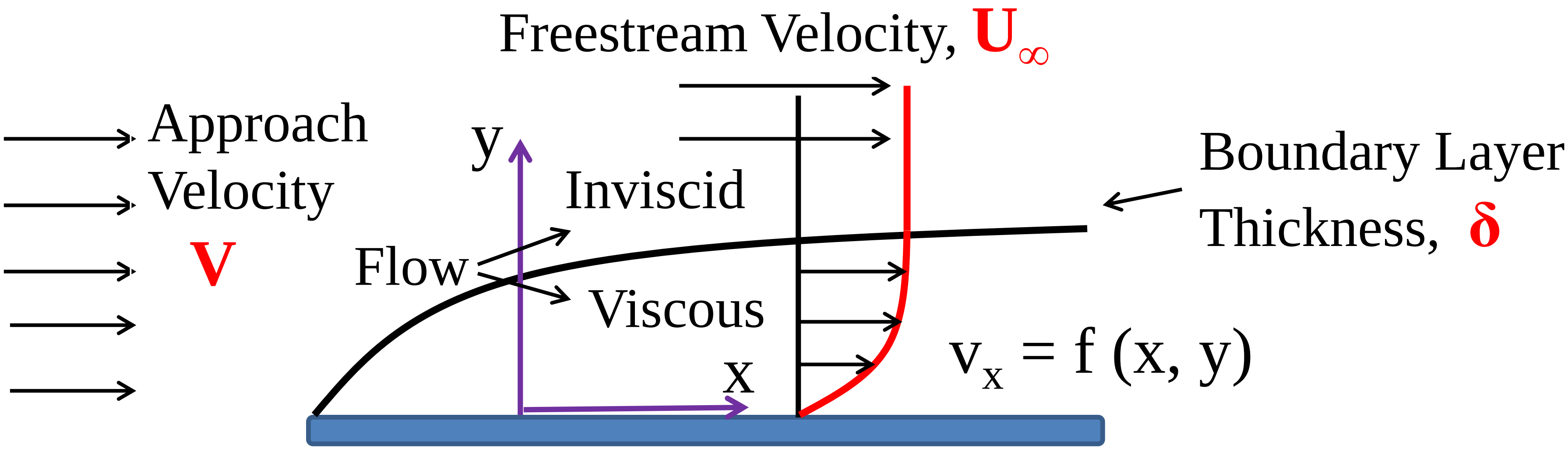
External Incompressible Viscous Flow – Boundary Layer



