Week5

04-02-2025

Analysis of population dynamics in discrete domain

$$\frac{dx}{dt} = ax \qquad \text{linear}$$

$$\frac{dx}{dt} = ax\left(1 - \frac{x}{N}\right) \qquad \text{logistic}$$

$$\frac{dx}{dt} = ax\left(1 - \frac{x}{N}\right) - h \qquad \text{logistic with harvesting}$$

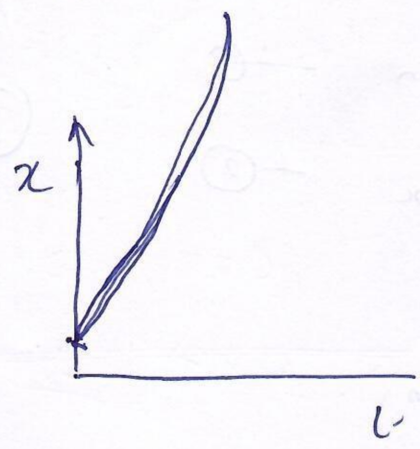
$$\frac{dx}{dt} = ax\left(1 - \frac{x}{\lambda_1}\right)\left(1 - \frac{x}{\lambda_2}\right) \qquad \text{logistic with threshold}$$

continuous model

x & t are continuous.

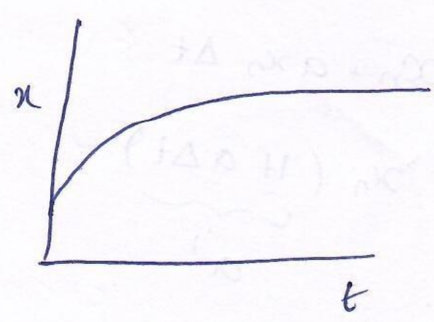
for linear

$$x(t) = x(0)e^{at}$$

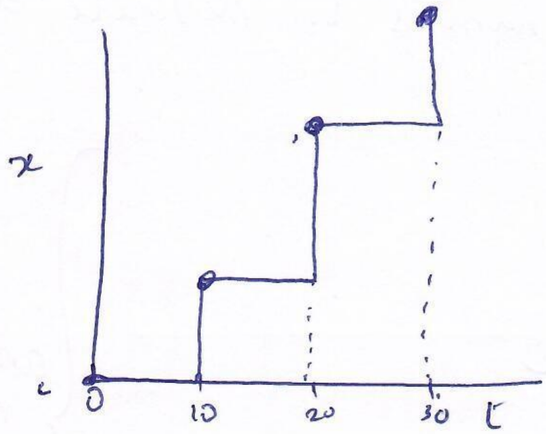


for logistic

$$x(t) = \frac{x(0)e^{at}}{1 - x(0) + x(0)e^{at}}$$



Discrete time domain



$t = 0 \quad 10 \quad 20 \quad 30 \quad 40$

$x = 1 \rightarrow 2 \rightarrow 4 \rightarrow 8 \rightarrow 16$ } $x(0) = 1.$

$x_{n+1} = 2x_n \quad \Leftarrow$ model. eq^n

$$x_{n+1} = a x_n \quad \text{--- (1)}$$

$$\frac{dx}{dt} = ax \quad \text{--- (2)}$$

(1) is discrete analogue of (2).

$$\frac{x_{n+1} - x_n}{\Delta t} = a x_n$$

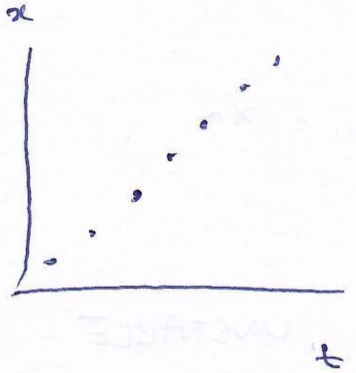
$$x_{n+1} = x_n + a x_n \Delta t$$

$$x_{n+1} = x_n \underbrace{(1 + a\Delta t)}_{a'}$$

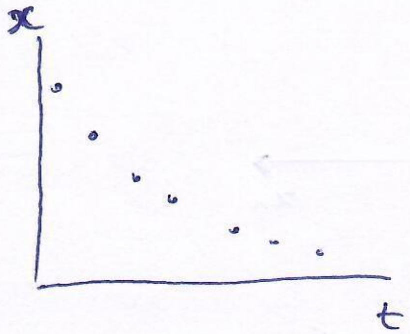
$$x_{n+1} = a' x_n$$

$x_{n+1} = a x_n$

$a > 1$



$a < 1$



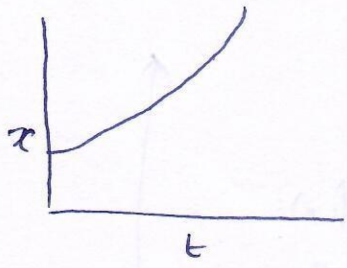
at equilibrium

$$x_n = x_{n+1} = x_{n+2} = \cdots$$

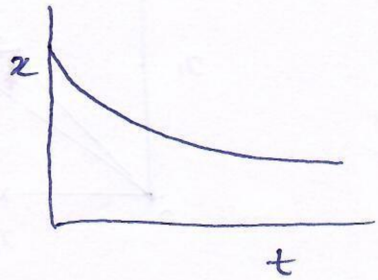
at fixed point, all discrete dynamical variable will be same.

$$\frac{dx}{dt} = ax$$

$a > 0$



$a < 0$



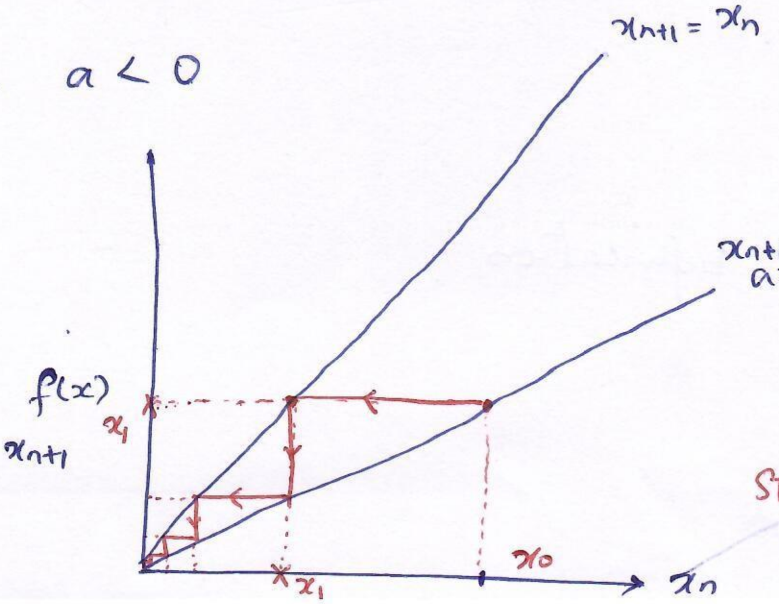
1 equilibrium solution.

$x_e = 0$

How to determine this fixed point?

