Classification of Partial Differential Equations

Classification of second order equations:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G(x, y),$$

Where A, B, C are constant. It is said to be

hyperbolic if $B^2 - 4AC > 0$,

parabolic if $B^2 - 4AC = 0$,

elliptic if $B^2 - 4AC < 0$.

Unsteady state heat conduction (1D, 2D, 3D): Parabolic $\quad \rho c\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}$

Steady state heat Conduction (2D, 3D): Elliptic $\quad \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$

Wave equation: Hyperbolic $\quad \frac{\partial^2 f}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0 \quad$ v = velocity of wa[redacted]

Non-Fourier Heat Conduction: Finite Speed of Heat Propagation

For heat conduction in a homogeneous and isotropic medium, the Fourier law of heat conduction:

From: $\rho c \dfrac{\partial T}{\partial t} + \nabla \cdot q'' - q''' = 0$

$$\mathbf{q}(\mathbf{r},t) = -k\nabla T(\mathbf{r},t)$$

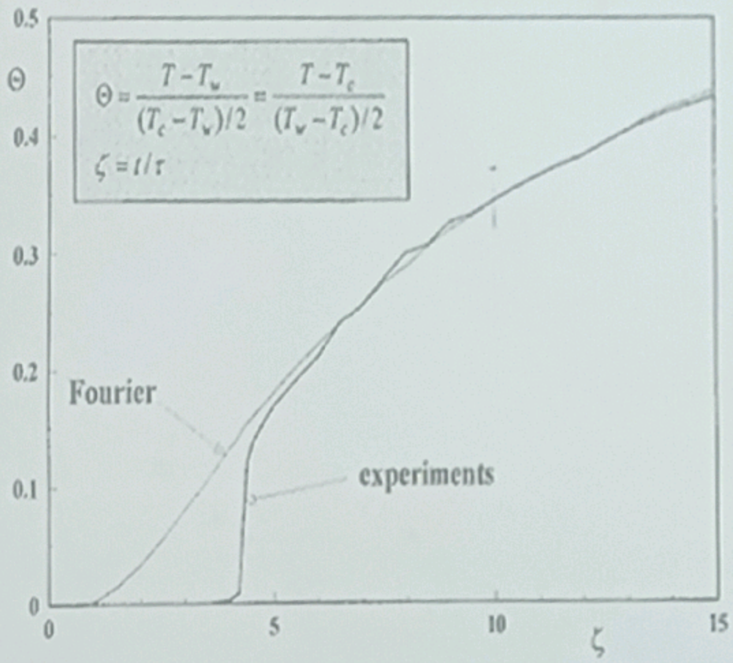
The relation between the heat flux **q** and the temperature gradient ∇T is called the *constitutive relation of heat flux*.

=> $\dfrac{1}{\alpha}\dfrac{\partial T}{\partial t} = \nabla^2 T + \dfrac{q'''}{k}$.

Parabolic heat-conduction equation

- The Fourier law of heat conduction is an early empirical law.
- It assumes that **q** and ∇T appear at the same time instant t and consequently implies that thermal signals propagate with an infinite speed

Non-Fourier Heat Conduction: Finite Speed of Heat Propagation



Mitra et al. (1995) "Experimental Evidence of Hyperbolic Heat Conduction in Processed Meat" ASME J. Heat Transfer, 117, pp. 568–573.

Two identical meat samples at different initial temperatures *Tc* , *Tw* (*c* = cold; *w* = warm) were brought into contact with each other.

One sample was refrigerated to *Tc* = 8.2°C and the other was left at room temperature of *Tw* = 23.1°C.

The above Figure shows the thermocouple reading at the position 6.3 mm deep in the initially warm part (at room temperature T_w = 23.1°C).

Non-Fourier Heat Conduction

Technology: Ultrafast pulse-laser heating on metal films ➔ heat conduction appears in the range of high heat flux and high unsteadiness.

Infinite heat propagation speed in the Fourier law becomes unacceptable. This has inspired the work of searching for new constitutive relations.

New constitutive relation proposed by Cattaneo (1958) and Vernotte (1958, 1961):

CV Constitutive Relation: $$\mathbf{q}(\mathbf{r},t) + \tau_0 \frac{\partial \mathbf{q}(\mathbf{r},t)}{\partial t} = -k\nabla T(\mathbf{r},t)$$

Here $\tau_0 > 0$ is a material property and is called the *relaxation time*.

Non-Fourier Heat Conduction

CV Constitutive Relation: $\mathbf{q}(\mathbf{r},t)+\tau_0\dfrac{\partial \mathbf{q}(\mathbf{r},t)}{\partial t}=-k\nabla T(\mathbf{r},t)$

Substitute **q** in: $\rho c\dfrac{\partial T}{\partial t}+\nabla\cdot \mathrm{q}''-q'''=0$

The corresponding heat-conduction equation:

$$\frac{1}{\alpha}\frac{\partial T}{\partial t}+\frac{\tau_0}{\alpha}\frac{\partial^2 T}{\partial t^2}=\nabla^2 T+\frac{1}{k}\left(q'''+\tau_0\frac{\partial q'''}{\partial t}\right)$$

This equation is of *hyperbolic* type, characterizes the combined diffusion and wave-like behavior of heat conduction, and predicts a *finite speed for heat propagation*:

$$V_{CV}=\sqrt{\frac{k}{\rho c\tau_0}}=\sqrt{\frac{\alpha}{\tau_0}}$$

Consider no heat generation

Standard Wave Equation:

$$\frac{\partial^2 q}{\partial t^2}=c^2\left(\frac{\partial^2 q}{\partial x^2}+\frac{\partial^2 q}{\partial y^2}+\frac{\partial^2 q}{\partial z^2}\right)$$

Non-Fourier Heat Conduction

CV Constitutive Relation: $$\mathrm{q}+\tau\frac{\partial \mathrm{q}}{\partial t}=-k\nabla T.$$

