Q2. $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0$ (2

Sub to. @ $r=0$, T = Bounded, for all θ.
@ $r=a$, $T=T_0$ for any θ
@ $\theta=0$, and @ $\theta=\pi$, $T=0$

Ans: Let us take: $\bar{r}=r/a$, $\bar{T}=T/T_0$

The governing eqn becomes: $\left[\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left(\bar{r}\frac{\partial \bar{T}}{\partial \bar{r}}\right) + \frac{1}{\bar{r}^2}\frac{\partial^2 \bar{T}}{\partial \theta^2}\right] = 0$

Now, consider: $\bar{T}(\bar{r},\theta) = R(\bar{r})P(\theta)$

So, $-\frac{\bar{r}}{R}\frac{d}{d\bar{r}}\left(\bar{r}\frac{dR}{d\bar{r}}\right) = \frac{1}{P}\left(\frac{d^2P}{d\theta^2}\right) = -\lambda_n^2$

$\Rightarrow P(\theta) = A_n\sin(\lambda_n\theta) + B_n\cos(\lambda_n\theta)$

@ $\theta=0$, $P(\theta)\big|_{\theta=0}=0$ gives: $A_n\sin(0) + B_n\cos(0) = 0 \Rightarrow B_n = 0$

@ $\theta=\pi$, $P(\theta)\big|_{\theta=\pi}=0$ gives: $A_n\sin(\lambda_n\pi) = 0$

For non-trivial solⁿ: $\sin(\lambda_n\pi) = 0 \Rightarrow \lambda_n = n$ $(n=0,1,2\ldots\infty)$

So $P(\theta) = A_n\sin(\lambda_n\theta)$ with $\lambda_n = n\cdot(n=0,1,2\ldots\infty)$

On the other hand, $\left[\frac{d}{d\bar{r}}\left(\bar{r}\frac{dR}{d\bar{r}}\right) - \frac{n^2}{\bar{r}}R(\bar{r})\right] = 0$

Let us consider, $R(\bar{r}) = \bar{r}^\alpha$

So, $\alpha^2\bar{r}^{\alpha-1} - \frac{n^2}{\bar{r}}\bar{r}^\alpha = 0$

$\Rightarrow \alpha = \pm n$

$R(\bar{r}) = \left[C_n\bar{r}^n + D_n\bar{r}^{-n}\right]$

@ $\bar{r}=0$, $R(\bar{r})\big|_{\bar{r}=0}$ = Bounded implies: $D_n = 0$

So, the combined solution is given as:

$\bar{T}(\bar{r},\theta) = \sum_{n=0}^{\infty} A_n r^n \sin(n\theta)$

@ $\bar{r}=1$, $\bar{T}(\bar{r},\theta)=1$ gives: $1 = \sum_{n=0}^{\infty} A_n\sin(n\theta)$

So, $A_n = \left[\frac{\int_0^\pi \sin(n\theta)\,d\theta}{\int_0^\pi \sin^2(n\theta)\,d\theta}\right]$

Now, $\int_0^\pi \sin(n\theta)\,d\theta = -\frac{\cos(n\theta)}{n}\Big|_0^\pi = \left[\frac{1-\cos(n\pi)}{n}\right]$

$\int_0^\pi \sin^2(n\theta)\,d\theta = \frac{1}{2}\int_0^\pi[1-\cos(2n\theta)]\,d\theta$

$= \frac{1}{2}\left[\theta - \frac{\sin(2n\theta)}{2n}\right]_0^\pi = \frac{1}{2}\left[\theta - \frac{2\sin(n\theta)\cos(n\theta)}{2n}\right]_0^\pi$

$= \frac{1}{2}\left[\pi - \cancel{\frac{2\sin(n\pi)\cos(n\pi)}{2n}} + 0 + \cancel{\frac{2\sin(0)\cos(0)}{2n}}\right]$

$= \pi/2$

So $A_n = \left[\frac{\int_0^\pi \sin(n\theta)\,d\theta}{\int_0^\pi \sin^2(n\theta)\,d\theta}\right] = \frac{2(1-\cos(n\pi))}{n\pi}$

The final solution is: $\bar{T}(\bar{r},\theta) = \sum_{n=0}^{\infty} A_n\bar{r}^n\sin(n\theta)$

or $T(r,\theta) = \left(\frac{2T_0}{a^n}\right)\sum_{n=0}^{\infty}\frac{2[1-\cos(n\pi)]}{n\pi}r^n\sin(n\theta)$