Linearity

$$\hat{L}(\underline{U}+\underline{V}) = \hat{L}(\underline{U}) + \hat{L}(\underline{V})$$

$$\hat{L}(\alpha\underline{U}) = \alpha\hat{L}(\underline{U})$$

$\hat{L}$ is operator operating on U & V.

Non-linear → convert to linear system using linearisation

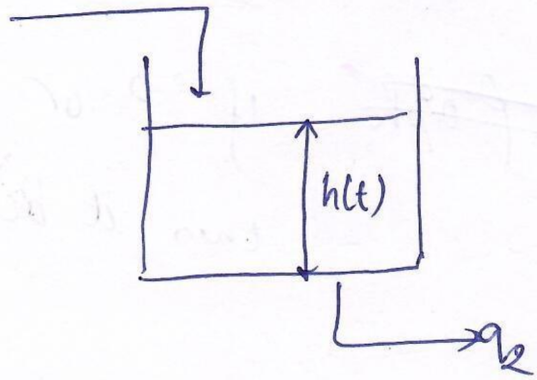
→ use non-linear dynamics to solve eqns

$\frac{dx}{dt} = ax$ is linear

$\frac{dx}{dt} = ax^2$ is non-linear

for Example

(I) $\frac{dh}{dt} = \frac{1}{A}(q_1 - q_2)$



Let $q_1 = 0$

$q_2 = A_p C_d \sqrt{2gh}$

A_p = Area of pipe

C_d = discharge rate

g = acceleration due to gravity

h = height (dynamical variable)

$$\frac{dh}{dt} = -\frac{A_p C_d \sqrt{2gh}}{A}$$

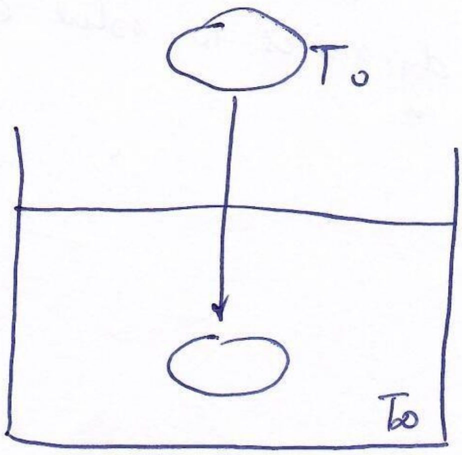
$$\frac{dh}{dt} = -\left(\frac{A_p C_d \sqrt{2g}}{A}\right)\sqrt{h} \quad \longleftarrow \quad a = \left(\frac{A_p C_d \sqrt{2g}}{A}\right)$$

$$\frac{dh}{dt} + a\sqrt{h} = 0$$

$\longrightarrow$ Non linear

↳ prove if it is linear or non-linear

(II)



$$\frac{dT}{dt} = \frac{-hA_s}{\rho V_c}(T - T_\infty)$$

$$\frac{dT^*}{dt} = \cdot \alpha (T^*)$$

↳ linear ?

~~if ρ or h~~ if ρ or h changes with time then it becomes non-linear.

linear dynamic system. — Non Autonomous

$$\begin{cases} \frac{d\underline{X}}{dt} = \underline{A}\underline{X} + \underline{B}\underline{U} \\ y = \underline{C}\underline{X} + \underline{D}\underline{U} \end{cases}$$

$\underline{X} = N \times 1$; $y = P \times 1$

$\underline{A} = N \times N$; $C = P \times N$

$\underline{B}$ $N \times M$; $\underline{D} = P \times M$

$\underline{U} = M \times 1$

For Non-linear system

$$\frac{dx_1}{dt} = f(x_1 \dots x_n; U_1 \dots U_m)$$

$$\frac{dx_2}{dt} = f(x_1, x_2 \dots x_n, U_1 U_2 \dots U_m)$$

$$\vdots$$

$$\frac{dx_N}{dt} =$$

Dynamical equations.

$$y_1 = g_1(x_1 \dots x_n, U_1 \dots U_M)$$

$$y_2 = g_2(x_1 \dots x_n, U_1 \dots U_m)$$

$$\vdots$$

$$y_p = g_p(x_1 \dots x_n; U_1 \dots U_m)$$

linearisation

$$\frac{dx_1}{dt} = x_1(x_1 + x_2)$$

$$\frac{dx_2}{dt} = x_2(x_1 + x_2)$$

at equilibrium solution / steady state

$$\left.\begin{aligned} \frac{dx_{1s}}{dt} &= 0 \\ \frac{dx_{2s}}{dt} &= 0 \end{aligned}\right\} \quad \begin{aligned} x_{1ss} &= 0 \\ x_{2ss} &= 0 \end{aligned}$$

linearise around steady state.

$$f_1(x_1, x_2) = f_1(x_{1s}, x_{2s}) + \left.\frac{\partial f_1}{\partial x_1}\right|_{x_{1s}} (x_1 - x_{1s}) +$$

$$\left.\frac{\partial f_1}{\partial x_2}\right|_{x_{2s}} (x_2 - x_{2s})$$

$$g_1(x_1 x_2) = g_1(x_{1s} x_{2s}) + \left.\frac{\partial g_1}{\partial x_1}\right|_{x_{1s}} (x_1 - x_{1s}) + \left.\frac{\partial g_1}{\partial x_2}\right|_{x_{2s}} (x_2 - x_{2s})$$

$$\boxed{\text{MULTI VARIABLE TAYLOR SERIES EXPANSION}}$$

Let $x_1 - x_{1s} = x_1^*$; $y_1 - y_{1s} = y_1^*$; $U_1 - U_{1s} = U_1^*$

$x_2 - x_{2s} = x_2^*$

$\vdots$

Deviation variables

In matrix form

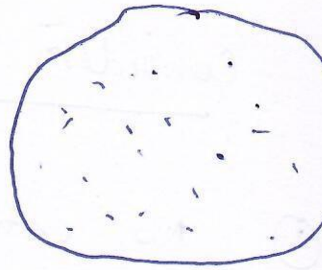
$$\frac{dx^*}{dt} = \underline{\underline{A}}\, \underline{x}^* + \underline{\underline{B}}\, U^*$$

$$y^* = \underline{\underline{C}}\, \underline{x}^* + \underline{\underline{D}}\, \underline{U}^*$$

Logistic population growth model

linear model for population growth.

