Numerical Questions Based on AHP Research Papers

Question 1: Pairwise Comparison Matrix and Consistency Analysis

Problem: A project manager needs to select the best software development methodology for a new project. Three methodologies are being considered: Agile (A), Waterfall (W), and Hybrid (H). The project manager makes the following pairwise comparisons:

- Agile is strongly more important than Waterfall (5)
- Agile is moderately more important than Hybrid (3)
- Hybrid is moderately more important than Waterfall (3)

a) Construct the pairwise comparison matrix
b) Calculate priority vector using the approximate method
c) Calculate the maximum eigenvalue (λ_{max})
d) Calculate the Consistency Index (CI) and Consistency Ratio (CR)
e) Determine if the judgments are acceptable

Solution:

a) **Pairwise Comparison Matrix:**

```
    | A  | W  | H  |
----|----|----|----|
A   | 1  | 5  | 3  |
W   | 1/5| 1  | 1/3|
H   | 1/3| 3  | 1  |
```

Converting to decimal form:

```
    | A    | W    | H    |
----|------|------|------|
A   | 1    | 5    | 3    |
W   | 0.2  | 1    | 0.333|
H   | 0.333| 3    | 1    |
```

b) **Priority Vector Calculation (Row Sum Method):**

Step 1: Find sum of each row

- Row A: 1 + 5 + 3 = 9
- Row W: 0.2 + 1 + 0.333 = 1.533

- Row H: 0.333 + 3 + 1 = 4.333

Step 2: Divide each row sum by total sum (9 + 1.533 + 4.333 = 14.866)

- Priority A: 9/14.866 = 0.605
- Priority W: 1.533/14.866 = 0.103
- Priority H: 4.333/14.866 = 0.292

Therefore, the priority vector is [0.605, 0.103, 0.292]

c) **Maximum Eigenvalue Calculation:**

Step 1: Multiply the pairwise comparison matrix by the priority vector

```
[1    5    3   ] [0.605] [1.842]
[0.2  1    0.333] × [0.103] = [0.310]
[0.333 3    1   ] [0.292] [0.888]
```

Step 2: Divide each element of the resulting vector by the corresponding element in the priority vector

- 1.842/0.605 = 3.044
- 0.310/0.103 = 3.009
- 0.888/0.292 = 3.041

Step 3: Calculate the average of these values to get λ_{max}
λ_{max} = (3.044 + 3.009 + 3.041)/3 = 3.031

d) **Consistency Index and Ratio:**

CI = (λ_{max} - n)/(n - 1) = (3.031 - 3)/(3 - 1) = 0.016

For n = 3, the Random Index (RI) is 0.58 (from Saaty's table)

CR = CI/RI = 0.016/0.58 = 0.027 = 2.7%

e) **Judgment Acceptability:**

Since CR = 0.027 < 0.10 (Saaty's threshold), the judgments are consistent and acceptable.

Question 2: Multi-Criteria Job Selection Using AHP

Problem: Consider the job selection problem described in Saaty's paper (Section 5). A person needs to choose between four job options: Domestic Company (DC), International Company (IC), College (C), and State University (SU) based on three criteria: Salary (S), Job Security (JS), and Work Flexibility (WF).

Given the following matrices:

Criteria comparison with respect to goal:

```
     | S   | JS  | WF  |
-----|-----|-----|-----|
S    | 1   | 2   | 5   |
JS   | 1/2 | 1   | 3   |
WF   | 1/5 | 1/3 | 1   |
```

Alternatives with respect to Salary:

```
     | DC  | IC  | C   | SU  |
-----|-----|-----|-----|-----|
DC   | 1   | 1/4 | 1/3 | 3   |
IC   | 4   | 1   | 2   | 7   |
C    | 3   | 1/2 | 1   | 5   |
SU   | 1/3 | 1/7 | 1/5 | 1   |
```

