Example: Separation of Variables: T(r, ϕ)

General Solution:

$$T(r,\phi) = (A_1 r^{\lambda} + A_2 r^{-\lambda})(B_1 \sin\lambda\phi + B_2 \cos\lambda\phi)$$

Determination of eigenvalues:

Using $\psi(\phi) = B_1 \sin\lambda\phi + B_2 \cos\lambda\phi$ and BCs $\begin{cases} \psi(\phi) = \psi(\phi + 2\pi) \\ \dfrac{d\psi(\phi)}{d\phi} = \dfrac{d\psi(\phi+2\pi)}{d\phi} \end{cases}$

$$\left.\begin{aligned} &[\sin\lambda\phi - \sin\lambda(\phi+2\pi)]B_1 + [\cos\lambda\phi - \cos\lambda(\phi+2\pi)]B_2 = 0 \\ &[\cos\lambda\phi - \cos\lambda(\phi+2\pi)]B_1 - [\sin\lambda\phi - \sin\lambda(\phi+2\pi)]B_2 = 0 \end{aligned}\right\}$$

In order to have nontrivial solutions for B_1 and B_2, the determinant of the coefficients must vanish, which yields

$$\cos 2\lambda\pi = 1$$

This is possible only if λ is equal to one of the values of $\lambda_n = n, \quad n = 0, 1, 2, \ldots$

Example: Separation of Variables: T(r, ϕ)

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-\lambda^2R=0$$

$$\left\{\begin{array}{l} \dfrac{d^2\psi}{d\phi^2}+\lambda^2\psi=0 \\ \psi(\phi)=\psi(\phi+2\pi) \\ \dfrac{d\psi(\phi)}{d\phi}=\dfrac{d\psi(\phi+2\pi)}{d\phi} \end{array}\right.$$

This is Cauchy-Euler Equation (<u>NOT</u> Bessel Eq)

Solution: $R(r)=A_1r^{\lambda}+A_2r^{-\lambda}$

Solution: $\psi(\phi)=B_1\sin\lambda\phi+B_2\cos\lambda\phi$

<u>NOTE:</u>

Bessel Differential Equation:

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(m^2x^2-v^2)y=0$$

Modified Bessel Differential Equation:

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-(m^2x^2+v^2)y=0$$

Cauchy-Euler Equation: $r^2\dfrac{d^2R}{dr^2}+a_0r\dfrac{dR}{dr}+b_0R=0$

<u>General Solution:</u> $T(r,\phi)=(A_1r^{\lambda}+A_2r^{-\lambda})(B_1\sin\lambda\phi+B_2\cos\lambda\phi)$

Example: Separation of Variables: $T(r, \phi)$

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-\lambda^2R=0$$

$$\frac{d^2\psi}{d\phi^2}+\lambda^2\psi=0$$

$$\psi(\phi)=\psi(\phi+2\pi)$$

$$\frac{d\psi(\phi)}{d\phi}=\frac{d\psi(\phi+2\pi)}{d\phi}$$

$T(0,\phi) = finite$

$T(r_0,\phi) = f(\phi)$

General Solution:

$$T(r,\phi)=(A_1r^{\lambda}+A_2r^{-\lambda})(B_1\sin\lambda\phi+B_2\cos\lambda\phi)$$

$$\lambda_n=n,\qquad n=0,1,2,\ldots$$

Set $A_2 = 0$ so that the solution will satisfy the finite BC.
Employ superposition:

$$T(r,\phi)=\sum_{n=0}^{\infty}r^n(a_n\sin n\phi+b_n\cos n\phi)$$

$$T(r,\phi)=b_0+\sum_{n=1}^{\infty}r^n(a_n\sin n\phi+b_n\cos$$

Example: Separation of Variables: T(r, ϕ)

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-\lambda^2R=0 \qquad T(r,\phi)=(A_1r^{\lambda}+A_2r^{-\lambda})(B_1\sin\lambda\phi+B_2\cos\lambda\phi)$$

$$T(0,\phi)=\textit{finite} \qquad T(r,\phi)=b_0+\sum_{n=1}^{\infty}r^n(a_n\sin n\phi+b_n\cos n\phi) \quad \textbf{Eq. (A)}$$

$$T(r_0,\phi)=f(\phi)$$

Note that for $\lambda = 0$ the product solution $T(r,\phi) = R(r)\psi(\phi)$ yields:

$$T_0(r,\pi)=(A_{10}+A_{20}\ln r)(B_{10}+B_{20}\phi)$$

- The boundedness, that is $[T(0,\phi) = \text{finite}]$, implies that $A_{20} = 0$.
- The condition of 2π periodicity, that is $[\psi(\phi) = \psi(\phi + 2\pi)]$ implies that $B_{20} = 0$.
- Therefore, the omission of the ϕ and ln r terms in **Eq. (A)** caused no problem as b_0 corresponds to $A_{10}B_{10}$.

Example: Separation of Variables: T(r, ϕ)

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-\lambda^2R=0$$

$$T(r,\phi)=(A_1r^{\lambda}+A_2r^{-\lambda})(B_1\sin\lambda\phi+B_2\cos\lambda\phi)$$

$$T(0,\phi)=finite$$

$$T(r,\phi)=b_0+\sum_{n=1}^{\infty}r^n(a_n\sin n\phi+b_n\cos n\phi)$$

$$T(r_0,\phi)=f(\phi)$$

Now apply non-homogeneous BC:

$$f(\phi)=b_0+\sum_{n=1}^{\infty}r_0^n(a_n\sin n\phi+b_n\cos n\phi)$$

This is the *complete Fourier series* representation of $f(\phi)$ on the interval $(0, 2\pi)$. The coefficients can be determined as:

$$b_0=\frac{1}{2\pi}\int_0^{2\pi}f(\phi)d\phi$$

$$a_n=\frac{1}{\pi r_0^n}\int_0^{2\pi}f(\phi)\sin n\phi\, d\phi,\quad n=1,2,3,\ldots$$

$$b_n=\frac{1}{\pi r_0^n}\int_0^{2\pi}f(\phi)\cos n\phi\, d\phi,\quad n=1,2,3,\ldots$$