Assumptions



1. Population confined to the region (No entry or exit)

2. Growth rate is function of instantaneous population

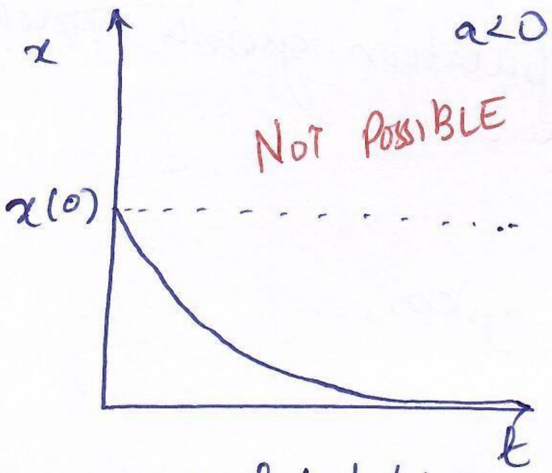
3. No death, Birth only from the present members ; No explicit birth rate term.

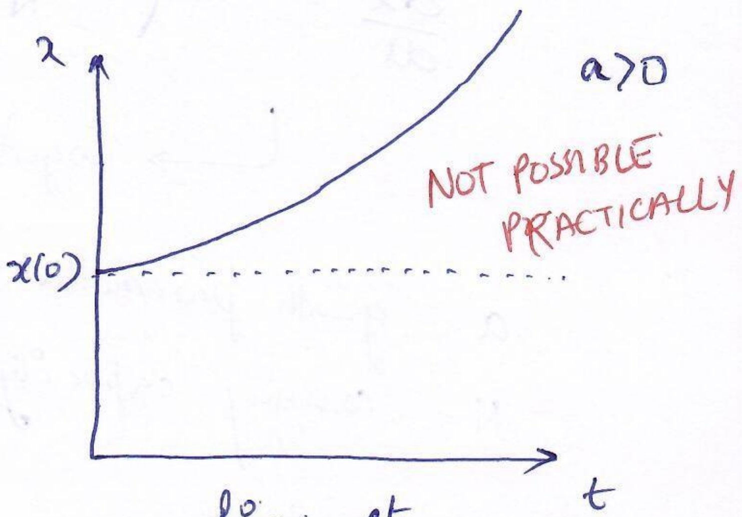
x = instantaneous population.

$\frac{dx}{dt} = f(x)$ for linearity $f(x) = ax$

$\frac{dx}{dt} = ax$

Bifurcation at $a = 0$





Decay of population
↳ contradicts assumption 3

$\lim_{t \to \infty} e^{at} \to \infty$ for $a > 0$

$\to$ population can't be ∞

Question → control population with time ??

Consider non-linear model

1. No entry or exit
2. growth rate is function of instantaneous population
3. No death; birth only from present members; no explicit birth rate term
4. Growth rate $\propto$ to instantaneous population only for small population
5. Negative growth rate at large population so as to limit the population

(point 4 & 5 will help in controlling the population)

$$\frac{dx}{dt} = ax\left(1 - \frac{x}{N}\right) \quad \text{--- (1)}$$

→ logistic population growth model

a = growth parameter
N = carrying capacity of system.

let $\frac{x}{N} = y$ — (2)

$$\frac{dx}{dt} = N\frac{dy}{dt} \quad \text{— (3)}$$

$$\frac{\cancel{N}dy}{dt} = a\cancel{N}y(1-y)$$

$$\frac{dy}{dt} = ay(1-y) \quad \text{— (4)}$$

Normalised form
$0 < y < 1$

Max population = 1

$$\frac{1}{y(1-y)}\frac{dy}{dt} = a$$

$$\frac{dy}{y(1-y)} = a\,dt$$

partial fraction

$$\left(\frac{1}{y} + \frac{1}{1-y}\right) dy = a\,dt$$

$$\boxed{y(t) = \frac{y(0)\,e^{at}}{1-y(0) + y(0)e^{at}}}$$

$y(0)$ = Initial condition.
$= \frac{x(0)}{N}$

$\frac{1}{y}dy = \ln|y|$

$\frac{1}{1-y}dy = -\ln|1-y|$

$$\left(\frac{y}{1-y}\right) = e^{at \cdot c}$$

$$\left(\frac{y}{1-y}\right) = ce^{at}$$

if $y(0) = 0$

$y(t) = 0$

If $y(0) = 1$

$y(t) = 1$ ← independent of time

⇒ if population of carrying capacity N,
& population has reached it max. & carrying capacity; then there won't be any Change in the population of system.

⇒ state of system ??

$y(t)$ is dynamical variable.
which means it should be function of time

but

$y(0) = 0 \Rightarrow y(t) = 0$
$y(0) = 1 \Rightarrow y(t) = 1$ } independent of time

Q. when does system become independent of time ?;
when there is no gradient

AT EQUILIBRIUM SOLUTION

$y(0) = 0$
$y(0) = 1$ } must be equilibrium solution.

system is in equili. state.

| There are 2 equilibrium solutions. |

what does it mean ??

