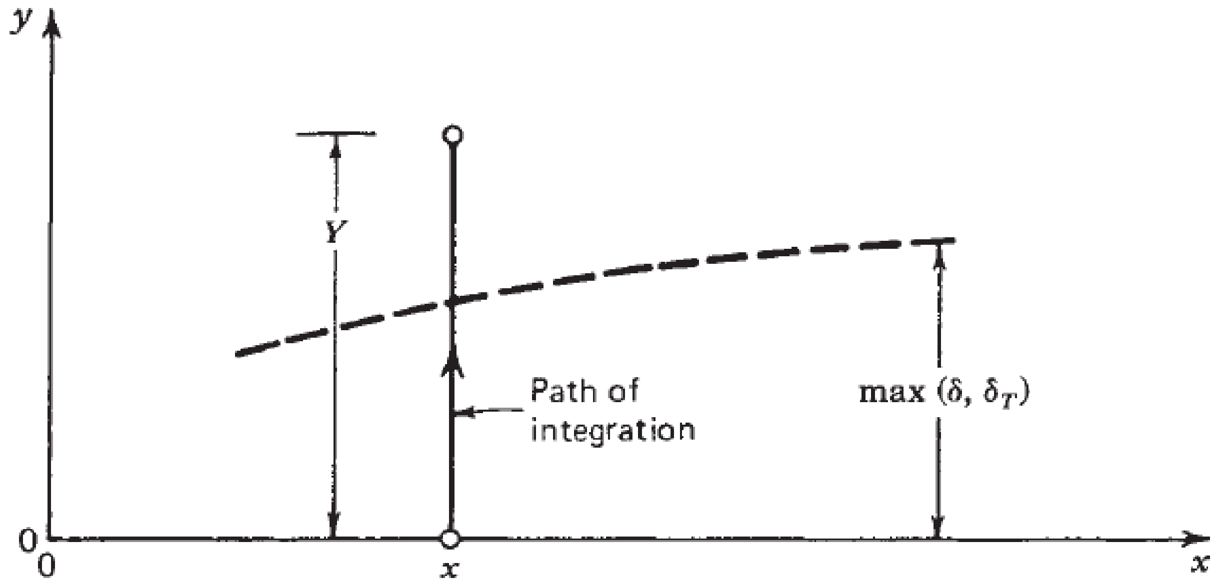
Integral solutions

Integral boundary layer equations for momentum and energy

$$\frac{d}{dx}\int_0^Y u(U_\infty - u)\,dy = \frac{1}{\rho}Y\frac{dP_\infty}{dx} + \frac{dU_\infty}{dx}\int_0^Y u\,dy + \nu\left(\frac{\partial u}{\partial y}\right)_0 \qquad (1)$$

$$\frac{d}{dx}\int_0^Y u(T_\infty - T)\,dy = \frac{dT_\infty}{dx}\int_0^Y u\,dy + \alpha\left(\frac{\partial T}{\partial y}\right)_0 \qquad (2)$$



Velocity profile

Assume uniform flow (U_∞, P_∞ = constants)

Assume that the shape of the longitudinal velocity profile is described by

$$u = \begin{cases} U_\infty m(n), \\ U_\infty, \end{cases} \qquad (3) \qquad n = \frac{y}{\delta}$$

Substituting into equation (1) gives

$$\delta \frac{d\delta}{dx} \left[\int_0^1 m(1 - m)\, dn \right] = \frac{\nu}{U_\infty} \left(\frac{dm}{dn} \right)_{n=0}$$

The resulting expressions for local boundary layer thickness and skin friction coefficient are

$$\frac{\delta}{x} = a_1 \, \mathrm{Re}_x^{-1/2}$$

$$C_{f,x} = \frac{\tau}{\frac{1}{2}\rho U_\infty^2} = a_2 \, \mathrm{Re}_x^{-1/2}$$

with the following notation:

$$a_1 = \left[\frac{2(dm/dn)_{n=0}}{\int_0^1 m(1-m)\, dn} \right]^{1/2}$$

$$a_2 = \left[2\left(\frac{dm}{dn}\right)_{n=0} \int_0^1 m(1-m)\, dn \right]^{1/2}$$