Example: Separation of Variables: T(r, ϕ)

Final Solution:

$$T(r,\phi)=\frac{1}{2\pi}\int_0^{2\pi} f(\phi')d\phi' + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{r_0}\right)^n\left[\sin n\phi\int_0^{2\pi} f(\phi')\sin n\phi' d\phi' + \cos n\phi\int_0^{2\pi} f(\phi')\cos n\phi' d\phi'\right]$$

At $r = 0$, the above equation reduces to: $T(0,\phi)=\frac{1}{2\pi}\int_0^{2\pi} f(\phi')d\phi'$

Hence, the centerline temperature is the *average* of the surface temperature distribution.

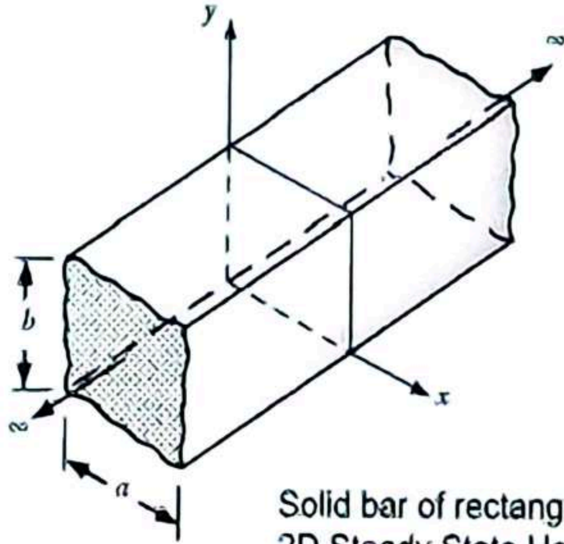
Example-2

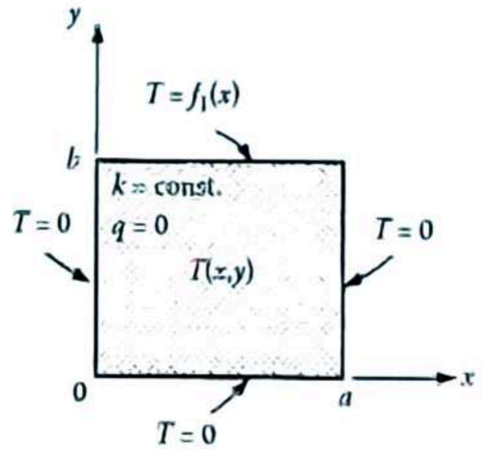
Finite Fourier Transforms

Rectangular Coordinate System: 2D Steady-State Problem

Finite Fourier Transforms: Example-2

2D Steady State Problem



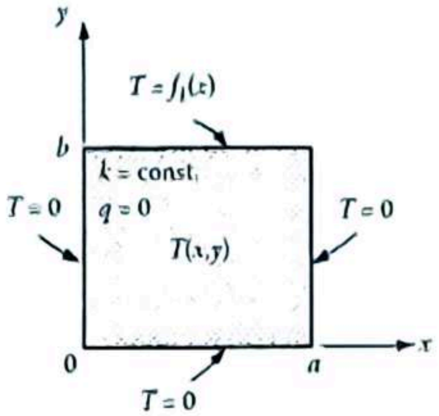


Solid bar of rectangular cross section.
2D Steady State Heat Conduction.

Solution by SOV:

$$T(x,y) = \frac{2}{a}\sum_{n=1}^{\infty}\frac{\sin(n\pi/a)x\,\sinh(n\pi/a)y}{\sin(n\pi/a)b}\int_0^a f_1(x')\sin\frac{n\pi}{a}x'\,dx'$$

Finite Fourier Transforms: Example-2



$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$T(0, y) = T(a, y) = 0$$

$$T(x, 0) = 0, \; T(x, b) = f(x)$$

In the finite interval $(0, a)$, the Fourier transform of the temperature distribution $T(x, y)$ with respect to variable x can be defined as

$$\bar{T}_n(y) = \int_0^a T(x, y) K_n(x) dx$$

Look at BC.
Take the Kernels as:

with the inversion formula

$$K_n(x) = \sqrt{\frac{2}{a}} \sin \lambda_n x$$

$$T(x, y) = \sum_{n=1}^{\infty} \bar{T}_n(y) K_n(x)$$

$$\lambda_n = \frac{n\pi}{a}, \; n = 1, 2, 3, \ldots$$

Finite Fourier Transforms: Example-2

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$T(0,y) = T(a,y) = 0$$

$$T(x,0) = 0,\ T(x,b) = f(x)$$

$$K_n(x) = \sqrt{\frac{2}{a}} \sin \lambda_n x$$

$$\lambda_n = \frac{n\pi}{a},\ n = 1, 2, 3, \ldots$$

Finite Fourier Transform of the heat conduction equation:

$$\int_0^a K_n(x) \frac{\partial^2 T}{\partial x^2} dx + \int_0^a K_n(x) \frac{\partial^2 T}{\partial y^2} dx = 0 \quad \Rightarrow \quad \int_0^a K_n(x) \frac{\partial^2 T}{\partial x^2} dx + \frac{d^2 \overline{T}_n}{dy^2} = 0$$

Integrating by parts twice:

$$\int_0^a K_n(x) \frac{\partial^2 T}{\partial x^2} dx = -\lambda_n^2 \overline{T}_n(y)$$