$f(x) \cdot x_{n+1} = a x_n$

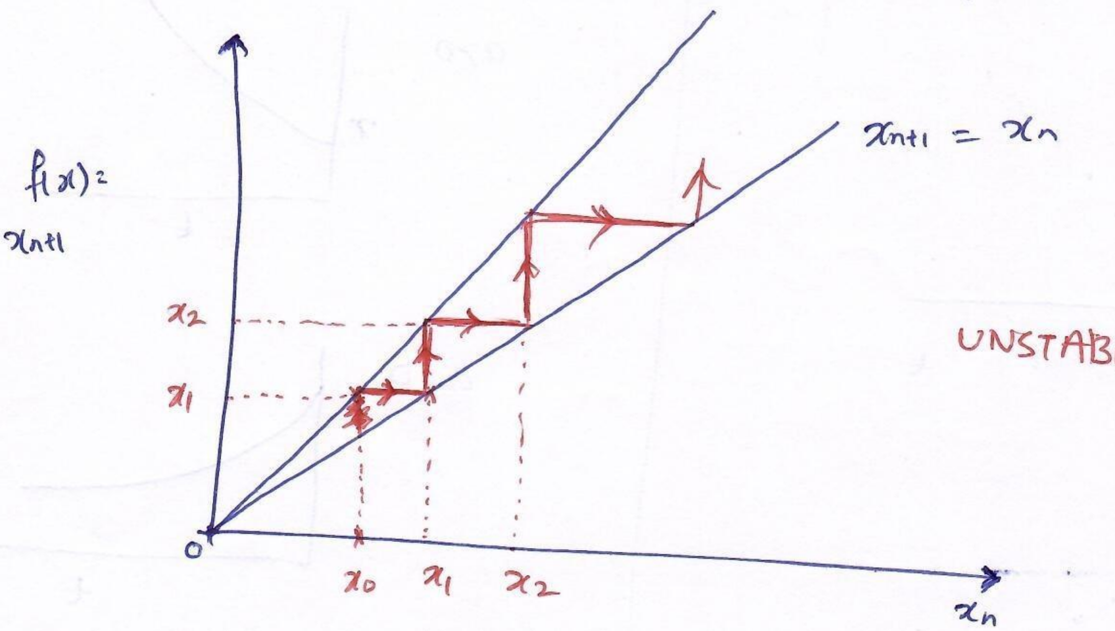
let $a < 0$



$a < 0$

STABLE SOLUTION

$x_{n+1} = a x_n \qquad a > 1$



UNSTABLE

$\because \left. \begin{array}{l} f(x_n) = x_{n+1} \\ x_{n+1} = x_n \end{array} \right\}$ intersection will tell you the fixed point / equilibrium point

