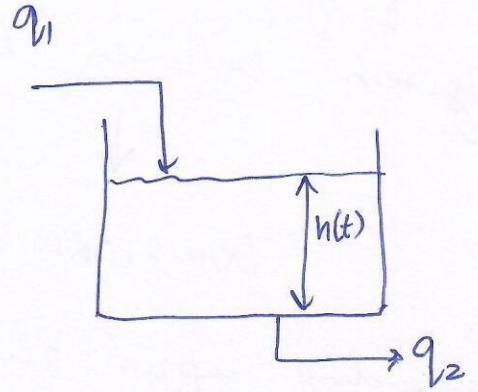
Analysis of system dynamics in transform domain

Q. Why transfer domain ?

Q When do you need it ?



$$\frac{dh}{dt} = \frac{1}{A}(q_1 - q_2)$$

let

$q_1 =$ constant $= C$

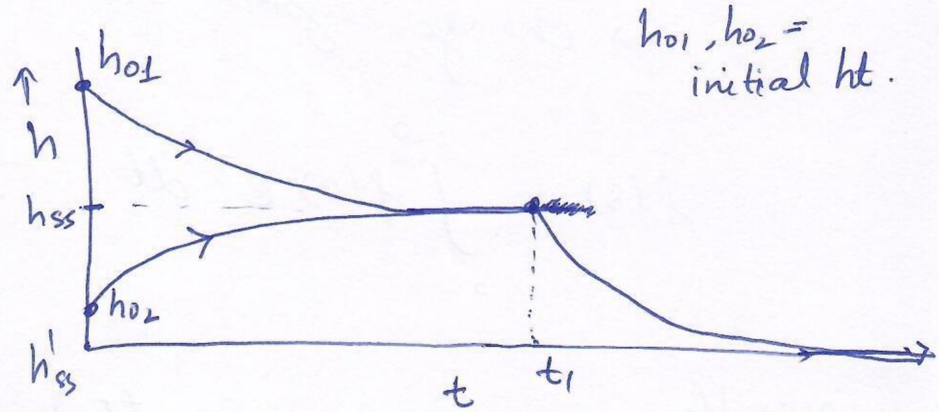
$q_2 = ah$

$$\frac{dh}{dt} = \frac{C}{A} - \frac{ah}{A}$$

for Steady state solution

$$\frac{dh}{dt} = 0$$

$$h_{ss} = \frac{C}{a}$$



$h_{01}, h_{02} =$ initial ht.

Assume @ $t = t_1$

$C = 0$

$$\frac{dh}{dt} = -\frac{a}{A} h$$

In this case $\Rightarrow$ $h'_{ss} = 0$

→ So everytime system state change, you need to solve ODE to see the process performance.

⇒ Also if you want to study dynamic of ~~them~~ system between two steady states.

⇓

TRANSFORM DOMAIN

Depending upon the solutions you are looking for choose transform domain or state space domain

LAPLACE TRANSFORM.

↳ change system from 't' domain to 's' domain.

$$\bar{f}(s) = \int_0^\infty f(t)\, e^{-st}\, dt \quad \text{——— (1)}$$

example

$$f(t) = A \quad \text{(constt.)}$$

$$\bar{f}(s) = \int_0^\infty A\, e^{-st}\, dt$$

$$= A\left(-\frac{1}{s}\right) e^{-st} \Big|_0^\infty$$

$$\bar{f}(s) = \frac{A}{s} \quad \text{——— (2)}$$

$$L\left[\frac{d^n f(t)}{dt^n}\right] = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2}\frac{df}{dt}(0) - s^{n-3}\frac{d^2 f}{dt^2}(0)$$

(3) ......

