

For $t < \tau_1$

$$T(x, t) = f(0) \phi'(x, t) + \int_{\tau=0}^t \phi(x, t-\tau) \left(\frac{df(\tau)}{d\tau} \right) d\tau$$

For $t \geq \tau_1$

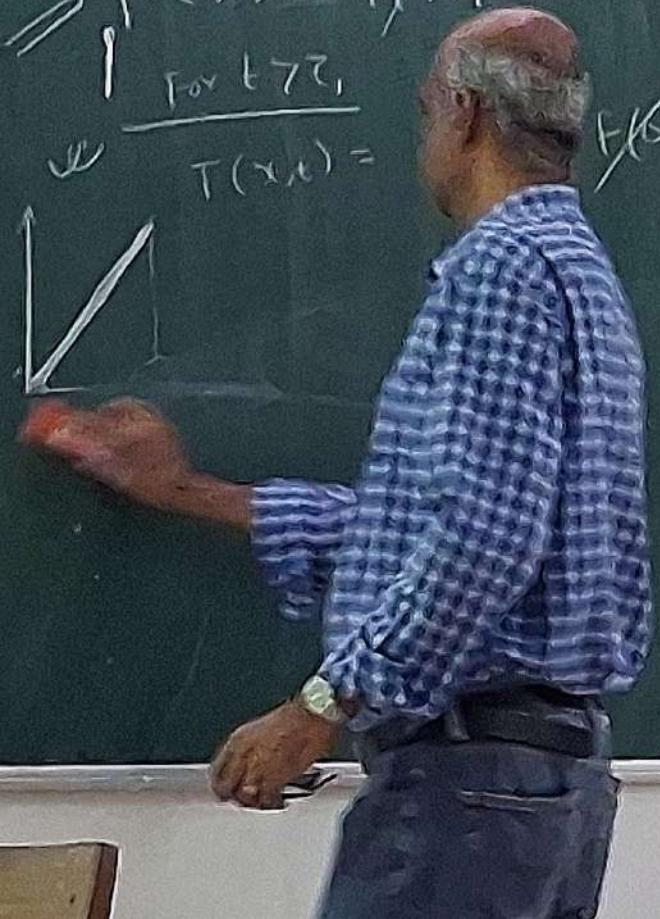
$$T(x, t) = \phi(x, t) f(0) + \phi(x, t-\tau_1) \Delta f_1 + \int_{\tau=\tau_1}^t \phi(x, t-\tau) \left(\frac{df(\tau)}{d\tau} \right) d\tau + \int_{\tau=\tau_1}^t \phi(x, t-\tau) \frac{df(\tau)}{d\tau} d\tau$$

$$\begin{matrix} \\ \\ \downarrow \\ 0 - b\tau_1 \\ = -b\tau_1 \end{matrix}$$

K.
t



$$\left\{ \begin{array}{l} \text{F0 例題} \\ T(x,t) = f(\phi(x,t)) + \int_0^t \phi(x,t-s) \frac{dF(s)}{ds} ds \end{array} \right.$$



Heat Transfer with Phase Change: Examples

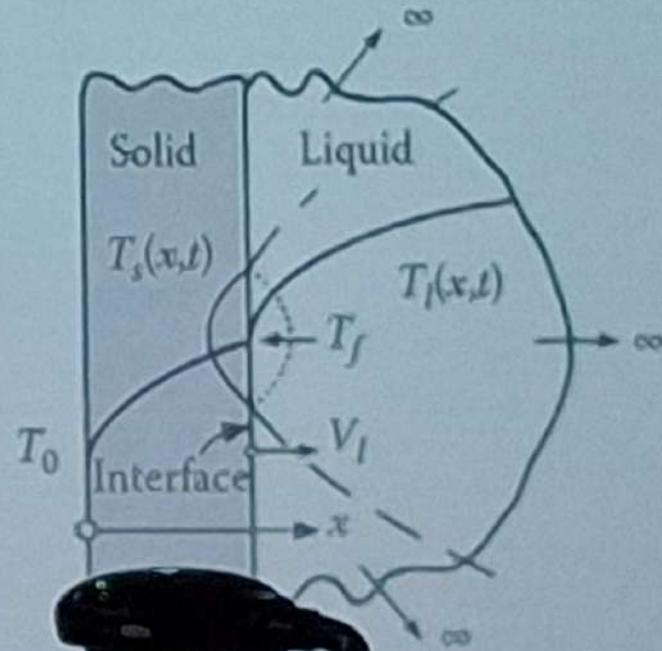
- Making of ice
- Freezing of food
- Solidification of metals in casting
- Crystal growth
- Aerodynamic ablation
- Casting and welding of metals and alloys
- Natural phenomena such as the cooling of large masses of igneous rock

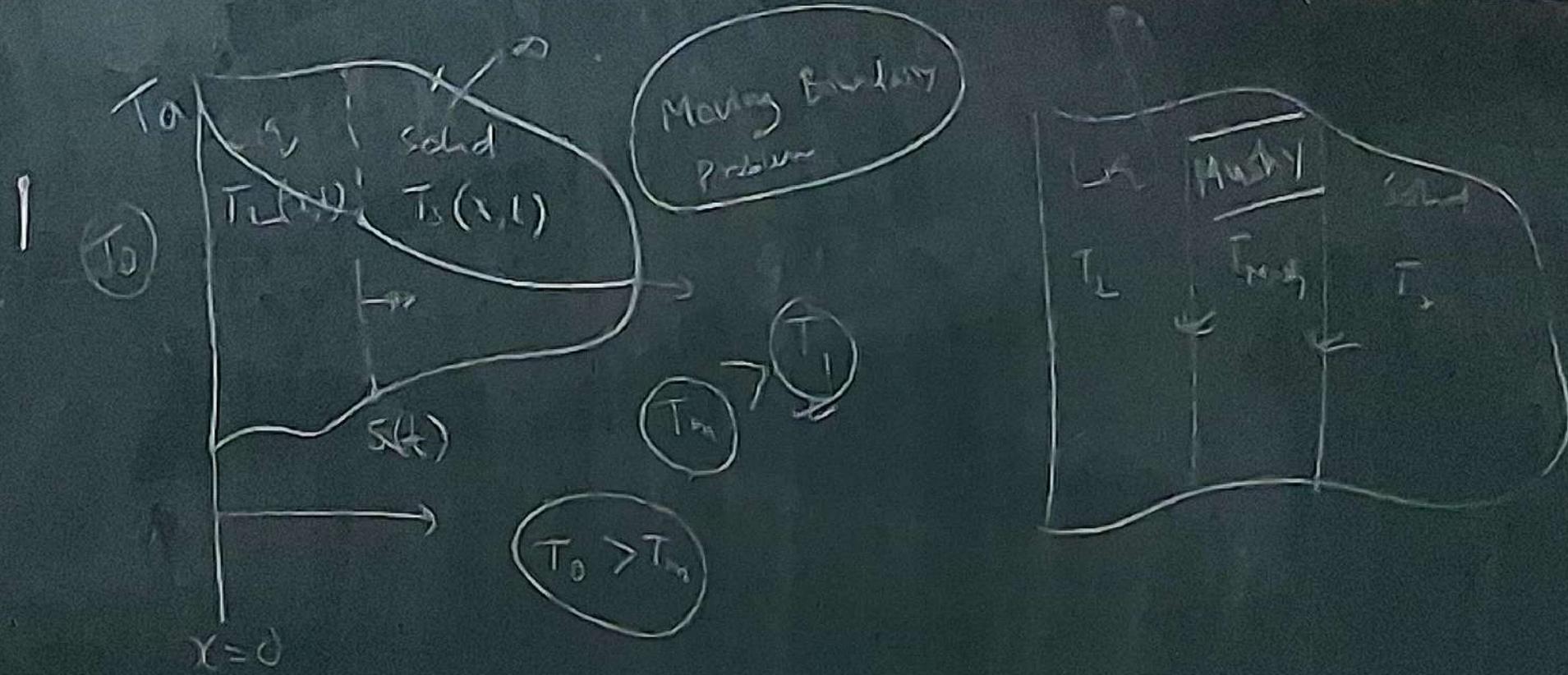
Solidification of Liquid: 1D

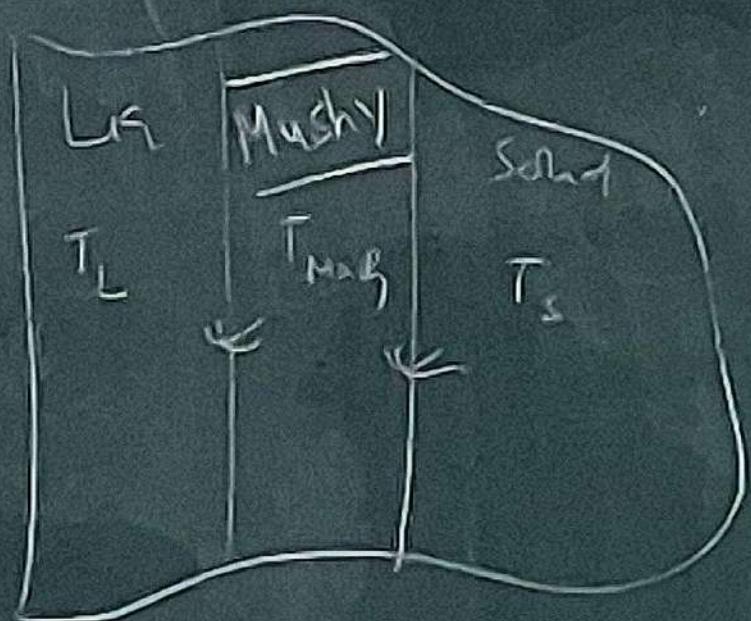
A liquid having a single phase-change temperature T_m (i.e., melting temperature, $T_m \equiv$ freezing temperature, T_f) is confined to a semi-infinite region $0 < x < \infty$.

Initially, the liquid is at a uniform temperature T_0 that is higher than the phase-change temperature T_m .

