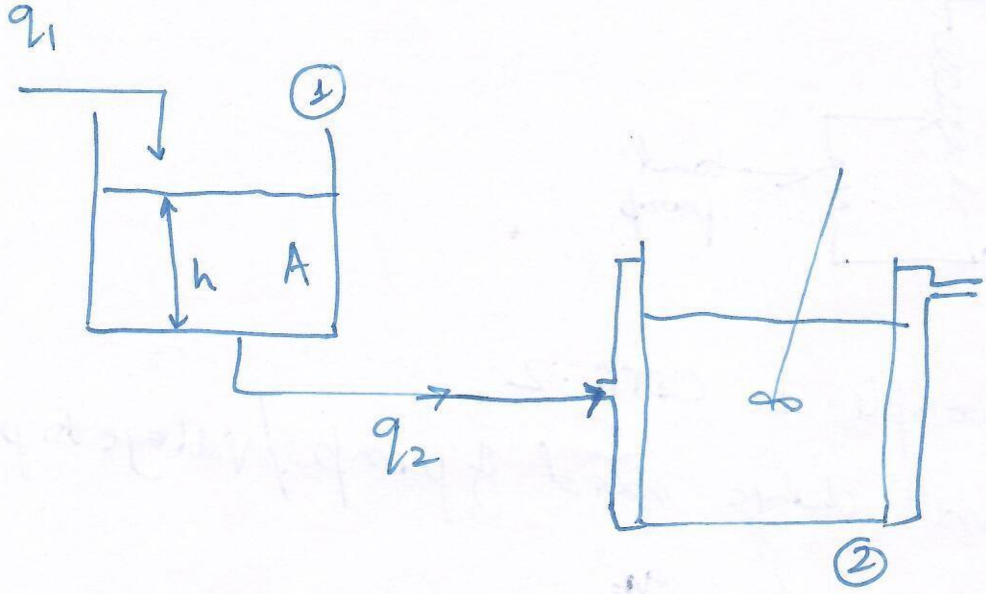
Week 4

Analysis of dynamics of discrete time systems

Continuous system — Done.

Interacting system — Done

Non-interacting system — Done.



$$\frac{dh}{dt} = \frac{1}{A}(q_1 - q_2)$$

let $q_1 = c$ [constt] & $q_2 = ah$

$$\frac{dh}{dt} + \frac{ah}{A} = \frac{c}{A}$$

at steady state

$C_{ss} = ah$; q_2 at ss $\Rightarrow q_{2ss} = ah_{ss} = C_{ss}$

let @ t ; CSTR 2 temp. Yes, we need to change

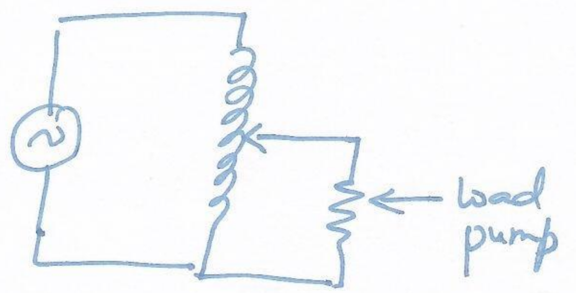
q_2 : ~~change~~ (increase q_2)

to change q_2, we need to change q_1.

How to setup a control system ??.

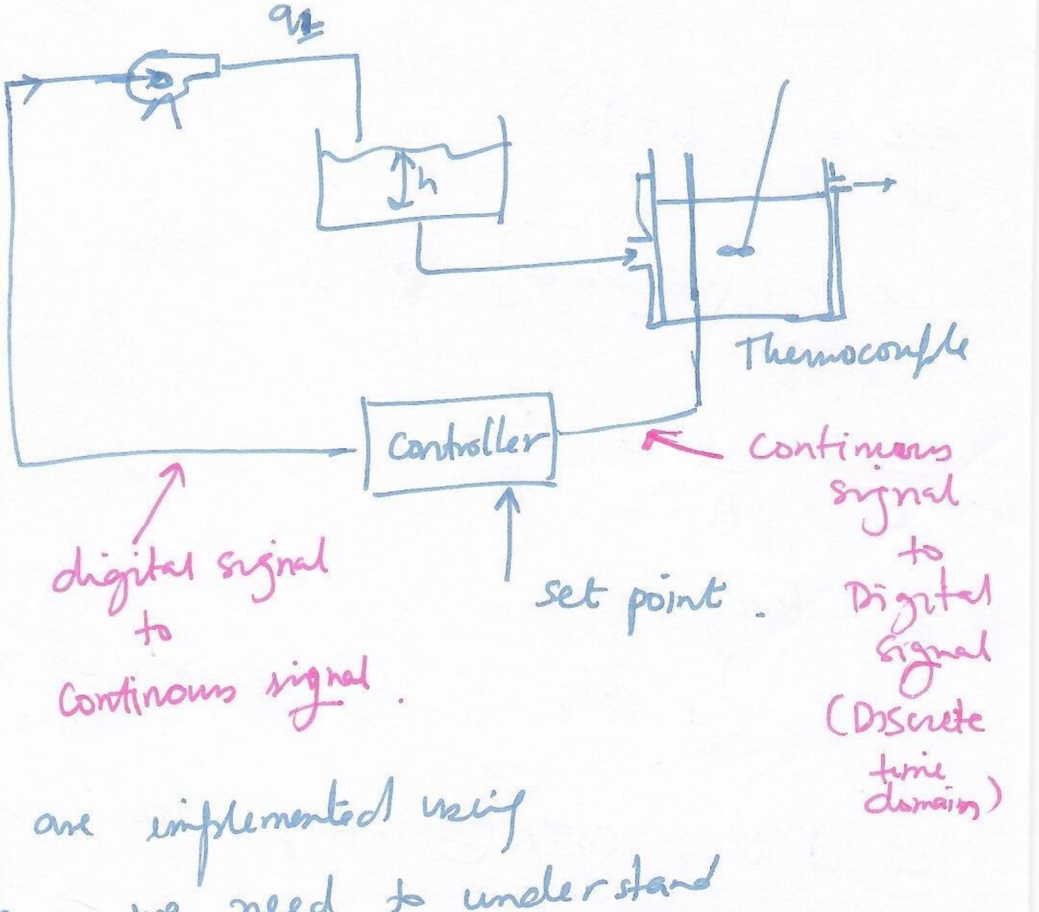
(1). Modulate C

- provide const. ~~sy~~ supply to pump

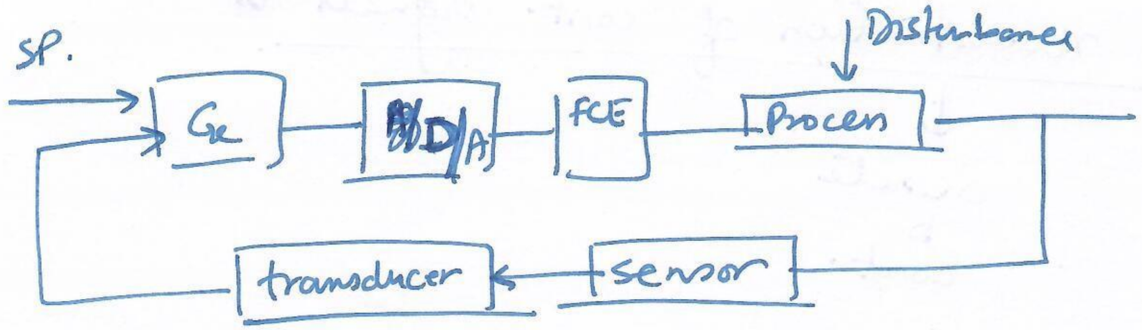


- add thermocouple to CSTR . 2

→ controller will change speed of pump / voltage to pump.



