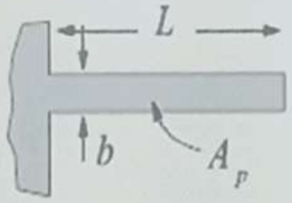
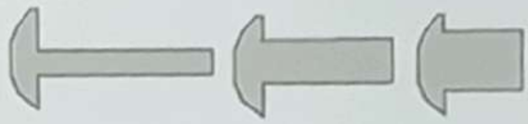
Fin Optimization: Rectangular Fin

For a given fin shape, fin material, and convection conditions, there exists an optimized design which transfers the maximum amount of heat for a given mass of the fin.





Consider: Adiabatic Fin Tip

$$q = \sqrt{hPkA_c}(T_b - T_\infty)\tanh N \qquad N^2 = \frac{hPL^2}{kA_c}$$

For a long fin (W ≫ b), P ≈ 2W and Ac = bW. Thus:

$$q' = \frac{q}{W} = \sqrt{2bhk}\,(T_b - T_\infty)\tanh N \qquad N^2 = \frac{2hL^2}{kb}$$

Fin Optimization: Rectangular Fin

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The length L can be eliminated using $A_P = bL$. The formula for N becomes

$$N^2 = \frac{2hA_P^2}{kb^3} \quad \Rightarrow \quad b = \left(\frac{2hA_P^2}{kN^2}\right)^{1/3}$$

$$q' = \left(4h^2k\,A_P\right)^{1/3}(T_b - T_\infty)\,N^{-1/3}\tanh N$$

$f(N) = N^{-1/3}\tanh N$ Set $\frac{df}{dN} = 0$ => $\cosh N \sinh N - 3N = 0$

Solve for N

$$N_{opt} = 1.419 = \left(\frac{2hA_P^2}{kb_{opt}^3}\right)^{1/2}$$

Once A_P is fixed, use the above formula to obtain b_{opt}, and find L from $L = A_P / b_{opt}$.

Fin Optimization: Rectangular Fin

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Rate of heat transfer:

$$q'_{opt} = \left(\frac{4h^2kA_P}{N_{opt}}\right)^{1/3}(T_b - T_\infty)\tanh N_{opt}$$

$$= 1.256\,\left(h^2kA_P\right)^{1/3}(T_b - T_\infty)$$

Fin Optimization: Rectangular Fin

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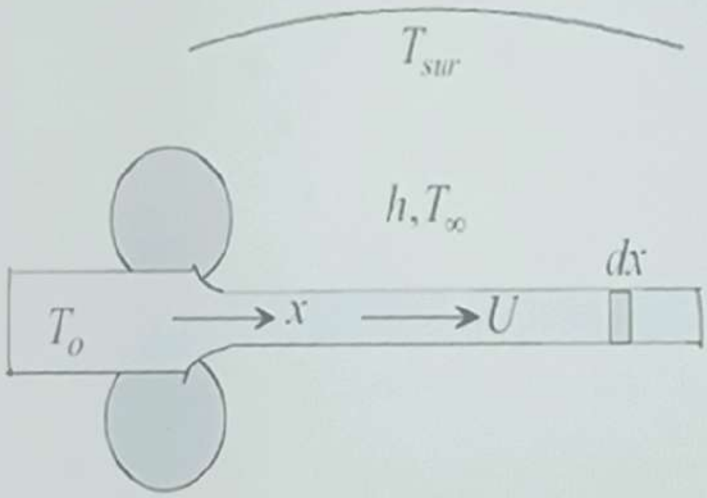
$$= 1.256\,\left(h^2kA_P\right)^{1/3}(T_b - T_\infty)$$

Triangular Fins:

$$q'_{opt} = 1.422\,\left(h^2kA_P\right)^{1/3}(T_b - T_\infty)$$

Which one is better – Rectangular or Triangular?

Example: Moving Fin



There are applications where a material exchanges heat with the surroundings while moving through a furnace or a channel.

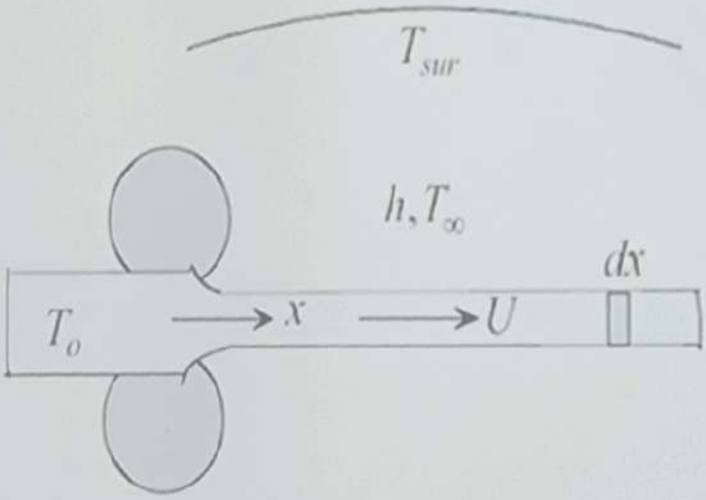
Examples

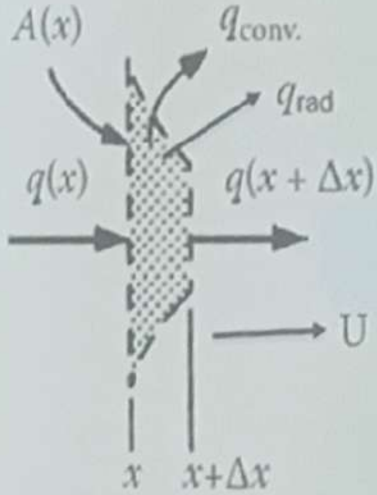
- the extrusion of plastics
- drawing of wires and sheets

Such problems can be modeled as moving fins as long as the criterion for fin approximation is satisfied.

- The Figure shows a sheet being drawn with velocity U through rollers.
- The sheet exchanges heat with the surroundings by radiation.
- It also exchanges heat with an ambient fluid by convection.
- Thus its temperature varies with distance from the rollers. Find $T(x)$.

Moving Fin



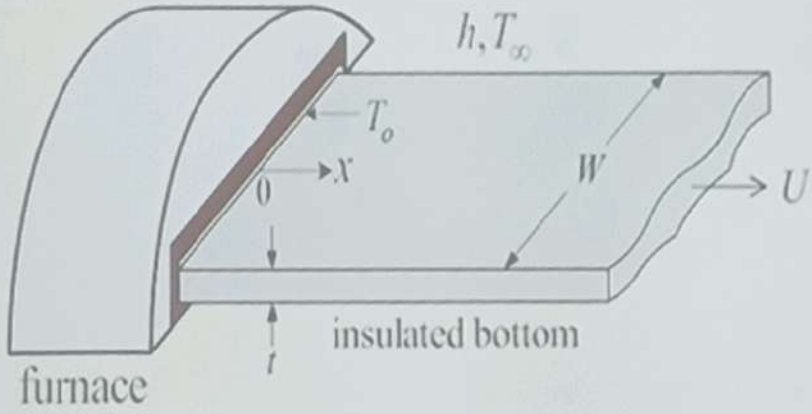


Assume steady state.

$$\frac{d^2T}{dx^2} - \frac{\rho c_p U}{k}\frac{dT}{dx} - \frac{hP}{kA_c}(T - T_\infty) - \frac{\varepsilon \sigma P}{kA_c}(T^4 - T_{sur}^4) = 0\,.$$

Here ε is emissivity, σ is Stefan-Boltzmann constant, P is perimeter, U is velocity.

Moving Fin with Surface Convection



A thin plastic sheet of thickness t and width W is heated in a furnace to temperature T_0.

The sheet moves on a conveyor belt traveling with velocity U. It is cooled by convection outside the furnace by an ambient fluid at T_∞.

The heat transfer coefficient is h. Assume steady state, $Bi < 0.1$, negligible radiation, and no heat transfer from the sheet to the conveyor belt.

Formulate the heat conduction problem.

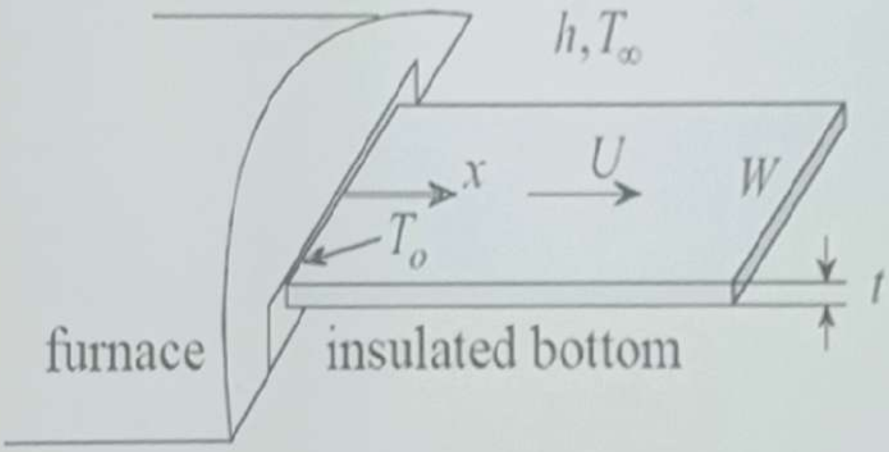
Moving Fin: Example

A thin plastic sheet of thickness t and width W is heated in a furnace to temperature T_0.

The sheet moves on a conveyor belt traveling with constant velocity U.

It is cooled by convection outside the furnace by an ambient fluid at T_∞.