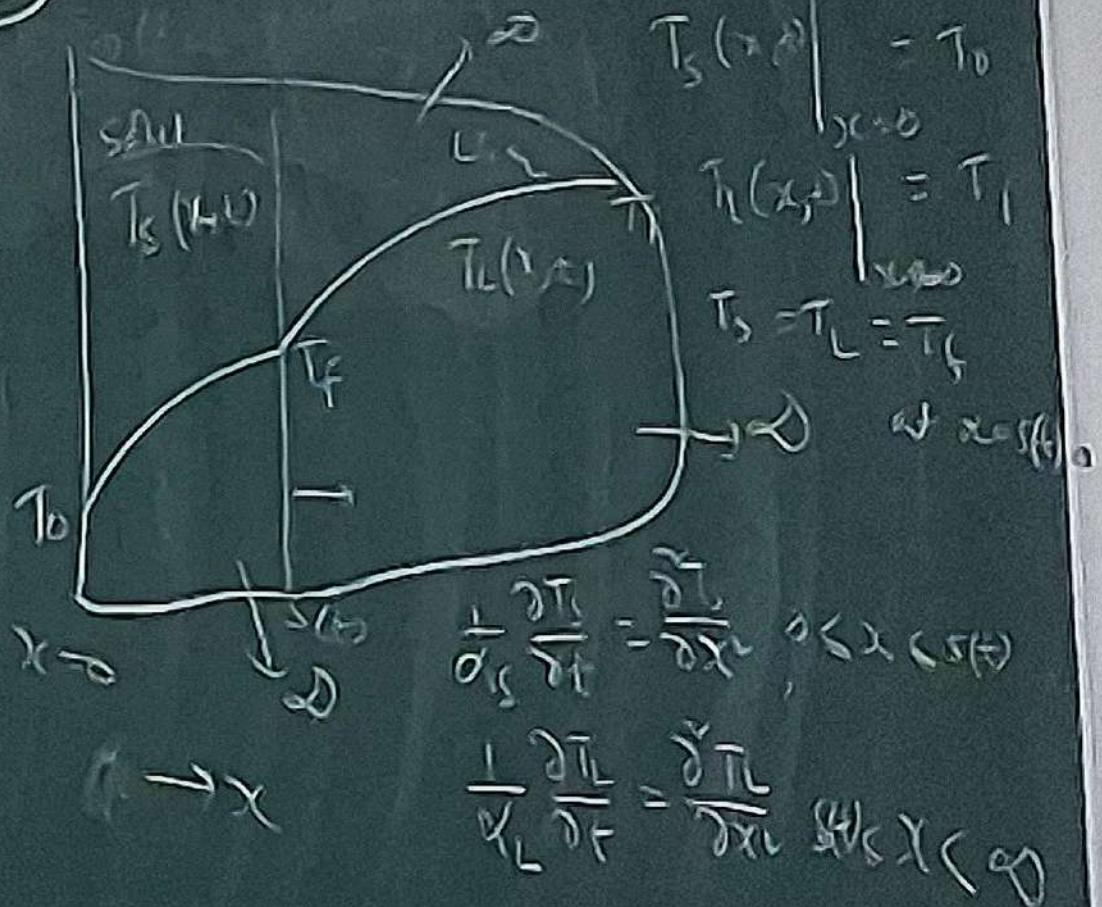
One-dimensional solidification problem







(1D)



$$T_S(x) \Big|_{x=0} = T_0$$

$$T_L(x_0) \Big|_{x=0} = T_1$$

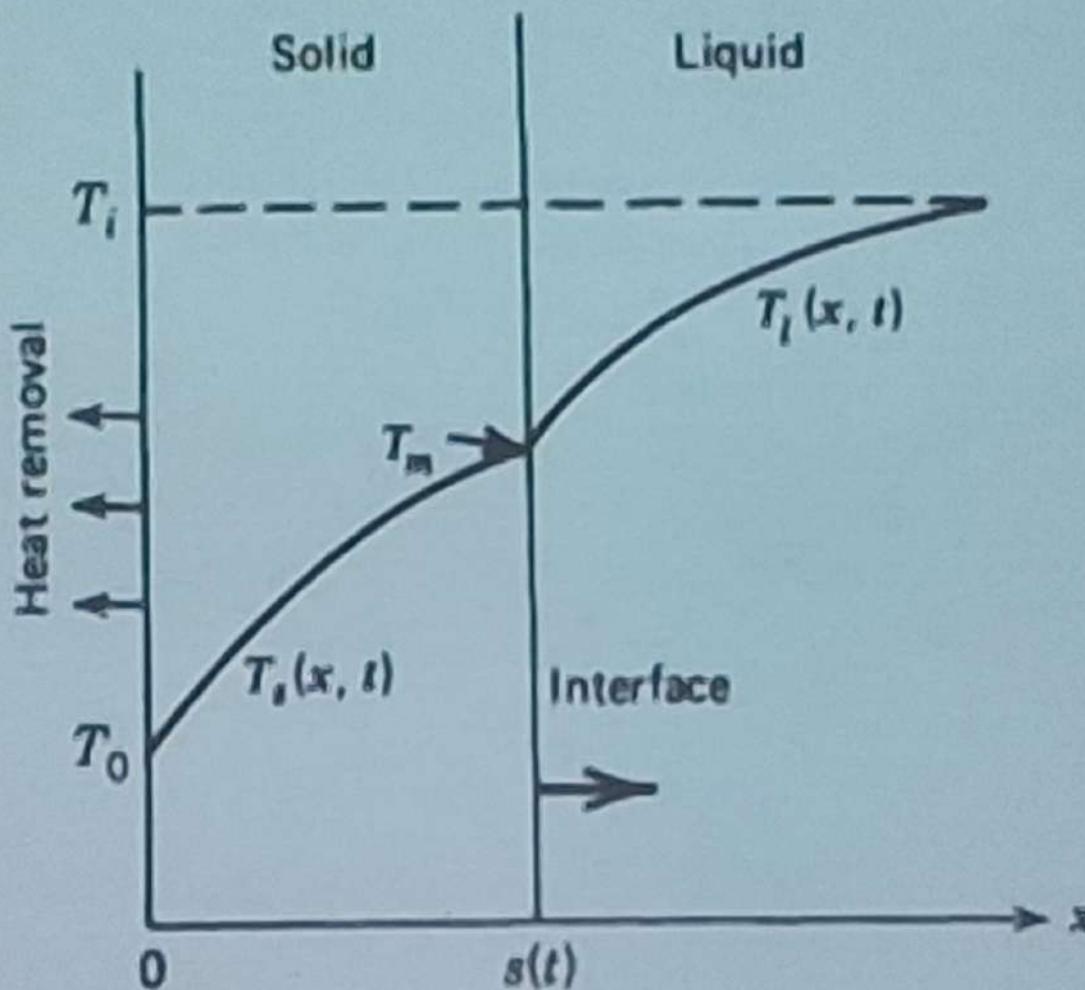
$$T_0 = T_L = T_f$$

at $x = x_0$

$$\frac{1}{\chi_L} \frac{\partial T_L}{\partial x} = \frac{\partial T_L}{\partial x}, \quad x < x_0$$

$$\frac{1}{\chi_L} \frac{\partial T_L}{\partial x} = \frac{\partial T_L}{\partial x}, \quad x > x_0$$

Solidification of Liquid: 1D



Solidification of liquid contained in semi-infinite area

Solidification of Liquid: Problem Formulation

$$\frac{\partial^2 T_s(x, t)}{\partial x^2} = \frac{1}{\alpha_s} \frac{\partial T_s(x, t)}{\partial t} \quad \text{in } 0 < x < s(t), \quad t > 0$$

$$\frac{\partial^2 T_l(x, t)}{\partial x^2} = \frac{1}{\alpha_l} \frac{\partial T_l(x, t)}{\partial t} \quad \text{in } s(t) < x < \infty, \quad t > 0$$

Standard diffusion equations

Assumed: Constant thermo-physical properties for the solid and liquid phases.

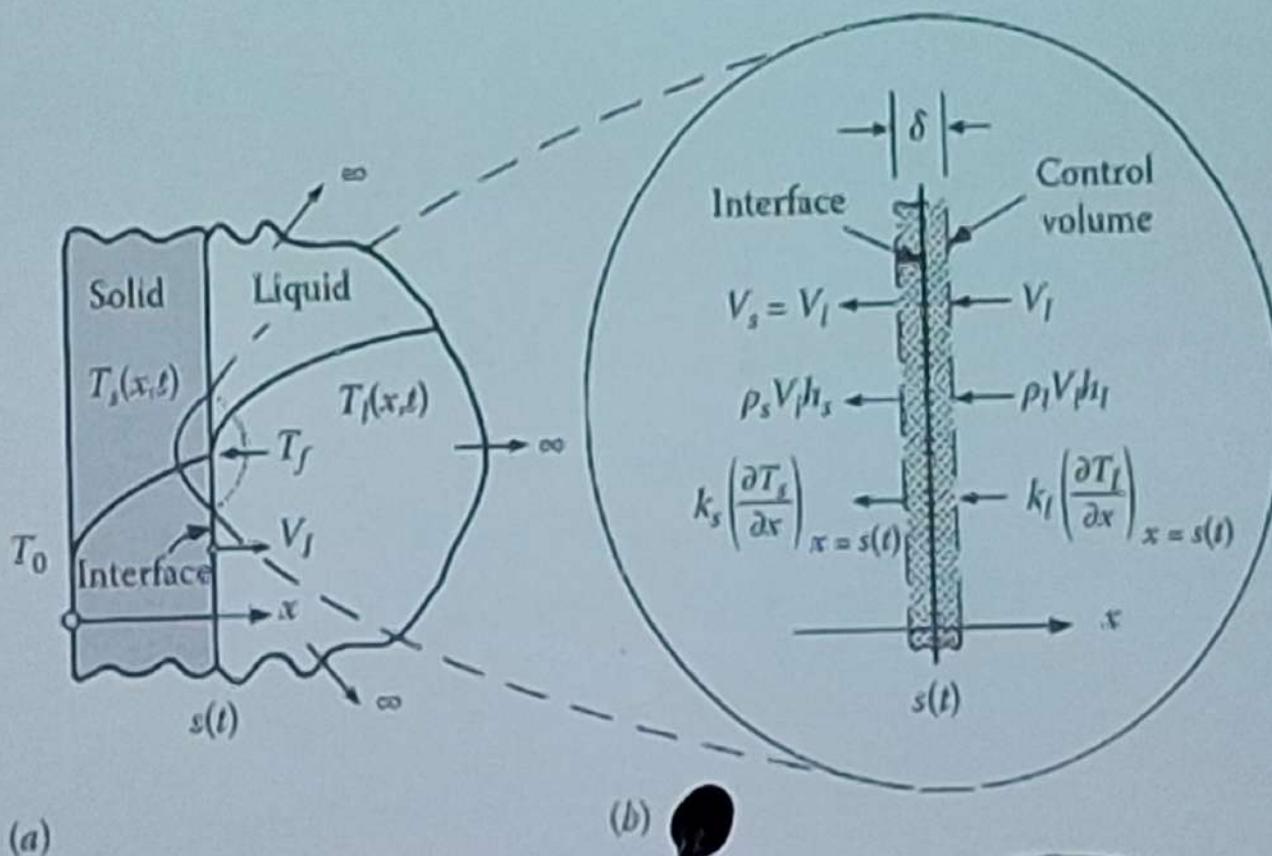
Here, $s(t)$ is the location of the solid–liquid interface which is not known a priori; hence it must be determined as a part of the solution.

