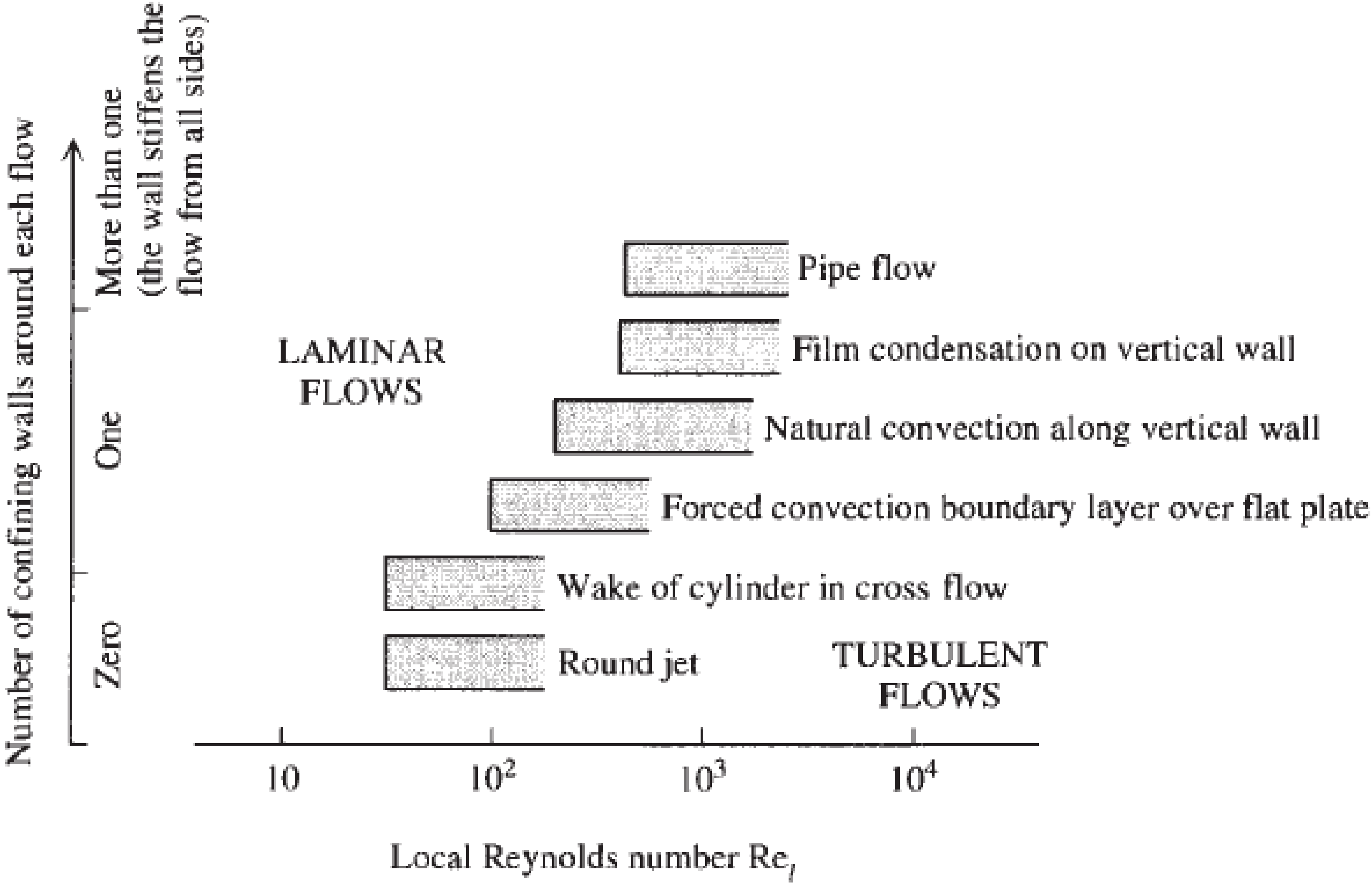
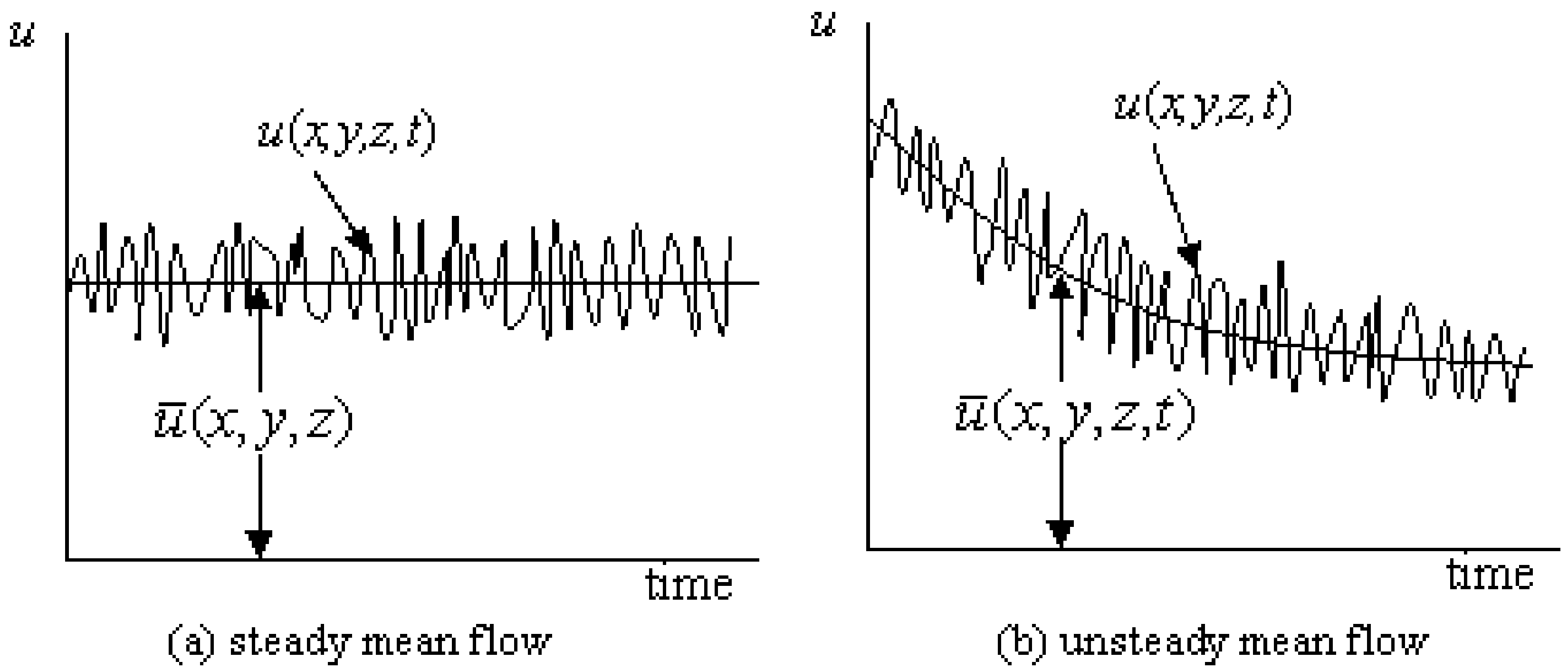
Summary of physical observations concerning the beginning of transition from laminar to turbulent flow