Flow over a flat plate

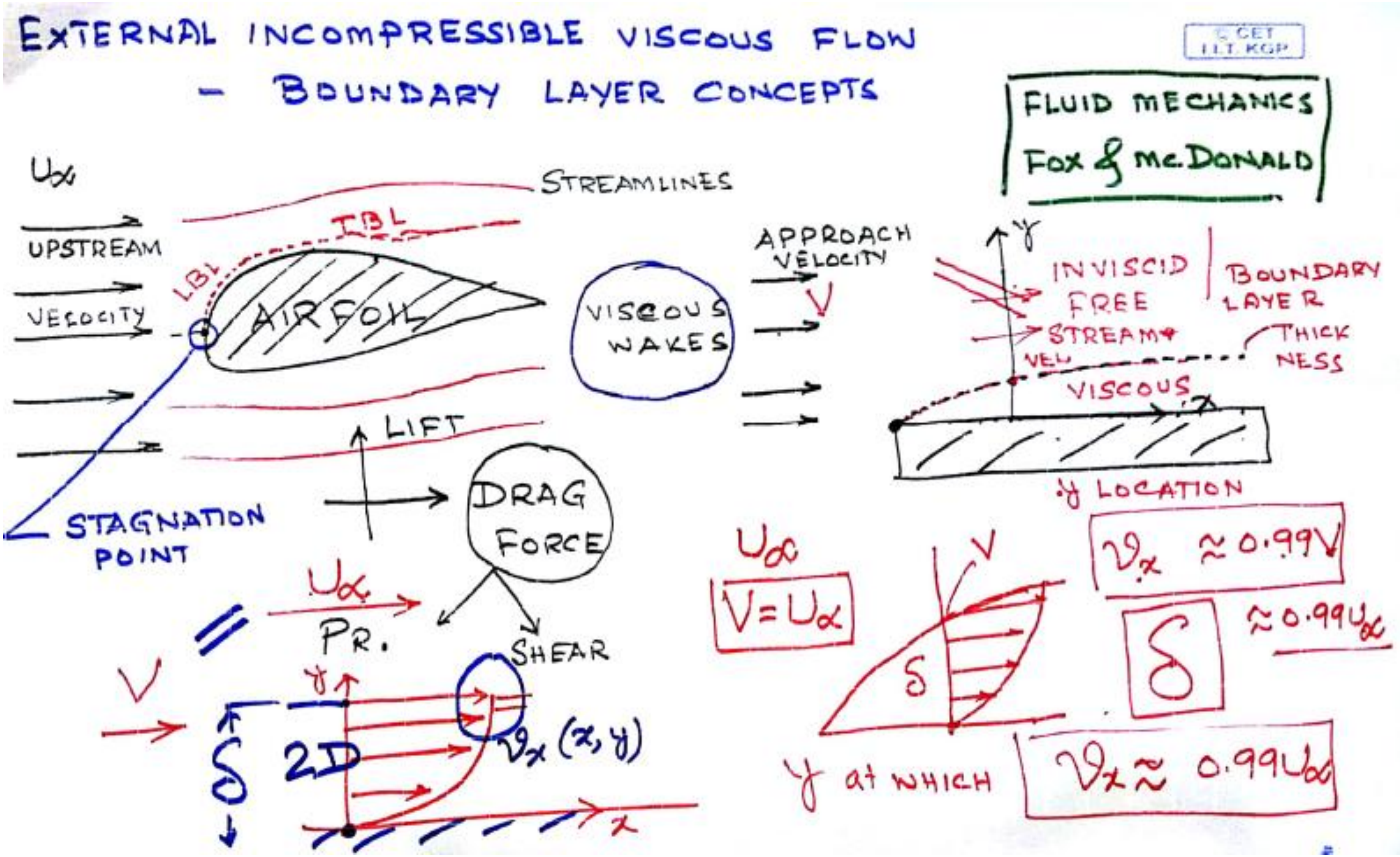
$v_x = f\,(x, y)$ Viscous 2D flow inside BL

$v_x = U_\infty$ Inviscid flow outside BL

$v_x = 0.99U_\infty$ at $y = \delta$

δ - boundary layer thickness

Boundary Layers



Flow Inside the Boundary Layer

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \gamma \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right)$$

Boundary Layer Approximations

$$v_x >> v_y \qquad \frac{\partial v_x}{\partial y} >> \frac{\partial v_x}{\partial x} \qquad \frac{\partial^2 v_x}{\partial y^2} >> \frac{\partial^2 v_x}{\partial x^2}$$

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \gamma \frac{\partial^2 v_x}{\partial y^2}$$

Exact Method – Blasius – Numerical Solution

Approximate Method – Momentum Integral Equation

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \gamma \frac{\partial^2 v_x}{\partial y^2}$$