Substitute in heat equation without heat generation: $$\frac{\partial T}{\partial t}+\tau\frac{\partial^2 T}{\partial t^2}=\alpha\nabla^2 T \qquad \alpha=\frac{k}{\rho c_p}$$

This equation is of *hyperbolic* type, characterizes the combined diffusion and wave-like behavior of heat conduction, and predicts a *finite speed for heat propagation*:

Standard Wave Equation: $$\frac{\partial^2 q}{\partial t^2}=c^2\left(\frac{\partial^2 q}{\partial x^2}+\frac{\partial^2 q}{\partial y^2}+\frac{\partial^2 q}{\partial z^2}\right)$$

Compare with hyperbolic heat equation: $$\tau\frac{\partial^2 T}{\partial t^2}=\alpha\nabla^2 T$$ **Consider no heat generation**

By comparing, the propagation speed is: $C=\sqrt{\alpha/\tau}$, **(Thermal wave speed)**

Single-Phase-Lagging Model

Consider the constitutive relation proposed by Cattaneo and Vernotte:

$$\mathbf{q}(\mathbf{r},t) + \tau_0 \frac{\partial \mathbf{q}(\mathbf{r},t)}{\partial t} = -k\nabla T(\mathbf{r},t)$$

Note that the CV constitutive relation is actually a first-order approximation of a more general constitutive relation:

$$\mathbf{q}(\mathbf{r},t+\tau_0) = -k\nabla T(\mathbf{r},t)$$

This is called Single-Phase-Lagging model (Tzou 1992).

Single-Phase-Lagging Model

$$\mathbf{q}(\mathbf{r}, t+\tau_0) = -k\nabla T(\mathbf{r}, t)$$

$$\mathbf{q}(\mathbf{r}, t) + \tau_0 \frac{\partial \mathbf{q}(\mathbf{r}, t)}{\partial t} = -k\nabla T(\mathbf{r}, t)$$

This suggests that the temperature gradient established at a point **r** at time *t* gives rise to a heat flux vector at **r** at a *later* time ($t + \tau_0$). There is a finite built-up time τ_0 for the onset of heat flux at **r** after a temperature gradient is imposed there.

Thus the τ_0 represents the time lag needed to establish the heat flux (the result) when a temperature gradient (the cause) is suddenly imposed.

Single-Phase-Lagging Model

$$\mathbf{q}(\mathbf{r}, t + \tau_0) = -k\nabla T(\mathbf{r}, t)$$

$$\mathbf{q}(\mathbf{r}, t) + \tau_0 \frac{\partial \mathbf{q}(\mathbf{r}, t)}{\partial t} = -k\nabla T(\mathbf{r}, t)$$

The value of τ_0 is material-dependent.

For most solid materials, τ_0 varies from 10^{-10} s to 10^{-14} s.

For gases, τ_0 is normally in the range of $10^{-8} \sim 10^{-10}$ s.

The value of τ_0 for some biological materials and materials with non-homogeneous inner structures can be up to 10^2 s.

Dual-Phase-Lagging Model

Single-Phase-Lagging Model: $\mathbf{q}(\mathbf{r}, t+\tau_0) = -k\nabla T(\mathbf{r}, t)$

The CV constitutive relation generates a more accurate prediction than the classical Fourier law. However, some of its predictions do not agree with experimental results.

The CV constitutive relation has only taken account of the fast-transient effects, but not the micro-structural interactions.

These two effects can be reasonably represented by the dual-phase-lag between **q** and ∇T:

$$\mathbf{q}(\mathbf{r}, t+\tau_0) = -k\nabla T(\mathbf{r}, t+\tau_T)$$

Dual-Phase-Lagging Model

$$\mathbf{q}(\mathbf{r}, t + \tau_0) = -k\nabla T(\mathbf{r}, t + \tau_T)$$

According to this relation, the temperature gradient at a point **r** of the material at time $(t + \tau_T)$ corresponds to the heat flux vector at **r** at time $(t + \tau_0)$.

The delay time τ_T is interpreted as being caused by the micro-structural interactions (small scale heat transport mechanisms occurring in the micro-scale, or small-scale effects of heat transport in space) such as phonon-electron interaction or phonon scattering, and is called the *phase-lag of the temperature gradient*.

Dual-Phase-Lagging Model

$$\mathbf{q}(\mathbf{r}, t+\tau_0) = -k\nabla T(\mathbf{r}, t+\tau_T)$$

Expanding both sides by using the Taylor series and retaining only the first-order terms of τ_0 and τ_T, we obtain the following constitutive relation that is valid at point **r** and time *t:*

$$q(\mathbf{r},t) + \tau_0 \frac{\partial q(\mathbf{r},t)}{\partial t} = -k\left\{\nabla T(\mathbf{r},t) + \tau_T \frac{\partial}{\partial t}[\nabla T(\mathbf{r},t)]\right\}$$

This is known as the Jeffreys-type constitutive equation of heat flux (Joseph and Preziosi 1989). In literature this relation is also called the *dual-phase-lagging constitutive relation*.

When $\tau_0 = \tau_T = 0$, this relation reduces to classical Fourier Law.
When $\tau_T = 0$, it reduces to the CV constitutive relation.