Temperature profile

Heat transfer coefficient information is extracted in a similar fashion from eq. (2) with $dT_\infty/dx = 0$

$$\begin{aligned} T_0 - T &= (T_0 - T_\infty)m(p), && 0 \le p \le 1 \\ T &= T_\infty, && 1 \le p \end{aligned} \qquad (4) \qquad p = \frac{y}{\delta_T}$$

1. For high-Pr fluids, $\delta_T << \delta$

Integral energy equation (2) reduces to

$$\mathrm{Pr} = \frac{2(dm/dp)_{p=0}}{(a_1\Delta)^2}\left[\int_0^1 m\,(p\Delta)\,[1 - m(p)]\,dp\right]^{-1} \qquad \Delta = \frac{\delta_T}{\delta}$$

1. For low-Pr fluids (liquid metals), $\delta_T >> \delta$

Integral energy equation (2) reduces to

$$\mathrm{Pr} = \frac{2(dm/dp)_{p=0}}{(a_1\Delta)^2}\left[\int_0^{1/\Delta} m\,(p\Delta)\,[1 - m(p)]\,dp + \int_{1/\Delta}^{1} [1 - m(p)]\,dp\right]^{-1} \qquad \Delta = \frac{\delta_T}{\delta}$$

The sum of two integrals stems from the fact that when $\delta_T >> \delta$, immediately next to the wall ($0 < y < \delta$), the velocity is described by the assumed shape $U_\infty m$, whereas for ($\delta < y < \delta_T$), the velocity is uniform, $u = U_\infty$.

Since Δ is much greater than unity, the second integral dominates

Impact of the assumed profile shape on the integral solution to the laminar boundary layer friction and heat transfer problem

Profile Shape $m(n)$ or $m(p)$ (Fig. 2.4)	$\frac{\delta}{x}\mathrm{Re}_x^{1/2}$	$C_{f,x}\,\mathrm{Re}_x^{1/2}$	$\mathrm{Nu}\,\mathrm{Re}_x^{-1/2}\,\mathrm{Pr}^{-1/3}$ Uniform Temperature ($\mathrm{Pr} > 1$)	$\mathrm{Nu}\,\mathrm{Re}_x^{-1/2}\,\mathrm{Pr}^{-1/3}$ Uniform Heat Flux ($\mathrm{Pr} > 1$)
$m = n$	3.46	0.577	0.289	0.364
$m = (n/2)(3 - n^2)$	4.64	0.646	0.331	0.417
$m = \sin(\pi n/2)$	4.8	0.654	0.337	0.424
Similarity solution	4.92[a]	0.664	0.332	0.453

Similarity solutions

- The basic idea in the construction of these solutions is the observation that from one location *x* to another, the *u* and *T* profiles look similar (hence, the name similarity solutions)