Flow Configuration	Condition[a] Necessary for the Existence of Laminar Flow	Source	Observations
Boundary layer flow (without longitudinal pressure gradient)	$\mathrm{Re} < 3.5 \times 10^5$ $\mathrm{Re} < 2 \times 10^4 - 10^6$	[3][b]	Re is the Reynolds number based on wall length and free-stream velocity
Duct flow	$\mathrm{Re} < 2000$		Re is based on hydraulic diameter and duct-averaged velocity
Free-jet flow (axisymmetric)	$\mathrm{Re} < 10-30$	[4]	Re is based on nozzle diameter and mean velocity through the nozzle
Wake flow (two-dimensional)	$\mathrm{Re} < 32$	[5]	Re is based on the cylinder diameter and free-stream velocity
Natural convection boundary layer flow			
Isothermal wall	$\mathrm{Gr} < 1.5 \times 10^9$ $(\mathrm{Pr} = 0.71)$ $\mathrm{Gr} < 1.3 \times 10^9$ $(\mathrm{Pr} = 6.7)$	[6][c] [6, 7]	The Grashof number Gr is based on wall height and wall–ambient temperature difference
Constant-heat-flux wall	$\mathrm{Gr}_* < 1.6 \times 10^{10}$ $(\mathrm{Pr} = 0.71)$ $\mathrm{Gr}_* < 6.6 \times 10^{10}$ $(\mathrm{Pr} = 6.7)$	[6] [8]	Gr_* is the Grashof number based on heat flux and wall height [see eq. (4.70), where $\mathrm{Gr}_* = \mathrm{Ra}_*/\mathrm{Pr}$]
Plume flow (axisymmetric)	$\mathrm{Ra}_q < 10^{10}$ $(\mathrm{Pr} = 0.71)$	[9]	Ra_q is the Rayleigh number based on heat source strength and plume height [see eq. (6.6)]
Film condensation on a vertical plate	$\dfrac{4\Gamma}{\mu} < 1800$	[10]	Γ is the condensate mass flow rate per unit of film width