Dual-Phase-Lagging Model: Heat Eq

Constitutive Relation:

$$q(\mathbf{r},t) + \tau_0 \frac{\partial q(\mathbf{r},t)}{\partial t} = -k\left\{\nabla T(\mathbf{r},t) + \tau_T \frac{\partial}{\partial t}[\nabla T(\mathbf{r},t)]\right\}$$

Substitute **q** in: $\rho c \frac{\partial T}{\partial t} + \nabla \cdot \mathrm{q}'' - q''' = 0$ to obtain:

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} + \frac{\tau_0}{\alpha}\frac{\partial^2 T}{\partial t^2} = \nabla^2 T + \tau_T \frac{\partial}{\partial t}\left(\nabla^2 T\right) + \frac{1}{k}\left(q''' + \tau_0 \frac{\partial q'''}{\partial t}\right)$$

Dual-Phase-Lagging Model

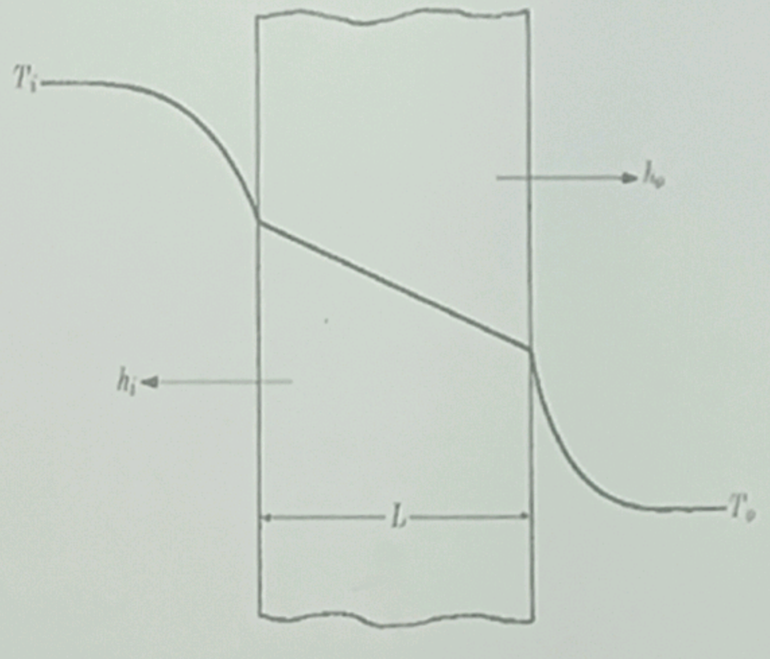
$$\frac{1}{\alpha}\frac{\partial T}{\partial t}+\frac{\tau_0}{\alpha}\frac{\partial^2 T}{\partial t^2}=\nabla^2 T+\tau_T\frac{\partial}{\partial t}\left(\nabla^2 T\right)+\frac{1}{k}\left(q'''+\tau_0\frac{\partial q'''}{\partial t}\right)$$

This equation is parabolic when $\tau_0 < \tau_T$

Although a wave term $(\tau_0/\alpha)\,\partial^2 T/\partial t^2$ exists in the equation, the mixed derivative $\tau_T\,\partial(\nabla^2 T)/\partial t$ completely destroys the wave structure. The equation, in this case, therefore predicts a nonwave-like heat conduction that differs from the usual diffusion predicted by the classical parabolic heat conduction.

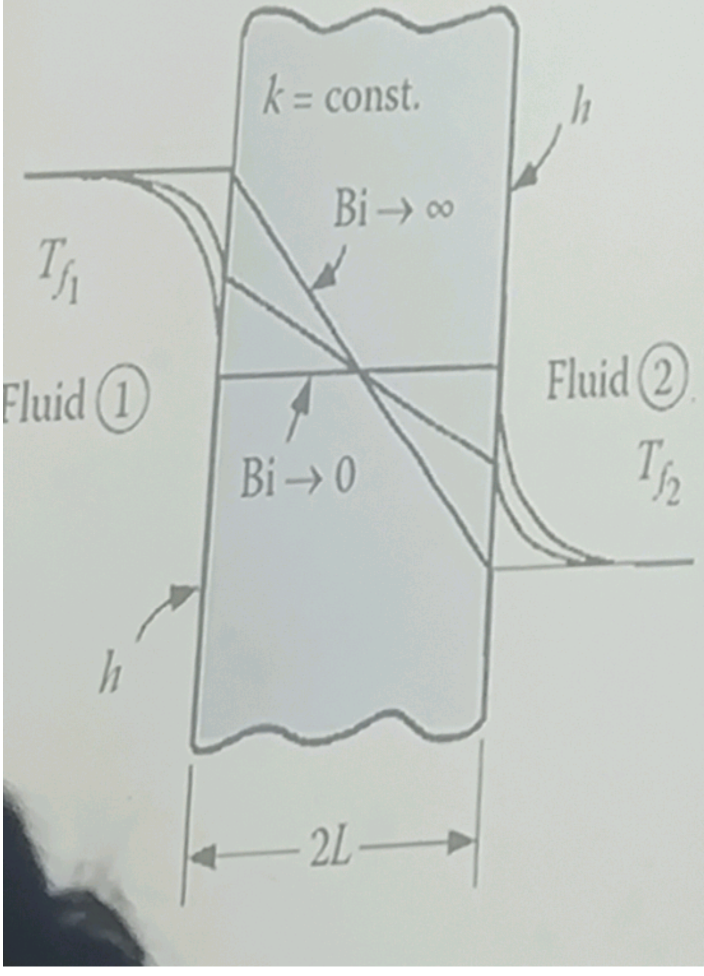
When $\tau_0 > \tau_T$, the Dual-Phase-Lagging equation can be approximated by Single-Phase-Lagging equation and then it predominantly predicts wave-like thermal signals.

