The program discussed above is therefore easily modified to deal with the specified wall heat flux case and a program, LAMBOUQ, for this situation can be obtained in the way indicated in the Preface.

3.7 VISCOUS DISSIPATION EFFECTS ON LAMINAR BOUNDARY LAYER FLOW OVER A FLAT PLATE

The effects of viscous dissipation on the temperature field have been ignored in the discussion up to this point in the chapter. However, viscous dissipation effects can be important particularly if the viscosity of the fluid is high or if the velocities are high. An example of the first case would be the flow of the lubricant through a journal bearing. In this case the lubricant temperature rise caused by viscous dissipation can be quite high. The second case usually involves the flow of gases at relatively high Mach numbers and it is to this case that attention will be given in the present section. In the present section, then, an analysis of the effects of dissipation on the laminar boundary layer flow over a flat plate which is aligned with the flow will be presented [9],[12],[13],[14],[15],[16]. The flow situation considered is, therefore, as shown in Fig. 3.25.

The effects of fluid property variations will again be ignored. It should, however, be noted that if viscous dissipation effects are important, fluid property changes are usually quite large. These changes can usually be adequately accounted for by evaluating the fluid properties at some suitable mean temperature and then treating them as constant in the analysis of the flow as is being done here. This will be discussed in the next section.

The equations governing the flow were discussed in Chapter 2. Because two-dimensional flow over a flat plate is being considered, these equations are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3.231}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \left(\frac{\mu}{\rho}\right)\frac{\partial^2 u}{\partial y^2} \tag{3.232}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \left(\frac{k}{\rho c_p}\right)\frac{\partial^2 T}{\partial y^2} + \left(\frac{\mu}{\rho c_p}\right)\left(\frac{\partial u}{\partial y}\right)^2 \tag{3.233}$$

The last term in the energy equation, i.e., in Eq. (3.233), is the viscous dissipation term.

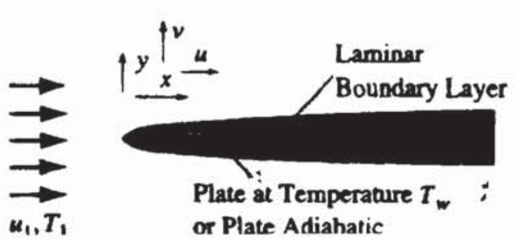

FIGURE 3.25
Laminar boundary layer flow with viscous dissipation

The boundary conditions on the velocity distribution are:

$$\begin{aligned} \text{At } y = 0: &\quad u = v = 0 \\ \text{For large } y: &\quad u \to u_1(x) \end{aligned} \tag{3.234}$$

Two possible boundary conditions on temperature at the wall will be considered in the present section, i.e., it will be assumed that either:

$$\text{At } y = 0: \quad \frac{\partial T}{\partial y} = 0 \tag{3.235}$$

or that:

$$\text{At } y = 0: \quad T = T_w \tag{3.236}$$

where T_w is the temperature of the wall which is assumed constant. The first of these boundary conditions, i.e., Eq. (3.235), applies when the wall is adiabatic because Fourier's law gives the heat transfer rate at the wall, in general, as:

$$q_w = -k \left.\frac{\partial T}{\partial y}\right|_{y=0} \tag{3.237}$$

Therefore, since $q_w = 0$ if the wall is adiabatic, the gradient of temperature at the wall must be zero if the wall is adiabatic.

Outside the boundary layer, the boundary condition on temperature is:

$$\text{For large } y: \quad T \to T_1 \tag{3.238}$$

Dissipation has no effect on the continuity and momentum equations because the fluid properties are being assumed constant. As a result, Eqs. (3.231) and (3.232) are the same as those considered earlier in this chapter in Section 3.2. As discussed in that section, a similarity solution to these equations can be obtained. To do this, the following "similarity" variable was introduced:

$$\eta = \frac{y}{x}\sqrt{Re_x} = y\sqrt{\frac{u_1}{\nu x}} \tag{3.239}$$

In terms of this variable, the distribution of u in the boundary layer is assumed to be:

$$\frac{u}{u_1} = f'(\eta) \tag{3.240}$$

Using the continuity equation, it was then shown that:

$$\frac{v}{u_1} = \frac{1}{2}\sqrt{\frac{\nu}{xu_1}}(\eta f' - f) \tag{3.241}$$

The momentum equation could then be written in terms of η as:

$$2f''' + ff'' = 0 \tag{3.242}$$

The primes, of course, denote differentiation with respect to η.

In terms of the similarity function, the boundary conditions on the solution become:

$$\eta = 0,\ f' = 0 \qquad \eta = 0,\ f = 0$$

large $f' \to 1$

The solution of Eq. (3.242) subject to these boundary conditions was discussed in Section 3.2 and given in graphical form in Figs. 3.4 and 3.5. This solution, it must be stressed, also applies when viscous dissipation is included.

