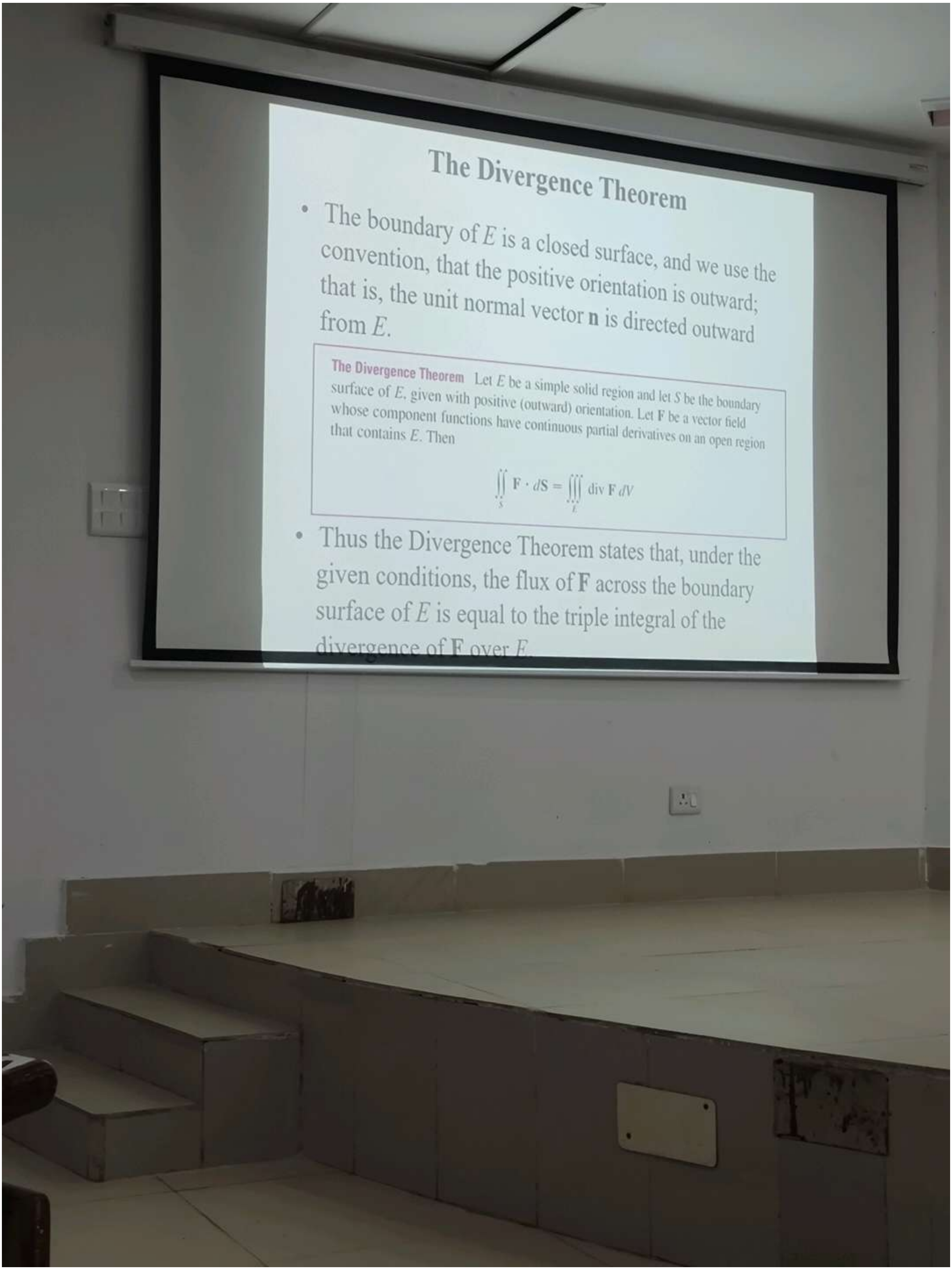
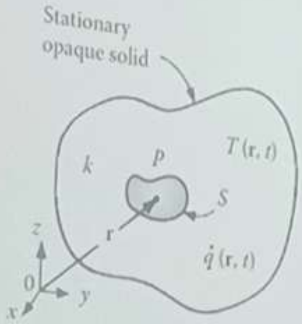

The Divergence Theorem
• The boundary of E is a closed surface, and we use the convention, that the positive orientation is outward; that is, the unit normal vector n is directed outward from E.
The Divergence Theorem Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let F be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then
$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV$$
• Thus the Divergence Theorem states that, under the given conditions, the flux of F across the boundary surface of E is equal to the triple integral of the divergence of F over E.

Heat Conduction Equation

Consider:



Control volume:

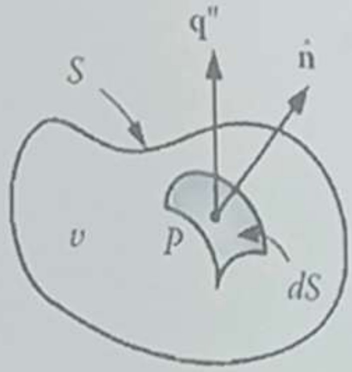
Mode of heat transfer: Only Conduction

$$\begin{bmatrix}\text{Rate of heat entering through} \\ \text{the bounding surfaces of } V\end{bmatrix} + \begin{bmatrix}\text{rate of energy} \\ \text{generation in } V\end{bmatrix} = \begin{bmatrix}\text{rate of storage} \\ \text{of energy in } V\end{bmatrix}$$

Energy Balance: $\dfrac{dE}{dt} = \dot{Q}$

Heat Conduction Equation

Control volume:



Energy Balance: $\frac{dE}{dt} = \dot{Q}$

$$\frac{dE}{dt} = \int_V \rho \frac{\partial e}{\partial t}\, dV = \int_V \rho c \frac{\partial T}{\partial t}\, dV$$

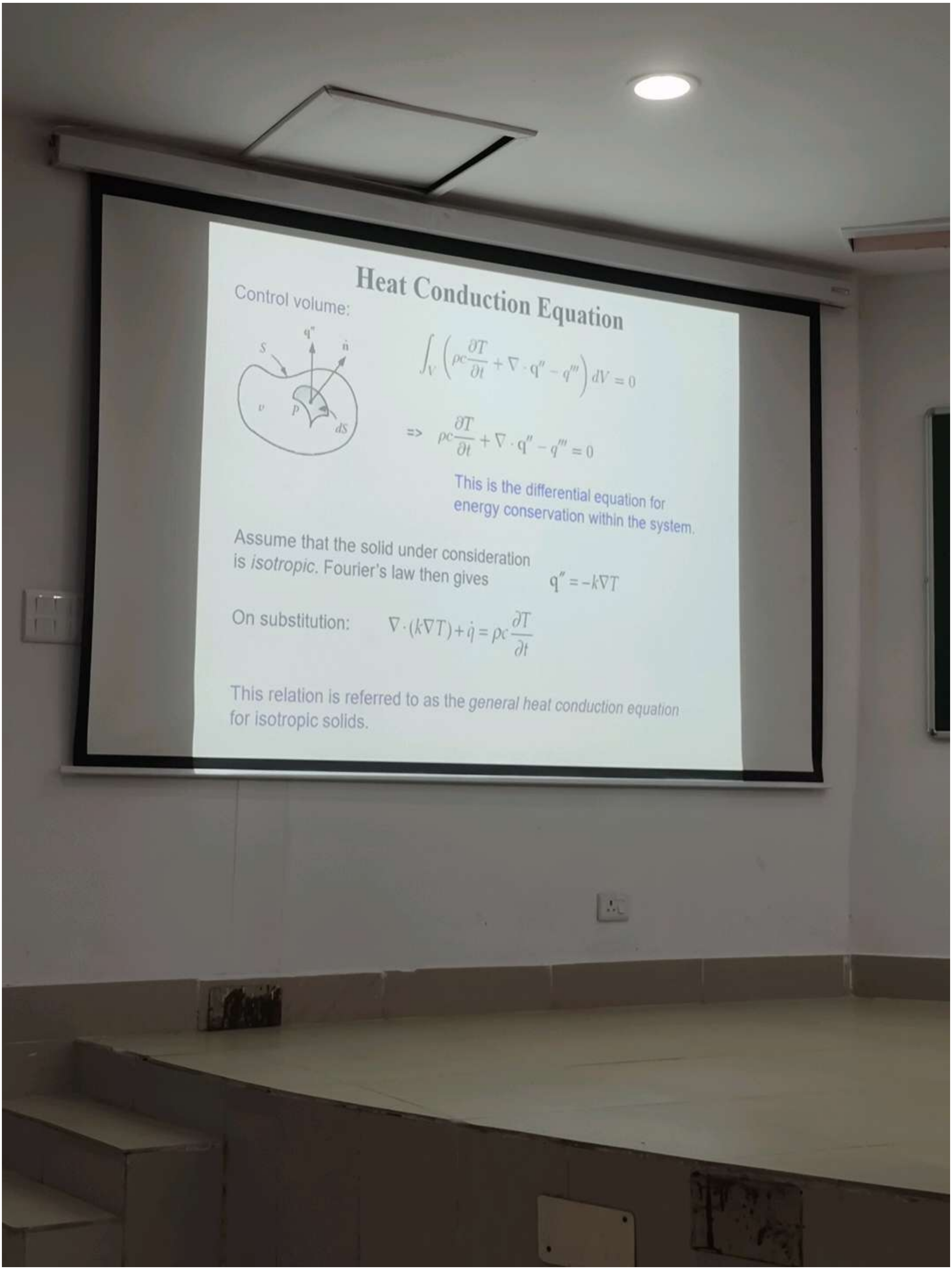
$$\dot{Q} = -\int_A \mathbf{q}'' \cdot \mathbf{n}\, dA + \int_V q'''\, dV$$

