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<u>Advanced Heat Transfer (CH61014)</u>

<u>Solution of Heat Conduction Problems by Integral Transform</u>

<u>Fourier Transforms in the Semi-Infinite and Infinite Regions</u>

We have discussed the **Finite Fourier Transforms** in the interval (0, L) which can be used for solution of steady-state/transient heat conduction problems on **Finite Domains** (finite regions). For problems posed in the **Semi-Infinite** region (0, ∞), we need transforms in the semi-infinite interval. Let us now discuss **Semi-Infinite Fourier Transform** for solution of steady-state/transient heat conduction problems on **Semi-Infinite Domain**.

<u>What is Semi-Infinite Region?</u>

A semi-infinite solid extends to infinity in all but one direction. It is a system that is bounded by one surface and that extends to infinity in other directions. As a result, a single identifiable surface characterizes a semi-infinite solid as shown below (Fig. 1).

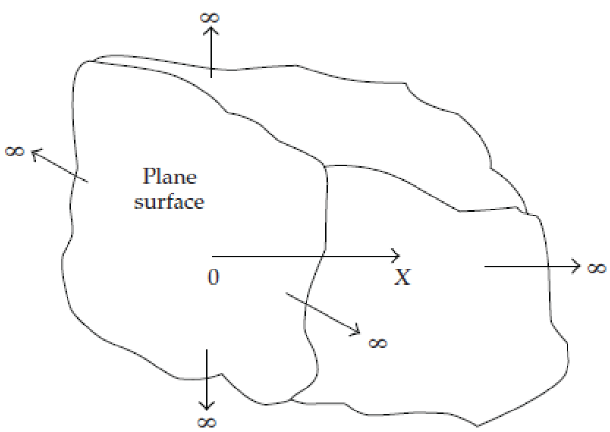

Figure 1: Schematic of a semi-infinite body.

<u>Note</u>: There is no such thing as a semi-infinite geometry in reality, it is only an idealization. If the edges of a real body are far enough from the point of interest on the surface for the time period of interest, the propagating energy front does not know that the material is actually finite. The process, therefore, works as if the body was infinite. This would be true for thin materials over short times or thick materials over longer times. For short periods of time, most bodies can be modelled as semi-infinite solids since heat does not have sufficient time to penetrate deep into the body, and the thickness of the body does not enter into the heat transfer analysis. A steel block can be treated as a semi-infinite solid when it is quenched rapidly to harden its surface. A body whose surface is heated by a laser pulse can be treated the same way.

The semi-infinite solid provides a useful idealization for certain types of practical problems in transient heat conduction. For example,

a) The earth, for example, can be considered to be a semi-infinite medium in determining the variation of temperature near its surface.

b) Also, a thick wall (finite solid) can be modelled as a semi-infinite medium if we are only interested in is the variation of temperature in the region near one of the surfaces, and the other surface is too far to have any impact on the region of interest during the time of observation.

c) A thermal burn occurs as a result of an elevation in tissue temperature above a threshold value for a finite period of time. The values of both the absolute temperature and the exposure time are crucial in determining the extent of injury. Ignoring the effects of blood flow and other physiological changes, we may analyze the temperature history and thermal injury for up to a third-degree burn, using semi-infinite region approximation.

For the slab application, the approximation would be reasonable for the early portion of the transient near the surface. In many applications, if a thermal change is imposed at the surface, a one-dimensional temperature wave will be propagated by heat conduction within the semi-infinite solid. Note that this idealized body is used to indicate that the temperature change in the part of the body in which we are interested (the region close to the surface) is due to the thermal conditions on a single surface.

<u>Question:</u> How will you solve the following problem (Fig. 2)? Can you solve it by Separation of variables?

