Separation of Variables: Generalization

Consider the following multidimensional, transient homogeneous problem:

$$\nabla^2 T(\hat{r},t) = \frac{1}{\alpha}\frac{\partial T(\hat{r},t)}{\partial t} \quad \text{in domain R} \quad t>0$$

$$k_i \frac{\partial T}{\partial n_i} + h_i T = 0 \quad \text{on boundary } S_i \quad i = 1 \text{ to } N \quad t>0$$

$$T(\hat{r}, t=0) = F(\hat{r}) \quad \text{in domain } R$$

Each boundary surface S_i fits the coordinate surface of the chosen orthogonal coordinate system.

Assume a separation in the form:

$$T(\hat{r},t) = \Psi(\hat{r})\Gamma(t)$$

Substitute this equation into the heat equation (PDE):

$$\frac{1}{\Psi(\hat{r})}\nabla^2\Psi(\hat{r}) = \frac{1}{\alpha\Gamma(t)}\frac{d\Gamma(t)}{dt} = -\lambda^2$$

(first separation constant)

$$\frac{d\Gamma}{dt} + \alpha\lambda^2\Gamma = 0$$

Separation of Variables: Generalization

$$\frac{1}{\Psi(\hat{r})}\nabla^2\Psi(\hat{r}) = \frac{1}{\alpha\Gamma(t)}\frac{d\Gamma(t)}{dt} = -\lambda^2$$

The spatial variable function $\psi(\hat{r})$ satisfies the following auxiliary problem:

$$\nabla^2\Psi(\hat{r}) + \lambda^2\Psi(\hat{r}) = 0 \qquad \text{in domain } R$$

$$k_i\frac{\partial\Psi}{\partial n_i} + h_i\Psi = 0 \qquad \text{on boundary } S_i$$

(Homogeneous Boundary Conditions)

- *This equation is called the Helmholtz equation.*
- *In general, it is a PDE in the three spatial variables.*

The Helmholtz equation can be solved by Separation of Variables, provided that its separation into a set of ODEs is possible.

A simple separation of the Helmholtz equation into ordinary differential equations is possible in 11 orthogonal coordinate systems.

Separation of Variables: Generalization

$$\nabla^2\Psi(\hat{r}) + \lambda^2\Psi(\hat{r}) = 0 \qquad \text{in domain } R$$

$$k_i\frac{\partial\Psi}{\partial n_i} + h_i\Psi = 0 \qquad \text{on boundary } S_i$$

Helmholtz equation

A simple separation of the Helmholtz equation into ordinary differential equations is possible in 11 orthogonal coordinate systems.

Coordinate System	Functions That Appear in Solution
1 Rectangular	Exponential, circular, hyperbolic
2 Circular cylinder	Bessel, exponential, circular
3 Elliptic cylinder	Mathieu, circular
4 Parabolic cylinder	Weber, circular
5 Spherical	Legendre, power, circular
6 Prolate spheroidal	Legendre, circular
7 Oblate spheroidal	Legendre, circular
8 Parabolic	Bessel, circular
9 Conical	Lamé, power
10 Ellipsoidal	Lamé
11 Paraboloidal	Baer

Separation of Variables: Generalization

$$\nabla^2\Psi(\hat{r}) + \lambda^2\Psi(\hat{r}) = 0 \qquad \text{in domain } R$$

$$k_i\frac{\partial\Psi}{\partial n_i} + h_i\Psi = 0 \qquad \text{on boundary } S_i$$

Helmholtz equation

The above system has nontrivial solutions only for certain values of the separation variable $\lambda = \lambda_m$, called *eigenvalues*.

The corresponding nontrivial solutions are called *eigenfunctions*:

$$\Psi(\lambda_m, \hat{r}) = \Psi_m(\hat{r})$$

Assuming the eigenfunctions and the eigenvalues λ_m are determined, the complete solution of the temperature function $T(\hat{r}, t)$ is obtained as:

$$T(\hat{r}, t) = \sum_{m=1}^{\infty} C_m\, \Psi_m(\hat{r})\, e^{-\alpha\lambda_m^2 t}$$

The summation is taken over all discrete spectrum of eigenvalues λ_m for the given problem.

Note that for three-dimensional problems (in finite regions) the summation in above expression for temperature is a triple infinite series.

Separation of Variables: Generalization

$$T(\hat{r},t) = \sum_{m=1}^{\infty} C_m \Psi_m(\hat{r})\, e^{-\alpha\lambda_m^2 t}$$

The solution contains the unknown coefficients C_m.

The above solution should satisfy the initial condition of the problem:

$$T(\hat{r}, t=0) = F(\hat{r})$$

Therefore, by substituting $t = 0$,

$$F(\hat{r},t) = \sum_{m=1}^{\infty} C_m \Psi'_m(\hat{r})$$

If the eigenfunctions $\Psi'_m(\hat{r})$ constitute an orthogonal set in the region considered, the unknown coefficients C_m are determined by making use of the orthogonality property of eigenfunctions $\Psi'_m(\hat{r})$; that is,

$$\int_R \Psi_m(\hat{r})\, \Psi_n(\hat{r})\, d\hat{r} = 0, \qquad \text{for } m \neq n$$

To determine C_m,

- Multiply both sides of $F(\hat{r},t) = \sum_{m=1}^{\infty} C_m \Psi'_m(\hat{r})$ by $\Psi'_m(\hat{r})$
- Integrate it over the region and make use of the orthogonality condition

