Ans. 1

1 $\dot{x} = Ax + b$.

$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ $\quad \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$ $\quad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$(1+\lambda^2-2\lambda)-4=0$

$\lambda^2-2\lambda-3=0 \quad \lambda^2-3\lambda+\lambda-3=0$

$\lambda(\lambda-3)+1(\lambda-3)=0$

$\lambda = 3, -1$

$u_1 \Rightarrow \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$

$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0 \quad \begin{bmatrix} -1 \\ 1 \end{bmatrix} = u^2$

$u' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Eigenvalues are orthogonal. Forms the basis.

(a) Represent b in terms of this basis:

$b = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\beta_1 = \frac{\langle u'. b \rangle}{\langle u'. u' \rangle} \quad \beta_2 = \frac{\langle u^2. b \rangle}{\langle u_2. u_2 \rangle}$

$b = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad = \frac{4}{2} = 2 \quad = \frac{-2}{2} = -1$

$$c_i(t) = c_i(0)e^{\lambda_i t} + e^{\lambda_i t}\int_0^t e^{-\lambda_i \tau} \beta_i(\tau)\, d\tau$$

$x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1^0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2^0 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad c_1^0 = \frac{\langle u'. x^0 \rangle}{\langle u', u' \rangle} \quad c_2 = \frac{\langle u^2, x^0 \rangle}{\langle u^2, u^2 \rangle}$

$= \frac{1}{2} \quad -\frac{1}{2}$

$c_1^0 = \frac{1}{2} \quad c_2^0 = -\frac{1}{2}$

$\therefore c_1(t) = \frac{1}{2}\cdot e^{3t} + e^{3t}\int_0^t e^{-3\tau}. 2. d\tau = \frac{1}{2}e^{3t} + e^{3t}\cdot 2\left[\frac{e^{-3\tau}}{-3}\right]_0^t$

$c_2(t) = -\frac{1}{2}e^{-t} + e^{-t}\int_0^t e^{-\tau}.(-1)\, d\tau = -\frac{1}{2}e^{-t} + e^{-t}(-1)\left[\frac{e^{-\tau}}{-1}\right]_0^t$

$c_1(t) = \frac{1}{2}e^{3t} + \frac{2e^{3t}}{-3}(e^{-3t}-1) = \frac{1}{2}e^{3t} - \frac{2}{3} + \frac{2}{3}e^{3t} = \frac{7}{3}e^{3t} - \frac{2}{3}$

$c_2(t) = -\frac{1}{2}e^{-t} + e^{-t}(e^{-t}-1) = -\frac{1}{2}e^{-t} + 1 - e^{-t} = 1 - \frac{3}{2}e^{-t}$

$$\begin{bmatrix} 5 & 2 & 2 \\ 1 & 4 & 4 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 0$$

$$5u_1 + 2u_2 + 2u_3 = 0$$
$$u_1 + 4u_2 + 4u_3 = 0$$
$$3u_1 + 3u_2 + 3u_3 = 0$$

$$u = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$A^t = \begin{bmatrix} -4 & 1 & 3 \\ 2 & -5 & 3 \\ 2 & 4 & -6 \end{bmatrix}$$

$$-4v_1 + v_2 + v_3 = 0$$
$$2v_1 - 5v_2 + 3v_3 = 0$$
$$2v_1 + 4v_2 - 6v_3 = 0$$

Set $v_3 = 1$

$$-4v_1 + v_2 = -1$$
$$\left\{ \begin{array}{l} 2v_1 - 5v_2 = -3 \\ 2v_1 + 4v_2 = 6 \end{array} \right.$$

$$\left. \begin{array}{l} 4v_1 - v_2 = 3 \\ 4v_1 - v_2 = 1 \end{array} \right\} \downarrow$$

Inconsistent.

Hence $\underline{v_3 = 0}$.

$$\left. \begin{array}{l} -4v_1 + v_2 = 0 \\ 2v_1 - 5v_2 = 0 \\ 2v_1 + 4v_2 = 0 \end{array} \right\} \text{Consistent}$$

$$\hat{x} = \left(\frac{7}{3}e^{3t} - \frac{2}{3}\right)\begin{bmatrix}1\\1\end{bmatrix} + 1 - \frac{3}{2}e^{-t}\begin{bmatrix}-1\\1\end{bmatrix}$$

$$= \begin{bmatrix}7/3\,e^{3t} - 2/3 - 1 + \frac{3}{2}e^{-t}\\ 7/3\,e^{3t} - 2/3 + 1 - \frac{3}{2}e^{-t}\end{bmatrix} = \begin{bmatrix}7/3\,e^{3t} - 5/3 + \frac{3}{2}e^{-t}\\ 7/3\,e^{3t} + \frac{1}{3} - \frac{3}{2}e^{-t}\end{bmatrix}.$$