Number of confining walls around each flow
Zero
One
More than one
(the wall stiffens the
flow from all sides)
LAMINAR
FLOWS
TURBULENT
FLOWS
Pipe flow
Film condensation on vertical wall
Natural convection along vertical wall
Forced convection boundary layer over flat plate
Wake of cylinder in cross flow
Round jet
10
10^2
10^3
10^4
Local Reynolds number Re_l

Turbulent velocity



Time dependence of velocity component in the x-direction in a turbulent flow

The derivation of the conservation laws for time-averaged flow begins with the transformation

$$u = \overline{u} + u', \qquad P = \overline{P} + P'$$

$$v = \overline{v} + v', \qquad T = \overline{T} + T'$$

$$w = \overline{w} + w'$$

where

$$\overline{u} = \frac{1}{\text{period}} \int_0^{\text{period}} u \, d(\text{time}) \qquad \int_0^{\text{period}} u' \, d(\text{time}) = 0$$

Definitions of time average property and fluctuating component are the foundation of a special kind of algebra that emerges in the process of substituting the $(\overline{\ }) + (\ )'$ decomposition into the mass, momentum, and energy equations and then time averaging these equations according to definition.

The rules (theorems) of this algebra are

$$\overline{u+v} = \overline{u} + \overline{v}$$

$$\overline{\overline{u}u'} = 0$$

$$\overline{uv} = \overline{u}\,\overline{v} + \overline{u'v'}$$

$$\overline{u^2} = \overline{u}^2 + \overline{u'^2}$$

$$\overline{\frac{\partial u}{\partial x}} = \frac{\partial \overline{u}}{\partial x}$$

$$\frac{\partial \overline{u}}{\partial t} = 0$$

$$\overline{\frac{\partial u}{\partial t}} = 0$$

Mass conservation equation

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial u'}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial v'}{\partial y} + \frac{\partial \overline{w}}{\partial z} + \frac{\partial w'}{\partial z} = 0$$

Integrating this equation term by term over time and applying rules

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0$$

The corresponding time-averaged forms of the momentum equations in the x, *y*, and *z* directions are

$$\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}=-\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x}+\nu\nabla^2\bar{u}-\frac{\partial}{\partial x}\left(\overline{u'^2}\right)-\frac{\partial}{\partial y}\left(\overline{u'v'}\right)-\frac{\partial}{\partial z}\left(\overline{u'w'}\right)$$

$$\bar{u}\frac{\partial \bar{v}}{\partial x}+\bar{v}\frac{\partial \bar{v}}{\partial y}+\bar{w}\frac{\partial \bar{v}}{\partial z}=-\frac{1}{\rho}\frac{\partial \bar{P}}{\partial y}+\nu\nabla^2\bar{v}-\frac{\partial}{\partial x}\left(\overline{u'v'}\right)-\frac{\partial}{\partial y}\left(\overline{v'^2}\right)-\frac{\partial}{\partial z}\left(\overline{v'w'}\right)$$

$$\bar{u}\frac{\partial \bar{w}}{\partial x}+\bar{v}\frac{\partial \bar{w}}{\partial y}+\bar{w}\frac{\partial \bar{w}}{\partial z}=-\frac{1}{\rho}\frac{\partial \bar{P}}{\partial z}+\nu\nabla^2\bar{w}-\frac{\partial}{\partial x}\left(\overline{u'w'}\right)-\frac{\partial}{\partial y}\left(\overline{v'w'}\right)-\frac{\partial}{\partial z}\left(\overline{w'^2}\right)$$

