Case

$$\underline{\underline{A}} = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \Rightarrow \begin{array}{l} \lambda_1 = ib \rightarrow v_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} \\ \lambda_2 = -ib \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1 e^{ibt} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$e^{ibt} = \cos bt + i \sin bt$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \cancel{c_1} \left[\cos bt + i \sin bt\right] \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \cancel{c_1} \begin{bmatrix} \cos bt + i \sin bt \\ -\sin bt + i \cos bt \end{bmatrix}$$

$$= \cancel{c_1} \begin{bmatrix} \cos bt \\ -\sin bt \end{bmatrix} + i \begin{bmatrix} \sin bt \\ \cos bt \end{bmatrix}$$

$$(Re + i \; Im)$$

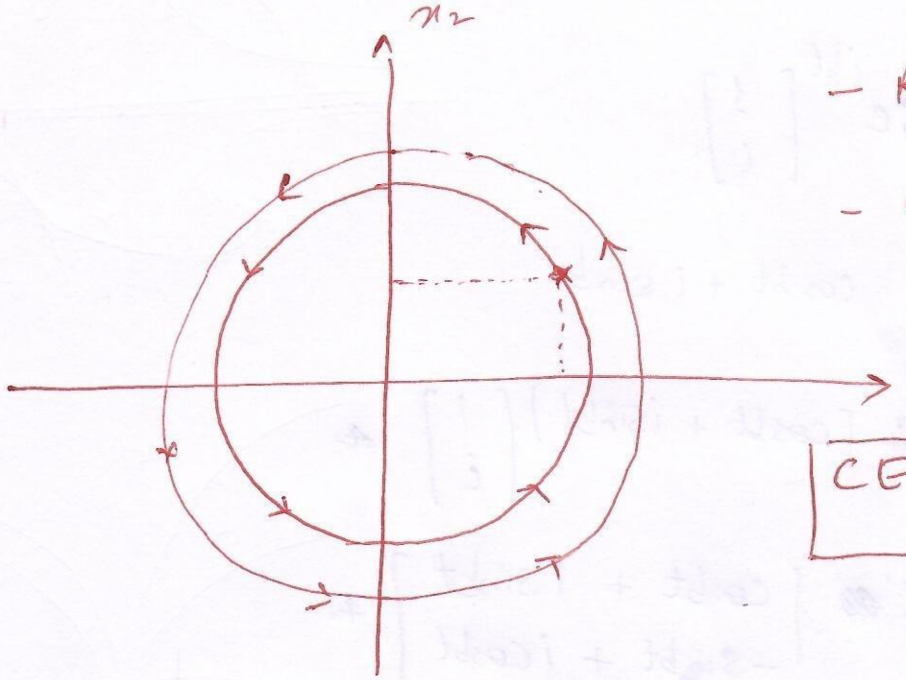
Both Re & Im part are solution to eqⁿ

$$\frac{dx}{dt} \begin{bmatrix} x \\ x_2 \end{bmatrix} = \underline{\underline{A}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

This implies $\begin{pmatrix} \cos bt \\ -\sin bt \end{pmatrix}$ & $\begin{bmatrix} \sin bt \\ \cos bt \end{bmatrix}$ are solutions.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_1 \cos bt + c_2 \sin bt \\ \cancel{-c_1 \cos bt} \\ -c_1 \sin bt + c_2 \cos bt \end{bmatrix} \rightarrow (1)$$

C_1 & C_2 → change radius of circle
b → circular nature of profile



- Repetitive system
- eigen values are purely Imaginary

$\boxed{\text{CENTRE SOLUTIONS}}$

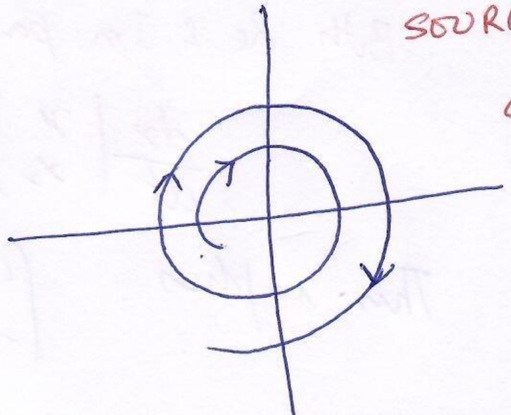
case

$$\underline{\underline{A}} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

$$\lambda_1 = a + ib \rightarrow v_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\lambda_2 = a - ib$$

SPIRAL SOURCE



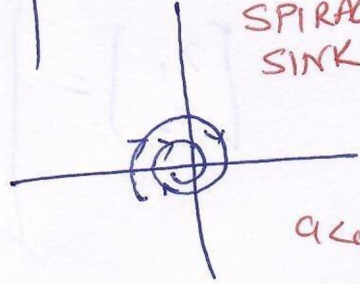
$a > 0$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{(a+ib)t} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$e^{a+ib} = e^{at} \cdot e^{ibt}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{at}(\cos bt + \sin bt) \\ e^{at}(\cos bt - \sin bt) \end{bmatrix}$$

