FFC	FBC

Adv.

1. No instability pb.
2. Takes action beforehand
3. Good for slow systems

Disadv.

1. Requires modelling of process (Plant / model)
2. Parameter identification
3. $\boxed{\text{FFC cannot take care of unmeasured load variable. Answer to last class ques}}$

→ To solve this we use FFC + FBC

→ Override/constraint control scheme

no. of measu > 1 manipulated variable = 1

- During normal operatn of the plant/shutdown/start-up condition, some abnormal/dangerous situations may arise & it may lead to destruction of equipment along with operating personnel.

∴ We use switch.

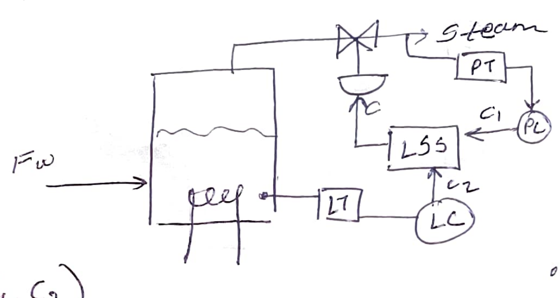
Switch

LSS : Lower selector switch
→ prevents to exceed lower limit. → constraint.

HSS : High selector switch

↓ $C = P_{sp} - P$

eg -



Control obj:

$P = P_{sp}$

$h \nless h_{min}$

∴ we use LSS

$C = \min(c_1, c_2)$

*** PC & LC are both direct action controllers provided the control valve is air to open.

situation :

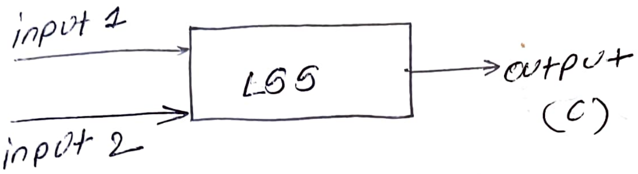
1. $C = C_1$ .... $(C_1 < C_2)$

suddenly → $h < h_{min}$

∴ set $C_2 = 0$

now, $(C_1 > C_2)$

2. $C = C_2$ ... (LC overrides pressure controller)



if (input 1 < input 2)

c = input 1

else

c = input 2

end.

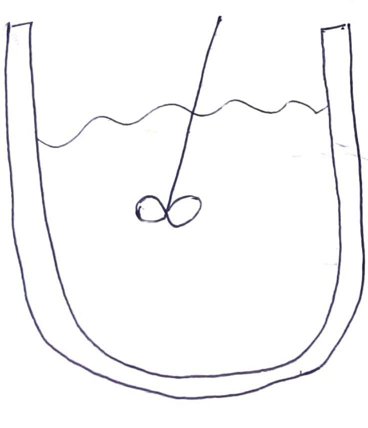
→ split range control.

no. of measurements = 1   MV > 1

• It provides additional safety and operational optimality

• split range control are not common in che.

Non-isothermal batch reactor:



• for unsteady state process T_{sp} changes dynamically.

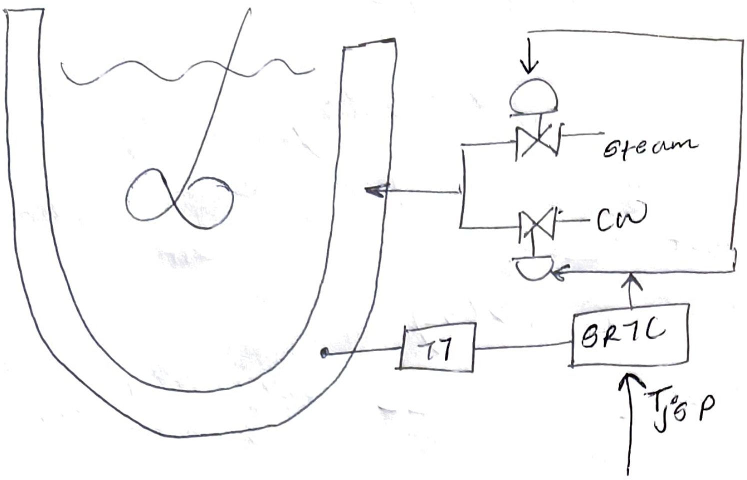
ex- $T_{sp} = 54 + 71 \exp(-2 \times 10^{-3} t)$

$T(t=0) = 15°C$ ← non-linearity in between

$T(t=t_f) = 100°C$

Similarly, we can generate jacket setpoint, T_{JSP}

eg



→ Ratio control scheme

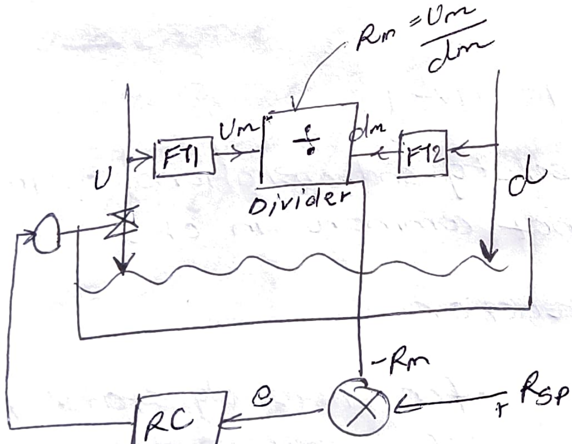
1. no. of measured V > 1
   MV = 1

2. Special type of FFC

3. $R = \frac{U}{d}$ ← MV, ← DV — 2 process streams (commonly flow rate)

d ⇒ wild stream

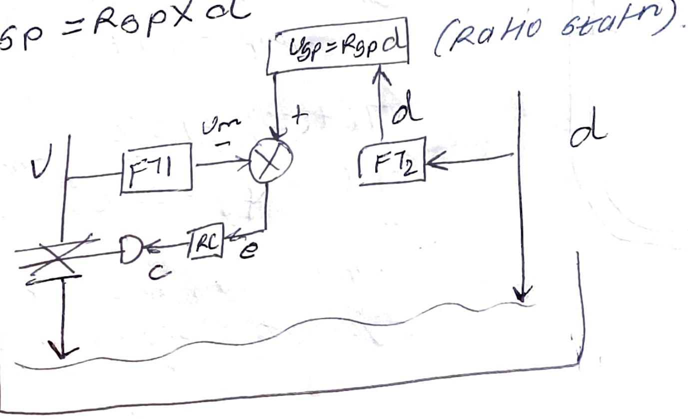
eg-



Ratio controller (RC) = P / PI / PID

∴ d is measured
∴ in essence Ratio Controller is a special type of FFC.

Another scheme: Ratio station
$U_{sp} = R_{sp} \times d$



example- Blending process, Distillation reflux, combustn chamber (fuel to air ratio)

which ratio control scheme is better?

→ Adaptive Control

PI : $u = K_c e + \frac{K_c}{\tau_i}\int e\,dt$

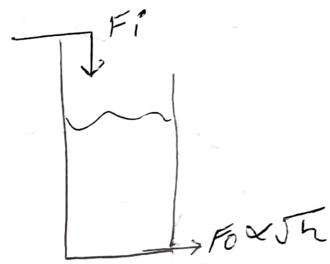
K_c, τ_i = constant.

- Controllers having variable parameters are called adaptive controllers

why do we need this?

1. process non-linearity
2. Non-stationary.

Model:



$$\rho F_i - \rho F_o = \frac{d(V\rho)}{dt}$$

$$F_i - F_o = A\frac{dh}{dt}$$

$$F_i - \beta\sqrt{h} = A\frac{dh}{dt}$$

$$F_i = A\frac{dh}{dt} + \beta\sqrt{h}$$

$$\beta\sqrt{h} = \beta\left[\sqrt{h_{ss}} + \frac{h \times 1}{2\sqrt{h_{ss}}} + \frac{h^2}{2!} \times \frac{-1}{4} h_{ss}^{-3/2}\right].$$

$\frac{dh^{1/2}}{dh}$

$\Rightarrow \frac{1}{2}(h)^{-1/2}$

$\frac{1}{2}(h)^{-1/2}$

$\frac{1}{2} \times \frac{-1}{2}(h)^{-3/2}$

$$\beta\sqrt{h} = \beta\left[\sqrt{h_{ss}} + \frac{(h-h_{ss})}{2\sqrt{h_{ss}}} - \frac{(h-h_{ss})^2}{2!}\frac{1}{4(h_{ss})^{3/2}}\right]$$

$$\beta\sqrt{h} = \beta\sqrt{h_{ss}} + \frac{\beta}{2\sqrt{h_{ss}}}(h-h_{ss})$$

in terms of deviation variables

$$A\frac{d}{dt}(h-h_s) + \beta\sqrt{h} - \beta\sqrt{h_s} = F_i - F_{is}.$$

$$A\frac{dh'}{dt} + \frac{\beta}{2\sqrt{h_s}}h' = F_i'$$

$$\frac{2A\sqrt{h_s}}{\beta}\frac{dh'}{dt} + h' = \frac{2\sqrt{h_s}}{\beta}F_i'$$

$$\tau_p \frac{dy}{dt} + y = k_p u$$

$$\tau_p = \frac{2A\sqrt{h_s}}{\beta} \quad k_p = \frac{2\sqrt{h_s}}{\beta}$$ ... τ_p, k_p are dependent on h_{ss} (s.s.)

$k_c, \tau_i = f(k_p, \tau_p)$

∴ controllers need to be adaptive.

- Criteria for changing k_c & τ_i
  1. ISE
  2. Phase margin, gain margin
  3. One-quarter decay ratio.

- Programmed/scheduled Adapting controller.
  eg - GSAC, combustion chamber

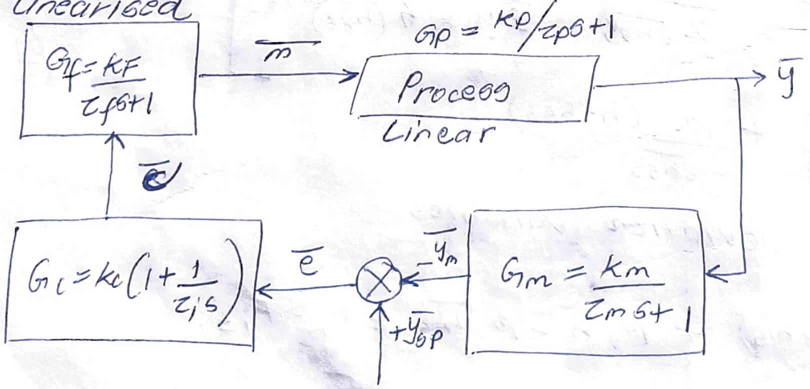
- Self Adapting controller
  - MRAC, STR

MRAC → Model reference adaptive control

STR → self tuning controller.

→ 14/02/25

Gain scheduling adaptive control.



$k_f, \tau_f = f$(ss condition)

$K_{overall}$ (overall gain) $= k_p k_f k_m k_c$ = constant ... [to keep stability margin constant]

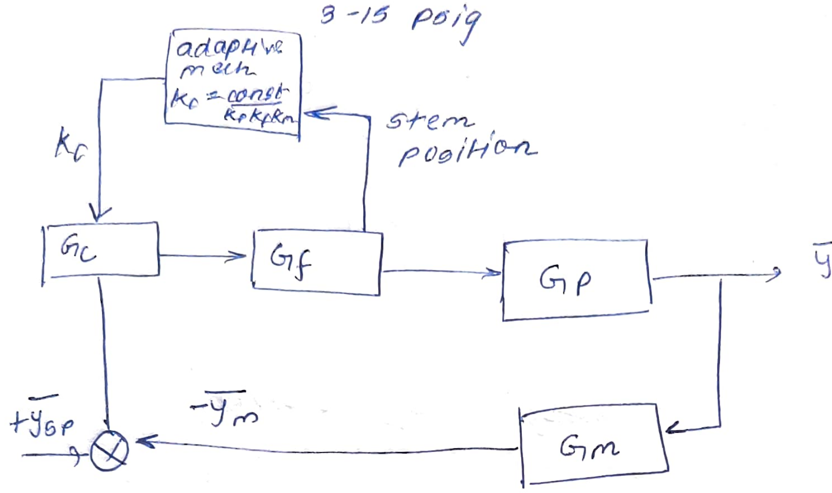
constant ← known experimentally

$$K_c = \frac{\text{const}}{K_p K_f K_m} \rightarrow K_p \,\&\, K_m \text{ are fixed}$$

control valve :

$$K_f = \frac{\text{stem position}}{3\text{-}15 \text{ psig}}$$

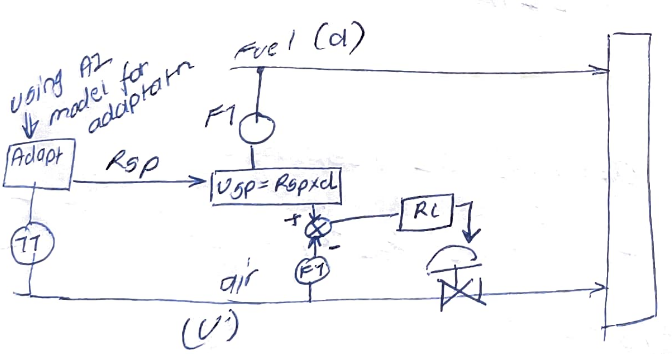
(gain) $K = \frac{\text{change in output}}{\text{change in input}}$



→ limitation:

1. Above can be only done for P-only controller, since we are finding K_c & not τ_i.
2. No way of correcting K_c.

• Combustn chamber (adaptive ratio control)



Fuel Air ↓ Incomplete combustion $\therefore T \downarrow$

Air ↑ Overheating & $T \uparrow$

$\therefore F/A = \text{optimal}$.

Air temperature changes continuously, therefore we need adaptive control.