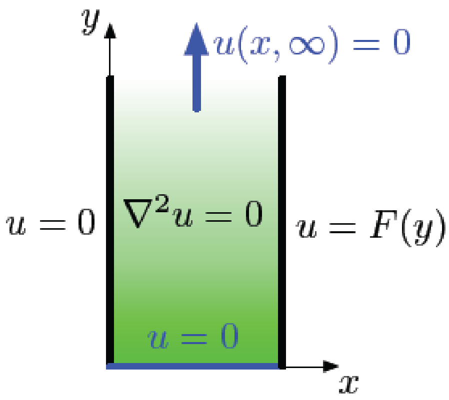

Figure 2

<u>Fourier Transforms in the Semi-Infinite Region: Fourier Sine Transform</u>

The Finite Fourier transforms in the interval (0, *L*) can be used for the solution of various heat conduction problems in finite regions. For problems posed in the semi-infinite region we need transforms in the semi-infinite interval (0, ∞). Such transforms can be obtained from finite Fourier transforms by taking limits as $L \to \infty$.

Recall the Finite Fourier Sine Transform of function *f*(*x*) as

$$\bar{f}_n = \sqrt{\frac{2}{L}} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx \tag{1}$$

with inversion formula as

$$f(x) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} \bar{f}_n \sin \frac{n\pi}{L} x \, dx \tag{2}$$

Combining the above two expressions, we have obtained which is the Fourier sine expansion of *f*(*x*) in the interval (0, L).

$$f(x) = \frac{2}{L}\sum_{n=1}^{\infty}\left\{\int_0^L f(x')\sin\frac{n\pi}{L}x'\,dx'\right\}\sin\frac{n\pi}{L}x \tag{3}$$

If we let $\Delta\omega = \pi/L$, then Eq. (3) can be written as

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\int_0^L f(x')\sin(n\Delta\omega)x'\,dx'\right\}\sin(n\Delta\omega)x\Delta\omega \tag{4}$$

$$f(x) = \frac{2}{\pi}\int_0^L f(x')\left\{\sum_{n=1}^{\infty}\sin(n\Delta\omega)x'\sin(n\Delta\omega)x\Delta\omega\right\}dx' \tag{5}$$

Now consider the infinite integral,

$$\int_0^{\infty}\sin\omega x'\sin\omega x\,d\omega \equiv \lim_{\omega\to 0}\sum_{n=1}^{\infty}\sin(n\Delta\omega)x'\sin(n\Delta\omega)x\Delta\omega \tag{6}$$

Now as we take limit $L \to \infty$ (or equivalently $\Delta\omega \to 0$), Eq. (5) can be written as

$$f(x) = \frac{2}{\pi}\int_0^{\infty} f(x')\int_0^{\infty}\sin\omega x'\sin\omega x\,d\omega\,dx', \qquad x > 0 \tag{7}$$

If we switch the order of integration in Eq. (7), we will get

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\sin\omega x\int_0^{\infty} f(x')\sin\omega x'\,dx'\,d\omega,\ x > 0 \tag{8}$$

This is called the Fourier Sine Integral representation of $f(x)$. This represents $f(x)$ at points of continuity in the interval $(0, \infty)$ and converges to the mean value $1/2\,[f(x+) + f(x-)]$ at points where $f(x)$ has finite jumps.

NOTE: Eq. (8) represents $-f(-x)$ when $x < 0$. Thus, if $f(x)$ is an *odd* function, Eq. (8) is valid for all values of x.

Now, if we let

$$\bar{f}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x \, dx \tag{9}$$

then Eq. (8) can be rewritten as

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{f}(\omega) \sin \omega x \, d\omega \tag{10}$$

Equation (9) is known as the **Fourier Sine Transform** of *f*(x) in the **Semi-Infinite Interval** (0, ∞), and Eq. (10 is the corresponding **Inversion Formula**. The kernel of this transform is

$$K(\omega, x) = \sqrt{\frac{2}{\pi}} \sin \omega x \tag{11}$$

which satisfies the following characteristic-value problem (CVP) with $\beta = 0$:

$$\frac{d^2 y}{dx^2} + \omega^2 y = 0$$

$$\alpha y(0) + \beta \frac{dy(0)}{dx} = 0, \; \alpha^2 + \beta^2 \neq 0$$

$$|y(x)| \leq M \text{ for } x > 0 \tag{12}$$

Here *M* is some finite constant.