The subscripts s and l refer, respectively, to the solid and liquid phases.

Solidification of Liquid: BC

Energy Balance at the Interface

An equation governing $s(t)$ is determined by considering an interface energy balance at $x = s(t)$.



Solidification of a liquid from a cold plane surface.

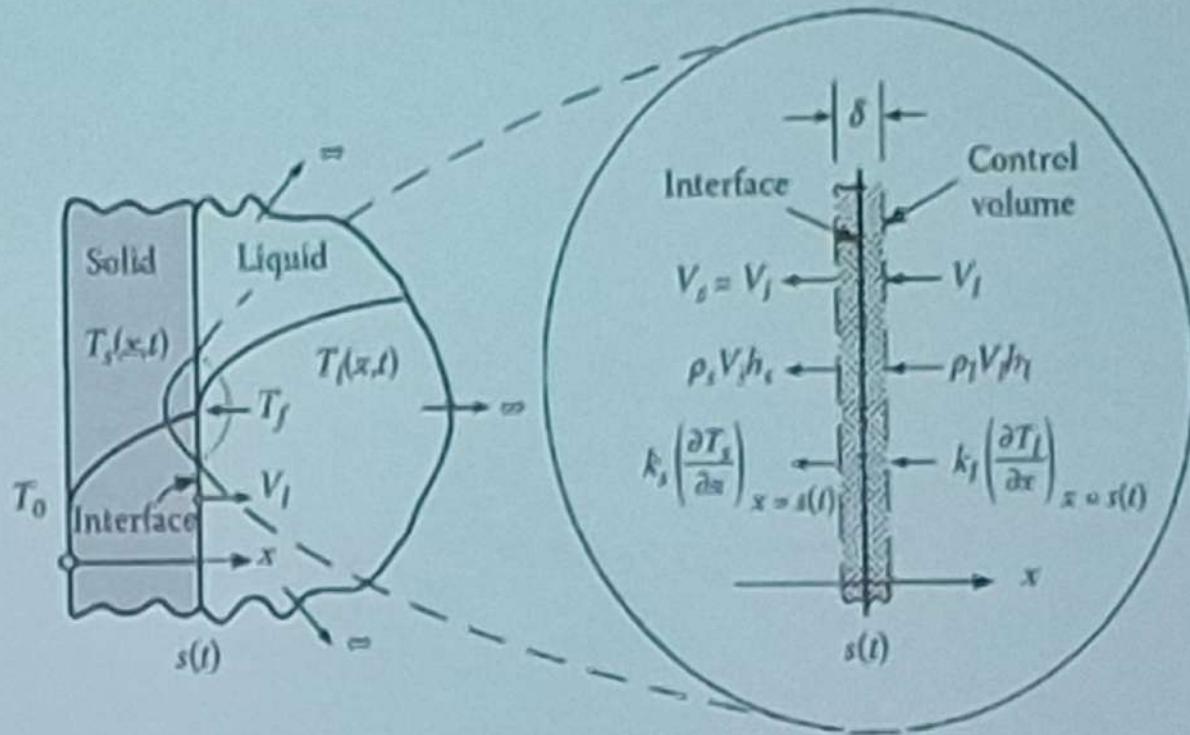
Solidification of Liquid: BC

The liquid starts to solidify at $\bar{x} = 0$ and the solid–liquid interface moves gradually in the positive x -direction with a velocity

$$V_I = \frac{ds}{dt}$$

The law of conservation of mass requires that

$$V_I = \frac{\rho_s}{\rho_l} V_s = \frac{\rho_s}{\rho_l} V_i$$



Referring to the above Figure an energy balance on the control volume per unit area give

$$\rho_l V_I h_l + k_l \left(\frac{\partial T_l}{\partial x} \right)_{x=s(t)} = \rho_s V_s h_s + k_s \left(\frac{\partial T_s}{\partial x} \right)_{x=s(t)}$$

Here k_l and k_s are the thermal conductivities, and h_l and h_s are the specific enthalpies, respectively, of the liquid and solid phases.

Solidification of Liquid: BC

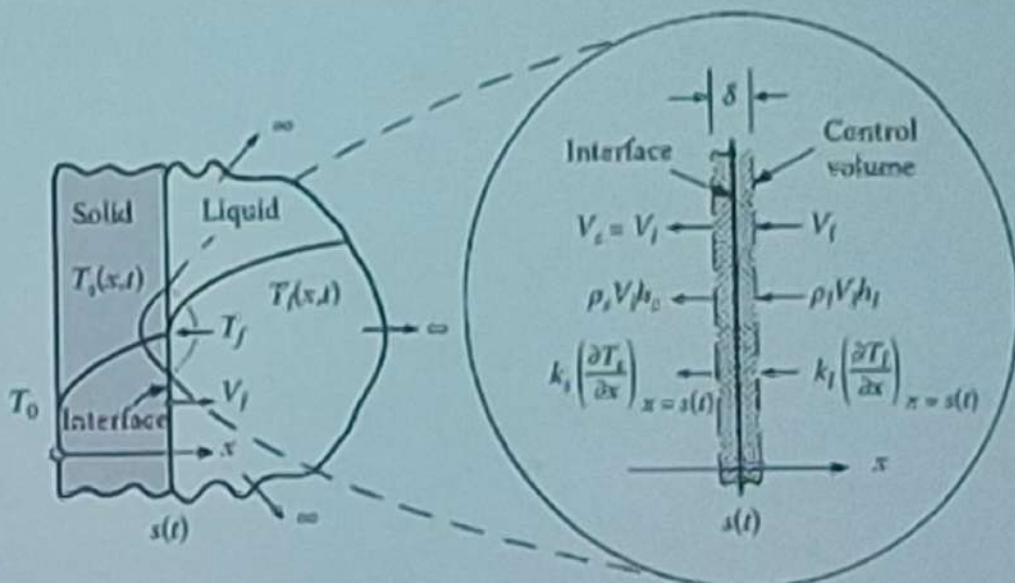
An energy balance on the control volume per unit area give

$$\rho_l V_l h_l + k_l \left(\frac{\partial T_l}{\partial x} \right)_{x=s(t)} = \rho_s V_s h_s + k_s \left(\frac{\partial T_s}{\partial x} \right)_{x=s(t)}$$

Use:

$$V_i = \frac{ds}{dt}$$

$$V_l = \frac{\rho_s}{\rho_l} V_s = \frac{\rho_s}{\rho_l} V_i$$



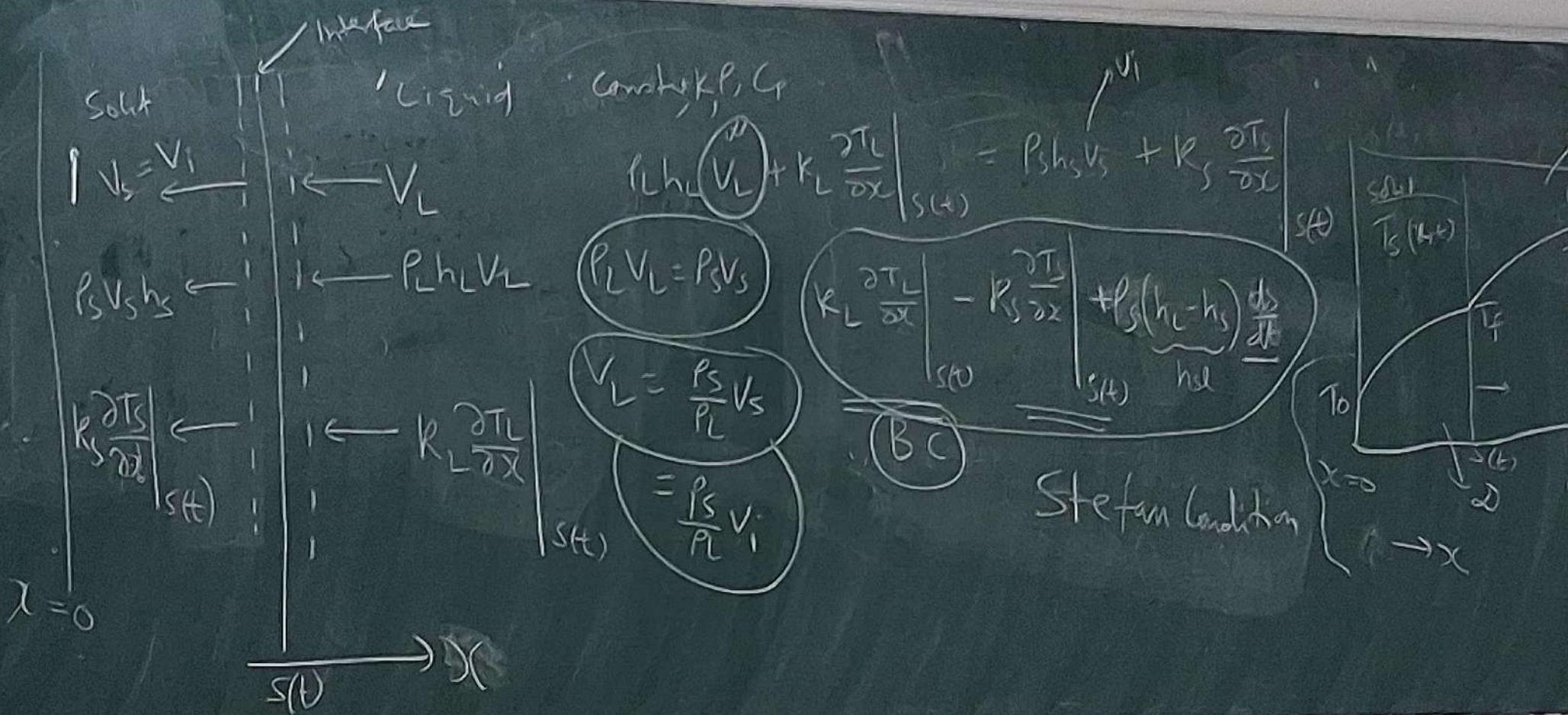
Combining:

$$k_l \left(\frac{\partial T_l}{\partial x} \right)_{x=s(t)} - k_s \left(\frac{\partial T_s}{\partial x} \right)_{x=s(t)} + \rho_s h_{sl} \frac{ds}{dt} = 0$$

where $h_{sl} = h_l - h_s$ is the latent heat of fusion.