Thus, Energy Balance:

$$\int_V \rho c \frac{\partial T}{\partial t}\, dV = -\int_A \mathbf{q}'' \cdot \mathbf{n}\, dA + \int_V q'''\, dV$$

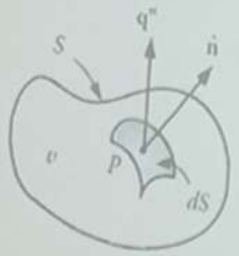
Use Divergence Theorem: $\int_A \mathbf{q}'' \cdot \mathbf{n}\, dA = \int_V \nabla \cdot \mathbf{q}''\, dV$

On substitution: $\int_V \left(\rho c \frac{\partial T}{\partial t} + \nabla \cdot \mathbf{q}'' - q''' \right) dV = 0$


Heat Conduction Equation
Control volume:
q"
n
S
v
P
dS
$\int_V \left(\rho c \frac{\partial T}{\partial t} + \nabla \cdot q'' - q''' \right) dV = 0$
=> $\rho c \frac{\partial T}{\partial t} + \nabla \cdot q'' - q''' = 0$
This is the differential equation for energy conservation within the system.
Assume that the solid under consideration is *isotropic*. Fourier's law then gives
$q'' = -k \nabla T$
On substitution:
$\nabla \cdot (k \nabla T) + \dot{q} = \rho c \frac{\partial T}{\partial t}$
This relation is referred to as the *general heat conduction equation* for isotropic solids.

Heat Conduction Equation

Control volume:



General heat conduction equation for isotropic solids:

$$\nabla \cdot (k\nabla T) + \dot{q} = \rho c \frac{\partial T}{\partial t}$$

$$\Rightarrow \quad k\nabla^2 T + \nabla k \cdot \nabla T + \dot{q} = \rho c \frac{\partial T}{\partial t} \qquad \text{(Eq. A)}$$

where $\nabla^2 = \nabla \cdot \nabla$ is the *Laplacian* operator.

If the thermo-physical properties k, ρ, and c are functions of space coordinates only, then Eq. (A) is a *linear* partial differential equation.

On the other hand, if any thermo-physical property, k, ρ, or c depends on temperature, Eq. (A) becomes a *nonlinear* partial differential equation.

For a *homogeneous isotropic* solid, k is constant:

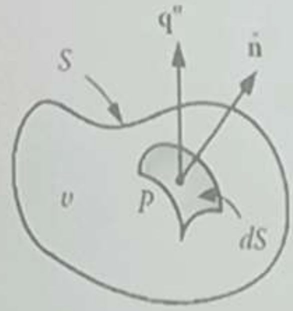
$$\nabla^2 T + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

where $\alpha = k/\rho c$ is the *thermal diffusivity* of the solid (Unit: m^2/s)

Fourier–Biot equation

Heat Conduction Equation

Control volume:



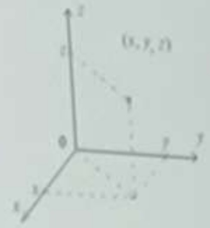
Constant thermo-physical properties:	$\nabla^2 T + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$	*Fourier–Biot equation*
Constant properties + No heat generation:	$\nabla^2 T = \frac{1}{\alpha}\frac{\partial T}{\partial t}$	*Heat diffusion equation*
Steady state, Constant *k*:	$\nabla^2 T + \frac{\dot{q}}{k} = 0$	*Poisson equation*
Steady state, No heat source:	$\nabla^2 T = 0$	*Laplace equation*

Laplacian in Different Coordinate Systems

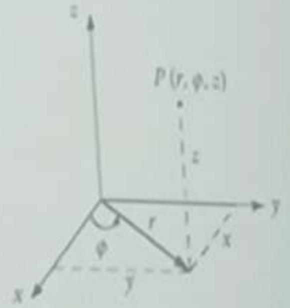
***In rectangular* coordinates:**

$T = T(x, y, z, t).$

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$



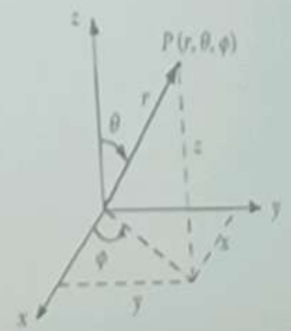
In *cylindrical* coordinates:



$T = T(r, \phi, z, t).$

$$\nabla^2 T = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}$$

In *spherical* coordinates:



$T = T(r, \theta, \phi, t).$

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r} + \left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}$$

Heat Conduction Equation in Different Coordinate Systems

General Heat Conduction Equation with Variable Thermal Conductivity in Various Coordinate Systems

Coordinate System	$\nabla \cdot (k \nabla T) + \dot{q} = \rho c \dfrac{\partial T}{\partial t}$
Rectangular	$\dfrac{\partial}{\partial x}\left(k \dfrac{\partial T}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(k \dfrac{\partial T}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(k \dfrac{\partial T}{\partial z}\right) + \dot{q} = \rho c \dfrac{\partial T}{\partial t}$
Cylindrical	$\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(kr \dfrac{\partial T}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial}{\partial \phi}\left(k \dfrac{\partial T}{\partial \phi}\right) + \dfrac{\partial}{\partial z}\left(k \dfrac{\partial T}{\partial z}\right) + \dot{q} = \rho c \dfrac{\partial T}{\partial t}$
Spherical	$\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(kr^2 \dfrac{\partial T}{\partial r}\right) + \dfrac{1}{r^2 \sin\theta}\dfrac{\partial}{\partial \theta}\left(k \sin\theta \dfrac{\partial T}{\partial \theta}\right) + \dfrac{1}{r^2 \sin^2\theta}\dfrac{\partial}{\partial \phi}\left(k \dfrac{\partial T}{\partial \phi}\right) + \dot{q} = \rho c \dfrac{\partial T}{\partial t}$

Heat Conduction Equation in Different Coordinate Systems: 1D Case

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \dot{q} = \rho c_p \frac{\partial T(x,t)}{\partial t}$$

Rectangular Coordinate

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) + \dot{q} = \rho c_p \frac{\partial T(r,t)}{\partial t}$$

Cylindrical Coordinate

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2k\frac{\partial T}{\partial r}\right) + \dot{q} = \rho c_p \frac{\partial T(r,t)}{\partial t}$$

Spherical Coordinate

In compact form:

$$\frac{1}{r^n}\frac{\partial}{\partial r}\left(r^nk\frac{\partial T}{\partial r}\right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

For rectangular coordinate system, replace r by x

$$n = \begin{cases} 0 & \text{for rectangular coordinates} \\ 1 & \text{for cylindrical coordinates} \\ 2 & \text{for spherical coordinates} \end{cases}$$

Heat Conduction Equation: Boundary Conditions

The differential equation of heat conduction will require two boundary conditions for each spatial dimension, as well as one initial condition for the non–steady-state problem

Initial condition: $T(\mathbf{r}, t)\big|_{t\to 0} = T_0(\mathbf{r})$ **r** is the position vector.

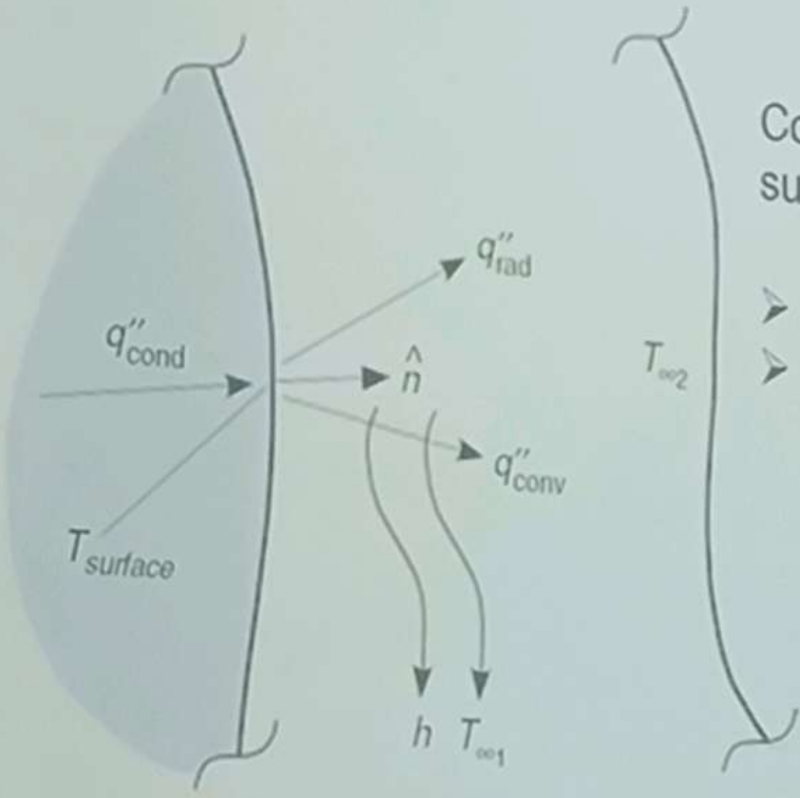
The Boundary Conditions specify the temperature or the heat flux at the boundaries of the region

For example, at a given boundary surface,

- the temperature distribution may be prescribed, or
- the heat flux distribution may be prescribed, or
- there may be heat exchange by convection and/or radiation with an environment at a prescribed temperature.