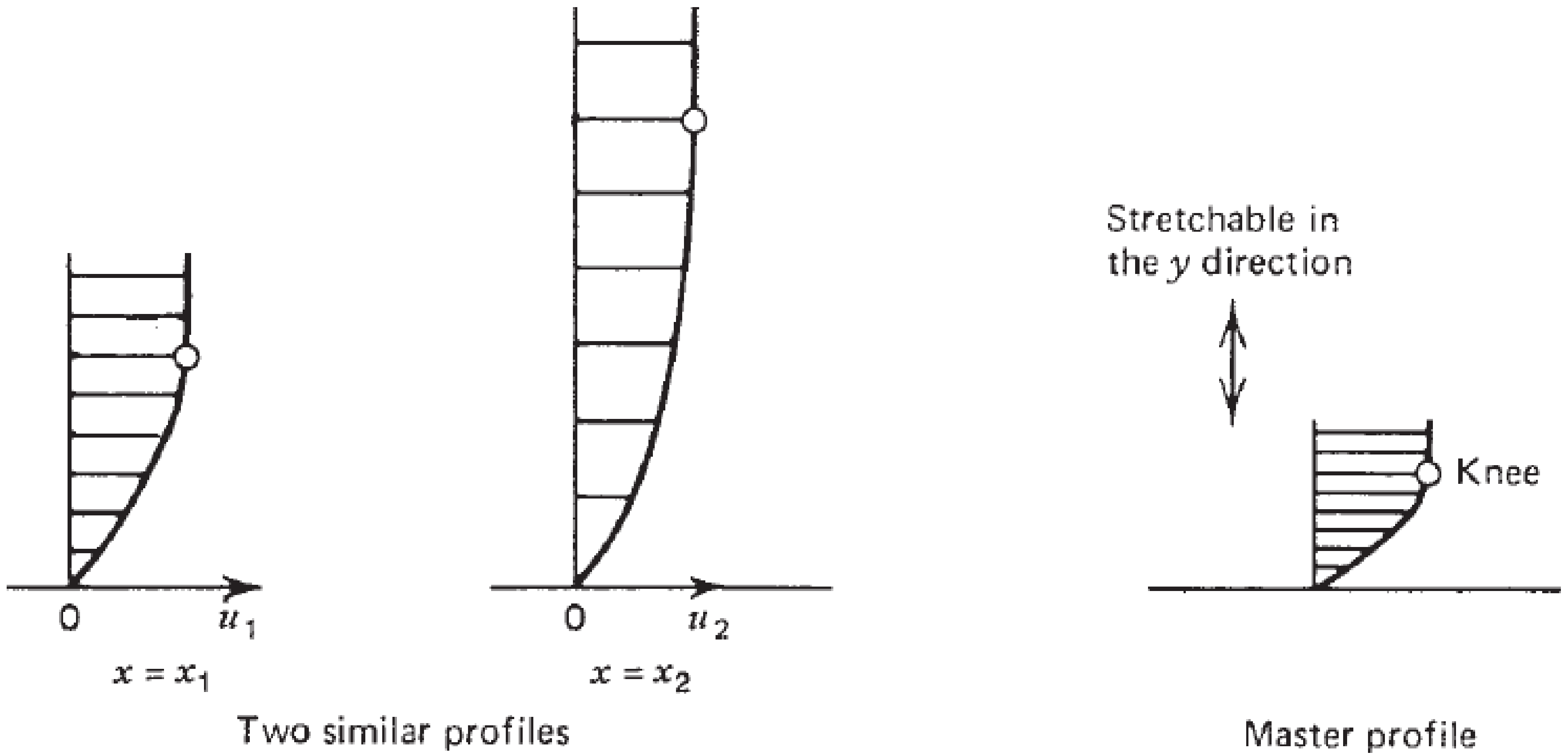
- Geometry, similarity, pattern and design (drawing) are at the core of science

Velocity profile

- Mathematically, the stretching of a master velocity profile amounts to writing

$$\frac{u}{U_\infty} = function(\eta)$$

where the similarity variable η is proportional to y and the proportionality factor depends on x.



Construction of similar profiles in the analysis of velocity boundary layers.

$$\frac{u}{U_\infty} = function(\eta)$$

where the similarity variable η is proportional to y and the proportionality factor depends on x.

Let,

$$\eta \propto y = y \times g(x)$$

Substituting into momentum BL equation we will eventually get the **Blasius equation** as

