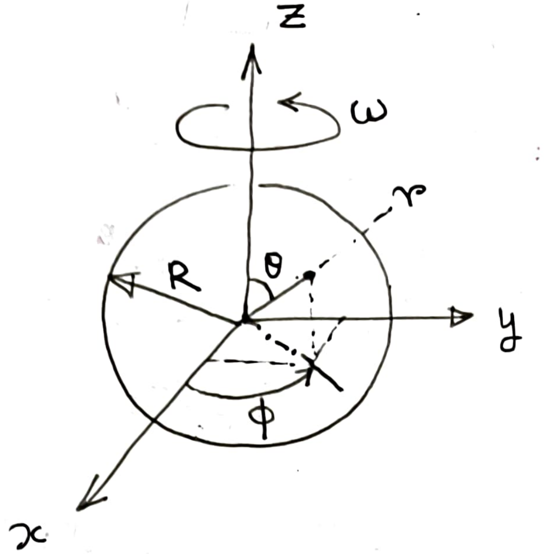
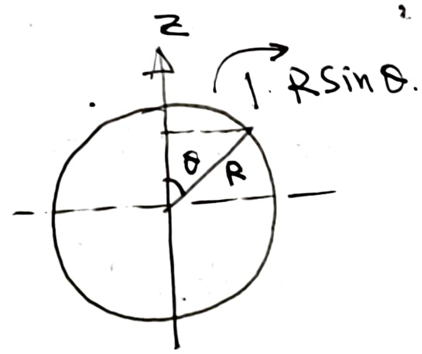
1. (a)





(b) $v_r = 0$; $v_\theta = 0$; Creeping flow:

$v_\phi = f(r)\sin\theta$

B.C: @ $r = R$: $v_r = 0$; $v_\theta = 0$; $v_\phi = R\omega \sin\theta$

@ $r = \infty$: $v_r = 0$, $v_\theta = 0$, $v_\phi = 0$.

(c) ϕ Component:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta} v_\phi \sin\theta\right) = 0$$

(Analysis of other components together with the information $v_r = v_\theta = 0$ yields $\mathcal{P}$ to be constant). [Must be shown]

(d) Assume $f(r) = r^n$ & substitute:

$$\frac{\sin\theta}{r^2}\cdot\frac{\partial}{\partial r}\left\{r^2 . n r^{n-1}\right\} + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\cdot\frac{\partial}{\partial\theta}(r^n \sin^2\theta)\right) = 0$$

or $$\frac{n(n+1)\, r^n}{r^2} + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{r^n}{\cancel{\sin\theta}} 2 \cancel{\sin\theta} \cos\theta\right) = 0$$

or $$\frac{n(n+1) r^{n-2}}{\sin\theta} - \frac{2r^n}{r^2}\sin\theta = 0$$

or $n(n+1) - 2 = 0$ or $n = 1, -2$

$\therefore f = c_1 r + c_2/r^2$

<u>Using B.C:</u>

$$v_\phi = \left(C_1 r + \frac{C_2}{r^2}\right)\sin\theta$$

@ $r \to \infty$ $\quad v_\phi \to 0 \Rightarrow C_1 = 0$

$$\therefore v_\phi = \frac{C_2 \sin\theta}{r^2}$$

@ $r \to R$ $\quad v_\phi = R\omega \sin\theta$

$$\frac{C_2 \sin\theta}{R^2} = R\omega \sin\theta$$

$\therefore C_2 = R^3\omega$ $\qquad \therefore v_\phi = \dfrac{R^3\omega\sin\theta}{r^2}$

$$= \left(\frac{R}{r}\right)^2 \cdot (r\omega\sin\theta)$$

2. This is the same creeping flow problem considered in class. Hence this is a 2D problem with v_r & v_θ. Inertial part of the N.S. eqn is set to zero since this is a creeping flow. Also, there is ϕ symmetry. Hence, using N.S eqns:

$$\frac{\partial p}{\partial r} - \rho g_r = \mu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right)\right] \quad (1)$$

$$\frac{1}{r}\frac{\partial p}{\partial\theta} - \rho g_\theta = \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\theta \sin\theta)\right) + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta}\right] \quad (2)$$

(a) Using the given expressions for v_r & v_θ:

(i) $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) = \frac{3}{2}\frac{v_\infty \sin\theta}{r^2}\left(\frac{R}{r}\right)^3$$

(ii) $$\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r) = \frac{v_\infty \cos\theta}{r^2}\left(2+\left(\frac{R}{r}\right)^3\right)$$

(iii) $$\frac{2}{r^2}\frac{\partial v_r}{\partial\theta} = -\frac{2 v_\infty \sin\theta}{r^2}\left[1-\frac{3}{2}\left(\frac{R}{r}\right)+\frac{1}{2}\left(\frac{R}{r}\right)^3\right]$$