As control action are implemented using computer ~~sy~~ systems — we need to understand discrete domain analysis



Disturbance — forcing function.

I/p to process — Disturbance
— control action

O/p of process — Temperature

sensor — Thermocouple

Transducer — converts cont. signal to discrete

Gc — Controller / digital computer

~~A/D~~ D/A — converter — convert discrete to Analog.

FCE — final control element.

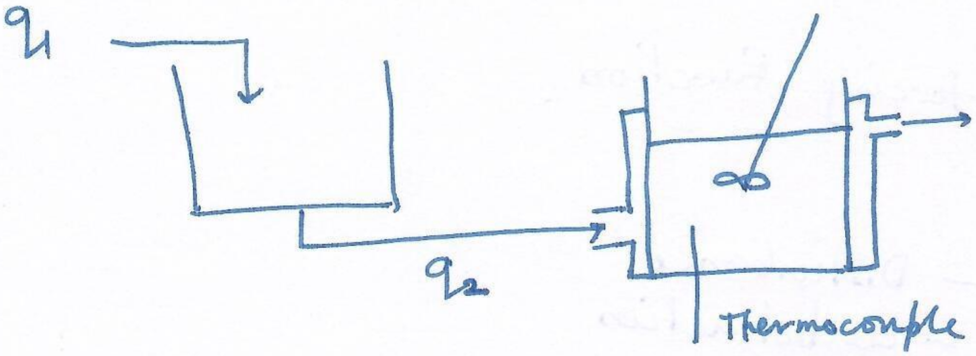
I. Analog I/p signal to Discrete signal
II. Cont. model to discrete time model
III. Discrete time signal to cont. signals

Three points to consider — conversion.

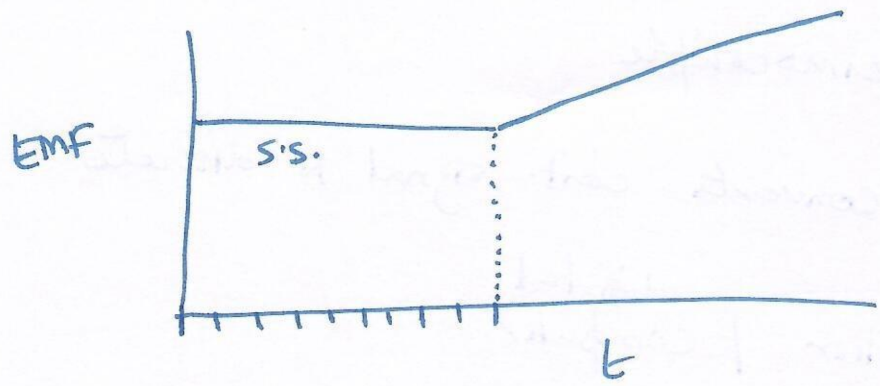
Sampling & reconstruction of cont. signals

↓ Cont. to Discrete

↓ discrete to cont.

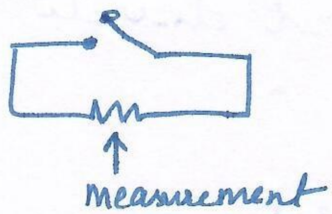


Let EMF be signal coming out of thermocouple



<u>Switch</u>

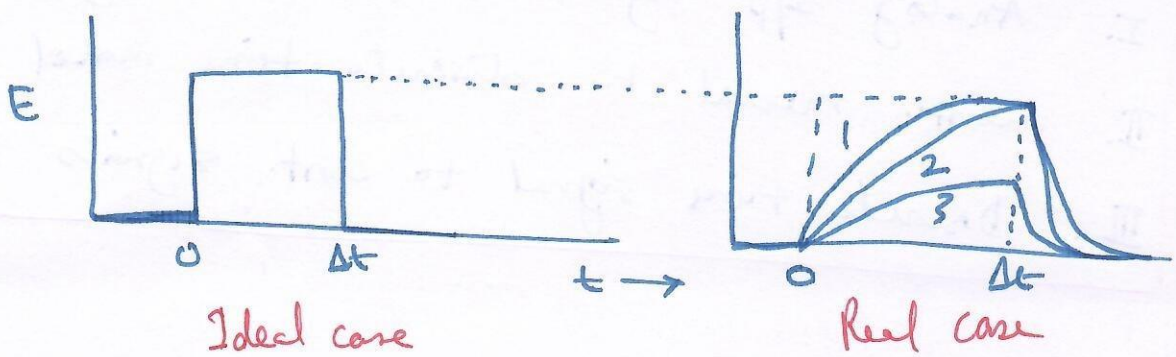
(Sampling)



close — reading

open — No reading.

BUT, switch will result in rectangular pulse.



In case 1. $\tau <<< \Delta t$

2. $\tau \sim$ 4-5 time Δt

3. $\tau > \Delta t$

So sampling is tough

Consider time constt. of system for more accurate results.

Mathematic model

$y^*(nT)$ = discrete signal

$y(nT)$ = conti. signal

$$\cancel{y^*(nT) = y(nT)\,\delta(t-nT)}$$

$$y^*(t) = y^*(0) + y^*(T) + y^*(2T) \ldots$$

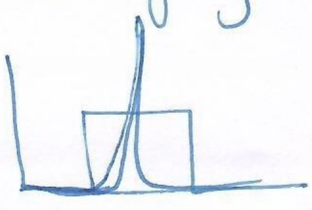
$$y^*(t) = y(0)\,\delta(t) + y(T)\,\delta(t-T) + y(2T)\,\delta(t-2T) \ldots$$

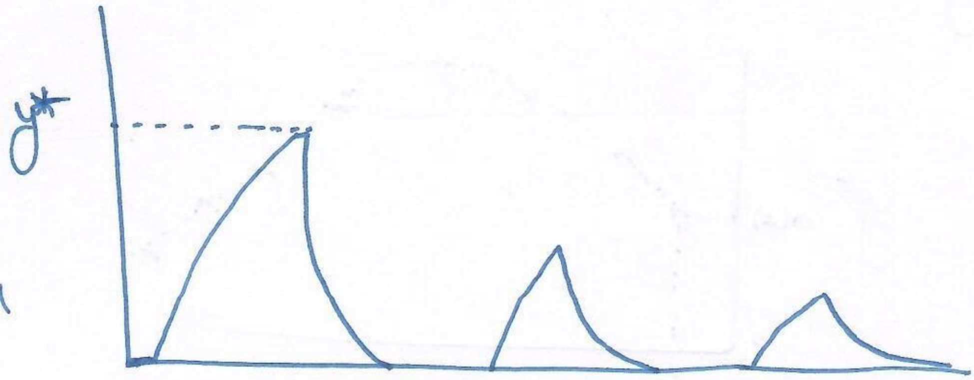
$$y^*(t) = \sum_{n=0}^{\infty} y(nT)\,\delta(t-nT)$$

$$y^*(s) = \sum_{n=0}^{\infty} y(nT)\,e^{-nTs}$$

Assumption

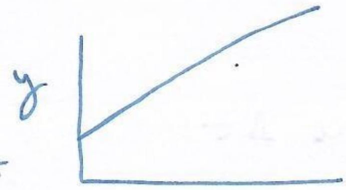
reduced pulse width to impulse function with area = area of signal

