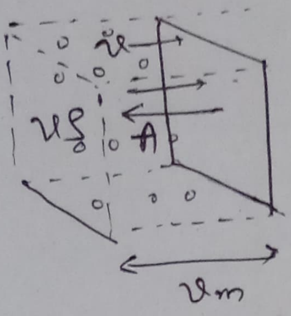
26/07/25 AFD JC

(mass number) Flux = Velocity × Density

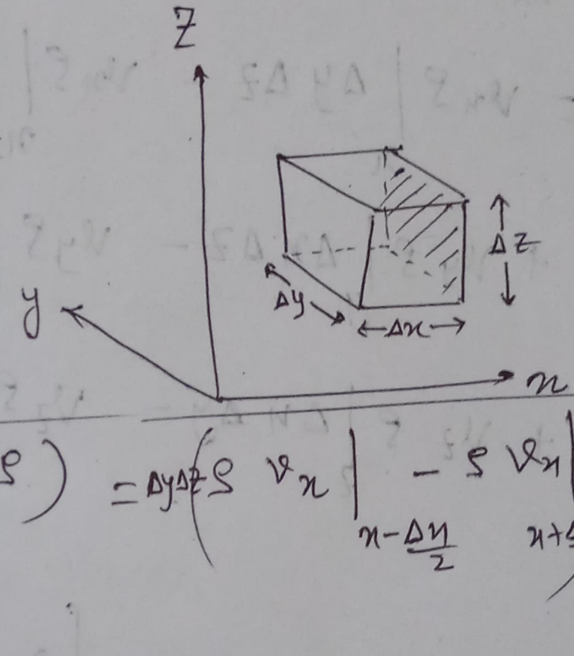
$v \times \rho$

$v \times N$



N particles/vol.

Continuity Equation



$$\Rightarrow \frac{d}{dt}(\Delta x \Delta y \Delta z \cdot \rho) = \Delta y \Delta z \left(\rho v_x \Big|_{x-\frac{\Delta x}{2}} - \rho v_x \Big|_{x+\frac{\Delta x}{2}} \right)$$

$$\Rightarrow \frac{d\rho}{dt} = \frac{\rho v_x \Big|_{x-\frac{\Delta x}{2}} - \rho v_x \Big|_{x+\frac{\Delta x}{2}}}{\Delta x}$$

$\Rightarrow$

$$v_x(x+h) = v_x + \frac{\partial v_x}{\partial x}\frac{h}{1!} + \cdots$$

$$= \lim_{\Delta x \to 0} \frac{\frac{\partial(\rho v_x)}{\partial x}\left(-\frac{\Delta x}{2}\right) - \frac{\partial(\rho v_x)}{\partial x}\frac{\Delta x}{2}}{\Delta x}$$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial \rho v_x}{\partial x}$$

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rate of change of mass = mass in − mass out

$$\frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z)$$

$$= v_x \rho\Big|_x \Delta y \Delta z - v_x \rho\Big|_{x+\Delta x} \Delta y \Delta z$$

$$+ v_y \rho\Big|_y \Delta x \Delta z - v_y \rho\Big|_{y+\Delta y} \Delta x \Delta z$$

$$+ v_z \rho\Big|_z \Delta x \Delta y - v_z \rho\Big|_{z+\Delta z} \Delta x \Delta y$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = -\left[\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z)\right]$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot(\rho \vec{V})$$

$$\nabla = \frac{\partial}{\partial x}\hat{\delta}_x + \frac{\partial}{\partial y}\hat{\delta}_y + \frac{\partial}{\partial z}\hat{\delta}_z$$

$$\vec{V} = v_x \hat{\delta}_x + v_y \hat{\delta}_y + v_z \hat{\delta}_z$$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial \rho v_x}{\partial x}$$

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rate of change of mass = mass in - mass out

$$\frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z)$$

$$= v_x \rho\Big|_x \Delta y \Delta z - v_x \rho\Big|_{x+\Delta x} \Delta y \Delta z$$

$$+ v_y \rho\Big|_y \Delta x \Delta z - v_y \rho\Big|_{y+\Delta y} \Delta x \Delta z$$

$$+ v_z \rho\Big|_z \Delta x \Delta y - v_z \rho\Big|_{z+\Delta z} \Delta x \Delta y$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = -\left[\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z)\right]$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = -\vec{\nabla}.(\rho \vec{v})$$

$$\nabla = \frac{\partial}{\partial x}\hat{\delta}_x + \frac{\partial}{\partial y}\hat{\delta}_y + \frac{\partial}{\partial z}\hat{\delta}_z$$

$$\vec{v} = v_x \hat{\delta}_x + v_y \hat{\delta}_y + v_z \hat{\delta}_z$$

$$\boxed{\frac{\partial \rho}{\partial t} = - \nabla \cdot (\rho \vec{v})}$$

tensors : quantities which relates dirn.

multiplication sign	order of result
None	Σ
$\times$	$\Sigma - 1$
$\cdot$	$\Sigma - 2$
$:$	$\Sigma - 4$