Alternatives with respect to Job Security:

```
     | DC  | IC  | C   | SU  |
-----|-----|-----|-----|-----|
DC   | 1   | 6   | 4   | 1/3 |
IC   | 1/6 | 1   | 1/4 | 1/8 |
C    | 1/4 | 4   | 1   | 1/5 |
SU   | 3   | 8   | 5   | 1   |
```

Alternatives with respect to Work Flexibility:

```
     | DC  | IC  | C   | SU  |
-----|-----|-----|-----|-----|
DC   | 1   | 1/5 | 1/7 | 1/6 |
IC   | 5   | 1   | 1/2 | 1/2 |
C    | 7   | 2   | 1   | 1   |
SU   | 6   | 2   | 1   | 1   |
```

a) Calculate the priority vector for the criteria
b) Calculate the priority vector for each alternative with respect to each criterion
c) Synthesize the results to determine the overall priorities for job selection
d) Which job should the person choose?

Solution:

a) **Priority Vector for Criteria:**

First, convert the fractions to decimals:

```
     | S   | JS  | WF   |
-----|-----|-----|------|
S    | 1   | 2   | 5    |
JS   | 0.5 | 1   | 3    |
WF   | 0.2 | 0.33| 1    |
```

Calculate row sums:

- S: 1 + 2 + 5 = 8
- JS: 0.5 + 1 + 3 = 4.5
- WF: 0.2 + 0.33 + 1 = 1.53

Total: 8 + 4.5 + 1.53 = 14.03

Normalizing:

- Priority S = 8/14.03 = 0.570
- Priority JS = 4.5/14.03 = 0.321
- Priority WF = 1.53/14.03 = 0.109

Criteria priority vector = [0.570, 0.321, 0.109]

b) **Priority Vectors for Alternatives:**

For Salary:

```
     | DC   | IC   | C    | SU    |
-----|------|------|------|-------|
DC   | 1    | 0.25 | 0.33 | 3     |
IC   | 4    | 1    | 2    | 7     |
C    | 3    | 0.5  | 1    | 5     |
SU   | 0.33 | 0.14 | 0.2  | 1     |
```

Row sums:

- DC: 1 + 0.25 + 0.33 + 3 = 4.58
- IC: 4 + 1 + 2 + 7 = 14
- C: 3 + 0.5 + 1 + 5 = 9.5
- SU: 0.33 + 0.14 + 0.2 + 1 = 1.67

Total: 4.58 + 14 + 9.5 + 1.67 = 29.75

Normalizing for Salary:

- DC: 4.58/29.75 = 0.154
- IC: 14/29.75 = 0.471
- C: 9.5/29.75 = 0.319
- SU: 1.67/29.75 = 0.056

Priority vector for Salary = [0.154, 0.471, 0.319, 0.056]

Similar calculations for Job Security:
Priority vector for Job Security = [0.283, 0.039, 0.111, 0.567]

Similar calculations for Work Flexibility:
Priority vector for Work Flexibility = [0.043, 0.176, 0.391, 0.390]

c) **Synthesis of Results:**

Multiply each alternative's priority by the corresponding criterion weight and sum:

For DC:
0.570 × 0.154 + 0.321 × 0.283 + 0.109 × 0.043 = 0.088 + 0.091 + 0.005 = 0.184

For IC:
0.570 × 0.471 + 0.321 × 0.039 + 0.109 × 0.176 = 0.268 + 0.013 + 0.019 = 0.300

For C:
0.570 × 0.319 + 0.321 × 0.111 + 0.109 × 0.391 = 0.182 + 0.036 + 0.043 = 0.261

For SU:
0.570 × 0.056 + 0.321 × 0.567 + 0.109 × 0.390 = 0.032 + 0.182 + 0.043 = 0.257

Overall priorities: DC = 0.184, IC = 0.300, C = 0.261, SU = 0.257

d) **Best Job Choice:**

The International Company (IC) has the highest priority (0.300), so it's the recommended choice.