Ques. 2

(a) $A = \begin{bmatrix} -4 & 2 & 2 \\ 1 & -5 & 4 \\ 3 & 3 & -6 \end{bmatrix}$

$$\det(A - \lambda I) = 0$$

$$\det \begin{pmatrix} -4-\lambda & 2 & 2 \\ 1 & -5-\lambda & 4 \\ 3 & 3 & -6-\lambda \end{pmatrix} = 0$$

$$(-4-\lambda)\{(-5-\lambda)(-6-\lambda)-12\} - 2\{1(-6-\lambda)-12\} + 2\{3 - 3(-5-\lambda)\} = 0$$

$$(-4-\lambda)\{30+5\lambda+6\lambda+\lambda^2-12\} - 2(-6-\lambda-12) + 2(3+15+3\lambda) = 0$$

$$(-4-\lambda)(\lambda^2+11\lambda+18) + 2(\lambda+18) + 2(3\lambda+18) = 0$$

$$-(4\lambda^2+44\lambda+72+\lambda^3+11\lambda^2+18\lambda) + 2\lambda + 36 + 6\lambda + 36 = 0$$

$$-(\lambda^3+15\lambda^2+62\lambda+72) + 8\lambda + 72 = 0$$

$$-\lambda^3 - 15\lambda^2 - 62\lambda - 72 + 8\lambda + 72 = 0$$

$$-\lambda^3 - 15\lambda^2 - 54\lambda = 0$$

$$\lambda^3 + 15\lambda^2 + 54\lambda = 0$$

$$\lambda(\lambda^2 + 15\lambda + 54) = 0$$

$$\lambda(\lambda+9)(\lambda+6) = 0$$

Eigen values are $\lambda_1 = 0$, $\lambda_2 = -6$, $\lambda_3 = -9$

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(b) $A = \begin{bmatrix} -4 & 2 & 2 \\ 1 & -5 & 4 \\ 3 & 3 & -6 \end{bmatrix}$ $A^T = \begin{bmatrix} -4 & 1 & 3 \\ 2 & -5 & 3 \\ 2 & 4 & -6 \end{bmatrix}$

For A^T, eigen values, $(A^T - \lambda I) = 0$, here $\lambda = 0$

for $\lambda = 0$

$-4v_1 + v_2 + 3v_3 = 0$ — ①

$2v_1 \cancel{+} -5v_2 + 3v_3 = 0$ — ②

$2v_1 + 4v_2 - 6v_3 = 0$ — ③

On solving equation ① & ③

& ② & ③

$v_1 = 1, v_2 = 1, v_3 = 1$

Eigen vector $(1, 1, 1)$

— for another $\lambda_2 = -6$ $(A^T + 6I) = 0$

Equations become

$2v_1 + v_2 + 3v_3 = 0$

$2v_1 + v_2 + 3v_3 = 0$

$2v_1 + 4v_2 + 0 = 0$

On solving above equations

$v_1 = 2, v_2 = -1, v_3 = -1$

Eigen vector $(2, -1, -1)$

For eigenvalue $\lambda_3 = -9$, $(A^T + 9I) = 0$

Equations become

$$5v_1 + v_2 + 3v_3 = 0$$

$$2v_1 + 4v_2 + 3v_3 = 0$$

$$2v_1 + 4v_2 + 3v_3 = 0$$

On solving above equations

$$v_1 = 1, \ v_2 = 1, \ v_3 = -2.$$

Eigen vector $(1, 1, -2)$

For matrix A, Eigen value $\lambda = 0$

$$(A + 0I) = 0$$

$$-4v_1 + 2v_2 + 2v_3 = 0$$

$$v_1 - 5v_2 + 4v_3 = 0$$

$$3v_1 + 2v_2 - 6v_3 = 0$$

On solving above equations, we get $v_1 = 1, v_2 = 1, v_3 = 1$

Eigen vector $(1, 1, 1)$

Eigen value $\lambda = -6$, $A + 6I = 0$

$$2v_1 + 2v_2 + 2v_3 = 0$$

$$v_1 + v_2 + 4v_3 = 0$$

$$3v_1 + 3v_2 + 0 = 0$$

On solving above equations, we get $v_1 = 1, v_2 = -1, v_3 = 0$

Eigen value, $\lambda = -9$, $(A + 9I = 0)$

$$5v_1 + 2v_2 + 2v_3 = 0$$

$$v_1 + 4v_2 + 4v_3 = 0$$

$$3v_1 + 3v_2 + 3v_3 = 0$$

On solving above equations, we get $v_1 = 0, v_2 = 1, v_3 = -1$

Eigen vectors $(0, 1, -1)$

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Eigen value —	0	−6	−9
Matrix			
A	1, 1, 1	1, −1, 0	0, 1, −1
A^T	1, 1, 1	2, −1, −1	1, 1, −2

Biorthogonality states that all eigen vectors of A multiplied by all eigen vectors of A^T would be zero except for same eigen values. (Theorem 3.5)

$$A \cdot A^T = \text{eigen value } (0, -6)$$

$$A \cdot A^T = (1, 1, 1)(2, -1, -1)$$

$$= 2 - 1 - 1 = 2 - 2 = 0.$$