$$\boxed{f(x) = x \quad \leftarrow \text{ in discrete systems}}$$

EXAMPLE - 2

$$\frac{dx}{dt} = a x^2 \quad \longrightarrow \quad x_e = 0$$

$$f(x) = a x^2$$

$$\frac{df}{dx} = 2 a x$$

@ $x_e = 0 \quad \left.\frac{df}{dx}\right|_{x_e = 0} = 0$ bifurication

$\cancel{x_{n+1} = ax}$

$$x_{n+1} = a x_n^2$$

@ fixed point

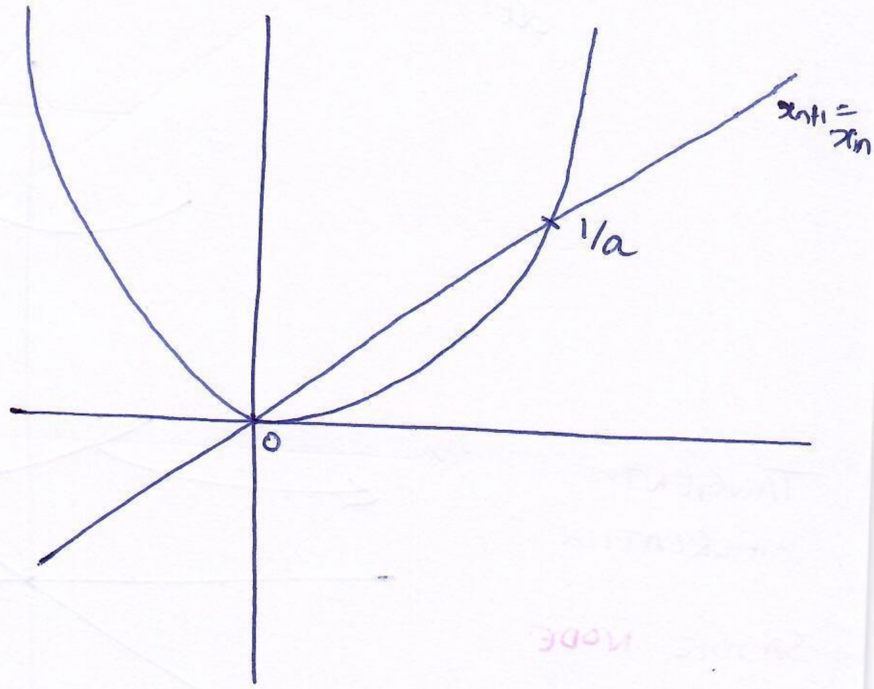
$$x_{n+1} = x_n$$

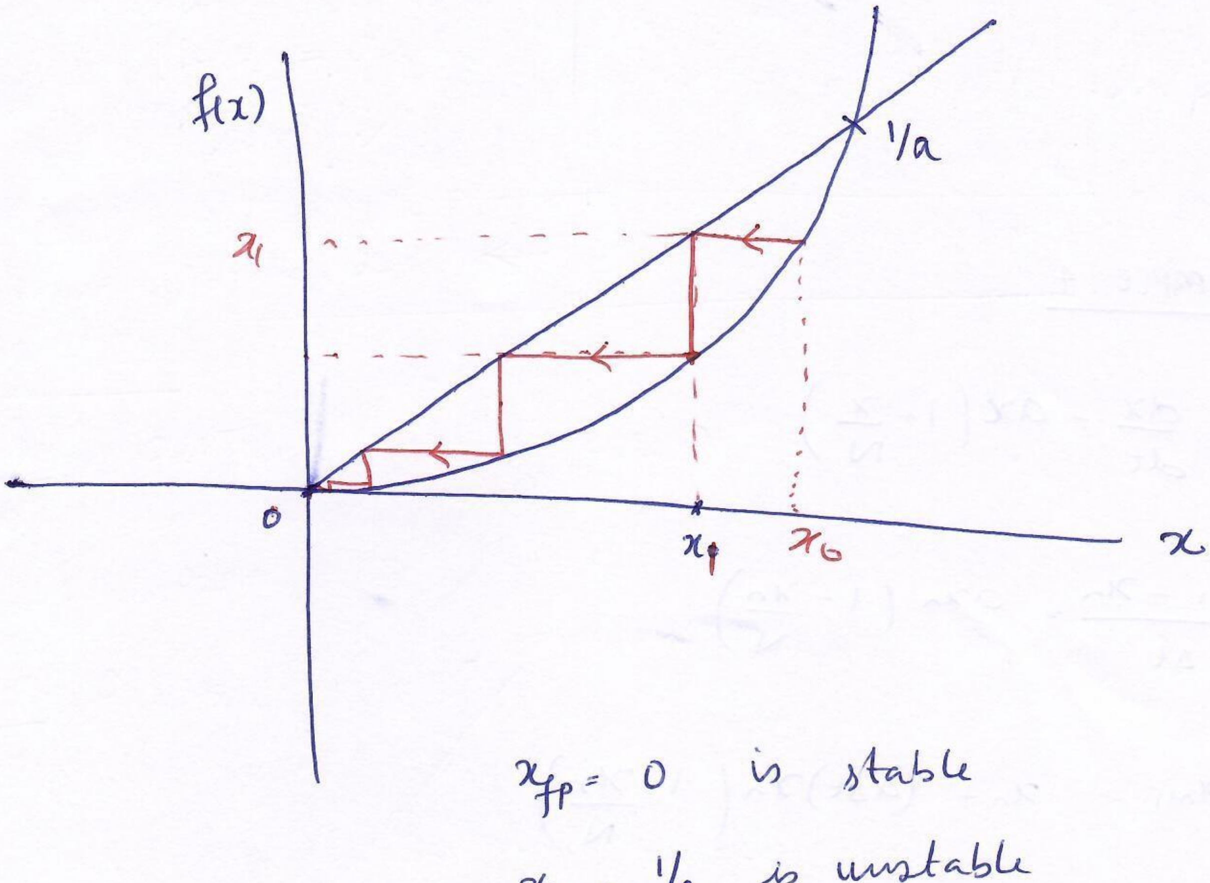
$$x_n = a x_n^2$$

$$(ax^2 - x) = 0$$

$$x(ax - 1) = 0$$

$$x_{fp} = 0 \quad ; \quad x_{fp} = 1/a$$





$x_{fp} = 0$ is stable

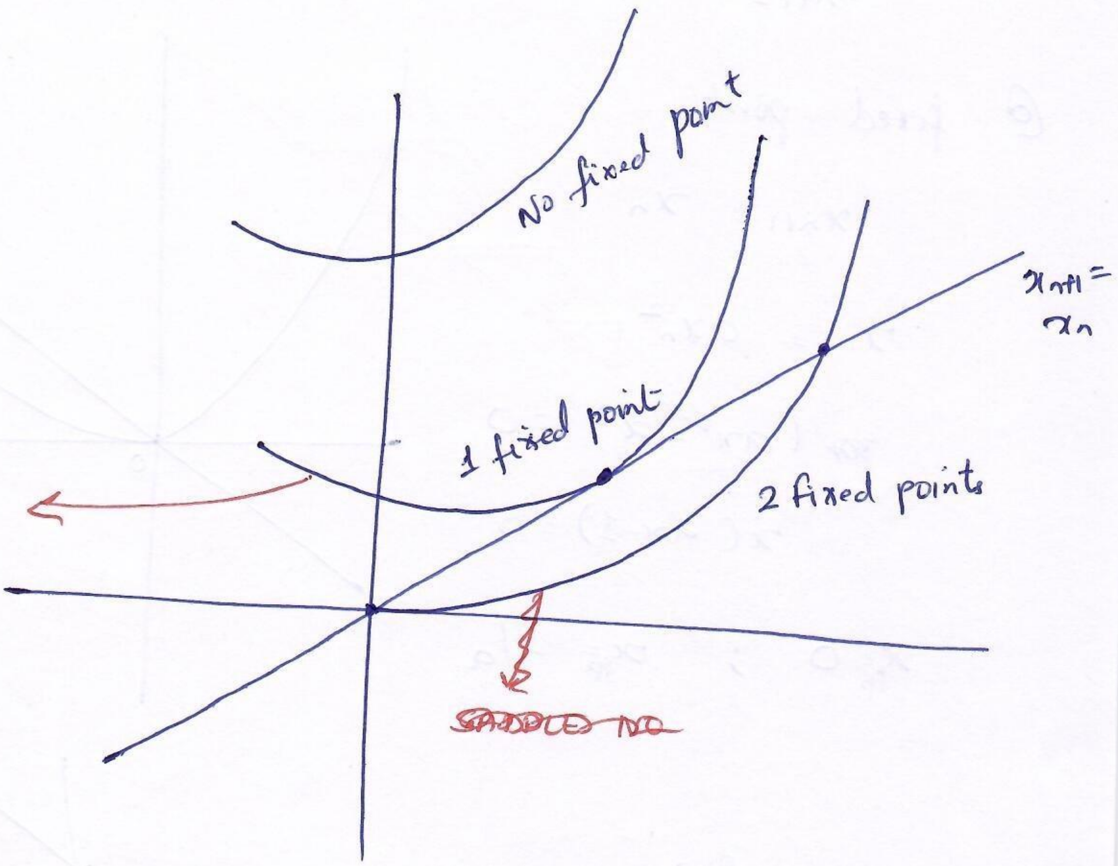
$x_{fp} = 1/a$ is unstable

Example 3.

$$\frac{dx}{dt} = ax^2 + C$$

TANGENT BIFURCATION

SADDLE NODE BIFURCATION



EXAMPLE 4

$$\frac{dx}{dt} = ax\left(1 - \frac{x}{N}\right)$$

$$\frac{x_{n+1} - x_n}{\Delta t} = a x_n \left(1 - \frac{x_n}{N}\right)$$

$$x_{n+1} = x_n + (a\Delta t) x_n \left(1 - \frac{x_n}{N}\right)$$

$$x_{n+1} = x_n \left[1 + a\Delta t \left(1 - \frac{x_n}{N}\right)\right]$$

$$x_{n+1} = x_n \left[1 + a\Delta t - a\Delta t \frac{x_n}{N}\right)$$

$$x_{n+1} = x_n \left[1 + a\Delta t\right] \left[1 - \frac{a\Delta t}{N(1+a\Delta t)} x_n\right]$$

$$x_{n+1} = a' x_n \left(1 - \frac{x_n}{N'}\right)$$

Where, $a' = 1 + a\Delta t$ ⟵ finite Growth parameter

$$N' = \frac{N(1+a\Delta t)}{a\Delta t}$$

$$x_{n+1} = a' x_n (1 - x_n)$$ for normalised population.

