1. Two parallel, plane circular disks (of radius R) lie one above the other a small distance apart. The space between them is filled with a fluid. The upper disk approaches the lower at a constant (small) velocity U, displacing the fluid. The pressure at r = R is p_o and p_r is not a function of z. The simplified forms of the basic equations (continuity and motion, r component) are given below.

$$\frac{1}{r}\frac{\partial}{\partial r}(r\,v_r) + \frac{\partial}{\partial z}(\,v_z\,) = 0$$

$$\mu\,\frac{\partial^2 v_r}{\partial z^2} = \frac{\partial p}{\partial r}$$

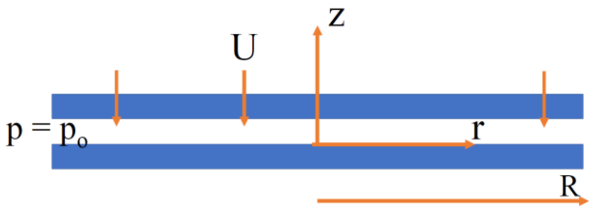
(a) Show how these simplifications are achieved using an analysis based on your understanding and estimate of the magnitudes of the different terms in the original equations. Please give clear logic for all your steps.

(b) Solve these equations with proper boundary conditions to derive an expression for that the resistance to motion on the moving disk in terms of **μ, U, R and h**, where h is the separation between the two plates. Take the origin to be at the centre of the lower (stationary) plate.

6+8=14

[Hint: Solve equation of motion first, then equation of continuity to obtain expression of pressure as a function of r and other system parameters]

Solution



Assumptions

1. R is very large compared to h
2. Leakage rate is small (Small ϑ_r)
3. $\frac{\partial p}{\partial z}$ is negligible
4. ϑ_z is very small as compared to ϑ_r
5. $\frac{\partial \vartheta_z}{\partial z}$ may not be small (as z is very small)
6. $\frac{\partial}{\partial z}(\frac{\partial v_z}{\partial z})$ can be appreciable
7. ϑ_r is not small as compared to ϑ_z
8. $\frac{\partial v_r}{\partial r}$ is small compared to $\frac{\partial v_r}{\partial z}$
9. $\vartheta_\theta = 0$, no θ dependence of ϑ_r, ϑ_z

The equation of motion (r component):

$$\rho\left(\frac{\partial\vartheta_r}{\partial t}+\vartheta_r\frac{\partial\vartheta_r}{\partial r}+\frac{\vartheta_\theta}{r}\frac{\partial\vartheta_r}{\partial\theta}+\vartheta_z\frac{\partial\vartheta_r}{\partial z}-\frac{\vartheta_\theta^2}{r}\right)=-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\vartheta_r)\right)+\frac{1}{r^2}\frac{\partial^2\vartheta_r}{\partial\theta^2}+\frac{\partial^2\vartheta_r}{\partial z^2}-\frac{2}{r^2}\frac{\partial\vartheta_\theta}{\partial\theta}\right]+\rho g_r \quad \textbf{(1)}$$

Steady state: 1st term on LHS = 0

Assumption 8: $\frac{\partial\vartheta_r}{\partial r}$ is small => 2nd term on LHS can be neglected.

Assumption 9: $\vartheta_\theta = 0$ => 3rd term on LHS can be neglected

Assumption 4; ϑ_z is very small => 4th term on LHS can be neglected.

Assumption 9: $\vartheta_\theta = 0$ => 5th term on LHS can be neglected.

$$\frac{\partial p}{\partial r} \neq 0$$

$\vartheta_r \neq f(\theta)$ => 3rd term on RHS can be neglected.

$\vartheta_r = f(z)$, $\therefore \frac{\partial^2\vartheta_r}{\partial z^2}$ cannot be neglected.

Assumption 9: $\vartheta_\theta = 0 \Rightarrow \therefore \frac{\partial\vartheta_\theta}{\partial\theta} = 0$

$g_r = 0$

Expanding 2nd term on LHS

$$0=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\gamma\left[\frac{1}{r}\frac{\partial\vartheta_r}{\partial r}-\frac{\vartheta_r}{r^2}+\frac{\partial^2\vartheta_r}{\partial r^2}+\frac{\partial^2\vartheta_r}{\partial z^2}\right] \quad \textbf{(Eq. 2)}$$

Assumption 8: As $\frac{\partial\vartheta_r}{\partial r}$ is small

$\frac{1}{r}\frac{\partial\vartheta_r}{\partial r}$ is even smaller and can be neglected.

ϑ_r is not large, but r is.

$\therefore \frac{\vartheta_r}{r^2}$ can be neglected.

$\frac{\partial^2\vartheta_r}{\partial r^2}$ is small as compared to $\frac{\partial^2\vartheta_r}{\partial z^2}$.

