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1. (a) SS of the system is given by:
(2)

(1) $x_s(\alpha_1 - \beta_1 x_s - \gamma_1 y_s) = 0$ &

(2) $y_s(\alpha_2 - \beta_2 y_s - \gamma_2 x_s) = 0$

(1) implies: $x_s = 0$ or $x_s = \dfrac{\alpha_1 - \gamma_1 y_s}{\beta_1}$ — ③

(2) implies: $y_s = 0$ or $y_s = \dfrac{\alpha_2 - \gamma_2 x_s}{\beta_2}$ — ④

∴ Four SS possible:

① $x_s = 0$ & $y_s = 0$

② $x_s = 0$ & $y_s = \alpha_2/\beta_2$

③ $y_s = 0$ & $x_s = \alpha_1/\beta_1$

From ③ & ④ (solving).

$$y_s = \frac{\alpha_2\beta_1 - \alpha_1\gamma_2}{\beta_1\beta_2 - \gamma_1\gamma_2}$$

$$x_s = \frac{\alpha_1\beta_2 - \alpha_2\gamma_1}{\beta_1\beta_2 - \gamma_1\gamma_2}$$

From physical consideration,
$x_s, y_s, \alpha, \beta, \gamma > 0$

(b) Jacobian Matrix:
(2)

$$\underline{\underline{J}} = \begin{bmatrix} \alpha_1 - 2\beta_1 x - \gamma_1 y & -\gamma_1 x \\ -\gamma_2 y & \alpha_2 - 2\beta_2 y - \gamma_2 x \end{bmatrix}$$

(c) S_1: $J = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix}$ $\lambda_1 = \alpha_1 > 0$, $\lambda_2 = \alpha_2 > 0$
(6)

Both eigenvalues are real and positive. Hence this steady state is an unstable node.

S_2: $J = \begin{bmatrix} \alpha_1 - \dfrac{\gamma_1\alpha_2}{\beta_2} & 0 \\ -\dfrac{\gamma_2\alpha_2}{\beta_2} & -\alpha_2 \end{bmatrix}$ $\lambda_1 = \alpha_1 - \dfrac{\gamma_1\alpha_2}{\beta_2}$

$\lambda_2 = -\alpha_2 < 0$

if $\alpha_1\beta_2 > \gamma_1\alpha_2$ Saddle

otherwise stable node

S_3:

$$J = \begin{bmatrix} -\alpha_1 & -\dfrac{\alpha_1\gamma_1}{\beta_1} \\ 0 & \alpha_2 - \dfrac{\alpha_1\gamma_2}{\beta_1} \end{bmatrix} \qquad \lambda_1 = \alpha_2 - \frac{\alpha_1\gamma_2}{\beta_1}$$

$$\lambda_2 = -\alpha_1 < 0$$

if $\alpha_2\beta_1 > \alpha_1\gamma_2$ unstable saddle
otherwise stable node

S_4: (Both population non-zero; called co-existance steady state)

$$J = \begin{bmatrix} -\beta_1 x_s & -\gamma_1 x_s \\ -\gamma_2 y_s & -\beta_2 y_s \end{bmatrix}$$

Evaluating the eigenvalue will give us lengthy equation and will be difficult to analyze. We should use the trace and determinant conditions.

$tr\, J = -\beta_1 x_s - \beta_2 y_s < 0$ (stability condition satisfied)

$$|J| = (\beta_1\beta_2 - \gamma_1\gamma_2)x_s y_s$$

$\therefore$ stability condition: $\beta_1\beta_2 > \gamma_1\gamma_2$

3. (a) (3)

SS: $\frac{1}{\tau}(1-x_s) - x_s y_s^2 = 0$ — ①

$\frac{1}{\tau}(y_0 - y_s) + x_s y_s^2 - k y_s = 0$ — ②

Adding ① & ② & rearranging:

$$y_s = \frac{1+y_0-x_s}{1+\tau k} \quad — ③$$

Substituting in ①

$$(1+\tau k)^2(1-x_s) - x_s \tau (1+y_0-x_s)^2 = 0 \quad — ④$$

$$[f(x_s,\tau)=0]$$

(b) (2) We need to explore the change in the SS as τ changes.

Let us see if we can obtain the SS solutions when $\tau = 0$

from ①: $(1-x_s) - \tau x_s y_s^2 = 0 \Rightarrow x_s = 1$ @ $\tau = 0$

from ③: $y_s = y_0$

Hence, the initial condition for homotopy continuation would be

@ $\tau = 0$: $x_s = 1$ (for use in eqn. ④).

(c) (5) Using the method:

$$\frac{x_s|_{\tau+\Delta\tau} - x_s|_{\tau}}{\Delta\tau} = \left(-\frac{f_\tau}{f_{x_s}}\right)_{\substack{x_s=1\\ \tau=0}}$$

$$f_\tau = 2(1+\tau k)k(1-x_s) - x_s(1+y_0-x_s)^2$$

$$f_{x_s} = -(1+\tau k)^2 - \tau(1+y_0-x_s)^2 + 2x_s\tau(1+y_0-x_s)$$

Substitute values to get

$$x_s|_{\tau+\Delta\tau} = 0.937$$