Separation of Variables: Generalization

$$C_m = \frac{\int_R \Psi_m(\hat{r})\, F(\hat{r})\, d\hat{r}}{N} \qquad N = \int_R \Psi_m^2(\hat{r})\, d\hat{r}$$

where N is called the *norm* of the eigenfunction

Having determined the coefficients C_m, the complete solution of the homogeneous boundary-value problem of heat conduction equation is given in the form

$$T(\hat{r},t) = \frac{\sum_{m=1}^{\infty} \Psi_m(\hat{r})\, e^{-\alpha\lambda_m^2 t}}{N} \int_R \Psi_m(\hat{r})\, F(\hat{r})\, d\hat{r}$$

Sometimes the eigenfunctions are so adjusted that the norm becomes unity. This is done if we define the normalized eigenfunctions $K(\lambda_m, \hat{r})$ as

$$K(\lambda_m, \hat{r}) = \frac{\Psi_m(\hat{r})}{\sqrt{N}}$$

Separation of Variables: Generalization

The final solution:

$$T(\hat{r},t) = \sum_{m=1}^{\infty} K(\lambda_m, \hat{r})\, e^{-\alpha \lambda_m^2 t} \int_R K(\lambda_m, \hat{r}) F(\hat{r})\, d\hat{r}$$

For finite regions if all h_i's simultaneously vanish, that is when the entire bounding surfaces of the region are insulated, the above solution should include the following term

$$\frac{1}{\text{Region}} \int_R F(\hat{r})\, d\hat{r}$$

The "region" in the denominator refers to the:

- Volume of the region for 3D problems
- Surface of the region for 2D problems
- Linear dimension for 1D problem

The physical significance of this term is that after the temperature transients have passed, the initial temperature distribution will tend to reach an average value over the region, since boundaries are insulated and there are no heat losses or gains.

Uniqueness of Solution for 3D Heat Equation

Prove that the solution of the following 3D Heat Problem is unique.

$$
\begin{aligned}
u_t &= \nabla^2 u, && \mathrm{x} \in D \\
u(\mathrm{x}, t) &= 0, && \mathrm{x} \in \partial D \\
u(\mathrm{x}, 0) &= f(\mathrm{x}), && \mathrm{x} \in D
\end{aligned}
$$

(Take $u = T$)

On the boundary

Let u_1, u_2 be two solutions. Define $v = u_1 - u_2$. Then v satisfies

$$
\begin{aligned}
v_t &= \nabla^2 v, && \mathrm{x} \in D \\
v(\mathrm{x}, t) &= 0, && \mathrm{x} \in \partial D \\
v(\mathrm{x}, 0) &= 0, && \mathrm{x} \in D
\end{aligned}
$$

Define:

$$V(t) = \int\int\int_D v^2 dV \geq 0$$

Uniqueness of Solution for 3D Heat Equation

Prove that the solution of the following 3D Heat Problem is unique.

$$
\begin{aligned}
u_t &= \nabla^2 u, && \mathrm{x} \in D && \text{(Take } u = T\text{)}\\
u(\mathrm{x}, t) &= 0, && \mathrm{x} \in \partial D && \text{On the boundary}\\
u(\mathrm{x}, 0) &= f(\mathrm{x}), && \mathrm{x} \in D
\end{aligned}
$$

Let u_1, u_2 be two solutions. Define $v = u_1 - u_2$. Then v satisfies

$$
\begin{aligned}
v_t &= \nabla^2 v, && \mathrm{x} \in D\\
v(\mathrm{x}, t) &= 0, && \mathrm{x} \in \partial D\\
v(\mathrm{x}, 0) &= 0, && \mathrm{x} \in D
\end{aligned}
$$

Define:

$$V(t) = \int\int\int_D v^2 dV \geq 0$$

$V(t) \geq 0$ since the integrand $v^2(x,t) \geq 0$ for all (x, t).

Uniqueness of Solution for 3D Heat Equation

$$V(t) = \int\int\int_D v^2 dV \geq 0$$

$$v_t = \nabla^2 v.$$

$$\frac{dV}{dt}(t) = \int\int\int_D 2vv_t dV$$

Substituting for v_t from the PDE yields

$$\frac{dV}{dt}(t) = \int\int\int_D 2v\nabla^2 v dV$$

Now use:

$$\begin{aligned}\int\int_S v\nabla v \cdot \hat{\mathbf{n}} dS &= \int\int\int_V \left(v\nabla^2 v + \nabla v \cdot \nabla v\right) dV \\ &= \int\int\int_V \left(v\nabla^2 v + |\nabla v|^2\right) dV.\end{aligned}$$

We get:

$$\frac{dV}{dt}(t) = 2\int\int_{\partial D} v\nabla v \cdot \hat{\mathbf{n}} dS - 2\int\int\int_D |\nabla v|^2 dV$$

Uniqueness of Solution for 3D Heat Equation

$$\frac{dV}{dt}(t) = 2\int\int_{\partial D} v\nabla v \cdot \hat{\mathbf{n}}\, dS - 2\int\int\int_D |\nabla v|^2\, dV$$

But on ∂D, $v = 0$, so that the first integral on the RHS vanishes. Thus,

$$\frac{dV}{dt}(t) = -2\int\int\int_D |\nabla v|^2\, dV \leq 0$$

At $t = 0$:

$$V(0) = \int\int\int_D v^2(\mathrm{x}, 0)\, dV = 0$$

Recall IC: $v(x, 0) = 0$

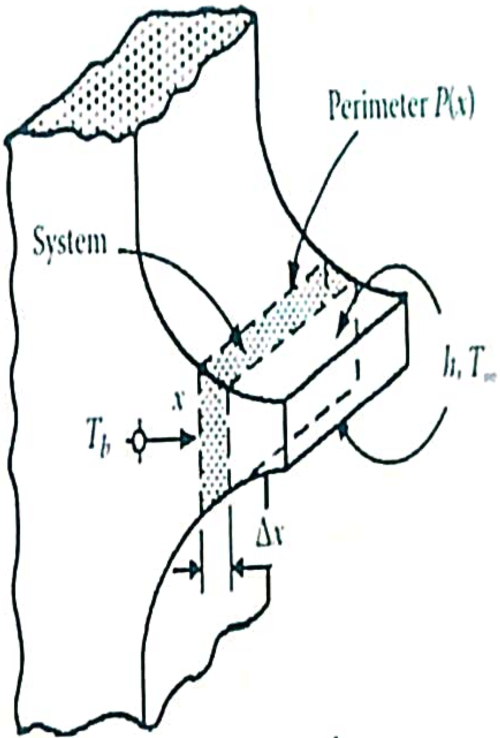
Thus $V(0)=0$, $V(t) \geq 0$ and $dV/dt \leq 0$