Example

$$\frac{dx}{dt} = ax + bu \quad \text{(4)}$$

$$s\bar{x}(s) - x(0) = a\bar{x}(s) + b\bar{u}(s)$$

$$(s-a)\bar{x}(s) - x(0) = b\bar{u}(s)$$

$$(s-a)\bar{x}(s) = b\bar{u}(s) + x(0)$$

$$\bar{x}(s) = \frac{b}{s-a}\bar{u}(s) + \frac{x(0)}{s-a}$$

$$x(t) = L^{-1}\left[\frac{b}{s-a}\bar{u}(s) + \frac{x(0)}{s-a}\right]$$

So for liquid level tank

$$\frac{dh}{dt} = \frac{1}{A}q_1 - \frac{1}{A}q_2$$

$q_1 = c$

$q_2 = ah$

$$\frac{dh}{dt} = \frac{c}{A} - \frac{ah}{A} \quad \text{(5)}$$

$$\frac{dh_{ss}}{dt} = \frac{c_{ss}}{A} - \frac{ah_{ss}}{A} \quad \text{(6)}$$

$$\frac{d}{dt}(h-h_{ss}) = \frac{1}{A}(c-c_{ss}) - \frac{a}{A}(h-h_{ss})$$

Let

$h-h_{ss} = y$

$c-c_{ss} = u$

Deviation variables

$$\frac{dy}{dt} = \frac{1}{A}u - \frac{a}{A}y$$

$$\frac{dy}{dt} + \frac{a}{A}y = \frac{1}{A}u$$

$$s\bar{y}(s) - y(0) + \frac{a}{A}\bar{y}(s) = \frac{1}{A}\bar{u}(s)$$

$y(0)$ = deviation at $t=0$

$\therefore \dot{y}(0) = 0$

$$\left(s + \frac{a}{A}\right)\bar{y}(s) = \frac{1}{A}\bar{u}(s)$$

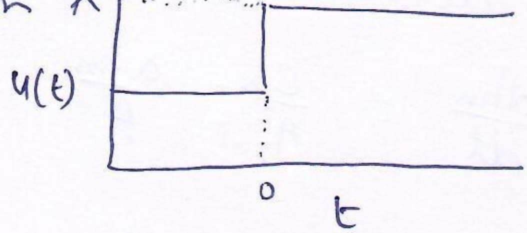
$$\frac{\bar{y}(s)}{\bar{u}(s)} = \frac{1/A}{s+a/A}$$

Transfer function of system.

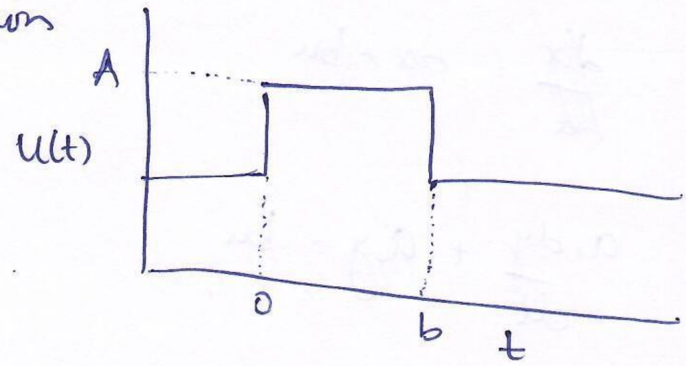
$$y(t) = L^{-1}\left\{\left(\frac{1/A}{s+a/A}\right)\bar{u}(s)\right\}$$

<u>Forcing functions (u)</u>

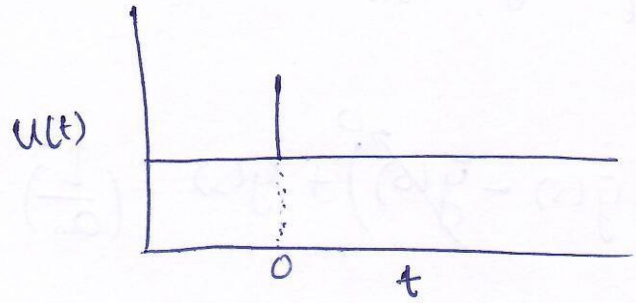
u = ideal step function



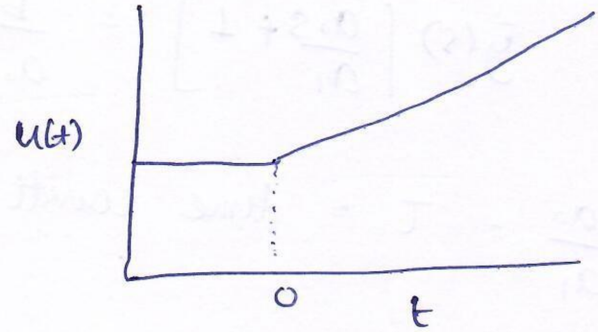
Ideal rectangular pulse function



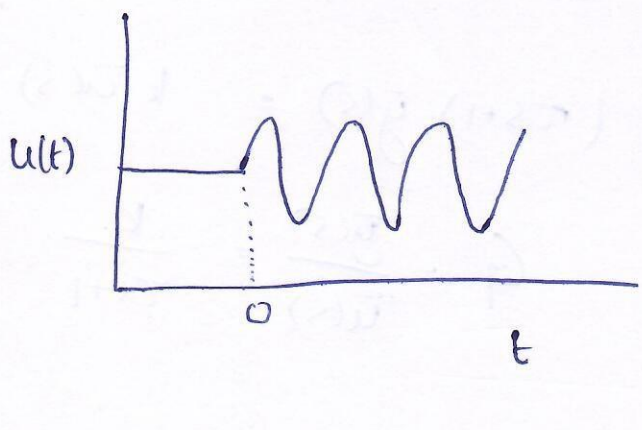
Ideal Impulse function



Ideal ramp function.