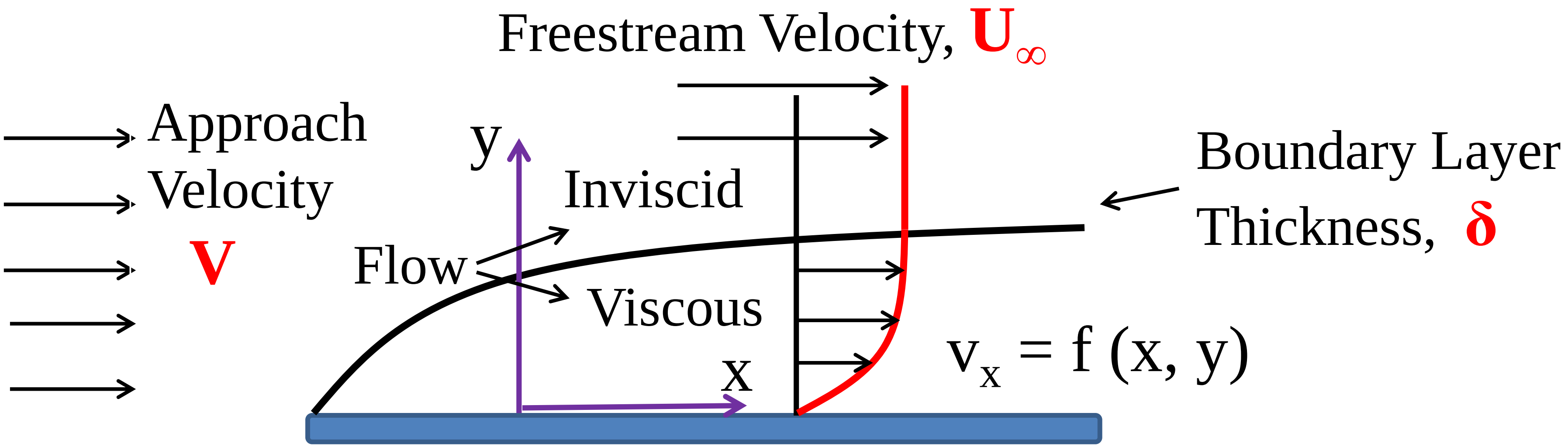
Boundary Conditions

$$At\ y = 0; \quad v_x\ , v_y = 0$$

$$At\ y = \delta; \quad v_x\ = U_\infty$$

$$At\ x = 0; \quad v_x\ = V$$

Boundary Layer Thicknesses



Flow over a flat plate

Boundary Layer Thickness, δ

y at which $v_x = 0.99\, U_\infty$

Also known as the disturbance thickness

Displacement Thickness

Amount of reduction in mass flow rate in viscous flow inside the boundary layer

$$= \int_0^\infty \rho\left(U_\infty - u\right) dy$$

For an inviscid flow the same reduction can be achieved by moving the plate up by a distance δ^*

$$\rho\, U_\infty\, \delta^* = \int_0^\infty \rho\left(U_\infty - u\right) dy$$

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U_\infty}\right) dy \approx \int_0^\delta \left(1 - \frac{u}{U_\infty}\right) dy$$

Momentum Thickness

Reduction in momentum for the fluid that actually flows in the boundary layer

$$= \int_0^\infty \rho\, u \left(U_\infty - u\right) dy$$

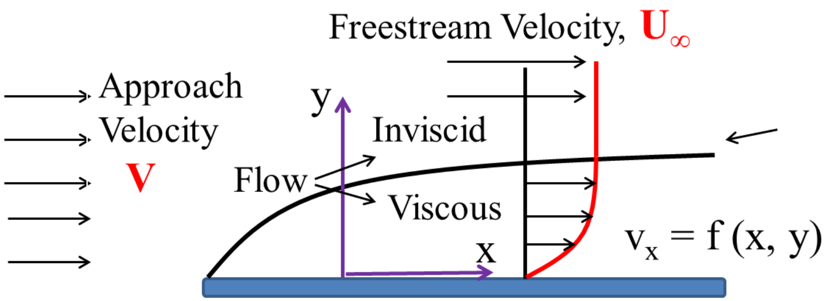
For an inviscid flow the same momentum reduction can be achieved by moving the plate up by a distance θ (Reduction = $\rho\, U_\infty\, \theta^*\, U_\infty$ **)**

$$\rho\, U_\infty^2\, \theta = \int_0^\infty \rho\, u \left(U_\infty - u\right) dy$$

$$\theta = \int_0^\infty \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy \approx \int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy$$

Blasius solution

$$\frac{v_x}{U} = g(\eta) \qquad \eta \approx \frac{y}{\delta(x)}$$



Flow over a flat plate

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \gamma \frac{\partial^2 v_x}{\partial y^2} \qquad \text{- (II)}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \qquad \text{- (I)}$$

$$v_x = \frac{\partial \psi}{\partial y} \qquad v_y = -\frac{\partial \psi}{\partial x}$$

ψ - Stream function

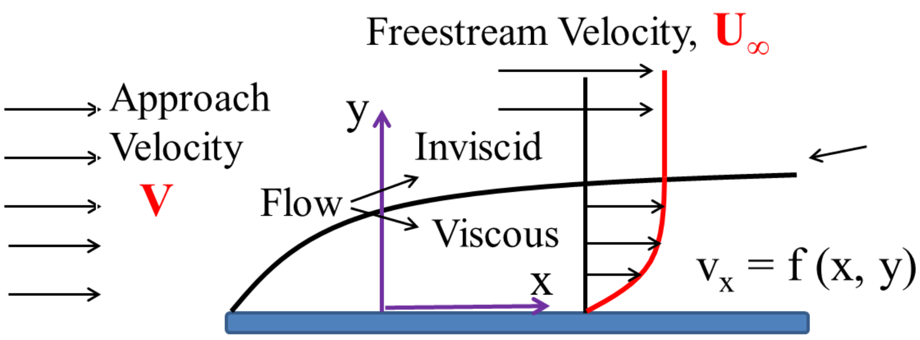
- Exact differential

$$\frac{\partial \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial y \partial x} = 0$$

Eq. I gets satisfied automatically

Blasius solution contd.