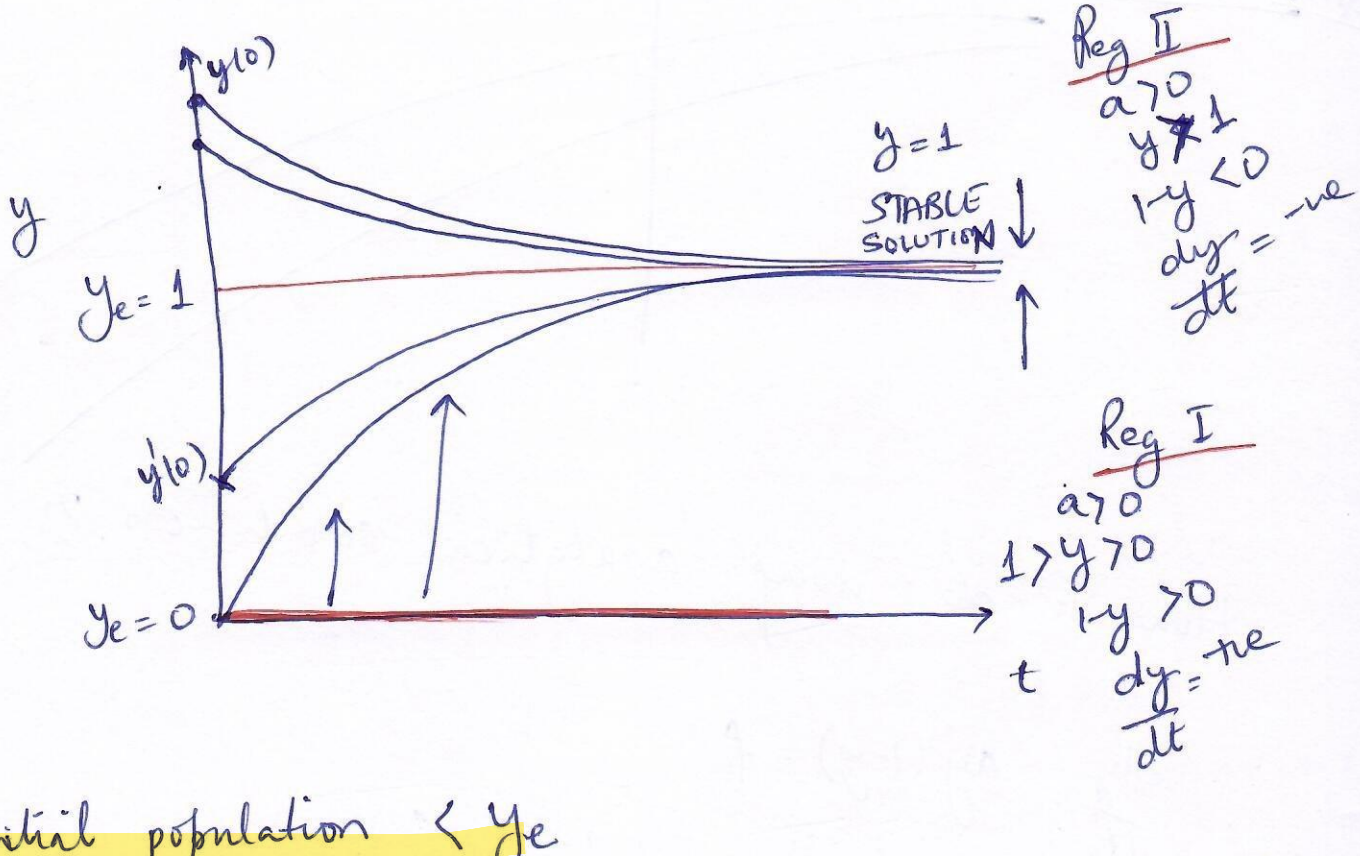
what does 2 equilibrium population means ??

if $\frac{dy}{dt} = ay(1-y) = 0$ $a > 0;\ y > 0,\ t > 0.$

$y_e = 0$ ← no initial population

$y_e = 1$ ← max carrying capacity is

what is stability of system ??

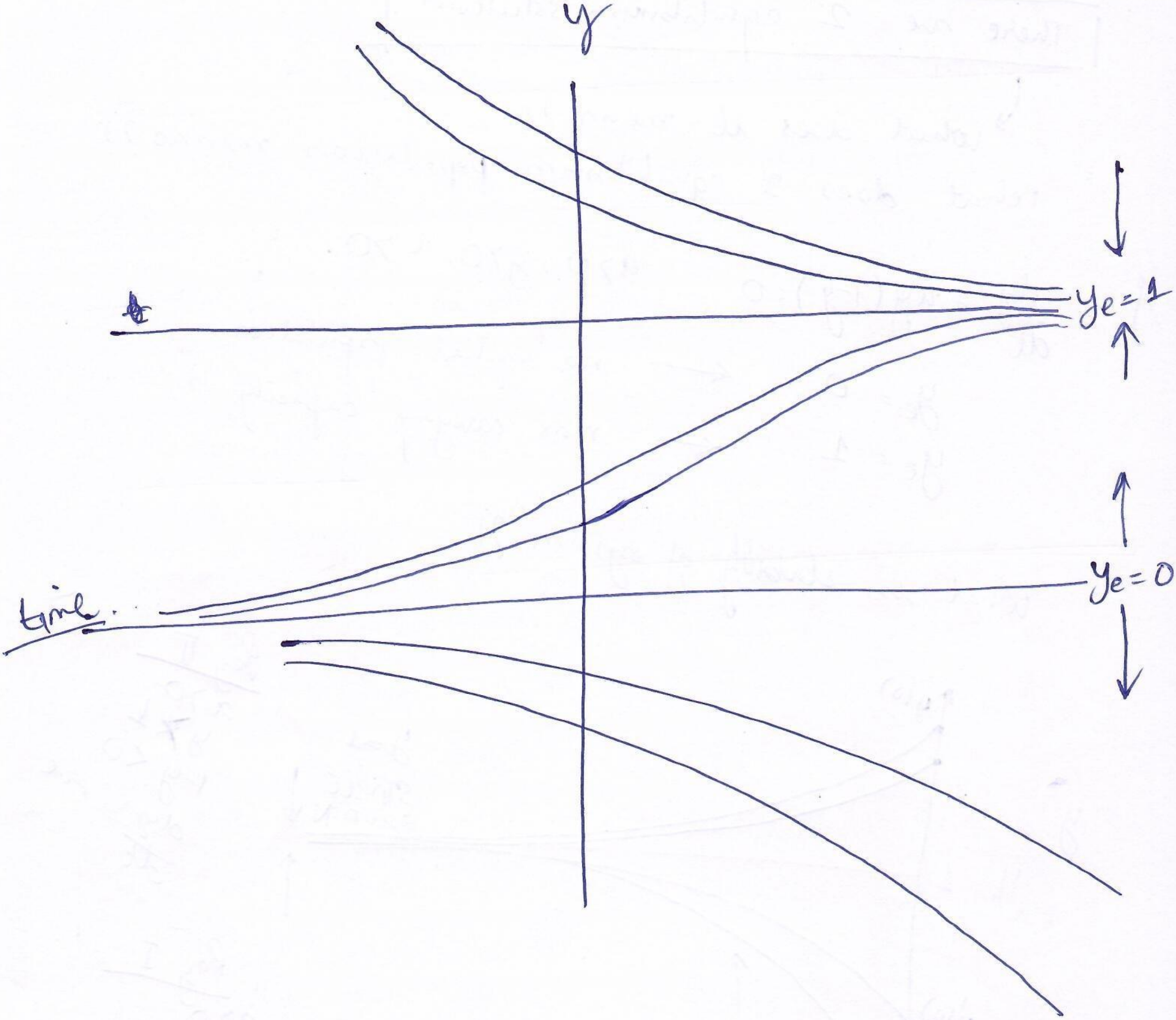


If initial population < y_e

then population ↑es till max-carrying capacity is reached.

$y_e = 1$ is stable equilibrium solution.