Ideal sine function



$$\cancel{\frac{dx}{dt}} = \cancel{bx + bu}$$

$$a_1 \frac{dy}{dt} + a_0 y = bu$$

$$\frac{a_1}{a_0} \frac{dy}{dt} + y = \frac{b}{a_0} u$$

laplace

$$\frac{a_1}{a_0}\left(s\,\bar{y}(s) - \cancelto{0}{\bar{y}(0)}\right) + \bar{y}(s) = \left(\frac{b}{a_0}\right) \bar{u}(s)$$

$$\frac{a_1}{a_0}\, s\,\cancel{\bar{y}(s)} +$$

$$\bar{y}(s) \left[\frac{a_0 s}{a_1} + 1 \right] = \frac{b}{a_0}\, \bar{u}(s)$$

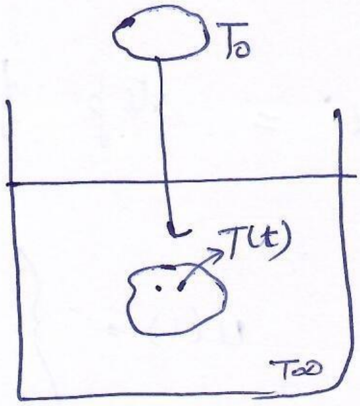
$$\frac{a_0}{a_1} = \tau = \text{time constt.}$$

$$\frac{b}{a_0} = k = \text{gain.}$$

$$(\tau s + 1)\, \bar{y}(s) = k\, \bar{u}(s)$$

$$G = \frac{\bar{y}(s)}{\bar{u}(s)} = \frac{k}{\tau s + 1}$$ ← 1st order system

determine k, τ - ?



$$\frac{dT}{dt} = \frac{-Ah}{\rho V_c}(T - T_\infty)$$

$$\frac{dT}{dt} + \left(\frac{hA_s}{\rho V_c}\right)T = \left(\frac{Ah_s}{\rho V_c}\right)T_\infty \quad — (1)$$

<u>at ss.</u>

$$\frac{dT_{ss}}{dt} + \left(\frac{hA_s}{\rho V_c}\right)T_{ss} = \left(\frac{Ah_s}{\rho V_c}\right)T_{\infty_{ss}} \quad — (2)$$

(1) − (2)

$$\frac{d}{dt}(T - T_{ss}) + \left(\frac{hA_s}{\rho V_c}\right)(T - T_{ss}) = \left(\frac{Ah_s}{\rho V_c}\right)(T_\infty - T_{\infty_{ss}})$$

$$\frac{d}{dt}y + \left(\frac{hA_s}{\rho V_c}\right)y = \left(\frac{A_s h}{\rho V_c}\right)u$$

$$\left(\frac{\rho V_c}{h_s A_s}\right)\frac{dy}{dt} + y = \cancel{\left(\frac{A_s h}{\rho V_c}\right)}u$$

$K = 1$

$$\tau = \frac{\rho V_c}{hA_s}$$

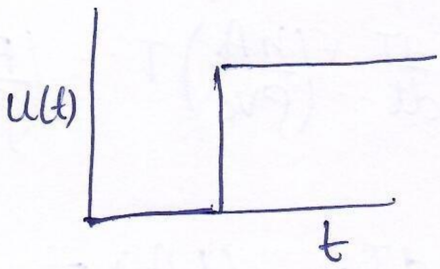
$$\frac{\bar{y}(s)}{u(s)} = \frac{K}{\tau s + 1}$$