General Boundary Condition

The boundary condition can be derived by writing an energy balance equation at the surface of the solid.



Consider conservation of energy at the surface.

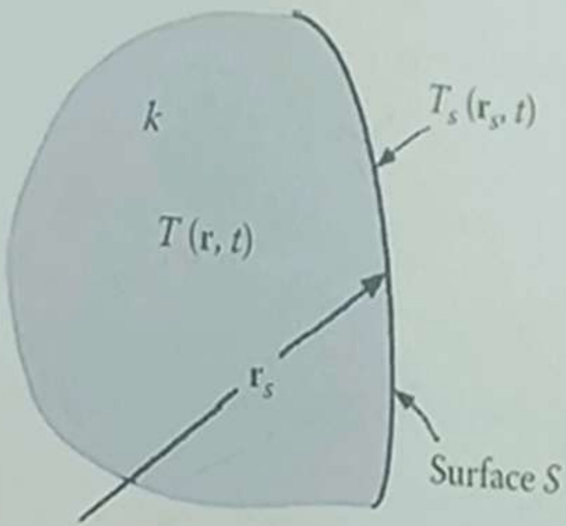
- Surface is assumed to be stationary.
- No energy can be accumulated at an infinitely thin surface.

$$q''_{in} = q''_{out}$$

Or, $$-k\left.\frac{\partial T}{\partial n}\right|_{surface} = h\left(T|_{surface} - T_{\infty_1}\right) + \varepsilon\sigma\left(T^4|_{surface} - T^4_{\infty_2}\right)$$

Boundary Condition of First Kind

- Prescribed Temperature
- Dirichlet Boundary Condition



$$T|_{\text{surface}} = T_0$$

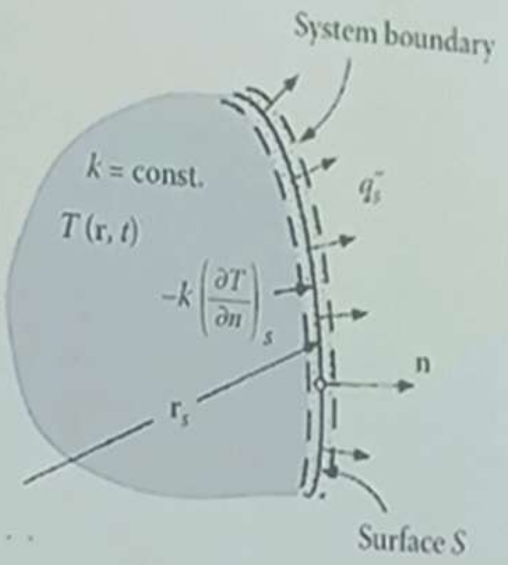
$$T|_{\text{surface}} = f(\hat{r}, t)$$

Homogeneous boundary condition of the first type:

$$T|_{\text{surface}} = 0$$

Boundary Condition of Second Kind

- Prescribed Heat Flux
- Neumann Boundary Condition



$$-k\frac{\partial T}{\partial n}\bigg|_{\text{surface}} = q_0''$$

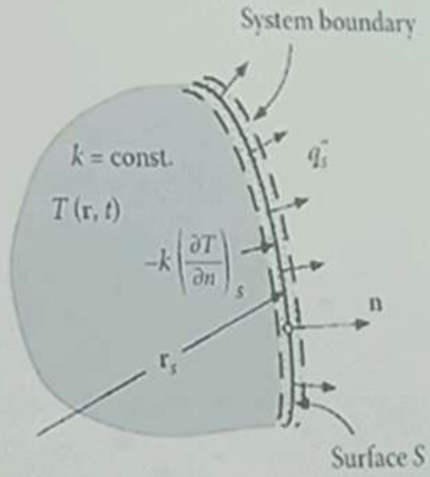
$$-k\frac{\partial T}{\partial n}\bigg|_{\text{surface}} = f(\hat{r}, t)$$

Homogeneous boundary condition of the second type:

$$\frac{\partial T}{\partial n}\bigg|_{\text{surface}} = 0 \quad \text{(perfectly insulated or adiabatic)}$$

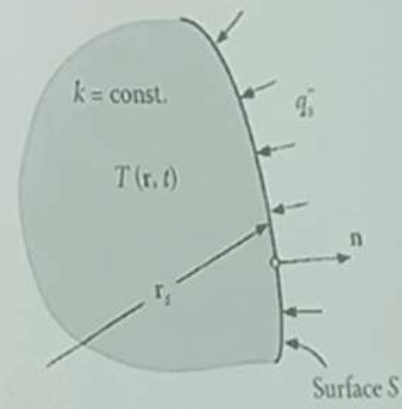
Boundary Condition of Second Kind

- Prescribed Heat Flux
- Neumann Boundary Condition



$$-k\left(\frac{\partial T}{\partial n}\right)_s = q''_s$$

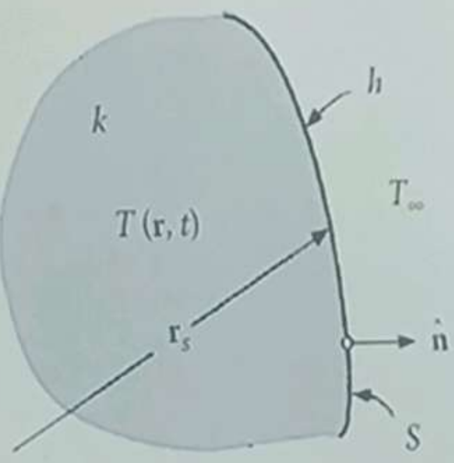
Sign Convention



$$k\left(\frac{\partial T}{\partial n}\right)_s = q''_s$$

Boundary Condition of Third Kind

- Convection
- Robbin's Boundary Condition



$$-k \left.\frac{\partial T}{\partial n}\right|_{\text{surface}} = h\left[T|_{\text{surface}} - T_\infty\right]$$

In general, the ambient fluid temperature T_∞ may assumed to be a function of position and time:

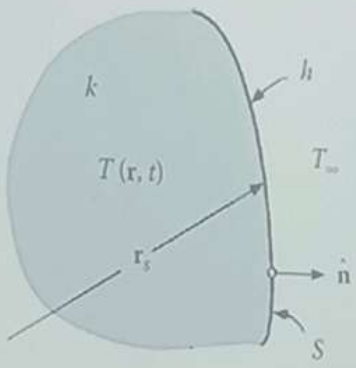
$$-k \left.\frac{\partial T}{\partial n}\right|_{\text{surface}} = h\left[T|_{\text{surface}} - T_\infty(\hat{r}, t)\right]$$

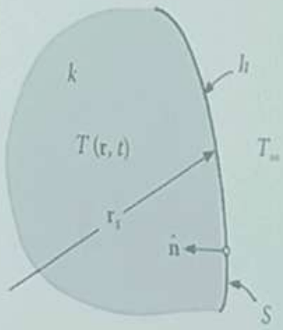
Homogeneous boundary condition of the third type:

$$-k \left.\frac{\partial T}{\partial n}\right|_{\text{surface}} = h\, T|_{\text{surface}}$$