Bi << 0.1

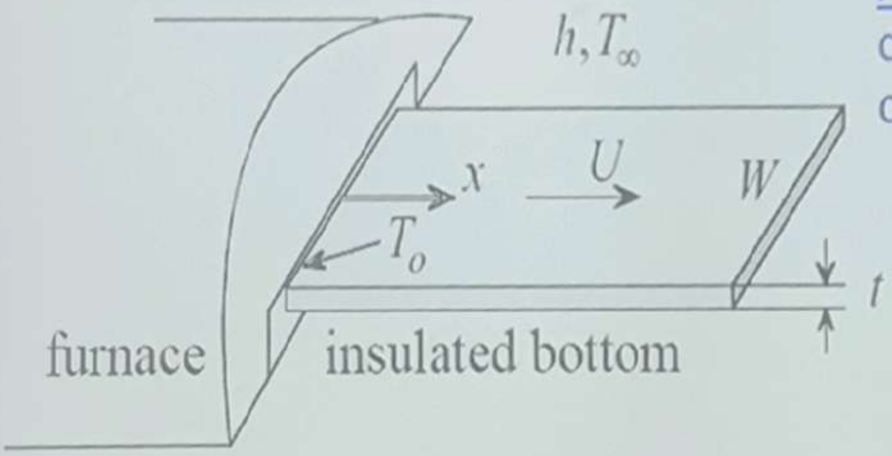


Determine the steady-state temperature distribution in the sheet.

<u>**Assumptions**</u>: One-dimensional, constant k, constant h, no radiation

$$\frac{d^2T}{dx^2} - \frac{\rho c_p U}{k}\frac{dT}{dx} - \frac{hP}{kA_c}(T - T_\infty) - \cancel{\frac{\varepsilon \sigma P}{kA_c}(T^4 - T_{sur}^4)} = 0.$$

Example: Moving Fin



Assumptions: One-dimensional, constant k, constant h, no radiation

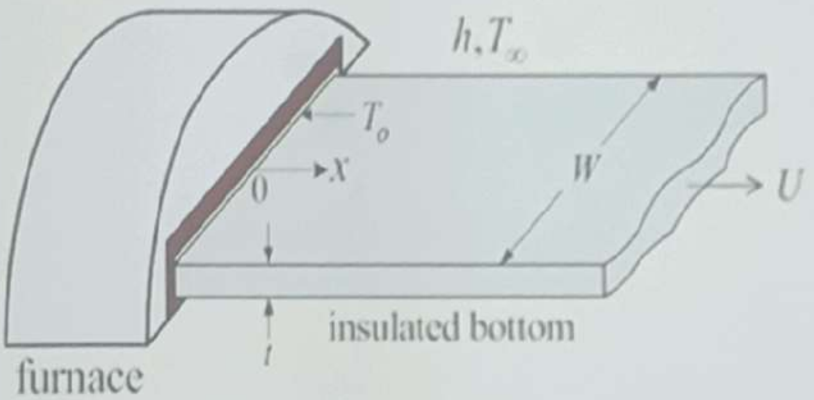
$$\frac{d^2T}{dx^2} - \frac{\rho c_p U}{k}\frac{dT}{dx} - \frac{hP}{kA_c}(T - T_\infty) - \cancel{\frac{\varepsilon \sigma P}{kA_c}(T^4 - T_{sur}^4)} = 0.$$

Boundary Conditions:

Fin base temperature is given. $T(0) = T_o$.

Far away from the furnace the temperature is T_∞ $\quad T(\infty) =$ finite.

Moving Fin with Surface Convection



$$\frac{d^2T}{dx^2} - \frac{\rho c_p U}{k}\frac{dT}{dx} - \frac{h P}{kA_c}(T - T_\infty) = 0.$$

$$T(0) = T_o. \qquad T(\infty) = \text{finite}.$$

Use

$A_c = Wt$,

$P = W + 2t$.

$$\frac{d^2T}{dx^2} + 2b\frac{dT}{dx} + m^2T = c$$

This is a linear, second-order differential equation with constant coefficients.

Where $b = -\frac{\rho c_p U}{2k}$, $c = -\frac{h(W+2t)}{kWt}T_\infty$, $m^2 = -\frac{h(W+2t)}{kWt}$.

Moving Fin with Surface Convection

$$\frac{d^2T}{dx^2} + 2b\frac{dT}{dx} + m^2T = c$$

$$T(0) = T_o.$$

$$T(\infty) = \text{finite}.$$

This is a linear, second-order differential equation with constant coefficients.

Solution: Note: $b^2 > m^2$

$$T = C_1 \exp\left(-bx + \sqrt{b^2 - m^2}\,x\right) + C_2 \exp\left(-bx - \sqrt{b^2 - m^2}\,x\right) + \frac{c}{m^2},$$

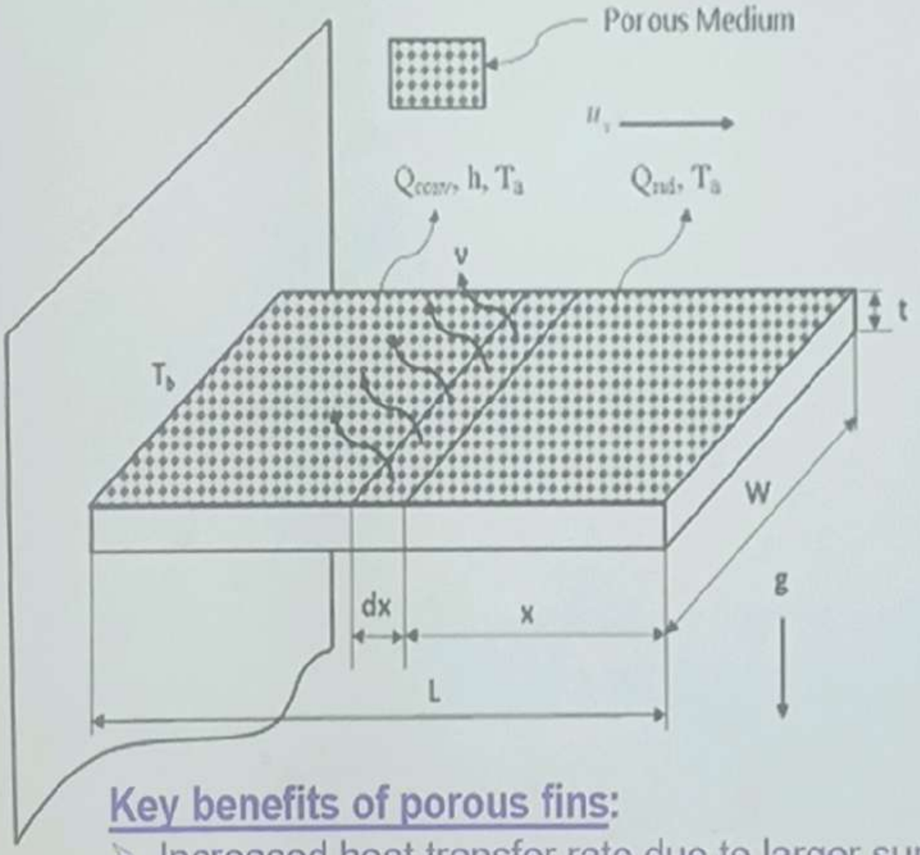
Since $m^2 < 0$, the BC $T(\infty) = \text{finite}$ requires $C_1 = 0$

From, the BC $T(0) = T_0$, $C_2 = T_o - \dfrac{c}{m^2}.$

On substitution, the temperature distribution in the sheet is:

$$\frac{T(x) - T_\infty}{T_o - T_\infty} = \exp\left[\frac{\rho c_p U}{2k}x - \sqrt{\left(\frac{\rho c_p U}{2k}\right)^2 + \frac{h(W + 2t)}{kWt}}\,x\right].$$

Porous Fin



As compared to a solid fin, a porous fin of equal weight gives better thermal performance by increasing the effective surface area of convective heat transfer.

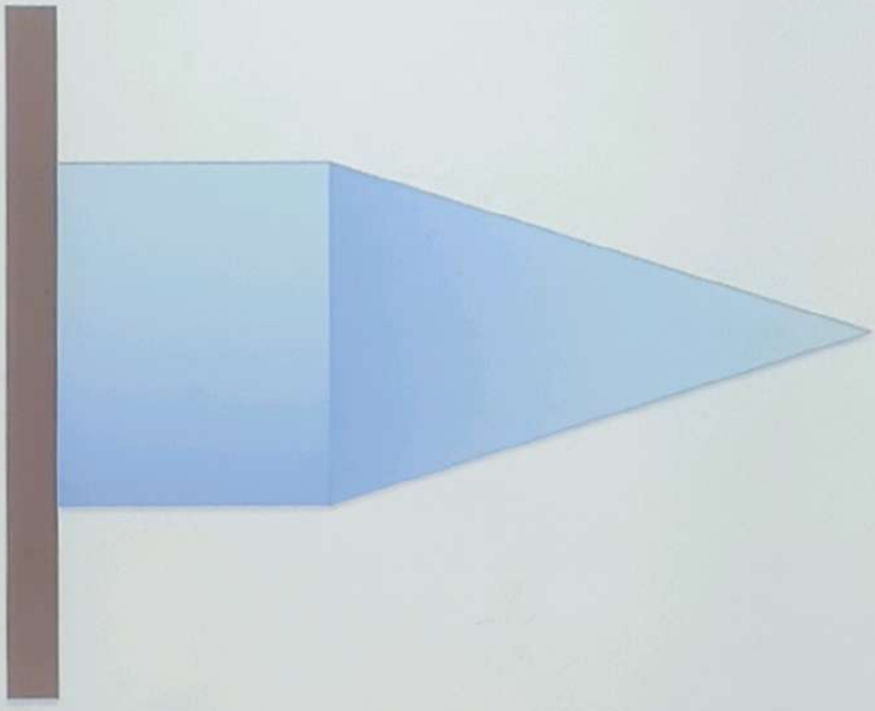
Key benefits of porous fins:

- Increased heat transfer rate due to larger surface area of porous structure
- Enhanced convection due to fluid flow through the pores within the fin
- Lightweight design

Applications of porous fins:

- Electronics cooling
- Automotive cooling
- Heat exchangers
- Aeronautical engineering

Fins with Combined Rectangular and Triangular Cross-sections



Orthogonal Functions
Sturm – Liouville Problem
(Characteristic – Value Problems)

Ordinary Fourier Series
Complete Fourier Series
Fourier Bessel Series
Legendre Fourier Series

Orthogonal Functions

Orthogonal Functions

Consider two real-valued functions $\phi_1(x)$and $\phi_2(x)$ that are both defined for all values of x on an interval $a \leq x \leq b$. These two functions are said to be *orthogonal functions* if

$$\int_{x=a}^{b} \phi_1(x)\phi_2(x)\, dx = 0$$

The definition of orthogonality can be extended to a general set of orthogonal functions $\phi_n(x)$ for $n = 1, 2, 3, \ldots$ by the following expression

$$\int_{x=a}^{b} \phi_n(x)\phi_m(x)\, dx = \begin{cases} 0 & \text{if } n \neq m \\ N(\lambda) & \text{if } n = m \end{cases}$$

where the interval of orthogonality is $a \leq x \leq b$, and where the term $N(\lambda)$ is called the norm or the normalization integral.