unity gain

$$\tau = \frac{\rho V_c}{A_s h}$$

I u = step func^n of magnitude A

$$u(t) = \begin{cases} 0 & t < 0 \\ A & t > 0 \end{cases}$$



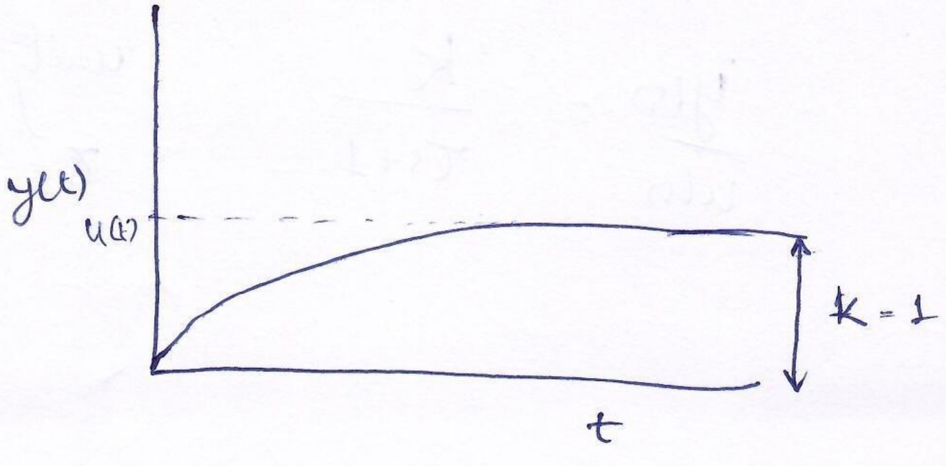
$$\bar{u}(s) = \frac{A}{s}$$

$$\frac{\bar{y}(s)}{\bar{u}(s)} = \frac{k}{\tau s + 1}$$

$$\bar{y}(s) = \left(\frac{k}{\tau s+1}\right)\left(\frac{A}{s}\right)$$

$$y(t) = L^{-1}\left\{\left(\frac{k}{\tau s+1}\right)\left(\frac{A}{s}\right)\right\}$$

$$y(t) = Ak(1 - e^{-t/\tau})$$ solve



time constt. = time required to reach 63.2% value of ~~devi~~ forcing function.

at 4 time constt. — 98.2% of forcing func^n value is achieved

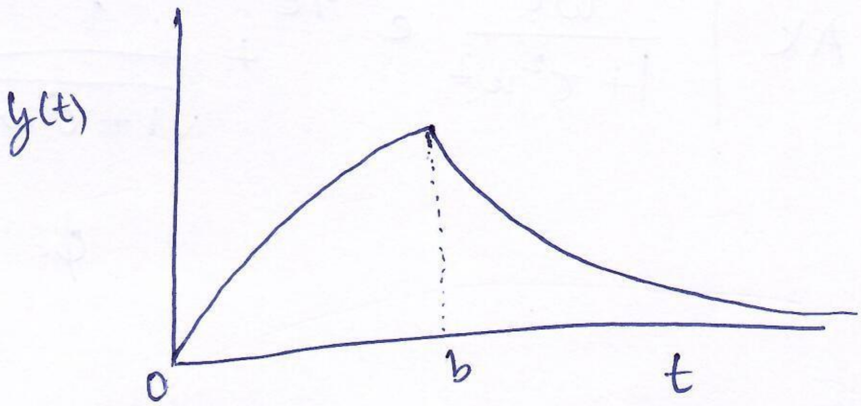
at 5τ — 99.3%

II. pulse function

Assignment 3

$$u(t) = \begin{cases} 0 & t<0 \\ A & 0<t<b \\ 0 & t>b \end{cases}$$

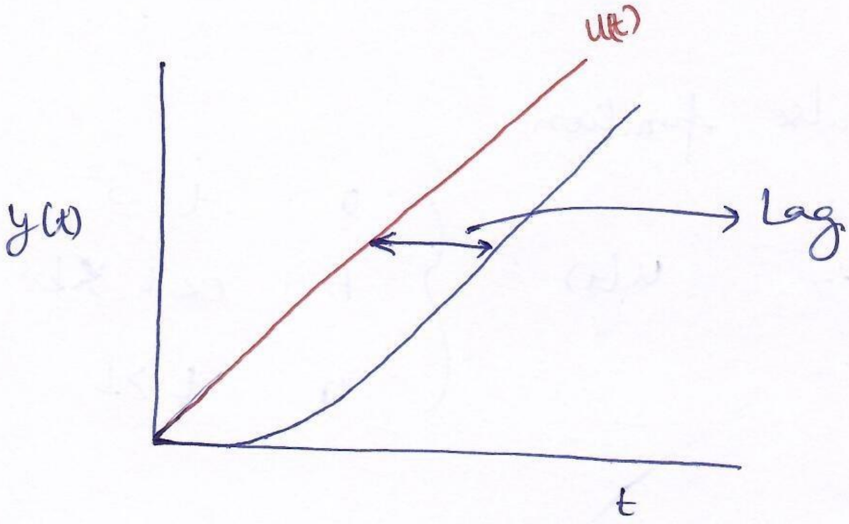
$$y(t) = \begin{cases} Ak\left(1-e^{-t/\tau}\right) & t<b \\ Ak\left[\left(1-e^{-t/\tau}\right) - \left(1-e^{-(t-b)/\tau}\right)\right] & t>b \end{cases}$$