$\Sigma \equiv$ Sum of orders

$(\ ) \equiv$ scalar

$[\ ] \equiv$ vector

$\{ \ \} \equiv 2^{nd}$ order tensor

scalar

$ab = ba$ commutative

$(ab)c = a(bc)$: Associative

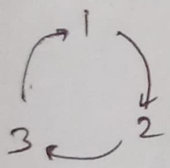
$a(b+c) = ab + ac$ Distributive

	Dot	Cross.
commutative	$\hat{a} \cdot \hat{b} = \hat{b} \cdot \hat{a}$ ✓	×
Associative	×	~~$(\hat{a} \times \hat{b}) \times \hat{c} = \hat{a} \times (\hat{b} \times \hat{c})$~~ ×
Distributive	✓	✓

Kronecker Delta $= \delta_{ij}$

$$\delta_{ij} = 1 \text{ if } i = j$$
$$= 0 \text{ if } i \neq j.$$

permutation symbol $\equiv \varepsilon_{ijk} = \pm\frac{1}{2}(i-j)(j-k)(k$



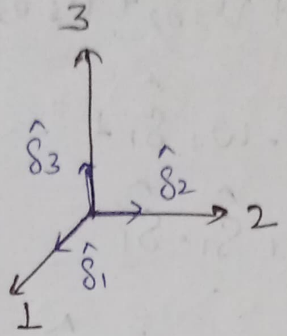
$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$$

$$\varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \sum_i \sum_j \sum_k \varepsilon_{ijk} \, a_{1i} a_{2j} a_{3k}$$

$\varepsilon_{ijk} = 0$ if any two indicies are alike.

$$\hat{v} = v_x \hat{\delta}_x + v_y \hat{\delta}_y + v_z \hat{\delta}_z$$
$$= v_1 \hat{\delta}_1 + v_2 \hat{\delta}_2 + v_3 \hat{\delta}_3$$
$$= \sum_{i=1}^{3} v_i \hat{\delta}_i = \sum v_i \hat{\delta}_i = v_i \hat{\delta}_i$$



$\hat{\delta} \equiv$ unit veetor

$$(\hat{\delta}_i \cdot \hat{\delta}_j) = \delta_{ij}$$
$$[\hat{\delta}_i \times \hat{\delta}_j] = \hat{\delta}_k$$
$$= \epsilon_{ijk} \hat{\delta}_k$$

$$\sum_j \sum_k \epsilon_{ijk} \epsilon_{hjk} = 2\delta_{ih}$$

$$\sum_k \epsilon_{ijk} \epsilon_{mnk} = \delta_{im} \delta_{jn} - \delta_{in} \cdot \delta_{jm}$$
$$= \delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}$$

$$(\hat{v}\cdot\hat{w}) = \hat{v}\cdot\left[\sum_j w_j \hat{\delta}_j\right]$$

$$= \hat{v}\cdot(w_1\hat{\delta}_1 + w_2\hat{\delta}_2 + w_3\hat{\delta}_3)$$

$$= \hat{v}\cdot w_1\hat{\delta}_1 + \hat{v}\cdot w_2\hat{\delta}_2 + \hat{v}\cdot w_3\hat{\delta}_3$$

$$= (v_1\hat{\delta}_1 + v_2\hat{\delta}_2 + v_3\hat{\delta}_3)\cdot w_1\hat{\delta}_1 \dots$$

$$= v_1\hat{\delta}_1\cdot w_1\hat{\delta}_1 + \dots$$

$$= v_1 w_1 \hat{\delta}_1\cdot\hat{\delta}_1$$

$$= \sum_i\sum_j v_i w_j \hat{\delta}_i\cdot\hat{\delta}_j$$

$$= \sum_i\sum_j v_i w_j \delta_{ij}$$

$$= \sum_i v_i w_i$$

$$[\hat{v}\times\hat{w}] = \hat{v}\times\left[\sum_j w_j\hat{\delta}_j\right]$$

$$= \hat{v}\times\left[w_1\hat{\delta}_1 + w_2\hat{\delta}_2 + w_3\hat{\delta}_3\right]$$

$$= (v_1\hat{\delta}_1 + v_2\hat{\delta}_2 + v_3\hat{\delta}_3)\times w_1\hat{\delta}_1$$

$$= v_2 w_1\left[\hat{\delta}_2\times\hat{\delta}_1\right] \dots$$

$$= v_2 w_1 \sum \varepsilon_{21k}\hat{\delta}_k \dots$$

$$= \sum_j\sum_k v_j w_k \sum_i \varepsilon_{jki}\hat{\delta}_i$$

$$= \sum_j\sum_k\sum_i v_j w_k\, \varepsilon_{ijk}\hat{\delta}_i$$

show that

$$[u_j \times [v \times w]_k]_i = v(u \cdot w) - w(u \cdot v)$$

$[v \times w]_k$

$$= \sum_i \sum_j v_i w_j \varepsilon_{kij} \hat{\delta}_k$$

$$[v \times w] = \sum_j \sum_k \sum_i v_j w_k \varepsilon_{ijk} \hat{\delta}_i$$

$$= \sum_j \sum_j v_i w_j \varepsilon_{kij} \hat{\delta}_k$$

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$$[u\times[v\times w]] = v(u\cdot w) - w(u\cdot v)$$

$$[v\times w] = \sum_i\sum_j\sum_k \varepsilon_{ijk}\,\hat{\delta}_i v_j w_k$$

$$\sum_k \varepsilon_{ijk}\varepsilon_{\underset{lm}{\cancel{mn}}k} = \delta_{i\underset{l}{\cancel{m}}}\delta_{j\underset{m}{\cancel{n}}} - \delta_{i\underset{m}{\cancel{n}}}\delta_{j\underset{l}{\cancel{m}}}$$

$$\varepsilon_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i)$$

$$[v\times w] = \sum_i\sum_j\sum_k \varepsilon_{ijk}\,\hat{\delta}_i v_j w_k \rightarrow \sum_j\sum_k \varepsilon_{ijk} v_j w_k$$