Orthogonal Functions

Extend the definition of an orthogonal set of functions to a more general case:

$$\int_{x=a}^{b} w(x)\phi_n(x)\phi_m(x)\,dx = \begin{cases} 0 & \text{if } n \neq m \\ N(\lambda) & \text{if } n = m \end{cases}$$

Weighted Inner Product

where $w(x)$ is the *weighting function* $[w(x) \geq 0]$.

Above Equation is interpreted as a set of functions $\phi_n(x)$ that is orthogonal over the interval $a \leq x \leq b$ with respect to weighting function $w(x)$.

$N(\lambda)$ is the norm, defined now as $$N(\lambda) = \int_{x=a}^{b} w(x)\left[\phi_n(x)\right]^2 dx$$

Orthogonal functions may take many forms:

- Trigonometric functions,
- Bessel functions,
- Legendre polynomials,
- Hermite polynomials,
- Tchebysheff polynomials,
- Laguerre polynomials

$$\int_{x=a}^{b} \phi_n(x)\phi_m(x)\,dx = \begin{cases} 0 & \text{if } n \neq m \\ N(\lambda) & \text{if } n = m \end{cases}$$

Weighting function $w(x)$ is unity here.

Sturm – Liouville Problems

Consider the following Characteristic-Value Problem, composed of the linear and homogeneous second-order differential equation of the general form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+\left[q(x)+\lambda w(x)\right]y=0$$

and the two homogeneous linear boundary condition prescribed at the ends of the finite interval (*a*, *b*).

$$\alpha_1 y(a)+\beta_1\frac{dy(a)}{dx}=0,\quad \alpha_1^2+\beta_1^2>0$$

$$\alpha_2 y(b)+\beta_2\frac{dy(b)}{dx}=0,\quad \alpha_2^2+\beta_2^2>0$$

Note:

1. The functions *p*(*x*), *q*(*x*) and *w*(*x*) are *real valued* and *continuous* (including *dp*/*dx*).
2. Also, *p*(*x*) and *w*(*x*) are *positive* over the entire interval (*a*, *b*), including the end points, while $q(x) \le 0$ in the same interval.
3. Furthermore, α_1, α_2, β_1 and β_2 are given real constants, and λ is an unspecified parameter, independent of x.

Characteristic-value problems of this type are known as *Sturm–Liouville problems*.

Sturm – Liouville Problems

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+\left[q(x)+\lambda w(x)\right]y=0 \qquad \alpha_1 y(a)+\beta_1\frac{dy(a)}{dx}=0,\quad \alpha_1^2+\beta_1^2>0$$

$$\alpha_2 y(b)+\beta_2\frac{dy(b)}{dx}=0,\quad \alpha_2^2+\beta_2^2>0$$

Nontrivial solutions of the above problem exist, in general, for a particular set of values $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the parameter λ. These are the characteristic values, (or the eigenvalues), and the corresponding solutions are the characteristic functions (or the eigenfunctions) of the problem.

Let λ_m and λ_n be any two distinct characteristic values, and $\phi_m(x)$ and $\phi_n(x)$ be the corresponding characteristic functions, respectively. These functions satisfy the above differential equation. It can be shown:

$$(\lambda_n-\lambda_m)\int_a^b \phi_m(x)\phi_n(x)w(x)dx=0 \quad \rightarrow \quad \int_a^b \phi_m(x)\phi_n(x)w(x)dx=0,\quad m\neq n$$

Thus, the characteristic functions of the Sturm–Liouville problem form an orthogonal set with respect to the weight function $w(x)$ on the interval (a, b).

Sturm – Liouville Problems: Singular End Points

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+\left[q(x)+\lambda w(x)\right]y=0 \qquad \alpha_1 y(a)+\beta_1\frac{dy(a)}{dx}=0,\quad \alpha_1^2+\beta_1^2>0$$

$$\alpha_2 y(b)+\beta_2\frac{dy(b)}{dx}=0,\quad \alpha_2^2+\beta_2^2>0$$

If it so happens that $p(a) = 0$, then the characteristic functions corresponding to different characteristic values will be orthogonal with respect to the weight function $w(x)$ on (a, b), provided that $y(x)$ and dy/dx are both finite at $x = a$.

In this case, the boundary condition at x = a is replaced by the requirement that $y(x)$ and dy/dx be finite at $x = a$ when $p(a) = 0$.

Sturm – Liouville Problems: Periodic BC

If $p(a) = p(b)$, then the boundary conditions can be replaced by

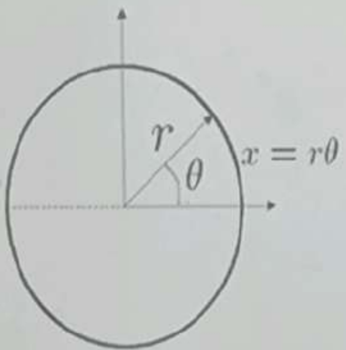
$$y(a) = y(b)$$

$$\frac{dy(a)}{dx} = \frac{dy(b)}{dx}$$

These are called *periodic boundary conditions* and they are satisfied, in particular, if the solution $y(x)$ is required to be periodic, of period $(b - a)$.

Heat Conduction in a Circular Ring:

We consider a thin wire of length 2L which is bent into the shape of a circle. Then at the endpoints we must match the temperature and the flux.



Thin circular wire - no radial temperature dependence. $u(r,\theta) = u(\theta)$

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$
$$= \frac{\partial^2 u}{\partial (r\theta)^2}$$

let $x = r\theta$ then $\frac{\partial^2 u}{\partial (r\theta)^2} = u_{xx}$.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad -L < x < L$$

$$u(-L,t) = u(L,t), \quad t > 0$$

$$\frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t), \quad t > 0$$

$$u(x,0) = f(x), \quad -L \le x \le L$$

Sturm – Liouville Problems

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+\left[q(x)+\lambda w(x)\right]y=0$$

$$\alpha_1 y(a)+\beta_1\frac{dy(a)}{dx}=0,\quad \alpha_1^2+\beta_1^2>0$$

$$\alpha_2 y(b)+\beta_2\frac{dy(b)}{dx}=0,\quad \alpha_2^2+\beta_2^2>0$$

<u>Note</u>:
The characteristic values of the Sturm–Liouville problem are all *real* and *nonnegative*, and the corresponding characteristic functions are *real.* The parameter λ can, therefore, be replaced by λ^2 with no loss in the generality of the problem.

In addition, there is *only one* characteristic function $\phi_n(x)$ which corresponds to each characteristic value λ_n, except when the periodic boundary conditions are involved.

More importantly, the characteristic functions form a *complete* orthogonal set.

Generalized Fourier Series

Consider a set of functions $\{\phi_n(x); n = 0, 1, 2, \ldots\}$ orthogonal with respect to a weight function $w(x)$ on the finite interval (a, b).

Expand an arbitrary function $f(x)$ in a series of these functions as

$$f(x) = A_0\phi_0(x) + A_1\phi_1(x) + \ldots + A_n\phi_n(x) + \ldots$$

Note:
Coefficients are unknown

$$\rightarrow \quad f(x) = \sum_{n=0}^{\infty} A_n\phi_n(x)$$

Now evaluate unknown coefficients:

$$\int_a^b f(x)\phi_n(x)w(x)\,dx = \int_a^b \left[\sum_{k=0}^{\infty} A_k\phi_k(x)\right]\phi_n(x)w(x)\,dx$$

Use orthogonal property: $$\int_a^b f(x)\phi_n(x)w(x)\,dx = A_n\int_a^b [\phi_n(x)]^2 w(x)\,dx$$

$$A_n = \frac{1}{N_n}\int_a^b f(x)\phi_n(x)w(x)\,dx \qquad N_n = \int_a^b [\phi_n(x)]^2 w(x)\,dx$$

Ordinary Fourier Series

The *ordinary Fourier series*, or simply the *Fourier series*, are developed from the characteristic functions of the following characteristic-value problem for different combinations of the boundary conditions:

$$\frac{d^2y}{dx^2}+\lambda^2 y(x)=0$$

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+\left[q(x)+\lambda w(x)\right]y=0$$

$$\alpha_1 y(0)+\beta_1\frac{dy(0)}{dx}=0,\quad \alpha_1^2+\beta_1^2>0$$

This characteristic-value problem is a special case of the Sturm–Liouville system with $p(x) = 1$, $q(x) = 0$, $w(x) = 1$, and λ replaced by λ^2.

$$\alpha_2 y(L)+\beta_2\frac{dy(L)}{dx}=0,\quad \alpha_2^2+\beta_2^2>0$$

Let the functions $\phi_n(x)$ are the characteristic functions (orthogonal) of the characteristic-value problem.