Finally, the energy equation expressed as, after a similar time-averaging procedure,

$$\bar{u}\frac{\partial \bar{T}}{\partial x}+\bar{v}\frac{\partial \bar{T}}{\partial y}+\bar{w}\frac{\partial \bar{T}}{\partial z}=\alpha\nabla^2\bar{T}-\frac{\partial}{\partial x}\left(\overline{u'T'}\right)-\frac{\partial}{\partial y}\left(\overline{v'T'}\right)-\frac{\partial}{\partial z}\left(\overline{w'T'}\right)$$

Boundary layer equations

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0 \qquad (1)$$

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} = -\frac{1}{\rho}\frac{d\overline{P}}{dx} + \nu\frac{\partial^2 \overline{u}}{\partial y^2} - \frac{\partial}{\partial y}\,(\overline{u'v'}) \qquad (2)$$

$$\overline{u}\frac{\partial \overline{T}}{\partial x} + \overline{v}\frac{\partial \overline{T}}{\partial y} = \alpha\frac{\partial^2 \overline{T}}{\partial y^2} - \frac{\partial}{\partial y}\,(\overline{v'T'}) \qquad (3)$$

Introduction of eddy shear stress and eddy heat flux

$$\tau_{\text{app}} = \mu \frac{\partial \overline{u}}{\partial y} - \rho \overline{u'v'} = \rho(\nu + \epsilon_M)\frac{\partial \overline{u}}{\partial y}$$ apparent shear stress

$$-q''_{\text{app}} = k \frac{\partial \overline{T}}{\partial y} - \rho c_P \overline{v'T'} = \rho c_P(\alpha + \epsilon_H)\frac{\partial \overline{T}}{\partial y}$$ apparent heat flux

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} = -\frac{1}{\rho}\frac{d\overline{P}}{dx} + \frac{\partial}{\partial y}\left[\left(\nu + \epsilon_M\right)\frac{\partial \overline{u}}{\partial y}\right]$$

$$\overline{u}\frac{\partial \overline{T}}{\partial x} + \overline{v}\frac{\partial \overline{T}}{\partial y} = \frac{\partial}{\partial y}\left[\left(\alpha + \epsilon_H\right)\frac{\partial \overline{T}}{\partial y}\right]$$

Mixing Length Model

- Simplest and oldest models that lead to concise formulas for wall friction and heat transfer

Based on the preliminary assumption

$$O(u') = l\frac{\partial \overline{u}}{\partial y} \quad \text{and} \quad O(v') = l\frac{\partial \overline{u}}{\partial y}$$

Leads to

$$\epsilon_M = l^2 \left| \frac{\partial \overline{u}}{\partial y} \right| \Rightarrow \epsilon_M = \kappa^2 y^2 \left| \frac{\partial \overline{u}}{\partial y} \right|$$

Where, $l = \kappa y$, and κ is von Karman's constant

Velocity distribution

- Introduction of wall coordinate based on the assumption that **apparent shear stress does not vary with *y***
- Non-dimensionalization of coordinate and relevant variables
- Prandtl's two-layer theory

1. The viscous sublayer (VSL), where $\nu \gg \epsilon_M$
2. The fully turbulent sublayer (or the turbulent core), where $\epsilon_M \gg \nu$

- The final velocity profile in the wall coordinate

$$u^+ = y^+ \qquad 0 < y^+ < 11.6$$
$$u^+ = 2.5 \ln y^+ + 5.5 \qquad y^+ > 11.6$$

Other theories and corresponding velocity distributions

Table 7.1 Summary of longitudinal velocity expressions for the inner region of a turbulent boundary layer