(i^{th} comp.)

$$[u\times[v\times w]]_i = \sum_j\sum_k \varepsilon_{ijk}\,u_j[v\times w]_k$$

$$[v\times w]_k = \sum_l\sum_m \varepsilon_{klm} v_l w_m$$

$$\sum_j\sum_k\sum_l\sum_m \varepsilon_{ijk}\varepsilon_{klm}\,u_j v_l w_m$$

$$\sum_j\sum_l\sum_m\left(\sum_k \varepsilon_{ijk}\varepsilon_{lmk}\right)u_j v_l w_m$$

$$\sum_j\sum_l\sum_m\left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)u_j v_l w_m$$

$\Rightarrow \sum_{j\ell m}$

$\Rightarrow \sum_j \sum_\ell \sum_m \delta_{i\ell} \delta_{jm} u_j v_\ell w_m$

$- \sum_j \sum_\ell \sum_m \delta_{im} \delta_{j\ell} u_j v_\ell w_m$

{δ_{ij} survives where $i=j$ only}

$\Rightarrow$

$\sum_j \sum_m \delta_{jm} u_j v_i w_m - \sum_j \sum_\ell \delta_{j\ell} u_j v_\ell w_i$

$\ell = i$ $\qquad$ $m = i$

$\Rightarrow v_i (\hat{u} \cdot \hat{w}) - w_i (\hat{u} \cdot \hat{v})$

$=$ R.H.S i

Prob. $(u \cdot [v \times w]) = v [w \times u]$

Unit dyads $\hat{\delta}_i \hat{\delta}_j$

$$\hat{\delta}_i \hat{\delta}_j : \hat{\delta}_k \hat{\delta}_l = \delta_{jk} \delta_{il}$$

$$= (\hat{\delta}_j \cdot \hat{\delta}_k)(\hat{\delta}_i \cdot \hat{\delta}_l)$$

$$\hat{\delta}_i \hat{\delta}_j \cdot \hat{\delta}_k = \hat{\delta}_i (\hat{\delta}_j \cdot \hat{\delta}_k) = \hat{\delta}_i \delta_{jk}$$

$$\hat{\delta}_i \cdot \hat{\delta}_j \hat{\delta}_k = (\hat{\delta}_i \cdot \hat{\delta}_j) \hat{\delta}_k = \delta_{ij} \hat{\delta}_k$$

$$\hat{\delta}_i \hat{\delta}_j \cdot \hat{\delta}_k \hat{\delta}_l = \hat{\delta}_i (\hat{\delta}_j \cdot \hat{\delta}_k) \hat{\delta}_l$$

$$= \delta_{jk} \hat{\delta}_i \hat{\delta}_l$$

$$\hat{\delta}_i \hat{\delta}_j \times \hat{\delta}_k = \hat{\delta}_i [\hat{\delta}_j \times \hat{\delta}_k]$$

$$= \hat{\delta}_i \sum_l \varepsilon_{jkl} \hat{\delta}_l$$

$$\hat{\delta}_i \times \hat{\delta}_j \hat{\delta}_k = [\hat{\delta}_i \times \hat{\delta}_j] \hat{\delta}_k$$

$$\overline{\overline{\tau}} = \sum_i \sum_j \tau_{ij} \hat{\delta}_i \hat{\delta}_j$$

if $\tau_{ij} = \tau_{ji} \Rightarrow$ symetric

$\tau_{ij} = -\tau_{ji} \Rightarrow$ Anti symetric

$$\hat{v}\hat{w} = \sum \hat{\delta}_i \hat{\delta}_j v_i w_j$$

Dyalic product of vectors

v & w.

unit tensor $(\hat{\delta}) = \sum \hat{\delta}_i \hat{\delta}_j \delta_{ij}$

$$\bar{\bar{\sigma}} : \bar{\bar{\tau}}$$

$$\bar{\bar{\sigma}} = \sum_i \sum_j \sigma_{ij} . \hat{\delta}_i \hat{\delta}_j$$

$$\bar{\bar{\tau}} = \sum_k \sum_l \tau_{kl} \hat{\delta}_k \hat{\delta}_l$$

$$\bar{\bar{\sigma}} : \bar{\bar{\tau}} = \sum_i \sum_j \sum_k \sum_l \sigma_{ij} \tau_{kl} \delta_{jk} \delta_{il}$$

$$= \sum_i \sum_j \sigma_{ij} \tau_{ji}$$

$$\bar{\bar{\tau}}^t = \sum_i \sum_j \tau_{ji} \hat{\delta}_i \hat{\delta}_j$$

$$\bar{\bar{\tau}} : \bar{\bar{\tau}}^t = \sum_i \sum_j \tau_{ij}^2$$

$$|\bar{\bar{\tau}}| = \sqrt{\frac{1}{2}\left(\bar{\bar{\tau}} : \bar{\bar{\tau}}^t\right)}$$