$\therefore$ The equation of motion reduced to

$$0=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\gamma\frac{\partial^2\vartheta_r}{\partial z^2}$$

Upon integration (as $\frac{\partial p}{\partial r}$ is not a function of z) $\vartheta_r = \frac{1}{2\mu}\frac{\partial P}{\partial r}z^2 + C_1 z + C_2$ **(Eq. 3)**

$BC:\quad z=0;\ v_r=0 \quad\Rightarrow\quad C_2=0$

$\qquad z=h;\ v_r=0 \quad\Rightarrow\quad C_1=-\frac{1}{2\mu}\frac{dp}{dr}h$

$$\vartheta_r=\frac{1}{2\mu}\frac{\partial P}{\partial r}(z-h)z \quad (C)$$

The equation of continuity becomes as $\vartheta_\theta = 0$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\,v_r) + \frac{\partial v_z}{\partial z} = 0 \qquad (A)$$

Putting the expression of ϑ_r from (C) to (A) and integrating w.r.t. z and as

$$BC: \quad z=0;\ v_r=0=v_z$$
$$z=h;\ v_r=0;\ v_z=-U$$

$$U = \frac{1}{r}\frac{d}{dr}\int_0^h r\,v_r\,dz = \frac{1}{r}\frac{d}{dr}\left(r\frac{dp}{dr}\right)\frac{1}{2\mu}\int_0^h z(z-h)\,dz = -\frac{h^3}{12\mu}\frac{1}{r}\frac{d}{dr}\left(r\frac{dp}{dr}\right)$$

Integrating w.r.t. r

$$-\frac{12\mu}{h^3}\frac{r^2}{2} = r\frac{dp}{dr} + C_1$$

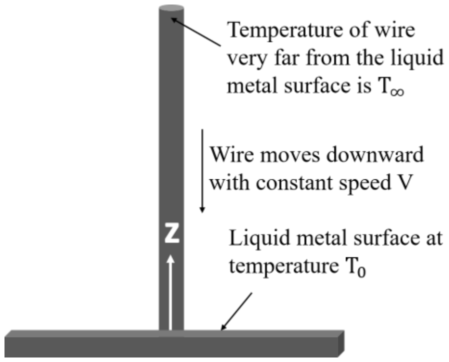
As $\frac{\mathrm{dp}}{\mathrm{dr}}$ is finite at $r = 0, C_1 = 0$ and $p = p_0$ at r = R

$$p - p_o = \frac{3\mu U}{h^3}\left(R^2 - r^2\right)$$

$$Force,\ F = \int_0^R 2\pi r\,dr\,(p - p_o) = 2\pi\int_0^R \frac{3\mu U}{h^3}\left(R^2 r - r^3\right)dr$$

$$Force,\ F = \frac{3\pi\mu U R^4}{2h^3}$$

2. A wire of constant density ρ moves downward with uniform speed V into a liquid metal bath at T_0. Assume, the resistance to radial heat conduction is negligible. Further assume that the portion of the wire just coming in contact with the liquid metal surface, instantaneously attains the liquid metal surface temperature. Derive the steady state temperature distribution in the axial direction for constant physical properties, taking the axial direction (z) as shown in the figure (vertically upward direction).



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Solution: The energy equation will be reduced to, $\rho C_p \vartheta_z \frac{\partial T}{\partial z} = k \frac{\partial^2 T}{\partial z^2}$

$T = f(z)$ only,

Hence, $\frac{d^2T}{dz^2} = \frac{\rho C_p \vartheta_z}{k}\frac{dT}{dz}$

Now, $\vartheta_z = -\vartheta$, $\frac{d^2T}{dz^2} = -\frac{\rho C_p \vartheta}{k}\frac{dT}{dz}$

Integrating, $\frac{dT}{dz} = -\frac{\rho C_p \vartheta}{k}T + C_1$

At, $z \to \infty, T = T_\infty \therefore \frac{dT}{dz} = 0$

Hence, $C_1 = \frac{\rho C_p \vartheta}{k}T_\infty$

The equation is reduced to,

$$\frac{dT}{dz} = -\frac{\rho C_p \vartheta}{k}T + \frac{\rho C_p \vartheta}{k}T_\infty$$

$$\frac{dT}{dz} = -\frac{\rho C_p \vartheta}{k}(T - T_\infty)$$

Let $\theta = \frac{T-T_\infty}{T_0-T_\infty} \therefore \frac{d\theta}{dz} = \frac{1}{T_0-T_\infty}\frac{dT}{dz}$

Applying, $\frac{d\theta}{dz}(T_0 - T_\infty) = -\frac{\rho C_p \vartheta}{k}(T - T_\infty)$

$$\therefore \frac{d\theta}{dz} = -\frac{\rho C_p \vartheta}{k}\theta$$

Integrating, $\ln\theta = -\frac{\rho C_p \vartheta}{k}z + C_2$

At, $z = 0, T = T_0 \therefore \theta = 1 \quad \therefore C_2 = 0$

Hence, $\ln\theta = -\frac{\rho C_p \vartheta}{k}z \quad \therefore \theta = \exp\left(-\frac{\rho C_p \vartheta}{k}z\right)$

$$\therefore \frac{T - T_\infty}{T_0 - T_\infty} = \exp\left(-\frac{\rho C_p \vartheta}{k}z\right)$$

$$\therefore T = T_\infty + (T_0 - T_\infty)\exp\left(-\frac{\rho C_p \vartheta}{k}z\right)$$