$u^+(y^+)$	Range	Reference
$u^+ = y^+$ $u^+ = 2.5 \ln y^+ + 5.5$	$0 < y^+ < 11.6$ $y^+ > 11.6$	Prandtl and Taylor [9]
$u^+ = y^+$ $u^+ = 5 \ln y^+ - 3.05$ $u^+ = 2.5 \ln y^+ + 5.5$	$0 < y^+ < 5$ $5 < y^+ < 30$ $y^+ > 30$	von Kármán [12]
$u^+ = 14.53 \tanh(y^+/14.53)$ $u^+ = 2.5 \ln y^+ + 5.5$	$0 < y^+ < 27.5$ $y^+ > 27.5$	Rannie [13]
$\dfrac{du^+}{dy^+} = \dfrac{2}{1 + \{1 + 4\kappa^2 y^{+2}[1 - \exp(-y^+/A^+)]^2\}^{1/2}}$ $\kappa = 0.4 \qquad A^+ = 26$	All y^+	van Driest [14]
$u^+ = 2.5 \ln(1 + 0.4y^+) + 7.8[1 - \exp(-y^+/11) - (y^+/11) \exp(-0.33y^+)]$	All y^+	Reichardt [15]
$\dfrac{du^+}{dy^+} = \dfrac{1}{1 + n^2u^+y^+[1 - \exp(-n^2u^+y^+)]}$ $n = 0.124$ $u^+ = 2.78 \ln y^+ + 3.8$	$0 < y^+ < 26$	Deissler [16]
$y^+ = u^+ + A[\exp Bu^+ - 1 - Bu^+ - \frac{1}{2}(Bu^+)^2 - \frac{1}{6}(Bu^+)^3 - \frac{1}{24}(Bu^+)^4]$ (last term in u^{+4} may be omitted)	All y^+ $A = 0.1108$ $B = 0.4$	Spalding [17]

Estimation of wall friction in boundary layer flow

- Prandtl's one-seventh power law as the fit for the $u_+(y_+)$ data,

- Assuming that fits the measurements sufficiently well near the high y_+ extremity of the profile (i.e., in the wake region), we can use the curve fit to define an outer boundary layer thickness δ such that the time-averaged velocity u calculated U_∞ when y equals δ. This leads to

$$\frac{U_\infty}{(\tau_0/\rho)^{1/2}} \cong f_u\left[\frac{\delta}{\nu}\left(\frac{\tau_0}{\rho}\right)^{1/2}\right] \qquad (1)$$

and

$$\frac{d}{dx}\int_0^\infty \overline{u}(U_\infty - \overline{u})\ dy = \frac{\tau_0}{\rho} \qquad (2)$$

Solution of Eq.(1) and (2) leads to

$$\frac{\tau_0}{\rho U_\infty^2} = 0.0225\left(\frac{U_\infty \delta}{\nu}\right)^{-1/4}$$

$$\frac{\delta}{x} = 0.37\left(\frac{U_\infty x}{\nu}\right)^{-1/5}$$

And

$$\frac{\tau_0}{\rho U_\infty^2} = \tfrac{1}{2} C_{f,\,x} = 0.0296\left(\frac{U_\infty x}{\nu}\right)^{-1/5}$$

$$\frac{\tau_{0-x}}{\rho U_\infty^2} = \tfrac{1}{2} C_{f,0-x} = 0.037\left(\frac{U_\infty x}{\nu}\right)^{-1/5}$$

Heat transfer in boundary layer flow

- Temperature distribution

- Defining wall coordinates based on the assumption that sufficiently close to the wall the apparent heat flux q'' app does not depend on y

- Defining Non-dimensional temperature

$$T^{+}(x^{+},\ y^{+}) = (T_0 - \overline{T})\frac{\rho c_P u_*}{q_0''}$$

This leads to the final temperature profile

$$T^{+} = \begin{cases} \Pr\ y^{+}, & y^{+} < y^{+}_{\mathrm{CSL}} \\ \Pr\ y^{+}_{\mathrm{CSL}} + \dfrac{\Pr_t}{\kappa} \ln \dfrac{y^{+}}{y^{+}_{\mathrm{CSL}}}, & y^{+} > y^{+}_{\mathrm{CSL}} \end{cases}$$

Where P_{rt} is the turbulent Prandtl number $(\Pr_t = \epsilon_M/\epsilon_H)$.