$$= \sqrt{\frac{1}{2}\sum_i \sum_j \tau_{ij}^2}$$

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1. If $\hat{\hat{\alpha}}$ is symmetric & $\hat{\hat{\beta}}$ is antisymmetric, show that $\hat{\hat{\alpha}} : \hat{\hat{\beta}} = 0$

$$\hat{\hat{\alpha}} : \hat{\hat{\beta}} = \sum_i \sum_j \alpha_{ij} \beta_{ji}$$

$$= \sum_i \alpha_{i1}\beta_{1i} + \alpha_{i2}\beta_{2i} + \alpha_{i3}\beta_{3i}$$

$$= \alpha_{11}\underline{\beta_{11}} + \cancel{\alpha_{12}\beta_{21}} + \cancel{\alpha_{13}\beta_{31}}$$

$$\cancel{\alpha_{21}\beta_{12}} + \alpha_{22}\underline{\beta_{22}} + \cancel{\alpha_{23}\beta_{32}}$$

$$\cancel{\alpha_{31}\beta_{13}} + \cancel{\alpha_{32}\beta_{23}} + \alpha_{33}\underline{\beta_{33}}$$

$\beta_{ij} = -\beta_{ji} \quad \forall\, i,j$

iff $\beta_{ii} = 0$

$$= 0$$

$$[\hat{\hat{\tau}} \cdot \hat{v}] = \sum_i \sum_j \hat{\delta}_i \hat{\delta}_j \tau_{ij} \cdot \sum_k \hat{\delta}_k v_k$$

$$= \sum_i \sum_j \sum_k \hat{\delta}_i \delta_{jk} \tau_{ij} v_k$$

$$= \sum_i \sum_j \hat{\delta}_i \tau_{ij} v_j$$

$$= \sum_i \hat{\delta}_i \left(\sum_j \tau_{ij} v_j \right)$$

Vector differential Operator:

$$\nabla = \frac{\partial}{\partial x_1}\hat{\delta}_1 + \frac{\partial}{\partial x_2}\hat{\delta}_2 + \frac{\partial}{\partial x_3}\hat{\delta}_3$$

$$= \sum_i \frac{\partial}{\partial x_i}\hat{\delta}_i$$

$$\nabla s = \left[\frac{\partial s}{\partial x_i}\hat{\delta}_i\right]$$

$$\nabla\cdot\hat{v} = \sum_i \frac{\partial v_i}{\partial x_i}$$

$$\nabla\times\hat{v} = \sum_i \frac{\partial}{\partial x_i}\hat{\delta}_i \times \sum_j v_j\hat{\delta}_j$$

$$= \sum_i\sum_j (\hat{\delta}_i\times\hat{\delta}_j)\frac{\partial v_j}{\partial x_i}$$

$$= \sum_i\sum_j\sum_k \varepsilon_{ijk}\,\hat{\delta}_k\frac{\partial v_j}{\partial x_i}$$

$$\nabla\hat{v} = \sum_i\sum_j \hat{\delta}_i\hat{\delta}_j\cdot\frac{\partial v_j}{\partial x_i}$$

$$\nabla\cdot\hat{\tau} = \sum_i \frac{\partial}{\partial x_i}\hat{\delta}_i\cdot\sum_j\sum_k \hat{\delta}_j\hat{\delta}_k\tau_{jk}$$

$$= \sum_i\sum_j\sum_k (\hat{\delta}_i\cdot\hat{\delta}_j\hat{\delta}_k)\frac{\partial \tau_{jk}}{\partial x_i}$$

$$= \sum_i \sum_j \sum_k \delta_{ij} \hat{\delta}_k \frac{\partial \tau_{jk}}{\partial x_i}$$

$$= \sum_k \left(\sum_i \frac{\partial \tau_{ik}}{\partial x_i} \right) \hat{\delta}_k$$

$$\nabla \cdot \nabla = \nabla^2$$

$$(\bar{\bar{\tau}} : \nabla \bar{v}) = (\nabla \cdot [\hat{\hat{\tau}} \cdot \hat{v}]) - (\hat{v} \cdot [\nabla \cdot \hat{\hat{\tau}}])$$

if $\bar{\bar{\tau}}$ is symetric

$$(\underline{\nabla \cdot [\hat{\hat{\tau}} \cdot \hat{v}]})$$

$$\downarrow$$

$$\sum_i \left(\sum_j \frac{\partial}{\partial x_i} \tau_{ij} v_j \right)$$

$$(\underline{\hat{v} \cdot [\nabla \cdot \hat{\hat{\tau}}]})$$

$$\downarrow$$

$$\sum_i \sum_j v_j \frac{\partial \tau_{ij}}{\partial x_i}$$

$$= \sum_i \sum_j \left[\frac{\partial}{\partial x_i} (\tau_{ij} v_j) - v_j \frac{\partial \tau_{ij}}{\partial x_i} \right]$$

$$= \sum_i \sum_j \tau_{ij} \frac{\partial v_j}{\partial x_i}$$

Assignment-1

Due on next Friday 9:00 AM @ NR222

1. $[u \times v] \cdot [w \times z] = (u \cdot w)(v \cdot z) - (u \cdot z)(v \cdot w)$

2. $[[u \times v] \times [w \times z]] = ([u \times v] \cdot z) w - ([u \times v] \cdot w) z$