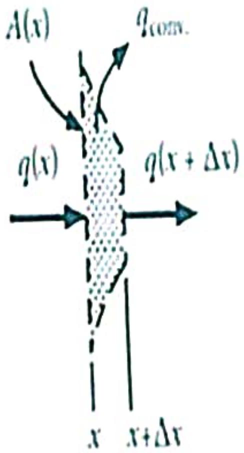
$V(t)$ is a non-negative, non-increasing function that starts at zero.

Thus $V(t)$ must be zero for all time t, so that $v(x,t)$ must be identically zero throughout the volume D for all time, implying the two solutions are the same, $u_1 = u_2$.

Thus, the solution to the 3D heat problem is unique.

Extended Surfaces: Fin Equation





Energy Balance:

$$q(x) = q(x+\Delta x) + q_{conv.}$$

Since Δx → 0,

$$q(x+\Delta x) = q(x) + \frac{dq}{dx}\Delta x$$

Also,

$$q_{conv} = hP(x)\Delta x(T - T_\infty)$$

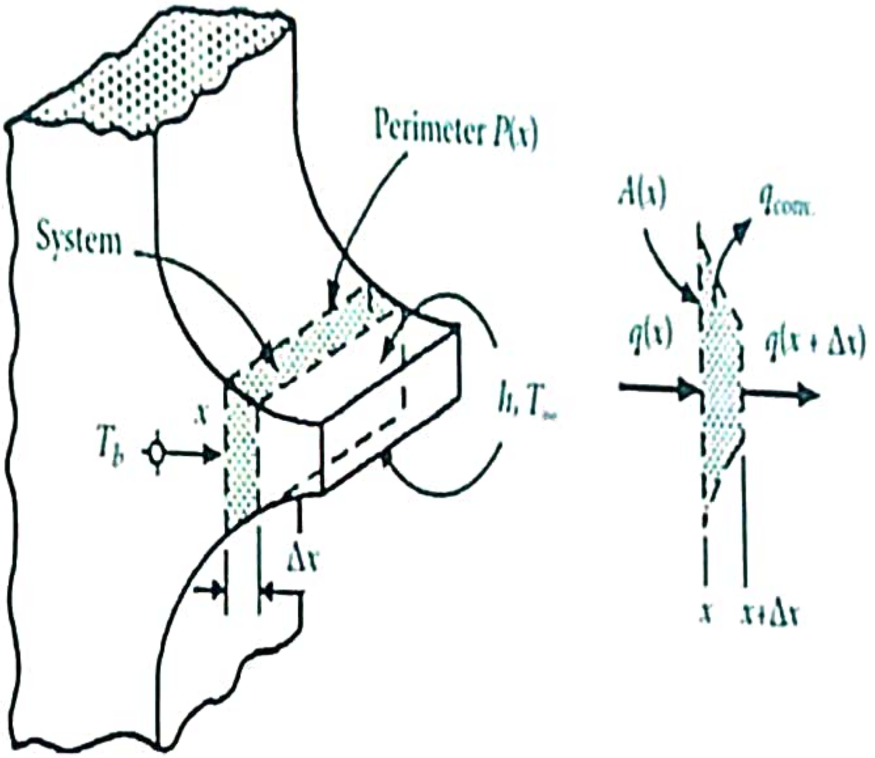
$$\frac{dq}{dx} + hP(x)(T - T_\infty) = 0$$

Use: $q(x) = -kA(x)\dfrac{dT}{dx}$

$$\frac{d}{dx}\left[A(x)\frac{dT}{dx}\right] - \frac{hP(x)}{k}(T - T_\infty) = 0$$

Fin Equation

Fin Equation: Non-dimensional Form



Fin Equation:

$$\frac{d}{dx}\left[A(x)\frac{dT}{dx}\right]-\frac{hP(x)}{k}(T-T_\infty)=0$$

$$\overline{T}=\frac{T-T_\infty}{T_B-T_\infty},\quad \overline{x}=\frac{x}{L}$$

$$\frac{d}{d\overline{x}}A_c\frac{d\overline{T}}{d\overline{x}}-\frac{hPL^2}{k}\overline{T}=0$$

One BC is at Fin Base: $T=T_b$ $\overline{T}(\overline{x}=0)=1$

Is it non-dimensional?