Steady-State Temperature Distribution in a Plane



Temperature Distribution in a Plane: Biot Number: Two Limiting Cases

Steady-State



$$q = \frac{T_{f_1} - T_{f_2}}{\sum R_t} \qquad \sum R_t = \frac{2}{hA} + \frac{2L}{kA}$$

$$\text{Biot number} = \text{Bi} = \frac{\text{internal resistance}}{\text{surfacer resistance}} = \frac{hL}{k}$$

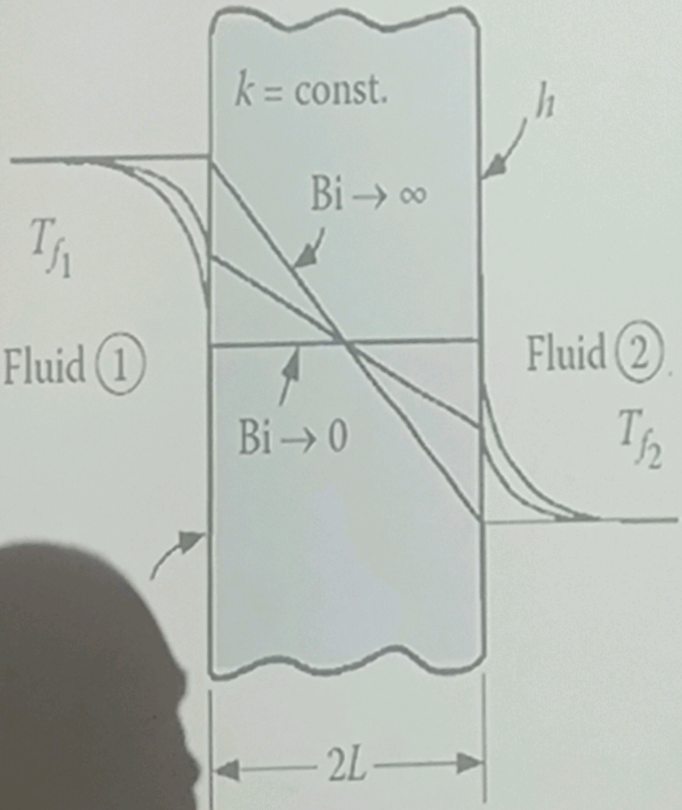
Case-1: The Biot number may be very large; that is, Bi → ∞

$$\sum R_t \cong \frac{2L}{kA}$$

Case-2: The Biot number may be very small; that is, Bi → 0

Temperature Distribution in a Plane: Biot Number: Two Limiting Cases

Steady-State



$$q = \frac{T_{f_1} - T_{f_2}}{\sum R_t} \qquad \sum R_t = \frac{2}{hA} + \frac{2L}{kA}$$

$$\text{Biot number} = \text{Bi} = \frac{\text{internal resistance}}{\text{surfacer resistance}} = \frac{hL}{k}$$

