Rheology

Aoka



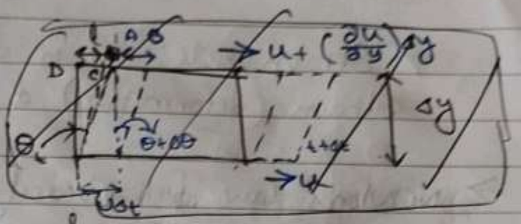
23rd July

→ Example of Gel → Curd

→ Defination of ... in polymers

→

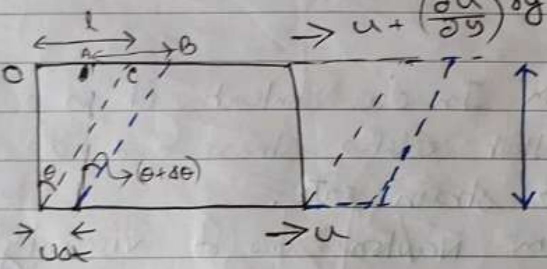
$$\sigma = \mu \frac{\partial v_x}{\partial y}$$



Strain at time 't' $(\gamma_t) = \theta_t = \frac{l}{\Delta y}$

~~$AB = l - u\Delta t$~~

$$\gamma_{t+\Delta t} = \theta_{t+\Delta t} = \frac{AB}{\Delta y\,\cancel{t+\Delta t}}$$



$$CB = \left(u + \left(\frac{\partial u}{\partial y}\right)\Delta y\right)\Delta t$$

$$AB = AC + CB$$
$$= OC - OA + CB$$
$$= (l - u\Delta t) + CB$$

$$\theta_{t+\Delta t} = \frac{AB}{\Delta y} = \frac{l - \cancel{u\Delta t} + \cancel{u\Delta t} + \frac{\partial u}{\partial y}\Delta y\,\Delta t}{\Delta y}$$

$$= \frac{l + \left(\frac{\partial u}{\partial y}\right)\Delta y . \Delta t}{\Delta y}$$

$$\text{shear rate} = \dot{\gamma} = \lim_{\Delta t \to 0} \frac{\gamma_{t+\Delta t} - \gamma_t}{\Delta t} = \frac{\frac{\frac{\partial u}{\partial y} . \Delta y . \Delta t}{\Delta y}}{\Delta t}$$

$$= \frac{\partial u}{\partial y} \quad \cancel{\frac{\partial u_x}{\partial y}}$$

$$\boxed{\dot{\gamma}_{xy} = \frac{\partial u_x}{\partial y}}$$

→ shear rate

# ~~Shear rate ramp up.~~ -

Note: $\sigma_{ii} = -P + \frac{\partial u_i}{\partial y}$ ; so normal stress has two components and Pressure is one of them.

→→→

~~For newtonian~~ fluid

Newton's law of viscosity:- $\text{stress} = \mu\left(\begin{matrix}\text{strain}\\ \text{rate}\end{matrix}\right)$

$\mu \rightarrow$ viscosity

→ Strain rate: it is defined ~~set~~ as the change in Strain (deformation) of a material w.r.t. time.

→ Newton's law of viscosity has nothing to do with newtonian fluid. Newtonian fluids are such whose viscosity doesn't change with strain rate.

→ Non-newtonian fluid also follow Newton's law of viscosity.

→ Constitutive relation: In our context, it is a mathematical law/ relation that relates stress and a function of strain and/or strain rate.

Exm: Newton's law of viscosity

Note: $\sigma_{ii} = \underbrace{-P}_{\text{pressure term}} + \frac{\partial u}{\partial y}$

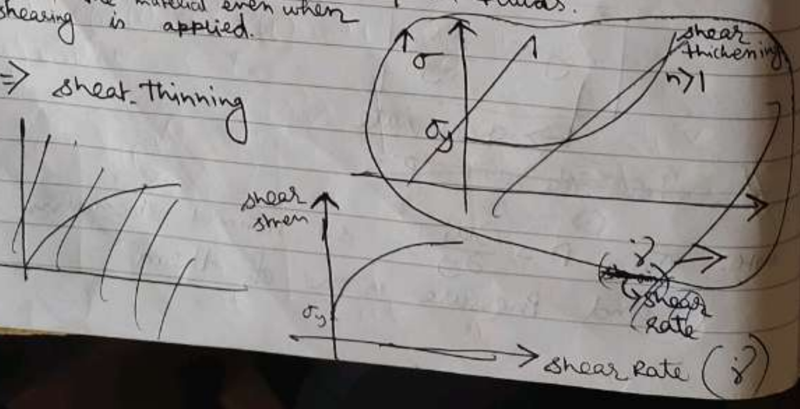
# Constitutive eq^n: Herschel Bulkley Model.

$$\underbrace{\sigma}_{\text{applied shear stress}} = \underbrace{\sigma_y}_{\text{yield stress}} + \underbrace{K}_{\text{Consistency coeff}} (\dot{\gamma})^{\overbrace{n}^{\text{flow index}}}$$

yield stress: The residual stress that ~~is the~~ remains in the material even when no shearing is applied.

~~n=1~~ For $n=1$; we get newton's law of viscosity which is valid for all fluids. → Bingham

$n<1 \Rightarrow$ shear Thinning



Exm of Shear Thickening:- ~~Corn stra~~ Cornstarch with water
" " " Thinning: Ketchup, blood

# $\sigma < \sigma_y$ → material will not flow on Practical timescale

# Shear thinning ⇒ the Phenomenon where the apparent viscosity of fluid decreases with increasing shear rate, causing fluid to flow faster under the same pressure-drop condition. It is ~~con~~ a non newtonian behaviour / Pseduo-plastic behaviour

# Stress Relaxation Test: Gradual reduction in stress over time while maintaining ~~the~~ a constant strain.

$\frac{d\gamma}{dt} = 0$

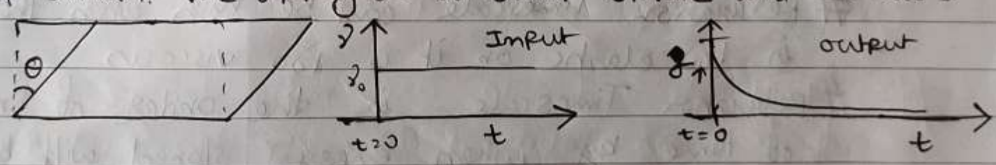
energy dissipation

For water-like liquid, stress relaxation is very fast.

→ To estimate: Time scale over which energy dissipates. This Time scale is known as Relaxation Time Scale and is material dependent.

↳ $\tau = \eta/G$

→ Step Strain: we apply a constant strain and we hold it

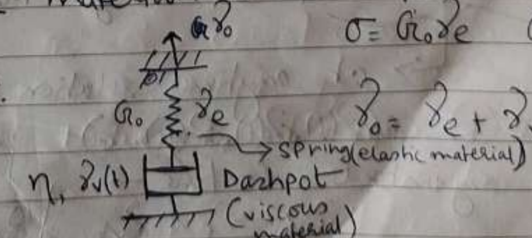



→ Mechanical Models: To Predict Rheological behaviour of viscoelastic material.

$\sigma = G_0\gamma_e$ ↳ modulus (unit Pa)

$\sigma = \eta\dot{\gamma}_v$ ↳ viscosity of dashpot, not the material.

# Maxwell Model:
Apply step strain $\gamma_0$




$\gamma_0 = \gamma_e + \gamma_v$

Assumption:-
i) Spring immediately experience the applied strain
ii) Dashpot element: strain evolves always starting from 0

at $t=0$, $\gamma_v = 0$, $\gamma_e = \gamma_0$

at $t=t$, $\gamma_0 = \gamma_e + \gamma_v$

# if two spring are in series then both of them feel same force $(\sigma_v = \sigma_e = \sigma)$

$$\gamma_0 = \gamma_e + \gamma_v$$

diff. w.r.t. time —

$$0 = \dot{\gamma}_e + \dot{\gamma}_v$$

$$\text{or, } 0 = \frac{\dot{\sigma}}{G_0} + \frac{\sigma}{\eta}$$

$\sigma_e = G_0 \gamma_e$ | $\sigma_v = \eta \dot{\gamma}_v = \sigma$

$\dot{\gamma}_e = \frac{\dot{\sigma}_e}{G_0} = \frac{\dot{\sigma}}{G_0}$

$$\text{or, } 0 = \frac{1}{G_0}\frac{d\sigma}{dt} + \frac{\sigma}{\eta} \Rightarrow \int_{\sigma_0}^{\sigma} \frac{d\sigma}{\sigma} = -\int_0^t \left(\frac{G_0}{\eta}\right) dt$$

$$\Rightarrow \ln\left(\frac{\sigma}{\sigma_0}\right) = -\left(\frac{G_0}{\eta}\right)t \Rightarrow \boxed{\sigma = \sigma_0 \exp\left(-\left(\frac{G_0}{\eta}\right)t\right)}$$

at $t=0$, $\sigma = \sigma_0 = G_0 \gamma_0$ $\quad \tau = \eta / G_0$

$[\because$ at $t=0$, $\gamma_e = \gamma_0]$ $\quad$ Relaxation time scale

$$\text{So, } \boxed{\sigma = G_0 \gamma_0 \exp(-t/\tau)} \Rightarrow \boxed{G(t) = G_0 \exp(-t/\tau)} \left(G(t) = \frac{\sigma}{\gamma_0}\right)$$

# Relaxation Timescale can be high when the material is too elastic or it is too viscous.

# Relaxation Timescale is the order of magnitude of time by when energy stored will be zero. It is used to basically represents the mobility of constituent elements.