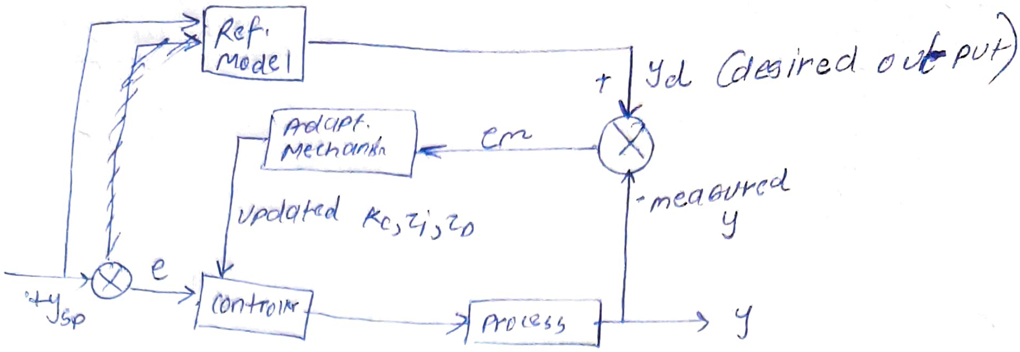
obj: maintain T of combustion chamber.

Air T ↑ then reduce flow rate of air ←

$\therefore R_{sp} = (F/A)_{optimal}$ needs to change

$\boxed{\text{• Make the above scheme for divider in ratio control scheme}}$

→ MRAC (Modell reference adaptive control)



Adaption mechanism,

$$ISE = \int e_m^2 \, dt \quad \text{--- integral square error}$$

$$= \int (y_d - y)^2 \, dt$$

$$\frac{dISE}{dk_c} = \frac{dISE}{d\tau_i} = \frac{dISE}{d\tau_D} = 0$$

} Ref model is required to use this method.

What is reference model?

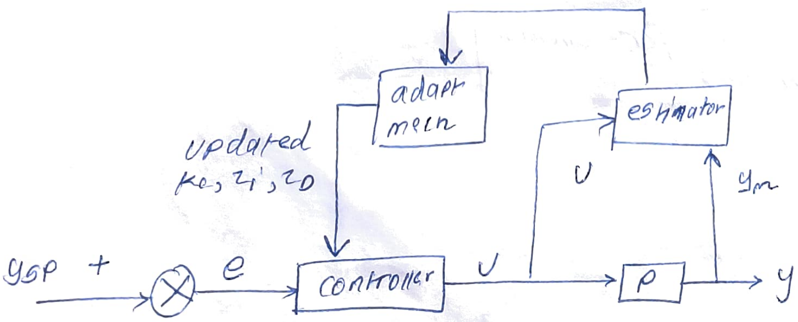
$$\bar{y}_d = \frac{G_c G_p G_f}{1 + G_c G_p G_f G_m} \bar{y}_{sp} \quad \text{--- CLTF}$$

$$\bar{y}_d = \frac{1}{\lambda s + 1} \bar{y}_{sp} \quad \text{--- Desired CL response}$$

└── does not contain k_c, τ_i, τ_D

∴ we make $k_c, \tau_i, \tau_D = f(\lambda)$

→ STR



Estimator:

$$\frac{y}{u} = \frac{k_p e^{-t_d s}}{\tau_p s+1}$$

y & u are known, ∴ k_p, τ_p & t_d can be estimated.
k_c, τ_i, τ_D can directly be corelated with k_p, τ_p & t_d using Cohen-Coon settings.

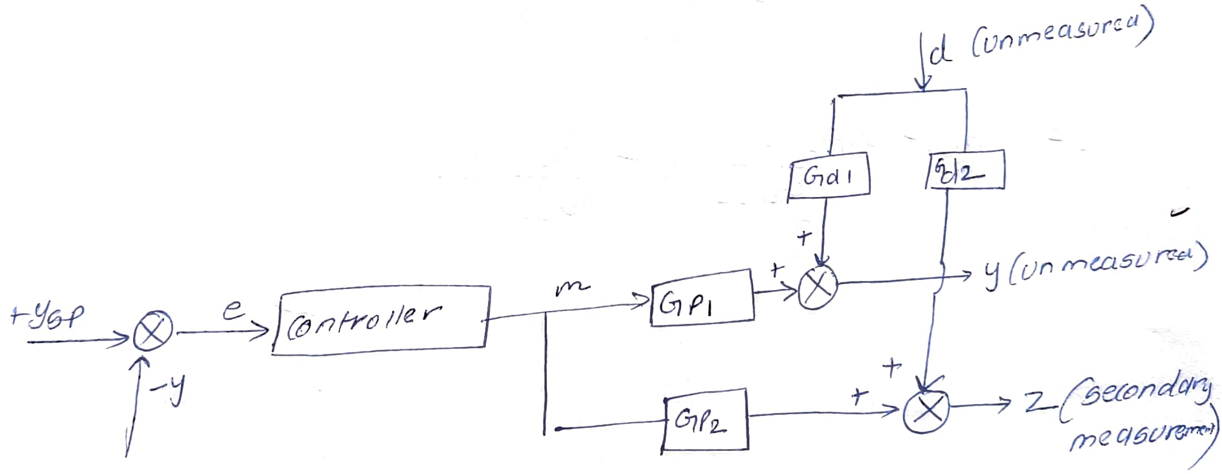
→ Inferential Control scheme

* FBC: P-family $u = u_s + K_c(y_{sp} - y)$ * CV ⇒ measured.

suppose CV is not measureable then FBC cannot be used.

* FFC: model $G_c = \frac{G_d}{G_P}$ * LV = measured

∴ we need inferential control scheme.



$$y = G_{P_1} m + d G_{d_1} \quad — ①$$

$$z = G_{P_2} m + d G_{d_2} \quad — ②$$

$d = ?$

$$\frac{y - G_{P_1} m}{G_{d_1}} = d$$

$$z = G_{P_2} m + \left(\frac{y - G_{P_1} m}{G_{d_1}}\right) G_{d_2}$$

$$\rightarrow z - \frac{y G_{d_2}}{G_{d_1}} = m\left(G_{P_2} - \frac{G_{P_1} G_{d_2}}{G_{d_1}}\right)$$

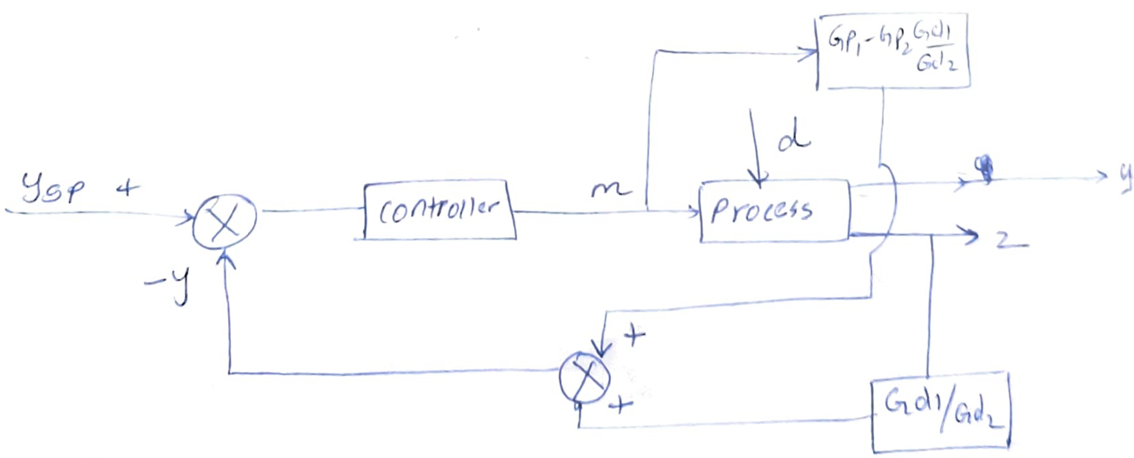
$$\frac{z - \frac{y G_{d_2}}{G_{d_1}}}{\left(G_{P_2} - \frac{G_{P_1} G_{d_2}}{G_{d_1}}\right)} = m$$

$$d = y - \frac{G_{p_1}\left\{\dfrac{z - y\dfrac{G_{d_2}}{G_{d_1}}}{G_{p_2} - G_{p_1}\dfrac{G_{d_2}}{G_{d_1}}}\right\}}{G_{d_1}}$$

$$d\,G_{d_1} = y - \frac{G_{p_1} z}{G_{p_2} - G_{p_1}\dfrac{G_{d_2}}{G_{d_1}}} + \frac{G_{p_1} y \dfrac{G_{d_2}}{G_{d_1}}}{G_{p_2} - G_{p_1}\dfrac{G_{d_2}}{G_{d_1}}}$$

$$\frac{d\,G_{d_1} + \dfrac{G_{p_1} z}{G_{p_2} - G_{p_1}\dfrac{G_{d_2}}{G_{d_1}}}}{\left(1 + \dfrac{G_{p_1}\dfrac{G_{d_2}}{G_{d_1}}}{G_{p_2} - G_{p_1}\dfrac{G_{d_2}}{G_{d_1}}}\right)} = y$$

$$y = \left(G_{p_1} - G_{p_2}\frac{G_{d_1}}{G_{d_2}}\right)m + \frac{G_{d_1}}{G_{d_2}} z$$



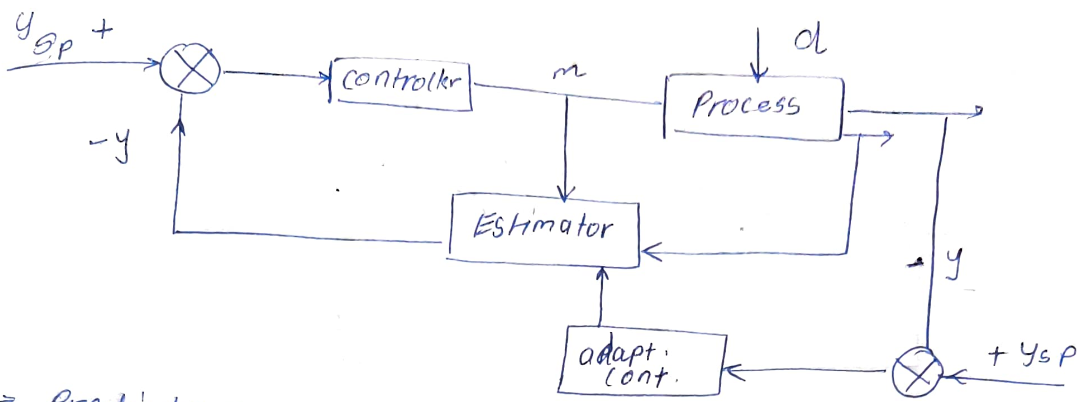
Remarks:

1. y is infered from measured z. ∴ It ~~it~~ called interential control scheme.
2. Imperfect model is a major limitation

3. example: Distillation column

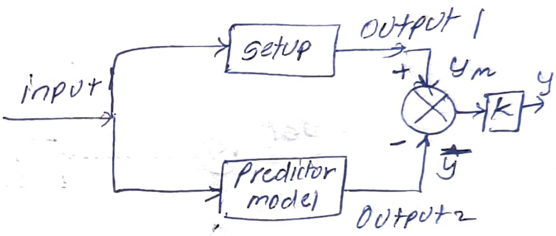
$z = T$

$y = x_D$



→ Predictor

$$\begin{cases} \dot{x} = f(x, u) \\ y = h(x) \end{cases}$$



$$\dot{\bar{x}} = \underbrace{f(\bar{x}, u)}_{\text{predictor}} + \underbrace{K(y_m - \bar{y})}_{\text{corrector}}$$

constant K ≡ Luenberger observer

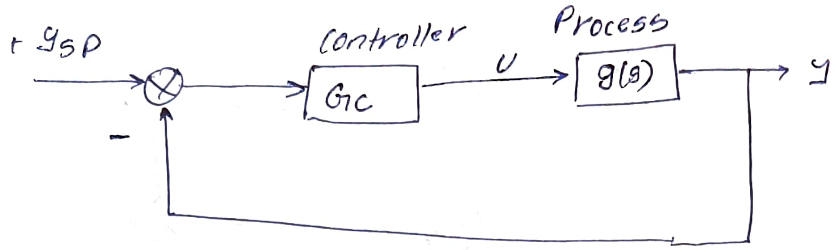
variable K ≡ extended Kalman filtering

→ Min loop interaction / RGA

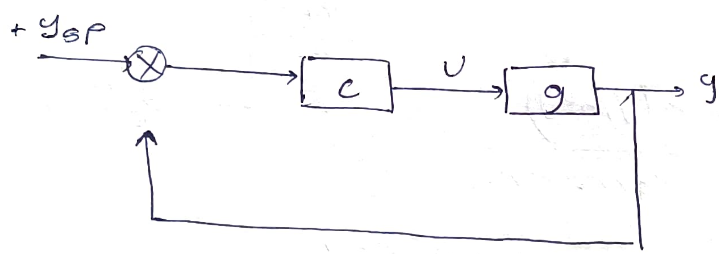
6/3/25

→ Internal Model Control (IMC)

• Preliminaries



$G_m = G_f = 1$



• Static controller: $c = K_c$

$$\tau_p \frac{dy}{dt} + y = k_p f(t)$$

$$G = \frac{\bar{y}}{\bar{f}} = \frac{k_p}{\tau_p s+1} \quad \text{or} \quad \frac{\bar{y}}{\bar{U}} = \frac{k_p}{\tau_p s+1}$$

for steady state

$$y = \frac{\bar{y}}{\bar{U}} = k_p \quad \ldots \quad \left[\tau_p \cancelto{0}{\frac{dy}{dt}} + y = k_p f(t)\right]$$

from diagram,

$$\bar{y} = gU = gc\,\bar{y}_{sp}$$

$$\bar{y} = \frac{k_c k_p}{\tau_p s+1} \bar{y}_{sp}$$

time constant

- open loop process (without controller) $= \frac{k_p}{\tau_p s+1}$ ... τ_p
- Open loop process (with controller) $= \frac{k_p k_c}{\tau_p s+1}$ ... τ_p

} speed of CL response does not change even if we employ the controller

$y_{sp} = A/s$ for $t > 0$.

$$\bar{y} = \frac{k_p k_c A}{s(\tau_p s+1)}$$

$$\bar{y} = \frac{k_p k_c A}{\cancel{\tau_p}} \left[\frac{1}{s} - \frac{1}{s+\frac{1}{\tau_p}} \right] \cancel{\times \tau_p}$$

$s+\frac{1}{\tau_p} - s$

$$y = A k_p k_c \left[1 - e^{-t/\tau_p}\right].$$

@ $t \to \infty$

$$y = A k_p k_c.$$

$\boxed{\text{offset} = A - k_p k_c A} \neq 0$ } offset is not eliminated

- offset can be eliminated if $k_c = 1/k_p$.