SPIRAL SINK

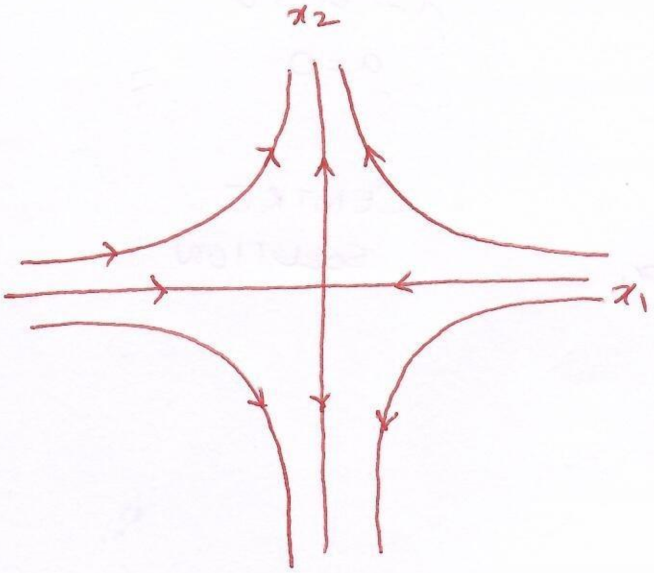


$a < 0$

$$\lim_{t \to \infty} e^{at} \to \infty \quad a > 0$$

$$\to 0 \quad a < 0$$

SUMMARY

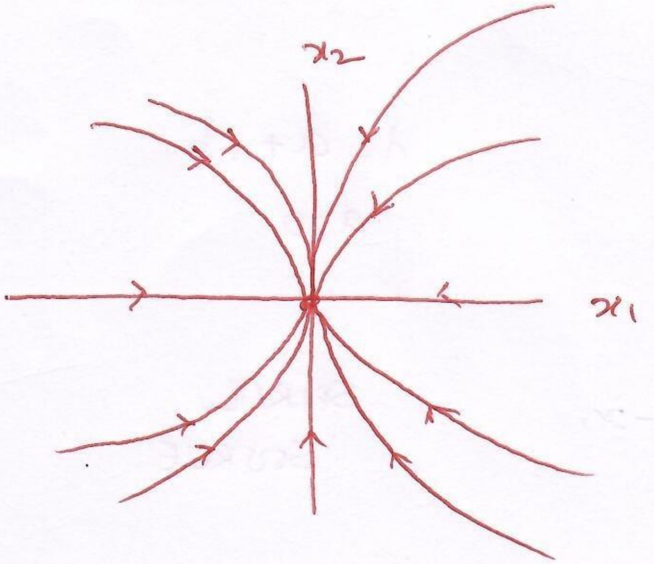


λ_1, λ_2 Both are Real

$\lambda_1 > 0$

$\lambda_2 < 0$

SADDLE SOLUTION

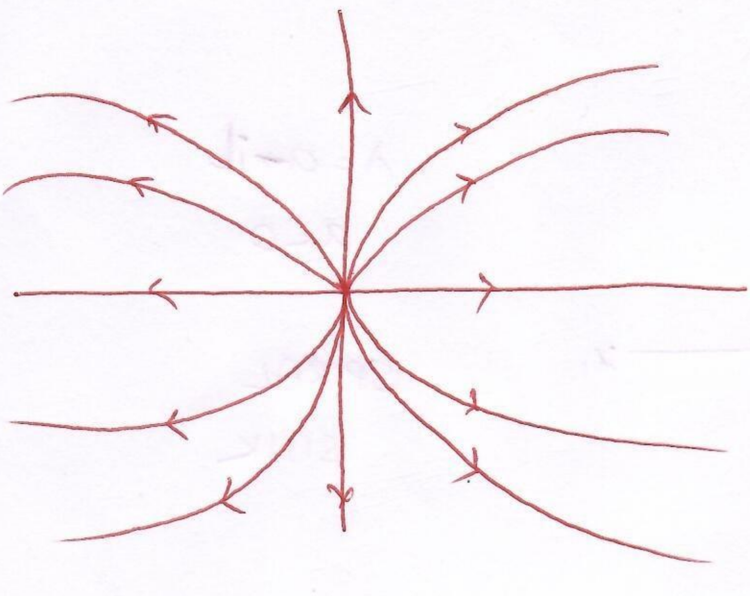


λ_1, λ_2 Both are Real

$\lambda_1, \lambda_2 < 0$

$[\lambda_2 > \lambda_1]$

SINK SOLUTION

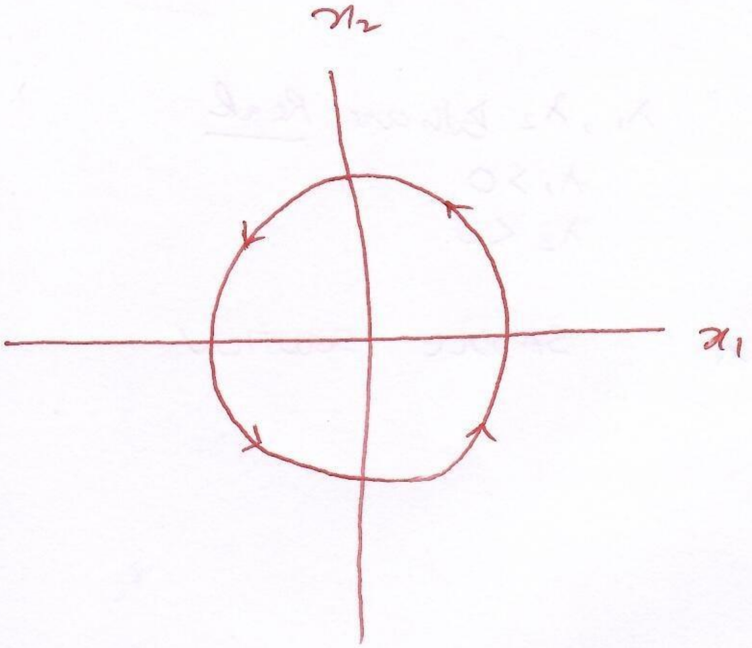


λ_1, λ_2 Both are Real