# Constitutive Eq$^n$ for Single Mode maxwell model:

→ at anytime relation between $\sigma$, $\gamma$, and so on...

$$\sigma(t) = f(\gamma(t), \dot{\gamma}(t), \ddot{\gamma}(t) \ldots)$$

At any instant of time —

$$\gamma = \gamma_e + \gamma_v \quad \Big| \quad \sigma = G_0 \gamma_e, \ \sigma = \eta \dot{\gamma}_v$$

$$\therefore \dot{\sigma} = G_0 \dot{\gamma}_e$$

$$\xrightarrow[\text{w.r.t. } t]{\text{diff}} \dot{\gamma} = \dot{\gamma}_e + \dot{\gamma}_v = \frac{\dot{\sigma}}{G_0} + \frac{\sigma}{\eta_0} =$$

$$\text{or, } \frac{d\gamma}{dt} = \frac{1}{G_0}\frac{d\sigma}{dt} + \frac{\sigma}{\eta_0}$$

# Creep Compliance: Assuming stress is constant, and solve the diff. eq. in strain



# Relaxation Modulus: Assume strain constant and solve for diff. eqⁿ in stress

$$\Rightarrow \quad G_0 \frac{d\gamma}{dt} = \frac{d\sigma}{dt} + \frac{G_0}{\eta_0}\sigma$$

$$\therefore \boxed{\frac{d\sigma}{dt} + \frac{\sigma}{\tau_0} = G_0 \frac{d\gamma}{dt}} \Rightarrow \frac{d\gamma}{dt} = \frac{1}{G}\frac{d\sigma}{dt} + \frac{\sigma}{\eta}$$

↳ General eqⁿ

~~e.g. if step strain $\gamma = 0 \Rightarrow$~~

~~Also~~,

→ Creep Experiment: Model is subjected to instantaneous tensile stress $\sigma_0$, which is held constant.

$$\frac{d\sigma_0}{dt} = 0$$

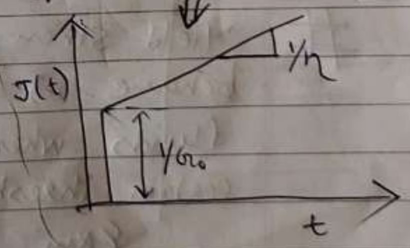
For Maxwell Model —

$$\therefore \frac{d\gamma}{dt} = \frac{\sigma_0}{\eta} \quad \Big| \quad \because \frac{d\sigma_0}{dt} = 0$$

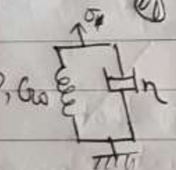
$$\text{or} \quad \frac{1}{\sigma_0}\int_{\gamma_0}^{\gamma} d\gamma = \frac{1}{\eta}\int_0^t dt \Rightarrow \boxed{\frac{\gamma(t)}{\sigma_0} = \frac{\gamma_0}{\sigma_0} + \frac{t}{\eta}}$$

$$\boxed{J(t) = 1/G_0 + t/\eta}$$

$\Downarrow$




# For Kelving Model:—




$$\sigma = G_0\gamma + \eta\frac{d\gamma}{dt}$$

Creep compliance:—

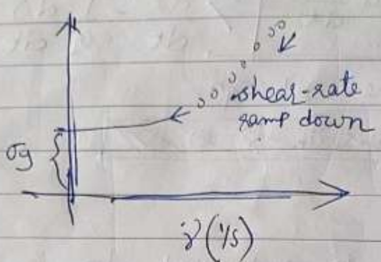
$$J(t) = \frac{1}{G_0}\left(1 - e^{-t/\tau}\right)$$

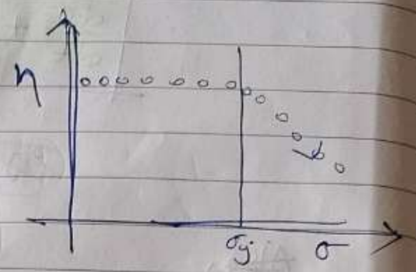
Storage moduli = $G' = K$

Loss " = $G'' = \eta\omega$

# Herschel - Bulkey Model:-

$$\sigma = \sigma_y + K(\dot{\gamma})^n$$




shear rate ramp up (low value to high value)

Dynamic state → Static → Dynamic stress
(liq. type material)

# ~~Shear~~ Thixotropic fluid: Non-newtonian fluids that exhibit a time-dependent decrease in viscosity when subject to shear stress.

After the shear stress is removed, they gradually recover their viscosity.

Examples: Paints, ketchup.

# Yield stress fluids are materials that behave as solids at low stresses but flow as fluids when the applied stress exceeds a certain threshold known as yield stress.

Exm: Toothpaste, mayonnaise

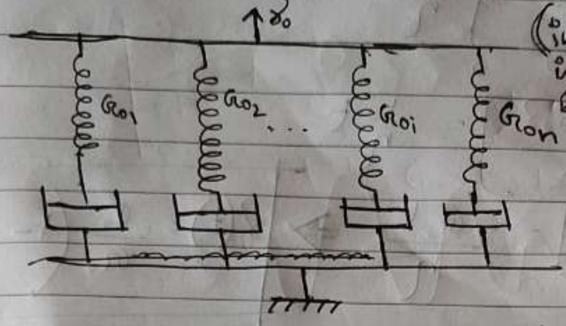
# Power law fluids: Known as shear-thinning or shear thickening fluids, follow a flow curve that can be described by the power law model, where viscosity is function of shear rate.

Shear-thinning ⇒ flow behaviour index $(n) < 1$

Shear thickening ⇒ " — — $(n) > 1$

$n = 1$, newtonian fluid

# Generalised Maxwell Model:—

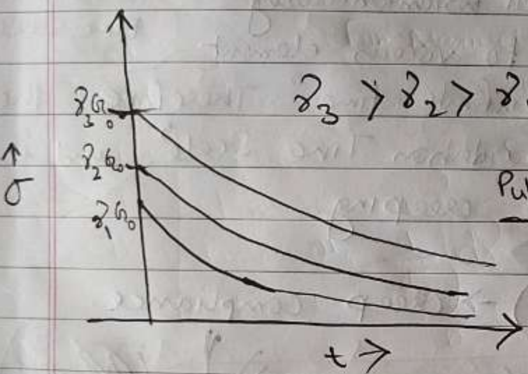



(in all individual elements, the strain is the same and total stress is summation of all individual stress enjoyed by each)

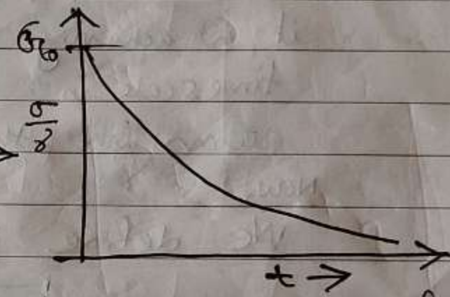
$$\tau_i = \frac{\eta_{0i}}{G_{0i}}$$

$$\sigma_i = (\gamma_0 G_{0i})\exp\left(-\frac{t}{\tau_i}\right) \Rightarrow \boxed{\sigma = \Sigma\sigma_i = \gamma_0 \Sigma\left[G_{0i}\exp\left(-\frac{t}{\tau_i}\right)\right]}$$

