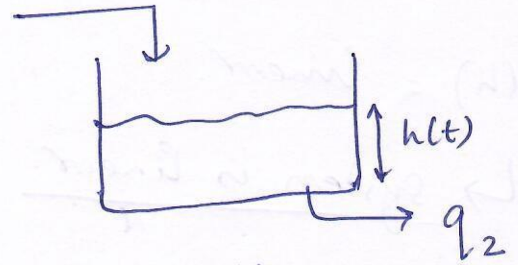
Dynamics of linear I[st] order autonomous systems in State space domain 18/01/2

1. State space domain = ??
   Dynamical system = ??
   State of system.

Example :- linear autonomous 1[st] order



$$\frac{dh}{dt} = \frac{1}{A}(q_1 - q_2)$$

order = 1 ; $h(t)$ = dynamical variable

q_1 = I/p vol. flow rate ; A = cross-sectional area of tank

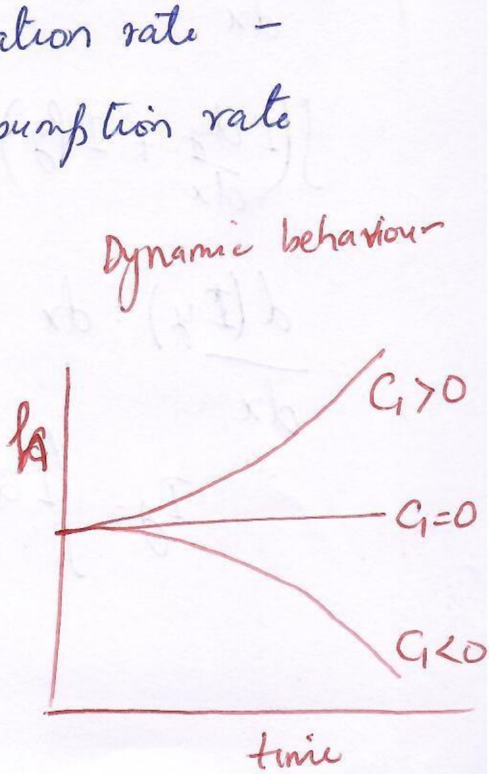
q_2 = O/p vol. flow rate

✓ time rate of change (mass / momentum / energy) = I/p − O/p + generation rate − consumption rate

Case I :- $q_1 = q_2$ = constant

$$\frac{dh}{dt} = \frac{1}{A}(C_1 - C_2) \Rightarrow \underbrace{\frac{dh}{dt} = C}_{\text{Linear system}}$$

Dynamic behaviour



Case 2 : q_1 = Constt. $= C_1$

$q_2 = f(h)$

$$\frac{dh(t)}{dt} = \frac{1}{A}\left(C_1 - f(h)\right)$$

$$\frac{dh}{dt} + \frac{1}{A} f(h) = \frac{C_1}{A}$$

if $f(h)$ = linear

↳ system is linear

Solve by integrating factor method

$$\frac{dy}{dx} + Py = Q \qquad I = e^{\int P dx}$$

$$I\frac{dy}{dx} + IPy = IQ$$

$$\int\left(I\frac{dy}{dx} + IPy\right) dx = \int IQ\, dx$$

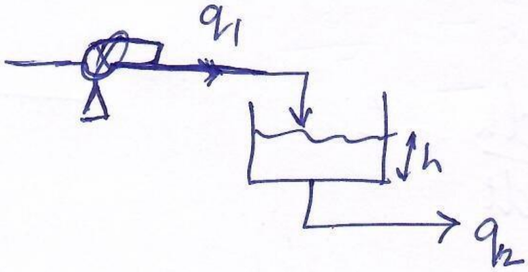
$$\frac{d(Iy)}{dx} \cdot dx = \int IQ\, dx$$

$$Iy = \int IQ\, dx$$

Case 3:-

$$q_1 = f(t)$$

$$q_2 = g(h)$$



$$\frac{dh}{dt} = \frac{1}{A} f(t) - \frac{1}{A} g(h)$$

$$\frac{dh}{dt} + \frac{1}{A} g(h) = \frac{1}{A} f(t)$$

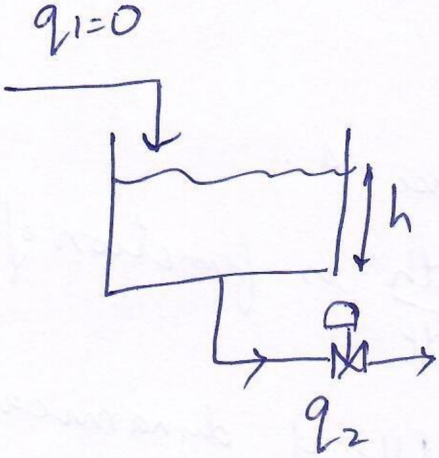
if $g(h)$ = linear then system is linear

Solve by If method

Case 4 -

$$q_1 = 0$$

$$q_2 = ah$$



$$\left.\frac{dh}{dt} = \frac{1}{A}(-ah)\right\} \quad \text{let } \frac{-a}{A} = b$$

$$\boxed{\frac{dh}{dt} = bh}$$

specific case

case 1:

$$\frac{dh}{dt} = C \longrightarrow \frac{dx}{dt} = C$$

case 2:

$$\frac{dh}{dt} + \frac{1}{A} f(h) = \frac{C_1}{A} \longrightarrow \frac{dx}{dt} + \alpha f(x) = \beta$$

$$\frac{dx}{dt} = -\alpha f(x) + \beta$$

case 3:

$$\frac{dh}{dt} + \frac{1}{A} g(h) = \frac{1}{A} f(t) \rightarrow \frac{dx}{dt} + \alpha g(x) = \beta f(t)$$

$$\frac{dx}{dt} = -\alpha g(x) + \beta f(t)$$

case 4:

$$\frac{dh}{dt} = bh \longrightarrow \frac{dx}{dt} = bx$$

In case 4.

$\frac{dh}{dt}$ is function of x only

$\Rightarrow$ RHS of dynamical eq^n is a function of dynamical variable only = autonomous system.

Phase ~~portraits~~ portraits

linear 1^{st} order autonomous system

$$\frac{dx}{dt} = ax \quad \left\{ \begin{pmatrix}\text{rate of change of} \\ \text{dynamical variable}\end{pmatrix} \text{ is } \begin{pmatrix}\text{function of} \\ \text{dynamical variable}\end{pmatrix} \right.$$

understand dynamical behaviours $\equiv$ value of a is important

if $a = 1 \Rightarrow \frac{dx}{dt} = x$

$a = -1 \Rightarrow \frac{dx}{dt} = -x$

$a \in \mathbb{R}$ (infinite no. of eqns)

$\uparrow$

process dynamics depends on this value.

$x(t) = x_0 e^{at}$ ← exponential function

$\uparrow$ const.

x_0 = initial condition

let $x_0 = 1$

$a > 0$

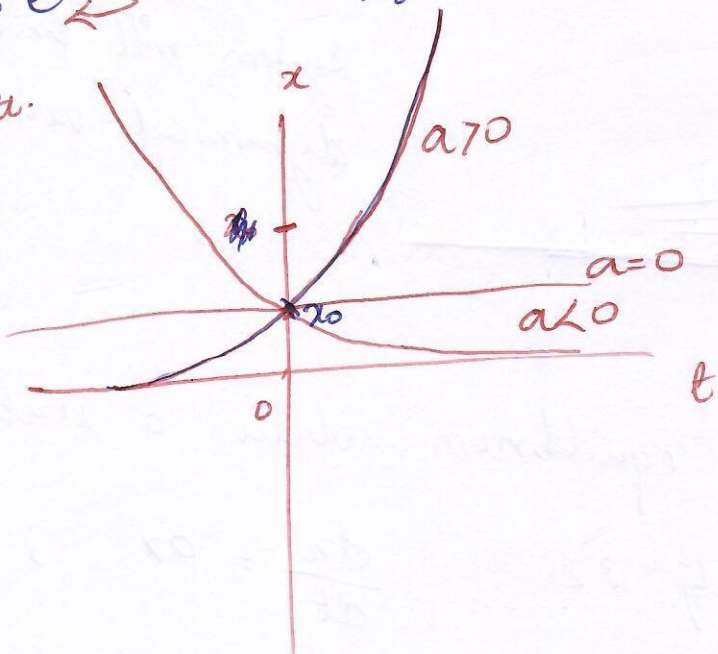
$x(t) \to \infty$

$a < 0$

$x(t) \to 0$

$a = 0$

$x(t) = x_0$



phase portrait

↳ helps to predict system performance with time.