On substitution:

$$\frac{d^2 \overline{T}_n}{dy^2} - \lambda_n^2 \overline{T}_n(y) = 0$$

Solution: $\overline{T}_n(y) = A_n \sinh \lambda_n y + B_n \cosh \lambda_n y$

Finite Fourier Transforms: Example-2

$$\frac{d^2\bar{T}_n}{dy^2} - \lambda_n^2\bar{T}_n(y) = 0 \qquad \bar{T}_n(y) = A_n \sinh\lambda_n y + B_n \cosh\lambda_n y$$

The transforms of the boundary conditions at $y = 0$ and at $y = b$ yield:

$$T(x, y = 0) = 0 \Rightarrow \bar{T}_n(0) = \int_0^a T(x,0)K_n(x)dx = 0$$

$$T(x, y = b) = f(x) \Rightarrow \bar{T}_n(b) = \int_0^a T(x,b)K_n(x)dx = \int_0^a f(x)K_n(x)dx = \bar{f}_n$$

Apply BC at $y = 0$: $B_n = 0$,

Apply BC at $y = b$: $A_n = \dfrac{\bar{f}_n}{\sinh\lambda_n b}$

Thus, the transform of the temperature distribution is given by:

$$\bar{T}_n(y) = \bar{f}_n \frac{\sinh\lambda_n y}{\sinh\lambda_n b}$$

Upon inversion:

$$T(x, y) = \sum_{n=1}^{\infty} \bar{f}_n \frac{\sinh\lambda_n y}{\sinh\lambda_n b} K_n(x)$$

Finite Fourier Transforms: Example-2

Upon inversion: $T(x,y) = \sum_{n=1}^{\infty} \bar{f}_n \frac{\sinh \lambda_n y}{\sinh \lambda_n b} K_n(x)$

$K_n(x) = \sqrt{\frac{2}{a}} \sin \lambda_n x$

$\lambda_n = \frac{n\pi}{a}, \; n = 1, 2, 3, \ldots$

which can also be written as (by expanding $\bar{f}_n$):

$$T(x,y) = \frac{2}{a} \sum_{n=1}^{\infty} \frac{\sin(n\pi/a)x \sinh(n\pi/a)y}{\sinh(n\pi/a)b} \int_0^a f(x') \sin \frac{n\pi}{a} x' \, dx'$$

This is the same result one obtains by applying SOV.

Alternative Approach:

Remove both x and y coordinates.
Define the Fourier transform with respect to variable y in the finite interval $(0, b)$ as

$$\bar{\bar{T}}_{mn} = \int_0^b \bar{T}_n(y) K_m(y) dy$$

Finite Fourier Transforms: Example-2

Alternative Approach:

$$T(x,y)=\sum_{n=1}^{\infty}\bar{T}_n(y)K_n(x)$$

STEP – 1:

FFT wrt x: $\bar{T}_n(y)=\int_0^a T(x,y)K_n(x)dx$

Kernel: $K_n(x)=\sqrt{\frac{2}{a}}\sin\lambda_n x$

$$\frac{d^2\bar{T}_n}{dy^2}-\lambda_n^2\bar{T}_n(y)=0 \quad \textbf{Eq. (1)}$$

$$\lambda_n=\frac{n\pi}{a},\ n=1,2,3,\ldots$$

STEP – 2:

Define the Fourier transform with respect to variable y in the finite interval $(0, b)$ as

$$\bar{\bar{T}}_{nm}=\int_0^b \bar{T}_n(y)K_m(y)dy$$

Inversion Formula: $\bar{T}_n(y)=\sum_{m=1}^{\infty}\bar{\bar{T}}_{nm}K_m(y)$

Kernel: $K_m(y)=\sqrt{\frac{2}{b}}\sin\beta_m y \qquad \beta_m=\frac{m\pi}{b},\ m=1,2,3,\ldots$

We now obtain the transform of **Eq. (1)** with respect to variable y:

$$\int_0^b K_m(y)\frac{d^2\bar{T}_n}{dy^2}dy-\lambda_n^2\int_0^b K_m(y)\bar{T}_n(y)dy=0$$

Finite Fourier Transforms: Example-2

Alternative Approach:

$$\int_0^b K_m(y)\frac{d^2\overline{T}_n}{dy^2}dy - \lambda_n^2\int_0^b K_m(y)\overline{T}_n(y)dy = 0$$

$$=> \quad \int_0^b K_m(y)\frac{d^2\overline{T}_n}{dy^2}dy - \lambda_n^2\overline{\overline{T}}_{mn} = 0 \qquad \text{Eq. (A)}$$

The integral in above Eq. can be evaluated by parts to yield

$$\int_0^b K_m(y)\frac{d^2\overline{T}_n}{dy^2}dy = -\beta_m^2\overline{\overline{T}}_{mn} + (-1)^{m+1}\sqrt{\frac{2}{b}}\beta_m\overline{f}_n \qquad \text{Eq. (B)}$$

Substitution of Eq. (B) into Eq. (A) yields the following **algebraic equation**:

$$(\lambda_n^2 + \beta_m^2)\overline{\overline{T}}_{mn} = (-1)^{m+1}\sqrt{\frac{2}{b}}\beta_m\overline{f}_n$$

Finite Fourier Transforms: Example-2

Alternative Approach:

$$(\lambda_n^2 + \beta_m^2)\bar{\bar{T}}_{nm} = (-1)^{m+1}\sqrt{\frac{2}{b}}\beta_m \bar{f}_n$$

=> $$\bar{\bar{T}}_{nm} = (-1)^{m+1}\sqrt{\frac{2}{b}}\frac{\beta_m \bar{f}_n}{\lambda_n^2 + \beta_m^2}$$

Invert using $T(x,y) = \sum_{n=1}^{\infty} \bar{T}_n(y) K_n(x)$ => $\bar{T}_n(y) = \sqrt{\frac{2}{b}}\sum_{m=1}^{\infty}(-1)^{m+1}\frac{\beta_m \bar{f}_n}{\lambda_n^2 + \beta_m^2}K_m(y)$