$\gamma_3 > \gamma_2 > \gamma_1$




Putting 3 curves into 1 $\longrightarrow$




$$\left.\begin{aligned} \sigma_1 &= G_0\gamma_1 \exp(-t/\tau_1) \\ \sigma_2 &= G_0\gamma_2 \exp(-t/\tau_2) \\ \sigma_3 &= G_0\gamma_3 \exp(-t/\tau_3) \end{aligned}\right\} \longrightarrow \frac{\sigma_i}{\gamma_i} = G_0 \exp\left(-\frac{t}{\tau_i}\right)$$

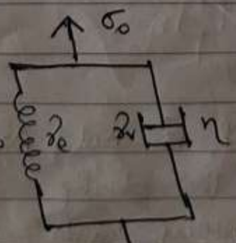
→ Relaxation modulus of material.

$$\Rightarrow \boxed{G(t) = G_0 \exp(-t/\tau_i)}$$

$$\boxed{G(t) = \Sigma G_{0i} \exp(-t/\tau_i)}$$

↳ generalised Maxwell model

# Creep Test:
(Kelvin Voight Model)




at $t=0$, $\gamma=0$. $\gamma_v=0 \Rightarrow \gamma_e=0$

at $t=t$, $\gamma_e=\gamma_v=\gamma(t)$

$$\sigma_0 = \sigma_e + \sigma_v = G_0\gamma_e + \eta\dot{\gamma}$$

$$\text{or, } \frac{\sigma_0}{\eta} = \frac{G_0\gamma_e}{\eta} + \frac{d\gamma}{dt}$$

$$\text{or, } \frac{\sigma_0}{\eta} = \frac{\gamma_e}{\tau} + \frac{d\gamma}{dt} \quad \Bigg| \quad \boxed{\eta/G_0 = \tau}$$

$$\text{or, } \frac{d\gamma}{dt} = \frac{\sigma_0 - G_0\gamma}{\eta}$$

or, $\int_0^{\gamma} \frac{d\gamma}{\sigma_0 - G_0\gamma} = \int_0^{t} \frac{dt}{\eta}$

or, $-\frac{1}{G_0}\ln\left[\frac{\sigma_0 - G_0\gamma}{\sigma_0}\right] = \frac{1}{\eta}t$

or, $\ln\left[\frac{\sigma_0 - G_0\gamma}{\sigma_0}\right] = -\frac{G_0 t}{\eta} \Rightarrow \boxed{\gamma(t) = \frac{\sigma_0}{G_0}\left(1 - \exp\left(-\frac{G_0 t}{\eta}\right)\right)}$

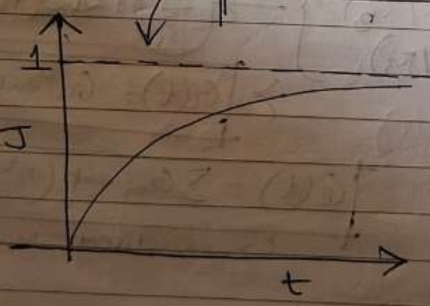
Note: if there is no viscous element present and we apply $\sigma_0$, the spring will instanteneously experience $\sigma_0/G_0$. But due to viscous element it is achieving $\sigma_0/G_0$ at infinite time. Therefore the timescale is called 'Retardation Time Scale' and it seems like the spring is creeping.

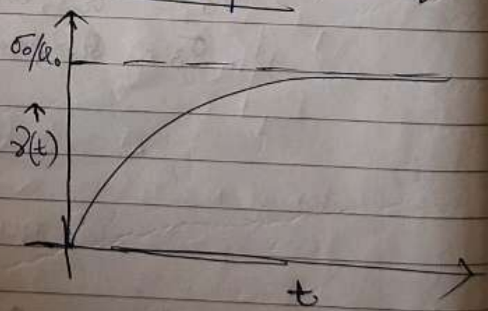
$\tau = \eta/G_0$ = Retardation time scale

Now,

We define $\gamma/\sigma_0 = J(t) \rightarrow$ creep compliance

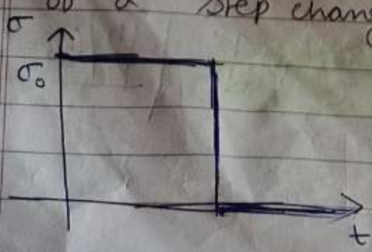
$$\boxed{J(t) = 1/G_0\left(1 - \exp(-t/\tau)\right)}$$

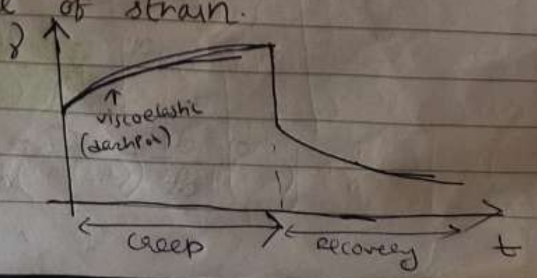



# Stress Relaxation Test is not possible for Voigh Element as the dashpot element would develope an indefinitely high force with the input of a step change of strain.

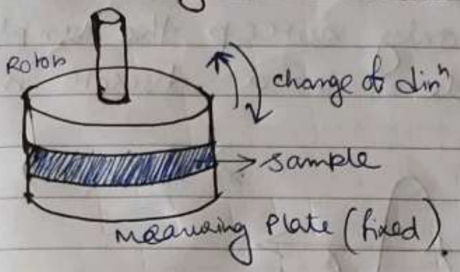
#

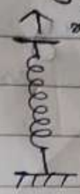


# $G'$ is stress at max$^n$ strain $\Rightarrow \boxed{\omega t = \frac{\pi}{2}}$
$G''$ is – at 0 strain $\gamma_0$ $\Rightarrow \boxed{\omega t = 0}$

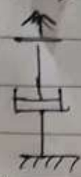
# Oscillatory Measurement (Storage and Loss modulus)



$\gamma = \gamma_0 \sin(\omega t)$

$\sigma = G_0 \gamma_0 \sin(\omega t)$



$\gamma = \gamma_0 \sin(\omega t)$

$\sigma = \eta_0 \gamma_0 \sin\left(\omega t + \frac{\pi}{2}\right)$

out of phase by $\pi/2$

$$\sigma = \eta \omega \gamma_0 \sin\left(\omega t + \frac{\pi}{2}\right)$$

$$\sigma = \sigma_0 \sin(\omega t + \pi/2)$$

Note: Here 'timescale' is user controlled. If time is high, frequency will be low and the rotor will shear the material very slowly. If the time is small, frequency will be high, and the rotor will ~~be~~ shear the material very fast.

For viscoelastic material – $\gamma = \gamma_0 \sin(\omega t)$ | $\sigma = \sigma_0 \sin(\omega t + \phi)$

$$\sigma = \underbrace{\sigma_0 \cos\phi \cdot \sin\omega t}_{\text{elastic Response (Solid like)}} + \underbrace{\sigma_0 \sin\phi \cdot \cos\omega t}_{\text{viscous response (Liquid like)}}$$

Storage modulus $G' = \dfrac{\sigma_0 \cos\phi}{\gamma_0}$ | Loss modulus: $G'' = \dfrac{\sigma_0 \sin\phi}{\gamma_0}$

Loss tangent or Damping factor $= \tan\delta = G''/G'$

Now, $G^* = G' + iG''$ | $G^* = \sigma/\gamma = \dfrac{\sigma_0 \exp[i(\omega t + \phi)]}{\gamma_0 \exp(i\omega t)}$

$G^*$ → complex modulus

$$G^* = \frac{\sigma_0}{\gamma_0} \exp(i\phi)$$

$$\therefore \frac{\sigma_0}{\gamma_0} \exp(i\phi) = G' + iG''$$

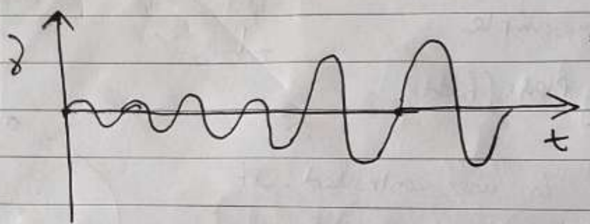
$$G' = \frac{\sigma_0}{\gamma_0} \cos(\phi)$$

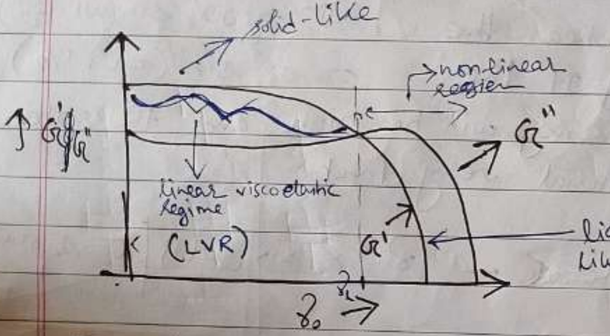
$$G'' = \frac{\sigma_0}{\gamma_0} \sin(\phi)$$

$$\frac{G''}{G'} = \tan\phi$$

# Strain Amplitude Sweep Test: This test is performed at fixed frequency. We only sweep the amplitude, that is ↑ the amplitude $\gamma_0$ in a discrete manner.