Question 3: Consistency Indices Comparison

Problem: Consider the following pairwise comparison matrix:

```
A = | 1    4    9  |
    | 1/4  1    3  |
    | 1/9  1/3  1  |
```

a) Calculate Saaty's Consistency Index (CI) and Consistency Ratio (CR)
b) Calculate the Geometric Consistency Index (GCI)
c) Calculate Koczkodaj's Consistency Measure (CM)
d) Determine if this matrix is acceptably consistent according to each method

Solution:

a) **Saaty's CI and CR:**

First, convert to decimal form:

```
A = | 1     4     9    |
    | 0.25  1     3    |
    | 0.111 0.333 1    |
```

Step 1: Calculate the priority vector using the geometric mean method:

- Row 1: $(1 \times 4 \times 9)^{(1/3)} = 3.302$
- Row 2: $(0.25 \times 1 \times 3)^{(1/3)} = 0.909$
- Row 3: $(0.111 \times 0.333 \times 1)^{(1/3)} = 0.333$

Sum = 3.302 + 0.909 + 0.333 = 4.544

Normalized priority vector:
w = [3.302/4.544, 0.909/4.544, 0.333/4.544] = [0.727, 0.200, 0.073]

Step 2: Calculate λ_{max}:
Multiply A × w:

```
| 1     4     9    | | 0.727 | = | 2.247 |
| 0.25  1     3    | | 0.200 |   | 0.610 |
| 0.111 0.333 1    | | 0.073 |   | 0.223 |
```

Divide each element of the resulting vector by the corresponding element in w:

- 2.247/0.727 = 3.091
- 0.610/0.200 = 3.050
- 0.223/0.073 = 3.055

λ_{max} = (3.091 + 3.050 + 3.055)/3 = 3.065

Step 3: Calculate CI:
CI = (λ_{max} - n)/(n - 1) = (3.065 - 3)/(3 - 1) = 0.0325

Step 4: Calculate CR:
For n = 3, RI = 0.58
CR = CI/RI = 0.0325/0.58 = 0.056 = 5.6%

b) **Geometric Consistency Index (GCI):**

GCI is defined as:
$GCI = (2/((n-1)(n-2))) * \sum(i<j)(\ln(a_{ij} \times w_j/w_i))^2$

First, calculate the error terms $e_{ij} = a_{ij} \times w_j/w_i$:

- $e_{12} = 4 \times (0.200/0.727) = 1.101$
- $e_{13} = 9 \times (0.073/0.727) = 0.903$
- $e_{23} = 3 \times (0.073/0.200) = 1.095$

$GCI = (2/((3-1)(3-2))) * ((\ln(1.101))^2 + (\ln(0.903))^2 + (\ln(1.095))^2)$
$GCI = (2/2) * (0.096^2 + 0.102^2 + 0.091^2)$
GCI = 1 * (0.009 + 0.010 + 0.008)
GCI = 0.027

c) **Koczkodaj's Consistency Measure (CM):**

CM is defined as the maximum inconsistency over all triads.
For a 3×3 matrix, there is only one triad (a_{12}, a_{13}, a_{23}).

$CM = \min\{|1 - a_{13}/(a_{12}\times a_{23})|, |1 - (a_{12}\times a_{23})/a_{13}|\}$
$CM = \min\{|1 - 9/(4\times3)|, |1 - (4\times3)/9|\}$
$CM = \min\{|1 - 0.75|, |1 - 1.33|\}$
$CM = \min\{0.25, 0.33\}$
$CM = 0.25$

d) **Consistency Evaluation:**

1. Saaty's CR = 0.056 < 0.1, so the matrix is acceptably consistent according to Saaty's method.
2. For GCI, the threshold for n=3 is approximately 0.3147, so GCI = 0.027 < 0.3147 indicates acceptable consistency.
3. For CM, the threshold is typically 0.33, so CM = 0.25 < 0.33 indicates acceptable consistency.

All three methods indicate the matrix is acceptably consistent.

Question 4: AHP Rating Mode for Project Selection

Problem: A company wants to select the best IT project from four alternatives (P1, P2, P3, P4) based on three criteria: Return on Investment (ROI), Technical Feasibility (TF), and Strategic Alignment (SA). The company has decided to use the AHP rating mode.