Three Types of BC at Fin Tip:

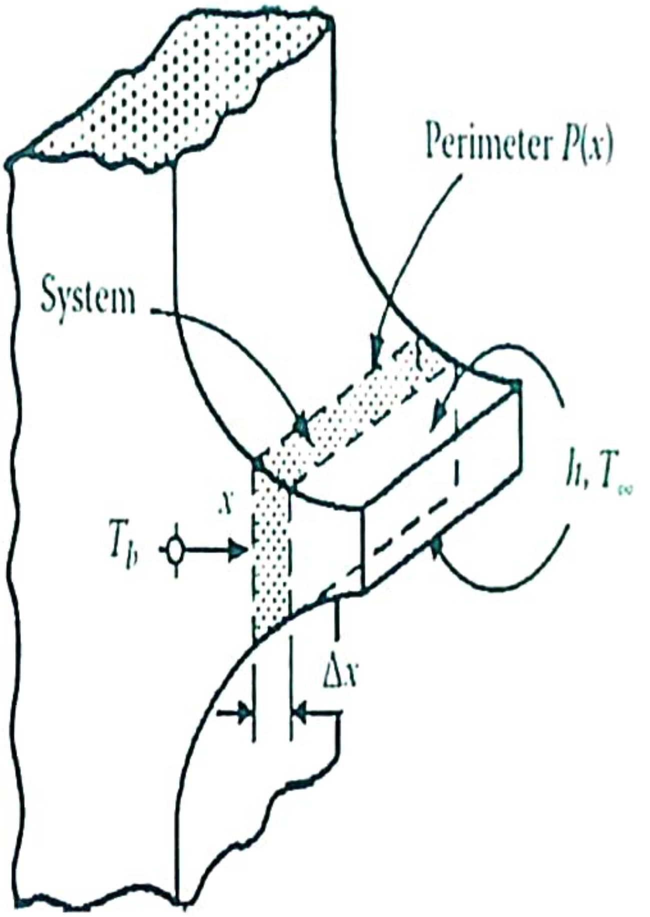
$$\overline{T}(\overline{x}=1)=\overline{T}_t, \qquad \text{fixed tip } T$$

$$\left.\frac{d\overline{T}}{d\overline{x}}\right|_{\overline{x}=1}=0, \qquad \text{insulated tip}$$

$$-\left.\frac{d\overline{T}}{d\overline{x}}\right|_{\overline{x}=1}=Bi_t,\overline{T}(\overline{x}=1) \qquad \text{tip convection}$$

No. Each term has unit of area. Further reduction cannot be made until the specific form of A_c has been set.

Fin Equation: Why 1D?

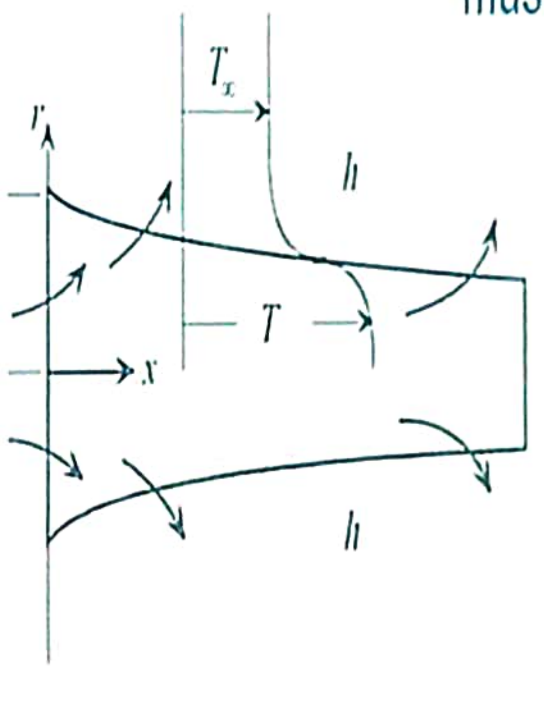


If y denotes the direction normal to the surface area, the energy balance at the surface would give

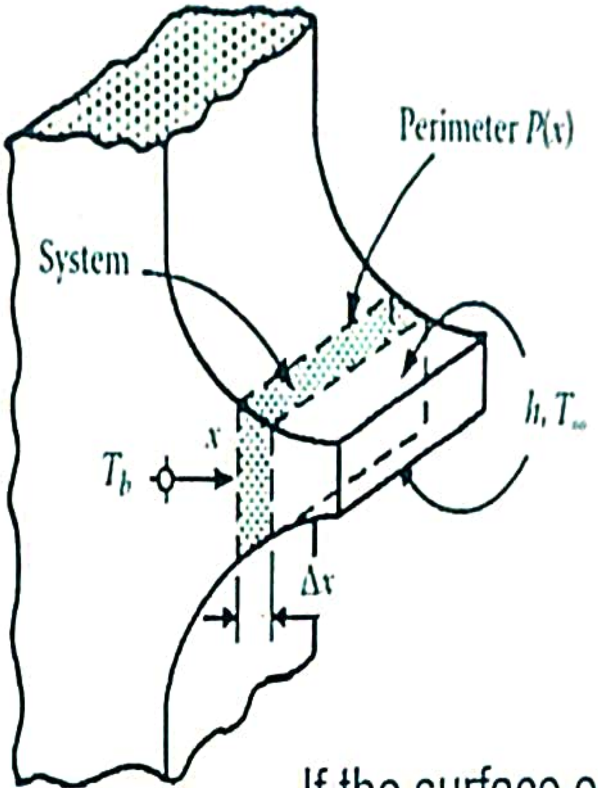
$$-k\left.\frac{\partial T}{\partial y}\right|_{y=b} = h(T - T_\infty)$$

where b denotes the thickness of the fin at a particular position x.

Thus, a temperature gradient must exist in the y direction



Fin Equation: Why 1D?



$$-k\frac{\partial T}{\partial y}\bigg|_{y=b} = h(T - T_\infty)$$

Approximate the derivative as: $\dfrac{\partial T}{\partial y}\bigg|_{y=b} \approx \Delta T/b,$

ΔT represents the average temperature difference across the fin in the y direction.