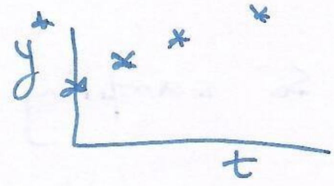




Example

$$y(t) = y_0 + mt$$





$$y^*(t) = y^*(0) + y^*(t) + y^*(2t) + \dots$$

$$y^*(nT) = y(nT)\,\delta(t-nT)$$

$$y^*(t) = \sum_{n=0}^{\infty} y(nT)\,\delta(t-nT)$$

Derived data func^n

laplace

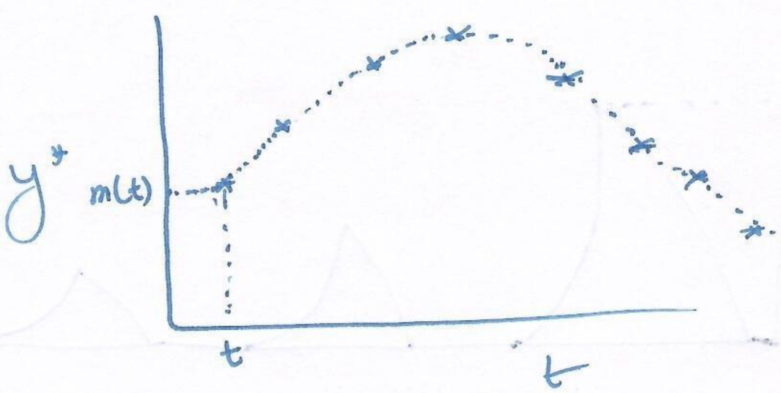
$$y^*(s) = \sum_{n=0}^{\infty} y(nT)\,e^{-sTn}$$

Convert discrete back to continuous

Hold element

$$m(t) = m(nT) + \left(\frac{dm}{dt}\right)_{nT}(t-nT) + \frac{1}{2!}\left(\frac{d^2m}{dt^2}\right)_{nT}(t-nT)^2 + \dots$$

Taylor series expansion



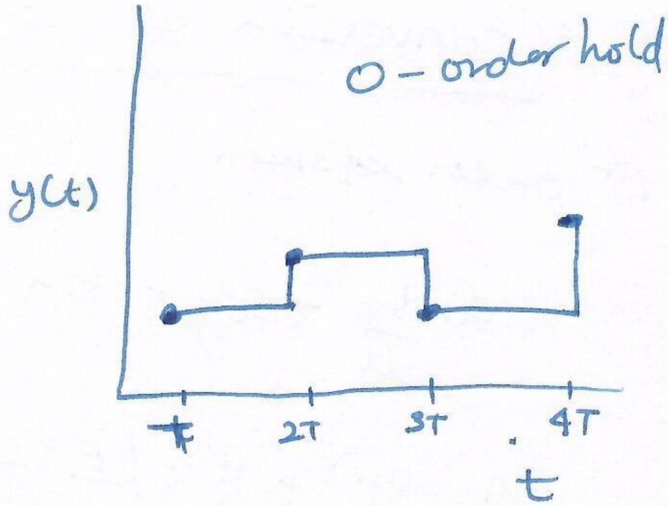
Zero order hold

$m(t) = m(nT)$

If I go from

$T \rightarrow 2T$

intensity of signal remains same from $T \rightarrow 2T$ then changes at $2T$



First order hold

$$m(t) = m(nT) + \left(\frac{dm}{dt}\right)_{nT} (t - nT)$$

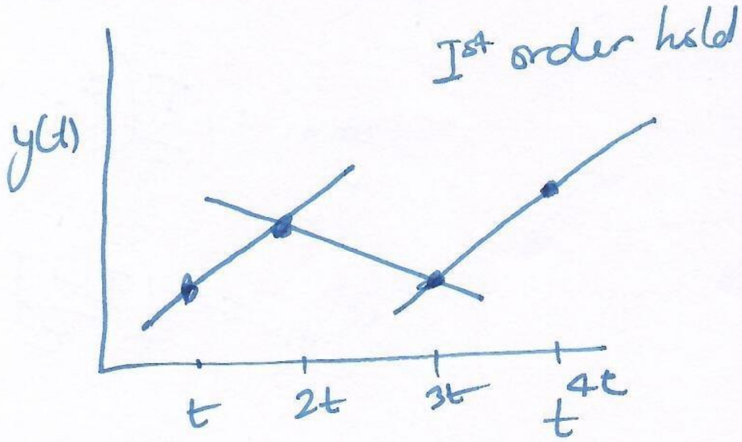
$\left(\frac{dm}{dt}\right)_{nT} \rightarrow \alpha$

$$m(t) = m(nT) + \alpha t - \alpha nT$$

$$m(t) = \alpha t + m(nT) - \alpha nT$$

$\rightarrow$ eq^n of straight line.

~~Second order hold~~



Conversion of cont. model to discrete model

1^{st} order system

$$a_1 \frac{dy}{dt} + a_0 y = bu$$

deviation variable form

$$\frac{a_1}{a_0}\left(\frac{dy}{dt}\right) + y = \left(\frac{b}{a_0}\right)u$$

← continuous model

$$\tau \frac{dy}{dt} + y = Ku$$

$$\tau\left(\frac{y_{n+1} - y_n}{\Delta t}\right) + y_n = K u_n$$

$\Delta t = T$

$$\frac{\tau}{T} y_{n+1} - \frac{\tau}{T} y_n + y_n = K u_n$$

$$\frac{\tau}{T} y_{n+1} = \left(\frac{\tau}{T} - 1\right) y_n + K u_n$$

$$y_{n+1} = \frac{T}{\tau}\left(\frac{\tau}{T} - 1\right) y_n + \frac{T}{\tau} K U_n$$

$$y_{n+1} = \left(\frac{\tau - T}{\tau}\right) y_n + \frac{TK}{\tau} U_n \quad \text{—— (1)}$$

← Discrete Model

Let $u(t) = A$

Cont. model $y(t) = AK\left(1 - e^{-t/\tau}\right)$ —— (2)

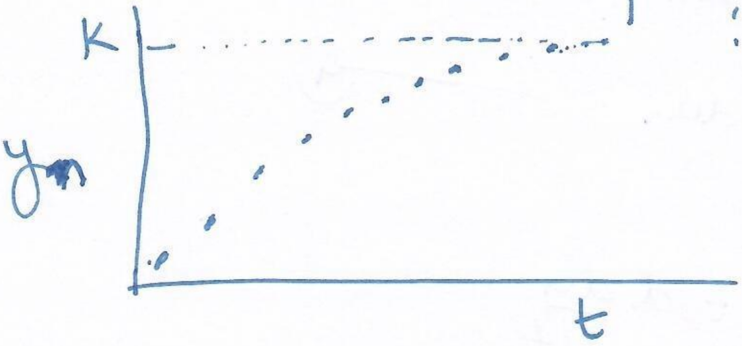
let $\tau = 1$

$T = 0.1$

$K = 1$

$u = 1$

y_n	y_{n+1} from eq^n 1
0	0.1
0.1	0.19
0.19	0.271
0.271	0.3439
⋮	⋮



Same as eq^n 2.

