22/07/2024

S.DE Part

- Experimenting virtually to optimise process conditions for scaleing up processes
- Analytical mathematics
- Modelling & simulation [governing eqn & solving eqn]

Evolution: MS → 30
ES → 50
CT → 20 (announced or surprise)

- Portion:

JC
Linear algebra & stability

S.DE
Partial differential eqn.

- Textbooks:

1. S. Pushpavanam 2. A. Gupta & Mordoboldi 3. Do & Rice

start →

$$h\left(\frac{dy}{dx}, y, x\right) = 0$$ ODE [one independent variable]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + h(x, t)$$ PDE [more than 1 independent variable]

→ order of PDE
[Highest degree]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$ ...... second order

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + \frac{\partial^2 u}{\partial x^2 \partial t}$$ .... O(3)

→ Linear/non-linear PDE

1. Product of dependent variables or its derivatives gives us a non-linear PDE.

ex-

$$\frac{\partial u}{\partial t} = u\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \times \frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

2. If power of dependent variable/its derivative is $\neq 1$

ex-

$$\frac{\partial u}{\partial t} = \sqrt{u} + \frac{\partial^2 u}{\partial x^2}$$

$$\left(\frac{\partial u}{\partial t}\right)^2 = \frac{\partial^2 u}{\partial x^2}$$

→ homogeneous/non-homogeneous :

A term in governing eqn should be constant / function of independent variable only.

ex - of NH eqn

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 4$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 3x^2$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + tx$$

$$\frac{\partial u}{\partial t} = x\frac{\partial^2 u}{\partial x^2} \Big\} \text{ Linear PDE}$$

$$\frac{\partial u}{\partial t} = x\frac{\partial^2 u}{\partial x^2} + xt \Big\} \text{ Linear \& NH}$$

→ Boundary condition :

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \Big\} \begin{array}{l} \text{2 BC for } x \\ \text{1 BC for } t \text{ [Initial condn]} \end{array}$$

- Dirichlet BC:

value of dependent variable is defined @ Boundary

ex - $x = 0 \quad u = 2$

$x = 0 \quad u = 0$

- Neuman BC

Derivative of dependent variable is specified

ex

at $x = 0, \ \frac{\partial T}{\partial x} = 0$

$x = L, \ -k\frac{\partial T}{\partial x} = q_0$

- Robin/mix BC

Dependent variable & its derivative is connected through an algebraic eqn.

$$-k\frac{\partial T}{\partial x} = h(T - T_\infty) \text{ at } x = L$$

• Cauchy BC:

If 2 BCs are specified at the same boundary.

at $x=0$, $\begin{cases} u=u_{01} \\ \dfrac{\partial u}{\partial x}=u_{02} \end{cases}$

• Physical BC:

This BC comes from the physics of the problem

at $r=R$ $\quad U_z=0$

at $r=0$ $\quad U_z=$ finite.

• Stokes first problem:

$t>0$

$t=0 \quad u(y)=0$

@ $y=0$, $u=U_0$ } DBC

@ $\underbrace{y\to\infty, \quad u=0}_{\text{Physical BC}}$

→ Classification of PDEs:

Second order PDE

$$\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} = f\left(x_1, x_2, x_3, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}\right).$$

↓

$$\sum_{i=1}^{3} a_{i1}\frac{\partial^2 u}{\partial x_i \partial x_1} + a_{i2}\frac{\partial^2 u}{\partial x_i \partial x_2} + a_{i3}\frac{\partial^2 u}{\partial x_i \partial x_3}$$

↓

$$a_{11}\frac{\partial^2 u}{\partial x_1^2} + a_{12}\frac{\partial^2 u}{\partial x_2 \partial x_1} + a_{13}\frac{\partial^2 u}{\partial x_3 \partial x_1}$$

$$+$$

$$a_{22}\frac{\partial^2 u}{\partial x_2 \partial x_1} + a_{22}\frac{\partial^2 u}{\partial x_2^2} + a_{23}\frac{\partial^2 u}{\partial x_2 \partial x_3}$$

$$+$$

$$a_{31}\frac{\partial^2 u}{\partial x_3 \partial x_1} + a_{32}\frac{\partial^2 u}{\partial x_3 \partial x_2} + a_{33}\frac{\partial^2 u}{\partial x_3^3}$$

• Coefficient matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$|A - \lambda I| = 0$ } Eigen values of A

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$$

$f(\lambda) = 0$ [3rd degree polynomial]

Case 1 : If at least 1 e-value (0) → parabolic PDE

Case 2 : If all e values are ve+ or all are ve- → elliptical PDE.

Case 3 : If e values are of mixed sign → hyperbolic PDE

Examples

1. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\lambda = 1, 1, 1$ → eliptic PDE.

2. $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ → parabolic PDE

3. $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ → hyperbolic PDE

→ For 2 independent variables:

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} = f\left(x, y, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$$

Compute ⇒ $B^2 - AC$

$B^2 - AC > 0 \rightarrow$ hyperbolic

$B^2 - AC = 0 \rightarrow$ parabolic

$B^2 - AC < 0 \rightarrow$ elliptical.

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→ operator.

$$A_{(m\times n)} X_{(n\times 1)} = B_{(m\times 1)}$$

A is an operators, which operate on R^n and gives R^m

$2 + 3 = 5$ ... $+ \rightarrow$ operator

$$\frac{d(y)}{dx} = \frac{dy}{dx}$$

$\frac{d}{dx} \Rightarrow$ operator.

$\int_a^b f(x)dx = R$ ... $\int_a^b \Rightarrow$ operator.

- Linear operator-

if $L(\alpha u + \beta v)$

$= \alpha L(u) + \beta L(v)$.

$\alpha, \beta \rightarrow$ scalars

$u, v \rightarrow$ continuous functions

ex

$$\frac{d}{dx}(y+z) = \frac{dy}{dx} + \frac{dz}{dx}.$$

$\therefore$ $\frac{d}{dx}$ is a linear operator.

$\int_a^b$ is a linear operator.

• Advantages of linear operator:
if 'L' is a linear operator then the principle of superposition is obeyed.