Invert using $T(x,y,t) = \sum_{n=1}^{\infty} \bar{T}_n(y,t) K_n(x)$ =>

$$T(x,y) = \sqrt{\frac{2}{b}}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{m+1}\frac{\beta_m \bar{f}_n}{\lambda_n^2 + \beta_m^2}K_m(y)K_n(x)$$

Finite Fourier Transforms: Example-2

Alternative Approach:

$$T(x,y)=\sqrt{\frac{2}{b}}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{m+1}\frac{\beta_m \bar{f}_n}{\lambda_n^2+\beta_m^2}K_m(y)K_n(x)$$

which can also be written as

$$T(x,y)=\frac{4}{ab}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{m+1}\frac{(m\pi/b)\sin(n\pi/a)x\sin(m\pi/b)y}{(n\pi/a)^2+(m\pi/b)^2}$$
$$\times\int_0^a f(x')\sin\frac{n\pi}{a}x'\,dx'$$

This expression is same as previously obtained expression (by using FFT of x only) because it can be shown that

$$\frac{\sinh(n\pi/a)y}{\sinh(n\pi/a)b}=\frac{2}{b}\sum_{m=1}^{\infty}(-1)^{m+1}\frac{(m\pi/b)\sin(m\pi/b)y}{(n\pi/a)^2+(m\pi/b)^2}$$

Finite Fourier Transforms

Example –1D Unsteady-State: Spherical Coordinate System

Finite Fourier Transforms
1D Spherical Coordinate System

One-dimensional linear heat conduction problems posed in spherical coordinates may be transformed into rectangular coordinate systems by introducing a new temperature function, $\theta(r, t) = rT(r, t)$

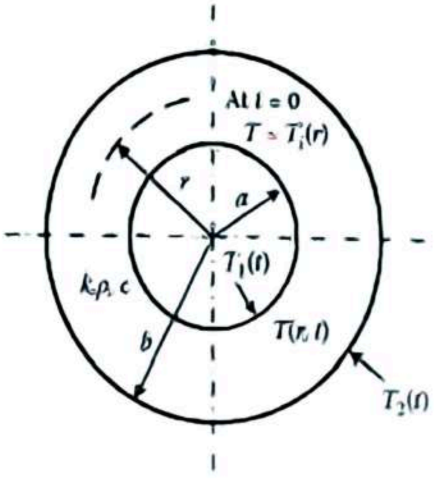
Fourier transforms can then be used to solve such problems.

Example:

Consider the hollow sphere shown in Figure, which is initially at temperature $T_i(r)$. The surfaces at $r = a$ and $r = b$ are maintained at temperatures $T_1(t)$ and $T_2(t)$, respectively, for times $t \geq 0$.

We wish to find the unsteady-state temperature distribution $T(r, t)$ in this spherical shell for $t > 0$.

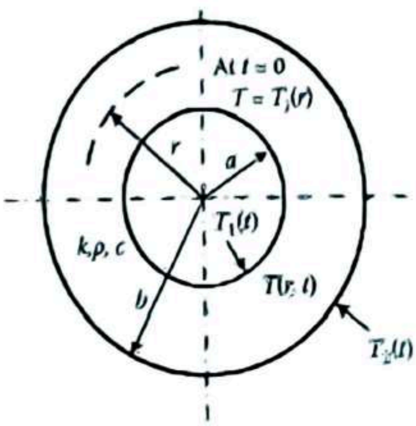
Assume constant thermo-physical properties.



Finite Fourier Transforms
1D Spherical Coordinate System

Example:

Consider the hollow sphere shown in Figure, which is initially at temperature $T_i(r)$. The surfaces at $r = a$ and $r = b$ are maintained at temperatures $T_1(t)$ and $T_2(t)$, respectively, for times $t \geq 0$.



We wish to find the unsteady-state temperature distribution $T(r, t)$ in this spherical shell for $t > 0$.

Assume constant thermo-physical properties.

Problem Formulation:

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$T(r,0) = T_i(r)$$

$$T(a,t) = T_1(t) \text{ and } T(b,t) = T_2(t)$$

Rewrite in terms of
$$\theta(r,t) = rT(r,t)$$

$$\frac{\partial^2 \theta}{\partial r^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t}$$

$$\theta(r,0) = rT_i(r)$$

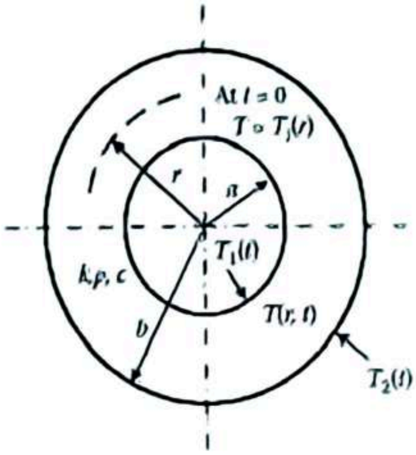
$$\theta(a,t) = aT_1(t) \text{ and } \theta(b,t) = bT_2(t)$$

$(r = a)$ $\qquad$ $(r = b)$

Finite Fourier Transforms
1D Spherical Coordinate System

Example:

$$\frac{\partial^2\theta}{\partial r^2}=\frac{1}{\alpha}\frac{\partial\theta}{\partial t}$$



$$\theta(r,0)=rT_i(r)$$

$$\theta(a,t)=aT_1(t) \text{ and } \theta(b,t)=bT_2(t)$$

Now, a change of variable

$r=x+a$

yields

$$\frac{\partial^2\theta}{\partial x^2}=\frac{1}{\alpha}\frac{\partial\theta}{\partial t}$$

$$\theta(x,0)=(x+a)T_i(x+a)$$

$$\theta(0,t)=aT_1(t) \text{ and } \theta(b-a,t)=bT_2(t)$$

(when $r=a$, $x=0$) (when $r=b$, $x=b-a$)