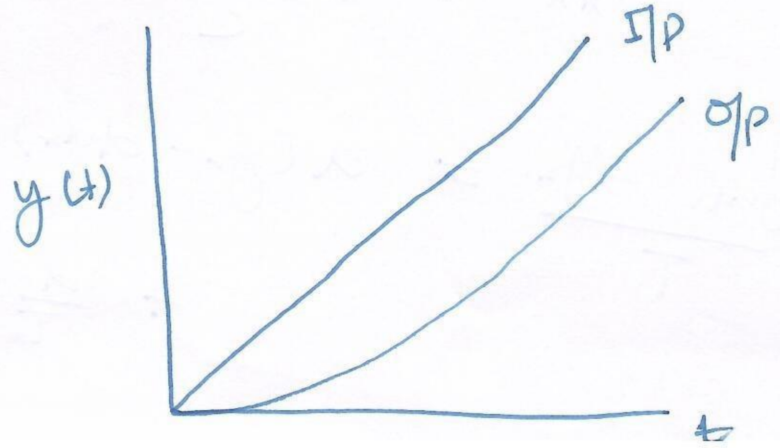
If $u(t) = At$

$$\frac{du}{dt} = A$$

$$\frac{u_{n+1} - u_n}{T} = A$$

$$u_{n+1} = u_n + AT \longleftarrow \text{ramp I/p}$$

$u_n = A, 2A, 3A \ldots$



SISO

$$\frac{dx}{dt} = ax + bu \quad \text{———} \ 1$$

$$y = cx + du \quad \text{———} \ 2$$

$$\frac{x_{n+1} - x_n}{T} = ax_n + bu_n$$

$$x_{n+1} = (1 + aT)\, x_n + bTu_n \quad \text{———} \ 3$$

$$y_n = cx_n + du_n \quad \text{———} \ 4$$

eq^n 2

$$\frac{dy}{dt} = c\frac{dx}{dt} + d\frac{du}{dt}$$

$$\frac{y_{n+1} - y_n}{T} = \frac{c}{T}(x_{n+1} - x_n) + \frac{d}{T}(u_{n+1} - u_n)$$

$$\frac{y_{n+1} - y_n}{T} = \frac{c}{\cancel{T}}(ax_n + bu_n) + \frac{d}{T}(u_{n+1} - u_n)$$

$$\frac{y_{n+1} - y_n}{T} = ac\, x_n + bcu_n - \frac{d}{T}u_n + \frac{d}{T}u_{n+1}$$

$$x_n = \frac{y_n - du_n}{c} \quad \text{from 4}$$

$$\frac{y_{n+1} - y_n}{T} = a(y_n - du_n) + \left(bc - \frac{d}{T}\right)u_n + \frac{d}{T}u_{n+1} \quad \text{———} \ 5$$

Z - transform of discrete time functions

laplace transform for dynamical response of ~~discrete~~ conti system — done.

If $y(0), y(T), y(2T) \ldots$ be the values of continuous function $y(t)$ sampled at uniform interval of period T then z-transform of sampled sequence is given by.

$$Z\{y(0), y(T), y(2T) \ldots\} = \sum_{n=0}^{\infty} y(nT) z^{-n} \qquad (1)$$

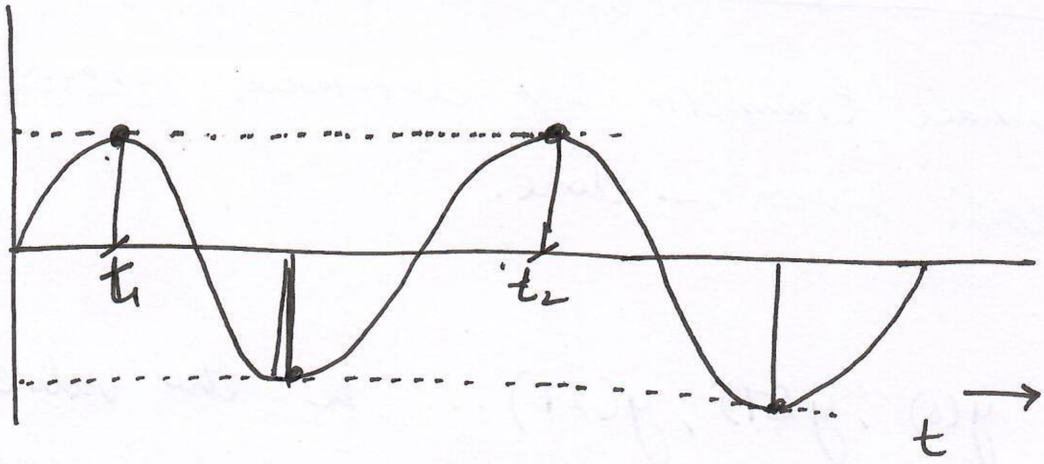
1. Z transform map discrete time signal from t-domain to z-domain

2. Depends on sampling interval.

3. Different cont. function exhibiting same sampled values at same discrete times will have the same z-transform

* z-transform is applied to sequence of some numbers

Example



If you sample at t_1, t_2 ; you will always get same value.

so sampling interval matters.

correspondance b/w laplace & Z-transform

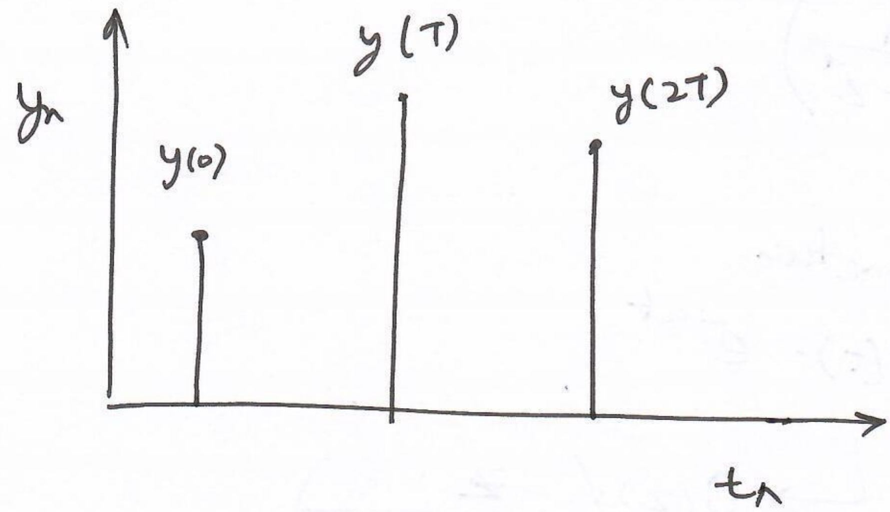
* Z-transform is special case of laplace