Case-1: The Biot number may be very large; that is, Bi → ∞

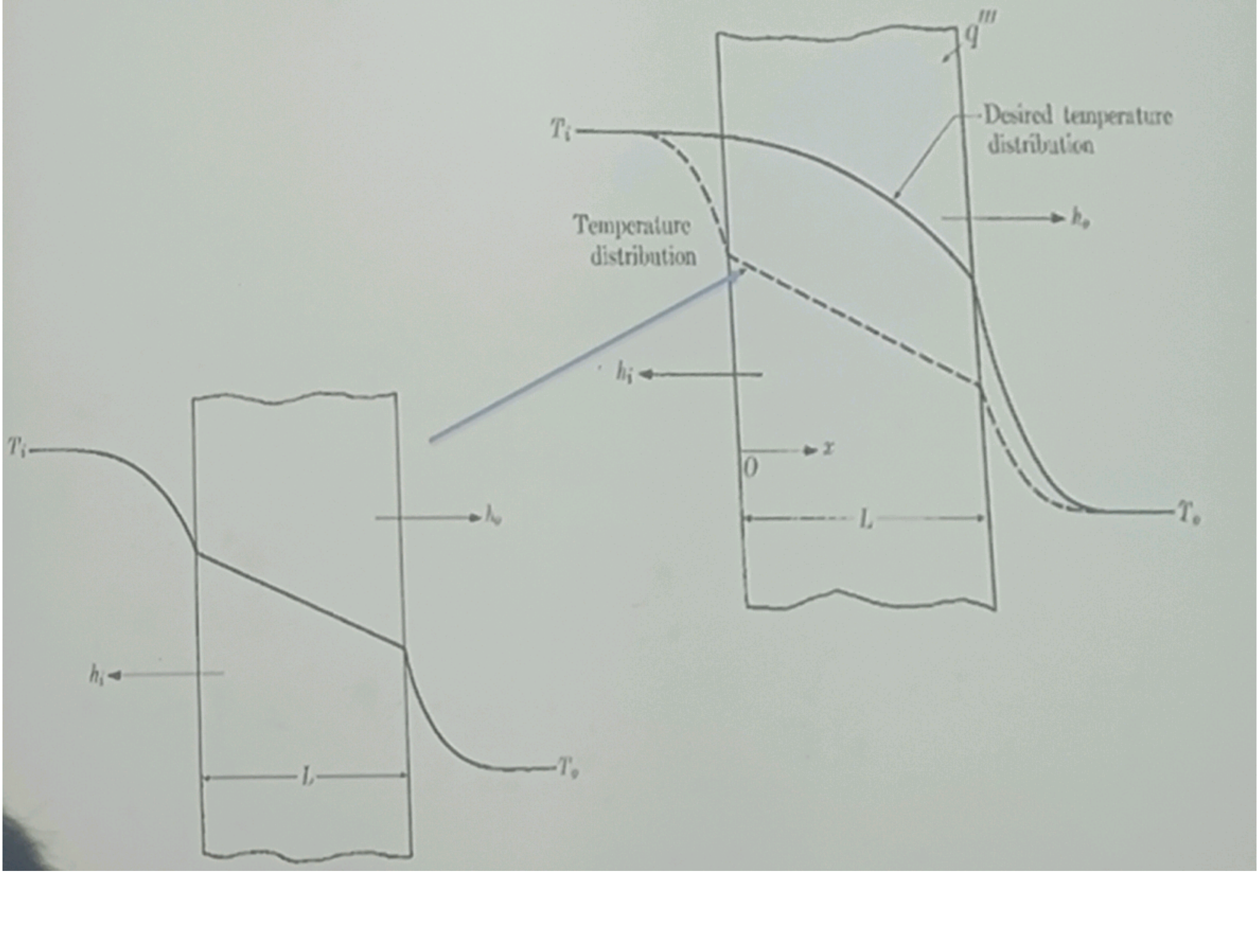
$$\sum R_t \cong \frac{2L}{kA}$$

Case-2: The Biot number may be very small; that is, Bi → 0

$$\sum R_t \cong \frac{2}{hA}$$

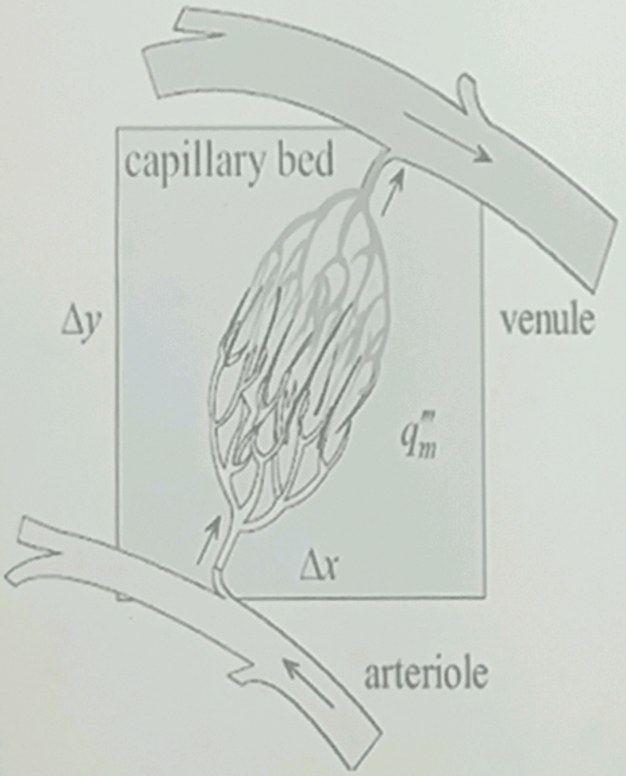
Eliminate Heat Loss from One Side ($x = 0$)

Steady-State



Heat Transfer in Living Tissue: Pennes' Bioheat Equation

The Pennes' bioheat equation is based on simplifying assumptions concerning the following four central factors:

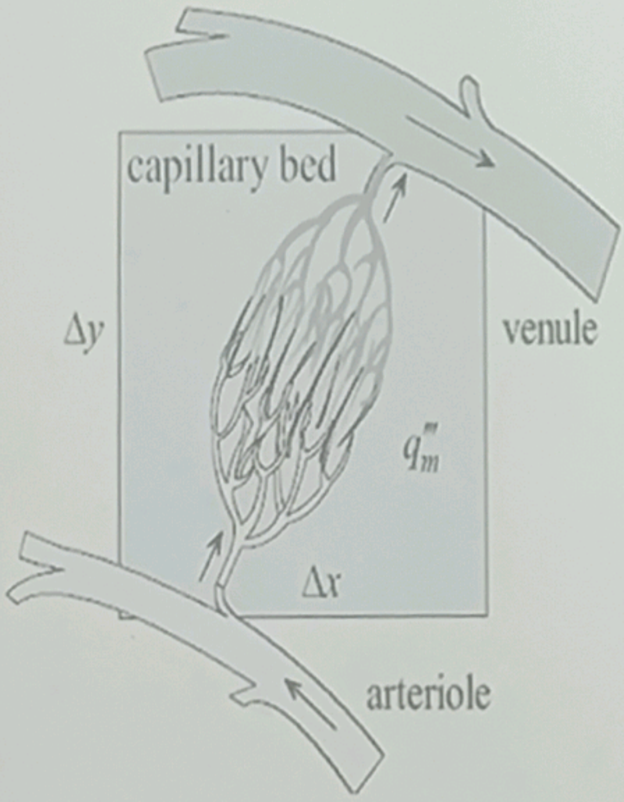


(1) *Equilibration Site*. The principal heat exchange between blood and tissue takes place in the capillary beds, the arterioles supplying blood to the capillaries, and the venules draining it. Thus, all pre-arteriole and post-venule heat transfer between blood and tissue is neglected.

(2) *Blood Perfusion*. The flow of blood in the small capillaries is assumed to be isotropic. This neglects the effect of blood flow directionality.

Heat Transfer in Living Tissue: Pennes' Bioheat Equation

The Pennes' bioheat equation is based on simplifying assumptions concerning the following four central factors:



(3) *Vascular Architecture*. Larger blood vessels in the vicinity of capillary beds play no role in the energy exchange between tissue and capillary blood. Thus, the Pennes' model does not consider the local vascular geometry.

(4) *Blood Temperature*. Blood is assumed to reach the arterioles supplying the capillary beds at the body core temperature T_{a0}. It instantaneously exchanges energy and equilibrates with the local tissue temperature T.

Heat Transfer in Living Tissue: Pennes' Bioheat Equation

$$\rho c\frac{\partial T}{\partial t}+\nabla\cdot q''-q'''=0$$

$q_b''' =$ net rate of energy added by the blood per unit volume of tissue
$q_m''' =$ rate of metabolic energy production per unit volume of tissue

According to Pennes', blood enters the element at the body core temperature T_{a0}, instantaneously equilibrates, and exists at the temperature of the element T. Thus,

$$q_b'''=\rho_b c_b \dot{w}_b\left(T_{a0}-T\right),$$

where

$c_b =$ specific heat of blood
$\dot{w}_b =$ blood volumetric flow rate per unit tissue volume
$\rho_b =$ density of blood

Thus, $$q'''=q_m'''+\rho_b c_b \dot{w}_b\left(T_{a0}-T\right).$$

Heat Transfer in Living Tissue: Pennes' Bioheat Equation

$$\rho c \frac{\partial T}{\partial t} + \nabla \cdot q'' - q''' = 0$$

$$q''' = q_m''' + \rho_b c_b \dot{w}_b \left(T_{a0} - T\right).$$

On substitution:

$$\nabla \cdot k \nabla T + \rho_b c_b \dot{w}_b \left(T_{a0} - T\right) + q_m''' = \rho c \frac{\partial T}{\partial t},$$

