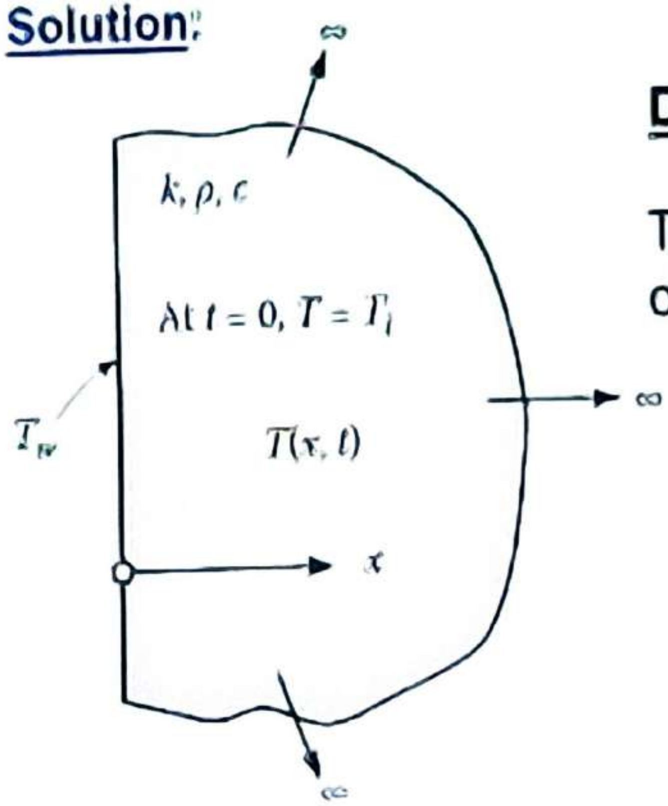
Laplace Transforms: Example-1

Consider the semi-infinite solid which is initially at a uniform temperature T_i.

The surface temperature is changed to T_w at $t = 0$ and is maintained constant at this value for times t > 0.

Assume constant thermo-physical properties. Find the unsteady-state temperature distribution $T(x, t)$ in the solid for $t > 0$.

Solution:



Define: $\theta(x, t) = T(x, t) - T_i$

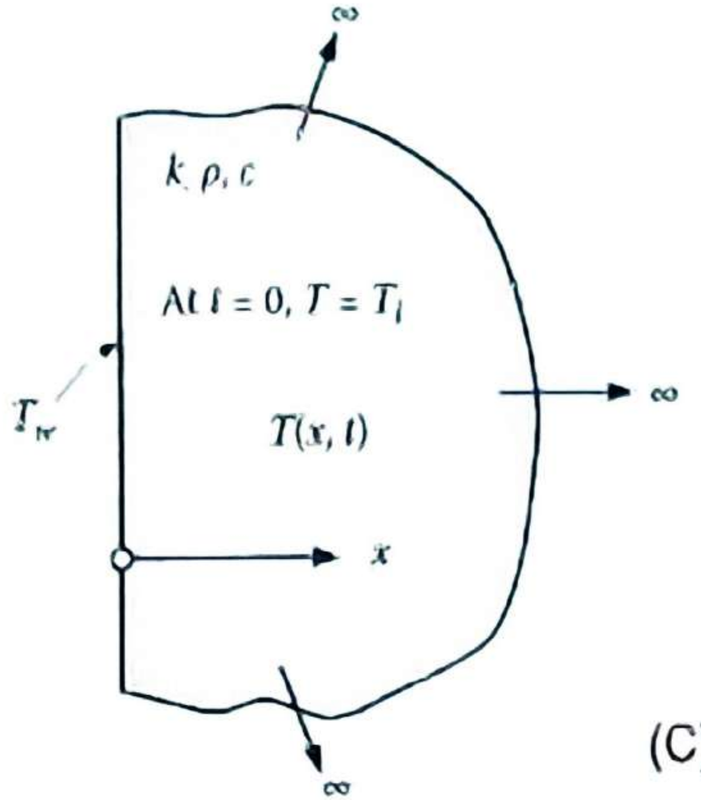
The formulation of the problem in terms of $\theta(x, t)$ is:

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}$$

$$\theta(x, 0) = 0$$

$$\theta(0, t) = T_w - T_i = \theta_w \quad \text{and} \quad \lim_{x\to\infty}\theta(x, t) = 0$$

Laplace Transforms: Example-1



$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t} \qquad \theta(x,0) = 0$$

$$\theta(0,t) = T_w - T_i = \theta_w \quad \text{and} \quad \lim_{x\to\infty}\theta(x,t) = 0$$

The transformed differential equation and boundary conditions are

$$\frac{d^2\bar{\theta}}{dx^2} - \frac{p}{\alpha}\bar{\theta} = 0$$

$$\text{(C)} \quad \bar{\theta}(0,p) = \frac{\theta_w}{p} \quad \text{and} \quad \lim_{x\to\infty}\bar{\theta}(x,p) = 0 \quad \text{(B)}$$

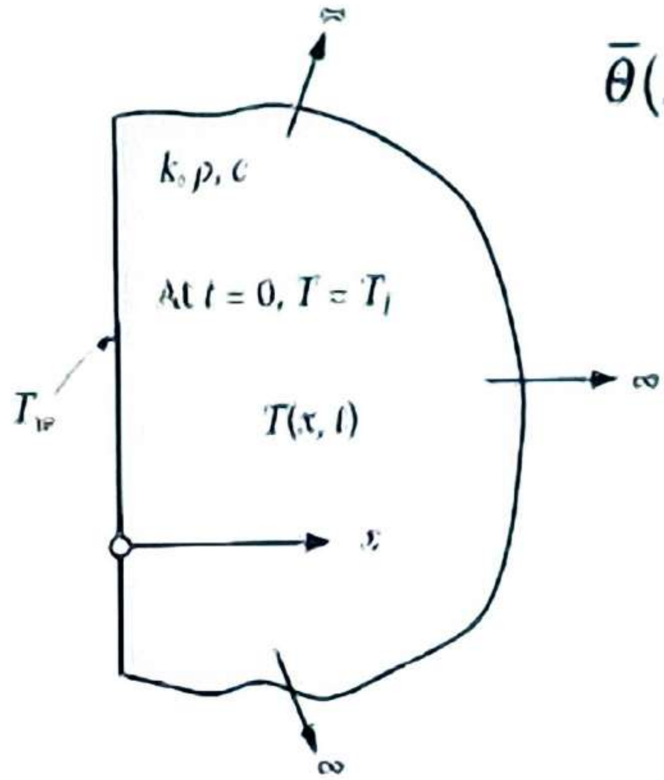
where $\bar{\theta}(x,p)$ is the Laplace transform of $\theta(x, t)$.

<u>**The general solution**</u>: $\bar{\theta}(x,p) = C_1 e^{-mx} + C_2 e^{mx}$ where C_1 and C_2 are the constants of integration, and $m^2 = p/\alpha$.

From BC (C): $C_2 = 0$

From BC (B): $C_1 = \theta_w/p$

Laplace Transforms: Example-1



$$\bar{\theta}(x,p) = C_1 e^{-mx} + C_2 e^{mx}$$

$C_2 = 0$

$C_1 = \theta_w/p$

$m^2 = p/\alpha$

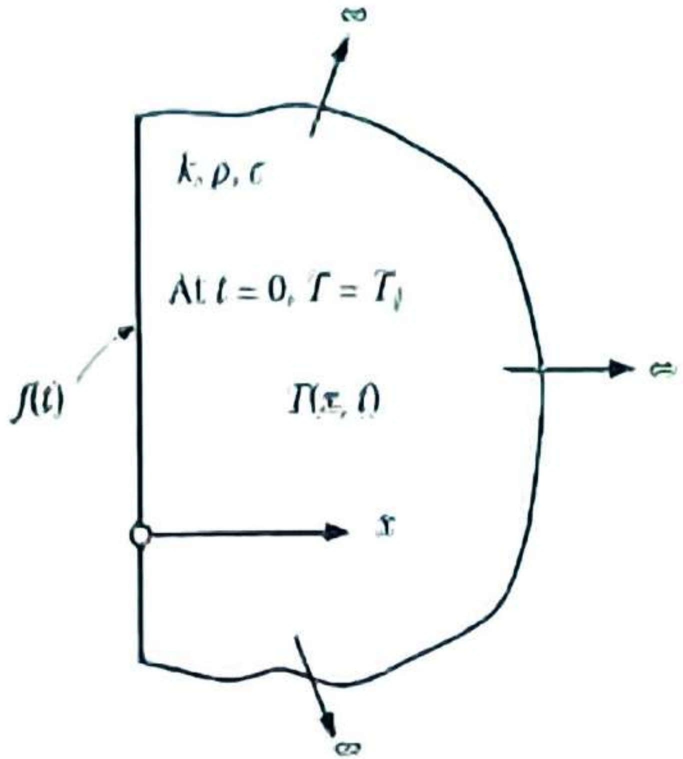
Solution:

$$\frac{\bar{\theta}(x,p)}{\theta_w} = \frac{e^{-mx}}{p} = \frac{-e^{x\sqrt{p/\alpha}}}{p}$$

