$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + f(x,t)$$
at $t=0$: $u=u_0$ & $\frac{\partial u}{\partial t}=0$
at $x=0$ & $u=u_0$

Conventional Causal Green's function and the IC solution expression

Get the solution expression of adjoint Green's function

$$g^*(x,y,t;x_1,t_1) = H(t_1-t)\,2\sum_{k=1}^{\infty}\frac{\sin(k\pi x)\sin(k\pi y)}{k\pi}\sin k\pi(t-t_1)$$

$$H(t_1-t_1) = 0 \text{ for } t_1<t$$
$$= 1 \text{ for } t_1 \geq t$$

g^* exists only for $t<t_1$

Governing equation for g^*:

① $\frac{\partial^2 g^*}{\partial t^2} = \frac{\partial^2 g^*}{\partial x^2} + \delta(x-x_1)\,\delta(t-t_1)$

$g^*=0$ & $\frac{\partial g^*}{\partial t}=0$

at $t=t_1$,

at $x=0\,\&\,1$, $g^*=0$

② $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + f(x,t)$

$\langle g^*, u\rangle$

$$\int_0^1\int_0^{t_1} u\frac{\partial^2 g^*}{\partial t^2}\,dt\,dx - \int_0^1\int_0^{t_1} u\frac{\partial^2 g^*}{\partial x^2}\,dt\,dx = u(x,t_1) - \int_0^1 g^*\frac{\partial u}{\partial t}\Big|_{t=0}dx$$

$$\int_0^1\int_0^{t_1} g^*\frac{\partial^2 u}{\partial t^2}\,dt\,dx + \int_0^1\int_0^{t_1} g^*\frac{\partial^2 u}{\partial x^2}\,dt\,dx = \int_0^1\int_0^{t_1} g^* f\,dt\,dx$$

$$\Rightarrow \int_0^1\left[u\frac{\partial g^*}{\partial t}\right]_0^{t_1} - \int_0^1\int_0^{t_1}\frac{\partial g^*}{\partial t}\frac{\partial u}{\partial t}\,dt\,dx - \int_0^{t_1}\left[u\frac{\partial g^*}{\partial x}\right]_0^1 - \int_0^{t_1}\int_0^1\frac{\partial g^*}{\partial x}\frac{\partial u}{\partial x}\,dx\,dt$$
$$- \int_0^{t_1}\left[g^*\frac{\partial u}{\partial x}\right]_0^1 - \int_0^{t_1}\int_0^1\frac{\partial g^*}{\partial x}\frac{\partial u}{\partial x}\,dx\,dt + \int_0^1\int_0^{t_1} g^* f\,dt\,dx$$
$$= u(x,t_1) - \int_0^1 g^*\frac{\partial u}{\partial t}\Big|_{t=0}dx$$

$$\Rightarrow \int_0^1\left[u(x,t)\frac{\partial g^*}{\partial t}\right]_{t=0}^{t=t_1} - u(0)\frac{\partial g^*}{\partial x}\Big|_{x=0}^{x=1} - \int_0^1\int_0^{t_1}\frac{\partial g^*}{\partial t}\frac{\partial u}{\partial t}\,dt\,dx$$
$$- \int_0^{t_1}\left[g^*(x,t)\frac{\partial u}{\partial x}\right]_{x=0}^{x=1} - \int_0^{t_1}\int_0^1\frac{\partial g^*}{\partial x}\frac{\partial u}{\partial x}\,dx\,dt + \int_0^1\int_0^{t_1} g^* f\,dt\,dx$$
$$- \int_0^1\left[g^*(x,t)\frac{\partial u}{\partial t}\right]_{t=0}^{t=t_1} - g^*(x,0)\frac{\partial u}{\partial t}\Big|_{t=0}dx + \int_0^1\int_0^{t_1}\frac{\partial g^*}{\partial t}\frac{\partial u}{\partial t}\,dt\,dx$$
$$+ \int_0^{t_1}\left[g^*(x,t)\frac{\partial u}{\partial x}\right]_{x=0}^{x=1} - \int_0^{t_1}\int_0^1\frac{\partial g^*}{\partial x}\frac{\partial u}{\partial x}\,dx\,dt = \text{RHS}$$

$$\Rightarrow -u_0\int_0^1\frac{\partial g^*}{\partial t}\Big|_{t=0}dx = u(x,t_1) - \iint_{R_1} g^* f\,dx\,dt$$

$$\boxed{u(x,t_1) = \iint_{R_1} g^* f\,dx\,dt - u_0\iint_{R_2}\frac{\partial g^*}{\partial t}\Big|_{t=0}dx}$$