Boundary Condition of Third Kind

- Convection
- Robbin's Boundary Condition

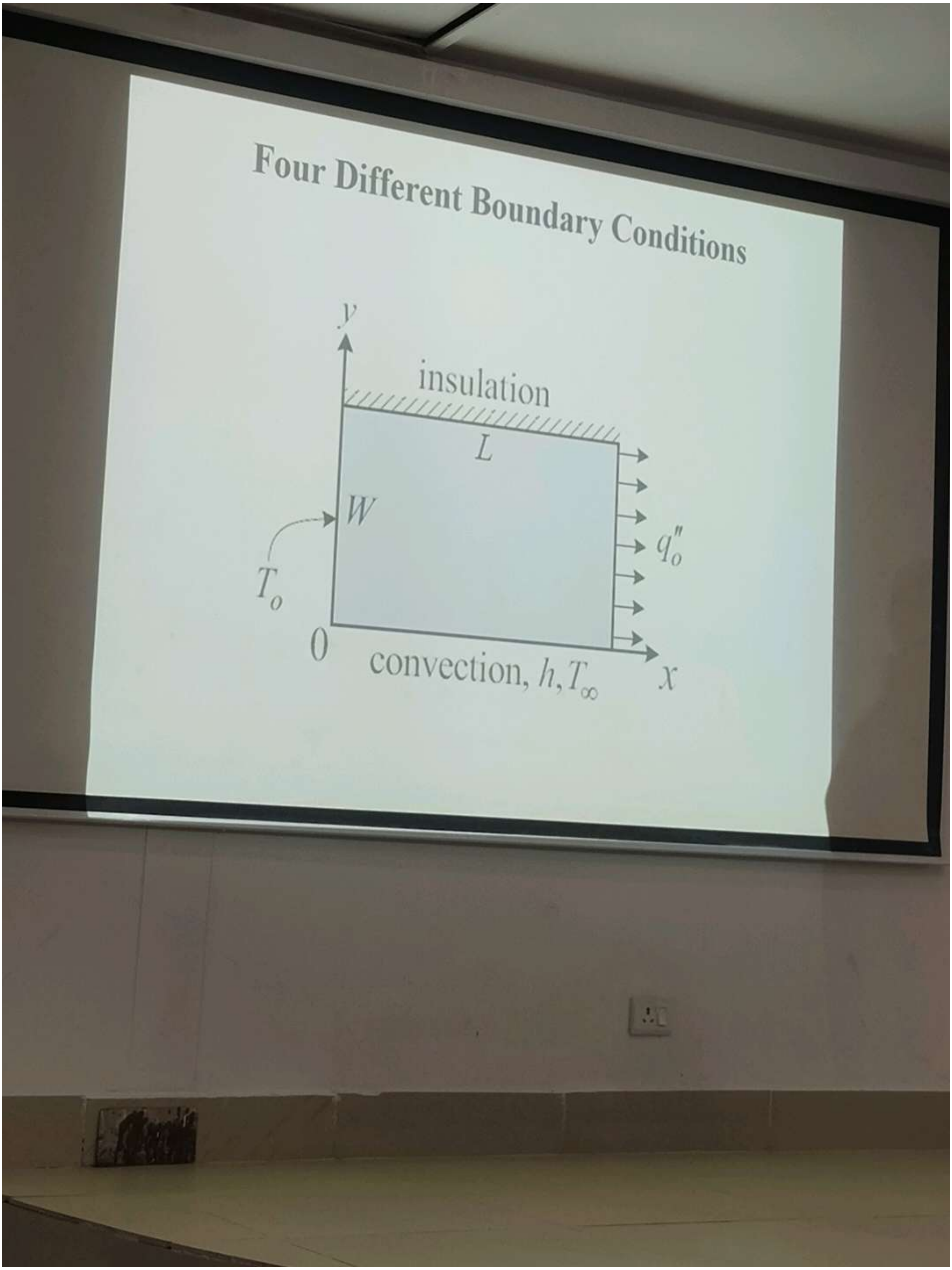




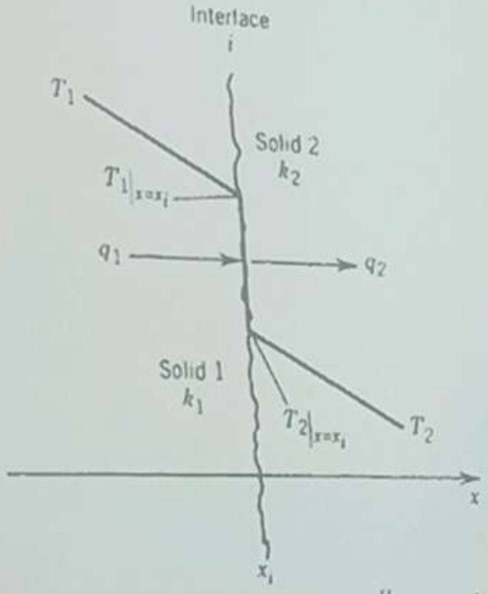
$$-k_s\left(\frac{\partial T}{\partial n}\right)_s = h[T(\mathbf{r}_s,t)-T_\infty]$$

$$k_s\left(\frac{\partial T}{\partial n}\right)_s = h[T(\mathbf{r}_s,t)-T_\infty]$$

$$\left[k\frac{\partial T}{\partial n}+hT(\mathbf{r},t)\right]_s = hT_\infty$$


Four Different Boundary Conditions
y
insulation
L
W
q''_o
T_o
0
convection, h, T_∞
x

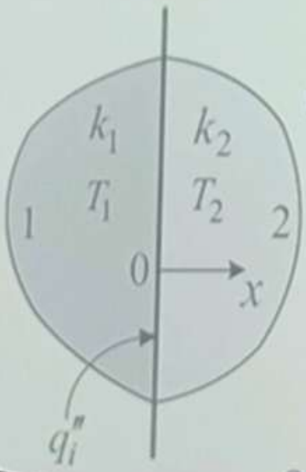
Boundary Condition at Interface



$$q_i'' = -k_1 \frac{\partial T_1}{\partial x}\bigg|_i = h_c(T_1 - T_2)_i = -k_2 \frac{\partial T_2}{\partial x}\bigg|_i$$

Here h_c W/($m^2 \cdot K$) is called the *contact conductance* for the interface

Boundary Condition: Interface with Heat Source



Example:

- An electrical heating element which is sandwiched between two non-electrically conducting materials
- Frictional heat generated by the relative motion between two surfaces

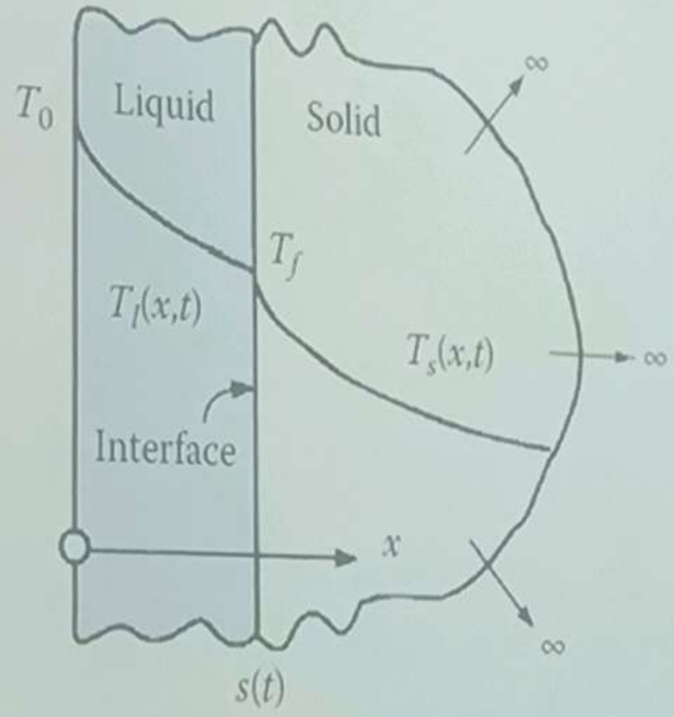
Energy dissipated at an interface may be conducted through one or both materials.

Consider that heat is conducted in the positive coordinate direction through both materials.

Continuity of temperature: $T_1(0,y) = T_2(0,y)$.

...nservation of energy: $\boxed{-k_1 \dfrac{\partial T_1(0,y)}{\partial x} + q_i'' = -k_2 \dfrac{\partial T_2(0,y)}{\partial x}}$

Boundary Condition at Sharp Solid-Liquid Interface: Phase Change



Heat Conduction Equation: Moving Solids

Assume rectangular coordinate system. Assume ρ, C_p constant.
Velocity components: u, v, w
Motion of solid will add convective (enthalpy) fluxes:

$$\rho c T\, u \qquad \rho c T\, v \qquad \rho c T\, w$$

Modify component heat flux vectors:

$$q''_x = -k\frac{\partial T}{\partial x} + \rho c T\, u \qquad q''_y = -k\frac{\partial T}{\partial y} + \rho c T\, v \qquad q''_z = -k\frac{\partial T}{\partial z} + \rho c T\, w$$

Substitute in: $\rho c\dfrac{\partial T}{\partial t} + \nabla \cdot \mathrm{q}'' - q''' = 0$

$$\nabla \cdot (k\nabla T) + q''' = \rho c\left[\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right] = \rho c\,\frac{DT}{Dt}$$

where $\dfrac{D}{Dt}$ is substantial (total) derivative

Temperature Dependent Thermal Conductivity

$$k(T) = k_0(1 + \beta T)$$

$$k_{\text{average}} = \frac{\int_{T_1}^{T_2} k_0(1+\beta T)dT}{T_2 - T_1} = k_0\left(1 + \beta\frac{T_2 + T_1}{2}\right)$$

$$= k\left(T_{\text{average}}\right)$$

Heat Conduction Equation: Nonlinearity

Non-linearity in conduction problems arises when thermo-physical properties are temperature dependent or when boundary conditions are non-linear.

Surface radiation and free convection are typical examples of non-linear boundary conditions.

In phase change problems the interface energy equation is non-linear.

Sources of Non-linearity: Non-linear Differential Equations

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+q'''=\rho c_p\frac{\partial T}{\partial t}.$$

If ρ and/or c_p vary with temperature, the transient term is non-linear. Similarly, if $k = k(T)$ the first term becomes non-linear.

This is evident if we rewrite the above equation as

$$k\frac{\partial^2 T}{\partial x^2}+\frac{dk}{dT}\left[\frac{\partial T}{\partial x}\right]^2+q'''=\rho c_p\frac{\partial T}{\partial t}.$$

Heat Conduction Equation: Review

Sources of Non-linearity: Non-linear Boundary Conditions

Free convection Boundary Condition:

$$-k\frac{\partial T}{\partial x} = \beta(T - T_{\infty})^{5/4}$$

Radiation Boundary Condition:

$$-k\frac{\partial T}{\partial x} = \varepsilon\sigma(T^{4} - T_{sur}^{4})$$

Phase-change Boundary Condition:

$$k_s\frac{\partial T}{\partial x} - k_L\frac{\partial T}{\partial x} = \rho_s L\frac{dx_i}{dt}$$

Variable Thermal Conductivity $k(T)$: Kirchhoff Transformation: 1D Case

Consider 1D case: $$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + q''' = \rho c_p \frac{\partial T}{\partial t}$$