Take inverse transform:

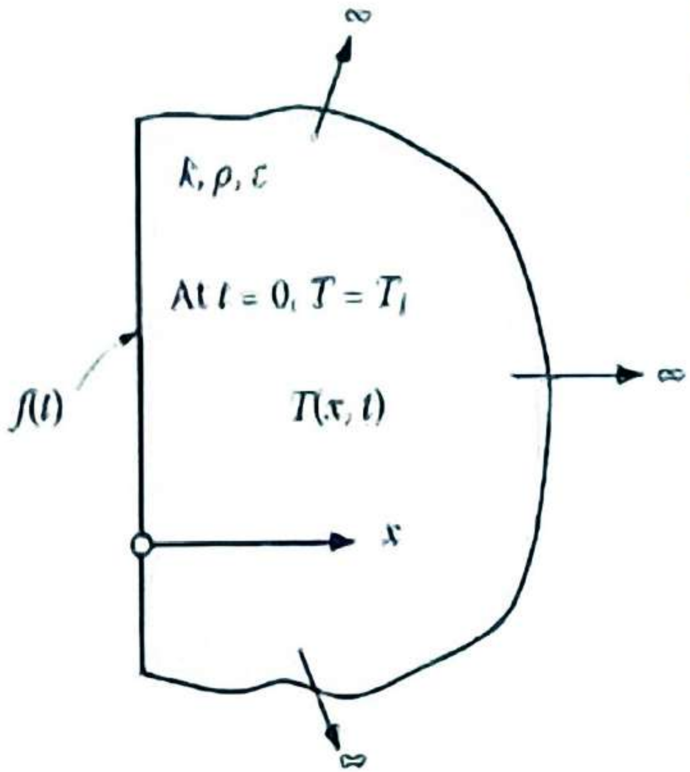
$$\frac{\theta(x,t)}{\theta_w} = \frac{T(x,t)-T_i}{T_w - T_i} = \mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

Laplace Transforms: Example-2



One-dimensional unsteady-state heat conduction in a semi-infinite solid with time-dependent surface temperature

Laplace Transforms: Example-2



Let us now consider the unsteady-state heat conduction in the same semi-infinite solid when the surface temperature variation is specified as a prescribed function $f(t)$ of time for $t \geq 0$.

The formulation of the problem in terms of $\theta(x, t) = T(x, t) - T_i$ is given by

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}$$

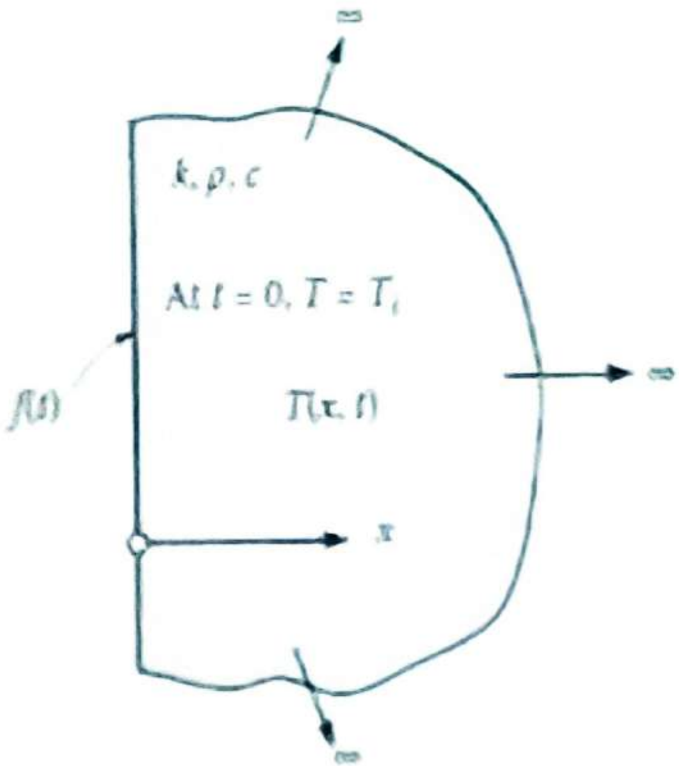
The semi-infinite solid is initially at a uniform temperature T_i. The surface temperature starts varying as $f(t)$ for $t \geq 0$. Assume constant thermo-physical properties. Find the unsteady-state temperature distribution $T(x, t)$ in the solid for $t > 0$.

$$\theta(x,0) = 0$$

$$\theta(0,t) = F(t) \quad \text{and} \quad \lim_{x\to\infty}(x,t) = 0$$

where $F(t) = f(t) - T_i$.

Laplace Transforms: Example-2



$$\frac{\partial^2\theta}{\theta x^2} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}$$

$$\theta(x,0) = 0$$

$$\theta(0,t) = F(t) \quad \text{and} \quad \lim_{t\to\infty}(x,t) = 0$$

where $F(t) = f(t) - T_i$.

The transformed differential equation and boundary conditions are given by

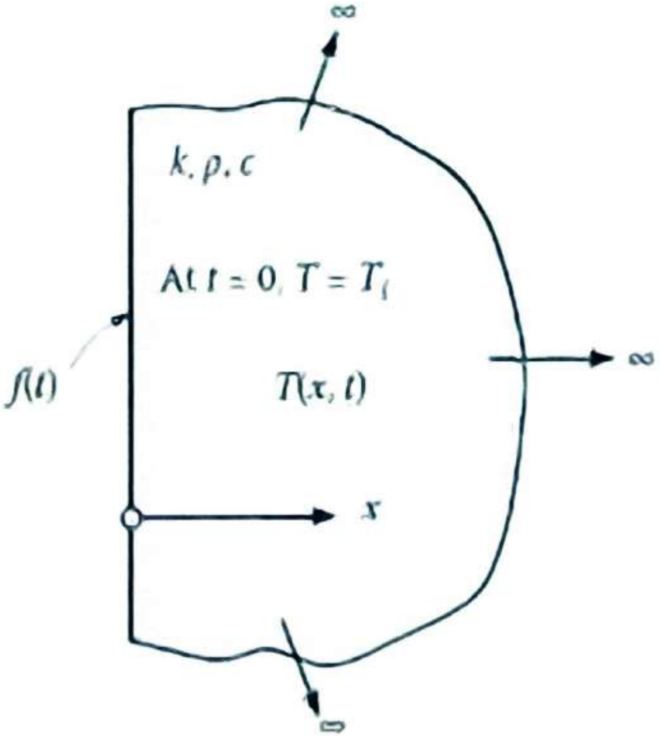
$$\frac{\partial^2\bar{\theta}}{\partial x^2} - \frac{p}{\alpha}\bar{\theta} = 0$$

$$\bar{F}(p) = \mathcal{L}[F(t)] = \bar{f}(p) - \frac{T_i}{p}$$

$$\bar{\theta}(0,p) = \bar{F}(p) \quad \text{and} \quad \lim_{x\to\infty}\bar{\theta}(x,p) = 0$$

$$\bar{f}(p) = \mathcal{L}[f(t)] = \int_0^\infty e^{-pt} f(t)\,dt$$

Laplace Transforms: Example-2



$$\frac{\partial^2 \bar{\theta}}{\partial x^2} - \frac{p}{\alpha}\bar{\theta} = 0$$

$$\bar{\theta}(0, p) = \bar{F}(p) \quad \text{and} \quad \lim_{x \to \infty} \bar{\theta}(x, p) = 0$$