If the surface energy balance is divided by $T_B - T_\infty$ and rearranged,

$$\Delta\overline{T} = \frac{\Delta T}{T_B - T_\infty} \approx \frac{hb}{k}\overline{T} = Bi_b\overline{T}$$

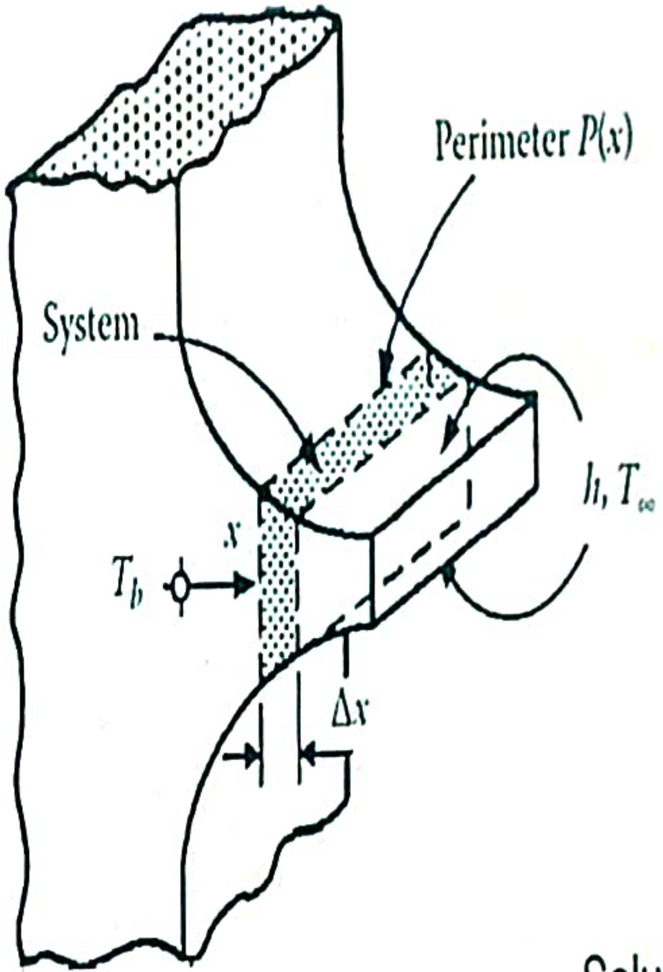
For the 1–D assumption to be correct we would expect that $\Delta\overline{T} \ll \overline{T}$,

i.e., the variation in temperature in the y direction is much smaller than the variation in the x direction. In other words, $Bi_b \ll 1$.

Consider aluminum fin of thickness 1 cm $\quad k \approx 400\ \mathrm{W/m{\cdot}K}.$

$h \approx 10\ \mathrm{W/m^2{\cdot}K}$

Fin Equation: Uniform Cross Section



$$\frac{d}{d\bar{x}} A_c \frac{d\overline{T}}{d\bar{x}} - \frac{hPL^2}{k}\overline{T} = 0$$

If A_c is constant:

$$\overline{T}'' - N^2\overline{T} = 0$$

$$N^2 = \frac{hPL^2}{kA_C}$$

Solution: $$\overline{T} = Ae^{N\bar{x}} + Be^{-N\bar{x}}$$

For adiabatic fin-tip:

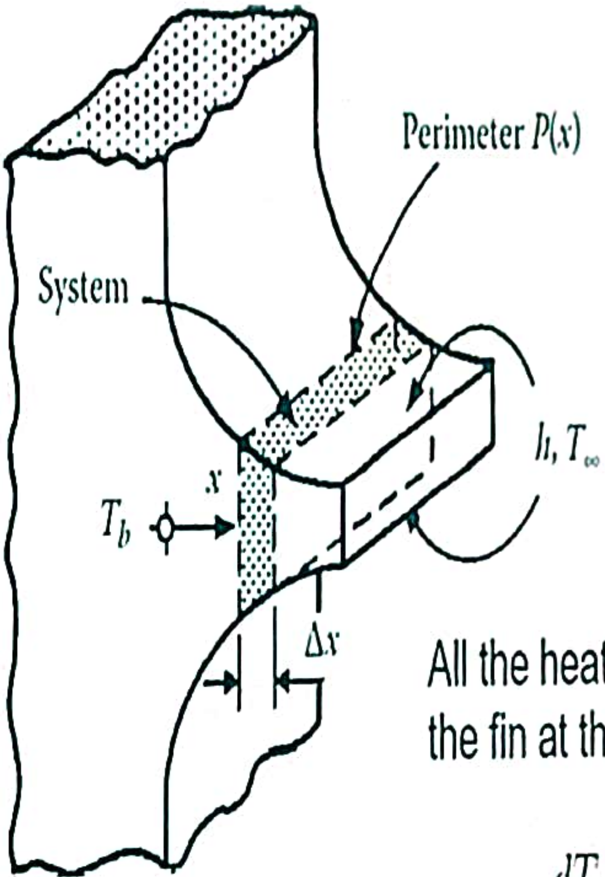
$$A = e^{-N} / \left(e^{N} + e^{-N}\right)$$

$$B = 1 - A = e^{N} / \left(e^{N} + e^{-N}\right)$$

$$\overline{T} = \frac{e^{N(1-\bar{x})} + e^{-N(1-\bar{x})}}{e^{N} + e^{-N}}$$

$$= \frac{\cosh\left[N(1-\bar{x})\right]}{\cosh(N)}$$

Fins: Uniform Cross Section: Heat Removal



For adiabatic fin-tip:

$$\overline{T} = \frac{e^{N(1-\overline{x})} + e^{-N(1-\overline{x})}}{e^{N} + e^{-N}}$$