R_1 R_2

$$T_2 = u_0\int_0^1\frac{\partial g^*}{\partial t}\Big|_{t=0}dx$$
$$= u_0\int_0^1 2\sum_{k=1}^{\infty}\frac{\sin(k\pi x)\sin(k\pi x)}{k\pi}(-k\pi)\cos k\pi(t-t_1)\Big|_{t=0}^{t=t_1}dx$$
$$= -2u_0\sum_{k=1}^{\infty}\frac{1-\cos k\pi}{k\pi}\cos(k\pi t_1)$$

$$T_1 = \int_0^1\int_0^{t_1} g^* f\,dx\,dt \qquad f = a \text{ dirac}$$
$$= a\int_0^1\int_0^{t_1}\sin x\pi\sum_{k=1}^{\infty}\frac{2\sin(k\pi x)\sin(k\pi x)}{k\pi}\cos k\pi(t-t_1)\,dx\,dt$$
$$= 2a\sum_{k=1}^{\infty}\frac{1-\cos k\pi}{(k\pi)^2}\sin(k\pi x)\int_0^{t_1}\int_0^1\sin k\pi(t-t_1)\sin x\pi\,dx\,dt$$
$$I = \frac{1}{2}\int_0^{t_1}\left[2\sin k\pi(t-t_1)\sin x\pi\,dx\right]$$
$$= \frac{1}{2}\int_0^{t_1}\left[\cos k\pi x\,t_1 + (\omega-\omega_0)t\right]_0^1 + \cos k\pi x\,t_1 + (\omega+\omega_0)t\Big|_0^1\,dt$$
$$= \frac{1}{2}\left[\frac{\sin k\pi x\,t_1 + (\omega-\omega_0)t}{\omega-\omega_0}\right]_0^{t_1} + \left[\frac{\sin k\pi x\,t_1 + (\omega+\omega_0)t}{\omega+\omega_0}\right]_0^{t_1}$$
$$= \frac{1}{2}\left[\frac{\sin(k\pi t_1) - \sin(k\pi t_1)}{\omega-\omega_0} + \frac{\sin(k\pi t_1) - \sin(k\pi t_1)}{\omega+\omega_0}\right]$$

$u(x_1,t_1)$ as a function of x_1, t_1

$x_1, t_1 \longrightarrow x, t$

$u(x,t) \longrightarrow$ solution expression

if $\omega \to k\pi \Rightarrow$ Amplitude $(u) \to \infty$

Resonance

Solution of the heat layer problem

Stokes Flow Problem

Newtonian Fluid $v_x = 0$

Velocity field $V_x(x,t)$ with hydrodynamic eq.

COC

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

Ass.: i) incompressible fluid (const) $\Rightarrow \rho = \text{const}$.

ii) 2-dimension is far long $\Rightarrow \frac{\partial}{\partial z}$ & $V_z = 0$

iii) Velocity is only in x-dir $\Rightarrow V_y = 0$ & $\frac{\partial}{\partial y}() = 0$

$$\boxed{\frac{\partial v_x}{\partial x} = 0}$$

Eqn: x - component

$$\rho\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z} = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \rho g_x$$

$v_z = 0$, $v_y = 0$, $g_x = 0$

$$\rho\frac{\partial v_x}{\partial t} = \mu\frac{\partial^2 v_x}{\partial x^2}$$

$$\Rightarrow \boxed{\frac{\partial V_x}{\partial t} = \nu\frac{\partial^2 V_x}{\partial x^2}}$$

at $t=0$, $V_x = 0$

at $y=0$, $V_x = u_0$

at $y=\infty$, $V_x = 0$

$\delta(t)$

No flow occurs until reaches to a boundary layer

Boundary layer problem are self-similar problem

Reducing the problem from 2 indep variables $\longrightarrow$ 1 indep variable

PDE $\longrightarrow$ ODE

Methods: Similarity Solution, Combined parameter solution

$$\frac{\partial V_x}{\partial t} = \nu\frac{\partial^2 V_x}{\partial y^2}$$

at $t=0$, $V_x = 0$

$y=0$, $V_x = u_0$

at $y=\infty$, $V_x = 0$

$y = \delta(t)$, $V_x^* = V_x/u_0$ ← V_x is that coeff.

$$\frac{\partial V_x^*}{\partial t} = \frac{1}{\delta}\frac{\partial^2 V_x^*}{\partial y^2}$$

$$\Rightarrow \boxed{\begin{aligned}&\frac{\partial V_x^*}{\partial t} = \frac{\nu}{\delta^2}\frac{\partial^2 V_x^*}{\partial \eta^2}\\ &\text{at } \tau=0,\ V_x^*=0\\ &\text{at } Y^*=0,\ V_x^*=1\\ &\quad Y^*=\infty,\ V_x^*=0\end{aligned}}$$