The solution of the above can be written as

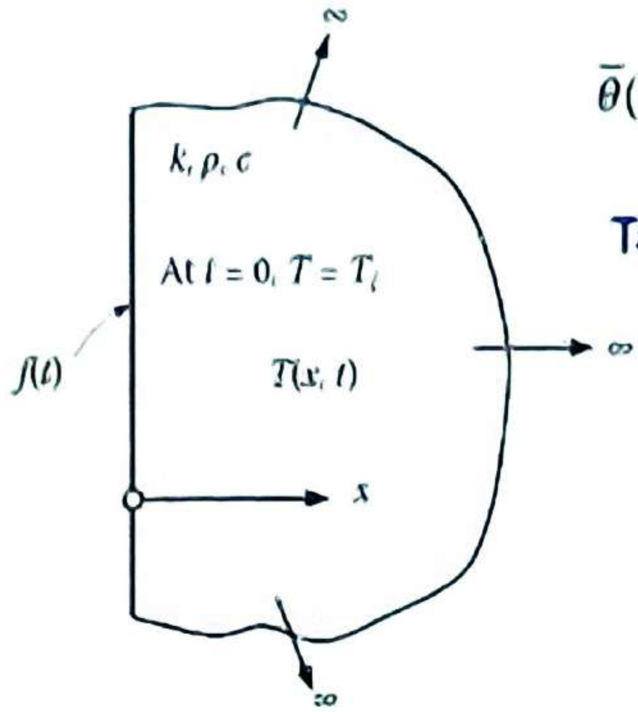
$$\theta(x, p) = C_1 e^{-mx} + C_2 e^{mx}$$

$$m^2 = p/\alpha.$$

$$C_2 = 0, \quad C_1 = \bar{F}(p).$$

Thus, $\bar{\theta}(x, p) = \bar{F}(p) e^{-mx} = \bar{F}(p) e^{-x\sqrt{p/\alpha}} = \left[\bar{f}(p) - \frac{T_i}{p}\right] e^{-x\sqrt{p/\alpha}}$

Laplace Transforms: Example-2



$$\bar{\theta}(x,p) = \bar{F}(p)e^{-mx} = \bar{F}(p)e^{-x\sqrt{p/\alpha}} = \left[\bar{f}(p) - \frac{T_i}{p}\right]e^{-x\sqrt{p/\alpha}}$$

Take inverse transform:

$$\theta(x,t) = T(x,t) - T_i$$

$$= \mathcal{L}^{-1}\left\{\bar{f}(p)e^{-x\sqrt{p/\alpha}}\right\} - T_i\mathcal{L}^{-1}\left\{\frac{1}{p}e^{-x\sqrt{p/\alpha}}\right\}$$

$$T(x,t) = T_i - T_i\,\mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) + \frac{x}{2\sqrt{\pi\alpha}}\int_0^t \frac{f(t-t')}{(t')^{3/2}}\exp\left(-\frac{x^2}{4\alpha t}\right)dt'$$

Initial Value Problems with Time-Dependent BCs and/or Sources

For example, consider a problem in which we have a unit–width slab with zero temperature maintained at x = 0, zero initial temperature, and at t = 0 the temperature at x = 1 is instantaneously brought to unity.

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$
$$T(0,t) = 0$$
$$T(1,t) = H(t)$$
$$T(x, t \leq 0) = 0$$

"Formally" the BC is now represented by a time–dependent function – although we have simply introduced some new mathematical terminology into a problem that can be readily solved with SOV and superposition methods.

We want that the condition at x = 1 would appear in the general form of

$$T(1,t) = F(t)$$

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$
$$T(0,t) = 0$$
$$T(1,t) = H(t)$$
$$T(x, t \leq 0) = 0$$

Problems with time–dependent BCs (or heat sources) can be solved by

- Integral (Fourier) Transform
- **Duhamel's Method**

where F is a prescribed function of t, and F = 0 when t <= 0

Duhamel Superposition Integral

Duhamel's theorem is an analytical method in which the solution for a forced system is obtained from an integral transformation of the solution to the fundamental problem that corresponds to the system.

The systems to which this method can be applied must have an initial state of zero temperature, and be completely homogeneous except for a single time–dependent forcing function appearing in a BC or the source.

This forcing function can be only a function of time (not of position). The corresponding fundamental solution, which is given the symbol $U(x, t)$, represents the solution to the same system except with the forcing function replaced by an equivalent unit step function at $t = 0$.

Duhamel Superposition Integral

Engineering problems where the boundary condition functions are time dependent:

- Heat transfer in internal combustion engine walls
- Space reentry problems
- Solar heating
- Periodic oscillation of temperature or flux
- In nuclear reactor fuel elements during power transients, the energy generation rate in the fuel elements varies with time

Duhamel Superposition Integral

Duhamel's theorem provides a convenient approach for

developing a solution to heat conduction problems with time-dependent boundary conditions and/or time-dependent energy generation,

by utilizing the

solution to the same problem, but with time-independent boundary conditions and/or time-independent energy generation.

In this method the solution of an auxiliary problem with a constant boundary condition is used to construct the solution to the same problem with a time dependent condition.

Duhamel's method is based on superposition of solutions, it follows that it is limited to linear equations.

Duhamel Superposition Integral

Consider a one-dimensional, heat conduction problem in a region $\Re$ with one homogeneous boundary condition and one non-homogenous, time-dependent boundary condition.

In addition, the problem is *initially at zero temperature*, which is a necessary restriction for Duhamel's theorem.

The formulation for such a problem in Cartesian coordinates is given as

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t} \qquad \text{in} \qquad 0 < x < L, \qquad t > 0$$

$$\text{BC1:} \qquad T(x=0,t) = f(t)$$

$$\text{BC2:} \qquad T(x=L,t) = 0$$

$$\text{IC:} \qquad T(x,t=0) = 0$$

Duhamel Superposition Integral

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t} \quad \text{in} \quad 0 < x < L, \quad t > 0$$

BC1: $T(x=0,t) = f(t)$

BC2: $T(x=L,t) = 0$

IC: $T(x,t=0) = 0$

Let us consider an auxiliary problem where $f(t)$ is now a stepwise disturbance of unity. The formulation of this auxiliary problem, which we will denote as $\Phi(x, t)$, follows as

$$\frac{\partial^2 \Phi(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \Phi(x,t)}{\partial t} \quad \text{in} \quad 0 < x < L, \quad t > 0$$

BC1: $\Phi(x=0,t) = 1$

BC2: $\Phi(x=L,t) = 0$

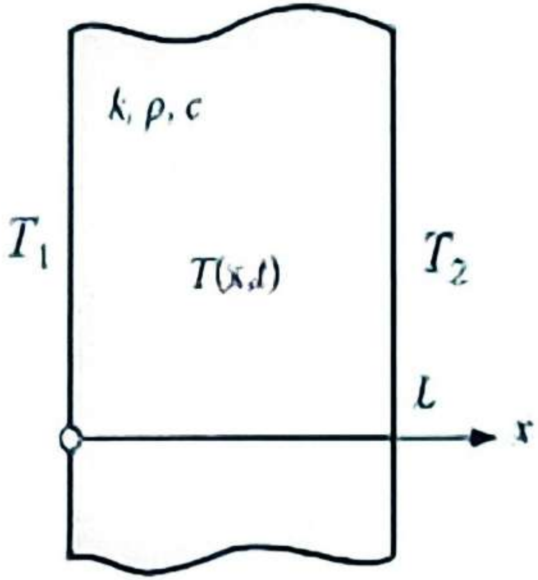
IC: $\Phi(x,t=0) = 0$

This formulation corresponds to the unit-step function for non-homogeneity in the original problem

$$H(t) = \begin{cases} 0, & t \le 0 \\ 1 & t > 0 \end{cases}$$

This can be solved by SOV

Duhamel Superposition Integral



$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$T(x,0) = f(x)$$

$$T(0,t) = T_1, \quad T(L,t) = T_2$$

Try: $T(x,t) = T_s(x) + T_t(x,t)$

$$\frac{d^2 T_s}{dx^2} = 0$$

$$T_s(0) = T_1, \quad T_s(L) = T_2$$

steady-state solution

$$\frac{\partial^2 T_t}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T_t}{\partial t}$$

$$T_t(x,0) = f(x) - T_s(x)$$

$$T_t(0,t) = 0, \quad T_t(L,t) = 0$$

transient solution

Duhamel Superposition Integral

$$\frac{\partial^2 \Phi(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \Phi(x,t)}{\partial t} \qquad \text{in} \qquad 0 < x < L, \qquad t > 0$$

BC1: $\Phi(x=0,t) = 1$

Auxiliary problem

BC2: $\Phi(x=L,t) = 0$

IC: $\Phi(x,t=0) = 0$

The solution $\Phi(x, t)$ satisfies the PDE of our original problem, as well as the homogeneous boundary condition of equation, and the initial condition of equation.