The system has opposite behaviours ~~at~~ ~~to~~ at both ends

Thus system has bifurcation

$$\because \quad \frac{dx}{dt} = ax \quad \begin{cases} a>0 & \lim_{t\to\infty} x(t) \to \infty \\ a<0 & \lim_{t\to\infty} x(t) \to 0 \end{cases}$$

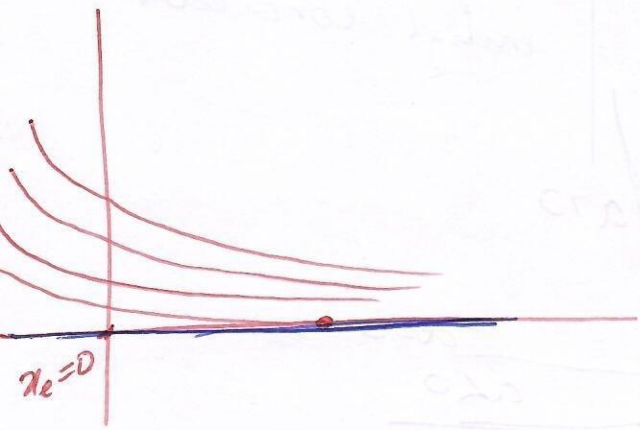
Thus system has bifurcation at $a=0$

Equilibrium solution @ gradient of system $=0$

$$\frac{dx}{dt} = ax$$

$$\left.\frac{dx}{dt}\right|_{x_e} = 0 \quad \Rightarrow \quad ax = 0$$

$$x_e = 0$$

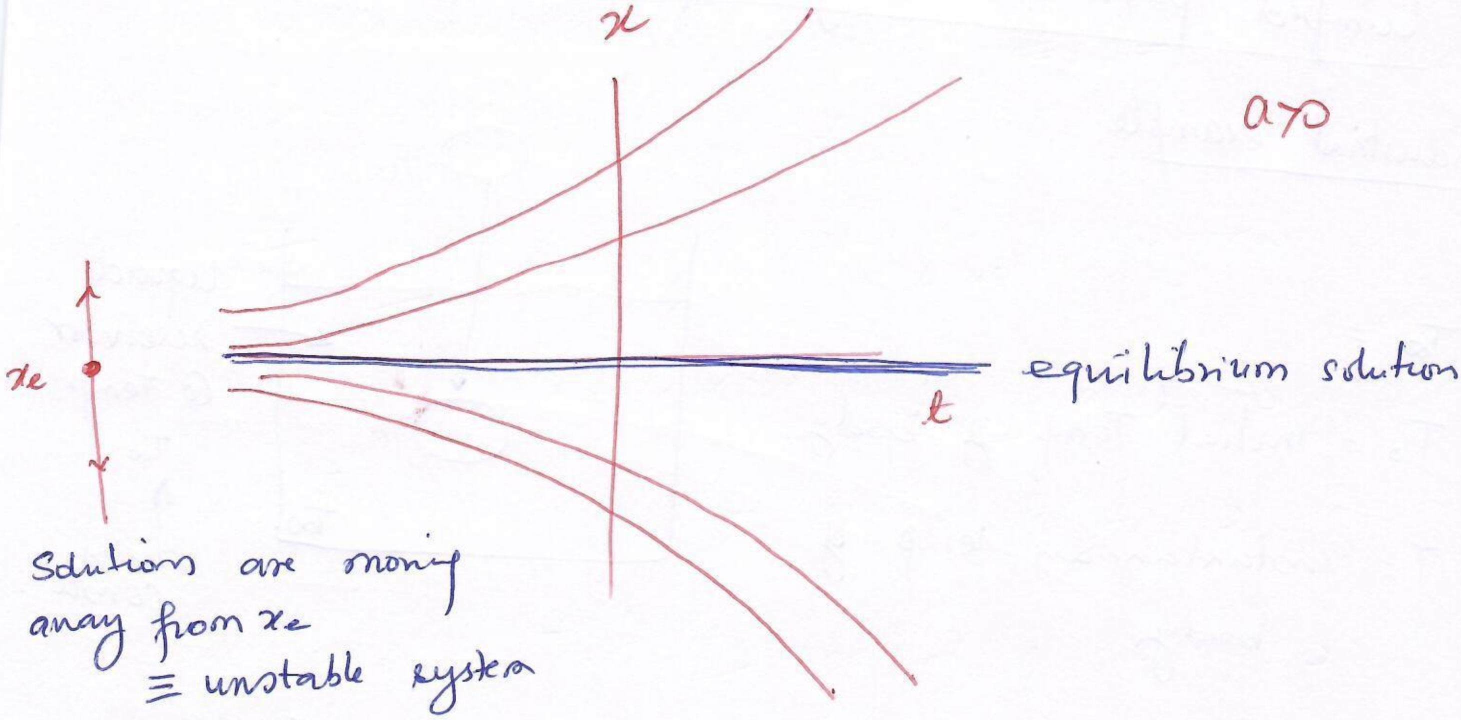


This means at point x_e the system will have same value of dynamical variable in future.

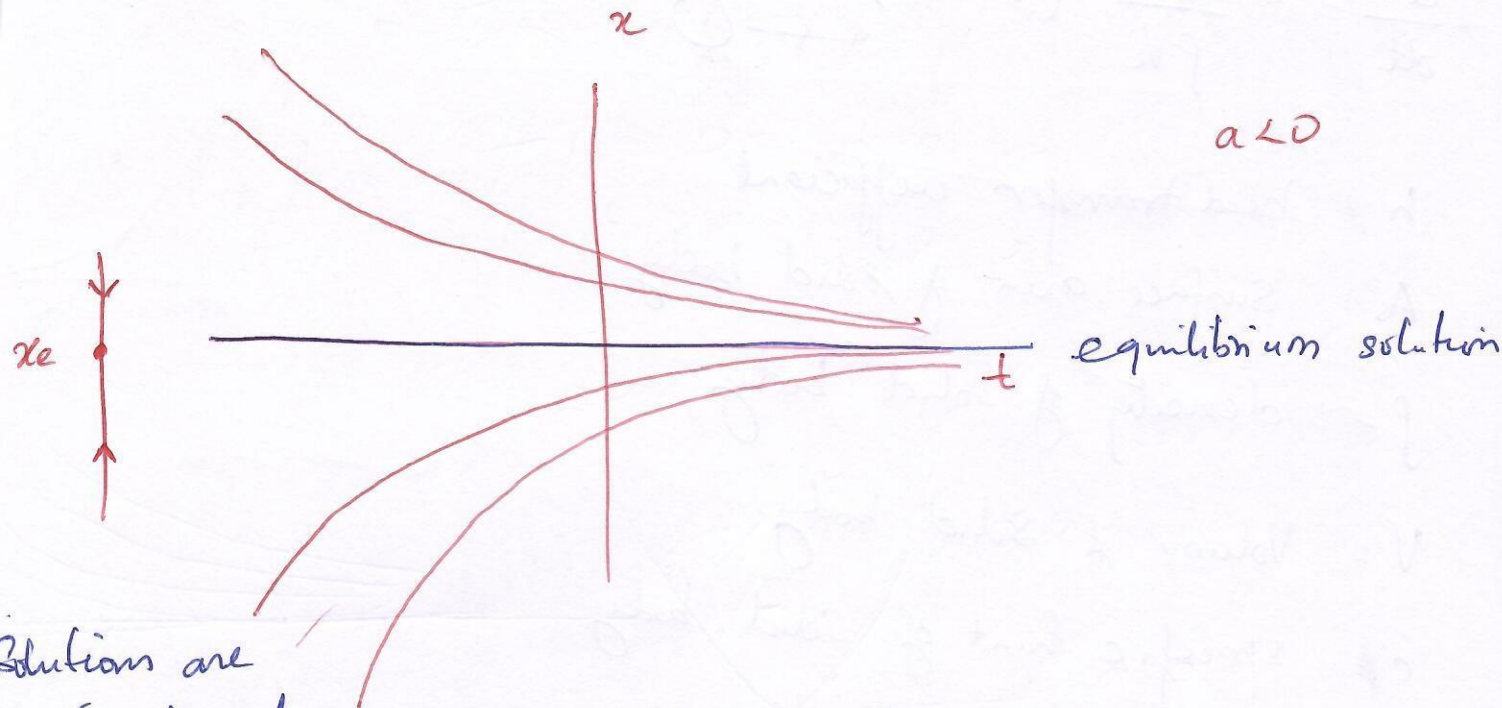
Is equilibrium solution a stable solution ??