→ Dynamic controller :

$$C = \frac{1}{g}$$

$$\bar{y} = C g \bar{y}_{sp}$$

$$\bar{y} = \bar{y}_{sp}$$

∴ No offset.

$C = \frac{\tau_p s+1}{k_p}$ ... First order process

$\bar{U} = C y_{sp} = \left(\frac{\tau_p s+1}{k_p}\right) \bar{y}_{sp}$ } improper sys. [not physically implementable]

$$\bar{U} = \frac{\tau_p \cancel{s} A}{k_p} + \frac{A}{k_p s}$$

$$U = \tau_p A\, \delta(t) + \frac{A}{k_p}$$

Making sys proper,

Dynamic controller with filter:

$$c = \frac{1}{g} = \frac{\tau_p s+1}{k_p} f = \underbrace{\frac{\tau_p s+1}{k_p} \times \frac{1}{(\lambda s+1)}}_{\text{lead-lag element}}$$

$$f = \frac{1}{(\lambda s+1)^n}$$

τ_f

~~1st order~~ lead-lag element

$\rightarrow$

$G_p = \frac{k_p}{\tau_p s+1}$ ... $\phi = -90$ ... 1st order lag transfer fn

Suppose,

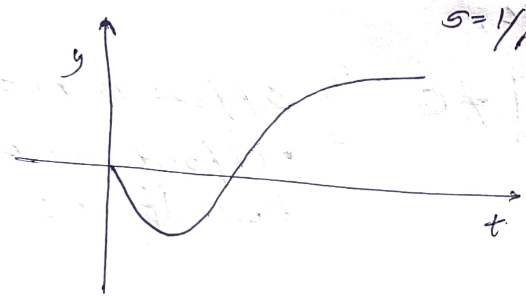
$$g = \frac{(-\beta s+1)}{(\tau_1 s+1)(\tau_2 s+1)} \sim \frac{e^{-\beta s}}{(\tau_1 s+1)(\tau_2 s+1)}$$ } dead time (delayed response)

~~Pole~~ zeros of g $\Rightarrow$ $-\beta s+1=0$

$s = 1/\beta$

Drawback:

(+ve zero gives inverse respose /delays respose)



} response of process having transfer fn g

Drawback:

* Controller performance is degraded.

$$c = \frac{1}{g} \times f = \frac{(\tau_1 s+1)(\tau_2 s+1)}{(-\beta s+1)(\lambda s+1)}$$ } unstable bcoz 1 pole is in right half plane.

$\therefore$ $\beta = 0$. — — for stability

$$c = \frac{(\tau_1 s+1)(\tau_2 s+1)}{(\lambda s+1)^2}$$

Static	Dynamic
$\bar{y} = cg\,\overline{y_{SP}}$	$\bar{y} = cg\,\overline{y_{SP}}$
$\bar{y} = K_C$	$c = f/g$
$\bar{y} = \dfrac{K_C K_P}{\tau_P s+1}\,\overline{y_{SP}}$	$\bar{y} = \dfrac{1}{(\lambda s+1)}\,\dfrac{\cancel{(\tau_P s+1)}}{\cancel{K_P}}\,\dfrac{y}{y_{sp}}$
$y(t) = AK_CK_P\left[1-e^{-t/\tau_P}\right]$	$y(t) = A\left[1-e^{-t/\lambda}\right]$
offset $= A(1-K_CK_P)$	offset $= 0$
time constant $= \tau_P$	time const $= \lambda < \tau_P$

plant : $g \Rightarrow$ actual

simple factorisation

Model : $\tilde{g} = \tilde{g}_- \cdot \tilde{g}_+ = \underbrace{\dfrac{1}{(\tilde{\tau}_1 s+1)(\tilde{\tau}_2 s+1)}}_{\tilde{g}_{-1}} \cdot \underbrace{(-\tilde{\beta}s+1)}_{\tilde{g}_+} \Rightarrow$ model

$\tilde{g}_-$: invertible part

$\tilde{g}_+$: non-invertible part

$$\tilde{g} = \frac{-\tilde{\beta}s+1}{(\tilde{\tau}_1 s+1)(\tilde{\tau}_2 s+1)}$$

$$c = \frac{1}{\tilde{g}_{-1}} f = \frac{(\tau_1 s+1)(\tau_2 s+1)}{1} \times \frac{1}{(\lambda s+1)^2}$$

$$\bar{y} = \frac{fg}{\tilde{g}_{-1}}\,\overline{y_{SP}} = f\tilde{g}_+\,\overline{y_{SP}} = \frac{(-\tilde{\beta}s+1)\,\overline{y_{SP}}}{(\lambda s+1)^2}$$

considering $g = \tilde{g}$

$$\bar{y} = \overline{y_{SP}}\,\frac{(-\tilde{\beta}s+1)}{(\lambda s+1)^2}$$

--- Controller cannot element inverse response

All pass factorisation :

$$\tilde{g} = \tilde{g}_{-1} \cdot \tilde{g}_+ = \frac{(\tilde{\beta}s+1)}{(\tilde{\tau}_1 s+1)(\tilde{\tau}_2 s+1)} \cdot \frac{(-\tilde{\beta}s+1)}{(\tilde{\beta}s+1)}$$

$$c = \frac{f}{\tilde{g}_{-1}}$$

- A transfer function having constant AR is called 'all pass', because it transfers all frequencies from input to output without any change in AR. → Proof?

find AR of $\tilde{g}_+$, it will come =1

12/08/25.

Process ⇒ g

Model ⇒ $\tilde{g}$

IMC ⇒ $c = \frac{f}{\tilde{g}_{-1}}$

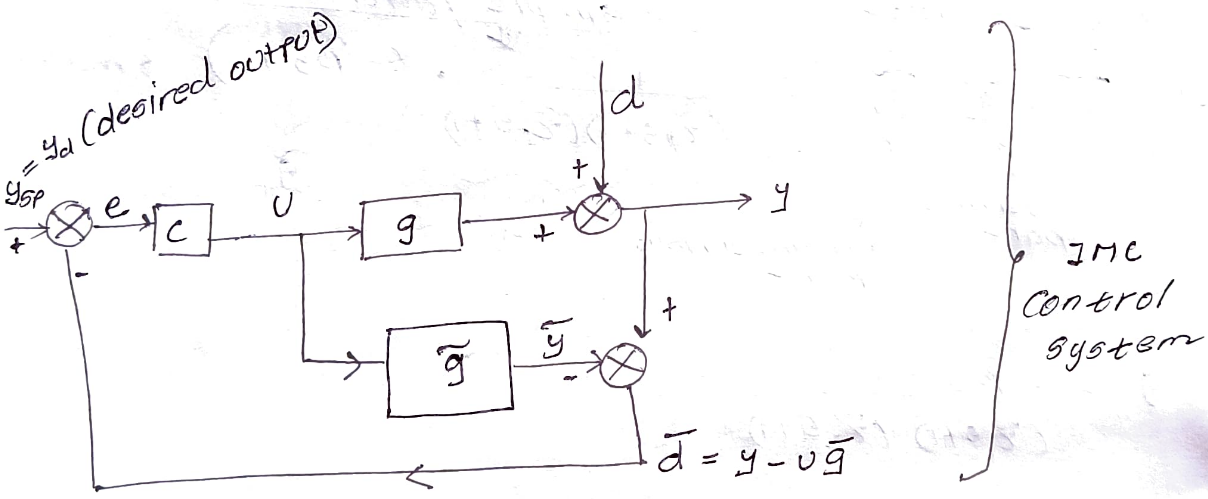


Fig-1

$e = y_d - \tilde{d}$

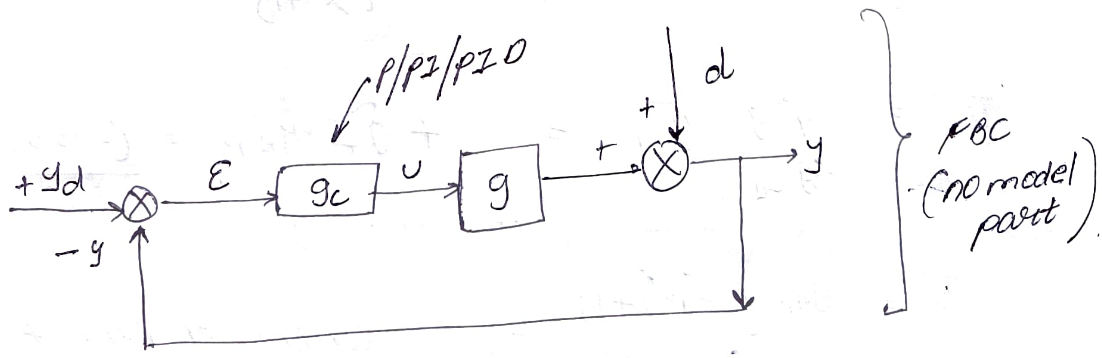


Fig-2

$\varepsilon = y_d - y$

$$g_c = \frac{c}{1 - c\tilde{g}}$$

Fig-1

$y = ug + d \longrightarrow y = u\bar{g} + \bar{d} \Rightarrow \bar{d} = y - u\bar{g}$

$y_d = ug + d$ .... True for perfect controller's case

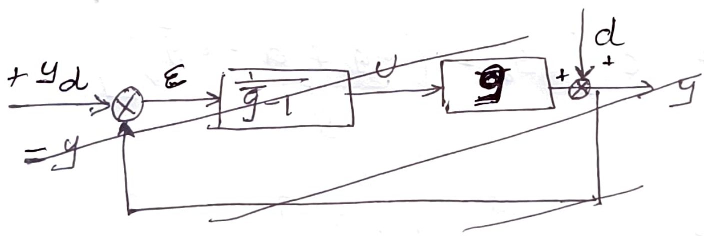
$u = \frac{1}{g}(y_d - d)$ .... ideal action

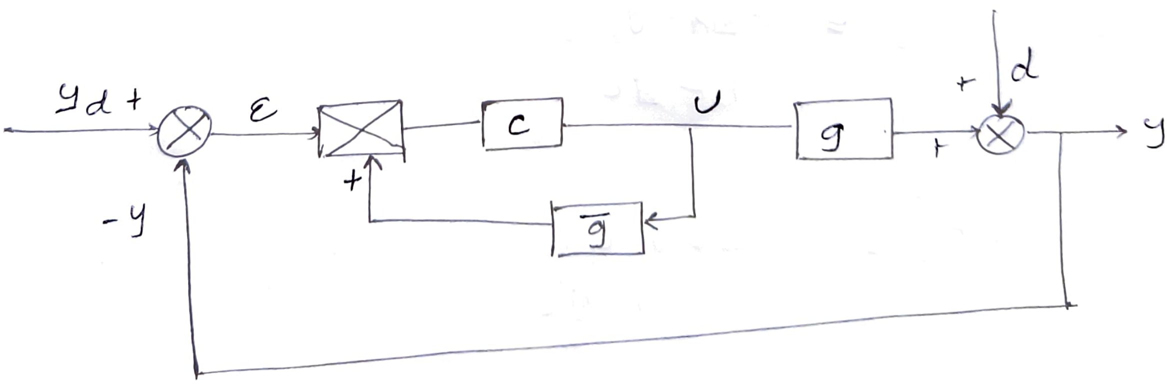
$u = \frac{1}{\bar{g}}(y_d - \bar{d})$ .... for our model. /Real action

$u = \frac{1}{\bar{g}}\left[y_d - (y - u\bar{g})\right]$

$u = c\left[y_d - y + u\bar{g}\right] \Rightarrow$

where, $c = \frac{1}{\bar{g}}$





$c \equiv g_c$

$u = c\left[y_d - y + u\bar{g}\right] = c\left[\varepsilon + u\bar{g}\right]$

$u - cu\bar{g} = c\varepsilon$

$u = \frac{c\varepsilon}{1 - c\bar{g}} \Longrightarrow \frac{u}{\varepsilon} = g_c = \frac{c}{1 - c\bar{g}} \Rightarrow FBC$

- if $c = \frac{1}{\bar{g}}$ then control action $\to \infty$ for FBC.