Only the time-dependent boundary condition of equation remains to be satisfied.

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t} \qquad \text{in} \qquad 0 < x < L, \qquad t > 0$$

BC1: $T(x=0,t) = f(t)$

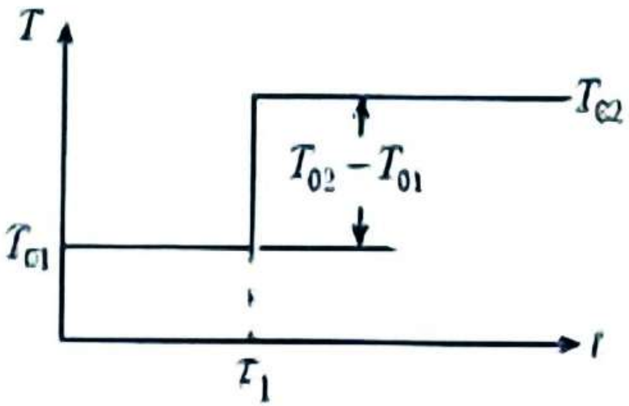
Original problem: BC2: $T(x=L,t) = 0$

IC: $T(x,t=0) = 0$

Duhamel Superposition Integral

To illustrate how solutions can be superimposed to construct a solution for a time dependent boundary condition, consider transient conduction in a plate which is initially at a uniform temperature equal to zero.

At time $t > 0$, one boundary is maintained at temperature T_{01}.
At time $t = \tau_1$, the temperature is changed to T_{02}. Find $T(x, t)$.



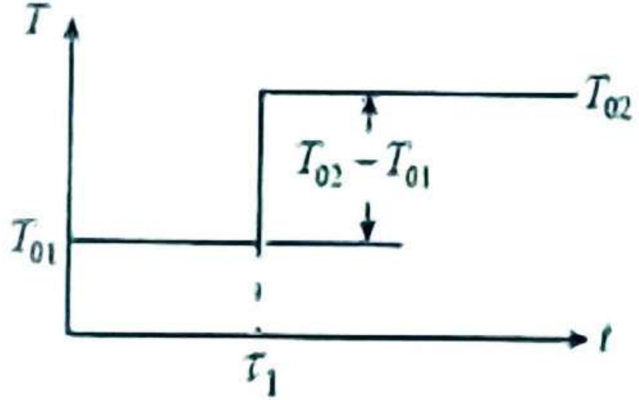
Assuming that the heat equation is linear, the problem can be decomposed into two problems:

- one which starts at $t = 0$, and
- a second one which starts at $t = \tau_1$

Note that each problem has a constant temperature boundary condition. Thus,

$$T(x,t) = T_1(x,t) + T_2(x,t-\tau_1)$$

Duhamel Superposition Integral



$$T(x,t) = T_1(x,t) + T_2(x,t-\tau_1). \quad \text{(A)}$$

Let $\bar{T}(x,t)$ be the solution to an auxiliary problem corresponding to constant surface temperature of magnitude unity. Thus the solution for $T_1(x, t)$ is

$$T_1(x,t) = T_{01}\,\bar{T}(x,t). \quad \text{(B)}$$

At $t = \tau_1$ the temperature is increased by (T_{02} -T_{01}). The solution to this problem is,

$$T_2(x,t-\tau_1) = (T_{02} - T_{01})\,\bar{T}(x,t-\tau_1). \quad \text{(C)}$$

Adding (B) and (C) gives the solution to the problem with the time dependent boundary condition (A)

$$T(x,t) = T_{01}\,\bar{T}(x,t) + (T_{02} - T_{01})\,\bar{T}(x,t-\tau_1).$$

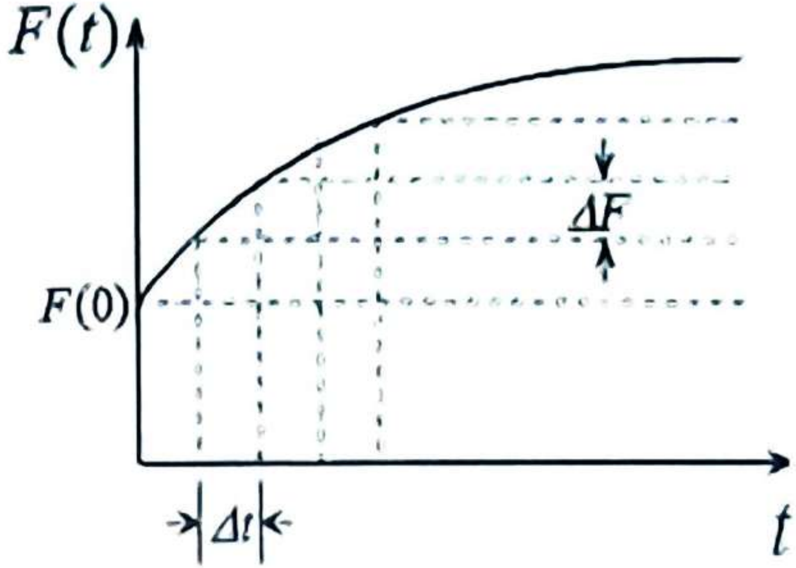
Duhamel Superposition Integral

Generalization:

We now generalize this superposition scheme to obtain solutions to problems with boundary conditions which change arbitrarily with time.

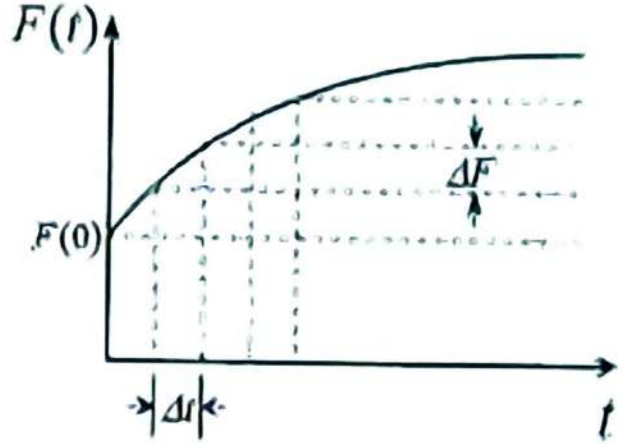
Note that $F(t)$ represents any time dependent boundary condition and that it is not limited to a specified temperature.

It could, for example, represent time dependent heat flux, ambient temperature or surface temperature.



Duhamel Superposition Integral

Generalization:



The function $F(t)$ is approximated by n consecutive steps of ΔF corresponding to time steps Δt.

The first problem starts at time $t = 0$ and has a finite step function of magnitude $F(0)$.

The contribution of this problem to the solution is

$$T_0(x,t) = F(0)\,\bar{T}(x,t)\,.$$

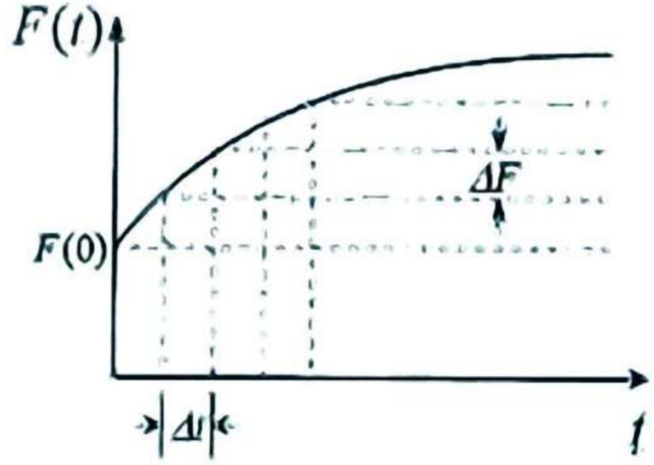
The contribution of the i-th problem, with input $\Delta F(\tau_i)$ to the solution is

$$T_i(x,t) = \Delta F(\tau_i)\,\bar{T}(x,t-\tau_i)\,.$$

Adding all solutions gives: $T(x,t) = F(0)\,\bar{T}(x,t) + \sum_{i=1}^{n} \Delta F(\tau_i)\bar{T}(x,t-\tau_i)\,.$