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \gamma \frac{\partial^2 v_x}{\partial y^2}$$



Flow over a flat plate

Near the B.L.

$$v_x \approx U \quad y \approx \delta \quad \frac{\partial v_x}{\partial y} \approx o$$

$$U \frac{U}{x} \approx \gamma \frac{U}{\delta^2} \qquad \delta^2 \approx \frac{\gamma x}{U}$$

$$\boxed{\delta \approx \sqrt{\frac{\gamma x}{U}}}$$

$$\eta = \frac{y}{\delta} = y\sqrt{\frac{U}{\gamma x}}$$

Blasius solution contd.

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \gamma \frac{\partial^2 v_x}{\partial y^2}$$

Introducing a stream function and by invoking the method of combination of variables

$$\eta = y\sqrt{\frac{U}{\gamma x}}$$

Dimensionless stream function

$$f = \frac{\psi}{\sqrt{\gamma x U}}$$

$$\boxed{f(\eta) \rightarrow \eta} \Rightarrow \text{ODE}$$

Blasius solution contd.

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \gamma \frac{\partial^2 v_x}{\partial y^2}$$

$$v_x = \frac{\partial \psi}{\partial y}$$

$$\eta = y\sqrt{\frac{U}{\gamma x}}$$

$$= \frac{\partial \psi}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}$$

$$f = \frac{\psi}{\sqrt{\gamma x U}}$$

$$= \sqrt{\gamma x U}\,\frac{df}{d\eta} \cdot \sqrt{\frac{U}{\gamma x}}$$

$$= U\frac{df}{d\eta}$$

$$\boxed{v_x = U\frac{df}{d\eta}}$$

$$v_y = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\left[f \cdot \sqrt{\gamma x U}\right]$$

$$= -\left[\sqrt{\gamma x U} \cdot \frac{\partial f}{\partial x} + \frac{1}{2}\sqrt{\frac{\gamma U}{x}}\,f\right]$$

Blasius solution contd.

$$v_y = -\left[\sqrt{\gamma x U}\,.\frac{df}{d\eta}.\frac{\partial \eta}{\partial x} + \frac{1}{2}\sqrt{\frac{\gamma U}{x}}\,f\right]$$

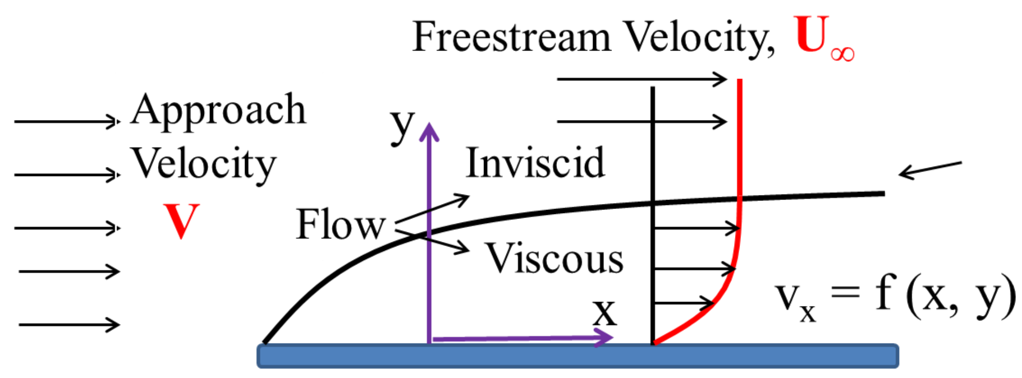
$$v_y = \frac{1}{2}\sqrt{\frac{\gamma U}{x}}\left[\eta\frac{df}{d\eta} - f\right]$$

$$v_x = U\frac{df}{d\eta} \longrightarrow v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} = \gamma\frac{\partial^2 v_x}{\partial y^2}$$

$$\boxed{2\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} = 0}$$

ODE

$$2\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} = 0$$



Flow over a flat plate

$$\eta = y\sqrt{\frac{U}{\gamma x}}$$

$$v_x, v_y = 0 \; at \; \eta = 0 \qquad f = \frac{df}{d\eta} = 0$$

$$v_x = U \; as \; \eta \to \infty \qquad f' = 1$$

Blasius Solution – contd.