• Principle of superposition

1. Identify no. of non-homogenities present in the system

Consider an ODE,

$$\frac{d^2y}{dx^2} = x \qquad @ \; x=0 \;\; y=1$$
$$@ \; x=1 \;\; y=2$$

3 sources of non-homogenity
⇒ 1 in gov. eqn & 2 in BC.

2. Breakdown this problem in 3 sub problems, considering 1 NH at a time.

$$y = \sum_{i=1}^{3} y_i$$
$$= y_1 + y_2 + y_3$$

y_1: $\frac{d^2y_1}{dx^2} = x \qquad @ \; x=0 \;\; y_1=0$
$@ \; x=1 \;\; y_1=0$

for y_1,
$$\frac{dy_1}{dx} = \frac{x^2}{2} + c_1$$
$$y_1 = \frac{x^3}{6} + c_1 x + c_2$$
$c_2 = 0$ $\qquad 0 = \frac{1}{6} + c_1$
$$y_1 = \frac{x^3}{6} - \frac{x}{6} + 0$$

y_2: $\frac{d^2y_2}{dx^2} = 0 \qquad @ \; x=0 \;\; y_2=1$
$@ \; x=1 \;\; y_2=0$

$$y_2 = -x+1$$

y_3: $\frac{d^2y_3}{dx^2} = 0 \qquad @ \; x=0 \;\; y_3=0$
$@ \; x=1 \;\; y_3=2$

$$y_3 = 2x$$

for y_2,
$$\frac{dy_2}{dx} = c_1$$
$$y_2 = c_1 x + c_2$$

$1 = 0 + c_2 \qquad c_2 = +1$
$0 = c_1 + 1 \qquad c_1 = -1$

$-\frac{x}{6} + \frac{6x}{6}$

for y_3,

$y_3 = c_1 x + c_2$

$c_2 = 0$

$2 = c_1 x$ @ $x = 1$

$c_1 = 2$

$\therefore\ y_3 = 2x$

$y = y_1 + y_2 + y_3$

$y = \frac{x^3}{6} - \frac{x}{6} - x + 1 + 2x$

$$\boxed{y = \frac{x^3}{6} + \frac{5x}{6} + 1}$$

$\frac{d^2y}{dx^2} = x$ @ $x = 0$ $y = 1$

@ $x = 1$ $\frac{dy}{dx} = 2$

$y = y_1 + y_2 + y_3 \ldots$ ($\because$ NH = 3)

y_1: $\frac{d^2y_1}{dx^2} = x$ @ $x = 0$ $y_1 = 0$

@ $x = 1$ $\frac{dy_1}{dx} = 0$

y_2: $\frac{d^2y_2}{dx^2} = 0$ @ $x = 0$ $y_2 = 1$

@ $x = 1$ $\frac{dy_2}{dx} = 0$

y_3: $\frac{d^2y_3}{dx^2} = 0$ @ $x = 0$ $y_3 = 0$

@ $x = 1$ $\frac{dy_3}{dx} = 2$

Eq-3

$$\frac{d^2y}{dx^2} = e^x$$

@ $x=0$ $y=1$

@ $x=1$ $\frac{dy}{dx} + 3y = 2$

$\boxed{3NH}$, $y = y_1 + y_2 + y_3$

y_1 : $\frac{d^2y_1}{dx^2} = e^x$ @ $x=0$ $y_1=0$

$x=1$ $\frac{dy_1}{dx} + 3y_1 = 0$.

$$\frac{dy_1}{dx} = e^x + c_1$$

$$y_1 = e^x + c_1x + c_2$$

y_2 : $\frac{d^2y_2}{dx^2} = 0$ @ $x=0$ $y_2 = 1$

$x=1$ $\frac{dy_2}{dx} + 3y_2 = 0$

$$y_2 = c_1x + c_2$$

y_3 : $\frac{d^2y_3}{dx^2} = 0$ @ $x=0$ $y_3=0$

$x=1$ $\frac{dy_3}{dx} + 3y_3 = 2$

$$y_3 = c_1x + c_2$$

$\Longrightarrow$ $y = y_1 + y_2 + y_3$

→ Linear superposition for PDE's :

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

@ $t=0$ $\quad u = u_{01}$

@ $x=0$ $\quad u = u_{02}$ $\quad \Bigg\}$ 3 NH

$x=1$ $\quad u = u_{03}$

$$u = \sum_{i=1}^{3} u_i$$

u_1 :

$$\frac{\partial u_1}{\partial t} = \frac{\partial^2 u_1}{\partial x^2}$$

@ $t=0$ $\quad u_1 = u_{01}$ $\quad$ NH → IC

@ $x=0$ $\quad u_1 = 0$ $\quad$ H → BC

@ $x=1$ $\quad u_1 = 0$

u_2 :

$$\frac{\partial u_2}{\partial t} = \frac{\partial^2 u_2}{\partial x^2}$$

@ $t=0$ $\quad u_2 = 0$ $\quad$ 1NH → BC

@ $x=0$ $\quad u_2 = u_{02}$ $\quad$ H → IC

@ $x=1$ $\quad u_2 = 0$

u_3 :

$$\frac{\partial u_3}{\partial t} = \frac{\partial^2 u_3}{\partial x^2}$$

@ $t=0$ $\quad u_3 = 0$ $\quad$ 1NH → BC

@ $x=0$ $\quad u_3 = 0$ $\quad$ H → IC

@ $x=1$ $\quad u_3 = u_{03}$

• $\underbrace{\text{Above problem}}_{(u_1)}$ is called as a well-posed problem, because 'variable seperation' method can be directly used.

• u_2 & u_3 are illposed problem.

• Convert ill-posed to well-posed problem:

for u_2,

at $t \to \infty$ $\quad \frac{\partial u_2}{\partial t} = 0$ $\quad$ [steady state problem)

$$u_2(x,t) = u_{2s}(x) + u_2^t(x,t)$$

$$\therefore \frac{\partial u_2}{\partial t} = \frac{\partial^2 u_2}{\partial x^2} \longrightarrow \frac{\partial u_2^t}{\partial t} = \frac{\partial^2 u_2^s}{\partial x^2} + \frac{\partial^2 u_2^t}{\partial x^2}$$

U_2^s: $\frac{d^2 U_2^s}{dx^2} = 0$

@ $x = 0$ $U_2 = U_{02}$

$U_2^s + U_2^t = U_{02}$

$U_2^s = U_{02}$

@ $x = 1$ $U_2 = 0$

$U_2^s + U_2^t = 0$

$U_2^s = 0$.