Duhamel Superposition Integral

Generalization:



Adding all solutions gives:

$$T(x,t) = F(0)\,\overline{T}(x,t) + \sum_{i=1}^{n} \Delta F(\tau_i)\overline{T}(x,t-\tau_i).$$

This result can be written as:

$$T(x,t) = F(0)\overline{T}(x,t) + \sum_{i=1}^{n} \frac{\Delta F(\tau_i)}{\Delta \tau_i}\overline{T}(x,t-\tau_i)\,\Delta\tau_i.$$

In the limit as $n \to \infty,\quad \Delta\tau_i \to d\tau,\quad \dfrac{\Delta F(\tau_i)}{\Delta \tau_i} \to \dfrac{dF(\tau)}{d\tau}$

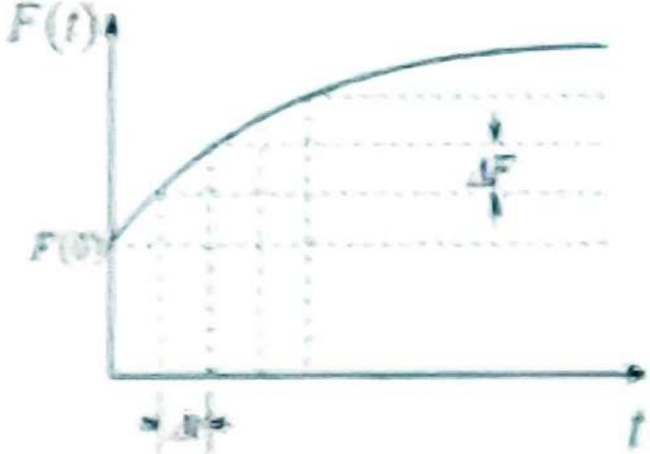
and the summation is replaced by integration, the above becomes

$$T(x,t) = F(0)\,\overline{T}(x,t) + \int_0^t \frac{dF(\tau)}{d\tau}\,\overline{T}(x,t-\tau)\,d\tau\,.$$

This is Duhamel's superposition integral

Duhamel Superposition Integral

Generalization:



$$T(x,t) = F(0)\,\bar{T}(x,t) + \int_0^t \frac{dF(\tau)}{d\tau}\,\bar{T}(x,t-\tau)\,d\tau\,.$$

Using integration by parts and recalling that $\bar{T}(x,0) = 0$,

$$T(x,t) = \int_0^t F(\tau)\,\frac{\partial \bar{T}(x,t-\tau)}{\partial t}\,d\tau\,.$$

Duhamel Superposition Integral

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t} \quad \text{in} \quad 0 < x < L, \qquad t > 0$$

BC1: $T(x=0,t) = f(t)$

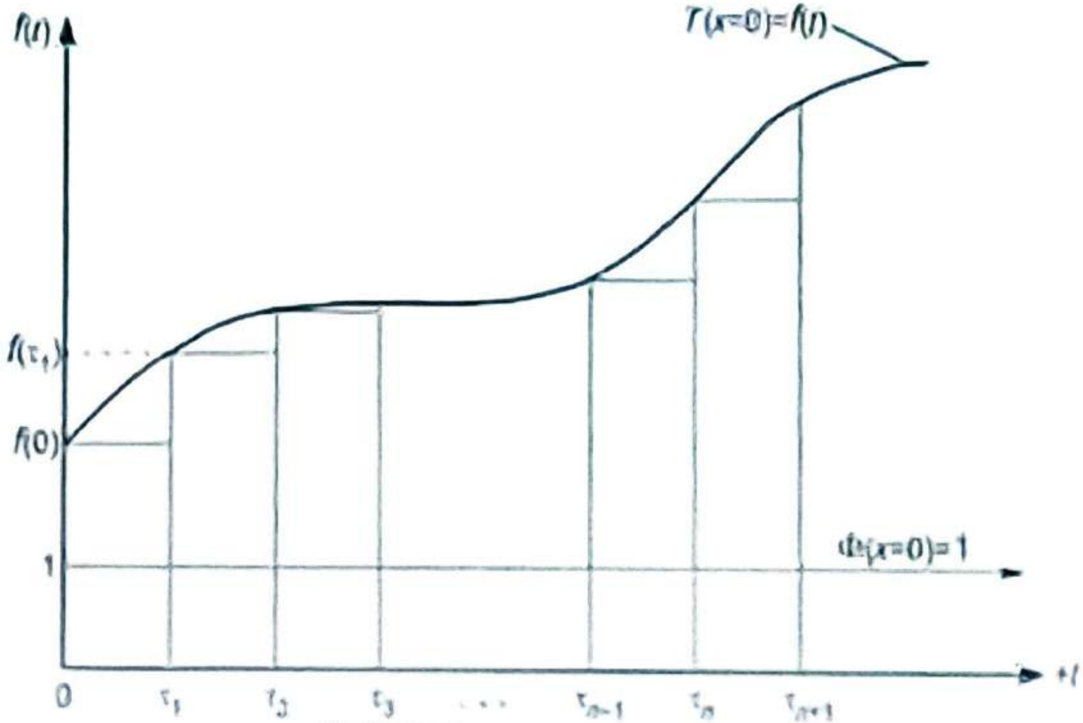
BC2: $T(x=L,t) = 0$

IC: $T(x,t=0) = 0$

Original problem

We will now seek a solution of our original problem in terms of the auxiliary problem $\Phi(x, t)$ by approximating the solution at the time-dependent boundary.

Time-dependent boundary condition at $x = 0$

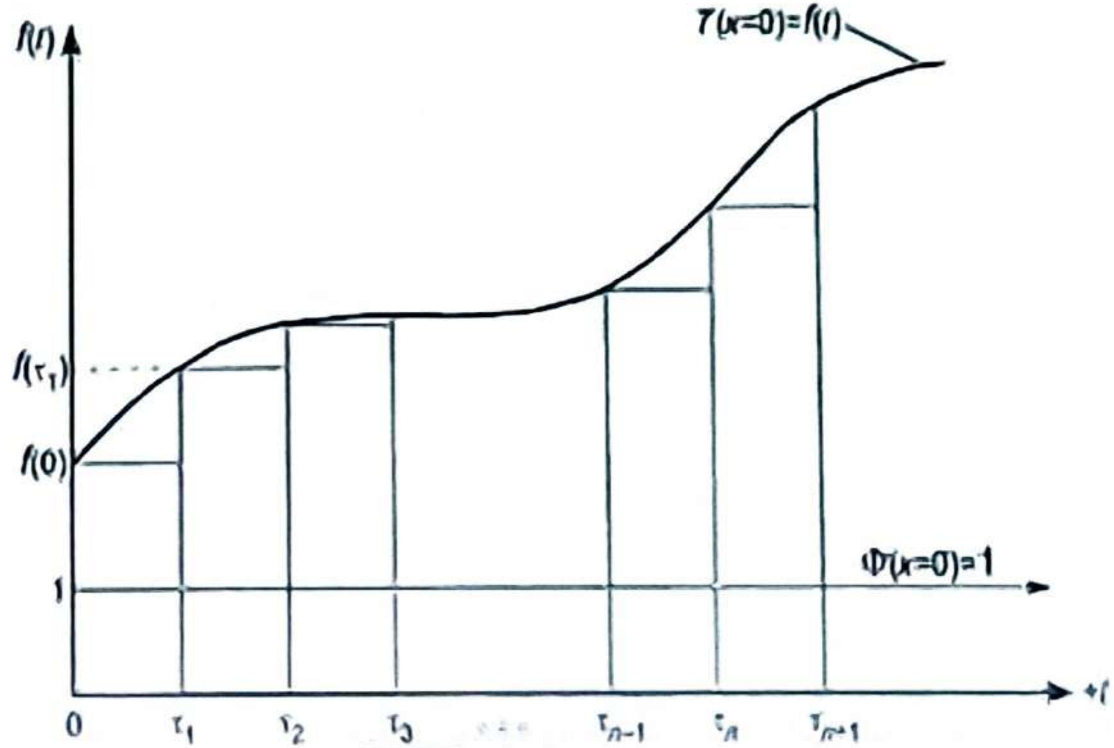


Duhamel Superposition Integral

Our approximate solution is now formed in terms of $\Phi(x, t)$, giving over the limited temporal domain $0 \le t \le \tau_1$