Stability ?? $\quad \frac{dx}{dt} = ax \quad ; \quad x_e = 0$

with increase in time, system diverges from x_e = ??



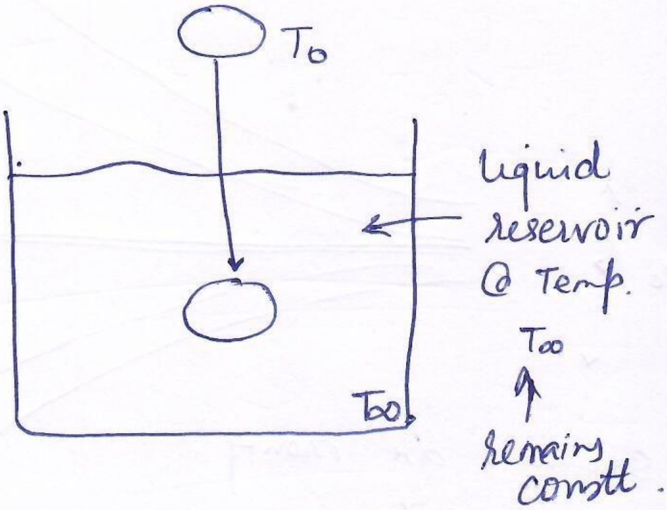
Solutions are moving away from x_e
$\equiv$ unstable system



Solutions are coming toward x_e
$\equiv$ stable solution.

Lumped parameter analysis of cooling of body

Industrial example



T_∞ =

T_o = initial Temp. of body

T = instantaneous temp. of body

$$\frac{dT}{dt} = \frac{-hA_s}{\rho V c}(T - T_\infty) \quad \text{--- (1)} \longleftarrow \text{governing eq}^n$$

h = heat transfer coefficient.

A_s = Surface area of solid body

ρ = density of solid body

V = Volume of Solid body

c = specific heat of Solid body

Question :- What are the equilibrium solution of system ??

- Solve model equation analytically to determine the time evolution of the system.
- Develop phase ~~pot~~ portrait for the system.

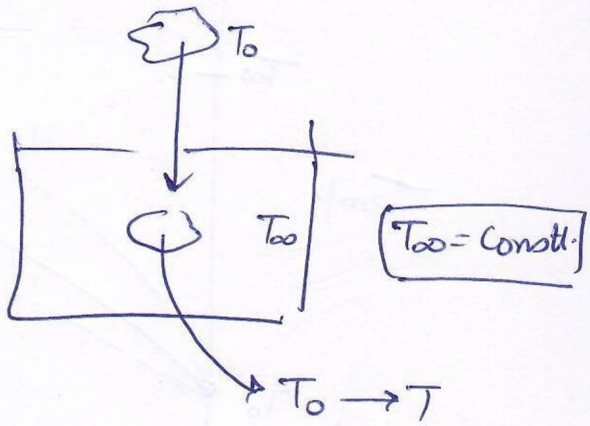
consider $\dfrac{hA_s}{\rho V c} = a$ (constt.) —— ②