$U_2^s = U_{02}(1-x)$.

U_2^t: $\frac{\partial^2 U_2^t}{\partial x^2} = \frac{\partial U_2^t}{\partial t}$.

at $x = 0$ $U_2^t = 0$

$x = 1$ $U_2^t = 0$

at $t = 0$ $U_2 = 0$

$U_2^s + U_2^t = 0$

$U_2^t = -U_2^s(x)$

$U_2^t = -U_{02}(1-x)$

well posed problem.

Note: Force U_2^s such that BC for U_2^t becomes homogeneous. ~~simply put $U_2^t = 0$ @ BC.~~ simply put expression of $U_2^t = 0$ @ BC.

ex

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$$

@ $t = 0$ $U = U_0$

$x = 0$, $U = U_0$,

$x = 1$, $\frac{\partial U}{\partial x} + U = 5$

well posed

U_1: $\frac{\partial U_1}{\partial t} = \frac{\partial^2 U}{\partial x^2}$

@ $t = 0$ $U = U_0$

$x = 0$ $U = 0$

$x = 1$, $\frac{\partial U}{\partial x} + U = 0$

Ill posed

U_2: $\frac{\partial U_2}{\partial t} = \frac{\partial^2 U_2}{\partial x^2}$

$t = 0$ $U = 0$

$x = 0$ $U = U_0$,

$x = 1$ $\frac{\partial U}{\partial x} + U = 0$

U_3: $\frac{\partial U_3}{\partial t} = \frac{\partial^2 U_3}{\partial x^2}$

@ $t = 0$, $U = 0$ @ $x = 0$, $U = 0$

@ $x = 1$ $\partial U/\partial x + U = 5$

$U_2:$ $(x,t)\, U_2 = U_2^s(x) + U_2^t(x,t)$.

$$\frac{\partial U_2}{\partial t} = \frac{\partial U_2^t}{\partial t}$$

$$\frac{\partial^2 U_2}{\partial x^2} = \frac{\partial^2 U_2^s}{\partial x^2} + \frac{\partial^2 U_2^t}{\partial x^2}.$$

$$\therefore \quad \frac{\partial U_2^t}{\partial t} = \frac{\partial^2 U_2^s}{\partial x^2} + \frac{\partial^2 U_2^t}{\partial x^2}.$$

$U_2^s:$ $\quad \dfrac{\partial^2 U_2^s}{\partial x^2} = 0$

@ $x=0$ $\quad U_2^s + U_2^t = \theta_{01}$

$U_2^t = 0$.

$U_2^s = \theta_{01}$

@ $x=1$ $\quad \dfrac{\partial U_2^s}{\partial x} + U_2^s = 0$

$U_2^t:$ $\quad \dfrac{\partial U_2^t}{\partial t} = \dfrac{\partial^2 U_2^t}{\partial x^2}$

@ $t=0$ $\quad U_2^t + U_2^s = 0$

$U_2^t = -U_2^s$

@ $x=0$ $\quad U_2^t = 0$

@ $x=1$ $\quad \dfrac{\partial U_2^t}{\partial x} + U_2^t = 0$

well posed

→ Special eqn & functions

· consider ODE

$$\frac{d^2y}{dx^2} = \lambda y \quad \rightarrow @ \left.\begin{matrix} x=0,1 \end{matrix}\right\} y=0$$

Case 1: $\quad \lambda = 0 \Rightarrow \dfrac{d^2y}{dx^2} = 0$

$y = c_1 x + c_2$

$\boxed{y=0}$ Trivial soln.

for non-trivial soln, $\lambda = 0$ is not possible

case 2: $\quad \lambda = +ve = \alpha^2$

$$\frac{d^2y}{dx^2} = \alpha^2 y$$

$$y = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$$

Applying BCs,

$$y = c_1(e^{\alpha x} - e^{-\alpha x})$$

$$0 = c_1(\underbrace{e^{\alpha} - e^{-\alpha}}_{ve+})$$

$$c_1 = 0.$$

$y(x) = 0$ (trivial solⁿ)

Case 3: $\lambda = ve - = -\alpha^2$

$$\frac{d^2y}{dx^2} + \alpha^2 y = 0$$

$$y(x) = c_1 \sin(\alpha x) + c_2 \sin(\beta x)$$

$$c_2 = 0$$

$$y = c_1 \sin \alpha x$$

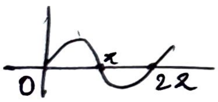
@ $x = 1$ $y = 0$

$$0 = c_1 \sin(\alpha)$$

For non trivial solⁿ

$\sin(\alpha) = 0.$

$\alpha = n\pi$ $n \in 1, 2, 3 \ldots \infty$



$y_n = c_n \sin(n\pi x)$ | $\alpha_n = n\pi$ (eigen values)
$y_n \to$ (eigen functions)

• Discrete Domain: $AX = \lambda X$ (finite eigen values) ---- eigen eqⁿ (eigen value problem).

• Continuous domain: $Ly = \lambda y$ (infinite eigen values) .... $L = \frac{d^2}{dx^2}$

~~for~~

$$y = \sum_{n=-\infty}^{\infty} c_n \sin(n\pi x).$$

$$\boxed{y = \sum_{n=1}^{\infty} A_n \sin(n\pi x)}$$

Note: for eigen value problem:

1. BC should be homogeneous
2. Governing eq^n should be homogeneous
3. Operator should be linear

→ Consider ODE:

$$\frac{d^2y}{dx^2} = \lambda y$$

@ $x = 0$ $y = 0$

$x = 1$ $\frac{dy}{dx} = 0$

Case 1: $\lambda = 0$ (trivial sol^n)

Case 2: $\lambda = ve+ = \alpha^2$ (trivial sol^n)

Case 3: $\lambda = ve- = -\alpha^2$

$\therefore$ $y = \cancel{c_1 e^{\alpha x}}$ $c_1 \sin(\alpha x) + c_2 \cos(\alpha x)$

