GENERAL THEORY OF ACADEMICS

42 SOLVED PREVIOUS YEAR PAPERS

THE Cookie PEOPLE

SHARP COOKIE

GENERAL THEORY OF ACADEMICS

(FIRST EDITION)

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*Since the authors are themselves students in particular and human beings in general
It should clearly resonate that the book is not error proof. In case of any doubts or ambiguities
get in touch with your professors.
*Some questions were not available and has been made from recollection only.
We deeply apologize for any other inconvenience caused.

ACKNOWLEDGEMENT

This book is a cumulative effort of lot of individuals. Practically it is not possible to include each of the concerned person's names but we extend our heartiest gratitude to each individual involved in the process. To all our beloved professors whose constant guidance and motivation facilitated the making of the book and to our lovely parents for being so approving, ***THANK YOU.***

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Preface

Students have been solving previous year question papers to prepare for exam throughout their school life. Solving previous year questions doesn't only help in revision but also boosts confidence. So why stop after school? This was the question in the authors' minds while starting this project. But unsupervised practice only embeds one's errors. So it is necessary to have a solution guide for checking the answers and also clear any misconceptions. Many freshers in IIT Kharagpur have felt the need of such a question bank-solution guide. This book was launched to fill this void.

This book is a compilation of 42 question papers and the solutions of previous semesters of IIT Kharagpur. It covers question papers of Physics, Maths, Electrical Technology, Chemistry, Programming and Data Structures and Mechanics of the 1st year course structure.

This book is meant for every 1st year student in IIT Kharagpur looking for well organized solutions for previous year question papers. Solving papers enhances a student's perspective about the pattern of the upcoming exams.Different exams give different weightage to different topics. This can be tackled by solving previous year question papers. This book is also meant for last-minute revision as each paper consists of questions and their solutions from the important topics.

THE DASSI FUNDAE ...

- *Always keep in mind that in absolute marking, one who gets 90 and one who gets 100 are same and equivalently in relative marking. To score a high CGPA, you have to score more no of EX rather than getting more no of full marks. So do not focus unnecessarily.*

- *This is not your home where you will get tutors from your coaching centre, your home-tutors and several others. Here professors are the only hope. So never miss classes as there will be none to help you. Be physically and mentally present at the class.*

- *Leaving some exceptions, professors are always quite friendly and approachable. Sometimes they are more like friends than teachers. They are always to help you. So never miss a chance to exploit their favour.*

- *This is not JEE where you have to solve hundreds of problems. Solve your tutorial sheets first to get an idea of the chapters and then jump for this book. There is no need of solving several problem books.*

MAKE YOU CONCEPTS CLEAR
AND
BEAT THE EXAMS!!!

CONTENTS

*KEY CONCEPT AND FORMULAE

MATHS I

MID SEMESTER **2017-18**

1. (a) Establish the following inequality using Mean Value Theorem:

$$(1-x) < e^{-x} < 1 - x + \frac{x^2}{2},\ x > 0.$$

(b) Let $f : [a, b] \longrightarrow \mathbb{R}$ be a differentiable function and $a > 0$. Using Cauchy's Mean Value Theorem show that there exist $c_1, c_2 \in (a, b)$ such that

$$\frac{f'(c_1)}{a+b} = \frac{f'(c_2)}{2c_2}.$$

(c) Find the value of a if the coefficient of x^{100} in the Maclaurin's series expansion of $(ax+5)e^x$ is $\frac{80}{100!}$, where $a \in \mathbb{R}$.

Ans 1a): Let $f(x) = (1-x) - e^{-x}$
$f'(x) = -1 + e^{-x} < 0$
$\Rightarrow f$ is a decreasing function for $x > 0$
Also $f(0) = 0$
$\therefore f(x) < f(0) \Rightarrow (1-x) < e^{-x}$ . . . . . . (1)
Let $g(x) = e^{-x} - (1 - x + \frac{x^2}{2})$
$g'(x) = -e^{-x} - (-1+x) \quad < 0$ [from (1)]
$\Rightarrow g$ decreasing for $x > 0$
Also $g(0) = 0$
$\therefore g(x) < g(0) = 0$
$\Rightarrow e^{-x} < 1 - x + \frac{x^2}{2}$
Hence,
$(1-x) < e^{-x} < (1 - x + \frac{x^2}{2})$ for $x > 0$

Ans 1b): By MVT we have,
$\frac{f(b)-f(a)}{b-a} = f'(c_1)$ . . . . . (1) for $c_1 \in (a, b)$
Let $g(x) = x^2$
Now by applying Cauchy's MVT on $f(x)$ and $g(x)$ we say that $\exists\ c_2 \in (a, b)$ such that,
$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c_2)}{g'(c_2)}$
$\Rightarrow \frac{f(b)-f(a)}{b^2-a^2} = \frac{f'(c_2)}{2c_2}$
$\Rightarrow \frac{f'(c_1)}{b+a} = \frac{f'(c_2)}{2c_2}$ [by applying (1)]

Ans 1c): $f(x) = (ax+5)e^x$
$f'(x) = (ax+5+a)e^x$
$f'(x) = (ax+5+2a)e^x$
$\vdots$

$f^n(x) = (ax + 5 + na)e^x$
$\therefore f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots$
Co-efficient of $x^{100} = \frac{80}{100!}$
$\Rightarrow \frac{5 + 100a}{100!} = \frac{80}{100!}$
$\Rightarrow a = \frac{80 - 5}{100} = \frac{75}{100} = 0.75$

2. (a) Using $\epsilon - \delta$ method, discuss the continuity of the following function at the point $(0, 0)$:

$$f(x,y) = \begin{cases} \frac{\sin^2(x-y)}{|x|+|y|}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

(b) Expand $f(x, y) = x^2 - 5xy + 2y^2$ in the powers of $(x - 1)$ and $(y - 1)$ using Taylor's theorems for two variables.

(c) Using Euler's theorem, find $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$ for the function $f(x, y) = e^{x^2+y^2}$.

Ans 2a): $f(x,y) = \begin{cases} \frac{\sin^2(x-y)}{|x|+|y|}, & f(x,y) \neq (0,0) \\ 0 \quad , & f(x,y) = (0,0) \end{cases}$

$|f(x, y) - f(0, 0)|$
$= \left|\frac{\sin^2(x-y)}{|x|+|y|} - 0\right|$
$\leq \left|\frac{(x-y)^2}{|x|+|y|}\right|$
$\leq \frac{|x-y||x-y|}{|x|+|y|}$
$\leq |x| + |y|$
$< 2\sqrt{x^2 + y^2}$
$< 2\delta < \epsilon$
$\therefore \delta < \frac{\epsilon}{2}$
Hence $f(x, y)$ is continuous at (0,0).

Ans 2b): $f(x, y) = x^2 - 5xy + 2y^2$
$f(1, 1) = -2$
$f_x(1, 1) = -3$
$f_y(1, 1) = -1$
$f_{xx}(1, 1) = 2$
$f_{xy}(1, 1) = -5$
$f_{yy}(1, 1) = 4$

$F(x, y) = -2 - 3(x - 1) - (y - 1) + (x - 1)^2 - 5(x - 1)(y - 1) + 2(y - 1)^2$

Ans 2c): $f(x, y) = e^{x^2+y^2}$
$u(x, y) = \ln f = x^2 + y^2$
$u(x, y)$ is homogeneous function of degree 2.
By using Euler's theorem

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u$$
$$x\frac{1}{f}\frac{\partial f}{\partial x}+y\frac{1}{f}\frac{\partial f}{\partial y}=2u$$
$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=2f\ln(f)$$

3. (a) For the function

$$f(x,y)=\begin{cases}2x^2\tan^{-1}\left(\frac{y}{3x}\right)-3y^2\tan^{-1}\left(\frac{x}{2y}\right), & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0),\end{cases}$$

compute $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ and $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ at the origin $(0,0)$.

(b) Let z be the function of u and v while $u=x^2-y^2-2xy$ and $v=y$. Given the relation

$$(x+y)\frac{\partial z}{\partial x}+(x-y)\frac{\partial z}{\partial y}=y(x-y),$$

find $\frac{\partial z}{\partial v}$ and $\frac{\partial^2 z}{\partial v^2}$ at the point $(x,y)=(1,2)$.

(c) Let

$$f(x,y)=\begin{cases}x^{5/2}\sin\left(\frac{1}{\sqrt{x}}\right)+y^{5/2}\sin\left(\frac{1}{\sqrt{y}}\right), & x\neq 0 \text{ and } y\neq 0\\ 0, & \text{elsewhere.}\end{cases}$$

i. Find $f_x(0,0)$ and $f_y(0,0)$.

ii. Write the definition of differentiability of a function of two variables at a point $(x_0,\, y_0)$.

iii. Using the definition, discuss the differentiability of $f(x,y)$ at the orgin $(0,0)$.

ANS **3 (a)**

$$f_y(0,0)=\lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{0-0}{\Delta y}=0$$

$$f_y(x,0)=\lim_{\Delta y\to 0}\frac{f(x,\Delta y)-f(x,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{2x^2\tan^{-1}\left(\frac{\Delta y}{3x}\right)-3\Delta y^2\tan^{-1}\left(\frac{x}{2\Delta y}\right)}{\Delta y}$$

$$=\lim_{\Delta y\to 0}\frac{2x\tan^{-1}\left(\frac{\Delta y}{3x}\right)}{3\left(\frac{\Delta y}{3x}\right)}=\frac{2}{3}x$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\lim_{\Delta x\to 0}\frac{f_y(\Delta x,0)-f_y(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\frac{2}{3}\Delta x-0}{\Delta x}=\frac{2}{3}$$

and

Similarly,

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\frac{3}{2}$$

(b) Compute

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = 2(x-y)\frac{\partial z}{\partial u}$$

and

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = -2(x+y)\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

Substituting these in the given relation:

$$2(x^2-y^2)\frac{\partial z}{\partial u} - 2(x^2-y^2)\frac{\partial z}{\partial u} + (x-y)\frac{\partial z}{\partial v} = y(x-y)$$

This implies

$$\frac{\partial z}{\partial v} = y, \qquad \text{at the points } x \neq y$$

Hence

$$\left.\frac{\partial z}{\partial v}\right|_{(1,2)} = 2$$

Further note that

$$\frac{\partial z}{\partial v} = y = v \qquad \Rightarrow \qquad \frac{\partial^2 z}{\partial v^2} = 1$$

(c) (i) $$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0-0}{\Delta x} = 0$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0-0}{\Delta y} = 0$$

(ii) The function $f(x,y)$ is said to be differentiable at (x_0, y_0) if

$$\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) = a\,\Delta x + b\,\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where ϵ_1 & ϵ_2 tend to 0 when $(\Delta x, \Delta y) \to (0,0)$ and a, b are independent of Δx and Δy.

OR

The function $f(x,y)$ is said to be differentiable at (x_0, y_0) if

$$\lim_{\Delta\rho \to 0} \frac{\Delta z - dz}{\Delta \rho} = 0$$

(iii) $$f(\Delta x, \Delta y) - f(0,0) = f(\Delta x, \Delta y) = \Delta x^{\frac{5}{2}} \sin\left(\frac{1}{\sqrt{\Delta x}}\right) + \Delta y^{\frac{5}{2}} \sin\left(\frac{1}{\sqrt{\Delta y}}\right)$$

$$= 0 \cdot \Delta x + 0 \cdot \Delta y + \Delta x \left(\Delta x^{\frac{3}{2}} \sin\left(\frac{1}{\sqrt{\Delta x}}\right)\right) + \Delta y \left(\Delta y^{\frac{3}{2}} \sin\left(\frac{1}{\sqrt{\Delta y}}\right)\right)$$

Thus $$\epsilon_1 = \Delta x^{\frac{3}{2}} \sin\left(\frac{1}{\sqrt{\Delta x}}\right), \qquad \text{note that } \epsilon_1 \to 0, \qquad \text{as } (\Delta x, \Delta y) \to 0$$

$$\epsilon_2 = \Delta y^{\frac{3}{2}} \sin\left(\frac{1}{\sqrt{\Delta y}}\right), \quad \text{note that} \quad \epsilon_2 \to 0, \quad \text{as} \quad (\Delta x, \Delta y) \to 0$$

Or one can use the definition 2 to show the differentiability.

4. (a) Using L'Hospital rule, evaluate $\lim_{x\to+\infty}\left[e^{(x+e^{-x})} - e^{x}\right]$.

(b) Find all the critical points of $f(x,y) = (x^2 - y^2)e^{(-x^2-y^2)/2}$. Further classify these points for local maximum, local minimum and saddle points.

(c) Using the method of Lagrange multipliers, find the absolute maximum and minimum of the function $f(x,y) = 2x^3 + y^4$ in the domain $D = \{(x,y) :\ x^2 + y^2 \le 1\}$.

Ans 4a): $\lim_{x\to+\infty}\left[e^{x+e^{-x}} - e^{x}\right]$ $\qquad (\infty - \infty \text{ form})$

$= \lim_{x\to+\infty} \dfrac{e^{e^{-x}} - 1}{e^{-x}}$ $\qquad (\frac{0}{0} \text{ form})$

$= \lim_{x\to+\infty} \dfrac{e^{e^{-x}}(-e^{-x})}{(-e^{-x})}$

$= \lim_{x\to+\infty} e^{e^{-x}} = e^0 = 1$

Ans 4b): $\dfrac{\partial f}{\partial x} = \big[2x - x(x^2 - y^2)\big]e^{(-x^2-y^2)/2}$ and

$\dfrac{\partial f}{\partial y} = \big[-2y - y(x^2 - y^2)\big]e^{(-x^2-y^2)/2}$

So critical points are solutions of the simultaneous equations.

$2x - x(x^2 - y^2) = 0$ and $-2y - y(x^2 - y^2) = 0$

i.e. $x[2 - (x^2 - y^2)] = 0$ and $y[-2 - (x^2 - y^2)] = 0$

They have solutions, (0,0) , ($\pm\sqrt{2}$,0) , (0,$\pm\sqrt{2}$)

Here, $\quad f_{xx} = (2 - 5x^2 + x^2(x^2 - y^2) + y^2)e^{(-x^2-y^2)/2}$

$f_{yy} = (5y^2 - 2 + y^2(x^2 - y^2) - x^2)e^{(-x^2-y^2)/2}$

$f_{xy} = xy(x^2 - y^2)e^{(x^2-y^2)/2}$

Taking D $= [f_{xx}(x_o, y_o)] \times [f_{yy}(x_o, y_o)] - [f_{xy}(x_o, y_o)]^2$

We calculate,

	Point	f_{xx}	f_{xy}	f_{yy}	D	Type
(i)	$(0,0)$	2	0	-2	-4	Saddle
(ii)	$(\pm\sqrt{2},0)$	$-4/e$	0	$-4/e$	$16/e^2$	Max
(i)	$(0,\pm\sqrt{2})$	$4/e$	0	$4/e$	$16/e^2$	Min

Ans 4c): <u>Case I</u>: Here $f_x = 6x^2, f_y = 4y^3$. So (0,0) is a candidate for absolut maximum or minimum inside $\{(x,y)|x^2 + y^2 < 1\}$

<u>Case II</u>: Let us find max^m or min^m value of $f(x,y)$ such that $\quad g(x,y) = (x^2 + y^2) = 1$

Consider $L(x,y,\lambda) = 2x^3 + y^4 + \lambda(x^2 + y^2 - 1)$

Here $L_x = 6x^2 + 2\lambda x \quad , \quad L_y = 4y^3 + 2\lambda y \quad , \quad L_\lambda = x^2 + y^2 - 1$

$L_x = 0 \ \Rightarrow x(3x + \lambda) = 0 \quad , \quad L_y = 0 \ \Rightarrow y(2y^2 + \lambda) = 0 \quad , \quad L_\lambda = 0 \ \Rightarrow x^2 + y^2 - 1 = 0$

If $x = 0 \ , \ y = \pm 1 \ , \ \lambda = -2$

If $y = 0 \ , \ x = \pm 1 \ , \ \lambda = \pm 3$

If $x \neq 0 \ , \ y \neq 0$ then $\ 3x = 2y^2 \ \Rightarrow y^2 = \dfrac{3}{2}x$,

Now, $x^2 + y^2 = 1 \ \Rightarrow x^2 + \frac{3}{2}x = 1 \ \Rightarrow x = -2$ or $x = \frac{1}{2}$
$x = -2$ is discarded
$\therefore x = \frac{1}{2}$ and$y = \pm\frac{\sqrt{3}}{2}$
Here, $f(0, \pm 1) = 1$, $f(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}) = \frac{13}{16}$, $f(-1, 0) = -2$, $f(1, 0) = 2$
$\therefore$ Absolute maximum at (1,0) and value is 2
and absolute minimum is at (-1,0) and value is -2.

MATHS I

MID-SPRING SEMESTER **2016**

1. (a) Let

$$f(x,y) = \begin{cases} \dfrac{x^\alpha y^\beta}{x^2+xy+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Discuss the conditions on α, $\beta \in \mathbb{R}$ under which the function $f(x,y)$ is
(i) continuous at $(0,0)$, and
(ii) discontinuous at $(0,0)$.

[3 marks]

(b) For the function

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0), \end{cases}$$

compute $\dfrac{\partial f}{\partial x}(x,y)$, $\dfrac{\partial f}{\partial y}(x,y)$ and discuss the continuity of $\dfrac{\partial f}{\partial x}$ at $(0,0)$.

[2 marks]

Ans 1a): $f(x,y) = \dfrac{x^a y^b}{x^2+xy+y^2}$

We need to check for continuity at $(x,y) \Rightarrow (0,0)$

Hence let $x = r\cos\theta \qquad y = r\sin\theta \qquad$ where $r \to 0$

$$\lim_{r\to 0} r^{a+b-2}\frac{(\cos\theta)^a(\sin\theta)^b}{1+\cos\theta\sin\theta}$$

$$\Rightarrow \lim_{r\to 0} r^{a+b-2}\left[\frac{(\cos\theta)^a(\sin\theta)^b}{1+\dfrac{\sin 2\theta}{2}}\right]$$

So if it must be continuous then conditions are $a+b \geq 2$ since r^{a+b-2} will make the function discontinuous.

Also the $[\ldots]$ part must be bounded. Since $1+\dfrac{\sin 2\theta}{2} \geq \dfrac{1}{2}$ it is bounded when $a \geq 0$, $b \geq 0$

Any other condition causes discontinuity.

Ans 1b): $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$

So $\left.\dfrac{df}{dx}\right|_{x=0} = \lim\limits_{h\to 0}\dfrac{f(h,0)-f(0,0)}{h} = 0$

$$\frac{df}{dx} = \lim_{h\to 0}\frac{\dfrac{(x+h)y}{\sqrt{(x+h)^2+y^2}} - \dfrac{(x)y}{\sqrt{x^2+y^2}}}{h}$$

$$= \frac{y^3}{(\sqrt{x^2+y^2})^3} \qquad \text{Similarly, } \frac{df}{dy} = \frac{x^3}{(\sqrt{x^2+y^2})^3}$$

2. (a) Let $f : [6, 15] \to \mathbb{R}$ be a differentiable function such that $f(6) = -2$ and $f'(x) \leq 10$ for all $x \in [6, 15]$. What is the largest possible value for $f(15)$?

[2 marks]

(b) Let $f : [-3, 3] \to \mathbb{R}$ be a differentiable function such that $f(-3) = -3$, $f(3) = 3$ and $|f'(x)| \leq 1$ for all $x \in [-3, 3]$. Find the value of $f(0)$. [2 marks]

Ans 2a): $\dfrac{f(15) - f(6)}{9} = f'(c)$

where c lies between (6,15) according to LMVT.

$\Rightarrow f(15) - f(6) \leq 9 \times (10)$ [since $f'(c) < 10$]

$\Rightarrow f(15) \leq 92$

92 is the max value.

Ans 2b): $\dfrac{f(3) - f(3)}{6} = f'(c)$ where c lies between (-3,3) according to LMVT.

$f'(c) = 1$

$\dfrac{f(3) - f(0)}{3} \leq 1 \Rightarrow 3 - f(0) \leq 3$ . . . . (1)

$\dfrac{f(0) - f(-3)}{3} \leq 1 \Rightarrow f(0) - 3 \leq 3$ . . . (2)

Hence $\Rightarrow f(0) = 0$ (from (1) and (2))

3. (a) Find all values of $t \in [0, 2\pi]$ such that the following equality holds

$$\lim_{x \to \infty} \left[1 + \frac{\log_e(1 + a^2)}{x}\right]^x = 2a\cos^2 t, \quad a > 0.$$

[2 marks]

(b) Find the first 4 terms (upto and including third order derivatives) of the Taylor's series expansion for the function $(x - 1)e^x$ in powers of $(x - 1)$ without remainder.

[2 marks]

Ans 3a): $\lim\limits_{x \to \infty} \left(\dfrac{1 + \ln(1 + a)}{x}\right)^x$

$= e^{x.\dfrac{ln(1 + a^2)}{x}}$

$= 1 + a^2$

Now, $1 + a^2 = 2a\cos^2 t$

$\Rightarrow 1 + a^2 \geq 2a$ (by AM-GM since a>0)

$\Rightarrow \cos^2 t = 1$

$\Rightarrow t = n\pi \pm 0$

Ans 3b): $(x - 1)e^x = f(x)$

According to Taylor's Theorem

$f(x) \Rightarrow f(a) + (x - a)f'(a) + \dfrac{(x - a)^2}{2!} f''(a) + \ldots$

So compute the terms till $\dfrac{(x - a)^3}{3!} f'''(a)$ to get the series.

4. (a) Define differentiability of a function $f(x,y)$ at a point (a,b). Using this definition, discuss the differentiability at the point $(0,0)$ of the function

$$f(x,y) = \begin{cases} xy\left(\dfrac{x^2-y^2}{x^2+y^2}\right), & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

[2 marks]

(b) Calculate the total differential dz to approximate the change in $z = x^2+2xy-y^2$ as (x,y) moves from the point $(1,1)$ to the point $(1.01, 1.01)$. Compute the error in this approximation with the exact change in z. [2 marks]

Ans 4a): $f(x,y) = \begin{cases} (xy)\left(\dfrac{x^2-y^2}{x^2+y^2}\right), if \quad (x,y) \neq (0,0) \\ 0 \quad if \quad (x,y) = (0,0) \end{cases}$

According to definition :

$\lim\limits_{\triangle f \to 0} \dfrac{\triangle}{z} - dz \triangle f$ should exist

$= \lim\limits_{(\triangle x, \triangle y)\to(0,0)} \dfrac{\triangle}{x} . \triangle\, y(\triangle x^2 - \triangle y^2)(\triangle x^2 + \triangle y^2)^{3/2}$

$\Rightarrow$ Let $\triangle y = m \triangle x$

$= \dfrac{m(1-m^2)}{(1+m^2)^{3/2}} \longrightarrow$ depends on m

$\Rightarrow$ not differentiable.

Ans 4b): exact $= f(1.01, 1.01) - f(1,1) = 4.02 \times 10^{-2}$ Total $= fx.dx + fy.dy = 0.04$

error $= \dfrac{4.02-4}{4.02} \times 100 = \dfrac{0.02}{4.02} \times 100 = 0.5\%$

5. (a) For the function

$$f(x,y) = \begin{cases} \dfrac{(x^2y+xy^2)\sin(x-y)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0), \end{cases}$$

compute $\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)$ and $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$ at $(0,0)$. [2 marks]

(b) Suppose that $x^2y^2z^3 + zx\sin y = 5$ defines z as a function of x and y. Then using implicit differentiation find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

[2 marks]

Ans 5a): $f(x,y) = \begin{cases} \dfrac{(x^2y+xy^2).\sin(x-y)}{x^2+y^2} & (x,y) \neq (0,0,) \\ 0 \end{cases}$

$\left.\dfrac{\partial f}{\partial x}\right|_{(0,y)} = \lim\limits_{h\to 0} \dfrac{f(h,y)-f(0,y)}{h}$

$$= \lim_{h\to 0} \frac{(h^2y + hy^2)\sin(h-y)}{h^2+y^2}$$
$$= 0$$
$$\left.\frac{\partial f}{\partial y}\right|_{0,0} = 0$$

Similarly solve the other.

Ans 5b): $x^2y^2z^3 + zx\sin y = 5$

$x^2y^2z^3 + zx\sin y - 5 = 0$

We have to find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$

$$\Rightarrow 2xy^2z^3 + 3x^2y^2x^2\frac{\partial z}{\partial x} + x\sin y\frac{\partial z}{\partial x} + z\sin y = 0$$
$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{2xy^2z^3 + z\sin y}{3x^2y^2z^2 + x\sin y}$$

Similarly solve the other part.

6. (a) Derive Taylor's series of $f(x,y) = \sin(xy)$ about the point $(x_0, y_0) = (1, \pi/2)$ upto and including second order terms (without remainder). [2 marks]

(b) Let $u(x,y) = x\phi(y/x) + \psi(y/x)$. Using Euler's theorem on homogeneous functions, find the function f such that the following relation holds

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = f(x)\phi(y/x).$$

[2 marks]

Ans 6a): $f(x,y) = \sin(x,y)$

$$f(x,y) = \sin(\pi/2) + (x-1)\sin'(y) + (y-\pi/2)\sin'(x\pi/2) + \frac{(x-1)^2}{2!}\sin''(y) + \frac{(y-\pi/2)^2}{2}\sin''(x\pi/2)$$
$$+2(x-1)(y-\pi/2)\frac{\partial}{\partial x}(\sin'(x\pi/2))$$

Ans 6b): $u(x,y) = x\phi(y/x) + \psi(y/x)$

$$\frac{\partial u}{\partial x} = \phi(y/x) + \phi'(y/x)(\frac{-1}{x})y + \psi'(y/x)(\frac{-1}{x^2})y$$
$$\frac{\partial u}{\partial y} = x\phi'(y/x)\frac{1}{x} + \psi'(y/x)\frac{1}{x}$$
$$y.\frac{\partial u}{\partial y} = y.\phi'(y/x) + \frac{y}{x}\psi'(y/x)$$
$$x.\frac{\partial u}{\partial x} = x\phi(y/x) - \phi'(y/x).y + \psi'(y/x)(\frac{-1}{x})$$
$$\Rightarrow y.\frac{\partial u}{\partial y} + x\frac{\partial u}{\partial x} = x\phi'(y/x)$$

f(x) = x

By using Euler theorem,

ϕ and ψ are homogeneous functions of order 0. Hence,

$$x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} = 0 \qquad x.\frac{\partial \psi}{\partial x} + y.\frac{\partial \psi}{\partial y} = 0$$

$\Rightarrow$ So the only part that gets differentiated is x part .

So lets take $\phi(y/x) = k$

$f = xk$ [differentiation of 'k' in the form $x\dfrac{\partial k}{\partial x} + y\dfrac{\partial k}{\partial y} = 0$]

Hence, $x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}$
$= k.x+\left(x\dfrac{\partial k}{\partial x}+y\dfrac{\partial k}{\partial y}\right)$
$= kx$
$\Rightarrow f(x)=x$

7. (a) Find all the critical points of

$$f(x,y)=x^2(2-y)-y^3+3y^2+9y.$$

Further, classify these critical points for local maximum, local minimum and saddle points.

[3 marks]

(b) Determine the absolute maximum and minimum values of the function

$$f(x,y)=x^2+y^2,$$

subject to the constraint

$$(x-2)^2+(y-1)^2\leq 20.$$ [2 marks]

Ans 7a): $f=x^2(2-y)-y^3+3y^2+3y$
$\dfrac{\partial f}{\partial x}=0\Rightarrow 2x(2-y)=0$
$\Rightarrow x=0$ or $y=2$
$\dfrac{\partial f}{\partial y}=0\Rightarrow x^2(-1)-3y^2+6y+9=0$
for $x=0\Rightarrow\dfrac{\partial f}{\partial y}\Rightarrow -y^2+2y+3=0$
$-y^2+3y-y+3=0$
$(y+1)(y-3)=0$
$\Rightarrow y=3$; $y=1$
(0,3) , (0,1)
for $y=2\Rightarrow -18+12+9-x^2=0$
$\Rightarrow -x^2+3=0$
$\Rightarrow x=\pm\sqrt{3}$
(2,$\sqrt{3}$) , (2,-$\sqrt{3}$)
Check the values for maximum and minimum.

Ans 7b): $(x-2)^2+(y-1)^2\leq 20$
$(x-2)^2+(y-1)^1-20+\lambda(x^2+y^2)=z$
find $\dfrac{dz}{dx}=0$ and $\dfrac{dz}{dy}=0$
and then check for which pair it is maximum.

MATHS I

MID SEMESTER **2015**

1(a). For what value of α, does $\dfrac{(\sin 2x + \alpha \sin x)}{x^3}$ tend to a finite limit as $x \to 0$? [2 M]

1(b). Show that between any two roots of $e^x \cos x - 1 = 0$, there exists at least one root of $e^x \sin x - 1 = 0$. [2 M]

1(c). Using Lagrange's Mean Value Theorem, prove the inequality $e^x > 1 + x$, $x > 0$. [1 M]

Ans 1a): $\lim\limits_{x \to 0} \dfrac{\sin 2x + a \sin x}{x^3}$
Using Taylor's series,
$$\lim_{x \to 0} \frac{2x + \dfrac{(2x)^3}{3!} + \cdots + ax + \dfrac{(ax)^3}{3!} + \ldots}{x^3}$$
rest of the terms will be zero.
$\Rightarrow 2^3 + a = 0$
$\Rightarrow a = -8$

Ans 1b): $e^x = \sec x$
$e^x - \sec x = 0$
$f(x) = e^{-x} - \cos x$
if 2 roots exist then
for $f'(x) = 0$ one root exists by Rolle's Theorem.
$\Rightarrow -e^{-x} + \sin x = 0$
Ans 1c): Let $f(x) = e^x$
$\dfrac{e^x - e^0}{x - 0} \geq e^c$ for c $\in$ (0,∞)
e^c is increasing function then
min $e^0 = 1$
$\therefore e^x - 1 > x$ [proved]

2(a). Find the points of inflexion of the curve $(1+x^2)y = 1 - x$. [2M]

2(b). Establish the inequality $a^x > x^a$ for $x \geq a \geq e$. [2M]

Question 2(a) is not in syllabus.
Ans 2b): To prove : $a^x \geq x^a$ for $x \geq a \geq e$
Tanking log of both sides
$\Rightarrow x \geq a \log_a x$
Lets take, $f(x) = x - a \log_a x$
$\therefore f'(x) = 1 - \dfrac{a}{x \ln a}$
Now, $x > a$ and $\ln a > 1$ (as $a > e$)
Thus, $\dfrac{a}{x \ln a} < 1$
$\therefore f'(x) > 1$ which means $f(x)$ is an increasing function.
$f(a) = 0$
$\therefore f(x) \geq 0$ for any $x \geq a$
$\therefore x - a \log_a x \geq 0$

$\therefore x \geq a \log_a x$
$\therefore a^x \geq x^a$ [proved]

Question 3 is not in syllabus.

4(a). Discuss the continuity of the following function at the origin: **[2M]**

$$f(x,y) = \begin{cases} \dfrac{x^4 - x^2y^6}{(x^2+y^6)^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

4(b). Discuss the differentiability of the following function at the origin: **[2M]**

$$f(x,y) = \begin{cases} \sqrt{xy} & \text{for } xy \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

Ans 4a): $\dfrac{x^4 - x^2y^6}{(x^2+y^6)^2}$
Let $x = my^3$
$\Rightarrow \dfrac{m^4y^12 - m^2y^12}{(y^6 + m^2y^6)^2}$
$\Rightarrow \dfrac{m^4 - m^2}{(1+m^2)^2}$
Value is dependent on m, thus functions is not continuous.
Ans 4b): $\lim\limits_{(x,y)\to(0,0)} \dfrac{\triangle z - dz}{\sqrt{x^2+y^2}}$

$$\lim_{(\triangle x, \triangle y)\to(0,0)} \frac{\sqrt{\triangle x. \triangle y} - \triangle x. f_x(0,0) - \triangle y. f_y(0,0)}{\sqrt{\triangle x^2 + \triangle y^2}}$$

$f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h} = 0$
$f_y(0,0) = 0$
$\triangle y = m \triangle x$
$\Rightarrow \lim\limits_{(\triangle x, \triangle y)\to(0,0)} \sqrt{\dfrac{m}{1+m^2}}$
Hence not differentiable.

5(a). Discuss the continuity of the first order partial derivatives of the function

$$f(x,y) = \begin{cases} x^3 \sin\left(\dfrac{1}{y^2}\right) & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

at the origin? **[3M]**

5(b). Find $\dfrac{\partial z}{\partial x}$ at $(0,0,0)$ if $x^3 + z^3 + x\,e^{yz} + z\cos x = 0$. **[1M]**

Ans 5a): $f = x^3 \sin(\frac{1}{y^2})$

$\frac{\partial f}{\partial x} = 3x^2 \sin(\frac{1}{y^2}) = f_x$

$\frac{\partial f}{\partial y} = 2x^3 \cos(\frac{1}{y^2})\frac{-1}{y^3} = f_y$

$\triangle x^2 + \triangle y^2 < \triangle^2$

Now, $3x^2 \sin(\frac{1}{y^2}) < 3x^2$

$\therefore 3x^2 \sin(\frac{1}{y^2}) < 3x^2 + 3y^2$

$\therefore 3x^2 \sin(\frac{1}{y^2}) < 3\triangle^2$

Hence f_x is continuous. [Proved by delta-epsilon method]

Ans 5b): $x^3 + z^2 + xe^{yz} + z\cos x = 0$

$\Rightarrow$ differentiating with respect to x

$\Rightarrow 3x^2 + 3z^2\frac{dz}{dx} + e^{yz} + xe^{yz}.y\frac{dy}{dx} + \frac{dz}{dx}\cos x - z\sin x = 0$

$\frac{dz}{dx}_{(0,0,0)} = -\frac{3x^2 + e^{yz} - z\sin x}{3z^2 + xye^{yz} + \cos x}$

6(a). Let the function $f(x,y) = x^3 - x^2y + y^3$ be approximated by the 2nd degree Taylor's polynomial about the point $(2,3)$.

Find δ such that the absolute value of the remainder R_2 (involving third order partial derivatives) is less than 0.019 whenever $|x-2| < \delta, |y-3| < \delta$. [3M]

6(b). Let $u = x^2 f\left(\frac{y}{x}\right) - g\left(\frac{y}{x}\right)$ where f and g are continuous functions and possess continuous partial derivatives of 2nd order. Using Euler's theorem for homogeneous function, find α such that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2x^\alpha f\left(\frac{y}{x}\right).$$

[2M]

Ans 6b): $u = x^2 f(y/x) - g(y/x)$

$\Rightarrow x^2.\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = 0$ (as n=0)

$\Rightarrow x^2.\frac{\partial^2 f}{\partial x^2} + y^2\frac{\partial^2 f}{\partial y^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} = 0$ (as n=0)

$\therefore u = h(x,y) - g(x,y)4$

So (n)(n-1) = (2)(2-1)

and $\alpha = 2$ according to Euler theorem.

7. Find the points of local maximum, local minimum and saddle points for the function

$$f(x,y) = (x^2 - y^2)\, e^{(-x^2-y^2)/2}$$ [4 M]

Ans 7): $\dfrac{\partial f}{\partial x} = \big[2x - x(x^2-y^2)\big]e^{(-x^2-y^2)/2}$ and
$\dfrac{\partial f}{\partial y} = \big[-2y - y(x^2-y^2)\big]e^{(-x^2-y^2)/2}$
So critical points are solutions of the simultaneous equations.
$2x - x(x^2-y^2) = 0$ and $-2y - y(x^2-y^2) = 0$
i.e. $x[2-(x^2-y^2)] = 0$ and $y[-2-(x^2-y^2)] = 0$
They have solutions, (0,0) , ($\pm\sqrt{2}$,0) , (0,$\pm\sqrt{2}$)
Here, $f_{xx} = (2 - 5x^2 + x^2(x^2-y^2) + y^2)e^{(-x^2-y^2)/2}$
$f_{yy} = (5y^2 - 2 + y^2(x^2-y^2) - x^2)e^{(-x^2-y^2)/2}$
$f_{xy} = xy(x^2-y^2)e^{(x^2-y^2)/2}$
Taking D = $[f_{xx}(x_o,y_o)] \times [f_{yy}(x_o,y_o)] - [f_{xy}(x_o,y_o)]^2$
We calculate,

	Point	f_{xx}	f_{xy}	f_{yy}	D	Type
(i)	$(0,0)$	2	0	-2	-4	Saddle
(ii)	$(\pm\sqrt{2},0)$	$-4/e$	0	$-4/e$	$16/e^2$	Max
(i)	$(0,\pm\sqrt{2})$	$4/e$	0	$4/e$	$16/e^2$	Min

This question was repeated in 2017-18 Midsem.

MATHS I

END SEMESTER EXAMINATION **2017**

1. (a) Solve the given differential equation

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = 0$$

(b) Solve and write the general solution of the given differential equation:

$$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 5 = e^x \cos x$$

Ans 1a): $x^2y'' - 3xy' + 5y = 0$

Let $x = e^z \quad \Rightarrow \quad z = \log x$

$x^2y'' = D_1(D_1 - 1)$ where $\frac{d}{dz} = D_1$

$xy' = D_1$

Thus the given Differential Equation gives,

$({D_1}^2 - 4D_1 + 5)y = 0$

$D_1 = 2 \pm i$

General solution: $e^{2z}(c_1 \cos z + c_2 \sin z)$

$=x^2(c_1 \cos(\log x) + c_2 \sin(\log x))$

Ans 1b): $(D^3 - D^2 + 3D + 5)y = e^x \cos x$

So, $m^3 - m^2 + 3m + 5 = 0$

$m = -1, (1 \pm 2i)$

$\Rightarrow$ Complementary solution: $c_1e^{-x} + e^x\{c_2 \cos 2x + c_3 \sin 2x\}$

for particular integral,

P.I. $= \frac{1}{D^3 - D^2 + 3D + 5}[e^x \cos x]$

$=e^x \frac{1}{(D_1 + 1)^3 - (D_1 + 1)^2 + 3(D_1 + 1) + 5} \cos x$

$=e^x \frac{1}{D^3 + 2D^2 + 4D + 8} \cos x$ [replace all D^2 with -1]

$=e^x \frac{1}{-D - 2 + 4D + 8} \cos x$

$=e^x \frac{1}{3D + 6} \cos x$

$=e^x \frac{3D - 6}{9D^2 - 36} \cos x$

$=\frac{e^x}{15}\{\sin x + \cos x\}$

General Solution: CF + PI

$= c_1e^{-x} + e^x\{c_2 \cos 2x + c_3 \sin 2x\} + \frac{e^x}{15}\{\sin x + \cos x\}$

2. (a) Solve the given equation,

$$(x^2 + y^2 + 2\alpha)dx + 2ydy = 0$$

(b) Solve:

$$\frac{dy}{dx} + y \cos x = y^3 \sin 2x$$

Ans 2a): $(x^2+y^2+2x).dx+2ydy=0$
$M=x^2y^2+2x$
$N=2y$
$\Rightarrow \dfrac{\partial M}{\partial y}=2y \qquad \dfrac{\partial N}{\partial x}=0$
$\Rightarrow \dfrac{1}{N}\left\{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}\right\}=1$
Integrating Factor $=e^{\int 1.dx}=e^x$
$\Rightarrow e^x(x^2+y^2+2x)dx+e^x.2y.dy=0$
$\Rightarrow \displaystyle\int e^x(x^2+y^2+2y).dx+0=c$
$\Rightarrow e^x(x^2+y^2)=c$

Ans 2b): $\dfrac{dy}{dx}+y\cos x=y^3\sin 2x$
$\Rightarrow y^3.\dfrac{dy}{dx}+y^{-2}\cos x=\sin 2x$
Let $y^{-2}=v \Rightarrow -2y^{-3}\dfrac{dy}{dx}=\dfrac{dv}{dx}$
$\dfrac{dv}{dx}(-2\cos x)v=\sin 2x$
IF $=e^{\int -2\cos x.dx}=e^{-2\sin x}$
$v.e^{-2\sin x}=\displaystyle\int e^{-2\sin x}\sin 2x.dx+c$
Let $\sin x=t \Rightarrow \cos x.dx=dt$
$ve^{-2\sin x}=\displaystyle\int e^{-2t}.2t.dt+c$
$\qquad\qquad =-te^{2t}-\dfrac{1}{2}e^{-2t}+c$
$y^{-2}e^{-2\sin x}=-\sin x.e^{-2\sin x}-\dfrac{1}{2}e^{-2\sin x}+c$

3. a)Test whether $f(z)$ is differentiable at x = 0 where,

$$f(z)=\begin{cases}\dfrac{Im(z^2)}{z} & z\neq 0\\ 0 & z=0\end{cases}$$

Are CR equations satisfied at z=0

(b) (i) Prove that $u(x,y)=e^{-x(x\sin y-y\cos y)}$
(ii) Find v(x,y) such that $f(z)=u+iv$ is analytic.
(iii) Find $f(z)$ in terms of z.

Ans 3a): f(z) $=\begin{cases}\dfrac{2xy}{x-iy} & (x,y)\neq(0,0)\\ 0 & (0,0)\end{cases}$

$$=\begin{cases}\dfrac{2xy(x+iy)}{x^2+y^2} & (x,y)\neq 0\\ 0 & z=0\end{cases}$$

Differentiability at z=0,
$\displaystyle\lim_{h\to 0}\frac{f(h)-f(0)}{h}$
$=\displaystyle\lim_{h\to 0}\frac{2xy}{x^2+y^2} \qquad$ Let $y=mx$
$=\displaystyle\lim_{(x,y)\to(0,0)}\frac{2m^2}{1+m^2}$
Limit depends on m, hence it does not exist.
C.R equations :

$u_x(0,0) = \lim_{h \to 0} \dfrac{(u(h,0) - u(0,0))}{h} = 0$
Similarly,
$u_y(0,0) = u_x(0,0) = v_y(0,0) = 0$ at z=0
C.R. equations are satisfied,
$u_x = v_y = 0$
$u_y = -v_x = 0$

Ans 3b(i): $u(x,y) = e^{-x}(x\sin y - y\cos y)$
$u_{xx} = -2e^{-x}\sin y + e^{-x}(x\sin y - y\cos y)$
$u_{yy} = e^{-x}(-x\sin y + 2\sin y + y\cos y)$
$u_{xx} + u_{yy} = 0$
Hence u is harmonic.
Ans 3b(ii): $dv = v_x.dx + v_y.dy$
$= -u_y.dx + u_x.dy$
$= e^{-x}(x\cos y - \cos y + y\sin y).dx + e^{-x}(\sin y - x\sin y + y\cos y).dy$
$v = -\int e^{-x}(x\cos y - \cos y + y\sin y).dx$
$= e^{-x}(x\cos y + y\sin y) + c$
Ans 3b(iii): $f(z) = f(x+iy) = u(x,y) + iv(x,y)$
$= u(z,0) + iv(z,0)$
$= e^{-z}(z\sin 0 - 0) + ie^{-z}(z\cos 0 + 0) + ie$
$= ize^{-z} + ie$

4. a) Let C be the arc of ellipse $\dfrac{(x-3)^2}{4} + \dfrac{y^2}{9} = 1$ lying on the quadrant oriented in counter clockwise direction. Evaluate $\int_C \dfrac{1}{z^4}dz$

(b) Let C be the circle $|z| = 3$ oriented in counter-clockwise direction. If $g(w) = \dfrac{z^3+2z}{(z-w)^3}dz$ then find $g(2)$ and $g(4i)$

Ans 4a): Let L br the joining point z=1 to z=5
Then $\int_c \dfrac{1}{z^4}.dz = -\int_L \dfrac{1}{z^4}.dz = \dfrac{1}{3}\left[\dfrac{1}{z^3}\right]_1^5 = -\dfrac{124}{375}$
Ans 4b): Let $f(z) = z^3 + 2z$ which is analytic in the domain.
$f''(z) = 6z$ and $g(z) = \int_c \dfrac{f(z)}{(z-2)^3}.dz$
$= \dfrac{2ni}{2!}f''(z)$
Scince $z = 4i$ lies outside domain
$g(4i) = 0$

5. (a) Find the Laurent series in the annulus $1 < |z| < 4$ for the following function about $z = 0$
$$f(z) = \frac{z+2}{z^2 - 5z + 4}$$
(b) Classify the singularities of the following function in the finite complex plane: $f(z) = \dfrac{ze^{\frac{1}{z^2+1}}}{\sin z}$

Ans 5a): $f(z) = \dfrac{z+2}{z^2 - 5z + 4}$
$= \dfrac{2}{(z-4)} + \dfrac{-1}{z-1}$

$$=-\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{z}{4}\right)^k-\sum_{j=1}^{\infty}\left(\frac{1}{z}\right)^j$$
$$=\sum_{k=-\infty}^{\infty}a_k z^k$$

where,

$$a_k=\begin{cases}-\frac{1}{2}4^{-k}, & k=0,1,2,\dots\\ -1, & k=-1,-2,\dots\end{cases}$$

Ans 5b): $f(z)=\dfrac{ze^{\frac{1}{z^2+1}}}{\sin z}$

(i) z=0 is removable singularity as,

$\lim\limits_{z\to 0}f(z)=e$

(ii) z= $n\pi$ where n∈I are simple poles since,

$\lim\limits_{z\to n\pi}f(z)=\infty$

(iii) z=$\pm i$ are essential singularities since,

$\lim\limits_{z\to \pm i}f(z)$ does not exis.t

6. (a) Prove that $5^x+12^x<13^x$ for all $x\in(2,\infty)$

(b) Find the value of the limit if it exists,

$\lim\limits_{x\to 0}\dfrac{e^x\sin x-x(1+x)}{x^3}$

Ans 6a): Let $f(x)=\left(\frac{5}{13}\right)^x+\left(\frac{12}{13}\right)^x-1$

$f(2)=0$

It has real root at say (2,α).

$\exists c\in(2,\alpha)$ such that $f'(c)=0$

but $f'(x)<0\quad\forall\quad x$

Ans 6b): $\lim\limits_{x\to 0}\dfrac{(1+x+\frac{x^2}{2!}+\dots)(x-\frac{x^3}{3!}+\dots)-x(1+x)}{x^3}$

$=\lim\limits_{x\to 0}\dfrac{x^3\left(\frac{1}{2}-\frac{1}{6}\right)}{x^3}=\dfrac{1}{3}$

7. (a) Solve the given differential equation by the method of variation of parameters,

$\dfrac{d^y}{dt^2}+2\dfrac{dy}{dt}+1=e^t\ln t$

(b) Solve the following system of linear differential equation,

$\dfrac{dx}{dt}=x+y$ Given y=5 and x=10 when t=0

$\dfrac{dy}{dt}=4x-2y$

Ans 7a): Auxiliary equation : $m^2+2m+1=0$

$m_1,m_2=-1$

$u_1=-\int t\ln t.dt=-\dfrac{t^2}{2}\ln t+\dfrac{t^2}{4}$

$u_2 = \int \ln t.dt = t\ln t - t$

$y_p = u_1y_1(t) + u_2y_2(t) = (\frac{t^2}{2}\ln t - \frac{3}{4}t^2)e^{-t}$

Ans 7b): Auxiliary equation:

$\begin{vmatrix} 1-k & 1 \\ 4 & -(2+k) \end{vmatrix} = 0 \qquad \Rightarrow \qquad k = -3, 2$

$x = c_1e^{-3t} + c_2e^{2t}$

$y = -4c_1e^{-3t} + c_2e^{2t}$

Using given condition,

$x = e^{-3t} + 9e^{2t}$

$y = -4e^{-3t} + 9e^{2t}$

8. a) $f(x,y) = x^2 - 2xy + 2y$ **Find the maximum value of** $f(x,y)$ **given** $0 \le x \le 3$ **and** $0 \le y \le 2$ **.**

(b) Given the function $f(x) = \frac{-xy^3}{x^2+y^2}$

(i)Prove that $f(x)$ **is continuous at the point (0,0)**

(ii) Check whether f_{xx} **is continuous at the point (0,0)**

(iii) Check whether the function $f(x,y)$ **is differentiable or not at the point (0,0)**

Ans 8a): Critical point is interior

$f_x = 0$ and $f_y = 0 \Rightarrow (1,1)$ is critical point.

$f(1,1) = 1$

Along boundaries,

$x = 0 \quad f(0,y) = 2y \quad 0 \le y \le 2 \quad \min f = 0 \text{ , } \max f = 4;$

$y = 2 \quad f(x,2) = x^2 - 4x + 4 \quad 0 \le x \le 3 \quad \min f = 0 \text{ , } \max f = 4;$

$x = 3 \quad f(3,y) = 9 - 4y, \quad 0 \le y < 2 \quad \min f = 0 \text{ , } \max f = 9;$

$y = 0 \quad f(x,0) = x^2, \quad 0 \le x \le 3 \quad \min f = 0 \text{ , } \max f = 9;$

Absolute Maximum = 9

Absolute Minimum = 0

Ans 8b(i): $f_x(0,0) = 0$

$f_z(x,y) = \frac{-x^2y^3 + y^5}{(x^2+y^2)^2}$

$\lim_{(x,y)\to(0,0)} f_x(x,y) = 0$

(ii) $f_{xx} = \frac{2xy^3(x^2-3y^2)}{(x^2+y^2)}$

$\lim_{(x,y)\to(0,0)}$ along $y = mx$ depends on m so limit does not extend.

(iii) Since f_x is continuous and f_y exists at (0,0) the function is differentiable .

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MATHS I

END SEMESTER **2015-16**

1(a). Find the positive real value of λ, for which the co-efficient of x^{64} in the Maclaurin expansion of $(x+\lambda^2)e^x$ is $\frac{5}{\lfloor 63}$. [2M]

1(b). Find the intervals for which the curve $y=e^x(\sin x+\cos x)$, $x\in(0,2\pi)$ is convex upwards (concave downwards) or convex downwards (concave upwards).

[3M]

1(c). Find the value(s) of α for which the radius of curvature of the curve $x=\alpha(t+\sin t)$, $y=\alpha(1-\cos t)$ at $t=2\pi$ is 1. [2M]

Ans 1a): $(x+\lambda^2)e^x$

$=(x+\lambda^2)(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots)$

$\therefore$ coefficient of $x^{64}=\frac{1}{63!}+\frac{\lambda^2}{64!}$

$\therefore \frac{5}{63!}=\frac{1}{63!}+\frac{\lambda^2}{64\times 63!}$

or $5=1+\left(\frac{\lambda}{8}\right)^2$

$\therefore \lambda=\pm 16$

Ans 1b): $y=e^x(\sin x+\cos x)$

or $\frac{dy}{dx}=e^x(\sin x+\cos x+\cos x-\sin x)=e^x(2\cos x)$

$\therefore$ For $\frac{dy}{dx}=0$

$x=\frac{\pi}{2},\frac{3\pi}{2}$

$\therefore \frac{d^2y}{dx^2}=2e^x(\cos x-\sin x)$

$\therefore \frac{d^2y}{dx^2}\left(\frac{\pi}{2}\right)=2e^{\pi/2}(-1)$ $\quad\therefore$ convex downwards

$\therefore \frac{d^2y}{dx^2}=2e^{3\pi/2}(1)$ $\quad\therefore$ convex upwards.

Question 1(c) is not in syllabus.

2(a). Find $f_{xy}(0,0)$ and $f_{yx}(0,0)$ for the function

$$f(x,y) = \begin{cases} \dfrac{xy(x^2+2y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

where $f_{xy} \equiv \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ and $f_{yx} \equiv \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$. [4M]

2(b). A container of capacity 64 m^3 in the form of a cuboid with an open top is to be made of thin sheet of metal. Calculate, with justification, the dimensions of the container if it has to use minimum possible amount of metal sheet.

[4M]

Ans 2a): $f_y(0,0) = \lim\limits_{k\to 0} \dfrac{f(0,k)-f(0,0)}{k-0} = 0$

$$f_y(h,0) = \lim_{k\to 0} \frac{f(h,k)-f(h,0)}{k-0}$$
$$= \lim_{k\to 0} \frac{1}{k} \times \frac{hk(h^2+2k^2)}{(h^2+k^2)}$$
$$= \lim_{k\to 0} \frac{h(h^2+2k^2)}{h^2+k^2} = h$$
$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_y(h,0)-f_y(0,0)}{h-0} = 1$$
$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h-0} = 0$$
$$f_x(0,k) = \lim_{h\to 0} \frac{f(h,k)-f(0,k)}{h-0}$$
$$= \lim_{k\to 0} \frac{1}{h} \times \frac{hk(h^2+2k^2)}{(h^2+k^2)}$$
$$= \lim_{k\to 0} \frac{k(h^2+2k^2)}{h^2+k^2} = 2k$$
$$f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(0,k)-f_x(0,0)}{k-0} = 2$$

Ans 2b): $f(x,y) = 2 \times \dfrac{64}{xz}(x+z) + xz$

$$= 128\left(\frac{1}{z}+\frac{1}{x}\right) + xz$$

$f_x = \dfrac{-128}{x^2} + z \qquad f_z = \dfrac{-128}{z^2} + x$

$z = \dfrac{128}{x^2} \qquad x = \dfrac{128}{z^2}$

$$z = \frac{128}{\left(\frac{128}{z^2}\right)\left(\frac{128}{z^2}\right)}$$
$$z = \frac{z^4}{128}$$
$$z^3 = 2^7$$
$$\therefore z = 2^{7/3}$$
$$\therefore x = \frac{2^7}{2^{14/3}}$$
$$\therefore x = 2^{7-\frac{14}{3}}$$
$$\therefore x = 2^{7/3}$$

Now , $x \times z \times y = 64$

$\therefore 2^{7/3} \times 2^{7/3} \times y = 2^6$

$\therefore y = 2^{\left(\dfrac{24-7-7}{4}\right)}$
$\therefore y = 2^{10/4} = 2^{5/2}$
Reason for minima :
$f_{xx} = \dfrac{-128}{z^3}(-2) \qquad f_{zz} = \dfrac{-128}{z^3}(-2)$
$f_{zx} = 1$
$f_{xx} = \dfrac{128 \times 2}{2^7} = 2$
$f_{zz} = \dfrac{128 \times 2}{2^7} = 2$
$\therefore f_{xx}f_{zz} - f_{zx} = 4 - 1 = 3 \;\; > \;\; 0$
Thus its a point of local minimum.
$\therefore x = 2^{7/3}, y = 2^{5/2}, z = 2^{7/3}$

3(a). Find the value of λ such that

$$(2xe^y + 3y^2)\frac{dy}{dx} + (3x^2 + \lambda e^y) = 0 \qquad \text{[3M]}$$

is exact. Using this value of λ, find the general solution of the differential equation.

3(b). Solve the differential equation

$$y + \frac{d}{dx}(xy) = x(\sin x + \ln x) \qquad \text{[2M]}$$

3(c). Solve the differential equation

$$\frac{dy}{dx} + \frac{y}{x}\ln y = \frac{y}{x^2}(\ln y)^2 \qquad \text{[2M]}$$

Ans 3a): $(2xe^y + 3y^2)\dfrac{dy}{dx} + (3x^2 + \lambda e^y) = 0$
For exact , $MdxNdy = 0$
$M_y = N_x \qquad$ where $2xe^y + 3y^2 = N$
$\qquad\qquad\qquad\qquad 3x^2 + \lambda e^y = M$
$\therefore M_y = \lambda e^y$
$\therefore N_x = 2e^y$
$\therefore \lambda = 2$
$\therefore (2xe^y + 3y^2)dy + (3x^2 + 2e^y)dx = 0$
or $3y^2dy + 3x^2dx + d(2xe^y) = 0$
or $d(x^3 + y^3) + d(2xe^y) = 0$
or $x^3 + y^3 + 2xe^y = c$ where c = integration constant

Ans 3b): $y + \dfrac{d}{dx}(xy) = x(\sin x + \ln x)$
or $ydx + xdy + ydx = x(\sin x + \ln x)dx$
or $\dfrac{dy}{dx} + \dfrac{2y}{x} = (\sin x + \ln x)$
$\therefore \text{IF} = e^{\left(2\int \dfrac{dx}{x}\right)}$
$\therefore yx^2 = \displaystyle\int x^2(\sin x + \ln x)dx$
$\therefore y = \dfrac{2\sin x}{x} + \dfrac{x \ln x}{3} + \left(\dfrac{2}{x^2} + 1\right)\cos x - \dfrac{x}{9} + \dfrac{c}{x^2}$

Ans 3c): $\frac{dy}{dx} + \frac{y}{x}\ln y = y\left(\frac{\ln y}{x}\right)^2$

or $\frac{1}{y}\frac{dy}{dx} + \frac{\ln y}{x} = \left(\frac{\ln y}{x}\right)^2$

Let $\ln y = z$

$\frac{dz}{dx} = \frac{1}{y}\frac{dy}{dx}$

$\therefore \frac{dz}{dx} + \frac{z}{x} = \left(\frac{z}{x}\right)^2$

Let $\frac{z}{x} = u$

$z = ux$

$\frac{dz}{dx} = u + x\frac{du}{dx}$

$u + x\frac{du}{dx} + u = u^2$

or $x\frac{du}{dx} = u^2 - 2u$

or $\frac{du}{u^2 - 2u} = \frac{dx}{x}$

or $\frac{du}{u(u-2)}(u - (u-2)) = \frac{dx}{x}$

or $du\left[\frac{1}{u-2} - \frac{1}{u}\right] = \frac{2dx}{x}$

or $\frac{du}{u-2} - \frac{du}{u} = 2\frac{dx}{x}$

or $\frac{du}{u-2} - \frac{du}{u} = \frac{2dx}{x}$

or $\ln\left|\frac{u-2}{u}\right| = 2\ln|x| + \ln|c|$

or $\ln\left|1 - \frac{2}{u}\right| = \ln|cx^2|$

$\therefore 1 - \frac{2}{u} = cx^2$

or $u = \frac{2}{1 - cx^2}$

or $\frac{\ln y}{x} = \frac{2}{1 - cx^2}$

or $\ln y = \frac{2x}{1 - cx^2}$

4(a). Using the method of variation of parameters, find the general solution of the following non-homogeneous differential equation

$$\frac{d^2y}{dx^2} + 4y = 3\,\mathrm{cosec}\,x$$

[4M]

4(b) Find the solution of the following homogeneous differential equation

$$\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0$$

with the conditions: $y(0) = 0,\ y'(0) = -1,\ y\left(\frac{\pi}{2}\right) = -1,\ y'\left(\frac{\pi}{2}\right) = -1.$ [3M]

Ans 4a): $\dfrac{d^2y}{dx^2} + 4y = 3\text{cosec}x$

$\therefore (D^2+4)y = 3\text{cosec}x$

$\therefore$ General solution are $\sin 2x$ and $\cos 2x$ (y_1 and y_2)

$\therefore w = \begin{vmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{vmatrix} = -2(\sin^2 2x + \cos^2 2x) = -2$

$y = c_1y_1 + c_2y_2 + y_1\displaystyle\int \frac{-y_2 f(x)}{w}dx + y_2\int \frac{y_1 f(x)}{w}dx$

$\quad = c_1\sin 2x + c_2\cos 2x + \sin 2x\displaystyle\int \frac{-\cos 2x}{-2}(\text{cosec}x)dx + \cos 2x\int \frac{\sin 2x}{-2}(\text{cosec}x)dx$

$\quad = c_1\sin 2x + c_2\cos 2x + \dfrac{\sin 2x}{2}\displaystyle\int(\text{cosec}x dx - 2\sin x dx) - \frac{\cos 2x}{2}\int 2\sin x\cos x\,\text{cosec}x dx$

$\quad = c_1\sin 2x + c_2\cos 2x + (\sin 2x)\left[\cos x - \dfrac{1}{2}\ln|\text{cosec}x + \cot x|\right] - \sin x\cos 2x$

Ans 4b): $(D^4 + 2D^2 + 1)y = 0$

Roots of auxiliary equation :

$m^4 + 2m^2 + 1 = 0$

or $(m^2+1)^2 = 0$

or $m = \pm i, \pm i$

$\therefore y = (c_1 + c_2x)\sin x + (c_3 + c_4x)\cos x$

$y(0) = 0 \qquad y(\pi/2) = -1$

or $c_3 = 0 \qquad$ or $c_1 + c_2(\pi/2) = -1$

$\qquad\qquad c_1 = c_2(-\pi/2) - 1$

$c_1 + c_4 = -1$ and $\quad c_2 - c_4(\pi/2) = -1$

$\therefore c_2(-\pi/2) - 1 + c_4 = -1$

or $c_2(-\pi/2) + c_4 = 0$

or $c_4 = \dfrac{\pi c_2}{2}$

$\therefore c_2 - \dfrac{\pi c_2}{2} \times \dfrac{\pi}{2} = -1$

or $c_2(\dfrac{\pi^2}{4} - 1) = 1$

or $c_2 = \dfrac{4}{\pi^2 - 4}$

or $c_4 = \dfrac{2\pi}{\pi^2 - 4}$

and $c_1 = -1 - \dfrac{2\pi}{\pi^2-4} = \dfrac{4 - 2\pi - \pi^2}{\pi^2 - 4}$

$\therefore y = \left(\left(\dfrac{4-2\pi-\pi^2}{\pi^2-4}\right) + \dfrac{4x}{\pi^2-4}\right)\sin x + \left(\dfrac{2\pi x}{\pi^2-4}\right)\cos x$

5(a). Find the general solution of the differential equation

$$(3+2x)^2\frac{d^2y}{dx^2} - 2(3+2x)\frac{dy}{dx} + 4y = 8x \qquad [3M]$$

5(b). Find the general solution of the following system of differential equations

$$\frac{dy_1}{dx} = y_1 + 4\,y_2$$

$$\frac{dy_2}{dx} = 2y_1 - y_2 \qquad [4M]$$

(WITHOUT reducing it to a differential equation of 2nd order)

Ans 5a): $(3+2x)^2\dfrac{d^2y}{dx^2}+2(3+2x)\dfrac{dy}{dx}+4y=8x$

Let $(3+2x)=u$

$\dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}=\dfrac{2dy}{du}$

$\dfrac{d^2y}{dx^2}=2\dfrac{d}{du}\left(\dfrac{dy}{du}\right)\dfrac{du}{dx}=\dfrac{4d^y}{du^2}$

$\therefore u^2\left(\dfrac{4d^2y}{duu^2}\right)+2(2u)\dfrac{dy}{du}+4y=8\left(\dfrac{u-3}{2}\right)$

or $u^2u''+uu'+y=u-3$

Let $u=e^z$

$\therefore (D(D-1)+D+1)y=e^z-3$

or $(D^2+1)y=e^z-3$

or $y=\dfrac{e^z}{2}-\dfrac{3}{1}=\dfrac{e^z-6}{1}$ [complimentary function]

$y=c_1\sin z+c_2\cos z+\dfrac{e^z-6}{2}$

$\quad=c_1\sin x(\ln(3+2x))+c_2\cos(\ln(3+2x))+x-\dfrac{3}{2}$ [integration constant]

Ans 5b): $\dfrac{dy_1}{dx}=y_1+4y_2$

$\dfrac{dy_2}{dx}=2y_1-y_2$

By the method of undetermined coefficients,

$\therefore \begin{vmatrix}1-\alpha & 4\\ 2 & -1-\alpha\end{vmatrix}=0$

or $(\alpha-1)(\alpha+1)-8=0$

or $\alpha=\pm3$

$\therefore y_1=c_1e^{3x}+c_2e^{-3x}$

$y_2=\dfrac{1}{4}\left(\dfrac{dy_1}{dx}-y_1\right)$

$\quad=\dfrac{1}{4}(3c_1e^{3x}-3c_2e^{-3x}-c_1e^{3x}-c_2e^{3x})$

$\quad=\dfrac{1}{4}\left(e^{3x}(2c_1)-(4c_2)e^{-3x}\right)$

6(a). Discuss the continuity of the function f(z) = Arg(z) (Principal Argument) at a point z = x, where x is real and negative. [1M]

6(b). If $u(x,y)=x^2-y^2$ and $v(x,y)=3x^2y-y^3$ are Harmonic functions in a domain D, then show that the function $F(x,y)=(u_y-v_x)+i(u_x+v_y)$ is analytic in D. [3M]

6(c). Let f(z) be defined as

$$f(z)=\begin{cases}\dfrac{\mathrm{Re}(z^3)+i\,y^3}{|z^2|}, & z\neq 0\\ 0, & z=0\end{cases}$$

Test the C-R equations at the origin. Is this function differentiable at the origin? Justufy.

Ans 6a): For function $f(z)=\arg(z)$

Let $z=x\pm iy$

For $x>0$

$\lim\limits_{y\to0}\arg(z)=0$

For $x < 0$

$\lim\limits_{y\to 0^+} \arg 1(z) = \pi$ and $\lim\limits_{y\to 0^-} \arg(z) = -\pi$

$\therefore$ function is not continuous for $z \to x$ where x is real.

Ans 6b): Given $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$

$\therefore F(x, y) = (u_y - v_x) + i(u_x + v_y)$

For $F(x, y)$ to be analytic,

$\dfrac{\partial}{\partial x}(u_y - v_x) = \dfrac{\partial}{\partial y}(u_x + v_y)$ and $\dfrac{\partial}{\partial y}(u_y - v_x) = -\dfrac{\partial}{\partial x}(u_x + u_y)$

For the 1st one:

$$\begin{array}{llll} u_y = -2y & v_x = 6xy & u_x = 2x & v_y = 3x^2 - 3y^2 \\ u_{xy} = 0 & v_{xx} = 6y & u_{yx} = 0 & v_{yy} = -6y \end{array}$$

$\therefore u_{xy} - v_{xx} = -6y = u_{yx} + v_{yy}$

For the 2nd one:

$$\begin{array}{llll} u_y = -2y & v_x = 6xy & u_x = 2x & v_y = 3x^2 - 3y^2 \\ u_{yy} = -2 & v_{yx} = 6x & u_{xx} = 2 & v_{xy} = 6x \end{array}$$

$\therefore u_{yy} - v_{yx} = -6x - 2 = -(u_{xx} + v_{xy})$

$\therefore F(x, y)$ is analytic [C-R equations are satisfied]

Ans 6c): $z^3 = (x + iy)^3$

$= x^3 + (iy)^3 + 3x^2(iy) + 3x(iy)^2$

$= x^3 - iy^3 + 3ix^2y - 3xy^2$

$= (x^3 - 3xy^2) + i(3x^2y - y^3)$

$\therefore f(x, y) = \dfrac{x^3 - 3xy^3 + iy^3}{x^2 + y^2}$

$\therefore u = \dfrac{x(x^2 - 3y^2)}{x^2 + y^2} \qquad \dfrac{y^3}{x^2 + y^2}$

$$\therefore u_x(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h}$$

$$= \lim_{h\to 0} \frac{h(h^2)}{h^2} = 0$$

$$v_y(0, 0) = \lim_{k\to 0} \frac{f(0, k) - f(0, 0)}{k} = 0$$

$$u_y(0, 0) = \lim_{k\to 0} \frac{f(0, k) - f(0, 0)}{k} = 0$$

$$u_x(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} = 0$$

$\therefore u_x = v_y$ and $u_y = -v_x$

For $\lim\limits_{z\to 0} \dfrac{f(z) - f(0)}{z - 0}$

$$= \lim_{(x,y)\to(0,0)} \frac{\dfrac{(x^3 - 3xy^2) + iy^3}{x^2 + y^2} - 0}{(x + iy)}$$

Let $y = mx$

$$\therefore \lim_{x\to 0} \frac{(x^3 - 3xm^2x^2) + i(m^3x^3)}{((1 + im)x)(x^2)(1 + m^2)}$$

$= \dfrac{(1 - 3m^2) + i(m^3)}{(1 + im)(1 + m^2)}$ which is a function of m.

$\therefore$ the function is non-differentiable at origin.

(7)(a). Use the M-L inequality to show that

$$\left|\int_{\Gamma} \frac{dz}{z^2+1}\right| \leq \frac{1}{2\sqrt{5}}$$

where Γ is the line segment joining the points 2 and 2+i. Show the details of your work.

[2M]

7(b). Evaluate the integral

$$\int_{\Gamma} \frac{e^{\pi i z}\ dz}{2z^2-5z+2}$$

where Γ is the unit circle $|z| = 1$ that orients counter-clockwise. [3M]

7(c). Show that the integral

$$\int_{\Gamma} \frac{z\, dz}{z+1}$$

is path independent and hence evaluate it. Here Γ is any curve lying in the upper half plane $\{z: \text{Im}(z)>0\}$ joining the points -1+2i and 1+2i. [2M]

Ans 7(a): To prove: $\left|\int_{\Gamma} \frac{dz}{z^2+1}\right| \leq \frac{1}{2\sqrt{5}}$

Length of $\Gamma = 1$

$M = Max\left|\frac{1}{z^2+1}\right|$

$$\left|\frac{1}{z^2+1}\right| = \frac{1}{z^2+1} = \frac{1}{|(z+i)(z-i)|} = \frac{1}{|(z+i)||(z-i)|} \leq max\left(\frac{1}{|z+i|}\right).max\left(\frac{1}{|(z-i)|}\right)$$

In the path between 2 to $2+i$ $max\left(\frac{1}{|z-i|}\right)$ occurs at $z = 2+i$

$\therefore max\left(\frac{1}{|z-i|}\right) = \frac{1}{|2|} = \frac{1}{2}$

$\therefore \left|\frac{1}{z^2+1}\right| \leq \frac{1}{2} \times \frac{1}{\sqrt{5}}$

$\therefore M = \frac{1}{2\sqrt{5}}$

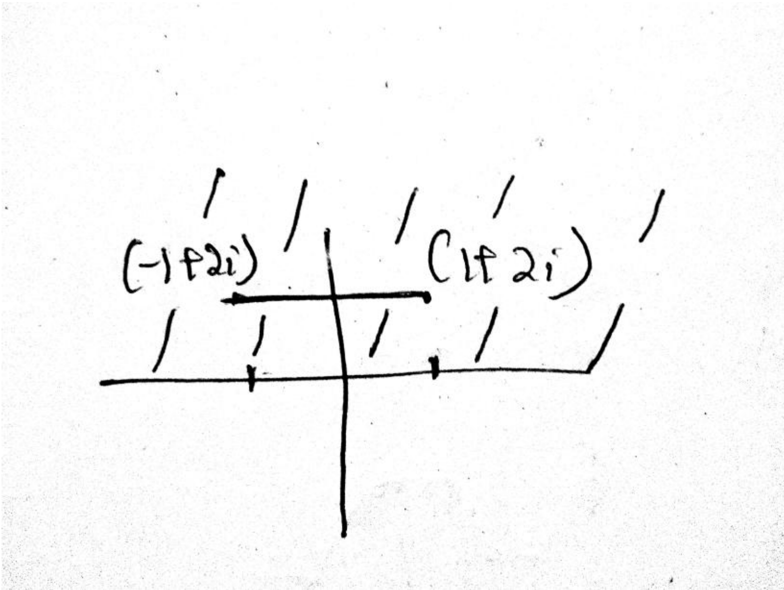
by ML theorem of inequality,

$\left|\int_{Gamma} \frac{dz}{z^2+1}\right| \leq \text{ML} \Rightarrow \left|\int \frac{dz}{z^2+1}\right| \leq \frac{1}{2\sqrt{5}}$

Ans 7b):
$$\begin{aligned}
\int_{\Gamma} \frac{e^{\pi iz}dz}{(2z^2-4z-z+2)} &= \int_{\Gamma} \frac{e^{\pi iz}dz}{(2z(z-2)-(z-2))} \\
&= \int_{\Gamma} \frac{e^{\pi iz}dz}{2(z-2)(z-1/2)} \\
&= \frac{1}{2} \times \frac{2}{3} \int_{\Gamma} e^{i\pi z} \frac{((z-1/2)-(z-2))}{(z-1/2)(z-2)} dz \\
&= \frac{1}{3} \int_{\Gamma} \frac{e^{i\pi z}dz}{z-2} - \frac{1}{3} \int_{\Gamma} \frac{e^{i\pi z}}{z-1/2} dz
\end{aligned}$$

$= 0 - \frac{1}{3}e^{i\pi(1/2)} = -i/3$ [There is only one point of discontinuity]

Ans 7c):



Function is not analytic at $z = -1$ which is outside the domain of the given curve. $D \equiv Im(z) > 0$
Let Γ be the straight line joining $(-1+2i)$ to $(1+2i)$
$\therefore z = x + 2i$ where $x \in [-1, 1]$

$\therefore \int_\Gamma \frac{z+1-1}{z+1} dz$

$= \int_\Gamma dz - \int_\Gamma \frac{dz}{z+1}$

$= [z]_{-1+2i}^{1+2i} - [ln|z+1|]_{-1+2i}^{1+2i}$

$= [1 + 2i + 1 - 2i] - \ln\left|\frac{2+2i}{2i}\right|$

$= 2 - \ln\left|\frac{i-1}{-1}\right|$

$= 2 - \ln|1 - i|$

MATHS I

END SEMESTER **2011-12**

1. (a) Let $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b).
If $\exists\, c \in (a,b)$ such that $f'(c)=0$, does it imply $f(a)=f(b)$?. Justify your answer.

(b) Find the values of p and q such that $\lim_{x\to 0}\dfrac{x(1-p\cos x)+q\sin x}{x^3}=\dfrac{1}{3}$.

(c) Find the radius of curvature for the curve $x^3+y^3=3axy$ at the point $\left(\dfrac{3a}{2},\dfrac{3a}{2}\right)$.

(d) Express $f(x)=3x^3-4x^2+5x-1$ in powers of $(x-3)$ using Taylor series expansion.

(2+2+2+2)

Ans 1a): By LMVT

$\dfrac{f(b)-f(a)}{b-a}=f'(c)$ where $c\in[a,b]$

For $\forall \quad c\in(a,b), f'(c)=0$

$\therefore \dfrac{f(b)-f(b)}{b-a}=0$

or $f(b)=f(a)$

Ans 1b): $\lim_{x\to 0}\dfrac{x(1-p(1-x^2/2!+x^4/4!))+q(x-x^3/3!+x^5/5!)}{x^3}=\dfrac{1}{3}$

or $\lim_{x\to 0}\left[(\dfrac{1-p}{x^2})+\dfrac{1}{2!}-\dfrac{x^2}{4!}+q(\dfrac{1}{x^2})-\dfrac{q}{3!}+q(\dfrac{x^2}{5!})\right]=\dfrac{1}{3}$

$\therefore$ coefficient of $\dfrac{1}{x^2}=0$

$\therefore 1-p+q=0$ and $\dfrac{1}{2!}-\dfrac{q}{3!}=\dfrac{1}{3}$

$\therefore q=1$

$\therefore p=2$

Part (c) is not in syllabus.

Ans 1d): $f(x)=3x^3-4x^2+5x-1$ $\quad f(3)=59$

$f'(x)=9x^2-8x+5$ $\quad f'(3)=62$

$f''(x)=18x-8$ $\quad f''(3)=46$

$f'''(x)=18$ $\quad f'''(3)=18$

$$f(x)=f(3)+f'(3)(x-3)+f''(3)\frac{(x-3)^2}{2!}+f'''(3)\frac{(x-3)^3}{3!}$$
$$=59+62(x-3)+23(x-3)^2+3(x-3)^3$$

2. (a) If $e^u = x^3 + y^3 + z^3 - 3xyz$ then find $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$.

(b) For a homogeneous function $z(x, y)$ of degree n, prove that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = nz$.

If $z = \sin^{-1}\left(\dfrac{x}{y}\right) + \tan^{-1}\left(\dfrac{y}{x}\right)$, for $x \neq 0, y \neq 0$, then evaluate $\dfrac{1}{y}\dfrac{\partial z}{\partial x} + \dfrac{1}{x}\dfrac{\partial z}{\partial y}$.