$$\frac{dT}{dt} = -a(T - T_\infty) \quad —— ③$$

@ x_e $\quad \dfrac{dx}{dt} = 0$

$$-a(T - T_\infty)\Big|_{T_e} = 0$$

$T_e = T_\infty \Rightarrow ??$



with time

$T \to T_\infty$

&

$T_e = T_\infty$

time evolution of system

$$\frac{dT}{dt} = -a(T - T_\infty)$$

$T(0) = T_0$

$T^* = T - T_\infty$ (say) —— ②

T_∞ is constt

$dT^* = dT$ —— ③

$$\frac{dT^*}{dt} = -aT^* \Rightarrow T^* = ce^{-at}$$

$$T - T_\infty = c'e^{-at} \quad ——$$

$$\left(\frac{dx}{dt} = -ax\right)$$

At $t = 0 \Rightarrow T = T_0$

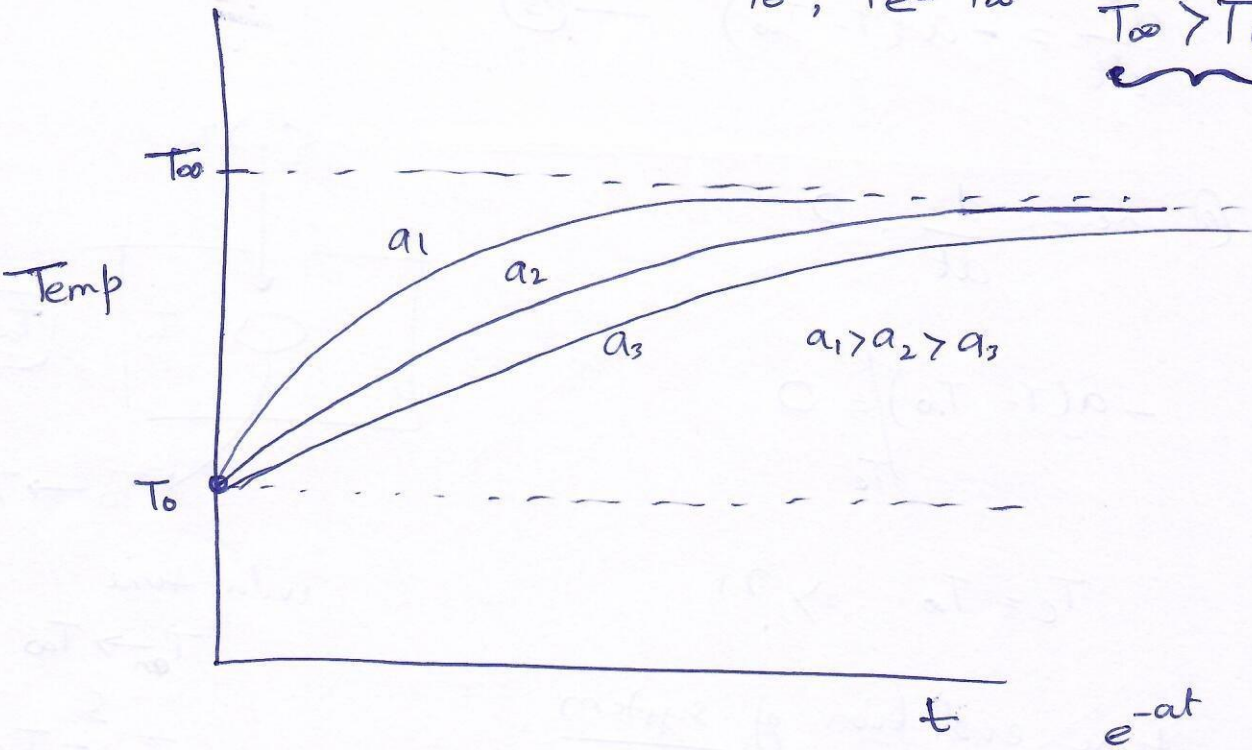
$T_0 - T_\infty = C$

$T - T_\infty = (T_0 - T_\infty)e^{-at}$

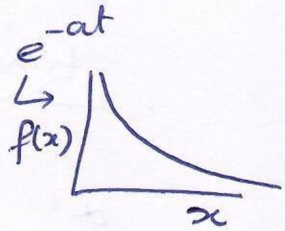
$$T(t) = T_\infty + (T_0 - T_\infty)e^{-at}$$ ← model eq^n solution.

T_0, $T_e = T_\infty$

$\underbrace{T_\infty > T_0}$



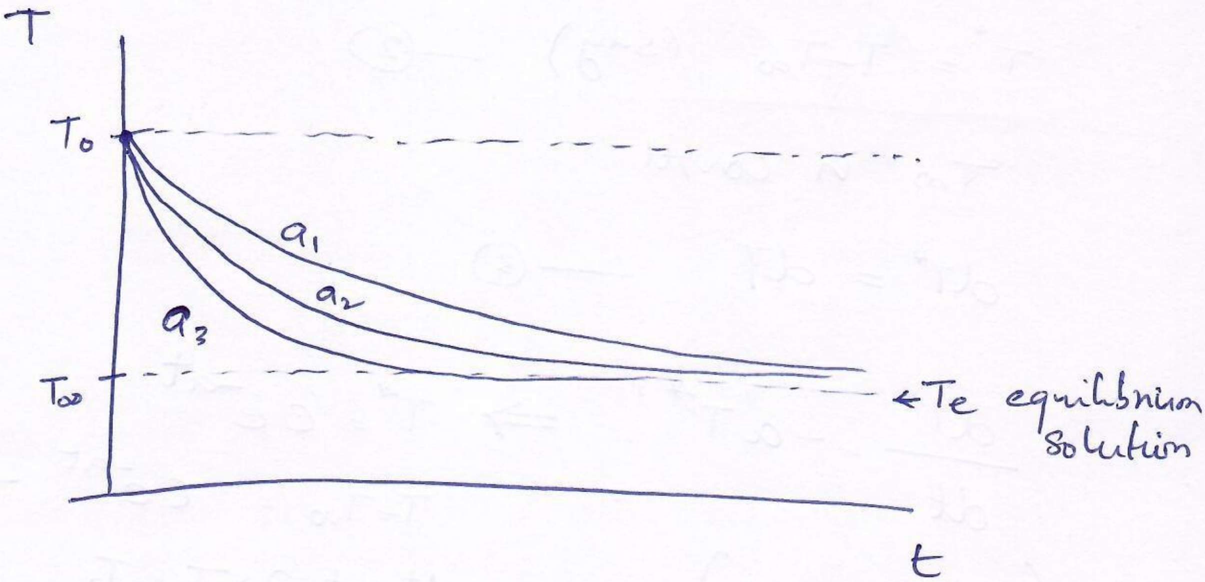
If a is large, faster response



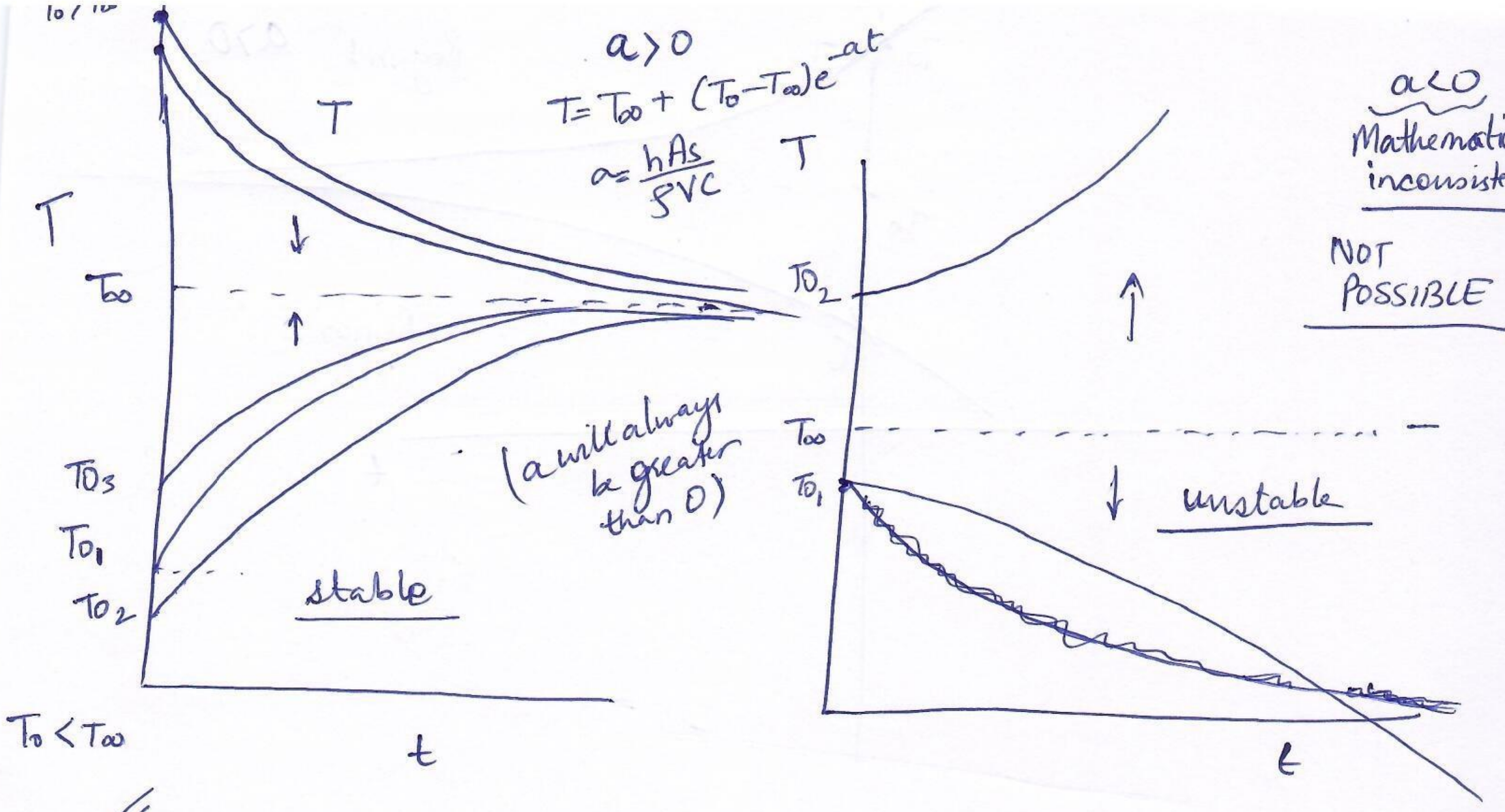
$$T(t) = T_\infty + (T_0 - T_\infty)e^{-at}$$