$\gamma = \gamma_0 \sin(\omega t)$

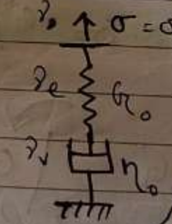



Beyond some point loss modulus will cross storage modulus as liq. like nature is more.

To get linear regime of material, we need to do Strain Amplitude Sweep (SAS) test and gradually $\gamma_0 \uparrow$.

# Consider Maxwell model and apply oscillatory response




$\gamma = \gamma_e + \gamma_v$

$\sigma = \sigma_e = G_0 \gamma_e$

$\sigma = \sigma_v = \eta_0 \dot{\gamma}_v$

$\dot{\gamma} = \dot{\gamma}_e + \dot{\gamma}_v =$

or $\frac{d\gamma}{dt} = \cancel{\frac{d\gamma_e}{dt} + \frac{d\gamma_v}{dt}}$

or, $\frac{d\gamma}{dt} = \frac{\dot{\sigma}_e}{G_0} + \frac{\sigma_v}{\eta_0} = \frac{\dot{\sigma}}{G_0} + \frac{\sigma}{\eta_0}$

$$\frac{d\gamma}{dt} = \frac{(\sigma_0 i\omega)\exp(i\omega t)}{G_0} + \frac{\sigma_0 \exp(i\omega t)}{\eta_0}$$

$\sigma = \sigma_0 e^{i\omega t}$

$\dot{\sigma} = (\sigma_0 i\omega).\exp(i\omega t)$

$$\text{or, } \int_{\gamma_t}^{\gamma_{t+\Delta t}} d\gamma = \frac{i\sigma_0\omega}{G_0}\int_t^{t+\Delta t} \exp(i\omega t)dt + \frac{\sigma_0}{\eta_0}\int_t^{t+\Delta t}\exp(i\omega t)dt$$

$$\text{or, } \gamma_{t+\Delta t} - \gamma_t = \frac{i\sigma_0\omega}{i\omega G_0}\left[\exp(i\omega(t+\Delta t)) - \exp(i\omega t)\right] + \frac{\sigma_0}{\eta_0 i\omega}\left[\exp(i\omega(t+\Delta t)) - \exp(i\omega t)\right]$$

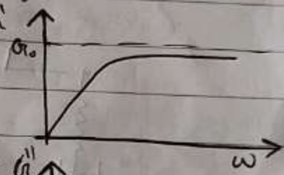
or, $\Delta\gamma = \sigma_0 \left[\exp(i\omega(t+\delta t)) - \exp(i\omega t)\right] \left\{ \frac{1}{G_0} + \frac{1}{i\eta_0\omega} \right\}$

$$\Delta\gamma = (\sigma_{t+\delta t} - \sigma_t)\left\{\frac{1}{G_0} + \frac{1}{i\eta_0\omega}\right\} = \Delta\sigma\left\{\frac{G_0 + i\eta_0\omega}{iG_0\eta_0\omega}\right\}$$

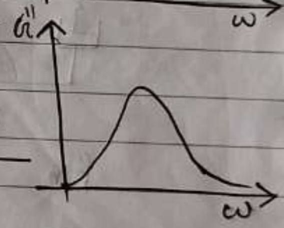
$$\boxed{\frac{\Delta\sigma}{\Delta\gamma} = G^* = \frac{iG_0\eta_0\omega}{G_0 + i\eta_0\omega}}$$

$$G^* = \frac{iG_0\eta_0\omega}{G_0 + i\eta_0\omega} \times \frac{G_0 - i\eta_0\omega}{G_0 - i\eta_0\omega} = \frac{iG_0^2\eta_0\omega + G_0\eta_0^2\omega^2}{G_0^2 + \eta_0^2\omega^2}$$

$$G^* = \frac{G_0\eta_0^2\omega^2}{G_0^2 + \eta_0^2\omega^2} + i\left(\frac{G_0^2\eta_0\omega}{G_0^2 + \eta_0^2\omega^2}\right)$$




or, $$\boxed{G^* = \underbrace{\frac{G_0\tau^2\omega^2}{1+\tau^2\omega^2}}_{G'} + i\underbrace{\frac{G_0\tau\omega}{1+\tau^2\omega^2}}_{G''}}$$




Note

→ $\tau$ is relaxation time scale which is material dependent

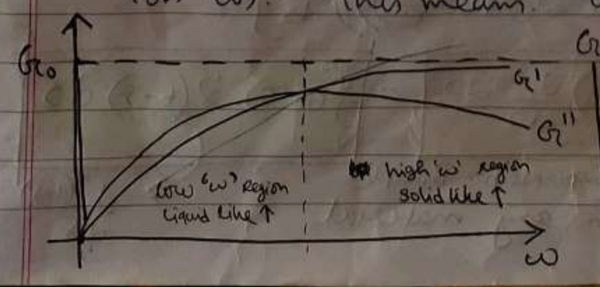
→ $\omega$ is basically $\frac{2\pi}{T}$ or we can say $\omega \propto \frac{1}{T}$ which is user controlled

→ To plot and analyze the behaviour of $G'$ and $G''$, we need to analyze $\tau\omega$ together as an entity.

→ $\tau\omega \ll 1 \Rightarrow \tau \ll T \Rightarrow$ Material will behave liquid like for us.

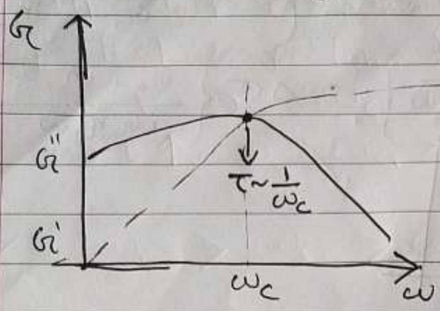
This means $G' \sim G_0(\tau\omega)^2 \Rightarrow G'' > G'$
$G'' \sim G_0(\tau\omega)$

→ $\tau\omega \gg 1 \Rightarrow \tau \gg T \Rightarrow$ Material will behave solid like for us. This means $G' \sim G_0$
$G'' \sim \frac{G_0}{\tau\omega} \Rightarrow G'' < G'$

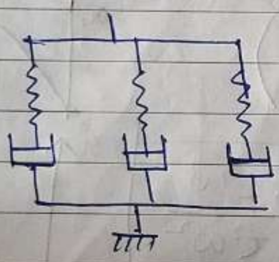



The transition from liquid like to solid like takes place at $\tau\omega \sim 1$ which gives $\omega_c \sim \frac{1}{\tau}$ where $\omega_c$ is crossover frequency. Here we are sweeping frequency, so we can call this as Frequency Sweep Test. This help us to roughly estimate "relaxation time scale".

→ ~~Note~~: Frequency Sweep experiments:




Note: For Generalised Maxwell Model —
(Generalised Maxwell model can capture any material)



$$G' = \sum \frac{G_{oi}\tau_i^2\omega^2}{1+\tau_i^2\omega^2}$$

$$G'' = \sum\left(\frac{G_{oi}\tau_i\omega}{1+\tau_i^2\omega^2}\right)$$

For GMM —

$$\frac{d\gamma}{dt} = \frac{\dot{\sigma}_i}{G_{oi}} + \frac{\sigma_i}{\eta_{oi}}$$

$$\text{or, } G_{oi}\frac{d\gamma}{dt} = \dot{\sigma}_i + \frac{\sigma_i}{\tau_i}$$

$$\text{or, } \frac{d\sigma_i}{dt} + \frac{\sigma_i}{\tau_i} = G_{oi}\frac{d\gamma}{dt}$$

$$\text{or, } \left[\sigma_i \exp\left(\frac{t}{\tau_i}\right)\right]_{-\infty}^{t} = \int_{-\infty}^{t} G_{oi}\frac{d\gamma}{dt}\exp\left(\frac{t}{\tau_i}\right)dt$$

$$\text{or, } \sigma_i \exp\left(\frac{t}{\tau_i}\right) = \int_{-\infty}^{t} G_{oi}\cdot\frac{d\gamma}{ds}\exp\left(\frac{s}{\tau_i}\right)ds$$

$$\text{or, } \sigma_i = \int_{-\infty}^{t} G_{oi}\left[\exp\left(-\frac{(t-s)}{\tau_i}\right)\right]\frac{d\gamma}{ds}\cdot ds$$

$$\therefore \sigma_1 = \sum\sigma_i = \int_{-\infty}^{t}\left[\sum_i \left(G_{oi}\left(\exp\left(-\frac{(t-s)}{\tau_i}\right)\right)\right)\frac{d\gamma}{ds}\right]ds$$

$$\therefore \boxed{\sigma = \int_{-\infty}^{t} G(t-s)\frac{d\gamma}{ds}\cdot ds} = \boxed{\int_{0}^{\infty} G(s)\cdot\dot{\gamma}(t-s)\,ds}$$