$\lambda_1, \lambda_2 > 0$

$\lambda_1 > \lambda_2$

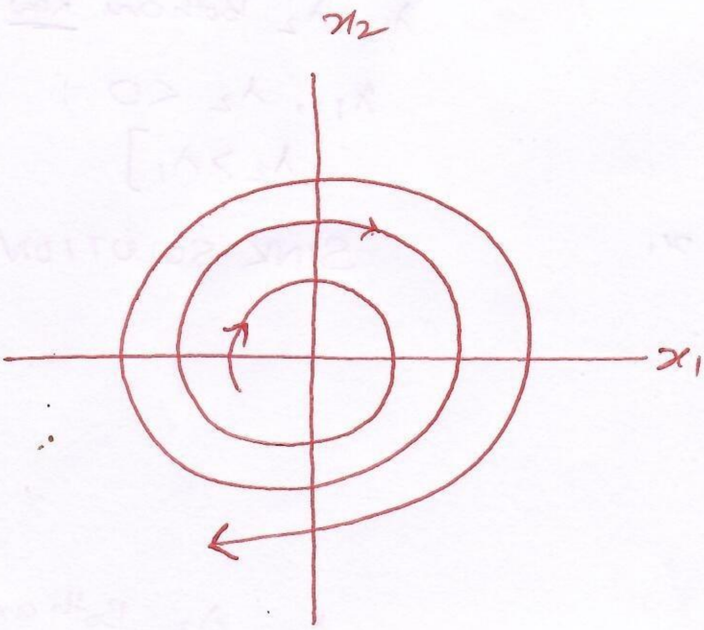
SOURCE SOLUTION



$\lambda = a + ib$

$a = 0$

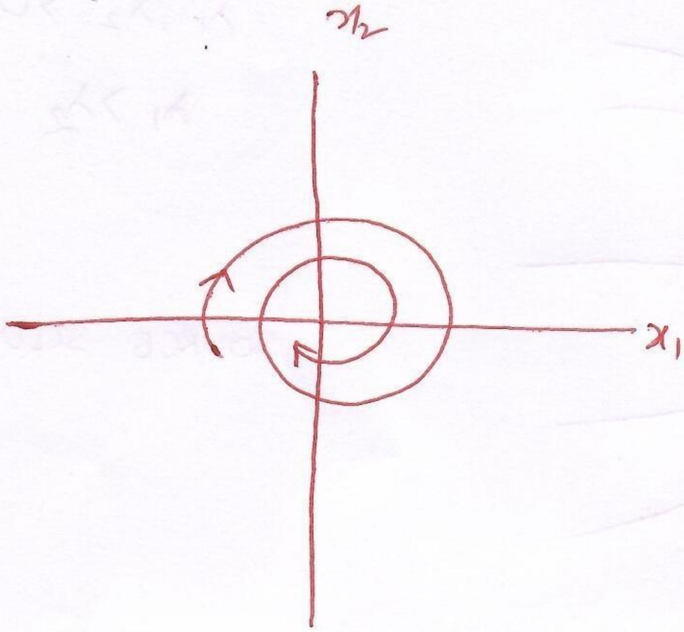
CENTRE SOLUTION



$\lambda = a + ib$

$a > 0$

SPIRAL SOURCE



$\lambda = a + ib$

$a < 0$

SPIRAL SINK

$\frac{d}{dt}$ ~~x~~

Eigen values of the system tells us about dynamical beh. of system & stability of system

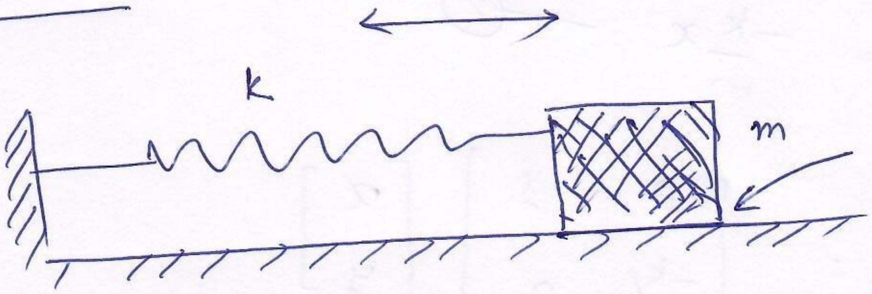
EXAMPLE

unidirectional movement.