Finite Fourier Transforms
1D Spherical Coordinate System

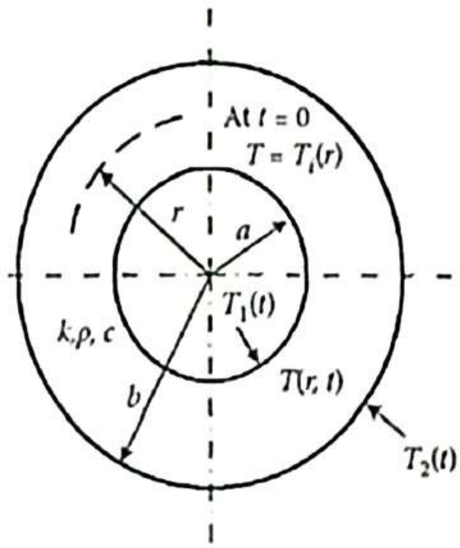
Example:

Reformulated Problem: Rectangular Coordinate System

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}$$

$$\theta(x,0) = (x+a)T_i(x+a)$$

$$\theta(0,t) = aT_1(t) \text{ and } \theta(b-a,t) = bT_2(t)$$



This problem can now be solved readily by using **Fourier transforms.** Note that the range of variable x is $(0, b-a)$.

Finite Fourier Transform in finite interval $(0, b\text{-}a)$

$$\bar{\theta}_n(t) = \int_0^{b-a} \theta(x,t)K_n(x)dx$$

Inversion Formula

$$\theta(x,t) = \sum_{n=1}^{\infty} \bar{\theta}_n(t)K_n(x)$$

Kernel of FFT

$$K_n(x) = \sqrt{\frac{2}{b-a}}\sin\lambda_n x$$

$$\lambda_n = \frac{n\pi}{b-a}, \; n = 1, 2, 3, \ldots$$

Finite Fourier Transforms
1D Spherical Coordinate System

Example:

The transform of the heat equation yields:

$$\frac{d\bar{\theta}_n}{dt} + \alpha\lambda_n^2\bar{\theta}_n(t) = \alpha\sqrt{\frac{2}{b-a}}\left[(-1)^n aT_1(t) - bT_2(t)\right]$$

Solution:

$$\bar{\theta}_n(t) = e^{-\alpha\lambda_n^2 t}\left\{\bar{\theta}_n(0) + \alpha\sqrt{\frac{2}{b-a}}\int_0^t\left[(-1)^n aT_1(t') - bT_2(t')\right]e^{\alpha\lambda_n^2 t'}dt'\right\}$$

$$\bar{\theta}_n(0) = \int_0^{b-a}(x+a)T_i(x+a)K_n(x)dx$$

Invert:

$$\theta(x,t) = \frac{2}{b-a}\sum_{n=1}^{\infty}e^{-\alpha\lambda_n^2 t}\sin\lambda_n x\left\{\int_0^{b-a}(x'+a)T_i(x'+a)\sin\lambda_n x'\,dx'\right.$$

$$\left. + \alpha\int_0^t\left[(-1)^n aT_1(t') - bT_2(t')\right]e^{\alpha\lambda_n^2 t'}dt'\right\}$$

Finite Fourier Transforms
1D Spherical Coordinate System

Example:

$$\theta(x,t) = \frac{2}{b-a}\sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t}\sin\lambda_n x\left\{\int_0^{b-a}(x'+a)T_i(x'+a)\sin\lambda_n x'\,dx'\right.$$

$$\left.+\alpha\int_0^t\left[(-1)^n aT_1(t') - bT_2(t')\right]e^{\alpha\lambda_n^2 t'}dt'\right\}$$

Write in terms of $\theta(r,t)$: Recall: change of variable: $r = x + a$

$$\theta(r,t) = \frac{2}{b-a}\sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t}\sin\lambda_n(r-a)\left\{\int_0^{b} r'T_i(r')\sin\lambda_n(r'-a)\,dr'\right.$$

$$\left.+\alpha\int_0^t\left[(-1)^n aT_1(t') - bT_2(t')\right]e^{\alpha\lambda_n^2 t'}dt'\right\}$$

The temperature distribution $T(r,t)$ then becomes: $\theta(r,t) = rT(r,t)$

$$T(r,t) = \frac{2}{r(b-a)}\sum_{n=1}^{\infty} e^{-\alpha\lambda_n^2 t}\sin\lambda_n(r-a)\left\{\int_0^{b} T_i(r')\sin\lambda_n(r'-a)r'\,dr'\right.$$

$$\left.+\alpha\int_0^t\left[(-1)^n aT_1(t') - bT_2(t')\right]e^{\alpha\lambda_n^2 t'}dt'\right\}$$

Hankel Transforms: Cylindrical Coordinate System

Integral transforms whose kernels are Bessel functions are called **Hankel transforms**, and they are obtained from the expansion of an arbitrary function in an infinite series of Bessel functions.

They are also referred to as **Bessel transforms**.

There are a great variety of these transforms because:

- there are variety of Bessel functions that are the solutions of Bessel's differential equation

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + (\lambda^2 r^2 - v^2)R = 0$$

- the Hankel transforms may be developed over either a finite interval or a semi-infinite region with various boundary conditions, or even over the infinite region.