$0 = c_2$

$y = c_1 \sin(\alpha x)$

$\frac{dy}{dx} = c_1 \alpha \cos(\alpha x)$

$0 = c_1 \alpha \cos(\alpha)$

for non-trivial sol^n

$\alpha \cos(\alpha) = 0$

$\cancel{\alpha = 0}$ $\cancel{or}$ $\alpha = (2n-1)\frac{\pi}{2}$

→ Interchanging BC,

@ $x = 0$ $\frac{dy}{dx} = 0$

$x = 1$ $y = 0$

$0 = c_1 \alpha$ ... from BC

$\therefore$ $\boxed{c_1 = 0}$

$0 = c_2 \cos(\alpha)$

$\boxed{y_n = c_{2n} \cos(\alpha_n x)}$

$\alpha_n = \frac{(2n-1)\pi}{2}$

→ Considering BC,

@ $x = 0$ $y = 0$

@ $x = 1$ $\frac{dy}{dx} + y = 0$

Case-1: $\lambda = 0$ [trivial solⁿ]

Case-2: $\lambda = ve^{+} = \alpha^2$

$$y = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$$

@ $x = 0$ $y = 0$

$$0 = c_1 + c_2 \qquad \boxed{c_2 = -c_1}$$

@ $x = 1$ $\frac{dy}{dx} + y = 0$

$$c_1 \alpha e^{\alpha x} + c_2 \times -\alpha e^{-\alpha x} + c_1 e^{\alpha x} + c_2 e^{-\alpha x} = 0$$

$$c_1 \alpha e^{\alpha x} + c_1 \alpha e^{-\alpha x} + c_1 e^{\alpha x} - c_1 e^{-\alpha x} = 0$$

$$c_1 \left(e^{\alpha x} (\alpha + 1) + e^{-\alpha x} (\alpha - 1) \right) = 0$$

$$c_1 \left(e^{\alpha} (\alpha + 1) + e^{-\alpha} (\alpha - 1) \right) = 0$$

$$\boxed{c_1 = 0}$$

$$\boxed{c_2 = 0}$$

Case-3: $\lambda = ve^{-} = -\alpha^2$

~~$x=0$ $y = c_1 \cos t$~~

$$y = c_1 \sin(\alpha x) + c_2 \cos(\alpha x)$$

@ $x = 0$ $y = 0$

$$0 = c_2$$

@ $x = 1$ $\frac{dy}{dx} + y = 0$

$$c_1 \alpha \cos(\alpha x) - c_2 \alpha \sin(\alpha x) + c_1 \sin(\alpha x) + c_2 \cos(\alpha x) = 0$$

$$c_1 \alpha \cos(\alpha) - c_2 \alpha \sin(\alpha) + c_1 \sin(\alpha) + c_2 \cos(\alpha) = 0$$

$c_1\alpha + c_1 \tan\alpha = 0$

$c_1(\alpha + \tan\alpha) = 0$.

for non-trivial solⁿ

$\boxed{\alpha + \tan\alpha = 0}$

$\therefore y_n = c_{1n} \sin(\alpha_n x)$

→ $\frac{d^2y}{dx^2} = \lambda y$

BC	Eigenvalues	Eigen fuⁿ
$\left.\begin{matrix} x=0 \\ x=1 \end{matrix}\right\} y=0$	$n\pi$	$\sin(n\pi x)$
$x=0$, $\frac{dy}{dx}=0$; $x=1$, $y=0$	$\frac{(2n-1)\pi}{2}$	$\cos\left(\frac{(2n-1)\pi}{2}x\right)$
$x=0$, $y=0$; $x=1$, $\frac{dy}{dx}+y=0$	Roots of $(\alpha_n + \tan\alpha_n = 0)$	$\sin(\alpha_n x)$

→ Eigen value problem in spherical co-ordinate system

$(1-x^2)y'' - 2xy' + n(n+1)y = 0$ .... ~~Leng~~ Legendary eqⁿ

n = non-zero integer.

BC's at $x = \pm y$ $y=0$ $-1 \le x \le 1$

solⁿ → $y(x) = c_1 P_n(x) + c_2 Q_n(x)$

↓ lengender Polynomial of order 'n'

↳ Legender fⁿ of order n

at $x = \pm 1 \Rightarrow Q_n \to \infty$

$\therefore c_2 = 0$

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = 1/2(3x^2-1)$

$P_3(x) = 1/2(5x^3-3x)$

- $P_n(1) = 1$
- $P_n(-1) = (-1)^n$
- $\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}$
- $P_n(x)$ & $P_m(x)$ are orthogonal functions

$$\int_{-1}^{1} P_n(x) P_m(x)\,dx = 0 \quad \text{for all } n \neq m$$

→ Cylindrical co-ordinate system:

- $$\frac{d^2y}{dr^2} + \frac{1}{r}\frac{dy}{dr} + \left(\lambda^2 - \frac{\alpha^2}{r^2}\right)y = 0$$

$0 \leq r \leq 1$

$x = \lambda r$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha^2)y = 0$$

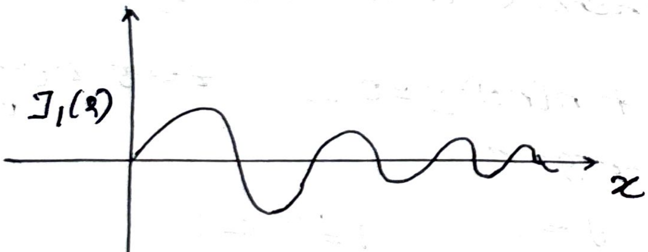
Bessel eqn of 1st kind.