(c) Obtain the expression for $\dfrac{\partial^2 \psi}{\partial x^2} + \dfrac{\partial^2 \psi}{\partial y^2}$ in terms of $\psi(r, \theta)$ using the relation $x = r\cos\theta$, $y = r\sin\theta$. **(3+3+3)**

Ans 2a): $e^u = x^3 + y^3 + z^3 - 3xyz$

$\therefore u = \ln(x^3 + y^3 + z^3 - 3xyz)$

$= \ln(x + y + z) + \ln(x + \omega y + \omega^2 z) + \ln(x + \omega^2 y + \omega z)$

$$\frac{du}{dx} = \frac{1}{x+y+z} + \frac{1}{x+\omega y+\omega^2 z} + \frac{1}{x+\omega^2 y+\omega z}$$
$$\frac{du}{dy} = \frac{1}{x+y+z} + \frac{\omega}{x+\omega y+\omega^2 z} + \frac{\omega^2}{x+\omega^2 y+\omega z}$$
$$\frac{du}{dz} = \frac{1}{x+y+z} + \frac{\omega^2}{x+\omega y+\omega^2 z} + \frac{\omega}{x+\omega^2 y+\omega z}$$
$$\frac{d^2u}{dx^2} = -\frac{1}{(x+y+z)^2} - \frac{1}{(x+\omega y+\omega^2 z)^2} - \frac{1}{(x+\omega^2 y+\omega z)^2}$$
$$\frac{d^2u}{dy^2} = -\frac{1}{(x+y+z)^2} - \frac{\omega^2}{(x+\omega y+\omega^2 z)^2} - \frac{\omega}{(x+\omega^2 y+\omega z)^2}$$
$$\frac{d^2u}{dz^2} = -\frac{1}{(x+y+z)^2} - \frac{\omega}{(x+\omega y+\omega^2 z)^2} - \frac{\omega^2}{(x+\omega^2 y+\omega z)^2}$$
$$\therefore \frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = -\frac{3}{(x+y+z)^2}$$

Ans 2b): $z(x, y) = x^n\phi(y/x)$

Let u be homogeneous function

$u = x^n\phi(y/x)$

differentiating u with respect to x $\quad (\dfrac{\partial}{\partial x})$

$u_x = nx^{n-1}\phi(y/x) + \dfrac{-y}{x^2} \times \phi'(y/x)x^n$

$xu_x = nx^n\phi(y/x) - yx^{n-1}\phi(y/x)$ . . . . (1)

differentiating with respect to y,

$u_y = x^{n-1}\phi(y/x)$

$yu_y = yx^{n-1}\phi(y/x)$ . . . . (2)

(1) + (2)

$yu_u + xu_x = nx^n\phi(y/x) - yx^{n-1}\phi(y/x) + yx^{n-1}\phi(y/x)$

$\Rightarrow xu_x + yu_y = nx^n\phi(y/x)$

$\Rightarrow xu_x + yu_y = nu$ [proved]

$z = \sin^{-1}\left(\dfrac{x}{y}\right) + \tan^{-1}\left(\dfrac{y}{x}\right)$

It is an homogeneous equation of order 0.

$\therefore nz_x + yz_y = 0$

Dividing by xy, we get , $z_x/y + z_y/x = 0$

Ans 2c): $\dfrac{d\psi}{d\gamma} = \dfrac{d\psi}{dx} \times \dfrac{dx}{d\gamma} + \dfrac{d\psi}{dy} \times \dfrac{dy}{d\gamma}$

$$= \frac{d\psi}{dx}\cos\theta + \frac{d\psi}{dy}\sin\theta$$

$$\frac{d^2\psi}{d\gamma^2} = \frac{d}{dx}\left(\frac{d\psi}{dx}\cos x\right)\cos x + \frac{d}{dy}(\frac{d\psi}{dy}\sin\theta)\sin\theta$$

$$= \frac{d^2\psi}{dx^2}\cos^2\theta + \frac{d^2\psi}{dy^2}\sin^2\theta$$

$$\frac{d\psi}{d\theta} = \frac{d\psi}{dx}(-\gamma\sin\theta) + \frac{d\psi}{dy}(\gamma\cos\theta)$$

$$\frac{d^2\psi}{d\theta^2} = \frac{d^2\psi}{dx^2}(\gamma^2\sin^2\theta) - \frac{d\psi}{dx}(\gamma\cos\theta) + \frac{d^2\psi}{dy^2}(\gamma^2\cos^2\theta) - \frac{d\gamma}{dy}(\gamma\sin\theta)$$

$$\therefore \frac{d^2\psi}{d\theta^2} + \gamma^2\frac{d^2\psi}{d\gamma^2} = \frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2} - \gamma\frac{d\psi}{d\gamma}$$

$$\therefore \frac{d^2\psi}{d\theta^2} + \gamma^2\frac{d^2\psi}{d\gamma^2} + \gamma\frac{d\psi}{d\gamma} = \frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2}$$

3. **(a) Show that the function** $f(z) = \begin{cases} \dfrac{(\bar{z})^2}{z}, & z \neq 0 \\ 0 \quad , & z = 0 \end{cases}$

is continuous and the Cauchy-Riemann equations are satisfied at origin. Does $f'(0)$ exist? Justify your answer.

(b) Show that the function $u = 2x(3-y)$ is harmonic. Find the conjugate harmonic function v and express $u + iv$ as an analytic function of z.

(c) If $f(z)$ is an analytic function of z, show that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2.$$

(d) Find $\lim\limits_{z\to 0}\dfrac{\text{Im}(z^2)}{|z|^2}$ if exists. **(3+2+2+1)**

Ans 3a): $f(Z) = \begin{cases} \dfrac{(\bar{Z})^2}{Z} \quad , & Z \neq 0 \\ 0 \quad , \quad Z = 0 \end{cases}$

$$\frac{(\bar{Z})^2}{Z} = \frac{(\bar{Z})^3}{|Z|^2}$$

$$(\bar{Z})^3 = (x - iy)^3 = (x^3 + 3xy^2) + i(-3x^2y + y^3)$$

$$\therefore \frac{(\bar{Z})^3}{|Z|^2} = \left(\frac{x^3 + 3xy^2}{x^2 + y^2}\right) + i\left(\frac{-3x^2y + y^3}{x^2 + y^2}\right)$$

$$u = \frac{x(x^2 + 3y^2)}{(x^2 + y^2)} \qquad v = \frac{y(-3x^2 + y^2)}{x^2 + y^2}$$

$$u_x(0,0) = \lim_{h\to 0}\frac{u(h,0) - u(0,0)}{h} = 1$$

$u_y(0,0) = \lim_{k \to 0} \dfrac{u(0,k) - u(0,0)}{k} = 0$
$v_x(0,0) = \lim_{h \to 0} \dfrac{v(h,0) - v(0,0)}{h} = 0$
$v_y(0,0) = \lim_{k \to 0} \dfrac{v(0,k) - v(0,0)}{k} = 1$
$\therefore u_x = v_y$ and $u_y = -v_x$ [C-R equations satisfied]
$\lim_{(x,y)\to 0} \left[x \left(\dfrac{x^2+3y^2}{x^2+y^2} \right) + iy \left(\dfrac{y^2-3x^2}{x^2+y^2} \right) \right]$
Let $x = \gamma \cos\theta$, $y = \gamma \sin\theta$
$\therefore \lim_{\gamma \to 0} [\gamma(\cos^3\theta + 3\sin^2\theta\cos\theta) + i\gamma(\sin^3\theta - 3\cos^2\theta\sin\theta)] = 0$
$\therefore$ continuous at origin.
For differentiability,
$$\lim_{\triangle Z \to 0} \frac{\dfrac{(\triangle \bar{Z})^3}{|\triangle Z|^2} - 0}{\triangle Z}$$
$$= \lim_{(x,y)\to(0,0)} \frac{\left[x(\dfrac{x^2+3y^2}{x^2+y^2}) + iy(\dfrac{y^2-3x^2}{x^2+y^2}) \right]}{(x+iy)}$$
$$= \lim_{(x,y)\to(0,0)} \frac{1}{(x+iy)} \left[x(\frac{x^2+3y^2}{x^2+y^2}) + iy(\frac{y^2-3x^2}{x^2+y^2}) \right]$$
Let $y = mx$
$$= \lim_{(x,y)\to(0,0)} \frac{1}{x(1+im)} \left[x(\frac{x^2+3m^2x^2}{x^2+m^2x^2}) + imx(\frac{m^2x^2-3x^2}{x^2+x^2m^2}) \right]$$
$$= \frac{1}{(1+im)} \left[\frac{1+3m^2}{1+m^2} + (im) \left(\frac{m^2-3}{1+m^2} \right) \right]$$
It is a function of m.
$\therefore$ not differentiable.

Ans 3b): $u = 6x - 2xy$
$u_x = 6 - 2y \qquad u_y = -2x$
$u_{xx} = 0 \qquad u_{yy} = 0$
$\therefore u_{xx} + u_{yy} = 0$
Also for analytic function $u_x = v_y$ and $u_y = -v_x$
$\therefore v_y = 6 - 2y$
or $v = 6y - y^2 + f(x)$
$v_x = f'(x) = -u_y = 2x$
or $f(x) = x^2$
$\therefore v = 6y - y^2 + x^2$
$\therefore u + iv = (6x - 2xy) + i(6y - y^2 + x^2)$

Ans 3c): $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} \right)(u^2 + v^2)$ $\qquad u_x = v_y$ and $u_y = -v_x$ where $f(Z) = u + iv$
$2uu_{xx} + 2u_x^2 + 2vv_{xx} + 2v_x^2 + 2uu_{yy} + 2u_y^2 + 2vv_{yy} + 2v_y^2$
$= 2u(u_{xx} + u_{yy}) + 2v(v_{xx} + v_{yy}) + 2(u_x^2 + v_x^2 + u_y^2 + v_y^2)$
As $f(Z)$ is analytic, then it is harmonic
$\therefore u_{xx} + u_{yy} = v_{xx} + v_{yy} = 0$
Also $u_x = v_y$ and $u_y = -v_x$
$\therefore 4(u_y^2 + v_y^2) = 4|f'(Z)|^2$

4. (a) Does the integral $\int\left(x^2 - iy^2\right) dz$ depend upon the path $y = 2x^2$ from $1+i$ to $2+8i$? Justify your answer and evaluate the integral.

(b) Using Cauchy Integral Formula find the value $\int_C \frac{z+3i}{\left(z^2 - iz + 2\right)^3} dz$ where C is $|z-1-2i| = 2$ which oriented in the anti-clockwise direction.

(c) Evaluate $\int_C \frac{z^2 - 5z + 3}{(z+1)(z-2)^2} dz,$ where C is the circle $|z-2| = 1$ in the anticlockwise direction. **(2+3+3)**

Ans 4a): $(x^2 - iy^2)$ is analytic in the domain.

$\therefore$ it is not path dependent.

$$\therefore z = x + iy$$
$$= x + i(2x^2)$$
$$\therefore dz = dx + i(4x)dx = dx(1 + i4x)$$
$$\therefore \int_1^2 (x^2 - i4x^4)(1 + i4x)dx$$
$$= \int_1^2 (x^2 + i4x^3 - i4x^4 + 16x^5)dx$$
$$= \frac{1}{3}[7] + \frac{16}{6}[64-1] + 4i\left[\frac{1}{4}[15] - \frac{1}{5}[32-1]\right]$$
$$= \frac{7}{3} + \frac{8}{3} \times 63 + 4i\left[\frac{15}{4} - \frac{31}{5}\right]$$
$$= \frac{511}{3} - \frac{49i}{5}$$

Ans 4b): $\int_c \frac{z+3i}{(z^2 - iz + 2)^3}$

$$z^2 - iz + 2 = z^2 + iz - 2iz + 2$$
$$= z(z+i) - 2i(z+i)$$
$$= (z+i)(z-2i)$$

$C : |z - 1 - 2i| = 2$

For $z = -i$

$|-i - 1 - 2i| = |1 + 3i| = \sqrt{10} > 2$

For $z = 2i$

$|2i - 1 - 2i| = 1 < \sqrt{2}$

$\therefore$ Not analytic $\sqrt{2}$

$$\int_c \frac{(z+3i)dz}{(z+i)^3(z-2i)^3}$$
$$f(z) = \frac{z+3i}{(z+i)^3} = \frac{z}{(z+i)^3} + \frac{3i}{(z+i)^3}$$
$$f'(x) = \frac{(z+i)^3 - 3z(z+i)^2}{(z+i)^6} - \frac{3i \times 3}{(z+i)^4}$$
$$= \frac{1}{(z+1)^3} - \frac{3z}{(z+i)^4} - \frac{9i}{(z+i)^4}$$
$$f''(z) = \frac{-3}{(z_i)^4} - \frac{9i(-4)}{(z+i)^5} - 3\left[\frac{(z+i)^4 - z^4(z+i)^3}{(z+i)^8}\right]$$
$$= \frac{36i}{(z+i)^5} - \frac{3}{(z+i)^4} - \frac{3}{(x+i)^4} + \frac{127}{(z+i)^5}$$
$$\therefore f^2(2i) = \frac{36i}{3^5 i^4} - \frac{3}{3^4 i^4} - \frac{3}{3^4 i^4} + \frac{12(3i)}{3^5 i^5}$$

$$= \frac{9\times 4}{9\times 27} + \frac{4\times 9}{9\times 27} - \frac{3}{3\times 27} - \frac{3}{3\times 27}$$
$$= \frac{8}{27} - \frac{2}{27} = \frac{2}{9}$$
$$\therefore f^2(2i) = \frac{2}{2\pi i}\phi_c \frac{z+3i}{(z+i)^3}dz$$
$$\therefore \phi_c \frac{(z+3i)dz}{(z^2-iz+2)^3} = \frac{2\pi i}{9}$$

Ans 4c): $|z-2| = 1$

For $z = -1 : |-1-2| = 3 > 1$

For $z = 2 : |2-2| = 0 < 1$

$$\int \frac{z^2-5z+3}{(z+1)(z-2)^2}dz$$
$$f(z) = \frac{z^2-5z+3}{z+1}$$
$$f'(z) = \frac{(2z-5)(z+1)-(z^2-5z+3)}{(z+1)^2}$$
$$= \frac{2z^2+2z-5z-5-z^2+5z-3}{(z+1)^2}$$
$$= \frac{z^2+2z-8}{(z+1)^2} = \frac{(z+1)^2-9}{(z+1)^2} = 1 - \frac{9}{(z+1)^2}$$
$$f'(2) = 1 - \frac{9}{9} = 0$$
$$f'(2) = \frac{1!}{2\pi}\oint_c c$$
$$\text{or } \oint_c \frac{z^2-5z+3}{(z+1)(z-2)^2}dz = 0$$

5. (a) Solve the differential equation $(y\log x - 1)y\,dx = x\,dy$.

(b) For the differential equation $x\,dy - y\,dx = (x^2+y^2)dx$ determine that particular solution $y(x)$ for which $y = \frac{\pi}{2}$ when $x = \frac{\pi}{2}$.

(c) Find the general solution of $\frac{d^5y}{dx^5} - \frac{dy}{dx} = 12e^x + 8\sin x - 2x$.

(d) Find the Wronskian of $f(x) = x^2$ and $g(x) = e^{-x^2}$. **(2+2+3+1)**

Ans 5a): $\frac{dy}{dx} = \frac{y}{x}(y\ln x - 1)$

$(y^2\ln x - y)dx - xdy = 0$

$Mdx + Ndy = 0$

$y^2 lnxdx = ydx + xdy$

or $\ln x dx = \frac{ydx+xdy}{y^2}$

$$\frac{(\ln x)}{x^2} = \frac{d(xy)}{(xy)^2}$$
$$\text{or } \frac{(xy)^{-1}}{-1} = \frac{-\ln x - 1}{x} - c$$
$$\text{or } \frac{1}{xy} = \frac{\ln x + 1}{x} + c$$

Ans 5b): $xdy - ydx = (x^2+y^2)dx$

or $\dfrac{xdy - ydx}{x^2} = (1 + \left(\dfrac{y}{x}^2\right))dx$

or $\dfrac{d(\dfrac{y}{x})}{(1 + \left(\dfrac{y}{x}\right)^2)} = dx$

or $\tan^{-1}\left(\dfrac{y}{x}\right) = x + c$

or $\tan^{-1}(1) = \dfrac{\pi}{2} + c$

or $c = -\dfrac{\pi}{4}$

$\therefore \tan^{-1}\left(\dfrac{y}{x}\right) = x - \dfrac{\pi}{4}$

Ans 5c): $(D^5 - D)y = |2e^x + 3\sin x - 2x|$
to get complimentary function we solve
$(D^5 - D)y = 0$
$m^4(m-1) = 0$
$\Rightarrow m = 1, 0, 0, 0, 0$
$\therefore$ C.F. $= c_1e^x + c_2 + c_3x + c_4x^2 + c_5x^3$
To find the particular integral,

P.I. $= \dfrac{1}{D^4(D-1)}[12e^x + 8\sin x - 2x]$

$= 12e^x \times x + \left(-\dfrac{1}{2} \times 8(\sin x + \cos x) + \dfrac{1}{D^4}(1 + D + D^2 + \ldots)(2x)\right)$

$= 12e^x.x - 4\sin x + \dfrac{1}{D^4}(2x+2) - 4\cos x$

P.I. $= 12e^x.x - 4\sin x - 4\cos x + 2\left(\dfrac{x^5}{120} + \dfrac{x^4}{24}\right)$

General Solution $= c_1e^x + c_2 + c_3x + c_4x^2 + c_5x^3 + 12e^x.x - 4\sin x - 4\cos x + 2\left(\dfrac{x^5}{120} + \dfrac{x^4}{24}\right)$

Ans 5d): Wronskian(W) $= \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix}$

$f(x) = x^2$, $g(x) = e^{-x^2}$
$f'(x) = 2x$, $g'(x) = -2xe^{-x^2}$

$\therefore$ W $= \begin{vmatrix} x^2 & e^{-x^2} \\ 2x & -2xe^{-x^2} \end{vmatrix} = -2x^3e^{-x^2} - 2xe^{-x^2}$

W $= -2xe^{-x^2}(x^2+1)$

6. (a) Find the general solution of $x^2y'' - 3xy' + y = \dfrac{\log x \sin(\log x)}{x}$.

(b) Using the method of variation of parameters, find the particular solution of the differential equation $y'' - y = \dfrac{2}{1+e^x}$.

(c) Solve the simultaneous system of differential equations $\dfrac{dx}{dt} + 2y + \sin t = 0, \dfrac{dy}{dt} - 2x - \cos t = 0$ subject to the conditions $x = 1,\ y = 1$ at $t = 0$. **(3+3+3)**

Ans 6a): $x^2y'' - 3xy' + y = \dfrac{\log x \sin(\log x)}{x}$

Put $x = e^z$ or $z = \log x$

$(D'(D'-1) - 3D' + 1)y = \dfrac{z \sin z}{e^z}$

Complimentary function,

$(D^2 - 4D + 1)y = 0$

$y = c_1 e^{(2+\sqrt{3})x} + c_2 e^{(2-\sqrt{3})x}$

Particular Integral:-

$\dfrac{1}{D^3 - 4D + 1}\left(\dfrac{z \sin z}{e^z}\right)$

Using $\dfrac{1}{f(D)}(e^{ax}V) = e^{ax}\dfrac{1}{f(D+a)}V$

$= (e^{-z})\dfrac{1}{D^2 - 6D + 6}(z \sin z)$

Using $\dfrac{1}{f(D)}(xV) = x.\dfrac{1}{f(D)}V - \dfrac{f'(D)}{\{f(D)\}^2}V$

$= \dfrac{1}{e^z}\left[z\dfrac{1}{D^2 - 6D + 6}(\sin z) - \dfrac{(2D-6)}{(D^2 - 6D + 6)^2}(\sin z)\right]$

$= \dfrac{1}{e^z}\left[z\dfrac{1}{5 - 6D}(\sin z) - \dfrac{(2D-6)}{(25 - 36D^2 - 60D)}(\sin z)\right]$

$= \dfrac{1}{e^z}\left[z\dfrac{(5+6D)}{25 - 36D^2}(\sin z) - \dfrac{(2D-6)}{(60D + 11)}(\sin z)\right]$

$= \dfrac{1}{e^z}\left[\dfrac{z}{61}(5 + 6D)\sin z - \dfrac{(2D-6)(60D-11)}{(3600D^2 - 121)}(\sin z)\right]$

$= \dfrac{1}{e^z}\left[\dfrac{z}{61}(5\sin z + 6\cos z) + \dfrac{(120D^2 - 382D + 66)}{-3721}\sin z\right]$

$= \dfrac{1}{e^z}\left[\dfrac{z}{61}(5\sin z + 6\cos z) - \dfrac{(66\sin z - 382\cos z - 120\sin z)}{3721}\sin z\right]$

$\therefore$ General Solution = C.F. + P.I.

$= c_1 e^{(2+\sqrt{3})x} + c_2 e^{(2-\sqrt{3})x} + \dfrac{1}{e^z}\left[\dfrac{z}{61}(5\sin z + 6\cos z) - \dfrac{(66\sin z - 382\cos z - 120\sin z)}{3721}\sin z\right]$

Ans 6b): $y'' - y = \dfrac{2}{1 + e^x}$

Complimentary solution :

$(D^2 - 1)y = 0$

$m^2 - 1 = 0$

$\Rightarrow m = \pm 1$

$y = c_1 e^x + c_2 e^{-x}$

$u = e^x$, $v = e^{-x}$

P.I. : $y = L_1 u + L_2 v$

W $= \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1 - 1 = -2$

$L_1' = \left(-\dfrac{2e^{-x}}{1 + e^x}\right) \times -\dfrac{1}{2}$

$\Rightarrow \displaystyle\int L_1' = \int \dfrac{e{-}x}{1 + e^x}dx$

$\Rightarrow L_1 = \ln(e^x + 1) - e^{-x} - x$

$L_2' = \dfrac{2e^x}{1 + e^x} \times -\dfrac{1}{2} = \dfrac{-e^x}{1 + e^x}$

$\Rightarrow \displaystyle\int L_2' = \int -\dfrac{e^x}{1 + e^x}$

$\Rightarrow L_2 = -\ln(e^x + 1)$

P.I. $= (\ln(e^x + 1) - e^{-x} - x)e^x + (-\ln(e^x + 1))e^{-x}$

$\quad = e^x \ln(e^x + 1) - 1 - xe^x - \ln(e^x + 1)e^{-x}$

Ans 6c): $\dfrac{dx}{dt} + 2y + \sin t = 0 \;\Big|\; \dfrac{dy}{dt} - 2x - \cos t = 0 \quad \longrightarrow (2)$

Differentiating with respect to t

$\dfrac{d^2x}{dt^2} + \dfrac{2dy}{dt} + \cos t = 0 \quad \longrightarrow (1)$

(1) - $2x$(2)

$\dfrac{d^2x}{dt^2} + 4x + 3\cos t = 0$

$(D^2 + 4)x = -3\cos t$

P.I. :-

$x = \dfrac{1}{(D^2+4)}(-3\cos t)$

$x = -\cos t$

Solution : $x = c_1 \cos 2t + c_2 \sin 2t - \cos t$

Putting $t = 0$ and $x = 1$

$\Rightarrow c_1 = 2$

$\therefore x = 2\cos 2t - cost + c_2 \sin 2t$

$y = -\dfrac{1}{2}\left(\dfrac{dx}{dt} + \sin t\right)$

$y = -\dfrac{1}{2}\left(-4\sin 2t + \sin t + 2c_2 \cos 2t + \sin t\right)$

$y = -\dfrac{1}{2}\left(2c_2 \cos 2t - 4\sin 2t + 2\sin t\right)$

Putting $t = 0$ and $y = 1$

$\Rightarrow c_2 = -1$

$\therefore x = 2\cos 2t - \sin 2t - \cos t$

$\therefore y = 2\sin 2t - \sin t + \cos 2t$

MATHS II

MID SEMESTER EXAMINATION **2017-2018**

1. (a) **Find whether the vectors $2x^3+x^2+x+1$, x^3+3x^2+x-2 and x^3+2x^2-x+3 of $P(x)$, the vector space of all polynomials over the real field, are linearly independent or not.**

   (b) **Let $V=\mathbb{R}^2$ be the set of all ordered pairs (x,y) of real numbers. Examine whether V is a vector space over $\mathbb{R}$ with the following two operations:**

$$(x,y)+(x_1,y_1)=(x+x_1,y+y_1),$$

$$c(x,y)=(|c|x,|c|y),$$

where $(x,y),(x_1,y_1)\in V$ and $c\in\mathbb{R}$. **[2 + 2 = 4 marks]**

Ans a): Let a,b,c be scalars (real) such that

$$a(2x^3+x^2+x+1)+b(x^3+3x^2+x-2)+c(x^3+2x^2-x+3)=0$$

$$(2a+b+c)x^3+(a+3b+2c)x^2+(a+b-c)x+(a-2b+3c)=0$$

Equating coefficients of like powers of x,

$2a+b+c=0$

$a+3b+2c=0$

$a+b-c=0$

$a-2b+3c=0$

$$\text{Coefficient matrix} \Rightarrow A=\begin{bmatrix}2&1&1\\1&3&2\\1&1&-1\\1&-2&3\end{bmatrix}\sim\begin{bmatrix}1&3&2\\2&1&1\\1&1&-1\\1&-2&3\end{bmatrix}\ldots\ldots\ldots\ldots[R_1\sim R_2]$$

$$\sim\begin{bmatrix}1&3&2\\0&-5&-3\\0&-2&-3\\0&-5&1\end{bmatrix}\qquad \begin{aligned}R_2&=R_2-2R_1\\R_3&=R_3-R_1\\R_4&=R_4-R_1\end{aligned}$$

$$\sim\begin{bmatrix}1&3&2\\0&1&3/5\\0&-2&-3\\0&-5&1\end{bmatrix}\qquad R_2=(-1/5)R_2$$

$$\sim\begin{bmatrix}1&3&2\\0&1&3/5\\0&0&1\\0&0&4\end{bmatrix}\qquad R_3=(-5/9)R_3$$

So clearly linearly independent.

b): Let V = $\mathbb{R}^2$ be the set of all ordered pairs (x,y) of real numbers.

Operations: $(x_1,y_1)+(x_2,y_2)=(x_1+x_2\ ,\ y_1+y_2)$

$c(x,y)=(|c|x,|c|y)$

In this case,

$(a+b)\alpha=a\alpha+b\alpha$ . . . . .[where $a,b\in\mathbb{R}$ and $\alpha\in V$]

So let $\alpha=(x,y)$ and [a,b$\in\mathbb{R}$]

$\Rightarrow (a+b)(x,y)=|a+b|(x,y)$

but,

$$\begin{aligned}(a+b)(x,y) &= a(x,y)+b(x,y)\\ &= (|a|x,|a|y)+(|b|x,|b|y)\\ &= (|a|+|b|)(x,y)\end{aligned}$$

But we know,

$|a+b| \le |a|+|b|$ triangle inequality

hence V($\mathbb{R}$) is not a vector space.

2. (a) Find the dimension of the subspace V of $\mathbb{R}^4$, spanned by $S = \{(1,-1,2,3),(11,24,29,61),(2,3,5,10)\}$. Also find a subset of S, which is a basis of V.(NO MARKS will be awarded if the basis is not a subset of S)

   (b) Let $T:\mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by $T(x,y,z) = (2x-3y+4z, -x-y-z)$. Then

   (i) find a basis of $N(T)$, the null space of T,

   (ii) find the dimension of the range space $R(T)$ of T. [2 + 2 = 4 marks]

Ans a): $A = \begin{bmatrix} 1 & -1 & 2 & 3\\ 11 & 24 & 29 & 61\\ 2 & 3 & 5 & 10\end{bmatrix}$

$\sim \begin{bmatrix} 1 & -1 & 2 & 3\\ 0 & 5 & 1 & 4\\ 0 & 0 & 0 & 0\end{bmatrix}$

$\therefore dim(V) = 2$ . . . .[since dim(V) = number of linearly independent vectors in the echelon form]

Any 2 vectors in S form basis; for example (1,-1,2,3),(2,3,5,10)

b): N(T) = span $\begin{bmatrix} -7/5\\ 1/5\\ 1\end{bmatrix}$

Basis of N(T) = $\begin{bmatrix} -7/5\\ 2/5\\ 1\end{bmatrix}$

any non-zero vector in N(T) is its basis.

By RN theorem , dim(R(T)) + dim(N(T)) = dim(V) $\Rightarrow$ dimR(T) = 2

3. (a) Using rank concept of matrices, find the real constants a,b,c,d, if exist, such that the graph of the function

$$f(x) = ax^3+bx^2+cx+d,$$

passes through the points $(-1,5),(-2,7),(2,11)$ and $(3,37)$.

   (b) Find a system of linear equations whose set of solutions is given by

$$S = \{(1-a,1+a,a) : a\in\mathbb{R}\}$$

Ans 3a): $f(-1) = -a+b-c+d = 5$

$f(-2) = -3a+4b-2c+d = 7$

$f(2) = 8a+4b+2c+d = 11$

$f(3) = 27a+9b+3c+d = 37$

$$\begin{bmatrix} -1 & 1 & -1 & 1 \\ -8 & 4 & -2 & 1 \\ 3 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 11 \\ 37 \end{bmatrix}$$

on solving the augmented matrix,

$$\begin{bmatrix} -1 & 1 & -1 & 1 & | & 5 \\ -8 & 4 & -2 & 1 & | & 7 \\ 8 & 4 & 2 & 1 & | & 11 \\ 27 & 9 & 3 & 1 & | & 37 \end{bmatrix}$$

we find; $a = 1, b = 2, c = 3, d = 1$

$f(x) = x^3 + 2x^2 - 3x + 1$

b): x = 1-a; y = 1+a; z = a;

$\Rightarrow$ x + y = 2

$\Rightarrow$ x + z = 1

4. (a) Check the diagonalizability of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If it is diagonalizable, then find a diagonal matrix D which is similar to A.

(b) Find all real values a and b such that the matrix

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & ia \\ \frac{-i}{\sqrt{2}} & b \end{bmatrix}$$

is unitary. [3 + 2 = 5 marks]

Ans 4a): Finding eigenvalues;

$$\begin{bmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} \Rightarrow (1-\lambda)(\lambda^2 - 1) = 0$$

$\therefore \lambda = 1, 1, -1$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

From here we get eigenvectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

And from the other eigenvalue we get the eigenvector $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

Thus $D = P^{-1}AP$ . . . . where P = $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \\ 0.5 & -0.5 & 0 \end{bmatrix}$

From here we get D = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

b): For A to be a unitary matrix A^* has to be equal to A^{-1}

$\Rightarrow \begin{bmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ -ia & b \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & ia \\ -i/\sqrt{2} & b \end{bmatrix} = I$

$\Rightarrow \begin{bmatrix} 1 & i(a/\sqrt{2}+b/\sqrt{2}) \\ -i(a/\sqrt{2}+b/\sqrt{2}) & (a^2+b^2) \end{bmatrix} = I$

$\Rightarrow a = -b$ and $a^2+b^2 = 1$

$\Rightarrow (a,b) = (1/\sqrt{2}, -1/\sqrt{2})$

$\Rightarrow (a,b) = (-1/\sqrt{2}, 1/\sqrt{2})$

5. (a) Rearrange the augmented matrix of the following system by only row-interchanges to place maximum-magnitude(absolute value) element of each row of the coefficient matrix in diagonal position, and then make these diagonal elements unity by appropriate elementary row-operations to augmented matrix(with entries 4 decimal places rounding):

$$\begin{aligned} 0.45x_1 + 0.30x_2 - 15.00x_3 &= 14.28 \\ 4.50x_1 + 0.15x_2 + 0.30x_3 &= 1.57 \\ 0.15x_1 - 10.50x_2 + 0.45x_3 &= -3.86. \end{aligned}$$

(b) Write down the three algebraic equations(not in matrix form) of Jacobi iteration scheme from the modified augmented matrix, obtained in (a), determining $x_i^{(n+1)}, i = 1,2,3; n = 0,1,2,\ldots.$

(c) Compute all iteration values up to 4 decimal places (round off) in tabular form under columns $n, x_1^{(n)}, x_2^{(n)}, x_3^{(n)}$, for $n = 0,1,2,3$ only, using the equations obtained in (b), with starting solution $x_i^{(0)} = 0$, for $i = 1,2,3$.

(d) From the modified augmented matrix obtained in (a), write down only the three algebraic equations(not in matrix form) of Gauss-Seidel method determining $x_i^{(n+1)}, i = 1,2,3; n = 0,1,2,\ldots.$ [1 + 0.5 + 2 + 0.5 = 4 marks]

Ans 5a): $\begin{bmatrix} 0.45 & 0.3 & -15.00 & 14.28 \\ 4.5 & 0.15 & 0.3 & 1.57 \\ 0.15 & -10.5 & 0.45 & -3.86 \end{bmatrix}$ . . . .[Diagonally dominant means $|a_{i1}| \geq \sum_{j=1}^{no.of col^n} a_{ij}$]

Making it diagonally dominant and solving,

$\begin{bmatrix} 1.00 & 0.03333 & 0.0667 & 0.3489 \\ -0.0143 & 1.00 & -0.0429 & 0.3676 \\ -0.0300 & -0.0200 & 1.00 & -0.9520 \end{bmatrix}$

5b): Using Gauss Jacobi

$x_1^{n+1} = 0.3489 - 0.0333x_2^{(n)} - 0.0667x_3^{(n)}$

$x_2^{n+1} = 0.3676 + 0.0143x_1^{(n)} + 0.0429x_3^{(n)}$

$x_3^{n+1} = 0.9520 + 0.0300x_1^{(n)} + 0.0200x_2^{(n)}$

5c):

n	$x_1^{(n)}$	$x_2^{(n)}$	$x_3^{(n)}$
0	0.0000	0.0000	0.0000
1	0.3489	0.3676	-0.9520
2	0.4002	0.3317	-0.9342
3	0.4002	0.3332	-0.9334

5b): Using Gauss Siedal method

$x_1^{n+1} = 0.3489 - 0.0333x_2^{(n)} - 0.0667x_3^{(n)}$
$x_2^{n+1} = 0.3676 + 0.0143x_1^{(n+1)} + 0.0429x_3^{(n)}$
$x_3^{n+1} = 0.9520 + 0.0300x_1^{(n+1)} + 0.0200x_2^{(n+1)}$

6. (a) Find a root of the equation $e^{2x} = x + 6$ correct up to 4 decimal places by using the Newton-Raphson method starting with initial guess $x_0 = 0.97$.

(b) The algebraic equation $x^4 + x - 1 = 0$ has a root α lying in the interval $[0.5, 1.0]$. Starting with this interval, use bisection method to find an interval of width 0.125 which contains α. [2 + 2 = 4 marks]

Ans a): Using Newton Rhapson,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{e^{2x_n} - x_n - 6}{2e^{2x_n} - 1}$$

Given $x_o = 0.97$
We get root $= 0.9705$

6b): $f(0.5) = -0.4375$
$f(1.0) = 1$
$(0.5 + 1)/2 = 0.75;$ $f(0.75) = 0.06640$
$\therefore$ root lies between 0.5 and 0.75
$(0.5 + 0.75)/2 = 0.625$ $f(0.75) = -0.2224$
So root lies between (0.625,0.75)

7. (a) Determine the interpolation polynomial by means of Lagrange interpolation formula for the points $(-2, -10), (0, 2), (1, 8)$.

(b) Suppose the function $f(x) = \ln(1 + x)$ is approximated using polynomial interpolation with $x = 0$ and $x = 1$. Let $p(x)$ denote the interpolating polynomial. Find an upper bound (as small as possible) for the error magnitude $\max\limits_{0 \le x \le 1} |f(x) - p(x)|$ without determining $p(x)$.

[2 + 3 = 5 marks]

Ans 7a): Usinmg Lagrange's Interpolation

$$P_x(x) = \frac{(x - x_1)(x - x_2)}{(x_o - x_1)(x_o - x_2)}y_o + \frac{(x - x_2)(x - x_o)}{(x_1 - x_2)(x_1 - x_o)}y_1 + \frac{(x - x_o)(x - x_1)}{(x_2 - x_o)(x_2 - x_o)}y_2$$

$\therefore$ we get,
f(x) = 6x + 2

b): $$\max_{0\le x\le 1}|f(x)-p(x)| \;\le\; \max_{0\le t\le 1}\frac{|f^{(2)}(t)|}{2!}\times max|(x-x_o)(x-x_1)|$$
$$= \frac{1}{2}\;\max_{0\le t\le 1}\frac{1}{(1+t)^2}\;\max_{0\le x\le 1}|(x-0)(x-1)|$$

$$\Rightarrow \max_{0\le t\le 1}\frac{1}{(1+t)^2} = 1$$

$$\Rightarrow \max_{0\le x\le 1}|(x-0)(x-1)| = \frac{1}{4}$$

[since x lies between 0 and 1 (x-0)(x-1) is -ve and min value of f(x)=-1/4 at x=1/2. Thus $|min|$ becomes the maximum value]

$$\therefore \max_{0\le x\le 1}|f(x)-p(x)| \;\le\; \frac{1}{8}$$

MATHS II

MID SEMESTER EXAMINATION **2017**

1. a) Find the dimension and basis of a cubic polynomial P(x) given that P(1)=P(2)=P(3).

Ans 1a): P(x) is a cubic polynomial with P(1)=P(2)=P(3). We can assume,
$P(x) = \alpha(x-1)(x-2)(x-3) + \beta$
So the basis of this polynomial space, [1,(x-1)(x-2)(x-3)]
Hence the dimension of the polynomial space = 2

Alternate Solution (lengthy process): Assume a cubic polynomial $P(x) = a_o + a_1x + a_2x^2 + a_3x^3$
Now, P(1)=P(2)=P(3)
$\Rightarrow a_o + a_1 + a_2 + a_3 = a_o + 2a_1 + 4a_2 + 8a_3 = a_o + 3a_1 + 9a_2 + 27a_3$
taking two at a time,
$\Rightarrow a_1 + 3a_2 + 7a_3 = 0$ . . . . .(1)
$\Rightarrow a_1 + 5a_2 + 19a_3 = 0$ . . . . . .(2)
$\Rightarrow 2a_1 + 8a_2 + 26a_3 = 0$ . . . .(3)
Solve this via Gaussian Elimination method or it is clear that (3)=(1)+(2). So there will be only two independent equations for 4 variables, namely, a_o, a_1, a_2 and a_3.
Thus Dimension = 2, and basis can be found via Gaussian Elimination as [1, (x-1)(x-2)(x-3)].

2. a) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear transformation defined by T(1,0,0)=(1,0,1), T(0,1,0)=(1,1,0) and T(0,0,1)=(0,1,1). Find the form of linear transformation T. Also find the null-space N(T) of T and its dimension.

Ans 2a): Let $(x, y, z) \in \mathbb{R}^3$ Then,
$(x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$

$$\begin{aligned} \text{Then } T(x,y,z) &= xT(1,0,0) + yT(0,1,0) + zT(0,0,1) \\ &= x(1,0,1) + y(1,1,0) + z(0,1,1) \\ &= (x+y, y+z, x+z) \end{aligned}$$

$\therefore T(x, y, z) = (x + y, y + z, x + z), \forall (x, y, z) \in \mathbb{R}^3$
Let $(x, y, z) \in N(T)$
Then $T(x, y, z) = \theta$, where θ is the zero element in $\mathbb{R}^3$.
$(x + y, y + z, x + z) = (0, 0, 0)$
$(x + y) = 0, (y + z) = 0, (x + z) = 0$
$\therefore$ the solution is x=0, y=0, z=0.
$\therefore$ null space of T, N(T) = $\{\theta\}$
$\therefore$ dimension of N(T) = 0.

2 b)Let $T : P_3(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$ be the linear transformation defined by

$$T(f(x)) = f'(x)$$

where the prime(') denotes derivative with respect to x. Find the matrix representation of T w.r.t the standard ordered basis $\{1, x, x^2, x^3\}$ for $P_3(\mathbb{R})$ and $\{1, x, x^2\}$ for $P_2(\mathbb{R})$ respectively.

Ans 2b): Given $T : P_3(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$ defined by T(f(x)) = f'(x)

$$\begin{aligned} \text{Now, } T(1) &= f'(1) = 0 = 0 \times 1 + 0x + 0x^2 \\ T(x) &= f'(x) = 1 = 1 \times 1 + 0x + 0x^2 \\ T(x^2) &= f'(x^2) = 2x = 0 \times 1 + 2x + 0x^2 \\ T(x^3) &= f'(x^3) = 3x^2 = 0 \times 1 + 0x + 3x^2 \end{aligned}$$

$\therefore$ Matrix representation of T wrt. the basis for both $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$ is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

3. a) Consider the following equations
$x + 2y + 3z = 6$
$y + (\lambda - 3)z = 3$
$y - z = \mu - 12$

What are the values of λ and μ for
(i) No solution
(i) Unique solution
(i) Infinitely many solutions

Ans 3a): Augumented matrix,

$$[A|b] = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & \lambda-3 & 3 \\ 0 & 1 & -1 & \mu-12 \end{bmatrix}$$

$R_3 \longrightarrow R_3 - R_2$

$$\Rightarrow [A|b] = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & \lambda-3 & 3 \\ 0 & 0 & 2-\lambda & \mu-15 \end{bmatrix} \qquad \text{(row echelon form)}$$

(i)No solution: $\lambda = 2, \mu \neq 15$
(ii)Unique solution: $\lambda \neq 2, \mu \in \mathbb{R}$
(iii)Infinitely many solutions: $\lambda = 2, \mu = 15$

3 b) Write the given matrix in row-echelon form and thus find its rank.

$$A=\begin{bmatrix} 3 & 0 & 1 & 2 \\ 6 & 1 & 0 & 0 \\ 12 & 1 & 2 & 4 \\ 6 & 0 & 2 & 4 \\ 9 & 0 & 1 & 2 \end{bmatrix}$$

Ans 3b): $A=\begin{bmatrix} 3 & 0 & 1 & 2 \\ 6 & 1 & 0 & 0 \\ 12 & 1 & 2 & 4 \\ 6 & 0 & 2 & 4 \\ 9 & 0 & 1 & 2 \end{bmatrix}$

$R_2 \longrightarrow R_2 - 2R_1, \quad R_3 \longrightarrow R_3 - 4R_1, \quad R_4 \longrightarrow R_4 - 2R_1, \quad R_5 \longrightarrow R_5 - 3R_1$

$$A=\begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 1 & -2 & -4 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{(row-echelon form)}$$

Rank of A = 3.

4 a)For the given matrix A, find all its eigen values and their respective geometric multiplicity.

$$A=\begin{bmatrix} -1 & -2 & 2 \\ -2 & -1 & 4 \\ -1 & 1 & 2 \end{bmatrix}$$

Ans 4a): $|A - \lambda I| = 0$

$$\begin{vmatrix} -1-\lambda & -2 & 2 \\ -2 & -1-\lambda & 4 \\ -1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda(\lambda^2 - 9) = 0$
$\Rightarrow \lambda = 0, -3, 3$
For $\lambda = 0$

$$\begin{bmatrix} -1 & -2 & 2 \\ -2 & -1 & 4 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving for x,y and z using Gaussian Elimination.

$R_2 \rightarrow R_2 - (R_1 + R_3)$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -2 \\ 0 & 0 & 0 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \longrightarrow R_3 - R_1$ and $R_2 \leftrightarrow R_3$

$$\Rightarrow \begin{bmatrix}1 & 2 & -2\\0 & -3 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \qquad \text{(row-echelon form)}$$

assuminf z to be free variable, let z = α

$3y = 0 \Rightarrow y = 0$

$x = 2z - 2y = 2\alpha$

$\therefore$ eigen vecor,

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \alpha\begin{bmatrix}2\\0\\1\end{bmatrix} \qquad \text{where } \alpha \in \mathbb{R} - \{0\}$$

foe $\lambda = 3$,

$$\begin{bmatrix}-4 & -2 & 2\\-2 & -4 & 4\\-1 & 1 & -1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Similarly solving (x,y,z) as shown in the case of $\lambda = 0$ via Gaussian Elimination,

eigen vector, (x,y,z) = β(0,1,1) where $\beta \in \mathbb{R} - \{0\}$

for $\lambda = 3$,

$$\begin{bmatrix}2 & -2 & 2\\-2 & 2 & 4\\1 & 1 & 5\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Similarly solving (x,y,z) as shown in the case of $\lambda = 0$ via Gaussian Elimination,

eigen vector, (x,y,z) = γ(0,1,1) where $\gamma \in \mathbb{R} - \{0\}$

4 b)The following matrix satisfies the equation, $A^3 = aA^2 + bA + c$

$$\begin{bmatrix}3 & 1 & 1\\-1 & 5 & -1\\1 & 1 & 3\end{bmatrix}$$

Find the values of a,b and c.

Ans 4b): We can solve this question by finding A^2 and A^3 and then putting them in the equation to get the desired value, but Caley Hamilton's method gives us an easier way out.

Characteristic equation: $\begin{bmatrix}3-\lambda & 1 & 1\\-1 & 5-\lambda & -1\\1 & 1 & 3-\lambda\end{bmatrix} = 0$

$\lambda^3 - 11\lambda^2 + 40\lambda - 44 = 0$

Every matrix satisfies its own characteristic equation.

$\Rightarrow A^3 - 11A^2 + 40A - 44I = 0$

$\Rightarrow A^3 = 11A^2 - 40A + 44I$

Comparing with $A^3 = aA^2 + bA + c$ we get,

a=11, b=-40 and c=44

5 a) Check the diagonizability of the matrix

A=$\begin{bmatrix}6 & -2 & 2\\-2 & 3 & -1\\2 & -1 & 3\end{bmatrix}$

If A is diagonizable then find an invertable matrix P such that $P^{-1}AP = D$ **where D is the diagonal matrix of A.**

Ans 5a): $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix}6-\lambda & -2 & 2\\-2 & 3-\lambda & -1\\2 & -1 & 3-\lambda\end{vmatrix} = 0$$

$\Rightarrow \lambda = 2, 2, 8$

for $\lambda = 2$,

$$\begin{bmatrix}4 & -2 & 2\\-2 & 1 & -1\\2 & -1 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_3 \longrightarrow R_3 + R_2$ and then $R_2 \longrightarrow R_2 + (1/2)R_1$

$$\begin{bmatrix} 4 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Assuming $y = \alpha$ and $z = \beta$

4x = 2y - 2z $\Rightarrow x = \dfrac{\alpha - \beta}{2}$

Eigen vectors,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix} \qquad \alpha, \beta \in \mathbb{R}$$

Algebraic multiplicity of $\lambda = 2$ is 2.

Geometric multiplicity of $\lambda = 2$ is 2. (2 eigen vectors)

for $\lambda = 8$,

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \longrightarrow R_3 - (R_2 - 2R_1)$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \longrightarrow R_2 - R_1$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Assuming $z = \gamma$

$y = -\gamma$

$x = \dfrac{2y - 2z}{-2} = z - y = 2\gamma$

$(x, y, z) = \gamma(2, -1, 1)$ where $\gamma \in \mathbb{R}$

Algebraic multiplicity for $\lambda = 8$ is 1.

Geometric multiplicity for $\lambda = 8$ is 1.

As the geometric multiplicity and algebraic multiplicity of the eigen values i.e. 2,2,8 are respectively same so the matrix A is diagonizable.

$$P = \begin{bmatrix} 1/2 & -1/2 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$ (Any scalar multiples of the eigen vectors as columns will also be acceptable)

(You can interchange the first two columns among themselves and also place the 3rd column first.)

Thus diagonal matrix,

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

and the invertible matrix,

$$P = \begin{bmatrix} 1/2 & -1/2 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

5 b)If A and B are similar matrices then show that A and B has the same eigen values.

Ans 5b): It is sufficient to show that A and B have the same characteristic polynomials.

Let $P^{-1}AP = B$ (A and B are similar)

$$\begin{aligned} \text{Now let, } (B - \lambda I) &= det(P^{-1}AP - \lambda P^{-1}P) \\ &= det(P^{-1}(A - \lambda I)P) \\ &= det(P^{-1})det(A - \lambda I)det(P) \\ &= det(A - \lambda I) \end{aligned}$$

$\therefore det(B - \lambda I) = det(A - \lambda I)$

Therefore A and B have same eigen values if they are similar.

6. a)Make the following matrix strictly diagonally dominant by rows :
$x_1 + 2x_2 - 8x_3 - 3x_4 = 6$
$x_1 + 7x_2 + 3x_3 - x_4 = 9$
$5x_1 - x_2 - x_3 - 8x_4 = 10$
$4x_1 + x_2 + x_3 + x_4 = 5$
Hence apply Jacobi's method to perform three iterations with the initial guess (0,0,0,0)
Ans 6a): diagonally dominant :

$$\begin{array}{rcl|l}
4x_1 + x_2 + x_3 + x_4 & = 5 & & x_1^{(k+1)} = \frac{1}{4}(5 - x_2^{(k)} - x_3^{(k)} - x_4^{(k)}) \\
2x_1 + 7x_2 + 3x_3 - x_4 & = 9 & & x_2^{(k+1)} = \frac{1}{7}(9 - 2x_1^{(k)} - 3x_3^{(k)} + x_4^{(k)}) \\
x_1 + 2x_2 - 8x_3 - 3x_4 & = 6 & & x_3^{(k+1)} = -\frac{1}{8}(6 - x_1^{(k)} - 2x_2^{(k)} + 3x_4^{(k)}) \\
5x_1 - x_2 - x_3 - 8x_4 & = 10 & & x_4^{(k+1)} = -\frac{1}{8}(10 - 5x_1^{(k)} + x_2^{(k)} + x_3^{(k)})
\end{array}$$

$$\begin{array}{l|l|l}
x_1^{(1)} = 1.25 & x_1^{(2)} = 1.4286 & x_1^{(3)} = 1.0670 \\
x_2^{(1)} = 1.2857 & x_2^{(2)} = 1.0714 & x_2^{(3)} = -0.7168 \\
x_3^{(1)} = -0.75 & x_3^{(2)} = 0.1964 & x_3^{(3)} = -0.1027 \\
x_4^{(1)} = -1.25 & x_4^{(2)} = -0.5357 & x_4^{(3)} = -0.5156
\end{array}$$

6. b) Consider $2x_1 - x_2 = 7$
$-x_1 + 2x_2 - x_3 = 1$
$-x_2 + 2x_3 = 1$
Use Gauss-Seidel method to perform 2 iterations.

Ans 6b): $x_1^{(k+1)} = \frac{1}{2}(7 + x_2^{(k)})$
$$x_2^{(k+1a)} = \frac{1}{2}(1 + x_1^{(k+1)} + x_3^{(k)})$$
$$x_3^{(k+1)} = \frac{1}{2}(1 + x_2^{(k+1)})$$
using initial guess as (0,0,0)

$$\begin{array}{ll}
x_1^{(1)} = 3.5 & x_1^{(2)} = 4.625 \\
x_2^{(1)} = 2.25 & x_2^{(2)} = 3.625 \\
x_3^{(1)} = 1.625 & x_3^{(2)} = 2.3125
\end{array}$$

7. (a) Using Newton-Raphson method, find a real root of the equation $\cos x = xe^x$, correct up to three decimal places with the initial approximation $x_0 = 0.5$, where x is in radians.

(b) Let f be a twice differentiable and bounded function in the small neighborhood I of the exact root of $f(x) = 0$. Then show that Newton-Raphson method will converge if $|f(x)f''(x)| < |f'(x)|^2$ holds for all x in I.

(c) Find the minimum number of iterations required to get the approximate root of an equation $f(x) = 0$ in the interval $[2, 5]$ with an accuracy less than $\epsilon = 0.005$ in bisection method.

Ans 7a): $f(x) = \cos x - xe^x$

$f'(x) = -(\sin x + (1 + x)e^x)$
Newton Raphson formula,
$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$
So $x_{n+1} = x_n + \dfrac{\cos x_n - x_n e^{x_n}}{\sin x_n + (1 + x_n)e^{x_n}}$
Let $x_o = 0.5$ then $x_1 = 0.5 + \dfrac{0.0532}{2.9525} = 0.5183$
Putting $x_1 = 0.5183$ we get, $x_2 = 0.5183 + \dfrac{-0.00082958}{3.043513} = 0.517757$
Putting $x_2 = 0.517757$ we get, $x_3 = 0.517757$
Now rounding off to 3 decimal places we get, $0.517757 \approx 0.518$
Ans : 0.518

Ans 7b): To find the root of $f(x) = 0$ by using Newton Raphson method, we have the formula
$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$
On comparing with fixed point iteration method (ie. $x = \phi(x)$) we see that,
$\phi(x) \equiv x - \dfrac{f(x)}{f'(x)}$
Hence this method will converge if $|\phi'(x)| < 1$
$\Rightarrow |f(x)f''(x)| < |f'(x)|^2$

Ans 7c): Let n be the number of iterations required by bisection method to achieve an accuracy of $> \epsilon$, then corresponding formula is
$\left|\dfrac{b-a}{2^n}\right| \leq \epsilon$
i.e. $n \geq \left[log_e\left(\dfrac{|b-a|}{\epsilon}\right)\right] \div log_e 2$
Here $|b - a| = 3$, $\epsilon = 0.005$
Thus $n \geq 9.2289$ i.e. n=10

MATHS II

MID SEMESTER EXAMINATION **2016**

1. By using row-echelon form, determine the value of k so that the following system of linear equations
$kx + y + z = 1$
$x + ky + z = 1$
$x + y + kz = 1$
has (i) a unique solution (ii) No solution (iii) Infinite number of solutions.

Ans 1): $kx + y + z = 1$
$x + ky + z = 1$
$x + y + kz = 1$
So we form augmented Matrix

$$A|B = \begin{bmatrix} k & 1 & 1 & | & 1 \\ 1 & k & 1 & | & 1 \\ 1 & 1 & k & | & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & k & 1 & | & 1 \\ 0 & 1-k & k-1 & | & 0 \\ 0 & 0 & -k^2-k+2 & | & 1-k \end{bmatrix}$$

(i) for k $\neq$ 1 and k $\neq$ -2
rank(A) = rank(A|B) = 3
Therefore unique solution.
(ii) For k = -2, rank(A) $\neq$ rank(A|B)
therefore no solution.
(iii) For k=1; rank(A) = rank(A,b) = 1<3

2. (i) Find the dimension of the subspace V of $\mathbb{R}^4$, where $V = [(2x{+}y{+}z,\ \ -x{+}y{-}2z,\ \ x{-}z,\ \ 2x{+}2y) \in \mathbb{R}^4]$ **where** $(x, y, z) \in \mathbb{R}$
(ii) Verify whether
W = $\{(x, y) : x, y \in \mathbb{R}$ either$x = 0$ or $y = 0$ or both $x = 0 = y\}$
is a subspace of $\mathbb{R}^2$

Ans 2(i): $V = (2x + y + z, -x + y - 2z, x + z, 2x + 2y)$
Thus, if $v \in V$, then,
$v = x(2, -1, 1, 2) + y(1, 1, 0, 2) + z(1, -2, 1, 0)$
[i.e. the subspaces are generated by vectors
$v_1 = (2, -1, 1, 2)$; $v_2 = (1, 1, 0, 2)$; $v_3 = (1, -2, 1, 0)$]
To check linear dependence,

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 & 2 \\ 1 & 1 & 0 & 2 \\ 1 & -2 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -1 & 1 & 2 \\ 0 & -3 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there are 2 non-zero rows $\Rightarrow dim(V) = 2$.
2(ii): Note that (0,0)$\in$W
Let (a,0), (0,b) $\in$ W . . . . [where $a \neq 0; b \neq 0$]
but (a,0) + (0,b) = (a,b) $\notin$ W

$\therefore$ W is not a subspace of $\mathbb{R}^2$.

3. Find the matrix of linear transformation on $\mathbb{R}^3 \leftrightarrow \mathbb{R}^3$ w.r.t the basis $(1,1,0),(0,1,0)$ and $(0,0,1)$ given $T(1,1,0)=(3,4,0)$; $T(0,1,0)=(2,1,0)$ and $T(0,0,1)=(5,-4,1)$.

Ans 3): $(x,y,z)=\alpha(1,1,0)+\beta(0,1,0)+\gamma(0,1,1)$
$\Rightarrow \alpha=x, \beta=y-x-z, y=z$
$\tau(x,y,z)=\alpha\tau(1,1,0)+\beta\tau(0,1,0)+\gamma\tau(0,1,0)=(x+2y+3z,3x+y-5z,z)$

(3,4,0) = 3(1,1,0) + 1(0,1,0) + 0(0,1,1)
(2,1,0) = 2(1,1,0) - (0,1,0) + 0(0,1,1)
(5,-4,1) = 5(1,1,0) - 10(0,1,0) + (0,1,1)

Matrix : $\begin{bmatrix} 3 & 2 & 5 \\ 1 & -1 & -10 \\ 0 & 0 & 1 \end{bmatrix}$

4. Find the algebraic and geometric multiplicities of the eigen values of the matrix
A = $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Ans 4): A = $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$
$|A-\lambda I|=0 \Leftrightarrow (\lambda-1)^2(\lambda-4)=0$
$\Rightarrow \lambda=1,1,4$
$\lambda=1 \Rightarrow$ algebraic multiplicity is 2 [Algebraic multiplicity is the no. of times λ has same value]
$\lambda=4 \Rightarrow$ algebraic multiplicity is 1
For $\lambda=1$; $(A-I)\lambda=0$

$$\Leftrightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank of coefficient matrix = 1
no of Liniearly independent eigen vectors = 3 - 1 = 2
Geometric multiplicity of λ=1 is 2.
Geometric multiplicity of λ=4 is 1.

5. (a) Find a matrix P for the given matrix A such that $P^{-1}AP$ is a diagonal matrix.
A = $\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$
(b) For a diagonalizable matrix A of the order n, prove that $\alpha A+\beta I_n$ is also diagonalizable where α and $\beta \in \mathbb{R}$.

Ans 5a): A = $\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$
$\Rightarrow C.E. \Rightarrow \lambda^3-12\lambda-16=0$
eigen values: λ = 4, -2, -2
Eigen vectors:
For λ=4 Ax=4x

$$\Rightarrow \begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reducing to row-echelon form,

$$\begin{bmatrix} -3 & -3 & 3 \\ 0 & -12 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Assume $z = \alpha$

$\Rightarrow 2y = z$

$\Rightarrow x + y - z = 0$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Thus the eigen vector corresponding to $\lambda = 4$ is $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

Similarly workout the eigen vectors for $\lambda = -2$ which would come out to be scalar multiples of $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\therefore$ the matrix P consisting of the eigen vectors as its column space is

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

or any matrix by interchange of vectors.

5b): A is diagonalizable

$\Rightarrow \exists$ a non-singular matrix P such that

$D = P^{-1}AP$ is diagonal

$\Rightarrow A = PDP^{-1}$

$$\begin{aligned} \alpha A + \beta I_n = \alpha PDP^{-1} + \beta I_n &= \alpha PDP^{-1} + \beta PP^{-1} \\ &= P\hat{D}P \Rightarrow \text{D is diagonal.} \end{aligned}$$

$\alpha A + \beta I_n$ is also diagonalizable.

6. Consider the system of linear equations,

$25x + 4y - z = 32$

$2x + 17y + 4z = 35$

$x + 3y + 10z = 24$

(i) To solve by Jacobi's iteration method, consider the above system in matrix form as $X^{K+1} = HX^K + C$. Write the matrices H and C with entries as fraction

(ii) Hence or otherwise perform 2 iterations (up to four decimal places) by Jacobi's method taking initial guess $x = y = z = 0$

(iii) For the same system , perform one iteration (upto four decimal places) by Gauss-Seidel method with initial guess $y = z = 0$

Ans 6(i): $A = \begin{bmatrix} 28 & 4 & -1 \\ 2 & 17 & 4 \\ 1 & 3 & 10 \end{bmatrix}$

$A = L + D + U$

$$L = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 3 & 0 \end{bmatrix} \qquad U = \begin{bmatrix} 0 & 4 & -1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 28 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

$$H = D^{-1}(L + U) = \begin{bmatrix} 0 & -1/7 & 1/28 \\ -2/17 & 0 & -4/17 \\ -1/10 & -3/10 & 0 \end{bmatrix}$$

$$C = D^{-1}b = \begin{bmatrix} 1/28 & 0 & 0 \\ 0 & 1/17 & 0 \\ 0 & 0 & 1/10 \end{bmatrix} \begin{bmatrix} 32 \\ 35 \\ 24 \end{bmatrix}$$
$$= \begin{bmatrix} 32/28 \\ 35/17 \\ 24/10 \end{bmatrix} = \begin{bmatrix} 8/7 \\ 35/17 \\ 12/5 \end{bmatrix}$$

(ii): $x^{(1)} = \begin{bmatrix} 1.1428 \\ 2.0588 \\ 2.4000 \end{bmatrix}$ $\quad x^{(2)} = \begin{bmatrix} 0.9344 \\ 1.3596 \\ 1.6681 \end{bmatrix}$

(iii) Gauss Siedal: $x^{(k+1)} = \frac{1}{28}(32 - 4y^{(k)} + z^{(k)})$

$$y^{(k+1)} = \frac{1}{17}(35 - 2x^{(k+1)} - 4z^{(k)})$$
$$z^{(k+1)} = \frac{1}{10}(24 - x^{(k+1)} - 3y^{(k+1)})$$

$x^{(1)} = 1.1428$
$y^{(1)} = 1.9244$
$z^{(1)} = 1.7084$

7. (a) Find the root of the equation $x - \ cosx = 0$ by the Newton Raphson method correct upto 7 decimal places. (take initial guess as 1)

7 (b) Write down the Newton Raphson expression for x_{n+1} for r times repeated roots.

Ans 7a): $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$= x_{n+1} - \frac{x_n - cos(x_n)}{1 + sin(x_n)}$$

$x_1 = 0.7503638$
$x_2 = 0.7391128$
$x_3 = 0.7390891$

b): $x_{n+1} = x_n - r\frac{f(x_n)}{f'(x_n)}$

MATHS II

END SEMESTER EXAMINATION **2018**

1. a) Using Cayley-Hamilton theorem only, find A^{100} and A^{-1}, if exists, and justify the reason if doesn't exist for the following matrix A:

$$A = \begin{pmatrix} 0 & \omega & \omega^2 \\ \omega & 0 & \omega^2 \\ \omega^2 & \omega & 0 \end{pmatrix}, \text{ where } \omega = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$$

b) Find the matrix A of a linear transformation $T: \mathbb{R}^4 \to M_{2X2}(\mathbb{R})$ defined by

$T(x_1, x_2, x_3, x_4) = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$, relative to the ordered bases

$S_1 = \{(1,1,1,1),\ (1,1,-1,0), (2,0,3,1),\ (-1,2,-1,1)\}$ of $\mathbb{R}^4$ and

$S_2 = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}$ of $M_{2X2}(\mathbb{R})$.

[NO MARKS IF THE BASES ARE TAKEN IN DIFFERENT ORDER]

Ans 1a): $A = \begin{bmatrix} 0 & \omega & \omega^2 \\ \omega & 0 & \omega^2 \\ \omega^2 & \omega & 0 \end{bmatrix}$ where $\omega = -\frac{1}{2} - \frac{\sqrt{3}}{2}$

$\Big[$ Cayley Hamilton theorem states that every matrix satisfies its identity matrix $\Big]$

By Cayley Hamilton theorem

$$\begin{bmatrix} -\lambda & \omega & \omega^2 \\ \omega & -\lambda & \omega^2 \\ \omega^2 & \omega & -\lambda \end{bmatrix} = -(1+\lambda^3) = -(1+A^3) = 0$$

$\Rightarrow A^3 = -1$

$\Rightarrow A^{-1} = -A^2$

$$\Rightarrow A^{-1} = \begin{pmatrix} 1 & -1 & -1 \\ -\omega & \omega & -1 \\ -\omega^2 & -1 & -(1+\omega) \end{pmatrix}$$

and $A^3 = -1$ $\quad \Rightarrow (A^3)^{33} = A^{99} = -1$

$\Rightarrow A^{100} = -A$

Ans 1b):

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$[T(1,1,1,1) \quad T(1,1,-1,0) \quad T(2,0,3,1) \quad T(-1,2,-1,1)]_{3\times 4} = [E_1 \quad E_3 \quad E_4 \quad E_2]_{1\times 4} A_{4\times 4}$$

$$T(1,1,-1,0) = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = 1.E_1 - 1.E_3 + 0.E_4 + 1.E_2 \text{ etc}$$

$$\Rightarrow \left[\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}\right]_{3\times 4} = [E_1\ E_3\ E_4\ E_2] \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

.

$$= \left[\begin{pmatrix} a_{11} & a_{41} \\ a_{21} & a_{31} \end{pmatrix} \begin{pmatrix} a_{12} & a_{42} \\ a_{22} & a_{32} \end{pmatrix} \begin{pmatrix} a_{13} & a_{43} \\ a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} a_{14} & a_{44} \\ a_{24} & a_{34} \end{pmatrix}\right]$$

$$\Rightarrow A = \begin{pmatrix} 1 & 1 & 2 & -1 \\ 1 & -1 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \end{pmatrix}$$

2. a) Consider the equation $x^2 - 4x + 3 = 0$. Which of the following equations (i) and (ii) below should be chosen so that fixed point iteration process converges to a root of the above equation for any initial guess x_o in the range (2,4)? Justify your answer.

(i) $x = \dfrac{x^2+3}{4}$ **(ii)**$x = \sqrt{4x-3}$

Ans 2a): For fixed point iteration

$|g'(x)| < 1$

Using this $x = \dfrac{x^2+3}{4} \Rightarrow g'(x) > 1$

but $x = \sqrt{4x-3} \Rightarrow g'(x) < 1$

So (ii) is the answer.

2b)Applying Newton's Forward interpolation formula determine the number of students who obtained less than 45 marks from the data given below.

$Marks:$	$30-40$	$40-50$	$50-60$	$60-70$	$70-80$
$No. of students:$	31	42	51	35	31

Ans 2b):

$Marks below$	$No. of students$	$\triangle f$	$\triangle^2 f$	$\triangle^3 f$	$\triangle^4 f$
40	31				
		42			
50	73		9		
		51		-25	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	190				

Using Newton's forward interpolation

$$f(x) = f(x_o) + k.\triangle f(x_o) + \frac{(k)(k-1)}{2!}\triangle^2 f + \dots\dots\dots \frac{(k)(k-1)(k-2)(k-3)}{4!}\triangle^4 f$$

where $k = \dfrac{45-40}{10} = 0.5$

$\Rightarrow f(45) = 47.867$ on solving

So $f(45)$ is 47 or 48.

3. a)Using Simpson's 1/3rd Rule evaluate the integral (3 decimal places) $\int_1^4 log_e(x)(3x^2+1)$ by dividing [1,4] into 6 equal length sub-intervals.

Ans 3a): Let $f(x) = log_e(x)(3x^2+1)$

n=6 h=1/2

x	1	1.5	2	2.5	3	3.5	4
$f(x)$	0	3.14235	9.0109	18.09674	30.7618	47.29180	87.92842

By Simpson's 1/3 rule

$$\int_1^4 f(x) \approx \frac{h}{3}\Big[f(x_o) + f(x_6) + 2\big[f(x_2) + f(x_4)\big] + 4\big[f(x_1) + f(x_3) + f(x_5)\big]\Big]$$

$= \dfrac{1}{6}\Bigg[421.59618\Bigg] = 70.266$

3. b)For a function f(x), given that

x	0	1/4	1/2	3/4	1
$f(x)$	8	a	b	3	-4

where a and b are real numbers. Given that for four equal sub-intervals of [0,1]
(i) the integral of $\int_0^1 f(x)dx$ value equal to 100, by using trapezoid rule.
(ii)the integral of $\int_0^1 f(x)dx$ value equal to 101, by using Simpson's 1/3rd rule.
Then find the values of a and b.

Ans: By trapezoid rule,

$$\int_0^1 f(x)dx = \frac{h}{2}\Big[f(x_o) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4)\Big]$$

$$100 = \frac{1}{8}\Big[8 + 2a + 6 + 2b - 4\Big]$$

$a + b = 315$ . . . .(i)

By Simpson's 1/3rd rule,

$$\int_0^1 f(x)dx = \frac{h}{2}\Big[f(x_o) + f(x_4) + 2f(x_2) + 4f(x_1) + 4f(x_3)\Big]$$

$2a + b = 598$ . . . . (ii)

$\Rightarrow$ from (i) and (ii)

a = 203

b = 192

4. a) For the improper integral $\displaystyle\int_0^2 \frac{2 + sin(\pi x)}{(1-x)^p}dx$**,**
(i) find all possible values of p such that the integral converges.
(ii) find all possible values of p such that the integral diverges.

b) Using Beta function evaluate the integral $\displaystyle\int_0^\infty \frac{t^3}{(1+t)^7}dx$

Ans 4a): If comparison test is used

Take $f(x) = \dfrac{1}{(1-x)^p}$ and show that,

$$\int_0^1 \frac{1}{(1-x)^p}dx = \begin{cases} \infty, & p \geq 1 \\ \dfrac{1}{1-p}, & p < 1 \end{cases}$$

$$0 < \frac{2 + sin(\pi x)}{(1-x)^p} \leq \frac{1}{(1-x)^p} \qquad 0 \leq x < 1$$

for $p < 1$ integral converges

$$0 < \frac{1}{(1-x)^p} \leq \frac{2 + sin(\pi x)}{(1-x)^p} \qquad 0 \leq x < 1$$

for $p \geq 1$, integral diverges.

Ans 4b): $\displaystyle\int_0^\infty \frac{t^3}{(1+t)^7} = \int_0^\infty \frac{t^{4-1}}{(1+t)^{4+3}}$

$= \beta(4, 3) \qquad \Big[\beta(m, n)$

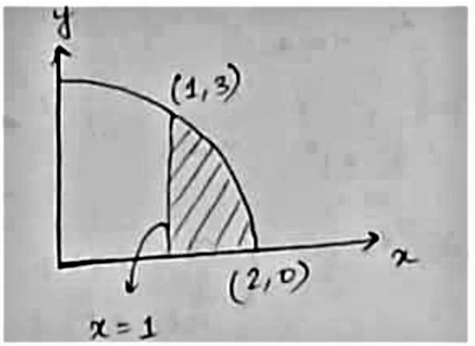
$= \dfrac{1}{60} \qquad = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\Big]$

5. a) Using double integration find the volume of the solid bounded above the surface $z = xy$ and bounded below by the region enclosed by $y = 4 - x^2, x = 1, x = 2$ and the x-axis.(sketch and shade the region of integration)

b) Use the transformation $x = u - v, y = u + v$ to evaluate $\int\int (x+y)dA$ where R is the region enclosed by $y = x, y = 3x, x + y = 4$.

c) Using Lebnittz Rule, find $\left(\frac{dI}{d\alpha}\right)_{\alpha=1}$, where $I = \int_{\alpha}^{2\alpha} \alpha x dx$

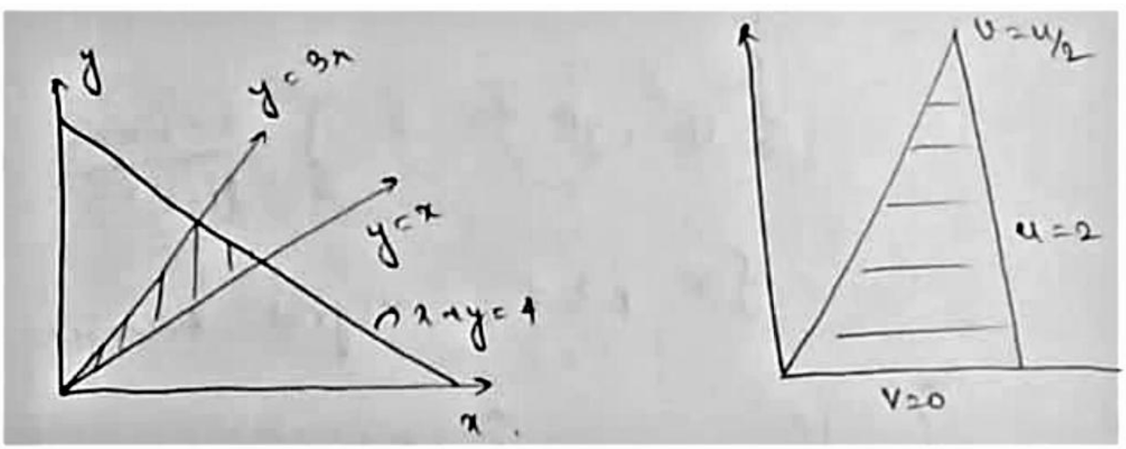
Ans 5a):



$$V = \int_{x=1}^{2} \int_{y=0}^{4-x^2} xydydx$$

On solving the integral we get, V=9/4.

Ans 5b):



$$y = x \quad \Rightarrow u + v = u - v \quad \Rightarrow v = 0$$
$$y = 3x \quad \Rightarrow u + v = 3(u - v) \quad \Rightarrow v = u/2$$
$$x + y = 4 \quad \Rightarrow u + v + u - v = 4 \quad \Rightarrow u = 2$$
$$\frac{\partial(x,y)}{\partial(u,v)} = 2$$
$$\Rightarrow \int\int (x+y)dA = \int_0^2 \int_0^{u/2} 4u.du.dv = \frac{16}{3}$$

Ans 5c): Lebnitz rule states that,

$$\frac{dI}{d\alpha} = \int_a^b \frac{\partial f}{\partial x} dx + \frac{\partial b}{\partial \alpha} f(b,\alpha) - \frac{\partial a}{\partial \alpha} f(a,\alpha)$$
$$\Rightarrow \int_{\alpha}^{2\alpha} x.dx + 2(2\alpha^2) - \alpha^2$$
$$= \frac{\alpha^2}{2}\Big]_{\alpha}^{2\alpha} + 3\alpha^2$$
$$= \frac{9}{2}\alpha^2$$
$$\frac{dI}{d\alpha}|_{\alpha=1} = \frac{9}{2}$$

Q6(A) Find the surface area of the paraboloid $2z = x^2 + y^2$ inside the cylinder $x^2 + y^2 = 1$.

3 marks

Solution:

$$S = \iint_D \sqrt{\left(1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right)}\, dx\, dy$$

where D is the region bonded by the circle $x^2 + y^2 = 1$.

Then

$$S = \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} \sqrt{(1 + x^2 + y^2)}\, dx\, dy \qquad OR \qquad S = 4\int_0^{\frac{\pi}{2}} \int_0^1 \sqrt{(1 + r^2)}\, r\, dr\, d\theta$$

$$S = 4\int_0^{\frac{\pi}{2}} \int_0^1 \sqrt{(1 + r^2)}\, r\, dr\, d\theta = \frac{2\pi}{3}\left(2^{\frac{3}{2}} - 1\right) = \frac{2\pi}{3}\,(2\sqrt{2} - 1)$$

Q6(B) The volume of the region D bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 2y$ is given by

$$\iiint_D dx\, dz\, dy$$

(i) Write the limits of the integration without changing the order

(ii) Find the volume of the region D

3 marks

Solution:

$$V = \int_0^2 \int_{y^2}^{2y} \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} dx\, dz\, dy \quad = \quad 2\int_0^2 \int_{y^2}^{2y} \sqrt{z - y^2}\, dz\, dy \quad = \quad \frac{4}{3}\int_0^2 (2y - y^2)^{\frac{3}{2}}\, dy$$

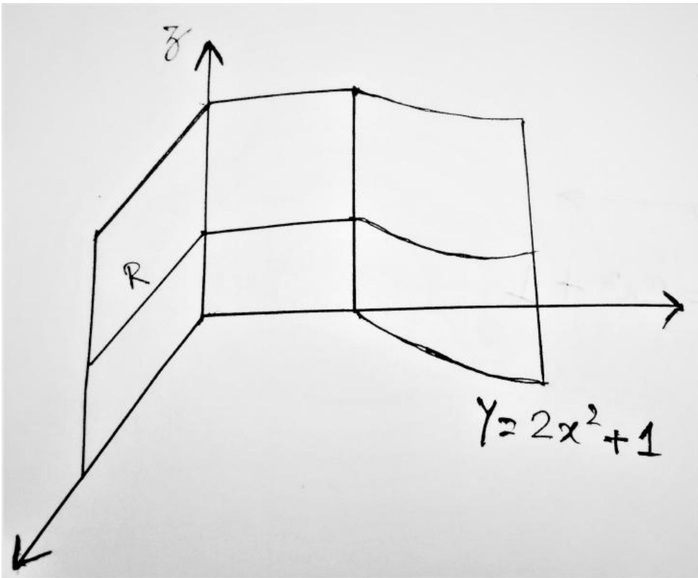
Substitute $y = 1 + \sin t$

$$V = \frac{8}{3}\int_0^{\frac{\pi}{2}} \cos^4 t \quad dt \quad = \quad \frac{\pi}{2}$$

7. a) Integrate $\int\int xz^2 dS$ in the region bounded by the cylinder $y = 2x^2 + 1$ where x varies from 0 to 2 and z varies from 4 to 8.

b) Find the directional derivative of $f(x, y, z) = x^2yz + 4xz^2$ at the point (1,-2,-1) in the direction of the maximum rate of change.