Hankel Transforms: Cylindrical Coordinate System

The **finite Hankel transform** of an arbitrary function $f(r)$ on the region (a, b) is defined as

$$\bar{f}_n = \int_a^b f(r) K_\upsilon(\lambda_n, r) r\, dr$$

with the inversion $f(r) = \sum_{n=1}^{\infty} \bar{f}_n K_\upsilon(\lambda_n, r)$

where the kernels $K_\nu(\lambda_n, r)$ are the normalized characteristic functions of the following characteristic-value problem:

$$r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} + (\lambda^2 r^2 - \upsilon^2) R = 0 \qquad \alpha_1 R(a) + \beta_1 \frac{dR(a)}{dr} = 0,\ \alpha_1^2 + \beta_1^2 \neq 0$$

$$\alpha_2 R(b) + \beta_2 \frac{dR(b)}{dr} = 0,\ \alpha_2^2 + \beta_2^2 \neq 0$$

The kernels $K_\nu(\lambda_n, r)$ and the characteristic values λ_n have been evaluated for the nine different combinations of the boundary conditions at $r = a$ and $r = b$.

Hankel Transforms: Cylindrical Coordinate System

When the region of the transform is $(0, r_0)$, the kernels $K_\nu(\lambda_n, r)$ are the normalized characteristic functions of the following characteristic-value problem:

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}+(\lambda^2r^2-\nu^2)R=0$$

$$R(0) = \textit{finite}$$

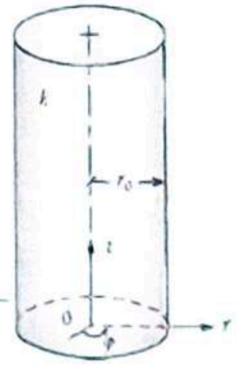
$$\alpha R(r_0)+\beta\frac{dR(r_0)}{dr}=0,\ \alpha^2+\beta^2\neq 0$$

Boundary condition at $r = r_0$	Kernel, $K_\nu(\lambda_n, r)$[†]	Characteristic values λ_n's are positive roots of[§]
Third kind[‡] ($\alpha \neq 0, \beta \neq 0$)	$\frac{\sqrt{2}}{r_0}\frac{1}{\left[1+1/\lambda_n^2\right]\left(H^2-\nu^2/r_0^2\right)^{1/2}}\frac{J_\nu(\lambda_n r)}{J_\nu(\lambda_n r_0)}$	$HJ_\nu(\lambda r_0)+\frac{dJ_\nu(\lambda r_0)}{dr}=0$
Second kind ($\alpha = 0, \beta \neq 0$)	$\frac{\sqrt{2}}{r_0}\frac{1}{\left(1-\nu^2/\lambda_n r_0^2\right)^{1/2}}\frac{J_\nu(\lambda_n r)}{J_\nu(\lambda_n r_0)}$	$\frac{dJ_\nu(\lambda r_0)}{dr}=0$[§]
First kind ($\alpha \neq 0, \beta = 0$)	$\frac{\sqrt{2}}{r_0}\frac{J_\nu(\lambda_n r)}{J_{\nu+1}(\lambda_n r_0)}$	$J_\nu(\lambda r_0)=0$

[†] $H = \alpha/\beta$.

[‡] See footnote ‡ in Table 4.2, and modify the transform accordingly.

[§] When $\nu = 0$, $\lambda_0 = 0$ is also a characteristic value for this case.



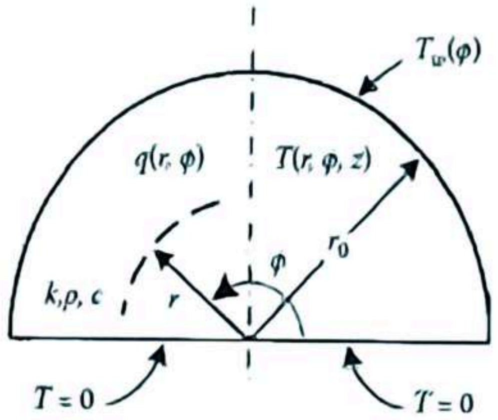
Hankel Transforms: Cylindrical Coordinate System

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}$$

The partial derivative with respect to z can be removed from the problem by applying Fourier transforms in the z direction.

Hankel Transforms: Example: T(r,ϕ,z)

Problem:



Consider a half cylinder of semi-infinite length, $0 \le r \le r_0$, $0 \le \phi \le \pi$ and $0 \le z < \infty$ as illustrated in cross-section in Figure.

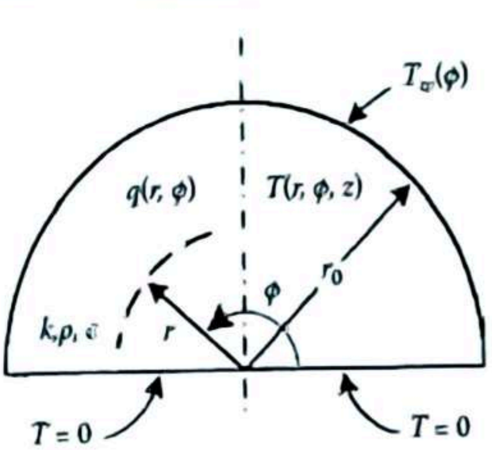
Internal energy is generated in this cylinder at a rate of $q(r, \phi)$ per unit volume.

The surfaces at $\phi = 0$, $\phi = \pi$, and $z = 0$ are at zero temperature, while the surface at $r = r_0$ is kept at temperature $T_w(\phi)$.

Find the steady-state temperature distribution $T(r, \phi, z)$ in the cylinder.