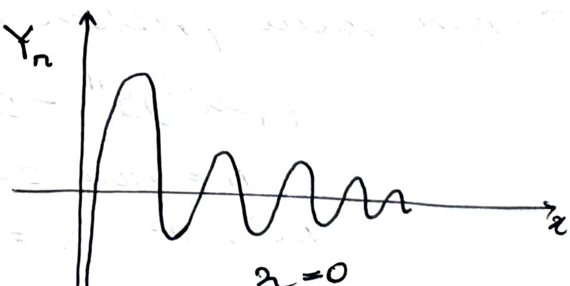
soln -

$$y(x) = c_1 J_\alpha(x) + c_2 Y_\alpha(x)$$

Bessel fn of 1st kind of order α

Bessel fn of 2nd kind of order α





$x = 0$

$Y_\alpha \to -\infty$

$\therefore c_2 = 0.$

$$\therefore y(x) = c_1 J_\alpha(x)$$

- $$\frac{d^2y}{dr^2} + \frac{1}{r}\frac{dy}{dr} - \left(\lambda^2 + \frac{\alpha^2}{r^2}\right)y = 0$$

~~x = λx~~ $\qquad x = \lambda r$

$$x^2 y'' + x y' - (x^2 + \alpha^2) y = 0$$ --- modified Bessel ~~fn~~ eqn

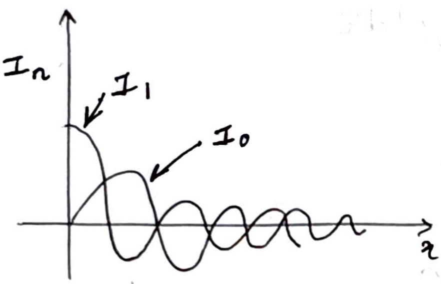
soln -

$$y(x) = C_1 I_\alpha(x) + C_2 K_\alpha(x)$$

Modified Bessel fn of 1st kind of order 'n'

Modified Bessel fn of 2nd kind of order n.

as $x \to 0 \quad K_\alpha \to -\infty$



→ Adjoint operator:

$L \to$ operator $\qquad L^* \to$ adjoint operator.

$Ax = b$ ... Discrete domain.

soln → $x = A^{-1}b$

↳ inverse operator

continuous domain,

$$LU = f$$

$$U = L^{-1} f$$

↳ adjoint operator

$$U = L^* f$$

- Evaluation of L^*, given L.

consider a generalised 2nd order ODE.

$$a_0(x) U'' + a_1(x) U' + a_2(x) U = 0$$

$$L(U) = 0.$$

$$L = a_0 \frac{d^2}{dx^2} + a_1 \frac{d}{dx} + a_2$$

BC,

@ $x = \alpha \qquad U = 0$

$x = \beta \qquad U = 0$

$$L^* = ?$$

- $\langle v, Lu \rangle = \int_\alpha^\beta v L u \, dx$ → .. v is another fn which has every characteristic of 'u'.

$$= \int_\alpha^\beta (a_0 u'' + a_1 u' + a_2 u) v \, dx$$

$$= \int_\alpha^\beta v a_0 u'' \, dx + \int_\alpha^\beta v a_1 u' \, dx + \int_\alpha^\beta v a_2 u \, dx$$

$$= v a_0 u' \Big|_\alpha^\beta - \int_\alpha^\beta (v a_0)' u' \, dx + v a_1 u \Big|_\alpha^\beta - \int_\alpha^\beta (v a_1)' u \, dx + \int_\alpha^\beta v a_2 u \, dx$$

$$= \left[v a_0 u' + v a_1 u \right]_\alpha^\beta - \int_\alpha^\beta (v' a_0 + v a_0') u' \, dx$$

$$- \int_\alpha^\beta (v' a_1 + v a_1') u \, dx + \int v a_2 u \, dx$$

$$= \left[v a_0 u' + v a_1 u \right]_\alpha^\beta - (v' a_0 + v a_0') u \Big|_\alpha^\beta + \int (v'' a_0 + v' a_0' + v' a_0' + v a_0'') u \, dx$$

$$- \int_\alpha^\beta (v' a_1 + v a_1') u \, dx + \int v a_2 u \, dx$$

$$= \left[v a_0 u' + v a_1 u_1 - v' a_0 u - v a_0' u \right]_\alpha^\beta + \int_\alpha^\beta \left[a_0 v'' + 2 a_0' v' + v a_0'' - a_1 v' - a_1' v + a_2 v \right] u \, dx$$

$$= J(u, v) + \int_\alpha^\beta \left[a_0 v'' + (2a_0' - a_1) v' + (a_0'' - a_1' + a_2) v \right] u \, dx$$

$$= J(u, v) + \int_\alpha^\beta (L^* v) u \, dx$$

$$= J(u, v) + \langle L^* v, u \rangle$$

$$L^* = a_0 \frac{d^2}{dx^2} + (2a_0' - a_1) \frac{d}{dx} + (a_0'' - a_1' + a_2)$$

$J(u, v)$ = Bi-linear concommitant.

$$= \left[v a_0 u' + v a_1 u - v' a_0 u - v a_0' u \right]_\alpha^\beta$$

$$= v(\beta) a_0(\beta) u'(\beta) - v(\alpha) a_0(\alpha) u'(\alpha) \text{ .... from BC}$$

→ forcing $J(u, v) = 0$

$$@\ x = \alpha \quad v = 0 \Big\} \quad J(u,v) \to 0 \quad , B^* \text{ (BC of adj Pb)}$$
$$\quad = \beta \quad v = 0$$

Note
B (set of BC of Original Pb)

for our example,

$L \neq L^*$ but $B = B^*$

If $L = L^*$ & $B = B^*$
then operator is self adjoint & problem is self adjoint

Example: $\quad Lu = \dfrac{d^2u}{dx^2}$; $\quad \underbrace{@\ x = 0, 1 \ \} \ u = 0}_{B}$

$L^* = ?$ $\quad B^* = ?$

$$\langle v, Lu \rangle = \int_0^1 v \frac{d^2u}{dx^2}\, dx$$

$$= vu' \Big|_0^1 - \int_0^1 v'u'\, dx$$

$$= vu' \Big|_0^1 - v'u \Big|_0^1 + \int_0^1 v'' u\, dx$$

$$= v\frac{du}{dx}\Big|_0^1 - v'u\Big|_0^1 + \int_0^1 \frac{d^2v}{dx^2} u\, dx$$

$$= [vu' - v'u]_0^1 + \int_0^1 (L^* v) u\, dx$$

$$= J(u, v) + \langle u, L^* v \rangle$$

$$\Rightarrow L^* = \frac{d^2}{dx^2} \quad , \quad L = L^*$$

$$J(u,v) = v(1)u'(1) - v'(1)\cancel{u(1)} - v(0)u'(0) + v'(0)\cancel{u(0)}$$

$$= v(1)u'(1) - v(0)u'(0).$$

$$\underbrace{@\ x = 0, 1 \ \} \ v = 0.}_{B^*}$$

Consider,

$$a_0(x)\frac{d^2 u}{dx^2} + a_1(x)\frac{du}{dx} + a_2(x)u + \lambda a_3 u = 0$$

$$L = a_0(x)\frac{d^2 u}{dx^2} + a_1(x)\frac{du}{dx} + a_2(x)$$

$$\boxed{Lu = -\lambda a_3 u} \rightarrow \text{General form} - (1)$$

Assume, Dirichlet BC / homogeneous

@ $x = 0, 1$ $u = 0$

eq (1) can be written as,

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u + \lambda r(x)u = 0 \quad \ldots (2)$$