Ans 7a):



Given that $\int_S\int xz^2 dS$

$\int_S\int G(x, y, z)dS = \int_R\int G(x, f(x, z), z)$

$= \sqrt{1 + y_x^2 + y_z^2}$

$f_x(x, z) = 4x$ and $f_z(x, z) = 0$

$\int_S\int xz^2 dS = \int_{x=0}^{2}\int_{z=4}^{8} xz^2\sqrt{1 + (4x)^2}dxdz$

On solving we get $\int_S\int xz^2 dS = \frac{28}{9}\left[65^{3/2} - 1\right] = 1627.3$

Ans 7b): $f(x, y, z) = x^2yz + 4xz^2$

for directional derivative

$\overrightarrow{\nabla} f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right).f(x, y, z)$

$= (2xyz + 4z^2)\hat{i} + x^2z\hat{j} + (x^2y + 8zx)\hat{k}$

$\overrightarrow{\nabla} f|_{(1,-2,-1)} = 8\hat{i} - \hat{j} - 10\hat{k}$

So max change occurs along $\overrightarrow{\nabla} f$

$\hat{n} = \frac{\overrightarrow{\nabla} f}{|\overrightarrow{\nabla} f|} = \frac{8\hat{i} - \hat{j} - 10\hat{k}}{\sqrt{165}}$

$\therefore$ directional derivative at (1,-2,-1)

$Df|_{(1,-2,-1)} = (8\hat{i} - \hat{j} - 10\hat{k}).\frac{8\hat{i} - \hat{j} - 10\hat{k}}{\sqrt{165}} = \sqrt{165}$

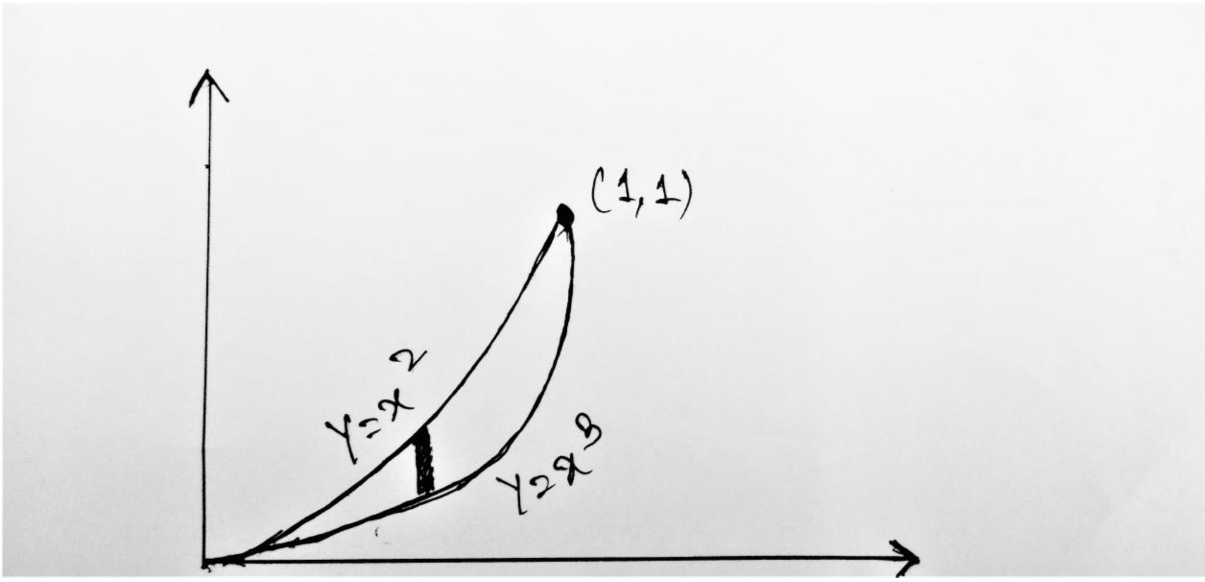
8. a) Using Green's Theorem evaluate $\oint_c (x^2 - y^2)dx - (2y - x)dy$, where C consists of the boundary of the region in the first quadrant that is bounded by $y = x^2, y = x^3$.

b) Consider the force field $\overrightarrow{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + xz^2\hat{k}$

(i) Verify wether $\overrightarrow{f}$ is a conservative force field or not. If so, then find the potential function of $\overrightarrow{F}$.

(ii) Find the work done to move an object in this field $\overrightarrow{F}$ from (1,-2,1) to (3,1,4).

Ans 8a):



Given , $\oint_C (x^2-y^2)dx + (2y-x)dy$

Let $P(x,y) = x^2 - y^2 \qquad Q(x,y) = 2y - x$

According to Green's Theorem,

$$\oint Pdx + Qdy = \int_R\int \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)dA$$

$$\Rightarrow \int_R\int (2y-1)dA$$

$$\Rightarrow \int_{x=0}^{x=1}\int_{y=x^3}^{y=x^2} (2y-1)dy.dx$$

On solving,

$$= -\frac{11}{420}$$

Ans 8b(i): Force can be conservative if $\overrightarrow{\nabla} \times \overrightarrow{F} = 0$ . . .[Necessary condition]

$$\overrightarrow{\nabla} \times \overrightarrow{F} = \begin{bmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy+z^2 & x^2 & 3xz^2 \end{bmatrix}$$

$$= -\hat{i}(0) - \hat{j}(3z^2 - 3z^2) + \hat{k}(2x - 2x) = 0$$

Thus $\overrightarrow{F}$ is a conservative force.

Now on comparing, we get,

$$\frac{\partial \phi}{\partial x} = 2xy + z^3 \ , \ \frac{\partial \phi}{\partial y} = x^2 \ , \ \frac{\partial \phi}{\partial z} = 3xz^2 \ . \ . \ . \ . \ . \ . \ \text{(1),(2),(3) respectively}$$

On Integrating we get,

$\phi = x^2y + xz^3 + f(y,z)$

Substituting ϕ in (2) from (4) we get,

$$\frac{\partial f}{\partial y} = 0$$

$\Rightarrow f(y,z) = h(z)$

Substituting in eq (3) we get,

$$\frac{\partial h}{\partial z} = 0$$

$\Rightarrow h(z) = c$

Thus $\phi = x^2y + xz^3 + c$ is the potential function.

Ans 8b(ii) : We know that,

$$\begin{aligned}\text{work done} &= \int_{P_1}^{P_2} \overrightarrow{F} dx \\ &= \int_{P_1}^{P_2} (2xy + z^3)dx + x^2dy + 3xz^2dz \\ &= \textbf{202} \text{ units of work.}\end{aligned}$$

MATHS II

END-SPRING SEMESTER **2017**

1. (a) Does there exist a linear transformation $T : \mathbb{R}^5 \longrightarrow \mathbb{R}^2$ such that the null space $N(T) = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 2x_2, x_3 = x_4 = x_5\}$? Justify your answer.

   (b) Let $\mathcal{P}_2(\mathbb{R})$ denote the vector space of all polynomials of degree ≤ 2 with real coefficients with a basis $\mathcal{B} = \{1, x, x^2\}$. Let $T : \mathcal{P}_2(\mathbb{R}) \longrightarrow \mathcal{P}_2(\mathbb{R})$ be a linear transformation defined by $T(f(x)) = f(1) + f'(0)x + (f'(0) + f''(0))x^2$. Find the matrix of the linear transformation T with respect to the basis $\mathcal{B}$.

   (c) Check the diagonalizability of the matrix

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}.$$

   If A is diagonalizable, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

Ans 1a): T: $\mathbb{R}^5 \longrightarrow \mathbb{R}^2$
N(T) = $[(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 2x_2, x_3 = x_4 = x_5]$
Let $x_1 = \alpha$ and $x_3 = \beta = x_4 = x_5$, $x_2 = \alpha/2$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1/2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Thus the dimension of N(T)= 2 as it has 2 basis (1,1/2,0,0,0) and (0,0,1,1,1).
$\therefore$ by rank-nulity theorem,
dim N(T) + dim R(T) = 5
$\Rightarrow$ dim R(T) = 3
but for the linear transformation T: $\mathbb{R}^5 \longrightarrow \mathbb{R}^2$
dim R(T) $\leq$ 2
So the linear transformation is not possible.
[remember dim R(T) $\leq$ 2 and not necessarily equal to 2 from T:$\mathbb{R}^5 \longrightarrow \mathbb{R}^2$]

Ans 1b): T: $P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$
$\beta = \{1, x, x^2\}$
$T(f(x)) = f(1) + f'(0)x + (f'(0) + f''(0))x^2$
when $f(x) = 1$
$T(1) = 1 + 0.x + 0.x^2$
When $f(x) = x$,
$T(x) = 1 + 1.x + (1 + 0)x^2$
$T(x) = 1 = 1.x + 1.x^2$

when $f(x) = x^2$,
$T(x^2) = 1 + 0.x + 2x^2$
$\therefore$ matrix of linear transformation is

$$[\mathrm{T}]_\beta^\beta = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Ans 1c): $\mathrm{A} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)((2-\lambda)(3-\lambda) - 2) + (-1)(2 - 2(2-\lambda)) = 0$
$\Rightarrow \lambda^3 - 6\lambda^2 + 5\lambda - 6 = 0$
$\Rightarrow (\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$
$\Rightarrow \lambda = 1, 2, 3$
The eigen values are 1 , 2 , 3 .
for $\lambda = 1$,

$$\Rightarrow \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$ and then $R_1 \leftrightarrow R_2$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

1 free variable so assuming $y = \alpha$
$\Rightarrow -1.z = 0$ $\quad$ $1.x + 1.y + 1.z = 0$
$\Rightarrow z = 0$ $\quad$ $\Rightarrow x = -z - y \Rightarrow x = -\alpha$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Similarly eigen vectors for $\lambda = 2$ and $\lambda = 3$ can be found,

Eigen vectors for $\lambda = 1$, $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Eigen vectors for $\lambda = 2$, $\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$

Eigen vectors for $\lambda = 3$, $\begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & -1 \\ 0 & -2 & -2 \end{bmatrix}$ [all the eigen vectors make up the coloumn space of P]

$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ [all the eigen values in diagonal]

2. (a) Using Simpson's $\frac{1}{3}$rd rule, evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\, d\theta$ by dividing the interval $[0, \frac{\pi}{2}]$ into six equal subintervals, where θ is in radians.

(b) Consider the following system of linear equations:

$$83x_1 + 11x_2 - 4x_3 = 95$$
$$7x_1 + 52x_2 + 13x_3 = 104$$
$$3x_1 + 8x_2 + 29x_3 = 71$$

Use Gauss-Seidel method, obtain next two iterations with the initial guess $(0, 0, 0)$ and round off the answer up to two decimal places.

(c) Following data for x and y is given.

x	5	14	21
y	1	3	4

Using Lagrange's interpolation, find the value of y when $x = 8$.

Ans 2a): $\int_0^{\pi/2} \sqrt{\cos\theta} d\theta = \frac{h}{3}[(y_o + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$

where,

x	0	$\pi/12$	$\pi/6$	$\pi/4$	$\pi/3$	$5\pi/12$	$\pi/2$
y	1	0.9828	0.9306	0.8409	0.7071	0.5087	0
	y_o	y_1	y_2	y_3	y_4	y_5	y_6

$h = \frac{\pi}{12}$

Ans: 1.1873

Ans 2b): 1st iterations:

$x_1^{(1)} = 1.14457831 \simeq 1.1446$

$x_2^{(1)} = 1.8459221501 \simeq 1.8459$

$x_3^{(1)} = 1.8206513055 \simeq 1.8206$

2nd iterations:

$x_1^{(2)} = 0.987680251 \simeq 0.9877$

$x_2^{(2)} = 1.41188022 \simeq 1.4119$

$x_3^{(2)} = 1.9547083 \simeq 1.9547$

Ans 2c): Lagrange's Interpolation formula

$$y(x) = \frac{(x - x_1)(x - x_2)}{(x_o)(x_o - x_2)} y_o + \frac{(x - x_o)(x - x_2)}{(x_1 - x_o)(x_1 - x_2)} y_1 + \frac{(x - x_o)(x - x_1)}{(x_2 - x_o)(x_2 - x_1)} y_2$$

where,
$\begin{array}{cccc} x & 5 & 1 & 21 \\ y & 1 & 3 & 4 \end{array}$

$$y(8) = \frac{(8 - 14)(8 - 21)}{(5 - 14)(4 - 21) \times 1} + \frac{(8 - 5)(8 - 21)}{(14 - 5)(14 - 21)}(3) + \frac{(8 - 5)(8 - 14)}{(21 - 5)(21 - 14)}(4) = \frac{295}{168} \simeq 1.7559$$

3. (a) Let the improper integral $I = \int_1^\infty \frac{\ln x}{(1+x^3)^{\frac{1}{p}}} dx$, for $p \neq 0$. Find the range of p for which I converges and find the range of p for which I diverges.

(b) Using differentiation under integral sign, evaluate the integral

$$\int_0^{\frac{\pi}{2}} \frac{\ln(1+\cos\alpha\cos x)}{\cos x} dx, \; 0 < \alpha < \pi.$$

(c) If $f = \alpha u + v - \beta,\; g = u^2 + \beta v + w,\; h = \gamma u - v + vw$, where α, β, γ are non zero constants, then compute the Jacobian $J = \frac{\partial(f,g,h)}{\partial(u,w,v)}$.

[3 +3+ 2 Marks]

Ans 3a): For $0 < p < 3$ choose q so that $p < q < 3$.

Then, $\lim\limits_{x\to\infty} \frac{x^{\frac{3}{q}} \ln x}{\sqrt[p]{x^3(1+x^{-3})}} = \lim\limits_{x\to\infty} \frac{\ln x}{x^{(\frac{3}{p}-\frac{3}{q})} \sqrt[p]{1+x^{-3}}}$

If $p < 0$ or $p \geq 3$, then $\lim = +\infty$

If $p \in (-\infty, 0)$ or $p \geq 3$ the integral diverges and $0 < p < 3$ it converges.

Ans 3b): $I(\alpha) = \int_0^{\pi/2} \frac{\ln(1+\cos\alpha\cos x)}{\cos x} dx$, where $0 < \alpha < \pi$

$I'(\alpha) = -\int_0^{\pi/2} \frac{\sin\alpha}{1+\cos\alpha\cos x}$ $\left[\int \frac{dx}{1+b\cos X} \text{type}\right]$

Hence , $I'(\alpha) = \frac{(-1)\times\sin\alpha}{\sqrt{1-\cos^2\alpha}} \cos^{-1}\left[\frac{\cos\alpha+\cos x}{1+\cos\alpha\cos x}\Big|_0^{\pi/2}\right]$

$= -\left(\cos^{-1}(\cos\alpha) - \cos^{-1}\left(\frac{\cos\alpha+1}{\cos\alpha+1}\right)\right)$

$= -(\alpha - 0)$

$\therefore I'(\alpha) = -\alpha \Rightarrow I(\alpha) = -\frac{\alpha^2}{2} + c$

$\alpha = \frac{\pi}{2} \Rightarrow I(\alpha) = 0$, $c = \frac{\pi^2}{8}$

$\therefore I(\alpha) = \frac{\pi^2}{8} - \frac{\alpha^2}{2}$

Ans 3c):

$$I = \frac{\partial(f,g,h)}{\partial(u,w,v)} = \begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial w} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial w} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial w} & \frac{\partial h}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \alpha & 0 & 1 \\ 2y & 1 & \beta \\ \gamma & u & w-1 \end{vmatrix} = 2uv + \alpha w - \alpha\beta v - \alpha - \gamma$$

4. (a) If $\dfrac{\beta(m, \frac{1}{2})}{\beta(m,m)} = 2^{\alpha}$, for any real $m > 0$, then find α.

(b) Evaluate the double integral by using the transformation $x = u(1-v)$, $y = uv$, $\displaystyle\int_{x=1}^{2}\int_{y=1}^{2-x} 12(x+y)^{10} e^{\frac{y}{x+y}}\, dy dx.$

(c) Evaluate $\displaystyle\int_{x=0}^{1}\int_{y=2x^2}^{3-x} 4xe^{3-y} dy dx$, by changing the order of integration. (no marks will be awarded, if evaluates directly).

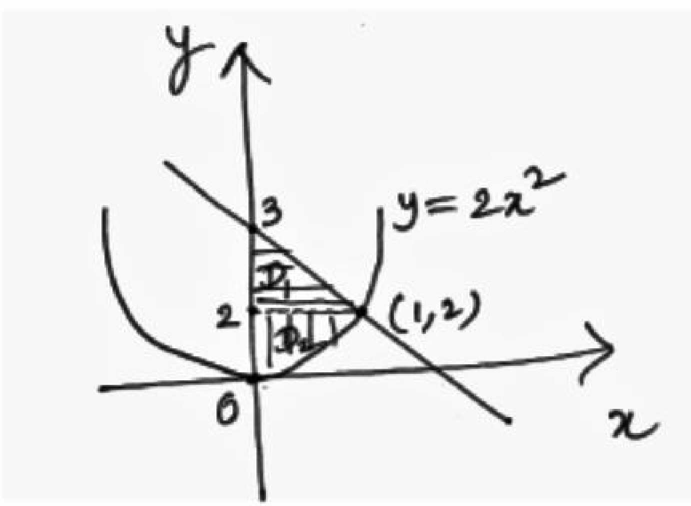
[1+3 +3 Marks]

Ans 4a): We have $\beta(m,n) = 2\displaystyle\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$,

$$\beta(m,n) = \frac{2}{2^{2m-1}}\int_0^{\pi/2} \sin^{2m-1}(2\theta) d\theta$$
$$= \frac{1}{2^{2m-1}}\beta(m, 1/2)$$
$$\Rightarrow \frac{\beta(m,1/2)}{\beta(m,m)} = 2^{2m-1}$$
$$\therefore \alpha = 2m-1$$

In **Qusetion 4(b)** the evaluation of the integral after the transformation is not possible.

Ans 4c):



$$\int_{x=0}^{1}\int_{y=2x^2}^{3-x} 4xe^{3-y} dy dx = \int\int_{D_1} 4xe^{3-y} dx.dy + \int_{D_2}\int 4xe^{3-y} dx.dy$$
$$= \int_{y=2}^{3}\int_{x=0}^{3-y} 4xe^{3-y} dx.dy + \int_{y=0}^{2}\int_{x=0}^{\sqrt{y/2}} 4xe^{3-y} dx.dy$$
$$= (2e-4) + (e^3 - 3e)$$
$$= e^3 - e - 4$$
$$\approx 13.36725509$$

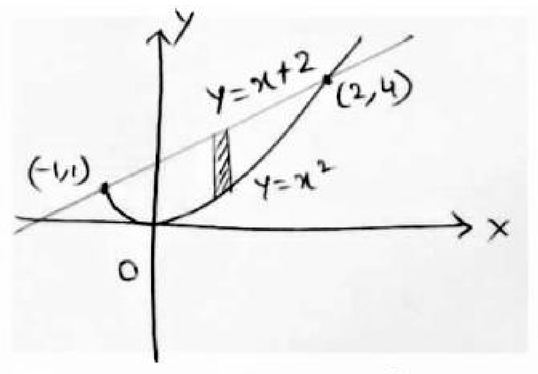
5. (a) Using a double integral, find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

(b) Find the surface area of the portion of the surface $z = \sqrt{4 - x^2}$ that lies above the rectangle $R:\ 0 \le x \le 1$ and $0 \le y \le 4$ in the XY-plane.

(c) Using a triple integral, find the volume of the solid enclosed by the cylinder $x^2 + y^2 = 9$, and bounded by the planes $z = 1$ and $x + z = 5$.

[2 + 2+ 3 Marks]

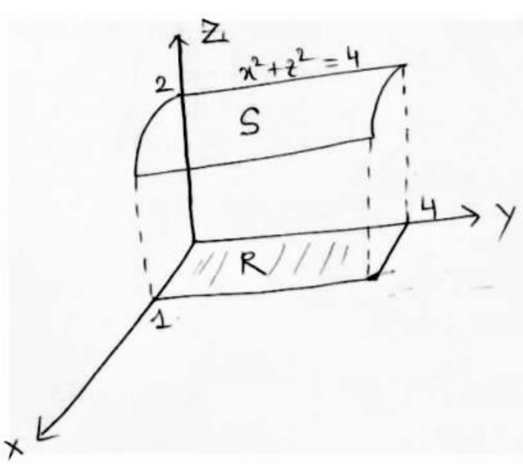
Ans 5a):



We have the required area as ,

$$A = \int_{x=-1}^{2} \int_{y=x^2}^{x+2} dy.dx$$
$$= \int_{-1}^{2} (x + 2 - x^2)dx$$
$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$
$$= (2 + 4 - 8/3) - (1/2 - 2 + 1/3)$$
$$= \frac{10}{3} + \frac{7}{6} = \frac{9}{2} = 4.5 \text{ sq.units}$$

Ans 5b):



As shown in the figure, the surface is a portion of the cylinder $x^2 + z^2 = 4$.
$\therefore$ the surface area is ,

$$S = \int\int_R \sqrt{1 + \left[\frac{\partial z}{\partial x}\right]^2 + \left[\frac{\partial y}{\partial x}\right]^2} dA \qquad \text{Here, } z = \sqrt{4 - x^2}$$

$$= \int\int_R \sqrt{1 + \left(\frac{-x}{\sqrt{4 - x^2}}\right)^2 + 0dA} \qquad \therefore \frac{\partial z}{\partial x} = \frac{(-2x)}{2\sqrt{4 - x^2}} = \frac{-x}{\sqrt{4 - x^2}}$$

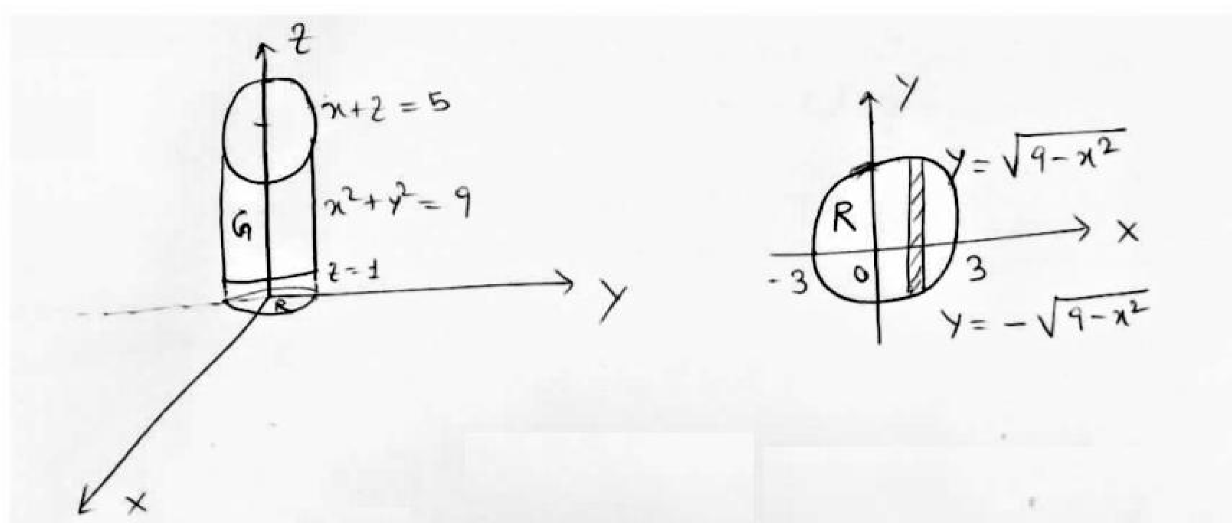
$$= \int_{y=0}^{4} \int_{x=0}^{1} \frac{2}{\sqrt{4 - x^2}} dx.dy \qquad \frac{\partial z}{\partial y} = 0$$

$$= 2\int_0^4 [\sin^{-1}(x/2)]_{x=0}^1 dy$$

$$= 2\int_0^4 (\pi/6)dy = \frac{\pi}{3}[y]_0^y$$

$$= 4\pi/3 = 4.1887 \text{ sq.units}$$

Ans 5c):



The solid G and its projection R on the XY-plane are shown in the above fighure. the lower surface of the solid is the place $z = 1$ and the upped surface is the plane $x + z = 5 \Rightarrow z = 5 - x$

$\therefore$ Volume of G $= \int\int\int_G dv = \int\int_R \left[\int_1^{5-x} dz\right] dA$

For the double integral over R we integrate w.r.t. y first

$$\therefore \text{Volume of G} = \int_{x=-3}^{3} \int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-x} dz.dy.dx$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} [z]_{z=1}^{5-x} dy.dx$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (4 - x)dy.dx$$

$$= \int_{-3}^{3} (4 - x)[y]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx$$

$$= \int_{-3}^{3} (8 - 2x)\sqrt{9 - x^2} dx$$

$$= 8\int_{-3}^{3} \sqrt{9 - x^2} dx - \int_{-3}^{3} 2x\sqrt{9 - x^2} dx$$

$= 8\left(\frac{9}{2}\pi\right) - \int_{-3}^{3} 2x\sqrt{9-x^2}dx$

$\Downarrow$

$= 0$ since the integrand is an odd function

$= 36\pi$ cubic units $= 113.0973$ cubic units

6. (a) If $\overrightarrow{\nabla} \times \vec{A} = 0$, evaluate $\overrightarrow{\nabla} \cdot (\vec{A} \times \vec{r})$ without using any identity, where $\vec{r} = x\hat{\mathrm{i}} + y\hat{\mathrm{j}} + z\hat{\mathrm{k}}$.

   (b) Find the constants a and b so that the surface $ax^2 - byz = (a+2)x$ is orthogonal to the surface $4x^2y + z^3 = 4$ at the point (1, -1, 2).

   (c) If $\overrightarrow{\nabla} \cdot (r^3\vec{r}) = \alpha r^\beta$, find the constants α and β, where $r = |\vec{r}|$.

[2+3+2 Marks]

Ans 6a): $\overrightarrow{A} \times \overrightarrow{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ x & y & z \end{vmatrix} = (zA_2 - yA_3)\hat{i} + (xA_3 - zA_1)\hat{j} + (yA_1 - xA_2)\hat{k}$

$\hat{\nabla}.(\hat{A} \times \hat{r}) = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(xA_3 - zA_1) + \frac{\partial}{\partial z}(yA_1 - xA_2)$

$= z\frac{\partial A_2}{\partial x} - y\frac{\partial A_3}{\partial x} + x\frac{\partial A_3}{\partial y} - z\frac{\partial A_1}{\partial y} + y\frac{\partial A_1}{\partial z} - x\frac{\partial A_2}{\partial z}$

$= x\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + y\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + z\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$

$= (x\hat{i} + y\hat{j} + z\hat{k})\left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\hat{k}\right]$

$= \hat{r}.(\overrightarrow{\nabla} \times \overrightarrow{A}) = \overrightarrow{r}.\overrightarrow{0} = 0$

Ans 6b): $\overrightarrow{\nabla}\phi = [2ax - (a+2)]\hat{i} - bz\hat{j} - by\hat{k}$

$\overrightarrow{\nabla}\psi = 8xy\hat{i} + 4x^2\hat{j} + 3z^2\hat{k}$

$(\overrightarrow{\nabla}\phi)_{1,-1,2} = [2a - (a+2)]\hat{i} - zb\hat{j} + b\hat{k}$

$(\overrightarrow{\nabla}\psi)_{1,-1,2} = -8\hat{i} + 4\hat{j} + 12\hat{k}$

$\because \phi = 0, \psi = 0$ are orthogonal at (1,-1,2).

$(\overrightarrow{\nabla}\phi).(\overrightarrow{\nabla}\psi) = 0$ at (1,-1,2)

$\Rightarrow 2a = b + 4 \quad \longrightarrow (1)$

$\because ax^2 - byz = (a+2)x$ passes through (1,-1,2) , $a + 2b = a + 2 \Rightarrow b = 1$

$\therefore$ From (1), $a = \frac{5}{2}$

7. (a) Find the directional derivative of $f(x, y, z) = x^2y^3z^4$ at the point $(2, 1, 1)$ in the direction $\hat{i} + 2\hat{j} + 2\hat{k}$.

   (b) Test whether the force field

$$\overrightarrow{F} = 2xyz^2\hat{i} + [x^2z^2 + z\cos(yz)]\hat{j} + [2x^2yz + y\cos(yz)]\hat{k}$$

   is conservative. If yes, find the scalar potential function ϕ and the work done in moving an object from $(0, 0, 1)$ to $(1, \frac{\pi}{4}, 2)$.

(c) Verify Green's theorem for

$$\oint_C [(3x^2 - 8y^2)dx + (4y - 6xy)dy],$$

where C is the boundary of the region bounded by $x = 0$, $y = 0$ and $x + y = 1$ and integration is taken in the anti clock wise direction.

[1 +3 +3 Marks]

Ans 7a): $\nabla f = 2xy^3z^4\hat{i} + 3x^2y^2z^4\hat{j} + 4x^2y^3z^3\hat{k}$
$= 4\hat{i} + 12\hat{j} + 16\hat{k}$ at (2,1,1)

Directional derivative is $\nabla f.\hat{b} = (4\hat{i} + 12\hat{j} + 16\hat{k}).\dfrac{\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{9}} = \dfrac{60}{3} = 20$

Ans 7b): curl F = 0
now, $F = \nabla\phi$

$$2xyz^2\hat{i} + [x^2z^2 + z\cos(yz)]\hat{j} + [2x^2yz + y\cos(yz)]\hat{k} = \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)$$

$\dfrac{\partial\phi}{\partial x} = 2xyz^2$ $\qquad\qquad$ $\phi = x^2yz^2 + f_1(y,z)$

$\dfrac{\partial\phi}{\partial y} = x^2z^2 + z\cos(yz)$ $\qquad\qquad$ $\dfrac{\partial\phi}{\partial y} = x^2z^2 + \dfrac{\partial f_1}{\partial y}$

$\dfrac{\partial\phi}{\partial z} = 2x^2yz + y\cos(yz)$

$\therefore \dfrac{\partial f_1}{\partial y} = z\cos(yz)$

$\therefore f_1 = \sin(yz) + f_2(z)$

$\therefore \phi = x^2yz^2 + \sin(yz) + f_2(z)$

$\therefore \dfrac{\partial\phi}{\partial z} = 2x^2yz + y\cos(yz) + \dfrac{\partial f_2}{\partial z}$

$\therefore \dfrac{\partial f_2}{\partial z} = 0$

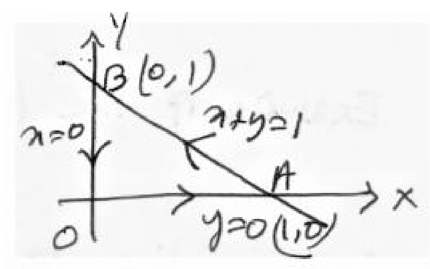
$\therefore f_2 = c$

$\phi = x^2yz^2 + \sin(yz) + c$

$\int_c F.dr = \int_c \nabla\phi.dr = \int d\phi = [x^2yz^2 + \sin(yz)]_{(0,0,1)}^{(1,\pi/4,2)} = \pi + 1$

Ans 7c): By Green's theorem , given line integral

$$= \int\int_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx.dy$$



$$= \int\int_R (-6y + 16y)dx.dy$$

$$= 10\int_0^1 \left[\int_0^{1-x} ydy\right] dx$$

$$= 10\int_0^1 \left(\frac{y^2}{2}\right)_0^{1-x} dx = 5\int_0^1 (1-x)^2 dx = \frac{5}{3}$$

Along OA, $y = 0, dy = 0, x$ varies from 0 to 1.

$$\int_{OA} = \int_0^1 3x^2 dx = 1$$

Along AB, $y = 1 - x, dy = -dx, x$ varies from 1 to 0.

$$\int_{AB} = \int_1^0 [3x^2 - 8(1-x)]^2 dx + \int_1^0 [4(1-x) - 6x(1-x)(-1)]dx$$

$$= \int_1^0 (-11x^2 + 26x - 12)dx = \frac{8}{3}$$

Along BO, $x = 0, dx = 0, y$ varies from 1 to 0

$$\int_{BO} = \int_1^0 4y dy = -2$$

$$\therefore \phi_c(f_1 dx + f_2 dy) = 1 + \frac{8}{3} - 2 = \frac{5}{3}$$

MATHS II

END SEMESTER EXAMINATION **2016**

1) (a) Let $T : \mathbb{R}^5 \longrightarrow \mathbb{R}^3$ be a linear mapping defined by $T(x, y, z) = (x + z, -x + 2y + z, y + z)$. Taking the usual basis of $\mathbb{R}^3$ i.e. (1,0,0) , (0,1,0) , (0,0,1) find the basis and dimension of the Range space and Null space.
(b) For the given matrix A, find its characteristic equation and findA^{-1} using Caley-Hamiltaon's theorem.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

Ans 1a): $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map
$T(e_1) = (-1, -1, 0)$
$T(e_2) = (0, 2, 1)$
$T(e_3) = (1, 1, 1)$
$\mathbb{R}^3 = span(e_1, e_2, e_3)$

$$\begin{aligned} R(T) &= span(T(e_1), T(e_2), T(e_3)) \\ &= span((1, -1, 0), (0, 2, 1), (1, 1, 1)) \\ &= span((1, -1, 0), (0, 2, 1)) \end{aligned}$$

$since (1, 1, 1) = (1, -1, 0) + (0, 2, 1)$
hence, $R(T)$ has a dimention 2 since it has 2 vectors as basis.

$$\begin{aligned} N(T) &= \big((x, y, z)/T(x, y, z) = (0, 0, 0)\big) \\ &= \big((x, y, z) \mid (x + z, -x + 2y + z, y + z) = (0, 0, 0)\big) \\ &= \big((x, x, -x)|x \in \mathbb{R}\big) \\ &= span\big((1, 1, -1)\big) \end{aligned}$$

$\big(1, 1, -1\big)$ is the basis of $N(T)$ hence dimension is 1.

1b): $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

$|A - \lambda I| = 0 \Leftrightarrow \lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0$

Now, $A^2 = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$

$A^3 = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$

Hence, $A^3 - 6A^2 + 7A + 2I = 0$
$A^3 - 6A^2 + 7A = -2I$
Multiplying A^{-1} on both sides
$\Rightarrow A^2 - 6A + 7I = -2A^{-1}$

$$\begin{aligned} \Rightarrow A^{-1} &= -\frac{1}{2}\Big[A^2 - 6A + 7I\Big] \\ &= \begin{bmatrix} -3 & 0 & 2 \\ -1 & 1/2 & 1/2 \\ 2 & 0 & -1 \end{bmatrix} \end{aligned}$$

2. (a) From the table given below construct Newton Forward difference table and estimate the value of $f(x)$ at x=0.15

x	0.1	0.2	0.3	0.4	0.5
$f(x)$	1.4	1.56	1.76	2.00	2.28

(b) In the interval [a,b] apply Simpson's 1/3 rd rule to estimate the integral $\int_a^b f(t)dt$ for two intervals $\in$ [a,b] where

$$f(t) = \int_a^b \left[200 \ln \left[\frac{14000}{14000 - 2100t} \right] - 9.8t \right]$$

(c) Using the expression for estimating the integral in 2(b). Find the valuie of the integral for a=30 , b=8.

Ans 2a): (i) Forward Difference method

x	$f(x)$	$\triangle f$	$\triangle^2 f$	$\triangle^3 f$	$\triangle^4 f$
0.1	1.4				
		0.16			
0.2	1.56		0.04		
		0.20		0	
0.3	1.76		0.04		0
		0.24		0	
0.4	2.00		0.04		
		0.28			
0.5	2.28				

(ii) $f = f_o + s \triangle f + \dfrac{s(s-1)}{2} \triangle^2 f$

$x = 0.15$, $h = 0.1$

since, $s = \dfrac{x - x_o}{h} = \dfrac{0.15 - 0.1}{0.1} = 0.5$

$f = 1.4 + 0.5 \times 0.16 + \dfrac{(0.5)(0.5-1)}{2} \times 0.04$

$= 1.4795$

2b): $\int_a^b f(t).\delta t = \dfrac{b-a}{6}\Big[f(a) + 4f(\dfrac{a+b}{2}) + f(b)\Big]$

2c): $f(8) = 177.27, \quad f(19) = 424.75, \quad f(30) = 901.67$

$A = \dfrac{22}{6}\Big[f(8) + 4f(19) + f(30)\Big] = 11065.78$

where $A = \int_8^{30} \Big(2000ln\Big[\dfrac{140000}{140000 - 2100t}\Big] - 9.8t\Big)\delta t$

3. (a) Show whether:

(i) $\displaystyle\int_0^2 \frac{\ln x}{\sqrt{x}} dx$ **converges or diverges.**

(ii) $\displaystyle\int_2^\infty \frac{dx}{\sqrt{x^2-1}}$ **converges or diverges.**

(b) Using Lebnitz rule of differentiation under integral evaluate

$$\int_0^{\pi/2} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$$

Ans 3a): $\displaystyle\lim_{x \to 0^+} (x-0)^{3/4} \frac{ln(x)}{x^{1/2}} = \lim_{x \to 0^+} \frac{ln(x)}{x^{-1/4}} = 0$

as $\mu = 3/4 < 1$, $\quad \int_0^2 \dfrac{ln(x)}{\sqrt{x}}$ converges . . . . [reffer to convergence via μ method]

(ii) $\displaystyle\int_2^\infty \frac{\delta x}{\sqrt{x^2-1}}$

$\displaystyle\lim_{x \to \infty} x \times \frac{1}{\sqrt{x^2-1}} = \lim_{x \to \infty} \frac{x}{x\sqrt{1 - 1/x^2}} = 1 \neq 0$

i.e. $\displaystyle\int_2^\infty \frac{dx}{\sqrt{x^2-1}}$ diverges.

3b): $I_1 = \int_0^{\pi/2} \dfrac{dx}{a^2cos^2(x) + b^2sin^2(x)} = \displaystyle\int_0^{\pi/2} \frac{sec^2(x).dx}{a^2 + b^2tan^2(x)}$

Let $tan(x) = t$; $\quad x = 0 \Rightarrow t = 0$

$x = \pi/2 \Rightarrow t = \infty$

$$I = \int_0^\infty \frac{dt}{a^2 + b^2t^2} = \frac{1}{b^2}\int_0^\infty \frac{dt}{t^2 + a^2/b^2} = \frac{\pi}{2ab}$$

$$\frac{\partial}{\partial a}\int_0^{\pi/2} \frac{dx}{a^2cos^2(x) + b^2sin^2(x)} = \int_0^{\pi/2} \frac{-2acos^2(x).dx}{(a^2cos^2(x) + b^2sin^2(x))^2}$$

i.e. $\displaystyle\int_0^{\pi/2} \frac{cos^2(x)}{(a^2cos^2(x) + b^2sin^2(x))^2} = \frac{\pi}{4a^3b}$

Similarly, $\displaystyle\int_0^{\pi/2} \frac{sin^2(x)}{(a^2cos^2(x) + b^2sin^2(x))^2}$

$$\int_0^{\pi/2} \frac{sin^2(\frac{\pi}{2} - x)}{(a^2sin^2(x) + b^2cos^2(x))^2}$$

$$= \frac{\pi}{4ab^3}$$

Hence, $\displaystyle\int_0^{\pi/2} \frac{dx}{(a^2cos^2(x) + b^2sin^2(x))^2} = \frac{\pi}{4ab}(\frac{1}{a^2} + \frac{1}{b^2})$

$$= \frac{\pi(a^2 + b^2)}{4a^3b^3}$$

4. (a) Find the value of the integral $\displaystyle\int_0^1 \frac{(1 - x^4)^{3/4}}{(1 + x^4)^2}dx$

(b) Find the value of $\dfrac{\beta(m + 1, n)}{\beta(m, n)}$

(c) Evaluate $\displaystyle\int_{x=0}^1 \int_{y=x}^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2 + y^2}}dy.dx$ by changing the order of integration.

Ans 4a): Finding the value of the integral $\displaystyle\int_0^1 \frac{(1 - x^4)^{3/4}}{(1 + x^4)^2}dx$ in terms of β function.

Let $z = \dfrac{1 - x^4}{1 + x^4}$

then $\displaystyle\int_0^1 \frac{(1 - x^4)^{3/4}}{(1 + x^4)^2}dx = \frac{1}{2^{9/4}}\int_0^1 z^{3/4}(1 - z)^{-3/4}dz$

Clearly of the form; $\displaystyle\int_0^1 x^{m-1}(1 - x)^{n-1}$

Hence $I = \dfrac{1}{2^{9/4}}\beta\Big(\dfrac{1}{4}, \dfrac{7}{4}\Big)$

Ans 4b): $\dfrac{m}{m + n}$

Ans 4c):

On changing order of Integration,

$R_1 : 1 \le y \le \sqrt{2}$

$0 \le x \le \sqrt{2 - y^2}$

$R_2 : 0 \le y \le 1$

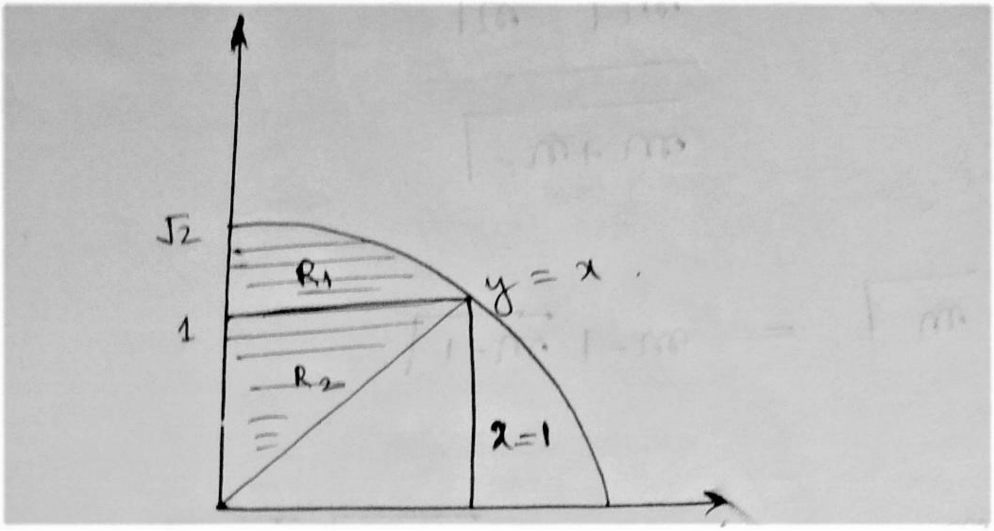
$0 \le x \le y$

$$I_1 = \int\int_{R_1} \frac{x}{\sqrt{x^2 + y^2}}.dxdy = \int_{y=1}^{\sqrt{2}} \int_{x=0}^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2 + y^2}}.dxdy$$

$$= \int_{y=1}^{\sqrt{2}} (\sqrt{2} - y).dy$$

$$= \frac{3 - 2\sqrt{2}}{2}$$

$$I_2 = \int\int_{R_2} \frac{x}{\sqrt{x^2 + y^2}}.dxdy = \int_{y=0}^1 \int_{x=0}^y \frac{x}{\sqrt{x^2 + y^2}}.dxdy$$

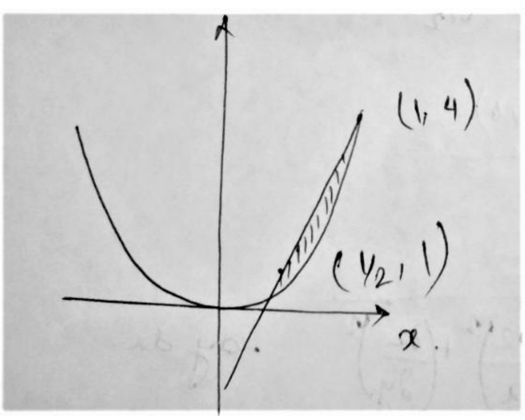


$$= \frac{\sqrt{2}-1}{2}$$

$$\int\int_R \frac{x}{\sqrt{x^2+y^2}}.dxdy = I_1 + I_2$$
$$= \frac{\sqrt{2}-1}{2}$$

5 (a) Use double integral to find the area of the region bounded by the parabola $y = 4x^2$ and the line $y = 6x - 2$
(b) Calculate the volume of the solid which is the part of the right circular cylinder $x^2 + y^2 = 25$ lying in the first octant between $z = 0$ and $z = 3$ using triple integral in cartesian co-ordinates. Also find the same volume using cylindrical co-ordinates.
(c) Using double integral, find the surface area of the part of the plane $2x + 3y + z = 6$ that lies in the first octant.

Ans 5a): $y = x^2$, $\quad y = 6x - 2$
Points of intersection
$\Rightarrow 4x^2 - 6x + 2 = 0$
$\Rightarrow 2x^2 - 3x + 1 = 0$
$x = 1/2 \quad or \quad x = 1$
$(1/2, 1)$, (1,4) are the points of intersection.



$$\text{Area} = \int\int_R dx.dy$$
$$= \int_{1/2}^{1}\int_{4x^2}^{6x-2} dx.dy$$

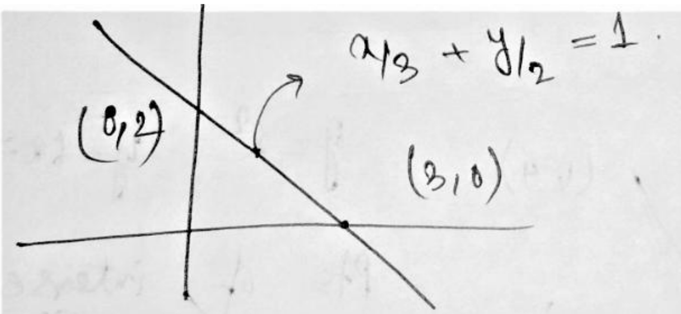
$$= \frac{1}{12}$$

5b): V $= \int_0^5 \int_0^{\sqrt{25-x^2}} \int_0^3 dzdydx$

$$= \int_0^5 3\sqrt{25 - xr}dx$$

$$= \frac{75\pi}{4}$$

5c):



Area $= \int\int \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 \left(\frac{\partial z}{\partial y}\right)^2}.dxdy$

$$= \sqrt{14} \int_0^3 \frac{6-2x}{3} dydx$$

$$= 3\sqrt{14}$$

6. (a) for the scalar function $f = x^2 + y^2$ find the maximum change in the rate ie. the maximum value of directional derivative at the point (2,1,-1).
(b) Given that $\overrightarrow{F} = (x + 2y + \alpha z)\hat{i} + (\beta x - 3y - z)\hat{j} + (4x + \gamma y + 2z)\hat{k}$ is irrational, find the values of α , β and γ.
(c) Given that $\overrightarrow{F}$ and $\overrightarrow{G}$ are two irrational vector functions prove that $\overrightarrow{F} \times \overrightarrow{G}$ is solenoidal.

Ans 6a): Directional derivative $= \frac{1}{|\overrightarrow{a}|}.(\overrightarrow{a}.\nabla f)$
where $\hat{a}$ is the direction
$f = x^2yz^3$
$\nabla \overrightarrow{f} = 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$
$(\nabla \overrightarrow{f})_p = -4\hat{i} - 4\hat{j} + 12\hat{k}$
Maximum in direction $-4\hat{i} - 4\hat{j} + 12\hat{k}$

6b): curl $\overrightarrow{F} = (\gamma + 1)\hat{i} + (\alpha - 4)\hat{j} + (\beta - 2)\hat{k}$

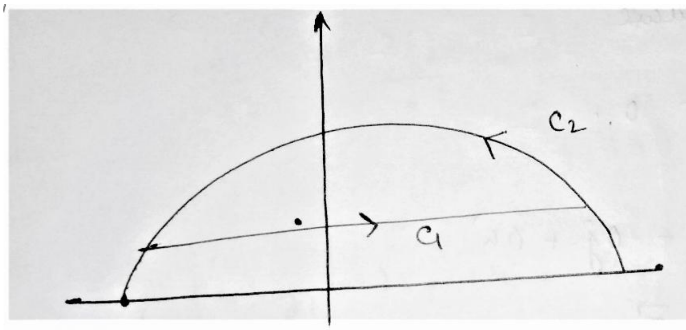
6c): $\overrightarrow{\nabla} \times \overrightarrow{F} = 0, \qquad \overrightarrow{\nabla} \times \overrightarrow{G} = 0$
$\overrightarrow{\nabla}\left(\overrightarrow{F} \times \overrightarrow{y}\right) = \overrightarrow{y}\left(\overrightarrow{\nabla} \times \overrightarrow{F}\right) - \overrightarrow{F}\left(\overrightarrow{\nabla} \times \overrightarrow{G}\right) = 0$
$\therefore \overrightarrow{\nabla}\left(\overrightarrow{F} \times \overrightarrow{y}\right) = 0$
Hence solenoidal.

7. (a) Verification of Green's theorem for $\phi(x^2 + y^2)dx + (x^2 - y^2)dy$.

(b) $\overrightarrow{F}(x, y, z) = (6x + 2y)\hat{i} + 2x\hat{j} + \hat{k}$

(i) To check whether $\overrightarrow{F}$ is conservative. If it is then find $f : \overrightarrow{F} = \overrightarrow{\nabla} f$
(ii) Find work done by $\overrightarrow{F}$ in moving a particle from (1,0,0) to (0,0,2).

Ans 7):



Line integral $= \phi_{C_1} \ldots\ldots + \phi_{C_2} \ldots\ldots$
$\phi_{C_1} = 8/3 \qquad \phi_{C_2} = -32/5$
So $\phi_C = \phi_{C_1} + \phi_{C_2} = -56/15$

$$\frac{\partial f}{\partial y} = 2x + \frac{\partial}{\partial y} f(y,z)$$

$$\frac{\partial f}{\partial x} = 6x + 2y$$

$$\frac{\partial}{\partial y} f_1(y,z) = 0$$

So $f(x,y,z) = 3x^2 + 2xy + f_2(z)$

$$\frac{\partial f}{\partial z} = \frac{\partial f_z}{\partial z} = 1$$

$\Rightarrow f_z(z) = z + c$
$f(x,y,z) = 3x^2 + 2xy + z + c$

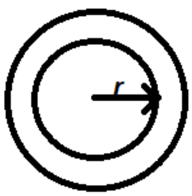
$$\text{work done} = \int_{(1,0,0)}^{(0,0,2)} F.dx = \int_{(1,0,0)}^{(0,0,2)} df = -1$$

PHYSICS

MID-SPRING SEMESTER EXAMINATION **2017-2018**

1 a. A thin circular disc of mass $\frac{1}{\pi}$kg and radius $\frac{1}{\pi^2}$m suspended by an elastic thread performs small torsional oscillations in a liquid. The moment of elastic force emerging in the thread is $\tau = \frac{12.5}{\pi^3}\theta$ N-m, where θ is the angle of rotation from the equilibrium position. The resistance force acting on a unit area of the disc is equal to $P = 3\pi^3\nu$ N/m^2 where the velocity, ν, in m/s, of a given element of the disc relative to the liquid. Find the frequency of the small oscillations.

Ans:



Torque due to resistive force = $(3\pi^3\nu \times 2\pi r\delta r) \times r$ .... [the term force×area×radius]
$=3\pi^3 r\theta \times 2\pi r^2\delta r = 6\pi^4 r^3\delta r = \tau_{res}$
Moment due to oscilating force (the thread) = $\frac{125}{\pi^3}\theta$

$$\tau_{res} = 6\pi^4\theta\frac{r^4}{4} \times 2 = 3\pi^4\theta R^4$$

.... [the 2 factor is multiplied as forces act on both upper and lower surfaces]

$$\tau_{ossc} = \frac{125}{\pi^3}\theta$$

Equation of Motion:

$$\frac{mR^2}{2}\ddot{\theta} + \frac{12.5}{\pi^3}\theta + 3\pi^4R^4\dot{\theta} = 0$$

Now R = $\frac{1}{\pi^2}$ and m = $\frac{1}{\pi}$

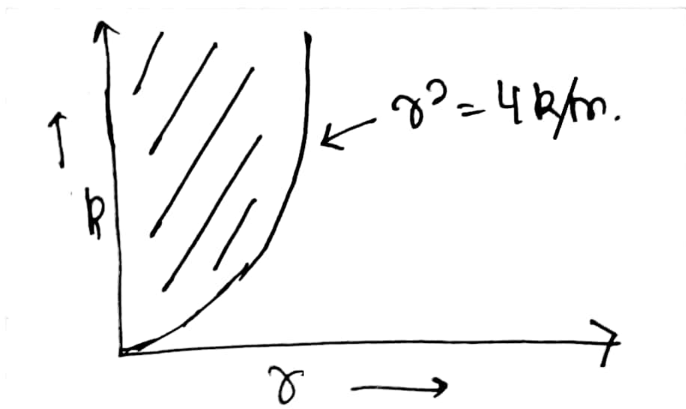
$$\frac{1}{2\pi^3}\ddot{\theta} + \frac{12.5}{\pi^3}\theta + \frac{3\pi^4}{\pi^8}\dot{\theta} = 0$$

$$\ddot{\theta} + 25\pi^2\theta + 6\pi\dot{\theta} = 0$$

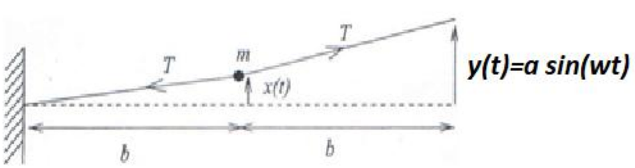
Comparing with: $\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = 0$
we get, $\omega_o = 5\pi, \beta = 3\pi$
$\Rightarrow \omega' = \sqrt{\omega_o^2 - \beta^2}$
T = $2\pi/\omega' = (1/2)s$. Thus $\nu = 2$Hz

1 b). Show the regions of undamped, underdamped, critically damped and overdamped oscillations in the k-r plane, where the restoring force, F_s = kr and the resistive force , $F_d = -r\dot{x}$.

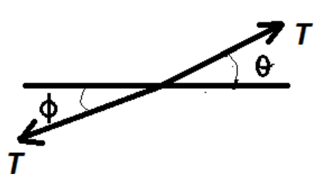
Ans: Equation of motion: m$\ddot{x}$ + r$\dot{x}$ + kx = 0 . . . . . . $\omega_o = k/m$
$\Rightarrow \omega' = \sqrt{\omega_o^2 - \beta^2} = \sqrt{k/m - r^2/4}$
Now for overdamped, ω' is complex . . . .[Please refer theory for complete explanation]
$\Rightarrow r^2 > 4\dfrac{k}{m}$
Now for underdamped, ω' isd real
$\Rightarrow r^2 < 4\dfrac{k}{m}$
Now for critically damped, $r^2 = 4\dfrac{k}{m}$



2. A mass, m, attached to the middle point of a stretched string of length, 2b, and tension, T, rests on a frictionless horizontal plane. One end of the string is fixed while the other end has a periodic motion of small amplitude, a, at right angles to the string in the horizontal plane such that the displacement, y, at time, t, is given by $y(t) = asin(\omega t)$ (sce Fig below). The displacement of the mass at time, t, is c(t). Assume T to be constant during the motion neglecting the effects of x^2 and a^2 . Find the initial displacement $x(0)$ and the velocity $\dot{x}(O)$ such that the resulting motion $x(t)$ of the mass is purely forced (that is there is no component of free oscillations)



Ans. Analyzing the Free Body Diagram



$Tcos\theta = Tcos\phi$ . . .and . . . $m\ddot{x} = Tsin\theta - Tsin\phi$
$\Rightarrow m_o\ddot{x} = T(\theta - \phi)$ . . . . Taking $\theta \sim \phi \sim 0$

$$\Rightarrow m_o\ddot{x} = \frac{T}{b}(y-2x) \ldots\ldots\ldots \text{ As } \phi = \frac{x}{b} and \theta = \frac{y-x}{b}$$
$$\Rightarrow m_o\ddot{x} = \frac{T}{b}(asin(\omega t) - 2x) \Rightarrow m_o\ddot{x} + (\frac{2T}{b})x = \frac{Ta}{b}sin(\omega t)$$
$$\Rightarrow \ddot{x} + (\frac{2T}{m_o b})x = \frac{Ta}{m_o b}sin(\omega t)$$

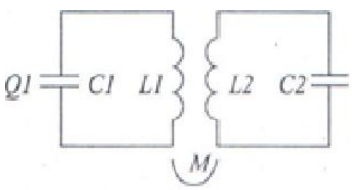
ON solving the differential equation we get $x(t) = \frac{f_o}{\omega_o^2 - \omega^2}sin(\omega t) + Csin(\omega_o t) + Dcos(\omega_o t)$

where $f_o = \frac{Ta}{m_o b}$ and $\omega_o = \frac{2T}{bm}$

Neglecting the transient part $\Rightarrow x(t) = \frac{f_o}{\omega_o^2 - \omega^2}sin(\omega t) \ldots\ldots$ i.e $C = D = 0$

$$\Rightarrow x(0) = 0 \qquad ; \qquad \dot{x}(0) = \frac{\omega a T}{(\omega_o^2 - \omega^2)mb} = \frac{\omega a T}{2T - \omega^2 mb}$$

Q 3 a) A coupled LC circuit is shown in Fig. below. The inductors have the same self inductance $L_l = L_2 = 2.5\mu F$ and the mutual inductance between them is $M1.5mH$. The capacitances $C_1 = C_2 = 2.5\mu F$. Q_1 and Q_2 arc thc instantaneous charges on the capacitors Cl and C2 respectively at any instant. Set up the equations for Q_1 and Q_2. Calculate the normal mode frequencies for this circuit. What would be the directions of the currents in the circuit in these two modes?

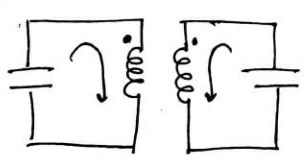


Ans. Writing the electrical equations, $\quad L_1\ddot{q}_1 + M\ddot{q}_2 + q_1/C_1 = 0 \ldots\ldots$ (1)
where, $L_1\ddot{q}_1 = L_1 I_1$ (self-inductance) $\quad q_2 M = MI_2$(mutual-inductance)
Similarly, $L_2\ddot{q}_2 + M\ddot{q}_2 + q_2/C_2 = 0 \ldots\ldots$ (2)
Now let $q_o' = (q_1 + q_2)/2$ and $q_1' = (q_1 - q_2)/2$
Adding and subtracting (1) and (2)
$(L+M)\ddot{q_o'} + q_o/C = 0$ and $(L-M)\ddot{q_1'} + q_1'/C = 0$

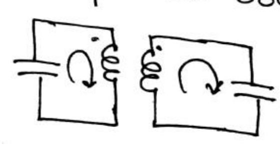
$$\Rightarrow \omega_o^2 = \frac{1}{C(L-M)} \qquad \Rightarrow \omega_1^2 = \frac{1}{C(L+M)}$$

$$\Rightarrow \omega_o^2 = 10^4 rad/s \qquad \Rightarrow \omega_1^2 = 2 \times 10^4 rad/s$$

In-phase oscillation.



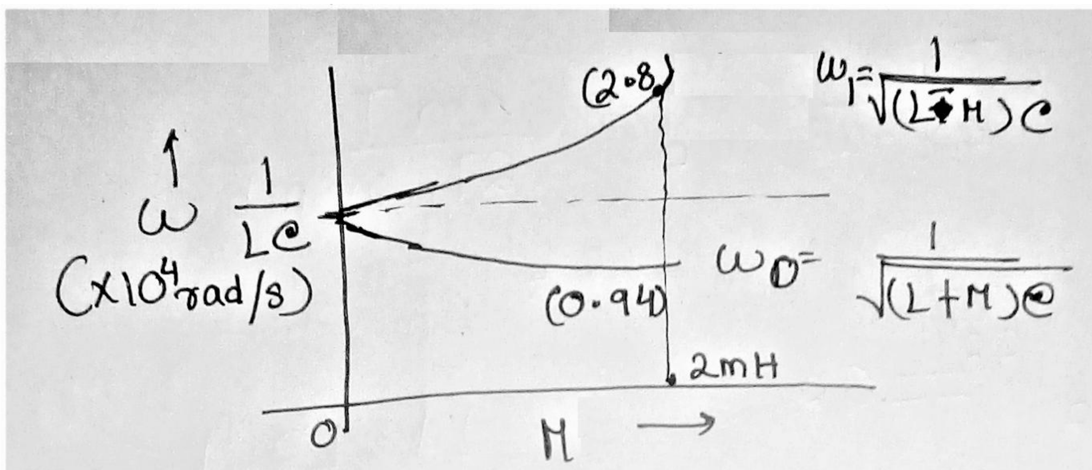
Out-phase oscillation



*Direction of winding is necessary to give exact answer.

3. b) Now suppose in the above problem the mutual inductance is varied from $M = 0$ to $M = 2.0mH$. Plot the variation of normal mode frequencies as a function of M roughly. [3]

Ans: $\omega_o^2 = \frac{1}{(L+M)C} \qquad \omega_1^2 = \frac{1}{(L-M)C}$
Now ω_o must decrease with increasing M, and conversely ω_1 must increase with increasing M.
Also for M=0, $\omega_o = \omega_1$



4. a) A 'wave packet' in a certain medium is represented by the following $\psi(x,t) = 4cos(5x204t)cos(2x72t)cos(x36t)$. **Find the group velocity and phase velocity for this packet.**

Ans:
$$\psi(x,t) = 4cos(5x - 204t)cos(2x - 72t)cos(x - 36t)$$
$$= 2[cos(7x - 276t) + cos(3x - 132t)]cos(x - 36t)$$
$$= cos(8x - 312t) + cos(6x - 240t) + cos(4x - 168t) + cos(2x - 96t)$$

$\omega_{avg} = 204$ rad/s $\qquad k_{avg} = 5\ m^{-1}$

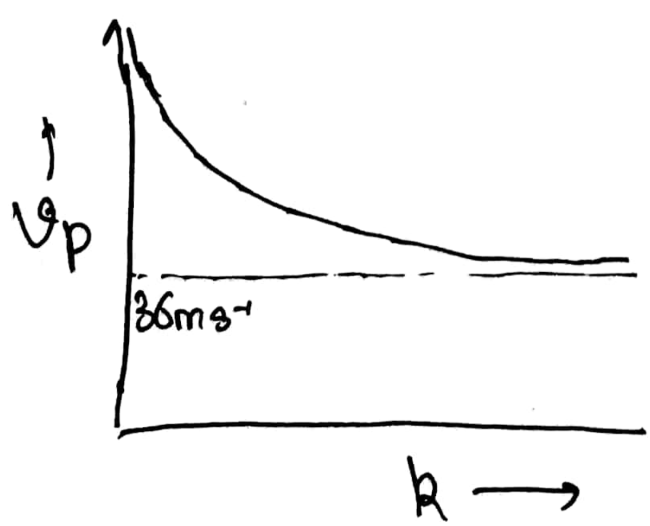
$\Rightarrow v_p = \omega_{avg}/k_{avg} = 40.8\ ms^{-1}$

$$\Rightarrow v_g = \Delta\omega/\Delta k = \frac{312 - 96}{8 - 2} = \frac{216}{6} = 36\ ms^{-1}$$

4 b) Plot the phase velocity in the medium as a function of wave number k, near $k = 5$. Also suggest a plausible dispersion relation near $k = 5$.

Ans: A possible dispersion relation:

$$\omega = 24 + 36k \Rightarrow \frac{\omega}{k} = \frac{24}{k} + 36 = v_p$$



Both side has asymtotic relations.

PHYSICS

MID - AUTUMN SEMESTER EXAMINATION**2017-2018**

1. For forced oscillation with a force$F = F_0 \cos wt$ of an oscillator of mass m and damping constant β , the amplitude of forced oscillation is given by .

$$A = \frac{F_0}{m} \frac{1}{\sqrt{(w_0^2 - w^2)^2 + 4w_0^2\beta^2}}$$

a. If damping constantβ of a driven system is $5s^{-1}$and time period of natural oscillation T_0 is $0.3sec$ find the ratio of maximum amplitude and the amplitude at very low driving frequency($w << w_0$)

b. if a system undergoing force oscillator has equal displacement amplitude at frequencies$w_1 = 400rad/sandw_2 = 600rad/s$, find its resonance frequency.

ans.

1. a. $A = \dfrac{F_0}{m} \dfrac{1}{\sqrt{(w_0^2 - w^2)^2 + 4w_0^2\beta^2}}$, Now $T_0 = 0.3secand\beta = 5Hzandw_0 = \frac{2\pi}{T_0}$

A_{max} means condition of resonance , maximum energy implying maximum amplitude.

$\Rightarrow w = w_{res} = \sqrt{w_0^2 - 2\beta^2}$ and $\Rightarrow A_{max} = \frac{F_0}{\sqrt{w_0^2-\beta^2}(2m\beta)}$ [try to take these as formula]

At low frequency $w << w_{res}$, the amplitude becomes$A_{low} = \frac{F_0}{k}$

$A = \frac{F_0}{mw_0^2} \frac{1}{\sqrt{(1-\frac{w^2}{w_0^2})^2+4(\frac{w}{w_0})^2\frac{\beta^2}{w_0^2}}}$

$\Rightarrow \frac{A_{max}}{A_{low}} = \frac{F_0}{2m\beta\sqrt{w_0^2-\beta^2}} \times \frac{k}{F_0} = \frac{k/m}{2\beta\sqrt{w_0^2-\beta^2}} \approx 2.157$

b. Given at$w_1 andw_2$, the amplitude is same $\frac{F_0}{m} \frac{1}{\sqrt{(w_0^2-w_1^2)^2+4w_1^2\beta^2}} = \frac{F_0}{m} \frac{1}{\sqrt{(w_0^2-w_2^2)^2+4w_2^2\beta^2}}$

or $(w_0^2 - w_1^2)^2 + 4w_1^2\beta^2 = (w_0^2 - w_2^2)^2 + 4w_2^2\beta^2$ solving we get

$w_{res} = w_0^2 - 2\beta^2 = \frac{w_1^2+w_2^2}{2} \approx 509.9rad/s$

2. Consider a one dimensional system depicted in the figure where two blocks of equal mass m are with springs whose opposite ends are fixed with respective walls. Two blocks are also inte- connected with a system of spring and velocity- damper. Each spring has spring constant k and the damping constant of the damper is $\gamma = \sqrt{km}$

a)Express the equations of motion for both the blocks.

b) Determine normal coordinates in terms of the displacements of the blocks , and find nature of the normal modes of oscillations.

c) Find the displacements of both the blocks as a function of time t=0 , one of the blocks is at rest in its mean position and the other blockhas velocity v_0 at the position x_0 from its mean.

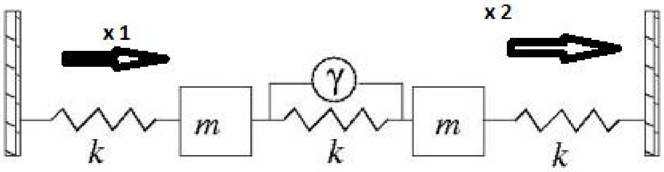


Figure 1

ans

a. Equation of motion $m\ddot{x_1} = -kx_1 - k(x_1 - kx_2) - r(\dot{x_1} - \dot{x_2})$

$m\ddot{x_2} = -kx_2 - k(x_2 - kx_1) - r(\dot{x_2} - \dot{x_1})$

$Takingq_0 = x_1 + x_2andq_1 = x_1 - x_2$

we get $\ddot{q_0} = -w_1^2 q_1 and\ddot{q_1} = -w_1^2 q_1 - 2\beta\dot{q_1}$ where $q_0 = x_1 + x_2$ and $q_1 = x_1 - x_2$ and $w_0 = \sqrt{k/m}$ and $w_2 = \sqrt{3k/m}$ and $\beta = r/m$

where q_0 is in phase oscillation and q_1 is out of phase oscillation , q_1 is an underdamped oscillation

$\beta = \sqrt{km}/m = \sqrt{k/m} \Rightarrow w_1 > beta(underdamped)$

$\Rightarrow x_1(t) = Acos(w_0 t + \phi_1) + Be^{-\beta t}cos(\sqrt{w_2^2 - \beta^2}(t) + \phi_2)$

$x_2(t) = Acos(w_0 t + \phi_1) - Be^{-\beta t}cos(\sqrt{w_2^2 - \beta^2}(t) + \phi_2)$

Imposing conditions $x_1(0) = x_0, x_2(0) = 0, \dot{x_1}(0) = v_0, \dot{x_2}(0) = 0$

$q_0 = Acos(w_1 t + \phi_1)andq_1 = Be^{-\beta t}cos(\sqrt{w_2^2 - \beta^2}(t) + \phi_2)$

$q_0(0) = x_0 \ q_1(0) = 0 \ \dot{q_0}(0) = v_0 \ \dot{q_1}(0) = v_0$

use this conditions to find q andx_1 and x_2

$$x_1(t) = 1/2[x_0 cos(w_1 t)] + e^{-\beta t}/2[x_0 cos(t\sqrt{w_0^2 - \beta^2} + \frac{v_0 + \beta x_0}{\sqrt{w_0^2 - \beta^2}} sin(t\sqrt{w_2^2 - \beta^2}$$

$$x_2(t) = 1/2[x_0 cos(w_1 t)] + e^{-\beta t}/2[x_0 cos(t\sqrt{w_0^2 - \beta^2} + \frac{v_0 + \beta x_0}{\sqrt{w_0^2 - \beta^2}} sin(t\sqrt{w_2^2 - \beta^2}$$

3. The electric field for an electromagnetic wave propagating in free space is given by

$$\vec{E} = \hat{n}E_0 e^{[i(\omega t - 2x + 4y - 4z)]},$$

where $\hat{n} = \frac{1}{\sqrt{18}}[4\hat{i} + \hat{j} - \hat{k}]$ is the unit vector along electric field $\vec{E}$.

(a) What is the wave-length and the frequency $f(= \omega/2\pi)$ of the wave?

(b) Obtain the unit vector in the direction of propagation of the wave and find the magnetic field $\vec{B}$.

(c) Show that the wave is transverse.

(2+2+2)

ANSWER:

Ans. $\vec{E} = \hat{n}E_0 e^{[i(\omega t - 2x + 4y - 4z)]} = \hat{n}E_0 e^{[i(\omega t - \vec{k}.\vec{r})]}$,

$\vec{k} = 2\hat{i} - 4\hat{j} + 4\hat{k}$, $\hat{n} = \frac{1}{\sqrt{18}}[4\hat{i} + \hat{j} - \hat{k}]$

$|\vec{k}| = 6 \text{ length}^{-1} = \frac{2\pi}{\lambda}$

(a) $\lambda = \frac{2\pi}{6} = \frac{\pi}{3}$ length

$f = \frac{2\pi}{\omega} = \frac{c}{\lambda} = \frac{6c}{2\pi} \text{ time}^{-1} = \frac{3c}{\pi} \text{ time}^{-1}$

(b) Direction of propagation$=\hat{k}=\frac{\vec{k}}{6} = \frac{1}{3}(\hat{i} - 2\hat{j} + 2\hat{k})$

$\vec{B} = \frac{\vec{k}\times\vec{E}}{\omega} = \frac{\hat{k}\times\vec{E}}{c} = \frac{\sqrt{18}}{6c}(\hat{j} + \hat{k})|\vec{E}|$

[Compare with $\overrightarrow{E} = \overrightarrow{E_0} e^{i(wt - \overrightarrow{k}\,\overrightarrow{x})}$ where $\overrightarrow{x} = xi + yj + zk$] c. For transverse waves, the direction of propagation

(b) Direction of propagation$=\hat{k}=\frac{\vec{k}}{6} = \frac{1}{3}(\hat{i} - 2\hat{j} + 2\hat{k})$

$\vec{B} = \frac{\vec{k}\times\vec{E}}{\omega} = \frac{\hat{k}\times\vec{E}}{c} = \frac{\sqrt{18}}{6c}(\hat{j} + \hat{k})|\vec{E}|$

is perpendicular to the direction of vibration
$\Rightarrow$ show $\overrightarrow{n}.\overrightarrow{k} = \overrightarrow{n}.\overrightarrow{B} = 0$ where $\hat{n}$ is the direction of electric field $\overrightarrow{B}$ is itself the magnetic field
4. a) Find equation of motion corresponding to the Lagarangian

$$L = \frac{1}{2}\dot{x}^2 - 10x^2 + x[cos4t]$$

using Euler Lagarangian equation $\frac{d}{dt}(\frac{\delta L}{\delta \dot{x}}) - \frac{\delta L}{\delta x} = 0$

b)Show that the steady state solution can be written as $x = \frac{1}{4}e^{i4t}$

ANS:
$L = \frac{1}{2}\dot{x}^2 - 10x^2 + x[cos4t]$
$\Rightarrow \frac{\delta L}{\delta \dot{x}} = \dot{x}\frac{\delta L}{\delta x} = -20x + cos4t$ and $\frac{d}{dt}(\frac{\delta L}{\delta \dot{x}}) = \ddot{x}$
$\Rightarrow$ using Euler Lagarangian form

$$\frac{d}{dt}(\frac{\delta L}{\delta \dot{x}}) - \frac{\delta L}{\delta x} = 0$$

we get $\ddot{x} + 20x = cos4t$ which is the solution of a forced harmonic oscillation .
comparing with $4m\ddot{x} + w_0^2 x = F_0 coswt$
we get $m = 1, w_0^2 = 20, F_0 = 1, w = 4$

Solving differential equation , we get $x(t) = Asinw_0 + Bcosw_0 t + \frac{F_0/m}{w_0^2 - w^2}e^{iwt}$ Note thet the particular solution contains a complex part but x(t) must be real function of t. But we can take the complex part as the solution.
Neglecting transient parts $x(t) = \frac{F_0/m}{w_0^2 - w^2}e^{iwt} = \frac{1}{4}e^{i4t}$ [substitute requiured values]

PHYSICS

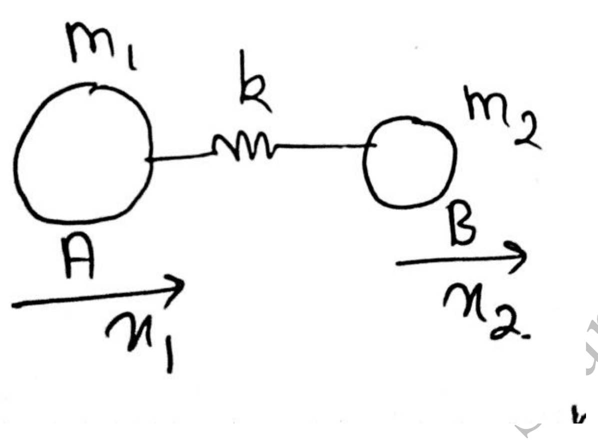
MID SEMESTER SPRING EXAMINATION **2016-17**

(1) Two particles of masses m_1 and m_2 are connected by a massless spring of spring constant k (shown in Figure). If the system is free to oscillate along the length of the spring, the frequencies and amplitudes of the normal modes of the oscillating system are given by

Ans: $\omega_s =$ __________ $\omega_a =$ __________ $a_1 =$ __________ $a_2 =$ __________

Ans 1): $m_1\ddot{x}_1 + k(x_1 - x_2) = 0$

$m\ddot{x}_2 + k(x_2 - x_1) = 0$



NOTE: The question is unclear. Please consult your professor.

