Transform Calculus (MA20202) End-sem Spring 2023 question and solution

1. a) Find the Laplace transform of the function $f(t) = \frac{\cos\sqrt{t}}{\sqrt{t}}$

   b) Find $L^{-1}\left\{\frac{e^{4-3s}}{(s+4)^{5/2}}\right\}$

   c) Find the Fourier Series representation of the periodic function

   $$f(x) = \begin{cases} 0 & -\pi \le x \le 0 \\ \sin x & 0 \le x \le \pi \end{cases}$$

   d) Find the Half-range cosine series of the function $f(x) = x^2$ in $(0, \pi)$ and hence find the sum of the series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$.

[4+4+4+5=17 M]

2. a) Find the complex form of the Fourier Series expansion of the periodic function $f(x) = e^{-x}$ in the interval $-1 < x < 1$.

   b) Derive the complex form of the Fourier Integral of the function $f(x)$. Mention the sufficient conditions for its existence.

   c) Find the Fourier Sine Transform of the function $f(x)\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$ for all $0 \le x < \infty$.

   d) Solve the integral equation $\int_0^\infty f(x)\cos ax\, dx = \begin{cases} 1-a & 0 \le a \le 1 \\ 0 & for\ a > 1 \end{cases}$.
   Hence evaluate $\int_0^\infty \frac{sin^2 t}{t^2} dt$.

[4+4+3+5=16 M]

3. a) Use Laplace transform to solve the wave equation for a finite string

   $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < 1, \qquad t > 0$$

   subject to the conditions (i) $u(0, t) = 0$ (ii) $u(1, t) = 0$
   (iii) $u_t(x, 0) = 0$ (iv) $u(x, 0) = \sin(\pi x)$

b) Apply appropriate Fourier transform with respect to x to solve the one dimensional heat equation

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} \qquad x > 0, \qquad t > 0$$

with (i) $u(0,t) = 0$ (ii) $u(x,0) = e^{-x}$ (iii) both u and $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$.

You can keep the final answer in integration form.

c) Use Fourier transform technique to solve the one-dimensional wave equation for an infinite string

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad -\infty < x < \infty, \qquad t > 0$$

subject to (i) $u(x,0) = e^{-|x|}$ (ii) $u_t(x,0) = 0$ (iii) both u and $\frac{\partial u}{\partial x} \to 0$ as $|x| \to \infty$. You can keep the final answer in integration form.

[5+6+6=17 M]

Solution:

1.a) $F(t) = sin\sqrt{t}\,,\ F'(t) = \frac{cos\sqrt{t}}{2\sqrt{t}},\ F(0) = 0$ [1 M]

Therefore, $L\{F'(t)\} = L\left\{\frac{cos\sqrt{t}}{2\sqrt{t}}\right\}$

$= sf(s) - F(0).$ [1 M]

$= \frac{1}{2}\sqrt{\frac{\pi}{s}}\, e^{-\frac{1}{4s}}$ [1 M]

Hence $L\left\{\frac{cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}}\, e^{-\frac{1}{4s}}$ [1 M]

b) $L^{-1}\left\{\frac{1}{(s+4)^{\frac{5}{2}}}\right\} = e^{-4t}L^{-1}\left\{\frac{1}{(s)^{\frac{5}{2}}}\right\}$ [1 M]

$= e^{-4t}\frac{t^{3/2}}{\Gamma(\frac{5}{2})} = \frac{4e^{-4t}.t^{3/2}}{3\sqrt{\pi}}$ [1 M]

$$L^{-1}\left\{\frac{e^{4-3s}}{(s+4)^{\frac{5}{2}}}\right\} = e^4 L^{-1}\left\{\frac{e^{-3s}}{(s+4)^{\frac{3}{2}}}\right\}$$ [1 M]

$= \frac{e^4.4e^{-4(t-3)}.(t-3)^{3/2}}{3\sqrt{\pi}}$ when $t > 3$ [1 M]

0 otherwise

or

$$= \frac{e^4.4e^{-4(t-3)}.(t-3)^{3/2}}{3\sqrt{\pi}} \; H(t-3)$$ where H is the Heaviside unit step function

c) $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}, a_1 = 0.$$ [1 M]