(III)

$$u(t) = \begin{cases} 0 & t < 0 \\ At & t > 0 \end{cases}$$ ramp func^n

$$y(t) = AK\tau\left(e^{-t/\tau} + \frac{t}{\tau} - 1\right)$$



(IV)

$$u(t) = \begin{cases} 0 & t < 0 \\ A \sin \omega t & t > 0 \end{cases}$$ Sine wave

$$y(t) = AK\left[\frac{\omega\tau}{1+\tau^2\omega^2} e^{-t/\tau} + \frac{1}{\sqrt{1+\tau^2\omega^2}} \sin(\omega t + \phi)\right]$$

$$\phi = -\tan^{-1}(\tau\omega)$$

Ex

$$a_1 \frac{dy}{dt} + a_0 y = bu$$

$$\frac{\bar{y}(s)}{\bar{u}(s)} = \frac{k}{\tau s + 1}$$

in case $\tau \approx 0$

$$\frac{\bar{y}(s)}{\bar{u}(s)} = k$$

$$\bar{y}(s) = k\,\bar{u}(s)$$

$\Rightarrow \quad y(t) = K\,u(t) \quad \leftarrow$ PURE GAIN SYSTEM

in case $a_0 \approx 0$

$$a_1 \frac{dy}{dt} = bu$$

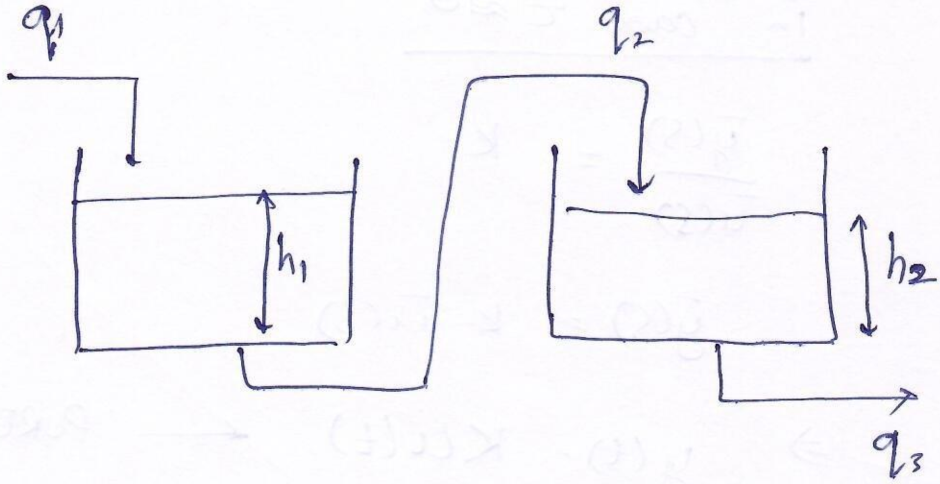
$$a_1 s \bar{y}(s) = b\,\bar{u}(s)$$

$\frac{\bar{y}(s)}{\bar{u}(s)} = \left(\frac{b}{a_1}\right)\frac{1}{s} \quad \leftarrow$ PURE CAPACITANCE

Q. what is response w.r.t forcing functions??

FOR SECOND ORDER SYSTEMS

$$a_2 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = bu$$



$$\frac{dh_1}{dt} = \frac{1}{A_1}\left[q_1 - q_2\right]$$

$$\frac{dh_2}{dt} = \frac{1}{A_2}\left[q_2 - q_3\right]$$

let $q_1 = c$

$q_2 = ah_1$

$q_3 = bh_2$

$$\frac{dh_1}{dt} = \frac{c}{A_1} - \frac{ah_1}{A_1} \Rightarrow \frac{dh_1}{dt} + \left(\frac{a}{A_1}\right)h_1 = \frac{c}{A_1}$$

$$\left(\frac{A_1}{a}\right)\frac{dh_1}{dt} + h_1 = \frac{c}{a} \quad \text{—(1)}$$

$$\frac{dh_2}{dt} = \frac{ah_1}{A_2} - \frac{bh_2}{A_2}$$

@ ss.

$$\left(\frac{A_1}{a}\right)\frac{dh_{1ss}}{dt} + h_{1ss} = \frac{c_{ss}}{a} \quad \text{—(2)}$$

(1) – (2)

$$\left(\frac{A_1}{a}\right)\frac{dy_1}{dt} + y_1 = \frac{u}{a}$$

$y = h - h_{ss}$

$u = C - C_{ss}$

laplace

$$\left[\frac{A_1}{a}s + 1\right]\bar{y}_1(s) = \frac{\bar{u}_1(s)}{a}$$

$$\frac{\bar{y}_1(s)}{u_1(s)} = \frac{(1/a)}{\left[A_1/a\, s + 1\right]} \quad \text{--- (3)}$$