One of the $\omega = \sqrt{\dfrac{k(m_1 + m_2)}{m_1 m_2}}$

(2) Two identical blocks, each of mass m and lying on a smooth horizontal platform (ignore friction) are connected to two identical massless springs of spring constant k (shown in Figure). The characteristic frequencies for normal modes of oscillations are given by:

Ans: $\omega_s =$ __________ $\omega_a =$ __________

Ans 2): $\therefore m\ddot{x}_1 + kx_1 + k(x_1 - x_2) = 0$

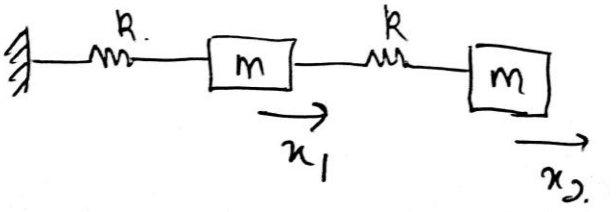
ans $m\ddot{x}_2 + k(x_2 - x_1) = 0$

Put $x_1 = A_1 \sin\omega t$ and $x_2 = A_2 \sin\omega t$

$\therefore -\omega^2 m x_1 + 2kx_1 - kx_2 = 0$

$\therefore -\omega^2 m x_2 + kx_2 - kx_1 = 0$

Solving $\begin{vmatrix} 2k - \omega^2 m & -k \\ -k & k - \omega^2 m \end{vmatrix} = 0$



$\therefore \omega = \dfrac{(\pm\sqrt{5} - 3)}{2}k$

(3) The equation of a travelling plane sound wave has the form: $\xi = 60\cos(1800t - 5.3x)$, where ξ is expressed in μm, t in seconds, and x in meters. **(i)** the phase velocity of the wave and **(ii)** the ratio of the displacement amplitude, with which the particles of medium oscillate, to the wavelength is

Ans: v_p ____________________ **Ans:** ________________________

Ans 3(i): $v_p = \dfrac{\omega}{k} = \dfrac{1800}{5.3} \approx 340\text{ms}^{-1}$
$\varepsilon = A\cos(\omega t - kx)$
Ans 3(ii): Displacement amplitude = A = 60μm
$\lambda = \dfrac{2\pi}{3.3} \approx 1.186$m
$\therefore$ ratio = 5.059×10^{-5}

(4) A plane elastic wave $\xi = ae^{-\gamma x}\cos(\omega t - kx)$, where a, γ, ω and k are constants, propagates in a homogeneous medium. The phase difference between the oscillations at the points where the particles' displacement amplitudes differ by $\eta = 1.0\%$, if $\gamma = 0.42\ m^{-1}$ and the wavelength is $\lambda = 50\ cm$ is

Ans: ________________________

Ans 4): Given $\dfrac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1} = 0.01$ [1%]
or $\varepsilon_1 - \varepsilon_2 = 0.01\varepsilon_1$
or $0.99\varepsilon_1 = \varepsilon_2$
or $\dfrac{\varepsilon_2}{\varepsilon_1} = 0.99$
Now let there be one point O, then there must be another point where this condition satisfies.
$\therefore \varepsilon_1 = ae^{-\gamma(O)}$ [All are
$\therefore \varepsilon_2 = ae^{-\gamma(x)}$ continious functions]
And required values = $\phi = k(x - O) = kx[(\omega t - k(O)) - (\omega t - k(x))]$
$\therefore 0.99 = e^{-\gamma(\frac{\phi}{R})} \cos\phi = e^{-(\frac{\gamma}{R})\phi}$

PHYSICS

END-AUTUMN SEMESTER EXAMINATION **2017-2018**

1. In a double-slits experimental set-up for observing Fraunhoffer diffraction and interference patterns, the width of each slit is b and the separation between the mid-points of two slits is a. Consider normal incidence of monochromatic light of wavelength from a distant source on the slits. The first order diffraction minima occur at = /6, where is the angle of observation making with an axis perpendicular to the slits and passing through the mid-point of separation between two slits.

(a) Find if b = 1600 nm.

(b) If a/b = 5, determine the value of sin for all the constructive interference patterns within the principal maxima.

Ans a) : The first minimum occurs at $\beta = \pm\pi$ where $\beta = \dfrac{\pi b sin\theta}{\lambda}$

$\Rightarrow \pm\pi = \dfrac{\pi b sin\theta}{\lambda} \quad \Rightarrow b sin\theta = \pm\lambda$

Now for $\theta = \pm\pi/6$, $\quad \Rightarrow b \times \dfrac{1}{2} = \lambda = 800$ nm $= 0.6\mu$m

Ans b) : $\beta = \dfrac{\pi b sin\theta}{\lambda}$ *and* $\gamma = \dfrac{\pi d sin\theta}{\lambda}$

where b and d are slit-width and period respectively.

Now $d/b = 5$ means that the $5th$ maxima disappears.

Thus 4 maximas are present.

Now for maxima $\gamma = m\pi \quad \Rightarrow d sin\theta = m\lambda$

$$\Rightarrow sin\theta = m(\frac{\lambda}{d}) = m(\frac{\lambda}{5b})$$

$$\Rightarrow sin\theta = m(\frac{800}{5 \times 1000}) = (\frac{m}{10})$$

$\Rightarrow sin\theta = 0, \pm\dfrac{1}{10}, \pm\dfrac{2}{10}, \pm\dfrac{3}{10}, \pm\dfrac{4}{10}$

2. In an infinite square well potential given by

$$f(n) = \begin{cases} 0 & for\ 0 \leq x \leq L \\ \infty & for\ x > L\ \ and\ \ x < 0 \end{cases}$$

the quantum state of a particle of mass m is described by an initial wave function

$$\Psi(x, 0) = \frac{\sqrt{3}}{2}\phi_1(x) + \frac{1}{2}\phi_2(x)$$

where $\phi_n(x)$ $(n \geq 1)$ are normalized eigenstates of Hamiltonian corresponding to this potential with E_n.

(a) What is the expectation value of Hamiltonian in the state $\Psi(x, 0)$

(b) Determine $\Psi(x, t)$ at time t.

(c) Calculate expectation value of momentum in the above state $\Psi(x, t)$ at time t, i.e. $\langle p \rangle = \int_{-\infty}^{\infty} \Psi^*(x, t) p \Psi(x, t) \delta x$

Ans a) : We know $H\psi = E\psi$

$$\langle E \rangle = \frac{c_1^2 E_1 + c_2^2 E_2}{c_1^2 + c_2^2} = \frac{3}{4}E_1 + \frac{1}{4}E_2$$

$$= (\frac{3}{4} + \frac{4}{4})E_1 = \frac{7}{4} \times \frac{h^2}{8mL^2} = \frac{7h^2}{32mL^2} \qquad [\text{where } E_n = \frac{n^2h^2}{8mL^2}]$$

Ans b) : $\Psi(x, t) = \dfrac{\sqrt{3}}{2}\phi_1(x) e^{\dfrac{-iE_1}{\hbar}z} + \dfrac{1}{2}\phi_2(x)^{\dfrac{-iE_2}{\hbar}z}$

where $\phi_n = \sqrt{\frac{2}{L}} sin(\frac{n\pi x}{L})$ and $E_n = \frac{n^2h^2}{8mL^2}$

Ans c) : $\hat{p} = -i\hbar\frac{\partial}{\partial x}$

$\Rightarrow \langle p\rangle = \int_0^L \delta x\Psi(x,t)(-i\hbar\frac{\partial}{\partial x})\phi(x,t)$

$= \int_0^L \delta x(-i\hbar)\frac{2\pi}{L^2}[sin(\frac{2\pi x}{L})cos(\frac{\pi x}{L})\frac{\sqrt{3}}{2}e^{\frac{i(E_2-E_1)}{\hbar}t} + \quad cos(\frac{2\pi x}{L})sin(\frac{\pi x}{L})\frac{2\sqrt{3}}{4}e^{\frac{-i(E_2-E_1)}{\hbar}t} \quad]$

$= \frac{\hbar}{L\sqrt{3}}sin(\frac{E_2-E_1}{\hbar}t)$

3. In the Young's double slit interference experiment, a very thin glass plate (refractive index $\mu = 1.5$) is placed between the slit S_1 and the screen, The light from slit S_2 do not pass through this glass plate.

(a) Find an expression for the shift of the central fringe from its original (without glass plate) position in terms of the thickness of the glass plate 't', distance 'D' between screen and the mid-point of the slits S_1 and S_2, $d = S_1 - S_2$ and μ.

(b) When a glass plate ($\mu = 1.5$) of thickness $t = 1.5 \times 10^{-6}$ m is introduced as above, the central fringe shifts to the position of the 2^{nd} dark fringe of the original pattern, find the wavelength of the light used.

(3+2)

Ans 3a) : From the knowledge of JEE

$(\mu-1)t = \frac{yd}{D}$

$\Rightarrow y = (\mu-1)\frac{tD}{d}$

Ans 3b) : For 2nd dark fringe, path difference $= \frac{3\lambda}{2}$

$\Rightarrow \frac{yd}{D} = \frac{3\lambda}{2} \qquad \Rightarrow y = \frac{3\lambda D}{2d}$

$\Rightarrow \frac{3\lambda D}{2d} = (\mu-1)\frac{tD}{d}$

$\Rightarrow \lambda = \frac{2t}{3}(\mu-1) = \frac{2}{3}\times 1.5\times 10^{-6}\times 0.5 = 0.5\ \mu$m

4. Starting from the time-dependent Schrödinger equation in 3 dimensions

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\psi + V\psi$$

(a) Show that the probability density ($\rho = |\psi|^2$) satisfies the equation

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0\,.$$

Find $\vec{J}$. (Hint: You can start with 1 dimensional Schrödinger equation and identify each component of $\vec{J}$)

(b) Calculate $\vec{J}$ for a plane wave form of wave function

$$\psi(\vec{r},t) = e^{i/\hbar(Et - \vec{p}\cdot\vec{r})}$$

(c) Find the expectation value of position x in the ground state of a harmonic oscillator

Ans 4a) : We know $E\psi = -i\hbar\frac{\partial}{\partial t}\psi$ and $H\psi = E\psi$

$\Rightarrow \frac{\partial\psi}{\partial t} = \frac{H\psi}{-i\hbar}$

$\Rightarrow \psi^*\frac{\partial\psi}{\partial t} = (\frac{i\hbar}{2m}\frac{\partial^2\psi}{\partial x^2} - \frac{i}{\hbar}v\psi) \times \psi^* \ldots\ldots\ldots\ldots (1)$

and $\psi\frac{\partial\psi^*}{\partial t} = (\frac{i\hbar}{2m}\frac{\partial^2\psi^*}{\partial x^2} - \frac{i}{\hbar}v\psi^*) \times \psi \ldots\ldots\ldots$ [Take compliment of (1)]

Adding $\frac{\partial}{\partial t}(\psi^*\psi) \ldots\ldots\ldots\ldots\ldots\ldots \left[\frac{\partial}{\partial t}|\psi|^2 = \frac{\partial}{\partial t}(\psi^*\psi) = \psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t}\right]$

$= \frac{i\hbar}{2m}\left[\psi^*\frac{\partial^2\psi}{\partial x^2} - \psi\frac{\partial^2\psi^*}{\partial x^2}\right]$

$= \frac{i\hbar}{2m}\left[\left(\psi^*\frac{\partial^2\psi}{\partial x^2} + \frac{\partial\psi^*}{\partial x}\frac{\partial\psi}{\partial x}\right) - \left(\psi\frac{\partial^2\psi^*}{\partial x^2} + \frac{\partial\psi^*}{\partial x}\frac{\partial\psi}{\partial x}\right)\right]$

$= \frac{i\hbar}{2m}\frac{\partial}{\partial x}\left[\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right]$

Converting to 3-D co-ordinates

$\frac{\partial}{\partial x}|\psi^2| = (-\overrightarrow{\nabla}\cdot\overrightarrow{J})$

$\Rightarrow \overrightarrow{J} = \left[\psi(\overrightarrow{\nabla}\cdot\psi^*) - \psi^*(\overrightarrow{\nabla}.\psi)\right]\frac{i\hbar}{2m}$

Ans 4b) : $J = \frac{i\hbar}{2m}\left[\psi(\nabla\cdot\psi^*) - \psi^*(\nabla.\psi)\right]$

Now $\psi = e^{(i/\hbar)(Et - p_x x - p_y y - p_z z)}$

and $\psi^* = e^{(-i/\hbar)(Et - p_x x - p_y y - p_z z)}$

$\Rightarrow \psi\frac{\partial}{\partial x}\psi^* = +\frac{i}{\hbar}p_x\psi\psi^* = \frac{ip_x}{\hbar}$

Similarly $\psi(\nabla\psi^*) = \frac{i}{\hbar}\overrightarrow{p}.(i+j+k) \ldots\ldots\ldots\ldots (1)$

and also $\psi^*(\nabla\psi) = \frac{-i}{\hbar}\overrightarrow{p}.(i+j+k)$ [Taking compliment of (1)]

$\Rightarrow J = \frac{i\hbar}{2m}\left[\frac{i}{\hbar}(p_x + p_y + p_z) \times 2\right] = \frac{-1}{m}(p_x + p_y + p_z) = -(v_x + v_y + v_z) = -v$

$\Rightarrow \hat{J} = -\hat{v}$

Ans 4c) : $\langle x\rangle = \frac{1}{N^2}\int_{-\infty}^{\infty} e^{-\alpha x^2} x e^{-\alpha^* x^2}\delta x = 0$ [Odd function]

$\langle x\rangle = \frac{1}{N^2}\int_{-\infty}^{\infty}\psi^*(x,t)x\psi(x,t)\delta x = 0$

N = Normalization Constant [You can also perform the integration to get the same results]

5. (a) Define: $\vec{L} = \vec{r}\times\vec{p}$, where $\vec{r} \equiv (r_1, r_2, r_3) \equiv (x, y, z)$ is the position vector and $\vec{p} \equiv (p_1, p_2, p_3)$ is the momentum, and derive an expression for the commutator $[L_1, L_3]$ in terms of L_2, using the operator relationships:

$$[r_i, p_j] = i\hbar\delta_{ij} \qquad [AB, C] = A[B, C] + [A, C]B$$

$$[A, BC] = B[A, C] + [A, B]C \qquad [A \pm B, C] = [A, C] \pm [B, C]$$

(b) Prove that the matrices

$$\sigma_1 = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfy the same commutation relations as L_i by evaluating the commutator $[\sigma_1, \sigma_3]$. Show that the vectors $|\beta_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $|\beta_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are orthonormal eigenstates of the operator σ_1. Find the eigenvalues of σ_1 and the expectation values $\langle\beta_2|\sigma_3|\beta_2\rangle$; $\langle\beta_1|\sigma_1|\beta_1\rangle$.

Ans 5a):
$$\begin{aligned}[L_1, L_3] &= [yp_z - zp_y, xp_y - yp_x] \\ &= [yp_z, xp_y] - [yp_z, yp_x] - [zp_y, xp_y] + [zp_y, yp_x] \\ &= y[p_z, xp_y] + [y, xp_y]p_z \\ &= y[p_z, x]p_y + y[p_z, p_y] + x[y, p_y]p_z + [y, x]xp_z \\ &= 0 + 0 + i\hbar xp_z\end{aligned}$$

Similarly
$$\begin{aligned}[yp_z, yp_x] &= y[p_z, yp_x] + [y, yp_x]p_z \\ &= y[p_z, y]p_x + y^2[p_z, p_x] + y[y, p_x]p_z + [y, y]p_x p_z \\ &= 0\end{aligned}$$

Similarly $[zp_y, xp_y] = 0$ and $[zp_y, yp_x] = -i\hbar zp_x$

$\Rightarrow [L_1, L_3] = i\hbar[xp_z - zp_x] = -i\hbar L_2$

Remembering the cyclic form
$$\begin{aligned}L_x &= yp_z - zp_y \\ L_y &= zp_x - xp_z \\ L_z &= xp_y - yp_x\end{aligned}$$

$\Rightarrow [l_x, L_z] = -i\hbar Ly$

Ans 5b) : $[\sigma_1, \sigma_2] = \dfrac{\hbar^2}{4}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \dfrac{\hbar^2}{4}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = i\hbar\sigma_2$

To prove othonormal eigen states, we have to show $\sigma_1|\beta_1\rangle = |\beta_1\rangle$ and $\sigma_2|\beta_2\rangle = |\beta_2\rangle$

$$\sigma_1|\beta_1\rangle = \frac{\hbar}{2\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{\hbar}{2\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = |\beta_1\rangle$$

$$\sigma_2|\beta_2\rangle = \frac{\hbar}{2\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{\hbar}{2\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

*There is probably some problem regarding the sign.

$$\begin{aligned}\langle\beta_2|\sigma_3|\beta_2\rangle &= \frac{1}{2}\begin{bmatrix} 1 & -1 \end{bmatrix}\frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \frac{\hbar}{4}\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{\hbar}{4}[0] = 0\end{aligned}$$

$$\langle\beta_2|\sigma_3|\beta_2\rangle = \frac{1}{2}\frac{\hbar}{2}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{\hbar}{4}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{\hbar}{2}$$

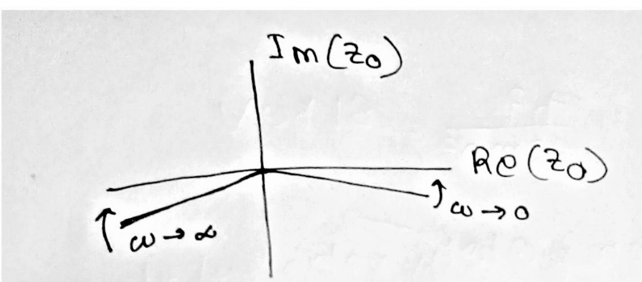
6. (a) A particle of mass m moves in one dimension and is subject to a restoring force which is proportional to its displacement x and a damping force which is proportional to its velocity. Derive the differential equation for its motion and assuming a trial solution $x = Ae^{i\omega t}$ describe the conditions for lightly damped, critically damped and over damped oscillations.

(b) Write down the differential equation of this system when it is also acted upon by a driving force $F_0 \cos \omega_F t$. If we can write the solution as $x = A\cos(\omega_F t + \phi)$, show that at low frequencies ω_F, (ignoring terms with ω_F^2 and ω_F) the phase ϕ is zero and the amplitude A is independent of the driving frequency ω_F, whereas at high frequencies (when ω_F^2 term dominates) $\phi = \pi$ and A depends on ω_F.

Ans 6a) : Resistive force $= c\dot{x}$
$\therefore$ Equation of motion $\Rightarrow m\ddot{x} + c\dot{x} + kx = 0 \quad \Rightarrow \ddot{x} + 2\beta\dot{x} + \omega_o x = 0$. . .where $\omega_o^2 = k/m$ and $\beta = c/2m$
Now $x = Ae^{i\omega t}$
$\therefore -A\omega^2 e^{i\omega t} + 2\beta(i\omega)Ae^{i\omega t} + \omega_o^2 Ae^{i\omega t} = 0$
or $\omega^2 - (2\beta i)\omega - \omega_o^2 = 0$
$\therefore \omega = (2\beta i \pm \sqrt{-4\beta^2 + 4\omega_o^2})/2 = i\beta \pm \sqrt{\omega_o^2 - \beta^2}$
$\therefore x = e^{i\omega t} = e^{(-\beta \pm i\sqrt{\omega_o^2 - \beta^2})t}$
$\therefore \omega_o > \beta$ underdamped
$\Rightarrow \omega_o >> \beta$ Lightly damped
$\therefore \omega_o < \beta$ overdamped
$\therefore \omega_o = \beta$ critically damped

Ans 6b) : Differential equation: $m\ddot{x} + c\dot{x} + kx = F_o cos(\omega_f t)$
or $\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = f_o cos(\omega_f t)$
$\therefore 2\beta = c/m$ and $\omega_o^2 = k/m$

$$\text{The complex amplitude is } z_o = \frac{f_o}{\omega_o^2 - \omega^2 + 2i\beta\omega} = \frac{f_o((\omega_o^2 - \omega^2) - 2i\beta\omega)}{(\omega_o^2 - \omega^2)^2 + 4\beta^2\omega^2}$$



$$\therefore \phi = tan^{-1}\Big(\frac{-2\beta\omega}{(\omega_o^2 - \omega^2)}\Big)$$

For $\omega \to 0$, Re$(z_o) > 0$ and Im$(z_o) < 0$
$\therefore$ the phasor approaches 0 $\quad \therefore \phi = 0$
For $\omega \to \infty$, Re$(z_o) < 0$ and Im$(z_o) > 0$
$\therefore$ the phasor approaches $-\pi$ $\quad \therefore \phi = -\pi$

$$|z_o| = \frac{f_o}{\sqrt{(\omega_o^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

$$\text{For } \omega \to 0 \quad |z_o| = \frac{f_o}{\omega_o^2}$$

$$\text{For } \omega \to \infty \quad |z_o| = \frac{f_o}{\omega^2\sqrt{(\frac{\omega_o^2}{\omega^2} - 1) + 4(\frac{\beta}{\omega})^2}} = \frac{f_o}{\omega^2} \text{ . . . . as } \frac{\omega_o}{\omega} \text{ and } \frac{\beta}{\omega} \text{ both tend to zero}$$

7. a) Consider an atomic system with two energy levels, $E_1 = -20$ eV, $E_2 = -14$ eV. Lowest energy state is a stable state and the upper level has a lifetime of $t_2 = 10$ ms. If a photon is emitted in transitions from the state 2 to state 1, what is the frequency and width of the observed spectral line?

b) A nucleus has a radius of 4×10^{-15} meter (4 Fermis). Use the uncertainty principle to estimate the kinetic energy (in units of eV) for a neutron localized inside the nucleus.
(mass of neutron $m_n = 1.675 \times 10^{-27}$ kg; $h = 6.626 \times 10^{-34}$ J-s; $e = 1.602 \times 10^{-19}$ C)

Ans 7a) : $\Delta E = h\Delta v$ or $\Delta v = \Delta E/h$
Also from uncertainty principal $\Delta E \Delta t = \hbar/2$ [for minimum uncertainty]
$\therefore \Delta v = \frac{\hbar}{2\Delta t} \times \frac{1}{h} = \frac{1}{4\pi\Delta t}$ $\Rightarrow \Delta t = 10\text{ms}$
$\therefore v = \frac{E_2 - E_1}{h} = \frac{6 \times 1.6 \times 10^{-19}}{6.634 \times 10^{-34}} = 1.45 \times 10^{15}\text{Hz}$
And $\Delta v = 8\text{Hz}$
Here ΔE is uncertain due to the uncertainty or specifically instability in the E_2 state. The lifetime t_2 cause a variation Δv
and $E = hv$
on partial differentiation we get $\Delta E = h\Delta v$

Ans 7b) : $\Delta x = 2\Delta r = 8 \times 10^{-15}\text{m}$
$\Delta x \Delta p = \hbar/2$ [minimum deviation]
$\therefore \Delta p = \frac{\hbar}{2\Delta x}$
$\therefore E = \frac{(\Delta p)^2}{2m} = \frac{\hbar^2}{m(\Delta x)^2} = \frac{\hbar}{32mr^2} = 81\text{Kev}$

8. a) We wish to use a plate of glass (refractive index $\mu = 1.5$) as a polarizer. What is the polarizing angle?

(1)

b) The indices of refraction of yellow light of 6,000 A in a doubly refracting medium are $\mu_o = 1.71$ for the ordinary ray and $\mu_e = 1.74$ for the extraordinary ray.
(i) Determine the velocity of each of these waves in this medium transverse to the optic axis.
(ii) Determine the minimum thickness of this material necessary to produce a path difference of half a wavelength for these two rays.

Ans 8a) : The reflected light is completely polarized at Brewster's Angle
$\therefore \theta_b = tan^{-1}(1.5) = 56.309^o$
Ans 8b(i)) : $v_o = c/\mu_o = 1.754 \times 10^8\text{ms}$
$v_e = c/\mu_e = 1.724 \times 10^8\text{ms}^{-1}$
Ans 8b(ii)) : $D = d(\mu_e - \mu_o)$ where d = thickness and D = $\lambda/2$
$\therefore d = \frac{\lambda}{2(\mu_e - \mu_o)} = 10\mu\text{m}$

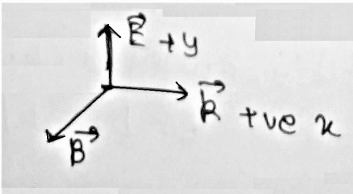
Multiple Choice Questions:

1. The electric field of an electromagnetic wave in vacuum is given by

$$E_x = 0; \quad E_y = 30\cos\left(2\pi \times 10^8 t - \frac{2\pi}{3}x\right); \quad E_z = 0\,.$$

Determine the direction of propagation of the wave and the magnetic field

Ans: **(B)) In x-direction and z-direction, respectively**
Propagation along +ve x $(\omega t - kx)$ type



$\vec{k} = c(\vec{E} \times \vec{B})$ . . . . . . where c is a real constant
$\therefore \vec{B}$ is along +ve z direction.

2. For an undamped simple harmonic oscillator with harmonic forcing, the equation of motion $\frac{d^2x}{dt^2} + \omega_0^2 x = f_0 \cos \omega t$ implies

Ans: **(D) Kinetic Energy is minimum at** $\omega = 0$
$\ddot{x} + \omega_o x = f_o cos(\omega t)$
For Forced underdamped oscillations $x(t) = \dfrac{f_o}{\omega_o^2 - \omega^2} cos(\omega t) + Bcos(\omega_o t) + Csin(\omega_o t)$
Neglecting transients $x(t) = \dfrac{f_o}{\omega_o^2 - \omega^2} cos(\omega t)$. [Steady state solution]
Complimentary function: $Bcos(\omega_o t) + Csin(\omega_o t)$
It is simply oscillating.
Particular integral is oscillating with ω and not ω_o.
v=0 is not true for all cases.It is only when t=0.

3. The P.E. of a particle is given by $V(x) = x^4 - 4x^3 - 8x^2 + 48x$. The points of stable and unstable equilibrium are:

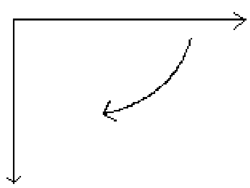
Ans: **(D)stable at x = -2;3. Unstable at x = 2**
$v(x) = x^4 - 4x^3 - 8x^2 + 48x$
$v'(x) = 4x^3 - 12x^2 - 16x + 48 = 4(x^3 - 3x^2 - 4x + 12) = 4(x + 2)(x - 2)(x - 3)$
Maxima at x=2 and mimina at x=-2,3
$\therefore$ stable at minima and unstable at maxima. Hence **(D)**.

4. What is the ground state energy of a quantum particle of mass m confined in a one-dimensional potential $V(x) = \begin{cases} \frac{1}{2}m\omega^2x^2 & \text{for } x \geq 0 \\ \infty & \text{for } x < 0 \end{cases}$?

Ans: **(A)** $E = (n + 1/2)\hbar\omega$
For $n = 0, E = \dfrac{1}{2}\hbar\omega$

5. If two electro-magnetic waves given by corresponding electric fields $\vec{E}_1 = \hat{i}\frac{E_0}{\sqrt{2}}\cos(\omega t - kz)$ and $\vec{E}_2 = \hat{j}\frac{E_0}{\sqrt{3}}\cos(\omega t - kz - 3\pi/2)$ superpose, what will be the polarization of the resultant wave ?

Ans: **(D)Right-elliptically polarized**
$\overrightarrow{E_2} = -\hat{j}\dfrac{E_o}{\sqrt{3}}sin(\omega t - kz)$
$\overrightarrow{E_1} = \hat{i}\dfrac{E_o}{\sqrt{2}}sin(\omega t - kz)$



Elliptically polarised as magnitudes of E_1 and E_2 are different.
Also it rotates clockwise.

PHYSICS

END - SPRING SEMESTER EXAMINATION **2017-2018**

Q 1.a) A cadmium red spectral line has maximum intensity , I_{max}, at wavelengh 643.847 nm. The intensity falls to the value $I_{max}/2$ at wavelength 643.84752 nm and 643.84648 nm. Estimate the coherence length of the electron , e= 1.6×10^{-19} C , 1eV= 1.6×10^{-19} J.[3]

Q 1.b) A beam of white light is normally incident on a transparent plate of refractive index 1.575 and thickness 0.5×10^{-6}m. What wavelengths lying within the limits of visible spectrum (400-700 nm) will be attenuated most in the transmitted beam? [4]

Q 1.c) An ideal young's double slit is illuminated with a source having two wavelengths, $\lambda_1 = 418.6$nm and $\lambda_2 = 421.4$nm. In the source the intensity at λ_1 is double of that at λ_2 . Determine the visibility of fringes near order m=0 and near order m=75 on the screen.[Visibility =$(I_{max} - I_{min})/I_{max} + I_{min})$] [5]

Q 1.d) A Fabry - Perot intereferometer is used to obtain circular fringes using monochromatic source. The separation between the mirrors is d= 6 nm. By moving one of the mirrors by 0.03 mm it has been found that 100 fringes cross the field of viw. If the coefficient of amplitude reflection, r, of the mirrors is 0.734 find the minimum wavelength separation(near the source wavelength) which could be resolved by the above interferometer. [6]

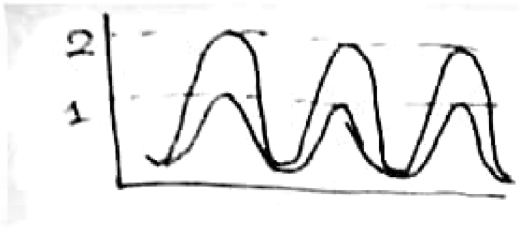
1.a) $l_c = c(\Delta T) = \frac{C}{|\Delta V|}$ $\quad \Delta T =$ time of wave packet
$V\lambda = c \quad \Delta V =$ bandwidth of source
or $(\Delta V)\lambda + V(\Delta\lambda) = 0$
$\Rightarrow \Delta V = \frac{\Delta}{\lambda}\lambda V = \frac{(\Delta\lambda)c}{\lambda^2}$
$\therefore l_c = \frac{c}{\Delta V} = \frac{\lambda^2}{(\Delta\lambda)} = \frac{(643.847\times10^{-9})^2}{1.04\times10^{-12}}m = 0.4m$
b) Minimum condition for the transmitted beam in thin files
$2\mu cos\theta = (m + 1/2)\lambda$ [Refer interference of transmitted beam in thin film interference]
Now $cos\theta = 1$ [normally incident]
$\therefore \lambda = \frac{2\mu t}{(m+1/2)}$
$= \frac{2\times1.575\times0.5\times10^{-6}}{(m+1/2)}m, m = 1, 2, 3, 4$
$m = 2$ gives $\lambda = 0.63\mu m$
$m = 3$ gives $\lambda = 0.45\mu m$

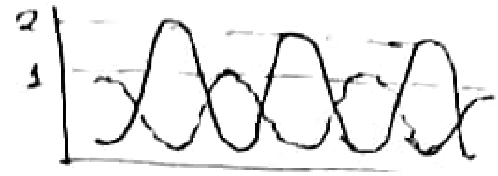
1.c) For concordance to occur
$(p + 1)\lambda_1 = p\lambda_2 =$ path difference
or $(p + 1)418.6 \times 10^{-9} = p \times 421.4 \times 10^{-9}$
$p = 149.5 \approx 150$
$\therefore$ For m=0 ,150,300,.....The maxima of λ_1 will lie on max of λ_2
For discordance to occur
$p\lambda_1 = (p + 1/2)\lambda_2 =$ Path difference
p = 75,(150+75),(300+75),..... The minima of λ_1 falls on maxima of λ_2

For m =0
$I_{max} = 3, I_{min} = 0$
$V = 1$

For m = 75
$I_{max} = 2, I_{min} = 1$
$V = 1/3$

1.d) 2d = m λ
or $2(\Delta d) = (\Delta m)\lambda$





$\rightarrow \lambda = \frac{2(\Delta d)}{\Delta m} = 0.6\mu m$
$\therefore m = \frac{2d}{\lambda} = 2 \times 10^4$
$\therefore CRP = \frac{m\pi f}{2} = \frac{m\pi}{2} \times \frac{2r}{1-r^2} = \frac{m\pi r}{1-r^2}$
$\therefore \frac{\lambda}{\Delta\lambda} = CRP = \frac{m\pi r}{1-r^2}$ [chromatic resolving power of Fabry - Perot interferometer]
$\rightarrow \Delta\lambda = \frac{\lambda(1-r^2)}{m\pi r} = 6pm$

Q 2.a) a double s;lit has individual slit width b= 5.74μm. The separation between centres of the slits is d= 22.96μm. A $\lambda = 0.6\mu$m and intensity 1 unit falls normally on this double slit. Drraw a neat Fraunhhofer intensity pattern as function of diffraction angle θ , showing the qualitative and quantitative details, i.e. the peaks inside the primary and first secondary lobes, their positions and intensities.(NO graph paper required) [5]

Q 2.b) A plane monochromatic light wave of intensity I_0 and wavelentgh 0.64μm along theb +Z axis is normally incident on an aperture placed at Z=0 plane with its centre, P, at origin . The aperture , as shown in the Fig 1, has three concentric zones withe radii $r_1 = 0.4mm$, $r_2 = \sqrt{2}r_1$ and $r_3 = \sqrt{3}r_1$. The inner opaque part is made of two semicircles of radius r_1, r_2 with their centres coinciding at P.only the white part captured between the shaded regions is open.Find the intensity on the z axis at a = 0.5m from the origin.

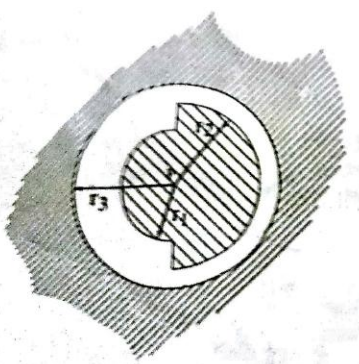


Fig 1: Problem 2 b)

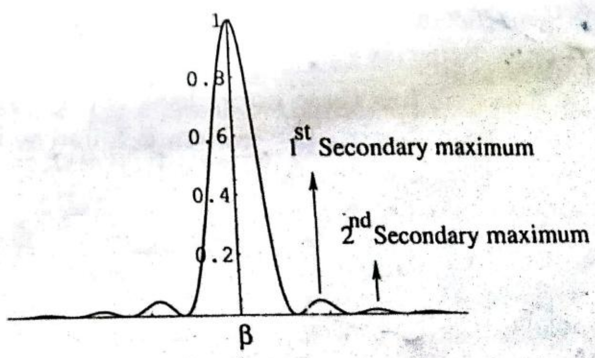
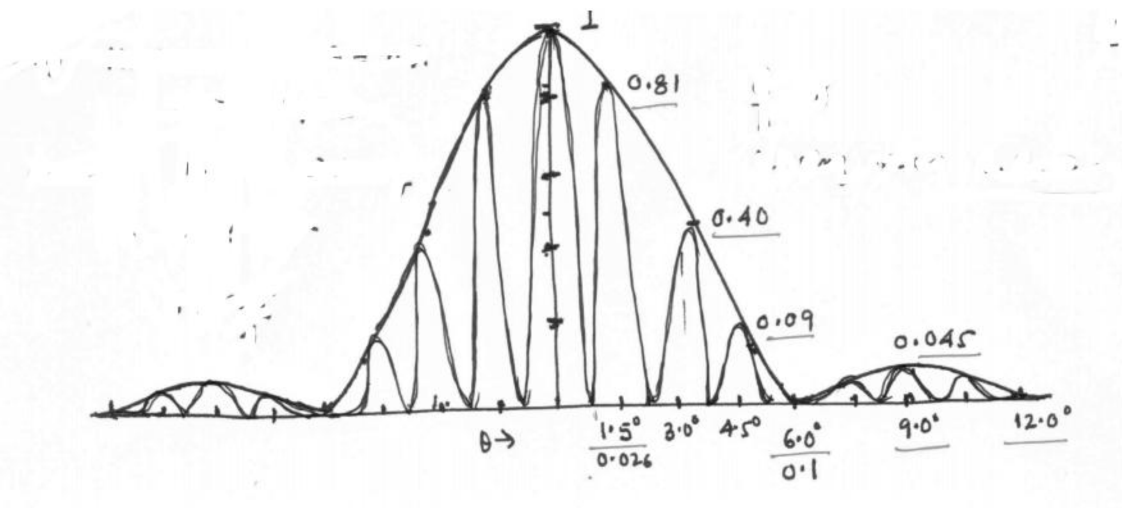


Fig 2: Problem 2 c)

2 c) Calculate the parameter $\beta = \pi b \sin\theta/\lambda$ for the first and the second maxima in a single slit Fraunhofer diffraction and hence find the ratio of their intensities (see Fig. 2)

2.a) First d/b = 4
$\therefore$ The missing orders are 4,8,12,......
$I = I_0\frac{sin^2\beta}{\beta^2} \times \frac{sin^2 Nr}{sin^2 r}$ Where $\beta = \frac{\pi b sin\theta}{\lambda}$ and r = $\frac{\pi d sin\theta}{\lambda}$
$\therefore$ The outer cover is of $I_0 sin^2\beta/\beta^2$ and inside is the wave of $sin^2 Nr/sin^2 r$

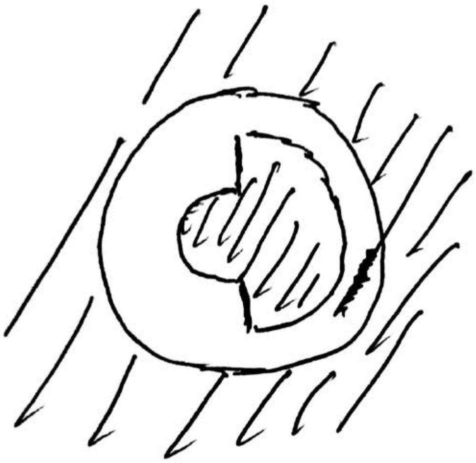
Maxima at $dsin\theta = n\lambda, n \neq 4, 8, 12, ...$

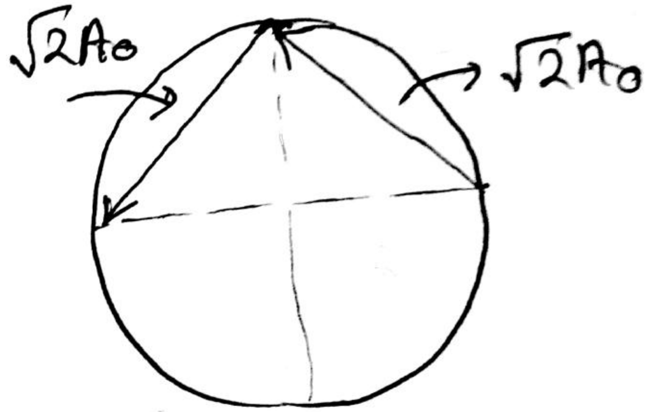


For central maxima ,$I_0 = 1 = I_0$
For 1st maxima , $dsin\theta = \lambda_1 \to sin\theta = \lambda/d \to \theta \approx 1.5°$
For 2nd maxima , $dsin\theta = 2\lambda \to sin\theta = 2\lambda/d \to \theta \approx 4.5°$
For 3rd maxima , $dsin\theta = 3\lambda \to \theta \approx 3°$
For 4th maxima (absent), $dsin\theta = 4\lambda \to \theta \approx 6°$
For 5th maxima,$\theta \approx 7.5°$
For 6th maxima,$\theta \approx 9°$
For 7th maxima,$\theta \approx 10.5°$
For 8th maxima,$\theta \approx 12°$
Also $I_0 = 1, I_1 = 0.81, I_2 = 0.40, I_3 = 0.09, I_4 = 0, I_5 = 0.032, I_6 = 0.045, I_7 = 0.016, I_8 = 0$
For maxima $I = I_0\frac{sin^2\beta}{\beta^2}$ as $\frac{si^2Nr}{sin^2r} \to 1$
$\beta = \frac{\pi bsin\theta}{\lambda}(d/b = 4, dsin\theta = n\lambda)$
$= \frac{\pi}{\lambda} \times \frac{d}{4}sin\theta = n\lambda \times \frac{\pi d}{4\lambda} = \frac{n\pi}{4}$
The 1st row is off the 1st lobe and 2nd row is of the 2nd lobe.

2.b) $r_n = \sqrt{nb\lambda}$
$n = \frac{rn^2}{b\lambda} \quad = n_1 = \frac{r_1^2}{b\lambda} = \frac{1}{2}$
$\therefore n_2 = 2n_1 = 1 \qquad n_3 = 3n_1 = 3/2$
[Read the concept of vibration spiral]
$\therefore$ 1/2 fresnel zone is covered next 1/2 is half open and 3rd 1/2 is full open.

$\therefore I_{net} = (\frac{\sqrt{2}A_0}{2})^2 + (\sqrt{2}A_0)^2 = \frac{5}{2}I_0$
The 2nd 1/2 is halved as the zone is half open. The 3rd 1/2 is full is at full open .
2.c) For maximum $tan\beta = \beta$
A maxima will occur near $\beta = (2m+1)\pi/2, m = 1, 2, 3$ as maxima occurs at .
Let first maxima occur at $\beta = \frac{3\pi}{2} + \delta_1, \delta_1$ is small
$\therefore tan(3\pi/2 + \delta_1) = \frac{3\pi}{2} + \delta_1$
$\to -cot\delta_1 = \frac{3\pi}{2} + \delta_1$
$\to -\frac{1}{tan\delta_1} = \frac{3\pi}{2} + \delta_1$
$\to \frac{1}{\delta_1} + \delta_1 = -\frac{3\pi}{2}$
Solving we get ,$\delta_1 = -0.07\pi$
$\therefore \beta = 1.430\pi$





Similarly for the 2nd one , $\beta = \frac{5\pi}{2} +_2, \delta_2$ is small
$\therefore \beta = 2.459\pi$ and $\delta_2 \approx -0.041\pi$
$\therefore \frac{I_1}{I_2} \approx \frac{0.047}{0.0164} \approx 3$

Q 3 a) A prism ABCD is made of Iceland spar ($n_o > n_e$) as shown in Fig 3. The optic axis on the left half (prism ABC) is parallel to the plane of the paper and parallel to AB. In the right half (prism ACD) the optic axis is also parallel to plane of the paper but perpendicular to the optic axis of the left prism. The side DC is parallel to the side AB. An unpolarised ray of light is incident normally on the surface AB. Complete the ray diagram showing appropriate wave-surfaces inside the crystal with proper polarisations of rays.(**No** graph paper required.) [5]

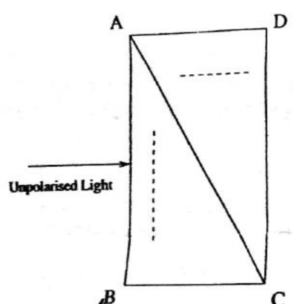


Fig 3: Problem 3 a)

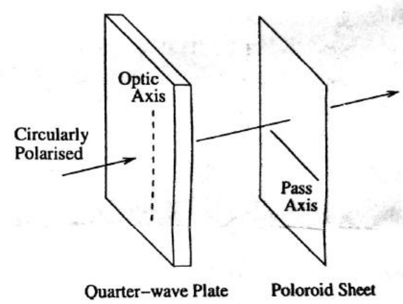
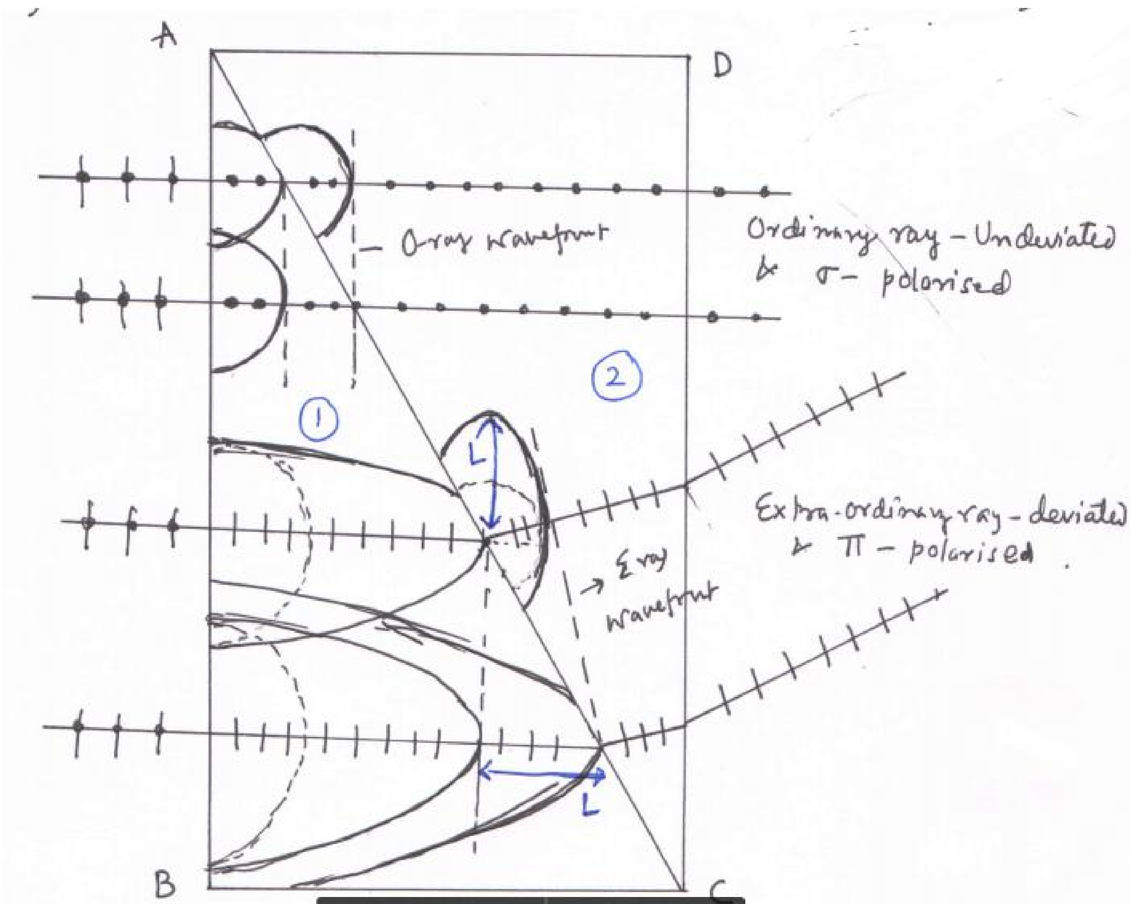


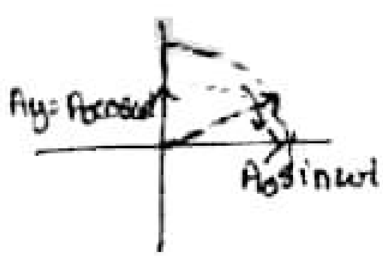
Fig 4: Problem 3 b)

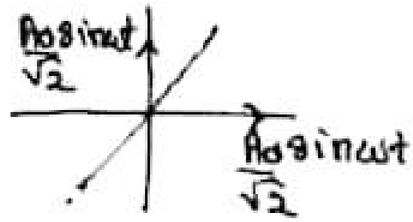
Q 3 b) A monochromatic light with circular polarisation falls normally on a quarter wave plate cut parallel to the optic axis as shown in Fig 4. Behind the plate there is a polaroid sheet whose pass-axis is perpendicular to the optic axis of the plate as shown in the same figure. Calculate the intensity of light transmitted by the polaroid sheet. [4]

3.a) $n_0 > n_e$ [-ve crystal]
(please read the theory of E - ray and O - ray in +ve or -ve crystals)
$\therefore \vartheta_0$ (ordinay ray) ¡ ϑ_e (extraordinay ray)
Also after the 2nd surface , the E-ray becomes the O-ray.



3.b) Let the light incident has intensity I_0
$A_x = A_0 sin\omega t$ or $A_x = A_0 cos\omega t$
$\therefore$ After the light passing through the quarter wavw plate , a phase difference of $\pi/2$ is introduced
$\therefore A_y = -A_0 sin\omega t$ or $A_x = A_0 cos\omega t$ any one of them occurs .
$\therefore$ The light becomes linearly or plane polarised at $\pi/4$ or $3\pi/4$





circularly polarised
$\therefore A_{net} = A_0$ always
$\therefore$ Intensity $= I_0$
Then after passing through polaroid, it becomes $I_0 cos^2\frac{\pi}{4}$ or $I_0 cos^2\frac{3\pi}{4} = I_0/2$
In first case, the optics axis itself acts as a polaroid ,As there is unpolarised light, the light entering as O-ray has its intensity halved while getting polarised and similarly the case of E- ray .
Thus $A_0 \rightarrow A_0/2$

Q 4 a) Find the kinetic energy of an electron, for which its Compton and de Broglie wavelengths are the same. Take rest mass energy of the electron as 0.51 MeV. [4]

Q 4 b) Estimate approximately the least energy of an electron confined in a box of 0.62 nm length by uncertainty principle, $\Delta x \Delta p \sim \hbar$. [4]

Q 4 c) Check whether the operator $-\frac{d}{dx}$ is Hermitian or not. [2]

Q 4 d) A system has three normalised energy eigenstates ψ_1, ψ_2 and ψ_3 with eigenvalues 1,4 and 9 units, respectively. The system is found to be in the following linear combination of the above eigenstates, $\psi = \frac{1}{2}\psi_1 - \frac{i}{\sqrt{2}}\psi_2 - \frac{1}{2}\psi_3$.

(i) What is the expectation value of the energy for this state? [3]

(ii) What is the probability of obtaining energy equal to 4 units when an energy measurement is made on this system? [2]

Q 4 e) Calculate the commutator $[xy, L_z]$. ($[x, p_x] = [y, p_y] = i\hbar$; $[x, y] = [x, p_y] = [p_x, p_y] = 0$, etc.., where p_x and p_y are the momenta along x and y directions respectively and L_z is the component of angular momentum along z direction.) [3]

....

4.a) $\lambda_c = \frac{h}{m_0 c}[Crompton]$ $\quad \lambda = \frac{h}{p}[Debroglie]$

equating we get $p = m_0 c$

also $p = m\vartheta = \frac{m_0\vartheta}{\sqrt{1-(\vartheta^2/c^2)}}$

$\therefore KE = \sqrt{p^2c^2 + m_0^2c^4} - m_0c^2$

$= \sqrt{m_0^2c^4 + m_0^2c^4} - m_0c^2 = (\sqrt{2}-1)m_0c^2 = 0.21MeV$

4.d) (i) $\langle E\rangle = \frac{|c_1|^2E_1+|c_2|^2E_2+|c_3|^2E_3}{|c_1|^2+|c_2|^2+|c_3|^2}$ Where $c_1 = \frac{1}{2}, c_2 = \frac{i}{\sqrt{2}}, c_3 = -\frac{1}{2}$

$= 4.5$ $\quad E_1 = 1, E_2 = 4, E_3 = 9$

$(ii) P_2 = \frac{|c_1|^2}{|c_1|^2+|c_2|^2+|c_3|^2} = \frac{1/2}{1/4+1/2+1/4} = 1/2$

4.b) $(\Delta\gamma)_{max} \sim x$ $\qquad (\Delta P)_{min} \sim \frac{\hbar}{x} = 1.7\times10^{-25}kgm/s$

$P_{min} \sim (\Delta P)_{min} \rightarrow E_{min} = \frac{P_{min}^2}{2me} = 0.16\times10^{-19} \approx 0.1eV$

4.c) We know P is hamiltonian operator

$\therefore P = P^T$

$\rightarrow i\hbar\frac{d}{dx} = (i\hbar\frac{d}{dx})^T = i\hbar(\frac{d}{dx})^T$

$\rightarrow (-\frac{d}{dx}) = (\frac{d}{dx})^T$

$\rightarrow (-\frac{d}{dx})^T = (\frac{d}{dx}) \neq -(\frac{d}{dx})$

$\therefore$ It is not an Homiltonian operator

4.e)$[xy, L_z]$

$= x[y, L_z] + [x, L_z]y$

$= x[y, xP_y - yP_x] + [x, xP_y - yP_x]y$

$= x[y, xP_y] - x[y, yP_x] + [x, xP_y]y - [x, yP_x]y$

Now $[y, yP_x] = [y, y]P_x + y[y, P_x] = 0$

Similarly $[x, xP_y] = 0$

$\therefore [x_y, L_z] = x[y, xP_y] - [x, yP_x]y$

$= x^2[y, P_y] + x[y, x]P_y - y[x, P_x]y - [x, y]yP_x$

$= x^2[y, P_y] + 0 - y^2[x, P_x] - 0$
$= (x^2 - y^2)i\hbar$
Use [A,BC] = [A ,B]C + B[A,C]
and the results given also.
Also $[x, P_y] = [y, P_x] = 0$

PHYSICS

END - SPRING SEMESTER EXAMINATION**2016-2017**

Q1. (a) Find the state of polarization of a light, which is moving in the positive x direction and having amplitudes of electric field in the y and z directions as $\sqrt{2}$ and $\sqrt{3}$, respectively in same units. The oscillating components of the electric field along y and z have the same frequency and wavelength and the z component is leading with a phase $\pi/3$.

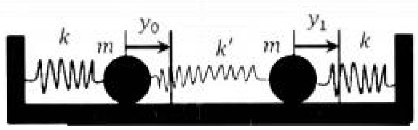
(b) In a Compton scattering with an electron of the incident of violet light photon (λ = 4000 Å)

(i) How much energy (in eV) is transferred to the electron, when difference in wavelength (Compton shift) is maximum?

(ii) Could this violet light eject electrons from a metal (having work function = 3.09 eV) by Compton collision?

(iii) In considering the Compton effect, how would you compare the scattering of photons from bound electrons and free electrons?

(c) A coupled oscillator has k = 9 Nm^{-1}, k' = 0.1 Nm^{-1} and m = 1 kg. Initially both particles have zero velocity with y_0= 5 cm and y_1= 0. After how many oscillations in y_0 does it completely die down?



1.a) Let y $=\sqrt{2}$ sin(Wt) and t $=\sqrt{3}$ sin(Wt + $\pi/3$)
By theory it is ellipse $\Rightarrow$ Polarization is elliptical

$\Rightarrow \frac{y^2}{2} + \frac{z^2}{3} - \frac{2xy}{\sqrt{6}} cos(\pi/3) = sin^2(\pi/3)$

(b). (1)$\Delta\lambda = \frac{h}{m_0 c}(1 - cos\phi)$
$(\Delta\lambda)_{max} = \frac{2h}{m_0 c} = 4.8pm \Rightarrow$ Energy transferred to the electron $= \frac{hc}{\lambda}$ - $\frac{hc}{\lambda+\Delta\lambda}$ $= \frac{hc}{\lambda(\lambda+\Delta\lambda)}(\Delta\lambda)$ (Approximately)
$= \frac{hc(\Delta\lambda)}{\lambda^2}$
$\Rightarrow \Delta E = \frac{6\times10^{-34}\times3\times10^{8}\times4.8\times10^{-12}}{4000\times4000\times10^{-20}} = 3.1725\times10^{-5}eV$

(2)Work function = 3.09eV

$\Rightarrow$ It cannot eject electrons from metal by Compton Collision .

(3)Please consult your professor

(c)$w_1 = \sqrt{\frac{k}{m}} = \frac{\sqrt{9}}{1} = 3$
$w_2 = \sqrt{\frac{9+0.2}{1}}$ Approximately = 3.32
$\Rightarrow q_0 = A_1 cosw_1 t$ and $q_2 = A_2 cosw_2$t
Where $q_0 = \frac{x_1+x_2}{2}$ and $q_1 = \frac{x_1-x_2}{2}$ [Please refer other papers for proper explanation]

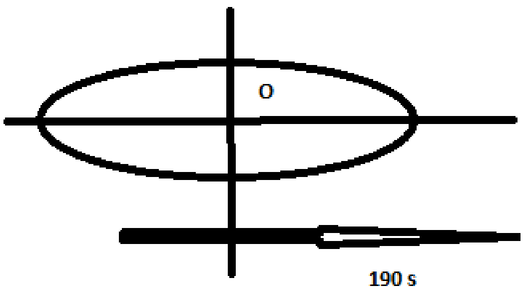
$x_1 = \frac{q_1+q_0}{2} = A_1 cosw_1 t + A_2 cosw_2 t$

$x_1 = \frac{q_0-q_1}{2} = A_1 cosw_1 t - A_2 cosw_2 t$
$\Rightarrow x_1(0) = 5 \quad \Rightarrow A_1 + A_2 = 5$

and $x_2(0) = 0 \quad and A_1 - A_2 = 0$

$\Rightarrow A_2 = A_1 = 2.5$
$\Rightarrow x_1 = 2.5(cosw_1t + cosw_2t)$
$x_2 = 2.5(cosw_1t - cosw_2t)$

$T_{beat} = \frac{2\pi}{w_1 - w_2}$ Approximate $= 190$s
This is the full span between which the particle oscillates [plot the function to have better visualization] $\Rightarrow$



Required time period= 95s
Average time period $= \frac{2\pi}{\frac{w_1+w_2}{2}} = 2.08s$
$\Rightarrow$ Approx 45 to 46 oscillations .

Q2. In a double slit experiment, the distance between two slits d is 0.5mm and the wavelength (λ) of the light is 600 nm.

a. If it is desired to have a fringe spacing of 1 mm on the screen, what is the proper screen distance from the slits? **(2 points)**

b. If a thin plate of glass (refractive index n=1.5) of thickness 100 µm is placed over one of the slits, what is the lateral fringe displacement (in terms of number of fringes) on the screen? **(3 points)**

c. What path difference corresponds to a shift in the double slit fringe pattern from a peak maximum to the (same) peak half-maximum? **(2 points)**

d. When one of the slits is blocked, you observe a single-slit Fraunhofer diffraction pattern using a light with mixture of wavelengths. For what wavelength does the second minimum coincide with the third minimum produced by a wavelength of λ = 600 nm. **(3 points)**

2.a) $\beta = \frac{\lambda D}{d}$
$\Rightarrow 10^{-3} = \frac{600\times10^{-9}\times D}{0.5\times10^{-3}}$
or D $= 0.5/0.6 = 0.833$m

b) $(\mu - 1)t = \frac{dy}{D}$
or y $= \frac{(\mu-1)tD}{d} = \frac{0.5\times100\times10^{-6}}{0.5\times10^{-3}} \times \frac{10^{-3}\times0.5\times10^{-3}}{600\times10^{-9}} = 0.5/6 = 0.083m$

c) Intensity $\alpha cos^2(\theta/2)$ Where ϕ is the phase difference.
At $\phi = 0$, maxima and at $\phi/2 = \frac{\pi}{4}$, it is half intensity .
$\Rightarrow \phi = \pi/2$
$\Rightarrow \phi = \frac{2\pi d sin\theta}{\lambda} = \pi/2$
or $sin\theta = \frac{\lambda}{4d} = \frac{600\times10^{-9}}{4\times0.5\times10^{-3}}$

$\frac{y}{D} = sin\theta = \theta = \frac{0.6}{4\times0.5} \times 10^{-3}$
or $y = 0.3 \times 10^{-3} \times 0.833 = 0.25mm$

d) For single slit maxima ,$\beta = m\pi, m\epsilon N$
Where $\beta = \frac{\pi b sin\theta}{\lambda} \Rightarrow m\pi = \frac{\pi b sin\theta}{\lambda}$ [Refer notes for better explanation]
or $m\pi = bsin\theta \Rightarrow 2\lambda_1 = 3\lambda$
or $\lambda_1 = 1.5 \times \lambda = 1.5 \times 600nm = 900nm$

Q3. (a) A dipole of length 2 m is oriented along y-axis and located at (0,0,0). The dipole is fed with a current $I(t) = I_0 \cos(\omega t)$ with I_0 =5 Amps and ω=6 x 10^9 rad/s. Find the magnitude and direction of flux density at (2,3,0). All coordinates are in km and ε_0 = 8.85 x 10^{-12} C^2 N^{-1} m^{-2} **(2 points)**

(b) Linearly polarized electromagnetic radiation of frequency (linear) 500 THz and flux density 100 Watt-m^2 is incident on a perfect mirror at an angle 30° to the mirror plane. Find the components of wave vector and electric field parallel and perpendicular to the mirror plane for the incident and reflected waves. **(2 points)**

(c) For an imperfect mirror a part of the wave is transmitted as well. For linearly polarized radiation, it is found that the intensity of the reflected wave falls to zero if the reflected wave is perpendicular to the transmitted wave. Given the refractive index of glass 1.5 and air 1.0, find the angle of incidence. **(2 points)**

(d) A metal rod of length 75 m is attached to the ground and connected to an AC oscillator characterized by $I(t) = I_0 \cos(\omega t)$ with $I_0 = 2$ Amps and ω=2π x 10^6 rad/s. Find the radiation resistance, R, defined through $\langle P \rangle = \frac{1}{2} I_0^2 R$, where $\langle P \rangle$ is the average power radiated. [Hint: Earth surface is conducting]. **(4 points)**

Probably not in syllabus.

Q4. (a) Consider the wave function $\Psi(x) = Ae^{-ax^2}$ defined between $x = \pm\infty$. Find the uncertainty in position Δx, and uncertainty in momentum Δp. **(2+2 Points)**

(b) Suppose a wave function is given by $\Psi(x) = \frac{1}{2}\psi_1(x) + \frac{1}{3}\psi_2(x)$, such that, $\hat{O}\psi_1(x) = o_1\psi_1(x)$ and $\hat{O}\psi_2(x) = o_2\psi_2(x)$, where $\hat{O}$ is a hermitian operator with only two eigenvalues o_1 and o_2. On measuring $\hat{O}$ on the wave function $\Psi(x)$ find the probabilities for finding the values o_1 and o_2, respectively. **(1+1 Points)**

(c) A particle of mass m is confined to the one-dimensional region $0 \ll x \ll L$ by two infinite potential walls (particle in a one-dimensional infinite potential box). The wave function for the particle at time $t = 0$ is given by $\Psi(x, t = 0) = \frac{3}{13}\Psi_1(x) + \frac{4}{13}\Psi_2(x) + \frac{12}{13}\Psi_3(x)$, where $\Psi_1(x), \Psi_2(x), \Psi_3(x)$ are the wave functions for the ground state, the first excited state, and the second excited state of the potential well, respectively. Find the average energy $\langle E \rangle$ of the particle at time $t = 0$. Find the wave function of the particle $\Psi(x,t)$ at a later time t. Find the average energy of the particle at the later time t. **(2+1+1 Points)**

ANSWER:
$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$
Now $\langle x \rangle = A^2 \int_{-\infty}^{\infty} xe^{-2ax^2}dx$
$\Rightarrow$ It is an odd function , it is zero
$\langle x \rangle = A^2 \int_{-\infty}^{\infty} xe^{-2ax^2}dx = I(let)$ [$\langle z \rangle = \int \Psi^* \hat{z}\Psi dx$ Where$\hat{z}$ is the operator of z
Let $x^2 = z \Rightarrow dt = 2xdx$
$I = A^2 \int_{-\infty}^{\infty} ze^{-2az}\frac{dz}{2\sqrt{z}}$
$= A^2 \int_{-\infty}^{\infty} \sqrt{z}e^{-2az}dz = \frac{A^2}{2}r(1/2) = \frac{A^2\sqrt{\pi}}{2}$

$\Rightarrow \Delta x = \sqrt{\langle x^2 \rangle} = \frac{A\pi^{1/4}}{\sqrt{2}}$
For P , $P = -i\hbar\frac{d\Psi}{dx} = -i\hbar(-2ax)Ae^{-ax^2} = A2ai\hbar xe^{-ax^2}$
$\Rightarrow$ Actually for gaussian wave function , the uncertainty is always $\hbar/2$ [minimum uncertainty]
$\Rightarrow \Delta x\Delta p = \hbar/2$
or $A/\sqrt{2}\pi^{1/4}(\Delta p) = \hbar/2$
$\Rightarrow (\Delta p) = \frac{\hbar}{A(\pi)^{1/4}\sqrt{2}}$

(b) $\langle O \rangle_{o_1} = \frac{(1/2)^2}{(1/2)^2+(1/3)^2} = 0.693$
$\langle O \rangle_{o_2} = \frac{(1/3)^2}{(1/2)^2+(1/3)^2} = 0.308$

(c) For 1 dimensional infinite potential well ,$E_n = (n+1)^2 E_0$ where $E_0 = \frac{h^2}{8mL^2}, E_n = \frac{(n+1)^2h^2}{8mL^2} \Rightarrow \langle E \rangle = \frac{c_1^2E_0+c_2^2E_1+c_3^2E_2}{c_1^2+c_2^2+c_3^2}$ Where $c_1 = 3/13, c_2 = 4/13, c_3 = 12/13$

See Ψ_1 is the wave fuction of the ground state of the particle .
$\langle E \rangle = \frac{(3^2+4^2\times 2^2+12^2\times 3^2)}{13^2}E_0 = 8.1E_0 = \frac{8.1\hbar^2}{8mL^2} = \frac{1.0125h^2}{mL^2}$

$\Psi(x,t) = \frac{3}{13}\Psi_1(x)e^{\frac{-iE_1t}{\hbar}} + \frac{4}{13}\Psi_2(x)e^{\frac{-iE_2t}{\hbar}} + \frac{12}{13}\Psi_3(x)e^{\frac{-iE_3t}{\hbar}}$
We know that all ϕ are orthogonal
$\Rightarrow \int_{-\infty}^{\infty}\phi_m(x)\phi_n(x)dx = 0$ for m $\neq$ n
Ignoring the time independent part $[e^{\frac{-iEt}{\hbar}}$
When we do $\Psi_1(x,t)^*\Psi_2(x,t)$ we get terms like $\Psi_1(x)^*\Psi_2(x)e^{-i(E_1-E_2)/\hbar}$ But averaging over space performing the integral , we get zero .So the average energy is time - independent.

Q5. (a) To listen to a radio-station, a certain home receiver must pickup a signal of at least 1.0×10^{-10} W. If the radio waves have a frequency of 96 MHZ, how many waves must the receiver absorb per second to get the station? **(2 points)**

(b) How much force is exerted on the receiving antenna for the case considered in part (a)? **(2 points)**

(c) How many 520 nm photons would have to be absorbed to raise the temperature of 1 g of water by 10^0C? (Specific heat of water is given as 4.2 J/g-K). **(2 points)**

(d) A 65 Kg student passes through a doorway of width 0.76 m. With what speed the student should move so that waves associated with him can give rise to a diffraction effects? **(2 points)**

(e) A Hydrogen atom absorbs a 486.2 nm photon at time t_0. A short time later at time t_1, the same atom emits a photon with wavelength of 97.23 nm. Find the change in energy of the Hydrogen atom. **(2 points)**

ANSWER: 5. a) E = hv
$\Rightarrow n = \frac{P}{E} = \frac{P}{hv} = \frac{10^{-10}}{6.6\times 10^{-34}\times 96\times 10^6} = 1.5783 \times 10^{15}$

b)$\lambda = \frac{h}{P}$
$\Rightarrow P = \frac{hv}{c} = \frac{E}{c}$

Or $\frac{dP}{dt} = (1/c)\frac{dE}{dt}$
or force $=(1/c) \times Power = \frac{10^{-10}}{3\times 10^8} = 3.3 \times 10^{-17}N$

c) Q = $1 \times 4.2 \times 10$

$E = hv = hc/\lambda$

$\Rightarrow h = \frac{4.2\times10\times520\times10^{-9}}{6.6\times10^{-34}\times3\times10^{8}} = 1.1 \times 10^{20}$

d) For diffraction , λ should be in the order of the Slit width , lets take it equal.
$\Rightarrow \lambda = \frac{h}{P} = \frac{h}{mv}, \lambda = 0.76$
$\Rightarrow v = \frac{h}{m\lambda} = \frac{6.6\times10^{-34}}{65\times0.76} = 1.34 \times 10^{-35}$

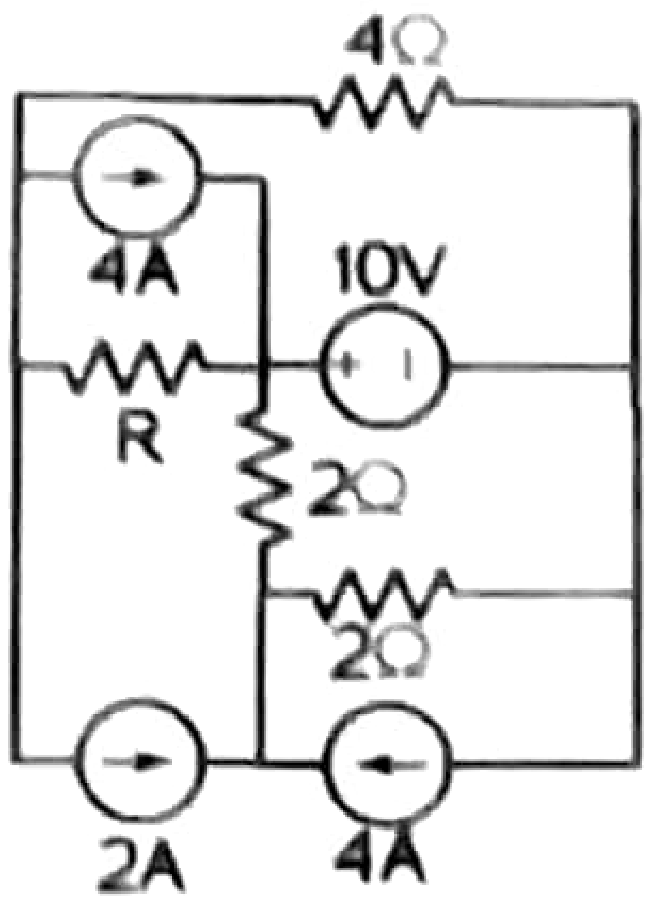
e) $\Delta E = h(\Delta v) = hc(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}) = \frac{6.6\times10^{-34}\times3\times10^{8}}{10^{-9}}(\frac{1}{97.23} - \frac{1}{486.2}) = 10eV$

$\Rightarrow$ The energy of the hydrogen atom decreased by 10eV

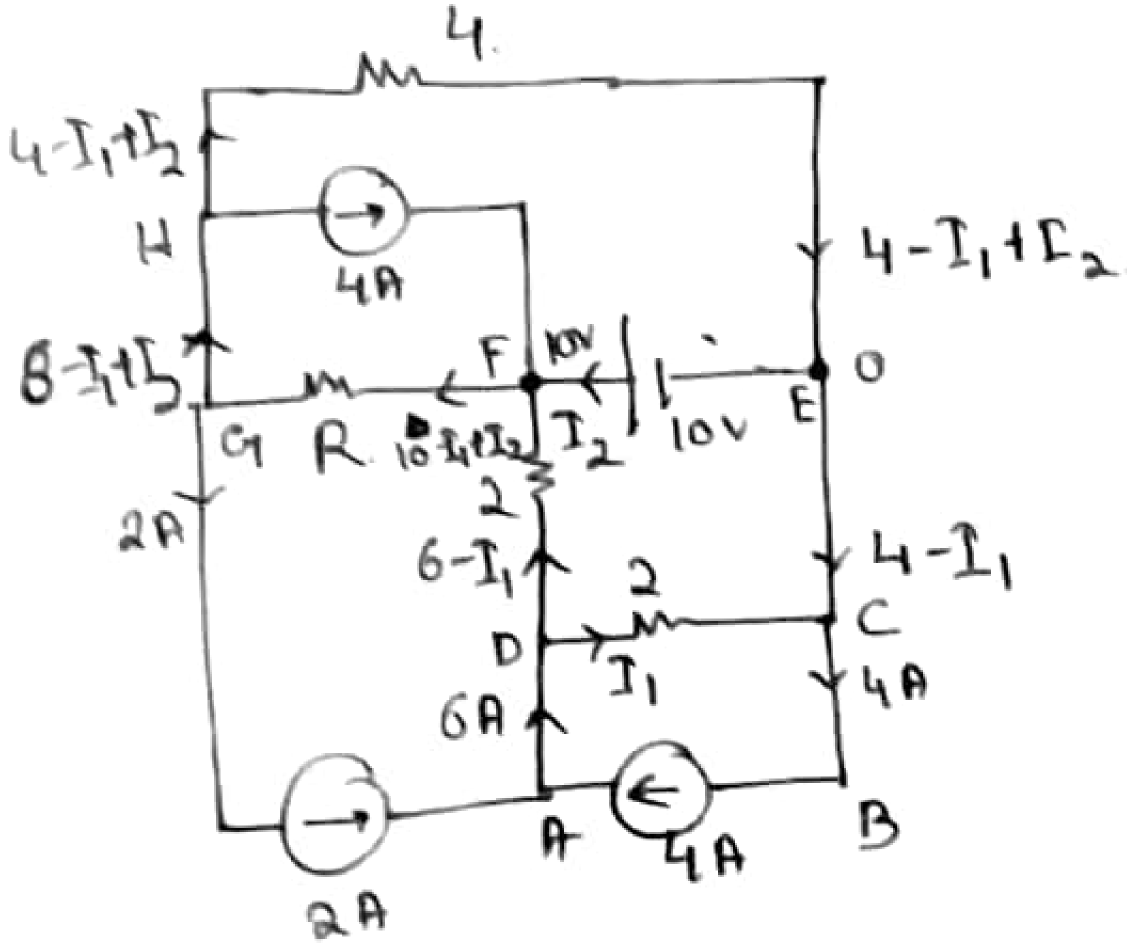
ELECTRICAL TECHNOLOGY

MID - AUTUMN SEMESTER EXAMINATION**2017-2018**

Q.1. a) **Find the value of the resistance R as shown in the figure below such that minimum power is transferred to the 4Ω resistor and also calculate the amount of power dissipated in the 4 Ω resistor under this power transfer condition.**



ANSWER:



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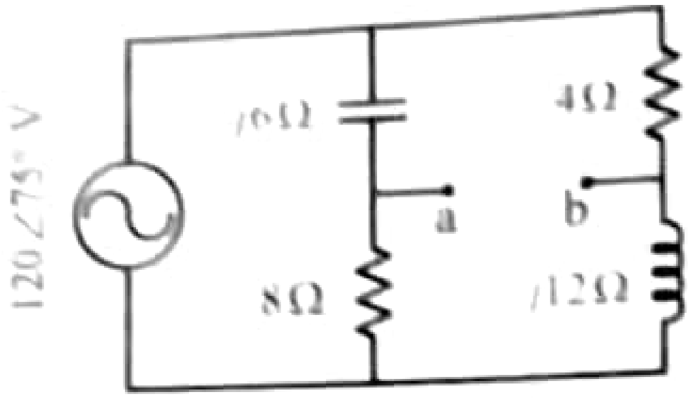
Taking currents I_1 in I_{DC} and I_2 in I_{EF} and passing other currents in other branches using KVL .
Using KVL in loop EFDC
$+10 + 2(6 - I_1) - 2I_1 = 0$
$\Rightarrow 10 + 12 = 4I_1$
$\Rightarrow I_1 = 5.5A$
$\therefore I(4\Omega) = 4 - I_1 + I_2$
$= \frac{10-6R}{R+4}$
Minimum power is 0 watt
$\therefore I = 0$ for R= 5/3
$\therefore I^2R = 0watt$
But I =0 For R = 5/3 . Thus minimum Power is 0.
Using KVL loop in FGE (4Ω)
$+\, 10 - R(10 - I_1 + I_2) - 4(4 - I_1 + I_2) = 0$
$\Rightarrow 10 - 10R - R(I_2 - I_1) - 16 - 4(I_2 - I_1) = 0$
$\Rightarrow -6 - 10R = (I_2 - 5.5)(R + 4)$
$\Rightarrow I_2 = 5.5 - \frac{6+10R}{R+4}$

Q. 1.b)Calculate the following AC network . Obtain the thevenin voltage and Thevenin impedence at the terminals a-b of the network.