Estimation of similarity parameter:

At the edge of boundary layer: at $y=\delta$, $V_x = V_\infty$, $\frac{\partial V_x}{\partial t}=0$ } Boundary condition; $T = T_\infty$, $\frac{\partial T}{\partial t}=0$ } Linear condition; $C = C_\infty$, $\frac{\partial C}{\partial t}=0$

Carry out an order of magnitude analysis at the edge of boundary layer from $\tau=0$ to τ

$$\frac{\Delta V_x^*}{\tau - 0} \simeq \frac{\Delta V_x^*}{\delta^2 - 0}$$

$$\Rightarrow \delta^* \sim \sqrt{\tau}$$

$$\eta = \frac{\text{distance}}{\text{Parameter}} = \frac{y^*}{\delta^*} = \frac{y^*}{\sqrt{\tau}}$$

$$\boxed{\eta = \frac{y^*}{\sqrt{\tau}}}$$

$$\frac{\partial V_x^*}{\partial \tau} = \frac{\partial V_x^*}{\partial \eta}\frac{\partial \eta}{\partial \tau}$$
$$= \frac{\partial V_x^*}{\partial \eta}\left(-\frac{1}{2}\right)\frac{y^*}{\sqrt{\tau^3}}$$
$$= -\frac{\eta}{2\tau}\frac{dV_x^*}{d\eta}$$

$$\frac{\partial V_x^*}{\partial y^*} = \frac{dV_x^*}{d\eta}\frac{\partial \eta}{\partial y^*}$$
$$= \frac{1}{\sqrt{\tau}}\frac{dV_x^*}{d\eta}$$

$$\frac{\partial^2 V_x^*}{\partial y^{*2}} = \frac{\partial}{\partial \eta}\left(\frac{\partial V_x^*}{\partial y^*}\right)\frac{\partial \eta}{\partial y^*}$$
$$= \frac{\partial}{\partial \eta}\left(\frac{1}{\sqrt{\tau}}\frac{dV_x^*}{d\eta}\right)\frac{\partial \eta}{\partial y^*}$$
$$= \frac{1}{\sqrt{\tau}}\left(\frac{1}{\sqrt{\tau}}\frac{d^2V_x^*}{d\eta^2}\right)\frac{1}{\sqrt{\tau}}$$
$$= \frac{1}{\tau}\frac{d^2V_x^*}{d\eta^2}$$

$$-\frac{\eta}{2\tau}\frac{dV_x^*}{d\eta} = \frac{1}{\tau}\frac{d^2V_x^*}{d\eta^2}$$

$$\Rightarrow \boxed{\frac{d^2V_x^*}{d\eta^2} = -\frac{\eta}{2}\frac{dV_x^*}{d\eta}}$$

at $\eta=0$, $V_x^*=1$
$\eta=\infty$, $V_x^*=0$ }

$$\frac{dV_x^*}{d\eta} = z$$

$$\frac{dz}{d\eta} = -\frac{\eta}{2}z$$

$$\Rightarrow \frac{dz}{z} = -\frac{\eta}{2}d\eta$$

$$\Rightarrow z = K_1\exp\left(-\frac{\eta^2}{4}\right)$$

$$\Rightarrow \frac{dV_x^*}{d\eta} = K_1\exp\left(-\frac{\eta^2}{4}\right)$$

$$\Rightarrow V_x^*(\eta) = K_1\int_0^\eta\exp\left(-\frac{\eta^2}{4}\right)d\eta + K_2$$

at $\eta=0$, $V_x^*=1$

$$1 = K_1\cancel{\int_0^0(\ )} + K_2$$

$$\Rightarrow K_2 = 1$$

$$V_x^* = 1 + K_1\int_0^\eta\exp\left(-\frac{\eta^2}{4}\right)d\eta$$

$$0 = 1 + K_1\int_0^\infty\exp\left(-\frac{\eta^2}{4}\right)d\eta$$

$$\Rightarrow K_1 = -\frac{1}{\int_0^\infty\exp\left(-\frac{\eta^2}{4}\right)d\eta}$$

$$\boxed{V_x^*(\eta) = 1 - \frac{\int_0^\eta\exp\left(-\frac{\eta^2}{4}\right)d\eta}{\int_0^\infty\exp\left(-\frac{\eta^2}{4}\right)d\eta}}$$

$$\eta = \frac{y^*}{\sqrt{\tau}}$$