$y_e = 0$ is unstable equilibrium solution



How to do using analytical condition ?!

$$\frac{dy}{dt} = ay(1-y) = f$$

$$f = ay - ay^2$$

$$\left.\frac{df}{dy}\right|_{y_e} = a - 2ay\Big|_{y_e}$$

@ $y_e = 0$ $\quad \frac{df}{dy} = a$ $\quad$ $a > 0$ in this case when $\left.\frac{df}{dy}\right|_{y_e} > 0 \leftarrow$ unstable

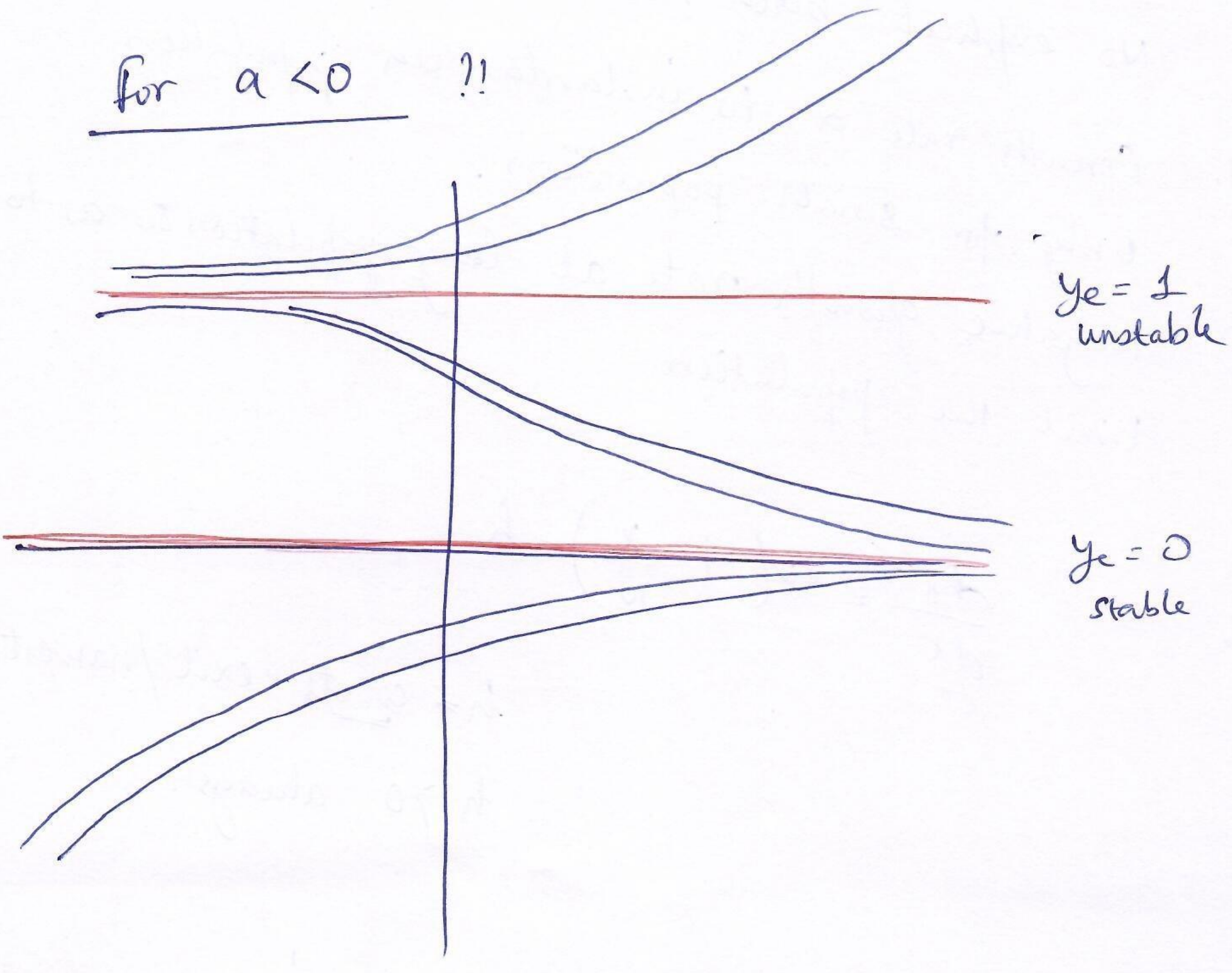
@ $y_e = 1 \quad \frac{df}{dy} = -a < 0$

when $\left.\frac{df}{dy}\right|_{y_e} < 0 \leftarrow$ stable

$\Big\}$ Condition for stability.

$$y(t) = \frac{y_0 e^{at}}{1 - y_0 + y_0 e^{at}}$$ $\rightarrow$ plot using desmos to confirm.

<u>For $a < 0$</u> ?!



Logistics population growth with harvesting

you want to harvest population at regular interval.

↳ Take away at constt. rate at regular time interval

Assumptions

1. No entry but exit of members at constt. rate.
2. Growth rate is the function of instantaneous population.
3. No death; birth only from present members, No explicit birth rate.
4. Growth rate $\propto$ to instantaneous population only for small population.
5. Negative growth rate at large population so as to limit the population.

SAME

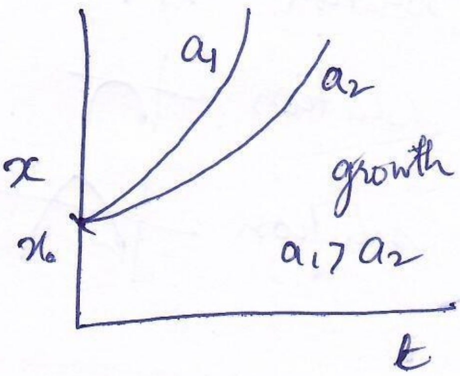
$$\frac{dx}{dt} = ax\left(1 - \frac{x}{N}\right) - h$$

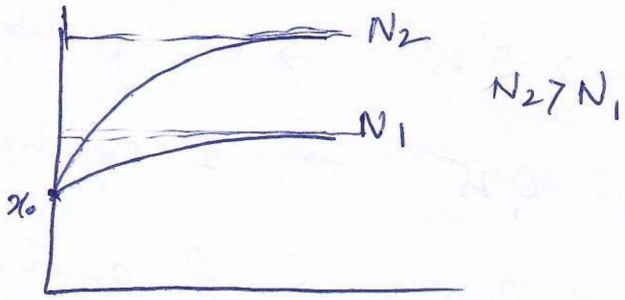
h = constt. exit / harvest

$h > 0$ always.

$$\frac{dx}{dt} = ax\left(1 - \frac{x}{N}\right) - h$$

3 parameter system
growth factor
+
carrying capacity
+
harvesting constt.