$K = 1/a$

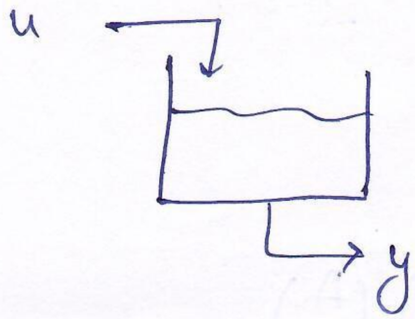
$\tau = \frac{A_1}{a}$

Similarly for tank 2

$$\frac{\bar{y}_2(s)}{\bar{y}_1(s)} = \frac{(a/b)}{\left(\frac{A_2}{b}\right)s + 1} \quad \text{--- (4)}$$

$$\frac{\bar{y}_2(s)}{\bar{u}_1(s)} = \frac{(1/a)}{\left(\frac{A}{a}\right)s + 1} \times \frac{(a/b)}{\left(\frac{A_2}{b}\right)s + 1}$$

overall transfer fcn = product of individual transfer functions.

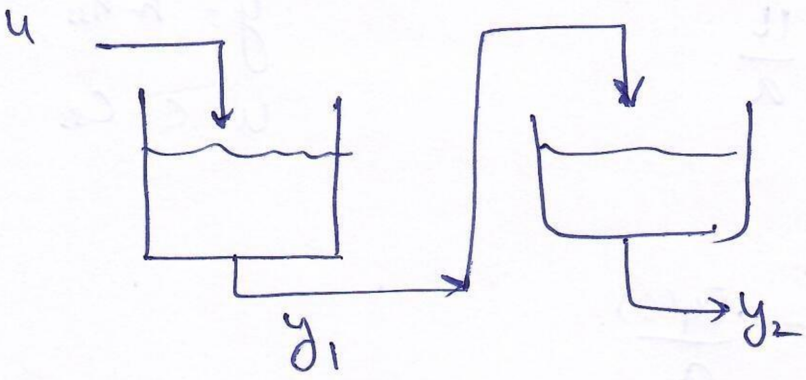


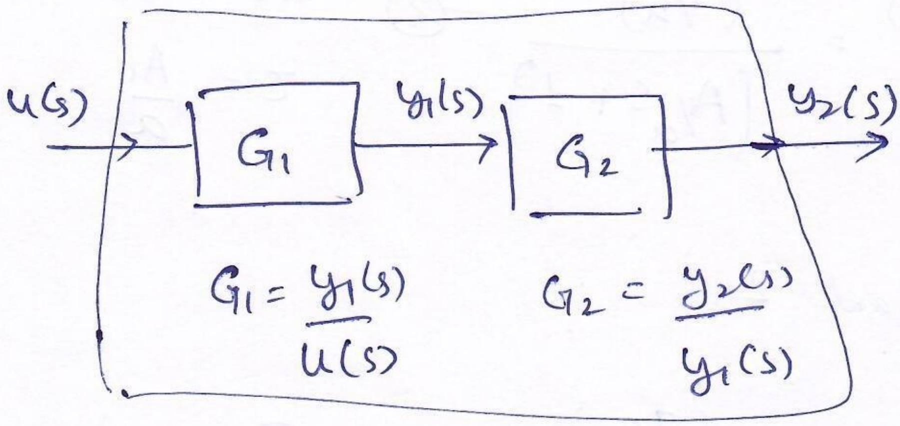
$\equiv$

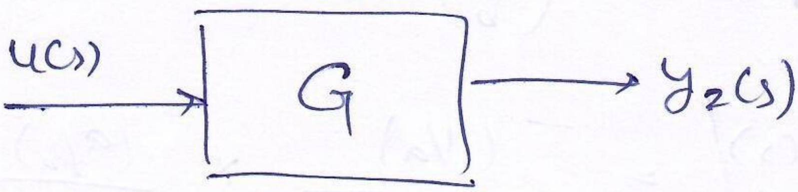
$$\bar{u}(s) \longrightarrow \boxed{G(s)} \longrightarrow \bar{y}(s)$$

$$G(s) = \frac{\bar{y}(s)}{\bar{u}(s)} = \frac{K}{\tau s + 1}$$

$= o/p / i/p$







$$G = G_1(s) \cdot G_2(s)$$

$$\frac{y_2(s)}{u(s)} = G = \frac{k_1 k_2}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

order = 2 (s^2)