$$y(t) = y(0) + y(T) + y(2T) \ldots$$

$$y(t) = y(0)\delta(t) + y(T)\delta(t-T) + y(2T)\delta(t-2T) + \ldots$$

$$y(t) = \sum_{n=0}^{\infty} y(nT)\,\delta(t-nT)$$

$$\bar{y}(s) = \sum_{n=0}^{\infty} y(nT)\,e^{-nTs}$$



let $e^{-nTs} = z^{-n}$

$$\hat{y}(z) = \sum_{n=0}^{\infty} y(nT)\, z^{-n}$$

For studying the dynamical response of discrete time systems — z- transform

z-transform of const. func^n

$$y(t) = c \quad \text{—— (1)}$$

$$\hat{y}\{y(0), y(T), y(2T) \dots\} = \sum_{n=0}^{\infty} y(nT)\, z^{-n}$$

for eq^n (1)

$$y(0) = c \ ; \ y(T) = c ; \ y(2T) = c$$

$$\hat{y}(z) = \sum_{n=0}^{\infty} \{c + cz^{-1} + cz^{-2} + cz^{-3} \dots\}$$

$$\hat{y}(z) = c \sum_{n=0}^{\infty} \{1 + z^{-1} + z^{-2} + z^{-3} \dots\}$$

$$\hat{y}(z) = c\left(\frac{1}{1-z^{-1}}\right)$$

for exponential function

$$y(t) = e^{-at}$$

$$\hookrightarrow \hat{y}(z) = \left(\frac{z}{z - e^{-aT}}\right)$$

<u>Q.1</u> $$\bar{y}(s) = \frac{s^2 - s - 6}{s^3 - 2s^2 - s + 2} = \frac{s^2 - s - 6}{(s-1)(s-2)(s+1)}$$

Calculate time response in continuous domain!

invert laplace.

Partial fracⁿ

$$\bar{y}(s) = \frac{3}{s-1} + \frac{(-4/3)}{s-2} + \frac{(-2/3)}{s+1}$$

$$y(t) = 3e^{t} - \frac{4}{3}e^{2t} - \frac{2}{3}e^{-t}$$

Q.2 $\hat{y}(z) = \dfrac{z}{z^2-4z+3}$

$$\hat{y}(z) = \frac{z^{-1}}{1-4z^{-1}+3z^{-2}}$$ $\Rightarrow$ divide by highest power z

$$\hat{y}(z) = \frac{z^{-1}}{(1-z^{-1})(1-3z^{-1})}$$

$$\hat{y}(z) = \frac{-1}{2}\left(\frac{1}{1-z^{-1}}\right) + \frac{1}{2}\left(\frac{1}{1-3z^{-1}}\right)$$

$$y(nT) = Z^{-1}\left\{\frac{-1}{2}\left(\frac{1}{1-z^{-1}}\right) + \frac{1}{2}\left(\frac{1}{1-3z^{-1}}\right)\right\}$$

$$y(nT) = \frac{-1}{2} + z^{-1}\left\{\frac{1}{2}\frac{1}{(1-3z^{-1})}\right\}$$

z transform of constt. func

$$Z^{-1}\left\{\frac{1}{2}\frac{1}{(1-3z^{-1})}\right\} = \frac{1}{2}\left\{Z^{-1}\left(\frac{z}{z-3}\right)\right\}$$

$\cancel{= \frac{1}{2} e}$

Z^{-1}

$$z(e^{-at}) = \frac{z}{z-e^{-at}}$$

$$Z^{-1}\left(\frac{z}{1-e^{at}}\right) = \left(e^{-a(0)},\ e^{-aT},\ e^{-a(2T)} \ldots\right)$$

$$e^{at} = 3$$

$$at = \ln 3 \approx 1.1$$

$$\cancel{y(nT) = \frac{1}{2}\left[e^{1.1n}\right]}$$

$$Z\left[\frac{1}{2}\left(\frac{1}{1-3z^{-1}}\right)\right] = \frac{1}{2}\left(e^{1.1n}\right) \qquad n \in 0, 1, 2$$

$$y(nT) = -\frac{1}{2} + \frac{1}{2}e^{1.1n} \quad ; \quad n \in 0, 1, 2 \ldots$$

Response of discrete time systems

Pulse transfer function. $(g(z))$

$$\hat{y}(z) = g(z)\,\hat{u}(z)$$

$g(z)$ relates sampled input to discretized o/p signal $\hat{y}(z)$.

- I/p / o/p model (SISO or MIMO)

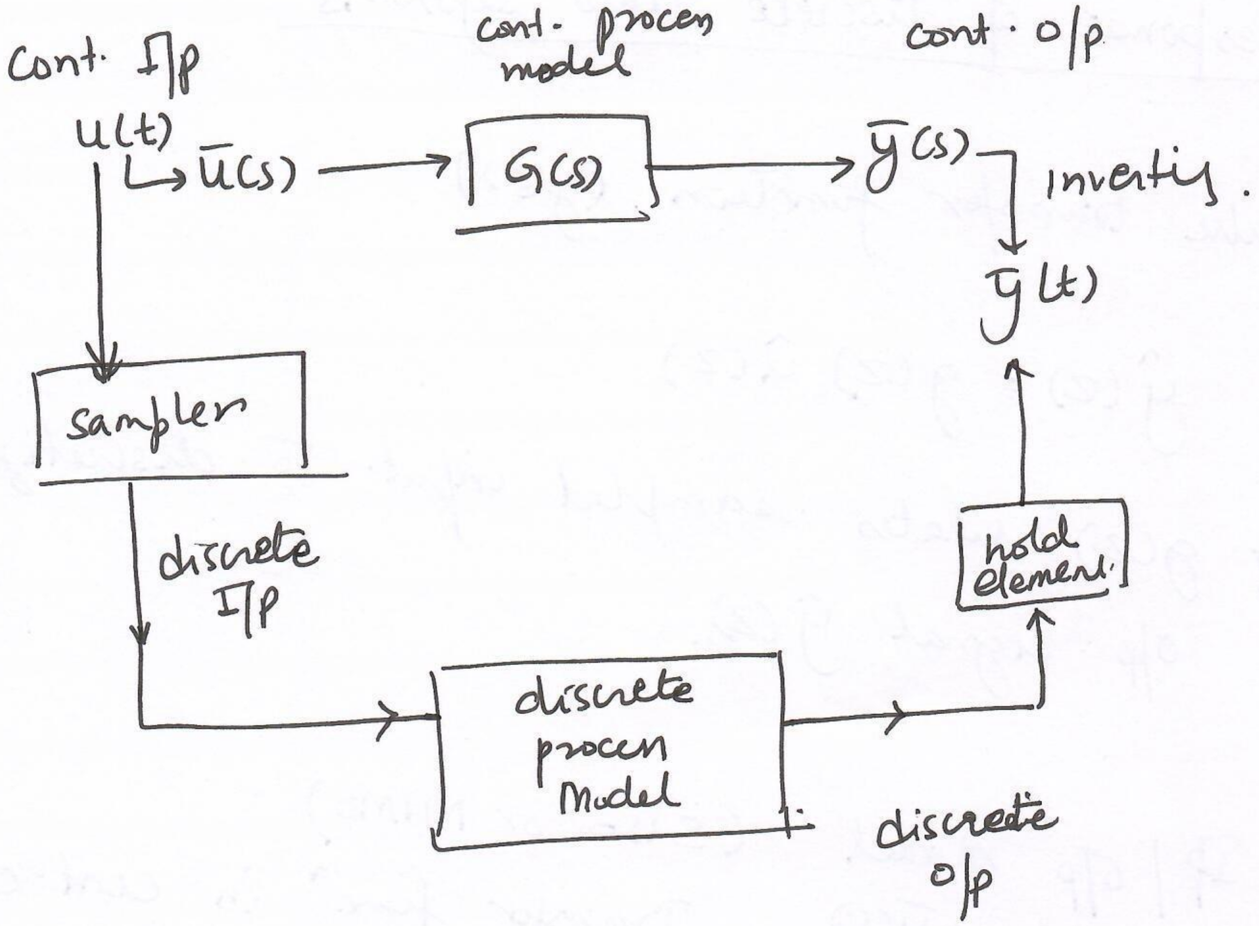
ie $G(s) = \dfrac{\bar{y}(s)}{\bar{u}(s)}$ - Transfer func^n in cont. domain