CLTF, fig-1

$$y = d + ug$$

$$y = d + g[c(y_d - \bar{a})]$$

$$y = d + gc(y_d - (y - u\bar{g}))$$

$$y = d + gcy_d - gcy + gcu\bar{g}$$

$$(gc+1)y = gc\bar{g}u + gcy_d + d.$$

$$U = ce = c[y_d - (\underbrace{ug + d}_{y} - u\bar{g})]$$

$$U = c[y_d - y + u\bar{g}]$$

$$U = \frac{c(y_d - y)}{1 - \bar{g}c}$$

$$(gc+1)y = \frac{gc\bar{g}c(y_d - y)}{1 - \bar{g}c} + gcy_d + d$$

$$\left(gc + 1 + \frac{gc^2\bar{g}}{1 - \bar{g}c}\right)y = \frac{gc^2\bar{g}y_d}{1 - \bar{g}c} + gcy_d + d.$$

$$\left(\frac{gc - \cancel{g\bar{g}c^2} + 1 - \bar{g}c + \cancel{gc^2\bar{g}}}{1 - \bar{g}c}\right)y = \frac{gcy_d}{(1 - \bar{g}c)} + d.$$

$$y = \frac{g_c\, y_d}{\cancel{(1-\bar{g}c)}(1-\bar{g}c+gc)} + \frac{d(1-\bar{g}c)}{1-\bar{g}c+gc}$$

servo case: $d = 0$

$$\boxed{y = \frac{g_c\, y_d}{(1-\bar{g}c+gc)}}$$

Questions:

1. find IMC: $c(s) = \frac{f}{\bar{g}_{-1}}$

2. $g_c = \frac{c}{1-c\bar{g}} \Rightarrow P, PI, PID$

   $K_c = ?$

   $\tau_i = ?$

   $\tau_D = ?$

Summary:

IMC: $c(s) = \frac{1}{\bar{g}_{-1}} \cdot f$

Model: $\bar{g} = \bar{g}_{-1} \cdot \bar{g}_{+}$ — Dead-time. / inverse response

FBC: $g_c = \frac{c}{1-c\bar{g}} \equiv P, PI, PID$ — Real / ideal

Ex-1

$$g = \bar{g} = \frac{k_p}{\tau_p s+1}\,\cancel{\frac{(-\beta s+1)}{(-\beta s+1)}}$$

$$\cancel{\bar{g} = \underbrace{\frac{k_p(-\beta s+1)}{(\tau_p s+1)}}_{\bar{g}_{-1}} \times \underbrace{\frac{1}{(-\beta s+1)}}_{\bar{g}_{+}}}$$

$$\cancel{\underbrace{\frac{k_p}{(\tau_p s+1)(-\beta s+1)}}_{\bar{g}_{-1}} \times \underbrace{(-\beta s+1)}_{\bar{g}_{+}}}$$

$\cancel{c = \dfrac{f}{\bar{g}_{-1}}}$

~~For dead time~~

$\cancel{\bar{g}_{-1} = \dfrac{k_p}{(\tau_p s+1)(}}$

Step 1: $\bar{g} = \bar{g}_{-1} \cdot \bar{g}_{+}$

$$= \underbrace{\frac{k_p}{\tau_p s+1}}_{g_{-1}} \cdot \underbrace{1}_{g_+}$$

Step 2: $c(s) = (\bar{g}_{-1})^{-1} f$

$$= \frac{(\tau_p s+1)}{k_p} \times \frac{1}{(\lambda s+1)}$$

Step 3: $g_c = \dfrac{c}{1 - c\bar{g}}$

$$g_c = \frac{\dfrac{(\tau_p s+1)}{k_p(\lambda s+1)}}{1 - \dfrac{\tau_p s+1}{k_p(\lambda s+1)} \cdot \dfrac{k_p}{\tau_p s+1}} = \frac{\tau_p s+1}{k_p \lambda s}$$

$$g_c = \frac{\tau_p}{k_p \lambda} \times \frac{(\tau_p s+1)}{\tau_p s} = k_c\left(1 + \frac{1}{\tau_i s}\right)$$

$$= k_c\left(\frac{\tau_i s+1}{\tau_i s}\right)$$

$$\boxed{k_c = \frac{\tau_p}{k_p \lambda} \qquad \tau_i = \tau_p}$$

$\Rightarrow$ Advantage: Reduction in tunning parameters. k_p & τ_p are already known.

Remark:

1. IMC ≡ tunning P-family Controller
2. $\lambda \uparrow$ (slow response)

$$C(s) = (\tilde{g}_-)^{-1} f = \frac{(\tau_p s+1)}{K_p} \cdot \frac{1}{\lambda s+1} \quad \text{—— (1)}$$

for PI controller,

$$u = K_c e + \int e\,dt \times \frac{K_c}{\tau_i} \quad \rightarrow \quad u(k) = K_c e(k) + E_{int}(k)$$

$$E_{int} = \frac{K_c}{\tau_i} \int e\,dt.$$

$$\frac{dE_{int}}{dt} = \cancel{\frac{K_c}{\tau_i}}\, e$$

$$\frac{E_{int}(k+1) - E_{int}(k)}{\Delta t} = e.$$

$$E_{int}(k+1) = E_{int}(k) + \Delta t\, e(k)$$

Convert eq (1) in time domain & write pseudo code.

$$\frac{u}{e} = \frac{(\tau_p s+1)}{K_p} \cdot \frac{1}{(\lambda s+1)}$$

$$\frac{\bar{y}}{\bar{u}} = \frac{K_p}{\tau_p s+1} \Rightarrow \tau_p \frac{dy}{dt} + y = K_p f(t).$$

$$\downarrow$$

$$\tau_p \frac{du}{dt} + u = K_p \left[\lambda \frac{de}{dt} + e\right]$$

$$\downarrow$$

$$\tau_p s \bar{u} + \bar{u} = K_p \lambda s \bar{e} + K_p \bar{e}$$

$$\frac{\bar{u}}{\bar{e}} = \frac{K_p(\lambda s+1)}{(\tau_p s+1)}$$

discretise this

$$\tau_p \left[\frac{u(k+1) - u(k)}{\Delta t}\right] + u(k) = K_p \left[\lambda \left(\frac{e(k+1) - e(k)}{\Delta t}\right) + e(k)\right]$$

$u(k+1) = ?$

ex-2

$$g = \bar{g} = \frac{k_p e^{-t_d s}}{\tau_p s+1} \qquad c = ? \quad g_c = ?$$

Pade approximation $\Rightarrow$ $e^{-t_d s} = \dfrac{1 - \frac{t_d s}{2}}{1 + \frac{t_d s}{2}}$

$$\bar{g} = \bar{g}_{-1} \times \bar{g}_{+1} = \frac{k_p\left(1 - \frac{t_d s}{2}\right)}{(\tau_p s+1)\left(1 + \frac{t_d s}{2}\right)}.$$

$$= \underbrace{\frac{k_p}{(\tau_p s+1)}}_{\bar{g}_{-1}} \times \underbrace{\frac{(1 - t_d s/2)}{(1 + t_d s/2)}}_{\bar{g}_{+1}}$$

$$c(s) = \left(\bar{g}_{-1}\right)^{-1} f = \frac{\tau_p s+1}{k_p} \times \frac{1}{(\lambda s+1)}.$$

$$g_c = \frac{c}{1 - c\bar{g}} = \cancel{\frac{\tau_p s+1}{k_p \lambda s}}$$

$$= \frac{(\tau_p s+1)}{k_p(\lambda s+1)\left[1 - \frac{\cancel{(\tau_p s+1)}}{\cancel{k_p}(\lambda s+1)} \times \frac{\cancel{k_p}(1 - t_d s/2)}{\cancel{(\tau_p s+1)}\left(1 + \frac{t_d s}{2}\right)}\right]}$$

$$g_c = \frac{(\tau_p s+1)}{k_p\cancel{(\lambda s+1)}\left[\frac{(1 + t_d s/2)(\lambda s+1) - (1 - t_d s/2)}{\cancel{(\lambda s+1)}\left(1 + \frac{t_d s}{2}\right)}\right]}$$

$$g_c = \frac{(\tau_p s+1)(1+t_d s/2)}{k_p\left[(\lambda s+1)(1+t_d s/2)-(1-\frac{t_d s}{2})\right]}$$

$$= k_c\left(1+\frac{1}{\tau_i s}+\tau_d s\right)\left(\frac{1}{\tau_f s+1}\right)$$

$k_c = ?$

$\tau_i = ?$

$\tau_D = ?$

$\tau_f = ?$

g	controller	Turning parameter
1. $\frac{k_p}{\tau_p s+1}$	PI	$k_c = \frac{\tau_p}{k_p\lambda}$; $\tau_i = \tau_p$
2. $\frac{k_p e^{-t_d s}}{\tau_p s+1}$	PID - 1st order filter	
3. $\frac{k_p}{(\tau_1 s+1)(\tau_2 s+1)}$	Ideal PID	$k_c = ?$ $\tau_i = ?$ $\tau_d = ?$

case-8:

$\Rightarrow$ $$\bar{g} = \underbrace{\frac{k_p}{(\tau_1 s+1)(\tau_2 s+1)}}_{\bar{g}_{-1}} \cdot \underbrace{1}_{\bar{g}_{+1}}$$

$$c = \frac{f}{\bar{g}_{-1}} = \frac{(\tau_1 s+1)(\tau_2 s+1)}{k_p} \times \frac{1}{(\lambda s+1)}$$

$$g_c = \frac{c}{1 - c\tilde{g}}$$

$$= \frac{(\tau_1 s+1)(\tau_2 s+1)}{k_p(\lambda s+1)^2 \left(1 - \frac{\cancel{(\tau_1 s+1)(\tau_2 s+1)}}{\cancel{k_p}(\lambda s+1)^2} \times \frac{\cancel{k_p}}{\cancel{(\tau_1 s+1)(\tau_2 s+1)}}\right)}$$

$$= \frac{(\tau_1 s+1)(\tau_2 s+1)}{k_p\left((\lambda s+1)^2 - 1\right)}$$

$$= \frac{(\tau_1 s+1)(\tau_2 s+1)}{k_p \lambda s \cancel{(\lambda s+2)}}$$

$$\cancel{= \frac{\tau_1 \tau_2 s^2 + s(\tau_1+\tau_2)+1}{k_p \lambda s \times 2\left(\frac{\lambda s}{2}+1\right)}}$$

$$= \frac{1}{k_p \lambda s}\left(\tau_1 \tau_2 s^2 + s(\tau_1+\tau_2)+1\right)$$

$$= k_c\left(\frac{\tau_i \tau_d s^2 + \tau_i s + 1}{\tau_i s}\right)$$

$$= \frac{(\tau_1+\tau_2)}{k_p \lambda}\left[\frac{\tau_1 \tau_2 s^2/(\tau_1+\tau_2) + s + \frac{1}{\tau_1+\tau_2}}{s}\right]$$

$$k_c = \frac{(\tau_1+\tau_2)}{k_p \lambda} \qquad \tau_i = \tau_1 + \tau_2 \qquad \tau_d = \frac{\tau_1 \tau_2}{\tau_1+\tau_2}$$

19/08/25.

- Model Predictive Control (MPC) { General name → Widely used in industry.

Process model :

$$\frac{dy}{dt} = f(y, u)$$

$$\frac{y_{k+1} - y_k}{\Delta t} = F[y(k), u(k)]$$

$$y(k+1) = y(k) + \Delta t F[y(k), u(k)]$$

$k \Rightarrow$ present time step

$k+1 \Rightarrow$ future time step.

- MPC use process model to predict output for a series of future time steps ($k+1, k+2, \dots k+P$). In case of MBC (model based control) we use model to predict output for a single future time step [$k+1$].

Organisation	MPC
Adersa	PFC
Aspen - Tech	DMC-plus
Shell	Shell Multivariable Organising Control - II
Honeywell	RMPCT

→ Dynamic Model Control scheme (DMC)

Models:

MPC is developed based on discrete model.

1. Step respose model

$$y(k) = \sum_{i=1}^{k} \beta(i) \underline{\Delta u(k-i)}$$

β = step response coeff.

$\Delta u(k)$ = control move = $u(k) - u(k-1)$.

↳ velocity form of control action.

2. Impulse response model / pulse response :

$$y(k) = \sum_{i=1}^{k} g(i) \underline{u(k-i)}$$

↓ impulse response coeff

Notations :

$$B = [\beta(1) \quad \beta(2) \ldots \ldots \beta(n)]^T$$

n = model horizon

$$\Delta U(k) = [\Delta u(k) \quad \Delta u(k+1) \ldots \Delta u(k+m-1)]^T$$

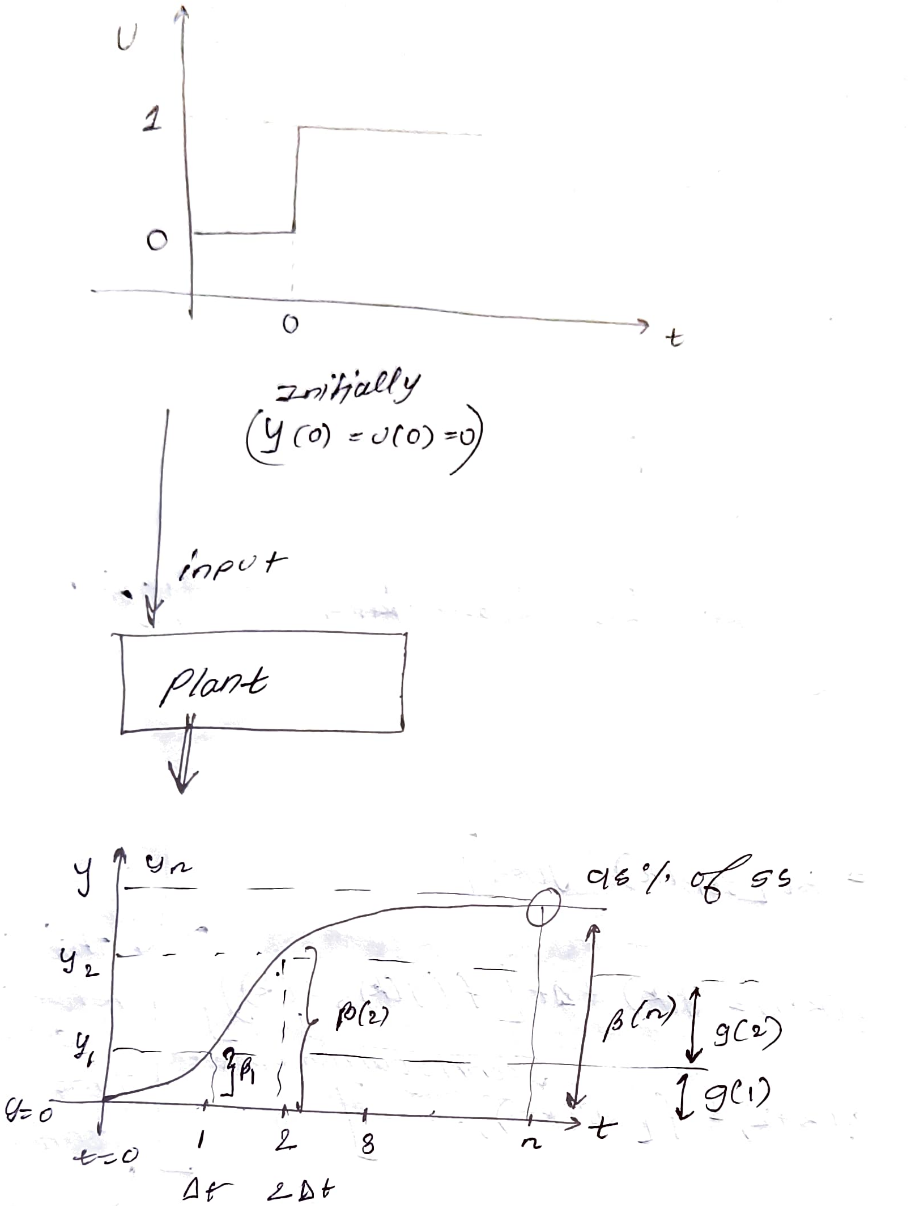
k+m-1

m = control horizon.