Interface

Solid $P \downarrow = V_i$ $P_S V_{ShS} \leftarrow$ $\lambda \left \frac{\partial T_S}{\partial x} \right \leftarrow$ $\lambda = 0$	$'\text{Liquid}$ $i \leftarrow V_L$ $i \leftarrow P_L h_L V_L$ $i \leftarrow K_L \frac{\partial T_L}{\partial x} \Big _{S(t)}$ $\frac{\partial S(t)}{\partial x}$	$\text{Const } k, P, C_p$ $(P_L h_L V_L) + K_L \frac{\partial T_L}{\partial x} \Big _{S(t)} = P_S h_S V_S + K_S \frac{\partial T_S}{\partial x} \Big _{S(t)}$ $P_L V_L = P_S V_S$ $V_L = \frac{P_S}{P_L} V_S$ $= \frac{P_S}{P_L} V_i$	V_i $\frac{dS(t)}{dt} = P_S h_S V_i - P_S h_L V_i \Big _{x=0}$ $= P_S (h_S - h_L) \cdot d$
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Solidification of Liquid: BC

An equation governing $s(t)$ is determined by considering an interface energy balance at $x = s(t)$, stated as

$$\left[\begin{array}{l} \text{Conduction} \\ \text{heat flux from the} \\ \text{the liquid phase} \\ \text{in the negative} \\ x \text{ direction} \end{array} \right] + \left[\begin{array}{l} \text{Rate of heat} \\ \text{liberated during} \\ \text{solidification} \\ \text{per unit area of} \\ \text{interface} \end{array} \right] = \left[\begin{array}{l} \text{Conduction} \\ \text{heat flux into} \\ \text{the solid phase} \\ \text{in the negative} \\ x \text{ direction} \end{array} \right]$$

$$k_l \frac{\partial T_l}{\partial x} + \rho L \frac{ds(t)}{dt} = k_s \frac{\partial T_s}{\partial x} \quad \text{at} \quad x = s(t), \quad t > 0$$



Solidification of Liquid: BC

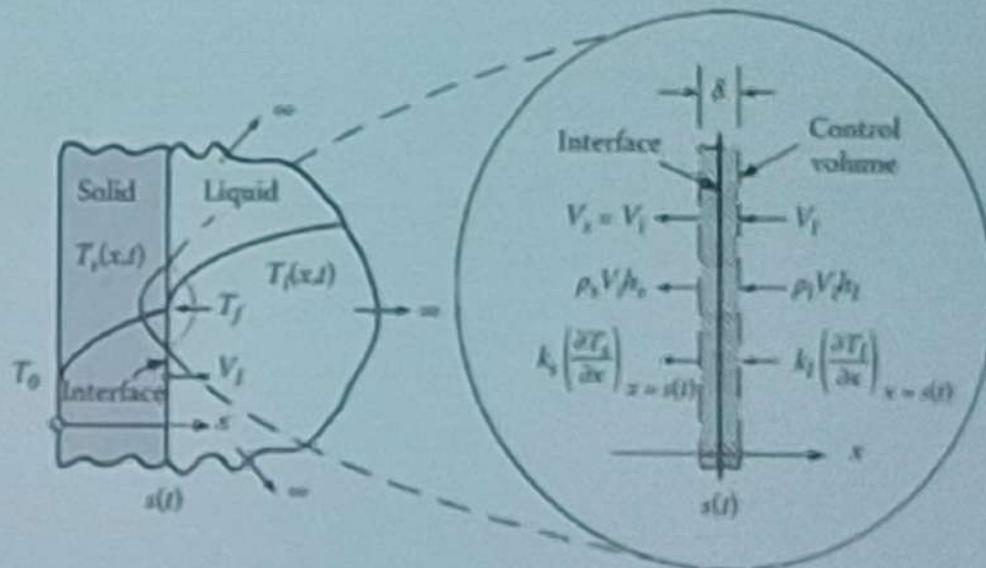
An energy balance on the control volume per unit area give

$$\rho_l V_l h_l + k_l \left(\frac{\partial T_l}{\partial x} \right)_{x=s(t)} = \rho_s V_s h_s + k_s \left(\frac{\partial T_s}{\partial x} \right)_{x=s(t)}$$

Use:

$$V_l = \frac{ds}{dt}$$

$$V_l = \frac{\rho_s}{\rho_l} V_s = \frac{\rho_s}{\rho_l} V_i$$



Combining:

$$k_l \left(\frac{\partial T_l}{\partial x} \right)_{x=s(t)} - k_s \left(\frac{\partial T_s}{\partial x} \right)_{x=s(t)} + \rho_s h_f \frac{ds}{dt} = 0$$

where $h_f = h_l - h_s$ is the latent heat of fusion.

This is Stefan condition, which states that the freezing front moves in a such way that its velocity is proportional to the jump in heat flux across the front.

Solidification of Liquid: Effect of Convection

Effect of Convection in the Liquid Phase:

$$k_l \left(\frac{\partial T_l}{\partial x} \right)_{x=s(t)} - k_s \left(\frac{\partial T_s}{\partial x} \right)_{x=s(t)} + \rho_s h_d \frac{ds}{dt} = 0$$

Stefan condition

How will it change?

where $h_d = h_l - h_s$ is the latent heat of fusion.

If the heat transfer in the liquid phase is controlled by convection,

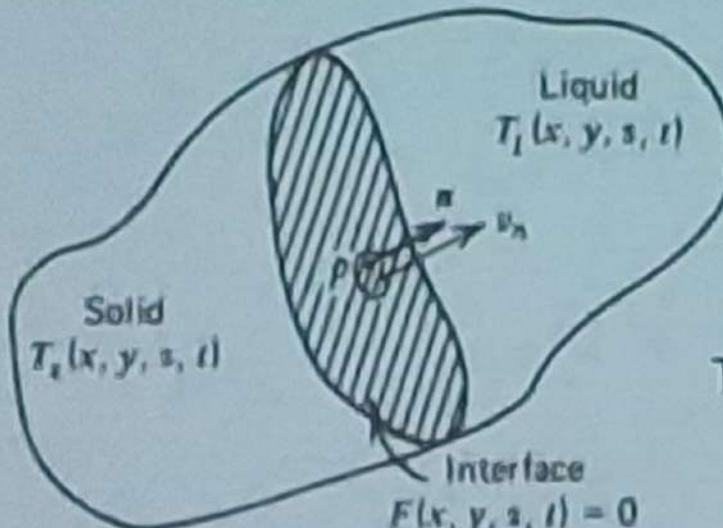
$$k_l \left(\frac{\partial T_l}{\partial x} \right)_{x=s(t)} = h(T_b - T_f)$$

where h is the heat transfer coefficient at the interface on the liquid side and T_b is the bulk temperature of the liquid phase.

Thus, the interface energy-balance relation (Stefan condition) can be rewritten as

$$h(T_b - T_f) - k_s \left(\frac{\partial T_s}{\partial x} \right)_{x=s(t)} + \rho_s h_{sl} \frac{ds}{dt} = 0$$

Stefan Condition: Multidimensional Generalization



The solid and liquid phases are separated by a sharp interface defined by the equation

$$F(x, y, z, t) = 0$$

The continuity of temperatures at the interface:

$$T_s(x, y, z, t) = T_l(x, y, z, t) = T_m$$

at $F(x, y, z, t) = 0$

The interface energy balance equation:

$$k_s \frac{\partial T_s}{\partial n} - k_l \frac{\partial T_l}{\partial n} = \rho L v_n \quad \text{at} \quad F(x, y, z, t) = 0$$

where $\partial/\partial n$ denotes the derivative along the interface along the normal direction vector \hat{n} at any location P on the interface and ρL is the latent heat of fusion of the liquid region, and v_n is the velocity of this interface at the location P in the direction \hat{n} .

$$\rightarrow k_L \frac{\partial T_L}{\partial x} - k_s \frac{\partial T_S}{\partial x} + \rho_s h_{SL} \frac{ds}{dt} \quad \text{at interface} \quad [x = s(t)] \quad S(t=0) = 0$$

$$T_S(x, t) = T_L(x, t) = T_f$$

$$\Rightarrow \frac{\partial T_S}{\partial x} dx + \frac{\partial T_S}{\partial t} dt = \frac{\partial T_L}{\partial x} dx + \frac{\partial T_L}{\partial t} dt = 0$$

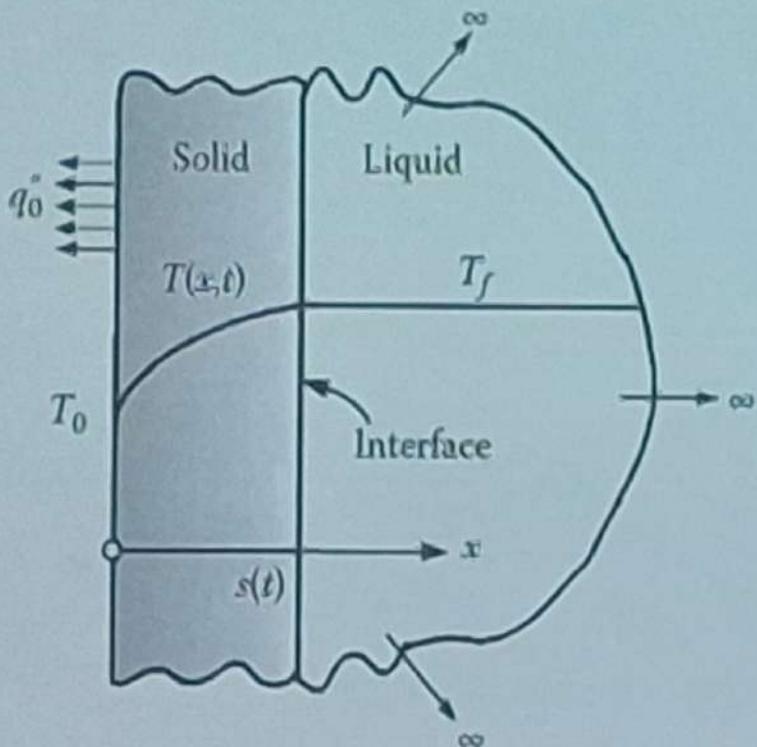
$$\Rightarrow \frac{\partial T_S}{\partial x} \frac{ds}{dt} + \frac{\partial T_S}{\partial t} = \frac{\partial T_L}{\partial x} \frac{ds}{dt} + \frac{\partial T_L}{\partial t} = 0$$

$$\Rightarrow \frac{ds}{dt} = - \frac{\partial T_S / \partial t}{\partial T_S / \partial x} = - \frac{\partial T_L / \partial t}{\partial T_L / \partial x}$$