The *ordinary Fourier series*, or simply a *Fourier series*, of $f(x)$ on the interval $(0, L)$:

$$f(x)=\sum_{n=0}^{\infty}A_n\phi_n(x),\quad 0<x<L \qquad A_n=\frac{1}{N_n}\int_0^L f(x)\phi_n(x)\,dx \qquad N_n\int_0^L\left[\phi_n(x)\right]^2 dx$$

Ordinary Fourier Series: How Many?

$$\frac{d^2y}{dx^2} + \lambda^2 y(x) = 0$$

$$\alpha_1 y(0) + \beta_1 \frac{dy(0)}{dx} = 0, \quad \alpha_1^2 + \beta_1^2 > 0$$

$$\alpha_2 y(L) + \beta_2 \frac{dy(L)}{dx} = 0, \quad \alpha_2^2 + \beta_2^2 > 0$$

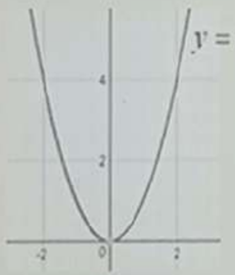
There are nine different combinations of the boundary conditions. Accordingly, corresponding to each combination there will be a series expansion of the form given by:

$$f(x) = \sum_{n=0}^{\infty} A_n \phi_n(x), \quad 0 < x < L$$

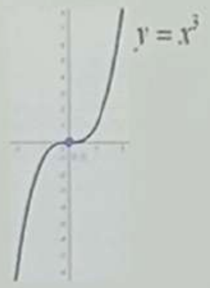
Odd Functions: Even Functions

Even Functions	Odd Functions
$f(-x) = f(x)$	$f(-x) = -f(x)$
Function is unchanged when reflected about the y-axis.	Function is unchanged when rotated 180° about the origin.

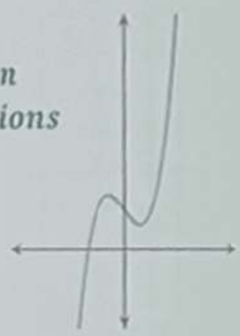
Example:



Example:



Neither Even nor Odd Functions



Trigonometric Functions	
Even Functions $f(-x) = f(x)$	Odd Functions $f(-x) = -f(x)$
$\cos(-x) = \cos x$ $\sec(-x) = \sec x$	$\sin(-x) = -\sin x$ $\csc(-x) = -\csc x$ $\tan(-x) = -\tan x$ $\cot(-x) = -\cot x$

Graphically

- A function is even if its graph is symmetric about the y-axis.
- A function is odd if its graph is symmetric about the origin.
- A function that is neither even nor odd is not symmetric about the y-axis or the origin.

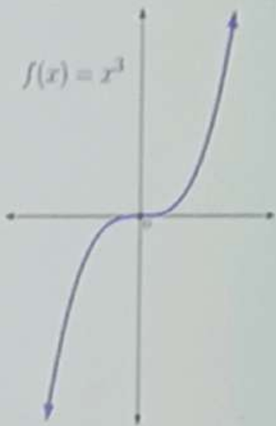
Fourier Sine Series

Consider the following characteristic-value problem:

A function is "odd" when:

$f(-x) = -f(x)\ \forall x$

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0 \qquad y(0) = 0 \quad \text{and} \quad y(L) = 0$$



Characteristic functions and characteristic values:

$$\phi_n(x) = \sin \lambda_n \quad \text{and} \quad \lambda_n = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots$$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}x, \quad 0 < x < L \quad \text{Eq. (A)}$$

***Fourier sine series* representation of *f*(*x*) on the interval (0, *L*)**

$$A_n = \frac{1}{N_n}\int_0^L f(x)\sin\frac{n\pi}{L}x\, dx \qquad N_n = \int_0^L \sin^2\frac{n\pi}{L}x\, dx = \frac{L}{2}$$

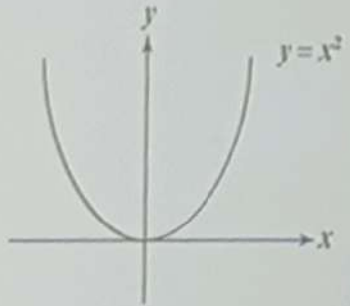
Note that the RHS of Eq. (A) is *a periodic function* of period 2*L*, and also is an *odd* function of *x*. Therefore, if Eq. (A) converges to *f*(*x*) in (0, *L*), it will also converge to – *f*(–*x*) in (–*L*, 0). In other words, if *f*(*x*) is an *odd* function of *x*, then Eq. (A) will represent *f*(*x*) not only in (0, *L*), but also in (– *L*, *L*).

Fourier Cosine Series

Consider the following characteristic-value problem:

A function is "even" when: f(x) = f(−x) for all x

$$\frac{d^2y}{dx^2}+\lambda^2 y=0 \qquad \frac{dy(0)}{dx}=0 \quad \text{and} \quad \frac{dy(L)}{dx}=0$$



Characteristic functions and characteristic values:

$$\phi_n(x)=\cos\lambda_n x \quad \text{and} \quad \lambda_n=\frac{n\pi}{L}, \quad n=0,1,2,\ldots$$

$$f(x)=\sum_{n=0}^{\infty}A_n\cos\frac{n\pi}{L}x=A_0+\sum_{n=1}^{\infty}A_n\cos\frac{n\pi}{L}x, \quad 0<x<L \qquad \text{Eq. (B)}$$

***Fourier cosine series* representation of *f*(x) on the interval (0, *L*)**

$$A_n=\frac{1}{N_n}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx \qquad N_n=\int_0^L\cos^2\frac{n\pi}{L}x\,dx=\begin{cases}L, & n=0\\ \frac{L}{2}, & n=1,2,3,\ldots\end{cases}$$

Note that the RHS of Eq. (B) is *a periodic function* of period 2*L*, and also is an *even* function of *x*. Therefore, if Eq. (B) converges to *f*(*x*) in (0, *L*), it will also converge to *f*(–*x*) in (–*L*, 0). In other words, if *f*(*x*) is an *even* function of *x*, then Eq. (B) will represent *f*(*x*) not only in (0, *L*), but also in (– *L*, *L*).

Complete Fourier Series

Any function of x, say F(x), can be written as:

$$F(x) = \frac{1}{2}\left[F(x) - F(-x)\right] + \frac{1}{2}\left[F(x) + F(-x)\right]$$

Odd function Even function

If $f(x)$ is odd:

$$f(-x) = -f(x)$$

If $f(x)$ is even:

$$f(-x) = f(x)$$

$$F(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{L}x + \sum_{n=0}^{\infty} b_n \cos\frac{n\pi}{L}x, \quad -L < x < L$$

or,

$$F(x) = b_0 + \sum_{n=1}^{\infty}\left[a_n \sin\frac{n\pi}{L}x + b_n \cos\frac{n\pi}{L}x\right], \quad -L < x < L$$

$$b_0 = \frac{1}{L}\int_0^L \left\{\frac{1}{2}[F(x) + F(-x)]\right\} dx = \frac{1}{2L}\int_{-L}^{L} F(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L \left\{\frac{1}{2}[F(x) - F(-x)]\right\} \sin\frac{n\pi}{L}x\,dx$$
$$= \frac{1}{L}\int_{-L}^{L} F(x)\sin\frac{n\pi}{L}x\,dx, \quad n = 1, 2, 3, \ldots$$

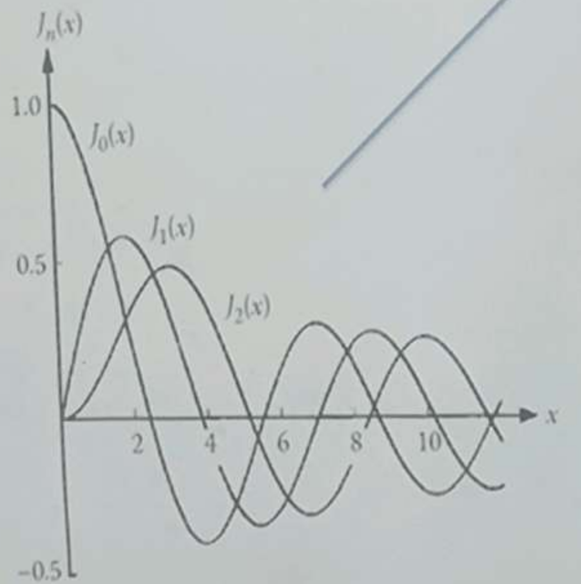
$$b_n = \frac{2}{L}\int_0^L \left\{\frac{1}{2}\left[F(x) + F(-x)\right]\right\} \cos\frac{n\pi}{L}x\,dx$$
$$= \frac{1}{L}\int_{-L}^{L} F(x)\cos\frac{n\pi}{L}x\,dx, \quad n = 1, 2, 3, \ldots$$

Bessel Functions

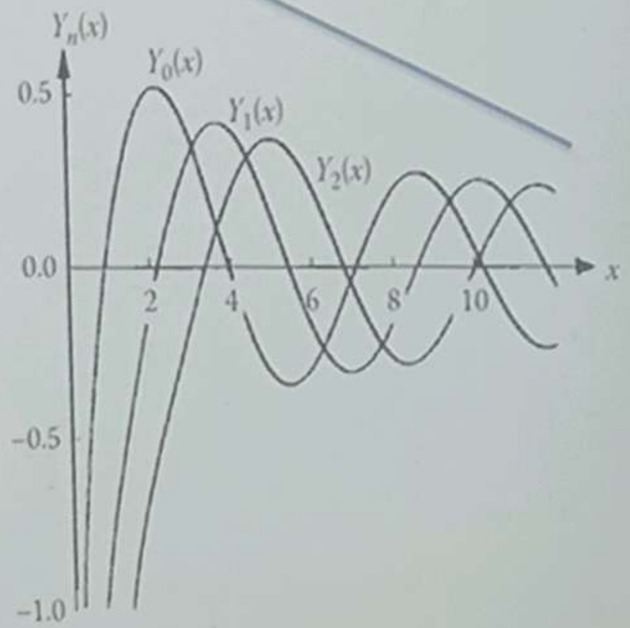
$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(m^2x^2-v^2)y=0$$

Bessel's differential equation of order v, where *m* is a *parameter* and *v* is any *real* constant.