If $a = 1$; $N = 1$. ← To see effect of h

$$\frac{dx}{dt} = x(1-x) - h$$

equilibrium solution ??

$$\frac{dx}{dt} = x(1-x) - h = 0$$

$$-x^2 + x - h = 0 \quad \text{——— (1)}$$

population are real numbers.
so if solution exist to eq^n (1) then
equilibrium solution ~~is there~~ would exist

for eqⁿ to have real solution

discriminant ≥ 0

$a = 1$
$b = -1$
$c = h$

$$b^2 - 4ac \geq 0$$

$$1 - 4h \geq 0$$

$$h \leq \frac{1}{4}$$

$$h \leq 0.25$$

for $h > 0.25$ $\rightarrow$ no equilibrium solution 

$h = 0.25$ $\rightarrow$ 1 equilibrium solution. 

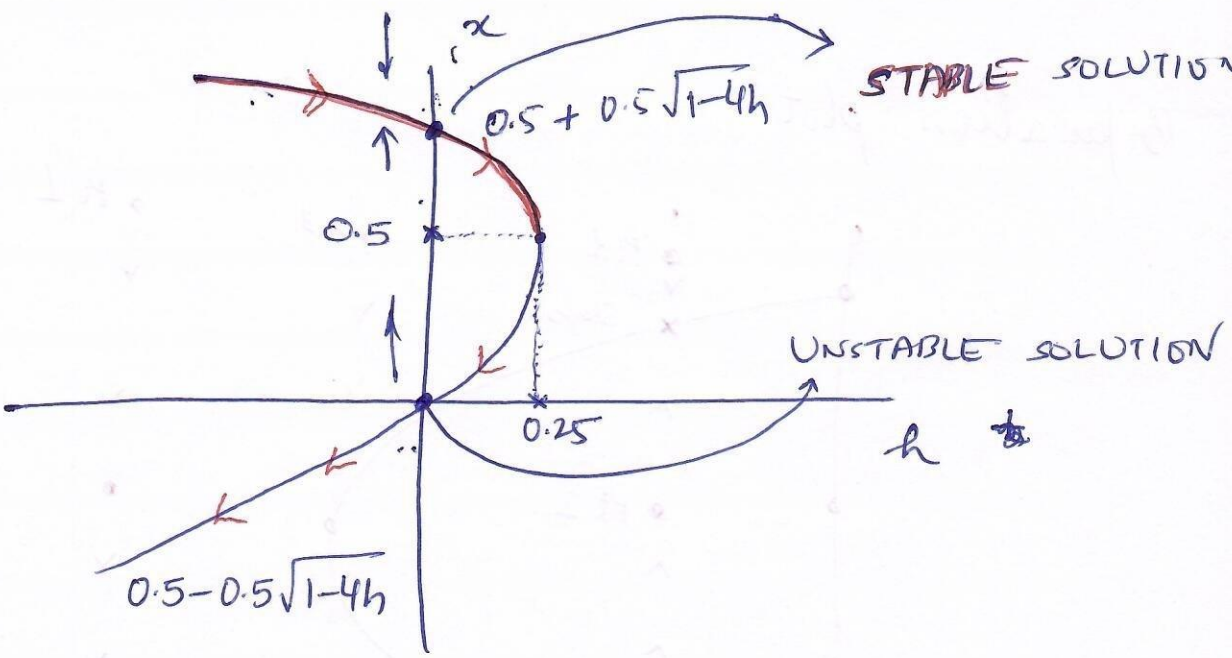
$h < 0.25$ $\rightarrow$ 2 equilibrium solution 

$$x_e = \frac{1 \pm \sqrt{1-4h}}{2}$$

for $h = 0.25$; $x_e = 0.5$

$h > 0.25$; $\sqrt{1-4h} < 0$ $\Leftarrow$ Not possible since population needs to be real no.

$$x_e = 0.5 \pm 0.5\sqrt{1-4h}$$



$$\frac{dx}{dt} = ax\left(1 - \frac{x}{N}\right) - h$$

~~$x_e = 0$; ← unstable~~

~~$x_e = N$ ← stable~~

System has bifurcation at $h = 0.25$

bifurcation parameter is 'h'