$\underbrace{T_0 > T_\infty}$

$a_3 > a_2 > a_1$



Example :- hot plate quenched in oil bath

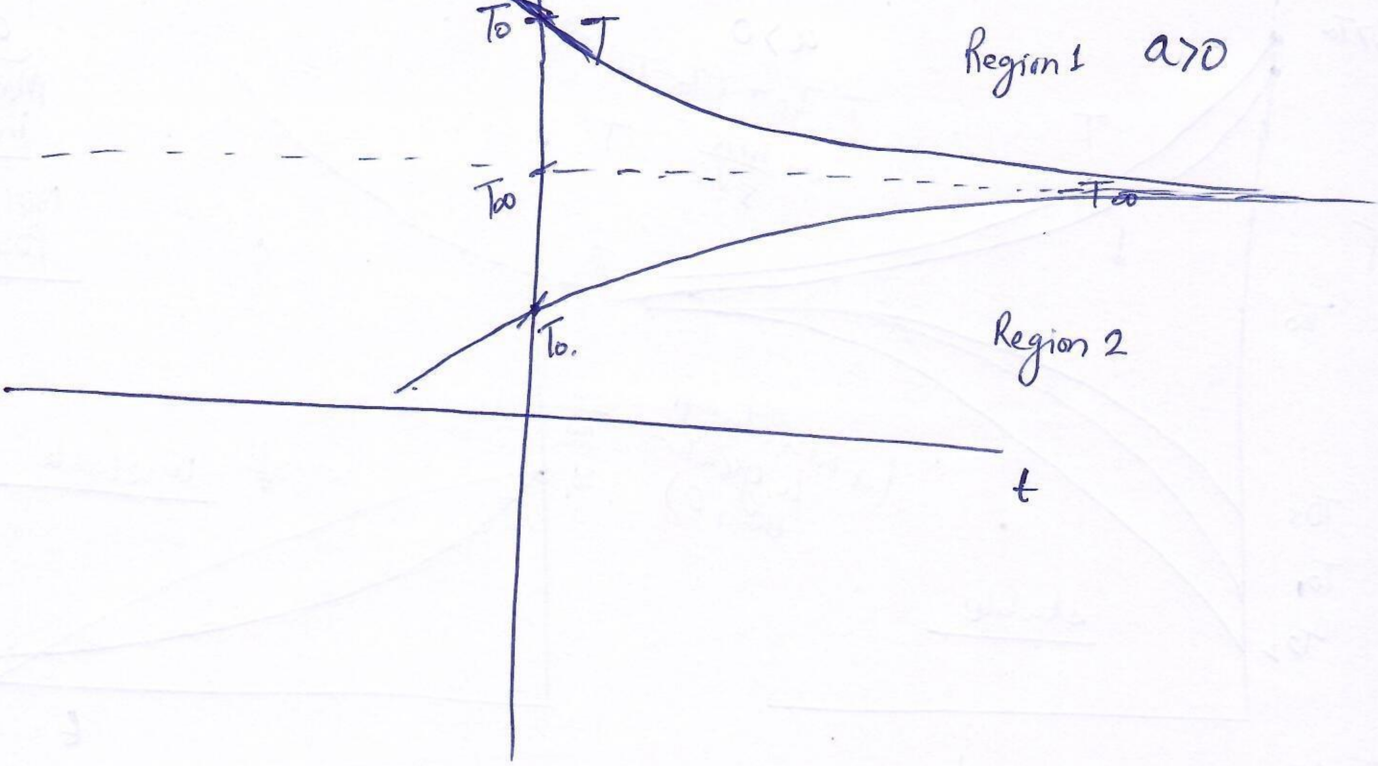


Q. How to develop phase portraits w/o solving governing eqⁿ ?!

- Analyse the solution & phase ~~patho~~ portraits for $T_0 < T_\infty$
  $T_0 = T_\infty$ & $T_0 > T_\infty$
- Study the effect of different system & material property on system dynamics
- Comment on bifurcation in the system

$$\frac{dT}{dt} = -a\,(T - T_\infty) \quad \text{—— (1)}$$

If $a>0$



identify equilibrium solution

$T_e = T_\infty$ (in this case)

$$\frac{dT}{dt} = -a(T - T_\infty)$$

In Region 1

$T > T_\infty$

we know $a > 0$

$T - T_\infty > 0$

$$\frac{dT}{dt} = (-ve)(\cdot X)$$

- gradient is going to be -ve.
- But you need to reach T_∞ as it is equilibrium solution

In Region 2

$T_\infty > T$

$a > 0$ we know

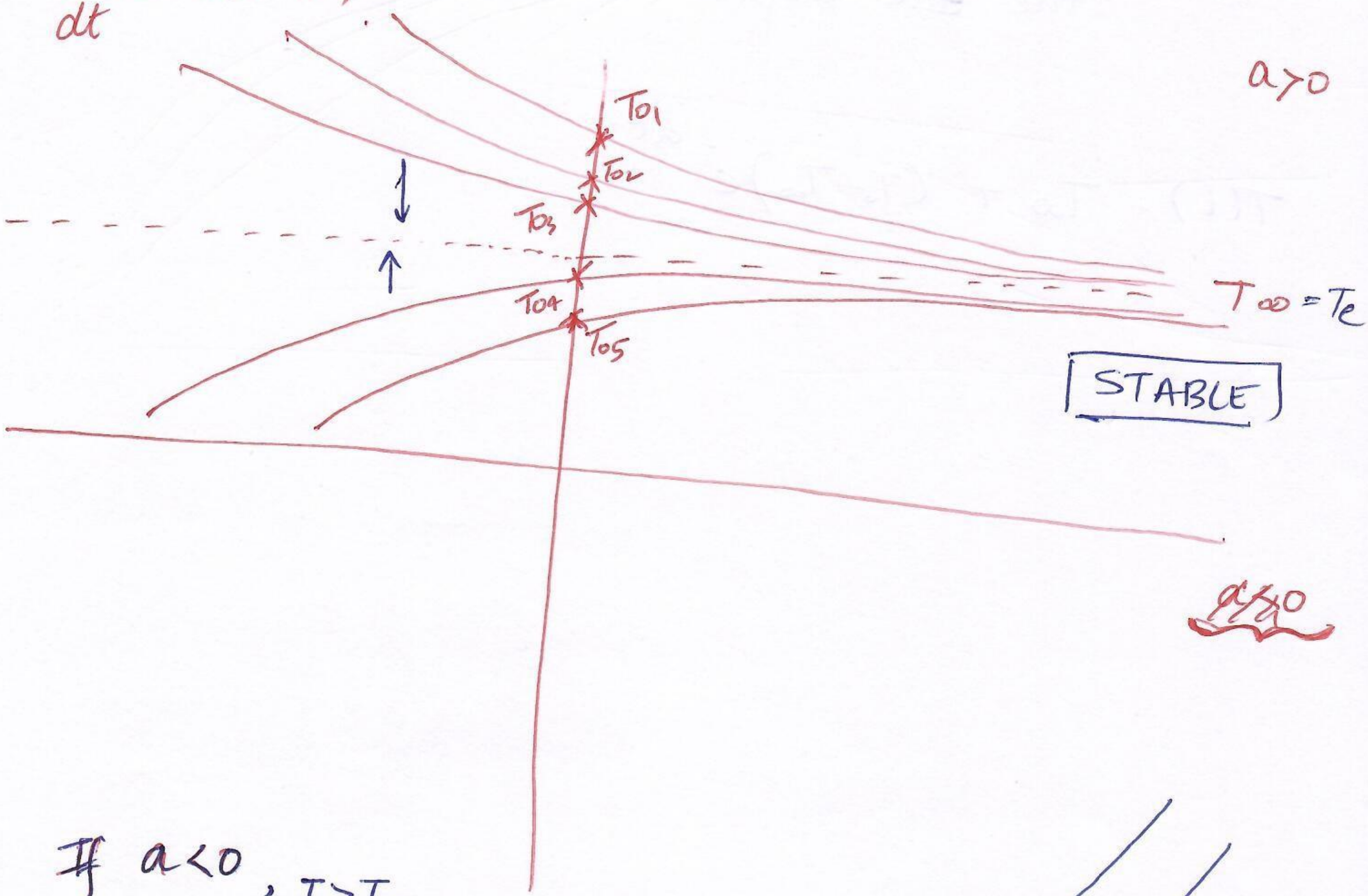
$$\frac{dT}{dt} = (-ve)(-ve) = +ve\ (X)$$

- gradient is always +ve
- reach T_∞ asymptotically

although -ve time ~~axis~~ does not exist in ~~pra~~ real but mathematically it does.