3. $[v \times w] \cdot [v \times w] + (v \cdot w)^2 = v^2 w^2$

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LHS:

$(\bar{\bar{\tau}} : \nabla \bar{v})$

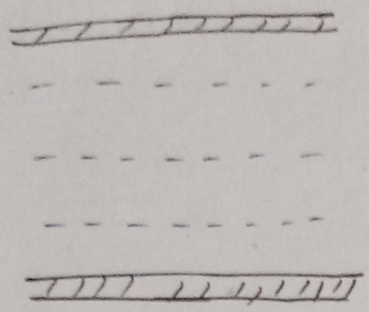
$$= \sum_i \sum_j \tau_{ij} \hat{\delta}_i \hat{\delta}_j : \sum_k \sum_l \hat{\delta}_k \hat{\delta}_l \frac{\partial v_l}{\partial x_k}$$

$$= \sum_i \sum_j \sum_k \sum_l \delta_{jk} \delta_{il} \tau_{ij} \frac{\partial v_l}{\partial x_k}$$

$$= \sum_i \sum_j \tau_{ij} \frac{\partial v_i}{\partial x_j}$$

$\because \bar{\bar{\tau}}$ is symmetric

$$= \sum_i \sum_j \tau_{ji} \frac{\partial v_j}{\partial x_i}$$

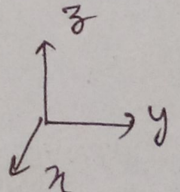


1. convective
2. molecular

co-ord dir^n	x mom.	y mom.	z mom.
x	$p+\tau_{xx}$ $+\rho v_x v_x$	τ_{xy} $+\rho v_y v_x$	τ_{xz} $+\rho v_z v_x$
y	τ_{yx} $+\rho v_x v_y$	$p+\tau_{yy}$ $+\rho v_y v_y$	τ_{yz} $+\rho v_z v_y$
z	τ_{zx} $+\rho v_x v_z$	τ_{zy} $+\rho v_y v_z$	$p+\tau_{zz}$ $+\rho v_z v_z$

$\equiv \hat{\hat{\Phi}}$

$$\tau_{yx} = -\mu \frac{dv_x}{dy}$$



$$\hat{\hat{\tau}} = -\mu\left(\nabla \hat{v} + (\nabla \hat{v})^t\right) + \left(\frac{2}{3}\mu - \kappa\right)(\nabla \cdot \hat{v})\hat{\hat{\delta}}$$

unit tensor

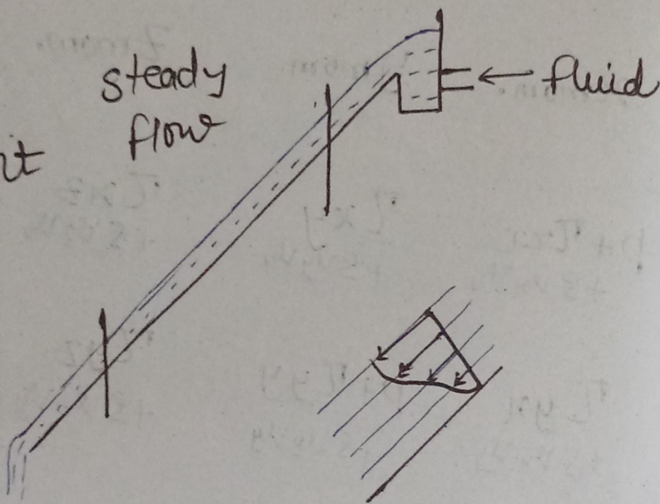
$\rho \hat{v} \hat{v}$

$$\hat{\phi} = p \hat{\delta} + \hat{\tau} + \rho \hat{v} \hat{v}$$

total mom. flux tensor.

ρ, μ constant

steady flow



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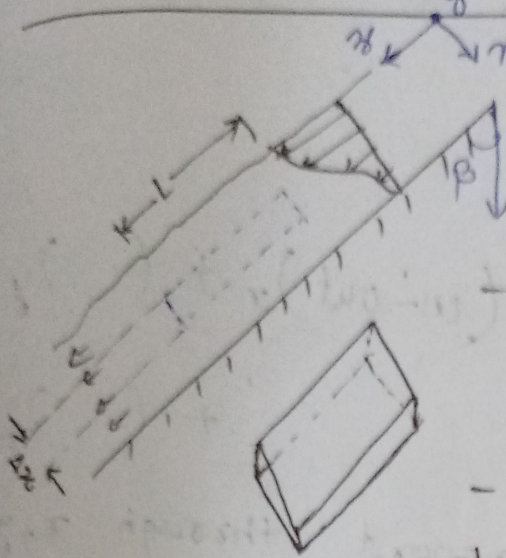
shell momentum balance :



$\hat{v}(x, y, z, t)$

~~fully developed~~

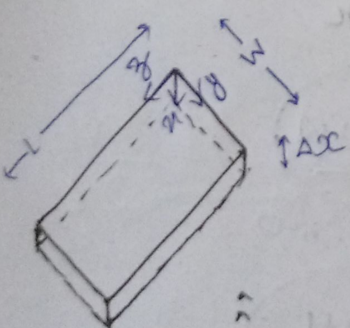
- momentum bal. for a shell.

- shell [small volumetric element] differential in one direcⁿ.

- mm bal. in 3 dirⁿ and 1 continuity eqn. = 4 eqn.