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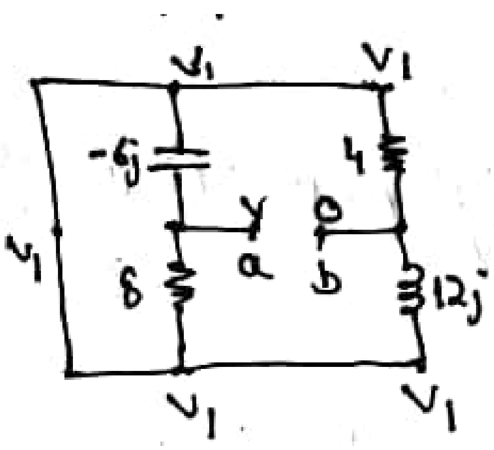
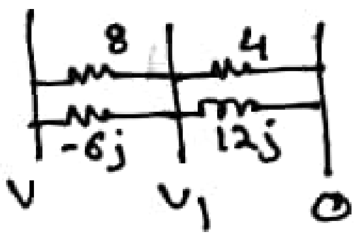
ANSWER:

1.b)For R_{Th}



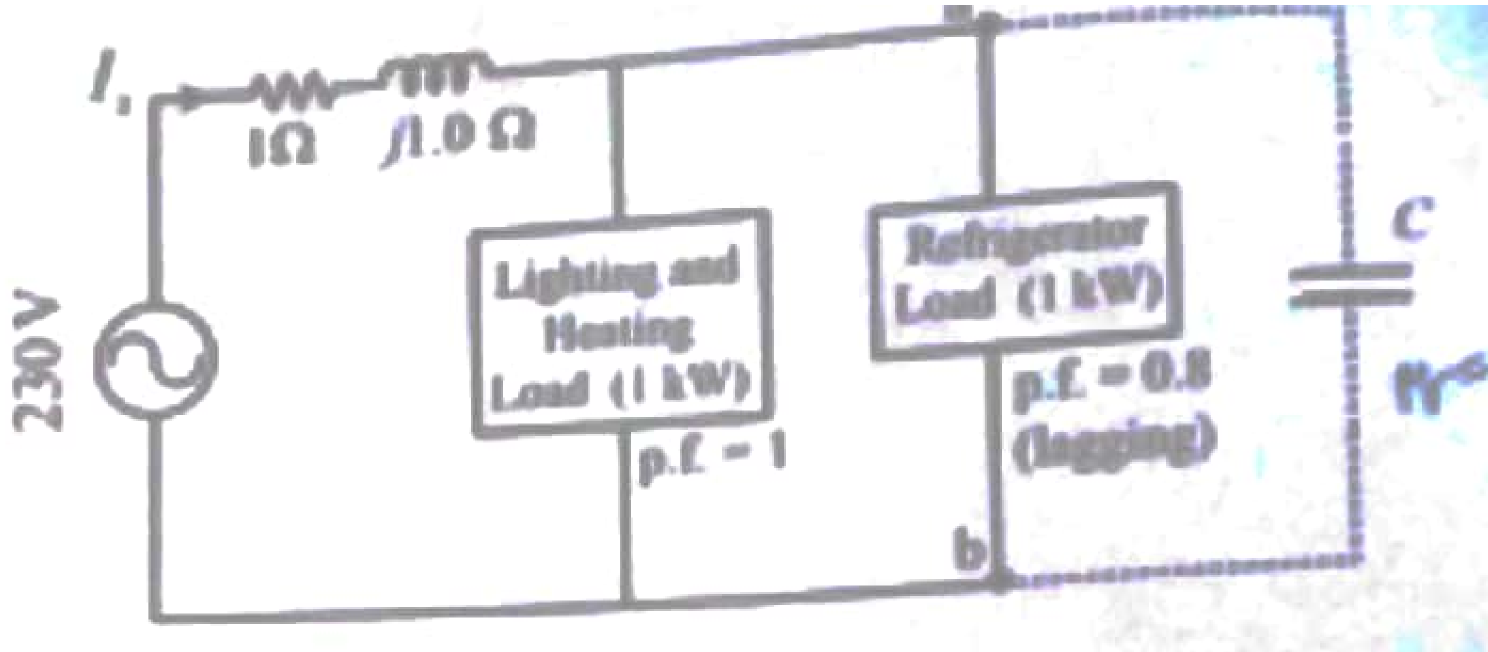


$Z_{net} = \frac{8(-6j)}{8-6j} + \frac{4(12j)}{4+12j} = 6.48 - 2.64j \approx 7\angle -22.17°$
$\therefore Z_{Th} = 6.48 - 2.64j\Omega$
For V_{Th}
$V_a = \frac{120\angle 75° \times 8}{10\angle -36.87°}$
$V_a = 96\angle 111.87° V$

$V_b = \frac{120\angle 75° \times 12\angle 90°}{12.65\angle 71.57°}$
$V_b = 113.834\angle 99.43° V$

$V_{Th} = V_b - V_a = 28.83\angle 53.6° V$

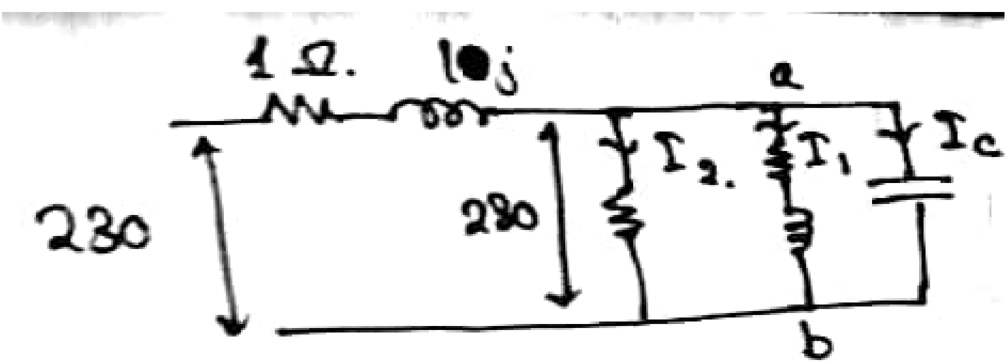
3.Q. Consider a housing load connected to a single - phase 50 Hz supply as follows .
(i) In the absence of the capacitor , the voltage across the terminals a-b is 200 V . Calculate the source current I_s and the overall power factor as seen by the source.
(ii) Calculate the value of the capacitance to be connected at the terminals a-b as shown . So that the magnitude of the source current I_s in the case?



ANSWER:

3. For Refrigerator .
Power = 1000 + 750j
For Heating and lightning
Power = 1000

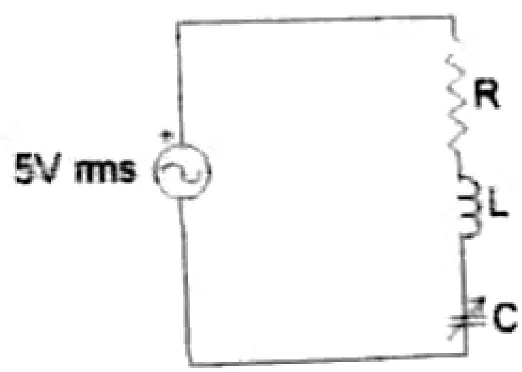
For Refrigerator $I = \frac{P}{V cos\phi} = \frac{1000}{200 \times 0.8} = 6.25\angle - 36.87°$
For Heater $I = \frac{P}{V cos\phi} = \frac{1000}{200} = 5\angle 0°$
$\therefore I_{source} = 10.68\angle - 20.56°$
Overall power factor = cos (20.56) = 0.936
$Z_{refrigerator} = 32\angle 36.87°$
$Z_{heater} = 40\angle 0°$



Let $V_{ab} = 230\angle 0_{\circ}$
and $V_{source} = 230\angle\delta$
$I_1 = \frac{P}{V cos\phi} = \frac{1000}{230 \times 0.8} = 5.435\angle - 36.87°$
$I_2 = \frac{P}{V cos\phi} = \frac{1000}{230 \times 1} = 4.348\angle 0°$
$\therefore I_1 + I_2 = 9.287\angle - 20.56°$
$\therefore I_1 + I_2 + I_c\angle 90°$

$\therefore 230\angle\delta = 230\angle 0^\circ + (9.287\angle -20.56^\circ + I_c\angle 90^\circ)(1.414\angle 45^\circ)$
$= 230\angle 0^\circ + 13.134\angle 24.44^\circ + \sqrt{2}I_c\angle 135^\circ$
$= (241.948I_c) + (5.43 + I_c)j$
$\therefore (230)^2 = (241.948I_c)^2 + (5.43 + I_c)^2$
$52900 = 58538.83 - 483.896I_c + I_c^2 + 29.48 + I_c^2 + 10.86I_c$
$0 = 5668.31 - 473.036I_c + 2I_c^2$
$I_c = 12.661, 223.86$
We know $V\omega C = I_c$
$\therefore (230)(2\pi(50))C = 12.661, 223.86$
$C = 175.2\mu F, 3.098mF$

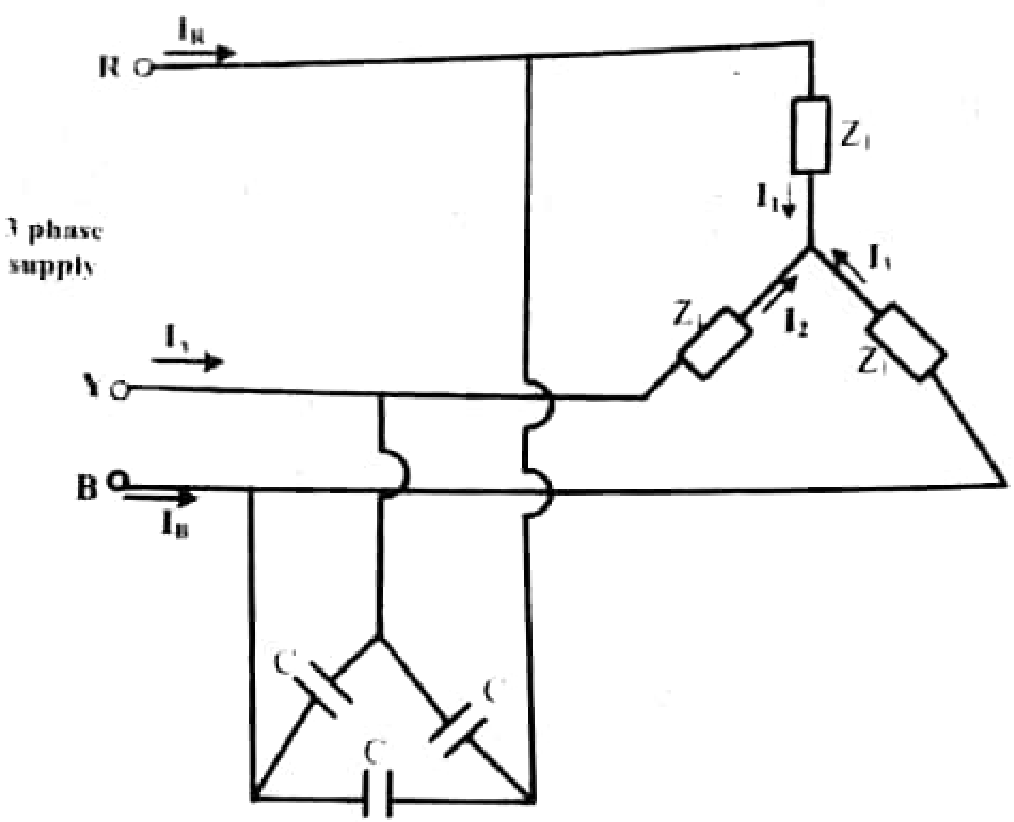
Q.4. A fixed and finite frequency sinusoidal voltage source of 5 V rms value is connected to a series RLC branch. This branch has fixed valued R and L and variable valued C as shown in the following figure . When C = $\frac{1}{5}$ farad the rms value of current flowing through the series RLC branch is 1A and the active power supplied by the voltage is 3W. When the value of capacitor is changed to $\frac{1}{45}$ farad the rms value of current flowing through the RLC branch is again 1A.
(i) Calculate the values of R and L.[2 + 6]
(ii)For what value of C, maximum power will be transferred to the RLC branch?[4]



ANSWER:
$V_{rms} = 5V$
Case I:
$C = 1/5F \qquad I_{rms} = 1A$
$\therefore$ Power $I_{rms}^2 R[I_{rms}^2|Z|cos\phi = I_{rms}^2 R]$
$3W = 1 \times R$
$R = 3\Omega$
since in both cases magnitude of current is same , thus impedence magnitude is same
as R is fixed
$\therefore |\omega L - \frac{1}{\omega C_1}| = |\omega L - \frac{1}{\omega C_2}|$
1 solution is $C_1 = C_2$
$\omega L - \frac{1}{\omega C_1} = \frac{1}{\omega C_2} - \omega L$
$\Rightarrow 2\omega L = \frac{1}{\omega}(\frac{1}{C_1} + \frac{1}{C_2})$
$\Rightarrow 2 \times \omega \times L = \frac{1}{\omega}(50) - - - - - - - - - - - - - - - - - -(1)$
$I = 1A$
$|V| = |I||Z| \qquad$ as $R = 1\Omega$
For capacitance $V_R = 3V$ in both cases
$\therefore$ Voltage across land C = 4V in both cases
$4 = 1 \times |\omega L - \frac{1}{\omega C_1}|$

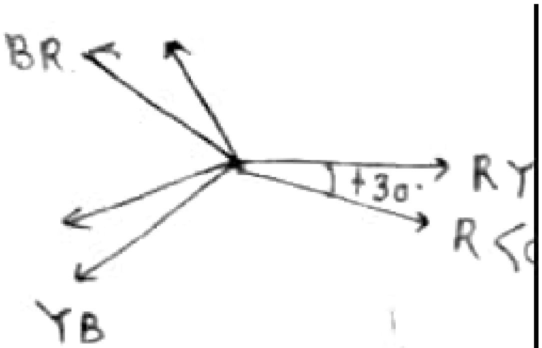
$C_1 > C_2$
$\therefore X_1 < X_2$ [X is capacitive reactance]
$\therefore$ Circuit is inductive in 1st case and opp in 2nd case.
$4 = 1 \times |\omega L - \frac{1}{\omega C_1}|$
$\Rightarrow 4 + \frac{5}{\omega} = \omega L = \frac{25}{\omega}$ [from (i)]
$4 = \frac{20}{\omega}$
$\Rightarrow \omega = 5Hz$
$\therefore 4 + \frac{5}{5} = 5L$
$\Rightarrow L = 1$
Maximum power transfer means resonance condition
$\therefore \omega L = \frac{1}{\omega C}$
$\Rightarrow C = \frac{1}{\omega^2 L} = \frac{1}{25 \times 1} = 0.04F$

Q.5. Each phase impedence of the star connected three phase load of resistance of 20 ohm and inductance of 0.2H . Three capacitors each of C = $90\mu F$, are connected in delta as shown in the figure .The arrangement is connected to a three phase 400 V , 50 Hz balanced voltage source . The pahse sequence RYB . Considering phase - R voltage as reference , obtain phasors (i) I_1, I_2 and I_3 and (ii) I_R, I_Y and I_B. Draw a phasor diagram showinh the line voltages and line currents at the source end

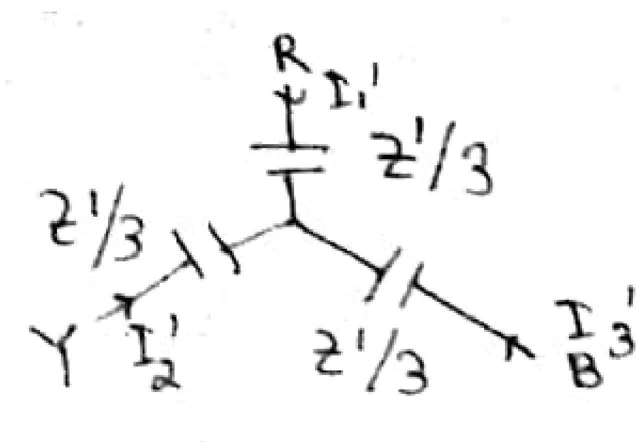


Answer: $Z_m = 20 + j(100\pi times 0.2)$ $\quad C = 90\mu F$ $\quad V_L = 400V$
$V_{RY} = 400\angle + 30°$ $\quad Z_{ph} = 20 + j(62.832) = 65.94\angle 72.34°$
$V_{YB} = 400\angle - 90°$
$V_{BR} = 400\angle - 210^{circ}$
$V_R =$ (Voltage across Z_1 in R_n branch)
$= \frac{400}{\sqrt{3}}\angle 0°$

Also $V_Y = \frac{400}{\sqrt{3}}\angle -120°$
$V_B = \frac{400}{\sqrt{3}}\angle -240°$

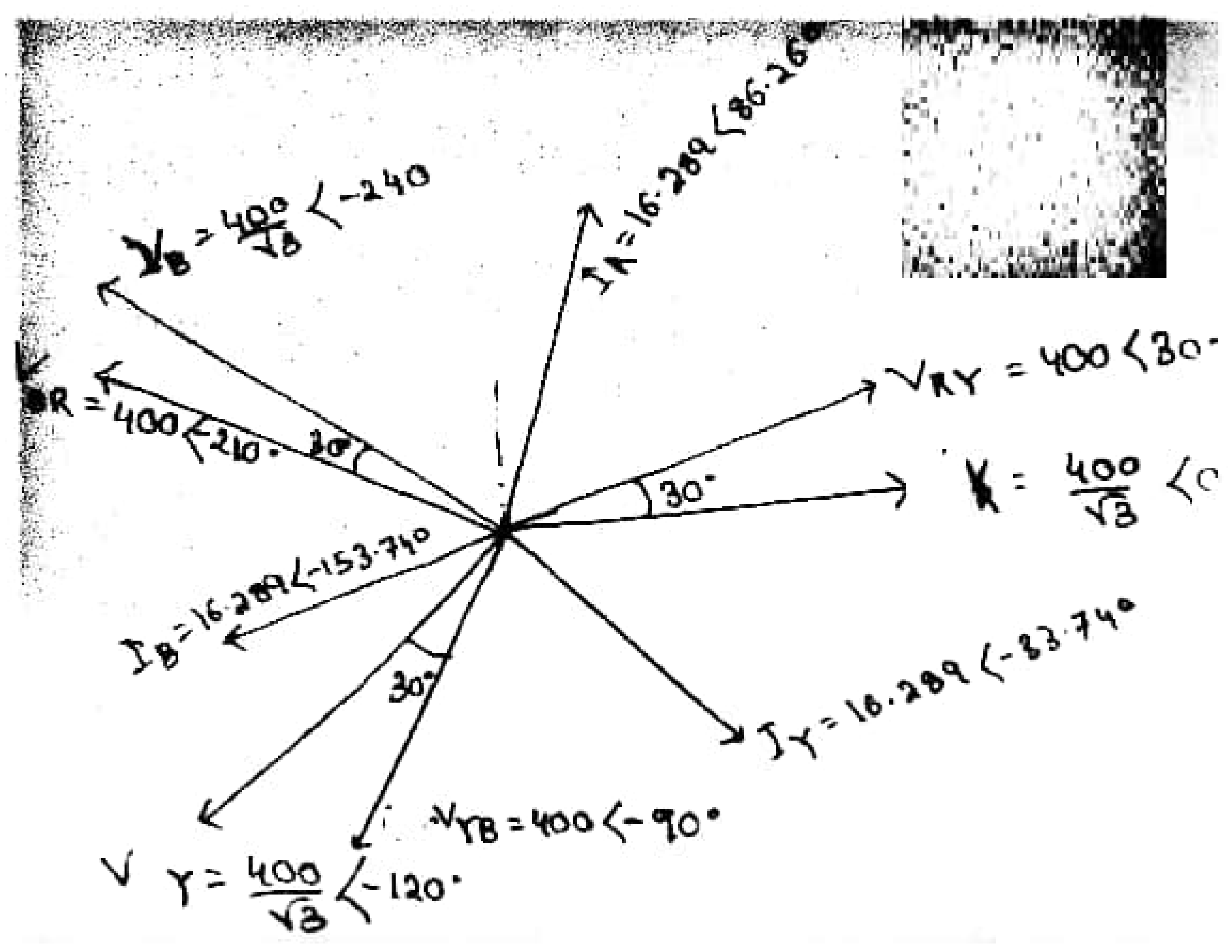


$\therefore I_1 = \frac{V_R}{Z} = (\frac{400}{\sqrt{3}}\angle 0°)/65.94\angle 72.34^{circ} \approx 3.5\angle -72.34°$
$\therefore I_2 = 3.5\angle -192.34°$
$I_3 = 3.5\angle -312.34°$
Converting Δ network to Y
Where $Z' = \frac{-j}{\omega C} = \frac{-j\times 10^6}{100\pi\times 90}$
$= \frac{-j\times 10^3}{9\pi} \approx -35.368j$

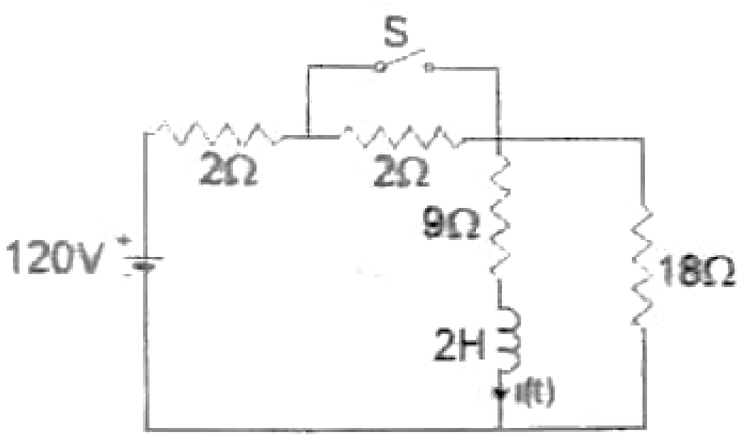


$\therefore Z'/3 \approx -11.789j = 11.789\angle -90°$

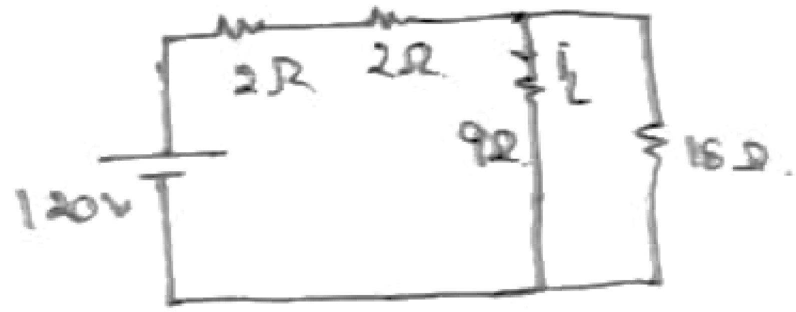
$\therefore I_1' = (\frac{400}{\sqrt{3}}\angle 0°)/11.789\angle -90° = 19.589\angle 90° A$
$\therefore I_2' = 19.589\angle -30° A$
$I_3' = 19.589\angle -150° A$
$\therefore I_R = I_1' + I_1 = 19.589\angle 90° + 3.5\angle -72.34° = 16.289\angle 86.26° A$
$I_Y = I_2' + I_2$
$= 19.589\angle -30° + 3.5\angle -192.34°$
$= 16.289\angle -33.74° A$
$\therefore I_B = 16.289\angle -153.74° A$



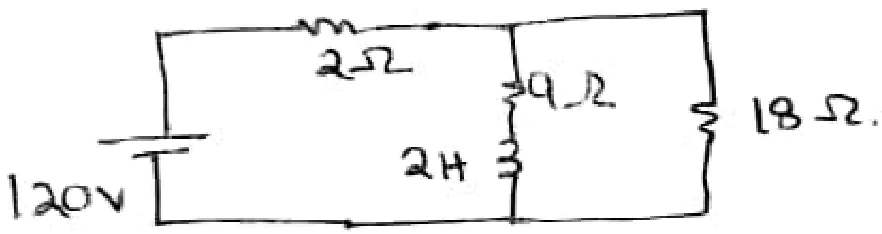
Q. 6. Assume that the circuit shown in the following figure is at steady state with switch " S " open . This switch closed at t =0 .The inductor current is i(t).
(i) calculate $i(0^+)$ and $\frac{di}{dt}(0^+)$
(ii) Calculate the steady state inductor current with switch " S " closed .
(iii) Obtain the expression for the inductor current i(t) for t>0.
Answer:



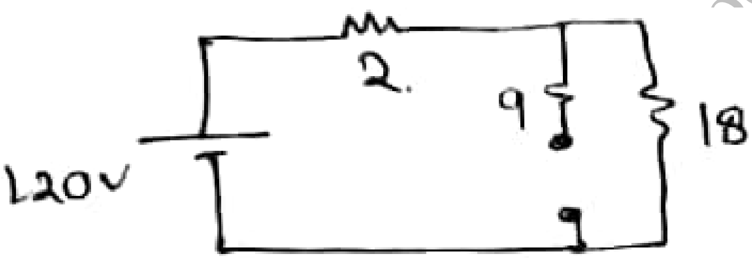
When switch is open
$R_{net} = \frac{9 \times 180}{27} + 4 = 10\Omega$
$\therefore$ Current (in net circuit)= $\frac{120}{10}A = 12A$
$\therefore i_L = \frac{18}{27} \times 12 = 8A$



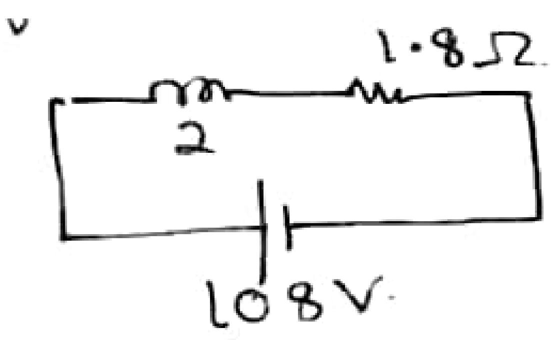
$\therefore i(0^+) = i(0^-) = 8A$ (properly of inductor)
When switch is closed



Using Thevenin theorem



$\therefore R_{Th} = 9 + \frac{18\times 2}{20} = 10.8\Omega$
$V_{Th} = \frac{18}{20} times 120V = 108V$
$\therefore$ Equivalent Thevenin circuit



$\therefore L\frac{di}{dt} + iR = V$
$2\frac{di}{dt} + i(\frac{108}{10}) = 108$
$\frac{di}{dt} + \frac{54}{10}i = 54$
or $10\frac{di}{dt} = 54(10 - i)$

or $10\int\frac{di}{10-i}=\int 54dt$
(Intergrating both sides and putting proper limits)
$10\times ln|\frac{10-i}{10-8}|=-54t$
$10-i=2e^{-5.4t}$
$10-2e^{-5.4t}=i$
$\therefore$ current through the inductor for t¿0 is $i=(10-2e^{-5.4t})A$
$\therefore$ steady state current is at t $=\infty$
$\therefore i=10A$
$\frac{di}{dt}=10.8e^{-5.4t}$
$\therefore \frac{di}{dt}(0^{+})=10.8$

ELECTRICAL TECHNOLOGY

MID-SPRING SEMESTER EXAMINATION **2017**

Q1. (a) Find the peak factor of the voltage signal $e(t) = E_m \sin(\omega t) + aE_m \sin(5\omega t)V$, where 'a' is a positive constant. **(4)**

(b) In the circuit, shown in Fig. Q1 (b), C is varying. Find the minimum value of the capacitive reactance such that the circuit will be operated at unity power factor. Draw to-scale phasor diagram on a graph paper of I, V, I_L, I_C, V_{R1}, V_{R2}, V_{XL} and V_{XC} when the circuit is operating at unity power factor. **(4+12)**

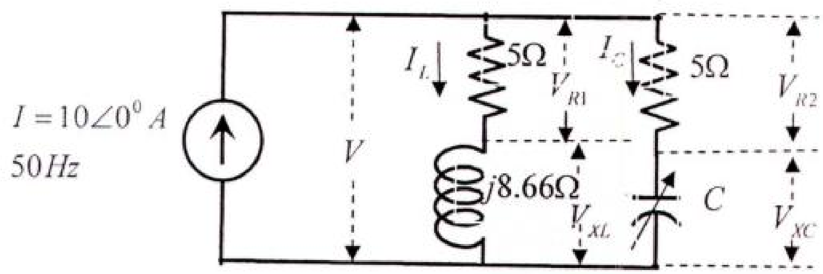


Fig. Q1 (b)

Ans 1a): Peak power factor = peak value/RMS value

RMS value $= \left[\int_0^T \frac{(\sin \omega t + a \sin 5\omega t)^2}{T}\right]^{1/2}$ where, $T = \frac{2\pi}{\omega}$

$= \frac{1}{\sqrt{2}}\sqrt{1+a^2}$

Peak value $= \sqrt{1+a^2}$

$\therefore$ peak factor $= \sqrt{2}$

Ans 1b): For the circuit to be resistive

$X_C = 8.66\Omega$ [Parallel circuits have same

or, $\frac{1}{\omega C} = 8.66$ voltage across them.

or, $C = \frac{1}{8.66 \times 2\pi(50)}$ As resistances are same, the shift in phasor up due to the X_L must

=36.76 mF be equal to the shift in phasor down due to the X_C]

$I_L + I_C = 10\angle 0^o$ A

$\therefore$ Impedance of inductive branch $\approx 10\angle 60^o \Omega$

$\therefore$ Impedance of capacitive branch $\approx 10\angle -60^o \Omega$

$I_C = \frac{X_L I}{X_L + X_C} = \frac{(10\angle 60^o)(10\angle 0^o)}{5}$

$= 20\angle 60^o$ A

$\therefore I_L = 10\sqrt{3}\angle -90^o$ A

$V = I_C X_C = (20\angle 60^o)(10\angle -60^o)$

$= 200\angle 0^o$ V

$V_{R_1} = 5 \times I_L = 50\sqrt{3}\angle -90^o$ V

$V_{R_2} = 5 \times I_C = 100\angle 60^o$ V

$V_{X_L} = 200\angle 0^o - V_{R_1} \approx 218\angle 23.4^o$ V

$V_{X_C} = 200\angle 0^o - V_{R_2} \approx 173.2\angle -30^o$ V

$I = 10\angle 0^o$ A

$\therefore$ Phasors can be drawn accordingly.

Q2. (a) For the circuit shown in Fig. Q2 (a), calculate the impedance $Z(j\omega)$. A sinusoidal current source $i_s(t) = 0.01cos(\omega t)$ A with variable frequency ω is connected between the terminals ab. Calculate the maximum possible rms current through the capacitor of $870\ pF$. Also calculate the corresponding value of ω.

(2+3+1)

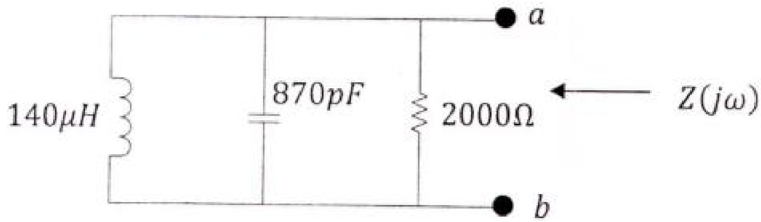


Fig. Q2 (a)

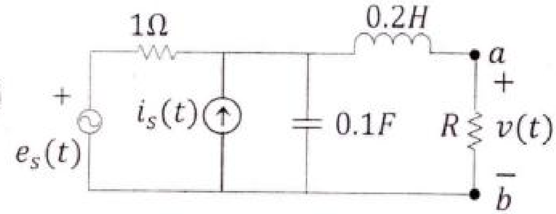


Fig. Q2 (b)

(b) The network shown in Fig. Q2 (b) is in steady state. $e_s(t) = 9cos(10t)\ V$ and $i_s(t) = 2\cos(10t - \pi/3)$ A.

(i) For the network on the left of the terminals ab (excluding R), obtain the Norton equivalent network.

(ii) Calculate a suitable value of R such that the average power drawn by R is maximum.

(iii) Derive the expression of instantaneous voltage $v(t)$ for this value of R.

(6+4+4)

Ans 2a): $X_L = j(\omega)(140) \times 10^{-6}$

$$X_C = \frac{-j}{\omega(870 \times 10^{-12})}$$

$R = 2000$

$$\therefore \frac{1}{Z_{net}} = \frac{1}{X_L} + \frac{1}{X_C} + \frac{1}{R}$$

$$= \left(\frac{1}{j\omega}\right)(7142.857) - \frac{1}{j}\omega(870 \times 10^{-12}) + \frac{1}{2000}$$

$$= \frac{1}{j}\left[\frac{7142.857}{\omega} - 870\omega \times 10^{-12}\right] + \frac{1}{2000}$$

$$= j\left[870\omega \times 10^{-12} - \frac{7142.857}{\omega}\right] + \frac{1}{2000}$$

$$\therefore Z(j\omega) = \frac{2000}{1 + 2000 \times (j)\left[870\omega \times 10^{-12} - \frac{7142.857}{\omega}\right]}$$

$$= \frac{2000}{1 + (j)\left[1.74\omega \times 10^{-6} - \frac{1.43 \times 10^{7}}{\omega}\right]}$$

Now, I_C is maximum when V_C is maximum.

As parallel circuit, V is same

$V = I \times X_{net}$

$\therefore |X_{net}|$ is maximum.

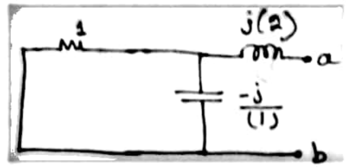
The denominator is 1

$$\therefore 1.74 \times 10^{-6}\omega = \frac{1.43 \times 10^{7}}{\omega}$$

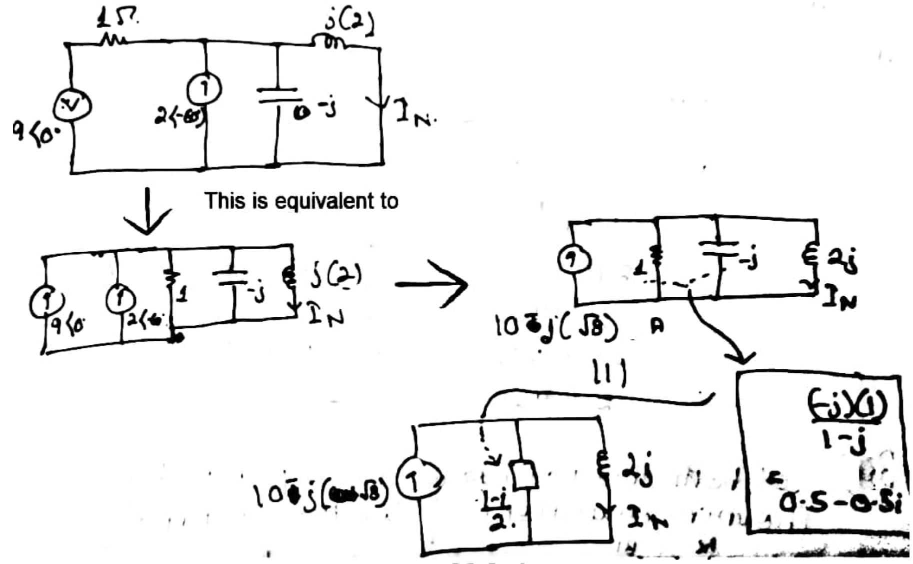
$$\text{or, } \omega^2 = \frac{1.43}{1.74} \times 10^{13}$$

or, $\omega = 2.87 \times 10^{7}$

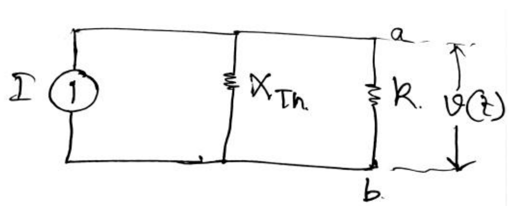
Ans 2b): For current sources opened and voltage sources shorted



$$X_{norton} = \frac{1(-j)}{1-j} + j(2) = (0.5 + 1.5j)\Omega$$



$$\therefore I_N = \left[\frac{0.5 - 0.5j}{(0.5-0.5j)+2j}\right][10 - \sqrt{3}j]$$
$$= [-0.2 - 0.4j][10 - \sqrt{3}j]$$
$$= 4.539\angle -126.4^o$$
$$\approx 4.54\angle -126.4^o$$



$X_{Th} = (0.5 + 1.5j)\Omega$
$I = 4.54\angle -126.4^o$ A
For maximum power drawn,
$R = [R_{Th}^2 + X_{Th}^2]^{1/2}$
$=\sqrt{(0.5)^2 + (1.5)^2} \approx 1.58\Omega$

$$I_{ab} = \left(\frac{Z_{Th}}{Z_{Th}+R}\right) I$$
$$\therefore V_{ab} = \left(\frac{Z_{Th}R}{Z_{Th}+R}\right) I$$
$$=(0.974\angle 35.77^o)(4.54\angle -126.4^o) \text{ V}$$
$$=4.422\angle -90.63^o$$
$$=4.422\cos(10t - 90.63^o) \text{ V}$$
$$=4.422\cos(10t - 1.582) \text{ V}$$

Ans 3a): Let both the networks be replaced by Thevenin equivalents. [See the terminals]

Q3. (a) The four configurations of the connections of two linear active DC networks A (with terminals A1 and A2) and B (with terminals B1 and B2) are given in the Fig. Q3 (a). The values of I_1, I_2 and I_3 are 16.5 A, 7.5 A and 1 A respectively. Find I_4. **(10)**

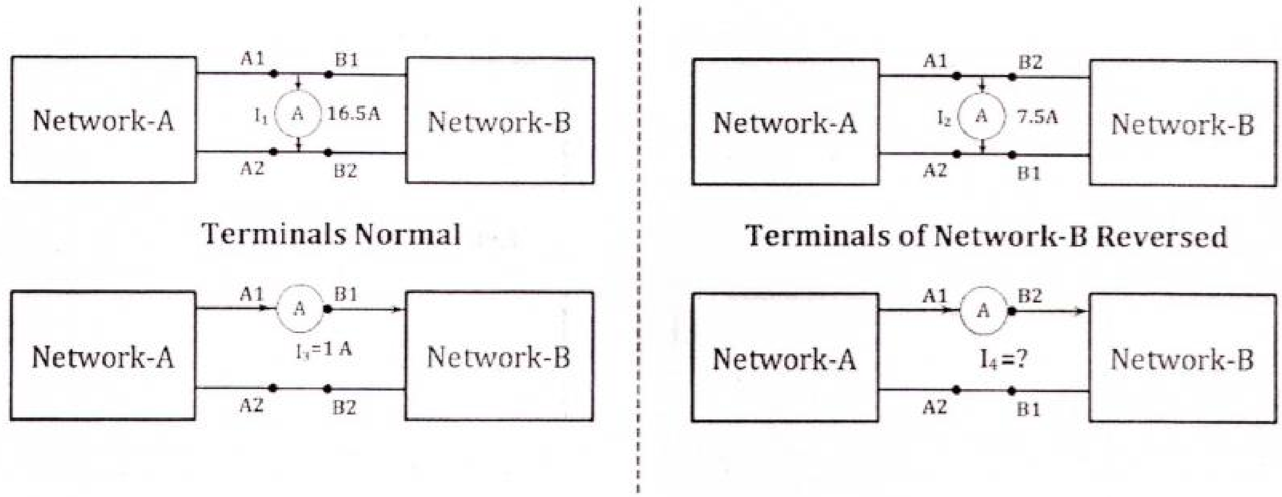


Fig. Q3 (a)

(b) In Fig. Q3 (b), find the values of ϕ $(0^0 \le \phi \le 90^0)$ for maximizing the active powers (i) P_1 and (ii) P_2. **(5+5)**

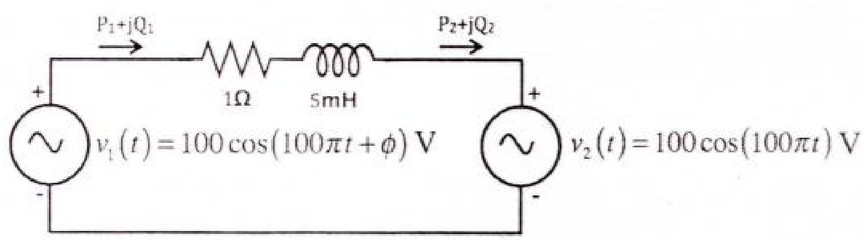


Fig. Q3 (b)

$\therefore \dfrac{V_A}{R_A} + \dfrac{V_B}{R_B} = 16.5$ A From 1

and, $\dfrac{V_A}{R_A} - \dfrac{V_B}{R_B} = 7.5$A From 2

$\therefore 2\dfrac{V_A}{R_A} = 24$

or, $V_A = 12R_A \ldots\ldots\ldots\ldots$ (i)

and, $\dfrac{V_A - V_B}{R_A + R_B} = 1$ From 3

or, $V_A - V_B = R_A + R_B$

or, $12R_A - V_B = R_A + R_B$

or, $11R_A = V_B + R_B$

or, $11\dfrac{R_A}{R_B} = \dfrac{V_B}{R_B} + 1$

or, $\dfrac{R_A}{R_B} = \dfrac{V_A}{R_A} - 7.5 + 1$ From 2

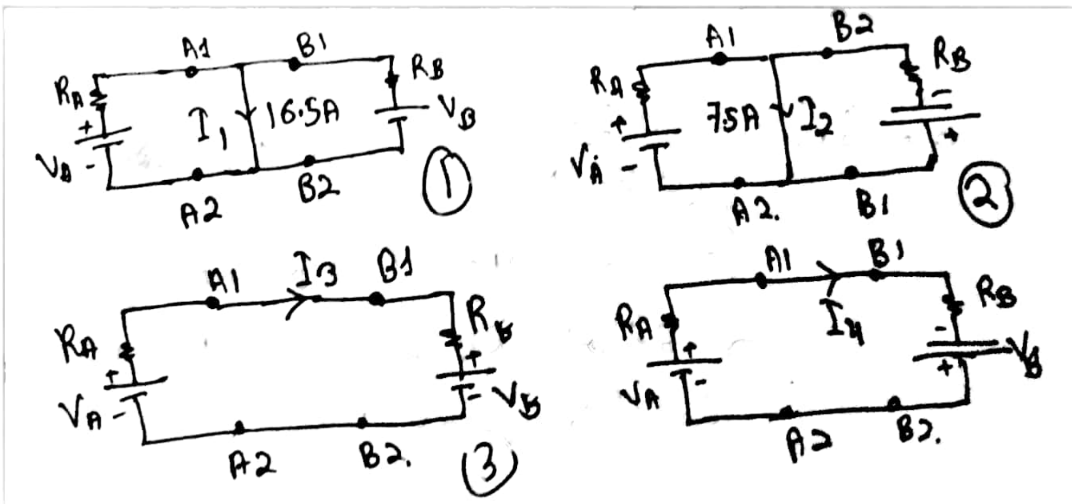
or, $\dfrac{R_A}{R_B} = 12 + 1 - 7.5$ From (i)

or, $R_A = 2R_B$

$\therefore$ From 3

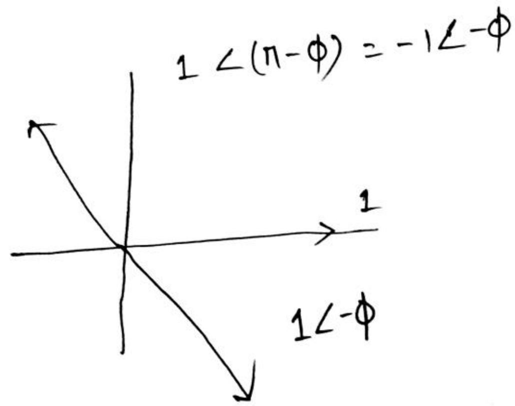
We need,

$$I_4 = \frac{V_A + V_B}{R_A + R_B}$$



$V_A - V_B = R_A + R_B$
or, $12R_A - V_B = R_A + 0.5R_A$
or, $V_B = 10.5R_A$
$\therefore I_4 = \dfrac{12R_A + 10.5R_A}{R_A + 0.5R_A} = 15$ A
Ans 3b(i): $P = VI*$

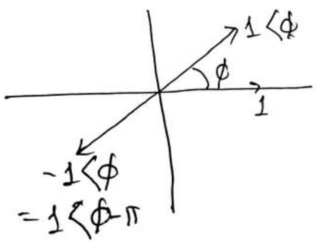
$$P_1 = Re\left[100 \times \frac{(100 - 100\angle\phi)}{1 + 0.5j}\right]$$
$$= Re\left[100 \times 100 \times \frac{(1 - 1\angle\phi)}{1 + 0.5j}\right]$$
$$= Re\left[\frac{10^4}{1.12}\left[\frac{(1 - 1\angle\phi)}{1\angle - 26.565^o}\right]\right]$$
$$= Re\left[\frac{10^4}{1.12} \times 2 \times \sin\frac{\phi}{2} \times e^{j(\frac{\pi}{2} - \frac{\phi}{2} + 26.565^o)}\right]$$



$$= Re\left[\frac{10^4}{1.12} \times 2 \times \sin\frac{\phi}{2} \times e^{j(116.565^o - \frac{\phi}{2})}\right]$$

$\therefore$ we have to maximize $\sin\dfrac{\phi}{2}\cos(116.565^o - \dfrac{\phi}{2})$
$= \dfrac{1}{2}\left[\sin(116.565^o) + \sin(\phi - 116.565^o)\right]$
$\therefore \phi = 116.565^o + 90^o$
$\therefore \phi = 206.565^o$
Ans 3b(ii):

$$P = VI*$$
$$\therefore P_2 = Re\left[\frac{(-100\angle\phi)(100-100\angle\phi)^*}{(1+0.5j)^*}\right]$$
$$I = \left[\frac{100-100\angle\phi}{1+0.5j}\right]$$
$$\therefore P_2 = Re\left[\frac{(100\angle\phi)(100\angle(-\phi)-100)}{1-0.5j}\right]$$
$$= Re\left[\frac{10^4-10^4\angle\phi}{1-0.5j}\right]$$
$$= Re\left[\frac{10^4-10^4\angle\phi}{1.12\angle-26.565}\right]$$
$$= Re\left[\frac{10^4}{1.12}\left[\frac{1-1\angle\phi}{1\angle-26.565}\right]\right]$$
$$= Re\left[\frac{10^4}{1.12}\left[2\sin\frac{\phi}{2}e^{j}(\frac{\phi}{1}\frac{\pi}{2}+26.565^o)\right]\right]$$



$$\therefore \text{we have to maximize } \sin\frac{\phi}{2}\cos(\frac{\phi}{2}-63.435^o)$$
$$= \frac{1}{2}\left[\sin(63.435^o)+\sin(\phi-63.435^o)\right]$$
$$\therefore \phi = 63.435^o = 90^o$$
$$\text{or, } \phi = 153.435^o$$

Q4. (a) In the circuit shown in Fig. Q4, the switch S was in neutral position (i.e., S is neither connected to position-1 nor to position-2) and the capacitor was uncharged. The switch is now moved to position-1 at t = 0. Write down the governing equation involving capacitor voltage v(t) and other parameters of the circuit. Get v(t) and the current through the capacitor i(t) and sketch them. What is the steady state voltage to which the capacitor will be charged? **(2+4+2)**

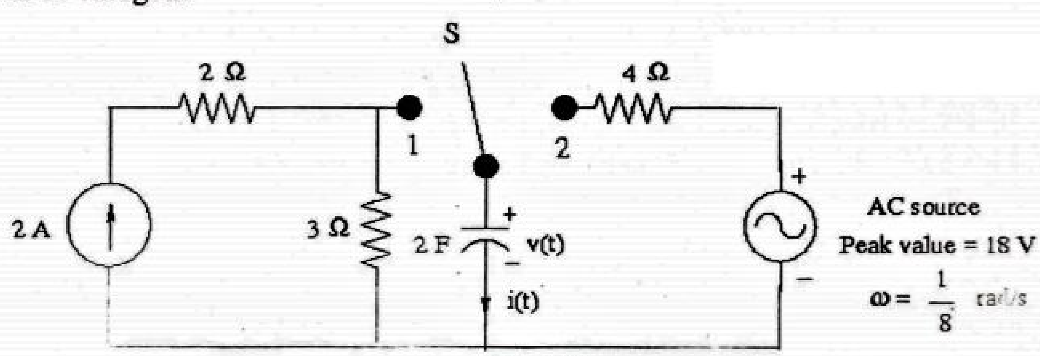
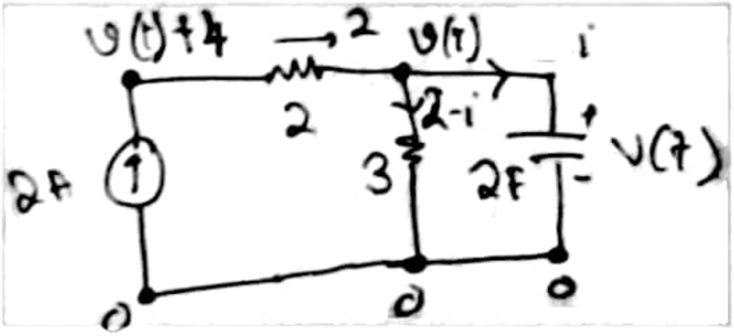


Fig. Q4

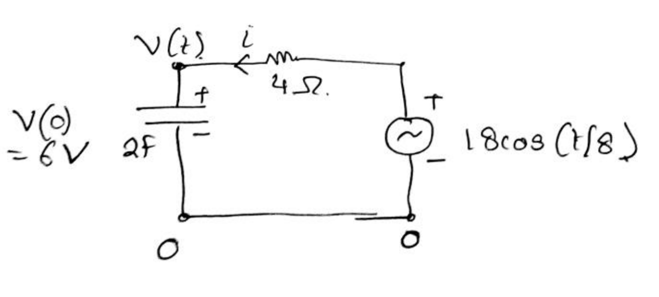
(b) Now assume that the circuit (Fig. Q4) reached steady state with S in position-1. The switch is now very quickly moved to position-2 at the instant when the a.c. supply voltage was at its positive peak. For this part of the problem, call this instant of switching to be t = 0. Derive the expressions for v(t) and i(t) for $t \geq 0$.

Ans 4a):



$\therefore q = VC$
or, $i = C\dfrac{d\theta(t)}{dt} = 2\dfrac{d\theta}{dt}$
Voltage across capacitor,
$3(2-i) = \theta(t)$
or, $6 - 3 \times 2\dfrac{dV}{dt} = V$
or, $1 - \dfrac{V}{6} = \dfrac{dV}{dt}$
or, $\displaystyle\int_0^t dt = \int_0^V \frac{dV}{1-\frac{V}{6}}$
or, $t = 6\left[\ln\left|1 - \dfrac{V}{6}\right|\right]_0^V$
$= -6\left[\ln\left|1 - \dfrac{V}{6}\right|\right]_0^V$
or, $e^{-t/6} = 1 - \dfrac{V}{6}$
or, $V(t) = 6(1 - e^{-t/6})$ volts
$\therefore i(t) = 2\dfrac{dV}{dt} = 2 \times 6(e^{-t/6})\dfrac{1}{6} = 2e^{-t/6}$ Amperes
Steady state voltage $\phi(\infty) = 6$ V

Ans 4b):



$\dfrac{dv}{dt} = \dfrac{-i}{2}$ where **i** is the current
or, $i = -2\dfrac{dv}{dt}$ As direction is opposite it is negative.
$\therefore 18\cos\left(\dfrac{t}{8}\right) - 4i - v = 0$
or, $18\cos\left(\dfrac{t}{8}\right) = 4i + v = v + 4\left(-2\dfrac{dv}{dt}\right) = v - 8\dfrac{dv}{dt}$

or, $8\dfrac{dv}{dt} = v - 18\cos\left(\dfrac{t}{8}\right)$

or, $18\cos\left(\dfrac{t}{8}\right) = v - 8\dfrac{dv}{dt}$

On solving we get, $v(t) = 9(1 - \sin\dfrac{t}{8})$V

$\therefore i = 2\dfrac{dv}{dt} = -18\cos\dfrac{t}{8} \times \dfrac{1}{8} = -\left(\dfrac{9}{4}\cos\dfrac{t}{8}\right)$ amperes

Q5. (a) Three single-phase voltage sources (V_a, V_b, V_c) are connected in delta to form a balanced three phase source. The values of two of these sources in volts are: $V_a = -20.92 - j239.09$ *and* $V_b = -196.6 + j137.66$; Find V_c (magnitude and phase) and identify the phase sequence. This three-phase source is used to supply two loads connected in parallel - one delta-connected and the other wye-connected. The total power supplied by the source is 8kVA at 0.6 power factor (lag) and the wye-connected load draws a total of 3 kW at unity power factor. Find the per phase impedance of the delta-connected load. **(1+2+5)**

(b) A balanced three-phase voltage source supplies a balanced three-phase delta-connected load through a transmission line having two equal segments (NM and MO), as shown in Fig. Q5. Each segment has a reactance of $j2\Omega$. The expression of the R-phase voltage $v_R = 240\sqrt{2}sin(100\pi t)$. Each phase of the balanced delta-connected load consists of a reactance of $-j84\Omega$with a current source in parallel, as shown in Fig. Q5. The source current connected between R and Y phase is $i_{RY} = 90\sqrt{2}sin(100\pi t)$. The phase sequence of both the sources is R-Y-B. Find the expression of the three line-to-line voltages at the mid-point (M) of the line. **(12)**

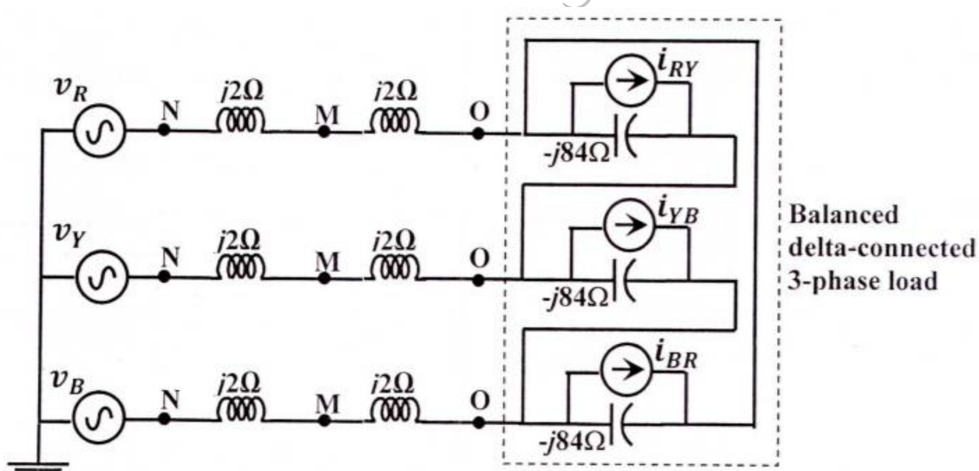


Fig.Q5

Ans 5a): $V_a = -20.92 - j(239.0.9)$

$\approx 240\angle -95^o$

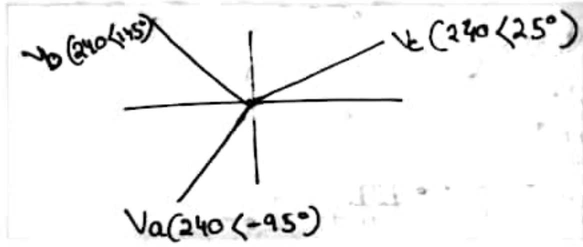
$V_b = -196.6 + j(137.66) \approx 240\angle 145^o$

Sequence is cab

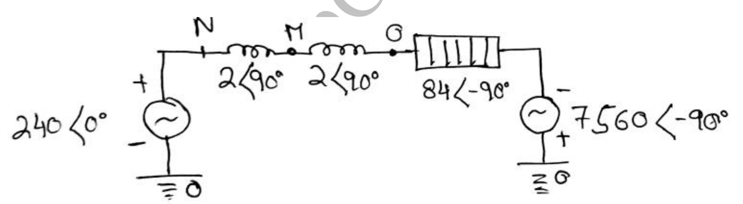
$V_a = 240\angle -95^o$

$V_b = 240\angle -215^o$

$V_c = 240\angle 25^o$



Power at wye load = 3kW at unity power factor.
Power at source = $4.8 + j(6.4)$
$\therefore$ Power at delta = $1.8 + j(6.4)$
$\therefore$ Apparent power = 6.65kVA $\qquad V_ph = 240$ V
$\therefore \dfrac{V^2}{|Z|} = 6.65 \times 10^3$
or, $|Z| = 0.115$
Also power factor = $\cos(\tan^{-1}\left(\dfrac{6.4}{1.8}\right)) = 0.271$
$\therefore Z = 0.115\angle 74.3^o$
Ans 5b): Working with rms values only
$i_{RY} = 90\angle 0^o \qquad V_R = 240\angle 0^o$
$i_{YB} = 90\angle -120^o \qquad V_Y = 240\angle -120^o$
$i_{BR} = 90\angle -240^o \qquad V_B = 240\angle -240^o$
We can replace the capacitor and current source with a capacitor and voltage source.
$\therefore$ those voltages $(P_RY = 90\angle 0^o)(84\angle -90^o) = 7560\angle -90^o$
$P_{YB} = 7560\angle -210^o$
$P_{BR} = 7560\angle -330^o$

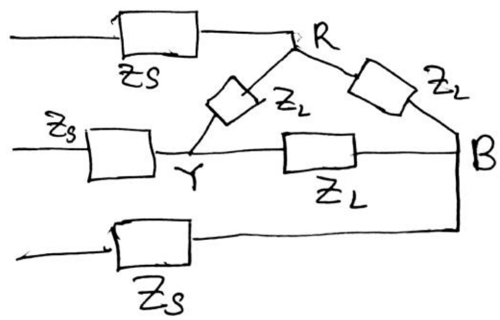


Only 1 phase of the line
Both sides are grounded as it is balanced
$I = 94.5 + 3j$
$$\begin{aligned}\therefore V_M &= 240 + (2j)(94.5 + 3j)\\ &= 240 + 189 - 6j\\ &= 429 - 6j = 429\angle -0.8^o\end{aligned}$$
$\therefore V_M = 429\sqrt{2}\sin(100\omega t - 0.8^o)$
$\therefore$ line voltages
$V_{RY} = 429\sqrt{6}\sin(100\pi + 29.2^o)$
$V_{YB} = 429\sqrt{6}\sin(100\pi + 149.2^o)$
$V_{BR} = 429\sqrt{6}\sin(100\pi + 269.2^o)$

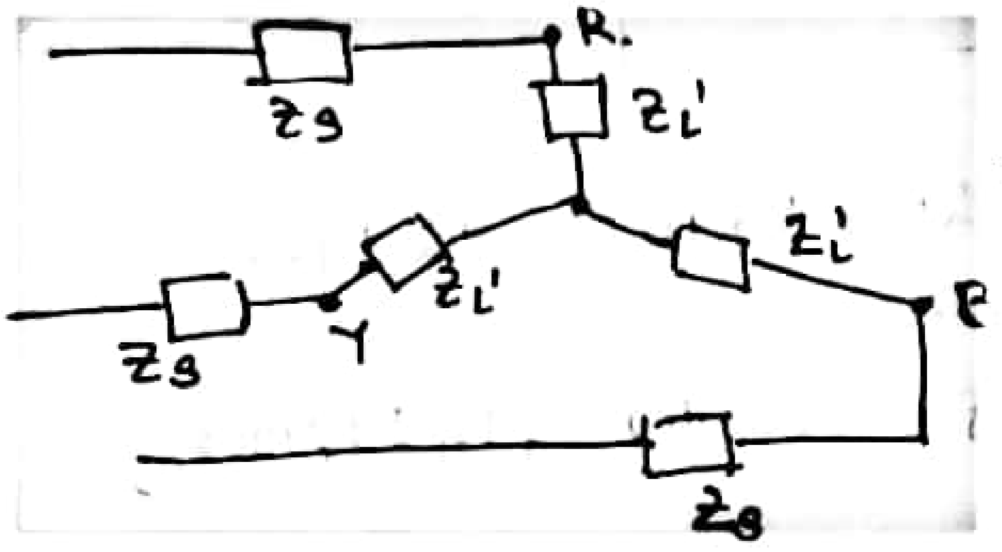
Q6. (a) A balanced delta-connected load having an impedance $Z_L = 300 + j210\ \Omega$ in each phase is supplied from 400 V, 3-phase supply (R-Y-B phase sequence) through a 3–phase line having an impedance of $Z_S = 4 + j8\ \Omega$ in each phase. The total power in the load is measured by means of two wattmeters with their current coils in lines R and B and their corresponding voltage coils across R and Y, and B and Y respectively. Calculate the reading on each wattmeter and the total power supplied to the load. **(15)**

(b) The current coil of a wattmeter is connected in series with an ammeter and an inductive load. A voltmeter and the voltage coil of the wattmeter are connected across a 400Hz supply. The ammeter, voltmeter, and wattmeter readings are 4.5A, 240V, and 29W respectively. The inductance and resistance of the voltage coil are 5mH and 4 kΩ respectively. If the voltage drops across the ammeter and current coil are negligible, what is the percentage error in wattmeter reading? **(5)**

Ans 6a):

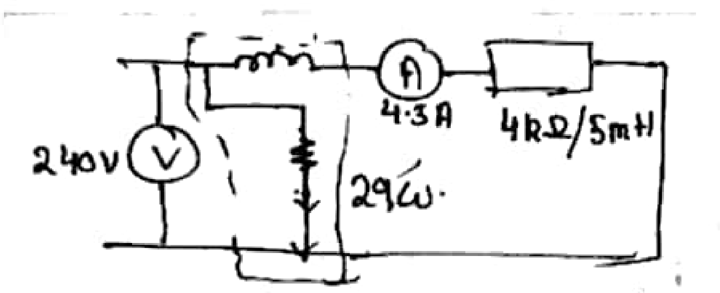


$Z_S = 4 + 8j$
$Z_L = 300 + 210j\ \Omega$
Phase voltage $= 400/\sqrt{3}$
Converting to star,



$Z'_L = 100 + 70j$
$\phi = \tan^{-1}\left(\dfrac{X'_L + X_S}{R'_L + R_S}\right) \approx 37^o$
Power $= V_{RY} I_R^*$

$\therefore$ the real part $= \left[\dfrac{V_{ph}}{|Z_S + Z_L'|}\right]^2 \times R_L'$
$= \dfrac{400 \times 400}{3 \times 130 \times 130} \times 100 = 315.582$ W
$\therefore$ Total power $= 946.746$ W
Let $V_{RY} = 400\angle 0^o$ $\quad V_{YB} = 400\angle -120^o$
$V_R = \dfrac{400}{\sqrt{3}}\angle -30^o$ $\quad V_B = \dfrac{400}{\sqrt{3}}\angle 90^o$
$\therefore I_R = \dfrac{400}{\sqrt{3} \times 130}\angle -67^o$ $\quad I_B = \dfrac{400}{\sqrt{3} \times 130}\angle 53^o$
$\therefore V_{R'} = V_R - I_R \times Z_S$
$= \dfrac{400}{\sqrt{3}}\angle(-30^o) - \dfrac{300}{\sqrt{3} \times 130} \times 8.9\angle -3.57$
$= 217\angle -31.86^o$
$\therefore V_{Y'} = 217\angle -151.86^o$
$\therefore V_{B'} = 217\angle -271.86^o$
$\therefore V_{R'Y'} = V_{R'} - V_{Y'} = 375.86\angle -1.86^o$
$\therefore V_{B'Y'} = V_{B'} - V_{Y'} = 375.86\angle 58.14^o$
$\therefore$ For 1st ammeter $P_{real} = Re\left[\left(\dfrac{(400)^2}{\sqrt{3} \times 130}\angle 67^o\right)(375.86\angle -1.86^o)\right] = 280.77$ W
For 2nd ammeter,
$P_{real} = Re\left[(375\angle 58.14^o)\left(\dfrac{400}{130\sqrt{3}}\angle -53^o\right)\right] = 665.03$ W
Note: the connection of the voltage coils of the wattmeters are not clear. Please verify from your professors.
Ans 6b):



$X_L = 400 \times 2\pi \times 5 \times 10^{-3} = 12.57\Omega$
$R_L = 4000\Omega$
$\therefore$ power factor $= \cos(\tan^{-1}(12.57/4000))$
Note : The datas are extremely unmatchable. A load of $(4000 + 12.57j)$ should not draw 4.5 A of current from a source pf 240 volts. please consult your professor regarding this.

ELECTRICAL TECHNOLOGY

MID-AUTUMN SEMESTER EXAMINATION **2015**

1. (a) For given three resistances R_A, R_B and R_C connected in delta, find the expressions for equivalent star resistances. [8]

(b) For the circuit shown in Figure 1, calculate the range of values of R such that the current from the battery can be adjusted from 50 mA to 100 mA. [12]

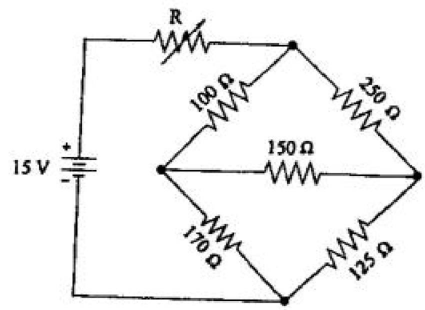


Figure 1: Pertaining to Q1(b)

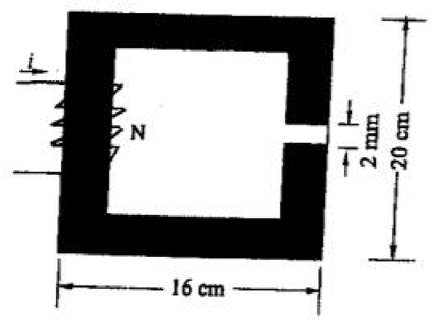
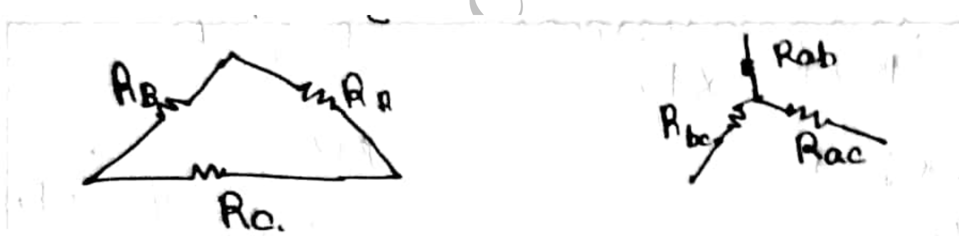


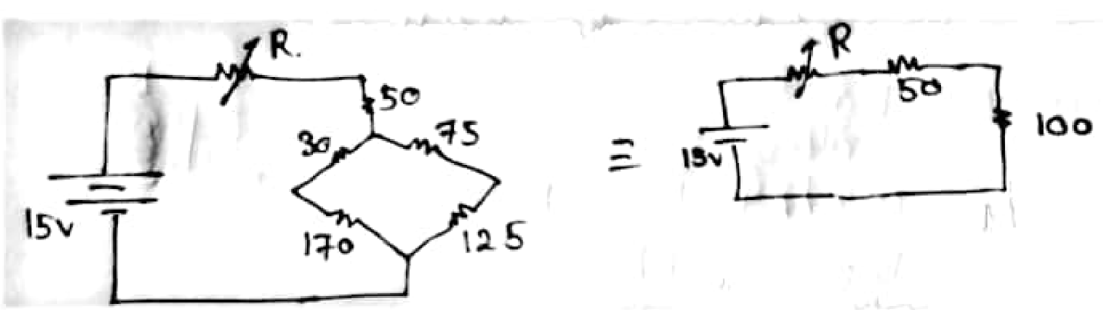
Figure 2: Pertaining to Q2(a)

Ans 1a): $R_{ab} = \dfrac{R_A R_B}{\sum R_A}$ $\qquad R_{bc} = \dfrac{R_B R_C}{\sum R_A}$ $\qquad R_{ca} = \dfrac{R_C R_A}{\sum R_A}$



* Refer from books, such questions do not come these days.

Ans 1b):



$\therefore I = \dfrac{15}{R+50+100} = \dfrac{15}{R+150}$

$\therefore 50 \times 10^{-3} < \dfrac{15}{R+150} < 100 \times 10^{-3}$

or $50 < \dfrac{15}{R+150} < 100$

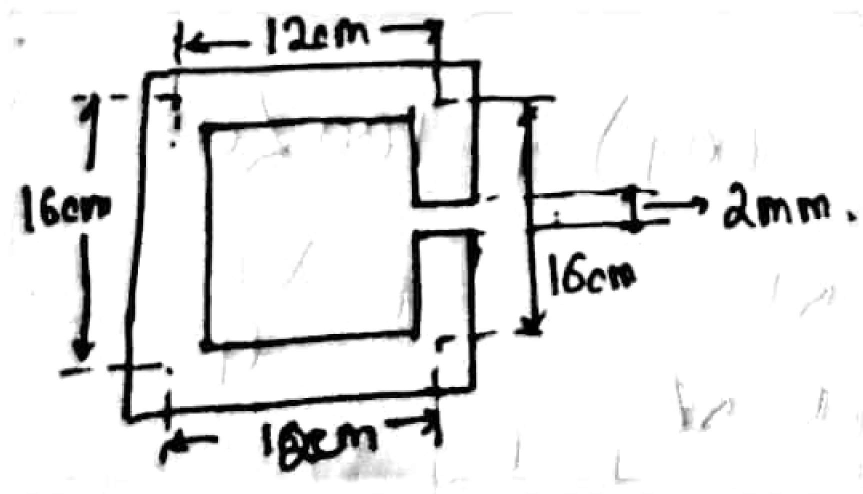
or $R + 150 < \dfrac{15 \times 100}{5}$ $\qquad \dfrac{15 \times 100}{10} > R + 150$

or $R + 150 < 300$ $\quad\quad$ $R > 0$
or $R < 150$ $\quad\quad$ $R > 0$
$\therefore R$ varies from 0 to 150 Ω

2. (a) For a rectangular magnetic core with an air gap shown in Figure 2, $N = 500$ turns; the cross section of the core is 4 cm × 4 cm; and μ_r (iron core) = 5000. Assuming uniform flux density, find the excitation current i needed to establish a flux density of 1.2 T in the air gap without considering the effect of fringing. [8]

   (b) Two identical 2500 turn coils A and B are in parallel planes such that 80% of the magnetic flux produced by one coil links the other. A current of 10 A in coil A produces a flux of 0.06 mWb in it. If the current in coil A changes from +10 A to -10 A in 0.02 s., what will be the magnitude of the voltage induced in the coil B? Calculate the self inductance of each coil and the mutual inductance. [4 + 4 + 4]

Ans 2a): $l_{net}(iron - core) = 0.12 \times 2 + 0.16 + (0.16 - 0.002) = 0.558 = l_c$



$l_{net}(air) = 0.002m = l_a$

Equivalent magnetic resistance $= \dfrac{l_c}{\mu_r \mu_o A} + \dfrac{l_a}{\mu_o A} = \dfrac{1}{\mu_o A}\left(\dfrac{l_c}{\mu_r} + l_a\right)$

$= \dfrac{1}{4\pi \times 10^{-7} \times 16 \times 10^{-4}}\left(\dfrac{0.558}{5000} + 0.002\right)$

$\therefore NI = 1.2 \times \dfrac{16 \times 10^{-4}}{4\pi \times 10^{-7} \times 16 \times 10^{-4}}\left(\dfrac{0.558}{5000} + 0.002\right)$

Flux density = 1.2
Total flux = Flux density × area
$= 16 \times 10^{-4} \times 1.2$
$\therefore I = 4.033$ A

Ans 2b): We know, $\phi =$ LI
or, $0.06 \times 10^{-3} = L_A \times 10$
or, $6 \times 10^{-6} = L_A = 6\mu$H
Also $V = -L\dfrac{di}{dt}$
$\phi_B = 0.8\phi_A$
$\therefore \dfrac{d\phi_B}{dt} = 0.8\dfrac{d\phi_A}{dt}$
or, $V_B = 0.8V_A = 0.8 \times 6 \times 10^{-6} \times \dfrac{20}{0.02} = 4.8$ mV

$\because$ they are identical
$\therefore L_A = L_B = 6\mu$H
By definition of mutual inductance
$\phi_B = M_{AB} I_A$
or, $10^{-3} \times 0.8 \times 0.06 = M_{AB} \times 10$
or, $M_{AB} = 4.8\mu$H

3. A single phase, 100V (rms), 50Hz source supplies a single phase load having impedance of 10 Ω and power factor of 0.8 (lag) through a line of impedance $\bar{Z}_{\text{line}} = (2 + j6)\,\Omega$, as shown in Figure 3. A pure capacitor is connected in parallel (shunt) with the load. Two voltmeters, V1 and V2, are connected at the source and load terminals respectively as shown in the figure. Find the *minimum value* of the capacitance C_{sh}, such that the readings of the two voltmeters are exactly same. Find the load current $\bar{I}_L$ and the source current $\bar{I}_S$ under this condition? [10 + 5 + 5]

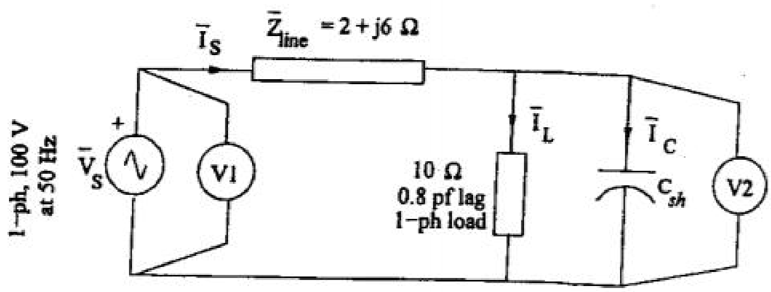
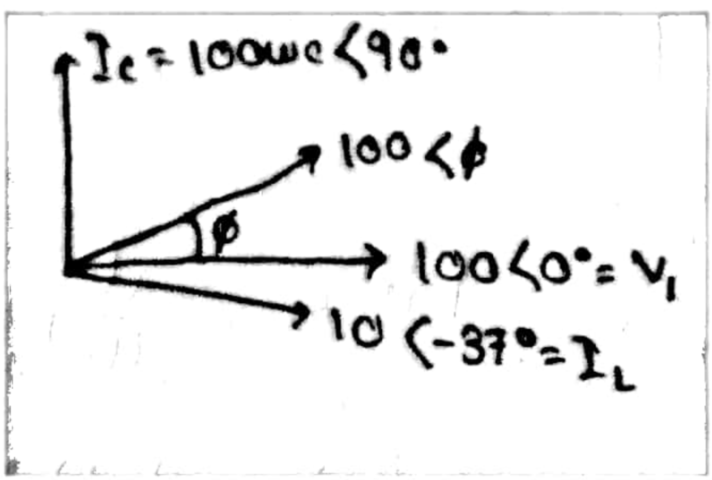


Figure 3: Pertaining to Q3.