Non-linear

Solidification of Liquid: Formulation

A Single-Region Phase-Change Problem:



$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

$$T(0, t) = T_0 \quad \text{and} \quad T(x, t) \Big|_{x=s(t)} = T_f$$

$$-k \frac{\partial T}{\partial x} \Big|_{x=s(t)} + \rho h_{sl} \frac{ds}{dt} = 0$$

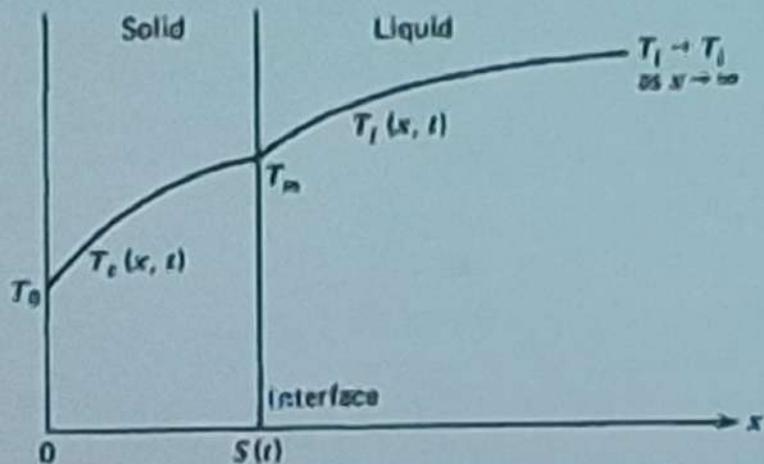
with the initial condition $s(0) = 0$.

Solidification of Liquid: Formulation

Two-Region Phase-Change Problem:

$$\frac{\partial^2 T_s}{\partial x^2} = \frac{1}{\alpha_s} \frac{\partial T_s}{\partial t}, \quad 0 < x < s(t)$$

$$\frac{\partial^2 T_l}{\partial x^2} = \frac{1}{\alpha_l} \frac{\partial T_l}{\partial t}, \quad s(t) < x < \infty$$



Solidification of Liquid

Stefan condition:

The conditions at $x = 0$ and as $x \rightarrow \infty$,

$$T_s(0, t) = T_0 \quad \text{and} \quad T_l(\infty, t) = T_i$$

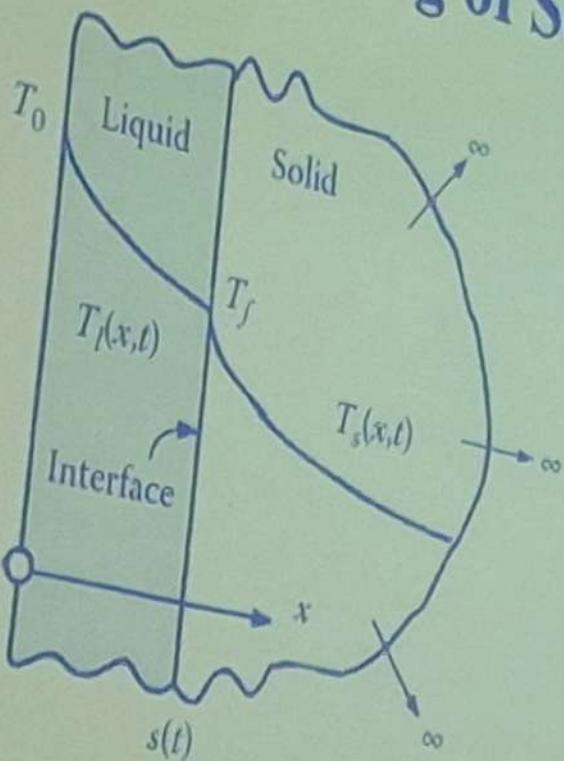
The interface conditions at $x = s(t)$,

$$T_s(s, t) = T_l(x, t) = T_f$$

$$k_l \left(\frac{\partial T_l}{\partial x} \right)_{x=s} - k_s \left(\frac{\partial T_s}{\partial x} \right)_{x=s} + \rho_s h_{sl} \frac{ds}{dt} = 0$$

The initial condition $s(0) = 0$

Melting of Solid: Stefan Condition



$$k_l \left(\frac{\partial T_l}{\partial x} \right)_{x=s(t)} - k_s \left(\frac{\partial T_s}{\partial x} \right)_{x=s(t)} + \rho_l h_{sl} \frac{ds}{dt} = 0$$

Homework!

$$\left[\begin{array}{l} \text{Conduction heat flux from the liquid phase in the positive } x \text{ direction} \end{array} \right] - \left[\begin{array}{l} \text{Conduction heat flux into the solid phase in the positive } x \text{ direction} \end{array} \right] = \left[\begin{array}{l} \text{Rate of heat absorbed during melting per unit area of interface} \end{array} \right]$$

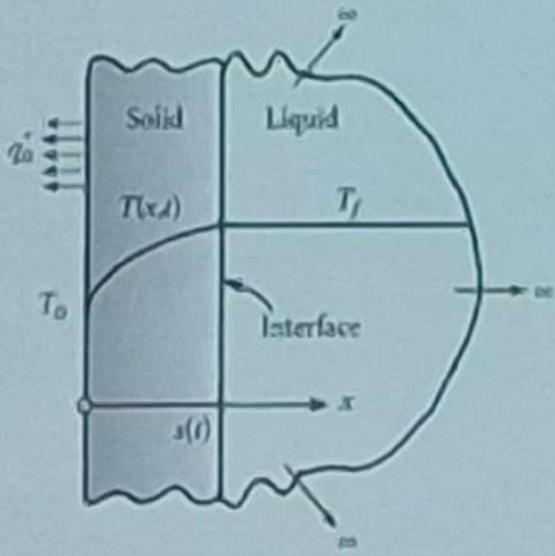
Solution of Phase Change Problems: Stefan's Exact Solution

Exact solutions are obtainable if the problem admits a *similarity solution*, allowing the two independent variables x and t to merge into a single similarity variable $\frac{x}{\sqrt{t}}$



Stefan's Exact Solution: Solidification

The solidification problem can be solved by utilizing the similarity method.



$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

$$T(0,t) = T_0 \quad \text{and} \quad T(x,t) \Big|_{x=s(t)} = T_f$$

$$-k \frac{\partial T}{\partial x} \Big|_{x=s(t)} + \rho h_{sl} \frac{ds}{dt} = 0$$

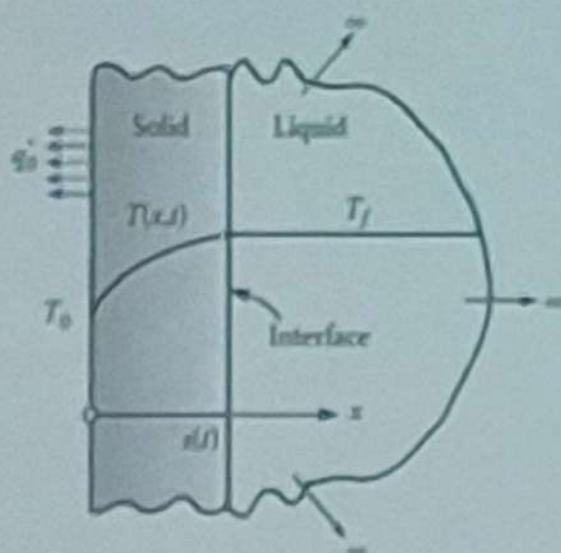
$$\text{Initial condition } s(0) = 0$$

It can be shown that this problem accepts a similarity solution in terms of the similarity variable

$$\eta = \frac{x}{2\sqrt{\alpha t}}$$

Define: $\theta(\eta) = T(\eta) - T_0$ and rewrite Heat Eq.

Stefan's Exact Solution: Solidification



$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

$$T(0,t) = T_0 \quad \text{and} \quad T(x,t) \Big|_{x=s(t)} = T_f$$

$$-k \frac{\partial T}{\partial x} \Big|_{x=s(t)} + \rho h_g \frac{ds}{dt} = 0 \quad \text{Initial condition } s(0) = 0$$

Define: $\theta(\eta) = T(\eta) - T_0$ and rewrite Heat Eq and BC

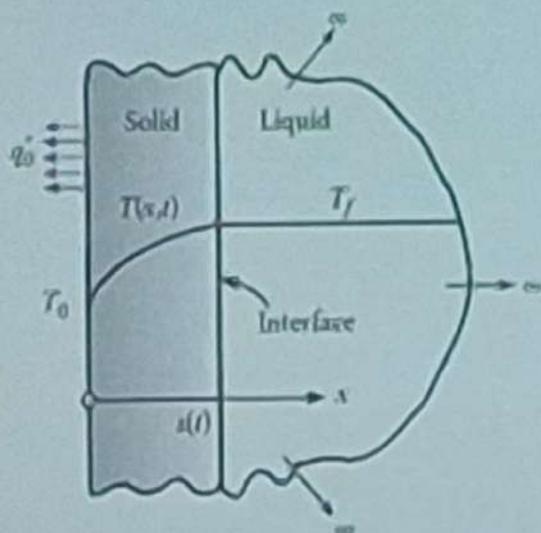
$$\frac{d^2\theta}{d\eta^2} + 2\eta \frac{d\theta}{d\eta} = 0 \quad \theta(0) = 0 \quad \text{and} \quad \theta(\lambda) = T_f - T_0 \quad \text{where: } \lambda = \frac{s(t)}{2\sqrt{\alpha t}}$$

The solution of the ordinary differential equation: $\theta(\eta) = A + B \operatorname{erf}(\eta)$

Imposing BC: $A = 0$ and $B = \frac{T_f - T_0}{\operatorname{erf}(\lambda)}$

where $\operatorname{erf}(\eta)$ is
the error function

Stefan's Exact Solution: Solidification



$$\theta(\eta) = A + B \operatorname{erf}(\eta)$$

$$\lambda = \frac{s(t)}{2\sqrt{\alpha t}}$$

$$A = 0 \quad \text{and} \quad B = \frac{T_f - T_0}{\operatorname{erf}(\lambda)}$$

Here note that, since B is to be a constant for all times, the parameter λ must also be a constant, independent of time t .