FREE SPRING MASS SYSTEM

NO EXTERNAL FORCE

NO FRICTION (Eq. 1)



$$m\frac{d^2x}{dt^2} + kx = 0 \quad (1)$$ (without damping)

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0 \quad (2)$$ (with damping)

Q. Equilibrium solution?

Q. phase ~~potrail~~ portraits?

Q. Stability of system?!

Q. Effect of different parameters on the dynamical beh. of system?!

$$m\frac{d^2x}{dt^2} + kx = 0 \qquad \text{I.}$$

Let $$\frac{dx}{dt} = y \qquad (1)$$

$$m\frac{dy}{dt} + kx = 0$$

$$\frac{dy}{dt} = -\frac{k}{m}x \qquad (2)$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0 \qquad \text{II.}$$

Let $$\frac{dx}{dt} = y \qquad (1)$$

$$m\frac{dy}{dt} + cy + kx = 0$$

$$\frac{dy}{dt} = -\frac{k}{m}x - \frac{c}{m}y \qquad (2)$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

(I). $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

Eigen Values $\lambda_1 = -i\sqrt{k/m}$ $\quad \underline{v}_1 = \begin{bmatrix} i\sqrt{\frac{m}{k}} \\ 1 \end{bmatrix}$

$\lambda_2 = i\sqrt{\frac{k}{m}}$ $\quad \underline{v}_2 = \begin{bmatrix} -i\sqrt{m/k} \\ 1 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = C_1 e^{-i\sqrt{\frac{k}{m}}t}\begin{bmatrix} i\sqrt{\frac{m}{k}} \\ 1 \end{bmatrix} + C_2 e^{i\sqrt{\frac{k}{m}}t}\begin{bmatrix} -i\sqrt{m/k} \\ 1 \end{bmatrix}$$

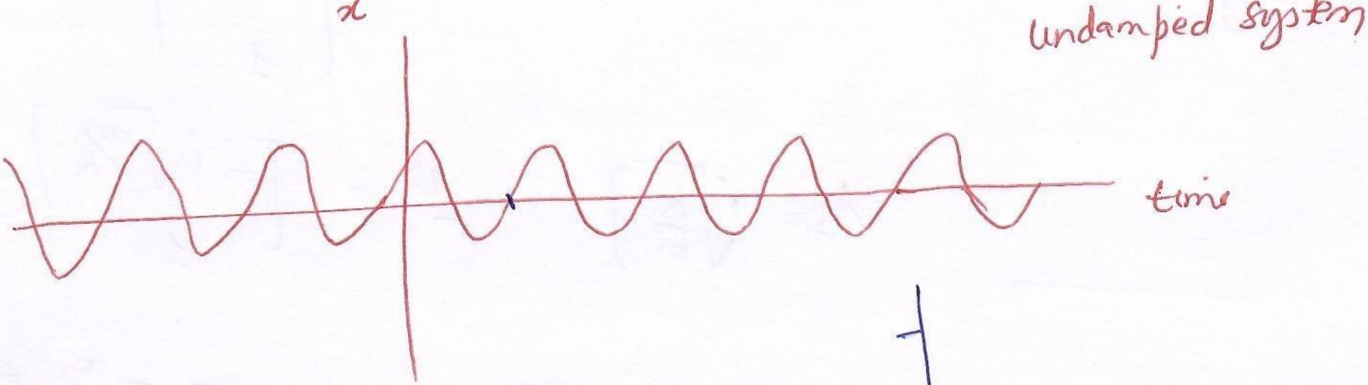
$$x(t) = C_1 e^{-i\sqrt{\frac{k}{m}}t} \cdot i\sqrt{\frac{m}{k}} + C_2 e^{i\sqrt{k/m}\,t} \cdot \left(-i\sqrt{\frac{m}{k}}\right)$$

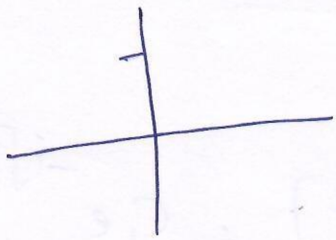
$$x(t) = C_1\left(\cos\sqrt{\tfrac{k}{m}}\,t - i\sin\sqrt{\tfrac{k}{m}}\,t\right)\left(i\sqrt{\tfrac{m}{k}}\right) +$$
$$C_2\left(\cos\sqrt{\tfrac{k}{m}}\,t + i\sin\sqrt{\tfrac{k}{m}}\,t\right)\left(-i\sqrt{\tfrac{m}{k}}\right)$$

$$= \left(C_1\cos\sqrt{\tfrac{k}{m}}\,t - C_2\cos\sqrt{\tfrac{k}{m}}\,t\right)\left(i\sqrt{\tfrac{m}{k}}\right)$$
$$+ \left(-C_1\, i\sin\sqrt{\tfrac{k}{m}}\,t - C_2 i\sin\sqrt{\tfrac{k}{m}}\,t\right)\left(i\sqrt{\tfrac{m}{k}}\right)$$