Ans 3): Let V_2 be $100\angle 0^o$
V_2 has same magnitude as $V_1 (V_s) = 100$V
$\therefore I_L = 10\angle -37^o$
and let V_1 be $100\angle\phi$
$\therefore 100\angle\phi = (I_L + I_C)(2 + 6j) + 100\angle 0^o$



$$
\begin{aligned}
100\angle\phi &= (8 - 6j + (100\omega c)j)(2 + 6j) + 100\angle 0^o \\
&= (8 + (100\omega c - 6)j)(2 + 6j) + 100\angle 0^o \\
&= (16 + 36 - 600\omega c) + j(48 + 200\omega c - 12) + 100 \\
&= (152 - 600\omega c) + j(36 + 200\omega c)
\end{aligned}
$$

Taking magnitude on both sides,
$(152 - 600\omega c)^2 + (36 + 200\omega c)^2 = (100)^2$

or, $(1.52 - 6\omega c)^2 + (0.36 + 2\omega c)^2 = 1$
or, $(1.52)^2 + (0.36)^2 + (2\omega c)^2 + (6\omega c)^2 + 2(0.36 \times 2 - 1.52 \times 6)\omega c = 1$
or, $1.44 + 40(\omega c)^2 - 17.28\omega c = 0$
$\Rightarrow \omega c = 0.31923$ or 0.11277
or, $2\pi(50)C = 0.31923$ or 0.11277
or, $C = 1.016\text{mF}$ or 0.359mF
$\therefore C_{sh} = 0.359\text{mF}$
$\therefore I_L = 10\angle 4 - 37^o$ A
$\therefore I_C = 11.277\angle 90^o$ A $\quad [I_C = 100\omega c]$

4. In the circuit shown in Figure 4, the switch is initially connected to point **A** for a long time such that the circuit reached steady state. At $t = 0$, the switch S, is moved very quickly to point **B**. Find $v_c(0^+)$ and $v_c(t)$ at $t = 0.08$ s. Obtain expression for $i_R(t)$ for $t \geq 0$ and sketch it. [3+10+3+4]

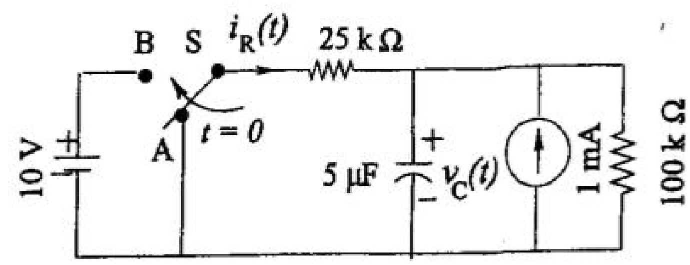
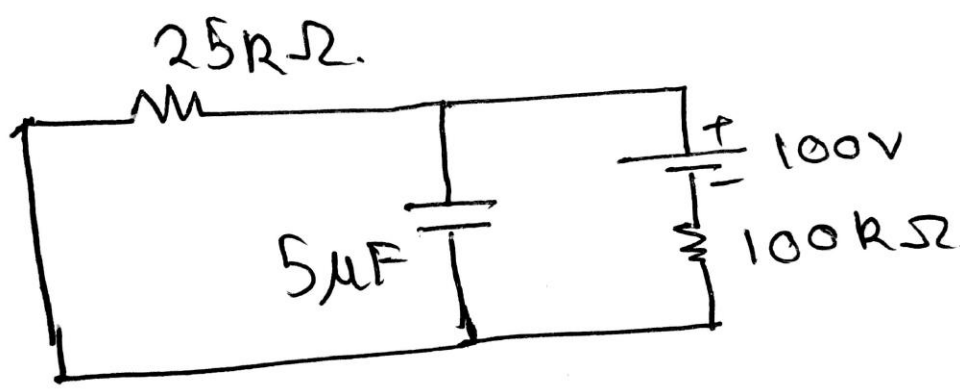
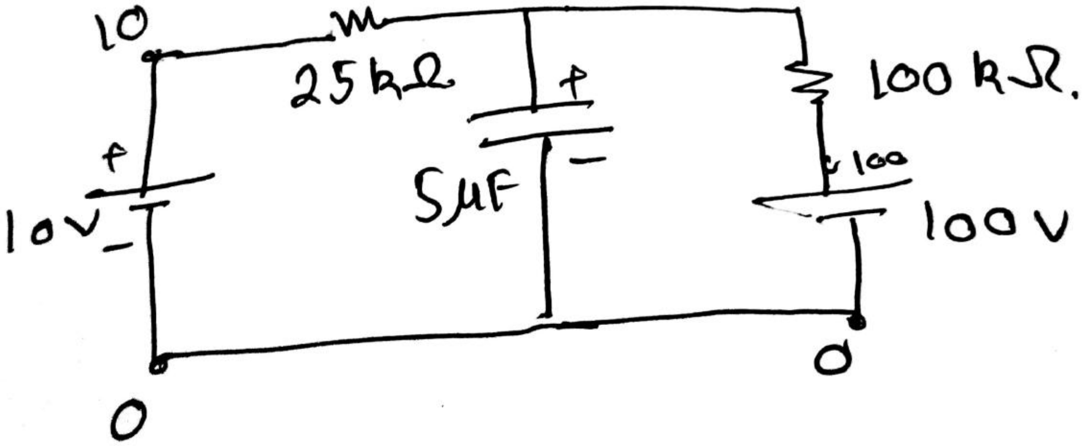


Figure 4: Pertaining to Q4

Ans 4): For $t < 0$



At steady state, I $= \dfrac{100}{125} = 0.8$ mA
$\therefore$ Voltage across capacitor $= 25 \times 10^3 \times 0.8 \times 10^{-3} = 20$ V
$V_c(0^+) = 20$ V
After, $t = 0$



[where $\tau = R_{Th}C$]

We know, $V_C(t) = V_\infty + (V_0 - V_\infty)e^{-t/\tau}$

$V_\infty = 283\text{V}$ [It can be easily found as at $t = \infty$, no current flows through the capacitors and simply voltage across capacitor is V_∞]

$V_o = 20\text{V}$

$\therefore R_{Th} = \dfrac{100 \times 25}{125} = 20\Omega$

$\therefore \tau = 20 \times 10^3 \times 5 \times 10^{-6}$

$=100 \times 10^3 \times 10^{-6} = 0.1Hz$

$\therefore V(t) = 28 + (20 - 28)e^{-10t}$

$= 28 - 8e^{-10t}$

$\therefore V(0.08) = 28 - 8e^{-0.8} = 24.41$ V

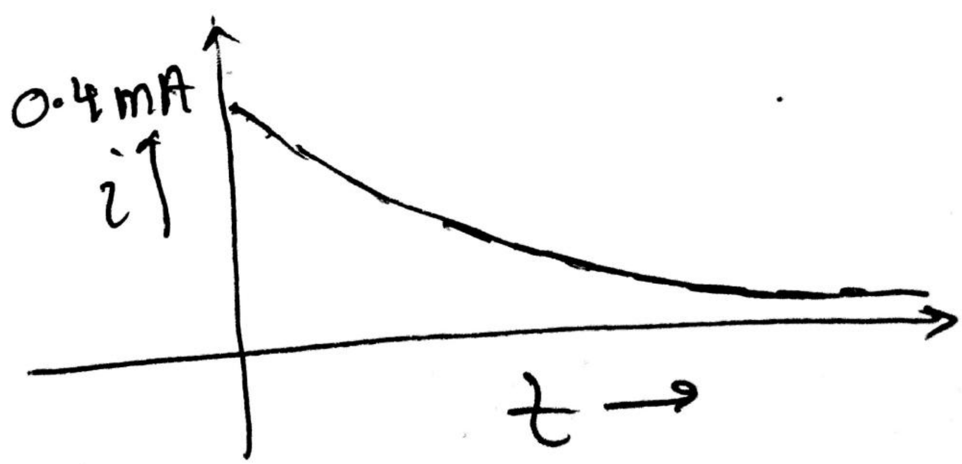
$q = VC$

$=(28 - 8e^{-10t})(5 \times 10^{-6})$

$\therefore I = (8 \times 10e^{-10t})(5 \times 10^{-6})$

$=(400 \times 10^{-6})e^{-10t}$

$=0.4e^{-10t}$ mA



5. A wattmeter is connected in the balanced three phase resistive network as shown in Figure 5. Find the reading of the wattmeter. Assume $\bar{V}_{RY} = 100\angle 0°$ V and show $\bar{I}_R$, $\bar{I}_Y$ & $\bar{I}_B$ in a neatly drawn phasor diagram. [10 + 10]

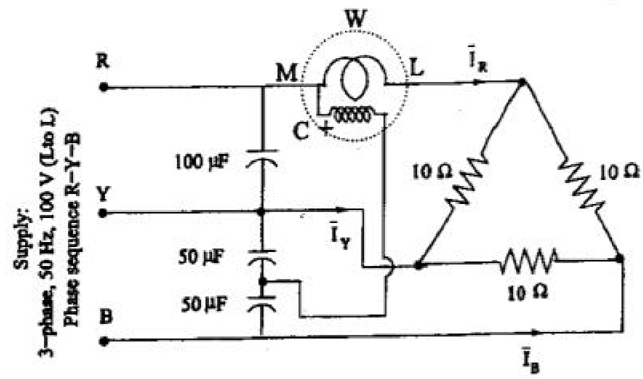


Figure 5: Pertaining to Q5

Ans 5): $V_{RY} = 100\angle 0^o$

$V_{YB} = 100\angle -120^o$

$V_{BR} = 100\angle -240^o$

Converting $\triangle$ to Y load, it gives, $R_Y = 10/\sqrt{3}\Omega$

$\therefore I_R = 30\angle 0^o$ $\quad V_R = \dfrac{100}{\sqrt{3}}\angle -30^o$

$\therefore I_Y = 30\angle -120^o$ $\quad V_Y = \dfrac{100}{\sqrt{3}}\angle -150^o$

$\therefore I_B = 30\angle 0^o$ $\quad V_B = \dfrac{100}{\sqrt{3}}\angle -270^o$

[All the currents marked are after the capacitor. So they are not affected by the capacitors as they are in parallel.]

Also $V_{YB} = V_Y - V_B$

$\therefore$ Voltage at the junction of two 50μF capacitors is

$$V = V_B + \frac{V_Y - V_B}{-2j/\omega C} \times \frac{-j}{\omega C}$$

$$= V_B + \frac{V_Y}{2} - \frac{V_B}{2} = \frac{V_Y + V_B}{2}$$

$$= \frac{100}{\sqrt{3}}[1\angle -270^o + 1\angle -150^o]$$

$$= \frac{100}{\sqrt{3}} \times 1\angle 150^o = \frac{100}{\sqrt{3}}\angle 150^o$$

$\therefore$ Reading of wattmeter $= VI_R^*$

$$P_{real} = Re\left[\left(\frac{100}{\sqrt{3}}\angle 150^o\right)(30\angle 0^o)\right]$$

= -1500 W

6. (a) Calculate the direction and magnitude of the current through the $5\,\Omega$ resistor between the points A and B in the circuit shown in Figure 6 by using nodal analysis method. [10]

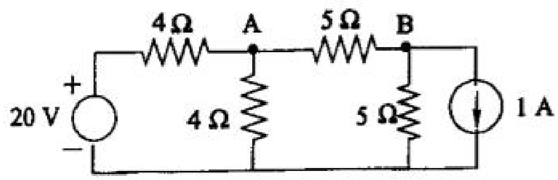


Figure 6: Pertaining to Q6(a)

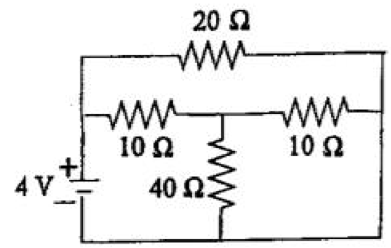
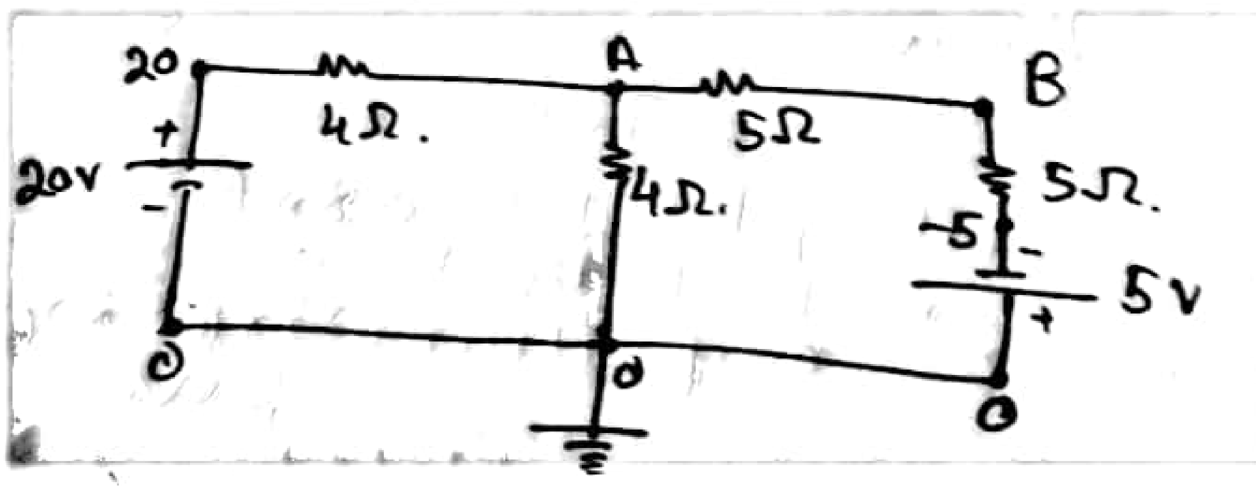


Figure 7: Pertaining to Q6(b)

(b) Calculate the current through the $40\,\Omega$ resistance in the network shown in Figure 7. [10]

Ans 6a): Changing the ammeter to equivalent voltmeter,



$\therefore$ At node A, $\dfrac{V_A - 20}{4} + \dfrac{V_A - (-5)}{10} + \dfrac{V_A}{4} = 0$

or, $\dfrac{V_A}{4} - 5 + \dfrac{V_A}{10} + \dfrac{1}{2} + \dfrac{V_A}{4} = 0$

or, $V_A\left(\dfrac{1}{2} + \dfrac{1}{10}\right) = 5 - \dfrac{1}{2} = \dfrac{9}{2}$

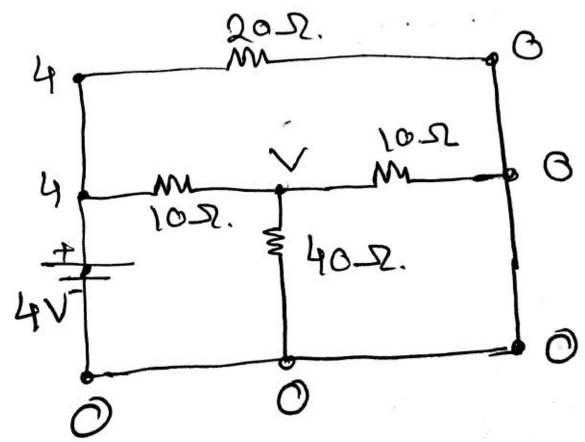
or, $V_A\left(\dfrac{5+1}{10}\right) = \dfrac{9}{2}$

or, $V_A = \dfrac{15}{2} = 7.5$ V

$\therefore I = \dfrac{V_A - (-5)}{10} = \dfrac{7.5 + 5}{10} = 1.25$ A

$\therefore$ 1.25A current flows from A to B.

Ans 6b):



By Nodal Law analysis,
$\frac{V-4}{10} + \frac{V}{10} + \frac{V}{40} = 0$
or, $V\left(\frac{2}{10} + \frac{1}{40}\right) = \frac{4}{10}$
or, $V(8+1) = 16$
or, $V = \frac{16}{9}$ V
$\therefore I_{40\Omega} = \frac{16}{9 \times 40}\text{A} = 0.044\text{A}$

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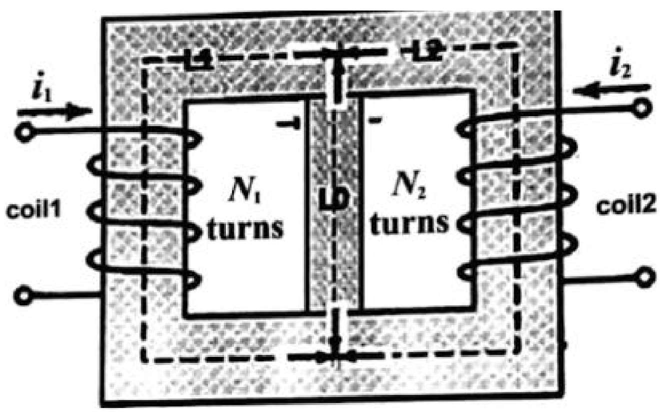
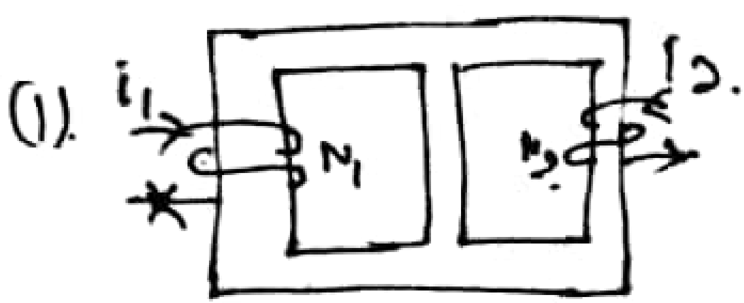
ELECTRICAL TECHNOLOGY

END - AUTUMN SEMESTER EXAMINATION**2017-2018**

Q. 1.a) The magnetic circuit as shown in the figure has rectangular cross section . The central limb has a cross section 1 cm $\times$ 2cm and all other limbs have cross section of $2cm \times 2cm$. Coil 1 has 500 turns(N_1) and coil 2 has (N_2) . Bothe the coils carry 1 A of current .In the figure , mean flux path lengths L_1, L_2 and L_0 are 20 cm ,16 cm and 8 cm respectively.Relative permeability of the material is 1600. Calculate
(i) the reluctance of the three sections having mean path lengths L_1, L_2 and L_0 [3]
(ii)the flux in the central limb.
(iii) self and mutual inductance of coil 1 and coil 2

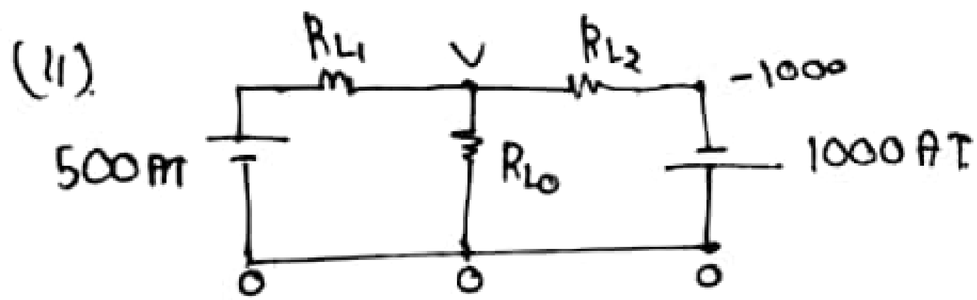


1.a)(i)



$L_1 = 20cm$
$L_2 = 16cm$
$L_0 = 8cm$
$R_{L_0} = \frac{l}{\mu_0 \mu_r A} = \frac{10^{-2} \times 8}{1600 \times 4\pi \times 10^{-7} \times 2 \times 10^{-4}} = 1.989 \times 10^5 AT/Wb$
$R_{L_1} = \frac{10^{-2} \times 20}{1600 \times 4\pi \times 10^{-7} \times 4 \times 10^{-4}} = 2.487 \times 10^5 AT/Wb$

$R_{L_2} = \frac{10^{-2}\times16}{1600\times4\pi\times10^{-7}\times4\times10^{-4}} = 1.989\times10^5 AT/Wb$
(ii)

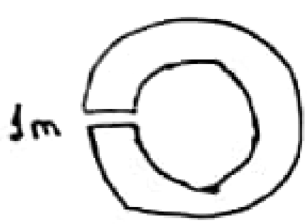


$\frac{V-500}{R_{L_1}} + \frac{V}{R_{L_0}} + \frac{V+1000}{R_{L_2}} = 0$
$\Rightarrow V(\frac{1}{R_{L_1}} + \frac{1}{R_{L_0}} + \frac{1}{R_{L_0}}) = \frac{500}{R_{L_1}} - \frac{1000}{R_{L_2}}$
$\Rightarrow V(\frac{1}{2.487} + \frac{2}{1.989}) = \frac{500}{2.487} - \frac{1000}{1.9889}$
$\Rightarrow V(1.4076) \approx -301.72$
$\Rightarrow V \approx -214.351 AT/m$
$\therefore \phi_{centrallimb} = \frac{V}{R_{L_0}} = \frac{214.351}{1.989\times10^5} \approx 1.078 mWb$

(iii)$L_1 = \frac{N^2}{R_{net}} \quad R_{net}(for L_1) = R_{L_1} + \frac{R_{L_1}\times R_{L_0}}{R_{L_2}+R_{L_0}} = 3.482\times10^3 AT/Wb$
$\therefore L_1 = 0.72H$
$L_2 = \frac{N^2}{R_{net}} \quad R_{net}(for L_2) = R_{L_2} + \frac{R_{L_1}\times R_{L_0}}{R_{L_1}+R_{L_0}} = 3.094\times10^3 AT/Wb$
Flux due 500At on other coil
$= \frac{R_{L_0}}{R_{L_1}+R_{L_0}} \times \frac{500}{R_{net}(for L_1)} = 0.718H$

Q.1.b) A ring of magnetic material with relative permeability 3000 has rectangular cross section of $5cm^2$ .The inner and outer diameters of the ring 20 cm and 25 cm respectively, gap of 1 mm length is cut across the ring . The ringis wound with 500 turns and a sinusoidal alternating current of 50Hz.
(i) What will be the value of the current when the magnitude of the induced voltage in the coil is 60V?
(ii) Find the maximum energy stored in the magnetic material and the air gap for a coil current of 2A at 50Hz.

1.b)



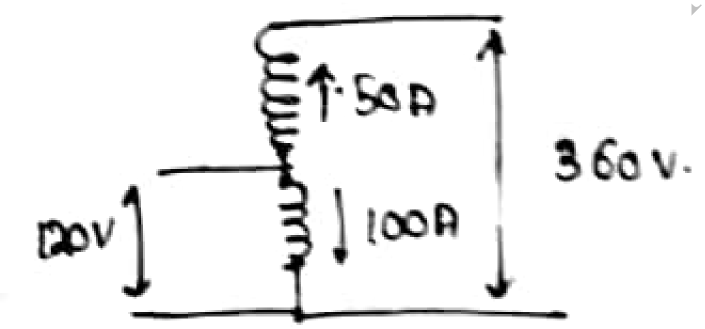
$\mu_r = 3000$
$A = 5cm^2$
$N = 500$
Mean diameter = 22.5cm
Mean length $= \pi d = 70.686cm$
Reluctance $\frac{l}{\mu_r\mu_0 A} = \frac{10^{-2}\times70.596}{3000\times4\pi\ times5\times10^{-7}\times10^{-4}} + \frac{10^{-3}}{4\pi\ times5\times10^{-4}\times10^{-7}} = 19.66\times10^5 At/Wb$

$L = \frac{N^2}{R} = 0.1272H$
let the current be $I_0 sin\omega t$ and induced voltage be $E_0 sin(\omega t + \phi)$
$\therefore L\frac{di}{dt} = E$
Both averaged over time , we can write
$\therefore \omega LI = E$
$\Rightarrow I = 60/(\pi 2 \times 50 \times 0.1272) = 1.5A$

(ii)Total energy $=\frac{1}{2}LI^2 = \frac{1}{2} \times 0.1272 \times 8 = 0.515J$
$I_{max} = 2\sqrt{2}A$
$LI = N\phi$
$\Rightarrow \phi_{max} = \frac{0.1272\times\sqrt{8}}{500} = 0.72mWb$
Air gap energy $=\frac{B_m^2}{2\mu_0} \times Ax = \frac{x\phi_{max}^2}{2\mu_0 A} = \frac{10^{-3}\times(7.2\times10^{-4})^2}{2\times4\pi\ times5\times10^{-7}\times10^{-4}} = 0.4133W$
$\therefore$ Energy in magnetic material =(0.51 - 0.413)J = 0.097J

Q.2.a) Consider a conventional two winding transformer of 240V/120V, 12kVA with full load efficiency % for a unity power factor load .It is reconnected as an auto transformer to provide power to a load at 360V from a 120 V supply.
(i) Show the winding connection diagram of the auto transformer .[2]
(ii) Calculate the kVA rating of the auto transformer[4]
Assuming negligible core loss , calculate the full load efficiency of the auto transformer for a 0.8(lagging) power factor load.[4]

ANSWER:
2.a) (i) $\eta = 96.2\%$
$P_{app} = 12kVA$



$I_H = \frac{12\times1000}{240} = 50A$
$I_G = \frac{12\times1000}{120} = 100A$
$\therefore kVA$ rating = $(150\ \times 120)kVA = 18000kVA$
Also $\eta = \frac{nScos\theta}{nScos\theta + x^2 P_{cu}}$
Put x=1 , $cos\theta = 1, S = 12kW$
$\therefore P_{cu} = 474W$
$\therefore$ For Auto-transformer $,\eta = \frac{18\times0.8}{18\times0.8+0.474} = 96.8\%$

Q2b. The following data were obtained for a 20 kVA, 50 Hz, 2000V/200V distribution transformer.

Open-circuit (OC) test with high voltage (HV) side open circuited			Short-circuit (SC) test with low voltage (LV) side short circuited		
V_{OC}	I_{OC}	P_{OC}	V_{SC}	I_{SC}	P_{SC}
200 V	4 A	120 W	60 V	10 A	300 W

Find the parameters of the approximate equivalent circuit of the transformer referred to the LV side and draw the corresponding circuit diagram. [8+2]

2.b) OC test (LV side)

$R_c = 333.33\Omega = V_{oc}^2/P_{oc}$

$R_c = V_{oc}/R_c = 0.6A$

$\therefore I_L = \sqrt{I_{net}^2 - I_R^2} = 3.955A$

$\therefore X_L = V_{oc}/I_L = 50.572\Omega$

SC test (HV Test)

$I_{sc}^2 \times R_c = 300$

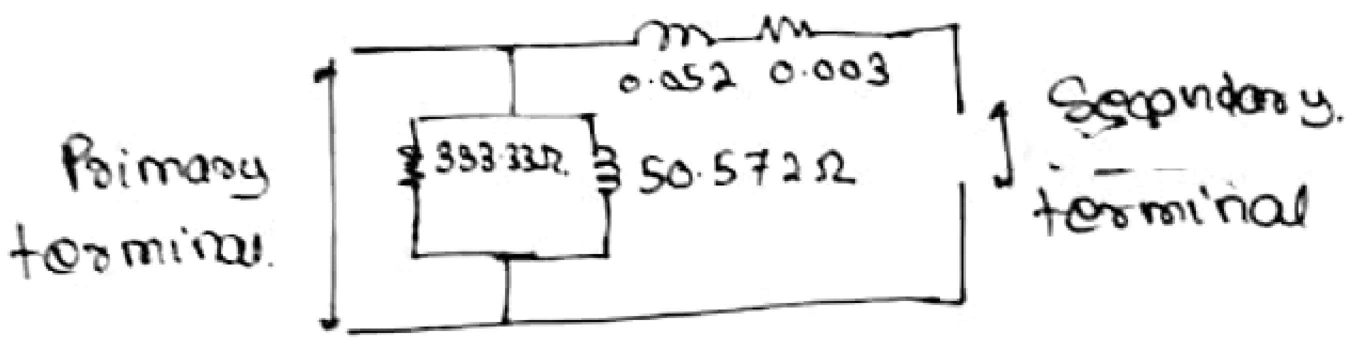
$\Rightarrow R_c = 3\Omega$

$|z| = V_{sc}/I_{sc}$

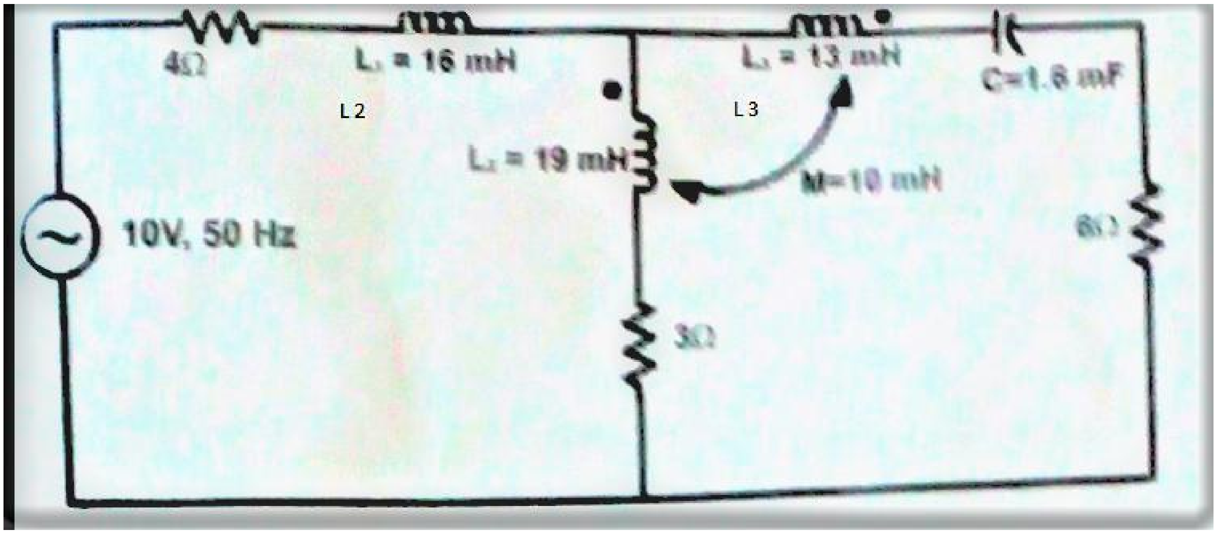
$X = \sqrt{|z|^2 - R_c^2} = 3\sqrt{3}\Omega$

SC parameters referred to LV side(a=10), R = 0.003Ω, $X = 0.052\Omega$

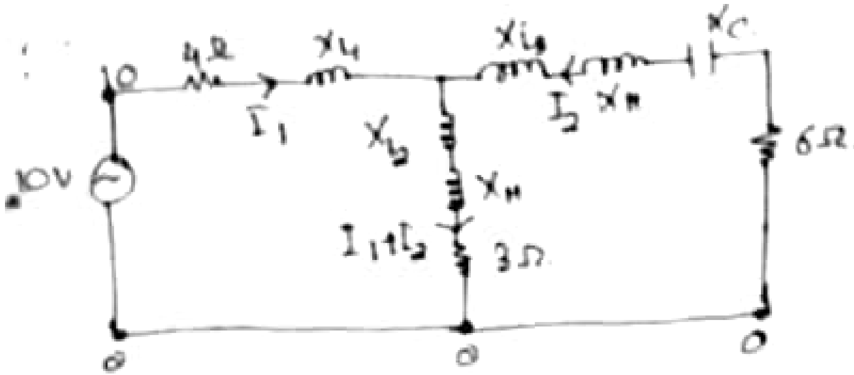
$\therefore$ Equivalent circuit



Q.3.a) The inductors L_2 and L_3 are coupled with mutual inductance of Mas shown in the circuit. Calculate the voltage across L_2 for a supply of 10 V, 50 Hz [10]



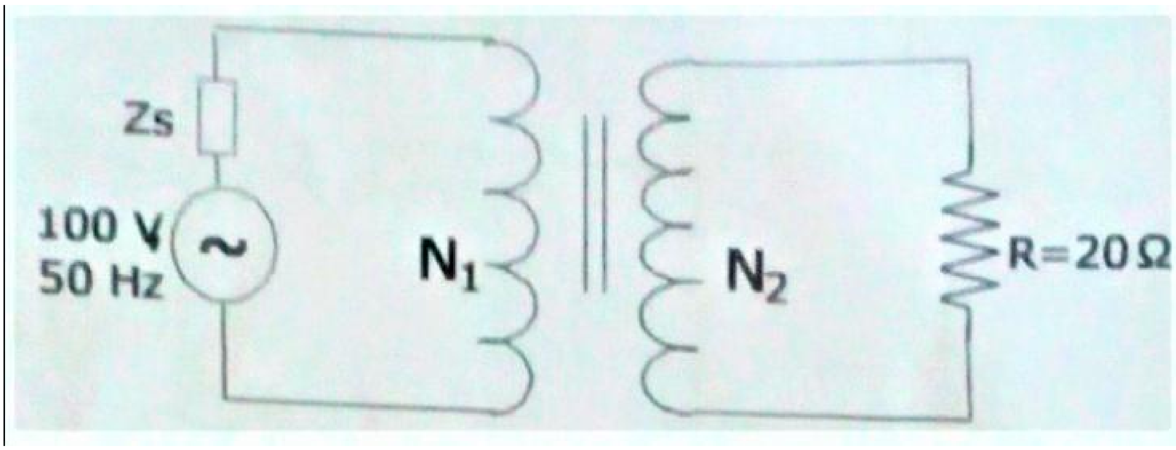
3.a)



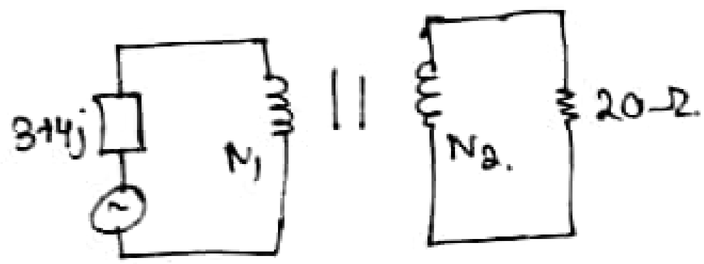
$X_{L_1} = 5.03\Omega \quad X_H = 3.14\Omega$
$X_{L_2} = 4.08\Omega \quad X_c = \frac{1}{100\pi \times 1.6 \times 10^{-3}} = 2\Omega$
$X_{L_3} = 5.97\Omega$
Mesh 1 :
$10 = (4 + 5.03j)I_1 + (I_1 + I_2)(3 + 4.03j + 3.14j)$
or $10 = I_1(7 + 12.25j) + I_2(3 + 7.22j)$
Mesh 2:
$(I_1 + I_2)(3 + 7.22j) + I_2(5.97j + 3.14j - 2j + 6) = 0$
or $I_1(9 + 14.33j) + I_2(6 + 7.11j) = 0$
$\therefore I_1 = 2.44 + 1.08j$
$I_2 = -4.12 - 2.56j$
$\therefore V_{L_2} = (I_1 + I_2)(X_{L_2} + X_H)$
$= (-1.68 - 1.48j)(7.22j)$
$= 10.69 - 12.13j \approx 16.17\angle - 48.62°$
$i = \frac{100}{\sqrt{8^2+4^2}} = 11.2$
$\therefore$ Real power $= i^2R = 1KW$
reactive power $= i^2X = (11.2)^2(4) = 502W$
Current through R = I/a= 22.4A

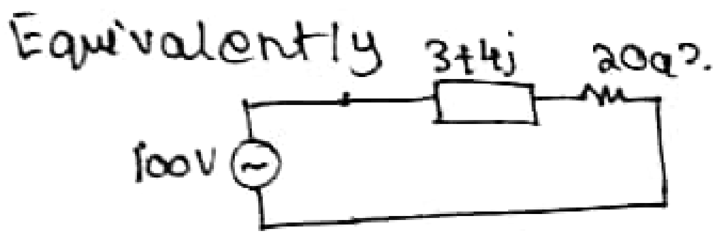
Q.3.b) For the circuit shown below the AC source has an internal impedance of Z= 3+ 4j Ω . If the transformer is ideal and has N = 10 turns .calculate.

(i) N for which maximum power transferred to the resistive load R [4]
(ii)the real and reactive power delivered by the source for the condition as in (i) above [4]
(iii)current through load R for the condition as in (i) above[2]



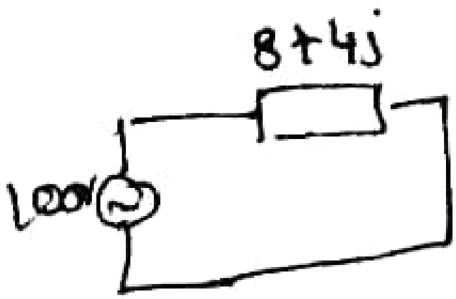
3.b)





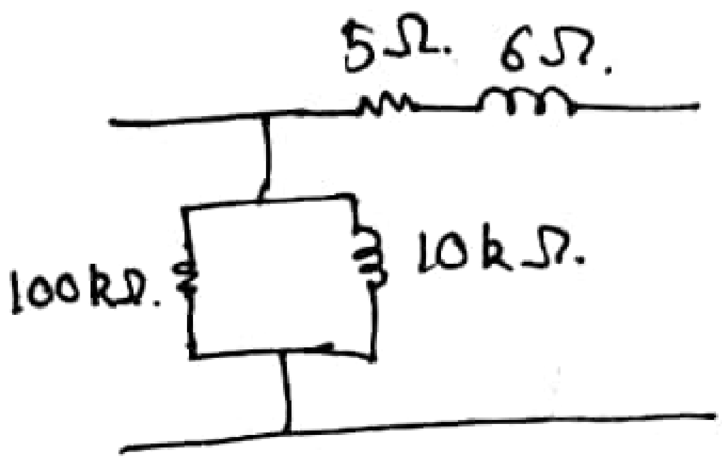
$N_1 = 10$
$i = \frac{100}{[(3+20a^2)^2+4^2]^{1/2}}$ Power across R= $\frac{(10^4)(20a^2)}{(3+20a^2)^2+16}$
$P_R = \frac{10^4 \times 20}{(\frac{3}{a}+20a)^2+\frac{16}{a^2}}$
P_R is maximum when a =0.5

$\therefore a = N_1/N_2$
$\Rightarrow N_2 = 20$



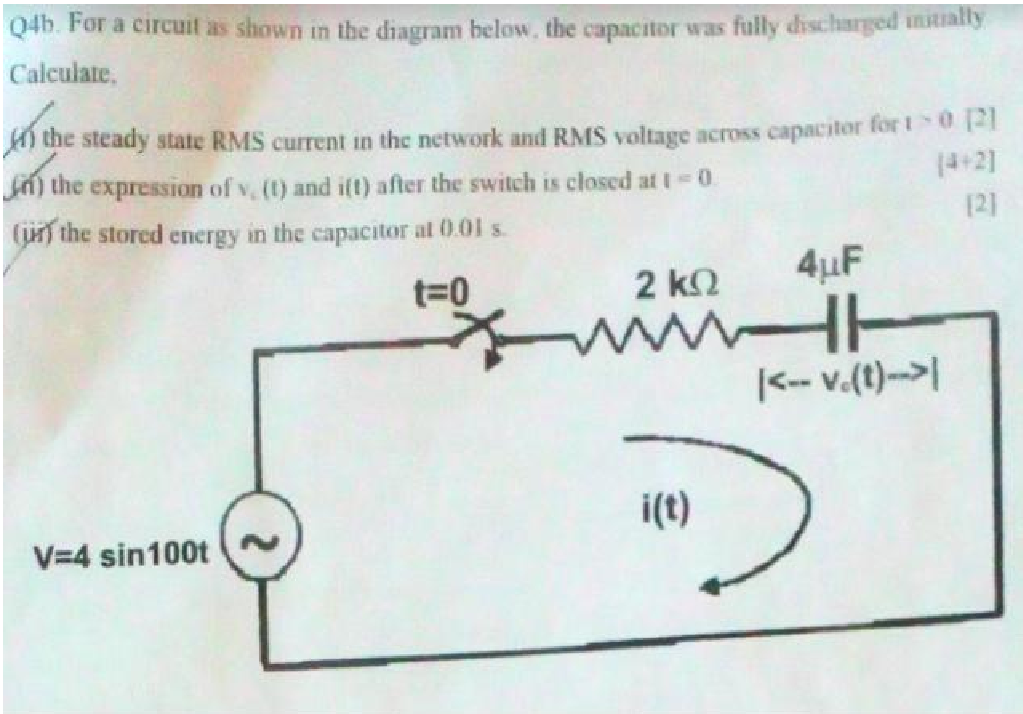
Q.4.a) Consider a 15kVA , 2300 V/230 v transformer having parameteras the high voltage side as core loss component R = 100 kΩmagnetizing reactance X = 10kΩ ,equivalent leakage reactance $X_{eq} = 6\Omega$ and equivalent resistance $R_{eq} = 5\Omega$. The load is connected to the low voltage side
(i) Calculate the voltage regulation at full load at 0.25 lagging power factor [4]
(ii)For a unity power factor load,calculate the load current at which the efficiency of the transformer is maximum.Calculate the maximum efficiency [4+2].

4.a)



I = $\frac{15\times1000}{2300} = 6.522A$
$cos\theta = 0.85 \;\; sin\theta = 0.5265$
$\therefore$ Voltage regulation $= I(Rcos\theta + Xsin\theta)$
$= 6.522(5 \times 0.85 + 6 \times 0.5265)$
$= 48.33V$

$\therefore$ Voltage regulation % $= \frac{48.33V}{2300} \times 100\% \approx 2.1\%$
(ii) Core loss $= \frac{V^2}{R} = \frac{(2300)^2}{100\times10^3} = 52.9W$
Efficiency is maximum when $P_i = P_c$
$I_L^2 \times (5) = 52.9W$
or $I_L = 3.253A$
$\therefore$ Efficiency is maximum at $I_L = 3.253A$


Q4b. For a circuit as shown in the diagram below, the capacitor was fully discharged initially

Calculate,

(i) the steady state RMS current in the network and RMS voltage across capacitor for t > 0 [2]

(ii) the expression of v_c (t) and i(t) after the switch is closed at t = 0 [4+2]

(iii) the stored energy in the capacitor at 0.01 s. [2]

4.b) (i) $X_L = 2000 \;\; R = 2000\Omega \;\; \omega = 100$
$X_c = \frac{1}{100 \times 4 \times 10^{-6}} = 2500$
Steady state RMS current
$= \frac{2\sqrt{2}}{\sqrt{(2000)^2 + (2500)^2}} = 0.883mH$
$\therefore$ rms Voltage $= I_{rms} X_c = 2.21V$
$RC = 2 \times 10^3 \times 4 \times 10^{-6} = 8 \times 10^{-3}s$

$(ii) \frac{dq}{dT} + \frac{q}{RC} = \frac{V_0}{R} sin\omega t$
$\therefore (D + \frac{1}{RC})q = \frac{V_0}{R} sin\omega t$
$\Rightarrow q = \frac{V_0}{R} \frac{sin\omega t}{D + \frac{1}{RC}}$
$= \frac{V_0}{R} \frac{(D - \frac{1}{RC})}{(D^2 - \frac{1}{R^2C^2})} sin\omega t$ [You must know the way of solving differential equation]
$= \frac{V_0}{R} \frac{\frac{1}{RC} sin\omega t - \omega cos\omega t}{\omega^2 + \frac{1}{R^2C^2}} + ke^{-t/RC}$
$q(0) = 0$
and V= q/C
$\therefore \vartheta(t) = \frac{V_0}{R} \frac{\frac{1}{RC} sin\omega t - \omega cos\omega t}{\omega^2 + \frac{1}{R^2C^2}} + \frac{V_0 \omega e^{-t/RC}}{RC(\omega^2 + \frac{1}{R^2C^2})}$

$= 500[\frac{125 sin100t - 100 cos100t}{25625}] + \frac{50000}{25625} e^{-t/0.008}$

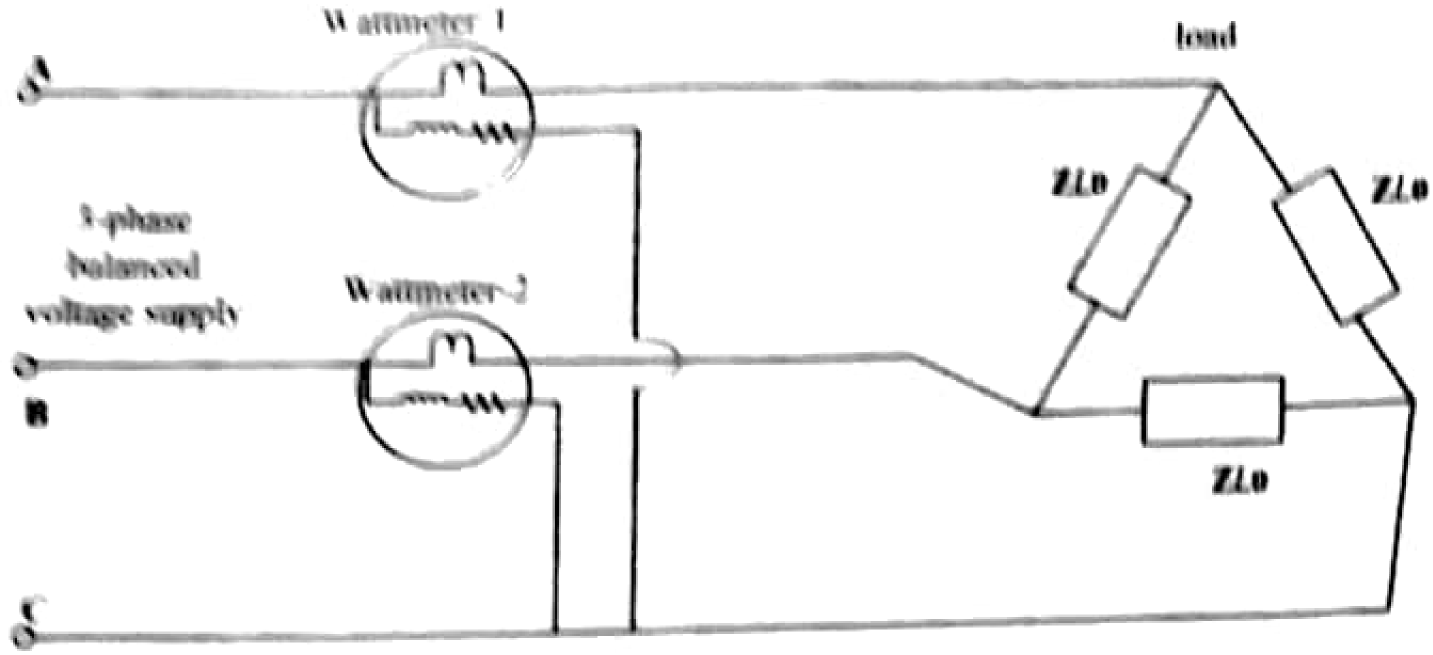
$= 2.44 sin100t - 1.95 cos100t + 1.95 e^{-125t}$

$i(t) = e\dot{\vartheta}(t) = 97.56 cos100t + 78 sin100t - 97.5e3^{-125t}$

$\therefore$ Energy $= \frac{1}{2} e (V(0.01))^2 = 4.85\mu J$

Q.5. In the circuit diagram below arrangement is made for three phase power measurement by 2 watt-meter method
(I) Derive and show that the sum of the two watt-meter readings is equal to three phase power delivered to the load ,consider phase A voltage as reference and ABC as phase sequence .

(II) with $Z\angle\theta = 20\angle 45°\Omega$ balanced voltage supply of 100 V ,50 Hz and ABC as phase sequence , obtain the 2 watt-meter readings for the connection as in the diagram and the total reactive power drawn form the source.



5. Power measured by wattmeter

$= (V_{AC})(I_A^*) + V_{BC}I_B^*$

$V_A = V_0/\sqrt{2}\angle 0° \quad V_c = V_0/\sqrt{2}\angle -240°$

$V_B = V_0/\sqrt{2}\angle -120° \quad V_0/\sqrt{2} = 100V$

$V_{AB} = \sqrt{3}V_0/\sqrt{2}\angle 30°$

$V_{BC} = \sqrt{3}V_0/\sqrt{2}\angle -90°$

$V_{CA} = \sqrt{3}V_0/\sqrt{2}\angle -210° \quad V_{AC} = \frac{-\sqrt{3}V_0}{\sqrt{2}}$

converting to equivalent star impedence $Z_{eq} = Z/3\angle\theta$

$I_A = \frac{3V_0}{\sqrt{3}Z}\angle -\theta - 120°$

$I_c = \frac{3V_0}{\sqrt{3}Z}\angle -\theta - 240°$

$I_B = \frac{3V_0}{\sqrt{3}Z}\angle -\theta - 120°$

$\omega_1 = Re(V_{AC}I_A^*) = Re(-100\angle -210°)(\frac{3\times 100}{\sqrt{3}\times 20}\angle 45°)$

$= 836.5W$

$Re(V_{BC}I_B^*) = Re(-100\angle -90°)(\frac{3\times 100}{\sqrt{3}\times 20}\angle 120° + 45°)$

$= 224.14W$

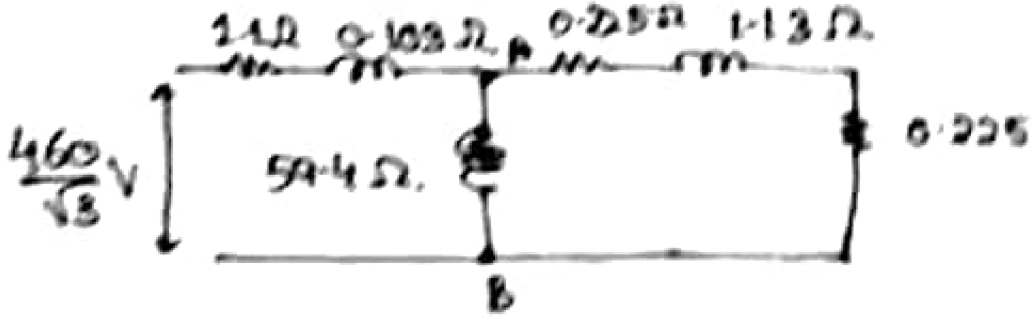
Also total reactive power $=3I_{reac}^2|X|$

$= 3(\frac{3\times 100}{\sqrt{3}\times 20})^2 \times \frac{20}{3\sqrt{2}} = 1060.66VAR$

Q. 6.a)

A 4-pole induction motor has the rotor parameters as R=.225Ω and X=1.13Ω and stator parameters as R=1.1Ω and X=.103Ω and core reactance as 59.4Ω with a full load operating slip of 2% . It is connected to a 3-ph 460V line supply of 60Hz. Find the mechanical power and torque developed(Neglect friction losses). Also find its efficiency.

6.a)



$(\frac{1}{S} - 1) = 11.025$

$\eta_s = \frac{120\times60}{6} = 1800rpm \quad \omega = \frac{1800}{60}\times 2\pi = 60\pi$

$s = 0.02$

$R_{Th} = \frac{(59.4j)(1.1+0.103j)}{(1.1+59.503j)} = 1.096 + 0.123j$

$V_{Th} = \frac{460}{\sqrt{3}} \times \frac{59.4j}{1.1+59.503j} \approx 265.076V$

$\therefore P_G = \frac{3V_{Th}^2}{(R_{Th}+\frac{r_2}{s})^2+(X_{Th}+x_2)^2} \times \frac{r_2}{s}$

$= \frac{3\times(265.076)^2\times(0.225/0.02)}{(1.096+11.25)^2+(0.123+1.13)^2} \approx 15.4KW$

$\therefore$ Torque $= \frac{(1-s)P_G}{\omega_r} = \frac{P_G}{\omega s} = 15399.68 = 81.7Nm$

Rotor loss $= 3I^2R = sP_a \approx 308W$

Mechanical Power $= P_G -$ Rotor loss
$= 15399.68 - 308)W \approx 15.1KW$

$Z_{net} = 1.1 + 0.103j + \frac{(59.4j)(11.25+1.13j)}{(11.25+60.53)}$

$= 11.572 + 3.158j = 11.995\angle 15.265°$

$\therefore P_{input} = \frac{3V_{Ph}^2}{R_{net}} = 3 \times \frac{(460)^2}{3\times11.572} \approx 18.286KW$

$\therefore \%efficiency \approx (15.1/18.286) \approx 82.6\%$

Q.6.b)A three phase squirrel cage induction machine operating at rated voltage and frequency has a starting torque of 135% and a maximum torque of 220% both with respect to its rated load torque . Neglecting the effects of stator resistance , core losses and rotational losses and assuming a fixed rotor resistance , determine the slip at maximum torque,with $0 \leq S_{max} \leq 1$ [8]

6.b) $T = \frac{2T_{max}}{\frac{S}{S_{Tmax}} + \frac{S_{Tmax}}{S}}$
$\therefore 1.35T_r = \frac{2 \times 2.2 \times T_r}{\frac{1}{S_{Tmax}} + \frac{S_m}{1}}$
or $S_{Tmax} = 2.9, 0.034$
$0 \leq S_{Tmax} < 1$
$\therefore S_{Tmax} = 0.34 = 34\%$

1. **(a)** Find the value of V_{TH} (Thevenin's equivalent voltage) and R_{TH} (Thevenin's equivalent resistance) between the open terminals A and B shown in Figure 1. [8]

(b) In the circuit, shown in Figure 2, determine the wattmeter reading. Assume ideal coils of the wattmeter. [12]

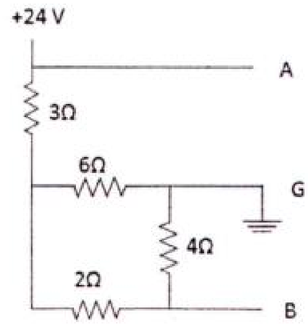


Figure 1: Pertaining to Q1(a)

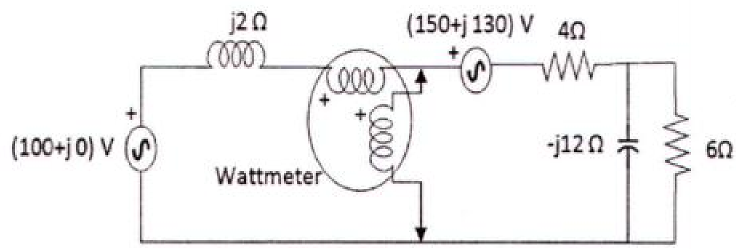
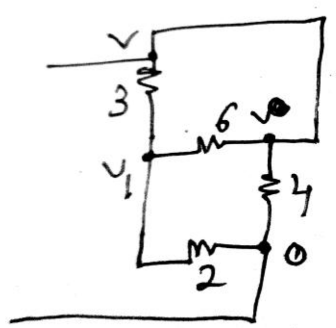
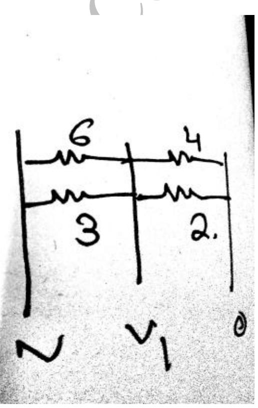


Figure 2: Pertaining to Q1(b)

ANSWER:

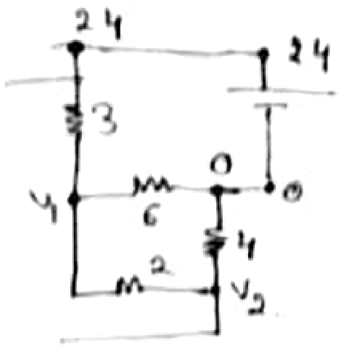
1.a)For R_{Th}

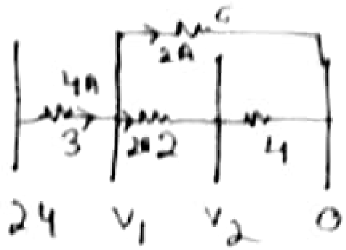




$$\therefore R_{th} = \frac{6\times3}{9} + \frac{4\times2}{6}$$
$$= 2 + \frac{4}{3} = \frac{10}{3} = 3.33\Omega$$

For V_{Th}





$R_{net} = 3 + \frac{6\times 6}{12} = 6$
$\therefore I = 4A$
$\therefore V_1 = 24 - 12 = 12$
$V_2 = V_1 - 2\times 2 = 12 - 4 = 8$
$\therefore V_{Th} = 24 - V_2 = 24 - 8 = 16V$

b)$I = \frac{100-(150+130j)}{2j+\frac{6(-12j)}{6-12j}} = \frac{-50-130j}{4.8-2.4j+2j} = 28.92\angle -106.27^\circ$
Voltage by wattmeter
$= 100 - (28.92\angle -106.27^\circ)(2\angle 90^\circ) \approx 47.34\angle 20.02^\circ$
$\therefore$ Wattmeter readings $= \text{Re}((47.34\angle 20.02^\circ)(28.92\angle -106.27)$
$= -810W$

2. **(a)** A cast steel magnetic structure made of a bar of section $2\,\text{cm} \times 2\,\text{cm}$ is shown in Figure 3. Determine the current that the 500 turn magnetizing coil on the left limb (AB) should carry so that a flux of 2 mWb is produced in the right limb (CD). Take μ_r = 600 and neglect leakage. [10]

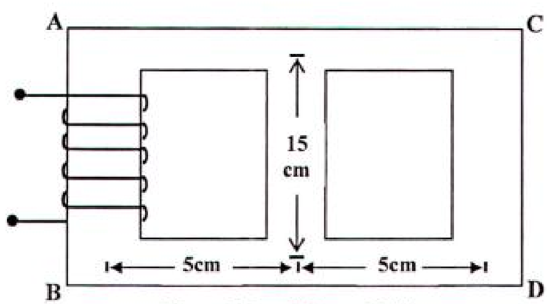
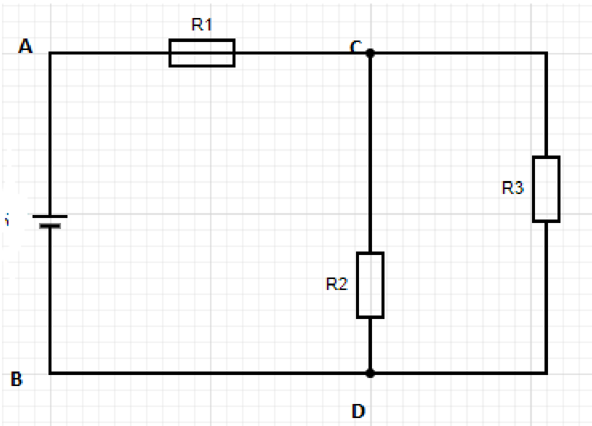


Figure 3: Pertaining to Q2(a)

(b) A 440 V, 3-ϕ, Δ-connected induction motor takes a total input power of 17.55 kW at a power factor of 0.82 (lag). Calculate the readings of two wattmeters connected to measure the input.
If another Y-connected load of 10 kW at 0.85 power factor (lead) is added in parallel to the induction motor, what will be the magnitude of the total current drawn and the total power taken from the line?

[4+6]

2.a)



$A = 4 \times 10^{-4} m^2$
$\phi = 2 \times 10^{-3}$
$\therefore R_1 = \frac{(5+5+15)\times10^{-2}}{\mu_r \mu_A}$
$R_2 = \frac{15^{-2}}{\mu_r \mu_A}, R_3 = \frac{(5+5+10)\times10^{-2}}{\mu_r \mu_A}$
$\therefore V_{CD} = R_3 \times \phi$
$= \frac{20\times10^{-2}}{\mu_r \mu_A} \times 2 \times 10^{-3}$
$\therefore \phi_{R_2} R_2 = \phi_{R_3} R_3$
Or $\frac{15\times10^{-2}}{\mu_r \mu_A} \times \phi_{R_2} = \frac{20\times10^{-2}}{\mu_r \mu_A} \times 2$
$\Rightarrow \phi_{R_2} = \frac{20}{15} \times 2 = 8/3$
$\therefore \phi_{R_1} = \phi_{R_2} + \phi_{R_3} = (2 + \frac{8}{3}) \times 10^{-3} = \frac{14}{3} \times 10^{-3}$
$\therefore V_{AB} = \phi_{R_1} \times R_1 + R_3 \times \phi_{R_3}$
$= \frac{14}{3} \times 10^{-3} \times \frac{20\times10^{-2}}{\mu_r \mu_A} + 2 \times 10^{-3} \times \frac{20\times10^{-2}}{\mu_r \mu_A}$
$= (\frac{400}{3} \times 10^{-5}) \times \frac{1}{4\times10^{-4}\times4\pi\times10^{-7}\times600}$
$I = \frac{1}{500}(\frac{4\times10^{-5}}{288\pi\times10^{-11}} = 8.842A$

b) Power = (17.55 + 12.25j)
Other Power = 10 -6.19j
$\therefore$ Total Power = 27.55 -6.058j
$\therefore \sqrt{3} V_L I_L cos\phi =$ Total power
or $\sqrt{3} \times 440 \times I_L \times cos(tan^{-1}(\frac{6.053}{27.55}) = 27.55 \times 10^3$
or $\sqrt{3} \times 440 \times I_L \times 0.9767 = 24.55 \times 10^3$
or $I_L = 38A$

3. (a) A 400/100 V, 5 kVA, single-phase two winding transformer is used as an auto-transformer to supply 300 V from a 100 V supply.
(i) Find the input and output current ratings of the auto-transformer.
(ii) The kVA rating of the auto-transformer and the transformed kVA.

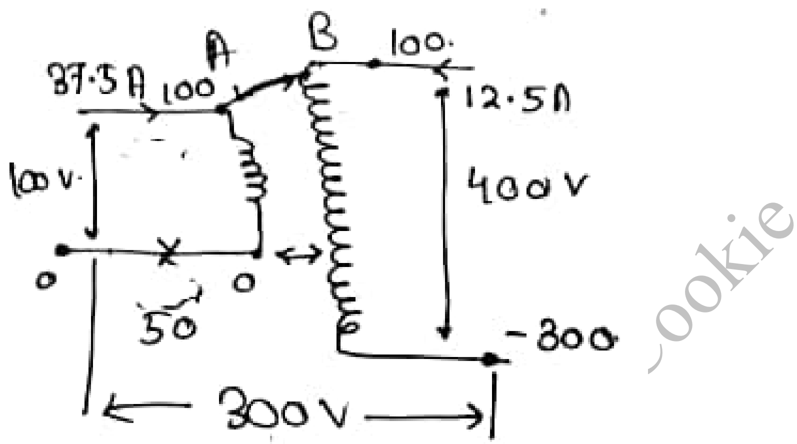
(iii) Find the currents in the two winding segments of the auto-transformer when operated at rated condition. Show the direction of the currents by drawing a connection diagram. [2+3+3]

3. (b) The nameplate of a single-phase transformer reads: 10MVA, 50Hz, 25kV/10kV, Z_{pu} = 0.12. During short-circuit test on the high voltage (HV) side of this transformer (with low voltage (LV) side shorted), the ammeter and wattmeter readings were 200A and 50kW respectively. During open circuit test on the LV side (HV side open) at rated frequency, the voltmeter and wattmeter readings were 10kV and 180kW.

(i) Find the values of total leakage reactance (x_{eq}) and winding resistance (r_{eq}) referred to HV side.

(ii) While supplying a load, the terminal voltages of the HV side and LV side of this transformer are found to be 25kV∠0° and 10kV∠6° respectively. Find the active power delivered by the transformer and the corresponding power factor. Also find the regulation and efficiency of the transformer under this operating condition. [2+2+2+2+4]

ANSWER: 3.a) $I_1 = \frac{5000}{100} = 50$



∴ (i) Total Input current= (50 - 12.5) = 37.5A Output current = 12.5A
$(ii) kVA$ rating $= 37.5 \times 100 = 3750V = 3.75kVA$
∴ kVA transformed = (5 -3.75)kVA = 1.25kVA
Current in primary coil = 50 A
current in the 2nd coil = 12.5A
(iii) Shown above. 3.b) Please consult your Professor.

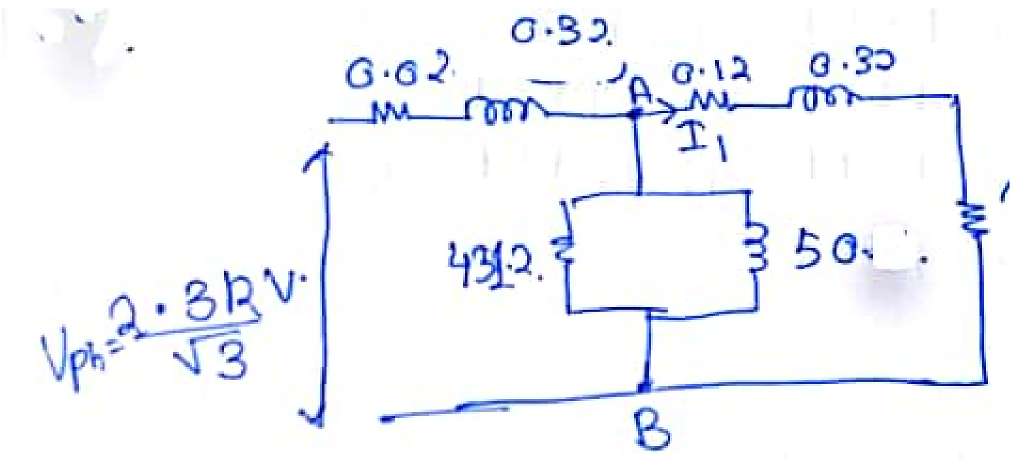
4. (a) A three phase induction motor has efficiency of 0.88, when load is 60 kW. At this load, per phase stator copper loss and rotor copper loss each equals the iron loss per phase. The mechanical loss is one fourth of no load loss. Calculate the slip. [8]

(b) A 2000 HP, 2300V, three phase, Y-connected, 4-pole, 50 Hz induction motor has the following specifications referred to the stator: $R_1 = 0.02\ \Omega$; $R_2 = 0.12\ \Omega$; $R_c = 451.2\ \Omega$; $X_\phi = 50\ \Omega$; $X_1 = X_2 = 0.32\ \Omega$. Find the full-load efficiency of the motor if the full-load slip is 3.5%. Neglect the mechanical losses. [12]

4.a) $\eta = 0.88$
$load = 60kW$
$P_{total} = 68.182kW$ $\quad P_G$ = air gap power
stator loss + iron loss =($P_{total} - P_G$)
rotor loss = sP_G
iron loss = sP_G
no load loss = core loss (iron loss) +stator loss

$\therefore$ mechanical loss $= \frac{1}{4}(P_{total} - P_G) = P_{mech}$
$\therefore P_{load} + P_{mech} + P_{rotor} = P_G$
$\Rightarrow 60kW + \frac{1}{4}(68.182 - P_G) + sP_G = P_G$
Also iron loss $= sP_G =$ stator loss
$\therefore P_{total} - P_G = 2sP_G$

$\Rightarrow P_G(2s+1) = P_{total}$
$\therefore 60kW + \frac{68.182}{4}kW - \frac{P_G}{4} = (1-s)P_G$
$\Rightarrow 77.0455 = P_G(1 - s + (1/4))$
$77.0455 = \frac{68.182}{2s+1}(\frac{5}{4} - s)$
$\Rightarrow 1.13(2s+1) = 1.25 - s$
$\Rightarrow 2.26s + 1.13 = 1.25 - s$
$\Rightarrow 3.26s = 0.12$
$\therefore s \approx 0.0368 = 3.68 percent$
4. b)



load $=0.12(\frac{1}{s} - 1) \approx 3.31$
$s = 0.035$
$P = 2000HP(output) = 1470998W$
$\therefore$ Net Impedence =(0.02 = 0.32j) + 451.2 $\|50j\|$(3.43 +0.32j) = 3.367 + 0.859j = $3.475\angle 14.32°$
$\therefore$ current $= \frac{2300}{\sqrt{3}\times 3.475\angle 14.32°} \approx \frac{662}{\sqrt{3}}\angle -14.32° = 382.2\angle -14.32°$
$\therefore$ Input power $= \sqrt{3} \times V_L \times I_L \times cos\phi$
$= \sqrt{3} \times 2300 \times 382.2 \times cos(14.32°) \approx 1475292W$

$V_{AB} = 2.3/\sqrt{3}\angle 0^0 - (382.2\angle -14.32°)(0.342\angle 69.44°) = 1257.74\angle -4.891°$
$\therefore I_1 = V_{AB}/(3.43 + 0.32j) = 365.10\angle -10.221°$
$\therefore P_{output} = 3 \times (365.1)^2 \times 3.31 = 1240320$
$\therefore$ efficiency $=\frac{1240320}{1470998} \approx 84.3 percent$

8AM to 1PM :65kW,45kVAR,79.1kVA ,$\eta = 0.791$
1PM to 6PM :80kW,50kVAR,94.34kVA ,$\eta = 0.9434$
6PM to 1AM :30kW,80kVAR,42.43kVA ,$\eta = 0.4243$
1AM to 8AM : No Load,$\eta = 0$
$\therefore$
$efficiency = \frac{65\times5+80\times5+30\times7}{65\times5+80\times5+30\times7+(0.791^2\times5+0.934^2\times5+0.4243^2\times7)\times1.2+0.37}$
$= \frac{935}{935+10.5+0.37} \approx 98.85 percent$

b) (i) $\eta_s = \frac{120\times f}{P} = \frac{120\times 50}{2k} = \frac{3000}{k}$

5. (a) The daily variation of load of a 100 kVA transformer is as follows:

8 AM to 1 PM: 65 kW, 45 kVAR
1 PM to 6 PM: 80 kW, 50 kVAR
6 PM to 1 AM: 30 kW, 30 kVAR
1 AM to 8 AM: No load

This transformer has no-load core-loss of 370 W and a full load ohmic loss of 1200 W. Determine the all-day efficiency of transformer. [10]

(b) (i) A 50 Hz induction motor is driving its rated load at 725 rpm. Find the number of poles and the slip of the motor? What is the frequency of the rotor current? [2+1+2]

(ii) A 3000V, 24-pole, 50 Hz, 3-ph, Y-connected induction motor has a slip-ring rotor of resistance 0.016 Ω and stand-still reactance of 0.265 Ω per phase. Full-load torque is obtained at a speed of 247 rpm. Neglecting stator impedance, calculate the ratio of maximum to full-load torque. [5]

Now P = 2k, where $k \epsilon N$
Nearest no. to 725 rpm $\approx$ 4
$\therefore \eta_s = 750$
$\therefore s = \frac{750-725}{750} \approx 3.33\%$
$\therefore$ No. of poles P = 8
frequency of rotor current = relative frequency = s(50Hz)
$= \frac{25}{750} \times 50 = 1.67Hz$
(ii) $\eta_s = \frac{120 \times f}{P} = \frac{120 \times 50}{24} = 250rpm$
$\because$ There is no stator resistance and core reactance,
$P_G = P_{supply}$
$s = \frac{250-247}{250} = 0.012$
$\therefore$ For $S_{T_{max}}, S = r_2/x_2 = \frac{0.016}{0.265} = 0.0604$
$\therefore T_c = \frac{2T_{cmax}}{\frac{s}{s_{max}} + \frac{s_{max}}{s}}$
$\therefore \frac{T_c}{T_{max}} = \frac{2}{\frac{0.012}{0.0604} + \frac{0.0604}{0.012}}$
or $\frac{T_{max}}{T_c} = \frac{5.232}{2} = 2.616$

6. (a) A DC series motor is rated 230 V, 12 hp, and 1200 rpm. It is connected to a 230 V supply and it draws a current of 40 A while rotating at 1200 rpm. The armature and series field winding resistances are 0.25 Ω and 0.1 Ω, respectively. Determine (i) the power and torque developed by the motor, and (ii) the speed, torque, and power if the motor draws 20 amperes. [4+6]

(b) A 230V DC shunt motor has an armature-circuit resistance of 0.23 ohms. When operating from a 230V supply and driving a constant-torque load, the motor is observed to be drawing an armature current of 60 A. An external resistance of 1.0 ohm is now inserted in series with the armature while the shunt field current is unchanged. Neglecting the effects of rotational losses, brush drop and armature reaction, calculate (i) the resultant armature current and (ii) the fractional speed change of the motor. [5+5]

NOT IN SYLLABUS

ELECTRICAL TECHNOLOGY

END - AUTUMN SEMESTER EXAMINATION**2015-2016**

1. (a) A series $R-L-C$ circuit consists of $R = 100\,\Omega$, $C = 15.91\,\mu F$ and a variable inductor (L). A 1000 V (r.m.s.), 50 Hz source is connected across the series combination and only the inductance L is varied to produce resonance in the circuit. (i) Find the voltage drop across inductor at resonance. (ii) Also find the value of L and the voltage drop across inductor when the drop across inductor is maximum. (iii) Sketch *the voltage drop across capacitor versus* L and *the voltage drop across inductor versus* L on the same plot. [3 + 8 + 4 = 15]

   (b) Find the value of I and the power absorbed by 60 V voltage source shown in Figure 1. [5]

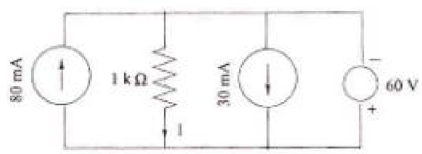


Figure 1: Pertaining to Q1(b).

ANSWER:

1.a) $\omega = 2\pi(50) = 100\pi$

(1) At resonance ,$\omega L = \frac{1}{\omega C}$

$\Rightarrow L = \frac{1}{\omega^2 C} = \frac{10^6}{(100\pi)^2 \times 15.91} = 0.63684H$

$I = \frac{V}{R} = 10A\angle 0°$

$\therefore V_L = (10\angle 0°)(\omega L)\angle 90°$

$= (10\angle 0°)(0.63684 \times 100\pi)\angle 90°$

$= 2000.7\angle 90° \approx 2kV\angle 90°$

(ii) At resonance ,maximum current flows.