$$f = x - x^2 - h$$

$$\frac{df}{dx} = 1 - 2x$$

at bifucation $\left.\frac{df}{dx}\right|_{x_e} = 0$

$$x_e = 0.5 \pm 0.5\sqrt{1-4h}$$

$$\left.\frac{df}{dx}\right|_{x_{e+}} = 1 - 2(0.5 + 0.5\sqrt{1-4h})$$

$$= \cancel{1} - \cancel{1} - \sqrt{1-4h}$$

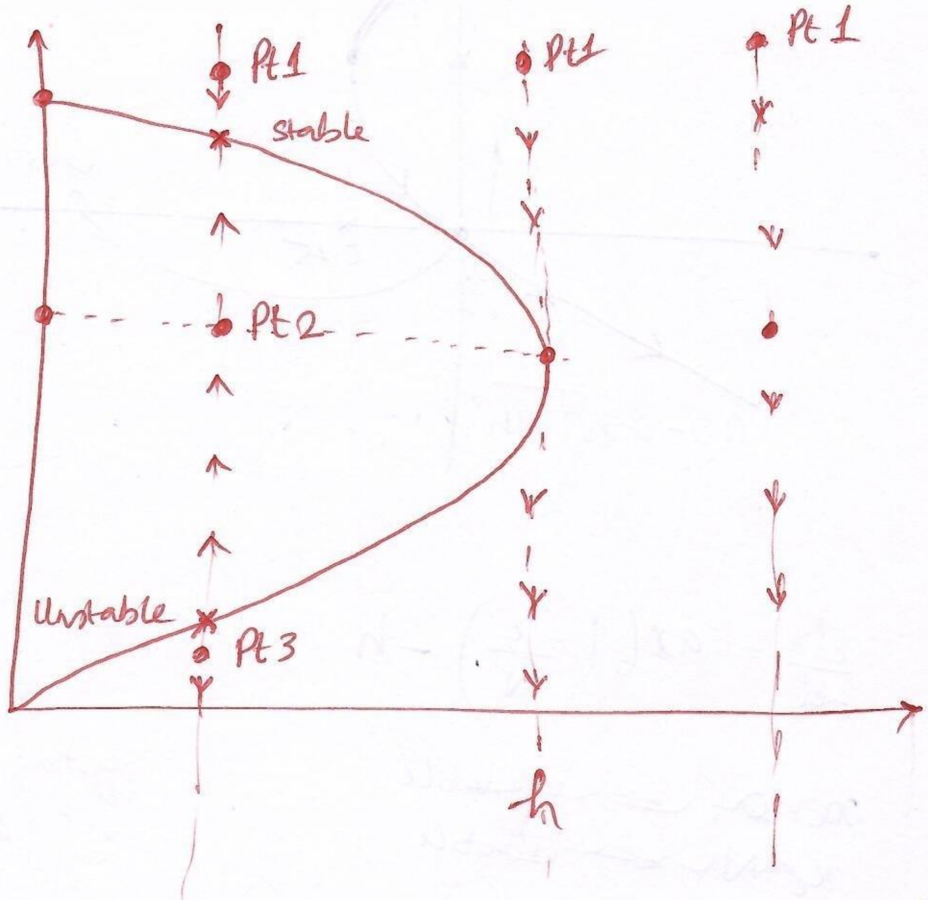
$$= -\sqrt{1-4h}$$

< 0 ← STABLE SOLUTION

$$\left.\frac{df}{dx}\right|_{x_{e-}} = +\sqrt{1-4h} > 0$$

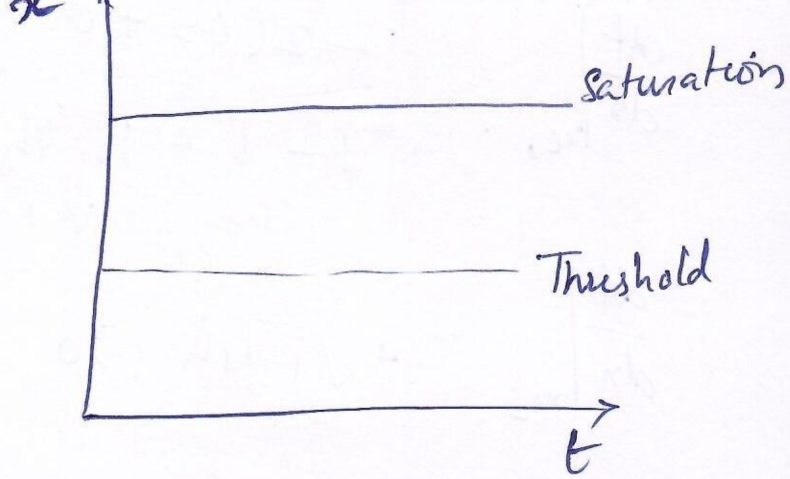
← UNSTABLE SOLUTION

Bifurcation plot



Q. What if we have threshold population for a particular species?

This means that if population is less than threshold population then it must die out to 0.



$$\frac{dx}{dt} = -ax\left(1-\frac{x}{\lambda_1}\right)\left(1-\frac{x}{\lambda_2}\right)$$

λ_1 = carrying capacity

λ_2 = threshold population

$$0 < \lambda_2 < \lambda_1$$

Assumptions

① upper limit on the population based on carrying capacity.

② Exponential growth at initial stages & saturation at later stages

③. Extintion when the initial population is less than the threshold population.

$$\frac{dx}{dt} = -ax\left(1-\frac{x}{\lambda_1}\right)\left(1-\frac{x}{\lambda_2}\right) \qquad \begin{array}{l} a>0 \\ \lambda_1>0 \\ \lambda_2>0 \end{array}$$

$$\frac{dx}{dt} = \frac{-ax}{\lambda_1 \lambda_2}(\lambda_1-x)(\lambda_2-x)$$

$$\frac{dx}{dt} = \left(\frac{-a}{\lambda_1\lambda_2}\right)x(\lambda_1-x)(\lambda_2-x)$$

$$\frac{dx}{x(\lambda_1-x)(\lambda_2-x)} = \left(\frac{-a}{\lambda_1\lambda_2}\right)dt$$

$$\left(\frac{A}{x} + \frac{B}{\lambda_1-x} + \frac{C}{\lambda_2-x}\right)dx = \left(\frac{-a}{\lambda_1\lambda_2}\right)dt$$

$$A = \frac{1}{\lambda_1\lambda_2}$$

$$B = \frac{1}{\lambda_2-\lambda_1}$$

$$C = \frac{1}{\lambda_1-\lambda_2}$$

$$A \ln|x| - B \ln|\lambda_1 - x| - C \ln|\lambda_2 - x| = -\frac{at}{\lambda_1\lambda_2} + C$$

$$(x^A)(\lambda_1 - x)^B (\lambda_2 - x)^C = e^{\frac{-a}{\lambda_1\lambda_2}t}.$$

$\downarrow$

Converting this to a form
$x = f(t)$ is difficult.

$$\frac{dx}{dt} = -ax\left(1-\frac{x}{\lambda_1}\right)\left(1-\frac{x}{\lambda_2}\right)$$

equilibrium solution

$$\frac{dx}{dt} = 0$$

$x = 0, \lambda_1, \lambda_2$ $\leftarrow$ 3 equilibrium solution.

let $\frac{dx}{dt} = f$

if $\frac{df}{dt} > 0$ @ equilibrium solution then unstable

< 0 @ equilibrium solution then stable

$$\frac{dx}{dt} = \left(\frac{-a}{\lambda_1\lambda_2}\right) x \cdot (\lambda_1\lambda_2 - (\lambda_1+\lambda_2)x + x^2)$$

$$f = \frac{dx}{dt} = \left(\frac{a}{\lambda_1\lambda_2}\right)\left[-x^3+(\lambda_1+\lambda_2)x^2-\lambda_1\lambda_2 x\right]$$

$$\frac{df}{dt} = \left(\frac{a}{\lambda_1\lambda_2}\right)\left(-3x^2+2(\lambda_1+\lambda_2)x-\lambda_1\lambda_2\right)$$