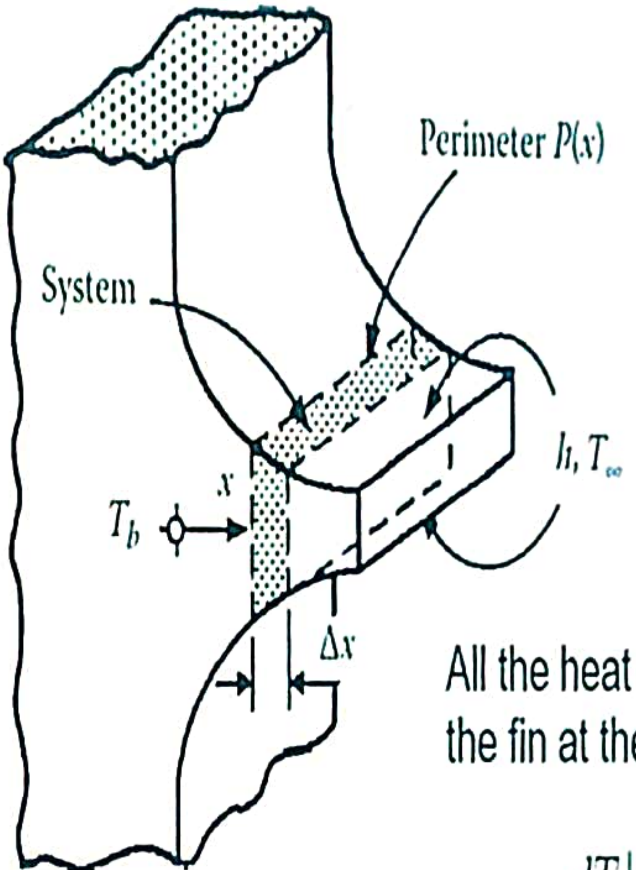
$$= \frac{\cosh[N(1-\overline{x})]}{\cosh(N)}$$

All the heat removed from the fin must be transported into the fin at the base by conduction. This gives

$$q = -kA_{C,B}\left.\frac{dT}{dx}\right|_{x=0} = -\frac{kA_{C,B}(T_B - T_\infty)}{L}\left.\frac{d\overline{T}}{d\overline{x}}\right|_{\overline{x}=0}$$

$$q = \frac{kA_C(T_B - T_\infty)}{L} N \tanh(N) = \sqrt{hPkA_C}\,(T_B - T_\infty)\tanh(N)$$

Fins: Uniform Cross Section: Heat Removal



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$$\overline{T} = \frac{e^{N(1-\overline{x})} + e^{-N(1-\overline{x})}}{e^{N} + e^{-N}}$$
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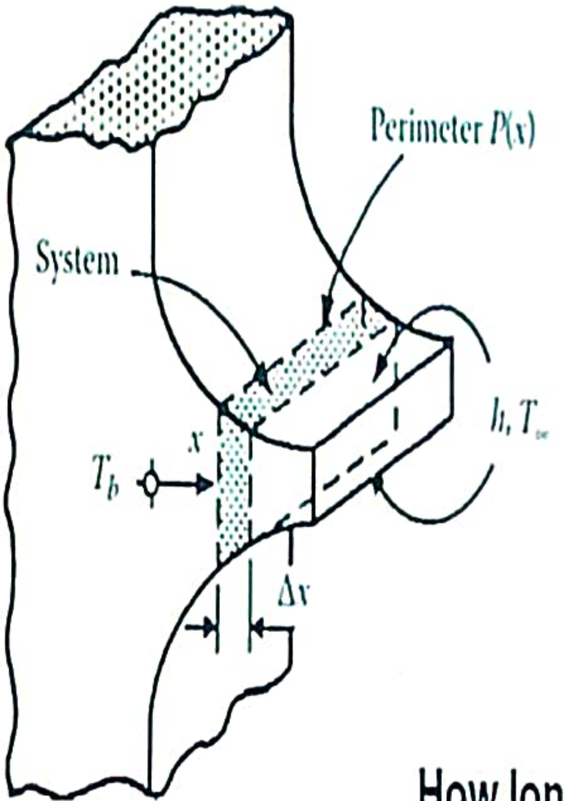
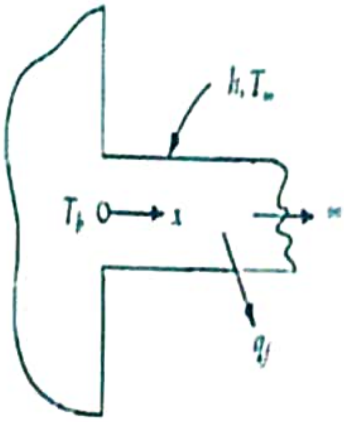
$$q = \frac{kA_C(T_B - T_\infty)}{L} N \tanh(N) = \sqrt{hPkA_C}\,(T_B - T_\infty)\tanh(N)$$

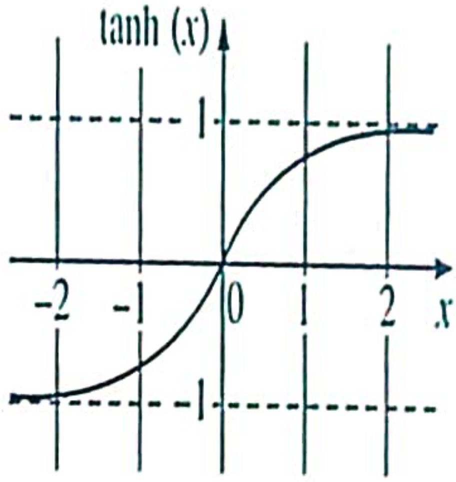
$$N^2 = \frac{hPL^2}{kA_C} \qquad \tanh(N) \to 1 \text{ for } N \gg 1,$$

Longer the fin – higher is the heat removal.
How long is LONG?

Fins: Uniform Cross Section: Heat Removal: Long Fin







How long is LONG?