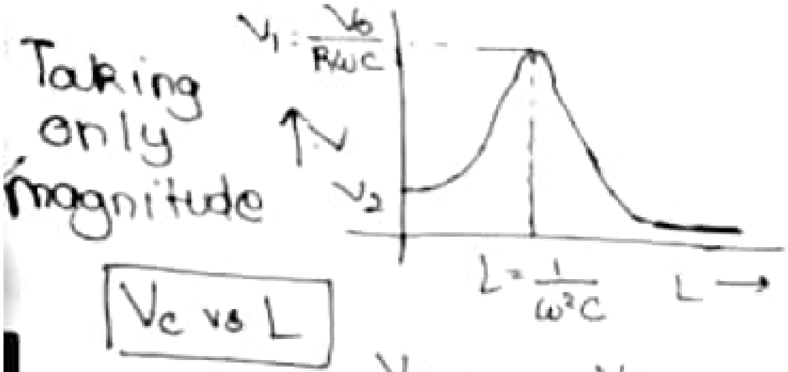
$\therefore$ Voltage V_L is maximum

$V_c = \frac{V_0}{\sqrt{R^2+(\omega L-\frac{1}{\omega C})^2}} \times \frac{1}{\omega C} \qquad V_L = \frac{V_0 \omega L}{\sqrt{R^2+(\omega L-\frac{1}{\omega C})^2}}$

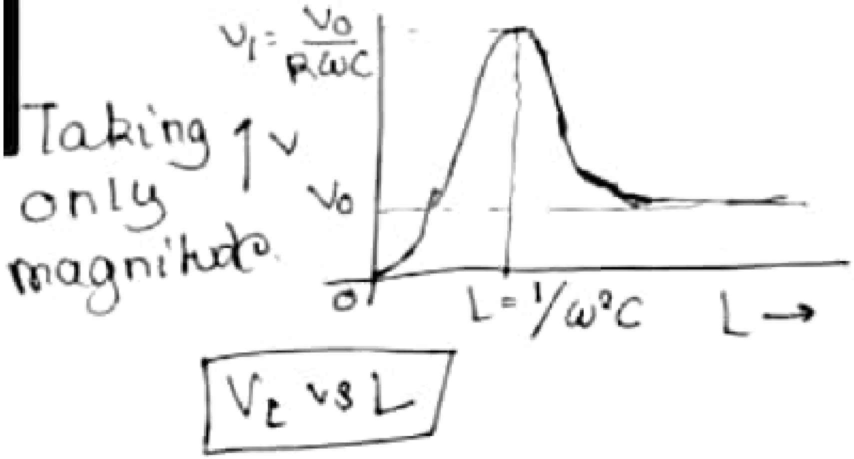
For $V_L = \frac{V_0 \omega}{\sqrt{R^2+(\omega L-\frac{1}{\omega C})^2}}$

$L \longrightarrow \alpha$

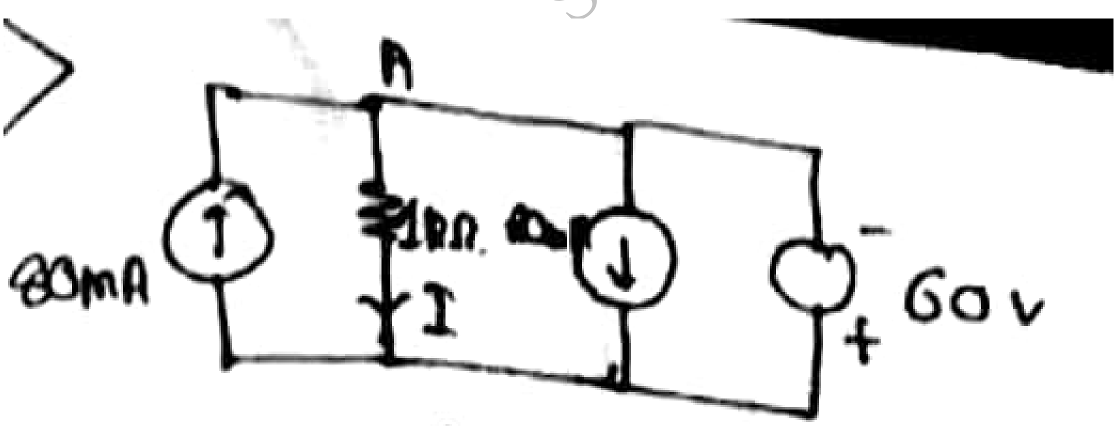
$\therefore V_\alpha = \frac{V_0 \omega}{\sqrt{\omega^2}} = V_0$



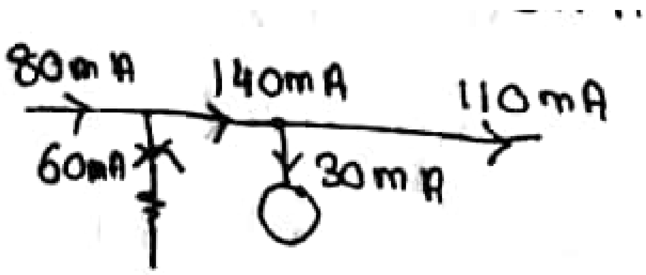
$$V_2 = \frac{V_0}{\sqrt{(\omega RC)^2+1}}$$



b) I (1000) 60



Or $I = \frac{60}{1000} = 0.006A = 60mA \quad I = -60mA \therefore$ At point A
$\therefore$ Power absorbed $= 60 \times 110 \times 10^{-3}W = 6.6W$



2. The following test results are obtained from *open circuit* (O.C.) test carried out from LV side and *short circuit* (S.C.) test carried out from HV side, on a 25 kVA, 440 V/220 V, 50 Hz single phase transformer:

O.C. test:	220 V	9.6 A	710 W
S.C. Test:	42 V	56.8 A	1030 W

(a) Draw the approximate equivalent circuits (i) referred to HV side and (ii) referred to LV side and insert all the parameter values. [4 + 4 = 8]

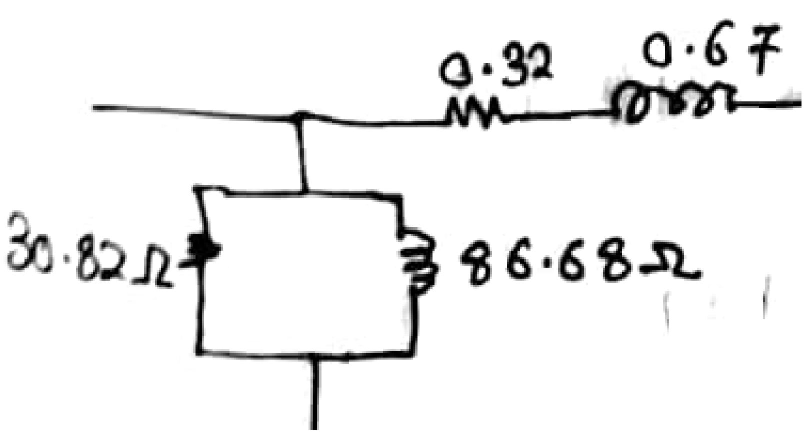
(b) If the HV side voltage is held fixed at 440 V, estimate the voltage across the LV side load which draws rated current at 0.8 p.f lagging. [4]

(c) What should be the applied voltage on the HV side, if the voltage across the LV side load is (same load as in part (b)) to be maintained at 220 V? [4]

(d) At what load power factor and kVA the transformer should be operated for maximum efficiency and what is its value? [4]

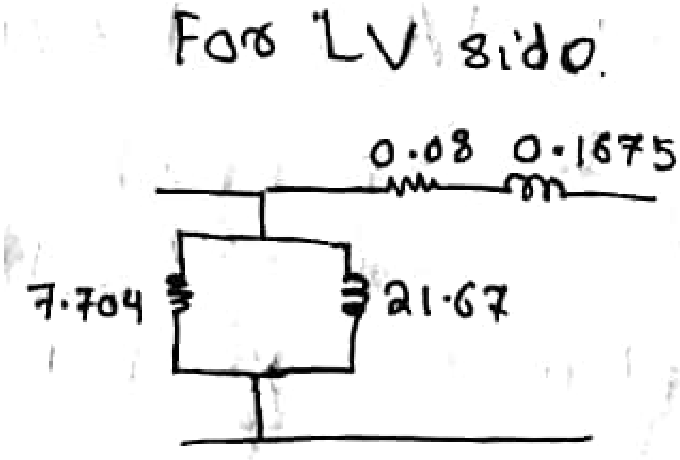
ANSWER:

For OC test a) $I^2R = W$
or $R = 710/(9.6)^2$
$= 7.704\Omega$
$Z = V/I \approx 23\Omega$
$X = 21.67\Omega$
$\therefore$ Referred to HV side
R= 7.704×4
$= 30.82\Omega$
and $X = 21.67 \times 4$
$= 86.68\Omega$
from HV side :

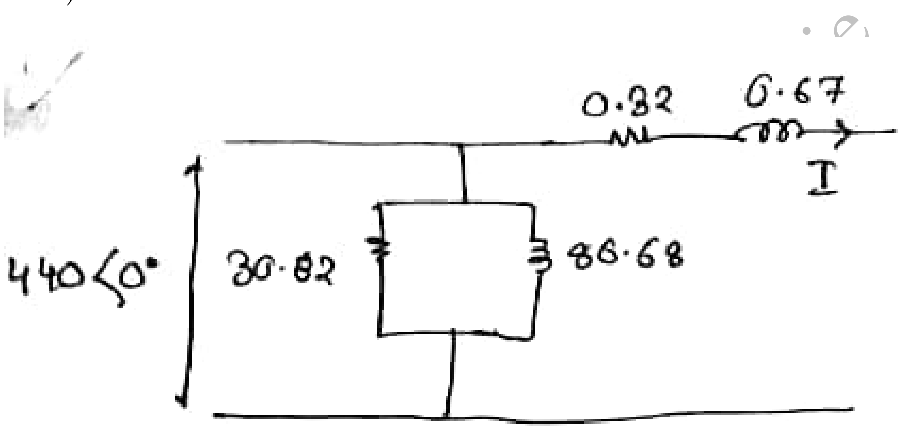


For SC test
$1030 = (56.8)^2 \times R$
or $R = 0.32\Omega$

$Z = 42/56.8 \approx 0.74\Omega$
$\therefore X = 0.67\Omega$
$\therefore$ Referred to LV
$R = 0.32/4 = 0.08\Omega$
$X = 0.67/4 = 0.1675\Omega$
For LV side



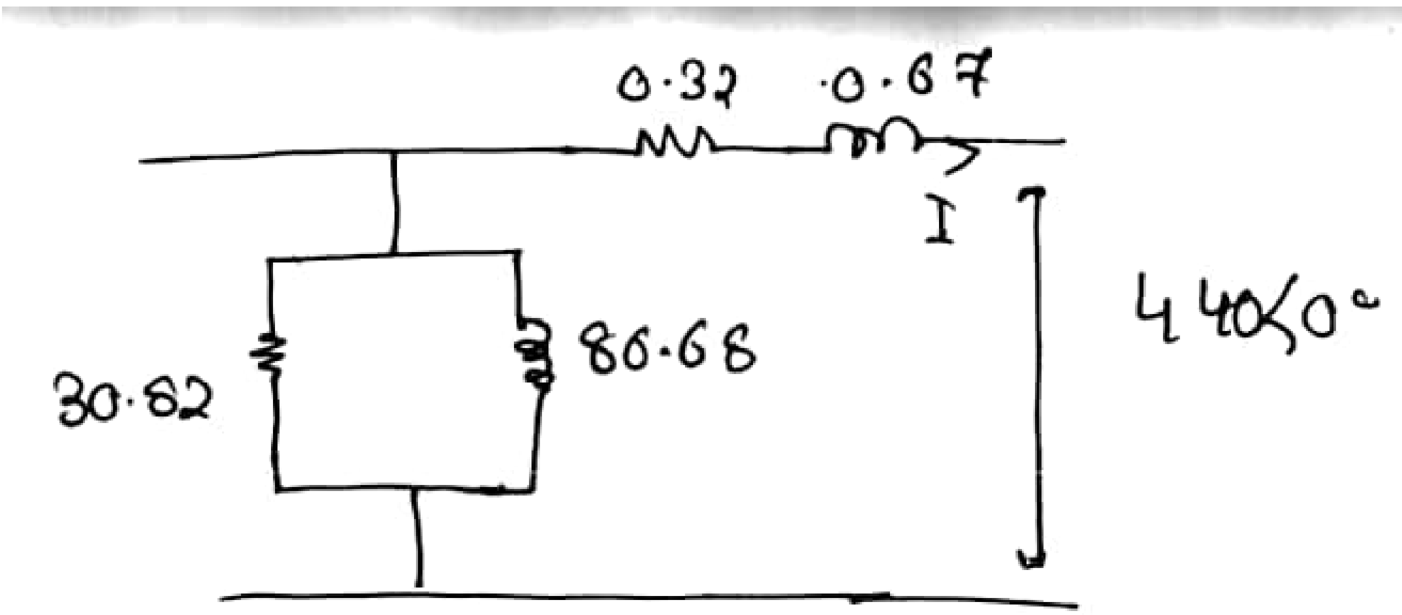
b)



$I = \frac{25 \times 10^3}{440} \approx 56.82$
$\phi = cos^{-1}(0.8) \approx 37°$
$\therefore I = 56.82\angle - 37°$

$V_{LV} = 440\angle 0° - (56.82\angle - 37°)(0.32 + 0.67j)$
$\approx 403.04\angle - 2.77°$
$\therefore V_{LV} = 403.04/4 \approx 100.76V$

c) [Every sum is referred to the HV side]



$V_{LV} = 440\angle 0° + (56.82\angle -37°)(0.32 + 0.67j)$
$\approx 447.83\angle 2.33°$
$\therefore V_{HV} = 477.83$
d) $\eta = \frac{xScos\theta}{xScos\theta + x^2 P_c + P_i}$
$maximum$ at $cos\theta = 1$
and $x = \sqrt{P_i/P_c} = \sqrt{710/1030} = 0.83$
$\therefore \eta = \frac{0.83\times 25}{0.83\times 25 + 2\times 0.71} \approx 93.6\%$
$kVA = 0.83 \times 25kVA = 20.75kVA$

4. (a) A 4-pole 50 Hz induction motor is driving a load at 1450 rpm.
   (i) What is the slip of the motor ?
   (ii) What is the frequency of the rotor current?
   (iii) What is the angular velocity of the stator field *with respect to the stator* and *with respect to the rotor*?
   (iv) What is the angular velocity of the rotor field *with respect to the rotor* and *with respect to the stator* ? [$4 \times 3 = 12$]

   (b) The rotor resistance and reactance per phase of a 4-pole, 50Hz, 3-phase induction motor are 0.025 Ω and 0.12 Ω respectively. Find the speed at maximum torque. Also find the additional rotor resistance per phase required to generate 75% of the maximum torque at starting. Neglect stator impedance. [$4 + 4 = 8$]

4.a) (i) $\eta_s = \frac{120\times f}{P} = \frac{120\times 50}{4} = 1500$
$\therefore s = \frac{1500-1450}{1500} = \frac{50}{1500} = \frac{1}{30} = 3.33\%$
(ii) frequency of the rotor current $= \frac{1}{30\times 1500} = 50$
(iii) velocity w.r.t stator $= \frac{1500}{60} \times 2\pi \approx 157rad/s[SW_s]$
(iv) velocity w.r.t rotor = 5.236 rad/s
velocity w.r.t stator = 157 rad/s
Both stator field and rotor field is the rotating magnetic field .

b) $n_s = \frac{120\times f}{p} = 1500rpm$
$ST_{max} = \frac{r_2}{x_2} = \frac{0.025}{0.12} = 0.208$

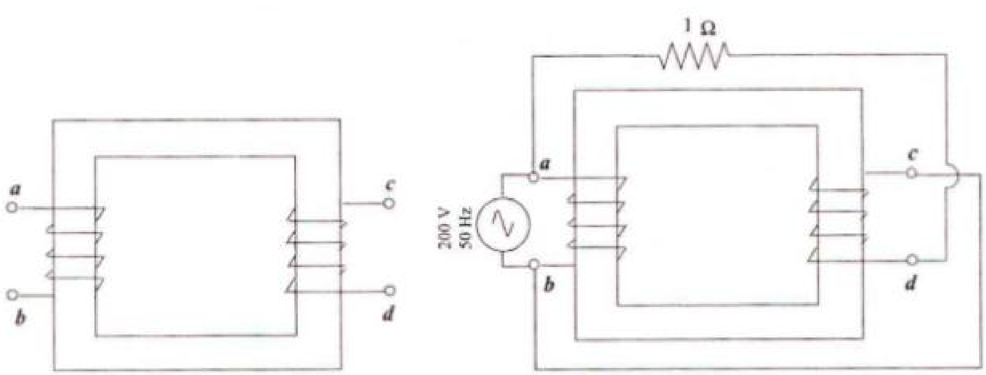
$\therefore n_r = (1-s)n_s = 1187.5rpm$
Neglecting stator resistance , and at start , s=1
$T_{max}(0.75) = T_{start}$
$\therefore T_c = \frac{2Tmax}{\frac{S}{S_{max}} + \frac{S_{max}}{S}}$
$\Rightarrow T_{start} = \frac{2T_{max}}{S_{max} + \frac{1}{S_{max}}}$
or $S_{max} + \frac{1}{S_{max}} = \frac{2}{0.75} = \frac{8}{3}$
or $2S_{max}^2 - 3S_{max} + 3 = 0$
$S_{max} = 2.22/0.45$
$\because S < 1$
$\therefore ST_{max} = 0.45$ or $r_2 = 0.45 \times 0.12 = 0.054\Omega$
$\therefore$ additional rotor resistance =0.029Ω

5. (a) The core and winding of a single phase, 2 kVA, 200 V / 200 V, 50 Hz transformer is shown in Figure 2.1. The terminals of the two windings are marked as **a**, **b**, **c** and **d**.

   (i) If terminal **a** is **dotted** then which of the other terminals (**c** or **d**) has to be **dotted**. [2]

   (ii) The value of the magnetizing reactance of the transformer is 500 Ω and the core (iron) loss under rated voltage and frequency is 20 W. The transformer is supplied from a 200 V, 50 Hz voltage source (between points **a** and **b**) as shown in Figure 2.2. The points **b** and **c** are shorted and the points **a** and **d** are connected through a 1 Ω resistance as shown in the figure. Neglecting the resistance and leakage reactance of the windings, draw the equivalent circuit of the transformer under this condition showing all connections and marking all four terminals. [4]

   (iii) Find the current flowing through the **1 Ω** resistance and the total current drawn from the source under the condition described in part(ii). [1 + 3 = 4]



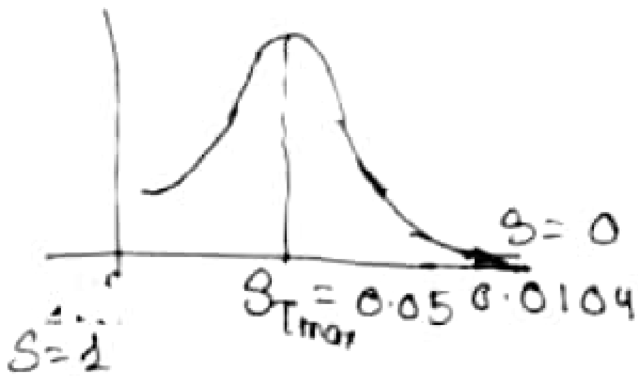
please consult your respective professor

(b) (i) A 3-phase, 50 Hz, squirrel-cage induction motor delivers full load at a slip of 4%. If, under this condition, the angular velocity of the rotor field with respect to the rotor structure is 3.1416 rad/s then find the number of poles of the machine. [2]

   (ii) When operated at rated voltage and frequency, a 3-ph squirrel cage induction motor delivers full load at a slip of 5% and develops a maximum torque of 250% of full load torque at a slip of 25%. Neglecting core loss and rotational losses and assuming that the resistances and inductances of the motor are constant, determine the starting torque (at rated voltage and frequency) of the motor expressed as percentage of the full load torque. [8]

(i) Given SW_s
$\therefore SW_s = 3.1416 = \pi$
or $W_s = 25\pi$

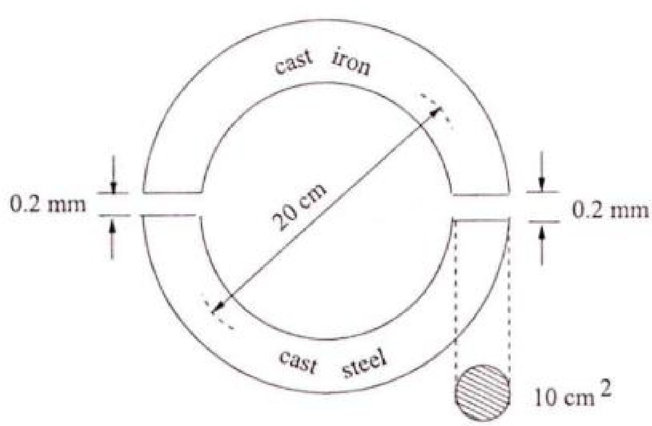
$\therefore \frac{rpm}{60} \times 2\pi = 25\pi$
or rpm $= 25 \times 30 = 750$
$\frac{120 \times f}{P} = 750$
$\Rightarrow P = \frac{120 \times 50}{750} = 8$
$\therefore 8 poles$ (ii) S=005
$T_{max} = 2.5 T_{operate}$
$T = \frac{2T_{max}}{\frac{S}{S_{T_{max}}} + \frac{S_{T_{max}}}{0.05}}$
or $\frac{0.05}{S_{max}} + \frac{S_{max}}{0.05} = 5$
or $(0.05)^2 + S_{max}^2 = 2 \times 0.05 \times S_{max}$
or $S_{max}^2 - 0.25 S_{max} + 0.0025 = 0$
or $S_{max} \approx 0.2396/0.0104$



Operating zone is on the right of $S_{T_{max}}$
$\therefore S_{T_{max}} = 0.0104$
$\therefore T_{start} = \frac{2 \times T_{max}}{\frac{1}{0.0104} + 0.0104}$ $\quad [s = 1]$
Or $\frac{T_{start}}{T_{max}} = 0.02079 \approx 2.08\%$

6. (a) A ring has a mean diameter of 20 cm and a cross sectional area of 10 cm^2. The ring is made up of semicircular sections of cast iron and cast steel with each joint having an air-gap of 0.2 mm as detailed in Figure 3. Find the ampere-turns required to produce a flux of 8×10^{-4} Wb. The relative permeabilities of cast steel and cast iron are 800 and 165 respectively. Neglect the effects of fringing and leakage.

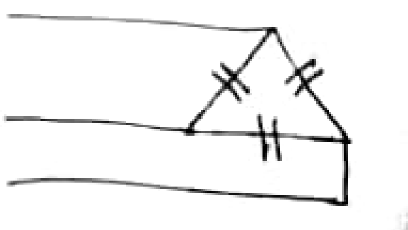
[10]



(b) A 3-phase, 37 kW, 440 V, 50 Hz induction motor operates on full load with an efficiency of 90% and at a power factor of 0.8 lagging. Calculate the total kVA rating of Δ connected capacitors required to raise the full load power factor to 0.9 lagging. What will be the value of the capacitance per phase? [6 + 4 = 10]

6.a) $l_{iron} = l_{steel} = \frac{\pi(20)}{2} - 0.02 = 31.396cm$
$\therefore R_{total} = \frac{l_{iron}}{\mu_{iron}\mu_0 A} + \frac{l_{steel}}{\mu_{iron}\mu_0 A} + \frac{l_{air}}{\mu_0 A}$
$\therefore NI = \frac{1}{\mu_0 A}(\frac{l_{iron}}{\mu_{iron}} + \frac{l_{steel}}{\mu_{steel}} + l_{air})\phi$
$= \frac{8\times10^{-4}}{4\pi\times10^{-7}\times10\times10^{-4}}(0.31396(\frac{1}{800} + \frac{1}{165}) + \frac{0.2\times10^{-3}\times2}{1})$
$= 1715.842$

b) Output power = 37kW
$\therefore$Input Power =41.11 kW
$\therefore \sqrt{V_L I_L cos\phi} = \frac{370}{9} \times 1000$
$\Rightarrow I_L = 67.43A$
$I_L = 67.43A\angle - 37°$



$I_c = \frac{440}{\sqrt{3}}\omega c\angle 90°$
$I_L + I_c\angle 90° = I_{net}\angle 25.842°$
$\Rightarrow 67.43\angle - 37° + I_c\angle 90° = I_{net}\angle 25.842°$
$\Rightarrow 53.852 - 40.58j + j(I_c) = I_{net}(0.8999 + j(0.43589))$
$\Rightarrow 53.852 + (I_c - 40.58) = I_{net}(0.8999 + j(0.43589))$
$\therefore \frac{I_c - 40.58}{53.852} = 0.4843$
$\Rightarrow I_c = 66.662$
$\therefore 66.662 = \frac{440}{\sqrt{3}} \times 2 \times \pi \times 50 \times c$
$\Rightarrow c = 0.8353mF$

Mid-sem 2017-2018
Spring Semester

MID-SEMESTER EXAMINATION

CS10001: Programming and Data Structure (Solutions)

Question 1: **[2.5+2.5+2=7]**

(a) Interpret **11011011** as a two's complement binary number, and give its decimal equivalent.
Answer:

First, note that the number is negative, since it starts with a 1.

Change the sign to get the magnitude of the number.

	1	1	0	1	1	0	1	1
¬	0	0	1	0	0	1	0	0
+								1
	0	0	1	0	0	1	0	1

Convert the magnitude to decimal: $00100101_2 = 25_{16} = 2\times16 + 5 = 37_{10}$.

Since the original number was negative, the final result is -37.

(b) Show how a computer would perform **10 + □3** using eight bit two's complement representation. Is there a carry? Is there an overflow?
Answer:

10 + -3 = 7:

1	1	1	1	1				
	0	0	0	0	1	0	1	0
+	1	1	1	1	1	1	0	1
	0	0	0	0	0	1	1	1

Carryout without overflow. Sum is correct.

(c) Convert the 8-bit signed two's complement hex number **0x3F** to decimal.

Answer: Easiest is to convert to binary: 0011 1111. Then convert to decimal: 32 + 16 + 8 + 4 + 2 + 1 = 63, which is the answer

Question 2: **[4+4=8]**

(a) Convert **□35.75** to IEEE 754 floating point format. What is the hex representation of the resultant bit pattern?

Answer: First, we need to write down the binary representation of 35.75: 100011.11
Next, we need to put normalize it: $1.0001111 * 2^5$
Now we can get the 3 pieces of the number we need:
S: 1 (negative)
E: 132 (127 + 5) = 1000 0100
M: 000 1111 0000 0000 0000 0000
Now we just put all the bits together: 1100 0010 0000 1111 0000 0000 0000 0000 and convert each group of 4 to a hex digit:
Answer: 0xC20F0000

(b) Convert the IEEE format floating point number **0x40200000** to decimal.

Answer: First, convert the hex to binary: 0100 0000 0010 0000 0000 0000 0000 0000
Then pull out each of the 3 pieces:
S: 0 (positive)
E: 1000 0000 = 128. Taking 128 - 127 = 1
M: 010 0000 0000 0000 0000 0000 So we have 1.01 * 2^1 = 10.1
Then convert that to decimal to get the answer: 2.5

Question 3:

What is the output of the following C Programs? **[3+3+3 = 9]**

(a)
```
#include <stdio.h>

int main()

{

  int a = 10, b = 20;

  int *p = &a, *q = &b;

  printf("%d ", (*p)++);

  printf("%d ", --(*q));

  printf("%d\n", a+b);

  return 0;

}
```
Answer: 10 19 30

Underlying Idea:(*checking whether you understand postfix prefix*):
1.*p is first printed and then incremented by 1 because the operator is postfix.
2. 2. q is first decremented then printed because the operator is prefix.
3. the updated values of a and b i.e. *p and *q are added which are 11 and 19 respectively.

(b)

```
#include<stdio.h>
int *fun(int p, int *q)
{
  p = 4;
  *q = 2;
  return (q);
}
int main()
{
  int x = 6, y = 9, z = 3;
  z = *fun(x, &y);
```

<u>*Note:*</u> * in the fuction signature implies the fuction will return a pointer.*]*

The function is returning 2 to variable z and also the value of y becomes 2 in the main function because q is a pointer to variable y

Key Concept: value of x is passed while address of y is passed so any changes made in the respective formal parameters(p and *q) will be reflected at y but not at x.*]*

```
    printf("%d %d %d\n", x, y, z);
    return 0;
  }
```

Answer: 6 2 2

Thus x remains unchanged i.e. 6 while z and y both become 2 as returned by the *fun function.

```
(c) #include<stdio.h>
  int main()
  {
    float arr[5] = {1.5, 2.5, 3.5, 4.5, 5.5};
    float *ptr1 = &arr[1];
    float *ptr2 = ptr1 + 3;
    float *ptr3 = ptr2 + 1;
    printf("%f ", *ptr2);
    printf("%ld ", ptr2 - ptr1);
    printf("%f\n", arr[ptr3 - ptr1]);
    return 0;
  }
```

ptr1 is a pointer to the 2nd element in the array i.e. *ptr1 will have the value 2.5

ptr2 is a pointer to the 5th element of the array i.e. *ptr2 will have the value 5.5

ptr3=ptr2+1*sizeof(float). Go through pointer operation to learn more.

*ptr2 will thus display 5.500000. (the extra 5 zeros are because float data type which in C displays 6 numbers after decimal point.

Ptr2-ptr1 is 3*sizeof(float) which will be scaled down to 3.

ptr3 contains address of ptr1+4*sizeof(float) therefore ptr3-ptr1 will result in 4*sizeof(float) which will be scaled down to 4. Read pointer operations to know more.

Answer: 5.500000 3 5.500000

Question 4:

A lazy programmer wants to write a program that will perform two activities together -- (a) reverse the elements of an integer array, and (b) sum the elements of the array. (S)he thought of writing an additional function to exchange the values of two variables. However, (s)he feels that it can be done without introducing any temporary variable within that function, exchange(int *p, int *q), which takes two pointers p and q containing the addresses/locations of two integer elements of an array and then it exchanges the values in those two locations. With this thought, (s)he started writing the program, but did not complete it. Help the lazy programmer to complete the program correctly. The part of the code is provided below. **[1.5 x 4 = 6]**

```
#include<stdio.h>
void exchange(int *p, int *q)
{
  *p = *p + *q;
  *q = *p - *q ;
```

This is a swapping function and since the addresses of the values to be swapped has been passed so the swapped values will reflect on the actual parameters as well.

```
  *p = *p - *q ;
```

Simple swapping without third variable logic has been used.

```
}
int main()
{
  // defining an array
  int a[] = {7,4,8,2,9,0,1,3,6,5};
  // marking the initial index of the array
  int i=0;
  // marking the final index of the array
  int f = sizeof(a)/sizeof(int) - 1;
```

f is storing the last index of the given array i.e. 9

```
  // initializing the sum
  int sum = 0;
  while ( i < f ) // converge from both ends of the array index
  {
```

Since the traversal is happening from both end the i<f marks that all the elements have been swapped.

```
    // updating the sum (adding two elements of the array at a time)
    sum = sum + a[i] + a[f] ;
```

adding elements from the initial end and the back end

```
    exchange( &a[i] , &a[f] ); // call to the exchange function
```

Swapping

```
    i = i + 1 ; // incrementing initial index
    f = f - 1 ; // decrementing final index
```

By incrementing i and decrementing f the array is being converged from both ends.

```
  }
  return 0;
}
```

Question 5:

The following program finds an approximate value of a definite integral. Let l be the left and r the right boundary for the integral. Also let h be the step size. The idea is to break the interval [l, r] into sub-intervals [l, l + h], [l + h, l + 2h], [l + 2h, l + 3h], . . . , [r – h, r]. Assume that r – l is an integral multiple of h. The program evaluates the function to be integrated at the center of each interval, multiplies these values by the width h and computes the sum of these products as the approximate integral. Suppose you are integrating the function x^2. Complete the following program. Each blank can have at most one statement. **[1 + 2 = 3]**

```
#include <stdio.h>
int main ()
```

```
{
  double l, r, h, x, s;
  printf("Enter left boundary : "); scanf("%lf", &l);
  printf("Enter right boundary : "); scanf("%lf", &r);
  printf("Enter step size : "); scanf("%lf", &h);
  s = 0;
  for (x = l; x < r ; x += h){
    s += h*(x + h/2)*(x + h/2) ;
  }
  printf("Integral = %lf\n", s);
  return 0;
}
```

for f(x)=x*x the functional value at the mid of the small regions is (x+h/2)*(x+h/2) and that multiplied by the width h and is totaled and stored in s. s+=h*(x+h/2)*(x+h/2) is equivalent to s=s+h*(x+h/2)*(x+h/2)

Question 6:

The following program attempts to find the roots of the polynomial $x^3 - 4x^2 - 4x + 16$. It uses the fact that the product of the roots is the negative of the constant term of the polynomial. For instance, $x^3 - 4x^2 - 4x + 16$ has roots, 2, −2, 4, and the product of the roots: $2 \times -2 \times 4 = -16$ (negative of the constant term 16). To find the roots, the program checks for each integer r if it is a factor of the constant term 16, and if so, whether replacing x by the integer r in the polynomial evaluates to zero. For instance, since 4 is a root, $4^3 - 4 \times 4^2 - 4 \times 4 + 16 = 0$.

Complete the following program. Each blank can have at most one statement. **[1 + 2 + 1 = 4]**

```
#include <stdio.h>
int main ()
{
  int r, nroot = 0;
  for (r=1; nroot<3; r++) {
    if (16 % r == 0) {
      if (r*r*r-4r*r-4*r+16==0) {
        printf("Root found : %d\n",r);
        nroot++ ;
      }
      if ( -r*r*r - 4r*r + 4*r + 16 == 0 ) {
```

nroot<3 is the limiting condition because there are 3 roots of the given equation.

if a root is found then the nroot is incremented by 1 .

Replacing -r in the function we must get 0 if -r is a root of the function

```
      printf("Root found : %d\n",-r);

     nroot++ ;
```

if a root is found then the nroot is incremented bv 1 .

```
    }

  }

 }

 return 0;

}
```

Question 7:

The following program randomly generates a sequence of integers between -8 and +91 and outputs the maximum and minimum values generated so far. It exits if a negative integer is generated. The program does not store the sequence generated (say, in an array), but updates the maximum and minimum values on the fly, as soon as a new entry is generated. Complete the following program. Each blank can have at most one statement. **[1×4=4]**

Hint: To generate a random number between 0 to n, you can use

rand()%(n+1)

So, for example, you can get a random number between 5 to 90 by -- rand()%86 + 5


```
#include <stdio.h>

#include <stdlib.h>

#include <time.h>

int main ()

{

  int a, i, max, min;

  i = 1;

while (1)
  {
  a = rand()%100 - 8 ;
```

use the logic in the example on how to generate random numbers between 5 and 90 given above. rand()%100 will generate random numbers between 0 and 99 and thus substracting 8 to it will generate random numbers between -8 and +91.

```
  //a stores a random number between -8 to +91

  printf("Iteration %d: new entry = %d, ", i, a);

  if ( a < 0 ) {
```

if the generated value is a negative number(a<0) then exiting.

```
   printf("...exiting...\n\n\n"); break;

    }
```

```
    if (i == 1) max = min = a ;
```

for the first iteration the value generated is assigned to both max and min for furthur comparisions.

```
    else {
      if (a > max) max = a;
```

if the new generated value is more than the current max then max is the new generated value i.e. max=a.

```
      if (a < min) min = a ;
```

if the new generated value is less than the current min then min is the new generated value i.e. min=a.

```
    }
    printf("max = %d, min = %d\n", max, min);
    ++i;
  }
  return 0;
}
```

Question 8:

What will be the outputs of the following programs? **[2 + 2 = 4]**

```
(a) #include<stdio.h>
    int main()
    {
      int x = 0, y = 10, z = 20;
      while (1) {
        x++;
        if (y > z) break;
        y += 4*x; z += 2*x;
      }
      printf("x = %d, y = %d, z = %d", x, y, z);
      return 0;
    }
```

iteration 1: x=1;y=14 and z=22
iteration 2: x=2;y=22 and z=26
iteration 3: x=3;y=34 and z=32
iteration 4: x=4 and since y>z so .breaks

Answer: x = 4, y = 34, z = 32

```
(b) #include<stdio.h>
int main()
{
  int s = 0;
  while (s++ < 10) {
```

first s will be compared with the logical statement s<10 and the s will be incremented. The highest number satisfying <10 is 9 but when s=9 after comparing with <10 s will be incremented and become 10.

```
    if ((s < 4) && (s < 9)) continue;
    printf("%d ", s);
  }
  return 0;
}
```

when (s<4) which implies s<9 as well the loop will interate without proceeding furthur.

therefore value of s>=4 upto the loop terminates will be printed. That is 4 5 6 7 8 9 10

Answer: 4 5 6 7 8 9 10

Question 9:

What will be the output of the following C programs? Only one of the 4 choices are correct. Circle only the correct choice. **[5x2 = 10]**

(a)
```
int fun(int *a,int n)
{
     n++;
    *a = *a + n;
    return *a;
}
int main()
{
    int n=10;
    n = fun(&n,++n);
    printf("%d",n);
    return 0;
}
```

The function fun returns *a and not a so its return type is int not int *. (don't get confused)

Value of n is incremented and is now 12

*a=11+12=23

this n in the formal parameter is a local variable to this function Therefore changes made to this n won't be reflected to n in main function and for that matter to *a.

n is incremented by 1 and address of n and value of n is passed to function fun

n receives the value of *a and is therefore 23

A. 20 B. 21 C. 22 D. 23

B.
```
#include <stdio.h>
int n = 17, r=35;
void my_disp_func( int r ) { printf("%d,%d
\n",n,r); } int main()
{
int n = 21, r=14;
```

here r is assigned 21.

here n is 17(the globally declared value) since there is no local variable with the name n.

GENERAL NOTE: If there is a global and a local variable with the same name then the value of the local variable will be considered while inside the function

```
    my_disp_func(n);
}
```

A. 17,21 B. 21,14 C. 17,35 D. 21,35

(c)
```
void h( int A[], int n )
{
  int i;
  for (i = 1; i < n; ++i)
    A[i] *= A[i-1];
}
int main ()
{
  int A[5] = {1,2,3,4,48};
  h(A,4);
  printf("%d", A[4]/A[3]);
  return 0;
}
```

Any changes made to A[] is reflected to A[] in main function as well because array is always called by reference type.

A[1]=A[1]*A[0]=2*1=2

A[2]=A[2]*A[1]=3*2=6

A[3]=A[3]*A[2]=4*6=24

the loop runs for i=1,2,3 since n=4.

A[4] is unchanged and is equal to 48 and the modified value of A[3]=24 as shown above so A[4]/A[3]=48/24=2

A. 16 B. 48 C. 2 D. 4

(d)
```
#include <stdio.h>
int f(int i)
{
  return i%2;
}
int main()
{
    int i=27;
    while(f(i))
    {
        printf("%d", i);
    }   i = i/2;
```

Will return 1 if i is odd else returns 0 if i is even.

As soon as i becomes a even number the function retuns 0 and the loop exits.

iteration 1: i=27
iteration 2: i=13
iteration 3: i=6 so I becomes even and therefore the function returns 0 and the loop exits.

notice no space is there after %d so 2713 will be printed and not 27 13

```
    return 0;

}
```

A. 27 B. 27 13 C. 2713 D. 1

(e)
```
#include <stdio.h>

void jumble(char A[], int size)
    {
        int i;
        for(i=0;i<size;i++)
            A[i] = A[size - i -1];
    }
int main()
{
    char A[] = "REMARKABLE";
    jumble(A, 10);
    printf("%s", A);
    return 0;
}
```

ideally to reverse the word completely i<size/2 but since i<size is given A[0]=A[9], A[1]=A[8], A[2]=A[7],A[3]=A[6] and A[4]=A[5] happens twice and so the second half of array is mirrored of the 1st half.

The first and the last element of the array will be swaped and since i doesnot stop at size/2 so after i=4 the array element will be mirrored.

A. REMARKABLE B. ELBAKRAMER C. ELBAKKABLE D. None of the above

Question 10:

A natural number N is called a 'perfect number' if the divisors of N that are less than N add to N. Example: 6 is a perfect number because the divisors of 6 that are less than 6 (, namely 1,2, and 3) add to 6. Given below is a program that prints all perfect numbers in a given range using functions. Fill in the blanks. **[1 x 5 = 5]**

```
/**
 * C program to print all perfect numbers in given range using function
 */
#include <stdio.h>


/* Function declarations */
int isPerfect(int num);
```

```c
void printPerfect(int start, int end);

int main()
{
    int start, end;

    /* Input lower and upper limit to print perfect numbers */
    printf("Enter lower limit to print perfect numbers: ");
    scanf("%d", &start);
    printf("Enter upper limit to print perfect numbers: ");
    scanf("%d", &end);

    printf("All perfect numbers between %d to %d are: \n", start, end);
    printPerfect(start, end);

    return 0;
}

/**
 * Check whether the given number is perfect or not.
 * Returns 1 if the number is perfect and 0 otherwise.
 */
int isPerfect(int num)
{
    int i, sum;
    /* Finds sum of all proper divisors */
    sum = 0;
    for(i=1; i<num; i++)
    {
        if( num%i == 0 )
        {
```

checking whether i is a factor of num

```c
                sum = sum + i ;
            }
        }
        if( sum == num )
            return 1;
        else
            return 0;
}


/**
 * Print all perfect numbers between given range start and end.
 */
void printPerfect(int start, int end)
{
    /* Iterates from start to end */
    while(start <= end)
    {
        if( isPerfect(start) )
        {
            printf("%d, ", start);
        }
        start++ ;
    }
}
```

if i is a factor then adding it to the sum

checking whether the sum of factors of num equals num . if they are found to be equal the then num is perfect and returns 1.

Checking whether start is perfect or not. if perfect then the function returns 1 and the if statement is executed i.e. the value of start is printed.

incrementing start.

MID-SEM 2017-2018
AUTUMN SEMESTER

Indian Institute of Technology Kharagpur
Programming and Data Structures (CS10001)
Autumn 2017-18: Mid-Semester Examination

Time: 2 Hours **Full Marks: 60**

INSTRUCTIONS

1. Answer ALL questions
2. Please write the answers either within the boxes provided or on the blank lines to be filled up. Any answer written elsewhere will not be evaluated.
3. You may use the last two blank pages for your rough works.

Q.1. Answer the following questions as directed.

a) What will get displayed when the following program is executed? [1]

```
#include <stdio.h>
int main() {
  int x = 2, y = 17, result = 5;
  result -= x/5 * 13 * y/3 * x;
  printf("result=%d\n", result);
  return 0;
}
```

result=5

Equivalent to result = result-(x/5 * 13 * y/3 * x). x=2 is an integer so when divided by 5 will result in 0.(integer data type truncates the decimal part). So the whole product becomes 0 and thus result=result-0. Thus ***result=5***

b) What will get displayed when the following program is executed? [1]

```
#include <stdio.h>
int main() {
  int x = -5, y = 10;
  if (x > y) x = 1;
  else if (y < 0) x = x * (-1);
  else x = 2 * x;
  printf("x=%d\n", x);
  return 0;
}
```

x=-10

c) What is the 8-bit two's complement representation of the decimal number -37? [2]

11011011

First write the 8bit binary equivalent of 37=00100101. Then compliment the entire binary equivalent which will then look like 11011010. Add 1 to this complimented number to get 2's compliment of -37=11011011.

d) What will get displayed when the following program is executed? [1]

```
#include <stdio.h>
int main() {
  char a = 'a';
  while ((a > 'a') && (a <= 'c')) a++;
  printf(“%c\n”, a);
  return 0;
}
```

a

a=’a’. so although a<=’c’ is true but a>’a’ is false. And since && operator is used so (a>’a’)&&(a<=’c’) results in false and the loop is thus not executed. So just the a is printed as it was i.e. **a.**
Note: *the a++ is a part of the loop. If a curly braces is not given then after a loop,only the first statement is considered a part of the loop by default.*

e) What will get displayed when the following program is executed? [2]

```
#include <stdio.h>
int main() {
  int sum = 1, index = 9;
  do {
          index = index - 1;
          sum = 2 * sum;
   } while (index > 9);
  printf(“sum=%d, index=%d\n”, sum, index);
  return 0;
}
```

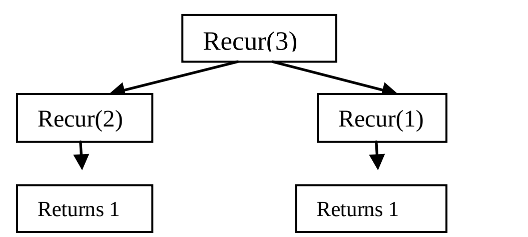
sum=2, index=8

It’s do while loop so at first 1 iteration is executed and then the condition is checked. Iteration1: index=8 and sum=2. Now index>9 is false so exits.

f) What value will the following function return when called as ***recur*(3)**? [2]

```
int recur(int data) {
  if (data > 2)
      return (recur(data - 1) - recur(data - 2));
  else return 1;
}
```

0

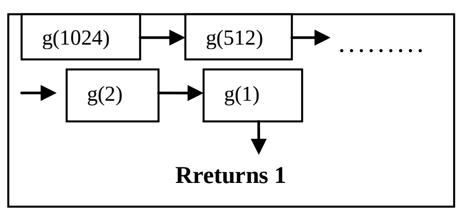


Thus 1-1=0

g) What value will the following function return when called as ***g*(1024)**? [1]

```
int g(int n) {
     if (n < 2) return n;
     return g(n/2);
}
```

1



h) What is the binary number corresponding to the hexadecimal number **C5.75?** [2]

11000101.01110101

Write 4 bit binary equivalent of each.
C(12)-1100;
5-0101;
7-0111; Write them in order as they are, replacing them by the 4 bit binary numbers respectively
11000101.01110101

i) Consider the program segment given below to read a letter from a..z and A..Z from the keyboard and convert it to uppercase if not already so. It is assumed that the user will only input a character from a..z and A..Z. Fill up the missing line with a single C expression so that the variable ***ch*** will contain the character in uppercase. Do not use any library functions. [2]

basic problem of ternary operator.

```
char ch;
ch = getchar();

ch = (ch>='A')&&(ch<='Z')?ch:'A'+ch-'a';
```

j) The following program segment is supposed to check whether the values stored by three integer variables ***a***, ***b***, and ***c*** are in ascending order. However, it contains an error. Encircle the part of the program that contains the error and write only that part corrected. [1]

```
if (a < b < c)
   printf("Numbers in ascending order \n");
else printf("Not in ascending order\n");
```

((a<b) && (b<c))

Q.2. Answer the following questions as directed.

a) What will get displayed when the following program is executed? [2]

```
#include <stdio.h>
int main() {
  int i;
  for (i = 1; i = -1; i++)
     if (i < 5) break;
  printf("%d\n", i);
  return 0;
}
```

-1

In the loop i is first initialized by 1 and then the control moves to the condition check area where instead of a condition i was assigned -1 .Then control enters the loop and the condition is checked and is satisfied i.e. -1 <5 so the loop breaks and then prints the current value of i i.e. **-1.**

b) What will get displayed when the following program is executed? [2]

The function returns nothing

```
#include <stdio.h>
void increment(int i) {
   i++;
}
int main() {
  int i = 0, j = 0;
  while (i++ < 10)  increment(j);
  printf("i=%d, j=%d\n", i, j);
  return 0;
}
```

i=11, j=0

TRICKY QUESTION!!
The variable i in the function increment is just given to confuse you. It's a local variable with scope just in the function increment. It will play no role to alter i in the main function. So when i will be equal to 10 it the loop will break but i will also get incremented by 1 as a postfix operator is used. So I will have value 10+1=11 and j remains same throughout the program i.e. 0.

i=11, j=0

c) What will get displayed when the following program is executed? [2]

```
#include <stdio.h>
int main() {
  float j = 1.0, i = 2.0;
  int n = 0;
  while (i/j > 0.05) {
     j = j + j;
     n++;
  }
  printf("n=%d, j=%f\n", n, j);
  return 0;
}
```

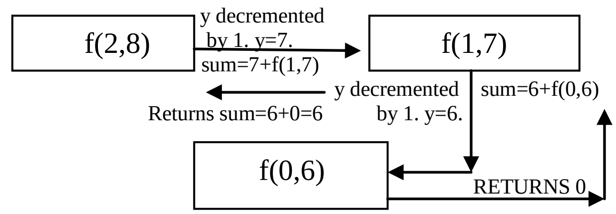
n=6, j=64.000000

When (i/j<0.05) then the loop will terminate. Therefore, $2/2^n<0.05$. This is true only for n=6(minimum value of n). Thus **n=6** and **j=2^6=64.000000**. The 6 zeros are a must for you to write or else marks won't be awarded. The six zeros after the decimal is because the data type of j is float.

d) What will get displayed when the following function is called as ***f*(2, 8)**? [2]

```
int f(int x, int y) {
    int sum = 0;
    y--;
    if (x == 0) return 0;
    else {
         printf("%d : ", x);
         sum = y + f(x - 1, y);
         printf ("%d : ", sum);
     }
     return sum;
}
```

2 : 1 : 6 : 13 :



Therefore sum=6+0=6 in the function f(1,7) and then it returns 6 to f(2,8) where sum =7+6=13..
At each step value of x is printed and at the end value of sum is printed all values are succeded by a : as given in the printf.

e) What will get displayed when the following program is executed? [2]

```
#include <stdio.h>
int main() {
  int sum = 0, i = 3;
  while (i < 100) {
     sum = sum + i;
     i = i + 3;
  }
  printf("sum=%d, i=%d\n", sum, i);
  return 0;
}
```

sum=1683, i=102

I will be a multiple of 3 and greatest value of i that will satisfy i<100 is 99. after that i=i+3=102 and the condition becomes false and loop exits. thus the value of i is **102**.The sum will be a A.P.
3+6+9+…..99
=3*(1+2+….+33)
=3*((33*34)/2)
=**1683**

Q.3.

a) A number is said to be *perfect* if it is the sum of all its factors (except itself). For example, 6 has factors 1, 2, 3 and 1+2+3 = 6, hence it is perfect. Also, 28 = 1+2+4+7+14 is perfect. In the following function ***checkPerfect*** fill up the missing lines so that it returns 1 if ***n*** is a perfect number and 0 if ***n*** is not a perfect number. [2 + 2]

```
int checkPerfect(int n) {
   int i, sum = 0;
   for (i = 1; i < n; i++) {
        if (n % i == 0)
            sum += i;
   }
   return (sum == n );
```

Checking whether i is a factor of n and if found true adding it to find the sum of the factors.

Checking if sum==n then the number is perfect and will result in 1 so the function will return 1 as asked else will return 0. Alternateive: n/sum is also a correct answer as sum of factors cannot exceed the number and will be equal to the number if it is perfect. so will return 1 if n is perfect else will return 0 as sum<n so n/sum will yield 0.

b) The following function ***strEqual*** takes two strings ***S1*** and ***S2*** as parameters. Fill up the missing lines in the function so that it returns 1 if the two strings are the same, 0 otherwise. [1 + 2 + 2]


```
int strEqual(char S1[], char S2[])
{
    int i = 0;

    /* Go on until the end of either of the strings */
    while (s1[i] != '\0' && s2[i] != '\0')
    {
        if (S1[i] != S2[i]) return 0;
        i++;
    }
    if (s1[i] == '\0'  && s2[i] =='\0')
        return 1;
    else
        return 0;
}
```

We have been asked to take two Strings as parameters. In C strings are array of characters.

Scanning through the strings letter by letter and checking whether they are same or not. If not immediately the strings are not same so returns 0.

If all the letters are same that does not mean the two strings are same. example S1[]="POST" and S2[]="POSTAL", in the loop both will be checked till i=3 in and the '\0' will be encountered in S1 and loop will quit. But S1[4]='\0' but S2[4]!='\0' so the strings are not same. so checking after the loop breaks if both have null character at the same i it implies they have same length and same letters so the two strings are same and returns 1 else 0.

Q.4.

a) The following **recursive** function ***find_power*** should return x^n when called as ***find_power(x,n),*** ***n*** being a non-negative integer. Fill up the missing lines in the function so that it returns x^n.

[1 + 1 + 2]

```
float find_power(float x, int n) {
    if (n == 0)
       return 1;
    else
       return x*find_power(x,n-1);
}
```

Straightforward simple problem of power calculation via recursion. continue till the base case of n==0 is reached and then return 1.

b) Fill up the missing lines in the following program so that it will display the sum of the elements of the array ***A*** when executed.

[2 + 2]

```
#include <stdio.h>
int main() {
    int i, n, k = 0, A[10], lim;
    printf("Enter number of elements ");
    scanf("%d", &n);
    printf("Enter the elements  ");
    for (i = 0; i < n; i++)
          scanf("%d", &A[i]);
    for (i = 0, lim = n/2; i < lim; i++) {
         /* Accumulate Sum */
         k = k + A[i]+A[n-i-1];
    }
    if (lim <(n-lim))   /* if middle element left out */
        k = k + A[i];
    printf("%d\n", k);
    return 0;
}
```

The first and the last elements are added and then the second and the second last and so on… This way we can do the sum in n/2 iterations. If n is odd then middle element is left.

If n is odd middle element is left out . if n is odd **lim<(n-lim)** or you can also do **n%2!=0** to check whether n is odd or not. Both are correct solutions.

Q.5.

a) The following program is supposed to insert a new integer value ***x*** into an already sorted (in ascending order) array ***A*** containing ***n*** distinct integers. You can assume that ***x*** does not already exist in ***A***, and there is space available to insert ***x*** in ***A***. For example, assume that ***n*** is 10, and ***A*** has the elements 10, 20, 30, 40, 60, 70, 80, 90, 100, 110, and ***x*** is 56. After insertion of ***x***, the array would become 10, 20, 30, 40, 56, 60, 70, 80, 90, 100, 110, and ***n*** would be 11. Fill up the missing lines in the program so that the program inserts ***x*** in the sorted array ***A***.

[2 + 2 + 2 + 2]

```
#include <stdio.h>
int main(){
  int x, i = 0, j, n, A[100];
  scanf("%d%d", &n, &x);
  for (j = 0; j < n; j++) scanf("%d", &A[j]);

  while (x > A[i] && i < n) i++; /* find position after which to insert */

  for (j=n; j>= i+1; j--) /* make space for inserting x */

          A[j] = A[j-1];

  n++;

  A[i] = x; /* insert the element at the required place */

  for (i = 0; i < n; i++) printf("%d ", A[i]);
  return 0;
```

Straightforward Insertion sort logic. The comments are self explanatory and is sufficient to understand and fill in the gaps.

b) The following **recursive** function ***reverse*** takes as parameters an integer array ***A*** and two other integers ***leftIndex*** and ***rightIndex*** which are indices of ***A***. After the function returns, only the part of the array from ***leftIndex*** to ***rightIndex*** (including both) should be reversed if ***leftIndex*** < ***rightIndex*****.** For example, if the elements of the array ***A*** are 1, 2, 3, 4, 5, 6, 7, and the function call ***reverse*****(*****A*****,2,6)** is made, then on return, the array ***A*** will contain 1, 2, 7, 6, 5, 4, 3 (i.e., ***A***[0] and *A*[1] remain unchanged, and ***A***[2] to ***A***[6] get reversed). Fill up the missing lines in the function so that it reverses the part of the array ***A*** between ***leftIndex*** and ***rightIndex***.

[1 + 3]

```
void reverse(int A[], int leftIndex, int rightIndex) {
    int temp;
    if (leftIndex < rightIndex) {
        temp = A[leftIndex];
        A[leftIndex] = A[rightIndex];
        A[rightIndex] = temp;
        reverse(A, leftIndex+1 , rightIndex-1);
    }
}
```

leftIndex<=rightIndex will also be accepted as a solution. just in that case if the number of elements to be swapped is odd then the middle element will be swapped by itself which is not required. so the given highlighted solution is better.

After swapping the leftmost element with the right most element increasing the index of the left and decreasing that of the right to get a converging swap of the corresponding elements of the array.

Q.6.

The function ***closest*** given below takes as parameters an integer array ***A***, the number of elements ***n*** in ***A***, and an integer ***val***. Assume that all integers in the array ***A*** are distinct and ***A*** is already sorted in ascending order. The function returns the index of the element in ***A*** with minimum absolute difference with ***val*** (i.e., it returns the index ***i*** such that |***A***[***i***] – ***val***| is minimum). If more than one element has the same minimum absolute difference with ***val***, then it returns the smallest index. For example, if ***A*** contains the elements 10, 13, 15, 19, 110 and ***val*** is 18, the function returns 3, which is the index of 19 (as |19 – 18| is the minimum). However, if ***val*** is 14, it returns 1 (as both |15 – 14| and |13 – 14| are the minimum, 13 occurs at index 1 and 15 at index 2, and 1 is the smaller index). Fill up the missing lines in the function so that it does the above. [2 + 2 + 2]

```
int closest(int A[], int n, int val) {
    int index, i;
    if (val < A[0]) /* smaller than the smallest element */
       index = 0;
    else if (val > A[n-1]) /* larger than the largest element */
            index = n - 1;
    else {  /* find the elements closest to val */

            for (i = 0; A[i]<val ; i++) ;

            if ((A[i]-val)<(val-A[i-1]))

                 /* if current element closest */
                  index = i;

            else index= i-1  ; /* set index to the closest element*/

         }
    return index;
}
```

Notice the ; given after the loop. That is there is no body of the loop. It is given to find the first i for which A[i] becomes >val. then the element closest to val will be i or i-1.

A[i-1]<val and A[i]>val. so checking the absolute differences to see which one is closer to val.

if (A[i]-val)>(val-A[i-1]) then i-1th index has element closer to val also if both are equal the answer is i-1 because it’s the smallest index.

MID SEMESTER 2016-2017
(SPRING SEMESTER)

1. (9 marks) Write C statements (program segments only) of a program that reads the lengths of the sides of a triangle to find the nature of the triangle. You are required to only write program segments for the following tasks only and not the complete program. **Marks:** $1+1+2+2+3$

   (a) Declare variables `a`, `b` and `c` of type `float`.

   ```
   float a,b,c;
   ```

   (b) Read `a, b, c`.

   ```
   scanf("%f %f %f",&a,&b,&c);
   ```

   (c) Check if `a` contains the largest value (larger than `b` and `c`). If not, print an error message.

   ```
   if(a<b) printf("Error:       a not largest n");
   else if(a<c) printf("Error:  a not largest n");
   ```

   (d) Write a program fragment to check and print whether `a, b, c` form the sides of a valid triangle. Assume that `a` has a value larger than `b` and `c`.

   ```
   if(b+c < a) printf("Error: not a valid triangle\n");
   ```

   (e) Print ”acute”, ”right-angled” or ”obtuse”, depending on the type of triangle formed by the sides `a, b, c`. Assume `a` is the largest side.

   ```
   if(b*b + c*c < a*a) printf("Obtuse triangle\n"); else
    if(b*b + c*c > a*a) printf("Acute triangle\n"); else
             printf("Right-angled triangle\n");
   ```

2. (7 marks) Complete the following C program so that it computes the sum of the following series upto *n* terms. **Marks:** $2+2+2+1$

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

```
/* Compute the sum of the series  [ 1-X^2/2!+X^4/4!- ...] */
#include < stdio.h >
 int main()
 \{
       float x, sum, term;
       int i, n;

       printf("Enter the value of x and the number of terms to sum\n");
       scanf("%f%d", &x, &n);

       // Initialize values

       sum=1;

       term=1;


       for (i = 1; i < n; i++)
       \{


             term = term * ((-1*x*x)/((2*i)*(2*i-1)));

             sum = sum + term;

         printf("\n Sum = %f\n");
         return 1;
\}
```

3. (6 marks) Complete the following C program which given an input string prints whether it is a palindrome or not. **Marks:** 1 + 2 + 1 + 1 + 1

```
#include <stdio.h >
int main() {
      char a[100];
      int i, j, length_a;

      printf("Enter the String(max length 100): ");
      // Read the string

      scanf("%s",a);
```

Standard format to scan a string i.e. an array of character.

```
      // Compute the length of string a using a loop and store it in
      // length_a. Do not use any library function.

      for(i=0;i<100;i++)
          {
          if(a[i]=='\0')
          break;
          }

      length_a=i;
```

Checking for the position at which null character appears. Null i.e. '\0' appears at the end of the string. So the index at which null character is present is the length of the string. So as soon as '\0' is encountered breaking the loop and assigning i to length.

```
      for (i = 0; i < length_a/2; i++) {

          if (a[i] == a[length_a-i-1])
           continue;
           else {

                  printf("Not a palindrome\n");
                   return 0;
          }

      }

      printf("String is a palindrome\n");
       return 0;
 }
```

Checking whether the characters from the front end matches their backend counterpart.

Even if one pair of corresponding characters are not same then they are not palindrome. So displaying the message that they are not palindrome.

else if all the corresponding elements match then displaying the message that the string is palindrome.

4. (5 marks) Write C program statements in the blanks such that the following function returns the minimum element in the array `a[ ]` between indices `start` and `end` (both inclusive): **Marks:** 1 + 1 + 3

```
int minv_arr (int a[], int start, int end) {
    int temp;

    if(start >= end) // base condition return

          a[start];

    else {
        // Make the recursive call and return the minimum element.
        // You are not allowed to use any loop

              temp=minv_arr(a,start,end-1);

              if(temp<a[end])

                   return temp;

              else return a[end];
    }
}
```

the function is being called each time with the end index decremented by one. This is done till end and start will point to the same element of the array i.e. the element start have been pointing to. Then it returns the element at index start and that is compared with the adjacent element which is pointed to by that particular called functions end index. The minimum among the two is returned to the calling function and again that returned element is compared with the next element in the sequence and the minimum of them is returned and so on until all the elements have been compared and a absolute minimum between start and end has been found. This minimum value is then returned to whatsoever other function has called the minv_arr function.

for example if start was 2 and end was 5 then first the functions will be recursively called until end is decremented and equals start that is 2(base condition). At this point arr[2]is returned to the calling function(itself)and assigned to temp and it will be compared to arr[3] and will return the minimum between them to its calling function(itself). Suppose arr[3] is minimum then temp will now be arr[3] and it will get compared with arr[4] and the minimum between them will be returned. Say arr[3] is the less than arr[4] so again temp will be assigned arr[3] and now arr[3] i.e. temp and arr[5] are compared and the minimum between the two is sent to some other function which initially called minv_arr function. Try and draw recursion tree(boxes) and keep track of which variable has what value in which box. That way you can solve problems based on recursion easily as this approach gives you a vivid pictorial view.

5. (10 marks) Write a program that takes as input `n`, followed by `n` integer numbers and store them in an array `A`. It then calls a function which copies the distinct elements of array `A` to an integer array `B` so that array `B` contains all elements of `A` but does not repeat any element. For example, if `A` stores {7, 2, 17, 19, 5, 2, 9, 9, 8, 2}, array `B` will contain {17, 2, 19, 5, 9, 8} after the function call. The program comprises of a `main( )`, the function `makeset( )` and the function `check( )` which is called by `makeset( )`. **Marks:** $3+5+2$

The function `printarray( )` is given which takes as input an array of integers `A` and its length `n` and prints the array.

```
void printarray (int A[], int n) {
      int i;
      for (i = 0; i < n; i++) printf ("%d ", A[i]) ;
      printf("\n") ;
}
```

(a) Write the function `check( )` which takes as input an integer `x`, an array `A` and its size `n`. It should return `1` if `x` is occurs in array `A` and `0` otherwise

```
int check(int x, int A[], int n)
{
int i;
for(i=0;i<n;i++)
{
if(a[i]==x)
return 1;
}
return 0;
}
```

```
Scanning through the
array and checking
whether the element
is present and if
found then returning
1 and the function is
terminated. Remember
as soon as a return
statement is
encountered the
function exits.
```

(b) Write the function `makeset( )` which takes as input an array of integers A, its size n1, and an array of integers B, The function must copy the unique elements of A into the array B and return the number of elements in B, by making use of calls to the function `check( )` defined above.

```
int makeset(int A[], int n1, int B[])
{
int i,j=0;
for(i=0;i<n1;i++)
{
if(check(A[i],B,j)==0)
B[j++]=A[i];
}
return j;
}
```

```
Inside the loop we are
calling check function to
see if the element A[i] is
already present in array B
or not. If it is not
present we are adding it
to array B and
incrementing its index via
j. If the element already
exists in array B then
check function returns 1
and the element is not
added to B.
```

(c) Complete the function `main( )`

```
int main ( ) {
        int A[100], int B[100] ;
        int i, nA, nB;
        scanf (``%d'', &nA) ;
        for (i=0; i<nA; i++)
             scanf (``%d'', &A[i]) ;
       // Call makeset

       nB=makeset(A,nA,B);

       printarray (A, nA) ;  printarray (B, nB) ;
       return 0;
 }
```

6. (11 marks) What will be printed when the following programs/ program segments execute? Write **only** the output that will be printed if the program is executed within the box.

Marks: 3 + 4 + 4

(a)

```
#include <stdio.h>
int main()
{
    int i = 12, j, last;
    while (i > 1) {
        j = 1;
        printf("%d: ", i);

        while (j < i) {

            if ((i % j) == 0) {
                printf("%d ", j);
                last = j;
            }
            j++;
        }
        i = last;
        printf("\n");
    }
    return 0;
}
```

SOLUTION:

```
12:  1 2 3 4 6
 6:  1 2 3
 3:  1
```

```
We need to figure out what the snippet is
trying to do. Its first printing all the
factors of twelve (excluding 12) using the
inner while loop. A variable named last
stores the highest factor of 12 i.e. 6 and
outside the inner while loop i is assigned
last that is i is assigned 6. Again the
inner while loop is iterated and all
factors of 6 is displayed. last stores the
highest factor of 6 i.e. 3 and i is thus
assigned 3. Factors of 3 are displayed and
since the highest factor of 3 is 1 so the
outer loop breaks once the inner loop has
completed printing the factors of 3.
```

(b)
```c
#include <stdio.h>
int main ()
{
  int a[] = { 6, 3, 2, 8 };
  int i, j;

  for (i = 0; i < 4; i++) {
      printf ("%d: ", a[i]);

      for (j = 0; j < 4; j++) {

          if ((a[i] % a[j]) == 0) {
              printf ("%d ", a[j]);
              continue;
            }

          if ((a[j] % a[i]) == 0) {
              printf ("%d ", a[j]);
              break;
            }
        }
        printf ("\n");
    }
  return 0;
}
```

SOLUTION:

6: 6 3 2
3: 6
2: 6
8: 2 8

What this snippet is trying to do is checking for the factors (present in the array) of a number in the array and also displaying the first multiple of a number if it's present in the array. so the factors of 6 in the array are 6,3,2 and for 8 is 2 and 8. And the first multiples of 2 and 3 are 6 so 6 is displayed against them.

(c)
```
void serve (int num_tasks)
{
        static int server = 1;
        int taskid = 1;

        printf("Starting %d tasks\n",num_tasks);

        for (int i = 0; i < num_tasks; i++) {
                  printf (Task %d - Server %d \n", taskid, server);
                 server++;
                 if (server > 5)
                        server = 1;
                 taskid++;
        }
        printf("Done\n");
}

int main ()
{
    serve (3);
    serve (4);
    return 0;
}
```

Solution:

```
Starting 3 tasks
Task 1 - Server 1
Task 2 - Server 2
Task 3 - Server 3
Done
Starting 4 tasks
Task 1 - Server 4
Task 2 - Server 5
Task 3 - Server 1
Task 4 - Server 2
Done
```

7. (7 marks) Consider the following functions: **Marks:** 3 + 4

(a)

```
int foo (int x, int y) {
  if (x < y)
    return x;
  else
    return foo (x - y, y);
}
```

```
foo(6,13):
x=6;y=13; thus
x<y so returns x
i.e. 6.
```

For each call below, indicate what value is returned:

```
foo(37,10) → foo(27,10)
                  ↓
foo(7,10) ← foo(17,10)

here x=7,y=10 so x<y and
the function returns x
i.e. 7.
```

`foo (6, 13)`	***6***
`foo (37, 10)`	***7***

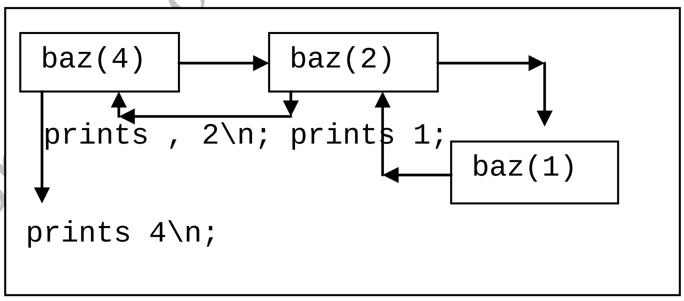
(b)

```
void baz (int n) {
  if (n <= 1)
    printf ("\%d ", n);
  else {
     baz (n/2);
     printf (",  \%d \\n",  n);
  }
}
```

For each call below, indicate what output is printed:

`baz(4)`

```
SOLUTION:
1, 2\n, 4\n
      OR
     1, 2
```



`baz(30)`

```
SOLUTION:
1, 3\n, 7\n, 15\n, 30\n
               OR
1, 3
, 7
, 15
, 30
```

```
follow similar kind of
recursion tree as shown
above.
```

END SEM SPRING 2018

1. Assume that on a certain machine an int variable is of size 32 bits (or 4 bytes), char variable is of size 8 bits (or 1 byte), and each memory address is of size 32 bits. Assume further that the sizeof() function call returns the size of its operand in bytes. Answer the following questions. **Marks:** $1+2=3$

   (a) Consider the following structure:

```
struct myStruct {
  char name[20];
  int account_number;
  struct myStruct *next;
};
```

   What does sizeof(struct myStruct) return on this machine?

A. 26	***B. 28***	C. 48	D. 46

   (b) What will be the output of the C program?

```
#include<stdio.h>
#include<stdlib.h>
int main()
{
    int *p;
    p = (int *)malloc(20);
    printf("%d\n", sizeof(p));
    free(p);
    return 0;
}
```

p is a pointer variable of integer. So it will have a size of 4bytes. The malloc(20) is assigned to confuse the students.

A. 40	B. 20	***C 4***	D. 80

2. Answer the following questions. **Marks:** $3\times2=6$

   (a) Consider the following sequence of push and pop operations on an initially empty stack *S*.

```
S = push(S,1);
S = pop(S);
S = push(S,2);
S = push(S,3);
S = pop(S);
S = push(S,4);
S = pop(S);
S = pop(S);
```

Stack follows the concept of LIFO i.e. Last In Fast Out

   Which of the following is the correct order in which elements are popped?

A. 1 2 4 3	B. 1 3 2 4	C. 1 2 3 4	***D. 1 3 4 2***

   (b) What is the output of the following program? **Marks:** 2

```
#include<stdio.h>
#include<stdlib.h>

int main()
{
    int *ptr;
    *ptr = 10;
    *ptr = 20;
    *ptr = 30;
```

Actually this will result in an error as memory for the pointer has not been alloted.

```
    printf("%d\n",*ptr);
    return 0;
}
```

A. 10	B. 20	C. 30	*D. None of the above*

(c) Under what condition will this program print the string "Hello"?

```
#include<stdio.h>
#include<stdlib.h>

int main()
{
    int *ptr;
    ptr = (int *)malloc(sizeof(int)*10);
    if (ptr == NULL)
        printf("Hello\n");
    return 0;
}
```

If due to some reason such as memory unavailability or something ptr cannot be assigned any memory then it will be assigned the value NULL. If this happens then the condition of if statement will be true and thus Hello will be displayed.

A. if the memory could not be allocated to the pointer "ptr"
B. if the memory has been allocated to the pointer "ptr" successfully
C. it will never print
D. none of the above

3. Fill in the blanks to complete a C program that creates a singly linked list by repeatedly calling the function push(). It then counts the number of nodes present in the singly linked list recursively using the function getCount(). Each blank has at most one statement.


```
// Recursive C program to find length or count of nodes in a linked list
#include<stdio.h>
#include<stdlib.h>

/* Link list node */
struct Node
{
    int data;
    struct Node* next;
};

/* Given a reference (pointer to pointer) to the head of a list and
   an int as parameters, the function pushes a new node on the front
   of the list. */
void push(struct Node** head_ref, int new_data)
{
    /* allocate node */
    struct Node* new_node =
            (__struct Node*__) malloc(______sizeof(struct Node)____);

    /* put in the data  */
    new_node->data  = new_data;

    /* link the old list off the new node */
    new_node->next = (*head_ref);
```

```
    /* move the head to point to the new node */
    ______(*head_ref) = new_node_______;

/* Counts the no. of occurences of a node
   (search_for) in a linked list (head)*/
int getCount(struct Node* head)
{
    // Base case
    if (________head == NULL__________)
        return 0;

    // count is 1 + count of remaining list
    return 1 +______getCount(head —> next)______;
}

/* the main function*/
int main()
{
    /* Start with the empty list */
    struct Node* head = NULL;
    int num;
    char flag = 'Y';

    // Use push() repeatedly to construct the list
    while(____flag == 'Y' (OR) flag != 'N'____)
        {
        printf("\n Enter the next number to be pushed into the stack: ");
        scanf("%d", &num);
            push(__________&head__________, num);
        printf("\n Do you want to push more numbers into the stack?
        (Answer Y to continue and N to stop pushing into the stack)");
        scanf("%c", &flag);
        }

     /* printing the size of the list created */
    printf("\n Number of nodes in the linked list is %d",
    ______getCount(head)_______);
    return 0;
}
```

The Solution is Self Explanatory. Comments are Provided for proper understanding.

4. Answer the following questions. **Marks:** $3+4+2=9$

(a) A quadratic algorithm with processing time $T(n) = cn^2$ spends 1 ms in processing 100 data items. Time spent for processing 5000 data items = **2500 ms (OR) 2.5 sec**

$\frac{1}{t}=\frac{100^2}{5000^2}$ or t=2.5s

(b) Consider the problem of exponentiation of integer x to the power of integer n (i.e., x^n). A straightforward way of doing this is to multiply x, n times. However a more efficient way to solve this problem would be to see that $x^n = x^{n/2} * x^{n/2}$. Assuming that $n = 2^k$, we can write a small recursive function to implement exponentiation.

```
 int power(int x, int n){
    if (n==0) return 1;
    if(n==1) return x;
    if ((n % 2) == 0) return power(x*x, n/2);
}
```

Let the time required to execute this program be $T(n)$. Assume $T(0) = c_1$ and $T(1) = c_2$.

(i) The recursive expression is given by $T(n) =$ ___*T(n/2)*___ $+ c_3$

(ii) The exact solution to the above recursive expression is ***c2 +(c3)log(n)***

> You can try with small values like 2 or 3. At first, The recursive function receives value each time the previous value divided by 2(like for 8: 8,4,2,1). So the time depends on log_2 . Now c3 is the time needed when whole function is executed.

(c) $log(n!) = \Theta($___*n*___$* log(n))$.

> log(n!)=log(n) + log(n-1) + ……. < log(n) +log(n)+ …..=nlog(n)

5. (a) What does the following function do on the elements of the array *arr*[]? **Marks:** 2

```
void whatdoIdo(int arr[], int size)
{
    int i=0;

    for(i=0; i < size; i++)
    {
        if( arr[i] % 2 == 0 )
            arr[i] = 0 ;
        else
            arr[i] = 1 ;
    }
}
```

> The solution is self explanatory

The functionwhatdoIdo ***replaces even elements in the array with 0 and odd elements with 1***.

(b) The following function computes the median of an array of floats *x*[]. Assume that all the entries in the array are distinct, and there is only a single digit after the decimal point for all the numbers. Fill the blanks.

Each blank can have only ONE statement.**Marks:** $0.5+0.5+0.5+0.5+0.5+0.5+1+1+1 = 6$

```
float median(int n, float x[]) {
    float temp;
    int i, j;
    for(i=0; i<n-1; i++) {
        for(j=_i+1_; j<__n__; j++) {
            if(___x[j] < x[i]___) {
                temp = ___x[i]___;
                  x[i] =___x[j]___;
                x[j] =___temp___;
            }
        }
    }

    if(___n%2 == 0___) {
        return
    } else {                              }
```

> To find the median firxt we are sorting the array using bubble sort.

> If the array contains even number of elements then the 2 middle elements averaged is the median or if the number of elements in the array is odd then the middle element is the median of the array.

```
}                    ( (x[n/2] + x[n/2 - 1]) /2.0 ) ;

                     x[n/2] ;
```

(c) Now suppose you have two already sorted arrays ar1[] and ar2[] of EQUAL size n. The following function attempts to find the median of the elements of the two arrays combined together. For instance if ar1 = $\{1.0, 12.0, 15.0, 26.0, 28.0\}$ and ar2 = $\{2.0, 13.0, 17.0, 30.0, 45.0\}$, you have to find the median of the elements $\{1.0, 12.0, 15.0, 26.0, 28.0, 2.0, 13.0, 17.0, 30.0, 45.0\}$. Fill in the blanks. Each blank has only ONE statement.

Marks: $(0.5 + 1) + (0.5 + 1) + 0.5 + (0.5 * 3) + (0.5 * 3) + 1.5 = 8$

```
/* This function returns median of ar1[] and ar2[].
          Assumptions in this function:
          Both ar1[] and ar2[] are sorted arrays
          Both have n elements */
float getMedian(float ar1[], float ar2[], int n)
{
    int i = 0;
    int j = 0;
    int count;
    float m1 = -1.0, m2 = -1.0; // contains medians from two arrays
    for (count = 0; count <= n; count++)
    {
        if (i == n)
        {
            m1 = ____m2____;
            m2 = ____ar2[0]____;
            break;
        }
        else if (j == n)
        {
            m1 = ____m2____;
            m2 = ____ar1[0]____;
            break;
        }
        if (ar1[i] ____< ar2[j]____)
        {
            m1 = ____m2____;
            m2 = ____ar1[i]____;
            i____++____;
        }
        else
        {
            m1 = ____m2____;
            m2 = ____ar2[j]____;
            j____++____;
        }
    }
    return ____(m1 + m2)/2.0____;
}
```


6. The following recursive function takes a string of given length as input and determines whether the string is a palindrome. It returns 0 if the string is not a palindrome and 1 if it is. Fill in the blanks. Each blank can have AT MOST one statement. **Marks:** $1 + 2 + 2 = 5$

```
int ispalindrome ( char A[], int n )
{
        if ( __n <= 1___ ) return 1;/*base case*/
        if ( A[0] != A[n-1] ) return 0;
        return ispalindrome( __&A[1], n-1__ );
}
```

We are checking whether the first and the last element of the string A is same or not. If they are found to be same we call the function once again with arguments that have the address of the second character of the string and the length of the actual string decremented by 1. So that when we are checking for the parity of the first and last characters of the passed string we are actually checking whether the 2nd and the 2nd last characters of the actual string are same or not. This goes on till we have checked for the parity of all corresponding characters and we return 1 if the string is palindrome else we return 0 if in between the parity is missed,

7. Complete the following program, where the main function takes three strings A, B, C as input from the user and determines whether the string A contains the regular expression $B * C$, where $*$ stands for any substring. For instance, if A = “abcdefg”, B = “bc” and C = “ef”, the function determines if an occurrence of “bc” followed (not necessarily immediately) by an occurrence of “ef” can be detected in “abcdefg”. In this case, the occurrence $B * C$ is detected at index position 1 in A, and the main function gives this output. Either of B or C can also be null. The main function makes use of another function *locateSubstr* that checks whether a string A contains another string B as a substring, and if so, returns the match index of B in A. Thus, when B and C are non-empty, the main function first finds if B is a substring of A, and if that is the case, whether C is a substring for the remaining portion of A, where the match for B ends. Fill in the blanks.

Each blank can have AT MOST one statement. **Marks:** 1 + 1 + 2 + 1 + 1 + 1 + 1 + 1 + 1 = 10

```
#include <stdio.h>
#include <stdlib.h>
#define MAXLEN 1024
int locateSubstr ( char A[] , char B[] )
{
        int i, j, match;
        if (strlen(B) == 0) return 0;
        for (i=0; i<=_strlen(A) - strlen(B)_; ++i) if (A[i] == B[0]) {
            match = 1;
            for (j=0; j<____Strlen(B)____; ++j) if (____A[i+j] != B[j]____)
                { match = 0; break; }
            if (match)______return i______;
        }
        return -1;
}
int main ()
{
        char A[MAXLEN], B[MAXLEN], C[MAXLEN];
        /* Assume the code to input strings from the users is here.
           You need not write anything here */
        /* i should store the matching index of B*C in A*/
        int i,j,k;
        if (strlen(B) == 0) i =_locateSubstr(A,C)_;
        else if (strlen(C) == 0) i = __locateSubstr(A,B)_;
        else{
             j = locateSubstr(A,B);
             if (j < 0) i=j;
             else k = locateSubstr(____&A[j+strlen(B)] , C____);
             if (k>=0) i=__________j__________;
             else i =_______—1_______;
        }
        if (i >= 0)
           printf("The pattern B*C is found in A at idx %d\n", i);
        else
           printf("The pattern B*C is not found in A\n");
        exit(0);
}
```


8. The following recursive function makes the base conversion. It reads two integers n and b from the terminal (with $n \geq 0$ and $b > 1$, both in base 10) and expresses n in base b. For example, the decimal

expansion of 345 in base 10 is $345 = 3 \times 10^2 + 4 \times 10 + 5$. Note that in this case $5 = 345\%10$ and $34 = 345/10$; The output as printed by the program should be: $(345)_{10} = (3, 4, 5)_{10}$. However, please note that for $n = 10$ and $b = 2$, the program should print $(10)_{10} = (1, 0, 1, 0)_2$ and NOT $(10)_2 = (1, 0, 1, 0)_2$.

Fill in the blanks. Each blank can have AT MOST one statement. **Marks:** $2 \times 5 = 10$

```
#include <stdio.h>

void baseconv ( int n , int b )
{
   /* n is too small. Simply print it and return. */
   if (____n < b______) { printf("%d",n); return; }

   /* Recursively print the more significant digits */
  ______baseconv(n/b,b)______;

   /* Finally print the least significant digit */
   printf(",%d",____n%b____);
}

int main ()
{
   int n, b;

   printf("n = "); scanf("%d", &n);
   printf("b = "); scanf("%d", &b);

   if ((n < 0)||(b < 2)) {
      fprintf(stderr, "Error: Invalid input...\n");
      exit(1);
   }

   printf("__(%d)_10 = (______",n);
   baseconv(n,b);
   printf("____)_%d_____",b);

   exit(0);
}
```

for converting a decimal number n to a number in base b we use recursion to continuously divide the number by its base till its smaller than the base b and we display this first followed by the subsequent display of the previous remainders (n%b) in the previous recursively called functions in down to top approach. That is in simple words we keep dividing the original number n by b till the quotient left is less than b, we display this quotient first and then we backtrack the remainders we got by continuously dividing n by b and print them in the reverse order. This gives us a number in base b equivalent to n base 10.

2	10	0
2	5	1
2	2	0
2	1	1

We stop here because the quotient achieved i.e 1 is less than 2. So now we write the remainders in reverse order.

9. Answer the following questions. **Marks:** $(4 \times 1) + 2 + (3 \times 2) + (3 \times 2) = 18$

(a) CIRCLE the correct choice.