So complete phase portrait is

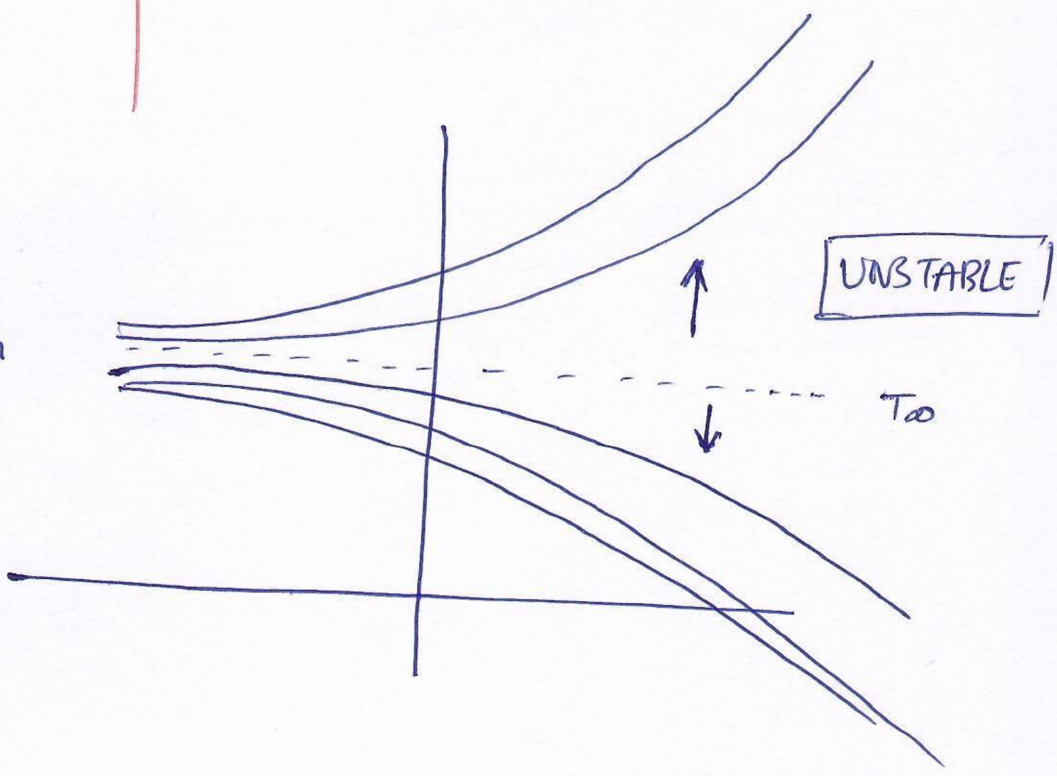
$\frac{dT}{dt} = -a(T-T_\infty)$



If $a<0$, $T>T_\infty$

$\frac{dT}{dt} = +ve$

gradient ↑ with time



$T<T_\infty$

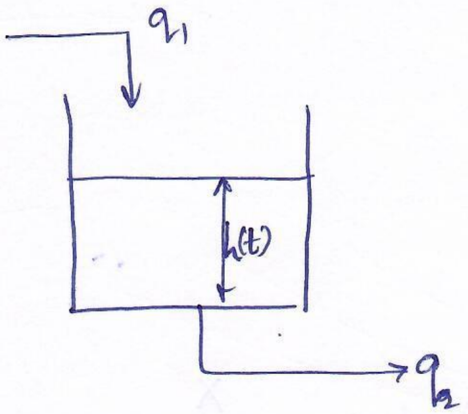
$\frac{dT}{dt} = -ve$

SYSTEM HAS BIFURCATION AT $a=0$

Question :- if we draw phase portrait w/o solving the model equations, How do we ensure that the ~~line~~ curve depict actual solution ?,

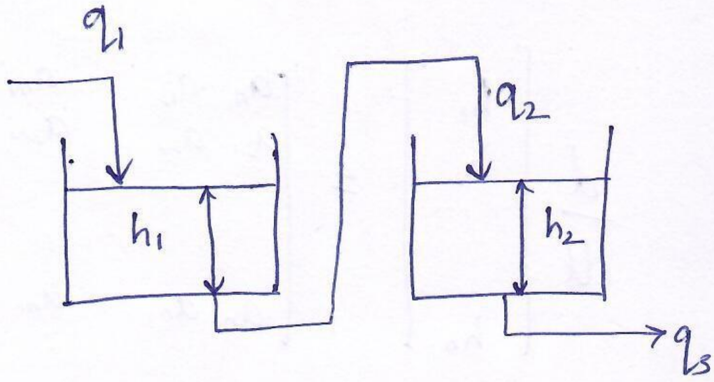
$$T(t) = T_\infty + (T_0 - T_\infty)e^{-at}$$

Higher order systems



1^{st} order

$$\frac{dh}{dt} = \frac{1}{A}(q_1 - q_2) \quad \text{—} ①$$



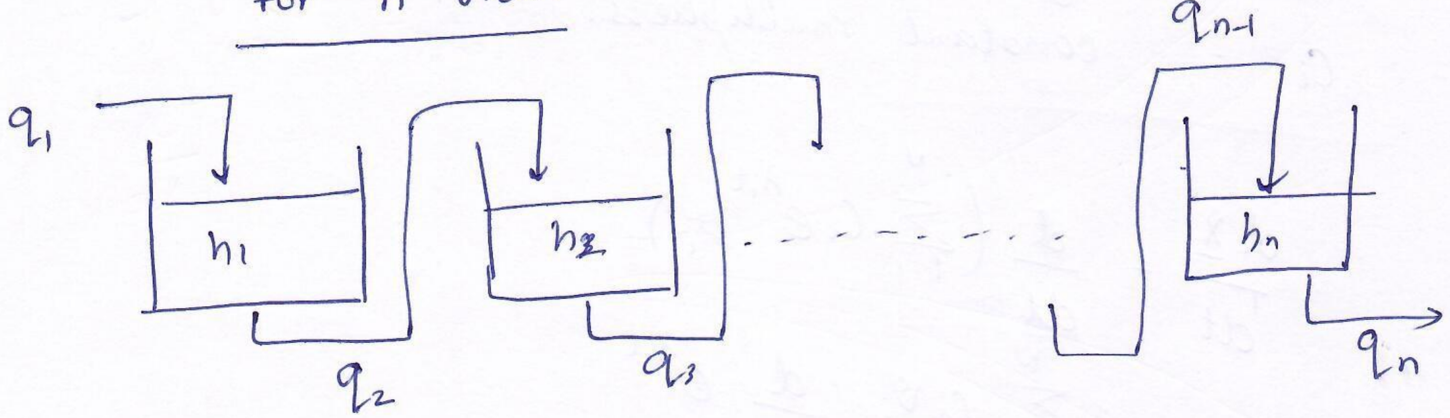
$$\frac{dh_1}{dt} = \frac{1}{A_1}(q_1 - q_2) \quad \text{—} ②$$