$$= (C_1 - C_2)\left(i\sqrt{\tfrac{m}{k}}\right)\cos\sqrt{\tfrac{k}{m}}\,t + (-C_1 - C_2)\,i\left(\sin\sqrt{\tfrac{k}{m}}\,t\right)\left(i\sqrt{\tfrac{m}{k}}\right)$$

$$x(t) = C_1 \cos at + C_2 \sin at$$

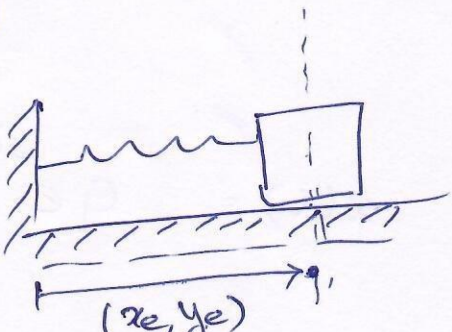




Equilibrium sol^n

$$\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0 & 1\\-k/m & 0\end{bmatrix}\begin{bmatrix}x_e\\y_e\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

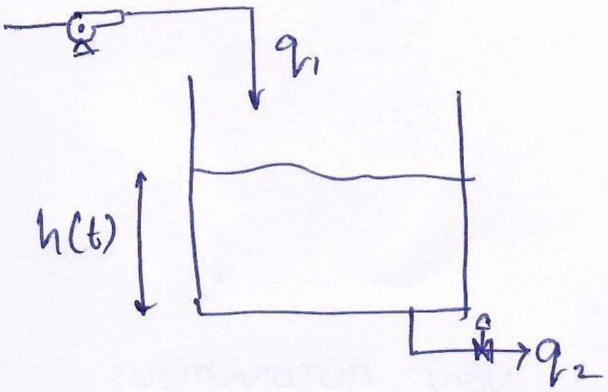
$$= \begin{bmatrix}x_e\\y_e\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$



free undamped oscillation

DYNAMICS OF NON - AUTONOMOUS SYSTEMS

Q. What does autonomous mean physically ??



$q_1 = 0$

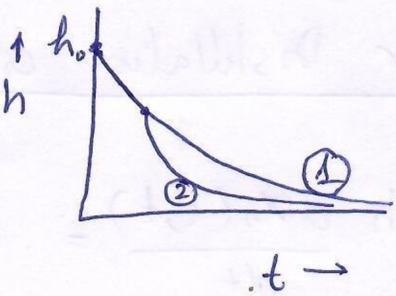
$q_2 = ah$

$$\frac{dh}{dt} = -\frac{a}{A} h$$

$$\frac{dh}{dt} = \frac{1}{A}(q_1 - q_2)$$

$$\frac{dh}{dt} = -bh \quad (1) \quad \leftarrow \text{autonomous}$$

$t \to \infty \, ; \, h \to 0$

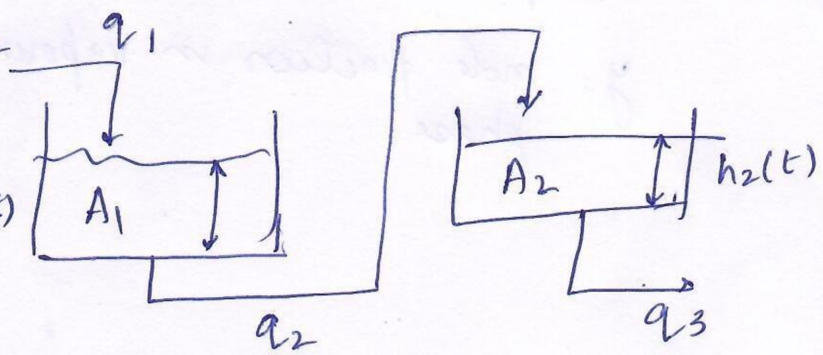


External force is required for system to divert from path 1 to 2
↳ This makes it non-autonomous

for example.

$$\frac{dh}{dt} = f(t) + ah \quad \leftarrow \text{Non-autonomous}$$

Case



$$\frac{dh_1}{dt} = \frac{1}{A_1} f(t) - \frac{1}{A_1} a h_1 \quad ; \quad \frac{dh_2}{dt} = \frac{1}{A_2} a h_1 - \frac{1}{A_2} b h_2$$

$$\frac{d}{dt}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} -\frac{a}{A_1} & 0 \\ \frac{a}{A_2} & -\frac{b}{A_2} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{A_1} \\ 0 \end{bmatrix} f(t)$$