Pennes' Equation

$$q = -k \nabla T.$$

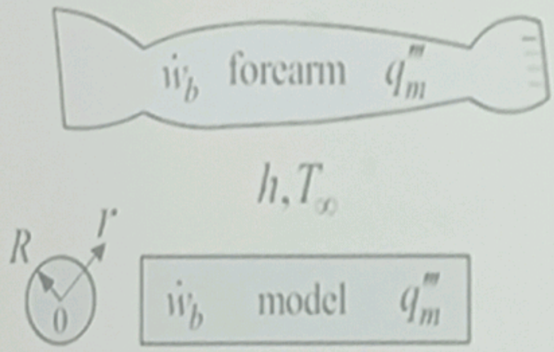
where

$c =$ specific heat of tissue
$k =$ thermal conductivity of tissue
$\rho =$ density of tissue

Write the Equation for Non-Fourier Heat Conduction:

$$q + \tau_q \frac{\partial q}{\partial t} = -k \nabla T$$

Heat Transfer in Living Tissue: Pennes' Bioheat Equation



- Steady-State
- 1D Case

$$\nabla \cdot k\nabla T + \rho_b c_b \dot{w}_b \left(T_{a0} - T\right) + q_m''' = \rho c \frac{\partial T}{\partial t},$$

Cylindrical coordinates:

$$\nabla \cdot k\nabla T = \frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right).$$

Heat Equation:

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}T}{\mathrm{d}r}\right) + \frac{\rho_b c_b \dot{w}_b}{k}\left(T_{a0} - T\right) + \frac{q_m'''}{k} = 0.$$

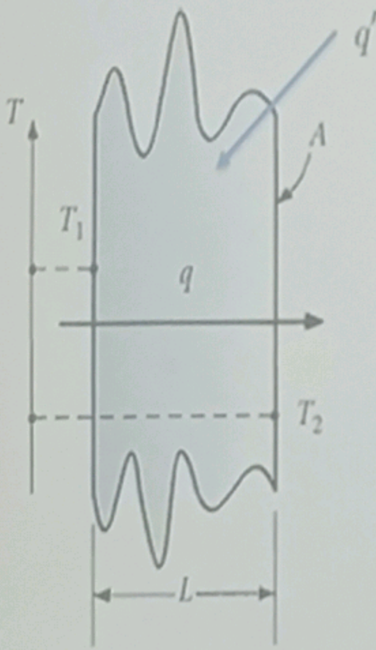
Boundary Conditions:

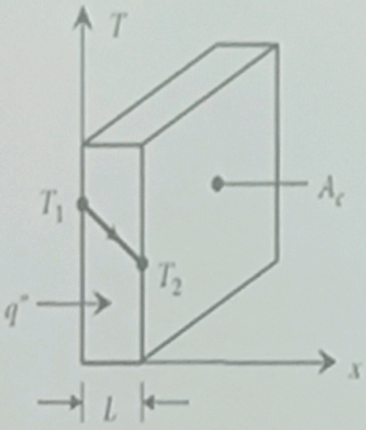
$$\frac{\mathrm{d}T(0)}{\mathrm{d}r} = 0, \text{ or } T(0) = \text{finite},$$

$$-k\frac{\mathrm{d}T(R)}{\mathrm{d}r} = h\left[T(R) - T_\infty\right].$$

Non-dimensional Form

Why non-dimensional form?





$$\frac{d^2T}{dx^2} = -\frac{q'''}{k}$$

Boundary conditions?

Define:

- Reference length
- Reference temperature
- Reference temperature difference

$$\overline{T} \equiv \frac{T - T_1}{T_2 - T_1} \qquad \overline{x} \equiv \frac{x}{L}$$

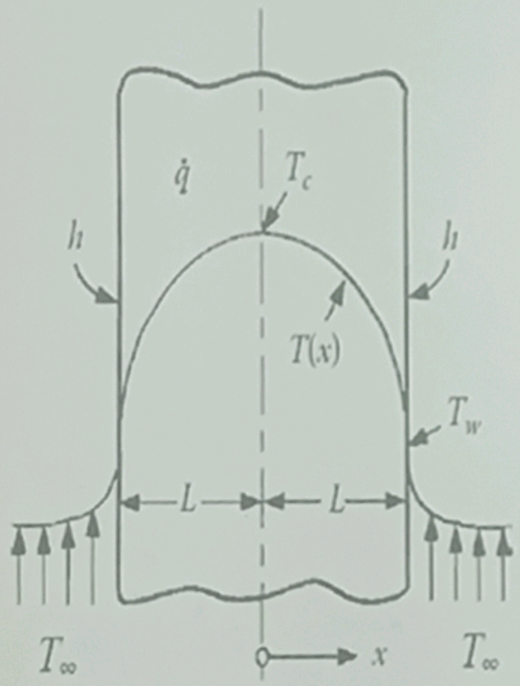
$$\frac{d^2\overline{T}}{d\overline{x}^2} = -\frac{q'''L^2}{k(T_2 - T_1)} = -S$$

$$\overline{T}(\overline{x} = 0) = 0, \quad \overline{T}(\overline{x} = 1) = 1$$

$$\overline{T} = -\frac{S\overline{x}^2}{2} + c_1\overline{x} + c_2$$

$$\overline{T} = \overline{x} + \frac{S\overline{x}}{2}(1 - \overline{x})$$

Non-dimensional Form: The Correct Way



Symmetric – solve half the problem

$$\frac{d^2T}{dx^2}+\frac{\dot{q}}{k}=0$$

$$\left.\frac{dT}{dx}\right|_{x=0}=0$$

$$-k\left.\frac{dT}{dx}\right|_{x=L}=h(T(x=L)-T_\infty)$$

Define:

- Reference length (L)
- Reference temperature (T_∞)
- Reference temperature difference (ΔT_C=?)

$$\overline{x}\equiv\frac{x}{L} \qquad \overline{T}\equiv\ ? \qquad \overline{T}\equiv\frac{T-T_\infty}{\Delta T_C}$$

Recall: $$\frac{d^2\overline{T}}{d\overline{x}^2}=-\frac{q'''L^2}{k(T_2-T_1)}$$ (Previous slide)

$$\overline{T}\equiv\frac{(T-T_\infty)k}{q'''L^2}$$

$$\frac{d^2\overline{T}}{d\overline{x}^2}=-1$$

$$\left.\frac{d\overline{T}}{d\overline{x}}\right|_{\overline{x}=0}=0,\quad \left.\frac{d\overline{T}}{d\overline{x}}\right|_{\overline{x}=1}=-Bi\,\overline{T}(\overline{x}=1)$$

Non-dimensional Form: 3D Heat Eq

A stationary, homogeneous, isotropic solid is initially at a const temp T_0. For time $t>0$, heat is generated within the solid and dissipated by convection from the bounding surfaces into a medium at const temp T_∞. Assume a rectangular geometry and a finite region, R

BVP of heat conduction eq^n:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad \text{in region } R,\ t>0$$

$$-k_i \frac{\partial T}{\partial n_i} = h_i (T_i - T_\infty) \quad \text{on } S_i \text{, boundary of } R,\ t>0$$

$i = 1, 2, \cdots$ no. of continuous bounding surfaces of the solid

$$T = T_0 \quad \text{in region } R,\ t=0$$

Non-dimensional Form: 3D Heat Eq

- Reference length = L = a characteristic dimension of the solid
- Define dimensionless length variables

$$\xi = \frac{x}{L}, \quad \eta = \frac{y}{L}, \quad \psi = \frac{z}{L}$$

$\frac{\partial}{\partial N} \equiv$ differentiation along outward-drawn normal in the new dimensionless coordinate system (ξ, η, ψ)

choose reference temp = T_∞

ref temp difference = $(T_0 - T_\infty)$

Define dimensionless excess temp $\theta = \dfrac{T - T_\infty}{T_0 - T_\infty}$

Non-dimensional Form: 3D Heat Eq

On substitution:

$$\frac{T_0 - T_\infty}{L^2}\left(\frac{\partial^2\theta}{\partial \xi^2} + \frac{\partial^2\theta}{\partial \eta^2} + \frac{\partial^2\theta}{\partial \psi^2}\right) + \frac{\dot{q}}{k} = \frac{T_0 - T_\infty}{\alpha}\frac{\partial\theta}{\partial t}$$

in R, $t>0$

$$-\frac{k_i}{L}\frac{\partial\theta}{\partial N_i} = h_i\,\theta$$

on S_i, $t>0$

$i = 1, 2, \dots$ = no. of [illegible] bounding surfaces

$$\theta = 1$$

in R, $t=0$

$$\frac{\partial^2\theta}{\partial \xi^2} + \frac{\partial^2\theta}{\partial \eta^2} + \frac{\partial^2\theta}{\partial \psi^2} + \frac{\dot{q}L^2}{(T_0 - T_\infty)k} = \frac{L^2}{\alpha}\frac{\partial\theta}{\partial t}$$

$$= \frac{\partial\theta}{\partial\tau} \quad \text{since } \tau = \frac{\alpha t}{L^2}$$

$$\Rightarrow \frac{\partial^2\theta}{\partial \xi^2} + \frac{\partial^2\theta}{\partial \eta^2} + \frac{\partial^2\theta}{\partial \psi^2} + \Phi = \frac{\partial\theta}{\partial\tau}$$

in R, $t>0$

in R, $\tau>0$

Non-dimensional Form: 3D Heat Eq

$$\frac{\partial^2\theta}{\partial\psi^2} + \frac{\partial^2\theta}{\partial\eta^2} + \frac{\partial^2\theta}{\partial\psi^2} + \frac{\dot{q}L^2}{(T_0-T_\infty)k} = \frac{L^2}{\alpha}\frac{\partial\theta}{\partial t}$$

$$= \frac{\partial\theta}{\partial\tau} \quad \text{where } \tau = \frac{\alpha t}{L^2}$$

in R, $t>0$

$$\Rightarrow \frac{\partial^2\theta}{\partial\psi^2} + \frac{\partial^2\theta}{\partial\eta^2} + \frac{\partial^2\theta}{\partial\psi^2} + \Phi = \frac{\partial\theta}{\partial\tau}$$

in R, $\tau>0$

$$\frac{\partial\theta}{\partial N_i} + \frac{h_i L}{k_i}\theta = 0$$

$$\Rightarrow \frac{\partial\theta}{\partial N_i} + B_i\theta = 0 \quad \text{on } S_i,\ \tau>0$$

$$\theta = 1 \quad \text{in } R,\ \tau = 0$$

Non-dimensional Form: 1D Unsteady Conduction

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$T(x, t = 0) = T_0$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$$

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h(T|_{x=L} - T_\infty)$$

Define: $x^* = \dfrac{x}{L}$

$$t^* = \frac{\alpha t}{L^2}$$

$$T^* = \frac{T - T_\infty}{T_0 - T_\infty}$$

Solution of Homogeneous Problem
by
Separation of Variables

Separation of Variables: Generalization

Consider the following multidimensional, transient homogeneous problem:

$$\nabla^2 T(\hat{r},t) = \frac{1}{\alpha}\frac{\partial T(\hat{r},t)}{\partial t} \qquad \text{in domain R} \qquad t>0$$

$$k_i \frac{\partial T}{\partial n_i} + h_i T = 0 \qquad \text{on boundary } S_i \quad i = 1 \text{ to } N \qquad t>0$$

$$T(\hat{r}, t=0) = F(\hat{r}) \qquad \text{in domain } R$$

Each boundary surface S_i fits the coordinate surface of the chosen orthogonal coordinate system.