Boltzmann Superposition Principle

→ Valid for any material

$\frac{d\gamma(t-s)}{ds}$

$$\boxed{\gamma(t) = \int_{-\infty}^{t} J(t-s)\,\frac{d\sigma}{ds}\,ds} \rightarrow \text{for generalised Kelvin-Vought model}$$

$$\hookrightarrow \gamma(t) = \int_0^\infty J(s)\,\frac{d\sigma(t-s)}{ds}\,ds.$$

Now, $\sigma = \int_0^\infty G(s)\,\frac{d\gamma(t-s)}{ds}\,ds$

$\gamma = \gamma_0 \sin(\omega t) \Rightarrow \dot{\gamma} = \gamma_0 \omega \cos(\omega t)$

$\dot{\gamma}(t-s) = \gamma_0 \omega \cos(\omega(t-s))$

$\therefore \sigma = \int_0^\infty G(s)\,\dot{\gamma}(t-s)\,ds = \int_0^\infty G(s)\,\gamma_0\,\omega \cos(\omega(t-s))\,ds$

or, $\sigma = \int_0^\infty G(s)\,\gamma_0\,\omega\left[\cos(\omega t).\cos(\omega s) + \sin(\omega t).\sin(\omega s)\right] ds$

or, $\sigma = \int_0^\infty G(s).\gamma_0\,\omega.\cos(\omega t).\cos(\omega s)\,ds + \int_0^\infty G(s).\gamma_0\,\omega \sin(\omega t)\sin(\omega s)\,ds$

or, $\sigma/\gamma_0 = \omega.\cos(\omega t)\int_0^\infty G(s).\cos(\omega s)\,ds + \omega \sin(\omega t)\int_0^\infty G(s)\sin(\omega s)\,ds$

$\therefore \sigma/\gamma_0 = \cos(\omega t).G'' + \sin(\omega t).G''$

$G'' = \omega\int_0^\infty G(s).\cos(\omega s)\,ds$ , $G' = \omega\int_0^\infty G(s).\sin(\omega s)\,ds.$

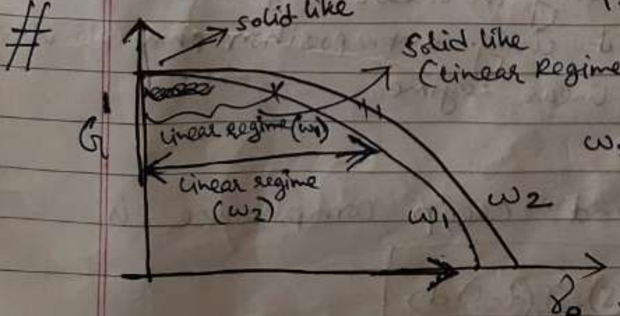
$G' = \omega\int_0^\infty G_0 \exp\left(-\frac{s}{\tau}\right).\sin(\omega s)\,ds$

$= \omega G_0 \underbrace{\int_0^\infty \sin(\omega s)\exp\left(-\frac{s}{\tau}\right) ds}_{I = \frac{\tau^2\omega}{1+\tau^2\omega}} \Longrightarrow \boxed{G' = \frac{G_0\,\omega^2\tau^2}{1+\omega^2\tau^2}}$

$\hookrightarrow$ relation btw. $G'$ and $G_0$

?

#




$\omega_2 > \omega_1$ (for same solid) : As relaxation timescale is larger than observational time scale (apparently) the material doesn't get time to relax. Therefore it behaves like a solid. Therefore linear regime is increased classmate

Date
Page

(imp) Polymer melt is given, put a crosslinker (which will start crosslinking). As time ↑, crosslinking density ↑. if test is performed (Frequency Sweep Test), $G', G''$ v/s $\omega$

at $t = 0$, $\omega_c = \omega_0$ (crossover frequency)

as $t \uparrow$, $\omega_c = \omega_{c_1} \rightarrow \omega_{c_2} \rightarrow \omega_{c_3} \ldots$ which one is large

$\Rightarrow \omega_{c_0} > \omega_{c_1} > \omega_{c_2}$ | As crosslinking frequency ↑, liquid like character ↑.

# Steady state shear test:— $\dot{\gamma}$ = constant

$$\cancel{\sigma = \int_0^\infty G(s)\dot{\gamma}\, ds} \Rightarrow \frac{\sigma}{\dot{\gamma}} = \int_0^\infty G(s)\, ds.$$

$$\sigma = \int_{-\infty}^{t} G(t-s)\,\dot{\gamma}(s)\, ds \quad \text{--- B.S.P.}$$

$$= \int_0^\infty G(s)\,\dot{\gamma}(t-s)\, ds$$

$$\eta = \frac{\sigma}{\dot{\gamma}} = \int_0^\infty G(s)\, ds$$

↓
viscosity

$$\eta = \int_0^\infty \sum G_{0_i} \exp\left(\frac{-s}{\tau_i}\right) ds = \sum G_{0i} \int_0^\infty \exp\left(\frac{-s}{\tau_i}\right) ds$$

$$= \sum G_{0_i} \left[\exp\left(\frac{-s}{\tau_i}\right)\right]_0^\infty (-\tau_i) = \sum G_{0i}\,(0-1)\,(-\tau_i)$$

$$\Rightarrow \boxed{\eta = \sum G_{0i}\tau_i} \rightarrow \text{in terms of maxwell's parameter}$$

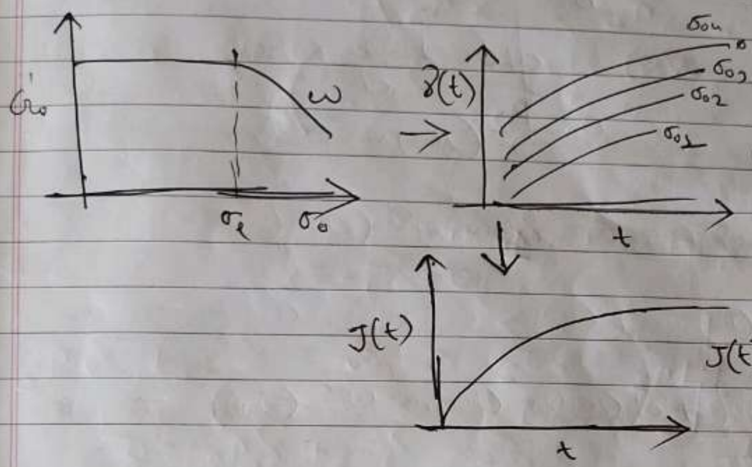
↳ valid only in linear regime as we are ~~der~~ deriving it using superposition principle which is valid in linear regime.

# Relaxation Modulus and creep compliance is related as

$$\boxed{t = \int_0^t G(s).\, J(t-s)\, ds}$$

# ~~Time sweep test~~

# For Creep Test ———

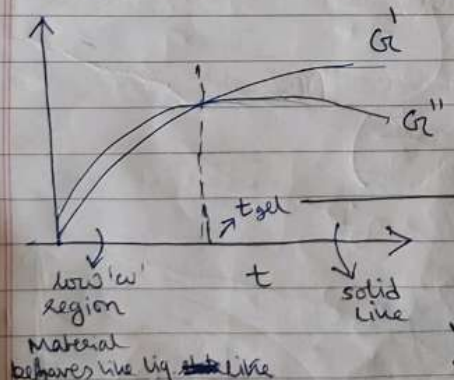



$\sigma_{04} > \sigma_{03} > \sigma_{02} > \sigma_{01}$

$$J(t) = \frac{\gamma(t)}{\sigma_0} = \frac{1}{G_0}\left(1 - \exp\left(\frac{-t}{\tau_r}\right)\right)$$

→ Time Sweep Test: we can either take $\gamma = \gamma_0 \sin(\omega t)$
or, $\sigma = \sigma_0 \sin(\omega t)$

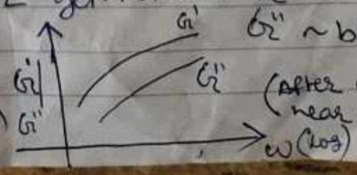
In frequency sweep test we are just probing the material at different frequency due to which it appears either solid-like or liquid like. However in time-sweep test, actual physical changes takes place.




