Solutions of Heat Conduction Problems
with
Integral Transforms

Integral Transform Techniques: Introduction

The ***integral transform*** of a function $f(x)$ defined in the interval $a \le x \le b$ is denoted by $\mathscr{I}\{f(x)\} = F(k)$, and defined by

$$\mathscr{I}\{f(x)\} = F(k) = \int_a^b K(x,k) f(x)\,dx,$$

where $K(x, k)$, given function of two variables x and k, is called the ***kernel*** of the transform.

Integral Transform Techniques: Introduction

$\mathscr{I}$ is a *linear operator* since it satisfies the property of ***linearity***.

$$\mathscr{I}\{f(x)\} = F(k) = \int_a^b K(x,k) f(x)\,dx,$$

$$\mathscr{I}\{\alpha f(x) + \beta g(x)\} = \int_a^b \{\alpha f(x) + \beta g(x)\} K(x,k)\,dx$$

$$= \alpha \mathscr{I}\{f(x)\} + \beta \mathscr{I}\{g(x)\},$$

where α and β are arbitrary constants.

In order to obtain $f(x)$ from a given $F(k) = \mathscr{I}\{f(x)\}$, we introduce the *inverse operator* $\mathscr{I}^{-1}$ such that

$$\mathscr{I}^{-1}\{F(k)\} = f(x).$$

Accordingly $\mathscr{I}^{-1}\mathscr{I} = \mathscr{I}\mathscr{I}^{-1} = 1$ which is the identity operator.

It can be proved that $\mathscr{I}^{-1}$ is also a linear operator as follows.

Integral Transform Techniques: Introduction

By expanding an arbitrary function in an infinite series of the characteristic functions of various Sturm–Liouville systems, several finite integral transforms have been introduced.

These transforms are also called *finite Sturm–Liouville transforms*:

- *Finite Fourier transforms*
- *Finite Hankel transforms*
- *Finite Legendre transforms*

Finite Fourier Transforms

Any function $f(x)$, which is piecewise differentiable on the interval $(0, L)$, can be expanded in a series of orthogonal eigen functions:

$$f(x) = \sum_{n=0}^{\infty} A_n \phi_n(x), \quad 0 < x < L \qquad A_n = \frac{1}{N_n}\int_0^L f(x)\phi_n(x)dx \qquad N_n = \int_0^L \left[\phi_n(x)\right]^2 dx$$

$$\bar{f}_n = \int_0^L f(x) K_n(x) dx$$

Finite integral transform of function $f(x)$

$$f(x) = \sum_{n=0}^{\infty} \bar{f}_n K_n(x)$$

The inversion

$$K_n(x) = \frac{\phi_n(x)}{\sqrt{N_n}}$$

Kernel of Transform

This is normalized characteristic (eigen) functions.

Finite Fourier Transforms: Sine

Any function $f(x)$, which is piecewise differentiable on the interval $(0, L)$, can be expanded in a Fourier sine series as:

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}x, \quad 0 < x < L \qquad A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$$

Substitute A_n:

$$f(x) = \frac{2}{L}\sum_{n=1}^{\infty}\left[\int_0^L f(x')\sin\frac{n\pi}{L}x'\,dx'\right]\sin\frac{n\pi}{L}x$$

Rearrange as:

$$\bar{f}_n = \int_0^L f(x)K_n(x)\,dx$$

$$f(x) = \sum_{n=1}^{\infty}\bar{f}_n K_n(x)$$

where: $K_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi}{L}x$ Kernel of Transform (Normalized eigenfunction)

$$\bar{f}_n = \int_0^L f(x)K_n(x)\,dx \quad \text{or} \quad \bar{f}_n = \sqrt{\frac{2}{L}}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$$

Finite Fourier sine transform of the function $f(x)$ over the interval $(0, L)$

Finite Fourier Transforms: Cosine

Any function $f(x)$, which is piecewise differentiable on the interval $(0, L)$, can be expanded in a Fourier cosine series as:

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi}{L}x = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi}{L}x, \; 0 < x < L$$

Substitute A_n:

$$A_n = \begin{cases} \dfrac{1}{L}\displaystyle\int_0^L f(x)dx, & n = 0 \\ \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx, & n = 1, 2, 3, \ldots \end{cases}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{N_n}\left[\int_0^L f(x')\cos\frac{n\pi}{L}x'\,dx'\right]\cos\frac{n\pi}{L}x$$

$$N_n = \begin{cases} L, & n = 0 \\ \dfrac{L}{2}, & n = 1, 2, 3, \ldots \end{cases}$$

$$\bar{f}_n = \int_0^L f(x)K_n(x)dx$$

Finite Fourier cosine transform of the function $f(x)$ over the interval $(0, L)$