Hankel Transforms: Example: T(r,φ,z)

Problem Formulation:



$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}(r,\phi)}{k}=0$$

$$T(0,\phi,z)=0 \text{ and } T(r_0,\phi,z)=T_\infty(\phi)$$

$$T(r,0,z)=T(r,\pi,z)=0$$

$$T(r,\phi,0)=0 \text{ and } T(r,\phi,\infty)=\textit{finite}$$

The range of ϕ is (0, π), and in this finite interval the finite Fourier transform $T_n(r, z)$ of $T(r, \phi, z)$ with respect to the variable ϕ can be defined as

$$\bar{T}_n(r,z)=\int_0^\pi T(r,\phi,z)K_n(\phi)d\phi \qquad \text{with the inversion} \qquad T(r,\phi,z)=\sum_{n=1}^{\infty}\bar{T}_n(r,z)K_n(\phi)$$

Hankel Transforms: Example: T(r,ϕ,z)

where the kernels $K_n(\phi)$ are the normalized characteristic functions of the following characteristic-value problem:

$$\frac{d^2\psi}{d\phi^2} + n^2\psi = 0 \qquad K_n(\phi) = \sqrt{\frac{2}{\pi}} \sin n\phi, \; n = 1, 2, 3, \ldots$$

See Table

$$\psi(0) = \psi(\pi) = 0$$

The transform of the heat conduction equation with respect to ϕ yields

$$\frac{\partial^2 \bar{T}_n}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{T}_n}{\partial r} - \frac{n^2}{r^2}\bar{T}_n(r,z) + \frac{\partial^2 \bar{T}_n}{\partial z^2} + \frac{\bar{\dot{q}}_n(r)}{k} = 0$$

Eq. (A)

where we have defined $\bar{\dot{q}}_n(r) = \int_0^\pi \dot{q}(r,\phi) K_n(\phi)\, d\phi$

Eq. (A) involves the differential operator (L): $\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2}$ **Use Hankel Transform to remove this.**

$$LT \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) - \frac{n^2}{r^2}T$$

$$\int_0^{r_0} f(r,z)\, K_0(\quad)\, r\, dr$$

$$T(r,\phi,z) \xrightarrow{\text{FFT}} \bar{T}_n(r,z) \xrightarrow{\text{F Hankel Tr}} \bar{\bar{T}}_n(z)$$

$\bar{\bar{T}}_n(z)$ → solve ODE

→ FFT → Solve Algebraic Eq

$$T(r,\phi,z) \xrightarrow{\text{FFT}} \bar{T}_n(r,z) \xrightarrow{\text{HK}(r)} \bar{\bar{T}}_n(z)$$

Hankel Transforms: Example: T(r,ϕ,z)

In order to remove this differential operator, we define the finite Hankel transform $\bar{\bar{T}}_n(\lambda_m, z)$ of the function $\bar{T}_n(r, z)$ in the finite interval $(0, r_0)$ as

$$\bar{\bar{T}}_n(\lambda_m, z) = \int_0^{r_0} \bar{T}_n(r, z) K_n(\lambda_m, r)\, r\, dr \quad \text{with the inversion} \quad \bar{T}_n(r, z) = \sum_{m=1}^{\infty} \bar{\bar{T}}_n(\lambda_m, z) K_n(\lambda_m, r)$$

where the kernels $K_n(\lambda_m, r)$ are the normalized characteristic functions of the following characteristic value problem:

$$\left.\begin{aligned} & r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + (\lambda^2 r^2 - n^2)R = 0 \\ & R(0) = \textit{finite} \text{ and } R(r_0) = 0 \end{aligned}\right\}$$

Kernel (see Table):

$$K_n(\lambda_m, r) = \frac{\sqrt{2}}{r_0} \frac{J_n(\lambda_m r)}{J_{n+1}(\lambda_m r_0)}$$

where the characteristic values λ_m are positive roots of

$$J_n(\lambda r_0) = 0$$

Hankel Transforms: Example: T(r,ϕ,z)

Now, the transform of **Eq. (A)** with respect to r, by Hankel transform yields:

$$\frac{d^2\bar{\bar{\bar{T}}}_n}{dz^2} - \lambda_m^2 \bar{\bar{\bar{T}}}_n(\lambda_m, z) = r_0 \frac{dK_n(\lambda_m, r_0)}{dr} \bar{T}_{wn} - \frac{1}{k} \bar{\bar{\dot{q}}}_n(\lambda_m)$$

where we have defined $\quad \bar{T}_{wn} = \int_0^{\pi} K_n(\phi) T_w(\phi) d\phi$

$$\bar{\bar{\dot{q}}}_n(\lambda_m) = \int_0^{r_0} \bar{\dot{q}}_n(r) K_n(\lambda_m, r) r\, dr$$

The solution of the ODE: $\quad \bar{\bar{\bar{T}}}_n(\lambda_m, z) = A_n^m e^{-\lambda_m z} + B_n^m e^{\lambda_m z} + \bar{\bar{\bar{T}}}_{pn}(\lambda_m)$

where the particular solution $\bar{\bar{\bar{T}}}_{pn}(\lambda_m)$ is given by

$$\bar{\bar{\bar{T}}}_{pn}(\lambda_m) = -\frac{1}{\lambda_m^2} \left[r_0 \frac{dK_n(\lambda_m, r_0)}{dr} \bar{T}_{wn} - \frac{1}{k} \bar{\bar{\dot{q}}}_n(\lambda_m) \right]$$

Hankel Transforms: Example: T(r,ϕ,z)

Since the temperature distribution $T(r, \phi, z)$ is to be finite as $z \to \infty$, we have

$$\lim_{z\to\infty} \bar{\bar{T}}_n(\lambda_m, z) = finite$$

which yields $B_n^m = 0$. On the other hand, since $T(r, \phi, 0) = 0$,

$$\bar{\bar{T}}_n(\lambda_m, z) = 0$$

which gives

$$A_n^m = -\bar{\bar{T}}_p(\lambda_m, n)$$

The solution for $\bar{\bar{T}}_n(\lambda_m, z)$ now becomes

$$\bar{\bar{T}}_n(\lambda_m, z) = (1 - e^{-\lambda_m z})\bar{\bar{T}}_{pm}(\lambda_m)$$

Hankel Transforms: Example: T(r,ϕ,z)