$$\hat{Y}(k+1) = [\hat{y}(k+1) \quad \hat{y}(k+2) \ldots \ldots \hat{y}(k+p)]$$ } predicted output

p = prediction horizon

$$\hat{Y}^o(k+1) = [\hat{y}^o(k+1) \quad \hat{y}^o(k+2) \ldots \ldots \hat{y}^o(k+p)]^T$$ } Predicted output based on past ~~data~~ control modes



$$n = \frac{\text{time req to reach } 95\% \text{ of new SS}}{\Delta t}$$

$$\beta(1) = y(1) - y(0) = y(1)$$

$$\beta(2) = y(2) - y(0) = y(2)$$

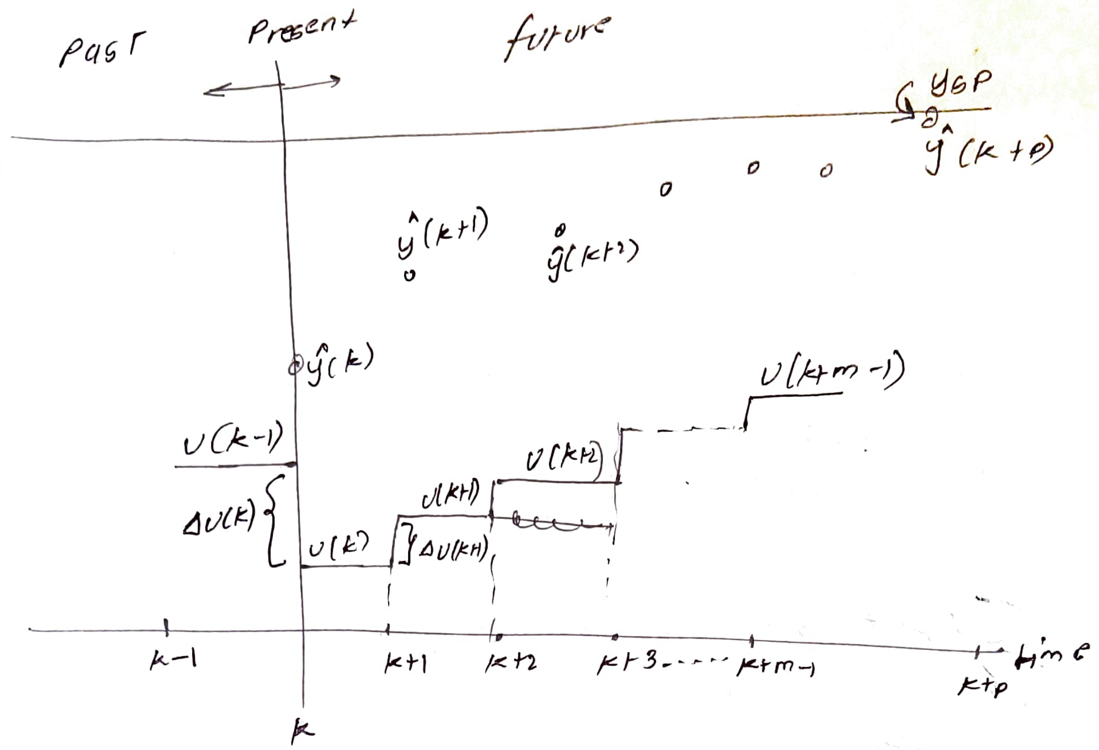
$\vdots$

$$\beta(n) = y(n) - y(0) = y(n)$$

$$\beta(n+1) = \beta(n+2) = \dots = \beta(n)$$

<u>Note:</u>

$$g(i) = \beta(i) - \beta(i-1) \quad \Rightarrow \quad g(1) = \beta(1) - \beta(0) = \beta(1)$$



P-only:

$$u(k) = K_c\left[y_{sp}(k) - y(k)\right]$$

Process:

$$y(k+1) = y(k) + \Delta t\left[f(y(k), u(k))\right]$$

control action updated

$$y(k+2) = f\left[u(k+1), \ldots\right]$$

This is ~~what~~ usually done. Time is updated continuously.

This is not the case for DMC. We stop at time step (k+m-1)

Step-response model:

$$y(k) = \sum_{i=\cancel{0}1}^{k} \beta(i)\, \Delta u(k-i)$$

$$y(k+l) = \sum_{i=\cancel{0}1}^{k+l} \beta(i)\, \Delta u(k+l-i)$$

$$y(k+l) = \underbrace{\beta(1)\Delta U(k+l-1) + \beta(2)\Delta U(k+l-2) \dots \beta(l)\Delta U(k)}_{\text{part-1}}$$

$$+ \underbrace{\beta(l+1)\Delta U(k-1) + \dots + \beta(n-1)\Delta U(k+l-n+1)}_{\text{part-2}}$$

$$+ \underbrace{\beta(n)\Delta U(k+l-n) + \dots + \beta(k+l)\Delta U(0)}_{\text{part-3}}$$

$$\therefore y(k+l) = \sum_{i=1}^{l} \beta(i)\Delta U(k+l-i) + \sum_{i=l+1}^{n-1} \beta(i) U(k+l-i)$$

$$+ \boxed{\beta(n) U(k+l-n)}$$

β does not change after nth time step.

$$\therefore \beta(n)\left[U(k+l-n) - \cancel{U(k+l-n-1)} + \cancel{U(k+l-n-1)} - \cancel{U(k+l-n-2)} + \dots - \cancel{\Delta U(0)} \right]$$

$$= \beta(n) U(k+l-n)$$

20/08/25

predicted output

$$\hat{y}(k+l) = \sum_{i=1}^{l} \beta(i)\Delta U(k+l-i) + \underbrace{\sum_{i=l+1}^{n-1} \beta(i)\Delta U(k+l-i) + \beta(n) U(k+l-n)}$$

- Models are not always accurate

$\therefore$ $\tilde{y}$ = corrected predicted y

$y = y_m$ = plant output.

$k+1-3$

2

$$l = 1 \quad \tilde{y}(k+1) / \hat{y}(k+1) = \beta(1)\Delta U(k+0) + \underbrace{\beta(2)\Delta U(k-1) + \beta(3)\Delta U(k-2)}$$

$$\underbrace{\dots + \beta(n-1)\Delta U(k+1-n+1)}$$

$$+ \underbrace{\beta(n) U(k+1-n)}.$$

Present & future ΔU

Past ΔU | $\hat{y}^{o}(k+1)$

$$+ \omega(k+1)$$

(Plant/model mismatch) correction part.

$$\hat{y}(k+1) = \beta(1)\Delta U(k) + \beta(2)\Delta U(k-1) + \beta(3)\Delta U(k-2) + \dots + \beta(k+1).\Delta U(0)$$

$\cancel{\Delta U(k-1)}$ $\cancel{\Delta U(k+1)}$ $\cancel{\beta(k+1)}$ $\cancel{\Delta U(k-k+1)}$ $\cancel{\Delta U(-1)}$

$$\hat{y}(k+1) = \hat{y}^{0}(k+1) + \beta(1)\Delta U(k)$$

$l = 2$

$$\tilde{y}(k+2) = \beta(1)\Delta U(k+1) + \beta(2)\Delta U(k) + \hat{y}^{0}(k+2)$$

$$= \underbrace{\hat{y}^{0}(k+2)}_{+} + \underline{\beta(2)\Delta U(k) + \beta(1)\Delta U(k+1)} + w(k+2)$$

$l = m$

$+m-m+1$

$$\tilde{y}(k+m) = \hat{y}^{0}(k+m) + \beta(m)\Delta U(k\cancel{+m-m+1}) + \beta(m-1)\Delta U(k+1)$$
$$\dots + \beta(2)\Delta U(k+m-2) + \beta(1)\Delta U(k+m-1)$$
$$+ w(k+m)$$

$$\tilde{y}(k+m+1) = \hat{y}^{0}(k+m) + \beta(m+1)\Delta U(k) + \beta(m)\Delta U(k+1)$$
$$\dots + \beta(2)\Delta U(k+m-1) + \beta(1)\cancelto{0}{\Delta U(k+m)}$$
$$+ w(k+m+1)$$

$\vdots$

$$\tilde{y}(k+P) = \hat{y}^{0}(k+P) + \beta(P)\Delta U(k) + \beta(P-1)\Delta U(k+1)$$
$$\dots \cancel{+\beta(2)\Delta U(k+P-2) + \beta(1)\Delta U(k+P-1)}$$
$$+ \beta(P-m+1)\Delta U(k+m-1) + w(k+P).$$

$$\boxed{\tilde{y}(k+1) = \hat{y}^{o}(k+1) + B\Delta U(k) + w(k+1)}$$

$$\Delta U = \begin{bmatrix} \Delta u(k) \\ \Delta u(k+1) \\ \vdots \\ \Delta u(k+m-1) \end{bmatrix}_{m \times 1}$$

$$B = \begin{bmatrix} \beta(1) & 0 & \cdots & 0 \\ \beta(2) & \beta(1) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \beta(p) & \beta(p-1) & \cdots & \beta(p-m+1) \end{bmatrix}_{p \times m}$$

$B \Rightarrow$ Dynamic Matrix

$$\hat{y}^{o}(k+1) = \underbrace{\begin{bmatrix} \quad \end{bmatrix}}_{\sum_{i=l+1}^{n-1}} + \beta(n)\underbrace{\begin{bmatrix} \quad \end{bmatrix}}_{\beta(n)u(k+l-n)}$$

→ Controller target / Designing controller

$$\tilde{y}(k+1) = \hat{y}^{o}(k+1) + B\Delta U(k) + w(k+1) = y_{sp}(k+1) = y^{*}(k+1)$$

$$B\Delta U(k) = y^{*}(k+1) - \left[\hat{y}^{o}(k+1) + w(k+1)\right] = E(k+1)$$

= projected error.

$$\beta(\Delta U(k)) = E(k+1).$$

DOF: $(f = V - E)$

$f = 0$ $\quad V = E$ $\quad$ specified system

$f > 0$ $\quad V > E$ $\quad$ underspecified $\xrightarrow{sol^n}$. Optimization

$f < 0$ $\quad V < E$ $\quad$ Over-specified $\xrightarrow{sol^n}$ Least-square

$$\underset{m \times 1}{B \, \Delta U(k)} = \underset{p \times 1}{E(k+1)} \qquad p > m \Rightarrow f < 0.$$

$$\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2} \quad \text{.... Euclidean Norm}$$

$$\| E(k+1) - B \Delta U(k) \|$$

$$\underset{\Delta U(k)}{\mathrm{Min}(\phi)} = \left[E(k+1) - B \Delta U(k) \right]^T \left[E(k+1) - B \Delta U(k) \right]$$

minimise error

$$\frac{\partial \phi}{\partial (\Delta U(k))} = 0$$

$$\cancel{2B} - B^T \left[E(k+1) - B \Delta U(k) \right] = 0$$

$$B^T B \Delta U(k) = B^T E(k+1)$$

$$\Delta U(k) = (B^T B)^{-1} B^T E(k+1)$$

$(B^T B)$ = nonsingular matrix

Modify

$$= (B^T B + k^2 I)^{-1} B^T E(k+1)$$

identity matrix

move suppression matrix

Objective function:

$m \times p \quad p \times m$

$m \times m$

$$\min_{\Delta U(k)} \phi = \left[E(k+1) - B\Delta U(k)\right]^T \left[E(k+1) - B\Delta U(k)\right]$$

$$+ k^2 (\Delta U(k))^T \Delta U(k)$$

26/08/25

$$\tilde{Y}(k+1) = \underbrace{\hat{Y}^0(k+1) + \underset{p \times m}{B}\Delta U(k)}_{\hat{Y}(k+1)} + w(k+1)$$

$$\Delta U(k) = (B^T B)^{-1} B^T E(k+1)$$

$$= (\underset{m \times p}{B^T}\underset{p \times m}{B} + \underbrace{k^2 I}_{m \times m})^{-1} B^T E(k+1) \text{ --- penalty against excessive control action}$$

$$\boxed{\begin{array}{l} \bullet \text{ Unconstrained DMC} \\ \bullet \text{ Developed for SISO system} \end{array}}$$

↓ formulate DMC for

- Contrained DMC
- Develop DMC for MIMO system

→ Calculate w (Plant/model mismatch)

Unmodeled Disturbance

$$W(k+1) = \left[w(k+1) \quad w(k+2) \; \cdots\cdots \; w(k+p)\right]^T$$

$w(k+1) = y_m(k+1) - \hat{y}(k+1)$ .... y_m cannot be known at future time step.

$w(k+1) = y_m(k) - \underbrace{\hat{y}(k+1)}_{\text{not known}}$ ... implicit eqn.