Substituting the constants A and B , the solution for the temperature distribution in the solid phase is obtained as

$$\frac{\theta(\eta)}{T_f - T_0} = \frac{\operatorname{erf}(\eta)}{\operatorname{erf}(\lambda)}$$

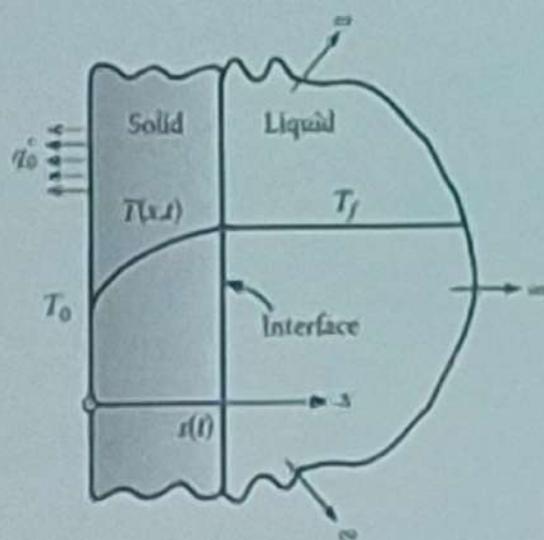
$$\frac{T(x,t) - T_0}{T_f - T_0} = \frac{\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)}{\operatorname{erf}(\lambda)}$$

The use of the Stefan condition: $\lambda e^{\lambda^2} \operatorname{erf}(\lambda) = \frac{1}{\sqrt{\pi}} \text{Ste}$ where $\text{Ste} = \frac{c(T_f - T_0)}{h_{sl}}$

where c is the specific heat of the solid phase.

Stefan Number
(Dimensionless)

Stefan's Exact Solution: Solidification



$$\frac{\theta(\eta)}{T_f - T_0} = \frac{\operatorname{erf}(\eta)}{\operatorname{erf}(\lambda)}$$
$$\frac{T(x,t) - T_0}{T_f - T_0} = \frac{\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)}{\operatorname{erf}(\lambda)}$$

The use of the Stefan condition: $\lambda e^{\lambda^2} \operatorname{erf}(\lambda) = \frac{1}{\sqrt{\pi}} \text{Ste}$

$$\text{Ste} = \frac{c(T_f - T_0)}{h_{sl}} \quad \text{Stefan Number}$$

where c is the specific heat of the solid phase.

The Stefan number represents the ratio of the *sensible heat* $c(T_f - T_0)$ to the *latent heat* h_{sl} .

The sensible heat is the energy added to a unit mass of solid to raise its temperature from T_0 to the fusion temperature T_f .

The latent heat is the energy released per unit mass during solidification at the fusion temperature.

Stefan's Exact Solution: Solidification

$$\text{Ste} = \frac{c(T_f - T_0)}{h_{sl}}$$

Stefan Number

where c is the specific heat of the solid phase.

The Stefan number represents the ratio of the sensible heat $c(T_f - T_0)$ to the latent heat h_{sl} .

The sensible heat is the energy added to a unit mass of solid to raise its temperature from T_0 to the fusion temperature T_f .

The latent heat is the energy released per unit mass during solidification at the fusion temperature.

Thus, one would expect that, for small values of Ste (for example, for most solidifying metals, $\text{Ste} < 1$), the solidification process is very rapid.

While for large values of Ste (for example, under laboratory conditions for water $\text{Ste} \sim 8$) the solidification process is much slower.

Stefan's Exact Solution: Solidification

$$Ste = \frac{c(T_f - T_0)}{h_{sl}}$$

Stefan Number

where c is the specific heat of the solid phase.

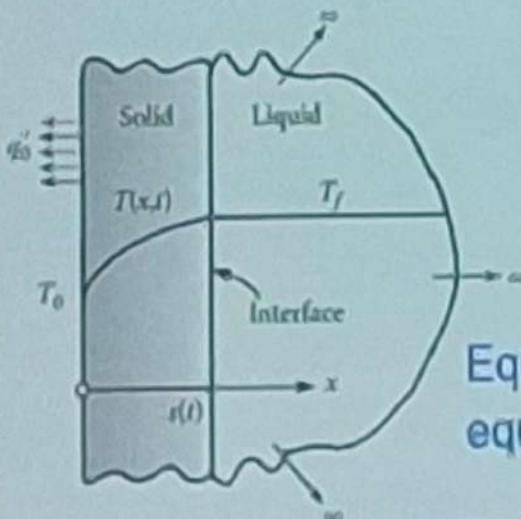
The Stefan number signifies the importance of sensible heat relative to the latent heat.

If the Stefan number is small, say, less than approximately 0.1, the heat released or absorbed by the interface during phase change is affected very little as a result of the variation of the sensible heat content of the material during the propagation of heat through the medium.

For materials such as aluminum, copper, iron, lead, nickel, and tin, the Stefan number based on a temperature difference between the melting temperature and the room temperature varies from about 1 to 3.

For melting or solidification processes taking place with much smaller temperature differences, the Stefan number is much smaller. For example, in phase-change problems associated with thermal energy storage, the temperature differences are small; as a result the Stefan number is generally smaller than 0.1.

Stefan's Exact Solution: Solidification



$$\frac{T(x,t) - T_0}{T_f - T_0} = \frac{\operatorname{erf}\left(\frac{x}{2\sqrt{at}}\right)}{\operatorname{erf}(\lambda)}$$
$$\lambda e^{\lambda^2} \operatorname{erf}(\lambda) = \frac{1}{\sqrt{\pi}} \text{Ste}$$
$$\text{Ste} = \frac{c(T_f - T_0)}{h_{sl}}$$

Equation (for λ) is a nonlinear transcendental algebraic equation and cannot be solved explicitly for the parameter λ .

For a given Ste, however, the solution for λ can be obtained numerically by means of a trial-and-error procedure.

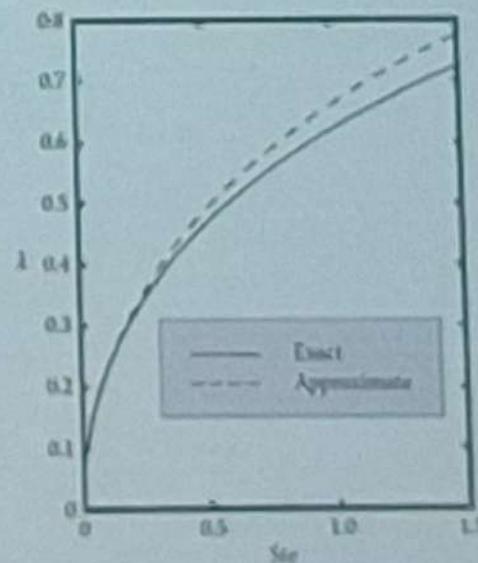
A plot of λ versus Ste is given in Figure.

Knowing λ , the instantaneous location of the interface $s(t)$ is obtained from

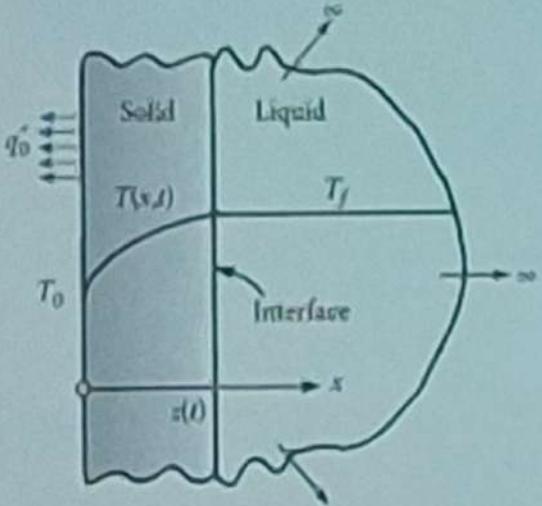
$$\lambda = \frac{s(t)}{2\sqrt{at}} \quad s(t) = 2\lambda\sqrt{at}$$

The temperature distribution $T(x, t)$ in the solid phase is given by

$$\frac{T(x,t) - T_0}{T_f - T_0} = \frac{\operatorname{erf}\left(\frac{x}{2\sqrt{at}}\right)}{\operatorname{erf}(\lambda)}$$



Stefan's Exact Solution: Solidification



$$\frac{T(x,t) - T_0}{T_f - T_0} = \frac{\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)}{\operatorname{erf}(\lambda)}$$

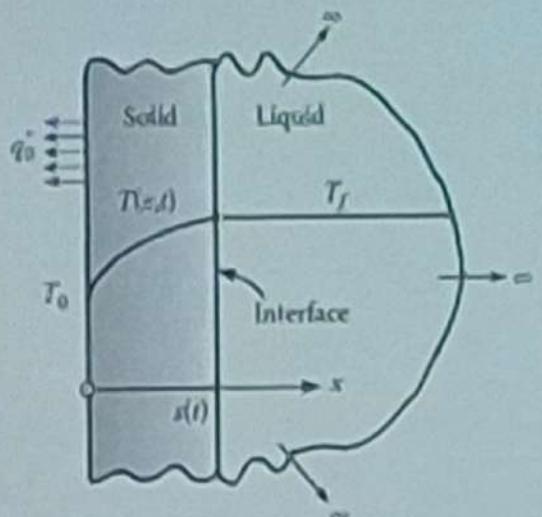
$$s(t) = 2\lambda\sqrt{\alpha t}$$

Once the temperature distribution is known, the heat flux removed from the surface at $x = 0$ to maintain its temperature at T_0 is obtained from Fourier's law:

$$\begin{aligned} q_0'' &= k \left(\frac{\partial T}{\partial x} \right)_{x=0} = \frac{k(T_f - T_0)}{\operatorname{erf}(\lambda)} \left[\frac{d}{d\eta} \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right) \right]_{x=0} \\ &= \frac{k(T_f - T_0)}{\sqrt{\pi\alpha\operatorname{erf}(\lambda)}} t^{-1/2} \end{aligned}$$

Clearly, the surface heat flux decreases as time increases. This is obviously the result of increasing thermal resistance due to the increasing solid phase thickness separating the liquid region from the surface at $x = 0$.