When this double transform is inverted successively through the use of inversion relations, we obtain the temperature distribution as

$$T(r,\phi,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(1-e^{-\lambda_m z})\bar{\bar{\bar{T}}}_{pN}(\lambda_m)K_n(\lambda_m,r)K_n(\phi)$$

which can also be written as

$$T(r,\phi,z)=\frac{4}{\pi r_0}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1-e^{-\lambda_m z}}{\lambda_m^2}\sin n\phi\,\frac{J_n(\lambda_m r)}{J_{n+1}(\lambda_m r_0)}\left[\lambda_m\int_0^{\pi}\sin n\phi' T_w(\phi')\,d\phi'\right.$$

$$\left.+\frac{1}{k r_0 J_{n+1}(\lambda_m r_0)}\int_0^{r_0}\int_0^{\pi}J_n(\lambda_m r')\sin n\phi' \dot{q}(r',\phi')r'\,dr'd\phi'\right]$$

Laplace Transforms: Example

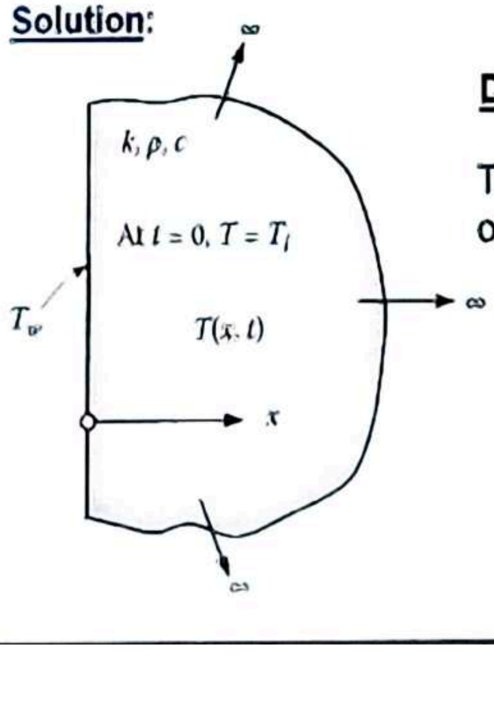
Laplace Transforms: Example-1

Consider the semi-infinite solid which is initially at a uniform temperature T_i.

The surface temperature is changed to T_w at $t = 0$ and is maintained constant at this value for times $t > 0$.

Assume constant thermo-physical properties. Find the unsteady-state temperature distribution $T(x, t)$ in the solid for $t > 0$.

Solution:



Define: $\theta(x, t) = T(x, t) - T_i$

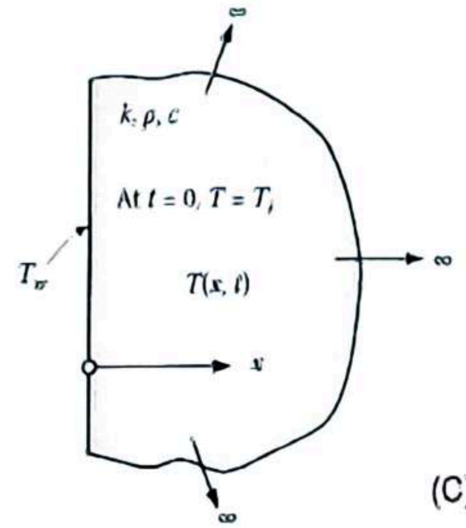
The formulation of the problem in terms of $\theta(x, t)$ is:

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t}$$

$$\theta(x, 0) = 0$$

$$\theta(0, t) = T_w - T_i = \theta_w \quad \text{and} \quad \lim_{x \to \infty} \theta(x, t) = 0$$

Laplace Transforms: Example-1



$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t} \qquad \theta(x,0) = 0$$

$$\theta(0,t) = T_\infty - T_i = \theta_w \quad \text{and} \quad \lim_{x\to\infty}\theta(x,t) = 0$$

The transformed differential equation and boundary conditions are

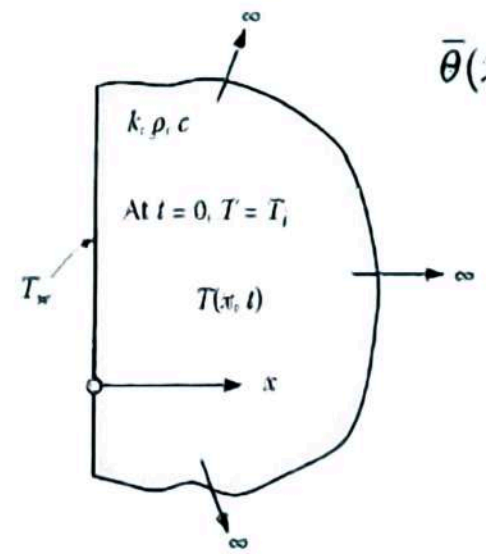
$$\frac{d^2\bar{\theta}}{dx^2} - \frac{p}{\alpha}\bar{\theta} = 0$$

$$\text{(C)} \quad \bar{\theta}(0,p) = \frac{\theta_w}{p} \quad \text{and} \quad \lim_{x\to\infty}\bar{\theta}(x,p) = 0 \quad \text{(B)}$$

where $\bar{\theta}(x,p)$ is the Laplace transform of $\theta(x, t)$.

<u>The general solution</u>: $\bar{\theta}(x,p) = C_1 e^{-mx} + C_2 e^{mx}$ where C_1 and C_2 are the constants of integration, and $m^2 = p/\alpha$.

Laplace Transforms: Example-1



$$\bar{\theta}(x,p) = C_1 e^{-mx} + C_2 e^{mx}$$

$C_2 = 0$

$C_1 = \theta_w / p$

$m^2 = p/\alpha$

Solution:

$$\frac{\bar{\theta}(x,p)}{\theta_w} = \frac{e^{-mx}}{p} = \frac{-e^{x\sqrt{p/\alpha}}}{p}$$

Take inverse transform:

$$\frac{\theta(x,t)}{\theta_w} = \frac{T(x,t) - T_i}{T_w - T_i} = \mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$