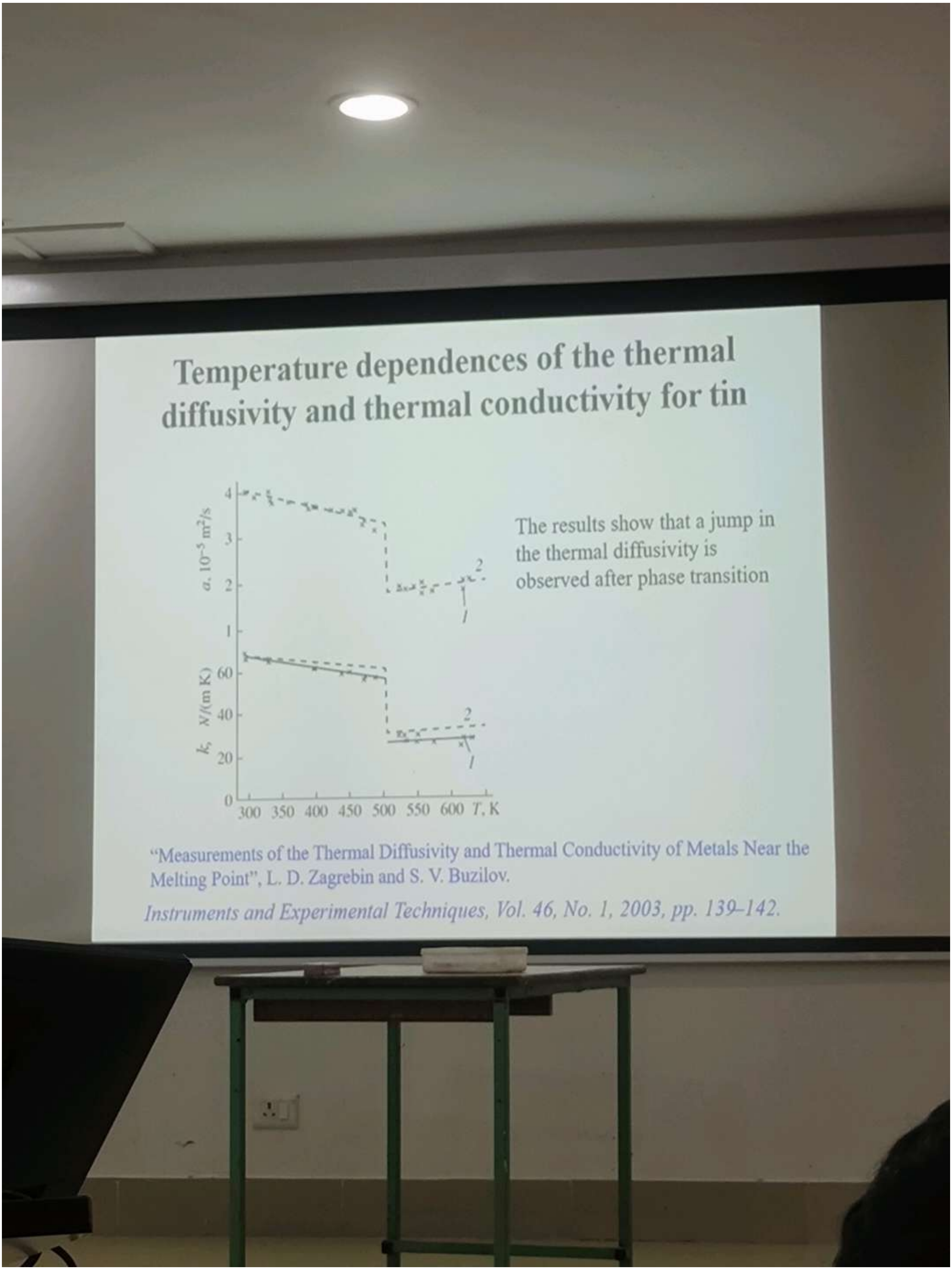
Introduce a new temperature variable θ(T) defined as $\theta(T) = \frac{1}{k_o}\int_0^T k(T)dT$.

Compute:

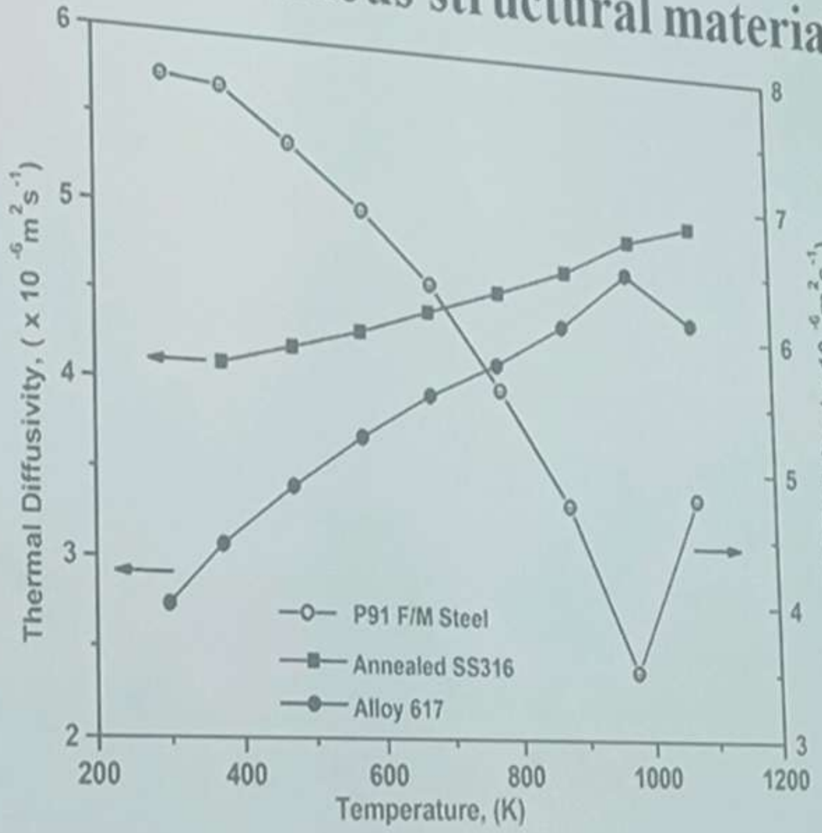
$$\frac{d\theta}{dT} = \frac{k}{k_o}. \qquad \frac{\partial T}{\partial t} = \frac{dT}{d\theta}\frac{\partial \theta}{\partial t} = \frac{k_o}{k}\frac{\partial \theta}{\partial t} \qquad \frac{\partial T}{\partial x} = \frac{dT}{d\theta}\frac{\partial \theta}{\partial x} = \frac{k_o}{k}\frac{\partial \theta}{\partial x}$$

On substitution:

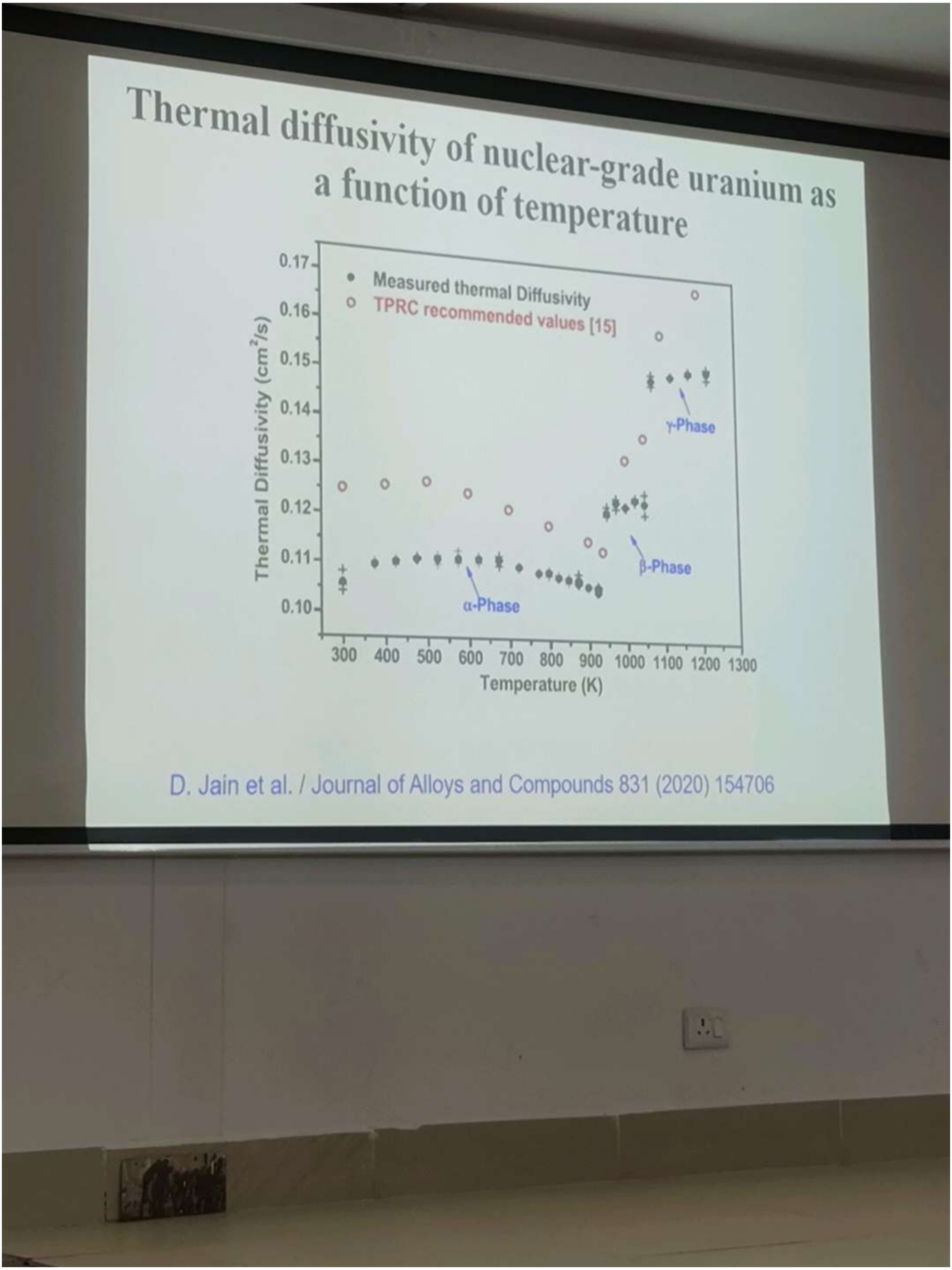
$$\frac{\partial^2 \theta}{\partial x^2} + \frac{q'''}{k_o} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t},$$ where α is the thermal diffusivity, defined as $\alpha = \alpha(T) = \frac{k}{\rho c_p}$.