↓

$w(k+1) = y_m(k) - \hat{y}(k)$

$w(k+1) = y_m(k) - \hat{y}(k|k-1) = y_m(k|k) - \hat{y}(k|k-1)$

predicted output @ kth time step w.r.t k-1 th

$w(k+i) = y_m(k) - \hat{y}(k|k-i)$ $\quad i = 1, \ldots, p.$

- Heuristic tunning Guidelines:

· Tunning parameter – $k^2, P, n, m, \underbrace{\Delta t}_{\text{sampling time}}$.

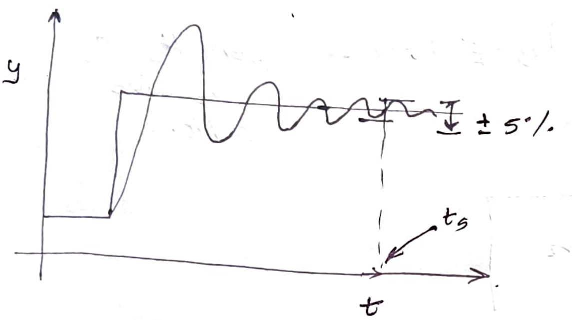
$\Delta t \downarrow$ we get better performance but computational load increa

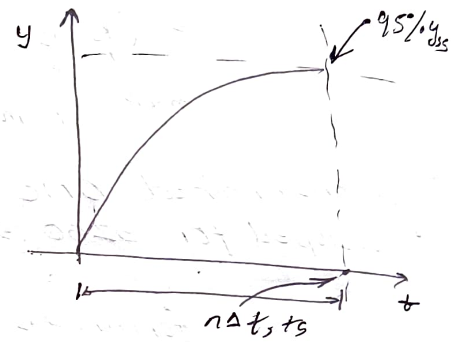
(1) $\Delta t = \frac{\tau_p}{10}$ ... generally

for multiple processes in series: $\Delta t = \frac{\tau_{p_1} + \tau_{p_2} + \tau_{p_3} \dots + \tau_{p_\alpha}}{\cancel{\alpha} \times 10}$

(2) select ~~...~~ n

$n\Delta t = t_s$ (settling time)





Range: $20 \le n \le 100$

Note: $n > P > m$ or $P > n > m$ [m must be smallest]

(3) $15 \le P \le 20$

(4) $5 \le m \le 15$ or $n/3 \le m \le n/2$

(5) $K^2 \equiv$ diagonal matrix b/w 1–10

(6) $\Gamma \equiv$ diagonal matrix b/w 1–100.

· DMC output: $\Delta U(k) = [\Delta U(k) \;\; \Delta U(k+1) \; \dots \dots \; \Delta U(k+m-1)]^T$

$$G_p = \frac{k_p}{\tau_p s + 1} = \frac{y}{U} \Rightarrow \tau_p \frac{dy}{dt} + y = k_p \, U$$

$$\tau_p\left(\frac{y(k) - y(k-1)}{\Delta t}\right) + y(k-1) = k_p U(k-1)$$

$$y(k) = \frac{1}{\tau_p}\left[(\tau_p - \Delta t)\, y(k-1) + k_p \Delta t\, U_{(k-1)}\right]$$

$$y(k+1) = \frac{1}{\tau_p}\left[(\tau_p - \Delta t)\, y(k) + k_p \Delta t\, U(k)\right]$$

$\Delta U(k) = U(k) - U(k-1)$

$U(k) = U(k-1) + \Delta U(k)$

known   known from DMC

• MIMO DMC controller:

$$E(k+1) = B \Delta U(k)$$

B_{ij} $i \to$ output
$j \Rightarrow$ input

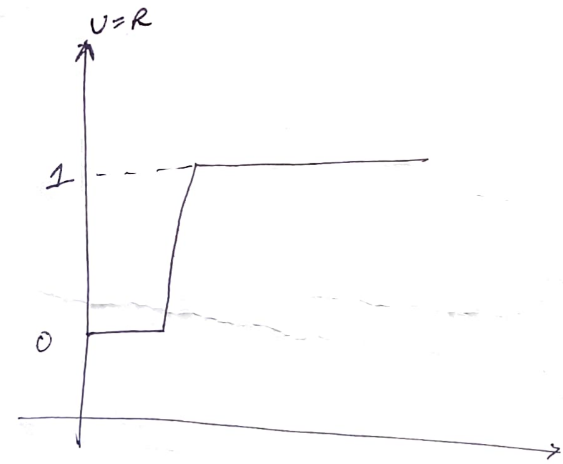
for a 2×2 system (2 input & 2 output)

$$\begin{bmatrix} E_1(k+1) \\ E_2(k+1) \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} \Delta U_1(k) \\ \Delta U_2(k) \end{bmatrix}$$

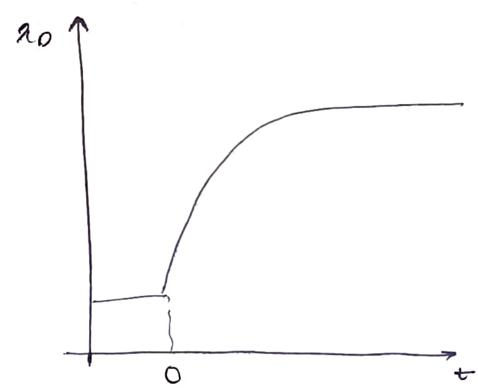
ex Distillation column:

CV	MV
x_D $i=1$	R $j=1$
x_B $(i=2)$	V_B $(j=2)$

Input	Output	B_{ij}
$R(j=1)$	$x_D(i=1)$	B_{11}
	$x_B(i=2)$	B_{21}

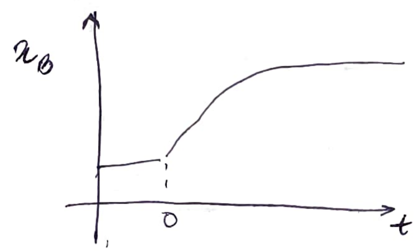


Change in R causes x_B to change as well



similarly,

$V_B(j=2)$	$x_B(i=1)$	B_{12}
	$x_B(i=2)$	B_{22}



$$\overset{\text{weight}}{S}\,E(k+1) = SB\,\Delta U(k)$$

$$\min_{\Delta U(k)} \phi \quad [E(k+1) - B\Delta U(k)]^T S^T S\,[E(k+1) - B\Delta U(k)] + k^2 \Delta U^T(k)\,\Delta U(k)$$

$$\frac{\partial \phi}{\partial \Delta U_k} = 0 \longrightarrow \Delta U(k) = (\underbrace{B^T}_{m\times p}\underbrace{\Gamma}_{p\times p}\underbrace{B}_{p\times m} + k^2 I)^{-1} B^T \Gamma\, E(k+1)$$

$\Gamma = S^T S.$

dimensions = $p \times p$

Note:

$$k^2 I = \begin{bmatrix} k_1^2 & 0 & 0 & \cdots & 0 \\ \vdots & k_1^2 & \ddots & & \\ 0 & & & & \\ \hline 0 & 0 & k_2^2 & \cdots & 0 \\ \vdots & & & k_2^2 & \ddots \; k_2^2 \\ 0 & \cdots & & & \end{bmatrix}$$

AKJ notes: 27/08/25.

- Constrained DMC / Quadratic DMC (QDMC) ↙ MIMO

OF: $\min_{\Delta U(k)} \phi = [E(k+1) - B\Delta U(k)]^T \Gamma^T [E(k+1) - B\Delta U(k)]$

$+ \underline{k^2 \Delta U(k)^T \Delta U(k)}$

penalty against excessive control action.

Constraints:

① on MV

$U_{min} \leq U(k+j) \leq U_{max}$

$j = 0, 1, \ldots, m-1$

② On ΔU
[control move]

$\Delta U_{min} \leq \Delta U(k+j) \leq \Delta U_{max}$

$j = 0, 1, 2, \ldots, m-1$

③ on y
[CV]

$\cancel{\tilde{y}_{min} \leq \tilde{y}(k+j) \leq \tilde{y}_{max}}$ $\quad y_{min} \leq \tilde{y}(k+j) \leq y_{max}$

$j = 1, 2, \ldots p$

Linear programing → Deals with L OF & L constraints

NLP → Deals with NL OF & either L or both (NL+L) contraints

Quadratic P → Deals with both L & NL OF

- Constraints on MV (Represent in terms of ΔU)

$U_{min} \leq U(k+j)$ $\quad j = 0, 1, \ldots, m-1$ $\quad U(k+j) \leq U_{max}$

$U(k+j) \geq U_{min}$

$U(k-1) + \sum_{j=0}^{m-1} \Delta U(k+1) \geq U_{min}$

$j=0$: $\quad U(k-1) + \Delta U(k) = U(k)$

$j=1$: $\quad U(k-1) + \Delta U(k) + \Delta U(k+1) = U(k+1)$ $\cancel{+U(k)}$

→

$- U(k-1) - \sum_{j=0}^{m-1} \Delta U(k+1) \leq - U_{min}$ $\quad$ —①

$$-\sum_{j=0}^{m-1} \Delta u(k+j) \leq -u_{min} + u(k-1) \quad \text{—} ①$$

for $u(k+j) \leq u_{max}$,

$$u(k-1) + \sum_{j=0}^{m-1} \Delta u(k+1) \leq u_{max}$$

$$\sum_{j=0}^{m-1} \Delta u(k+1) \leq u_{max} - u(k-1) \quad \text{—} ②$$

- combining ① & ② in matrix form:

$$\underbrace{\begin{bmatrix} -1 & 0 & \cdots & 0 & \cdots & 0 \\ -1 & -1 & \cdots & & \cdots & 0 \\ \vdots & & & & & \vdots \\ -1 & -1 & \cdots & & \cdots & -1 \\ 1 & 0 & \cdots & & \cdots & 0 \\ 1 & 1 & \cdots & & \cdots & 0 \\ \vdots & & & & & \vdots \\ 1 & 1 & \cdots & & \cdots & 1 \end{bmatrix}_{2m \times m} \begin{bmatrix} \Delta u(k) \\ \Delta u(k+1) \\ \vdots \\ \Delta u(k+m-1) \end{bmatrix}_{m \times 1} \leq \begin{bmatrix} -u_{min} + u(k-1) \\ \vdots \\ -u_{min} + u(k-1) \\ u_{max} - u(k-1) \\ \vdots \\ u_{max} - u(k-1) \end{bmatrix}_{2m \times 1}}_{\text{for SISO.}}$$

- Matrix form for MIMO (2×2) system

$$\begin{bmatrix} -1 & 0 & \cdots & 0 & \cdots & 0 \\ -1 & -1 & \cdots & & \cdots & 0 \\ \vdots & & & & & \vdots \\ -1 & -1 & \cdots & & \cdots & -1 \\ \hline 1 & 0 & \cdots & & \cdots & 0 \\ 1 & 1 & \cdots & & \cdots & 0 \\ \vdots & & & & & \vdots \\ 1 & 1 & \cdots & & \cdots & 1 \end{bmatrix}_{2m_1 \times m_1} \begin{bmatrix} \Delta u_1(k) \\ \Delta u_1(k+1) \\ \vdots \\ \Delta u_1(k+m_1-1) \end{bmatrix}_{m_1 \times 1} \leq \begin{bmatrix} -u_{1min} + u_1(k-1) \\ \vdots \\ -u_{1min} + u_1(k-1) \\ \hline u_{1max} - u_1(k-1) \\ \vdots \\ u_{1max} - u_1(k-1) \end{bmatrix}_{2m_1 \times 1}$$

for $i=2$

$$\begin{bmatrix} -1 & 0 & \cdots & 0 \\ -1 & -1 & \cdots & 0 \\ \vdots & & & \\ -1 & -1 & \cdots & -1 \\ \hline 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & & & \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{2m_2 \times m_2} \begin{bmatrix} \Delta U_2(k) \\ \Delta U_2(k+1) \\ \vdots \\ \Delta U(k+m_2-1) \end{bmatrix}_{m_2 \times 1} \leq \begin{bmatrix} -U_2 min + U_2(k-1) \\ \vdots \\ -U_2 min + U_2(k-1) \\ \hline U_2 max - U_2(k-1) \\ \vdots \\ \end{bmatrix}_{2m_2 \times 1}$$

overall -

$$\begin{bmatrix} -I & 0 & \cdots & 0 \\ -I & -I & \cdots & 0 \\ \vdots & & & \\ -I & -I & \cdots & -I \\ \hline I & 0 & \cdots & 0 \\ I & I & \cdots & 0 \\ \vdots & & & \\ I & \cdots I & \cdots & I \end{bmatrix}_{2(m_1+m_2)\times(m_1+m_2)} \begin{bmatrix} \Delta U(k) \\ \vdots \\ \Delta U(k+M-1) \end{bmatrix}_{(m_1+m_2)\times 1} \leq \begin{bmatrix} -U min + U(k-1) \\ \vdots \\ \hline U max - U(k-1) \\ \vdots \end{bmatrix}_{2(m_1+m_2)\times 1}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Delta U(k) = \begin{bmatrix} \Delta U_1(k) \\ \Delta U_2(k) \end{bmatrix}$$

• Constraint on ΔU [already in form of ΔU]

