Ques. 1

Assumptions.

- Flow is in x-direction & velocity varies as a function of y.
- Density is constant (Incompressible flow) - No slip condition at wall.
- viscosity is constant. - Laminar flow

Equation of Navier-Stokes equation:

$$\frac{\partial}{\partial t}\rho v + \nabla \cdot \rho v v = -\nabla p + \mu \nabla^2 v + \rho g$$

From Appendix B.4, on page 846, the continuity equation becomes

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

From appendix B.6, the Navier Stokes equation yields

$$\rho\left(\cancel{\frac{\partial v_x}{\partial t}} + \cancel{v_x}\frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \cancel{\frac{\partial v_x}{\partial z}}\right) = -\frac{\partial p}{\partial x} + \mu\left[\cancel{\frac{\partial^2 v_x}{\partial x^2}} + \frac{\partial^2 v_x}{\partial y^2} + \cancel{\frac{\partial^2 v_x}{\partial z^2}}\right] + \rho g_x$$

$$\rho\left(\cancel{\frac{\partial v_y}{\partial t}} + v_x\cancel{\frac{\partial v_y}{\partial x}} + v_y \cancel{\frac{\partial v_y}{\partial y}} + v_z \cancel{\frac{\partial v_y}{\partial z}}\right) = -\frac{\partial p}{\partial y} + \mu\left[\cancel{\frac{\partial^2 v_x}{\partial x^2}} + \cancel{\frac{\partial^2 v_y}{\partial y^2}} + \cancel{\frac{\partial^2 v_y}{\partial z^2}}\right] + \cancel{\rho g_y}$$

$$\rho\left(\cancel{\frac{\partial v_z}{\partial t}} + v_x\cancel{\frac{\partial v_z}{\partial x}} + v_y \cancel{\frac{\partial v_z}{\partial y}} + v_z \cancel{\frac{\partial v_z}{\partial z}}\right) = -\frac{\partial p}{\partial z} + \mu\left[\cancel{\frac{\partial^2 v_z}{\partial x^2}} + \cancel{\frac{\partial^2 v_z}{\partial y^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}\right] + \rho g_z$$

$$\rho v_y \frac{\partial v_x}{\partial y} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 v_x}{\partial y^2} + \rho g_x$$

$$\frac{d^2 v_x}{dy^2} - \frac{\rho v_0}{\mu}\frac{dv_x}{dy} = -\frac{dP}{dx}$$

This ODE is first order in $\frac{dv_x}{dy}$, so we can use integrating factor I

$$I = \exp\left(\int^y -\frac{\rho v_0}{\mu}\, ds\right) = e^{-\frac{\rho v_0 y}{\mu}}$$

Multiply both sides of the ODE by I

$$e^{-\rho v_0/y\mu}\frac{d^2 v_x}{dy^2} - \frac{\rho v_0}{\mu} e^{-\frac{\rho v_0 y}{\mu}}\frac{dv_x}{dy} = \frac{P_L - P_0}{L} e^{-\rho v_0 y/\mu}$$

By product rule left side can be written as $\frac{d}{dy}(Iv_x')$

$$\frac{d}{dy}\left(e^{-\frac{\rho v_0 y}{\mu}}\frac{dv_x}{dy}\right) = \frac{P_L - P_0}{L} e^{-\frac{\rho v_0 y}{\mu}}$$

Integrating both sides with respect to y

$$e^{-\frac{\rho v_0 y}{\mu}}\frac{dv_x}{dy} = \frac{P_L - P_0}{L}\left(\frac{-\mu}{\rho v_0}\right) e^{-\frac{\rho v_0 y}{\mu}} + C_1$$

Multiply both sides by $e^{\rho v_0 y/\mu}$

$$\frac{dv_x}{dy} = \frac{P_L - P_0}{L}\left(\frac{-\mu}{\rho v_0}\right) + C_1 e^{\frac{\rho v_0 y}{\mu}}$$

Integrate both sides w.r.t y once more

$$v_x(y) = \frac{P_L - P_0}{L}\left(\frac{-\mu}{\rho v_0}\right) y + C_1\left(\frac{\mu}{\rho v_0}\right) e^{\frac{\rho v_0 y}{\mu}} + C_2$$

Using B.Cs.

$$v_x(0) = C_1\left(\frac{\mu}{\rho v_0}\right) + C_2 = 0$$

$$v_x(B) = \frac{P_L - P_0}{L}\left(\frac{-\mu}{\rho v_0}\right)B + C_1\left(\frac{\mu}{\rho v_0}\right)e^{\rho v_0 B/\mu} + C_2 = 0$$

Solving this system of equations yields

$$C_1 = \frac{B}{L}\,\frac{P_0 - P_L}{1 - e^{\rho v_0 B/\mu}}, \quad C_2 = \frac{-\mu B}{\rho v_0 L}\,\frac{P_0 - P_L}{1 - e^{\rho v_0 B/\mu}}$$

On substituting these C_1 & C_2 in main equation

$$v_x(y) = \frac{\mu B}{\rho v_0 L}(P_0 - P_L)\left[\frac{y}{B} + \frac{1}{1 - e^{\rho v_0 B/\mu}}e^{\rho v_0 y/\mu} - \frac{1}{1 - e^{\rho v_0 B/\mu}}\right]$$

$$= \frac{\mu B}{\rho v_0 L}(P_0 - P_L)\left[\frac{y}{B} - \frac{e^{\rho v_0 y/\mu} - 1}{e^{\rho v_0 B/\mu} - 1}\right].$$

$\therefore$ $A = \dfrac{\rho v_0 B}{\mu}$

$$v_x(y) = \frac{(P_0 - P_L)B^2}{\mu L}\,\frac{1}{A}\left(\frac{y}{B} - \frac{e^{Ay/B} - 1}{e^A - 1}\right).$$

Ques 2. $[U\times[V\times W]] = V(U\cdot W) - W(U\cdot V)$

$$[U\times[V\times W]]_i = \varepsilon_{ipq}\, U_p\, (V\times W)_q$$

$$= \varepsilon_{ipq}\, U_p\, \varepsilon_{qlm} V_l W_m$$

$$= \varepsilon_{ipq}\, \varepsilon_{qlm}\, U_p V_l W_m \quad -(1)$$

We know,

$$\varepsilon_{ipq}\, \varepsilon_{qlm} = \delta_{il}\,\delta_{pm} - \delta_{im}\,\delta_{pl} \quad -(2)$$

Substituting eq^n (2) in (1)

$$[U\times[V\times W]]_i = (\delta_{il}\,\delta_{pm} - \delta_{im}\,\delta_{pl})\, U_p V_l W_m$$

$$= \underbrace{\delta_{il}\,\delta_{pm}\, U_p V_l W_m}_{(A)} - \underbrace{\delta_{im}\,\delta_{pl}\, U_p V_l W_m}_{(B)} \quad -(3)$$

Solving term A

$$\delta_{il}\,\delta_{pm}\,U_p V_l W_m = U_p V_i W_p$$
$$= (U \cdot W) V$$

Solving term B

$$\delta_{im}\,\delta_{pl}\,U_p V_l W_m = U_p \cancel{V_p}\, U_p W_i$$
$$= (U \cdot V) \cdot W$$

$\therefore$ eqn 3 becomes

$$[U \times [V \times W]] = (U \cdot W) V - (U \cdot V) \cdot W$$

Hence proved.