To determine fixed point.

$$a x_n (1 - x_n) = x_n$$

at $x_n = x$.

$$a x (1 - x) - x = 0$$

$$x\left[a - ax - 1\right] = 0$$

$$x_{fp} = 0$$

$$x_{fp} = \frac{a-1}{a}$$

at $a = 1$ ⇒ One fixed point

at $a > 1$ ⇒ 2 fixed point

$$f(x) = x_{n+1} = a x_n (1 - x_n)$$

$$x_{fp} = 0 \; ; \; x_{fp} = \frac{a-1}{a}$$

$$\frac{df}{dx} = a - 2ax$$

at $x_{fp} = 0 \Rightarrow \frac{df}{dx} = a$

slope < 1 attracting fixed point (stable) $\Rightarrow a < 1$

slope > 1 repelling fixed point (unstable) $\Rightarrow a > 1$

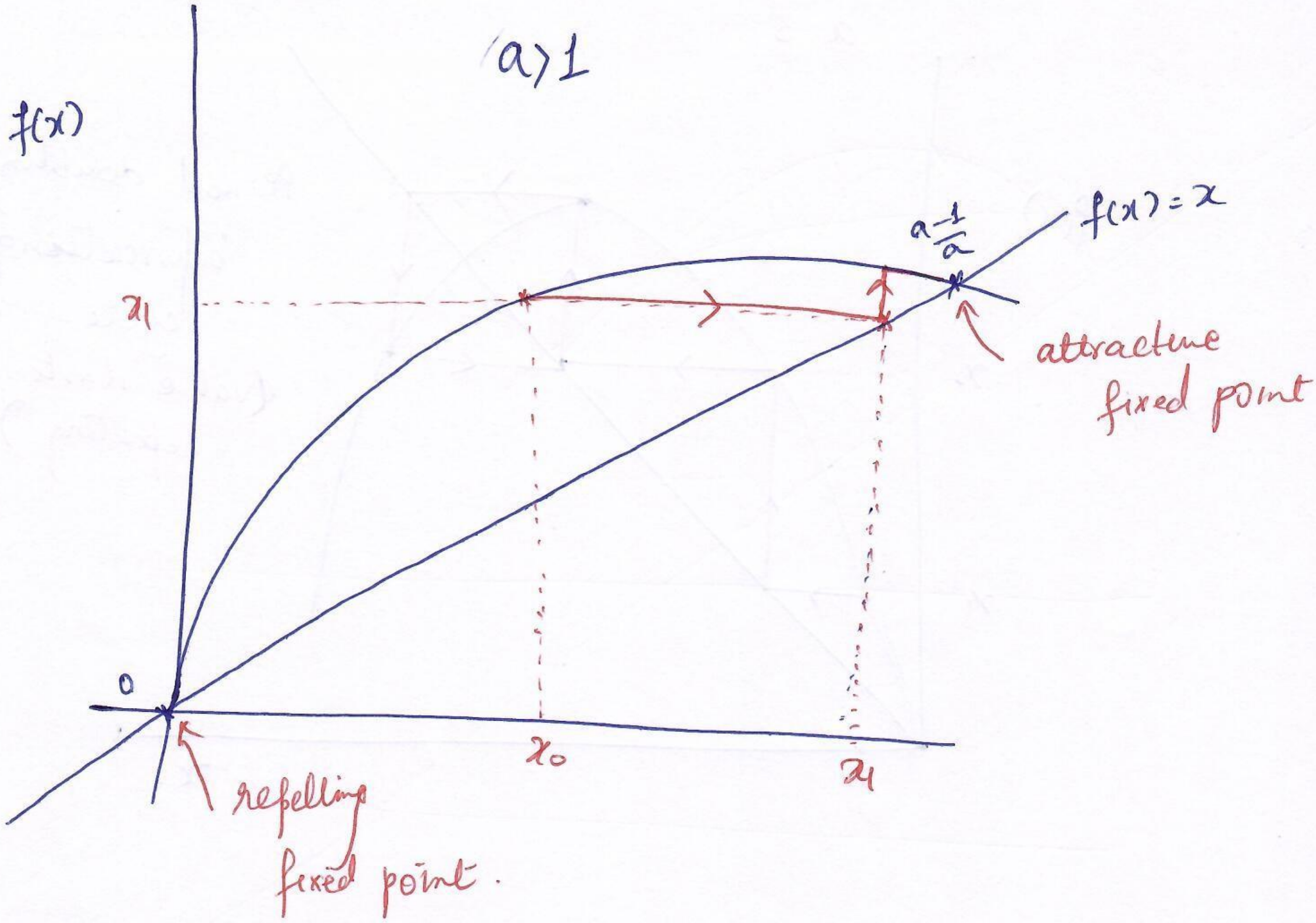
at $x_{fp} = \frac{a-1}{a} \rightarrow \frac{df}{dx} = a - 2\cancel{a} \cdot \frac{(a-1)}{\cancel{a}} = 2 - a$

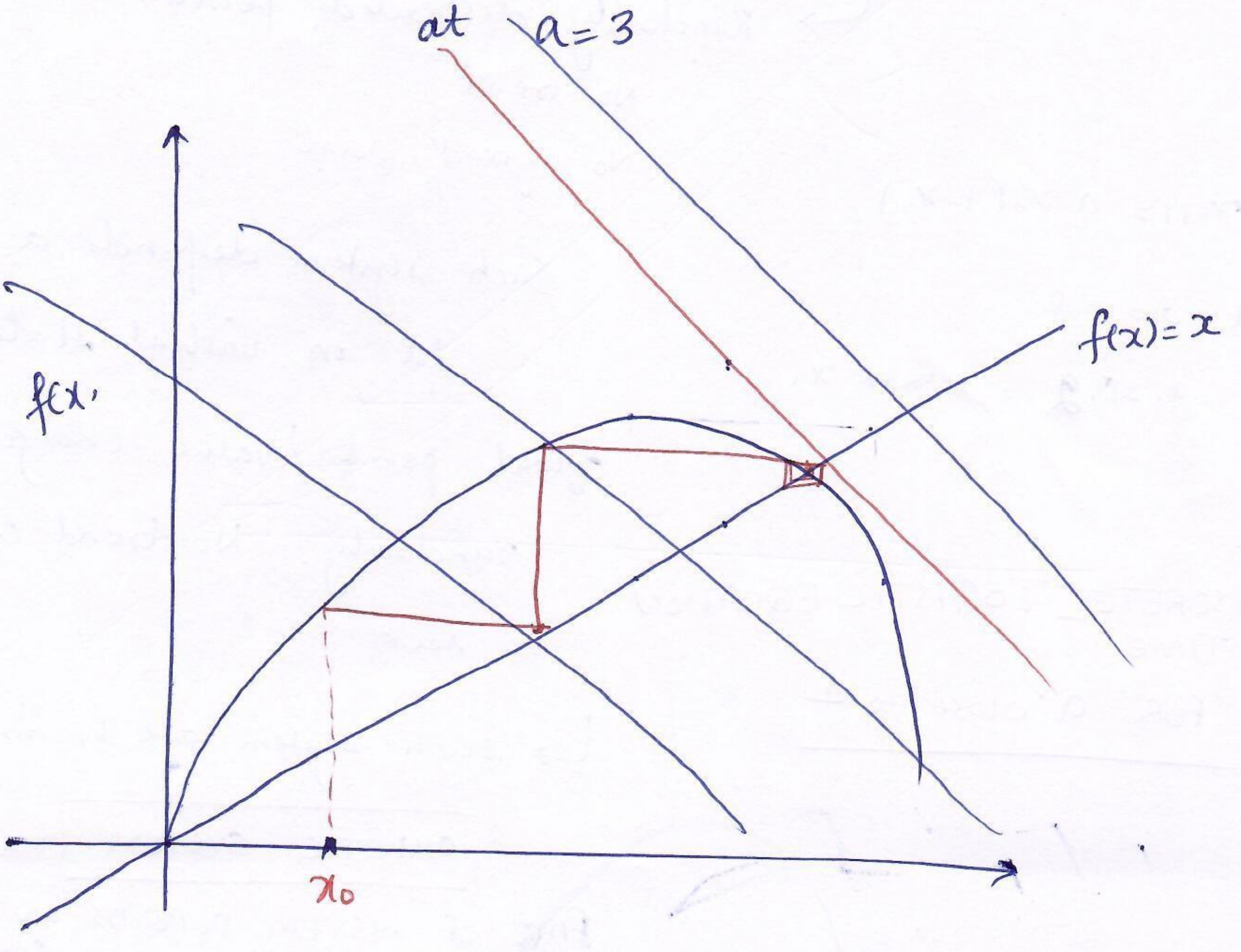
$\frac{df}{dx} = \underline{2-a}$

> 1 repelling fixed point (unstable)

