Q. Determine $\mathcal{L}\left\{\operatorname{erf}\left(\frac{1}{\sqrt{t}}\right)\right\}$.

Solution: By the definition of Laplace transform we have

$$L\left[\operatorname{erf}\left(\frac{1}{\sqrt{t}}\right)\right] = \int_0^\infty e^{-st}\frac{2}{\sqrt{\pi}}\int_0^{\frac{1}{\sqrt{t}}} e^{-u^2}\,\mathrm{d}u\,\mathrm{d}t$$

Changing the order of integration and evaluating the inner integral, we get

$$L\left[\operatorname{erf}\left(\frac{1}{\sqrt{t}}\right)\right] = \frac{2}{\sqrt{\pi}}\int_0^\infty\int_0^{\frac{1}{u^2}} e^{-st}e^{-u^2}\mathrm{d}t\,\mathrm{d}u = \frac{2}{\sqrt{\pi}}\frac{1}{s}\int_0^\infty e^{-u^2}\left(1-e^{-s\frac{1}{u^2}}\right)\mathrm{d}u$$

Using the value of Gaussian integral we have

$$L\left[\operatorname{erf}\left(\frac{1}{\sqrt{t}}\right)\right] = \frac{2}{\sqrt{\pi}}\frac{1}{s}\left[\frac{\sqrt{\pi}}{2} - \int_0^\infty\left(e^{-u^2-s\frac{1}{u^2}}\right)\mathrm{d}u\right]$$

Let us assume

$$I(s) = \int_0^\infty e^{-u^2-s\frac{1}{u^2}}\mathrm{d}u$$

By differentiation under integral sign

$$\frac{dI}{ds} = \int_0^\infty e^{-u^2-s\frac{1}{u^2}}\left(-\frac{1}{u^2}\right)\mathrm{d}u$$

Substitution $\frac{\sqrt{s}}{u} = x \Rightarrow -\frac{\sqrt{s}}{u^2}du = dx$ leads to

$$\frac{dI}{ds} = -\frac{1}{\sqrt{s}}\int_0^\infty e^{-x^2-s\frac{1}{x^2}}\mathrm{d}x = -\frac{1}{\sqrt{s}}I$$

Solving the above differential equation we get

$$\ln I(s) = -2\sqrt{s} + \ln c \quad\Rightarrow\quad I(s) = ce^{-2\sqrt{s}}$$

Further note that

$$I(0) = \int_0^\infty e^{-u^2}\mathrm{d}u = \frac{\sqrt{\pi}}{2} \quad\Rightarrow\quad c = \frac{\sqrt{\pi}}{2}$$

Therefore, we get

$$I(s) = \frac{\sqrt{\pi}}{2}e^{-2\sqrt{s}}$$

Substituting this value in the equation (0.1), we obtain

$$L\left[\operatorname{erf}\left(\frac{1}{\sqrt{t}}\right)\right] = \frac{2}{s\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2}e^{-2\sqrt{s}}\right] = \frac{1-e^{-2\sqrt{s}}}{s}$$