$$\frac{dh_2}{dt} = \frac{1}{A_2}(q_2 - q_3) \quad \text{—} ③$$

2^{nd} order

$$\text{Dynamical Variable} \equiv \begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix}$$

for n^{th} order



$$\frac{dh_1}{dt} = \frac{1}{A_1}(q_1 - q_2)$$

Let $q_1 = 0$; $q_2 = ah_1$, $q_3 = bh_2$

$$\frac{dh_2}{dt} = \frac{1}{A_2}(q_2 - q_3)$$

$$\Rightarrow \frac{dh_1}{dt} = \frac{-ah_1}{A_1} \quad ; \quad \frac{dh_2}{dt} = \frac{1}{A}(ah_1 - bh_2)$$

$$\frac{d}{dt}\begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix}$$

$(n \times n)$

$$\frac{d\underline{X}}{dt} = \underline{\underline{A}}\,\underline{X}$$

Vector $= \underline{X}$

Matrix $= \underline{\underline{A}}$

Solution

$$\underline{X} = \sum_{i=1}^{N} c_i e^{\lambda_i t} v_i \quad \text{— (2)}$$

λ_i = eigen ~~vectors~~ values of $\underline{\underline{A}}$

v_i = eigen vectors.

c_i = constant multipliers.

$$\frac{d\underline{X}}{dt} = \frac{d}{dt}\left(\sum_{i=1}^{N} c_i e^{\lambda_i t} v_i\right)$$

$$= \sum_{i=1}^{N} c_i \underline{v}_i \frac{d}{dt} e^{\lambda_i t}$$

$$= \sum_{i=1}^{N} c_i \underline{v}_i \lambda_i e^{\lambda_i t}$$

$$= \cancel{\sum c_i v_i} = \sum_{i=1}^{N} c_i e^{\lambda_i t} (\lambda_i \underline{v}_i) \quad \text{— (3)}$$

eigen values

eigen vector

$$\underline{\underline{A}}\,\underline{v}_i = \lambda_i \underline{v}_i \longrightarrow (4)$$

$$\frac{d\underline{x}}{dt} = \sum_{i=1}^{N} c_i e^{\lambda_i t} (\underline{\underline{A}}\, \underline{v}_i)$$

$$= \sum_{i=1}^{N} \underline{\underline{A}}\left[c_i e^{\lambda_i t} \underline{v}_i\right]$$

$$= \underline{\underline{A}} \sum_{i=1}^{N} c_i e^{\lambda_i t} \underline{v}_i$$

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x} \quad (5)$$

Example =

$$\left.\begin{aligned} \frac{dx_1}{dt} &= -2x_1 - 4x_2 + 2x_3 \\ \frac{dx_2}{dt} &= -2x_1 + x_2 + 2x_3 \\ \frac{dx_3}{dt} &= 4x_1 + 2x_2 + 5x_3 \end{aligned}\right\} \text{1st order autonomous coupled}$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x}$$

$\lambda_1 = 3$ | $v_1 = [2 \quad -3 \quad -1]'$

$\lambda_2 = -5$ | $v_2 = [2 \quad -1 \quad 1]'$

$\lambda_3 = 6$ | $v_3 = [1 \quad 6 \quad 16]'$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 e^{3t}\begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} + c_2 e^{-5t}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{6t}\begin{bmatrix} 1 \\ 6 \\ 16 \end{bmatrix}$$

If we know initial conditions

then C_1, C_2, C_3 can be determined.

Assume at $t=0$, $x_1(0) = x_2(0) = x_3(0) = 0$

$c_1 =$

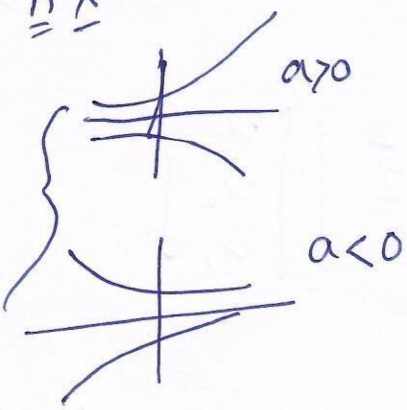
$C_2 =$ <u>Do it Yourself.</u>

$C_3 =$

<u>Phase plane analysis</u>

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x}$$

$$\frac{dx}{dt} = ax$$



eigen value of $\frac{dx}{dt}$

$\hookrightarrow$ bifurcation at 'a'

Case 2 If $a > 0$; $b < 0$ $\longrightarrow$ DIY

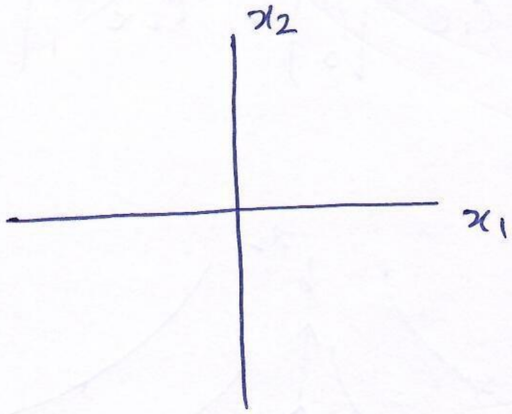
Saddle solution

One direction is stable other unstable

Case 3

$a > 0$ $b > 0$ $\longrightarrow$ DIY

$a > b > 0$



Both axes unstable

case 4

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{at} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{bt} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$a > b > 0$

$$\lim_{t \to \infty} e^{at} \to \infty$$

$$\lim_{t \to \infty} e^{bt} \to \infty$$

$x_1 = c_1 e^{at} \Rightarrow dx_1 = c_1 a e^{at}$

$x_2 = c_2 e^{bt} \Rightarrow dx_2 = c_2 b e^{bt}$

$$\cancel{\frac{dx_2}{dx_1} = \frac{c_1 b}{c_2 b} e^{a-b}} \qquad \frac{dx_2}{dx_1} = \left(\frac{c_2 b}{c_1 a}\right) e^{\underbrace{(b-a)t}_{\text{slope}}}$$

now we have idea about how slope varies with time

$b - a < 0$

In case of ~~A~~ 2^{nd} order system $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\frac{d\underline{X}}{dt} = \underline{A}\,\underline{X}$

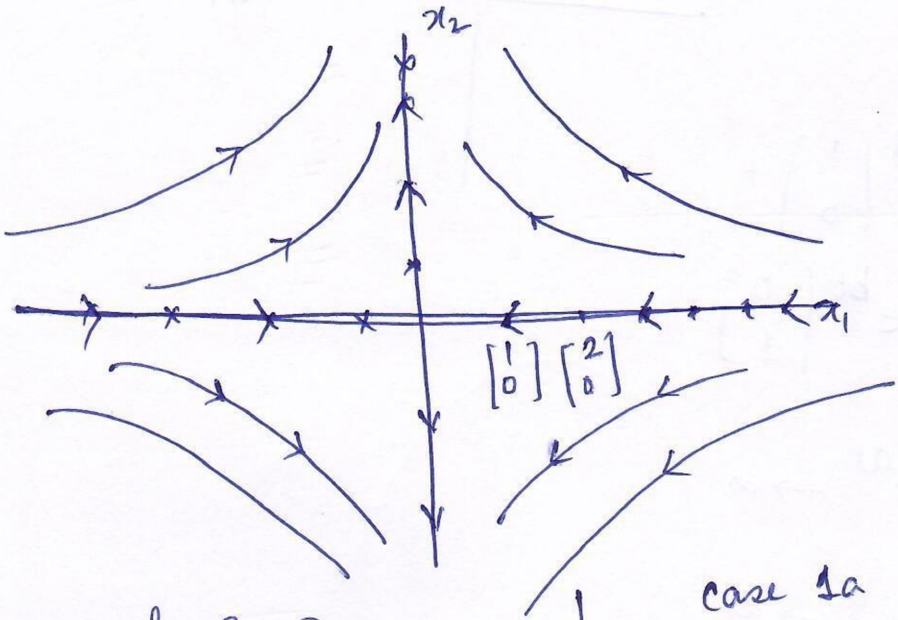
let $\underline{\underline{A}} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ ~~diag matrix~~