4 var.s v_x, v_y, v_z and p

- assumptions : - Fully developed
1. steady, incompressible, μ & ρ const.



2. $v_y = 0$, $v_x = 0$, $v_z \neq 0$,
$v_z = v_z(x)$

mm. bal. eqn. :

$\hat{\phi}$ = momentum flux tensor

ϕ_{ij} = j direction momentum across i surface.

rate of change of momentum

= mom. generation + (in - out)

∵ steady flow

$$0 = WL\Delta x \rho g_z + (\text{in}-\text{out})_x + (\quad)_y + (\quad)_z$$

z momentum transport through x, y and z surfaces

$$0 = WL\Delta x \rho g_z + \left(\phi_{xz} LW\Big|_x - \phi_{xz} LW\Big|_{x+\Delta x}\right)$$
$$+ \left(\phi_{yz}\Delta x L\Big|_y - \phi_{yz}\Delta x L\Big|_{y+W}\right)$$
$$+ \left(\phi_{zz}\Delta x W\Big|_z - \phi_{zz}\Delta x W\Big|_{z+L}\right)$$

$$0 = \rho g \cos\beta + -\frac{\partial \phi_{xz}}{\partial x}$$
$$+ \frac{\phi_{yz}\big|_y - \phi_{yz}\big|_{y+W}}{W}$$
$$+ \cancel{\frac{\phi_{zz}\big|_z - \phi_{zz}\big|_{z+L}}{L}} \rightarrow 0$$

BSL Appendix - B

surface ↓ / Momentum →	x	y	z
x	$p+\tau_{xx}+\rho v_x v_x$	$\tau_{xy}+\rho v_x v_y$	τ_{xz}
y	$\tau_{yx}+\rho v_x v_y$	$p+\tau_{yy}+\rho v_y v_y$	$\tau_{yz}+\rho v_y v_z$
z	$\tau_{zx}+\rho v_z v_x$	$\tau_{zy}+\rho v_y v_z$	$p+\tau_{zz}+\rho v_z v_z$

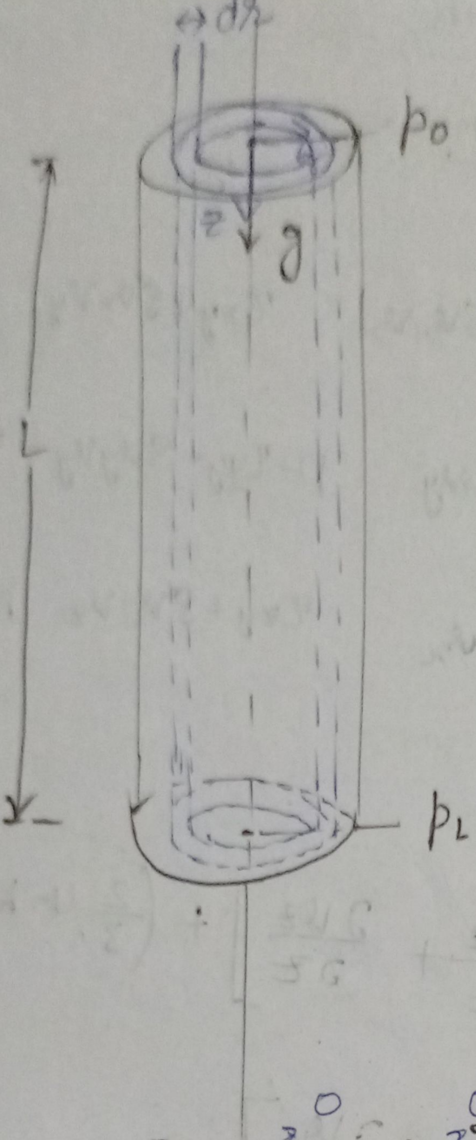
$$\tau_{zz} = -\mu\left[\cancel{\frac{\partial v_z}{\partial z}} + \cancel{\frac{\partial v_z}{\partial z}}\right] + \left(\frac{2}{3}\mu - \kappa\right)\overset{0}{\cancel{(\nabla\cdot v)}}$$

$$\tau_{yz} = -\mu\left[\cancel{\frac{\partial v_z}{\partial y}} + \cancel{\frac{\partial v_y}{\partial z}}\right]$$

$$\tau_{xz} = -\mu\left[\cancel{\frac{\partial v_x}{\partial z}} + \frac{\partial v_z}{\partial x}\right]$$

$$\nabla\cdot v = \overset{0}{\cancel{\frac{\partial v_x}{\partial x}}} + \overset{0}{\cancel{\frac{\partial v_y}{\partial y}}} + \overset{0}{\cancel{\frac{\partial v_z}{\partial z}}}$$

pressure is not func. of z.



$$\tau_{zz} = -\mu\left[\cancelto{0}{\frac{\partial v_z}{\partial z}} + \cancelto{0}{\frac{\partial v_z}{\partial z}}\right] + \cancel{\left(\frac{2}{3}\mu - \kappa\right)(\nabla\cdot v)}$$

$$\tau_{\theta z} = -\mu\left[\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right]$$

$$\tau_{rz} = -\mu\left[\cancelto{0}{\frac{\partial v_r}{\partial z}} + \frac{\partial v_z}{\partial r}\right]$$

$$\nabla\cdot v = \frac{1}{r}\frac{\partial}{\partial r}(r\cdot v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