Assume a separation in the form:

$$T(\hat{r},t) = \Psi(\hat{r})\Gamma(t)$$

Substitute this equation into the heat equation (PDE):

$$\frac{1}{\Psi(\hat{r})}\nabla^2\Psi(\hat{r}) = \frac{1}{\alpha\Gamma(t)}\frac{d\Gamma(t)}{dt} = -\lambda^2$$

(first separation constant)

$$\frac{d\Gamma}{dt} + \alpha\lambda^2\Gamma = 0$$

Separation of Variables: Generalization

$$\frac{1}{\Psi(\hat{r})}\nabla^2\Psi(\hat{r}) = \frac{1}{\alpha\Gamma(t)}\frac{d\Gamma(t)}{dt} = -\lambda^2$$

The function $\Gamma(t)$ satisfies the following ODE.

$$\frac{d\Gamma}{dt} + \alpha\lambda^2\Gamma = 0$$

Its solution is taken as $\exp(-\alpha\lambda^2 t)$

The negative sign chosen for λ^2 and the positive nature of λ^2 implies that the solution asymptotically approaches to zero as time increases indefinitely.

This result is expected from the physical nature of the problem, that is, for a homogeneous boundary condition of the third kind it implies that a solid dissipates heat from its bounding surfaces by convection into a surrounding at zero temperature.

Separation of Variables: Generalization

$$\frac{1}{\Psi(\hat{r})}\nabla^2\Psi(\hat{r}) = \frac{1}{\alpha\Gamma(t)}\frac{d\Gamma(t)}{dt} = -\lambda^2$$

The spatial variable function $\psi(\hat{r})$ satisfies the following auxiliary problem:

$$\nabla^2\Psi(\hat{r}) + \lambda^2\Psi(\hat{r}) = 0 \quad \text{in domain } R$$

$$k_i\frac{\partial\Psi}{\partial n_i} + h_i\Psi = 0 \quad \text{on boundary } S_i$$

(Homogeneous Boundary Conditions)

- This equation is called the *Helmholtz equation.*
- In general, it is a PDE in the three spatial variables.

The Helmholtz equation can be solved by Separation of Variables, provided that its separation into a set of ODEs is possible.

A simple separation of the Helmholtz equation into ordinary differenti equations is possible in 11 orthogonal coordinate systems.

Separation of Variables: Generalization

$$\nabla^2 \Psi(\hat{r}) + \lambda^2 \Psi(\hat{r}) = 0 \qquad \text{in domain } R$$

$$k_i \frac{\partial \Psi}{\partial n_i} + h_i \Psi = 0 \qquad \text{on boundary } S_i$$

Helmholtz equation

A simple separation of the Helmholtz equation into ordinary differential equations is possible in 11 orthogonal coordinate systems.

Coordinate System	Functions That Appear in Solution
1 Rectangular	Exponential, circular, hyperbolic
2 Circular cylinder	Bessel, exponential, circular
3 Elliptic cylinder	Mathieu, circular
4 Parabolic cylinder	Weber, circular
5 Spherical	Legendre, power, circular
6 Prolate spheroidal	Legendre, circular
7 Oblate spheroidal	Legendre, circular
8 Parabolic	Bessel, circular
9 Conical	Lamé, power
10 Ellipsoidal	Lamé
11 Paraboloidal	Baer

Separation of Variables: Generalization

$$\nabla^2\Psi(\hat{r}) + \lambda^2\Psi(\hat{r}) = 0 \qquad \text{in domain } R$$

$$k_i\frac{\partial\Psi}{\partial n_i} + h_i\Psi = 0 \qquad \text{on boundary } S_i$$

Helmholtz equation

The above system has nontrivial solutions only for certain values of the separation variable $\lambda = \lambda_m$, called *eigenvalues*.

The corresponding nontrivial solutions are called *eigenfunctions*:

$$\Psi(\lambda_m, \hat{r}) = \Psi_m(\hat{r})$$

Assuming the eigenfunctions and the eigenvalues λ_m are determined, the complete solution of the temperature function $T(\hat{r}, t)$ is obtained as:

$$T(\hat{r}, t) = \sum_{m=1}^{\infty} C_m\, \Psi_m(\hat{r})\, e^{-\alpha\lambda_m^2 t}$$

The summation is taken over all discrete spectrum of eigenvalues for the given problem.

Note that for three-dimensional problems (i[illegible]finite regions) the [illegible]tion in above expression for temper[illegible]ture is a [illegible]iple [illegible] series.

Separation of Variables: Generalization

$$T(\hat{r},t) = \sum_{m=1}^{\infty} C_m \Psi_m(\hat{r})\, e^{-\alpha\lambda_m^2 t}$$

The solution contains the unknown coefficients C_m.

The above solution should satisfy the initial condition of the problem:

$$T(\hat{r}, t=0) = F(\hat{r})$$

Therefore, by substituting $t = 0$,

$$F(\hat{r},t) = \sum_{m=1}^{\infty} C_m \Psi_m(\hat{r})$$

If the eigenfunctions $\Psi_m(\hat{r})$ constitute an orthogonal set in the region considered, the unknown coefficients C_m are determined by making use of the orthogonality property of eigenfunctions $\Psi_m(\hat{r})$; that is,

$$\int_R \Psi_m(\hat{r})\, \Psi_n(\hat{r})\, d\hat{r} = 0, \qquad \text{for } m \neq n$$

To determine C_m,

- Multiply both sides of $F(\hat{r},t) = \sum_{m=1}^{\infty} C_m \Psi_m(\hat{r})$ by $\Psi_m(\hat{r})$
- Integrate it over the region and make use of the orthogonality conditi