<u>NOTE</u>: Here, the characteristic values, ω, are continuous from zero to ∞ rather than discrete.

Fourier Transforms in the Semi-Infinite Region: Fourier Cosine Transform

In a similar way, we can obtain the Fourier cosine integral representation of $f(x)$.

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x \int_0^\infty f(x')\cos\omega x'\, dx'\, d\omega,\ x > 0 \tag{13}$$

NOTE: Eq. (13) represents $f(-x)$ when $x < 0$. Therefore, if $f(x)$ is an *even* function, then Eq. (13) is valid for all values of x.

From the Fourier cosine integral representation of $f(x)$, Eq. (13), the **Fourier Cosine Transform** of $f(x)$ in the **Semi-Infinite Interval** $(0, \infty)$ can be written as

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{f}(\omega)\cos\omega x\, d\omega \tag{14}$$

The corresponding inversion formula is written as

$$\bar{f}(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\omega x\, dx \tag{15}$$

The kernel of this transform is

$$K(\omega, x) = \sqrt{\frac{2}{\pi}}\cos\omega x \tag{16}$$

which satisfies the characteristic-value problem (Eq. 12) with $\alpha = 0$.

NOTE: Another **Fourier Transform in the Semi-Infinite Interval** $(0, \infty)$ can be developed in a similar way by considering the case where the kernel satisfies the characteristic-value problem (Eq. 12) for both $\alpha \neq 0$ and $\beta \neq 0$. Table 1 summarizes these transforms.

Table 1: Kernels for use in Fourier Transforms in the Semi-Finite Interval

Kernels For Use in Fourier Transforms in the Semi-Finite Interval

Transform: $\bar{f}(\omega) = \int_0^\infty f(x)K(\omega, x)dx$

Inversion: $f(x) = \int_0^\infty \bar{f}(x)K(\omega, x)d\omega$

$$\left\{ \begin{array}{c} \dfrac{d^2y}{dx^2} + \omega^2 y = 0 \\ \alpha y(0) + \beta \dfrac{dy(0)}{dx} = 0 \\ |y(x)| \le M \text{ for } x > 0 \end{array} \right\}$$

Boundary condition at $x = 0$	Kernel, $K(\omega, x)^\dagger$
Third kind ($\alpha \ne 0, \beta, \ne 0$)	$\sqrt{\dfrac{2}{\pi}} \dfrac{\omega \cos \omega x - H \sin \omega x}{\sqrt{\omega^2 + H^2}}$
Second kind ($\alpha_1 = 0, \beta \ne 0$)	$\sqrt{\dfrac{2}{\pi}} \cos \omega x$
First kind ($\alpha_1 \ne 0, \beta, = 0$)	$\sqrt{\dfrac{2}{\pi}} \sin \omega x$

† $H = \alpha/\beta$.

Fourier Transforms in the Infinite Region:

If the function $f(x)$ is defined for all values of x, then write it as a sum of odd function and even function as follows.

$$f(x) = \underbrace{\frac{1}{2}\left[f(x) + f(-x)\right]}_{f_e(x)} + \underbrace{\frac{1}{2}\left[f(x) - f(-x)\right]}_{f_o(x)}, \; -\infty < x < \infty \tag{17}$$

Such treatment will finally lead to the following function $f(\omega)$ which is the **Fourier Transform** of $f(x)$ in the **Infinite Interval** $-\infty < x < \infty$ (Eq. 18).

$$\overline{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x'} f(x') dx', \ -\infty < \omega < \infty$$

(18)

The corresponding **Inversion Formula** is given by Eq. (19).

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \overline{f}(\omega) d\omega, \ -\infty < x < \infty$$

(19)