1. $v_r = 0,\ v_\theta = 0 \quad v_z = f(r)$

shell mom. bal.

$$0 = (2\pi r\, dr\, L \rho g)$$

$$+ \phi_{rz} 2\pi r L \Big|_r - \phi_{rz} 2\pi r L \Big|_{r+dr}$$

$$+ \phi_{zz} 2\pi r\, dr \Big|_z - \phi_{zz} 2\pi r\, dr \Big|_{z+L}$$

Momentum $\rightarrow$

Surface $\downarrow$

$$0 = r \rho g + \frac{\phi_{rz} r \big|_r - \phi_{rz} r \big|_{r+\Delta r}}{dr}$$

$$+ \frac{\phi_{zz} r \big|_z - \phi_{zz} r \big|_{z+L}}{L}$$

$$\Rightarrow 0 = \rho g r - \frac{\partial}{\partial r}(r \phi_{rz}) + r \frac{P_0 - P_L}{L}$$

$$\phi_{rz} = -\mu \frac{\partial v_z}{\partial r} \qquad \{\because v_r = 0\}$$

$$0 = \mu \frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r}\right) + \frac{r}{L}\left\{(p_0 - p_L) + \rho g L\right.$$

$$\mathcal{P}_0 = p_0 + \rho g \cdot 0 \qquad \mathcal{P}_z = p_z + \rho g z$$

$$\mathcal{P}_L = p_L + \rho g L$$

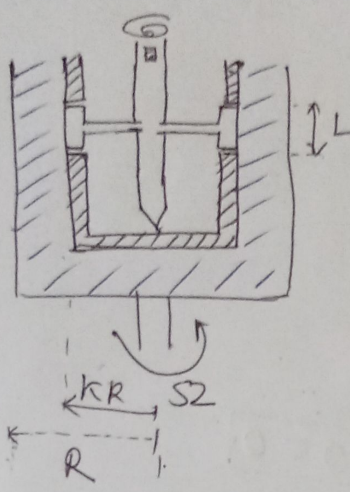
$$0 = \mu \frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r}\right) + \frac{r}{L}\left\{\mathcal{P}_0 - \mathcal{P}_L\right\}$$

$$\frac{\partial}{\partial r}\left(r \tau_{rz}\right) = r \frac{\Delta P}{L}$$

$$\tau_{rz} = \frac{\Delta P}{L} \frac{r}{2}$$

30/08/24

Couette flow viscometer



$T = k_t \theta_b$ → torque

Assumptions

1. μ, ρ = const. steady, incompressible, Newtonian fluid
2. Fully developed
3. $v_r, v_z = 0$ $v_\theta = f(r)$
4. $p = p(r, z)$

continuity eqn (cylindrical co-ordinates):

$$\cancel{\frac{\partial \rho}{\partial t}}^{0} + \frac{1}{r}\frac{\partial}{\partial r}\cancel{(\rho r v_r)}^{0} + \frac{1}{r}\frac{\partial}{\partial \theta}\cancel{(\rho v_\theta)}^{0} + \frac{\partial}{\partial z}\cancel{(\rho v_z)}^{0} = 0$$

Equation of motion:

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right)$$

$$= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_\theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}\right.$$

$$\left. + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right] + \rho g_\theta$$

$\frac{\rho}{r}\frac{d}{dr}(r$

BCs

@ $r=R$ $v_\theta = R\Omega$

@ $r=\kappa R$ $v_\theta = 0$

from continuity eqn $\boxed{0=0}$

Eqn of motion: Pg 848

*r component:

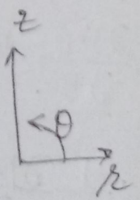
$$\boxed{\rho\left(0-\frac{v_\theta^2}{r}\right)=\frac{-\partial p}{\partial r}}$$

*θ:

$$\boxed{\rho(0)=\mu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right)}$$

*z

$$\boxed{0=-\frac{\partial p}{\partial z}+\rho g_z}$$



$0 = 0$: continuity

$$-\rho \frac{v_\theta^2}{r} = -\frac{\partial p}{\partial r} \quad : r$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial r v_\theta}{\partial r}\right) = 0 \quad : \theta$$

$$-\frac{\partial p}{\partial z} - \rho g = 0 \quad : z$$

$\Rightarrow$ $v_\theta = f(r)$ only :

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d(r v_\theta)}{dr}\right) = 0$$

$$\Rightarrow \cancel{\frac{d}{dr}\left[\frac{1}{r}\left(r\frac{dv_\theta}{dr} + v_\theta\right)\right] = 0}$$

$$\cancel{\frac{d}{dr}\left(\frac{dv_\theta}{dr} + \frac{v_\theta}{r}\right) = 0}$$

$$\frac{d^2 v_\theta}{dr^2} +$$

$$\frac{1}{r}\frac{d}{dr}(r v_\theta) = C_1$$

$$\frac{d}{dr}(r v_\theta) = C_1 r$$

$$r v_\theta = C_1 \frac{r^2}{2} + C_2$$

$$\boxed{v_\theta = C_1 \frac{r}{2} + \frac{C_2}{r}}$$

$$\cancel{v_\theta = C_1 \frac{r}{2}}$$

$$R\Omega = C_1 \frac{R}{2} + \frac{C_2}{R} \quad \text{—— (1)}$$

$$0 = C_1 \frac{kR}{2} + \frac{C_2}{kR}$$

$$0 = C_1 \frac{k^2 R}{2} + \frac{C_2}{R} \quad \text{—— (2)}$$

(1) − (2):