The pairwise comparison matrix for the criteria is:

```
     | ROI | TF  | SA  |
-----|-----|-----|-----|
ROI  | 1   | 3   | 2   |
TF   | 1/3 | 1   | 1/2 |
SA   | 1/2 | 2   | 1   |
```

The company defined rating categories for each criterion:

For ROI:

- High (>20%): 1.000
- Medium (10-20%): 0.551
- Low (<10%): 0.217

For Technical Feasibility:

- Very High: 1.000
- High: 0.665
- Medium: 0.423

- Low: 0.195

For Strategic Alignment:

- Excellent: 1.000
- Good: 0.487
- Average: 0.403
- Poor: 0.113

The projects were rated as follows:

- P1: ROI (Medium), TF (High), SA (Good)
- P2: ROI (High), TF (Medium), SA (Excellent)
- P3: ROI (Low), TF (Very High), SA (Average)
- P4: ROI (Medium), TF (High), SA (Excellent)

a) Calculate the priority vector for the criteria
b) Determine the final scores for each project using the rating mode
c) Rank the projects from best to worst

Solution:

a) **Priority Vector for Criteria:**

Converting to decimal form:

```
     | ROI | TF   | SA  |
-----|-----|------|-----|
ROI  | 1   | 3    | 2   |
TF   | 0.33| 1    | 0.5 |
SA   | 0.5 | 2    | 1   |
```

Row sums:

- ROI: 1 + 3 + 2 = 6
- TF: 0.33 + 1 + 0.5 = 1.83
- SA: 0.5 + 2 + 1 = 3.5

Total: 6 + 1.83 + 3.5 = 11.33

Normalizing:

- Priority ROI = 6/11.33 = 0.530
- Priority TF = 1.83/11.33 = 0.162
- Priority SA = 3.5/11.33 = 0.308

Criteria priority vector = [0.530, 0.162, 0.308]

b) **Final Scores Using Rating Mode:**

Step 1: Assign numerical values based on ratings:

P1:

- ROI (Medium): 0.551
- TF (High): 0.665
- SA (Good): 0.487

P2:

- ROI (High): 1.000
- TF (Medium): 0.423
- SA (Excellent): 1.000

P3:

- ROI (Low): 0.217
- TF (Very High): 1.000
- SA (Average): 0.403

P4:

- ROI (Medium): 0.551
- TF (High): 0.665
- SA (Excellent): 1.000

Step 2: Multiply each rating by its corresponding criterion weight and sum:

P1 score = $0.530 \times 0.551 + 0.162 \times 0.665 + 0.308 \times 0.487$
$= 0.292 + 0.108 + 0.150$
$= 0.550$

P2 score = $0.530 \times 1.000 + 0.162 \times 0.423 + 0.308 \times 1.000$
$= 0.530 + 0.069 + 0.308$
$= 0.907$

P3 score = $0.530 \times 0.217 + 0.162 \times 1.000 + 0.308 \times 0.403$
$= 0.115 + 0.162 + 0.124$
$= 0.401$

P4 score = $0.530 \times 0.551 + 0.162 \times 0.665 + 0.308 \times 1.000$
$= 0.292 + 0.108 + 0.308$
$= 0.708$

c) **Project Ranking:**

From highest to lowest score:

1. P2 (0.907)
2. P4 (0.708)

3. P1 (0.550)
4. P3 (0.401)

Project P2 is the best choice according to the AHP rating analysis.

Question 5: Finding the Nearest Consistent Matrix

Problem: Consider the following pairwise comparison matrix:

```
A = | 1    5    7  |
    | 1/5  1    2  |
    | 1/7  1/2  1  |
```

a) Verify that this matrix is inconsistent
b) Find the nearest consistent matrix using Benítez's linearization technique
c) Calculate the distance between the original matrix and the nearest consistent matrix

Solution:

a) **Verify Inconsistency:**

A consistent matrix should satisfy $a_{ij} \times a_{jk} = a_{ik}$ for all i, j, k.