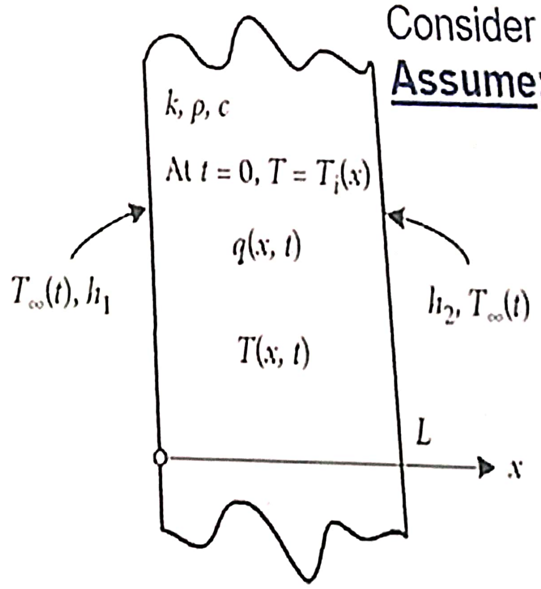
$$f(x) = \sum_{n=0}^{\infty} \bar{f}_n K_n(x)$$

Corresponding inversion formula

$$K_n(x) = \frac{1}{\sqrt{N_n}}\cos\frac{n\pi}{L}x$$

Kernel of Transform

Finite Fourier Transforms: Example-1



Consider the plane wall having thickness L in the x-direction.
Assume: k and α are constants, h_1 and h_2 are very large.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}(x,t)}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

This problem cannot be solved readily by the method of separation of variables.

$$T(x,0) = T_i(x)$$

$$T(0,t) = T(L,t) = T_\infty(t)$$

Basic Idea: Remove the partial derivative with respect to the space variable x from the formulation of the problem.

Note that x varies from zero to L. Accordingly, an integral transform of the temperature distribution $T(x, t)$ with respect to the space variable x is defined on the finite interval $(0, L)$ as

$$\bar{T}_n(t) = \int_0^L T(x,t) K_n(x)\, dx \qquad T(x,t) = \sum_{n=1}^{\infty} \bar{T}_n(t) K_n(x)$$

How to determine $K_n(x)$?

Finite Fourier Transforms: Example-1

$$\frac{\partial^2 T}{\partial x^2}+\frac{\dot{q}(x,t)}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \quad \textbf{Eq. (1)}$$

$$T(x,0)=T_i(x)$$

$$T(0,t)=T(L,t)=T_\infty(t)$$

To determine $K_n(x)$, we consider the following auxiliary problem (remove nonhomogeneous terms from the DE, BC)

$$\frac{\partial^2 \psi}{\partial x^2}=\frac{1}{\alpha}\frac{\partial \psi}{\partial t}$$

$$\psi(x,0)=T_i(x)$$

$$\psi(0,t)=\psi(L,t)=0$$

Eigenvalue problem:

$$\frac{d^2X}{dx^2}+\lambda^2 X=0 \qquad X(0)=X(L)=0$$

$$K_n(x)=\frac{\phi_n(x)}{\sqrt{N_n}}=\sqrt{\frac{2}{L}}\sin\lambda_n x$$

$$\lambda_n=\frac{n\pi}{L},\ n=1,2,3,\ldots$$

Now, we obtain the finite Fourier sine transform of the Heat Equation (**Eq. 1**) by first multiplying it by $K_n(x)$ and then integrating the resultant expression over x from *zero* to L:

$$\int_0^L K_n(x)\frac{\partial^2 T}{\partial x^2}dx+\frac{1}{k}\int_0^L K_n(x)\dot{q}(x,t)dx=\frac{1}{\alpha}\int_0^L K_n(x)\frac{\partial T}{\partial t}dx$$

Finite Fourier Transforms: Example-1

$$\int_0^L K_n(x)\frac{\partial^2 T}{\partial x^2}dx + \frac{1}{k}\int_0^L K_n(x)\dot{q}(x,t)dx = \frac{1}{\alpha}\int_0^L K_n(x)\frac{\partial T}{\partial t}dx$$

RHS: $$\int_0^L K_n(x)\frac{\partial T}{\partial t}dx = \frac{d}{dt}\int_0^L K_n(x)T(x,t)dx = \frac{d\overline{T}_n}{dt}$$

First Term on LHS:
(Integration by parts)

$$\int_0^L K_n(x)\frac{\partial^2 T}{\partial x^2}dx = \underbrace{K_n(L)}_{=0}\frac{\partial T}{\partial x}\bigg|_{x=L} - \underbrace{K_n(0)}_{=0}\frac{\partial T}{\partial x}\bigg|_{x=0}$$

$$-\int_0^L \frac{dK_n}{dx}\frac{\partial T}{\partial x}dx = -\int_0^L \frac{dK_n}{dx}\frac{\partial T}{\partial x}dx$$

