Transform Calculus (MA20202) Mid-Sem question and solution
(exam held on 15.2.23)

1. a) Find the Laplace Transform of the following function

$$f(t) = \begin{cases} t & if\ 0 < t < 1 \\ e^t & if\ t > 1 \end{cases}$$

b) Use Shifting property to get the Laplace Transform of

$$f(t) = e^{-t} \cos t\ \sin t.$$

c) Find Laplace Transform of $f(t) = \frac{\sin t}{t}$.

d) Find Laplace Transform of $f(t) = \int_0^t \sinh at \sin bt\ dt$.

[3+2+2+3=10M]

2. (a) If $L\{erf\sqrt{t}\}$ is given by $\frac{1}{s\sqrt{s+1}}$, find $L\{t\ erf 2\sqrt{t}\}$ mentioning the properties of Laplace Transform that you are using.

(b) Let $f(t) = \sin t,\ 0 < t < \pi$

$= \sin 2t,\ \pi < t < 2\pi$

$= \sin 3t,\ t > 2\pi$

Express $f(t)$ in terms of the Heaviside unit step function defined by

$u(t-a) = 0,\ \ t < a$

$= 1,\ \ t > a$

Hence find the Laplace Transform of $f(t)$.

(c) If $L^{-1}\left\{\frac{7s+13}{s(s^2+4s+13)}\right\} = 1 - e^{-at}\cos(bt) + c\ e^{-at}\sin(bt)$,
find a, b, c.

[3+4+3=10M]

3. (a) State the convolution theorem for the Inverse Laplace Transform.

(b) Find the solution of the O D E

$t\ y'' + y' + t\ y = 0,\ y(0) = 10,\ \ y'(0) = A$ using the Laplace

Transform technique. Hence obtain the value of A.

(c) Find the solution of the Integral equation

$$f(t) = t + \exp(-2t) + \int_0^t f(\tau)\ \exp[2\,(t-\tau)]\ d\tau\ .$$

[1+4+5=10M]

1. a) Find Laplace Transform of the following function

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b) Use Shifting property to get the Laplace Transform of $F(t) = e^{-t} \cos t\ \sin t$.

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[3+2+2+3=10M]

Ans.

a) $F(t) = \begin{cases} t & if\ 0 < t < 1 \\ e^t & if\ t > 1' \end{cases}$

$$L\{F(t)\} = \int_0^1 e^{-st} t\, dt + \int_1^\infty e^{-st} e^t dt$$

$$= \left[t.\frac{e^{-st}}{-s}\right]_0^1 - \int_0^1 \frac{e^{-st}}{-s}\, dt + \left[\frac{e^{-t(s-1)}}{-(s-1)}\right]_1^\infty \quad \text{[1 mark]}$$

$$= -\frac{e^{-s}}{-s} - \left[\frac{e^{-st}}{s^2}\right]_0^1 + \left[\frac{e^{-t(s-1)}}{-(s-1)}\right]_1^\infty$$

$$= -\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} + \left[\frac{e^{-t(s-1)}}{-(s-1)}\right]_1^\infty$$

$$= -\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} + \frac{e^{-(s-1)}}{-(s-1)}$$

$$= \frac{1}{s^2} + e^{-s}\left(\frac{e}{s-1} - \frac{1}{s} - \frac{1}{s^2}\right) \quad for\ s > 1. \quad [2\ marks]$$

b) $F(t) = e^{-t} \cos t\ \sin t.$

$L\{\cos t\ \sin t\} = \frac{1}{2} L\{\sin 2t\} = \frac{1}{s^2+4}$ [1 mark]

$L\{e^{-t} \cos t\ \sin t\} = \frac{1}{(s+1)^2+4} = \frac{1}{s^2+2s+5}$ [1 mark]

c) $F(t) = \frac{\sin t}{t}$

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1} ds\ [1\ mark]$$

$= \frac{\pi}{2} - tan^{-1}s$ [1 mark]

d) $F(t) = \int_0^t \sinh at \sin bt$

$$L\{F(t)\} = L\left\{\int_0^t \sinh at \sin bt\right\}$$

$$\sinh at \sin bt = \frac{e^{at}\sin bt\ - e^{-at}\sin bt}{2}$$

$L\{\sinh at \sin bt\} = \frac{L\{e^{at}\sin bt\} - L\{e^{-at}\sin bt\}}{2}$ [1 mark]

$L\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$ $L\{e^{-at}\sin bt\} = \frac{b}{(s+a)^2+b^2}$ [

$$L\{\sinh at \sin bt\} = \frac{b}{2}\left[\frac{1}{(s-a)^2+b^2} - \frac{1}{(s+a)^2+b^2}\right]$$

$$= \frac{2sab}{\{(s-a)^2+b^2\}\{(s+a)^2+b^2\}} = f(s)$$

[1 mark]

$L\left\{\int_0^t \sinh at \sin bt\right\} = \frac{f(s)}{s} = \frac{2ab}{\{(s-a)^2+b^2\}\{(s+a)^2+b^2\}} = \frac{2ab}{(s^2+a^2-2as+b^2)(s^2+a^2+b^2+2as)}$

$= \frac{2ab}{(s^4+a^4+b^4+2a^2b^2-2a^2s^2+2b^2s^2)}$ [1 mark]

Or :

$\sinh at \sin bt = \left(\frac{e^{at}-e^{-at}}{2}\right).\left(\frac{e^{ibt}-e^{-ibt}}{2i}\right)$ [1 mark]

$f(s) = \frac{e^{at+ibt} - e^{at-ibt} - e^{-at+ibt} + e^{-at-ibt}}{4i} = \frac{1}{4i}\left(\frac{1}{s-a-ib} + \frac{1}{s+a+ib} - \frac{1}{s-a+ib} - \frac{1}{s+a-ib}\right)$