$$\cancel{R}\Omega = \frac{C_1 \cancel{R}}{2}(1-k^2)$$

$$\frac{2\Omega}{1-k^2} = C_1$$

$$C_2 = -(kR)^2 \frac{C_1}{2}$$

$$C_2 = -(kR)^2 \frac{\Omega}{1-k^2}$$

$$v_\theta = \frac{\Omega r}{1-k^2} - \frac{(kR)^2 \Omega}{r \, (1-k^2)}$$

$$= \Omega R \frac{\left(\frac{r}{R}\right)}{k\left(\frac{1}{k}-k\right)} - \Omega R \frac{kR}{r\left(\frac{1}{k}-k\right)}$$

$$\boxed{v_\theta = \Omega R \frac{\left(\frac{r}{kR} - \frac{kR}{r}\right)}{\left(\frac{1}{k}-k\right)}}$$

(Pg. 844)

$$\tau_{r\theta} = -\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + 0\right]$$

$$\tau_{r\theta} = -\mu\, r\frac{d}{dr}\left(\frac{v_\theta}{r}\right)$$

$$= -\mu r \frac{d}{dr}\left[\frac{\Omega R}{\left(\frac{1}{k}-k\right)}\left(\frac{1}{kR} - \frac{kR}{r^2}\right)\right]$$

$$= \frac{-\mu r\, \Omega R}{\left(\frac{1}{k}-k\right)}\left(0 + \frac{kR\,2}{r^3}\right)$$

$$\tau_{r\theta} = -\frac{\mu\Omega}{\left(\frac{1}{k}-k\right)}\frac{2R^2 k}{r^2}$$

$$=$$

EOM

$$\frac{\partial \rho \hat{v}}{\partial t} = -\nabla \cdot \rho \hat{v}\hat{v} - \nabla p - \nabla \cdot \tau + \rho \hat{g}$$

$$\hat{v} \cdot \frac{\partial \rho \hat{v}}{\partial t} = -\hat{v} \cdot (\nabla \cdot \rho \hat{v}\hat{v}) - \hat{v} \cdot (\nabla p) - \hat{v} \cdot (\nabla \cdot \tau) + \hat{v} \cdot (\rho \hat{g})$$

$$\frac{\partial \left(\frac{1}{2}\rho v^2\right)}{\partial t} = -\nabla \cdot \left(\frac{1}{2}\rho v^2 \hat{v}\right) + \cdots$$

Appendix A-4 1.4 or 4.14

$$\nabla \hat{v}\hat{v} = \sum_k \hat{\delta}_k \sum_j \frac{\partial}{\partial x_j}(v_j v_k)$$

$$\Rightarrow \hat{v} \cdot (\nabla \hat{v}\hat{v})$$

$$= \sum_i v_i \hat{\delta}_i \cdot \sum_k \sum_j \hat{\delta}_k \frac{\partial}{\partial x_j} v_j v_k$$

$$= \sum_i \sum_j \sum_k v_i \delta_{ik} \frac{\partial}{\partial x_j} v_j v_k$$

$$= \sum_i \sum_j v_i \frac{\partial}{\partial x_j} v_i v_j$$

$$= \sum_i \sum_j \left(v_i^2 \frac{\partial v_j}{\partial x_j} + v_i v_j \frac{\partial v_i}{\partial x_j} \right)$$

$$= \sum_j \left(\sum_i v_i^2\right) \frac{\partial v_j}{\partial x_j} = \nabla \cdot (v^2 \hat{v})$$

$$\hat{g} = -\nabla\phi$$

$$\hat{v} \cdot \frac{\partial \rho \hat{v}}{\partial t} = -\hat{v} \cdot (\nabla \cdot \rho \hat{v}\hat{v}) - \hat{v} \cdot (\nabla p) - \hat{v} \cdot (\nabla \cdot \tau)$$

$$+ \underbrace{\hat{v} \cdot (\rho \hat{g})}_{\downarrow}$$

$$- \hat{v} \cdot (\rho \nabla \phi)$$

$$-\left(\nabla \cdot (\phi \hat{v} \rho) - \underbrace{\phi \nabla \cdot (\rho \hat{v})}_{\downarrow}\right)$$

$$+ \phi \frac{\partial \rho}{\partial t}$$

$$-\nabla \cdot (\phi \hat{v} \rho) - \phi \frac{\partial \rho}{\partial t} \quad \text{— } 5$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho\phi\right) = -\nabla \cdot \left(\frac{1}{2}\rho v^2 \hat{v}\right) - \nabla \cdot (\rho \phi \hat{v})}$$

$$\rho \frac{D\hat{v}}{Dt} = \mu \nabla . \nabla \hat{v} - \nabla p + \rho \hat{g}$$

if viscous force is neglidible than inertial force.

$$\rho \frac{D\hat{v}}{Dt} = -\nabla p + \rho \hat{g}$$ → Euler's Eqn

$$0 = \mu \nabla^2 \hat{v} - \nabla p + \rho \hat{g}$$ → stokes flow eqn

inertial force is negligible.