Estimation of the heat transfer coefficient

Assumption

A wall heat flux formula consistent with the temperature distribution developed (for a fully turbulent sublayer) holds well enough near the outer edge of the boundary layer (i.e., at the edge of the wake region).

Leads to

$$\rho c_P u_* \frac{T_0 - T_\infty}{q_0''} = \Pr\ y_{\mathrm{CSL}}^{+} + \frac{\Pr_t}{\kappa} \ln \frac{\delta u_*/\nu}{y_{\mathrm{CSL}}^{+}} \qquad (1)$$

and

$$\frac{U_\infty}{u_*} = \frac{1}{\kappa} \ln \frac{\delta u_*}{\nu} + B \qquad (2)$$

Solution of Eq.(1) and (2) leads to

Solution of Eq.(1) and (2) leads to

$$\frac{h}{\rho c_P U_\infty} = \frac{\frac{1}{2}C_{f,x}}{\mathrm{Pr}_t + \left(\frac{1}{2}C_{f,x}\right)^{1/2}\left[\mathrm{Pr}\, y^+_{\mathrm{CSL}} - B\,\mathrm{Pr}_t - (\mathrm{Pr}_t/\kappa)\ln\, y^+_{\mathrm{CSL}}\right]}$$

Substituting all the constant's

$\mathrm{Pr}_t \cong 0.9, \qquad \kappa \cong 0.41, \qquad y^+_{\mathrm{CSL}} \cong 13.2 \qquad$ taking B ~= 5.1

$$\mathrm{St}_x = \frac{\frac{1}{2}C_{f,x}}{0.9 + \left(\frac{1}{2}C_{f,x}\right)^{1/2}(13.2\mathrm{Pr} - 10.25)}$$

Local Stanton number expression produced by von Karman's three-region velocity profile

$$\mathrm{St}_x = \frac{\frac{1}{2}C_{f,x}}{1 + 5\left(\frac{1}{2}C_{f,x}\right)^{1/2}\left\{\mathrm{Pr} - 1 + \ln\left[1 + \frac{5}{6}(\mathrm{Pr} - 1)\right]\right\}}$$

- Derived expression of the *Stanton number* leans to the logical *empirical formula* suggested by Colburn

$$\mathrm{St}_x \ \mathrm{Pr}^{2/3} = \tfrac{1}{2} C_{f,x}$$

The Colburn analogy can be restated in terms of Nu_x and Re_x, by the wall friction estimated using Prandtl's one-seventh power law

$$\begin{aligned} \mathrm{Nu}_x &= \tfrac{1}{2} C_{f,x} \mathrm{Re}_x \ \mathrm{Pr}^{1/3} \\ &= 0.0296 \mathrm{Re}_x^{4/5} \ \mathrm{Pr}^{1/3} \qquad (\mathrm{Pr} \gtrsim 0.5) \\ \overline{\mathrm{Nu}}_L &= 0.037 \mathrm{Re}_L^{4/5} \ \mathrm{Pr}^{1/3} \qquad (\mathrm{Pr} \gtrsim 0.5) \end{aligned}$$

***Reynolds analogy* between heat transfer and wall friction in turbulent flow**

$$\mathrm{St}_x = \frac{1}{2} C_{fx}$$

This can be derived theoretically with the assumption P_r *and* P_{rt} *will be equal to 1*

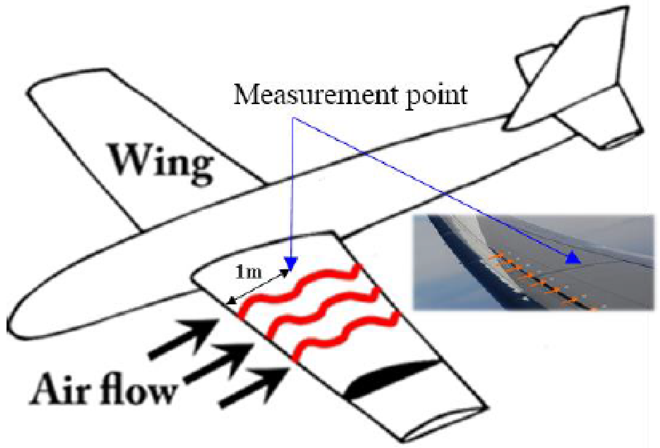
and solving the equations within the TBL