→ This is apparent gel point because this depends on probing frequency. Increasing frequency will increase solid-like properties thus gel-point will be achieved faster.
"True-gel point" is fixed and independent of all factors.

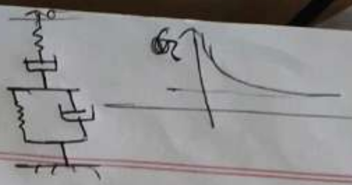
→ Unique feature of near gel-point : $G' \sim a\omega^n$
$G'' \sim b\omega^n$

both $G', G'' \rightarrow$ Parallel



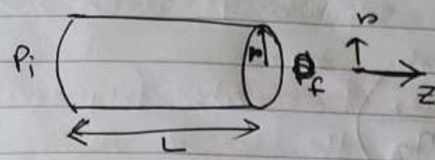

(After crossover near gel-point)

Burger's Model:-



## Asgn 1

1)




$$-\frac{dP}{dz} = \frac{P_i - P_f}{L}$$

Using cauchy Mom$^n$ balance eq$^n$:—

$$\rho\left(\underset{0}{\cancel{\frac{\partial v_z}{\partial t}}} + \underset{0}{\cancel{v_r}}\frac{\partial v_z}{\partial r} + \underset{0}{\cancel{\frac{v_\theta}{r}}}\frac{\partial v_z}{\partial \theta} + v_z \underset{0}{\cancel{\frac{\partial v_z}{\partial z}}}\right) = -\frac{\partial P}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz})$$

$$+ \frac{1}{r}\frac{\partial}{\partial \theta}\underset{0}{\cancel{(\tau_{\theta z})}} + \frac{\partial}{\partial z}\underset{0}{\cancel{(\tau_{zz})}} + \rho \underset{0}{\cancel{g_z}}$$

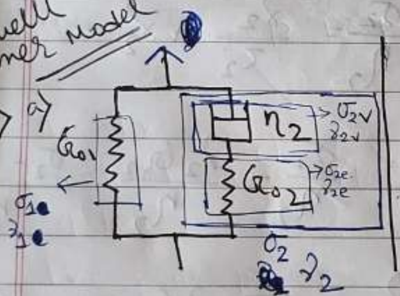
Or, $\frac{\partial P}{\partial z} = \frac{1}{r}(r\tau_{rz})$

For ~~newtonian~~ fluid, $\tau_{rz} = \eta \frac{\partial v_z}{\partial r}$

" Power law fluid, $\tau_{rz} = \eta\left(\frac{\partial v_z}{\partial r}\right)^n$

Maxwell-Zener Model

2) a)




$\cancel{\sigma}$ $\gamma = \gamma_1 = \gamma_2$

for 1 —

$\sigma_1 = \gamma_1 G_{01}$ | $\cancel{\sigma_2 \neq \eta_2}$

for 2 —

$\gamma_2 = \gamma_{2e} + \gamma_{2v} = \gamma$

$\Rightarrow \dot{\gamma} = \dot{\gamma}_{2e} + \dot{\gamma}_{2v} = \frac{\dot{\sigma}_{2e}}{G_{02}} + \frac{\sigma_{2v}}{\eta_2}$ | $\sigma_{2v} = \sigma_2$, $\dot{\sigma}_{2e} = \dot{\sigma}_2$

or, $\dot{\gamma} = \frac{\dot{\sigma}_2}{G_{02}} + \frac{\sigma_2}{\eta_2}$

overall

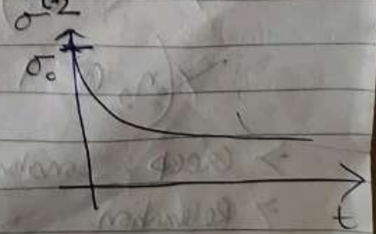
$\sigma = \sigma_1 + \sigma_2$

$\dot{\sigma} = \dot{\sigma}_1 + \dot{\sigma}_2$

or, $\frac{\dot{\sigma}}{G_{02}} = \frac{\dot{\sigma}_1}{G_{02}} + \frac{\dot{\sigma}_2}{G_{02}}$

or, $\frac{\dot{\sigma}}{G_{02}} = \frac{\dot{\gamma}_1 G_{01}}{G_{02}} + \frac{\dot{\sigma}_2}{G_{02}} \Rightarrow \frac{\dot{\sigma}}{G_{02}} = \frac{G_{01}\dot{\gamma}_1}{G_{02}} + \dot{\gamma} - \frac{\sigma_2}{\eta_2}$

or, $\frac{\dot{\sigma}}{G_{02}} = \frac{G_{01}\dot{\gamma}}{G_{02}} + \dot{\gamma} - \frac{(\sigma - \sigma_1)}{\eta_2}$




or, $\frac{\dot{\sigma}}{G_{02}} = \frac{G_{01}\dot{\gamma}}{G_{02}} + \dot{\gamma} - \frac{(\sigma - \gamma G_{01})}{\eta_2}$

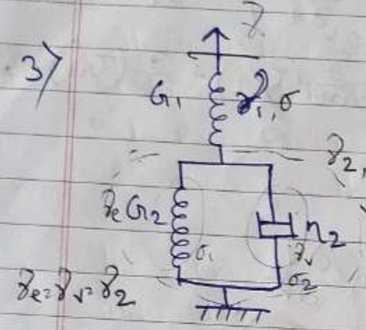
or, $\frac{\dot{\sigma}}{G_{02}} + \frac{\sigma}{\eta_2} = \dot{\gamma}\left(1 + \frac{G_{01}}{G_{02}}\right) + \gamma\left(\frac{G_{01}}{\eta_2}\right)$ Ans

2) $\gamma = \gamma_0$ : constant

$$\frac{\dot{\sigma}}{G_{R0_2}} + \frac{\sigma}{\eta_2} = \gamma_0\left(\frac{G_{R0_2}}{\eta_2}\right)$$

or, $\frac{1}{G_{R0_2}}\frac{d\sigma}{dt} + \frac{\sigma}{\eta_2} = \gamma_0\left(\frac{G_{R0_2}}{\eta_2}\right)$

3)




$\sigma_1 = \gamma_1 G_1 = \sigma$

$\sigma = \sigma_1 + \sigma_2$

$\uparrow \gamma = \gamma_1 + \gamma_2$

$\rightarrow \sigma_2 = \sigma_e + \sigma_v = \gamma_e G_2 + \dot{\gamma}_v \eta_2$

$\sigma_2 = \gamma_2 G_2 + \dot{\gamma}_2 \eta_2 = \sigma$

overall $\Rightarrow \gamma = \gamma_1 + \gamma_2$

$\Rightarrow \frac{\sigma_2}{\eta_2} = \frac{\gamma_2}{\tau_2} + \dot{\gamma}_2$

$\Rightarrow \dot{\gamma}_2 = \frac{\sigma_2}{\eta_2} - \frac{\gamma_2 G_2}{\eta_2} = \frac{\sigma_2 - \gamma_2 G_2}{\eta_2}$

$\therefore \frac{d\gamma_2}{dt} = \frac{\sigma_2 - \gamma_2 G_2}{\eta_2}$

$\Rightarrow$

$G_1\gamma_1 = \gamma_2 G_2 + \dot{\gamma}_2 \eta_2$

or, $G_1\gamma_1 = \gamma_2 G_2 + \frac{d\gamma_2}{dt}\eta_2$

or, $\sigma = G_2(\gamma - \gamma_1) + \eta_2(\dot{\gamma} - \dot{\gamma}_1)$

or, $\sigma = G_2\left(\gamma - \frac{\sigma}{G_1}\right) + \eta_2\left(\dot{\gamma} - \frac{\dot{\sigma}}{G_1}\right)$

$\Rightarrow \sigma + \frac{\eta_2}{G_1}\dot{\sigma} + \frac{G_2 \sigma}{G_1} = G_2\gamma + \eta_2\dot{\gamma}$

$\Rightarrow \sigma\left(1 + \frac{G_2}{G_1}\right) + \frac{\eta_2}{G_1}\dot{\sigma} = G_2\gamma + \eta_2\dot{\gamma}$

$\Rightarrow \left(\eta_2 \dot{\sigma} + (G_1 + G_2)\sigma = G_1 G_2 \gamma + G_1 \eta_2 \dot{\gamma}\right)$

→ Creep : constant stress – $\eta_2 \dot{\sigma} = 0$

→ Relaxation: " strain : $\dot{\gamma} = 0$

# Asgn 3

1) a) Strain Amplitude Sweep Test is used to determine the linear viscoelastic regime (LVR) for a given soft material by gradually increasing the strain or stress applied to the material at constant frequency while measuring its response.