(iv) $$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right) = -\frac{2 v_\infty \cos\theta}{r^2}\left[1-\frac{3}{2}\left(\frac{R}{r}\right)+\frac{1}{2}\left(\frac{R}{r}\right)^3\right]$$

(v) $$\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\theta \sin\theta)\right) = \frac{2 v_\infty \sin\theta}{r^2}\left(1-\frac{3}{4}\left(\frac{R}{r}\right)-\frac{1}{4}\left(\frac{R}{r}\right)^3\right)$$

Substituting in (1):

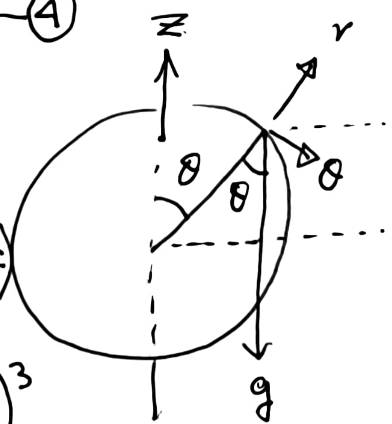
$$\frac{\partial p}{\partial r} - \rho g_r = \mu\frac{v_\infty \cos\theta}{r^2}\left[2+\left(\frac{R}{r}\right)^3 - 2 + 3\left(\frac{R}{r}\right) - \left(\frac{R}{r}\right)^3\right]$$

$$= \frac{3\mu v_\infty \cos\theta}{r^2}\left(\frac{R}{r}\right) \quad (3)$$

Substituting in 2

$$\frac{1}{r}\frac{\partial p}{\partial \theta} - \rho g_\theta = \mu \frac{v_\infty \sin\theta}{r^2}\left[\frac{3}{2}\left(\frac{R}{r}\right)^3 + 2 - \frac{3}{2}\left(\frac{R}{r}\right) - \frac{1}{2}\left(\frac{R}{r}\right)^3 - 2 + 3\left(\frac{R}{r}\right) - \left(\frac{R}{r}\right)^3\right]$$

$$= \frac{3}{2}\mu \frac{v_\infty \sin\theta}{r^2}\left(\frac{R}{r}\right) \quad \text{— } 4$$



(b) From 3:

$$\frac{\partial p}{\partial r} - \rho(-g\cos\theta) = 3\mu \frac{v_\infty \cos\theta}{r^2}\left(\frac{R}{r}\right)$$

$$\frac{\partial p}{\partial r} + \rho g \cos\theta = 3\mu \frac{v_\infty \cos\theta}{R^2}\left(\frac{R}{r}\right)^3$$

$g_\theta = g\sin\theta$

$g_r = -g\cos\theta$

Define

$$\mathcal{P} = p + \rho g r \cos\theta \quad \text{— } 5$$

$$\therefore \frac{\partial \mathcal{P}}{\partial r} = \frac{\partial p}{\partial r} + \rho g \cos\theta$$

$$\therefore \boxed{\frac{\partial \mathcal{P}}{\partial r} = 3\left(\frac{\mu v_\infty}{R^2}\right)\left(\frac{R}{r}\right)^3 \cos\theta.}$$

From 5:

$$\frac{\partial \mathcal{P}}{\partial \theta} = \frac{\partial p}{\partial \theta} - \rho g r \sin\theta = \frac{\partial p}{\partial \theta} - \rho g_\theta \cdot r$$

$$= r\left(\frac{1}{r}\frac{\partial p}{\partial \theta} - \rho g_\theta\right)$$

From 4

$$= \cancel{r}\left\{\frac{3}{2}\frac{\mu v_\infty \sin\theta}{r^2}\left(\frac{R}{\cancel{r}}\right)\right\}$$

$$\boxed{\frac{\partial \mathcal{P}}{\partial \theta} = \frac{3}{2}\left(\frac{\mu v_\infty}{R}\right)\left(\frac{R}{r}\right)^2 \sin\theta}$$

$$\therefore \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) \qquad v_\theta = -V_\infty \sin\theta\left(1-\frac{3}{4}\left(\frac{R}{r}\right)-\frac{1}{4}\left(\frac{R}{r}\right)^3\right).$$

$$\frac{\partial v_\theta}{\partial r} = -V_\infty \sin\theta\left(-\frac{3}{4}R.\frac{(-1)}{r^2}-\frac{1}{4}R^3\frac{(-3)}{r^4}\right)$$

$$r^2\frac{\partial v_\theta}{\partial r} = -V_\infty \sin\theta\left(\frac{3}{4}\frac{R}{r^2}+\frac{3}{4}\frac{R^3}{r^4}\right)r^2$$