Stefan's Exact Solution: Solidification

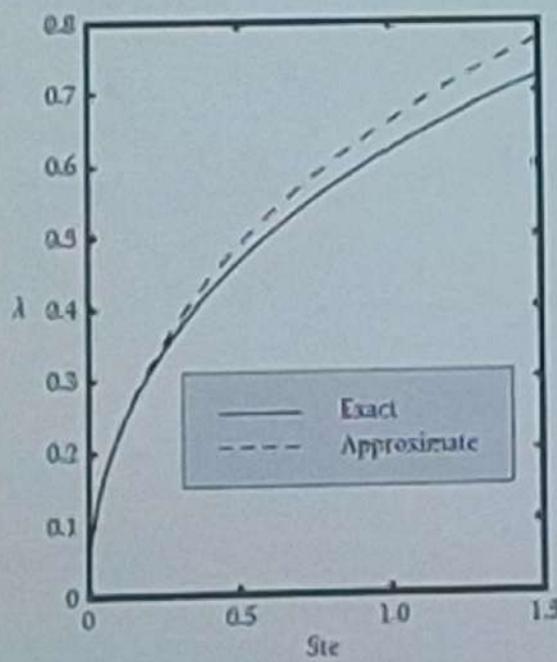


$$\frac{T(x,t) - T_0}{T_f - T_0} = \frac{\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)}{\operatorname{erf}(\lambda)}$$

$$s(t) = 2\lambda\sqrt{\alpha t}$$

Here we note from Figure that when Ste is small, λ is also small. On the other hand, when λ is small,

$$e^{\lambda^2} = 1 + \lambda^2 + \frac{1}{2!} \lambda^4 + \dots \approx 1$$



$$\operatorname{erf}(\lambda) = \frac{2\lambda}{\sqrt{\pi}} \left(1 - \frac{\lambda^2}{3 \times 1!} + \frac{\lambda^4}{5 \times 2!} + \dots \right) \approx \frac{2\lambda}{\sqrt{\pi}}$$

Therefore, from $\lambda e^{\lambda^2} \operatorname{erf}(\lambda) = \frac{1}{\sqrt{\pi}} \text{Ste}$

$$\lambda \approx \frac{\sqrt{\text{Ste}}}{\sqrt{2}}$$

for small Ste

Note that this result is accurate to the first order.

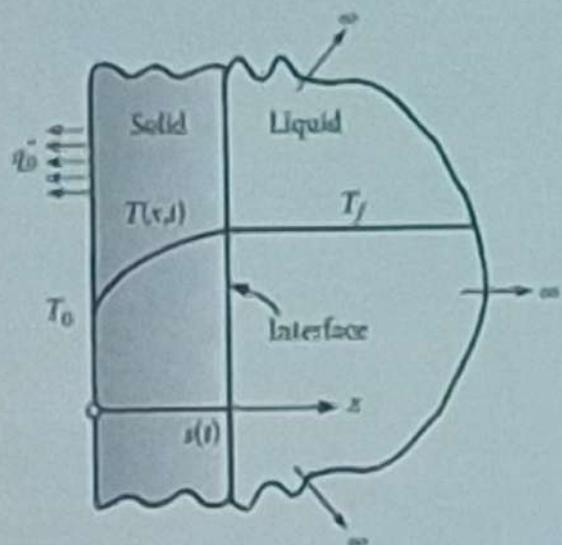
On substitution:

A black computer mouse is visible in the bottom right corner of the slide.

$$s(t) = \sqrt{2\text{Ste}t}$$

for small Ste

Stefan's Exact Solution: Solidification



$$\frac{T(x,t) - T_0}{T_f - T_0} = \frac{\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)}{\operatorname{erf}(\lambda)}$$

$$s(t) = 2\lambda\sqrt{\alpha t}$$

$$\lambda \equiv \sqrt{\frac{\text{Ste}}{2}} \text{ for small Ste} \quad s(t) \equiv \sqrt{2\text{Ste}\alpha t} \text{ for small Ste}$$

This result is consistent with the previous discussion that the case of small Ste corresponds to a slow solidification process.

For small values of Ste, the temperature distribution and heat flux are given by:

$$\frac{T(x,t) - T_0}{T_f - T_0} \approx \frac{x}{s(t)} \text{ for small Ste}$$

$$q_0'' \equiv \frac{k(T_f - T_0)}{s(t)} \text{ for small Ste}$$

Stefan's Condition in Dimensionless Form

The interface energy balance equation (Stefan's Condition) can be expressed in the dimensionless form as

$$\frac{\partial \theta_s}{\partial \eta} - \frac{k_l}{k_s} \frac{\partial \theta_l}{\partial \eta} = \frac{1}{\text{Ste}} \frac{d\delta(\tau)}{d\tau}$$

where

$$\theta_i(\tau, \eta) = \frac{T_i(x, t) - T_m}{T_m - T_0}, \quad i = s \text{ or } l; \quad \eta = x/b$$

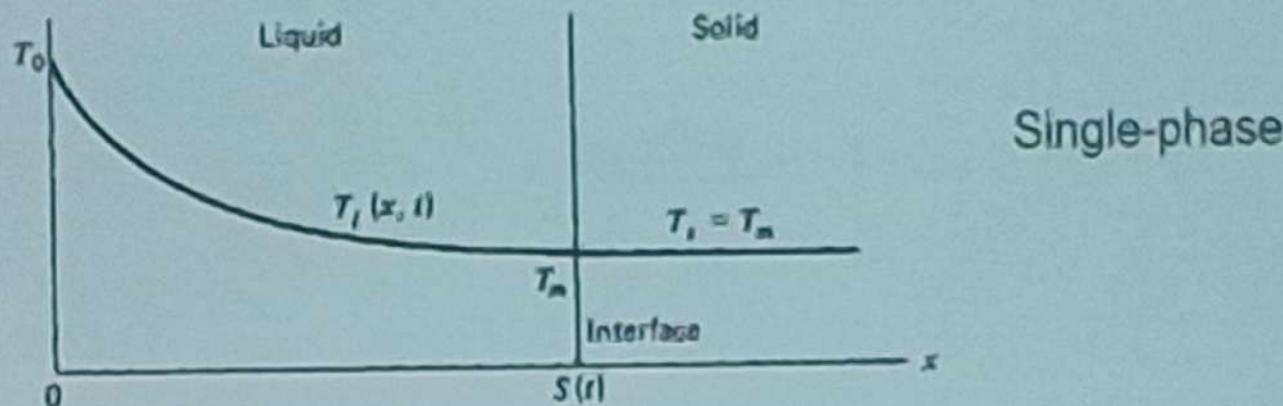
$$\delta(\tau) = \frac{s(t)}{b}; \quad \tau = \frac{\alpha_s t}{b^2}; \quad \text{Ste} = \frac{C_s(T_m - T_0)}{L}$$

Here, b is a reference or characteristic length, L is the latent heat, C_s is the specific heat, T_m is the melting temperature, T_0 is a reference temperature, $s(t)$ is the location of the solid-liquid interface, and Ste is the Stefan number.

Solution of Phase Change Problems: Integral Method

The Integral method provides a relatively simple and straightforward approach for the solution of one-dimensional transient phase-change problems.

Melting of Solid: Single Phase: Integral Method



$$\frac{\partial^2 T_l}{\partial x^2} = \frac{1}{\alpha_l} \frac{\partial T_l(x, t)}{\partial t} \quad \text{in } 0 < x < s(t), \quad t > 0$$

$$T_s = T_m \quad \text{in } x > s(t), \quad t > 0$$

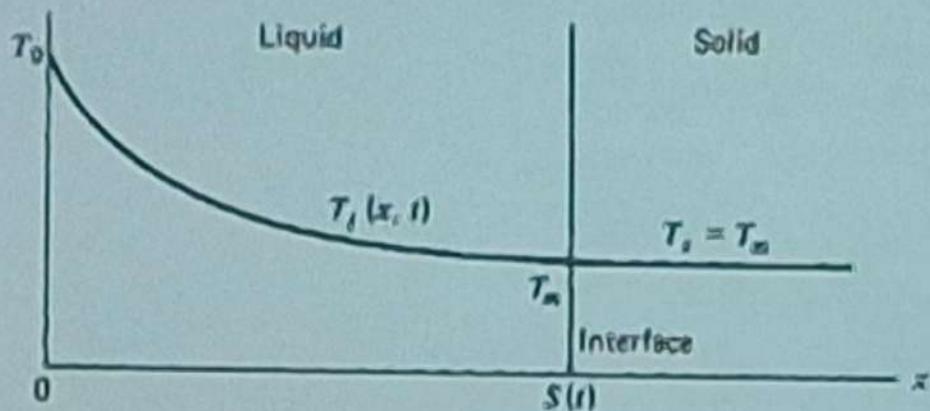
Boundary condition: $T_l(x = 0, t) = T_0$

At the interface: $T_l(x = s, t) = T_m \quad \text{at } x = s(t)$

$$-k_l \left. \frac{\partial T_l}{\partial x} \right|_{x=s(t)} = \rho L \frac{ds(t)}{dt} \quad \text{at } x = s(t)$$

$$s(0) = 0$$

Melting of Solid: Single Phase: Integral Method



First, define a *thermal layer thickness*:

We choose the region $0 \leq x \leq s(t)$ as the thermal layer

Next, integrate the heat conduction equation from $x = 0$ to $x = s(t)$,

$$\int_{x=0}^{s(t)} \frac{\partial^2 T}{\partial x^2} dx = \frac{1}{\alpha} \int_{x=0}^{s(t)} \frac{\partial T}{\partial t} dx$$