In general

$$\frac{d}{dt}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + b(t)$$

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x} + \underline{b}(t) \longleftarrow \text{NON AUTONOMOUS DYNAMICAL SYSTEM}$$

for Distillation columns

$$h\,\frac{dx_n(i,t)}{dt} = L_{n-1}X_{n-1}(i,t) + V_{n+1}(t)\,y_{n+1}(i,t) - V_n(t)\,y_n(i,t) - L_n(t)\,x_n(i,t)$$

i = index for component ; h = liquid holdup

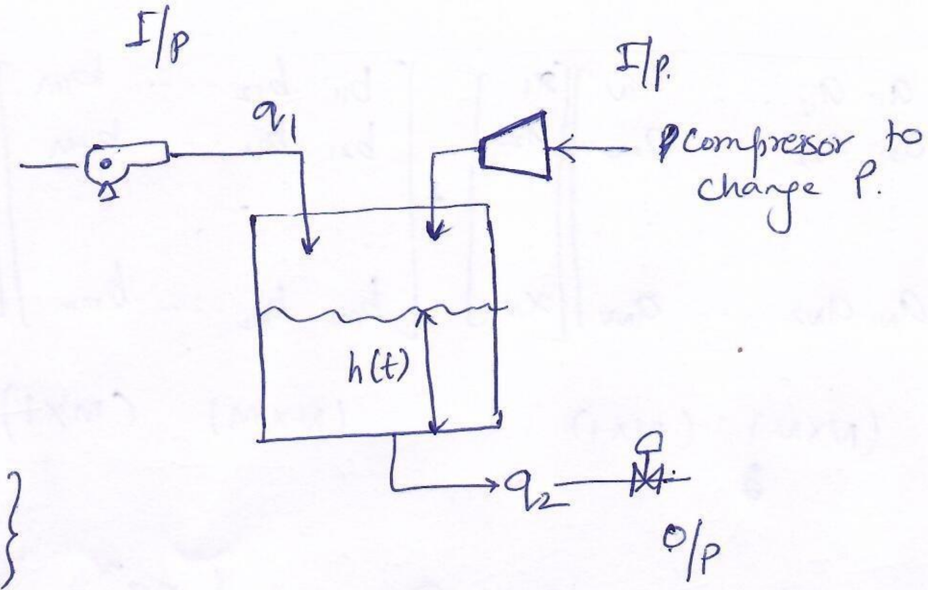
n = index for plate

x = mole fraction in liquid phase

L = Liquid flowrate

y = mole fraction in vapour phase

V = Vapour flowrate.



2 I/p
1 O/p

Distillation column is MIMO.

L & V are I/p's

x, y are o/p's

Generalising MIMO system

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \dots + a_{1N}x_N + b_{11}U_1 + b_{12}U_2 \dots + b_{1M}U_M$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \dots + a_{2N}x_N + b_{21}U_1 + b_{22}U_2 + \dots + b_{2M}U_M$$

$$\frac{dx_3}{dt} \quad \vdots \qquad \vdots \qquad \vdots$$

$$\vdots$$

$$\frac{dx_N}{dt} = a_{N1}x_1 + a_{N2}x_2 + \dots + a_{NN}x_N + b_{N1}U_1 + b_{N2}U_2 + \dots b_{NM}U_M$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1M} \\ b_{21} & b_{22} & \cdots & b_{2M} \\ \vdots & & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mm} \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_m \end{bmatrix}$$

$N \times 1$ $\quad (N \times N)$ $\quad (N \times 1)$ $\quad (N \times M)$ $\quad (M \times 1)$

Just like Water tank system

$$\frac{dh}{dt} = f(t) - ah \quad \text{①} \quad \text{Dynamical eq}^n$$

$$q_2 = ah \quad \text{②} \quad \text{output eq}^n$$

MIMO system representation

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{U} \quad \text{is dynamical eq}^n$$

$$\underline{y} = \underline{\underline{C}}\,\underline{x} + \underline{\underline{D}}\,\underline{U} \quad \text{is o/p eq}^n$$

SIMILAR MATRICES.

→ If P is non-singular such that $P^{-1}AP = B$ then A & B are called <u>similar matrices</u>

→ The operation $\boxed{B = P^{-1}AP}$ is called similarity transformation

→ Similar matrices have <u>same Eigen values.</u>

→ If $\underline{X}$ is an eigenvector of $\underline{\underline{A}}$ with eigen value λ then $\underline{\underline{P}}^{-1}\underline{X}$ will be the eigen vector of $\underline{\underline{B}}$ with the same eigenvalue λ.

$$\underline{\underline{B}} = \underline{\underline{P}}^{-1} \underline{\underline{A}}\, \underline{\underline{P}}$$ Condition for similarity

$$\underline{\underline{B}}\,\underline{\underline{P}}^{-1} = \underline{\underline{P}}^{-1} \underline{\underline{A}}\,\underline{\underline{P}}\,\underline{\underline{P}}^{-1}$$

$$\underline{\underline{P}}\,\underline{\underline{P}}^{-1} = \underline{\underline{I}}$$

$$\underline{\underline{B}}\,\underline{\underline{P}}^{-1} = \underline{\underline{P}}^{-1} \underline{\underline{A}}.$$

$$(\underline{\underline{B}}\,\underline{\underline{P}}^{-1})\underline{x} = (\underline{\underline{P}}^{-1}\underline{\underline{A}})\underline{x}$$

$$\underline{\underline{B}}(\underline{\underline{P}}^{-1}\underline{x}) = \underline{\underline{P}}^{-1}(\underline{\underline{A}}\,\underline{x})$$

If $\underline{x}$ is an eigen vector of $\underline{\underline{A}}$ with eigen value of λ

Then $\underline{\underline{A}}\,\underline{x} = \lambda \underline{x}$

$$\underline{\underline{B}}\,(\underline{\underline{P}}^{-1}\underline{x}) = \underline{\underline{P}}^{-1}(\lambda \underline{x})$$

$$\underline{\underline{B}}\,(\underline{\underline{P}}^{-1}\underline{x}) = \lambda(\underline{\underline{P}}^{-1}\underline{x})$$

let $\underline{\underline{P}}^{-1}\underline{x} = Y$

$\underline{\underline{B}}(Y) = \lambda(\underline{Y}) \Rightarrow \lambda$ is eigen value of B. with the corresponding eigen vector as $\underline{\underline{P}}^{-1}\underline{x}$.