@ $x_e = 0$

$$\frac{df}{dt} = -a \qquad < 0 \qquad \text{stable solution}$$

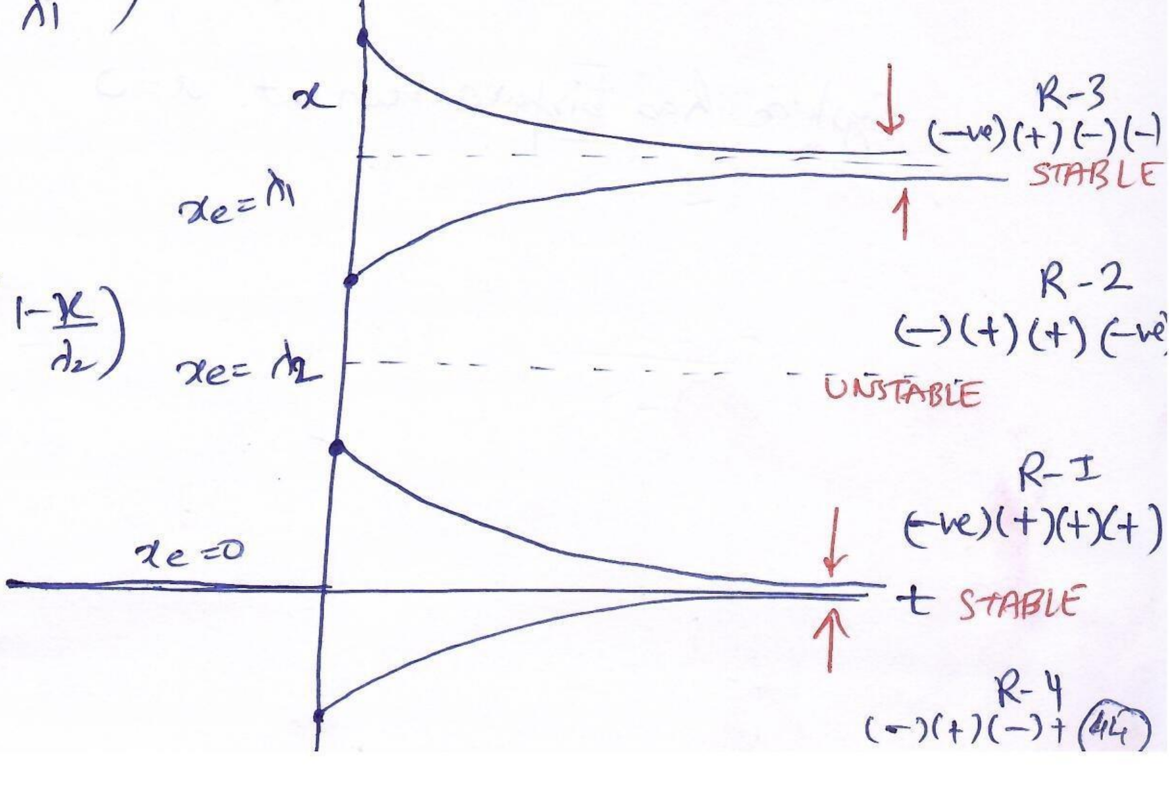
@ $x_e = \lambda_1$

$$\frac{df}{dt} = \frac{a(-\lambda_1+\lambda_2)}{a\lambda_2} \qquad < 0 \qquad \text{stable solution.}$$

@ $x_e = \lambda_2$

$$\frac{df}{dt} = a\left(\frac{\lambda_1+\lambda_2}{\lambda_1}\right) \qquad > 0 \qquad \text{unstable solution}$$

$$\frac{dx}{dt} = -ax\left(1-\frac{x}{\lambda_1}\right)\left(1-\frac{x}{\lambda_2}\right)$$



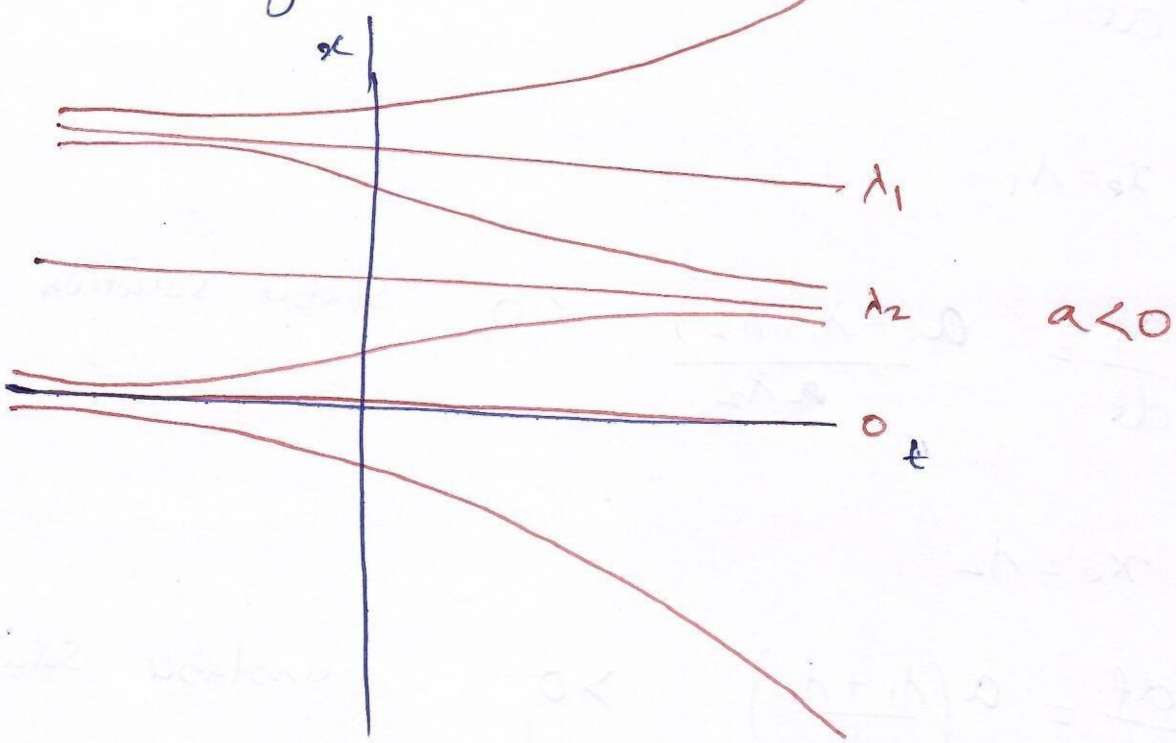
Bifurcation ??

$$\frac{dx}{dt} = -ax\left(1-\frac{x}{\lambda_1}\right)\left(1-\frac{x}{\lambda_2}\right) \qquad a>0;\ \lambda_1\lambda_2>0;\ \lambda_1>\lambda_2$$

a = +ve always

what will happen if a = -ve ??