$$\hat{u}(z) \longrightarrow \boxed{g(z)} \longrightarrow \hat{y}(z)$$

** I/p & o/p signals must be sampled at same time.

No Hold pulse transfer function

In cont. domain analysis we have $u(t) \rightarrow$ we get $u(s) \longrightarrow$ we feed to $\boxed{G(s)} \longrightarrow$ to get $\bar{y}(s) \longrightarrow$ inverting this we get $y(t)$.



$$\underbrace{\text{Discrete I/p} \rightarrow \boxed{\text{Model}} \rightarrow \text{Discrete o/p}}_{\text{No hold pulse transfer func}^n}$$

Laplace domain TF $\quad g(s)$
No hold pulse TF $\quad g(z)$

$$\begin{array}{ccc} g(s) & \longrightarrow & g(t) \\ & & \big\downarrow \text{sampling} \\ g(z) & \longleftarrow & g(T),\ g(2T),\ g(3T) \ldots \end{array}$$

Ex. 1

$$g(s) = \frac{k}{\tau s + 1} = \frac{k}{\tau}\left(\frac{1}{s + 1/\tau}\right)$$

$$g(t) = \frac{k}{\tau} e^{-t/\tau}$$

No Hold $$g(z) = \frac{k}{\tau}\left(\frac{1}{1 - e^{-t/\tau} z^{-1}}\right)$$

$$g(0) = \frac{k}{\tau}$$

$$g(T) = \frac{k}{\tau}\left(e^{-T/\tau}\right)$$

$$g_{NH}(z) = \frac{k}{\tau} + \frac{k}{\tau}\left(e^{-T/\tau}\right) z^{-1} + \frac{k}{\tau} e^{-2T/\tau} z^{-2} + \ldots$$

$$= \frac{k}{\tau}\left(1 + e^{-T/\tau} z^{-1} + e^{-2T/\tau} z^{-2} + \ldots\right)$$

$$g_{NH} = \frac{k}{\tau}\left(\frac{1}{1 - e^{-T/\tau} z^{-1}}\right)$$

ZERO ORDER HOLD

discrete signal $\rightarrow$ [Hold element] $\rightarrow$ continuous signal.

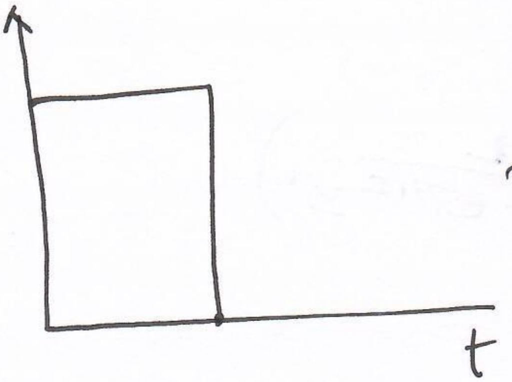
discrete I/p $\rightarrow$ [process] $\xrightarrow[\text{o/p}]{\text{Discrete}}$ [Hold] $\rightarrow$ Conti signal.

$g(s)$

$\cancel{g(s)} = g(H)\,g(P)$ H = Hold
P = process

come from discrete model.

H(t)



rectangular pulse

$$g(s) = \frac{1}{s} - \frac{1}{s}e^{-Ts}$$

$$g(H) = \frac{1}{s}\left(1 - e^{-Ts}\right)$$ Laplace of Hold element.

$$g(p) = \frac{k}{\tau s + 1}$$

$$g(s) = \frac{k}{s(\tau s + 1)}\left(1 - e^{-Ts}\right)$$

$$g(t) = \begin{cases} k\left(1 - e^{-t/\tau}\right) & t < T \\ k\left(1 - e^{-t/\tau}\right) - k\left(1 - e^{-\frac{t-T}{\tau}}\right) & t > T \end{cases}$$

$$g(z) = \frac{k}{1 - z^{-1}} - \frac{k}{1 - e^{-T/\tau} z^{-1}} - \frac{k z^{-1}}{(1 - z^{-1})} + \frac{k z^{-1}}{(1 - e^{-T/\tau} z^{-1})}$$

using shifting property.

$$g(z) = \frac{K(1-e^{-T/\tau})z^{-1}}{1-e^{-T/\tau}z^{-1}}$$ ← 1st order ~~Hold~~ pulse transfer func^n with Zero order Hold

Response of 1st order pulse TF with Zero order Hold.

$$g(p) = \frac{K}{\tau s+1} \quad ; \quad g(H) = \frac{1}{s}(1-e^{-Ts})$$

$$g(p)\,g(H) = \left(\frac{K}{\tau s+1}\right)\frac{1}{s}(1-e^{-Ts})$$

$$= \frac{K}{s(\tau s+1)} - \frac{Ke^{-Ts}}{s(\tau s+1)}$$

shift function

$$= K(1-z^{-1})\left[\frac{1}{1-z^{-1}} + \frac{1}{\tau}\left(\frac{1}{1+e^{-T/\tau}z^{-1}}\right)\right]$$

$$g(z) = \frac{K(1-e^{-T/\tau})z^{-1}}{(1-e^{-T/\tau}z^{-1})}$$

$$\hat{u}(z) = \frac{A}{1-z^{-1}} \qquad \text{step I/p}$$

$$\hat{y}(z) = g(z)\cdot\hat{u}(z)$$

$$\hat{y}(z) = AK\left\{\left(\frac{1}{1-z^{-1}}\right)\left(\frac{(1-e^{-T/\tau})z^{-1}}{1-e^{-T/\tau}z^{-1}}\right)\right.$$