Response Measurement: Rheometers measure's the material response.

→ Storage Modulus ($G'$): Reflects material's elastic or solid like behaviour

→ Loss Modulus ($G''$): Reflects material's viscous or liquid-like behaviour.

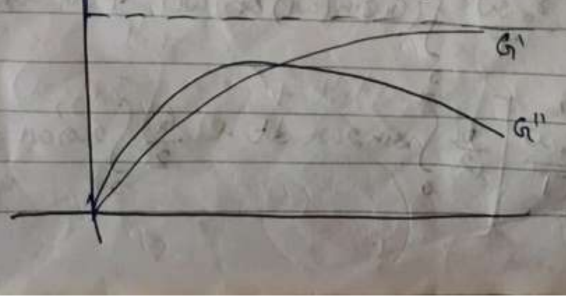
In LVR both $G'$, $G''$ should remain constant as strain or stress increases. It indicates material structure remains intact, and it deforms elastically (reversible behaviour).

Determine LVR limits: As strain or stress increases beyond a critical point, the material starts to deform. At this stage $G'$, $G''$ deviate from their constant values, indicating material is yielding or undergoing irreversible deformation.

b) At low frequency, the material has more time to relax, and viscous behaviour dominates

At high frequency, the material has ~~no~~ less time to ~~relax~~ respond, so solid-like behaviour dominates.

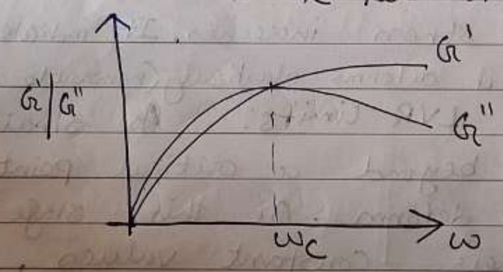
In LVR, the materials $G'$, $G''$ property remain constant regardless of the frequency or strain.



c) Frequency Sweep Test: experiment which is used to study the viscoelastic behaviour of a material over a range of ~~timescale~~ frequencies. This test is conducted in the linear viscoelastic Regime (LVR) to asses how the material's response change over varying frequencies.

At low frequencies, the material has time to relax between deformation, so $G'' > G'$.
at high frequencies, the material behaves more elastically with $G' > G''$.

The crossover point (where $G' = G''$) gives an indication of characteristic relaxation time.




$\gamma = \gamma_0 \sin(\omega t)$
$\sigma = \sigma_0 \sin(\omega t + \phi)$
$= \sigma_0 \cos\phi \sin\omega t + \sigma_0 \sin\phi \cos\omega t$

2) $\sigma(t) = G' \sin(\omega t) + G'' \cdot \cos(\omega t)$

Power loss $= \frac{1}{T} \int_0^T \sigma(t) \dot{\gamma}(t) \, dt$ | $\gamma(t) = \gamma_0 \sin(\omega t)$

or, Power loss $= \frac{1}{T} \int_0^T \left( G' \sin(\omega t) + G'' \cos(\omega t) \right) \left( \gamma_0 \cdot \omega \cos(\omega t) \right) dt$

$$= \frac{\gamma_0 \omega}{T} \int_0^T \left( G' \sin(\omega t) \cos(\omega t) + G'' \cos^2(\omega t) \right) dt$$

$$= \frac{\gamma_0 \omega}{T} \left[ \frac{G'}{2} \int_0^T \sin 2\omega t \, dt + \frac{G''}{2} \int_0^T 2\cos^2 \omega t \, dt \right]$$

$$= \frac{\gamma_0^2 \omega}{2T}\left[ G'\left(\frac{\cos 2\omega t}{2\omega}\right)_T^0 + G''\int_0^T (\cos 2\omega t + 1)\,dt \right]$$

$$= \frac{\gamma_0^2 \omega}{2T}\left[ G'\left(\frac{1 - \cos 2\cancel{\omega} \times \frac{2\pi}{\cancel{\omega}}}{2\omega}\right) + G''\left(\frac{\sin 2\omega t}{2\omega} + t\right)_0^T \right]$$

$$= \frac{\gamma_0^2 \omega}{2T}\left[ G'\left(\frac{1 - \overset{1}{\cos 4\pi}}{2\omega}\right) + G''\left(\frac{\overset{0}{\sin 4\pi}}{2\omega} + T\right) \right]$$

$$= \frac{\gamma_0^2 \omega}{2\cancel{T}} \cdot G'' \cancel{T} \Rightarrow \text{Powerloss} = \frac{\gamma_0^2 \omega G''}{2}$$

$\therefore$ Powerloss $\propto G''$

$$\frac{1}{T}\int_0^t \sigma(t)\,\dot{\gamma}(t)\,dt$$

## Asgn → 4

1) For Generalised Maxwell Model —

$$\sigma = -\int_0^\infty G(s)\, \frac{d\gamma(t-s)}{ds}\, ds$$

$$= \int_0^\infty G(s)\,\gamma_0\, \omega \cos(\omega(t-s))\, ds$$

$\gamma = \gamma_0 \sin(\omega t)$

$\dot{\gamma} = \gamma_0 \omega \cos(\omega t)$

$\dot{\gamma}(t-s) = -\gamma_0 \omega \cos(\omega(t-s))$

$$= \int_0^\infty G(s)\,\gamma_0\,\omega \left[\cos(\omega t).\cos(\omega s) + \sin(\omega t).\sin(\omega s)\right] ds$$

$$\sigma = \int_0^\infty G(s)\,\gamma_0\,\omega\cos(\omega t)\cos(\omega s)\, ds + \int_0^\infty G(s)\,\gamma_0\,\omega \sin(\omega t)\sin(\omega s)\, ds$$

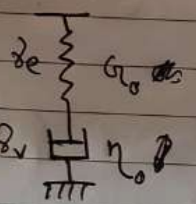
or, $$\frac{\sigma}{\gamma_0} = \omega.\cos(\omega t)\int_0^\infty G(s)\cos(\omega s)\, ds + \omega.\sin(\omega t)\int_0^\infty G(s)\sin(\omega s)\, ds$$

or, $$\frac{\sigma}{\gamma_0} = \cancel{\omega}\cos(\omega t).G'' + \sin(\omega t).G'$$

$$\therefore G' = \omega\int_0^\infty G(s).\overset{\sin}{\cancel{\cos}}(\omega s)\, ds$$

$$G'' = \omega\int_0^\infty G(s)\overset{\cos}{\cancel{\sin}}(\omega s)\, ds$$

2)




$\sigma = \sigma_e = \sigma_v$ | $\sigma_e = G\gamma_e$ $\sigma_v = \eta\dot{\gamma}_v$

$\gamma = \gamma_e + \gamma_v \Rightarrow \dot{\gamma} = \dot{\gamma}_e + \dot{\gamma}_v$

$\cancel{\dot{\gamma} = \frac{\dot{\sigma}}{G_0} + \sigma}$

or, $$\frac{d\gamma}{dt} = \frac{\dot{\sigma}_e}{G_0} + \frac{\sigma_v}{\eta_0}$$

or, $$\frac{d\gamma}{dt} = \frac{\dot{\sigma}}{G_0} + \frac{\sigma}{\eta_0}$$ | $\sigma = \sigma_0 \exp(i\omega t)$

Or, $$\frac{d\gamma}{dt} = \frac{(i\sigma_0\omega)\exp(i\omega t)}{G_0} + \frac{\sigma_0 \exp(i\omega t)}{\eta_0}$$