General solution: $y(x)=C_1J_v(mx)+C_2Y_v(mx)$



Bessel functions of the first kind



Bessel functions of the second kind

Modified Bessel Functions

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-(m^2x^2+v^2)y=0$$

Modified Bessel's differential equation of order v.

General solution: $y(x)=C_1J_v(mx)+C_2K_v(mx)$

Modified Bessel Functions

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-(m^2x^2+v^2)y=0$$

Modified Bessel's differential equation of order v.

General solution: $y(x)=C_1J_v(mx)+C_2K_v(mx)$

Numerical Values of $J_n(x)$, $Y_n(x)$, $I_n(x)$, and $K_n(x)$

X	$J_0(x)$	$J_1(x)$	$Y_0(x)$	$Y_1(x)$	$I_0(x)$	$I_1(x)$	$K_0(x)$	$K_1(x)$
0.0	1.0000	0.000	$-\infty$	$-\infty$	1.000	0.000	∞	∞
0.1	0.9975	0.0499	−1.5342	−6.4590	1.0025	0.0501	2.4271	9.8538
0.2	0.9900	0.0995	−1.0811	−3.238	1.0100	0.1005	1.7527	4.7760
0.3	0.9776	0.1483	−0.8073	−2.2931	1.0226	0.1517	1.3725	3.0560
0.4	0.9604	0.1960	−0.6060	−1.7809	1.0404	0.2040	1.1145	2.1844
0.5	0.9385	0.2423	−0.4445	−1.4715	1.0635	2.2579	0.9244	1.6564
0.6	0.9120	0.2867	−0.3085	−1.2604	1.0920	0.3137	0.7775	1.3028
0.7	0.8812	0.3290	−0.1907	−1.1032	1.1263	0.3719	0.6605	1.0503
0.8	0.8463	0.3688	−0.0868	−0.9781	1.1665	0.4329	0.5653	0.8618
0.9	0.8075	0.4059	0.0056	0.8731	1.2130	0.4971	0.4867	0.7165
1.0	0.7652	0.4401	0.0883	−0.7812	1.2661	0.5652	0.4210	0.6019

Fourier Bessel Series

Series expansions in terms of Bessel functions arise most frequently in connection with the following characteristic-value problem (r-direction):

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}+(\lambda^2r^2-v^2)R=0 \qquad \alpha_1R(a)+\beta_1\frac{dR(a)}{dr}=0,\quad \alpha_1^2+\beta_1^2>0$$

This is a special case of the Sturm–Liouville system with:

$$\alpha_2R(b)+\beta_2\frac{dR(b)}{dr}=0,\quad \alpha_2^2+\beta_2^2>0$$

$p(r)=r,\ q(r)=-\dfrac{v^2}{r},$ and $w(r)=r$ **Compare:** $\dfrac{d}{dx}\left[p(x)\dfrac{dy}{dx}\right]+\left[q(x)+\lambda w(x)\right]y=0$

(Divide the DE throughout by r)

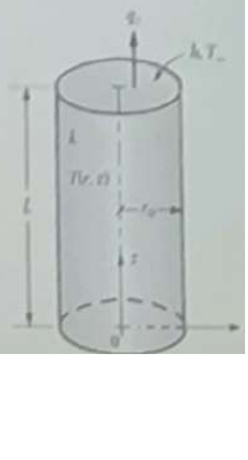
Hence, the characteristic functions of this problem form a complete orthogonal set with respect to the weight function r on the interval (a, b).

For Solid Cylinders (r-direction):

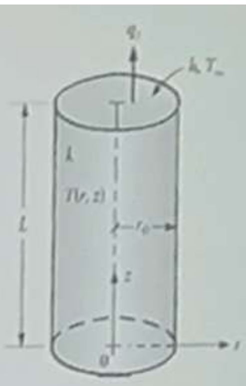
$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}+(\lambda^2r^2-v^2)R=0$$

$$R(0)=\textit{finite}$$

$$\alpha R(r_0)+\beta\frac{dR(r_0)}{dr}=0,\quad \alpha^2+\beta^2>0$$



Fourier Bessel Series



(A) $\left[\begin{array}{l} r^2\dfrac{d^2R}{dr^2}+r\dfrac{dR}{dr}+(\lambda^2r^2-v^2)R=0 \end{array}\right.$ $\quad R(0)=finite$

$$\alpha R(r_0)+\beta\frac{dR(r_0)}{dr}=0,\quad \alpha^2+\beta^2>0$$

The general solution of above **Eq. (A)** can be written as:

$$R(r)=AJ_v(\lambda r)+BY_v(\lambda r)$$

The first boundary condition yields $B\equiv 0$. Hence, the characteristic functions are $J_v(\lambda r)$ and the characteristic values λ_n are the roots of the *characteristic-value Eq:*

$$\alpha J_v(\lambda r_0)+\beta\frac{dJ_v(\lambda r_0)}{dr}=0 \quad\Rightarrow\quad \lambda_n, n=1,2,3,\ldots$$

In view of the fact that $p(0)=0$ and $R(0)$ = finite (and also $dR(0)/dr$ = finite), the characteristic functions of the system **(A)** form an orthogonal set with respect to the weight function $w(r)=r$ over the interval $(0, r_0)$; that is,

$$\int_0^{r_0} J_v(\lambda_m r)J_v(\lambda_n r)r\,dr=0,\quad \lambda_m\neq\lambda_n$$

and the set $\{J_v(\lambda_n r);\ n=1,2,\ldots\}$ is a complete orthogonal set.

Thus, we can expand an arbitrary function $f(r)$, which is piecewise differentiable on the interval $(0, r_0)$, in a series of these characteristic functions in the same interval.

Fourier Bessel Series: Table

Fourier–Bessel Series in the Finite Interval $(0, r_0)$

Fourier-Bessel expansion: $f(r) = \sum_{n=1}^{\infty} A_n J_\nu(\lambda_n r), \quad 0 < r < r_0$

Expansion coefficients: $A_n = \frac{1}{N_n} \int_0^{r_0} f(r) J_\nu(\lambda_n r) r\, dr$

$$\left\{ \begin{array}{c} r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + (\lambda^2 r^2 - v^2)R = 0 \\ R(0) = \textit{finite} \\ \alpha R(r_0) + \beta \frac{dR(r_0)}{dr} = 0 \end{array} \right\}$$

Boundary condition at $r = r_0$	$N_n = \int_0^{r_0} J_\nu^2(\lambda_n r) r\, dr$ [†]	Characteristic values λ_n are positive roots of[†]
Third kind[‡] $(\alpha \neq 0, \beta \neq 0)$	$\frac{r_0^2}{2}\left[1 + \frac{1}{\lambda_n^2}\left(H^2 - \frac{v^2}{r_0^2}\right)\right] J_\nu^2(\lambda_n r_0)$	$HJ_\nu(\lambda r_0) + \frac{dJ_\nu(\lambda r_0)}{dr} = 0$
Second kind $(\alpha \neq 0, \beta \neq 0)$	$\frac{r_0^2}{2}\left[1 - \left(\frac{v}{\lambda_n r_0}\right)^2\right] J_\nu^2(\lambda_n r_0)$	$\frac{dJ_\nu(\lambda r_0)}{dr} = 0$ [§]
First kind $(\alpha \neq 0, \beta \neq 0)$	$\frac{r_0^2}{2} J_{\nu+1}^2(\lambda_n r_0)$	$J_\nu(\lambda r_0) = 0$

[†] $H = \alpha/\beta$.

[‡] When $v > 0$ and $r_0 = -\nu/H > 0$, $\lambda_0 = 0$ is a characteristic value for this case with the corresponding characteristic function $\phi_0(r) = r^\nu$.

[§] When $v = 0$, $\lambda_0 = 0$ is also a characteristic value for this case with the corresponding characteristic function $\phi_0(r) = 1$.

Separation of Variables: Rectangular Coordinate System

Sign of Separation Constant

$$\frac{\partial^2\Theta}{\partial x^2} + \frac{\partial^2\Theta}{\partial y^2} = 0 \quad \text{in} \quad 0 < x < L, \quad 0 < y < W$$

2D Steady-State Heat Conduction

BC1: $\Theta(x = 0, y) = 0$ BC2: $\Theta(x = L, y) = 0$

BC3: $\Theta(x, y = 0) = 0$ BC4: $\Theta(x, y = W) = T_2 - T_1 = T_3$

Try: $\Theta(x, y) = X(x)Y(y)$

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = 0$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -\frac{1}{X}\frac{d^2X}{dx^2} = \pm\lambda^2 \quad ??$$

How to choose sign of λ?

Always select the sign of the separation constant to produce a boundary value problem (i.e., Sturm–Liouville problem) in the dimension that corresponds to all homogenous boundary conditions.

Sign of Separation Constant

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0$$

BC1: $X(x=0)=0$ BC2: $X(x=L)=0$

$$X(x) = C_1 \cos\lambda x + C_2 \sin\lambda x$$

$$\frac{d^2Y}{dy^2} - \lambda^2 Y = 0$$

BC3: $Y(y=0)=0$

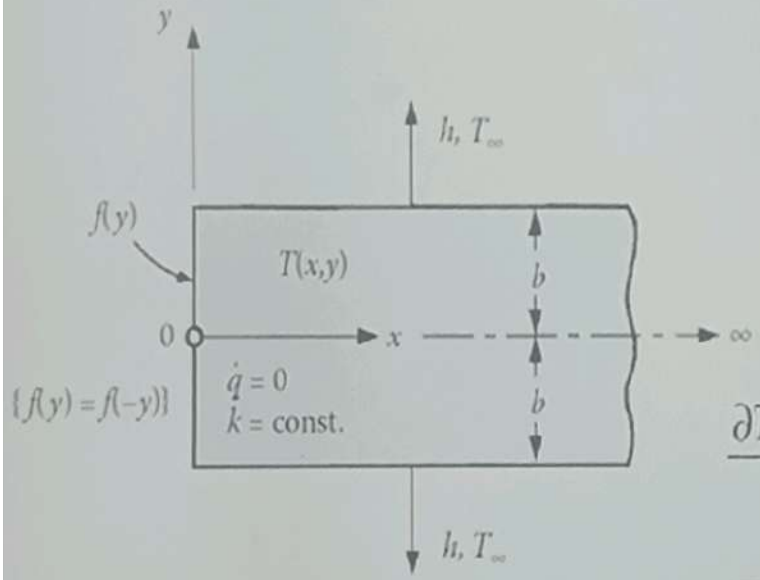
$$Y(y) = C_3 e^{\lambda y} + C_4 e^{-\lambda y}$$

$$Y(y) = C_3 \cosh\lambda y + C_4 \sinh\lambda y$$

For Algebraic Simplicity:
For the case of a *finite domain*, use the hyperbolic solutions.