take inverse z-transform

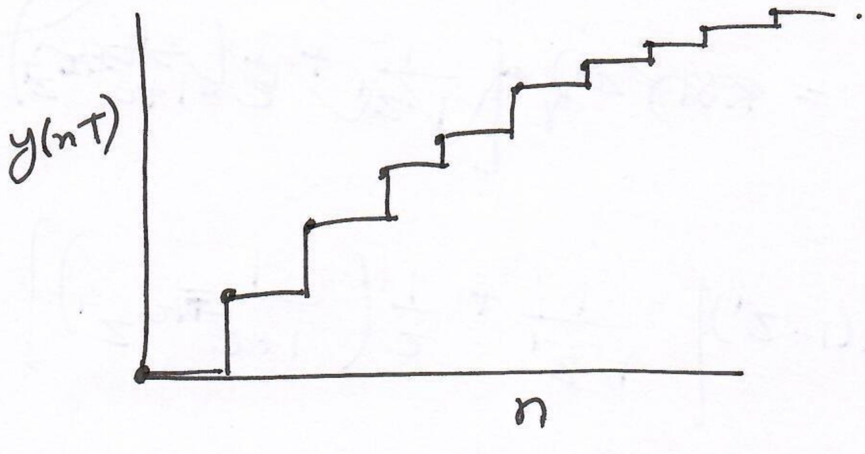
$$y(nT) \; \cancel{g(z)} = AK\left[1 - e^{-nT/\tau}\right]$$

Time evolution of discrete ~~[illegible]~~ system to step I/p.

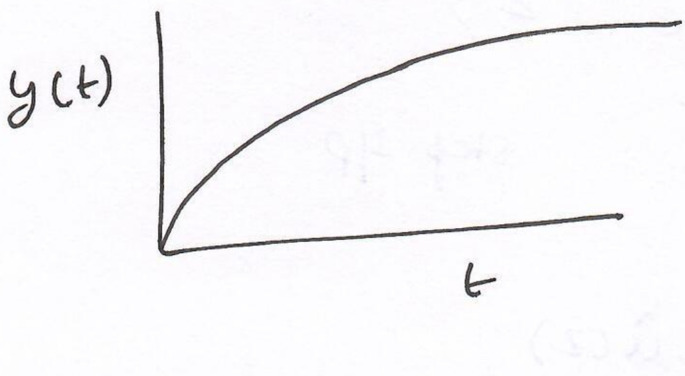
Let

$A = 1$	
$K = 1$	
$T = 0.1$	
$\tau = 1$	

n	$y(nT)$
0	0
1	0.095
2	0.181
3	0.259
4	0.329
5	0.393



DISCRETE



CONT.

$$g(p) = \frac{k_1 k_2}{(\tau_1 s+1)(\tau_2 s+1)}$$

2nd order system with zero hold

$$g(H) = \frac{1}{s}\left(1 - e^{-Ts}\right)$$

$$g(s) = g(p) \cdot g(H) = \frac{k_1 k_2}{s(\tau_1 s+1)(\tau_2 s+1)}\left(1-e^{-Ts}\right)$$

$$g(z) = K(1-z^{-1})\left[\frac{1}{1-z^{-1}} + \frac{\tau_1}{\tau_2-\tau_1}\cdot\frac{1}{1-e^{-T/\tau_1} z^{-1}} - \frac{\tau_2}{\tau_2-\tau_1}\cdot\frac{1}{1-e^{-T/\tau_2} z^{-1}}\right]$$

– ~~a~~

– <u>Invert</u>

$$\hat{u}(z) = \frac{A}{1-z^{-1}}$$

$$y(nT) = AK\left\{1 + \left(\frac{\tau_1}{\tau_2-\tau_1}\right)e^{-\frac{nT}{\tau_1}} - \left(\frac{\tau_2}{\tau_2-\tau_1}\right)e^{-\frac{nT}{\tau_2}}\right\}$$

① For N discrete system in series. with laplace domains TF

$$G(z) = Z(G_1(s), G_2(s) \cdots G_N(s))$$

$$Z(G_1(s) \times G_2(s) \times G_3(s) \cdots G_N(s)) \neq G_1(z) \cdot G_2(z) \cdots G_N(z)$$

② OVERALL G :

$$u \rightarrow \boxed{g_1(s)} \xrightarrow{y_1} \boxed{g_2(s)} \rightarrow y_2$$

$$g_1(s) = \frac{\bar{y}_1(s)}{\bar{u}(s)} \; ; \quad g_2(s) = \frac{\bar{y}_2(s)}{\bar{y}_1(s)}$$

$$G(s) = \frac{\bar{y}_2(s)}{\bar{u}(s)}$$

overall $G(s) = g_1(s)\, g_2(s)$

(product of individual g_p)

$$u \rightarrow \boxed{g_1(z)} \xrightarrow{y_1} \boxed{g_2(z)} \rightarrow y_2$$

$$\hat{y}(z) = g(z) \cdot \hat{u}(z)$$

$$g_{overall}(z) = g_1(z) \cdot g_2(z) \cdots g_N(z)$$

---

s domain $\longrightarrow$ t domain $\longrightarrow$ z domain

$$g(s) \rightarrow g(t) \rightarrow g(nT) \rightarrow g(z)$$

FOR MIMO

$$\frac{dx}{dt} = ax+bu$$
$$y = cx+du$$
} for SISO ; CONT. DOMAIN

$$\frac{x_{n+1} - x_n}{T} = ax_n + bu_n$$

$$\frac{x_{n+1}}{T} = \left(\frac{1}{T} + a\right) x_n + bu_n$$

$$x_{n+1} = \cancel{T}(1+Ta)x_n + bTu_n = a_1 x_n + b_1 u_n$$

$$y_n = cx_n + du_n$$

We know $u \rightarrow u(0) ; u(T) ; u(2T) ; \ldots$

$\qquad\qquad u_1 \quad u_2 \quad u_3 \ldots$

$x(0) \rightarrow x_0$; ~~y0~~

We can calculate

$x_{n+1} ; y_n$ using eqn

Also $y = cx+du$

$$\frac{dy}{dt} = c\frac{dx}{dt} + d\left(\frac{du}{dt}\right)$$

$$y_{n+1} - y_n = c(x_{n+1} - x_n) + d(u_{n+1} - u_n)$$

$$y_{n+1} = y_n + cx_{n+1} - cx_n + du_{n+1} - du_n$$

$$= y_n + c\left[a_1 x_n + b_1 u_n\right] - cx_n + du_{n+1} - du_n$$

$$y_{n+1} = y_n + c(a_1 - 1)x_n + (cb_1 - d)u_n + du_{n+1}$$

$$x_n = \frac{1}{c} y_n - \frac{d}{c} u_n$$

$$y_{n+1} = y_n + c(a_1 - 1)\left[\frac{1}{c} y_n - \frac{d}{c_1} u_n\right] + (cb_1 - d) u_n + d u_{n+1}$$