Let $u(s) = \frac{A}{s}$

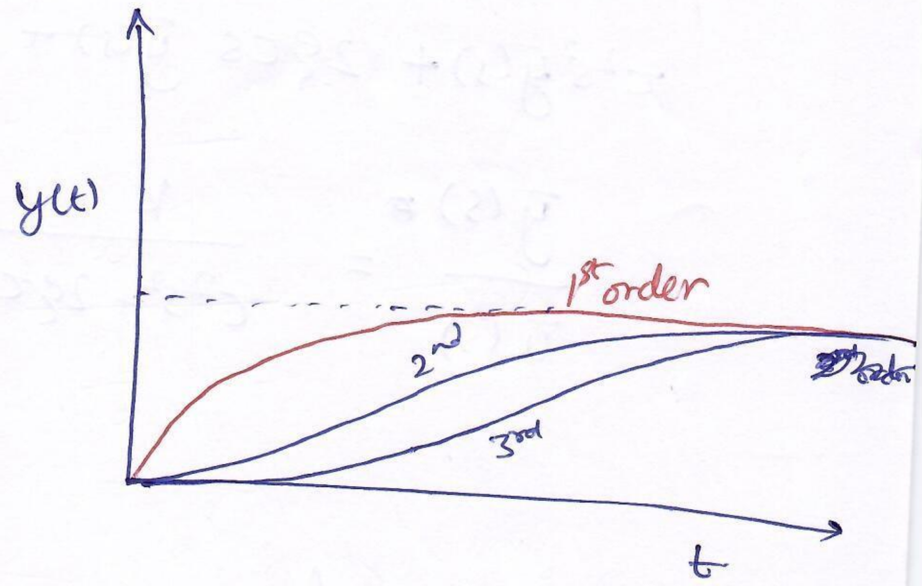
$$y_2(s) = \left[\frac{k_1 k_2}{(\tau_1 s + 1)(\tau_2 s + 1)}\right]\left[\frac{A}{s}\right]$$

Inverse laplace

$$y(t) = L^{-1}\left[\frac{Ak_1k_2}{s(\tau_1 s+1)(\tau_2 s+1)}\right]$$

$$y(t) = Ak_1k_2\left[1 - \left(\frac{\tau_1}{\tau_1-\tau_2}\right)e^{-t/\tau_1} - \left(\frac{\tau_2}{\tau_2-\tau_1}\right)e^{-t/\tau_2}\right]$$

No oscillations.



In general

$$a_2\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0 y = bu$$

$$\frac{a_2}{a_0}\frac{d^2y}{dt^2} + \frac{a_1}{a_0}\frac{dy}{dt} + y = \left(\frac{b}{a_0}\right)u$$

$$\frac{a_2}{a_0} = \tau^2 = \text{natural gain of oscillation}$$

$\frac{a_1}{a_0} = 2\xi\tau$ ξ = damping coefficient

$\frac{b}{a_0} = k$ Static gain

$$\tau^2 \frac{d^2y}{dt^2} + 2\tau\xi \frac{dy}{dt} + y = ku$$

laplace

$$\tau^2 s^2 \bar{y}(s) + 2\xi\tau s\, \bar{y}(s) + \bar{y}(s) = k\, \bar{u}(s)$$

$$\frac{\bar{y}(s)}{\bar{u}(s)} = \frac{k}{\tau^2 s^2 + 2\xi\tau s + 1}$$

2^{nd} order T.F.

let $u(s) = \frac{A}{s}$

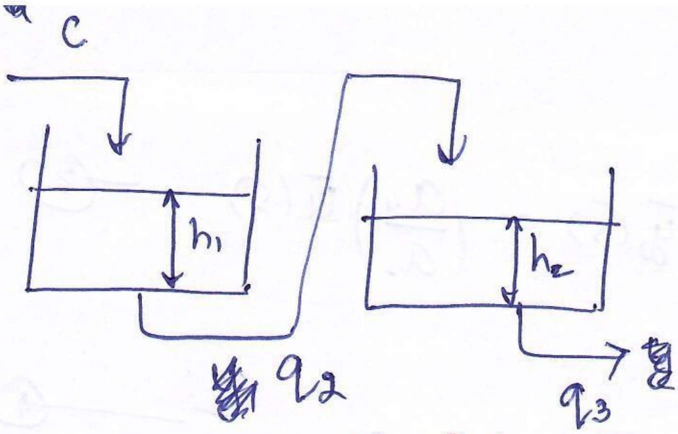
$$y(t) = L^{-1}\left[\frac{AK}{s(\tau^2 s^2 + 2\xi\tau s + 1)}\right]$$

$$y(t) = L^{-1}\left[\frac{AK}{s(s-a)(s-b)}\right]$$

depend on a, b $\rightarrow$ system behavior changes

complex conjugate $\rightarrow$ oscillatory
Real $\rightarrow$ monotonous
equally $\rightarrow$ critically damped.