for ~~from (1) & (2)~~,

$$p(x) = e^{\int \frac{a_1(x)}{a_0(x)} dx}$$

$$q(x) = \frac{a_2(x)}{a_0(x)} p$$

$$r(x) = \frac{a_3(x)}{a_0(x)} p$$

$$\Downarrow$$

$$\ln(p) = \int \frac{a_1(x)}{a_0(x)} dx$$

$$\frac{1}{p}\frac{dp}{dx} = \frac{a_1(x)}{a_0(x)}$$

$$\frac{dp}{dx} = \frac{a_1(x)}{a_0(x)} p.$$

putting p, q & r in (2) $\rightarrow$

$$\frac{d}{dx}\left(\cancel{\frac{a_1(x)}{a_0(x)}} p \frac{du}{dx}\right) + \frac{a_2(x)}{a_0(x)} pu + \lambda \frac{a_3(x)}{a_0(x)} pu = 0$$

$$\frac{\lambda d^2 v}{dx^2} + \frac{a_1(x)}{a_0(x)} \frac{\lambda dv}{dx} + \frac{a_2(x)}{a_0(x)} \lambda v + \lambda \frac{a_3(x)}{a_0(x)} \lambda v = 0$$

$$a_0(x) \frac{d^2 v}{dx^2} + a_1(x) \frac{dv}{dx} + a_2(x) v + \lambda a_3 v = 0 \cdots$$ same as eq 1

operator for (2),

$$L = \frac{d}{dx}\left(P \frac{d}{dx}\right) + q(x)$$

if $\quad Lv = a_0 v'' + a_1 v' + a_2 v \quad \cdots (3)$

$$L^* v = a_0 v'' + (2a_0' - a_1) v' + (a_0'' - a_1' + a_2) v$$

$$\therefore L^* = a_0 \frac{d^2}{dx^2} + (2a_0' - a_1) \frac{d}{dx} + (a_0'' - a_1' + a_2)$$

$$L = \frac{d}{dx}\left(P \frac{d}{dx}\right) + q = P\frac{d^2}{dx^2} + \frac{dP}{dx}\frac{d}{dx} + q \cdots (5)$$

Comparing (3) & (5),

$$a_0 = P \; ; \; a_1 = \frac{dP}{dx} \; ; \; q = a_2$$

$$L^* = P\frac{d^2}{dx^2} + \left(2\frac{dP}{dx} - \frac{dP}{dx}\right)\frac{d}{dx} + \left(\cancel{\frac{d^2P}{dx^2}} - \cancel{\frac{d^2P}{dx^2}} + q\right)$$

$$L^* = P\frac{d^2}{dx^2} + \frac{dP}{dx}\frac{d}{dx} + q$$

$L = L^*$ $\rightarrow$ self adjoint operator.

• Sturm-Louiville eqn [standard eigen value problem]

$$a_0 \frac{d^2u}{dx^2} + a_1 \frac{du}{dx} + a_2 u = -\lambda a_3 u$$

@ $x = a$ $\quad \alpha_1 \frac{du}{dx} + \alpha_2 u = 0$

@ $x = b$ $\quad \beta_1 \frac{du}{dx} + \beta_2 u = 0$

$$L = a_0(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_2(x)$$

for self adjoint system $L = L^*$ & $B = B^*$,
from previous page we know that $L = L^*$.
To see whether $B = B^*$ we must examine the bilinear concomitant.

$$J(u,v) = \left[v a_0 u' - (v a_0)' u + a_1 v u\right]_a^b$$

$$= \left[v a_0 u' - v' a_0 u - v a_0' u + a_1 v u\right]_a^b$$

$a_0' = \frac{dp}{dx} = p'$; $a_1 = \frac{dp}{dx} = p' = a_0'$

$a_0' = a_1$; $a_0 = p$; $a_1 = p'$

$$J(u,v) = \left[v a_0 u' - v' a_0 u\right]_a^b$$

$$J(u,v) = a_0\left[v u' - v' u\right]_a^b$$

$$= a_0\left[v(b)u'(b) - v'(b)u(b) - v(a)u'(a) + v'(a)u(a)\right]$$

@ $x = a$, $\alpha_1 u' + \alpha_2 u = 0$

@ $x = b$, $\beta_1 u' + \beta_2 u = 0$

$$J(u,v) = a_0\left[v(b)\left(-\frac{\beta_2}{\beta_1}u(b)\right) - v'(b)u(b) - v(a)\left(-\frac{\alpha_2 u(a)}{\alpha_1}\right) + v'(a)u(a)\right]$$

$$J(u,v) = a_0\left[-\frac{\beta_2}{\beta_1} u(b)v(b) - u(b)v'(b) + \frac{\alpha_2}{\alpha_1} u(a)v(a) + v'(a)u(a) \right]$$

$$J(u,v) = a_0\left[-\frac{u(b)}{\beta_1}\left(\beta_2 v(b) + \beta_1 v'(b)\right) \right] + a_0\left[\frac{u(a)}{\alpha_1}\left(\alpha_2 v(a) + \alpha_1 v'(a)\right)\right]$$

for $J(u,v) = 0$

@ $x=a$ $\quad \alpha_2 v + \alpha_1 v' = 0$

@ $x=b$ $\quad \beta_2 v + \beta_1 v' = 0$

$\therefore$ $\boxed{B = B^*}$

→ Characteristics of S-L problem:

1. $Lu = -\lambda r(x)\, u$
2. BC are homogeneous

Theorem 1: There is a countable infinity of eigenvalues λ.