$$\tau_0 = (\mu + \rho \epsilon_M) \frac{d\overline{u}}{dy}$$

$$-q''_0 = (k + \rho c_P \epsilon_H) \frac{d\overline{T}}{dy}$$

Problem 1

1. An experiment is conducted in a laboratory to examine the wall shear stress on an aeroplane wing, maintained at a constant temperature of 30 °C (T_w = 30 °C). An air stream of uniform velocity (u_∞ = 10 m/s) and temperature (T_∞ = 10 °C) at a distance from the wing is blown over it. Roughness present near the leading edge of the wing triggers the flow so that the flow is assumed to be turbulent from x = 0. At distance x (= 1 m) from the plate leading edge in the direction of the flow, the wall shear stress τ_w equal to 0.23 N m^{-2} is measured. In the same location, a wire of diameter 1 μm is placed at 0.19 mm above the wall to measure the fluid mean temperature, which is found to be equal to 25 °C.



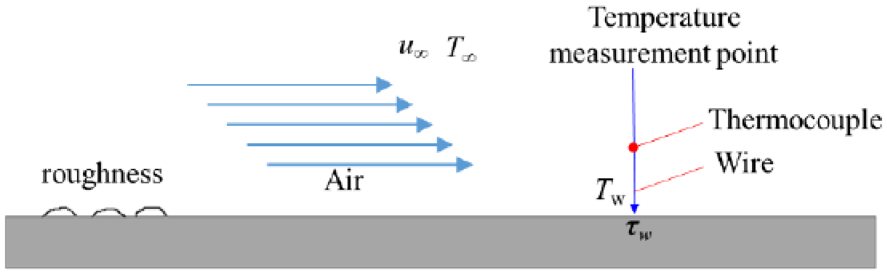


Figure 1. Schematic of the problem

(a) With these experimental results, you are asked to calculate the wall heat flux at the location of the measurement.

(b) Compare with the Colburn correlation and find the % difference relative to the direct measurement.

(c) Now a change in the mean temperature measurements is performed and this time the temperature is measured using a thermocouple. The thermocouple sensor has a spherical-shaped diameter d ≈ 50 μm and is placed at 4.5 mm above the wall. What is the expected temperature?

(d) What is the sensor size-induced uncertainty in the mean temperature measurement at this distance from the wall? If the uncertainty is less than ±0.5°C, it can be ignored.

The aeroplane wing in the direction of the airflow can be assumed as a flat plate for simplicity.

Use *Prandtl mixing model* for the velocity distribution and for the temperature distribution use the 2- layers model deduced using *Prandtl mixing model* given as.

$$T^+ \begin{cases} = \Pr y^+ & \text{for} \quad y^+ < y^+_{CSL} \\ = \Pr y^+_{CSL} + \dfrac{\Pr_t}{\kappa} \ln \dfrac{y^+}{y^+_{CSL}} & \text{for} \quad y^+ > y^+_{CSL} \end{cases}$$

The empirical constants are given as $Pr_t \approx 0.9$; $\kappa \approx 0.41$ and for the above mention conditions and $y^+_{CSL} \approx 13.2$

Assume the wall co-ordinte is valid till $y^+ < 200$

Physical properties of air:

$\rho = 1.13\ kg\ m^{-3}$; $\nu = 16.7\times 10^{-6}\ m^2\ s^{-1}$; $k = 0.026\ W\ m^{-1}\ K^{-1}$; $Pr = 0.7$; $C_p = 10^3\ J\ kg^{-1}\ K^{-1}$

(e) The uniform velocity of air is now changed 50 m/s. Do you expect any change in the measured temperature? (For both measured temperature, initially by the wire located 0.19 mm above the wall and then by the thermocouple located 4.5 mm above the wall) Explain your answer with proper justification.

Problem 2

1. Water flows with the velocity $U_\infty = 0.2$ m/s parallel to a plane wall. The following calculations refer to the position $x = 6$ m measured downstream from the leading edge. The water properties can be evaluated at 20◦C.