$$T(x,t) \cong f(0)\Phi(x,t) \qquad \text{for} \qquad 0 \le t \le \tau_1$$

Time-dependent boundary condition at x = 0

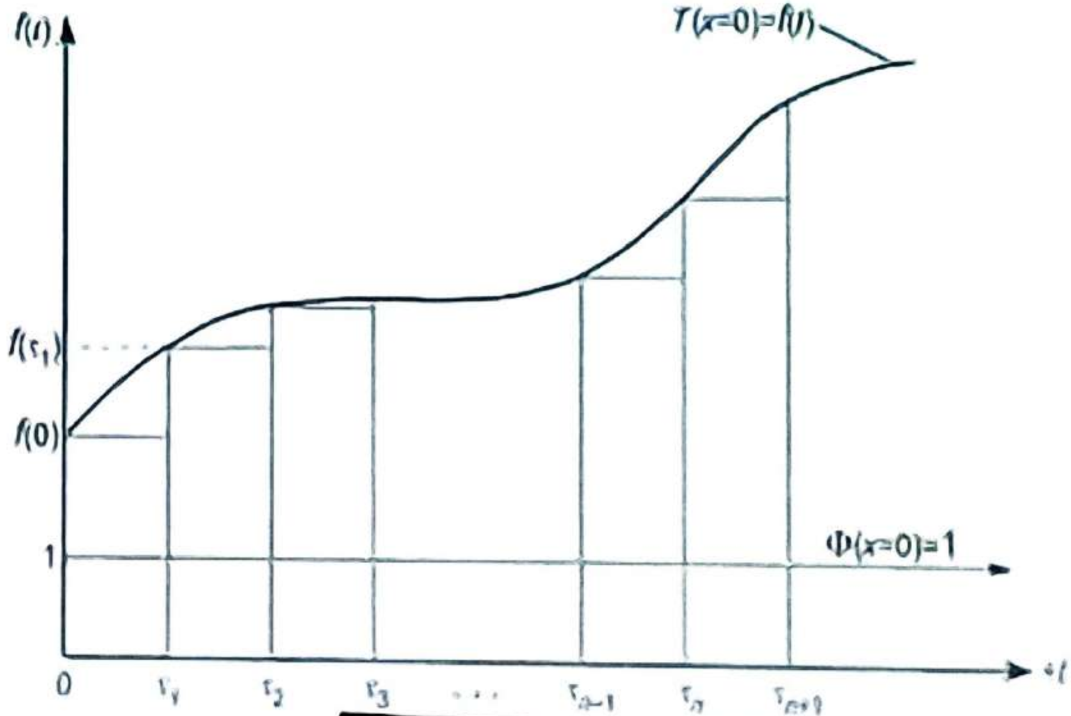


Duhamel Superposition Integral

We extend our approach by adding additional increments by shifting our unit-step solution

$$T(x,t) \cong f(0)\Phi(x,t) + \left[f(\tau_1) - f(0)\right]\Phi(x, t-\tau_1) + \left[f(\tau_2) - f(\tau_1)\right]$$
$$\times\, \Phi(x, t-\tau_2) + \cdots + \left[f(\tau_n) - f(\tau_{n-1})\right]\Phi(x, t-\tau_n)$$

Time-dependent boundary condition at x = 0



Duhamel Superposition Integral

$$T(x,t) \cong f(0)\Phi(x,t) + \left[f(\tau_1) - f(0)\right]\Phi(x,t-\tau_1) + \left[f(\tau_2) - f(\tau_1)\right]$$
$$\times\, \Phi(x,t-\tau_2) + \cdots + \left[f(\tau_n) - f(\tau_{n-1})\right]\Phi(x,t-\tau_n)$$

Introducing the following notation,

$$\Delta f_m = f(\tau_m) - f(\tau_{m-1})$$

$$\Delta \tau_m = \tau_m - \tau_{m-1}$$

our approximate solution becomes

$$T(x,t) \cong f(0)\Phi(x,t) + \sum_{m=1}^{N} \Phi(x,t-\tau_m)\left(\frac{\Delta f_m}{\Delta \tau_m}\right)\Delta \tau_m$$

Duhamel Superposition Integral

$$T(x,t) \cong f(0)\Phi(x,t) + \sum_{m=1}^{N} \Phi(x, t-\tau_m)\left(\frac{\Delta f_m}{\Delta \tau_m}\right)\Delta\tau_m$$

We may now let our number of increments tend to infinity, $N \to \infty$, and therefore $\tau_m \to 0$, in which case the term in parenthesis becomes a derivative and the summation becomes an integral. This gives an exact solution to our original problem of the form.

$$T(x,t) = f(0)\Phi(x,t) + \int_{\tau=0}^{t} \Phi(x, t-\tau)\frac{df(\tau)}{d\tau}d\tau$$

This is Duhamel's superposition integral

<u>Note:</u>

τ is a variable of integration. We have evaluated the time-dependent non-homogeneity in terms of this variable τ.

Duhamel Superposition Integral

Duhamel's superposition integral:

$$T(x,t) = f(0)\Phi(x,t) + \int_{\tau=0}^{t} \Phi(x,t-\tau)\frac{df(\tau)}{d\tau}d\tau \qquad \textbf{(A)}$$

Alternative form of Duhamel's superposition integral:

$$T(x,t) = \int_{\tau=0}^{t} f(\tau)\frac{\partial\Phi(x,t-\tau)}{\partial t}d\tau$$

The alternative form is based on integration by parts of **(A)** and

$\Phi(x,t=0) \equiv 0$ based on the original problem statement (i.e, requirement of zero initial temperature)

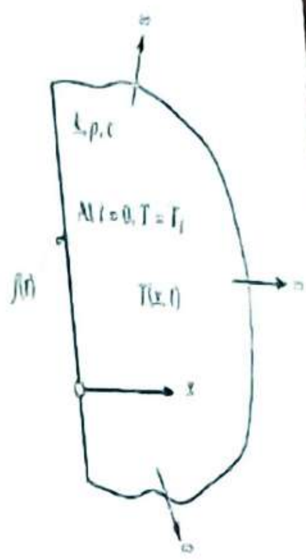
and using the relation $\dfrac{\partial\Phi(x,t-\tau)}{\partial t} = -\dfrac{\partial\Phi(x,t-\tau)}{\partial\tau}$

Duhamel Superposition Integral

Another Example

Original problem:

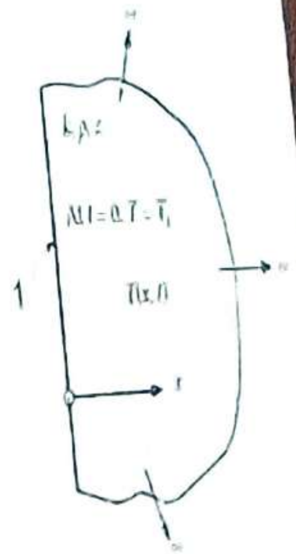
A semi-infinite solid, $0 \leq x < \infty$, initially at a uniform temperature T_i. For times $t \geq 0$, the surface temperature at $x = 0$ is specified as a prescribed function of time as $T(0, t) = f(t)$.



Assuming constant thermo-physical properties, find the unsteady-state temperature distribution $T(x, t)$.

