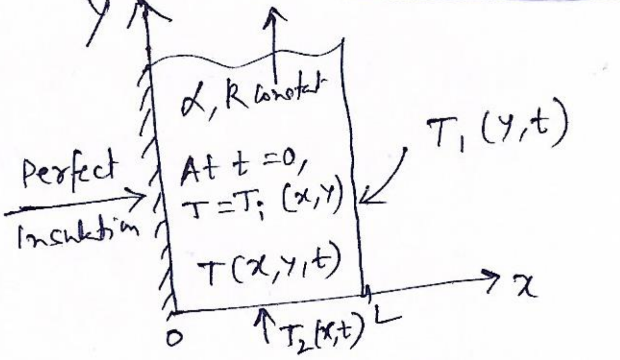
• Solution of unsteady-state heat conduction problem by Integral Transform

• Unsteady-State heat conduction



Problem Formulation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$T(x,y,0) = T_i(x,y)$$

$$\frac{\partial T(0,y,t)}{\partial x} = 0, \quad T(L,y,t) = T_1(y,t)$$

$$T(x,0,t) = T_2(x,t)$$

Assume, both $T_i(x,y)$ and $T_1(y,t)$ vanish as $y \to \infty$

Semi-infinite rectangular strip

Solve by Semi-infinite Fourier Transform technique.

Range of space variable x is finite $(0, L)$

" " " y " semi-infinite $(0, \infty)$

Define (First remove x from PDE)

$$\bar{T}_n(y,t) = \int_0^L T(x,y,t)\, K_n(x)\, dx$$

$$\text{Inversion } T(x,y,t) = \sum_{n=1}^{\infty} \bar{T}_n(y,t)\, K_n(x)$$

Kernels are the normalized characteristic functions of the following CVP:

$$\frac{d^2 y}{dx^2} + \lambda^2 y = 0$$

$$\frac{d\, y(0)}{dx} = 0 \quad \text{and} \quad y(L) = 0$$

$$K_n(x) = \sqrt{\frac{2}{L}} \cos \lambda_n x, \quad \lambda_n = \frac{(2n-1)\pi}{2L}$$

$n = 1, 2, \ldots$

Now, we obtain the transform of the heat conduction eqn wrt variable x as

$$\int_0^L K_n(x)\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right]dx = \frac{1}{\alpha}\int_0^L K_n(x)\frac{\partial T}{\partial t}\,dx$$

$$\Rightarrow \int_0^L K_n(x)\frac{\partial^2 T}{\partial x^2}\,dx + \int_0^L K_n(x)\frac{\partial^2 T}{\partial y^2}\,dx = \frac{1}{\alpha}\frac{\partial \bar{T}_n}{\partial t}$$

$$\Rightarrow \underbrace{\int_0^L K_n(x)\frac{\partial^2 T}{\partial x^2}\,dx}_{\text{integration by parts twice}} + \frac{\partial^2 \bar{T}_n}{\partial y^2} = \frac{1}{\alpha}\frac{\partial \bar{T}_n}{\partial t}.$$

$$= (-1)^n\sqrt{\frac{2}{L}}\,\lambda_n T_1(y,t) - \lambda_n^2\,\bar{T}_n(y,t)$$

On substitution,

$$\frac{\partial^2 \bar{T}_n}{\partial y^2} - \lambda_n^2\,\bar{T}_n(y,t) + (-1)^n\sqrt{\frac{2}{L}}\,\lambda_n T_1(y,t) = \frac{1}{\alpha}\frac{\partial \bar{T}_n}{\partial t}$$

This is still PDE $[\bar{T}_n(y,t)]$ from this eqn, we now remove the space variable y from this eqn.

Note the range of variable y is $(0,\infty)$.

$$\frac{\partial^2 \bar{T}_n}{\partial y^2} - \lambda_n^2 \bar{T}_n(y,t) + (-1)^n \sqrt{\frac{2}{L}}\, \lambda_n T_1(y,t) = \frac{1}{\alpha}\frac{\partial \bar{T}_n}{\partial t} \quad \text{--- (A)}$$

$y \rightarrow$ semi infinite region $(0, \infty)$

$$\bar{T}_n(y,t) \xrightarrow{\text{Integral transform}} \bar{\bar{T}}_n(\omega,t)$$

$$\bar{\bar{T}}_n(\omega,t) = \int_0^\infty \bar{T}_n(y,t)\, K(\omega, y)\, dy$$

Inversion

$$\bar{T}_n(y,t) = \int_0^\infty \bar{\bar{T}}_n(\omega,t)\, K(\omega,y)\, d\omega$$