$$2\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} = 0$$

$$v_x = U\frac{df}{d\eta} \qquad v_y = \frac{1}{2}\sqrt{\frac{\gamma U}{x}}\left[\eta\frac{df}{d\eta} - f\right]$$

$$\eta = y\sqrt{\frac{U}{\gamma x}}$$

$$at\,\eta = 0 \quad f = \frac{df}{d\eta} = 0$$

$$as\,\eta \to \infty \quad f^{/} = 1$$

Numerical Solution by Howarth

Table 9.1 The function $f(\eta)$ for the laminar boundary layer along a flat plate at zero incidence. (After L. Howarth [5].)

$\eta = y\sqrt{\frac{U_\infty}{\nu x}}$	f	$f' = \frac{u}{U_\infty}$	f''
0	0	0	0.33206
0.4	0.02656	0.13277	0.33147
1.0	0.16557	0.32979	0.32301
1.4	0.32298	0.45627	0.30787
2.0	0.65003	0.62977	0.26675
2.4	0.92230	0.72899	0.22809
3.0	1.39682	0.84605	0.16136
3.4	1.74696	0.90177	0.11788
4.0	2.30576	0.95552	0.06424
4.4	2.69238	0.97587	0.03897
5.0	3.28329	0.99155	0.01591
5.4	3.68094	0.99616	0.00793
6.0	4.27964	0.99898	0.00240
6.4	4.67938	0.99961	0.00098
7.0	5.27926	0.99992	0.00022
7.4	5.67924	0.99998	0.00007
8.0	6.27923	1.00000	0.00001
8.4	6.67923	1.00000	0.00000

Blasius solution contd.

$$\eta = y\sqrt{\frac{U}{\gamma x}}$$

$$f^{/} = 0.991 \quad \rightarrow \frac{v_x}{U} = 0.99$$ η=5.0 Edge of the B.L.

$$5.0 = \delta\sqrt{\frac{U}{\gamma x}}$$

$$\delta = \frac{5.0}{\sqrt{\frac{U}{\gamma x}}} = \frac{5.0x}{\sqrt{\mathrm{Re}_x}}$$

$$\frac{\delta}{x} = \frac{5.0}{\sqrt{\mathrm{Re}_x}}$$

Table 9.1 The function $f(\eta)$ for the laminar boundary layer along a flat plate at zero incidence. (After L. Howarth [5].)

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Blasius solution contd.

$$\tau_w = \mu \frac{\partial v_x}{\partial y}\Bigg]_{y=0} = \mu \frac{\partial}{\partial y}\frac{\partial \psi}{\partial y}\Bigg]_{y=0}$$

$$= \mu \frac{\partial}{\partial y} U \frac{df}{d\eta}\Bigg]_{\eta=0} = \mu U \frac{d^2 f}{d\eta^2}\frac{d\eta}{dy}\Bigg]_{\eta=0}$$

0.332

$$\boxed{\tau_w = \mu U \sqrt{\frac{U}{\gamma x}} \frac{d^2 f}{d\eta^2}_{\eta=0}}$$

$\eta = 0$ $\quad f^{//} = 0.332$

Table 9.1 The function $f(\eta)$ for the laminar boundary layer along a flat plate at zero incidence. (After L. Howarth [5].)