For adiabatic fin-tip:

tanh(3) = 0.995

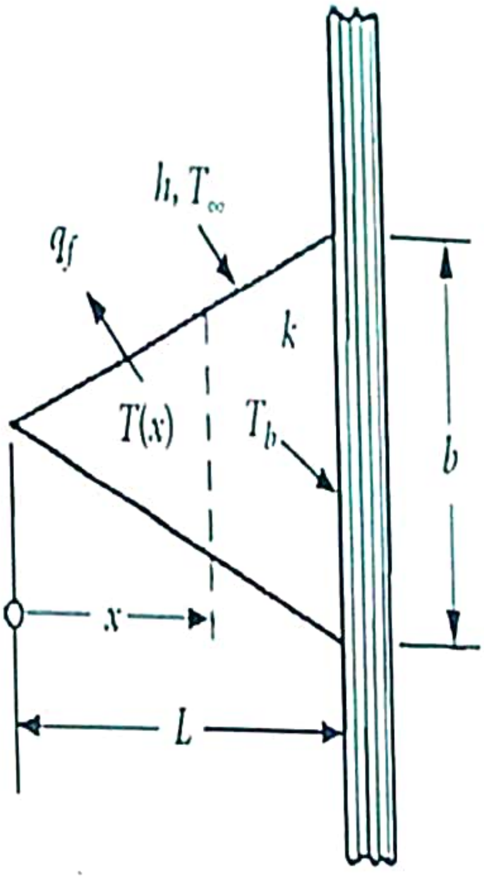
$$q = \frac{kA_C(T_B - T_\infty)}{L} N \tanh(N) = \sqrt{hPkA_C}\,(T_B - T_\infty)\tanh(N)$$

Consequently a fin with N > 3 is essentially 'infinite' in length. Adding additional length to the fin (and thus increasing N) will not significantly increase the heat transfer from the fin.

From a design viewpoint, Rule of Thumb: N > 2 to 2.5.

Fins: Non-Uniform Cross Section: Triangular Fin

Fins of non–uniform cross section can usually transfer more heat for a given mass than those of a constant cross section.



Width of fin = l

$$\frac{d}{dx}\left[A(x)\frac{dT}{dx}\right]-\frac{hP(x)}{k}(T-T_\infty)=0$$

Define: $\theta(x) = T(x) - T_\infty$,

$$\frac{d}{dx}\left[A(x)\frac{d\theta}{dx}\right]-\frac{hP(x)}{k}\theta=0$$

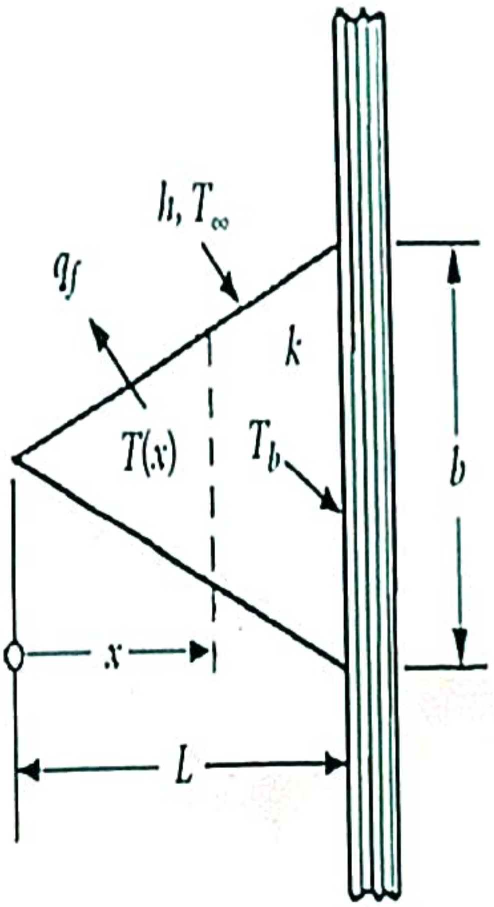
$$A(x)=\frac{bx}{L}l \quad \text{and} \quad P(x)=2\left(\frac{bx}{L}+l\right)$$

If we assume that $b \ll l$, then $P(x) \cong 2l$.

$$\frac{d}{dx}\left(x\frac{d\theta}{dx}\right)-m^2\theta=0$$

where $m^2 = 2hL/kb$.

Fins: Non-Uniform Cross Section: Triangular Fin



$$\frac{d}{dx}\left(x\frac{d\theta}{dx}\right) - m^2\theta = 0$$

where $m^2 = 2hL/kb$.

$$x^2\frac{d^2\theta}{dx^2} + x\frac{d\theta}{dx} - m^2 x\theta = 0$$

Multiplying both sides by x

Define: $\eta = \sqrt{x}$

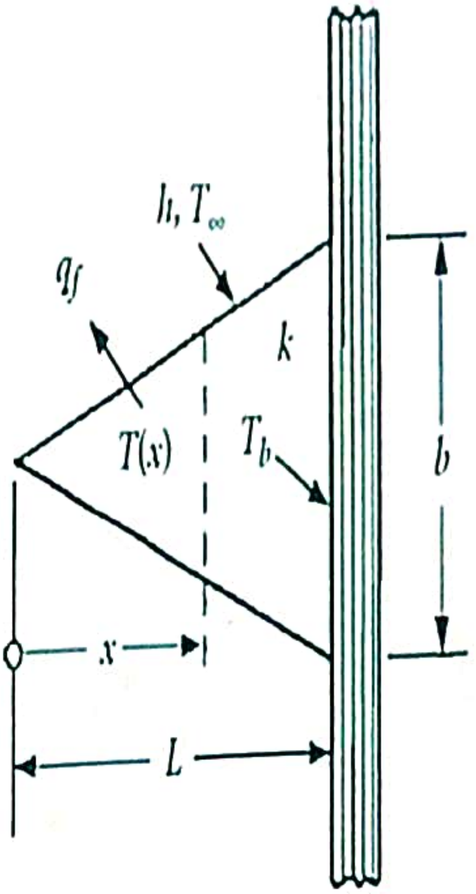
$$\eta^2\frac{d^2\theta}{d\eta^2} + \eta\frac{d\theta}{d\eta} - 4m^2\eta^2\theta = 0$$