Kernel $K(\omega,y)$ satisfies

$$\frac{d^2\phi}{dy^2} + \omega^2\phi = 0$$

$$\phi(0) = 0 \quad \text{and} \quad |\phi(y)| \le M \text{ for } y > 0$$

$\uparrow$ finite constant

$$K(\omega,y) = \sqrt{\frac{2}{\pi}} \sin \omega y$$

The Kernel $K(\omega,y)$ satisfies the CVP for all values of ω from 0 to ∞

$$\left\{\begin{aligned} &\frac{d^2\phi}{dy^2} + \omega^2\phi = 0 \\ &\phi(0) = 0 \\ &|\phi(y)| \le M \text{ for } y > 0 \end{aligned}\right.$$

Now, obtain the transform of above (1st eqn on this page) eqn (A):

$$\int_0^\infty K(\omega,y) \frac{\partial^2 \bar{T}_n}{\partial y^2} dy - \lambda_n^2 \int_0^\infty K(\omega,y)\, \bar{T}_n(y,t)\, dy + (-1)^n \sqrt{\frac{2}{L}}\, \lambda_n \int_0^\infty K(\omega,y)\, T_1(y,t)\, dy = \frac{1}{\alpha}\int_0^\infty K(\omega,y) \frac{\partial \bar{T}_n}{\partial t} dy$$

$$\Rightarrow \int_0^\infty K(\omega,y) \frac{\partial^2 \bar{T}_n}{\partial y^2} dy - \lambda_n^2 \bar{\bar{T}}_n(\omega,t) + (-1)^n \sqrt{\frac{2}{L}} \lambda_n \bar{T}_1(\omega,t) = \frac{1}{\alpha} \frac{d\bar{\bar{T}}_n}{dt} \quad \text{---- (B)}$$

where we have defined

$$\bar{T}_1(\omega,t) = \int_0^\infty T_1(y,t) K(\omega,y) dy$$

The integral on the LHS can be evaluated by integrating it by parts twice:

$$\int_0^\infty K(\omega,y) \frac{\partial^2 \bar{T}_n}{\partial y^2} dy = -\omega^2 \bar{\bar{T}}_n(\omega,t) - \sqrt{\frac{2}{\pi}} \omega \bar{T}_{2n}(t)$$

$$\text{where } \bar{T}_{2n}(t) = \bar{T}_n(0,t) = \int_0^L T_2(x,t) K_n(x) dx$$

Because $T_i(x,y)$ and $T_1(y,t)$ both vanish as $y \to \infty$, we note that the temperature and its 1st derivative wrt y also vanish as $y \to \infty$

On substitution of above $\int_0^\infty K(\omega,y) \frac{\partial^2 \bar{T}_n}{\partial y^2} dy$ into the 1st eqn on this page (Eq. B)

$$\frac{d\bar{\bar{T}}_n}{dt} + \alpha(\omega^2 + \lambda_n^2)\bar{\bar{T}}_n(\omega,t) = F_n(\omega,t)$$

where $F_n(\omega,t) = \alpha\left[(-1)^n \sqrt{\frac{2}{L}}\lambda_n \bar{T}_1(\omega,t) + \sqrt{\frac{2}{\pi}}\,\omega\,\bar{T}_{2n}(t)\right]$

$$\bar{T}_{2n}(t) = \bar{T}_n(0,t) = \int_0^L T_2(x,t)K_n(x)\,dx$$

This is 1st order ODE.
solve ~~this~~ ODE and
invert twice to obtain $T(x,y,t)$:

$$\bar{\bar{T}}_n(\omega,t) = e^{-\alpha(\omega^2+\lambda_n^2)t}\left[\bar{\bar{T}}_{in}(\omega) + \int_0^t e^{\alpha(\omega^2+\lambda_n^2)t'} F_n(\omega,t')\,dt'\right]$$

where $\bar{\bar{T}}_{in}(\omega) = \int_0^L K_n(x)\left\{\int_0^\infty K(\omega,y)T_i(x,y)\,dy\right\}dx$

Invert twice ; $\bar{\bar{T}}_n(\omega,t) \xrightarrow{\text{invert}} \bar{T}_n(y,t) \xrightarrow{\text{invert}} T(x,y,t)$

$$T(x,y,t) = \sum_{n=1}^{\infty} K_n(x)\left\{\int_0^\infty K(\omega,y)\,e^{-\alpha(\omega^2+\lambda_n^2)t} \times \left[\bar{\bar{T}}_{in}(\omega) + \int_0^t e^{\alpha(\omega^2+\lambda_n^2)t'} F_n(\omega,t')\,dt'\right]d\omega\right\}$$