$\eta = y\sqrt{\frac{U_\infty}{\nu x}}$	f	$f' = \frac{u}{U_\infty}$	f''
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$$\boxed{\tau_w = 0.332 U \sqrt{\rho \mu \frac{U}{x}} = \frac{0.332 \rho U^2}{\sqrt{\mathrm{Re}_x}}}$$

Blasius solution contd.

$$\tau_w = \frac{0.332\rho U^2}{\sqrt{\mathrm{Re}_x}}$$

Shear stress coefficient C_f

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$$

$$C_f = f\left(x\right)$$

$$\overline{\tau_w} = \int_0^L C_f \frac{1}{2}\rho U^2 dx$$

Drag coefficient C_D

$$\frac{\delta}{x} = \frac{5.0}{\sqrt{\mathrm{Re}_x}}$$

$$C_D = \frac{\int_A \tau_w}{\frac{1}{2}\rho U^2 A} = \frac{\int_A C_f \frac{1}{2}\rho U^2}{\frac{1}{2}\rho U^2 A} = \frac{1}{L \times 1}\int_0^L \frac{0.664}{\sqrt{\mathrm{Re}_x}}\, dx = 2\,\overline{C_{f\ L}}$$

$$C_D = \frac{1}{L \times 1}\int_0^L \frac{0.664}{\sqrt{\mathrm{Re}_x}}\, dx = 2\,\overline{C_{f\ L}}$$

$$F = \int_0^L 0.664\, C_f\, \sqrt{\mathrm{Re}_x}\, W\, dx$$

Use the numerical results of Howarth to evaluate the following quantities for laminar boundary layer flow on a flat plate

Table 9.1 The function $f(\eta)$ for the laminar boundary layer along a flat plate at zero incidence. (After L. Howarth [5].)

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(a) δ^*/δ evaluate for $\eta = 5$ and $\eta \to \infty$

(b) v_y / U at the edge of the BL

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U_\infty}\right) dy \approx \int_0^\delta \left(1 - \frac{u}{U_\infty}\right) dy$$

$$\eta = y\sqrt{\frac{U}{\gamma x}}$$

(a) $\dfrac{\delta^*}{\delta}$ evaluate for $\eta = 5$ and $\eta \to \infty$

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U_\infty}\right) dy \approx \int_0^\delta \left(1 - \frac{u}{U_\infty}\right) dy$$

$$\eta = y\sqrt{\frac{U}{\gamma x}} \qquad dy = d\eta\sqrt{\frac{\gamma x}{U}}$$

$$\delta^* = \int_0^\eta \left(1 - \frac{v_x}{U}\right)\sqrt{\frac{\gamma x}{U}}\, d\eta$$

$$\delta^* = \sqrt{\frac{\gamma x}{U}} \int_0^\eta \left(1 - \frac{df}{d\eta}\right) d\eta$$

$$\delta^* = \sqrt{\frac{\gamma x}{U}} \int_0^\eta \left(1 - \frac{df}{d\eta}\right) d\eta$$

$$\frac{\delta}{x} = \frac{5.0}{\sqrt{\mathrm{Re}_x}}$$

$$\delta^* = \frac{\delta}{5}\left(\int_0^\eta \left(1 - f^{/}\right) d\eta\right)$$

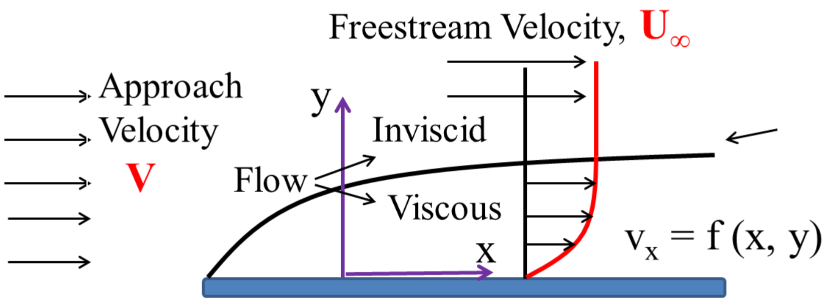
$$\frac{\delta^*}{\delta} = \frac{1}{5}\left[\eta - f\right]_0^\eta$$

$$\eta = 5, \eta \to \infty$$

$$\frac{\delta^*}{\delta} = 0.34334\left(\eta = 5\right)$$

$$= 0.34415(\eta \to \infty)$$

$\dfrac{v_y}{U}$ At the edge of B.L.