For the case of a *semi-infinite domain*, we will generally use the exponential form.

Separation of Variables: Example



$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$T(0,y) = f(y), \quad T(x,y)\big|_{x\to\infty} = T_\infty$$

$$\frac{\partial T(x,0)}{\partial y} = 0, \quad -k\frac{\partial T(x,b)}{\partial y} = h[T(x,b) - T_\infty]$$

The method of separation of variables would not be directly applicable.

Introduce: $\theta(x,y) = T(x,y) - T_\infty$

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0 \qquad \theta(0,y) = f(y) - T_\infty = F(y), \quad \theta(x,y)\big|_{x\to\infty} = 0$$

SOV applicable

$$\frac{\partial \theta(x,0)}{\partial y} = 0, \quad -k\frac{\partial \theta(x,b)}{\partial y} = h\theta(x,b)$$

Separation of Variables: Example

$\theta(x,y) = T(x,y) - T_\infty$

$\theta(0,y) = f(y) - T_\infty = F(y), \quad \theta(x,y)\big|_{x\to\infty} = 0$

DE: $\dfrac{\partial^2\theta}{\partial x^2} + \dfrac{\partial^2\theta}{\partial y^2} = 0$

BC:

$$\frac{\partial\theta(x,0)}{\partial y} = 0, \quad -k\frac{\partial\theta(x,b)}{\partial y} = h\theta(x,b)$$

Try product solution:

$\theta(x,y) = X(x)Y(y)$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \pm\lambda^2$$

Homogeneous direction: y
Choose + sign

$$\frac{d^2X}{dx^2} - \lambda^2 X = 0$$

$$\frac{d^2Y}{dy^2} + \lambda^2 Y = 0$$

Always select the sign of the separation constant to produce a characteristic value problem (i.e., Sturm–Liouville problem) in the dimension that corresponds to all homogenous boundary conditions.

General Solution: $\theta(x,y) = (Ae^{-\lambda x} + Be^{\lambda x})(C\cos\lambda y + D\sin\lambda y)$

Separation of Variables: Example

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=0 \qquad \theta(0,y)=f(y)-T_\infty=F(y), \quad \theta(x,y)\big|_{x\to\infty}=0$$

$$\frac{d^2X}{dx^2}-\lambda^2X=0$$

$$\frac{\partial\theta(x,0)}{\partial y}=0, \quad -k\frac{\partial\theta(x,b)}{\partial y}=h\theta(x,b)$$

$$\frac{d^2Y}{dy^2}+\lambda^2Y=0$$

$$\theta(x,y)=(Ae^{-\lambda x}+Be^{\lambda x})(C\cos\lambda y+D\sin\lambda y)$$

$$\frac{dT(0)}{dy}=0$$

$$k\frac{dY(b)}{dy}+hY(b)=0$$

Sturm-Liouville problem: y-direction. Look at the Table.

Eigen functions are: $\phi_n(y)=\cos\lambda_n y$

Eigenvalues are +ve root of: $\lambda_n\tan\lambda_n b=\frac{h}{k}, \quad n=1,2,3,\ldots$
(apply BC)

Transcendental equation. Solve graphically or numerically.

Separation of Variables: Example

General Solution: $\theta(x,y) = (Ae^{-\lambda x} + Be^{\lambda x})(C\cos\lambda y + D\sin\lambda y)$

Eigen functions: $\phi_n(y) = \cos\lambda_n y$ **Eigenvalues are +ve root of:** $\lambda_n \tan\lambda_n b = \frac{h}{k}, \quad n = 1,2,3,\ldots$

Apply the homogeneous boundary conditions: $\theta(x,y) = \sum_{n=1}^{\infty} \alpha_n e^{-\lambda_n x} \cos\lambda_n y$

Apply nonhomogeneous BC: $F(y) = f(y) - T_\infty = \sum_{n=1}^{\infty} a_n \cos\lambda_n y$

where $a_n = \frac{2\lambda_n}{\lambda_n b + \sin\lambda_n b \cos\lambda_n b} \int_0^b [f(y) - T_\infty] \cos\lambda_n y \, dy$

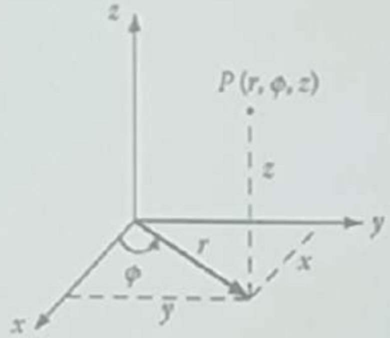
See Table

Hence, the temperature distribution is:

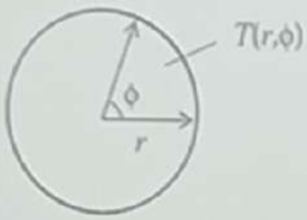
$$\theta(x,y) = T(x,y) - T_\infty$$

$$= 2\sum_{n=1}^{\infty} \frac{\lambda_n e^{-\lambda_n x} \cos\lambda_n y}{\lambda_n b + \sin\lambda_n b \cos\lambda_n b} \int_0^b \left[f(y') - T_\infty\right] \cos\lambda_n y' \, dy'$$

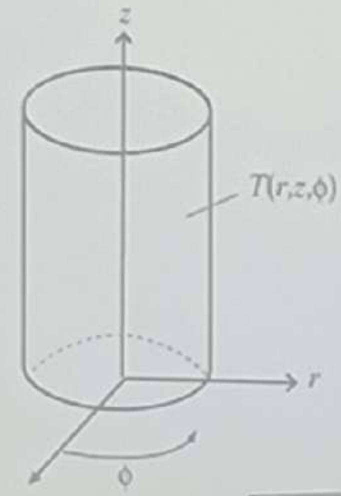
Separation of Variables: Cylindrical Coordinate System

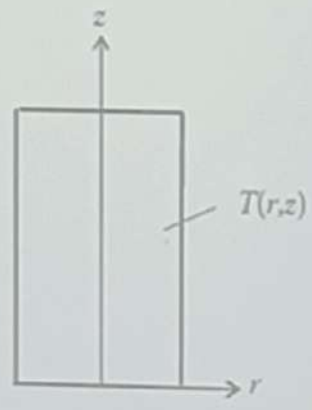


- $T(r, z)$
- $T(r, \varphi)$
- $T(\varphi, z)$: **Not Important**



$$\nabla^2 T + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

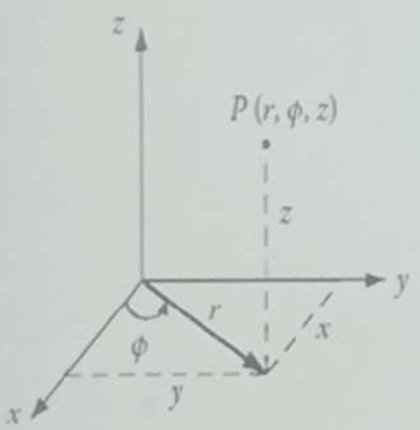


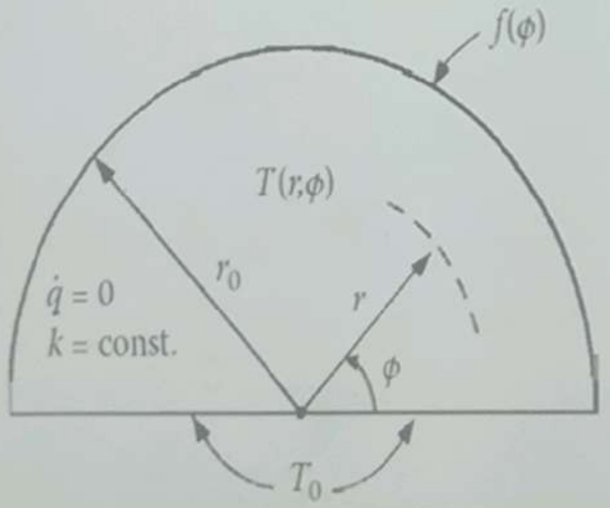


$$\nabla^2 T = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}$$

Example: Separation of Variables: $T(r, \phi)$

Steady State 2D Problem.





Consider a long solid cylinder of semicircular cross section.

The cylindrical surface at $r = r_0$ is held at an arbitrary temperature $f(\phi)$.

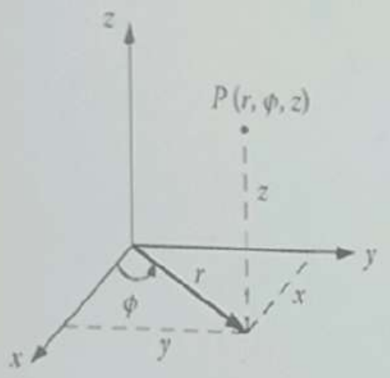
The planar surfaces at $\phi = 0$ and $\phi = \pi$ are both maintained at the same constant temperature T_0.