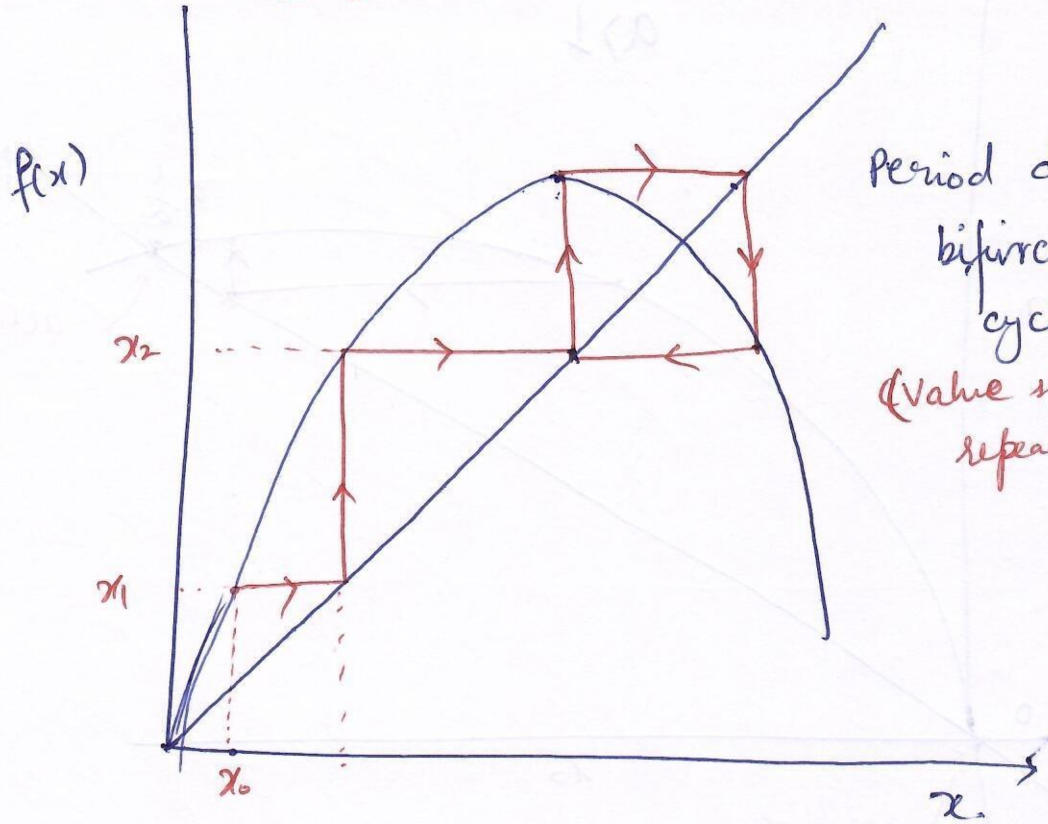
$\underline{< 1}$ attracting fixed point (stable)

if $1 < a < 3 \leftarrow$ attracting fixed point


a>1
f(x)
f(x)=x
a-1/a
x1
attractive fixed point
0
x0
x1
repelling fixed point.


at a=3
f(x)=x
f(x)
x0

$a=3$



Period doubling bifurcating cycle.
(Value starts repeating)

for $a > 3$; $a = 3.8$

↳ Randomly distributed points.
No trend.
No fixed point

$x_{n+1} = a x_n (1 - x_n)$

Let $a = 3.8$
$x_0 = 0.9$ x_1

Such system depends a lot on initial state x_0.

final point / Value changes randomly. No trend can be seen

↳ such system are known as

$\boxed{\text{CHAOTIC SYSTEM}}$

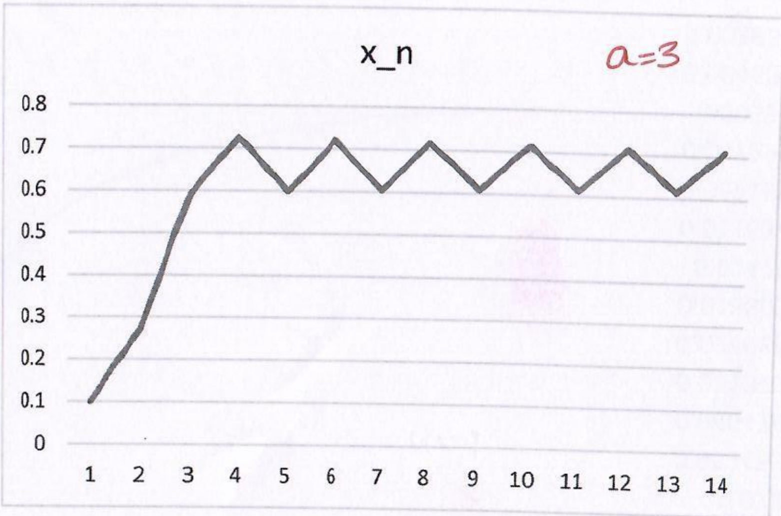
DISCRETE_ TIME LOGISTIC EQUATION

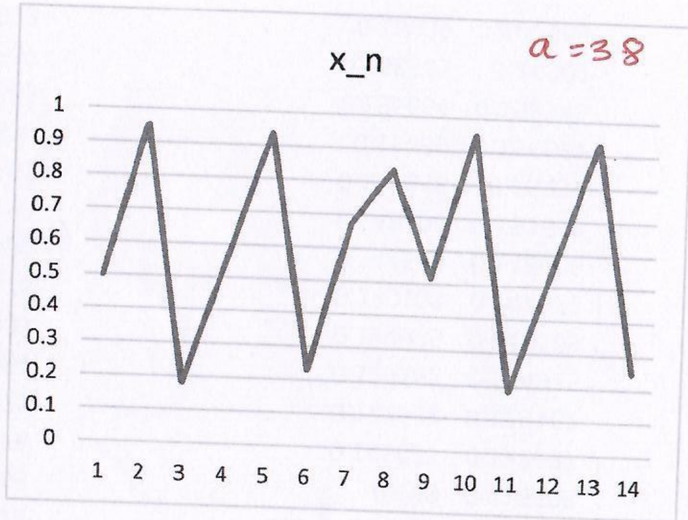
FOR a close to 4

FATE OF SYSTEM DEPENDS ON INITIAL CONDITION

a	x_n	x_n+1
3	0.1	0.27
	0.27	0.5913
	0.5913	0.724993
	0.724993	0.598135
	0.598135	0.721109
	0.721109	0.603333
	0.603333	0.717967
	0.717967	0.607471
	0.607471	0.71535
	0.71535	0.610873
	0.610873	0.713121
	0.713121	0.613738
	0.613738	0.711191
	0.711191	0.616195