(a) A probe is to be inserted in the viscous sublayer to the position represented by $y^+ = 2.7$. Calculate the actual spacing y (mm) between the probe and the wall.

(b) Calculate the boundary layer thickness δ, and compare this value with the estimate based on the assumption that the length x is covered by turbulent boundary layer flow.

(c) Calculate the heat transfer coefficient averaged over the length x.

Problem 3

By holding and rubbing the ball in his hand, the pitcher warms the leather cover of the baseball to 30°C. The outside air temperature is 20°C and the ball diameter is 7 cm. The pitcher throws the ball at 50 miles/h (22.35 m/s) to the catcher, who is stationed 18.5m away.

(a) Assume that the ball surface temperature remains constant, and calculate the heat transferred from the ball to the surrounding air during the throw.

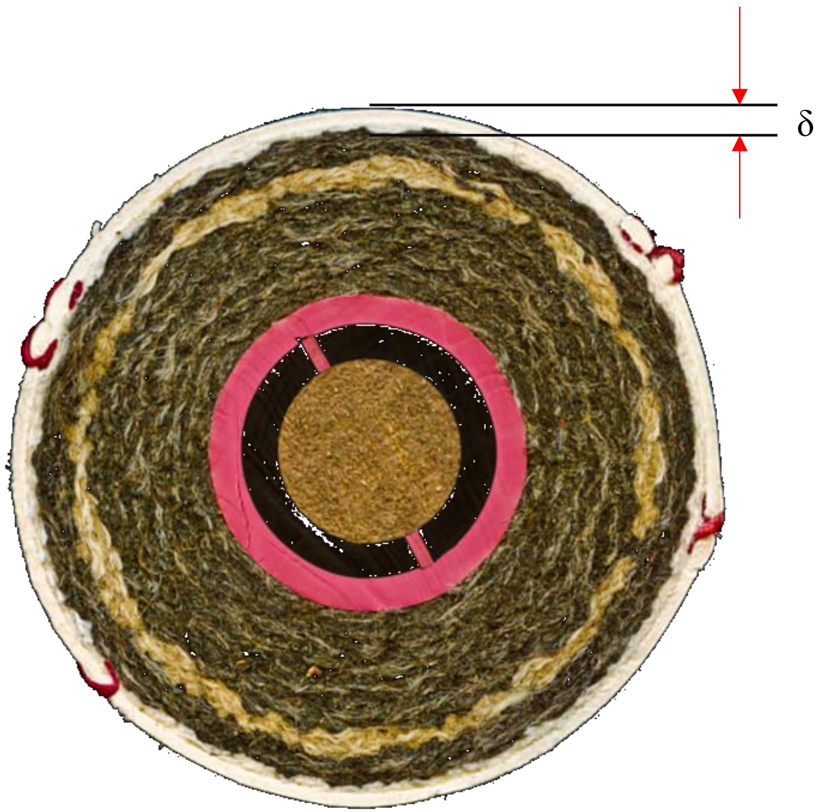
(b) Calculate the temperature drop experienced by the leather cover to account for the heat transfer calculated in part (a). Assume that the thickness of the layer of leather that experiences the air cooling effect is comparable to the conduction penetration depth $\delta \sim (\alpha t)^{1/2}$, where α is the thermal diffusivity of leather. Validate the correctness of the constant surface temperature assumption made in part (a).

Data: $\alpha_{leather}$= 0.001 cm^2/s; $c_{leather}$ = 1.5 kJ/kg.K ; $\rho_{leather}$= 860 kg/m^3 ; at 20°C

k_{air}=0.025 W/mK; ν_{air} = 0.15 cm^2/s; Pr_{air}=0.72

Churchill and Bernstein correlation, for all Re_D and $Pr>0.2$

$$\overline{Nu}_D = 0.3 + \frac{0.62 Re_D^{1/2} Pr^{1/3}}{\left[1+\left(0.4/Pr\right)^{2/3}\right]^{1/4}}\left[1+\left(\frac{Re_D}{282{,}000}\right)^{5/8}\right]^{4/5}$$



Baseball ball cut section