$\underline{\underline{P}}$ = augmentation of v_i (eigen vector) of $\underline{\underline{A}}$ ← How to decide P ??

$$\underline{\underline{P}} = [\underline{v}_1 \,|\, \underline{v}_2 \,|\, \underline{v}_3 \,|\, \ldots\, \underline{v}_N]$$

$$\underline{\underline{P}} = \left[\underline{v}_1 \mid \underline{v}_2 \mid \underline{v}_3 \ldots \underline{v}_n \right]$$

$B = P^{-1}AP$ condition of similarity.

$P^{-1} \rightarrow P^{-1}AP = B$.

$$\underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{A}}\left[\underline{v}_1 \mid \underline{v}_2 \mid \ldots \underline{v}_n\right]$$

$$= \left[\underline{\underline{A}}\underline{v}_1 \mid \underline{\underline{A}}\underline{v}_2 \ldots \mid \underline{\underline{A}}\underline{v}_n\right]$$

$B = \Lambda$

$$= \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \lambda_2 & 0 \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

$$\underline{\underline{A}}\,\underline{\underline{P}} = \left[\lambda_1 \underline{v}_1 \mid \lambda_2 \underline{v}_2 \mid \ldots \lambda_n \underline{v}_n\right]$$

$$\underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{P}}\,\underline{\underline{\Lambda}}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & \cdots & \lambda_n \end{bmatrix}$$

Condition for diagonalisation

An $N \times N$ matrix is diagonalisable if it has N linearly independent eigen vectors.

Example :-

$$\frac{dx_1}{dt} = -2x_1 - 4x_2 + 2x_3$$

$$\frac{dx_2}{dt} = -2x_1 + x_2 + 2x_3$$

$$\frac{dx_3}{dt} = 4x_1 + 2x_2 + 5x_3$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underline{\underline{\begin{bmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix}}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\lambda_1 = 3 \rightarrow \underline{v}_1 = [2 \;\; 3 \;\; 1]^T$$

$$\lambda_2 = -5 \rightarrow \underline{v}_2 = [2 \;\; -1 \;\; 1]^T$$

$$\lambda_3 = 6 \rightarrow \underline{v}_3 = [1 \;\; 6 \;\; 16]^T$$

are these eigen vectors linearly independent ??

[if the only solution to $a\underline{v}_1 + b\underline{v}_2 + c\underline{v}_3 = 0$ should be $a = b = c = 0$ the v_1, v_2, v_3 are linearly independent]

→ Condition for linear independent

$$a\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + b\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + c\begin{bmatrix} 1 \\ 6 \\ 16 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solve for a, b, c

$$a = b = c = 0$$

If linearly independent eigen vectors are not present then

there exist a non-singular matrix $\underline{\underline{P}}$ such that

$\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{J}}$ where $\underline{\underline{J}}$ is called Jordan matrix

$$\underline{\underline{J}} = \begin{bmatrix} \underline{\underline{J}}_1 & 0 & 0 & \cdots & 0 \\ 0 & \underline{\underline{J}}_2 & 0 & 0 \cdots & 0 \\ \vdots & \vdots & \underline{\underline{J}}_3 & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \underline{\underline{J}}_N \end{bmatrix}$$

$\underline{\underline{J}}_i$ are called Jordan Blocks

let $N = 3$

If 3 eigen vectors which are linearly independent.

$$\underline{\underline{J}} = \begin{bmatrix} \boxed{\lambda_1} & 0 & 0 \\ 0 & \boxed{\lambda_2} & 0 \\ 0 & 0 & \boxed{\lambda_3} \end{bmatrix} \leftarrow \text{3 Jordan Blocks}$$

If 2 eigen vectors

$$\underline{\underline{J}} = \begin{bmatrix} \boxed{\lambda_1} & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

2 Jordan Blocks

If 1 Eigen vector

$$\underline{\underline{J}} = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

1 Jordan Block

So, we have diagonal elements + super diagonal element

Example

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{U}$$

$(N\times 1)$ $(N\times N)(N\times 1)$ $(N\times M)(M\times 1)$

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\underline{x} + \underline{g}(t) \quad \text{—— (0)}$$

multiply by P^{-1}

$$~~d^{-1}AP~~ \quad \frac{d}{dt}(P^{-1}\underline{x}) = \underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{x} + \underline{\underline{P}}^{-1}g(t) \quad \text{—— (1)}$$

$$\frac{d}{dt}(\underline{\underline{P}}^{-1}\underline{x}) = \underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}}\,\underline{\underline{P}}^{-1}\underline{x} + \underline{\underline{P}}^{-1}g(t)$$

$$\frac{d}{dt}(\underline{\underline{P}}^{-1}\underline{x}) = (P^{-1}AP)\,P^{-1}\underline{x} + P^{-1}g(t)$$