• Constraints of CV

$$y_{min} \le \tilde{y}(k+j) \qquad j = 1, 2, \cdots, P$$

$$Y_{min} \le \tilde{Y}(k+j) = \hat{Y}^{o}(k+1) + B\Delta U(k) + W(k+1)$$

$$B\Delta U(k) \ge Y_{min} - \hat{Y}^{o}(k+1) - W(k+1)$$

$$-B\Delta U(k) \le -Y_{min} + \hat{Y}^{o}(k+1) + W(k+1)$$

• MIMO (2×2) system

$$\underbrace{\begin{bmatrix} \overset{P_1 \times m_1}{-B_{11}} & \overset{P_1 \times m_2}{-B_{12}} \\ \underset{P_2 \times m_1}{-B_{21}} & \underset{P_2 \times m_2}{-B_{22}} \end{bmatrix}}_{(P_1+P_2)\times(m_1+m_2)} \underbrace{\begin{bmatrix} \Delta u_1(k) \\ \Delta u_1(k+1) \\ \vdots \\ \Delta u_1(k+m_1-1) \\ \Delta u_2(k) \\ \Delta u_2(k+1) \\ \vdots \\ \Delta u_2(k+m_2-1) \end{bmatrix}}_{(m_1+m_2)\times 1} \le \underbrace{\left[\begin{array}{l} -y_{min} + \hat{y}^{o}(k+1) + w_1(k+1) \\ \vdots \\ -y_1 min + \hat{y}^{o}(k+P_1) + w_1(k+P_1) \\ \hline -y_2 min + \hat{y}_2^{o}(k+1) + w_2(k+1) \\ \vdots \\ -y_2 min + \hat{y}_2^{o}(k+P_2) + w_2(k+P_2) \end{array}\right]}_{(P_1+P_2)\times 1}$$

for $\tilde{y}(k+j) \le y_{max}$

$$\tilde{Y}(k+j) = \hat{Y}^{o}(k+1) + B\Delta U(k) + W(k+1) \le Y_{max}$$

$$B\Delta U(k) \le Y_{max} - \hat{Y}^{o}(k+1) - W(k+1)$$

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} \Delta u_1(k) \\ \Delta u_1(k+1) \\ \vdots \\ \Delta u_1(k+m_1-1) \\ \hline \Delta u_2(k) \\ \Delta u_2(k+1) \\ \vdots \\ \Delta u_2(k+m_2-1) \end{bmatrix} \le \left[\begin{array}{l} y_{1max} - \hat{y}_1^{o}(k+1) - w_1(k+1) \\ \vdots \\ y_{1max} - \hat{y}_1^{o}(k+P_1) - w_1(k+P_1) \\ \hline y_{2max} - \hat{y}_2^{o}(k+1) - w_2(k+1) \\ \vdots \\ y_2(max) - \hat{y}_2^{o}(k+P_1) - w_2(k+P_2) \end{array}\right]$$

2/04/25

DMC : Ex. find $\Delta U(k)$ @ $t = 1, 2$ min with $y_{sp} = 1$ step change

$$G_p = \frac{1}{5s+1} = \frac{\bar{y}}{\bar{U}} = \frac{k_p}{\tau_p s + 1} \qquad k_p = 1, \quad \tau_p = 5 \text{ min} \quad \text{--- plant}$$

Process is initially @ ss.

O & I are deviation variables

$\therefore \; y(0) = U(0) = 0$

$n = 6 \quad p = 5 \quad m = 1 \qquad \Delta t = 1 \text{min}, \; k^2 = 0$

$$\Delta U(k) = \cancel{\sum_{i=1}^{k} B(i)\Delta U(k)} \cdot \underbrace{[B^T B + \cancel{k^2} I)^{-1} B^T}_{r} E(k+1) \text{ --- DMC}$$

$$\Delta U(k) = r E(k+1)$$

Step-1: find r

$B \Rightarrow ?$

· introduce step change in U

$$\bar{y} = \frac{k_p}{\tau_p s + 1} \times \frac{1}{s}$$

$$\bar{y} = \frac{k_p}{\tau_p(s + 1/\tau_p)} \times \frac{1}{s}$$

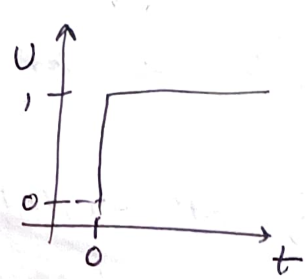
$$\bar{y} = \frac{k_p}{\cancel{\tau_p}}\left[-\frac{1}{s + 1/\tau_p} + \frac{1}{s}\right] \times \cancel{\tau_p}$$

$$y = k_p - k_p e^{-t/\tau_p}$$

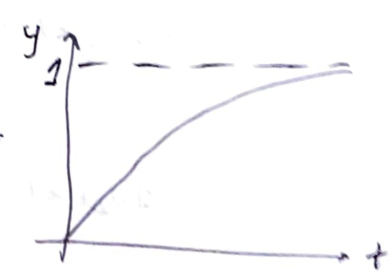
$$\boxed{y = 1 - e^{-t/5}}$$

Discretising

$$y(i\Delta t) = 1 - e^{-\frac{i\Delta t}{5}}$$



$\longrightarrow$ Plant $G_p = \frac{1}{5s+1}$ $\longrightarrow$



$$y(k+l) = \sum_{i=1}^{k+l} \beta(i)\, \Delta u(k+l-i)$$

$i=1$ $\quad \beta(1) = 0.181$

$i=2$ $\quad \beta(2) = 0.3294 = 0.33$

$i=3$ $\quad \beta(3) = 0.451$

$i=4$ $\quad \beta(4) = 0.551$

$i=5$ $\quad \beta(5) = 0.632$

$$\begin{bmatrix} 0.181 \\ 0.33 \\ 0.451 \\ 0.551 \\ 0.632 \end{bmatrix}_{\substack{p \times m \\ 5 \times 1}} = B$$

$i=6$ $\quad \beta(6) = 0.699$

$$\gamma = (B^T B)^{-1} \times B^T$$

$$\gamma = \left(\begin{bmatrix} 0.181 & 0.33 & 0.451 & 0.551 & 0.632 \end{bmatrix} \begin{bmatrix} 0.181 \\ 0.33 \\ 0.451 \\ 0.551 \\ 0.632 \end{bmatrix} \right)^{-1} B^T$$

$$\gamma = \frac{1}{[1.048]} \begin{bmatrix} 0.181 & 0.33 & 0.451 & 0.551 & 0.632 \end{bmatrix}$$

$$\gamma = \cancel{\begin{bmatrix} 0.19 & 0.05 & 0.082 & 0.0997 & 0.114 \end{bmatrix}}$$

$$\begin{bmatrix} 0.173 & 0.314 & 0.43 & 0.525 & 0.603 \end{bmatrix}$$

Step-2: $\quad E(k+1) = Y^*(k+1) - \left[\hat{Y}(k+1) + \omega(k+1)\right]$

$$Y^*(k+1) = \begin{bmatrix} y^*(k+1) & y^*(k+2) & \cdots & y^*(k+p) \end{bmatrix}^T$$

$p=5$ $$= \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \end{bmatrix}^T$$

$$\therefore \quad = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$$

$\hat{y}^{o}(k+1)$ = Based on passed ΔU

$\Delta U(t) = 0 \qquad t \leq 0.$

$\hat{Y}^{o}(k+1) = \left[\hat{y}^{o}(k+1) \quad \hat{y}^{o}(k+2) \quad \cdots\cdots \quad \hat{y}^{o}(k+P)\right]^T$

(prediction based on past)

$= \left[0 \quad 0 \quad 0 \quad 0 \quad 0\right]^T$

$w(k+1) = y_m(k) - \hat{y}(k)$

y_m can be found from plant:

$G_p = \dfrac{1}{5s+1} = \dfrac{y}{U} \qquad \tau_p \dfrac{dy}{dt} + y = k_p U$

$\tau_p \left[\dfrac{y(k) - y(k-1)}{\Delta t}\right] + y(k-1) = k_p U(k-1)$

$y(k) = \dfrac{1}{\tau_p}\left[(\tau_p - \Delta t)\, y(k-1) + k_p \Delta t\, U(k-1)\right]$

$y(k) = \dfrac{1}{5}\left[4y(k-1) + U(k-1)\right]$

$y(k=2) = \frac{1}{5}\left[4 \times 0 + 2.045\right]$

$y(1\,min) = \frac{1}{5}\left[4 \times 0 + 0\right] = 0$

$\hat{y}(k|k-1) = 0.$

$\therefore\ w(k+1) = 0.$

$E(k+1) = Y^{*}(k+1) = \left[1 \quad 1 \quad 1 \quad 1 \quad 1\right]^T$

@ t=1min

$\Delta U(k) = \gamma E(k+1)$

$= \begin{bmatrix} 0.173 & 0.314 & 0.43 & 0.525 & 0.603 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$= 2.045$

$U(k) = \Delta U(k) + U(k-1) = 2.045$

$\tilde{Y}(k+1) = \hat{Y}^{0}(k+1) + \beta \Delta U(k) + w(k+1)$

$\hat{Y}(k+1) = \cancel{\hat{Y}^{0}(k+1)}^{\,0} + \beta \Delta U(k)$

$= 2.045 \begin{bmatrix} 0.181 \\ 0.33 \\ 0.451 \\ 0.551 \\ 0.632 \end{bmatrix}$

$= \begin{bmatrix} 0.3701 \\ 0.675 \\ 0.922 \\ 1.127 \\ 1.292 \end{bmatrix}$ $\begin{matrix} \hat{y}(k+1) \\ \hat{y}(k+2) \rightarrow \hat{y}(3) \end{matrix}$

@ t=2min

$\hat{y}(k+l) = \sum_{i=1}^{l} \beta(i) \Delta U(k-i) + \sum_{i=l+1}^{n-1} \beta(i) \Delta U(k-i) + \beta(n) \Delta U(k+l-n)$

$l=1$

$\hat{y}(k+1) = \beta(1) \Delta U(k-1) + \underbrace{\beta(2) \Delta U(k-2) + \beta(3) \Delta U(k-3) + \cdots \beta(5) \Delta U(k-5) + \beta(6) \Delta U(k-5)}_{\hat{y}^{0}(k+1)}$

$\hat{Y}^{0}(k+1) = \beta \Delta U_{past} + \beta(n) U_{past}$

$$\beta_{past} = \begin{bmatrix} \beta(2) & \beta(3) & \beta(4) & \beta(5) \\ \beta(3) & \beta(4) & \beta(5) & 0 \\ \beta(4) & \beta(5) & 0 & 0 \\ \beta(5) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{p\times(n-2)}$$

$$\Delta U_{past} = \begin{bmatrix} \Delta U(k-1) = 2.046 \\ \Delta U(k-2) = 0 \\ \Delta U(k-3) = 0 \\ \Delta U(k-4) = 0 \end{bmatrix}_{(n-2)\times 1}$$

$\beta(n) = 0.699$

$$U_{past} = \begin{bmatrix} U(k-5) = 0 \\ U(k-4) = 0 \\ U(k-3) = 0 \\ U(k-2) = 0 \\ U(k-1) = 2.046 \end{bmatrix}_{p\times 1}$$

$$\hat{y}^{0}(k+1) = \beta_{past}\,\Delta U_{past} + \beta(n) U_{past}.$$

$$w(k+1) = y_m(k) - \hat{y}(k)$$

$$= 0.409 - 0.37$$

$$w(k+2) = 0.409 - 0.675$$

$\vdots$

$$w(k+5) = 0.409 - 1.292.$$

$$W(k+1) = \begin{bmatrix} w(k+1) & w(k+2) & w(k+3) & w(k+4) & w(k+5) \end{bmatrix}$$

$\Delta U (t=2min) = ?$

$w(k+1) = 0.089 \quad = y_m(k) - \hat{y}(k)$

$w(k+2) = \cancel{0.266}\ 0.409$

$w(k+3) = \cancel{0.513}\ 0.409$

$w(k+4) = \cancel{0.718}\ 0.409$

$w(k+5) = \cancel{0.883}\ 0.409.$

$w(k+i)$

$= y_m(k) - \hat{y}(k/k-i)$

$$E(k+1) = Y^*(k+1) - \left(\hat{Y}^o(k+1) + W(k+1)\right)$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} \hat{y}^o(k+1) \\ \hat{y}^o(k+2) \\ \hat{y}^o(k+3) \\ \hat{y}^o(k+4) \\ \hat{y}^o(k+5) \end{bmatrix} + \begin{bmatrix} 0.039 \\ 0.409 \\ 0.409 \\ 0.409 \\ 0.409 \end{bmatrix} \right)$$

$$\hat{Y}^o(k+1) = \underbrace{\begin{bmatrix} 0.33 & 0.451 & 0.551 & 0.632 \\ 0.451 & 0.551 & 0.632 & 0 \\ 0.551 & 0.632 & 0 & 0 \\ 0.632 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{5\times4} \underbrace{\begin{bmatrix} 2.046 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{4\times1} + B(6) U_{past}$$

$$= \underbrace{\begin{bmatrix} 0.675 \\ 0.923 \\ 1.127 \\ 1.293 \\ 0 \end{bmatrix}}_{5\times1} + 0.699 \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2.046 \end{bmatrix}}_{5\times1}$$

$$= \underbrace{\begin{bmatrix} 0.675 \\ 0.923 \\ 1.127 \\ 1.293 \\ 1.43 \end{bmatrix}}_{5\times1}$$

$$E(k+1) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 0.675 \\ 0.923 \\ 1.127 \\ 1.293 \\ 1.43 \end{bmatrix} + \begin{bmatrix} 0.039 \\ 0.409 \\ 0.409 \\ 0.409 \\ 0.409 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0.286 \\ -0.332 \\ -0.536 \\ -0.702 \\ -0.839 \end{bmatrix}$$

$$\Delta u(k) = \gamma E(k+1) = -1.1597$$

• 8/04/25 Generic Model Control (GMC)

- linear / non-linear control
- simplest non-linear control scheme

state-space form of non-linear system:

$$\frac{dx}{dt} = f(x, \overset{\text{MV+DV}}{\downarrow}{u})$$

$y = h(x)$ ... CV

Alternative way to represent

$\dot{x} = f(x) + g(x)u \leftarrow$ MV + DV

$y = h(x)$ ---- CV

$\dot{x} = f(x, \underset{\text{MV}}{u}, \underset{\text{DV}}{d}, \theta, t)$

Ref trajectory

$$\frac{dy_r}{dt} = k_1(y_{sp} - y) + k_2\int(y_{sp} - y)dt \quad — (1)$$

Target: to achieve $y = y_r$

$$s\bar{Y}_r = k_1(\bar{y}_{sp} - \bar{y}) + k_2\frac{1}{s}(\bar{y}_{sp} - \bar{y}) \quad — (2)$$

$$\bar{y} = \frac{k_1 s + k_2}{s^2 + k_1 s + k_2}\bar{y}_{sp} \quad \text{----- CLTF} \quad — (3)$$

$\frac{dx}{dt} = f(x, u, d)$ --- SISO case

$y = x$

$$\therefore \frac{dy}{dt} = \frac{dx}{dt} = f(x, u, d) \quad — (4)$$

$y \equiv$ CV $u \equiv$ MV $d \equiv$ DV

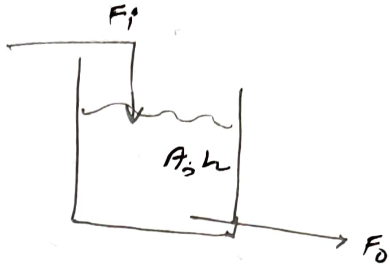
Now,
$y = y_r$

$\therefore$ substituting (2) in (1),

$f(y, u, d) = k_1 e + \underbrace{k_2\int e\,dt}_{\text{(integral action 2)}} \Rightarrow$ find u from this, it will give GMC design eqn.