Flow over a flat plate

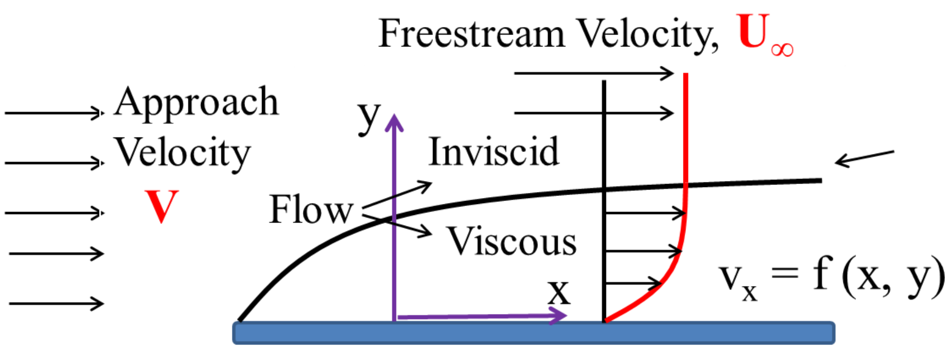
$$v_y = \frac{1}{2}\sqrt{\frac{\gamma U}{x}}\left(\eta f' - f\right)$$

$$\frac{v_y}{U} = \frac{1}{2\sqrt{\mathrm{Re}_x}}\left(\eta f' - f\right)$$

At the edge of B.L., $\eta = 5, f = 3.28$

$$\frac{v_y}{U} = \frac{0.84}{\sqrt{\mathrm{Re}_x}}$$

v_x is a function of x.



Flow over a flat plate

A laboratory wind tunnel has a test section that is 305 mm square. Boundary-layer velocity profiles are measured at two cross-sections and displacement thicknesses are evaluated from the measured profiles. At section 1 , where the freestream speed is $U_1 = 26$ m/s, the displacement thickness is $\delta_1^* = 1.5$ mm. At section 2 , located downstream from section 1 , $\delta_2^* = 2.1$ mm. Calculate the change in static pressure between sections 1 and 2 .

Express the result as a fraction of the freestream dynamic pressure at section 1. Assume standard atmosphere conditions.

Blasius solution contd.

Inviscid flow

Area 1 = (305- (2*1.5))*(305- (2*1.5))

Area 2 = (305- (2*2.1))*(305- (2*2.1))

Bernoulli's Equation

$$\frac{p_1}{\rho}+\frac{v_1^2}{2}+gz_1=\frac{p_2}{\rho}+\frac{v_2^2}{2}+gz_2$$

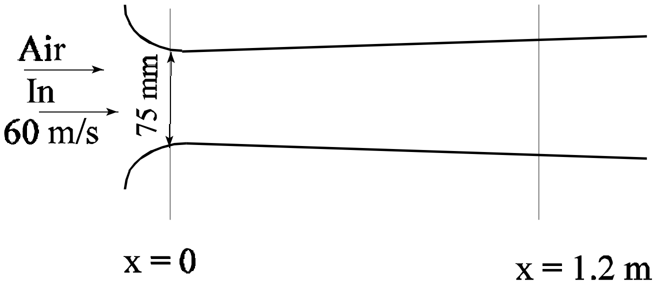
$$p_1-p_2=\frac{1}{2}\rho U_1^2\left[\left(\frac{U_2}{U_1}\right)^2-1\right]$$

$$\frac{p_1 - p_2}{\frac{1}{2}\rho U_1^2} = \left[\left(\frac{U_2}{U_1} \right)^2 - 1 \right]$$

0.016

$$\frac{A_1}{A_2} = \frac{U_2}{U_1} = \frac{\left(L - 2\delta_1^*\right)^2}{\left(L - 2\delta_2^*\right)^2}$$

A uniform flow of standard air ($\mu/\rho = 1.5\mathrm{x}10^{-5}\ \mathrm{m}^{2}/\mathrm{s}$) at 60 m/s enters a plane wall diffuser with negligible boundary layer thickness at the inlet. The inlet width is 75 mm. The diffuser walls diverge slightly to accommodate the boundary layer growth so that the pressure gradient is negligible. Flat plate boundary-layer behavior may be assumed for each plate. Explain why the Bernoulli equation is applicable to this flow. Estimate the diffuser width 1.2m downstream from the entrance for this condition.



Here, the flow through the diffuser is assumed as steady and incompressible. The in and out elevations of diffuser lie in same datum. Since the diffuser is open at both ends, the pressures at entrance and exit are equal. Based on these assumptions, the diffuser is modeled for Bernoulli's theorem.