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_0^{\pi} \sin x \cos nx\, dx$$

$$= -\frac{1}{\pi}.\frac{1+(-1)^n}{n^2-1} \; for\; n > 1.$$ $[1\ M]$

$$a_n = \begin{cases} 0 \;\; if\; n\; is\; odd \\ -\frac{2}{\pi}.\frac{1}{n^2-1} \;\; if\; n\; is\; even \end{cases}$$

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin^2 x\, dx = \frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin x \sin nx\, dx = \frac{1}{2\pi}\int_0^{\pi} [\cos(n-1)x - \cos(n+1)x]dx$$

$$= 0.$$

[1 M]

Therefore, $f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2-1}\cos 2nx$ [1M]

d) $a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} x^2 dx = \frac{\pi^2}{3}$ [1 M]

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx \, dx$$

$$\Longrightarrow a_n = \frac{2}{\pi} \left[\frac{x^2 \sin nx}{n} + \frac{2x \cos nx}{n^2} - \frac{2 \sin nx}{n^3} \right]_0^{\pi} \qquad [1 \text{ M}]$$

$$= \frac{2}{\pi} . \frac{2x \cos nx}{n^2} = \frac{4(-1)^n}{n^2} \qquad [1\ M]$$

$$\therefore\ x^2 = a_0 + \sum_{n=1}^{\infty} a_n \cos nx = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

[1 M]

$$x^2 = \frac{\pi^2}{3} + 4\left[- \frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \cdots \right]$$

For $x = 0,\ 0 = \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots \right].$

$$\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots \right] = \frac{\pi^2}{12} \qquad [1 \text{ M}]$$

2a). $f(x) \sim \sum_{n=-\infty}^{\infty} C_n^* e^{\frac{in\pi x}{l}}$

$C_n^* = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-\frac{in\pi x}{l}}\, dx$ — (1M)

$f(x) = e^{-x}, \; -1<x<1$

$2l = 2, \; l = 1$

$= \frac{1}{2}\int_{-1}^{1} e^{-x}\cdot e^{-in\pi x}\, dx = \frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi)x}\, dx$

$= \frac{1}{2} \left. \frac{e^{-(1+in\pi)x}}{-(1+in\pi)} \right|_{-1}^{1} = \frac{1}{2}\left\{\frac{-1}{(1+in\pi)}\left[e^{-1}e^{-in\pi} - e^{1}\cdot e^{in\pi}\right]\right\}$

$e^{in\pi} = (e^{i\pi})^n = (-1)^n$

$1 - i^2 n^2\pi^2 \; (= 1 + n^2\pi^2)$

$= \frac{1}{2}\frac{(-1)^n(1-in\pi)}{(1+n^2\pi^2)}\left[(e - e^{-1})\right]$ — (2M)

$\therefore f(x) = e^{-x} \sim \sum_{n=-\infty}^{\infty} \frac{1}{2}(-1)^n (e - \frac{1}{e})\frac{(1-in\pi)}{1+n^2\pi^2}\cdot e^{in\pi x}$ — (1M)

Total — 4M)

Complex form of the Fourier Integral:

2b) Start with the F.S of $f_L(x) \sim \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{L}\sum_{n=1}^{\infty}\left[\cos w_n x \cdot \int_{-L}^{L} f_L(v)\cos w_n v\, dv + \sin w_n x \cdot \int_{-L}^{L} f_L(v)\sin w_n v\, dv\right]$ — (1M)

Where $\Delta w = w_{n+1} - w_n = \pi/L$.

As $L \to \infty$, Using the suff. condition of Absolute integrability of the resulting non-periodic fn $f(x) = \lim_{L\to\infty} f_L(x)$ on $(-\infty, \infty)$),

obtain $f(x) = \int_0^{\infty}\left[A(w)\cos wx + B(w)\sin wx\right] dw$

Where $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\, dv$ and $B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\, dv$ (1M)

Then obtain $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)\cos(wx - wv)\, dv\right] dw$

and $0 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)\sin(wx - wv)\, dv\right] dw$

adding these two: we get $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\, e^{iw(x-v)}\, dv\, dw$ (1M)

Suff. Conditions: 1. $f(x)$ is piecewise cont. on every finite interval

2. $f(x)$ is absolutely integrable on the x-axis. (1M)

2 d) $f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(\alpha) \cdot \cos \alpha x \cdot d\alpha$

$= \sqrt{\frac{2}{\pi}} \int_0^1 \sqrt{\frac{2}{\pi}} (1-\alpha) . \cos\alpha x \, d\alpha = \frac{2}{\pi x^2}(1-\cos x)$