Temperature dependences of the thermal diffusivity and thermal conductivity for tin
a, 10⁻⁵ m²/s
4
3
2
1
k, W/(m K)
60
40
20
0
300 350 400 450 500 550 600 T, K
2
1
2
1
The results show that a jump in the thermal diffusivity is observed after phase transition
"Measurements of the Thermal Diffusivity and Thermal Conductivity of Metals Near the Melting Point", L. D. Zagrebin and S. V. Buzilov.
Instruments and Experimental Techniques, Vol. 46, No. 1, 2003, pp. 139–142.

Temperature dependent variation in thermal diffusivity for various structural materials



"Thermal Transport and Thermal Diffusivity by Laser Flash Technique: A Review",
R. Sundar and C. Sudha
International Journal of Thermophysics (2025) 46:13


Thermal diffusivity of nuclear-grade uranium as a function of temperature
Measured thermal Diffusivity
TPRC recommended values [15]
Thermal Diffusivity (cm²/s)
0.17
0.16
0.15
0.14
0.13
0.12
0.11
0.10
γ-Phase
β-Phase
α-Phase
300 400 500 600 700 800 900 1000 1100 1200 1300
Temperature (K)
D. Jain et al. / Journal of Alloys and Compounds 831 (2020) 154706

Variable Thermal Conductivity $k(T)$: Kirchhoff Transformation: General Case

Consider:

$$\nabla \cdot [k(T)\nabla T] + \dot{q}(\mathbf{r},t) = \rho(T)c(T)\frac{\partial T}{\partial t} \qquad \text{Eq. (A)}$$

Kirchhoff transformation: $$\theta(\mathbf{r},t) = \frac{1}{k_R}\int_{T_R}^{T(\mathbf{r},t)} k(T')dT' \qquad \text{Eq. (B)}$$

where T_R is a reference temperature and $k_R = k(T_R)$.

From Eq. (B):

$$\nabla\theta = \frac{k(T)}{k_R}\nabla T \qquad \frac{\partial\theta}{\partial t} = \frac{k(T)}{k_R}\frac{\partial T}{\partial t}$$

Using these with Eq. (A): $$\nabla^2\theta + \frac{\dot{q}(\mathbf{r},t)}{k_R} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

where $\alpha(T) = k(T)/\rho(T)c(T)$ is the thermal diffusivity

Variable Thermal Conductivity $k(T)$: Kirchhoff Transformation: General Case

$$\nabla^2\theta + \frac{\dot{q}(\mathbf{r},t)}{k_R} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

where $\alpha(T) = k(T)/\rho(T)c(T)$ is the thermal diffusivity

Still Nonlinear!

Use of transformation?

For many solids, however, the dependence of α on temperature can usually be neglected compared to that of k. If the variation of α with temperature is not significant and. hence, it can be approximated to be constant, then the above transformed equation becomes linear.

For steady-state problems, since the right-hand side vanishes identically, the transformed equation is a linear differential equation regardless of whether α is temperature dependent or not.

The position of nonlinear term shifted:

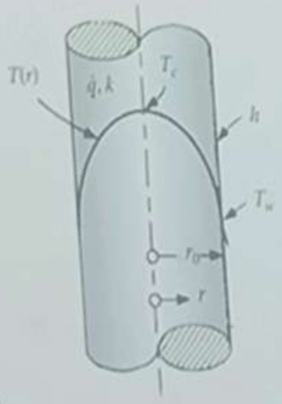
$$\nabla \cdot [k(T)\nabla T] + \dot{q}(\mathbf{r},t) = \rho(T)c(T)\frac{\partial T}{\partial t}$$

Variable Thermal Conductivity $k(T)$: Kirchhoff Transformation: Example

Find the rate of heat generation per unit volume in a rod that will produce a centerline temperature of 2000°C for the following conditions:

$$r_0 = 1\,\text{cm},\ T_w = 350°\text{C and } k = \frac{3167}{T + 273}$$

where T is in °C and k in W/(m · K). Also, calculate the surface heat flux.



Solution:

Differential equation:

$$\frac{1}{r}\frac{d}{dr}\left[rk(T)\frac{dT}{dr}\right] + \dot{q} = 0$$

Boundary conditions:

$$\left(\frac{dT}{dr}\right)_{r=0} = 0 \quad \text{and} \quad T(r_0) = T_w$$

Variable Thermal Conductivity $k(T)$: Kirchhoff Transformation: Example

Solution:

Differential equation:

$$\frac{1}{r}\frac{d}{dr}\left[rk(T)\frac{dT}{dr}\right]+\dot{q}=0$$

Boundary conditions:

$$\left(\frac{dT}{dr}\right)_{r=0}=0 \quad \text{and} \quad T(r_0)=T_w$$

Define a new temperature function $\theta(r)$ as:

$$\theta(r)=\frac{1}{k_w}\int_{T_w}^{T(r)}k(T')dT'$$

where $k_w = k(T_w)$

Transform the differential equation and the boundary conditions.

Variable Thermal Conductivity $k(T)$: Kirchhoff Transformation: Example

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right)+\frac{\dot{q}}{k_w}=0 \qquad \left(\frac{d\theta}{dr}\right)_{r=0}=0 \text{ and } \theta(r_0)=0$$

The solution of this problem for $\theta(r)$ is given by:

$$\theta(r)=\frac{\dot{q}r_0^2}{4k_w}\left[1-\left(\frac{r}{r_0}\right)^2\right]$$

Use: $\theta(r)=\frac{1}{k_w}\int_{T_w}^{T(r)}k(T')dT'$ We get: $\int_{T_w}^{T(r)}k(T')dT'=\frac{\dot{q}r_0^2}{4}\left[1-\left(\frac{r}{r_0}\right)^2\right]$

This relation can be written explicitly for $T(r)$ when the relation $k = k(T)$ is given. At $r = 0$, this equation yields

$$\int_{T_w}^{T_c}k(T)dT=\frac{\dot{q}r_0^2}{4}$$

where T_c is the centerline temperature.

Variable Thermal Conductivity $k(T)$: Kirchhoff Transformation: Example

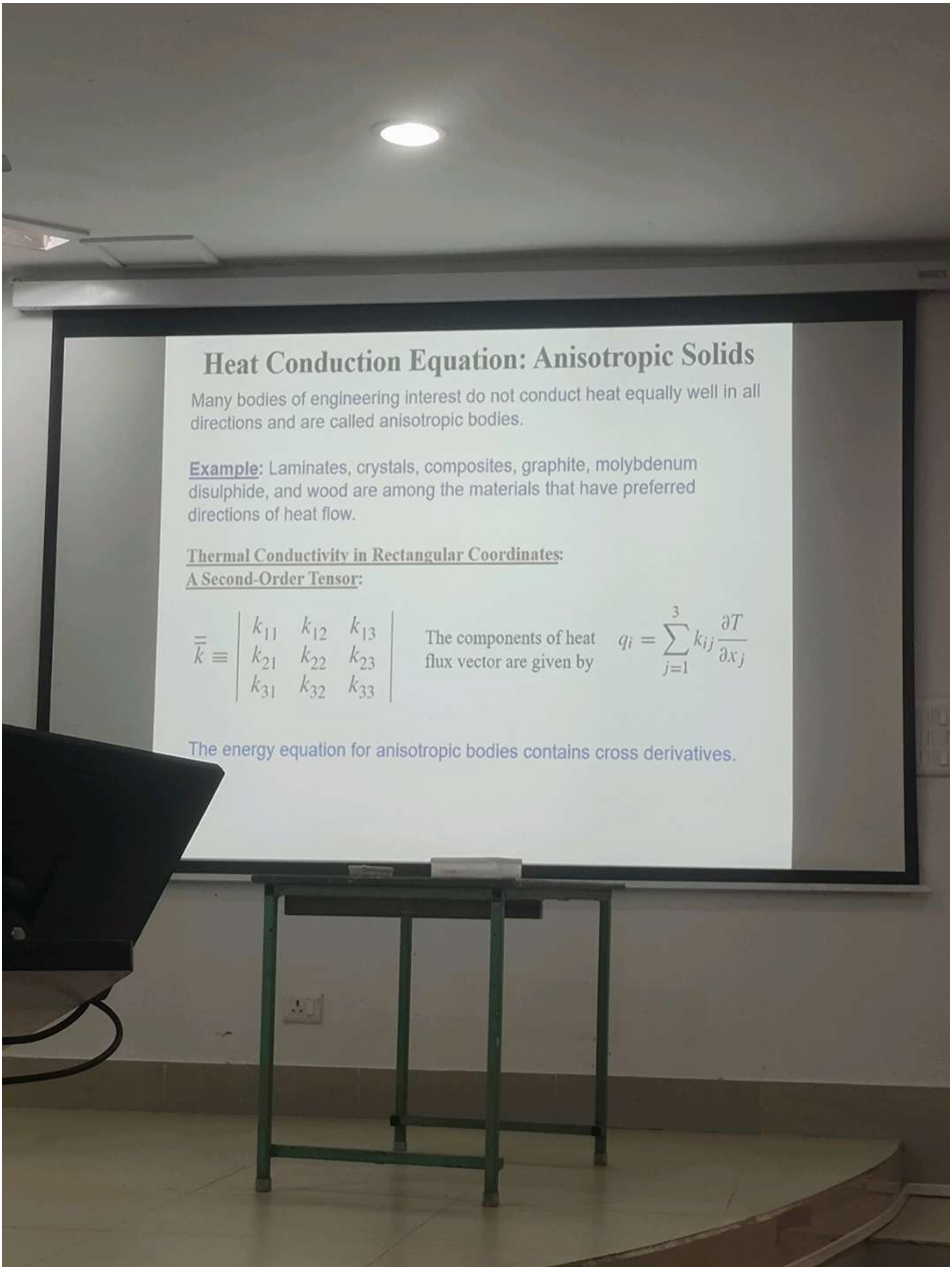
$$\int_{T_w}^{T_c} k(T)\,dT = \frac{\dot{q} r_0^2}{4}$$ where T_c is the centerline temperature.