$$\boxed{\xi'''(\eta) + \frac{1}{2}\xi(\eta)\xi''(\eta) = 0}$$

With boundary condition as ,

u=0 *at* y=0

v=0 *at* y=0

$u \rightarrow u_\infty$ *at* $y \rightarrow \infty$

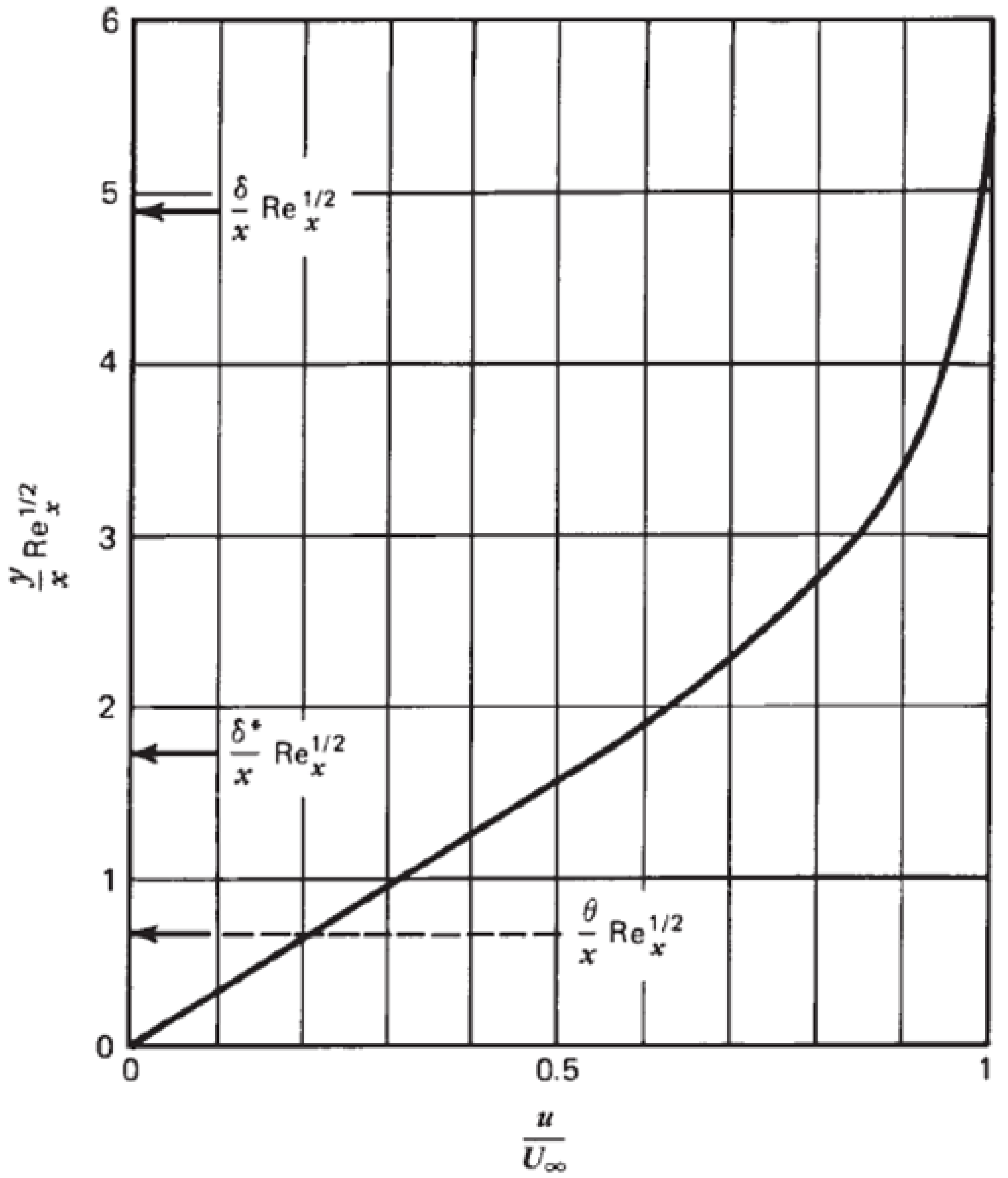
Where,

$$\eta = \frac{y}{\sqrt{\nu x / U_\infty}} \quad \text{and} \quad \xi'(\eta) = \frac{u}{U_\infty}$$

Solutions to the laminar constant-property boundary layer with an impermeable wall and $u_\infty =$ constant