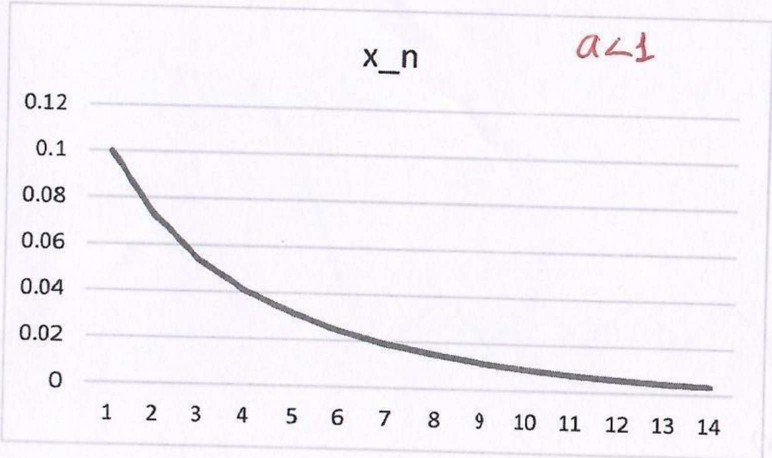
a	x_n	x_n+1
3.8	0.5	0.95
	0.95	0.1805
	0.1805	0.562095
	0.562095	0.935348
	0.935348	0.229794
	0.229794	0.672557
	0.672557	0.836851
	0.836851	0.518819
	0.518819	0.948654
	0.948654	0.185096
	0.185096	0.573174
	0.573174	0.929653
	0.929653	0.248514
	0.248514	0.709668



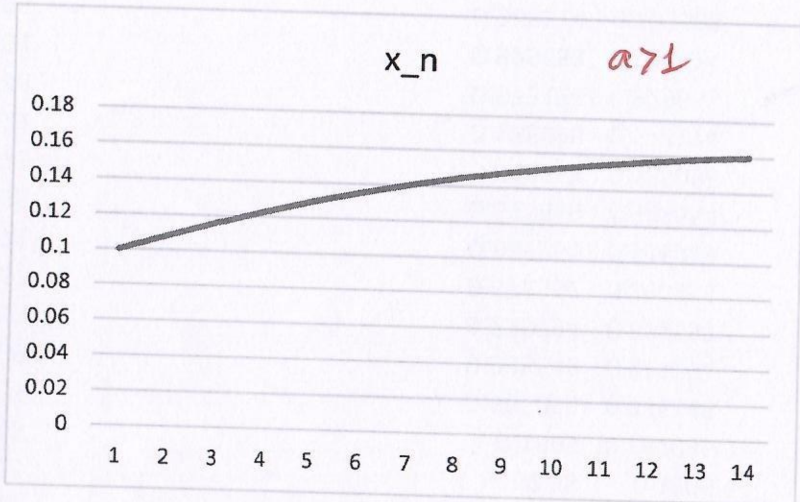


X_n+1 = x_n *a *(1-x_n)

a		x_n	x_n+1
	0.8	0.1	0.072
		0.072	0.053453
		0.053453	0.040476
		0.040476	0.031071
		0.031071	0.024084
		0.024084	0.018803
		0.018803	0.01476
		0.01476	0.011634
		0.011634	0.009199
		0.009199	0.007291
		0.007291	0.00579
		0.00579	0.004605
		0.004605	0.003667
		0.003667	0.002923

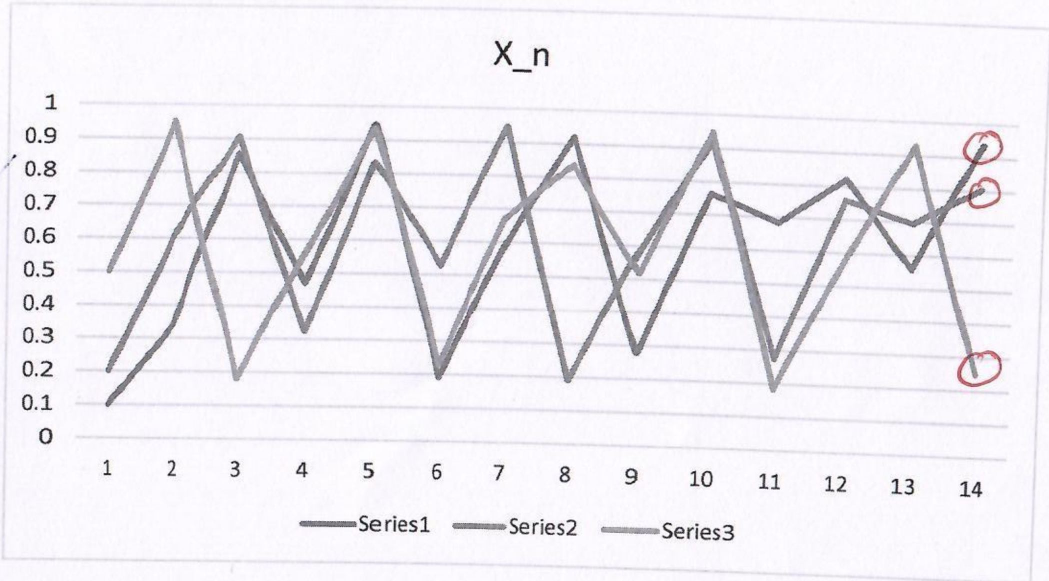


a	x_n	x_n+1
1.2	0.1	0.108
	0.108	0.115603
	0.115603	0.122687
	0.122687	0.129162
	0.129162	0.134975
	0.134975	0.140108
	0.140108	0.144573
	0.144573	0.148406
	0.148406	0.151658
	0.151658	0.15439
	0.15439	0.156664
	0.156664	0.158545
	0.158545	0.16009
	0.16009	0.107569



x_n

0.1	0.2	0.5
0.342	0.608	0.95
0.855137	0.905677	0.1805
0.470736	0.32462	0.562095
0.946746	0.833119	0.935348
0.191589	0.52832	0.229794
0.588555	0.946952	0.672557
0.9202	0.190888	0.836851
0.27904	0.586909	0.518819
0.764472	0.921298	0.948654
0.684208	0.275531	0.185096
0.821056	0.758531	0.573174
0.558307	0.696014	0.929653
0.937081	0.803998	0.248514



For small change in initial condition final output of the system changes — Random

— No trend

Bifurcation diagrams : Assignment 2

$$\frac{dx}{dt} = ax - ax^2 \longrightarrow \begin{matrix} x_e = 0 \\ x_e = 1 \end{matrix} \rightarrow \left.\frac{df}{dx}\right|_{x_e=0} = \cancel{a}\ a \begin{cases} a>0 & \text{unst} \\ a<0 & \text{stable} \end{cases}$$

$$\left.\frac{df}{dx}\right|_{x_e=1} = -a \begin{cases} a>0 & \text{stable} \\ a<0 & \text{Unstable} \end{cases}$$

$$\frac{dx}{dt} = a - x^2$$

$$\frac{dx}{dt} = ax - x^2$$

$$\frac{dx}{dt} = ax - x^3$$

Non linear system

$$\frac{dx_1}{dt} = -x_1$$

$$\frac{dx_2}{dt} = x_1^2 + x_2$$

When should we linearize ??