(1)

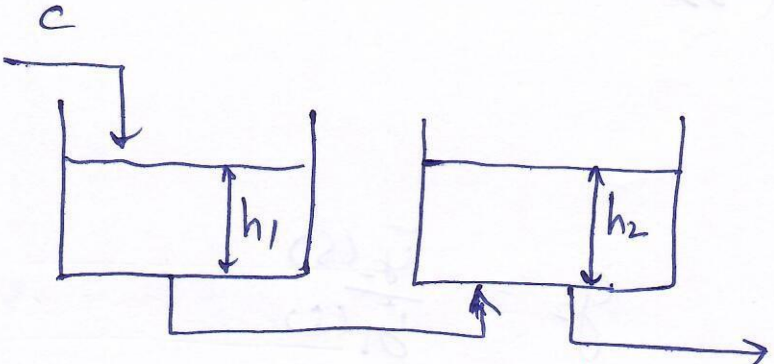


$$\frac{dh_1}{dt} = f(c, h_1)$$

$$\frac{dh_2}{dt} = g(h_1, h_2)$$

Non-Interacting system

(2)



$$\frac{dh_1}{dt} = f(c, h_1, h_2)$$

$$\frac{dh_2}{dt} = g(h_1, h_2)$$

Interacting system

for case 2 in deviation variable form

$$a_1 \frac{dy_1}{dt} + a_2 y_1 + a_3 y_2 = a_4 u \quad \text{--- (1)}$$

$$b_1 \frac{dy_2}{dt} + b_2 y_2 - b_3 y_1 = 0 \quad \text{--- (2)}$$

$$\left(\frac{a_1}{a_2}\right) \frac{dy_1}{dt} + y_1 + \left(\frac{a_3}{a_2}\right) y_2 = \left(\frac{a_4}{a_2}\right) u$$

$$\left(\frac{b_1}{b_2}\right) \frac{dy_2}{dt} + y_2 - \left(\frac{b_3}{b_2}\right) y = 0$$

in laplace form

$$\left(\left(\frac{a_1}{a_2}\right)s+1\right)\bar{y}_1(s) + \left(\frac{a_3}{a_2}\right)\bar{y}_2(s) = \left(\frac{a_4}{a_2}\right)\bar{u}(s) \quad (3)$$

$$\left(\frac{b_1}{b_2}s+1\right)\bar{y}_2(s) - \left(\frac{b_3}{b_2}\right)\bar{y}_1(s) = 0 \quad (4)$$

Transfer function.

$$g_1 = \frac{\bar{y}_1(s)}{\bar{u}(s)} \qquad g_2 = \frac{\bar{y}_2(s)}{\bar{y}_1(s)}$$

$$G = \frac{\bar{y}_2(s)}{\bar{u}(s)}$$

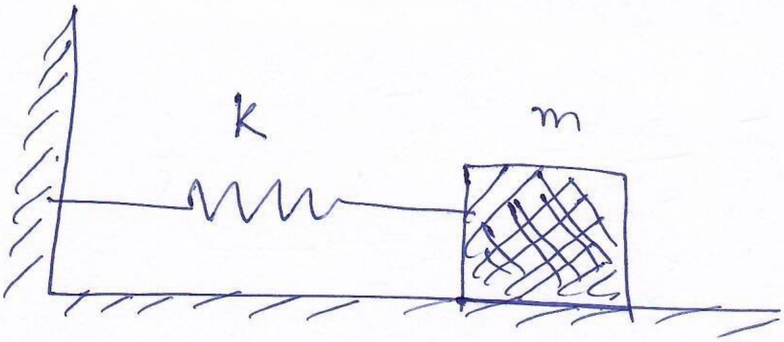
$$g_1(s) = \frac{\left(\dfrac{a_4 b_1 s + b_2 a_4}{a_2 b_2 + a_3 b_3}\right)}{\left(\dfrac{a_1 b_1}{a_2 b_2 + a_3 b_3}\right)s^2 + \left(\dfrac{a_1 b_2 + a_2 b_1}{a_2 b_2 + a_3 b_3}\right)s + 1}$$

2nd order

1st order??

$$G(s) = \frac{\dfrac{a_4 b_3}{a_2 b_2 + a_3 b_3}}{\left(\dfrac{a_1 b_1}{a_2 b_2 + a_3 b_3}\right)s^2 + \left(\dfrac{a_1 b_2 + a_2 b_1}{a_2 b_2 + a_3 b_3}\right)s + 1}$$

2nd order



Forced Vibration w/o damping

$$m\frac{d^2x}{dt^2} + kx = f_0 \sin \omega t$$

with damping.