It was also shown in Section 3.2 that a similarity solution could be obtained for the temperature distribution for the case where the wall temperature is constant. To do this, a dimensionless temperature defined as follows was introduced:

$$\phi = \frac{T_w - T}{T_w - T_1} \tag{3.243}$$

and it was assumed that ϕ depended only on η. In terms of this variable, the energy equation without viscous dissipation became:

$$\phi'' + \frac{Pr}{2}\phi' f = 0 \tag{3.244}$$

While the boundary conditions on the solution can be written as:

$$\begin{aligned} \eta = 0&: \quad \phi = 0 \\ \eta \text{ large}&: \quad \phi \to 1 \end{aligned} \tag{3.245}$$

Thus, as was the case with the velocity distribution, the partial differential equation governing the temperature distribution was shown to reduce to an ordinary differential equation.

It seems reasonable to assume that similar temperature profiles will also exist when viscous dissipation is important. Attention will first be given to the adiabatic wall case. If the wall is adiabatic and viscous dissipation is neglected, then the solution to the energy equation will be $T = T_1$ everywhere in the flow. However, when viscous dissipation effects are important, the work done by the viscous forces leads to a rise in fluid temperature in the fluid. This temperature will be related to the kinetic energy of the fluid in the freestream flow, i.e., will be related to $u_1^2/2c_p$. For this reason, the similarity profiles in the adiabatic wall case when viscous dissipation is important are assumed to have the form:

$$\frac{T - T_1}{u_1^2/2c_p} = \theta_a(\eta) \tag{3.246}$$

Substituting this and the velocity profile results into the energy equation, i.e., Eq. (3.233), then gives:

$$u_1 f'\theta_a' \frac{\partial \eta}{\partial x} + \frac{u_1}{2}\sqrt{\frac{\nu}{x u_1}}(\eta f' - f)\theta_a' \frac{\partial \eta}{\partial y} = \frac{k}{\rho c_p}\theta_a'' \frac{\partial \eta^2}{\partial y} + \frac{\mu}{\rho c_p}(f'')^2 \frac{u_1^2}{u_1^2/2c_p}\left(\frac{\partial \eta}{\partial y}\right)^2$$

i.e.:

$$-\frac{f'\theta_a'}{2x} y\sqrt{\frac{u_1}{x\nu}} + \frac{1}{2}\sqrt{\frac{\nu}{x u_1}}(\eta f' - f)\theta_a'\sqrt{\frac{u_1}{x\nu}} = \frac{k}{\rho c_p}\,\theta_a''\,\frac{u_1}{x\nu}\,\frac{1}{u_1} + \frac{\mu}{\rho c_p}(f'')^2\frac{2c_p}{u_1}\left[\frac{u_1}{x\nu}\right]$$

i.e.:

$$-\frac{f'\theta_a'\eta}{2} + \frac{\eta f'\theta_a'}{2} - \frac{f\theta_a'}{2} = \frac{\theta_a''}{Pr} + 2(f'')^2$$

i.e.:

$$\theta_a'' + \frac{Pr}{2} f\theta_a' + 2\,Pr(f'')^2 = 0 \tag{3.247}$$

This is an ordinary differential equation which shows that a similarity solution does, indeed, exist. This equation contains the Prandtl number as a parameter, i.e., the variation of θ_a with η depends on Pr.

The boundary conditions on θ_a are:

$$\eta = 0: \quad \theta_a' = 0$$
$$\eta \text{ large:} \quad \theta_a \to 0$$

The solution of Eq. (3.247) subject to these boundary conditions can be obtained by numerical integration or an analytical solution can be obtained by introducing an integrating factor. The latter procedure leads to the following equation for the value of θ_a at the wall:

$$\theta_a(0) = \frac{\int_0^\eta \exp\left(\frac{Pr}{2}\int_0^\eta f\,d\eta\right) 2\,Pr(f'')^2\,d\eta}{\int_0^\infty \exp\left(\frac{Pr}{2}\int_0^\eta f\,d\eta\right) d\eta} \tag{3.248}$$

This equation can be integrated numerically to give the value of $\theta_a(0)$ for any chosen value of the Prandtl number. Some typical results are shown in Fig. 3.26.

It will be seen from the results given in Fig. 3.26 that for Prandtl numbers between approximately 0.5 and 10, the variation of $\theta_a(0)$ with Pr is approximately described by:

$$\theta_a(0) = Pr^{1/2} \tag{3.249}$$

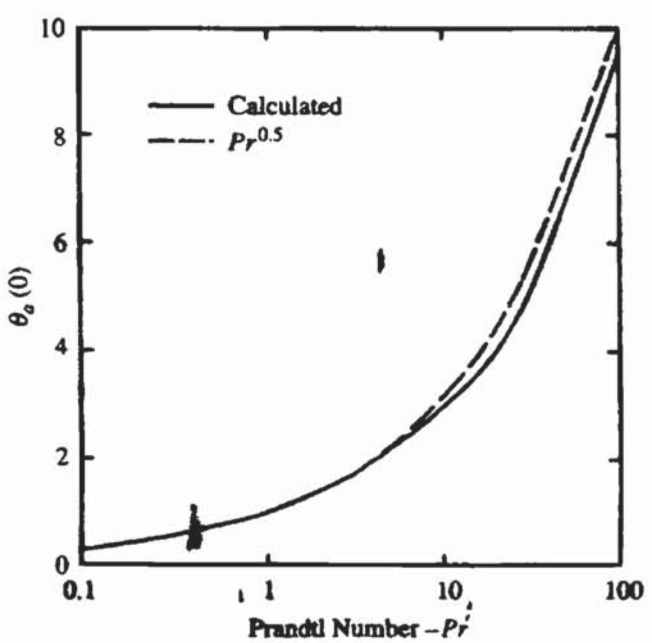

FIGURE 3.26
Variation of dimensionless adiabatic wall temperature with Prandtl number.

Considering the definition of θ_a as given in Eq. (3.246), it will be seen that:

$$\theta_a(0) = \frac{T_{w_{ad}} - T_1}{u_1^2/2c_p} \tag{3.250}$$

where $T_{w_{ad}}$ is the adiabatic wall temperature. This equation gives:

$$T_{w_{ad}} = T_1 + \theta_a(0)u_1^2/2c_p \tag{3.251}$$

Now for a perfect gas:

$$R = c_p - c_v = c_p(1 - 1/\gamma)$$

where $\gamma = c_p/c_v$ is the specific heat ratio of the gas involved and R is the gas constant. From this it follows that:

$$c_p = \frac{\gamma R}{\gamma - 1}$$

Hence:

$$\frac{u_1^2}{2c_p} = \frac{\gamma - 1}{2}\left(\frac{T_1 u_1^2}{\gamma R T_1}\right) \tag{3.252}$$

But:

$$a = \sqrt{\gamma R T}$$

where a is the speed of sound in the gas. Therefore Eq. (3.252) gives:

$$\frac{u_1^2}{2c_p} = \frac{\gamma - 1}{2} T_1 M_1{}^2 \tag{3.253}$$

where the Mach number, M, is defined by:

$$M = \frac{u}{a} = \frac{u}{\sqrt{\gamma R T}} \tag{3.254}$$

Therefore, Eq. (3.251) gives:

$$\frac{T_{w_{ad}}}{T_1} = 1 + \theta_a(0)\frac{\gamma - 1}{2} M_1^2 \tag{3.255}$$

It is conventional to write this equation for the adiabatic wall temperature as:

$$\frac{T_{w_{ad}}}{T_1} = 1 + r\left(\frac{\gamma - 1}{2}\right) M_1^2 \tag{3.256}$$

where:

$$r = \theta_a(0) \tag{3.257}$$

r being termed the recovery factor. The results given above therefore show that for Prandtl numbers between approximately 0.5 and 10, r is approximately given by:

$$r = Pr^{1/2} \tag{3.258}$$

This result agrees very well with experimental results for air as shown in Fig. 3.27.

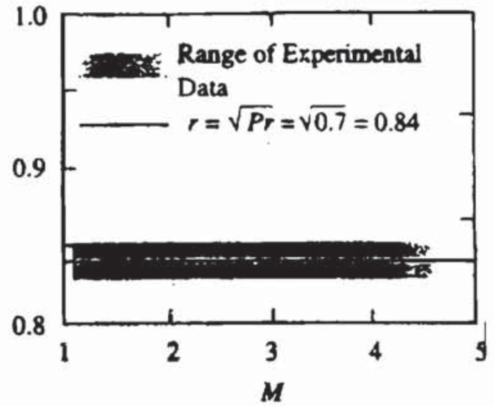

FIGURE 3.27
Typical effect of Mach number on the recovery factor for a laminar boundary layer.

EXAMPLE 3.8. Consider the flow of air which has a freestream temperature of 0°C over an adiabatic flat plate as shown in Fig. E3.8. If the flow in the boundary layer can be assumed to be laminar, determine how the temperature of the plate surface varies with Mach number.

Solution. Since the boundary layer flow is laminar and since it can be assumed that for air $Pr = 0.7$, it follows that here:

$$r = 0.7^{1/2} = 0.837$$

In this case then:

$$\frac{T_{w_{ad}}}{T_1} = 1 + 0.167M_1^2$$

But here, $T_1 = 0°C = 273$ K so:

$$T_{w_{ad}} = 273(1 + 0.167M_1^2)$$

Using this equation, the following are obtained:

M	$T_{w_{ad}}/T_1$	$T_{w_{ad}}$ K	$T_{w_{ad}}$ C
0	1.000	273	0
1	1.167	319	46
2	1.669	456	183
3	2.506	684	441
4	3.677	1004	731

It will be seen, therefore, that high surface temperatures can exist at Mach numbers above about 2. These surface temperatures may, in fact, be so high that special high-temperature materials such as titanium alloys rather than conventional aluminum alloys must be used for structural components in aircraft designed to fly at high Mach numbers.