$$\frac{dx_1}{dt} = -x_1 \Rightarrow x_1 = c_1 e^{-t} \quad \text{--- (1)}$$

$$\frac{dx_2}{dt} = x_1^2 + x_2 \Rightarrow \frac{dx_2}{dt} = c_1^2 e^{-2t} + x_2$$

$$\frac{dx_2}{dt} - x_2 = c_1^2 e^{-2t}$$

$$e^{-t}\frac{dx_2}{dt} - e^{-t} x_2 = c_1^2 e^{-3t}$$

$$\Rightarrow \frac{d}{dt}\left(e^{-t} x_2\right) = c_1^2 e^{-3t}$$

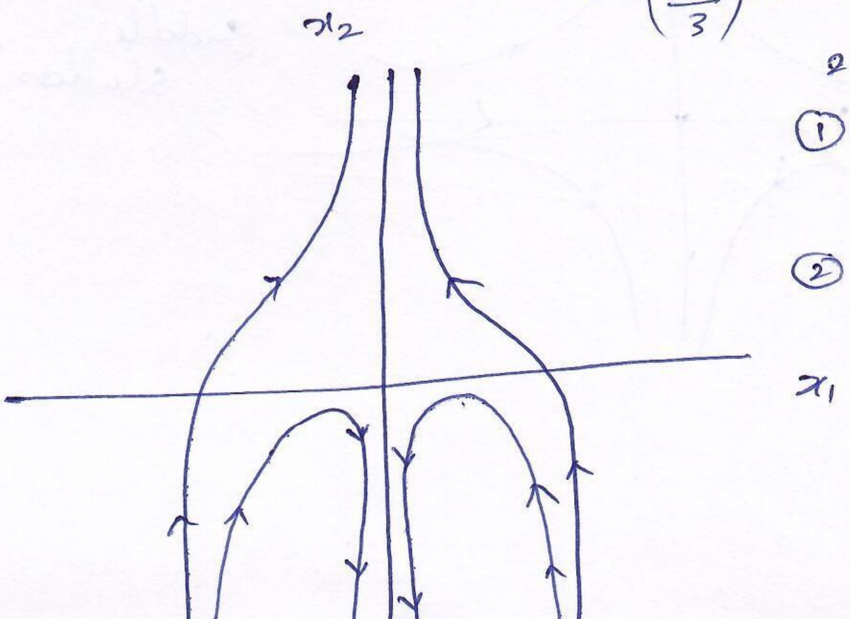
$$e^{-t} x_2 = \left(\frac{-c_1}{3}\right) e^{-3t} + C_2$$

$\Rightarrow$ EASY TO SOLVE ANALYTICALLY

$$x_2 = \left(\frac{-c_1}{3}\right) e^{-2t} + C_2 e^{t} \quad \text{--- (2)}$$

2 options to plot

1. $\dfrac{dx_2}{dx_1} = \dfrac{x_1^2 + x_2}{-x_1}$
2. parametric plot



Lets linearize

(0,0) is solution to the eq^n

$$\frac{dx_1}{dt} = -x_1$$

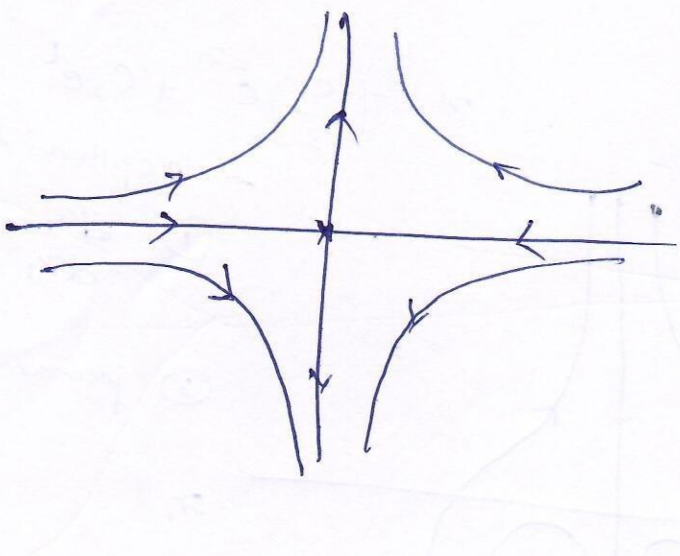
$$\frac{dx_2}{dt} = x_1^2 + x_2$$

as (0,0) is solution
So in close proximity of (0,0)
$x_1 = 0$
for linearizing. lets assume
$x_1 = 0$

$$\frac{dx_2}{dt} = x_2$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1 e^{-t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^{t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



Saddle
solution

$$\frac{dx_1}{dt} = x_1^2 \qquad x_1 = \frac{1}{c_1 - t}$$

$$\frac{dx_2}{dt} = -x_2 \qquad x_2 = c_2 e^{-t}$$

①

linearizing. @ (0,0)

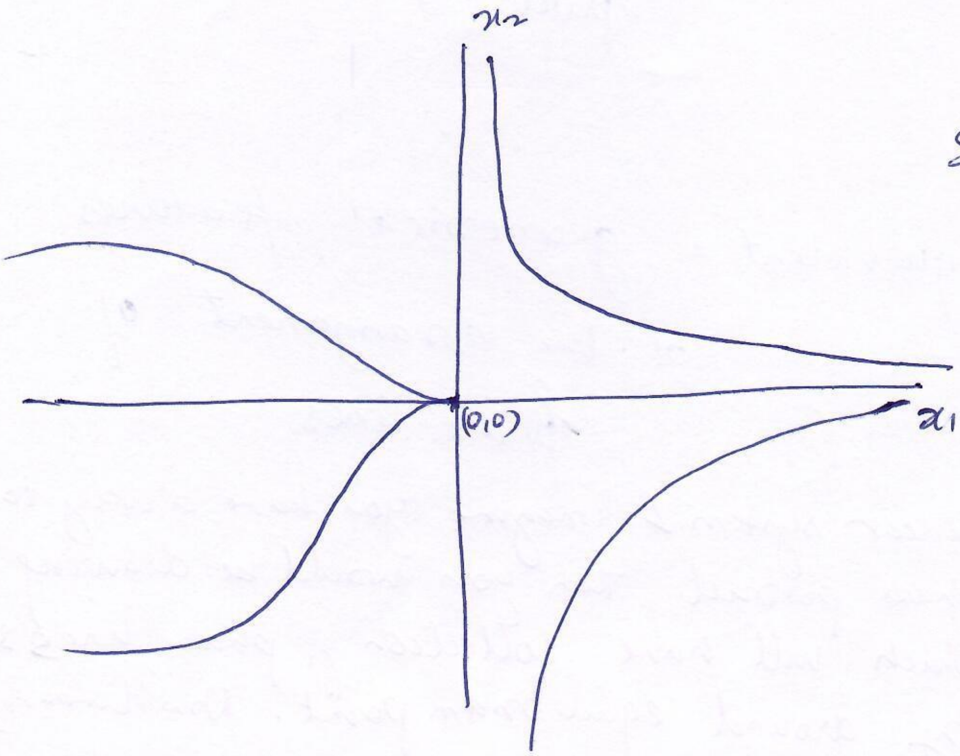
$$\frac{dx_1}{dt} = 0 \quad ; \quad \frac{dx_2}{dt} = -x_2$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\lambda_1 = 0$

$\lambda_2 = -1$

Based on 1st solution



Since $\lambda_1 = 0$ the behaviour of the system is very different

Hartman - Grobman theorem

The orbit structure of a dynamical system in the neighbourhood of a hyperbolic equilibrium point is topologically equivalent to the ~~sb~~ orbit structure of its linearized system.

orbit structure = arrangement of different phase lines in a particular region.