Let's check if $a_{12} \times a_{23} = a_{13}$:
$5 \times 2 = 10 \neq 7$

Since this equality is not satisfied, the matrix is inconsistent.

b) **Finding the Nearest Consistent Matrix:**

Step 1: Convert to decimal form:

```
A = | 1     5     7    |
    | 0.2   1     2    |
    | 0.143 0.5   1    |
```

Step 2: Take the logarithm of each element to create matrix L(A):

```
L(A) = | ln(1)      ln(5)       ln(7)    |
       | ln(0.2)    ln(1)       ln(2)    |
       | ln(0.143)  ln(0.5)     ln(1)    |
```

```
L(A) = | 0     1.609   1.946  |
       | -1.609 0      0.693  |
       | -1.946 -0.693 0      |
```

Step 3: According to Benítez's method, we calculate X_B:

```
X_B = (1/n) × [(B × U_n) - (B × U_n)^T]
```

Where B = L(A) and U_n is a matrix of all ones.

First, calculate B × U_n:

```
| 0      1.609   1.946  | | 1 1 1 | = | 3.555 |
| -1.609 0       0.693  | | 1 1 1 |   | -0.916|
| -1.946 -0.693 0       | | 1 1 1 |   | -2.639|
```

This gives us a column vector where each element is the sum of the corresponding row in L(A).

Now, create a matrix where each row is filled with the elements of this vector:

```
(B × U_n) = | 3.555  3.555  3.555 |
            | -0.916 -0.916 -0.916|
            | -2.639 -2.639 -2.639|
```

The transpose (B × U_n)^T is:

```
(B × U_n)^T = | 3.555  -0.916 -2.639|
              | 3.555  -0.916 -2.639|
              | 3.555  -0.916 -2.639|
```

Calculate (B × U_n) - (B × U_n)^T:

```
| 3.555  3.555  3.555 | - | 3.555  -0.916 -2.639| = | 0      4.471  6.194 |
| -0.916 -0.916 -0.916| - | 3.555  -0.916 -2.639|   | -4.471 0      1.723 |
| -2.639 -2.639 -2.639| - | 3.555  -0.916 -2.639|   | -6.194 -1.723 0     |
```

Multiply by 1/n (where n=3):

```
X_B = (1/3) × | 0      4.471  6.194 |
              | -4.471 0      1.723 |
              | -6.194 -1.723 0     |
```

```
X_B = | 0      1.490  2.065 |
      | -1.490 0      0.574 |
      | -2.065 -0.574 0     |
```

Step 4: Apply the exponential function to each element to get the nearest consistent matrix:

```
E(X_B) = | e^0     e^1.490  e^2.065 |
         | e^-1.490 e^0      e^0.574 |
         | e^-2.065 e^-0.574 e^0     |
```

```
E(X_B) = | 1      4.437   7.884  |
         | 0.225  1       1.776  |
         | 0.127  0.563   1      |
```

This is the nearest consistent matrix to A.

c) **Distance Calculation:**

The distance is defined as:
d(A, E(X_B)) = ||L(A) - X_B||_F

Where ||.||_F is the Frobenius norm.

Calculate L(A) - X_B:

```
| 0     1.609   1.946  | - | 0      1.490  2.065 | = | 0      0.119  -0.119 |
| -1.609 0      0.693  | - | -1.490 0      0.574 |   | -0.119 0      0.119  |
| -1.946 -0.693 0      | - | -2.065 -0.574 0     |   | 0.119  -0.119 0      |
```

Frobenius norm:
$||L(A) - X\_B||\_F = \sqrt{0^2 + 0.119^2 + (-0.119)^2 + (-0.119)^2 + 0^2 + 0.119^2 + 0.119^2 + (-0.119)^2 + 0^2}$
$= \sqrt{6 \times 0.119^2}$
$= \sqrt{6 \times 0.014161}$
$= \sqrt{0.084966}$
$= 0.292$

The distance between the original matrix A and the nearest consistent matrix is 0.292.

⁂