$= \frac{s}{2i}\left(\frac{1}{(s^2-(a+ib)^2)} - \frac{1}{(s^2-(a-ib)^2)}\right) = \frac{s}{2i}\left(\frac{1}{(s^2-a^2+b^2-2abi)} - \frac{1}{(s^2-a^2+b^2+2abi)}\right) =$

$\frac{2sab}{(s^2-a^2+b^2)^2+4a^2b^2}$ [1 mark]

$L\left\{\int_0^t \sinh at \sin bt\right\} = \frac{f(s)}{s} = \frac{2ab}{(s^2-a^2+b^2)^2+4a^2b^2}$ [1 mark]

$= \frac{2ab}{(s^4+a^4+b^4+2a^2b^2-2a^2s^2+2b^2s^2)}$

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Ans. 2(a) Given $L\{\operatorname{erf}\sqrt{t}\} = \frac{1}{s\sqrt{s+1}}$

$\therefore L\{\operatorname{erf}2\sqrt{t}\} = L\{\operatorname{erf}\sqrt{4t}\} = \frac{1}{4}F\left(\frac{s}{4}\right)$ $\left[\text{Using } L\{F(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)\right]$ $\frac{1}{2}$M

$= \frac{1}{4}\frac{1}{\frac{s}{4}\sqrt{\frac{s}{4}+1}} = \frac{2}{s\sqrt{s+4}}$ 1M

$L\{t.\operatorname{erf}(2\sqrt{t})\} = -\frac{d}{ds}L\{\operatorname{erf}2\sqrt{t}\}$

$= -2\frac{d}{ds}\left[\frac{1}{s\sqrt{s+4}}\right]$ $\left[\text{Using } L\{tf(t)\} = -F'(s)\right]$ $\frac{1}{2}$M

$= \frac{3s+8}{s^2(s+4)^{3/2}}$ 1M

$\left[\text{or } \frac{2}{s^2(s+4)^{1/2}} + \frac{1}{s(s+4)^{3/2}} \text{ or any other correct form}\right]$

2. (b) $F(t) = \sin t\, u(t) + (\sin 2t - \sin t)u(t-\pi) + (\sin 3t - \sin 2t)u(t-2\pi)$ 1M

$\left[\text{or } F(t) = \sin t\{u(t) - u(t-\pi)\} + \sin 2t\{u(t-\pi) - u(t-2\pi)\} + \sin 3t\{u(t-2\pi)\}\right]$

Using $L[f(t-a)u(t-a)] = e^{-as}F(s)$

$L\{(\sin 2t - \sin t)u(t-\pi)\} = e^{-\pi s}\left[\frac{2}{s^2+4} + \frac{1}{s^2+1}\right]$

$[\because \sin t\, u(t-\pi) = -\sin(t-\pi)u(t-\pi)]$

Similarly $L\{(\sin 3t - \sin 2t)u(t-2\pi)\} = e^{-2\pi s}\left[\frac{3}{s^2+9} - \frac{2}{s^2+4}\right]$

$\therefore L\{F(t)\} = \frac{1}{s^2+1} + e^{-\pi s}\left[\frac{2}{s^2+4} + \frac{1}{s^2+1}\right] + e^{-2\pi s}\left[\frac{3}{s^2+9} - \frac{2}{s^2+4}\right]$

1Mark is for writing $e^{-\pi s}\frac{1}{s^2+1}$ (+ve sign) because this is the only trick in the problem. Any other 2 terms correctly, another 1M. Rest 2 terms correctly, another 1M. NO MARKS IS AWARDED IF DONE BY DIRECT INTEGRATION / WITHOUT USING PROPERTY OF UNIT STEP FUNCTION. Because in the question, 'hence' is written.

2. (c) $L^{-1}\left[\frac{7s+13}{s(s^2+4s+13)}\right] = L^{-1}\left[\frac{1}{s} + \frac{-s+3}{s^2+4s+13}\right]$

$= L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{-(s+2)+5}{(s+2)^2+3^2}\right\} = 1 - e^{-2t}\cos(3t) + \frac{5}{3}e^{-2t}\sin(3t)$

$a = 2$ 1M $b = 3$ 1M $c = \frac{5}{3}$ 1M

Q3: Part Marking.

a). Write the statement : $\mathcal{L}^{-1}[F(s).G(s)] = f*g$

— (1M).

b) Obtain $(s^2+1)Y'(s) + sY(s) = 0$

[If this equation is wrong, 0 Marks] — (2M)

Solve it, $y(t) = C\,J_0(t)$ or $10\,J_0(t)$ — (1M)

Write $A = 0$ explicitly to get (1M)

(4M).

$\boxed{*}$ Those who did not identify $\mathcal{L}^{-1}\left[\frac{1}{\sqrt{1+s^2}}\right] = J_0(t)$, no mark is awarded *].

c) $F(s) = \left[\frac{1}{s^2} + \frac{1}{s+2}\right]\left[\frac{s-2}{s-3}\right]$ — (1M).

Do the partial fractions correctly, to get

$$F(s) = \frac{2}{3s^2} - \frac{1}{9s} + \frac{4}{5(s+2)} + \frac{14}{45(s-3)}$$

— get (2M).

* If these partial fraction is not correct, no marks are awarded.

(3M)

Inverse it, to get

$$f(t) = \frac{2}{3}t - \frac{1}{9} + \frac{4}{5}e^{-2t} + \frac{14}{45}e^{3t}.$$

If all these 4 terms are correct, then — (2M).

(**) If more than 3 terms are correct, then part marking is given out of these 2 M