Modified Bessel Equation

Solution: $\theta(\eta) = C_1 I_0(2m\eta) + C_2 K_0(2m\eta)$

$$\theta(x) = C_1 I_0(2m\sqrt{x}) + C_2 K_0(2m\sqrt{x})$$

Fins: Non-Uniform Cross Section: Triangular Fin



Solution: $\theta(\eta) = C_1 I_0(2m\eta) + C_2 K_0(2m\eta)$

$$\theta(x) = C_1 I_0(2m\sqrt{x}) + C_2 K_0(2m\sqrt{x})$$

$$T(0) = finite \Rightarrow \theta(0) = finite$$

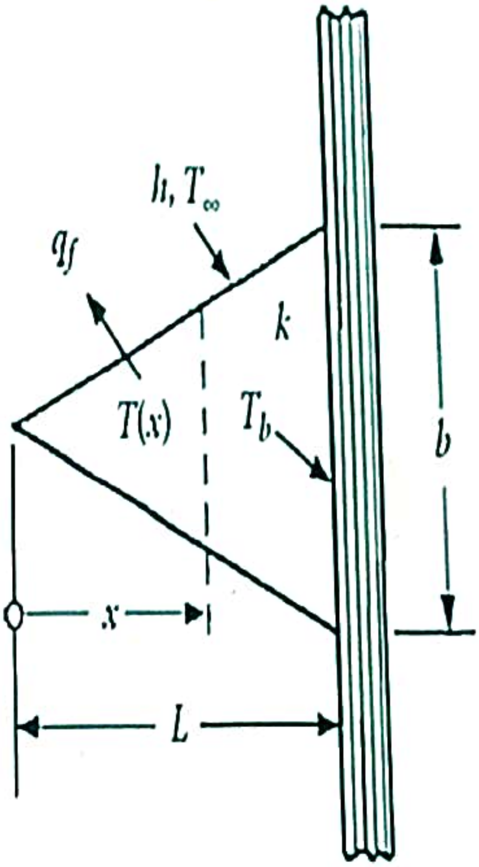
$$T(L) = T_b \Rightarrow \theta(L) = T_b - T_\infty = \theta_b$$

Since $K_0(0) \rightarrow \infty$, $C_2 = 0.$

$$C_1 = \frac{\theta_b}{I_0\left(2m\sqrt{L}\right)}$$

$$\frac{\theta(x)}{\theta_b} = \frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{I_0\left(2m\sqrt{x}\right)}{I_0\left(2m\sqrt{L}\right)}$$

Triangular Fins: Rate of Heat Transfer



$$\frac{\theta(x)}{\theta_b}=\frac{T(x)-T_\infty}{T_b-T_\infty}=\frac{I_0\left(2m\sqrt{x}\right)}{I_0\left(2m\sqrt{L}\right)}$$

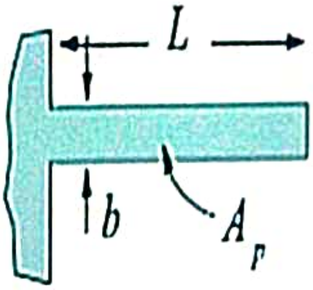
$$q_f=kA\left(\frac{dT}{dx}\right)_{x=L}=kA\left(\frac{d\theta}{dx}\right)_{x=L}$$

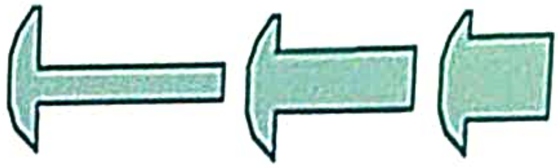
$$\frac{d}{dx}\left[I_0(\alpha x)\right]=\alpha I_1(\alpha x)$$

$$q_f=l\sqrt{2hkb}\,\theta_b\frac{I_1(2m\sqrt{L})}{I_0(2m\sqrt{L})}$$

Fin Optimization: Rectangular Fin

For a given fin shape, fin material, and convection conditions, there exists an optimized design which transfers the maximum amount of heat for a given mass of the fin.





Consider: Adiabatic Fin Tip

$$q = \sqrt{hPkA_c}(T_b - T_\infty)\tanh N \qquad N^2 = \frac{hPL^2}{kA_c}$$

For a long fin ($W \gg b$), $P \approx 2W$ and $A_c = bW$. Thus:

$$q' = \frac{q}{W} = \sqrt{2bhk}\,(T_b - T_\infty)\tanh N \qquad N^2 = \frac{2hL^2}{kb}$$

Fin Optimization: Rectangular Fin

$$q' = \frac{q}{W} = \sqrt{2bhk}\,(T_b - T_\infty)\tanh N \qquad N^2 = \frac{2hL^2}{kb}$$

The length L can be eliminated using $A_P = bL$. The formula for N becomes

$$N^2 = \frac{2hA_p^2}{kb^3} \quad \Rightarrow \quad b = \left(\frac{2hA_p^2}{kN^2}\right)^{1/3}$$

$$q' = \left(4h^2k\,A_P\right)^{1/3}(T_b - T_\infty)\,N^{-1/3}\tanh N$$

$$f(N) = N^{-1/3}\tanh N$$

Set $\frac{df}{dN} = 0$ => $\cosh N \sinh N - 3N = 0$

Solve for N