Auxiliary problem:

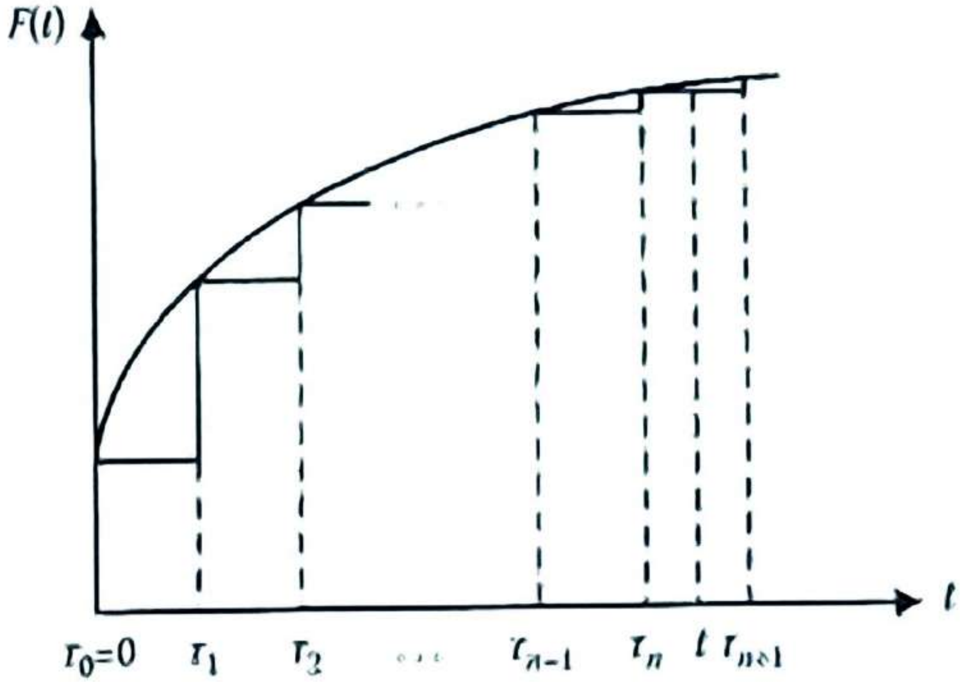
A semi-infinite solid, $0 \leq x < \infty$, initially at a uniform temperature T_i. For times $t \geq 0$, the surface temperature at $x = 0$ is specified as $T(0, t) = 1$.



Assuming constant thermo-physical properties, find the unsteady-state temperature distribution $T(x, t)$.

Duhamel Superposition Integral

Another Example



$$F(t) \cong F(0)H(t) + \sum_{i=1}^{n} \underbrace{[F(\tau_i) - F(\tau_i - 1)]}_{f(\tau_i) - f(\tau_i - 1)} H(t - \tau_i)$$

where $\tau_0 = 0 < \tau_1 < \ldots < \tau_{n-1} < \tau_n < t < \tau_{n+1}$,

$$H(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

Duhamel Superposition Integral

Another Example

The solution of the "original problem" using the solution of the "auxiliary problem" can be written as:

$$\theta(x,t) \cong F(0)\phi(x,t) + \sum_{i=1}^{n}[f(\tau_i) - f(\tau_i - 1)]\phi(x,t-\tau_i)$$

$$\theta(x,t) \cong F(0)\phi(x,t) + \sum_{i=1}^{n}\frac{\Delta f_i}{\Delta \tau_i}\phi(x,t-\tau_i)\Delta\tau_i$$

where $\Delta f_i = f(\tau_i) - f(\tau_{i-1})$ and $\Delta\tau_i = \tau_i - \tau_{i-1}$

In the limit as $n \to \infty$,

$$\theta(x,t) = F(0)\phi(x,t) + \int_0^t \frac{df}{d\tau}\phi(x,t-\tau)d\tau$$

Duhamel Superposition Integral

Another Example

$$\theta(x,t) = -T_i \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) + \frac{x}{2\sqrt{\pi\alpha}} \int_0^t \frac{f(\tau)}{(t-\tau)^{3/2}} \exp\left[-\frac{x^2}{4\alpha(t-\tau)}\right] d\tau$$

which can also be written as

$$T(x,t) = T_i - T_i \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) + \frac{x}{2\sqrt{\pi\alpha}} \int_0^t \frac{f(t-t')}{(t')^{3/2}} \exp\left(-\frac{x^2}{4\alpha t'}\right) dt'$$

Approximate Analytic Method :

Integral Method

Approximate Analytic Method: Integral Method

Scope of Approximate Methods:

- When exact analytic solutions are impossible or too difficult to obtain

- The resulting analytic solutions are too complicated for computational purposes (analytical solutions may be too complex, implicit, or require numerical integration)

The *integral method* is used extensively in fluid flow, heat transfer, and mass transfer.

Because of the mathematical simplifications associated with this method, it can deal with such complicating factors as phase change, temperature-dependent properties, and nonlinearity.

Approximate Analytic Method: Integral Method

Scope of Approximate Methods:

When the differential equation of heat conduction is solved exactly in a given region subject to specified boundary and initial conditions, the resulting solution is satisfied exactly at each point over the considered domain; however, with the integral method the solution is satisfied only *on the average* over the region.

The results are approximate, but several solutions obtained with this method when compared with the exact solutions have confirmed that the accuracy is generally acceptable for many engineering applications.

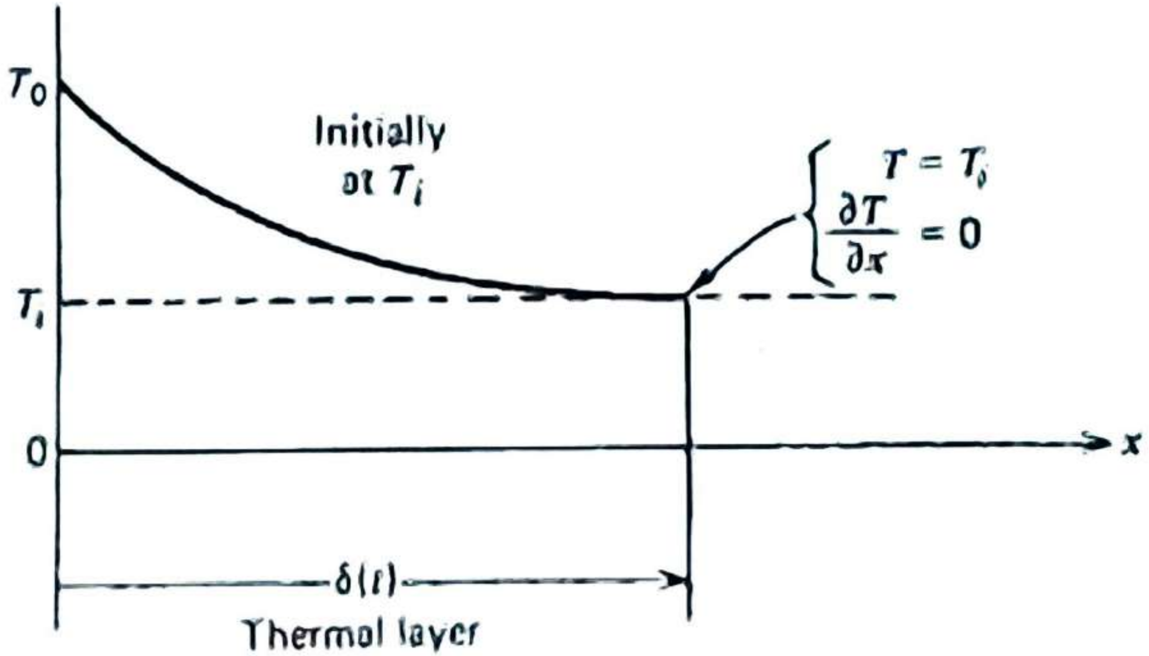
Approximate Analytic Method: Integral Method

Basic Steps:

First of all

Define a phenomenological distance $\delta(t)$, called the *thermal layer (or penetration depth)*.

The thermal layer is defined as the distance beyond which, for practical purposes, there is no heat flow; hence the initial temperature distribution remains unaffected beyond $\delta(t)$.



Definition of thermal penetration layer for heat conduction in a semi-infinite medium.

Approximate Analytic Method: Integral Method

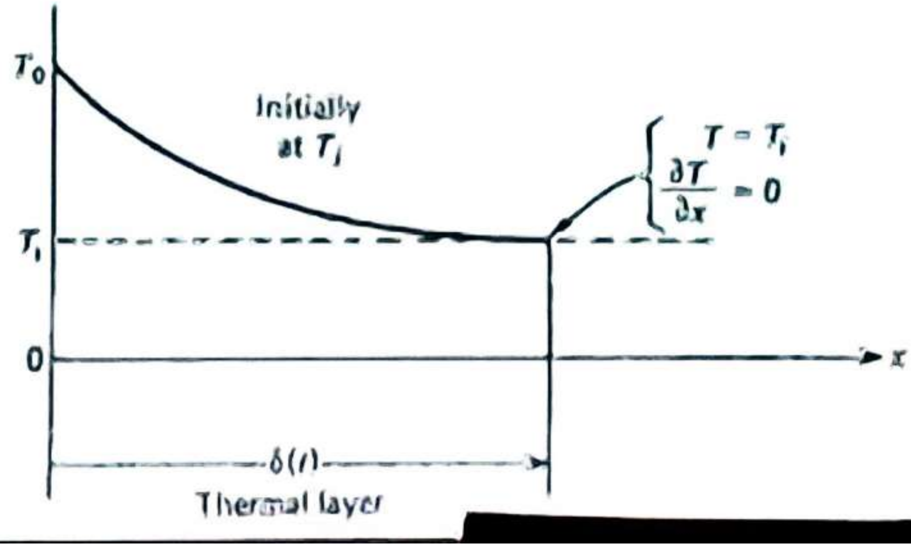
Basic Steps:

Step-1:

The differential equation of heat conduction is integrated over a phenomenological distance $\delta(t)$, called the *thermal layer* in order to remove from the differential equation the derivative with respect to the space variable.