$-\infty < \lambda < \infty$ such that $\lambda_n \to \infty$ if $n \to \infty$

(infinite no. of eigen values exist in n-dimensional space)

Theorem 2: if λ_m & λ_n are 2 distinct eigenvalues corresponding to eigenfunctions y_m & y_n then the eigen functions are orthogonal functions with respect to weight fn $r(x)$

S-L eqn

$$Ly = -\lambda r(x) y$$

subject to $B=0$ for $a<x<b$

Assume, λ_m & λ_n are distinct.

$$\lambda_m \downarrow y_m \qquad \lambda_n \downarrow y_n$$

$$L y_m = -\lambda_m r y_m \quad \text{--- } (1)$$

$$L y_n = -\lambda_n r y_n \quad \text{--- } (2)$$

$$\cancel{\langle y_m, y_n \rangle = \int_a^b + \lambda_m \lambda_n r^2 y_m y_n \, dx}$$

· inner products,

$$\langle y_n, L y_m \rangle = \int_a^b -\lambda_m r y_m y_n \, dx$$

$$= -\lambda_m \langle r y_m, y_n \rangle \quad \text{--- } (3)$$

we take,

$$\langle y_m, L y_n \rangle = \langle + y_m, -\lambda_n r y_n \rangle$$

$$= -\lambda_n \langle y_m, r y_n \rangle \text{ ---}$$

$$= -\lambda_n \langle r y_n, y_m \rangle \quad \text{--- } (4)$$

(3) − (4),

$$\langle y_n, L y_m \rangle - \langle y_m, L y_n \rangle$$

$$\cancel{= -\lambda_m \langle r y_m,}$$

$$= -\lambda_m \int r y_m y_n \, dx + \lambda_n \int r y_m y_n \, dx$$

$$\langle u, L v \rangle = \langle L^* u, v \rangle + J(u, v)$$

$$\langle L^* y_n, y_m \rangle + J(y_m, y_n) - \langle y_m, L y_n \rangle$$

$$= (\lambda_n - \lambda_m) \int r y_m y_n \, dx$$

for S-L,

$$L = L^* \quad \& \quad J(u, v) = 0$$

$$\therefore \quad (\lambda_n - \lambda_m) \int r y_m y_n \, dx = 0$$

$\lambda_n - \lambda_m = 0$ only when $n = m$

$\therefore \int r\, y_m y_n\, dx = 0$ when $n \neq m$

* Theor: Eigen fn are orthogonal fn if eigen value are distinct.

$Ly = -\lambda r y.$ $\qquad$ $Ly_m = -\lambda_m r y_m$

$y_m, y_n \rightarrow \lambda_m, \lambda_n.$ $\qquad$ $Ly_n = -\lambda_n r y_n$

$\langle y_n, Ly_m \rangle = \int -\lambda_m y_n \cancel{*} y_m r\, dx = -\lambda_m \int y_n r \cancel{*} y_m\, dx \quad \text{.... (1)}$

$\langle Ly_n, y_m \rangle = -\lambda_n \int y_m \times r \times \cancel{*} y_n\, dx \quad \text{..... (2)}.$

(1) - (2),

$\langle y_n, Ly_m \rangle - \langle Ly_n, y_m \rangle \cancel{*}$

$= -\lambda_m \int y_n r \cancel{*} y_m\, dx + \lambda_n \int y_m r \cancel{*} y_n\, dx$

now,

$\langle y_n, Ly_m \rangle = \underbrace{\cancel{J(y_n, y_m)}}_{0 \text{ for S-L eq}^n} + \langle L^* y_n, y_m \rangle$

&

$L = L^*$ for S-L eq^n

$\therefore \langle y_n, Ly_m \rangle = \langle Ly_n, y_m \rangle$

$\therefore -\lambda_m \int y_n r \cancel{*} y_m\, dx + \lambda_n \int y_m r \cancel{*} y_n\, dx = 0.$

$(\lambda_n - \lambda_m) \int y_m y_n r\, dx = 0.$

$\lambda_n - \lambda_m = 0$ only for $n = m$

$\therefore \boxed{\int y_m y_n r\, dx = 0 \text{ for } n \neq m}$

* Soln of PDE's

Parbolic PDE → 2D problem, 3D problem, 4D problem } well posed pb } Rectangular, cylindrical & spherical co-ord sys.

Actual problem

Similarly, elliptical & hyperbolic PDE

ex-1

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

@ $t=0$; $u = u_0$

@ $x=0$; $u=0$

@ $x=1$; $u=0$.

$u = T(t)\, X(x)$ ... variable seperatn

$$X\frac{dT}{dt} = T\frac{d^2X}{dx^2}$$

$$\underbrace{\frac{1}{T}\frac{dT}{dt}}_{\text{fn of } t \text{ only}} = \underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{fn of } x \text{ only}} = C$$

$C \to -\lambda^2$ (non-trivial soln).

$C \to \underbrace{0, \lambda^2}_{\text{trivial soln}}$

$$\frac{d^2X}{dx^2} = -\lambda^2 X \qquad @ \ x = 0, =1 \ \} \ X = 0$$

$$X_n = \underbrace{C_1}_{\text{eigen value}} \underbrace{\sin(n\pi x)}_{\text{eigen fn}}$$

At $1, 2, \ldots \infty$

$T_n = C_2 \exp(-n^2\pi^2 t)$

$$u = \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} T_n X_n.$$

$$\boxed{u = \sum_n C_n \exp(-n^2\pi^2 t)\sin(n\pi x)}$$

at $t=0$, $u = u_0$

$$u_0 = \sum_n C_n \sin(n\pi x)$$

multiply both sides by $\sin(m\pi x)\,dx$ & integrate

$$\int_0^1 U_0 \sin(m\pi x)\,dx = \int_0^1 \sum C_n \sin(m\pi x)\sin(n\pi x)\,dx$$

$$\int_0^1 U_0 \sin(m\pi x)\,dx = \int_0^1 C_n \sin^2(n\pi x)\,dx \quad \dots \text{for } n = m$$