η	ζ	ζ'	ζ''
0	0	0	0.3321
0.2	0.00664	0.06641	0.3320
0.4	0.02656	0.13277	0.3315
0.6	0.05974	0.19894	
0.8	0.10611	0.26471	
1.0	0.16557	0.32979	
1.2	0.23795	0.39378	
1.4	0.32298	0.45627	
1.6	0.42032	0.51676	
1.8	0.52952	0.57477	
2.0	0.65003	0.62977	
2.2	0.78120	0.68132	
2.4	0.92230	0.72899	
2.6	1.07252	0.77246	
2.8	1.23099	0.81152	
3.0	1.39682	0.84605	
3.2	1.56911	0.87609	
3.4	1.74696	0.90177	
3.6	1.92954	0.92333	
3.8	2.11605	0.94112	
4.0	2.30576	0.95552	
4.2	2.49806	0.96696	
4.4	2.69238	0.97587	
4.6	2.88826	0.98269	
4.8	3.08534	0.98779	
5.0	3.28329	0.99155	

For higher values of η, $\zeta = \eta - 1.72$.



Temperature profile

The heat transfer part of the problem was solved along similar lines. Introducing the dimensionless similarity temperature profile

$$\theta(\eta) = \frac{T - T_0}{T_\infty - T_0}$$

The boundary layer energy equation assumes the form

$$\theta''(\eta) + \frac{\Pr}{2}\xi(\eta)\theta'(\eta) = 0$$

With, boundary condition as

$$\theta = 0 \qquad \text{at } \eta = 0$$

$$\theta \to 1 \qquad \text{as } \eta \to \infty$$

Solution gives

$$\theta(\eta) = \frac{\int_0^{\eta} \exp\left[-\frac{\Pr}{2}\int_o^{\eta} \xi(\beta)\,d\beta\right]}{\int_0^{\infty} \exp\left[-\frac{\Pr}{2}\int_o^{\gamma} \xi(\beta)\,d\beta\right] d\gamma}$$

$$h_x = \frac{\dot{q}_0''}{t_0 - t_\infty} \qquad \text{(positive } \dot{q}'' \text{ in the positive } y \text{ direction)}$$

$$\dot{q}_0'' = -k\left(\frac{\partial t}{\partial y}\right)_0 = -k(t_\infty - t_0)\left(\frac{\partial \theta}{\partial y}\right)_0 = k(t_0 - t_\infty)\left(\frac{d\theta}{d\eta}\frac{\partial \eta}{\partial y}\right)_0 = \frac{k(t_0 - t_\infty)}{\sqrt{\nu x / U_\infty}}\theta'(0)$$

The local *Nu* can be defined as $\mathrm{Nu} = \dfrac{hx}{k} = \theta'(0)\,\mathrm{Re}_x^{1/2}$

Pohlhausen calculated several *θ'(0)* values that for Pr > 0.5 are correlated accurately by

$$\theta'(0) = 0.332\mathrm{Pr}^{1/3}$$

Gives

$$\mathrm{Nu} = 0.332\mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2} \qquad (\mathrm{Pr} > 0.5)$$

The average heat flux obtained in this manner can be non-dimensionalized as the overall Nusselt number:

$$\mathrm{Nu}_{0-x} = \frac{q''_{0-x}}{T_0 - T_\infty}\frac{x}{k} = \frac{h_{0-x}x}{k}$$

Gives

$$\mathrm{Nu}_{0-x} = \begin{cases} 0.664\mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2} & (\mathrm{Pr} > 0.5) \\ 1.128\mathrm{Pr}^{1/2}\,\mathrm{Re}_x^{1/2} & (\mathrm{Pr} < 0.5) \end{cases}$$

Limitations

- In concluding this section, it is worth noting the imperfect character of boundary layer theory and the approximation built into the exact similarity solution.

- Examination of the Blasius solution for the velocity normal to the wall shows that v tends to a finite value, $0.86U_\infty \mathrm{Re}_x^{-1/2}$, as η tends to infinity.

- Because in boundary layer theory $v/U_\infty \sim \mathrm{Re}_x^{-1/2}$ as $\eta \to \infty$, this theory becomes "better" as $\mathrm{Re}_x^{1/2}$ increases, that is, as the boundary layer region becomes more slender.

- Other limitations of the theory is the breakdown of the slenderness feature in the region near the tip.