$\therefore f(x) = \frac{2}{\pi x^2}(1-\cos x)$ — (3M)

Now $\frac{2}{\pi} \int_0^\infty \frac{1-\cos x}{x^2} \cos\alpha x \, dx = \frac{4}{\pi} \int_0^\infty \frac{\sin^2(\frac{x}{2})}{x^2} \cos\alpha x dx$

$\because \cos x = 1 - 2\sin^2\frac{x}{2}$

$\boxed{\text{No Part Marking}}$

$= \frac{1}{\pi} \int_0^\infty \frac{\sin^2(x/2)}{(x/2)^2} \cos\alpha x \, dx.$

Taking $\alpha = 0$, we get $\frac{1}{\pi} \int_0^\infty \frac{\sin^2(x/2)}{(x/2)^2} dx$

put $\frac{x}{2} = t$, $dx = 2dt$.

$\therefore \frac{2}{\pi} \int_0^\infty \frac{\sin^2 t}{t^2} dt = 1 - 0 = 1 \quad \left(\begin{matrix} \because \alpha = 0 \\ 1-\alpha = 1 \end{matrix}\right)$

$\therefore \int_0^\infty \frac{\sin^2 t}{t^2} dt = \pi/2$ — (2M)

Total 5M.

2 e) $\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin\frac{x}{2} \cos\frac{x}{2} \cdot \sin\alpha x \, dx$

$= \sqrt{\frac{1}{2\pi}} \int_0^\infty f(x) \sin x . \sin\alpha x \, dx$ — (1M)

$= \frac{1}{\sqrt{2\pi}} \int_0^\infty f(x) \left[\cos(\alpha+1)x \mp \cos(\alpha-1)x\right] dx$ — (1M)

$= \frac{1}{4} \left[F_c(\alpha+1) \mp F_c(\alpha-1)\right] dx$ — (1M)

(3M).

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3. (a) Use L.T. to solve the wave equation for a finite string $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ $\quad 0<x<1, \quad t>0$

subject to the conditions (i) $u(0,t)=0$ (ii) $u(1,t)=0$ (iii) $u_t(x,0)=0$ (iv) $u(x,0) = \sin \pi x$ $\boxed{5M}$

Sol^n: $s^2\bar{u} - s\,u(x,0) - u_t(x,0) = \frac{d^2\bar{u}}{dx^2}$

$$\Rightarrow s^2\bar{u} - s\sin \pi x = \frac{d^2\bar{u}}{dx^2}$$

$$\Rightarrow \frac{d^2\bar{u}}{dx^2} - s^2\bar{u} = -s\sin\pi x \qquad 1M$$

$$\therefore \bar{u} = Ae^{sx} + Be^{-sx} - s\frac{1}{D^2-s^2}\sin \pi x$$

$$= Ae^{sx} + Be^{-sx} - s\left(\frac{1}{-\pi^2 - s^2}\right)\sin\pi x$$

$$= Ae^{sx} + Be^{-sx} + \frac{s}{s^2+\pi^2}\sin\pi x \qquad 1M$$

$u(0,t)=0 \quad \therefore A+B=0$

$u(1,t)=0 \quad Ae^{s}+Be^{-s}=0 \quad \therefore A=0 \qquad 1M$

$B=0 \qquad 1M$

$$\therefore \bar{u} = \frac{s\sin\pi x}{s^2+\pi^2}$$

$$\therefore u = \sin(\pi x)\cos(\pi t) \qquad 1M$$

3. (b) Apply appropriate F.T. w.r.t. x to solve the one-dimensional heat equation $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$ $x>0, t>0$

with (i) $u(0,t)=0$ (ii) $u(x,0)=e^{-x}$ (iii) both u and $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$. $\boxed{6M}$

Sol^n: u at $x=0$ is given. So F.S.T. is appliable. 1M

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial u}{\partial t} \sin\alpha x \, dx = 2\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin\alpha x \, dx$$

$$\frac{d}{dt}\hat{u}_s + 2\alpha^2 \hat{u}_s = 0 \quad \text{1M} \quad \therefore \hat{u}_s = Ae^{-2\alpha^2 t} \quad \text{1M}$$