$$= -\frac{3}{4}V_\infty \sin\theta\left(R+\frac{R^3}{r^2}\right).$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) = -\frac{3}{4}V_\infty \sin\theta\left(R^3.\frac{(-2)}{r^3}\right)$$

$$= \frac{3}{2}V_\infty \sin\theta\left(\frac{R}{r}\right)^3$$

$$\therefore \frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) = \frac{3}{2}\frac{V_\infty \sin\theta}{r^2}\cdot\left(\frac{R}{r}\right)^3.$$

$$\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r) \qquad v_r = V_\infty \cos\theta\left(1-\frac{3}{2}\left(\frac{R}{r}\right)+\frac{1}{2}\left(\frac{R}{r}\right)^3\right)$$

$$r^2 v_r = V_\infty \cos\theta\left(r^2-\frac{3}{2}Rr+\frac{1}{2}\frac{R^3}{r}\right)$$

$$\frac{\partial}{\partial r}(\quad) = V_\infty \cos\theta\left(2r-\frac{3}{2}R+\frac{R^3}{2}\left(-\frac{1}{r^2}\right)\right)$$

$$\frac{\partial^2}{\partial r^2}(\quad) = V_\infty \cos\theta\left(2-\frac{R^3}{2}\frac{(-2)}{r^3}\right)$$

$$= V_\infty \cos\theta\left(2+\left(\frac{R}{r}\right)^3\right)$$

$$\therefore \frac{1}{r^2}\cdot\frac{\partial^2}{\partial r^2}(r^2 v_r) = \frac{V_\infty \cos\theta}{r^2}\left(2+\left(\frac{R}{r}\right)^3\right).$$

$$\frac{2}{r^2}\cdot\frac{\partial v_r}{\partial\theta}$$

$$= \frac{2}{r^2}\cdot\frac{\partial}{\partial\theta}\cdot\left[V_\infty \cos\theta . f(r)\right]$$

$$= \frac{2V_\infty f(r)}{r^2}\cdot\frac{\partial}{\partial\theta}\cos\theta.$$

$$= -\frac{2V_\infty \sin\theta}{r^2} f(r)$$

$$= -\frac{2V_\infty \sin\theta}{r^2}\left[1-\frac{3}{2}\left(\frac{R}{r}\right)+\frac{1}{2}\left(\frac{R}{r}\right)^3\right]$$

$$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right) \qquad v_r = V_\infty\cos\theta\underbrace{\left(1-\frac{3}{2}\left(\frac{R}{r}\right)+\frac{1}{2}\left(\frac{R}{r}\right)^3\right)}_{f(r)}$$

$$\frac{\partial v_r}{\partial\theta} = -V_\infty \sin\theta f(r)$$

$$\sin\theta\frac{\partial v_r}{\partial\theta} = -V_\infty\sin^2\theta f(r)$$

$$\frac{\partial}{\partial\theta}\left(\qquad\right) = -V_\infty 2\sin\theta\cos\theta f(r)$$

$$\frac{1}{r^2\sin\theta}\left(\qquad\right) = -\frac{2V_\infty\cos\theta\left(1-\frac{3}{2}\left(\frac{R}{r}\right)+\frac{1}{2}\left(\frac{R}{r}\right)^3\right)}{r^2}$$

$$\frac{1}{r^2}\cdot\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(v_\theta\sin\theta\right)\right)$$

$$= \frac{1}{r^2}\cdot\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\cdot\left(-V_\infty\sin^2\theta\, f(r)\right)\right)$$

$$= \frac{1}{r^2}\left(-V_\infty f(r)\right)\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\cdot\frac{\partial}{\partial\theta}\cdot\sin^2\theta\right)$$

$$= \frac{-V_\infty f(r)}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\cancel{\sin\theta}}\cdot 2\cancel{\sin\theta}\cos\theta\right)$$

$$= \frac{-2V_\infty f(r)}{r^2}\cdot(-\sin\theta)$$

$$= \frac{2V_\infty\sin\theta}{r^2}\left(1-\frac{3}{4}\left(\frac{R}{r}\right)-\frac{1}{4}\left(\frac{R}{r}\right)^3\right)$$

$$\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2} = 0$$

$\therefore$ Add:

$$\frac{2V_\infty\cos\theta}{r^2} + \frac{V_\infty\cos\theta}{r^2}\left(\frac{R}{r}\right)^3$$

$$-\frac{2V_\infty\cos\theta}{r^2} + \frac{3V_\infty\cos\theta}{r^2}\left(\frac{R}{r}\right) - \frac{V_\infty\cos\theta}{r^2}\left(\frac{R}{r}\right)^3$$

$$= \frac{3V_\infty\cos\theta}{r^2}\cdot\left(\frac{R}{r}\right)$$