If P is made from augmented eigen vectors of $\underline{\underline{A}}$ then

$$P^{-1}AP = \Lambda \quad \text{—— (2)}$$

$$\text{Let } \underline{\underline{P}}^{-1}\underline{x} = \underline{y} \;\&\; \underline{\underline{P}}^{-1}\underline{g}(t) = \underline{b}(t) \quad \text{—— (3)}$$

$$\frac{d(\underline{y})}{dt} = \underbrace{\underline{\underline{\Lambda}}}_{\text{diagonal matrix}}\underline{y} + \underline{b}(t) \quad \text{—— (4)}$$

$$\frac{d\underline{y}}{dt} = \underline{\underline{\Lambda}}\,\underline{y} + \underline{b}(t)$$

$$\Rightarrow \frac{d\underline{y}}{dt} - \underline{\underline{\Lambda}}\,\underline{y} = \underline{b}(t)$$

$$e^{-\underline{\underline{\Lambda}}t}\frac{d\underline{y}}{dt} - \Lambda e^{-\underline{\underline{\Lambda}}t}\,\underline{y} = e^{-\underline{\underline{\Lambda}}t}\,\underline{b}(t)$$

$$\frac{d}{dt}\left(\underline{y}\,e^{-\underline{\underline{\Lambda}}t}\right) = e^{-\underline{\underline{\Lambda}}t}\,\underline{b}(t)$$

$$d\left(\underline{y}\,e^{-\underline{\underline{\Lambda}}t}\right) = e^{-\underline{\underline{\Lambda}}t}\,\underline{b}(t)\,.\,dt$$

$$\underline{y}\,e^{-\underline{\underline{\Lambda}}t} = \int e^{-\underline{\underline{\Lambda}}t}\,\underline{b}(t)\,dt + \underline{c}$$

$$\underline{y} = \left(e^{-\underline{\underline{\Lambda}}t}\right)^{-1}\int e^{-\underline{\underline{\Lambda}}t}\,\underline{b}(t)\,dt + \left(e^{-\underline{\underline{\Lambda}}t}\right)^{-1}\underline{c}$$

??

If we can determine $\underline{\underline{\Lambda}}$ then we can solve.

$$e^{-\underline{\underline{\Lambda}}t} = ??$$

$$e^{x} = 1 + x + \frac{x^2}{2!} + \cdots$$

$$e^{-\underline{\underline{\Lambda}}t} = I + (-\underline{\underline{\Lambda}}t) + \frac{1}{2!}(-\underline{\underline{\Lambda}}t)^2 + \cdots$$

$$\frac{dy}{dt} = \lambda y + b(t)$$

$$\frac{dy}{dt} - \lambda y = b(t)$$

solve using I.f

$$\text{If} = e^{-\lambda t}$$

$$e^{-\lambda t}\frac{dy}{dt} - \lambda e^{-\lambda t} y = e^{-\lambda t} b(t)$$

$$\frac{d}{dt}\left(y e^{-\lambda t}\right) = e^{-\lambda t} b(t)$$

$$d\left(y e^{-\lambda t}\right) = e^{-\lambda t} b(t)\,dt$$

$$y e^{-\lambda t} = \int e^{-\lambda t} b(t)\,dt + C$$

$$y = e^{\lambda t}\int e^{-\lambda t} b(t)\,dt + e^{\lambda t} C$$

What is $\Lambda t = ??$

$$\underline{\underline{\Lambda}} t = \begin{bmatrix} \lambda_1 t & \cdots & & 0 \\ \vdots & \lambda_2 t & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_n t \end{bmatrix}$$

$$(\underline{\underline{\Lambda}} t)^2 = \begin{bmatrix} \lambda_1^2 t^2 & & \\ & \lambda_2^2 t^2 & \\ & & \ddots \ \lambda_n^2 t^2 \end{bmatrix}^2$$

← This is why we diagonalised the matrix

$$e^{-\Lambda t} = \begin{bmatrix} 1 + (-\lambda_1 t) + \left(\frac{\lambda_1^2 t^2}{2!}\right) + \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 - \lambda_2 t + \frac{\lambda_2^2 t^2}{2!} + \cdots & 0 & 0 & \cdot \end{bmatrix}$$

$$= \begin{bmatrix} e^{-\lambda_1 t} & 0 & 0 & \cdots & 0 \\ 0 & e^{-\lambda_2 t} & 0 & \cdots & 0 \\ \vdots & \vdots & e^{-\lambda_3 t} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & e^{-\lambda_n t} \end{bmatrix}$$

$$e^{-\underline{\underline{\Lambda}} t} =$$

$$\int \underline{\underline{A}} dt = \begin{bmatrix} \int a\,dt & \int b\,dt & \cdots \\ \vdots & & \\ \vdots & & \\ \int a_n dt & \int & \end{bmatrix}$$

Integral of matrix is integral of Individual element.