Melting of Solid: Single Phase: Integral Method

$$\int_{x=0}^{s(t)} \frac{\partial^2 T}{\partial x^2} dx = \frac{1}{\alpha} \int_{x=0}^{s(t)} \frac{\partial T}{\partial t} dx$$

LHS: $\int_{x=0}^{s(t)} \frac{\partial^2 T}{\partial x^2} dx = \left. \frac{\partial T}{\partial x} \right|_0^{s(t)}$

RHS: $\frac{1}{\alpha} \int_{x=0}^{s(t)} \frac{\partial T}{\partial t} dx = \frac{1}{\alpha} \left[\frac{d}{dt} \left(\int_{x=0}^{s(t)} T dx \right) - \left. \frac{ds(t)}{dt} T \right|_{x=s(t)} \right]$



$$\left. \frac{\partial T}{\partial x} \right|_{x=s(t)} - \left. \frac{\partial T}{\partial x} \right|_{x=0} = \frac{1}{\alpha} \left[\frac{d}{dt} \left(\int_{x=0}^{s(t)} T dx \right) - \left. \frac{ds(t)}{dt} T \right|_{x=s(t)} \right]$$

Melting of Solid: Single Phase: Integral Method

$$\frac{\partial T}{\partial x} \Big|_{x=s(t)} - \frac{\partial T}{\partial x} \Big|_{x=0} = \frac{1}{\alpha} \left[\frac{d}{dt} \left(\int_{x=0}^{s(t)} T \, dx \right) - \frac{ds(t)}{dt} T \Big|_{x=s(t)} \right]$$

$$\rightarrow -\frac{\rho L}{k} \frac{ds(t)}{dt} - \frac{\partial T}{\partial x} \Big|_{x=0} = \frac{1}{\alpha} \frac{d}{dt} [\theta - s(t)T_m]$$
$$\theta(x) \equiv \int_{x=0}^{s(t)} T(x, t) \, dx$$

This equation is the *energy integral equation* for this problem.

$$\int_{x=0}^{s(t)} \frac{\partial T_L}{\partial x} dx = \frac{1}{\alpha} \int_{t=0}^{s(t)} \frac{\partial T_L}{\partial t} dt$$

$$T_L|_{x=s(t)} = T_{L_0}$$

$$-k_L \frac{\partial T_L}{\partial x} \Big|_{s(t)} = \rho_L \frac{ds}{dt} + x \dot{s}(t)$$

Lagathet

$$\frac{10}{N} - k \frac{ds}{dt}$$

$$- \frac{1}{N} [\theta - s(t) T_{L_0}]$$

Melting of Solid: Single Phase: Integral Method

$$\frac{\partial T}{\partial x} \Big|_{x=s(t)} - \frac{\partial T}{\partial x} \Big|_{x=0} = \frac{1}{\alpha} \left[\frac{d}{dt} \left(\int_{x=0}^{s(t)} T \, dx \right) - \frac{ds(t)}{dt} T \Big|_{x=s(t)} \right]$$

$$\rightarrow -\frac{\rho L}{k} \frac{ds(t)}{dt} - \frac{\partial T}{\partial x} \Big|_{x=0} = \frac{1}{\alpha} \frac{d}{dt} [\theta - s(t)T_m]$$
$$\theta(x) \equiv \int_{x=0}^{s(t)} T(x, t) \, dx$$

This equation is the energy integral equation for this problem.

We choose a second-degree polynomial approximation for the temperature in the form

$$T(x, t) = a + b(x - s) + c(x - s)^2$$

Melting of Solid: Single Phase: Integral Method

$$T(x, t) = a + b(x - s) + c(x - s)^2$$

Three conditions are needed to determine these three coefficients.

$$T(x = 0, t) = T_0$$

First and
Second
conditions

$$T(x = s(t), t) = T_m$$

Use of Stefan Condition is inconvenient

Derive Third Condition:

Differentiate BC: $T_l(x = s, t) = T_m$ at $x = s(t)$

$$dT \equiv \left[\frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial t} dt \right]_{x=s(t)} = 0 \quad \rightarrow \quad \frac{\partial T}{\partial x} \frac{ds(t)}{dt} + \frac{\partial T}{\partial t} = 0$$

Eliminate $ds(t)/dt$ using
Stefan Condition

$$\left(\frac{\partial T}{\partial x} \right)^2 = \frac{\rho L}{k} \frac{\partial T}{\partial t} \quad \text{at } x = s(t)$$

$$\text{Eliminate } \frac{\partial T}{\partial t} \text{ using Heat Eq.} \quad \left(\frac{\partial T}{\partial x} \right)^2 = \frac{\alpha \rho L}{k} \frac{\partial^2 T}{\partial x^2} \quad \text{at } x = s(t)$$

Third Condition

Melting of Solid: Single Phase: Integral Method

The resulting temperature profile becomes

$$T(x, t) = T_m + b(x - s) + c(x - s)^2$$

where

$$b = \frac{\alpha \rho L}{ks} [1 - (1 + \mu)^{1/2}]$$

$$c = \frac{bs + (T_0 - T_m)}{s^2}$$

$$\mu = \frac{2k}{\alpha \rho L} (T_0 - T_m) = \frac{2C(T_0 - T_m)}{L}$$

Melting of Solid: Single Phase: Integral Method

Substituting the temperature profile into the energy integral equation, we obtain the following ordinary differential equation for the determination of the location of the solid–liquid interface $s(t)$, that is,

$$s \frac{ds}{dt} = 6\alpha \frac{1 - (1 + \mu)^{1/2} + \mu}{5 + (1 + \mu)^{1/2} + \mu} \quad \text{with } s(t=0) = 0$$

The solution of the above ODE:

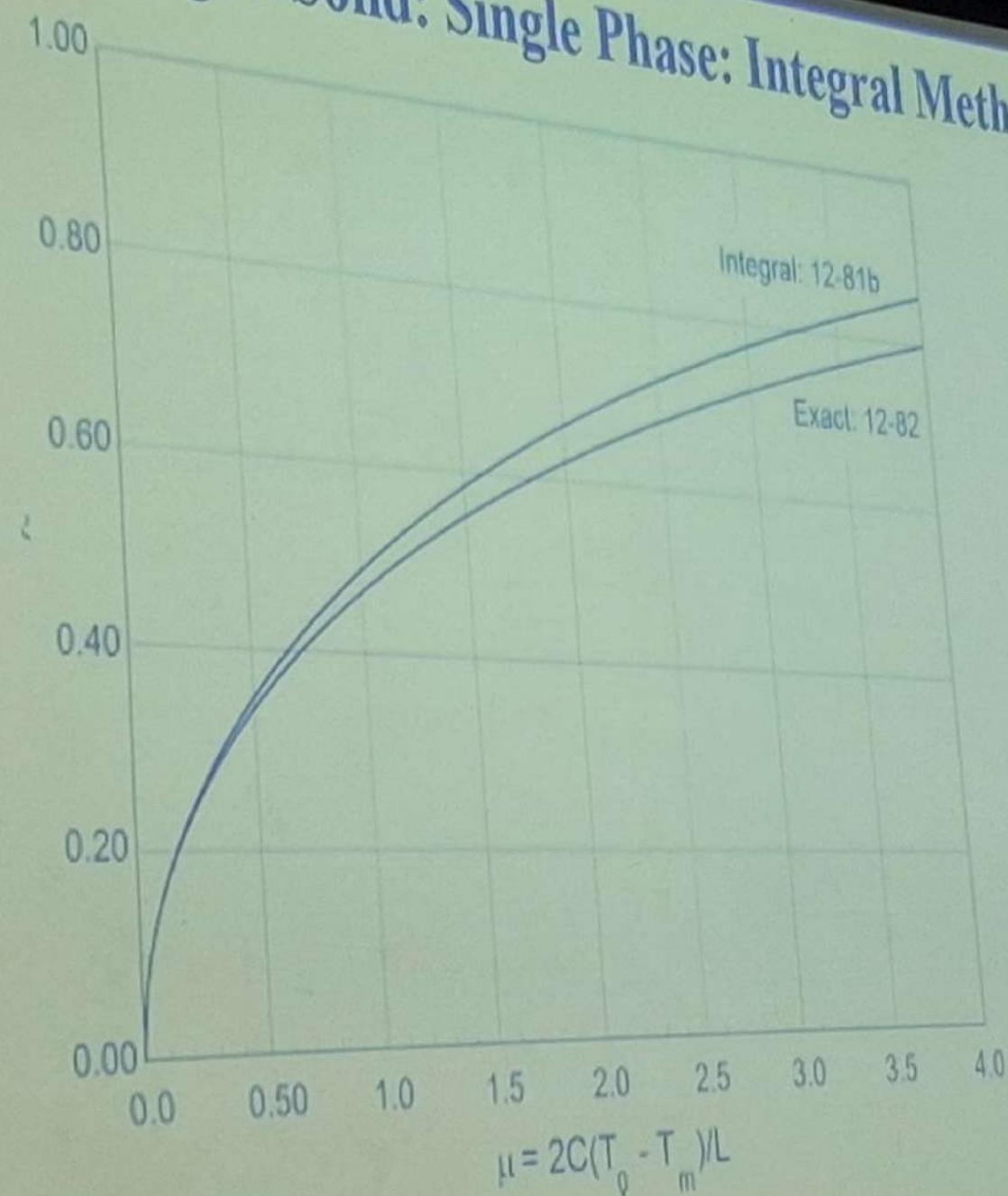
$$s(t) = 2\lambda\sqrt{\alpha t} \quad \text{where} \quad \lambda \equiv \left[3 \frac{1 - (1 + \mu)^{1/2} + \mu}{5 + (1 + \mu)^{1/2} + \mu} \right]^{1/2}$$

Exact Solution (Similarity Solution) for the Same Problem:

$$s(t) = 2\lambda(\alpha_1 t)^{1/2} \quad \text{where } \lambda \text{ is the root of} \quad \lambda e^{\lambda^2} \operatorname{erf}(\lambda) = \frac{C(T_0 - T_m)}{L\sqrt{\pi}}$$

NOTE: Our approximate solution for $s(t)$ is of the same form as the exact solution of the same problem. However, definition of λ is different.

Melting of Solid: Single Phase: Integral Method



$\mu = 2C(T_0 - T_m)/L$

Comparison of exact and approximate solutions