Hyperbolic equilibrium point = No Eigen value will be zero or will have zero as real part.

topologically equivalent = geometrical features of the arrangement of phase lines.

∵ If you take a non-linear system & imagine you have a way to determine complete phase portrait. Then you would be drawing the phase portrait which will have collection of phase lines & a particular geometry around equilibrium point. Now linearize the system & again determine the topological features

Now if the ~~typo~~ equilibrium point is hyperbolic then the nature of dynamical system will be similar according to the Theorem.

This will help us to figure out if system can be linearized or not.

Determine ~~equib~~ equilibrium point, eigen values if none of the eigen value are ~~zeor~~ zero or real part is zero, then system ~~is~~ have hyperbolic equilibrium point & system can be linearized.

if $(0,0)$ is not a solution to equations then perform proper taylor series expansion for linearization.

$$\frac{dx_1}{dt} = x_1^2 - x_2^2 - 1 = f_1$$

$$\frac{dx_2}{dt} = 2x_2 = f_2$$

$f_1 = 0 \qquad x_{1e}^2 - x_{2e}^2 - 1 = 0$

$f_2 = 0 \qquad x_{2e} = 0$

$x_{1e}^2 = 1$

$x_{1e} = \pm 1$

$$\begin{bmatrix} x_{1e} \\ x_{2e} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

TAYLOR SERIES

$$\frac{dx_1}{dt} = x_1^2 - x_2^2 - 1$$

$f_1 = x_1^2 - x_2^2 - 1$

$\frac{\partial f_1}{\partial x_1} = 2x_1 \qquad \frac{\partial f_1}{\partial x_2} = -2x_2$

$\frac{\partial f_2}{\partial x_1} = 0 \qquad \frac{\partial f_2}{\partial x_2} = 2$

$$\underline{J} = \begin{bmatrix} 2x_1 & -2x_2 \\ 0 & 2 \end{bmatrix}$$

$$\underline{J}\Big|_{\begin{bmatrix} -1 \\ 0 \end{bmatrix}} = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} ; \quad J\Big|_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

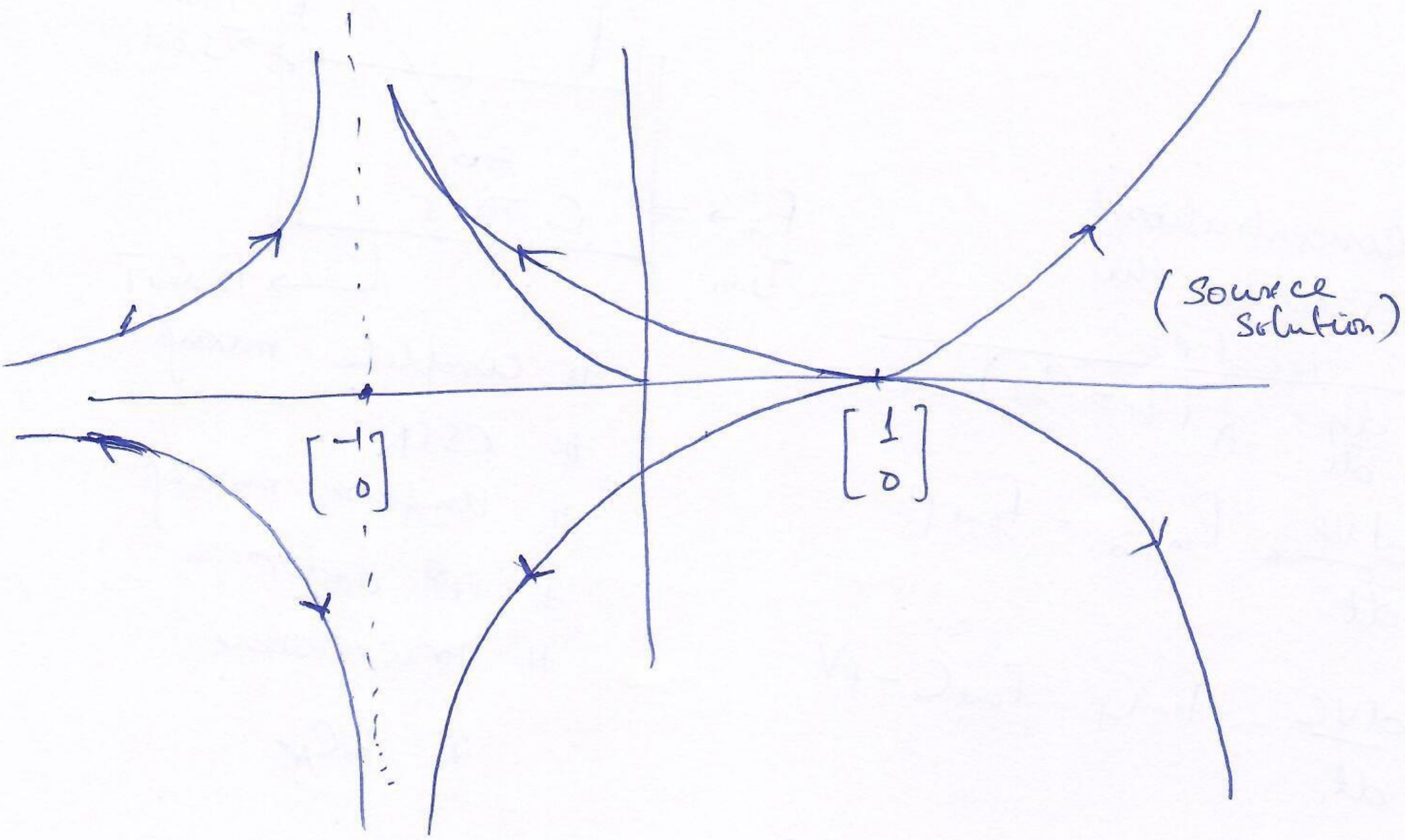
Eigen Values of Both are $(-2,2)$ & $(2,2)$ respectively.

So it can be linearized

$$\begin{bmatrix} x_{1e} \\ x_{2e} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} -1 \\ 0 \end{bmatrix} \rightarrow$ E.V. $(-2,2)$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow$ E.V. $(2,2)$



(Saddle solution)