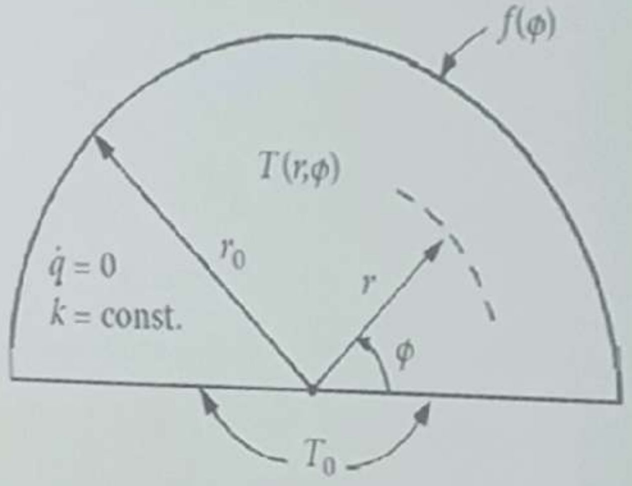
There is no internal energy sources or sinks.

Find steady-state $T(r, \phi)$.

Example: Separation of Variables: $T(r, \phi)$

Steady State 2D Problem.





Governing differential equation:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} = 0$$

Boundary conditions:

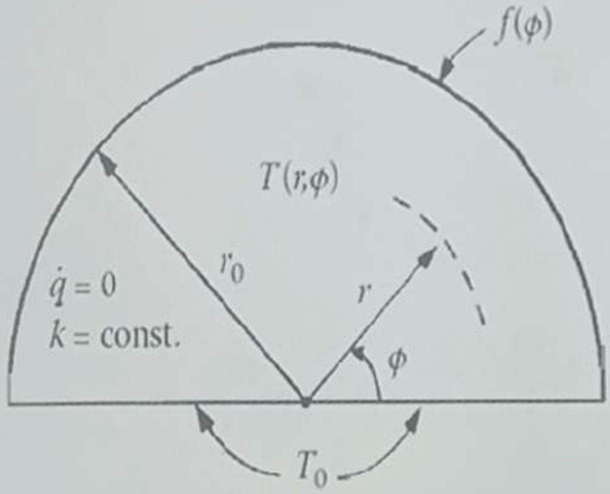
$$T(0,\phi) = T_0, \quad T(r_0,\phi) = f(\phi)$$

$$T(r,0) = T_0 \quad T(r,\pi) = T_0$$

Define: $\theta(r, \phi) = T(r, \phi) - T_0$

Try Product Solution: $\theta(r,\phi) = R(r)\psi(\phi)$

Example: Separation of Variables: $T(r, \phi)$



The ϕ direction is the homogeneous direction for $\theta(r, \phi)$.

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} - \lambda^2 R = 0$$

Cauchy–Euler equation

$$\frac{d^2\psi}{d\phi^2} + \lambda^2\psi = 0$$

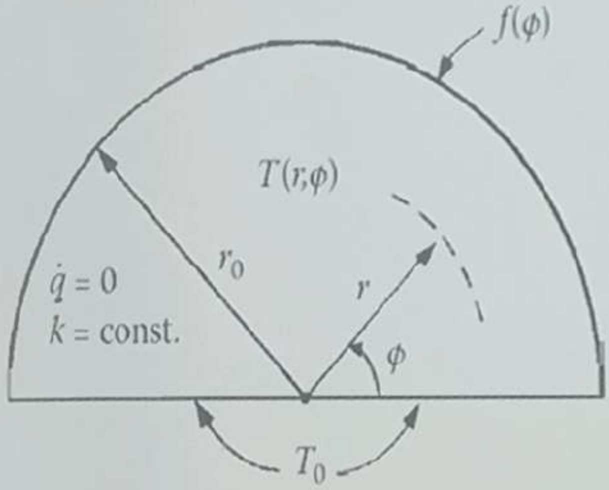
General solution: $\theta(r,\phi) = (A_1 r^{\lambda} + A_2 r^{-\lambda})(B_1 \sin\lambda\phi + B_2 \cos\lambda\phi)$

Since $T(0, \phi) = T_0$ and, therefore, $\theta(0, \phi) = 0$, then $r^{-\lambda}$ cannot exist in the solution as it is not bounded at $r = 0$. Set $A_2 = 0$ so that $\theta(r, \phi)$ is bounded as $r \to 0$.

Employing the other boundary conditions, we get

$$\theta(r,\phi) = \sum_{n=1}^{\infty} a_n r^{\lambda_n} \sin\lambda_n\phi, \quad 0 < \phi < \pi \quad \text{with} \quad \lambda_n = \frac{n\pi}{\pi} = n, \quad n = 1,2,3,\ldots$$

Example: Separation of Variables: T(r, ϕ)



$$\theta(r,\phi) = \sum_{n=1}^{\infty} a_n r^{\lambda_n} \sin \lambda_n \phi, \quad 0 < \phi < \pi$$

with $\lambda_n = \dfrac{n\pi}{\pi} = n, \quad n = 1,2,3,\ldots$

Impose the nonhomogeneous boundary condition:

$$F(\phi) = \sum_{n=1}^{\infty} a_n r_0^n \sin n\phi, \quad 0 < \phi < \pi$$

where $F(\phi) = f(\phi) - T_0$

This is the Fourier sine expansion of $F(\phi)$ on the interval $(0, \pi)$, with the expansion coefficients given by

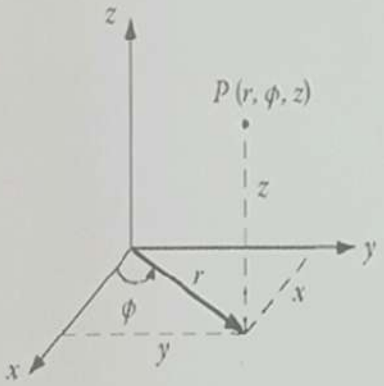
$$a_n = \frac{2}{\pi r_0^n} \int_0^{\pi} F(\phi')\sin n\phi' \, d\phi'$$

Thus, the solution for $T(r, \phi)$:

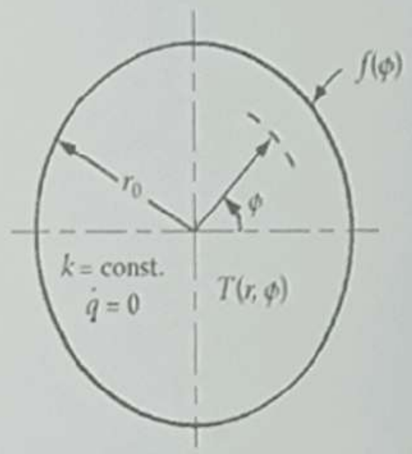
$$T(r,\phi) = T_0 + \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^n \sin n\phi \int_0^{\pi} [f(\phi') - T_0] \sin n\phi' \, d\phi'$$

Example: Separation of Variables: T(r, ϕ)

Steady State 2D Problem.



Consider a long solid cylinder of circular cross section. Assume that the surface of the cylinder is held at an arbitrary temperature $f(\phi)$. Find steady state temperature distribution $T(r, \phi)$. No heat generation. Constant k.



$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2}=0$$

$$T(0,\phi)= finite$$

$$T(r_0,\phi)= f(\phi)$$

$$T(r,\phi)=T(r,\phi+2\pi)$$

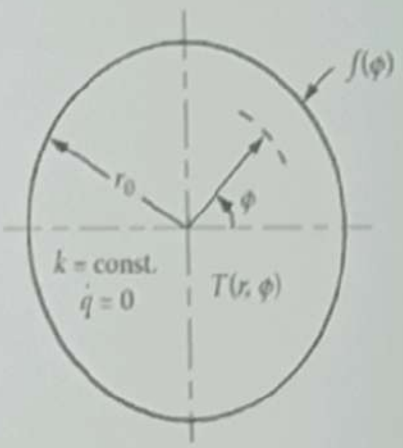
$$\frac{\partial T(r,\phi)}{\partial \phi}=\frac{T(r,\phi+2\pi)}{\partial \phi}$$

Try Product Solution: $T(r,\phi)=R(r)\psi(\phi)$

Example: Separation of Variables: T(r, ϕ)

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2}=0$$

$$\begin{cases} T(0,\phi)=\mathit{finite} \\ T(r_0,\phi)=f(\phi) \\ T(r,\phi)=T(r,\phi+2\pi) \\ \dfrac{\partial T(r,\phi)}{\partial \phi}=\dfrac{T(r,\phi+2\pi)}{\partial \phi} \end{cases}$$



Try: $T(r,\phi)=R(r)\psi(\phi)$

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-\lambda^2 R=0$$

$$\begin{cases} \dfrac{d^2\psi}{d\phi^2}+\lambda^2\psi=0 \\ \psi(0)=\psi(\phi+2\pi) \qquad \dfrac{d\psi(0)}{d\phi}=\dfrac{d\psi(\phi+2\pi)}{d\phi} \end{cases}$$

Choose the sign of the separation constant such that the homogeneous ϕ direction results in a Sturm–Liouville type characteristic value problem.