Or, $$\int_{\gamma_t}^{\gamma_{t+\delta t}} d\gamma = \frac{i\sigma_0\omega}{G_0}\int_t^{t+\delta t} \exp(i\omega t)\, dt + \frac{\sigma_0}{\eta_0}\int_t^{t+\delta t}\exp(i\omega t)\, dt$$

or, $\gamma_{t+\delta t} - \gamma_t = \frac{i\sigma_0 \omega}{G_0} \left[ \frac{\exp(i\omega t)}{i\omega} \Big|_t^{t+\delta t} \right] + \frac{\sigma_0}{\eta_0 i\omega} \left[ \exp(i\omega t) \right]_t^{t+\delta t}$

or, $\gamma_{t+\delta t} - \gamma_t = \frac{\sigma_0}{G_0} \left[ \exp(i\omega(t+\delta t)) - \exp(i\omega t) \right] + \frac{\sigma_0}{\eta_0 i\omega} \left[ \exp(i\omega(t+\delta t)) - \exp(i\omega t) \right]$

or, $\Delta\gamma = \sigma_0 \left( \exp(i\omega(t+\delta t)) - \exp(i\omega t) \right) \left( \frac{1}{G_0} + \frac{1}{i\eta_0 \omega} \right)$

or, $\Delta\gamma = (\sigma_{t+\delta t} - \sigma_t) \left\{ \frac{i\eta_0\omega + G_0}{i\eta_0 \omega G_0} \right\} = \Delta\sigma \left\{ \frac{G_0 + i\eta_0\omega}{i\eta_0\omega G_0} \right\}$

$\Rightarrow G^* = \frac{\Delta\sigma}{\Delta\gamma} = \frac{i\eta_0\omega G_0}{G_0 + i\eta_0\omega} = i\eta_0\omega G_0 \times \frac{(G_0 - i\eta_0\omega)}{G_0^2 + \eta_0^2\omega^2}$

$$G^* = \frac{G_0 \eta_0^2 \omega^2}{G_0^2 + \eta_0^2\omega^2} + i \frac{G_0^2 \eta_0 \omega}{G_0^2 + \eta_0^2 \omega^2} \qquad G' + iG''$$

$$= \frac{G_0 \omega^2 \tau^2}{1 + \tau^2\omega^2} + i \frac{G_0 \tau\omega}{1+\tau^2\omega^2} = G' + iG''$$

(vii) # Steady-State Shear test: used to study how a material behaves when subjected to a constant steady shear rate ($\dot{\gamma}$ = const.).

The test provides information about whether material behaves like a newtonian-fluid (with constant viscosity) or exhibits non-newtonian behaviour (viscosity changes with shear rate).

The primary output of this test is material's viscosity $(\eta) = \frac{\sigma}{\dot{\gamma}}$

4) For Multimodel Maxwell's Model :—

$$G(t) = \sum G_{0i} \exp(-t/\tau_i) \qquad \Big|\ \dot{\gamma}(t-s) = \text{cons.}$$

Now, $\sigma = \int_0^\infty G(s)\, \dot{\gamma}(t-s)\, ds$

$$\Rightarrow \frac{\sigma}{\dot{\gamma}} = \int_0^\infty \sum G_{0i} \exp(-s/\tau_i)\, ds = \eta$$

or, $\eta = \sum G_{0i} \int_0^\infty \exp(-s/\tau_i)\, ds = \sum G_{0i} \left.\dfrac{\exp(-s/\tau_i)}{-1/\tau_i}\right|_0^\infty$

$$= -\sum G_{0i}\tau_i (0-1) \Rightarrow \boxed{\sum G_{0i}\tau_{0i} = \eta}$$

# Asgn → 5

1) Subject to step stress, where creep compliance exhibits two-step exponential increases followed by a plateau, a generalised Kelvin-Voight model can be used.

Creep Test

↳ $\frac{d\gamma}{dT} > 0$

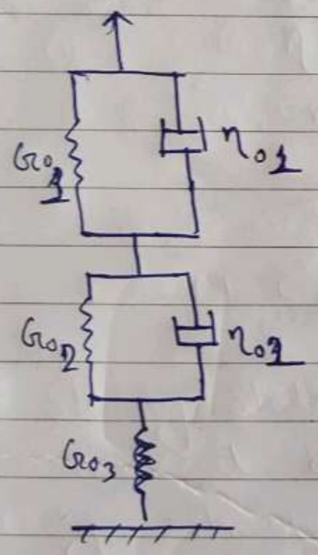
$\gamma(t) = \int_0^\infty J(s)\, \sigma(t-s)\, ds$

$\sigma(t) = \int_0^\infty G(s)\, \gamma(t-s)\, ds$

Suggested Model:

i) Two Kelvin-Voight element connected in series: Capture short term viscoelastic behaviour with different time constants.

ii) One spring in series with there Kelvin-Voight elements to capture the instantaneous elastic response.




$$\gamma = \gamma_1 + \gamma_3 + \gamma_3$$

$$\gamma = \frac{\sigma_0}{G_{o1}}\left(1 - \exp\left(\frac{-t}{\tau_1}\right)\right) + \frac{\sigma_0}{G_{o2}}\left(1 - \exp\left(-\frac{t}{\tau_2}\right)\right) + \frac{\sigma_0}{G_{o3}}$$

For 1

$\sigma = \sigma_e + \sigma_v$

$= G_0 \gamma_e + \eta_0 \dot{\gamma}_v \quad [\because \sigma_e = \sigma_v]$

$\frac{\sigma}{\eta_0} = \frac{G_0 \gamma_e}{\eta_0} + \frac{d\gamma_v}{dt}$

$\therefore \frac{d\gamma_v}{dt}$

$$\Rightarrow \frac{\gamma}{\sigma_0} = \frac{1}{G_{o1}}$$

Asgn→4

3) $G = G_1 e^{-t/\tau_1} + G_2 e^{-t/\tau_2}$ | $\gamma = \gamma_0$

$$\sigma = \int_0^{\infty} G(s)\, \dot{\gamma}(t-s)\, ds$$

$$= \int_0^{t_1} \left(G_1 e^{-t/\tau_1} + G_2 e^{-t/\tau_2}\right) \dot{\gamma}(t-s)\, ds + \int_{t_1}^{t_2} \left(G_1 e^{-t/\tau_1} + G_2 e^{-t/\tau_2}\right) \dot{\gamma}(t-s)\, ds$$

$$+ \int_{t_2}^{\infty} \left(G_1 e^{-t/\tau_1} + G_2 e^{-t/\tau_2}\right) \dot{\gamma}(t-s)\, ds$$

$$= \int_0^{t_1} (\ )\, \cancel{K}\, ds + \int_{t_1}^{t_2} \left(G_1 e^{-t/\tau_1} + G_2 e^{-t/\tau_2}\right)(-K)\, ds + 0$$

$$\sigma = K \int_0^{t_1} \left(G_1 e^{-t/\tau_1} + G_2 e^{-t/\tau_2}\right) ds - K \int_{t_1}^{2t_1} \left(G_1 e^{-t/\tau_1} + G_2 e^{-t/\tau_2}\right) ds$$

$$= K\left[G_1 \frac{e^{-t/\tau_1}}{-1/\tau_1}\Bigg|_0^{t_1} + G_2 \frac{e^{-t/\tau_2}}{-1/\tau_2}\Bigg|_0^{t_1}\right]$$

$$- K\left[G_1 \frac{e^{-t/\tau_1}}{-1/\tau_1}\Bigg|_{t_1}^{2t_1} + G_2 \frac{e^{-t/\tau_2}}{-1/\tau_2}\Bigg|_{t_1}^{2t_1}\right]$$

$$= K\left[G_1\left(\frac{e^{-t_1/\tau_1}}{-1/\tau_1} + \tau_1\right) + G_2\left(\frac{e^{-t_1/\tau_2}}{-1/\tau_2} + \tau_2\right)\right]$$

$$- K\left[G_1\left(\frac{e^{-2t_1/\tau_1} - e^{-t_1/\tau_1}}{-1/\tau_1}\right) + G_2\left(\frac{e^{-2t_1/\tau_2} - e^{-t_1/\tau_2}}{-1/\tau_2}\right)\right]$$

$$= K\Big[-\cancel{G_1 e^{-t_1/\tau_1} \cdot \tau_1} + G_1\tau_1 - \cancel{G_2\tau_2 e^{-t_1/\tau_2}} + G_2\tau_2$$

$$+ G_1 e^{-2t_1/\tau_1} \cdot \tau_1 + \cancel{G_1\tau_1 e^{-t_1/\tau_1}} + G_2 e^{-2t_1/\tau_2} \cdot \tau_2 + \cancel{G_2 e^{-t_1/\tau_2} \cdot \tau_2}\Big]$$

$$\sigma = K\left[G_1\tau_1 + G_2\tau_2 + G_1\tau_1 e^{-2t_1/\tau_1} + G_2\tau_2 e^{-2t_1/\tau_2}\right]$$