Now use the data in problem:

$$\dot{q} = \frac{4}{r_0^2} \int_{350}^{2000} \frac{3167}{T+273} dT = \frac{4 \times 3167}{(0.01)^2} \ln \frac{2273}{623} = 1.64 \times 10^8 \, W/m^3$$

Surface heat flux:

$$q_s'' = \frac{\dot{q} V}{A}$$


Heat Conduction Equation: Anisotropic Solids
Many bodies of engineering interest do not conduct heat equally well in all directions and are called anisotropic bodies.
Example: Laminates, crystals, composites, graphite, molybdenum disulphide, and wood are among the materials that have preferred directions of heat flow.
Thermal Conductivity in Rectangular Coordinates:
A Second-Order Tensor:
$\bar{\bar{k}} \equiv \begin{vmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{vmatrix}$
The components of heat flux vector are given by
$q_i = \sum_{j=1}^{3} k_{ij} \frac{\partial T}{\partial x_j}$
The energy equation for anisotropic bodies contains cross derivatives.

Heat Conduction Equation: Anisotropic Solids

At any point in the medium, each component q_x , q_y, and q_z of the heat flux vector is considered a linear combination of the temperature gradients $\partial T/dx$, $\partial T/dy$, and $\partial T/dz$:

$$q''_x = -\left(k_{11}\frac{\partial T}{\partial x} + k_{12}\frac{\partial T}{\partial y} + k_{13}\frac{\partial T}{\partial z}\right)$$

$$q''_y = -\left(k_{21}\frac{\partial T}{\partial x} + k_{22}\frac{\partial T}{\partial y} + k_{23}\frac{\partial T}{\partial z}\right)$$

$$q''_z = -\left(k_{31}\frac{\partial T}{\partial x} + k_{32}\frac{\partial T}{\partial y} + k_{33}\frac{\partial T}{\partial z}\right)$$

$$k_{11}\frac{\partial^2 T}{\partial x^2} + k_{22}\frac{\partial^2 T}{\partial y^2} + k_{33}\frac{\partial^2 T}{\partial z^2} + (k_{12}+k_{21})\frac{\partial^2 T}{\partial x\,\partial y} + (k_{13}+k_{31})\frac{\partial^2 T}{\partial x\,\partial z}$$
$$+ (k_{23}+k_{32})\frac{\partial^2 T}{\partial y\,\partial z} + \dot{q}(x,y,z,t) = \rho c\frac{\partial T(x,y,z,t)}{\partial t}$$

where $k_{12} = k_{21}$, $k_{13} = k_{31}$, and $k_{23} = k_{32}$ by the reciprocity relation.

Heat Conduction Equation: Anisotropic Solids

Orthotropic Bodies:

The conductivity matrix depends on the orientation of the coordinate system in the body.

If the coordinate system is parallel to three mutually perpendicular preferred directions of heat conduction, then the geometry is said to be orthotropic and the coordinate system lies along the principal axes of heat conduction.

An orthotropic body has direction-dependent thermal conductivity whose principal values are aligned with the coordinate axes. In an orthotropic body the conductivity matrix has a diagonal form,

$$\begin{bmatrix} k_{11} & 0 & 0 \\ 0 & k_{22} & 0 \\ 0 & 0 & k_{33} \end{bmatrix}$$

Wood is an example of an orthotropic body

In the case of non-crystalline anisotropic solids, such as wood, the thermal conductivities k_{11}, k_{22}, and k_{33} are in the mutually perpendicular directions.

Heat Conduction Equation: Anisotropic Solids

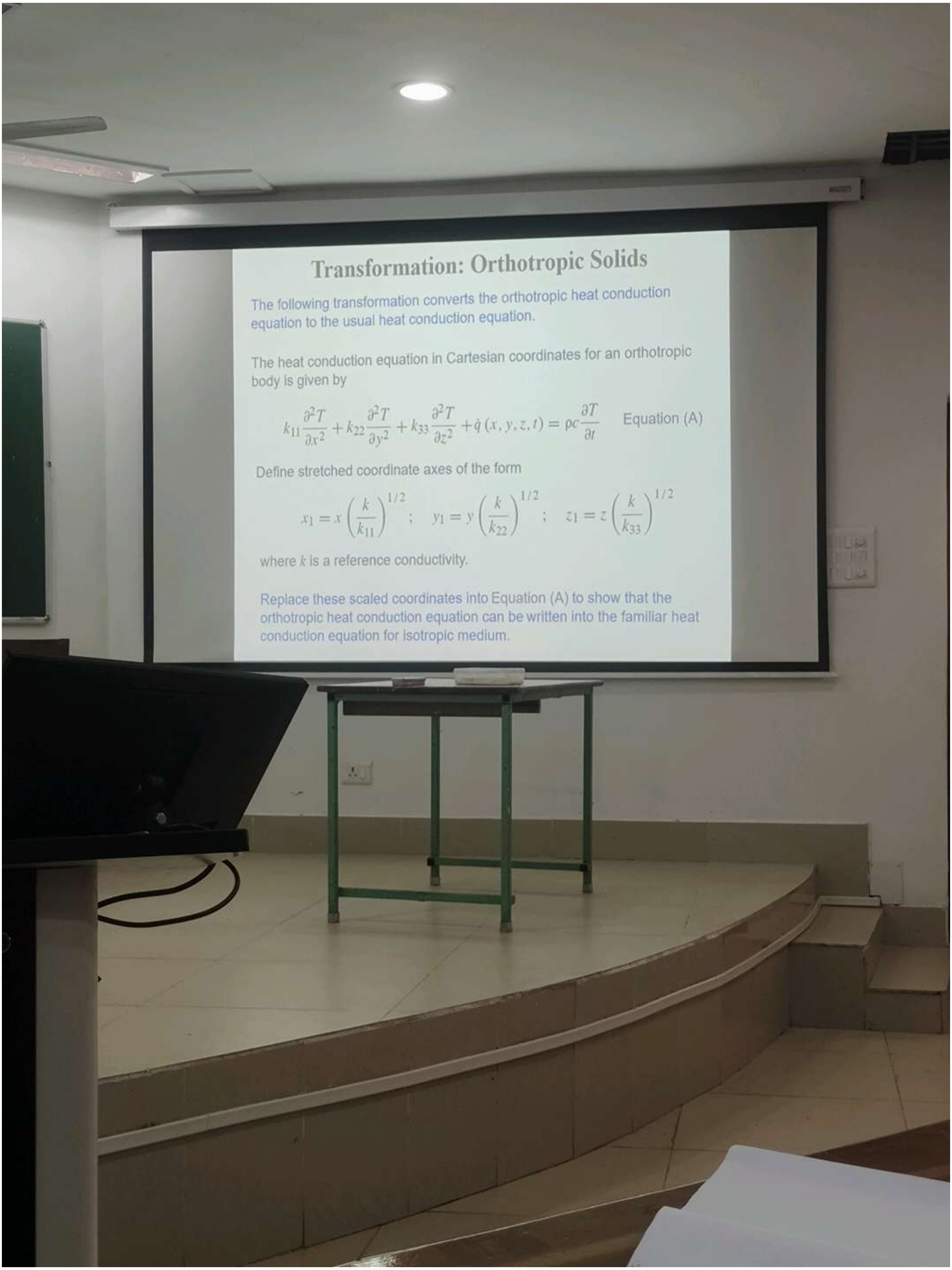
Orthotropic Bodies:

$$\begin{bmatrix} k_{11} & 0 & 0 \\ 0 & k_{22} & 0 \\ 0 & 0 & k_{33} \end{bmatrix}$$

The heat conduction equation in Cartesian coordinates for an orthotropic body is given by

$$k_{11}\frac{\partial^2 T}{\partial x^2} + k_{22}\frac{\partial^2 T}{\partial y^2} + k_{33}\frac{\partial^2 T}{\partial z^2} + \dot{q}(x, y, z, t) = \rho c\frac{\partial T}{\partial t}$$

The energy equation for orthotropic bodies does not contain any cross derivatives and it can be transformed into the standard isotropic energy equation by a suitable choice of new spatial coordinates.