--- (5)

Ex - liq tank system :



$CV \Rightarrow h$

$MV = F_i$

$$\rho F_i - \rho F_o = \frac{d(hA\rho)}{dt}$$

$$F_i - F_o = A\frac{dh}{dt} \quad \text{---}$$

$$\frac{dh}{dt} = \frac{F_i - F_o}{A} \quad ....$$

Substitute above in eqn (5),

$$\frac{F_i - F_o}{A} = k_1 e + k_2 \int e\,dt \qquad e = h_{sp} - h$$

$$F_i = F_o + A\left[k_1 e + k_2 \int e\,dt\right] \quad \text{--- GMC eqn}$$

Properties :

1. Directly incorporate non-linear model.

ex -

$$A\frac{dh}{dt} = F_i - \beta\sqrt{h} \quad \text{.....} \quad \text{non-linear eqn}$$

$$\frac{dh}{dt} = \frac{F_i - \beta\sqrt{h}}{A}$$

$$F_i = \beta\sqrt{h} + A\left[k_1 e + k_2 \int e\,dt\right]$$

2. I action secures additional compensation for inevitable modelling error.

3. k_1 & k_2 are tunning parameters & are is easy to tune

4. Provides better control than PID

GMC ≡ P-family controllers under certain conditions

$$\frac{dy_r}{dt} = k_1 e + k_2 \int e\,dt \quad ; k_2 = 0$$

$$\frac{dy_r}{dt} = k_1 e$$

$$y_r = k_1 \int e\,dt$$

1st order process:

$$G_P = \frac{k_P}{\tau_P s + 1}$$

$$\tau_P \frac{dy}{dt} + y = k_P f(t) \text{ or } U$$

$y = y_r$

$$\therefore \quad \tau_P k_1 e + k_1 \int e\,dt = k_P U$$

$$\underbrace{U = \frac{\tau_P k_1 e}{k_P} + \frac{k_1}{k_P}\int e\,dt = \frac{k_1 \tau_P}{k_P}\left[e + \underbrace{\frac{1}{\tau_P}}_{I_1}\int e\,dt\right]}_{\text{PI controller}}$$

$$K_c = \frac{k_1 \tau_P}{k_P} \qquad \tau_i = \tau_P$$

Eq-2

$$G_P = \frac{k_P}{\tau^2 s^2 + 2\varphi s \tau + 1}$$

$$k_2 = 0$$

$$\frac{dy_r}{dt} = k_1 e \quad \Big| \quad y = y_r.$$

$$\bar{y}\,\tau^2 s^2 + \bar{y}\,2\varphi s\tau + \bar{y} = k_P \bar{U}$$

$$\tau^2\left[\frac{d^2 y}{dt^2}\right] + 2\varphi\tau\frac{dy}{dt} + y = k_P U$$

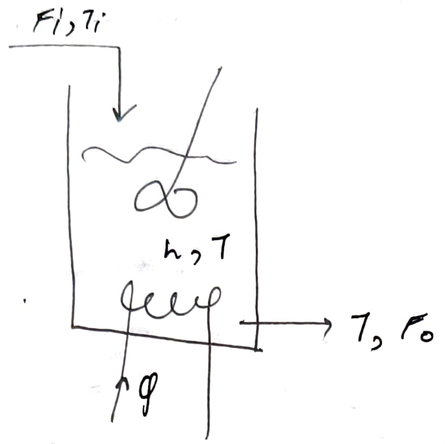
$$\tau^2 k_1 \frac{de}{dt} + 2\varphi\tau k_1 e + \int k_1 e\,dt = k_P U$$

$$U = \frac{2\varphi\tau k_1 e}{K_P} + \frac{k_1}{k_P}\int e\,dt + \frac{\tau^2 k_1}{k_P}\cancel{\int e\,dt}\frac{de}{dt}$$

$$v = \frac{2\varphi\tau k_1}{k_P}\left[e + \frac{1}{2\varphi\tau}\int e\,dt + \frac{\tau}{2\varphi}\frac{de}{dt}\right] \quad \text{--- PID controller}$$

$$k_c = \frac{2\varphi\tau k_1}{k_P} \qquad \tau_i = 2\varphi\tau \qquad \tau_D = \frac{\tau}{2\varphi}$$

eg -



$$\cancel{F_i - F_o = A\frac{dh}{dt}} \rightarrow 0 \quad - (1)$$

$$F_i \rho C_p T_i - F_o \rho C_p T + \varphi = \frac{d(\rho C_p T h A)}{dt}$$

$$F_i T_i - F_o T + \frac{\varphi}{\rho C_p} = \frac{A\,d(Th)}{dt} \quad - (2)$$

const h ; $V = hA$

$$\therefore \quad F_i T_i - F_o T + \frac{\varphi}{\rho C_p} = \frac{V\,dT}{dt}$$

for const V $\quad F_i(t) = F_o(t)$ --- neglecting dynamics

$$\frac{V\,dT}{dt} = F_i(T_i - T) + \frac{\varphi}{\rho C_p}$$

$$\frac{dT}{dt} = \frac{F_i}{V}(T_i - T) + \frac{\varphi}{\rho C_p V}$$

CV = T

MV = φ

$$\frac{dT}{dt} = \frac{F_i}{V}(T_0 - T) + \frac{Q}{\rho c_p V} = k_1 e + \int k_2 e\,dt \quad \text{--- GMC}$$

$$Q = V c_p \rho \left[\frac{F_i}{V}(T - T_i) + k_1 e + k_2 \int e\,dt \right] \qquad e = T_{sp} - T$$

ex-liq tank

Plant: $A\frac{dh}{dt} = F_i - \beta h^{7/3}$ ..... experimental setup

Model: $A\frac{dh}{dt} = F_i - \beta\sqrt{h}$

* Globally Linearizing Control (GLC) ..... Non-linear control scheme

Motivation:

1. We need non-linear control scheme for a non-linear process

- Linear controller can be used when degree of non-linearity is low in the process.

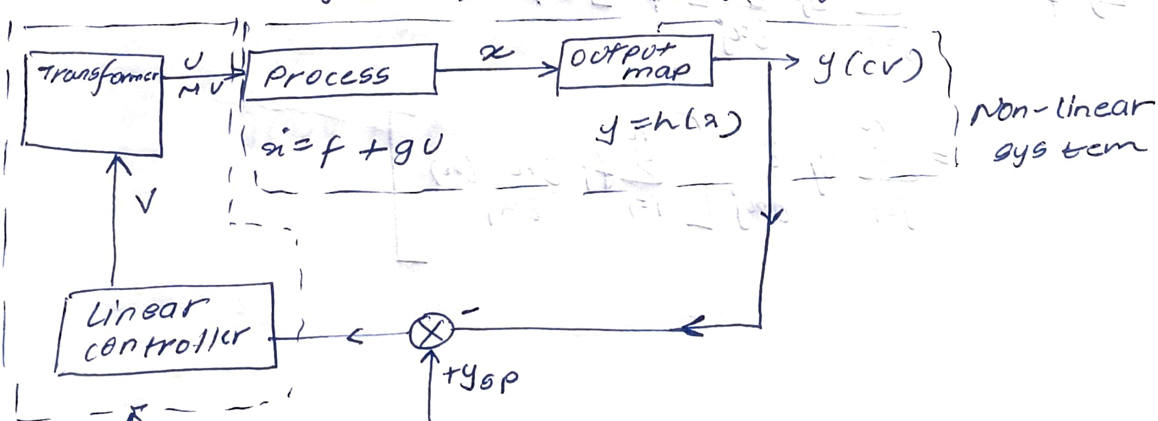
→ In this method we linearize the model @ each & every operating condition.

- State-space representation:

$$\dot{x} = f(x) + g(x)U$$

$$y = h(x)$$

Nonlinear U-y ⟶ Linear V-y system



GLC structure

→ Properties of GLC

1. Directly incorporate non-linear model
2. Ensures global linearization.
3. 4 tunning parameters per control loop: we can reduce it to 1.
4. Continuous & discrete time domain
5. GLC → (FB + FF)

• Lie derivative

L_f (Lie operator)

$L_f h(x)$ = lie derivative of 'h' wrt non-linear fn 'f'

$$L_f h(x) = \sum_{i=1}^{n} f_i \frac{\partial h}{\partial x_i}(x)$$

$\dot{x} = f + gu$; $y = h$

$x \to n \times 1$
$f \to n \times 1$
$g \to n \times m$
$u \to m \times 1$
$y \to m \times 1$

for SISO $m = 1$

• Properties of Lie operator

$L_f(h+t) = L_f(h) + L_f(t)$

$L_f(ht) = t L_f h + h L_f t$

$L_{[f+g]} = L_f + L_g$

$L_{hf} = h L_f$

$$L_g L_f h = \sum_{j=1}^{n} g_j \frac{\partial}{\partial x_j}[L_f h](x)$$

$$= \sum_{j=1}^{n} g_j \frac{\partial}{\partial x_j}\left[\sum_{i=1}^{n} f_i \frac{\partial h}{\partial x_i}(x)\right]$$

→ Relative order (γ)

γ is the smallest integer for which $L_g (L_f)^{r-1} h(x) \neq 0$.

for $r=1$ $\quad L_g h(x) \neq 0$

for $r=2$ $\quad L_g h(x) = 0, \; L_g L_f (h(x)) \neq 0$

for $\gamma = 3$ $\quad L_g h(x) = L_g L_f h(x) = 0, \; L_g L_f^2 h(x) \neq 0$

$$\frac{dy}{dt} = L_f h(x)$$

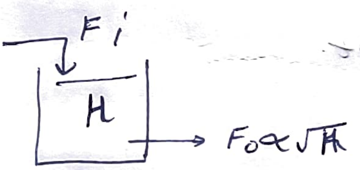
$$\vdots$$

$$\frac{d^{r-1} y}{dt^r} = L_f^r h(x)$$

$$\frac{d^r y}{dt^r} = L_f^r h(x) + L_g L_f^{r-1} h(x) u$$

$\gamma \Rightarrow$ smallest order of derivative of y that explicitly depends on u.

ex-



$$F_i \rho - F_o \rho = A \frac{dH}{dt} \times \rho$$

$$F_i - \text{cons}\sqrt{H} = A\frac{dH}{dt}$$

$$F_i - \beta\sqrt{H} = A\frac{dH}{dt}$$

$$\frac{dH}{dt} = \frac{-\beta}{A}\sqrt{H} + \frac{F_i}{A}.$$

$L_g L_f h(x)$

$$\dot{x} = g(x)u + f(x)$$

$$F = \frac{-\beta\sqrt{H}}{A} \; ; \; g = 1/A. \; ; \; y = H = h; \quad u = F_i$$

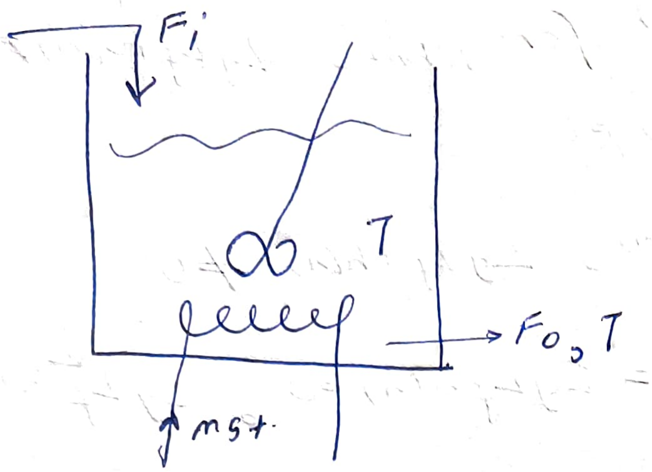
Find γ,

$r=1$ $\qquad L_g h(x) = \sum_{i=1}^{n} g_i \frac{\partial h}{\partial x_i}(x)$ $\qquad n=1$

$$= \frac{1}{A} \times \frac{\partial H}{\partial H} = g = 1/A \neq 0$$

$\therefore \underbrace{r=1}_{\text{relative order.}}$

Ex - 2



const V

$$\rho C_p T_i F_i - \rho C_p T F_o + \frac{m_{st}\lambda}{\rho C_p} = \frac{d}{dt}(V\rho C_p T)$$

$$\frac{dT}{dt} = \frac{F}{V}(T_i - T) + \frac{m_{st}\lambda}{\rho C_p V}$$

$$\dot{x} = g(x)u + f(x).$$

$$g(x) = \cancel{\frac{F}{V}} \frac{\lambda}{\rho C_p V} \qquad f(x) = \cancel{\frac{m_{st}\lambda}{\rho C_p V} + \frac{T_i F}{V}} \frac{F}{V}(T_i - T)$$

$$y = h = T \qquad u = m_{st}.$$

$\gamma = 1$

$\gamma = 1$

$$L_g h(x) = \cancel{-\frac{F}{V}\lambda \frac{\partial T}{\partial T}}$$

$$= \frac{\lambda}{\rho C_p V}\frac{\partial T}{\partial T} = \frac{\lambda}{\rho C_p V} \neq 0.$$