$u(x,0)=e^{-x}$ $\quad \hat{u}_s(\alpha,0) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x} \sin\alpha x \, dx = \sqrt{\frac{2}{\pi}} \frac{\alpha}{1+\alpha^2}$

$$\therefore A = \sqrt{\frac{2}{\pi}} \frac{\alpha}{1+\alpha^2} \quad \text{1M}$$

$$\hat{u}_s(\alpha,t) = \sqrt{\frac{2}{\pi}} \frac{\alpha}{1+\alpha^2} e^{-2\alpha^2 t} \quad \text{1M}$$

$$\therefore u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{\alpha e^{-2\alpha^2 t} \sin\alpha x}{1+\alpha^2} d\alpha \quad \text{1M}$$

Zero marks awarded if F.C.T./exp.F.T. is applied because these are not applicable here

3.(c) Use F.T. to solve the one-dimensional wave equation for an infinite string $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ $-\infty < x < \infty, t>0$
s.t. (i) $u(x,0) = e^{-|x|}$ (ii) $u_t(x,0)=0$ (iii) $u, \frac{\partial u}{\partial x} \to 0$ as $|x| \to \infty$. $\boxed{6M}$

Sol^n: Applying F.T. w.r.t. x and simplifying

$$\frac{d^2\hat{u}(\alpha,t)}{dt^2} = -\alpha^2 \hat{u}(\alpha,t) \quad \text{1M}$$

$$\therefore \hat{u}(\alpha,t) = A(\alpha)\cos\alpha t + B(\alpha)\sin\alpha t \quad \text{1M}$$

$\frac{d}{dt}\hat{u}(\alpha,0)=0$ $\quad \therefore -A(\alpha)\alpha\sin\alpha t + B(\alpha)\alpha\cos\alpha t\Big|_{t=0} = 0$

$$\therefore \alpha B(\alpha)=0 \quad \therefore B(\alpha)=0 \quad \text{1M}$$

$$\hat{u}(\alpha,t) = A(\alpha)\cos\alpha t$$

$$\hat{u}(\alpha,0) = F[e^{-|x|}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-|x|} e^{i\alpha x} dx = \sqrt{\frac{2}{\pi}}\frac{1}{1+\alpha^2}$$

$\therefore A(\alpha) = \sqrt{\frac{2}{\pi}} \frac{1}{1+\alpha^2}$ 1M $\quad \hat{u}(\alpha,t) = \sqrt{\frac{2}{\pi}}\frac{1}{1+\alpha^2}\cos\alpha t$ 1M

$$\therefore u(x,t) = \frac{1}{\pi}\int_{-\infty}^\infty \frac{1}{1+\alpha^2}\cos\alpha t \, e^{-i\alpha x} d\alpha \quad \text{1M}$$

Solⁿ: (3c) Aliter

$$\frac{d^2\hat{u}}{dt^2} + \alpha^2 \hat{u}(\alpha,t) = 0$$ 1M

$$\hat{u} = Ae^{i\alpha t} + Be^{-i\alpha t}$$ 1M

$$\frac{\partial \hat{u}}{\partial t} = A(i\alpha)e^{i\alpha t} + B(-i\alpha)e^{-i\alpha t}$$

$$= i\alpha\left[Ae^{i\alpha t} - Be^{-i\alpha t}\right]$$

At $t=0$, $0 = i\alpha(A-B)$ $\therefore A = B$ 1M

Also at $t=0$, $u(x,0) = e^{-|x|}$

$$\therefore \hat{u}(\alpha,0) = F\left[e^{-|x|}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|x|}e^{i\alpha x}\,dx = \sqrt{\frac{2}{\pi}}\frac{1}{1+\alpha^2}$$

$\therefore A + B = \sqrt{\frac{2}{\pi}}\frac{1}{1+\alpha^2}$ $\therefore A = B = \frac{1}{\sqrt{2\pi}}\frac{1}{1+\alpha^2}$ 1M

$$\therefore \hat{u} = \frac{1}{\sqrt{2\pi}}\frac{1}{1+\alpha^2}\left[e^{i\alpha t} + e^{-i\alpha t}\right]$$ 1M

$$\therefore u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\frac{1}{1+\alpha^2}\left[e^{i\alpha t} + e^{-i\alpha t}\right]e^{-i\alpha x}\,d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\alpha(t-x)} + e^{-i\alpha(t+x)}}{1+\alpha^2}\,d\alpha$$ 1M

or any other equivalent form