$$y_{n+1} = \alpha y_n + \beta u_n + \gamma u_{n+1}$$

<u>FOR MIMO</u> 2 i/p ; 2 o/p , 2nd order System

$$\frac{dx_1}{dt} = a_{11} x_1 + a_{12} x_2 + b_{11} u_1 + b_{12} u_2$$

$$\frac{dx_2}{dt} = a_{21} x_1 + a_{22} x_2 + b_{21} u_1 + b_{22} u_2$$

$$y_1 = c_{11} x_1 + c_{12} x_2 + d_{11} u_1 + d_{12} u_2$$

$$y_2 = c_{21} x_1 + c_{22} x_2 + d_{21} u_1 + d_{22} u_2$$

In discrete form

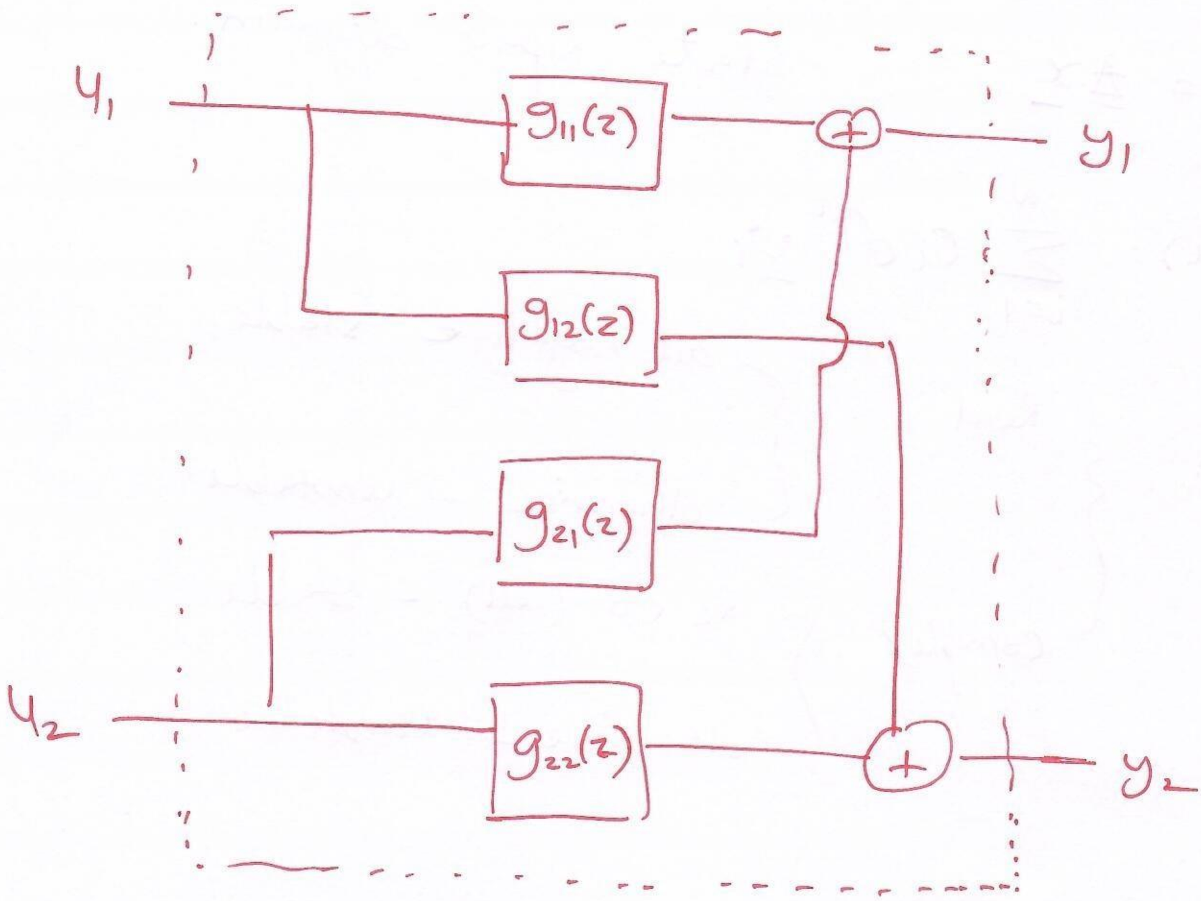
$$\frac{x^1_{n+1} - x^1_n}{T} = a_{11} x^1_n + a_{12} x^2_n + b_{11} u^1_n + b_{12} u^2_n$$

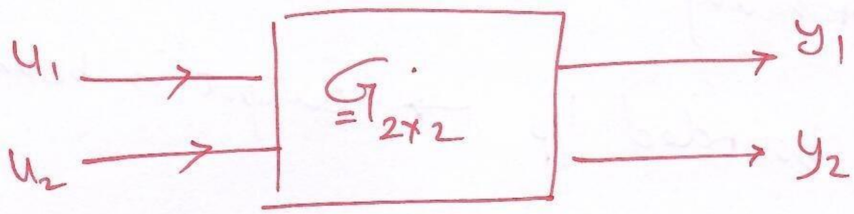
$$\frac{x^2_{n+1} - x^2_n}{T} = a_{21} x^1_n + a_{22} x^2_n + b_{21} u^1_n + b_{22} u^2_n$$

$$y^1_n = c_{11} x^1_n + c_{12} x^2_n + d_{11} u^1_n + d_{12} u^2_n$$

$$y^2_n = c_{21} x^1_n + c_{22} x^2_n + d_{21} u^1_n + d_{22} u^2_n$$

MIMO





STABILITY

$$\frac{dx}{dt} = \underline{\underline{A}}\,\underline{x}$$ State space domain.

$$x(t) = \sum_{i=1}^{N} c_i e^{\lambda_i t} \underline{v}_i$$

$$\lambda \begin{cases} \text{Real} & \begin{cases} \text{all negative stable} \\ \text{otherwise - unstable} \end{cases} \\ \text{complex} & \begin{cases} \text{Re} < 0 \text{ (all) - Stable} \\ \text{otherwise - Unstable} \end{cases} \end{cases}$$

In transfer domain

what is stability ?

for bounded I/P → output is bounded.

Then stable

If pole = negative or negative real part

Then stable

~~gp~~ In discrete time (p.q) order

pulse TF

$$g(z) = \left(\frac{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_q z^{-q}}{1 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_p z^{-p}}\right) K$$

Step I/P

$$\hat{y}(z) = A k \left(\frac{1}{1-z^{-1}}\right)\left(\frac{a_0 + a_1 z^{-1} \cdots + a_q z^{-q}}{(1-r_1 z^{-1})(1-r_2 z^{-1}) \cdots (1-r_p z^{-1})}\right)$$

factorize

$$\hat{y}(z) = Ak\left[\frac{c_0}{1-z^{-1}} + \frac{c_1}{1-r_1 z^{-1}} + \frac{c_2}{1-r_2 z^{-1}} + \cdots + \frac{c_p}{1-r_p z^{-1}}\right]$$

Inverse

$$y(nT) = Ak\left[c_0 + c_1 e^{n \ln r_1} + c_2 e^{n \ln r_2} + \cdots + c_p e^{n \ln r_p}\right]$$

r = real or complex

$$e^{n \ln(r)} = e^{n \ln[|r| e^{i\theta}]} = e^{n i\theta} \cdot e^{\ln |r|}$$

Complex-~~no~~ $|r| < 1$ stable

Real No. $r < 0$ stable