General Solution: $T(r,\phi)=(A_1 r^{\lambda}+A_2 r^{-\lambda})(B_1\sin\lambda\phi+B_2\cos\lambda\phi)$

Example: Separation of Variables: T(r, ϕ)

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-\lambda^2R=0$$

$$\frac{d^2\psi}{d\phi^2}+\lambda^2\psi=0$$

$$\psi(0)=\psi(\phi+2\pi)$$

$$\frac{d\psi(0)}{d\phi}=\frac{d\psi(\phi+2\pi)}{d\phi}$$

General Solution: $T(r,\phi)=(A_1r^{\lambda}+A_2r^{-\lambda})(B_1\sin\lambda\phi+B_2\cos\lambda\phi)$

Determination of eigenvalues:

$$\left.\begin{aligned}&[\sin\lambda\phi-\sin\lambda(\phi+2\pi)]B_1+[\cos\lambda\phi-\cos\lambda(\phi+2\pi)]B_2=0\\&[\cos\lambda\phi-\cos\lambda(\phi+2\pi)]B_1-[\sin\lambda\phi-\sin\lambda(\phi+2\pi)]B_2=0\end{aligned}\right\}$$

In order to have nontrivial solutions for B_1 and B_2, the determinant of the coefficients must vanish, which yields

$$\cos 2\lambda\pi=1$$

This is possible only if λ is equal to one of the values of

$$\lambda_n=n,\quad n=0,1,2,\ldots$$

Example: Separation of Variables: T(r, ϕ)

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-\lambda^2R=0$$

$$\frac{d^2\psi}{d\phi^2}+\lambda^2\psi=0$$

$$\psi(0)=\psi(\phi+2\pi)$$

$$\frac{d\psi(0)}{d\phi}=\frac{d\psi(\phi+2\pi)}{d\phi}$$

$$T(0,\phi)=finite$$

$$T(r_0,\phi)=f(\phi)$$

General Solution:

$$T(r,\phi)=(A_1r^{\lambda}+A_2r^{-\lambda})(B_1\sin\lambda\phi+B_2\cos\lambda\phi)$$

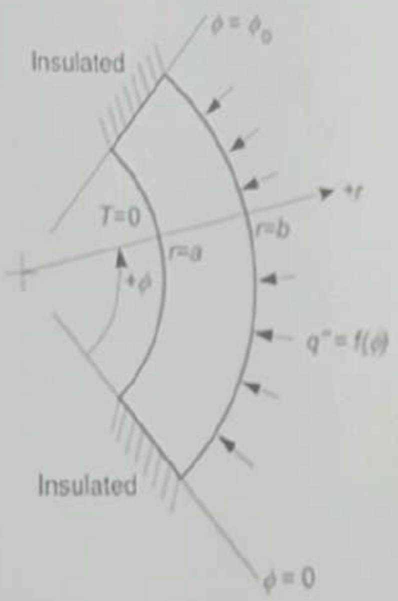
$$\lambda_n=n,\quad n=0,1,2,\ldots$$

Set $A_2 = 0$ so that the solution will satisfy the finite BC.
Employ superposition:

$$T(r,\phi)=\sum_{n=0}^{\infty}r^n(a_n\sin n\phi+b_n\cos n\phi)$$

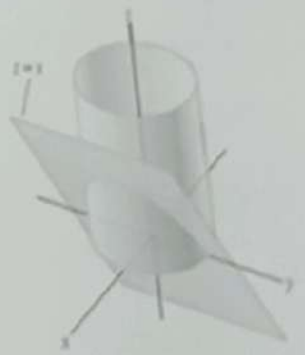
$$T(r,\phi)=b_0+\sum_{n=1}^{\infty}r^n(a_n\sin n\phi+b_n\cos n\phi)$$

Another Example: SOV: T(r, ϕ)

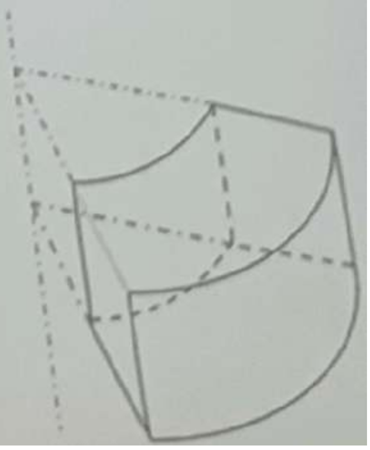


A 2-D, long cylindrical wedge defined by angle φ_0, inner radius a, and outer radius b is maintained at steady-state conditions with a angularly varying incident heat flux on the outer surface, with prescribed temperature of zero on the inner surface, and insulated on the two sides of the wedge. Find steady state temperature T(r,φ).

A cylindrical wedge can be created from a cylinder by slicing it with a plane that intersects the base of the cylinder.



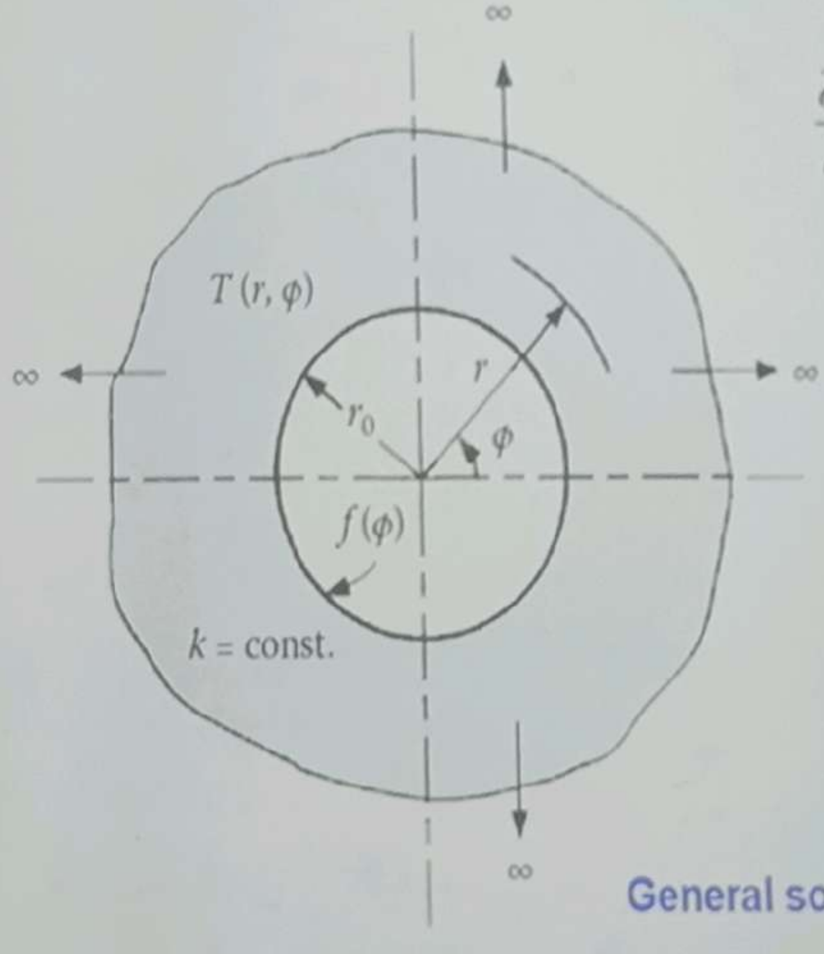
$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} = 0 \quad \text{in} \quad a < r < b, \quad 0 < \phi < \phi_0$$



BC1: $T(r = a) = 0$ BC2: $-k \left.\frac{\partial T}{\partial r}\right|_{r=b} = -f(\phi)$

BC3: $\left.\frac{\partial T}{\partial \phi}\right|_{\phi=0} = 0$ BC4: $\left.\frac{\partial T}{\partial \phi}\right|_{\phi=\phi_0} = 0$

Another Example: SOV: T(r, ϕ)



$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2}=0$$

$$T(r_0,\phi)=f(\phi)$$

$$T(r\to\infty,\phi)=\text{finite}$$

$$T(r,\phi)=T(r,\phi+2\pi)$$

$$\frac{\partial T(r,\phi)}{\partial \phi}=\frac{T(r,\phi+2\pi)}{\partial \phi}$$

General solution:

$$T(r,\phi)=(A_1 r^{\lambda}+A_2 r^{-\lambda})(B_1\sin\lambda\phi+B_2\cos\lambda\phi)$$

$$\lambda_n=n,\qquad n=0,1,2,\ldots$$

Example: Separation of Variables: T(r, θ)

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\phi}{d\theta}\right)+\lambda^2\phi=0$$

This can be transformed into Legendre's equation by setting $x = \cos\theta$

$$\frac{d}{dx}\left[(1-x^2)\frac{d\phi}{dx}\right]+\alpha(\alpha+1)\phi=0$$

where we replaced $\lambda^2 = \alpha(\alpha + 1)$. If $\alpha = n$, where n is *zero* or a *positive integer*, we know that the solutions of Legendre's equation, which are finite at $x = \pm 1$ (i.e., at $\theta = 0$ and $\theta = \pi$), are the Legendre polynomials. Thus,

$$\phi_n(x) = A_n P_n(x), \quad n = 0,1,2,\ldots \qquad \Rightarrow \qquad \phi_n(\theta) = A_n P_n(\cos\theta), \quad n = 0,1,2,\ldots$$

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}-\frac{\lambda^2}{r^2}R=0$$

Cauchy–Euler equation:

Solution:

$$R_n(r) = B_n r^n + C_n r^{-(n+1)}$$

Superposition: Splitting A Problem

Non-homogeneity in DE.

$$T(x,y) = \psi(x,y) + \phi(x)$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$\frac{d^2\phi}{dx^2} + \frac{\dot{q}}{k} = 0 \qquad \frac{d\phi(0)}{dx} = 0, \quad \phi(a) = 0$$

$$\frac{\partial \psi(0,y)}{\partial x} = 0, \quad \psi(a,y) = 0$$

$$\phi(x) = \frac{\dot{q}a^2}{2k}\left[1 - \left(\frac{x}{a}\right)^2\right]$$

$$\frac{\partial \psi(x,0)}{\partial y} = 0, \quad \psi(x,b) = -\phi(x)$$

$$\psi(x,y) = -\frac{2\dot{q}}{ak}\sum_{n=0}^{\infty} \frac{(-1)^n}{\lambda_n^3} \frac{\cos\lambda_n x \cosh\lambda_n y}{\cosh\lambda_n b} \qquad \lambda_n = \frac{(2n+1)\pi}{2a}, \quad n = 1,2,3,\ldots$$

$$\left.\begin{aligned} T(x,y) &= \psi(x,y) + \phi(x) \\ T(x,y) &= \psi(x,y) + \phi(y) \end{aligned}\right\}$$

Can we use both?
How to select?

Example: Principle of Superposition: Separation of Variables