The thermal layer is defined as the distance beyond which, for practical purposes, there is no heat flow; hence the initial temperature distribution remains unaffected beyond $\delta(t)$.

The resulting equation is called the *energy integral equation* (it is also called the *heat balance integral*).



Approximate Analytic Method: Integral Method

Basic Steps:

Step-2:

A suitable profile is chosen for the temperature distribution over the thermal layer.

A polynomial profile is generally preferred for this purpose, noting that experience has shown that there is no significant improvement in the accuracy of the solution to choose a polynomial greater than the fourth degree.

The coefficients in the polynomial are determined in terms of the thermal layer thickness $\delta(t)$ by utilizing the actual (or if necessary, derived) boundary conditions.

Approximate Analytic Method: Integral Method

Basic Steps:

Step-3:

When the temperature profile thus constructed is introduced into the energy integral equation and the indicated operations are performed, an ordinary differential equation is obtained for the thermal layer thickness $\delta(t)$ with time as the independent variable.

The solution of this differential equation subject to the appropriate initial condition [i.e., in this case $\delta(t) = 0$ for $t = 0$] gives $\delta(t)$ as a function of time.

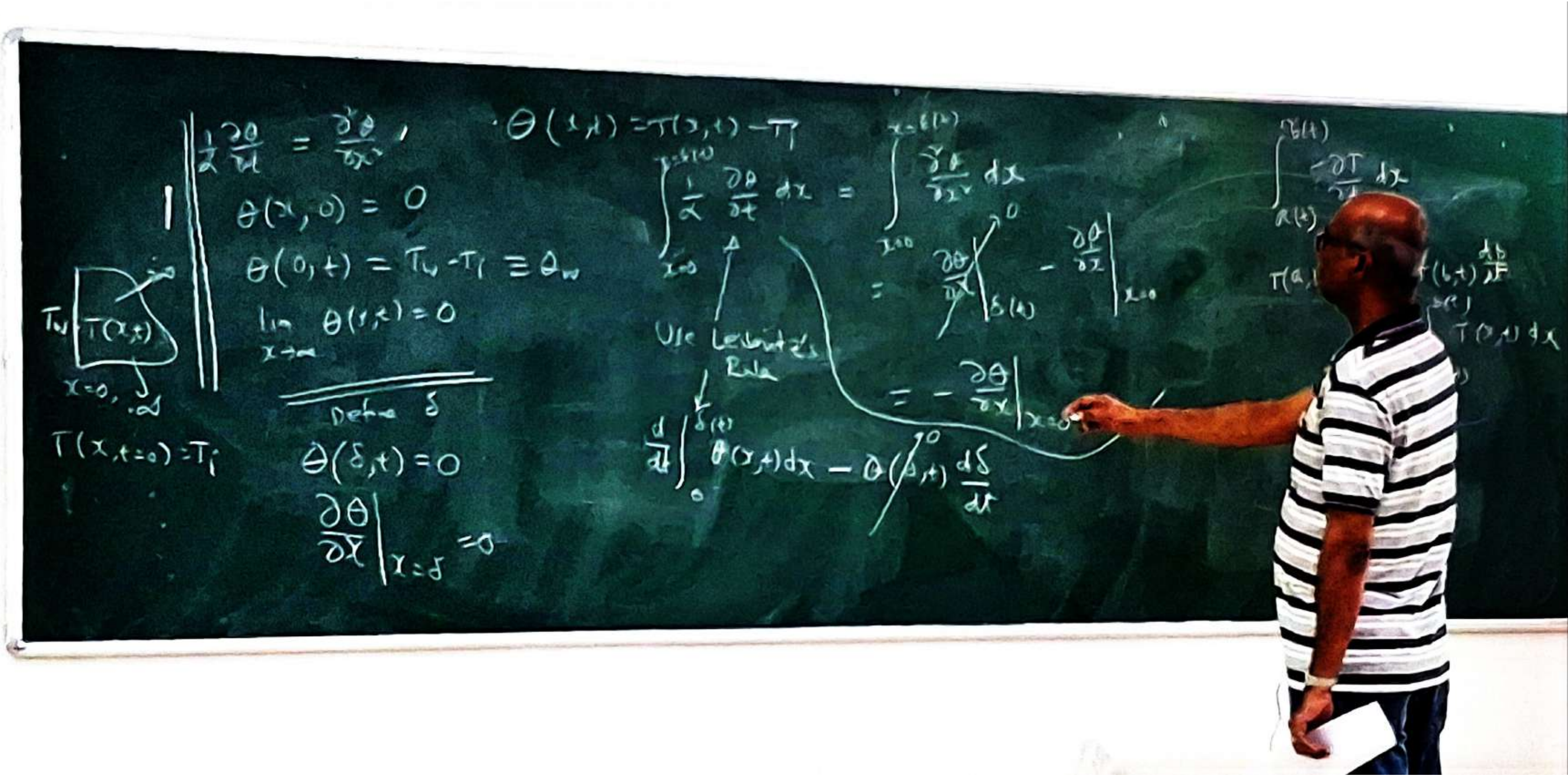
Approximate Analytic Method: Integral Method

Basic Steps:

Step-4:

Once $\delta(t)$ is available from step 3, the temperature distribution $T(x, t)$ is known as a function of time and position in the medium, and the heat flux at the surface is readily determined.

Experience has shown that the method is more accurate for the determination of heat flux than the temperature profile.



$\frac{1}{\alpha}\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial x^2}$
$\theta(x,t) = T(x,t) - T_i$
$\theta(x,0) = 0$
$\theta(0,t) = T_w - T_i \equiv \theta_w$
$\lim_{x\to\infty} \theta(x,t) = 0$
$T(x,t=0) = T_i$
Define δ
$\theta(\delta,t) = 0$
$\left.\frac{\partial\theta}{\partial x}\right|_{x=\delta} = 0$
Use Leibnitz's Rule
$= -\left.\frac{\partial\theta}{\partial x}\right|_{x=0}$

$\frac{\partial \theta}{\partial x^2}$,

$= 0$

$) = T_w - T_i \equiv \theta_w$

$\theta(x,t) = 0$

define δ

$(\delta, t) = 0$

$\left.\frac{\partial \theta}{\partial x}\right|_{x=\delta} = 0$

$$\theta(x,t) = T(x,t) - T_i$$

$$\frac{d}{dt}\int_0^{\delta(t)} \theta(x,t)\,dx = -\alpha \left.\frac{\partial \theta}{\partial x}\right|_{x=0}$$

This is heat balance integral or energy integral

$$\theta(x,t) = a + bx + cx^2$$

$$a = \theta_w \qquad b = -2\frac{\theta_w}{\delta} \qquad c = \frac{\theta_w}{\delta^2}$$

$$\Longrightarrow \frac{\theta(x,t)}{\theta_w} = \frac{T(x,t) - T_i}{T_w - T_i} = 1 - 2\left(\frac{x}{\delta}\right) + \left(\frac{x}{\delta}\right)^2$$

Substitute and integrate

$$\delta \frac{d\delta}{dt} = 6\alpha, \qquad \delta(0) = 0$$

solve

$$\delta = \sqrt{12\alpha t}$$

$$\int_{a(t)}^{b(t)} \frac{\partial T}{\partial t}\,dx$$

$$T(a,t)\frac{da}{dt} - T(b,t)\frac{db}{dt} + \frac{d}{dt}\int_{a(t)}^{b(t)} T(x,t)\,dx$$