Transformation: Orthotropic Solids
The following transformation converts the orthotropic heat conduction equation to the usual heat conduction equation.
The heat conduction equation in Cartesian coordinates for an orthotropic body is given by
$k_{11}\frac{\partial^2 T}{\partial x^2} + k_{22}\frac{\partial^2 T}{\partial y^2} + k_{33}\frac{\partial^2 T}{\partial z^2} + \dot{q}(x, y, z, t) = \rho c \frac{\partial T}{\partial t}$ Equation (A)
Define stretched coordinate axes of the form
$x_1 = x\left(\frac{k}{k_{11}}\right)^{1/2}; \quad y_1 = y\left(\frac{k}{k_{22}}\right)^{1/2}; \quad z_1 = z\left(\frac{k}{k_{33}}\right)^{1/2}$
where k is a reference conductivity.
Replace these scaled coordinates into Equation (A) to show that the orthotropic heat conduction equation can be written into the familiar heat conduction equation for isotropic medium.

Transformation: Orthotropic Solids

Replacing the scaled coordinates into Equation (A)

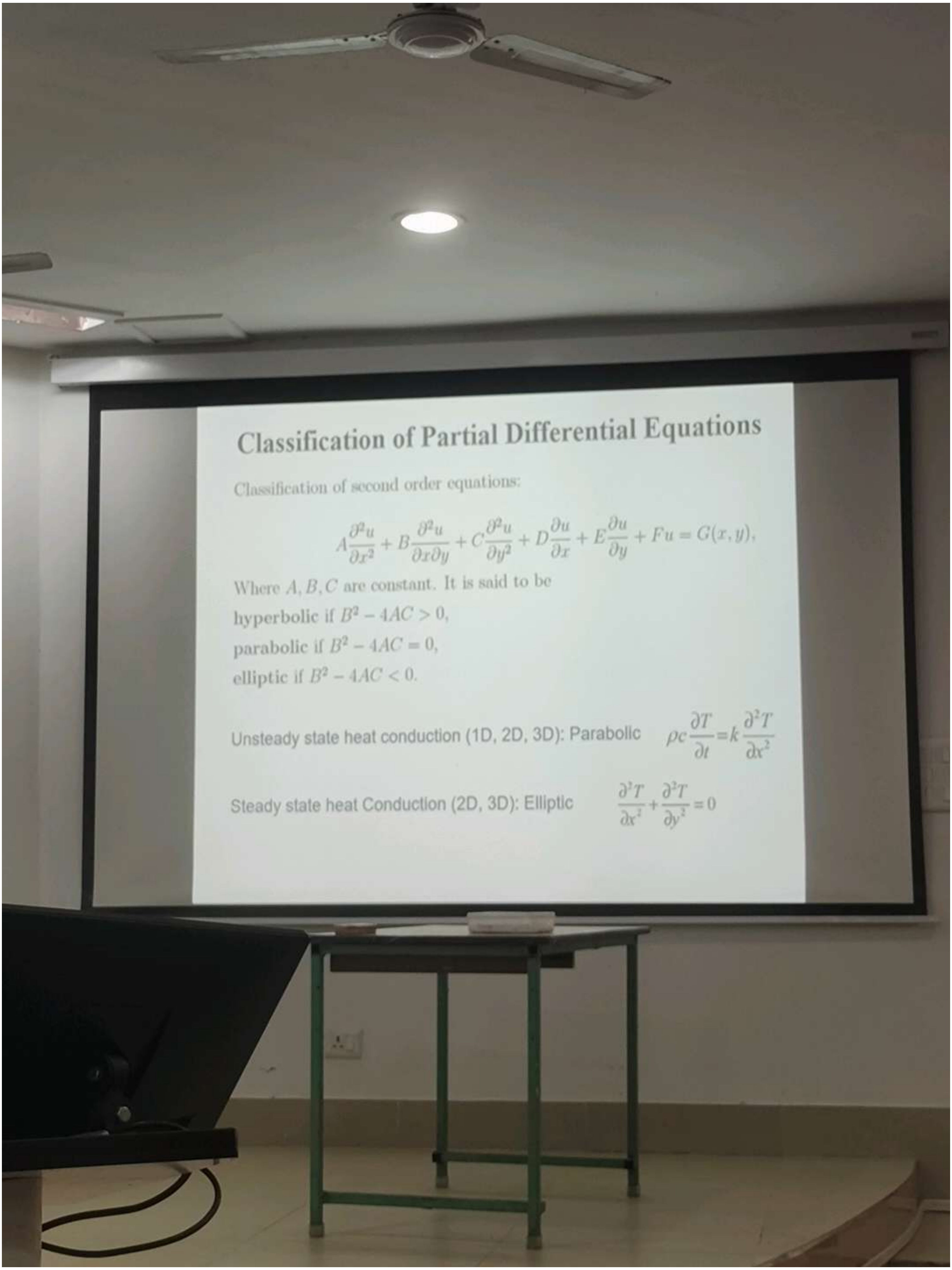
$$k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\dot{q}(x,y,z,t)=\rho c\frac{\partial T}{\partial t}$$

The reference conductivity is not arbitrary, it must be chosen so that the original differential volume is equal to the scaled differential volume.

For the 3D Cartesian case, the differential volume scales according to

$$dx\,dy\,dz=\frac{(k_{11}k_{22}k_{33})^{1/2}}{k^{3/2}}dx_1\,dy_1\,dz_1$$

and the requirement that $dv = dv_1$ causes $\quad k=(k_{11}k_{22}k_{33})^{1/3}$


Classification of Partial Differential Equations
Classification of second order equations:
$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G(x, y),$
Where A, B, C are constant. It is said to be
hyperbolic if $B^2 - 4AC > 0$,
parabolic if $B^2 - 4AC = 0$,
elliptic if $B^2 - 4AC < 0$.
Unsteady state heat conduction (1D, 2D, 3D): Parabolic $\rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$
Steady state heat Conduction (2D, 3D): Elliptic $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$

Heat Propagation

For heat conduction in a homogeneous and isotropic medium, the Fourier law of heat conduction:

From: $\rho c \dfrac{\partial T}{\partial t} + \nabla \cdot q'' - q''' = 0$

$$\mathbf{q}(\mathbf{r},t) = -k \nabla T(\mathbf{r},t)$$

The relation between the heat flux **q** and the temperature gradient ∇T is called the *constitutive relation of heat flux*.

=> $\dfrac{1}{\alpha}\dfrac{\partial T}{\partial t} = \nabla^2 T + \dfrac{q'''}{k}$

Parabolic heat-conduction equation

- The Fourier law of heat conduction is an early empirical law.
- It assumes that **q** and ∇T appear at the same time instant t and consequently implies that thermal signals pro with an infinite speed.

Non-Fourier Heat Conduction: Finite Speed of Heat Propagation

If the material is subjected to a thermal disturbance, the effects of the disturbance will be felt instantaneously at distances infinitely far from its source.

Although this result is physically unrealistic, it has been confirmed by many experiments that the Fourier law of heat conduction holds for many media in the <u>usual range of heat flux **q** and temperature gradient</u>.

What about heat conduction appearing in the <u>range of high heat flux and high unsteadiness</u>?

Non-Fourier Heat Conduction

Technology: Ultrafast pulse-laser heating on metal films → heat conduction appears in the range of high heat flux and high unsteadiness.

Infinite heat propagation speed in the Fourier law becomes unacceptable. This has inspired the work of searching for new constitutive relations.

New constitutive relation proposed by Cattaneo (1958) and Vernotte (1958, 1961):

CV Constitutive Relation: $\mathbf{q}(\mathbf{r},t) + \tau_0 \dfrac{\partial \mathbf{q}(\mathbf{r},t)}{\partial t} = -k\nabla T(\mathbf{r},t)$

Here $\tau_0 > 0$ is a material property and is called the relaxation time.

Non-Fourier Heat Conduction

CV Constitutive Relation: $\mathbf{q}(\mathbf{r},t) + \tau_0 \dfrac{\partial \mathbf{q}(\mathbf{r},t)}{\partial t} = -k\nabla T(\mathbf{r},t)$

Substitute $\mathbf{q}$ in: $\rho c \dfrac{\partial T}{\partial t} + \nabla \cdot \mathrm{q}'' - q''' = 0$

The corresponding heat-conduction equation:

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} + \frac{\tau_0}{\alpha}\frac{\partial^2 T}{\partial t^2} = \nabla^2 T + \frac{1}{k}\left(q''' + \tau_0 \frac{\partial\, q'''}{\partial t}\right)$$

Non-Fourier Heat Conduction

CV Constitutive Relation: $\mathbf{q}(\mathbf{r},t) + \tau_0 \frac{\partial \mathbf{q}(\mathbf{r},t)}{\partial t} = -k\nabla T(\mathbf{r},t)$

Substitute **q** in: $\rho c \frac{\partial T}{\partial t} + \nabla \cdot \mathrm{q}'' - q''' = 0$

The corresponding heat-conduction equation:

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} + \frac{\tau_0}{\alpha}\frac{\partial^2 T}{\partial t^2} = \nabla^2 T + \frac{1}{k}\left(q''' + \tau_0 \frac{\partial q'''}{\partial t}\right)$$

This equation is of *hyperbolic* type, characterizes the combined diffusion and wave-like behavior of heat conduction, and predicts a *finite speed for heat propagation:*

$$V_{CV} = \sqrt{\frac{k}{\rho c \tau_0}} = \sqrt{\frac{\alpha}{\tau_0}}$$

Consider no heat generation

Standard Wave Equation:

$$\frac{\partial^2 q}{\partial t^2} = c^2\left(\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} + \frac{\partial^2 q}{\partial z^2}\right)$$