One more integration by parts yields:

$$\int_0^L K_n(x)\frac{\partial^2 T}{\partial x^2}dx = -\underbrace{T(L,t)}_{T_\infty(t)}\frac{dK_n}{dx}\bigg|_{x=L}$$

$$+\underbrace{T(0,t)}_{T_\infty(t)}\frac{dK_n}{dx}\bigg|_{x=0} + \int_0^L \frac{d^2K_n}{dx^2}T(x,t)dx$$

Finite Fourier Transforms: Example-1

$$\int_0^L K_n(x)\frac{\partial^2 T}{\partial x^2}dx + \frac{1}{k}\int_0^L K_n(x)q(x,t)dx = \frac{1}{\alpha}\int_0^L K_n(x)\frac{\partial T}{\partial t}dx$$

RHS: $$\int_0^L K_n(x)\frac{\partial T}{\partial t}dx = \frac{d}{dt}\int_0^L K_n(x)T(x,t)dx = \frac{d\overline{T}_n}{dt}$$

First Term on LHS: $$\int_0^L K_n(x)\frac{\partial^2 T}{\partial x^2}dx = -\underbrace{T(L,t)}_{T_{\infty}(t)}\left.\frac{dK_n}{dx}\right|_{x=L} + \underbrace{T(0,t)}_{T_{\infty}(t)}\left.\frac{dK_n}{dx}\right|_{x=0} + \int_0^L \frac{d^2K_n}{dx^2}T(x,t)dx$$

Note that $\frac{dK_n}{dx} = \sqrt{\frac{2}{L}}\lambda_n \cos\lambda_n x$ and $\frac{d^2K_n}{dx^2} = -\lambda_n^2 K_n(x)$

Finally: $$\int_0^L K_n(x)\frac{\partial^2 T}{\partial x^2}dx = \sqrt{\frac{2}{L}}\lambda_n\left[1-(-1)^n\right]T_{\infty}(t) - \lambda_n^2\overline{T}_n(t)$$

Finite Fourier Transforms: Example-1

$$\frac{1}{\alpha}\frac{d\bar{T}_n}{dt}+\lambda_n^2\bar{T}_n(t)=\sqrt{\frac{2}{L}}\lambda_n\left[1-(-1)^n\right]T_\infty(t)+\frac{1}{k}\bar{\dot{q}}_n(t)$$ Eq. (A)

The original problem is now converted to ODE

Where $\bar{\dot{q}}_n(t)=\int_0^L \dot{q}(x,t)K_n(x)dx$ Finite Fourier sine transform of $\dot{q}(x,t)$

Rewrite Eq. (A) as:

$$\frac{1}{\alpha}\frac{d\bar{T}_n}{dt}+\lambda_n^2\bar{T}_n(t)=F_n(t) \quad \text{where:} \quad F_n(t)=\sqrt{\frac{2}{L}}\lambda_n\left[1-(-1)^n\right]T_\infty(t)+\frac{1}{k}\bar{\dot{q}}_n(t)$$

To solve the above ODE

1. Multiply both sides of this equation by IF $\exp(\alpha\lambda_n^2 t)$

2. Then integrate the resultant expression from zero to t

$$\int_0^t e^{\alpha\lambda_n^2 t'}\left[\frac{1}{\alpha}\frac{d\bar{T}_n}{dt'}+\lambda_n^2\bar{T}_n(t')\right]dt'=\int_0^t e^{\alpha\lambda_n^2 t'}F_n(t')dt'$$

Finite Fourier Transforms: Example-1

$$\int_0^t e^{\alpha\lambda_n^2 t'}\left[\frac{1}{\alpha}\frac{d\bar{T}_n}{dt'}+\lambda_n^2\bar{T}_n(t')\right]dt'=\int_0^t e^{\alpha\lambda_n^2 t'}F_n(t')dt'$$

Where: $F_n(t)=\sqrt{\frac{2}{L}}\lambda_n\left[1-(-1)^n\right]T_\infty(t)+\frac{1}{k}\bar{q}_n(t)$

Rewrite above Eq. as:

$$\frac{1}{\alpha}\int_0^t \frac{d}{dt'}\left[e^{\alpha\lambda_n^2 t'}\bar{T}_n(t')\right]dt'=\int_0^t e^{\alpha\lambda_n^2 t'}F_n(t')dt'$$

This yields:

$$\bar{T}_n(t)=e^{-\alpha\lambda_n^2 t}\left[\bar{T}_n(0)+\alpha\int_0^t e^{\alpha\lambda_n^2 t'}F_n(t')dt'\right]$$

Where:

$$\bar{T}_n(0)=\int_0^L T_i(x)K_n(x)dx$$

Finite Fourier sine transform of the initial condition

Now invert $T_n(t)$ by using the inversion formula to obtain the temperature distribution $T(x,t)$:

$$T(x,t)=\sum_{n=1}^{\infty}e^{-\alpha\lambda_n^2 t}\left[\bar{T}_n(0)+\alpha\int_0^t e^{\alpha\lambda_n^2 t'}F_n(t')dt'\right]K_n(x)$$