↳ slopes of all phase lines will change the sign



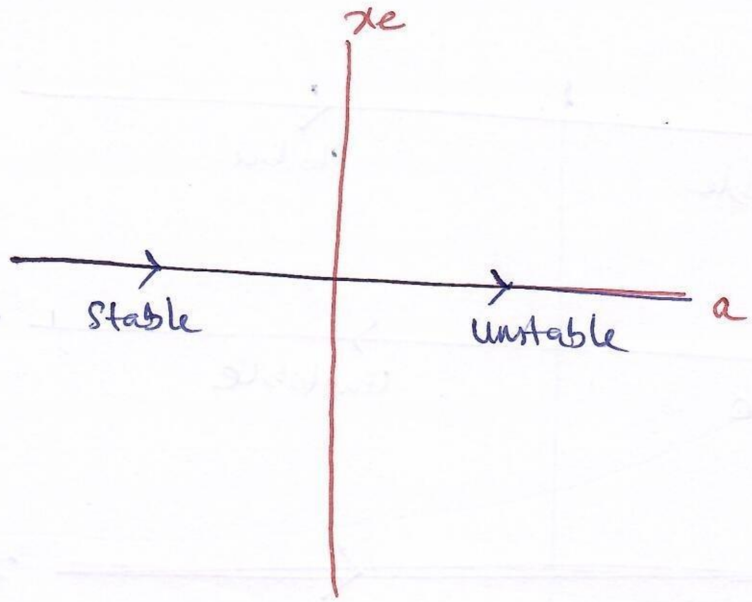
System has bifurcation at a=0

Bifurcation Diagram :

I. $\frac{dx}{dt} = ax \Rightarrow x_e = 0$

$a > 0$ unstable

$a < 0$ stable



II. $\frac{dx}{dt} = ax\left(1 - \frac{x}{N}\right)$ $N = 1$

for $a > 0$

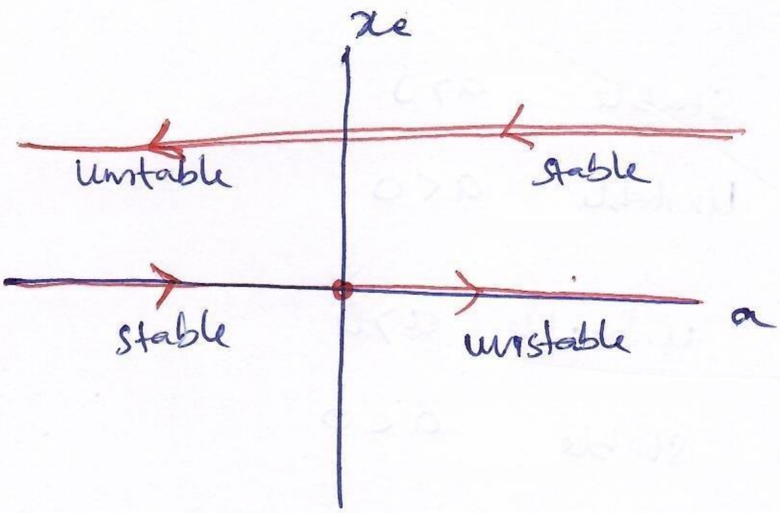
$x_e = 0$ unstable

$x_e = 1$ stable

for $a < 0$

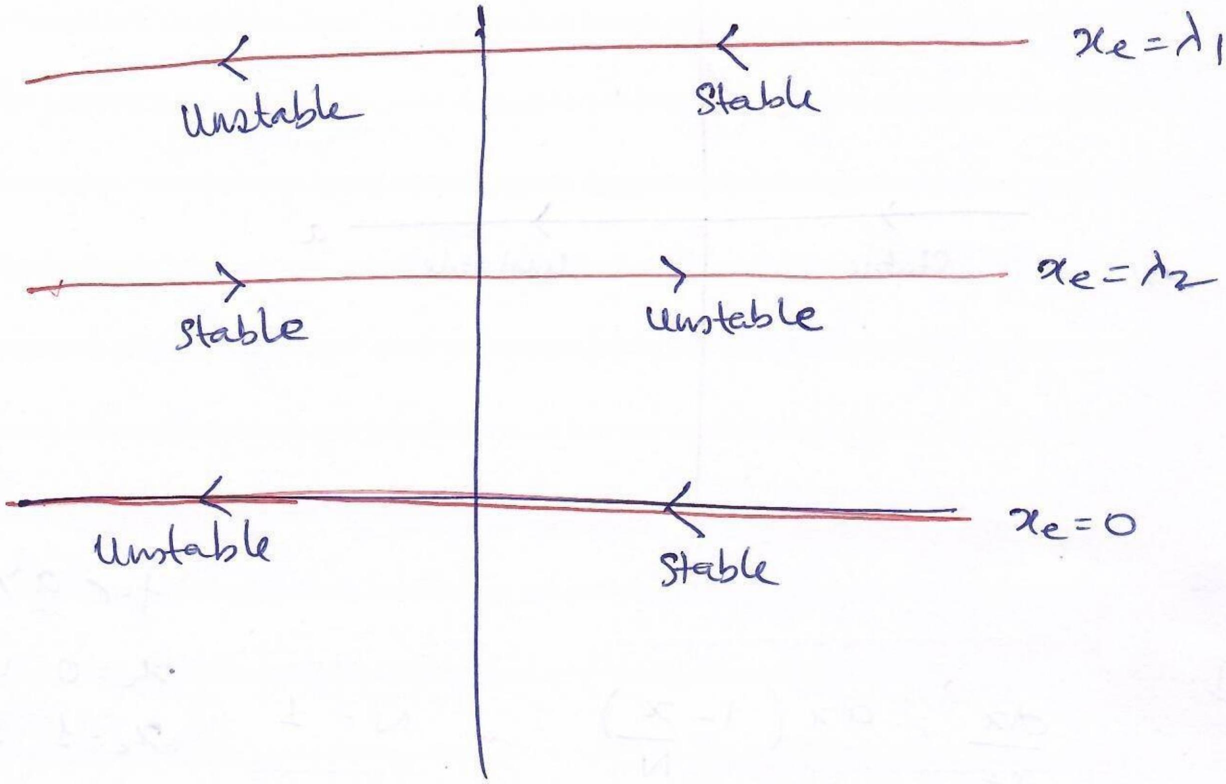
$x_e = 0$ Stable

$x_e = 1$ unstable



III.

$$\frac{dx}{dt} = -ax\left(1-\frac{x}{\lambda_1}\right)\left(1-\frac{x}{\lambda_2}\right) \qquad x_e = 0, \lambda_1, \lambda_2$$



$x_e = 0$	Stable	$a > 0$
	Unstable	$a < 0$
$x_e = \lambda_2$	Unstable	$a > 0$
	Stable	$a < 0$
$x_e = \lambda_1$	Stable	$a > 0$
	Unstable	$a < 0$