(diag. matrix)

$\lambda_1 = a \qquad v_1 = [1 \quad 0]'$

$\lambda_2 = b \qquad v_2 = [0 \quad 1]'$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{at}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{bt}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



direction of Arrow gives time evolution.

at $t=0$; for $c_2=0$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

at $t=0$ for $c_1=0$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

case 1a

at $t = t$

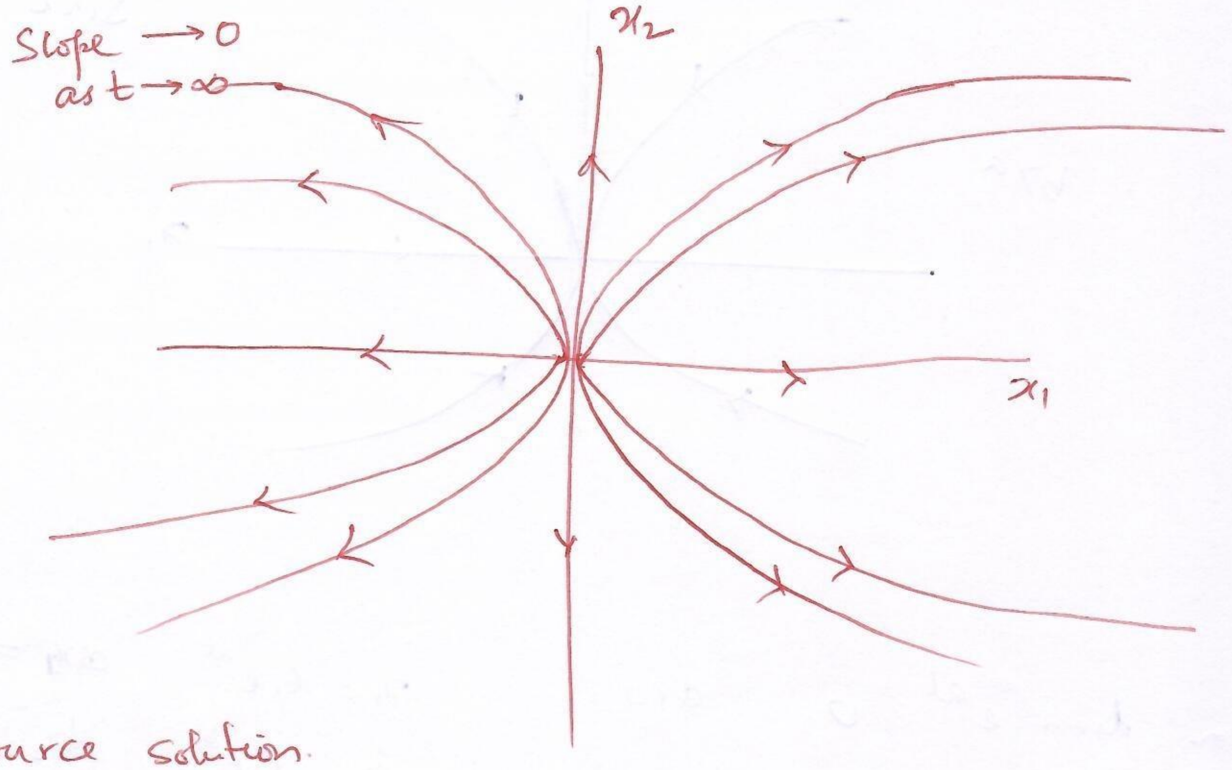
$\boxed{a<0 \; ; \; b>0}$

$\lim_{t\to\infty} e^{at} = 0$ (stable)

$\lim_{t\to\infty} e^{bt} \to \infty$ (unstable)

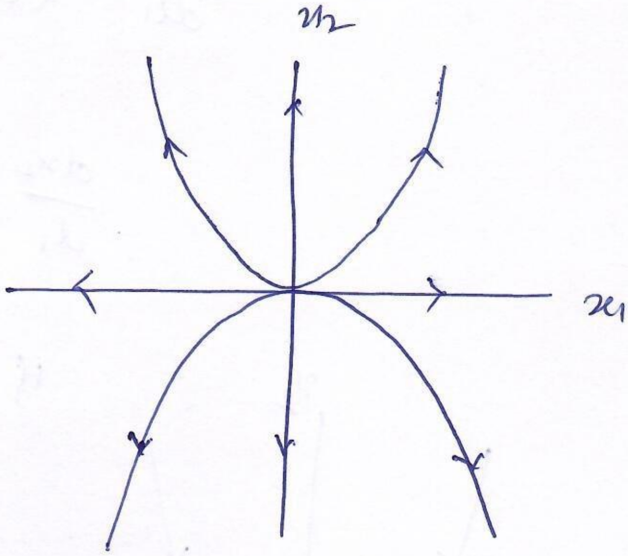
at equlibrium

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$



Source solution.

Case 5 $\quad b > a > 0$

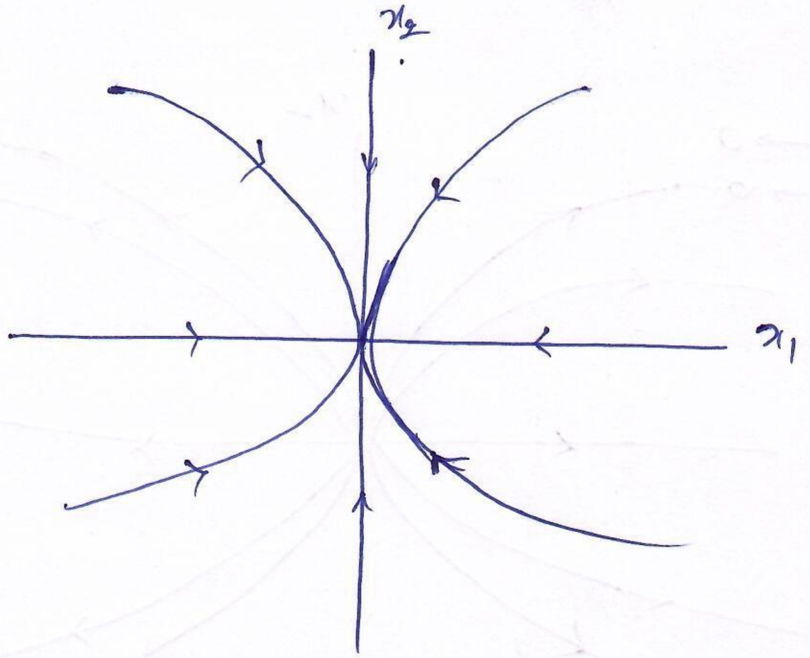


Case 6

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$a < 0 \qquad b < 0$

Sink solution



$b > a$

both axis are stable

$\lim_{t\to\infty} e^{at} = 0$ $\quad a < 0$

$\lim_{t\to\infty} e^{bt} = 0$ $\quad b < 0$

$x_1 = c_1 e^{at}$ $\quad dx_1 = c_1 a e^{at}$

$x_2 = c_2 e^{bt}$ $\quad dx_2 = c_2 b e^{bt}$

$$\frac{dx_2}{dx_1} = \left(\frac{c_2 b}{c_1 a}\right) e^{(b-a)t}$$

if $b > a$

$$\frac{dx_2}{d_1} \to \infty$$

if $a > b$