$\downarrow$

$\cos 2\theta = 1 - 2\sin^2\theta$

$$\int_0^1 U_0 \sin(n\pi x)\,dx = \int_0^1 C_n \sin^2(n\pi x)\,dx$$

$$-\frac{U_0 \cos(n\pi x)}{n\pi}\Big|_0^1 = C_n \int_0^1 \frac{1 - \cos(2n\pi x)}{2}\,dx$$

$$-\frac{U_0 \cos(n\pi)}{n\pi} + \frac{U_0}{n\pi} = C_n \left[\frac{1}{2}x - \frac{\sin(2n\pi x)}{2n\pi}\right]_0^1$$

$$\frac{-U_0 \cos(n\pi) + U_0}{n\pi} = C_n\left[\frac{1}{2} - 0\right]$$

$$C_n = \frac{2U_0}{n\pi}(1 - \cos(n\pi)) = \frac{2U_0}{n\pi}(1 - (-1)^n)$$
$$= \frac{2U_0}{n\pi}(1 + (-1)^{n+1}).$$

$$\therefore \boxed{U = 2\sum_n U_0 \frac{(1 - \cos(n\pi))}{n\pi} \exp(-n^2\pi^2 t)\sin(n\pi x)}$$

eq-2

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}$$

@ $\quad t=0 \quad v=v_0$

$x=0. \quad \frac{\partial v}{\partial x}=0$

$x=1 \quad v=0$

$v = T(t)\,X(x)$.

$$\frac{1}{T_n}\frac{dT_n}{dt} = \frac{1}{X_n}\frac{d^2X_n}{dx^2} = \text{cont}$$
$\hookrightarrow -\alpha^2$

$$X_n = C_1 \cos\left((2n-1)\frac{\pi}{2}x\right).$$

$$T_n = C_2 \exp\left(-\frac{(2n-1)^2\pi^2}{4}t\right).$$

$$v = \sum_n C_n \exp\left(-\frac{(2n-1)^2\pi^2}{4}t\right)\cos\left((2n-1)\frac{\pi}{2}x\right)$$

@ $\quad v(t=0)=v_0$

$$v_0 = \sum_n C_n \cos\left((2n-1)\frac{\pi}{2}x\right)$$

$\rightarrow$

$$\int_0^1 v_0 \cos\left((2m-1)\frac{\pi}{2}x\right)dx = \sum_n \int_0^1 C_n\cos\left[\frac{(2n-1)\pi x}{2}\right]\cos\left(\frac{(2m-1)\pi x}{2}\right)dx$$

$n=m$

$$\left.\frac{v_0 \sin\left[\frac{(2m-1)\pi}{2}\right]x}{\frac{(2m-1)\pi}{2}}\right|_0^1 = C_n/2$$

$$\frac{C_n}{2} = \frac{v_0 \sin(\alpha_n x)}{\alpha_n}$$

$$C_n = \frac{2v_0 \sin(\alpha_n x)}{\alpha_n}$$

$$\boxed{U = \sum_n \frac{2U_0 \sin(\alpha_n x)}{\alpha_n} \exp(-\alpha_n^2 t) \cos(\alpha_n x)}$$

$$\alpha_n = \frac{(2n-1)\pi}{2}.$$

ex-8

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$$

@ $t=0$ $U=U_0$

$x=0$ $U=0$

$x=1$ $\frac{\partial U}{\partial x} + \beta U = 0.$

$$U = X(x) T(t)$$

$$X_n = C_1 \cancel{\sin(\alpha_n x)} \quad ; \quad \cancel{\alpha_n + \tan(\alpha_n)} = 0.$$

$$T_n = C_2 \cdot \exp(-\alpha_n^2 t)$$

$$U = \sum_n C_n \exp(-\alpha_n^2 t) \sin(\alpha_n x).$$

$$\cancel{@ \quad t=0 \quad U = U_0}$$

$$\cancel{U_0 = \sum_n}$$

$$\frac{1}{T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = \text{const}.$$

$$X_n = C_1 \sin(\alpha_n x) + C_2 \cos(\alpha_n x)$$

$$0 = C_2$$

$$C_1 \alpha_n \cos(\alpha_n x) + \beta C_1 \sin(\alpha_n x)\Big|_{x=1} = 0$$

for eigen values

$$\alpha_n + \beta \tan(\alpha_n x) = 0$$

$$C_1 \alpha_n + \beta C_1 \cancel{\sin} \tan(\alpha_n x) = 0$$

$$C_1 (\alpha_n + \beta \tan(\alpha_n x)) = 0$$

$$\boxed{U = \sum_n C_n \exp(-\alpha_n^2 t) \sin(\alpha_n x).}$$

$$U_0 = \sum_n C_n \sin(\alpha_n x).$$

$$\int_0^1 U_0 \sin(\alpha_m x)\, dx = \sum_n \int_0^1 C_n \sin(\alpha_n x) \sin(\alpha_m x)\, dx$$

$$\int_0^1 U_0 \sin(\alpha_n x)\, dx = \int_0^1 C_n \sin^2(\alpha_n x)\, dx.$$

$$v_0\left(\frac{1-\cos(\alpha_n)}{\alpha_n}\right) = \frac{c_n}{2}\left[x - \frac{1}{2\alpha_n}\sin(2\alpha_n x)\right]_0^1$$

$$= \frac{c_n}{2}\left[1 - \frac{\sin(2\alpha_n)}{2\alpha_n}\right].$$

$$v_0\frac{(1-\cos\alpha_n)}{\alpha_n} = \frac{c_n}{2}\left[1 - \frac{1}{\cancel{2}\alpha_n} \times \frac{\cancel{2}\tan\alpha_n}{1+\tan^2\alpha_n}\right]$$

$$\frac{v_0(1-\cos\alpha_n)}{\alpha_n} = \frac{c_n}{2}\left[1 - \frac{1}{\alpha_n}\frac{(-\alpha_n/\beta)}{1+\alpha_n^2/\beta^2}\right]$$

$$\frac{v_0(1-\cos\alpha_n)}{\alpha_n} = \frac{c_n}{2}\left[\frac{\alpha_n^2+\beta+\beta^2}{\alpha_n^2+\beta^2}\right].$$

$$c_n = 2v_0\frac{(1-\cos\alpha_n)}{\alpha_n} \cdot \frac{(\alpha_n^2+\beta^2)}{[\alpha_n^2+\beta+\beta^2]}$$

$$\therefore \quad u = \sum_n c_n \exp(-\alpha_n^2 t)\sin(\alpha_n x)$$