

INDIAN INSTITUTE OF TECHNOLOGY KHARAGPUR

Department of Chemical Engineering

End-semester (Autumn) Examination 2018-2019

Subject: Advanced Mathematical Techniques in Chemical Engineering (CH61015)

Remarks:

1. This question paper contains two parts: **Part A** and **Part B**. Attempt both parts.
2. Write all the answers of a part together.
3. Unless otherwise stated, usual mathematical notations apply.
4. Time = 3 h; maximum marks = 100; total number of printed pages = 3.

Part A: Differential equations

1. Solve the following equation using separation of variables.

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial z^2} = 0$$

At $r = 1, u = 0$; at $z = 0, u = u_0$; at $z = 1, u = 0$. Assume other physical boundary conditions.

... 10 marks

2. Solve the following equation.

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2} \frac{1}{\sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = 0$$

At $r = 1, u = 1$. Assume other physical boundary conditions associated with spherical coordinates.

... 10 marks

3. Solve the following equation completely using Green's function method.

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + x$$

At $t = 0, T = T_0$; at $x = 0, T = 0$; At $x = 1, \frac{\partial T}{\partial x} + T = 0$.

... 20 marks

4. Find eigenvalues and eigenfunctions of the following problem.

$$(1+x)^2 \frac{d^2 u}{dx^2} + \lambda u = 0$$

subject to $u(x=0) = u(x=1) = 0$.

Hint: Use $e^{\pm i\theta} = \cos \theta \pm i \sin \theta$.

... 10 marks

Part B: Linear algebra

5. For the following two systems of equations in unknowns x_i 's and y_i 's, determine whether the corresponding range spaces are identical using Fredholm's alternative theorem.

System 1

$$\begin{aligned}x_1 + 3x_2 + 3x_3 + 2x_4 &= a_1 \\2x_1 + 6x_2 + 9x_3 + 7x_4 &= a_2 \\-x_1 - 3x_2 + 3x_3 + 4x_4 &= a_3\end{aligned}$$

System 2

$$\begin{aligned}y_1 + 2y_3 + 3y_4 &= b_1 \\y_2 + 4y_3 + 5y_4 &= b_2 \\3y_1 + y_2 + 10y_3 + 14y_4 &= b_3\end{aligned}$$

... 10 marks

6. Determine an orthonormal set from the following set.

$$T = \{\underline{u}_1, \underline{u}_2, \underline{u}_3\} = \left\{ \begin{bmatrix} 1 \\ 1+i \\ 1 \end{bmatrix}, \begin{bmatrix} -2-3i \\ 1-i \\ 2+5i \end{bmatrix}, \begin{bmatrix} -3-i \\ 1+3i \\ -1-i \end{bmatrix} \right\}$$

... 10 marks

7. Consider the following non-autonomous, non-separable, non-linear ODE.

$$\frac{dy}{dt} = y^2 + 16t^2 - 8yt + 2y - 8t + \lambda$$

(a) Verify whether you obtain an autonomous equation by the following transformation:

$$u(t) = y(t) - 4t.$$

(b) For the new equation, sketch a well-labelled bifurcation diagram.

(c) For $\lambda = 5$, sketch a well-labelled phase portrait for the autonomous system.

... 15 marks

8. Consider the nonhomogeneous equation

$$\underline{X}' = \underline{A}X + \underline{G}(t)$$

where \underline{A} is $n \times n$ matrix and $\underline{G}(t)$ is a continuous function of t . The solution of this matrix equation satisfying $\underline{X}(0) = \underline{X}_0$ is given by

$$\underline{X}(t) = \exp(t\underline{A}) \left(\underline{X}_0 + \int_0^t \exp(-s\underline{A}) \underline{G}(s) ds \right)$$

Using the above result, determine the general solution of the following equation.

$$\frac{d^2x}{dt^2} + x = \cos(t)$$

...15 marks