[i] What value will be assigned to the variable a after the following two statements are executed?

```
int a = 7, b = 5, c = -3;
a = a - a % b * c;
```

A. 0	B. 9	***C. 13***	D. 14

[ii] What value is assigned to the variable var?

```
#define T 10+10
var = T * T;
```

Here T is 10+10 and not 20. So wherever T occurs in the snippet replace T by 10+10 and not 20. var=10+10*10+10=120. This is the way #define works.

A. 400	B. 210	C. 200	***D. 120***

[iii] Which of the following is NOT a legal name of a C variable?

<u>A. 12_pds</u>	**B. _12pds**	**C. pds_12**	**D. pds12_**

[iv] What is the 8-bit 2's-complement representation of -49?

A. 11001110	***<u>B. 11001111</u>***	C. 10110001	D. 11010001

(b) Find the 32-bit (single-precision) floating point representation of +41.6 in the IEEE 754 format. Put only ONE binary digit in each gap/space provided below.

Sign Bit:

0

Exponent:

1	0	0	0	0	1	0	0

Mantissa:

0	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1

0	0	1	1	0

(c) Consider the following for loop:

```
for (k=1; k<100; ++k) k *= k+1;
```

[i] How many times is the statement "$k* = k+1$;" executed? ***3***

[ii] How many times is the loop condition "$k < 100$" checked in the loop? ***4***

[iii] What is the value stored in the variable "k" immediately after the for loop terminates? ***183***

(d) What will be the output of the following programs?

[i]

```
#include<stdio.h>
int main()
{
   int x = 4, y = 6, z = 0;
   while (y != 0) {
      if (y % 2) z += x;
      x *= 2;
      y /= 2;
   }
   printf("%d\n", z);
   return 0;
}
```

iteration 1:
if statement is not satisfied because y is even.
x=4*2=8; y=6/2=3;
iteration 2:
if statement is satisfied as y is odd, so
z=z+x=0+8=8
x=8*2=16;y=3/2=1(as y is int)
iteration 3:
if statement is satisfied as y is odd, so
z=z+x=8+16=24
x=16*2=32;y=1/2=0(as y is int)
now since y is 0 the loop terminates and the final value of **z=24**

Answer: ***24***

[ii]

```
#include <stdio.h>
int main ()
{
   int a = 5, b = 5, c;
   char p = 'p', q = 'q';
   c = !( (a>=b) || ((a<=b)&&(p>q)) );
   printf("%d\n", c);
   return 0;
}
```

(a>=b) is true so results in 1. a<=b is true again as a==b so it again results in 1 but p>q is false so results in 0. so when ((a<=b)&&(p>q) is evaluated it is (1&&0) which results in 0. So when ((a>=b)||((a<=b)&&((p>q))) is evaluated it is 1||0 which results in 1 and when negated i.e. !1 is done it gives 0.Thus c=0.

Answer: ***0***

[iii]

```
#include <stdio.h>
int main ()
```

```
{
    int p, q;
    for (p=q=0; p<10; ++p) {
        q = p + q;
        ++p;
    }
    printf("%d\n", q);
    return 0;
}
```

If you see the loop closely what is happening is p is being incremented by 2 after each iteration and that is being summed over and stored in q till p<10. So q=2+4+6+8=20.

Answer: ___*20*___

10. The following program computes the sum of the square of digits in the decimal representation of a non-negative integer. For example, the sum of the square of digits for 320127 is $3^2 + 2^2 + 0^2 + 1^2 + 2^2 + 7^2 = 67$. Fill in the blanks with appropriate C constructs.

Each blank can have AT MOST one statement. **Marks:** $1 + 1 + 1 + 1 + 2 + 1 = 7$

```
#include <stdio.h>
int main ()
{
    unsigned int n, d, sum;

    /* Read the unsigned integer n*/
    scanf( ___"%u"___ , &n );

    /* Initialize sum */
    sum =___0___;

    /* Loop as long as n is not reduced to zero*/
    while (___n > 0 (OR) n != 0___) {

        /* Store in d the least significant digit of n*/
        d =___n%10___;

        /* Add the square of this least significant digit to sum*/
        sum =___sum + d*d___;

        /* Remove this digit from n */
        n =___n/10___;
    }

    /* Print the sum of the square of digits of the input integer*/
    printf("The sum of the square of digits is %d \n", sum );

    return 0;
```

The solution is Self Explanatory

Indian Institute of Technology Kharagpur
Programming and Data Structures (CS10001)
Autumn 2017-18: End-Semester Examination

Time: 3 Hours **Full Marks: 100**

INSTRUCTIONS

1. Answer ALL questions.
2. Please write the answers either within the boxes provided or on the blank lines to be filled up. Any answer written elsewhere will not be evaluated.
3. You may use the blank pages at the end for your rough work.
4. In the C programs, **assume that all appropriate header files are included.**

QUESTION 1. **Answer the following questions as directed.**

a) Given that the ASCII codes for lowercase letters in the English alphabet ‘a’ to ‘z’ are 97 to 122 (in decimal), what will be displayed when the following C program is executed? [1 Mark]

```
int main() {
  int a;
  a = 'd' * 2 + 4;
  printf("%d", a);
  return 0;
}
```

204

The ASCII code of ‘d’ is 100. Thus on performing the operation ‘d’*2+4 its evaluates 100*2+4=204.

b) What will be displayed when the following C program is executed? [1 Mark]

```
int main() {
  int a = 5;
  printf("%d", a == 5 ? 0 : 5);
  return 0;
}
```

0

Simple application of ternary operator. Since a==5 is true so the first value i.e 0 is printed. If the condition was to be false then 5 would have been printed.

c) What will be displayed when the following C program is executed? [1 Mark]

```
int main() {
  int a = 0;
  printf("%d", a == 0);
  return 0;
}
```

1

a is first assigned 0. So when a==0 it results in ‘true’ but since the data type of the printed value is integer so ‘true’ results in 1 which is displayed.

d) What will be displayed when the following C program is executed? [1 Mark]

```
int main() {
  int a, sum = 0;
  for (a = 0; a < 5; a++)
    sum = sum + a++;
  printf("%d", sum);
  return 0;
}
```

6

iteration 1: sum=0; a=1; (a is post incremented)
a is again incremented by 1 in the loop definition so a becomes 2.
iteration 2: sum=2; a=3; (a is post incremented)
a is again incremented by 1 in the loop definition so a becomes 4.
iteration 3: sum=2+4=6; a=5; (a is post incremented)
a is again incremented by 1 in the loop definition so a becomes 6.
now since 6<5 is not satisfied in the loop condition so the loop terminates and the **sum 6** is displayed.

e) What will be displayed when the following C program is executed? [1 Mark]

```
int main() {
  int a[4] = {10, 20, 30, 40}, count, sum = 0;
  for (count = 0; count < 4; count++)
    sum = sum + a[count] + a[3-count];
  printf("%d", sum);
  return 0;
}
```

200

If we observe closely what the a[count]+a[3-count] is doing when count is looped from 0 to 4 is is equal to 2*(sum of the elements of the array)=2*(10+20+30+40)=200

f) What will be displayed when the following C program is executed? [1 Mark]

```
int a = 1;
void myProc() {
  if ( a != 2 ) {
    int a = 3; printf ("%d\n", a);
  }
  else { int a = 4; }
  printf ( "%d\n", a);
}
int main() {
  myProc(); printf ("%d\n", a);
  return 0;
}
```

3
1
1

This questions test your knowledge of scope and understanding of global and local variables. First, a=1 is assigned(this a is global) then myProc() function is called and the if condition is checked. a here is 1 so a!=2 is true so control enters the if block. here a local variable a is assigned 3 which is then displayed. Then the control exits the if block and then after else block the value of a is printed which is 1(here the value of the global variable a is printed as outside the if block there is no existence of local variable a=3.) then again in the main function the global value of a is displayed i.e. 1.

g) What will be displayed when the following C program is executed? [1 Mark]

```
int main() {
  int *a, **b, c;
  b = &a; a = &c; c = 20;
  printf ("%d %d", **b, *a);
  return 0;
}
```

20 20

Very simple use of single and double pointers. and b are single and double pointer respectively. a is pointer to c and b is pointer to a. So *a and **b will display the same value as that of c which is 20. SO 20 will be displayed twice for **b and *a respectively.

h) What will be displayed when the following C program is executed? [1 Mark]

```
int main() {
  int *a, b[5] = {1, 2, 3, 4, 5};
  a = b;
  printf ("%d %d", *(b+2), *(a+3));
  return 0;
}
```

3 4

The array name i.e. b is itself a pointer to the array and here after b is assigned to a, a becomes a pointer to the array as well. so *(b+2) and *(a+3) will display the 3rd and the 4th element of the array i.e. 3 and 4 respectively.

i) Write a **<u>single</u>** C statement that declares **a** to be an integer type variable, **b** to be an integer pointer, and **b** is initialized to contain the address of **a**. [1 Mark]

```
int a, *b = &a;
```

j) Consider that a text file named **file1.txt** contains a single line with just one word **Avishek** in it. What will be displayed when the following C program is executed? [1 Mark]

```
int main() {
  FILE *fptr; char line[6];
  if ((fptr = fopen("file1.txt", "r")) != NULL)
    while (fgets(line, 3, fptr) != NULL)
      printf ("%s\n",line);
  return 0;
}
```

```
Av
is
he
k
```

QUESTION 2. <u>**Answer the following questions as directed.**</u>

a) Write a <u>**single**</u> C statement that declares **str1** to be an array of three pointers pointing to constant strings “Sarthak”, “Anita” and “Dinesh”, respectively. [2 Marks]

```
char *str1[ ] = {“Sarthak”, “Anita”, “Dinesh”}; OR
char *str1[3] = {“Sarthak”, “Anita”, “Dinesh”};
```

b) What will be displayed when the following C program is executed? [2 Marks]

```
int main() {
  int i;
  for (i = 0; i < 4; i++) {
    int x = 0;
    if (x==2) break;
    else printf ("%d\n",x);
    x++;
  }
  return 0;
}
```

```
0
0
0
0
```

in the loop in the first statement x is always assigned 0 so the if statement is never true so the printf statement in else block is executed where x is displayed which is always 0. Then x is incremented by 1 but again when the loop is reiterated x is assigned 0 and same process follows until the loop is terminated i.e. as a result four 0s are printed.

c) What will be displayed when the following C program is executed? [2 Marks]

```
struct abc {int a; int b;};
struct abc def[5];
int main() {
  def[0].a = 5; def[0].b = 10;
  def[1].a = 15; def[1].b = 20;
  printf ("%d\n", ++(*def).a);
  return 0;
}
```

```
6
```

(*def).a will result in 5 so ++(*def).a will be ++5 which will result in 6.

d) What will be displayed when the following C program is executed? [2 Marks]

```
int main() {
  char *abc = "bef";
  if (*abc + 1 == 'c') printf ("%c", *abc++);
  else printf ("%s", abc);
  return 0;
}
```

b

abc is a pointer to the string "bef" so *abc will be 'b' and thus *abc+1 will be 'c' so the if statement will be true and will print *abc++ i.e. b. Note here post increment is used so first *abc i.e. 'b' will be printed then it will be incremented.

e) What is the hexadecimal representation of the decimal number 140.5? [2 Marks]

8C.8

f) What is the decimal representation of the result if 11101100 is subtracted from 01010000? Assume that both the binary numbers are represented in 8-bit 2's complement form. [2 Marks]

100

g) What will be the displayed when the following C program is executed? [2 Marks]

```
int main() {
  char b = 'a';
  switch (b) {
    default: printf ("10\n");
    case 'a':
    case 'A': printf ("20\n");
    case 'b':
    case 'B': printf ("30\n");
              break;
  }
  return 0;
}
```

20
30

This is a case of fall through. The control in switch case is first directed to case 'a' and since there is no break statement for case 'a' so the control moves on to the next case i.e. case 'A' and prints 20 and since again there is no break control moves to case 'b' then to case 'B' and prints 30 then encounter counter break statement and the control breaks from the switch case. Note That default will only be executed when no of the cases match which is not the scene here.

h) What will be displayed when the following C program is executed if each address is of size 8 bytes in the computer? [2 Marks]

```
int main() {
  char a[5] = {'d', 'e', 'f', 'g', 'h'};
  printf ("%u %u\n", (a + 3) - a, *(a + 3) - *a);
  return 0;
}
```

3 3

i) What will be displayed when the following C program is executed from the command line as:
./a.out A 5 [2 Marks]

```
int main(int argc, char *argv[]) {
  char c; int a;
  sscanf(argv[1], "%c", &c);
  sscanf(argv[2], "%d", &a);
  printf ("%c", c + a);
  return 0;
}
```

F

j) What will be displayed when the following C program is executed? [2 Marks]

```
struct node {
   int data; struct node *next;
};
int main() {
  struct node a, b, c;
  a.data = 10; b.data = 20; c.data = 30;
  a.next = &b; b.next = &c; c.next = NULL;
  printf ("%d %d", a.data, a.next→next→data);
  return 0;
}
```

10 30

Answer the following questions as directed.

QUESTION 3.

a) The following C function is used to compute the product of two matrices, with each matrix represented as a 2-d array. The function takes as parameters a 2-d array A with n_r_A rows and n_c_A columns and a 2-d array B with n_r_B rows and n_c_B columns. The function returns a pointer to the first element of the product matrix C. Fill in the missing lines. [1+1+1+1+1+1 = 6 Marks]

```
int **mult(int **a, int n_r_a, int n_c_a, int **b, int n_r_b, int n_c_b) {

  int **c;                    //Declare suitable variable for returning result
  int sum, i, j, k;

  c = (int **) malloc (sizeof(int *)*n_r_a);
  for (i = 0; i < n_r_a; i++) {

     c[i] = (int *) malloc(sizeof(int)*n_c_b);
    for (j = 0; j < n_c_b; j++) {
      c[i][j] = 0;
      for (k = 0; k < n_c_a; k++)

        c[i][j] = c[i][j] + a[i][k] * b[k][j];
      }
  }
  return c;
}
```

b) The following C program is written to compute the sum of all the elements of a 2D array. However, when compiled, a **syntax error** is detected. Encircle the line where there is a syntax error and write it correctly in the box provided. Even after correcting the syntax error, when the program is executed, the sum is not printed correctly since there is also a **logical error** in one of the statements in the **get_sum** function. Encircle the statement which has a logical error and write it correctly in the box provided. [2+2=4 Marks]

```
int get_sum(int x[2][]) {
int sum = 0, i, j;
  for (i = 0; i < 2; i++)
    for (j = 0; j < 3; j++)
      sum =+ x[i][j];
      return sum;
}
int main() {
  int A[2][3] = {{1, 2, 3}, {4, 5, 6}};
  printf ("%d\n", get_sum(A));
  return 0;
}
```

```
int get_sum(int x[ ][3]) {

sum += x[i][j];
```

QUESTION 4.

a) The following program uses quicksort to sort an array of integers in decreasing order. What will be displayed when the program is executed? [5 Marks]


```
int partition (int *A, int lt, int rt) {
  int pivot, i, j, temp;
  pivot = A[lt]; i = lt; j = rt;
  while (i < j) {
    while (A[j] < pivot) j--; while (A[i] >= pivot && i <= rt) i++;
    if (i < j) {
      temp = A[j]; A[j] = A[i]; A[i] = temp; j--;
    }
  }
  temp = A[lt]; A[lt] = A[j]; A[j] = temp;
  return j;
}
void quicksort (int *A, int lt, int rt) {
  int ind;
  if (lt >= rt) return;
  ind = partition(A,lt,rt); printf ("%d\n", A[ind]);
  quicksort(A,ind+1,rt); quicksort(A,lt,ind-1);
}
int main() {
  int X[9] = {45, 65, 24, 38, 17, 101, 4, 203, 115};
  quicksort (X, 0, 8);
  return 0;
}
```

```
45
17
38
101
115
```

b) The following **recursive** function is used to implement insertion sort. Fill in the missing lines.

[1+1+1+1+1=5 Marks]

```
void recur_insertion_sort (int A[], int n) {
  int j, key;

  if (n <= 1) return;                         //Base case

       //Recursively sort first n-1 elements

  recur_insertion_sort (A, n-1);
      // Insert the last element at its correct position
  key = A[n-1];
      // Move each array element greater than key one position to its right

  j = n-2;
  while (j >= 0 && A[j] > key) {

    A[j+1] = A[j];
    j--;
  }
  A[j+1] = key;
}
```


QUESTION 5.

a) When the following C program is executed, it produces seven lines of output. The first, third, fourth and fifth lines of the output are shown in the box below (the addresses are printed in hex). Write what will be printed as the second, sixth and seventh lines of the output. Assume that each address is of size 4 bytes in the computer.

[2+2+2=6 Marks]

```
int **allocate (int h, int w) {
  int **p, i, j;
  p = (int **) malloc(h*sizeof (int *));
  printf ("p = %p\n", p);
  printf ("p+1 = %p\n", p+1);
  for (i = 0; i < h; i++) p[i] = (int *) malloc(w * sizeof(int));
  for (i = 0; i < h; i++) printf("p[%d] = %p\n", i, p[i]);
  return(p);
}

int main() {
  int **q;
  int M = 3, N = 3;
  q = allocate (M, N);
  printf ("&q[1][2] = %p\n", &q[1][2]);
  printf ("&q[0] = %p\n", &q[0]);
  return 0;
}
```

```
p = 00980FF8

p+1 = 00980FFC
p[0] = 00981010
p[1] = 00981028
p[2] = 00981040

&q[1][2] = 00981030

&q[0] = 00980FF8
```

b) What will be displayed when the following C program is executed? [2+2=4 Marks]

```
struct abc {int a; int b;} *def;
int main() {
  char *ch = (char *) malloc (4*sizeof(int));
  def = (struct abc *) malloc(5*sizeof(struct abc));
  def→a = 5; def→b = 10;
  def[1].a = 10; def[1].b = 11;
  ++def→a;
  ++def;
  printf ("%d\n", def→a);
  printf ("%d\n", sizeof(def) - sizeof(ch));
  return 0;
}
```

```
10
0
```

QUESTION 6.

a) What will be displayed when the following C program is executed? [1+1+2+2=6 Marks]

```
int rec1 (int);
int rec2 (int);
int main() {
  printf("%d : %d : %d : %d\n", rec1(0), rec1(1), rec2(1022), rec2(2001));
  return 0;
}
int rec1 (int n) {
  if (n == 0) return 1;
  else return ( rec2(n-1) );
}
int rec2 (int m) {
  return ( !rec1(m) );
}
```



```
1 : 0 : 0 : 1
```

b) The following C function **recursively** computes the product of two input parameters **a** and **b**. Fill in the missing lines. [2+2=4 Marks]

```
int recur (int a, int b) {
  if (b == 0) return 0;

  if (b % 2 == 0) return recur (a+a, b/2);

  return recur (a+a, b/2) + a;
}
```

QUESTION 7.

a) We want to convert the cartesian form (**a**, **b**) of a complex number **a** + i**b** to the corresponding polar form **r**(cosθ + i sinθ) representation denoted as (**r**, **θ**). Here **r** and **θ** are related to **a** and **b** as: $\mathbf{r} = (\mathbf{a}^2 + \mathbf{b}^2)^{1/2}$ and $\boldsymbol{\theta} = \tan^{-1}(\mathbf{b}/\mathbf{a})$ in radian. For example, if the complex number 3 + i4 is represented in cartesian form as (3, 4), its polar form representation is (5, 0.927295) since $(3^2+4^2)^{1/2} = 5$ and $\tan^{-1}(4/3) = 0.927295$ radian.

The following C program uses a function **cart2pol** to convert from cartesian to polar form of a complex number. Fill in the missing lines so that the main function can print the correct polar form corresponding to a complex number taken as input in Cartesian form. [1+1+1+1+1=5 Marks]

```
typedef struct {
   float rel; float img;
} cartesian;

typedef struct {
   float mag; float theta;
} polar;

polar *cart2pol (cartesian a) {
  //Dynamically allocate memory

  polar  *b = (polar *) malloc(sizeof(polar));
  //Assign value for magnitude and theta in structure

   b→mag            = sqrt(a.rel*a.rel + a.img*a.img);

   b→theta          = atan(a.img/a.rel);

  return b;
}

int main() {
  cartesian x; polar * y; scanf (“%f%f”, &x.rel, &x.img);

  y = cart2pol(x);                          //Call cart2pol function
  printf("%f %f", y→mag, y→theta);
  return 0;
}
```

b) The following function takes an array of students as input along with the length of the array and sorts the array in lexicographically ascending order of the **lastname**. Fill in the missing lines. You may use appropriate string library function(s). [1+1+1+1+1= 5 Marks]

```
typedef struct  {
   char firstname[30]; char lastname[30];
} fullname;

typedef struct {
   int roll; fullname name;
} student;

void selectionsort (student A[], int size) {
  int i, j, pos;
  student min, temp;
  for (i = 0; i < size - 1; i++) {
    pos = i;

    min = A[i]; //Save current minimum
    for (j = i + 1; j < size; ++j) {
      //Does lastname come before lastname in current minimum

      if (strcmp(A[j].name.lastname,min.name.lastname) < 0                    )
      {

        min = A[j];                          //Update current minimum
        pos = j;
      }
    }
   //Bring the smallest data to the current location of i

    temp = A[i];

    A[i] = A[pos];
    A[pos] = temp;
  }
}
```

QUESTION 8.

a) The following C recursive function **reverse_list** is used to reverse the contents of a linked list. It takes as parameter a pointer to a pointer to the first element of the list. Thus, starting from the head, if the list contains the following sequence of data: {56, 78, 20, 88, 80}, after the function is completed, the list will contain the following sequence of data: {80, 88, 20, 78, 56}. Fill in the missing lines.

[1+1+1+2=5 Marks]

```
struct node {
   int data; struct node *next;
};
void reverse_list (struct node **head) {
  struct node *previous   = NULL;

  struct node *current = *head;
  struct node *next;
  while (current != NULL) {

    next  = current→next;
    current→next = previous;
    previous = current;

    current = next;
  }
    *head = previous;
}
```

This question had a small mistake. The problem statement says "following C recursive function" whereas the code given is non-recursive. This may have created a confusion for some students, though chances are less as the code was quite clearly non-recursive and we have not received a single question or comments on the copy. Nevertheless, it is decided to give full marks to ALL students in this question.

b) The following recursive function **merge_sorted** takes as input the head pointers to two linked lists. In each linked list, the elements are in ascending order. The function returns the head pointer to a new linked list which contains all the elements of the two input lists in ascending order. Fill in the missing lines. Assume that the same definition of **struct node** as given in Question 8(a) is used here.

[1+1+1+1+1=5 Marks]

```
struct node *merge_sorted (struct node *a, struct node *b) {
  struct node *result = NULL;
  // Base cases

  if (a == NULL) return (b);

  else if (b==NULL) return (a);
 //Pick either from a or from b and recur

  if (a→data <= b→data ) {
    result = a;
    result→next = merge_sorted (a→next, b);
  }
  else {
    result = b;
    result→next = merge_sorted (a, b→next);
  }
  return(result);
}
```

QUESTION 9.

a) The following C program uses a stack of size 100 characters with the given definitions of **push()** and **pop()** to determine if an input string is a palindrome or not. A string is said to be a palindrome if it reads the same forward and backward. For example, **abcba** is a palindrome while **abcbc** is not. Assume that **push()** will never be called on a full stack and **pop()** will never be called on an empty stack. Fill in the missing lines. Use appropriate C string library function(s) if needed. [1+2+2+1= 6 Marks]

```
char stack[100]; int top = -1;
void push (char c) { stack[++top] = c; }
char pop() { return (stack[top--]); }

int main() {
  char input_string[30];    int i, count = 0, length;
  scanf("%s", input_string);

  length = strlen(input_string);______________________

  //Push characters in stack

  for (i = 0; i < length; i++) push(input_string[i]);__________________
  for (i = 0; i < length; i++) {
      //Compare with original string by popping from stack

       if (input_string[i] == pop())_______________
             count++;
  }
  if (count == length)__________________          //Check condition for palindrome
    printf("Palindrome");
  else printf("Not palindrome");
  return 0;
}
```


b) What will be the displayed when the following C program is executed? [4 Marks]

```
typedef struct { int list[10]; int front, rear; } queue;
void enqueue(queue *q, int e)  {
  q→rear = (q→rear + 1);
  q→list[q→rear] = e;
}
void dequeue (queue *q) { q→front = (q→front + 1); }
int main() {
  int i = 0; int e = 4; queue *newqueue;
  newqueue = (queue *)malloc(sizeof(queue));
  newqueue→front = 0; newqueue→rear = -1;
  for (i = 0; i < 7; i++) { ++e; enqueue(newqueue,e); }
  for (i = 0; i < 3; i++) dequeue(newqueue);
  for (i = newqueue→front; i < newqueue→rear; i++) printf ("%d ", newqueue→list[i]);
  return 0;
}
```

8 9 10

END SEMESTER AUTUMN 2016-2017

1. Answer in **one** word. **Marks:** $10 \times 1 = 10$

(a) What would be returned to the pointer if dynamic memory allocation fails?

NULL

When no memory is allocated, it does not find the address and thus no address is returned

(b) What is the minimum array size required for storing a string "IIT"?

4

3 bytes and 1 for NULL character

(c) What is the name of the operation for deleting an element from a stack?

POP

(d) Address of which element of an array is also denoted by its name?

First element of index 0

(e) Write the octal code of the binary string 11100001.

341

Make groups of three from the right hand side. Here is
011 100 001 when converted to decimal gives 341
011-3 100-4 001-1

(f) Which header file is required to be included for dynamic memory allocation?

stdlib.h malloc.h

(g) What is the maximum unsigned integer represented by a 16 bit binary number?

$\mathbf{2^{16}-1}$

All the bits are 1. So when converted to decimal it gives the sum of the GP series $2^{15}+2^{14}+2^{13}+2^{12}+\ldots\ldots+2^{0}$

(h) What should be passed as parameter to the function that is called-by-reference?

Address of the variable or the data passed

(i) Which statement should be preferably used for conditional branching at different values of a variable?

switch statement

switch statement passes the value and matches the condition according to the passed value

(j) Name the operator for determining the size of a data type.

sizeof(<data type>)

2. State whether the following statements are True/False. Justify your answer in at most two simple sentences. **Marks:** $5 \times 2 = 10$

(a) A structure should preferably be passed as a parameter to a function using call-by-reference.

TRUE	If a structure is passed by call-by-value, then the changes made in the structure will be local to the function and changes will not be reflected in the function form where the structure was passed.

(b) A string cannot be assigned to another string.

TRUE	Normal equal operators do not allow such assignment operation. But string can be copied using string.h library using strcpy(target,source) function.

(c) Iterative function call is more efficient than a recursive function call for the same computation.

TRUE	Iterative call uses much less memory than recursive function call. When recursion occurs, when function call occurs, the previous function remain in the memory and thus the memory consumption is much higher. This do not occur in recursion.

(d) Dynamically allocated memory should be freed after its use.

TRUE	If memory is kept allocated, then once the reference of that memory is lost the data remains allocated and cannot be used further. But in ANSI C the memory gets free after the code stops execution

(e) Linked list is a more dynamic data structure than a list implemented by an array through dynamic memory allocation.

TRUE	For dynamic link list, we can always increase the memory or decrease the memory. There is no extra memory allocation and memory.

3. Write statements (corresponding to a C program segment) for the following: **Marks:** $5 \times 2 = 10$

(a) Declare an integer variable *i* and a pointer to the corresponding data type *p*. Assign address of *i* to *p*.

```
int i;
int *p;
p=&i;
```

(b) Define a structure consisting of an integer variable *x* and a real variable *y*. Declare a variable *s* corresponding to the structure.

```
typedef struct data
{
    int x; float y;
}s;
```

(c) Declare *p* as a pointer to a pointer of data type int, and allocate an array of 20 pointers to *p* by dynamic memory allocation.

```
int **p;
p=malloc(20*sizeof(int*))
```

As p is a pointer to a pointer, it is a double pointer. As it is an array of pointers.so the sizeof is of an int pointer.

(d) Define a node of a linked list which stores a name of a student as a string not exceeding 50 characters and the student's CGPA as a real number. Declare a variable for a node using the definition.


```
typedef struct
{
    char name[50];
    float cgpa;
    node* next;
}list;
```

(e) Declare a function prototype named *swap* for swapping two integer variables passed as parameters to it. No need to define the function.

```
void swap(int* x, int* y)
```

The parameters must be passed by reference as if it is passed by call-by-value, then the swapping done will not be reflected in the caller function

4. What will be printed when the following programs/ program segments execute? **Marks:** $3+3+4=10$

(a)
```
char frname[12]="Pineapple", drname[12];
int i=0,j=0;

while(frname[i]!='\0') {
    if(i%2==0)
        drname[j]=frname[i];
    i++;   j++;
}
drname[j]='\0';
printf("i=%d j=%d drname=%s \n",i,j,drname);
```

i=9 j=9 drname=Pn

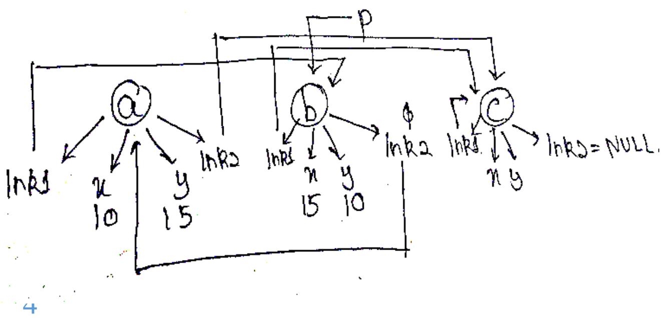
The while loop exits out when the null character is reached. Also j runs along with i. So the odd characters are not assigned. Depending on the compiler, junk characters are assigned. So if NULL character is assigned, then the drname is terminated.

(b)
```
int compute( int n)
{
    if(n<1)
        return(1);
    return(n*compute(n-2));
}

void main()
{

    printf("val1=%d val2=%d val3=%d\n",compute(5),compute(4),compute(-5));
}
```


val1=15 val2=8 val3=1

(c)
```
struct _st {
  int x,y;
  struct _st *lnk1,*lnk2;
  }  a,b,c, *p;

   a.x=b.y=10;   a.y=b.x=15;
   c.x=a.x+b.x;  c.y=a.x+b.y;
   a.lnk1=&b; b.lnk1=&c; a.lnk2=&c; b.lnk2=&a;
   c.lnk1=c.lnk2=NULL;  p=&b;
   printf("val1=%d  val2=%d \n",p->lnk1->x,p->lnk2->y);
   printf("val3=%d val4=%d \n",p->lnk2->lnk1->x, p->lnk2->lnk1->y);
```

val1=25 val2=15
val3=15 val4=10



5. Write C program statements for the following operations. **Marks:** 4 + 6 = 10

(a) Define a node of a circular linked list which contains a complex number.

```
typedef  struct node
{
       float real,imaginary;
       node *next;//pointer to the next node
}node,*list;
```

(b) Assume that the head of the above circular list is pointed by a pointer named `head`. Write a function which takes the head of the list as argument and returns the sum of complex numbers in the list.

```
node sum(list head)
{
    node sum;
    sum.real=sum.imaginary=0.0f;
    while(head->next != NULL)  //Traversing to the next node until NULL found
    {
        sum.real+=head->real;
        sum.imaginary+=head->imaginary;
    }
    return sum;
}
```

6. Write C program segments/statements to serve the following purposes. **Marks:** 3 + 3 + 4 = 10

(a) Define a structure for implementing a stack of integers using an array.

```
typedef struct {
  int data[SIZE]; //the array of the data
  int tos; //the pointer of the top
}stack;
```

(b) Write a function for creating a stack.

```
void create()
{
      stack* s; //creating the stack
      s->tos = -1;  //initializing its head to -1
}
```

(c) Write a function for pushing an integer into the stack.

```
int push(stack *s,int n)
{
  if((s->tos == SIZE-1) // header reached its maximum
    {
      printf("Sorry stack is FULL\n");
      return ERROR;
    }
  ++(s->tos); //increasing the head pointer
  s->data[s->tos]=n; //entering the data
  return 1;
}
```

7. Write a C function which takes an array of *N* integers and returns the range of its value (i.e., maximum value - minimum value). **Marks:** 10

```
int range(int a[],int n)
{
        int max,min;
        max=min=a[0];  //Initializing the both max and min to the first element of the array
        for (int i = 1; i < n; ++i)
         {
           if (a[i]>max)
                max=a[i]; //Initializing the maximum to ar[i] when greater element found
           if (a[i]<min)
                min=a[i]; //Initializing the minimum to ar[i] when lesser element found
         }
         return (max-min);
}
```

8. Write C program statements for the following operations. **Marks:** $2 \times 5 = 10$

(a) Dynamically allocate a 2-D array of characters to a pointer for storing 100 strings each of maximum character length 80.

```
char **p; //Declaring the two dimensional array
p=(char**)malloc(100*sizeof(char*)); //Array of 100 strings
for(int i=0;i<100;i++)
  p[i]=(char*)malloc(80*sizeof(char));//Array of 80 characters
```

(b) Read four command line arguments and print them as strings in the same order.

```
#include <stdio.h>
int main(int argc, char const *argv[])
{
   //The first one is the executive file ./filename
   printf("%s\n",argv[1] );
   printf("%s\n",argv[2] );
   printf("%s\n",argv[3] );
   printf("%s\n",argv[4] );
   return 0;
}
```

9. Write a C program with loop to evaluate the following series summation with an accuracy of 5th decimal place. **Marks:** 10

$$x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots$$

```
#include <stdio.h>
#include <math.h>

int main()
{
        float sum=0.0f;
        float diff=1.0f,x;  //The difference between the consecutive terms
        int cnt=0;   // the counted no of terms
        float fact=1;  //Calculating the factorial of the terms
        float termPrev=0;  // As a copy for the previous term
        printf("Enter value of x\n");
        scanf("%f",&x);
        while(diff>.00001)
        {
                cnt++;
                fact=fact*cnt;   //finding the factorial
                float term=pow(x,cnt)*pow(-1,cnt-1)/fact;  //producing the terms
                diff = fabs(term-termPrev);   //difference between the consecutive terms
                termPrev=term;
                sum+=term;
        }
        printf("required sum = %f\n",sum);
}
```

10. Fill in the gaps of the following statements for the operations mentioned in parantheses beside the questions. **Marks:** $10 \times 1 = 10$

(a) `int x=` ***(float)*** `5.0;` (type cast).

(b) `struct cplx{ float a,b}; struct cplx m=` ***cplx(4.5,4.9)*** .
(Initialize to values of member variables a and b to 4.5 and 4.9, respectively.)

(c) `int x,*p; p=&x;` ***&p*** `= 10;`
(Assign 10 to the variable `x` accessed through the pointer `p`.)

(d) `char name[20]="Kharagpur",dst[20];` ***strcpy(dst , name)***
(Copy the content of the string `name` to `dst` using string library function.)

(e) `int x; float y;` ***scanf(" %d %f " ,x , y)***
(Read variables x and y from keyboard.)

(f) `char line[80];` ***scanf("%[^\n]", line)***
(Read a line in the character array line as a string from the keyboard. The line may or may not contain space(s). You need to store space(s).)

(g) `char *p, string="Mango"; p=` ***&string[0] ;***
(The string should be accessed through the pointer p.)

(h) `#define sqr(x)` ***x*x***
(Implement squaring of x using the macro definition.)

(i) `int p[3][2]=` **Please clarify from your professor**
(Initialize the array with natural numbers starting from 1.)

(j) `FILE *fp=` ***fopen("input.txt","r")*** ;
(Open the file named "input.txt" in read mode.)

CHEMISTRY

MID - SPRING SEMESTER EXAMINATION**2017-2018**

1. **Each question carries 1 mark.**
2. **Write the correct option for each question in the *Table provided.***

i) $T - p\left(\frac{\partial T}{\partial p}\right)_S$ is equal to

A. $\left(\frac{\partial U}{\partial S}\right)_V$

* B. $\left(\frac{\partial U}{\partial S}\right)_p$

C. $\left(\frac{\partial U}{\partial V}\right)_S$

D. $\left(\frac{\partial U}{\partial p}\right)_S$

Explanation: $T - p(\delta T/\delta p)_S = T - p(\delta V/\delta S)_p$
$\delta U = -p\delta V + T\delta S$

$(\delta U/\delta S)_p = -P(\delta V/\delta S)_p + T$

ii) Carnot's engines working between two fixed temperatures of the source a

* **A. Will have same efficienfy**

B. Efficiency will depend on the working substance

C. Efficiency will depend on the explicit values of the temperatures

D. Efficiency cannot be defined

$\eta = 1 - (T_c/T_h)$ It depends just only on Source and the Sink, T_c is temperature of sink and T_h is the temperature of the source.

iii) Which of the following is an incorrect definition of chemical potential of one component (i) in a mixture?

* A. $\left(\frac{\partial U}{\partial n_i}\right)_{T,p,n_{j\neq i}}$

B. $\left(\frac{\partial H}{\partial n_i}\right)_{S,p,n_{j\neq i}}$

C. $\left(\frac{\partial A}{\partial n_i}\right)_{T,V,n_{j\neq i}}$

D. $\left(\frac{\partial G}{\partial n_i}\right)_{T,p,n_{j\neq i}}$

iv) If the chemical potential of a component in a binary mixture increases, then that of the other component will

*A. Decrease

B. Increase

C. May increase or decrease

D. Remains constant

For binary mixture , $n_1\delta\mu_1 + n_2\delta\mu_2 = 0$

$\Rightarrow \delta\mu_1 = -(n_2/n_1)\delta\mu_2$

Thus $\delta\mu_1$ increases then $\delta\mu_2$ has to decrease (increase in -ve value)

v) A heat engine operates in cycles to keep a room at 20° C when the outside temperature is 3° C. What is the minimum work that must be done to supply 100 kJ of heat per cycle to the room?

A. 94.2 J

B. 94.2 kJ

* C. 5.8 kJ

D. 5.8 J

$\eta = \frac{workdone(w)}{heatgiven(Q)}$ = 1 - $\frac{T_{sink}}{T_{source}}$

$\Rightarrow$ work done (w) = Q(1 -$\frac{T_{sink}}{T_{source}}$)

$\Rightarrow$ W = 100 (1 -$\frac{3+273}{20+273}$)

$\Rightarrow$ W = (100$\times\frac{17}{293}$)KJ $\Rightarrow$ W = 5.8KJ

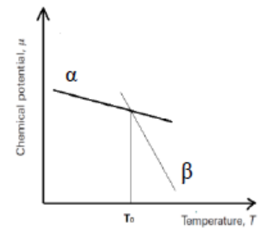
vi. Calculate the change in entropy of a mole of aluminium (Atomic weight 27) which is heated from 600 to 700^0 C. The melting point of Al is 660^0 C, the heat of fusion is 393 Jg^{-1} and the heat capacities of the solid and liquid may be taken as 31.8 and 34.3 $JK^{-1}mol^{-1}$ respectively.

*A. 14.92 J $K^{-1}mol^{-1}$

B. 3.97 J $K^{-1}mol^{-1}$

C. 21.12 J $K^{-1}mol^{-1}$

vii. The chemical potential, μ of a pure substance in the phases α and β is shown as a function of temperature, T at the pressure p_1. Given, the molar volume of phase α is larger than that of phase β. Identify the **INCORRECT** statement about the phase equilibrium, $\alpha \rightleftharpoons \beta$.



A. At the temperature T_0, the phases α and β coexist in equilibrium.

B. For pressures $p_2 > p_1$, $\mu_i(T, p_1) < \mu_i(T, p_2)$ for $i = \alpha, \beta$.

* C. **On decreasing the pressure, the equilibrium temperature, T_0 would shift to a lower value.**

D. At temperatures $T \gg T_0$, the system exists in the phase β.

glhkgk

viii. The equilibrium constant, K_{eq} for a particular chemical reaction between 300-800 K is found to obey the equation. $\ln K_{eq} = -a - \frac{b}{T} + \frac{c}{T^2}$, where a, b and c are positive constants over the given range of temperature. What is the value of standard reaction enthalpy, $\Delta_r H^0$?

A. $\Delta_r H^0 = -b,$

* B. $\Delta_r H^0 = R\left[b - \frac{2c}{T}\right]$

C. $\Delta_r H^0 = R\left[b - \frac{c}{T^3}\right]$

D. $\Delta_r H^0 = R\left[b + \frac{2c}{T}\right]$

ix. Comparing the densities of the solid, liquid and vapour phases of a substance, it was found that the densities (ρ) were such that $\rho_{solid} > \rho_{liquid} > \rho_{gas}$. Under what conditions is it impossible for the liquid to exist?

A. All pressures above the triple point and all temperatures below the critical point

* B. **All pressures below the triple point and all temperatures above the critical point**

C. All pressures above the triple point and all temperatures above the critical point

D. All pressures below the triple point and all temperatures below the critical point

x. One mole of an ideal gas initially at 25 °C and 1 atm is heated at constant pressure until it expands to a volume of 35L. You are given that $C_P = 20.8\ JK^{-1}mol^{-1}$. What are the final temperature of the gas and the entropy change of the gas?

A. T_{final} = 208 K; ΔS = - 7.48 $JK^{-1}mol^{-1}$

B. T_{final} = 427 K; ΔS = 7.48 $kJK^{-1}mol^{-1}$

C. T_{final} = 208 K; ΔS = 7.48 $JK^{-1}mol^{-1}$

*D. **T_{final} = 427 K; ΔS = 7.48 $JK^{-1}mol^{-1}$**

2. Show that $\left(\frac{\partial S}{\partial V}\right)_T = \frac{R}{V_m - b}$ for a van der Waals gas with molar volume V_m. [5]

Ans. $(\delta S/\delta V)_T = (\delta P/\delta T)_V$ [Maxwell Relation]

P = nRT/(V-nb) - n^2a/V^2 = RT/($V_m - b$) - a/V_m^2

Hence, $(\delta S/\delta V)_T = (\delta P/\delta T)_V$ = R/ (V_m - b)

3. (a) Calculate the difference in slope of the chemical potential against temperature on either side of the normal freezing point of water. Enthalpy of fusion of water is 6.0 kJ/mol.

(b) At 50.14 K a substance has a vapour pressure of 258.9 torr. Calculate its heat of vaporization in kJ/mol if it has a vapour pressure of 161.2 torr at 277.5 K. [3+2]

(a) Diff in slope of μ vs T,

$(\delta\mu/\delta T)_{P,liq} - (\delta\mu/\delta T)_{P,solid}$

$= (-S_{liq}) - (-S_{solid}) = -\Delta S_{trs}$

$= -\Delta H_{\Delta} trs/T_{trs}$

Slope = -(6 KJ/mol)/273K = - 21.97 J/mol . K

(b) ln ($P_2/P_1 = -\Delta H_{vap}/R(\frac{1}{T_2} - \frac{1}{T_1})$

$p_1 = 258.9 torr$ $\quad$ $T_1 = 50.14K$

$p_2 = 161.2 torr$ $\quad$ $T_1 = 277.5K$

$\Delta H_{vap} = -0.241$ KJ/mol

4. (a) One kilogram of air is heated reversibly at constant pressure from an initial state of 300 K and 1bar until its volume triples. Calculate w, q, ΔU **for the process. Assume air obeys the relation** $PV/T = 83.14$ **bar cm^3 mol^{-1}K^{-1} and that $C_p = 29$ Jmol^{-1}K^{-1}. Molecular mass of air at STP = 28.97 g mol^{-1} [1 J = 10 bar cm^3].** [3]

ANS: Molecular mass of air = 28.97 g/mol
$\Rightarrow$ number of moles = 1000g /28.97 g/mol
= 34.52
Isobaric process $\Rightarrow$ P constant $\Rightarrow \frac{V_1}{T_1} = \frac{V_2}{T_2}$
Final volume $V_2 = 3V_1 \Rightarrow \frac{V_1}{300} = \frac{3V_1}{T_2}$
Final temperature = 900K
q = n$C_p\Delta T$ =34.52 × 29 × (900 – 300)
= 600.65 KJ
$\Delta U = nC_v\Delta T$ =34.52 × (29 – 8.314) × (900 – 300) = 428.45KJ
From first law $\Delta U = q + W \Rightarrow W = \Delta U - q$ = (428.45 - 600.65)KJ = -172.2KJ [Alternatively , W may be calculated from using then expression W =PΔV]

(b) Write an expression for the differential of the enthalpy, $H(p,T)$ of a pure substance terms of partial derivatives of H with respect to p and T. Use this to fill in the blanks i expression for $\left(\frac{\partial H}{\partial T}\right)_V$ with system properties such as C_V, C_p, α, κ etc. [2]

ANS: $dH = (\delta H/\delta P)_T dP + (\delta H/\delta T)_P dT$

$(\delta H/\delta T)_V = (\delta H/\delta P)_T(\delta P/\delta T)_V + C_P($ from definition of $C_P = (\delta H/\delta T)_P)$

Also $\alpha = \frac{1}{V}(\delta V/\delta T)_P$ and $\kappa = \frac{-1}{V}(\delta V/\delta P)_T$

From the cyclic rule applied to P,V,T

$(\delta V/\delta P)_T(\delta V/\delta T)_P(\delta T/\delta P)_V = -1$

$\frac{\alpha}{\kappa} = \frac{1}{V}(\delta V/\delta T)_P.(-V)(\delta P/\delta V)_T = (\delta P/\delta T)_V$

$\Rightarrow (\delta H/\delta T)_V = (\delta H/\delta P)_T\frac{\alpha}{\kappa} + C_P$

$(\delta H/\delta T)_V = (\delta H/\delta P)_T\frac{\alpha}{\kappa} + C_P$

5. Consider the chemical reaction: $A_2 \rightleftharpoons 2A$ at a temperature, $T = 298$ K and pressure, p for which the equilibrium constant, $K_{eq} = \frac{1}{e}$.

An experiment is designed so that (i) the reaction is arrested at a given stage (ξ), (ii) the degree of dissociation of A_2 determined in the arrested reaction mixture and (iii) the reaction Gibbs energy, $\frac{\Delta_r G}{RT}$ estimated for the given ξ .

(i)	The standard reaction Gibbs energy, $\frac{\Delta_r G^0}{RT}$	**+1.0**
(ii)	The reaction Gibbs energy, $\frac{\Delta_r G}{RT}$ for 10% dissociation of A_2	**-2.124 (option 1)** **-2.219 (option 2)**
(iii)	The reaction Gibbs energy, $\frac{\Delta_r G}{RT}$ for 90% dissociation of A_2	**+4.478 (option 1)** **+3.836 (option 2)**

(iii)	The reaction Gibbs energy, $\frac{\Delta_r G}{RT}$ for 90% dissociation of A_2	**+4.478 (option 1)** **+3.836 (option 2)**
(iv)	The spontaneous chemical reaction for 10% dissociation of A_2	$\boldsymbol{A_2 \rightarrow 2A}$
(v)	The spontaneous chemical reaction for 90% dissociation of A_2	$\boldsymbol{2A \rightarrow A_2}$

1X5=5

(i) $\frac{\Delta_r G^0}{RT} = -\ln K_{eq} = -\ln\left(\frac{1}{e}\right) = +1$

(ii - v) For the reaction Gibbs energy, $\frac{\Delta_r G}{RT} = \frac{\Delta_r G^0}{RT} + \ln Q$

<u>Option 1:</u> $Q = \frac{4\alpha^2}{1-\alpha}$ if α = degree of dissociation per mole of A_2 at a given stage of the forward reaction

<u>Option 2:</u> $Q = \frac{4\alpha^2}{1-\alpha^2}$ if α = degree of dissociation per mole of A_2 at a given stage of the forward reaction and K_{eq} is calculated in terms of mole fraction

	α	Q	$\frac{\Delta_r G}{RT} = 1 + lnQ$	$\Delta_r G$	Spontaneous reaction
Option 1	0.1	0.044	-2.124	< 0	$A_2 \rightarrow 2A$
	0.9	32.4	4.478	> 0	$2A \rightarrow A_2$
Option 2	0.1	0.040	-2.219	<0	$A_2 \rightarrow 2A$
	0.9	17.05	3.836	>0	$2A \rightarrow A_2$

CHEMISTRY

MID - SEMESTER EXAMINATION **2016-2017**

PART – A

Q1. Write the correct option/ options (A/B/C/D) in the Answer Script ....(1 × 10 = 10)

(i) Which of the following statements is correct for an ideal gas

(A) $\left(\frac{\partial H}{\partial V}\right)_T = 0$; $\left(\frac{\partial U}{\partial V}\right)_T \neq 0$; (B) $\left(\frac{\partial H}{\partial V}\right)_T = 0$; $\left(\frac{\partial U}{\partial V}\right)_T = 0$;

(C) $\left(\frac{\partial H}{\partial V}\right)_T \neq 0$; $\left(\frac{\partial U}{\partial V}\right)_T = 0$; (D) $\left(\frac{\partial H}{\partial V}\right)_T \neq 0$; $\left(\frac{\partial U}{\partial V}\right)_T \neq 0$

ANS: (B)

(ii) Activity of a '***m***' molal solution of $FeCl_3$ electrolyte is

(A) $9(\gamma_\pm\ m)^4$; (B) $(\gamma_\pm\ m)^4$; (C) $(\gamma_\pm\ m^4)$; (D) $27(\gamma_\pm\ m)^4$

ANS: (D)

Reason: $mFeCl_3 \rightleftharpoons mFe^{3+} + 3mCl^-$

Activity = $(\gamma_\pm\ m) \times (3(\gamma_\pm\ m))^3 = 27(\gamma_\pm\ m)^4$

(iii) Which of the following expressions is correct for a closed system involved in expansion-compression work?

(A) $dH = TdS + VdP$; (B) $dH = TdS - PdV$; (C) $dH = TdS - VdP$; (D) $dH = TdS + PdV$

ANS: (A)

(iv) Which of the following relations is true for the mixing of two ideal gases?

(A) $(\partial \Delta_{mix}G/\partial P)_T = 0$; (B) $\Delta_{mix}G = 0$; (C) $\Delta_{mix}A = 0$; (D) $(\partial\Delta_{mix}G/\partial T)_P = 0$

ANS: (A)

Reason: $dG = -SdT + VdP$ $(\partial \Delta_{mix}G/\partial P)_T = \Delta_{mix}V$

For ideal mixing $\Delta_{mix}V$ is 0

(v) At the triple-point in the phase diagram of a pure substance the vapour pressures are related as

(A) $p_{solid} = p_{liq} \neq p_{vap}$; (B) $p_{solid} \neq p_{liq} = p_{vap}$; (C) $p_{solid} \neq p_{liq} \neq p_{vap}$; (D) $p_{solid} = p_{liq} = p_{vap}$

Ans: (D)

(vi) Which of the following expression does not represent the chemical potential (μ_i) of a component i in a homogeneous mixture:

(A) $\mu_i = \left(\frac{\partial H}{\partial n_i}\right)_{T,P,nj\neq i}$; **(B)** $\mu_i = \left(\frac{\partial G}{\partial n_i}\right)_{T,P,nj\neq i}$; **(C)** $\mu_i = \left(\frac{\partial A}{\partial n_i}\right)_{T,V,nj\neq i}$; **(D)** $\mu_i = \left(\frac{\partial U}{\partial n_i}\right)_{S,V,nj\neq i}$

ANS: (A)

(vii) What will be the thermal efficiency of a Carnot heat engine that receives 500 kJ of heat per cycle from a high temperature heat reservoir at 627 °C and rejects heat to a low temperature heat reservoir at 27 °C?

(A) 0.333; **(B)** 0.6; **(C)** 0.4; **(D)** 0.6667

ANS: (D)

(viii) In what proportion of mole fraction (x) should hexane and heptane be mixed to obtain the highest value of entropy of mixing (ΔS_{mix})?

(A) $x_{\text{heaxane}} = 0.4$, $x_{\text{heptane}} = 0.6$; **(B)** $x_{\text{heaxane}} = 0.6$, $x_{\text{heptane}} = 0.4$;
(C) $x_{\text{heaxane}} = 0.5$, $x_{\text{heptane}} = 0.5$; **(D)** $x_{\text{heaxane}} = 0.25$, $x_{\text{heptane}} = 0.75$

ANS: (C)

(ix) To experience cooling under Joule-Thomson expansion, the gas must have the initial temperature set at

(A) Above the upper inversion temperature; **(B)** Between the upper and lower inversion temperatures;
(C) Below the lower inversion temperature; **(D)** Any temperature

ANS: (B)

(x) The lowering of the chemical potential of a species in an ideal mixture (vapor or liquid) is a consequence of

(A) Enthalpy of mixing; **(B)** Change in volume on mixing;
(C) Entropy of mixing; **(D)** Le Châtelier's principle

ANS: (C)

Q2. (a) Two moles of an ideal gas occupying volume of 44.8 litres at 273 K and 1 atm pressure have been expanded reversibly and isothermally. Calculate the final volume of the gas if the heat absorbed during the process is 3 kJ.

[3]

(b) Two moles of an ideal monatomic gas initially at 1 atm and 300 K are put through the following cycle consisting of three steps, all of which are reversible.

Step-I: Isothermal compression to 2 atm.

Step-II: Isobaric temperature change to 400 K

Step-III: Return to the initial state following the path $P = a + bT$, where a, b are constants.

(i) Sketch the cycle on a P-T diagram.

(ii) Calculate the entropy change (ΔS) associated with each of the three steps of the cycle (given: molar heat capacity of the gas, C_P=2.5$\boldsymbol{R}$).

[2 + 3 =5]

(c) The vapour pressure of solid ammonia in torr is found to obey the equation:

$$\ln P = -\frac{4124.4}{T} - 1.82 \ln T + 34.48$$

Use the Clausius-Clapeyron equation to determine the molar enthalpy of sublimation of ammonia at 170 K.

[2]

Ans. 2.(a) n = 2moles

$V_i = 44.8lit$

$T_i = 27.3K$

$P_i = 1atm$

expanded reversibly and isothermally

So $\Delta U = q + W$

$W = -nRTln(\frac{v_2}{v_1}$

$\rightarrow q = -W$

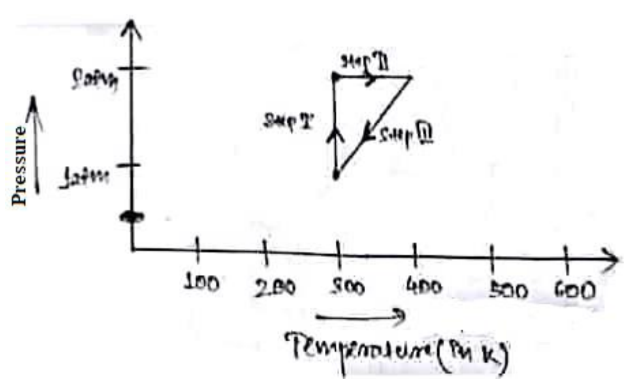
$\rightarrow 3000 = nRTln(\frac{v_2}{v_1}$

$\rightarrow ln(\frac{v_2}{v_1} = \frac{3000}{2\times8.314\times273}$

$v_2 = 86.755lit$

Final volume of the gas = 86.755lit.

2.b) (I)



(II) $dS = \frac{dq_{rev}}{T} = \frac{dU - W_{rev}}{T} = \frac{C_v dT + PdV}{T} = \frac{C_v dT}{T} + nR\frac{dV}{V}$
$\therefore \int dS = C_v \int \frac{dT}{T} + nR\int \frac{dV}{V}$
$\rightarrow \Delta S = C_v ln(\frac{T_2}{T_1}) + nRln(\frac{P_1}{P_2})$
For step (I)
n = 2
$C_p = 2.5R$ $\quad P_2 = 2atm$
$\therefore \Delta S_1 = 2.5Rln(1) + nRln(1/2) = -11.53J/K$
$P_1 = 1atm$
$T(constant)$
$\Delta S_1 = -11.53J/K$

2.c) lnP $= \frac{-4124.4}{T} - 1.82(lnT) + 34.48$
Clausius clayperon Equation :
$\frac{d(lnP)}{dP} = \frac{\Delta H_{vap}}{RT^2}$
$\therefore \frac{d(lnP)}{dT} = \frac{4124.4}{T^2} - \frac{1.82}{T} = \frac{\Delta}{H}_{sub}RT^2$
$\rightarrow \frac{4}{T^2}[4124.4 - \frac{\Delta H_{sub}}{R}] = \frac{1.82}{T}$
$\rightarrow 4124.4 - \frac{H_{sub}}{8.314} = 1.82 \times 170$
$\rightarrow \frac{H_{sub}}{8.314} = 4124.4 - 309.4$
$= 3815$
$\rightarrow \Delta H_{sub} = 3815 \times 8.314 = 8172KJ$
molar enthalpy of sublimation of ammonia at 170K = 8172KJ

Q3. (a) For a real gas with molar volume and molar enthalpy of V_m and H_m, respectively, show: $\left(\frac{\partial H_m}{\partial P}\right)_T = V_m(1 - \alpha T)$, where α is the isobaric expansion coefficient.

[3]

(b) For a van der Waals gas, the compressibility factor Z is given by
$Z = 1 + \frac{(b - a/RT)P}{RT}$, where a and b are constants.

If $C_{P,m}$ and μ_{JT} are the molar heat capacity and Joule-Thomson coefficient of the gas, respectively, then show that $C_{P,m} \times \mu_{JT} = \left(\frac{RT^2}{P}\right)\left(\frac{\partial Z}{\partial T}\right)_P$.

[3]

*(c) Two moles of supercooled water at – 10 °C is converted into ice at – 10 °C and 1 atm. State whether the process is reversible or irreversible. Calculate the value of ΔS_{sys}, assuming $C_{P,m}$ values of water and ice in the temperature range –10 °C to 0 °C are 75 $JK^{-1}mol^{-1}$ and 38 $JK^{-1}mol^{-1}$, respectively. $\Delta_{fus}H = 6006\ J\ mol^{-1}$.

[4]

2

3.(a) To prove,
$(\frac{\delta H_m}{\delta P})_T = V_m(1 - \alpha T)$
Using maxwell relation we know,

$\delta H = V\delta P + T\delta S$
$\rightarrow (\frac{\delta H}{\delta P})_T = V + T(\frac{\delta S}{\delta P})_T$
We know ,
$(\frac{\delta S}{\delta P})_T = -(\frac{\delta V}{\delta T})_P$
$\therefore (\frac{\delta H}{\delta P})_T = V - T(\frac{\delta V}{\delta T})_P$
but $\frac{1}{V_m}(\frac{\delta V_m}{\delta T})_P = \alpha \rightarrow (\frac{\delta V_m}{\delta T})_P = \alpha V_m$
$\therefore (\frac{\delta H_m}{\delta P})_T = V_m - V_m \alpha T$
$\therefore (\frac{\delta H_m}{\delta P})_T = V_m(1 - \alpha T)$
(Proved)
3.b) To prove: $Cp_m \mu_{gT} = (RT^2)(\frac{\delta z}{\delta T})_P$
We know , $Cp_m \times \mu_{gT} = (\frac{\delta H}{\delta T})_P(\frac{\delta T}{\delta P})_T$ as [$(\frac{\delta H}{\delta T})_P(\frac{\delta T}{\delta P})_H(\frac{\delta P}{\delta H})_T = -1$]
$\delta H = T\delta S + V\delta P$
$(\frac{\delta H}{\delta P})_T = T(\frac{\delta S}{\delta P})_T + V$
from maxwell's relation $(\frac{\delta S}{\delta P})_T = -(\frac{\delta V}{\delta T})_P$
$\rightarrow (\frac{\delta H}{\delta P})_T = V - T(\frac{\delta V}{\delta T})_P$
$= Cp_m \mu_{gT} = T(\frac{\delta V}{\delta T})_P - V$
now Z $= \frac{PV}{RT} = 1 + \frac{(b - a/RT)P}{RT}$
$\rightarrow PV = RT + (b - a/RT)P$

3.(c) The process is irreversible
$\Delta S_{sys} = \int \frac{dq_{rev}}{T}$
$-$ 10°(supercooled water) $(Q_1) \longrightarrow$ 0°(water)$Q_2 \longrightarrow (Q_3) -$ 10°(ice)
$dQ_1 = 2lp_m dT$
$\Delta l_1 = 2\int \frac{Cp_m dT}{T} = 150ln(T)]_{263}^{273}$
$\Delta S_1 = 150ln(\frac{273}{263})J/K$
$dQ_2 = -2\Delta H_{fus} = -2 \times 6006 = -12012J$
$\Delta S_2 = \frac{-12012}{273} = -44J/K$
$dQ_3 = 2Cp_m dT$
$\Delta S_3 = 2\int \frac{Cp_m dT}{T} = 76ln(T)]_{237}^{263} = 76ln(\frac{263}{273})J/K$
Entropy is a state function,
$\Delta S_{rev} = \Delta S_{irrev} = 150ln(\frac{273}{263}) - 76ln(\frac{273}{263}) - 44$
$\rightarrow \Delta S_{sys} = 74ln(\frac{273}{263}) - 44$
$= -41.24J/K$
$Answer : \Delta S_{sys} = -41.24J/K$
$\rightarrow V = \frac{RT}{P} + (b - \frac{a}{RT})$
$\rightarrow (V - b + \frac{a}{RT}) = \frac{RT}{P}$
differentiating both sides w.r.t T at constant P.
$\rightarrow (\frac{\delta V}{\delta T})_P - \frac{a}{RT^2} = \frac{R}{P}$
multiplying both sides by T.
$\rightarrow T(\frac{\delta V}{\delta T})_P - \frac{a}{RT} = \frac{RT}{P}$
$\rightarrow V - T(\frac{\delta V}{\delta T})_P = V - \frac{a}{RT} - \frac{RT}{P} - - - - - - - - -$
replacing V as in equation(2) in (1)
$\rightarrow V - T(\frac{\delta V}{\delta T})_P = b - \frac{2a}{RT}$
$\therefore Cp_m \times \mu_g T = \frac{2a}{RT} - b(L.H.S)$
$(R.H.S)$
$z = 1 + \frac{(b - a/RT)p}{RT}$
$(\frac{\delta z}{\delta T})_P = \frac{-bT}{RT^2} + \frac{2aP}{R^2T^3}$
$\frac{RT^2}{P}(\frac{\delta z}{\delta T})_P = -b + \frac{2a}{RT}$
$\therefore \frac{RT^2}{P}(\frac{\delta z}{\delta T})_P = \frac{2a}{RT} - b(R.H.S)$
$(L.H.S) = (R.H.S)$

$\therefore Cp_m \times \mu_g T = (\frac{RT^2}{P})(\frac{\delta z}{\delta T})_P$
$(proved)$

Q4. (a) In a fuel cell, methane gas undergoes the same reaction as the combustion process to produce CO_2 (*g*) and $H_2O(l)$ and generate electricity. Calculate the maximum electrical work that can be obtained from 1 mole of methane gas at 25 °C, if $\Delta_f G°$ values of $CH_4(g)$, $CO_2(g)$, and $H_2O(l)$ at 25 °C are given to be −50.8, −394.5, and −237.3 kJ mol^{-1}, respectively.

[2]

(b) For the reaction: $\mathbf{U}(s) + \frac{3}{2}\mathbf{H_2}(g) \leftrightarrow \mathbf{UH_3}(s)$; the equilibrium pressure of H_2 (g) over solid uranium and uranium hydride (UH_3) at 500 K is 1.04 Torr. Calculate the standard Gibbs energy of formation of UH_3 (s) at 500 K. Assume that the H_2 gas behaves like an ideal gas at this low pressure.

[4]

(c) Determine the standard electrode potential of a cell for which the total reaction is as follows:

$$Co^{3+}(aq.) + 3Cl^-(aq.) + 3Ag(s) \rightarrow 3AgCl(s) + Co(s)$$

[Given: $E°_{(AgCl/Ag, Cl^-)} = +0.22$ V; $E°_{(Co^{3+}/Co^{2+})} = +1.81$ V; $E°_{(Co^{2+}/Co)} = -0.28$ V]

[4]

....

4.(a) Combustion reaction of methane :
$CH_4 + 2O_2 \longrightarrow 2H_2O + CO_2$
$\Delta G°_{reaction} = \Delta G°_{H_2O} + \Delta G°_{CO_2} - \Delta G°_{CH_4}$
$\Delta G°_{O_2} = 0$(pure element)
$\rightarrow \Delta G°_{reaction} = [2 \times (-237.3) + (-394.5) + 50.8] KJ/mol$
$= -818.3 KJ/mol$
Thus maximum electrical work that can be obtained from 1 mole of methane gas = 818.3KJ.
4.a) $CH_4 + 2O_2 \longrightarrow CO_2 + 2H_2O$
$\therefore \Delta G_{net} = -394.5 + 2(-273.3) - (-50.8)$
$[\Delta G_{product} - \Delta G_{reactants} \longrightarrow Hess$ Law
=-818.3KJ 4.b) $U(s) + \frac{3}{2}H_2(g) \longrightarrow UH_3(s)$
$K_p = \frac{P_{UH_3}(s)}{P_{H_2}^{3/2} P_U(s)}$
$P_{UH_3} = 1 = P_U$ as U and UH_3 are solid
$K_p = (P_{H_3})^{-2/3} = 0.97 Torr^{-2/3}$
$\Delta G = -RTln(K_p) = -R \times 500 ln(0.97)$
$\therefore \Delta G = 15.23R$
4.c)$Co^{3+} + e^- \longrightarrow Co^{+2} \qquad E_1 = 1.81V$
$Co^{2+} + 2e^- \longrightarrow Co \qquad E_2 = 0.28V$
$\therefore E_{net} = \frac{E_1 + 2E_2}{3} = 0.417$
$\therefore Co^{3+} + 3e^- \longrightarrow Co \qquad 0.417V$
$3Cd^- \longrightarrow 3Cd + e^- \qquad -0.22V$

$\therefore E_{net} = (0.417 - 0.22)V = 0.197V$
When half cells are $E_{net} = \frac{n_1 E_1 + n_2 E_2}{n_1 + n_2}$
When cell reactions are added $E_{net} = E_+ + E_-$ without multiplication of coefficient

CHEMISTRY

MID-AUTUMN SEMESTER **2016-17**

Q1. Write the correct option (a/b/c/d) in the Answer Script ....(1 × 10 = 10)

(i) Chemical potential μ_i of a component i in a homogeneous mixture can be defined as:

(a) $\mu_i = \left(\frac{\partial G}{\partial n}\right)_{T,P,n_j}$; **(b)** $\mu_i = \left(\frac{\partial H}{\partial n}\right)_{T,P,n_j}$; **(c)** $\mu_i = \left(\frac{\partial A}{\partial n}\right)_{T,P,n_j}$; **(d)** $\mu_i = \left(\frac{\partial U}{\partial n}\right)_{T,P,n_j}$

(ii) What will be the activity for 'm' molal solution of an AB_2 electrolyte?

(a) $(\gamma_{\pm}\ m^3)$; **(b)** $(\gamma_{\pm}\ m)^3$; **(c)** $4(\gamma_{\pm}\ m)^3$; **(d)** $(\gamma_{\pm}\ m_{\pm}^{\ 3})$

(iii) Which of the following statements is always true for a liquid mixture of two components A and B in equilibrium with a mixture of their vapours?

(a) $\mu_A(l) = \mu_B(l)$ and $\mu_A(g) = \mu_B(g)$; **(b)** $\mu_A(l) = \mu_A(g) = \mu_B(l) = \mu_B(g)$
(c) $\mu_A(l) = \mu_A(g)$ and $\mu_B(l) = \mu_B(g)$; **(d)** $\mu_A(l) \neq \mu_A(g) \neq \mu_B(l) \neq \mu_B(g)$

(iv) Which of the following is a reversible process?

(a) Melting of ice at 0°C and 1 atm.; **(b)** Melting of ice at 25°C and 1 atm.;
(c) Evaporation of water at 25°C and 1 atm. **(d)** Freezing of water at –10°C and 1 atm.

(v) At inversion temperature of a gas, the value of Joule-Thomson coefficient (μ_{JT}) will be:

(a) $\mu_{JT} = 0$; **(b)** $\mu_{JT} > 0$; **(c)** $\mu_{JT} < 0$; **(d)** $\mu_{JT} \approx \infty$

(vi) What will happen to the chemical potential of O_2 when 1.0 mole of O_2 gas is added to a container that already contained 1.0 mole of O_2 gas?

(a) $\mu(O_2)$ will increase; **(b)** $\mu(O_2)$ will decrease; **(c)** $\mu(O_2)$ will remain unchanged

(vii) Which one is the correct condition for spontaneous reaction in an electrochemical cell

(a) $\Delta G_{T,P} < 0$, $E_{cell} > 0$; **(b)** $\Delta G_{T,P} > 0$, $E_{cell} < 0$
(c) $\Delta G_{T,V} < 0$, $E_{cell} < 0$; **(d)** $\Delta G_{T,P} = 0$, $E_{cell} = 0$

(viii) What fraction of the total quantity of heat (q_h) taken from the source that is at temperature T_h can be converted into work in a reversible cyclic process? (ΔT is the temperature difference between the source and the sink)

(a) $\Delta T \times T_h$; **(b)** $\Delta T / T_h$; **(c)** zero ; **(d)** $T_h / \Delta T$.

(ix) Which one of the following fundamental equations is/ are applicable for any open system?

(a) $dG = VdP - SdT$; **(b)** $dG = VdP - SdT + \Sigma\mu_i dn_i$;
(c) $dG = PdV + TdS + \Sigma\mu_i dn_i$; **(d)** $dG = PdV - SdT + \Sigma\mu_i dn_i$

(x) $\Delta S > 0$ is a condition for spontaneity for which of the following systems:

(a) Closed system; **(b)** Open system; **(c)** Isolated system; **(d)** All systems

Ans 1)

(i) Ans: (a) $\mu_1 = \left(\dfrac{\partial G}{\partial n}\right)_{T,P,n_j}$
Reason: Definition

(ii) Ans: (c) $4(\gamma_{\pm}m)^3$
Reason: $mAB_2 \rightleftharpoons mA + 2mB$
activity $= (\gamma_{\pm}m)(\gamma_{\pm}2m)^2$

(iii) (c) $\mu_A(l) = \mu_A(g)$ and $\mu_B(l) = \mu_B(g)$
Reason: For the components to be in equilibrium the chemical potential must be same in liquid and vapour state.

(iv) Ans: (a) Melting of ice at 0^oC and 1 atm.

(v) Ans: (a) $\mu_{JT} = 0$
Reason: Definition

(vi) Ans: (c) $\mu(O_2)$ will remain unchanged.
Reason: μ is an intensive property, does not depend on moles.

(vii) Ans: (a) $\triangle G_{T,P} < 0$, $E_{cell} > 0$
Reason: $\triangle G < 0$ means spontaneous.
$\triangle G = -nFE_{cell}$
$\therefore$ for $\triangle G$ to be -ve E_{cell} has to be > 0

(viii) Ans: (b) $\triangle T/T_h$
Reason: $\eta = 1 - \dfrac{T_{sink}}{T_{source}}$ η = efficiency
$\eta = \dfrac{\triangle T}{T_h}$
$\Rightarrow \dfrac{w}{q_h} = \dfrac{\triangle T}{T_h}$
Fraction of heat converted to work is $\dfrac{\triangle T}{T_h}$

(ix) Ans: (b) $dG = VdP - Sdt + \sum \mu dn$
Reason: Definition

Q2. (a) Show that: $$U = G - P\left(\frac{\partial G}{\partial P}\right)_T - T\left(\frac{\partial G}{\partial T}\right)_P$$

(b) Prove the following relation: $$\left(\frac{\partial H}{\partial V}\right)_T = -V^2\left(\frac{\partial P}{\partial T}\right)_V\left(\frac{\partial (T/V)}{\partial V}\right)_P$$

(c) Two moles of an ideal gas at **45 ^{o}C** are compressed adiabatically and reversibly from **75.0** litres to **20.0** litres. Calculate **q**, **w**, **ΔU**, and **ΔH** for the process. [Given: $\mathbf{C_V = 2.5}$ ***R*** for the gas].

Ans 2a): To prove:

$$U = G - P\left(\frac{\partial G}{\partial P}\right)_T - T\left(\frac{\partial G}{\partial T}\right)_P \ldots\ldots\ldots\ldots (1)$$

From the Gibbs equation,

$$\partial G = V\partial P - S\partial T$$

$$\left(\frac{\partial G}{\partial P}\right)_T = V \ldots\ldots\ldots\ldots (2)$$

$$\left(\frac{\partial G}{\partial T}\right)_P = -S \ldots\ldots\ldots\ldots (3)$$

Putting (2) and (3) in RHS of (1)

$$U = G - PV + ST$$

Now substituting in $G = H - TS$

$$\Rightarrow U = H - TS - PV + ST$$

$$\Rightarrow U = H - PV$$

$$\Rightarrow U = G - P\left(\frac{\partial G}{\partial P}\right)_T - T\left(\frac{\partial G}{\partial T}\right)_P$$ [proved]

Ans 2b): To Prove: $\left(\frac{\partial H}{\partial V}\right)_T = -V^2\left(\frac{\partial P}{\partial T}\right)_V\left(\frac{\partial (T/V)}{\partial V}\right)_P$

RHS:

$$-V^2\left(\frac{\partial P}{\partial T}\right)\left(\frac{\partial (T/V)}{\partial V}\right)_P$$

$$= -V^2\left(\frac{\partial P}{\partial V}\right)_V\left(\frac{V\partial T - T\partial V}{V^2\partial V}\right)_P$$

$$= \left(\frac{\partial P}{\partial T}\right)_V\left[T - V\left(\frac{\partial T}{\partial V}\right)_P\right]$$

$$= T\left(\frac{\partial P}{\partial T}\right)_V - V\left(\frac{\partial T}{\partial V}\right)_P\left(\frac{\partial P}{\partial T}\right)_V$$

Using cyclic rule,

$$\left(\frac{\partial V}{\partial P}\right)_T\left(\frac{\partial P}{\partial T}\right)_V\left(\frac{\partial T}{\partial V}\right)_P = -1$$

$$\therefore \left(\frac{\partial T}{\partial V}\right)_P\left(\frac{\partial P}{\partial T}\right)_V = -\left(\frac{\partial P}{\partial V}\right)_T$$

$$= T\left(\frac{\partial P}{\partial T}\right)_V + V\left(\frac{\partial P}{\partial V}\right)_T$$

from Maxwell relation, $\left(\frac{\partial P}{\partial T}\right)_V = \left(\frac{\partial S}{\partial V}\right)_T$

$$= T\left(\frac{\partial S}{\partial V}\right)_T + V\left(\frac{\partial P}{\partial V}\right)_T$$

from Gibbs equation,

$$\partial H = T\partial S + V\partial P$$

$\Rightarrow \left(\frac{\partial H}{\partial V}\right)_T = T\left(\frac{\partial S}{\partial V}\right)_T + V\left(\frac{\partial P}{\partial V}\right)_T$

$\therefore$ LHS = RHS [proved]

Ans 2c): $V_i = 75$ litres

$V_f = 20$ litres

$T_i = 45^oC = 318$K

q = 0 (adiabatic process)

$C_v = 2.5R$

$C_p = 3.5R$

$\gamma = \frac{C_p}{C_v} = \frac{7}{5}$

$PV^\gamma = c$

$\Rightarrow TV^{\gamma-1} = c'$ [where $c' = \frac{c}{cR}$]

$\Rightarrow T_iV_i^{\gamma-1} = c'$

$\Rightarrow 318 \times (75)^{2/5} = c'$

$\Rightarrow c' = 1788.35$ lit$^{2/5}$ K

Work done W $=-\int_{V_i}^{V_f} PdV$

$= -\int_{V_i}^{V_f} \frac{cdV}{V^\gamma}$

$= -\frac{c[V_{1-\gamma}]_{V_i}^{V_f}}{1-\gamma}$

$= \frac{c\big[(V_f)^{1-\gamma} - (V_i)^{1-\gamma}\big]}{\gamma - 1}$

$= \frac{c'}{nR}\frac{\big[(20)^{-2/5} - (75)^{-2/5}\big]}{0.4}$

$= \frac{1788.35}{2 \times 0.082}\frac{(0.1239)}{0.4}$

= 3377.69 atm lit

= 342.23 KJ

$\therefore$ W = 342.23 KJ

$\triangle$ U = q + W

$\therefore \triangle$U = 342.23 KJ

$T_fV_f^{\gamma-1} = T_iV_i^{\gamma-1}$

$T_f = T_i\left(\frac{V_i}{V-f}\right)^{\gamma-1} = 539.56$ K

$\triangle H = \triangle U + \triangle(VP)$

$=\triangle U + nR \triangle T$

$=342.23 + 2 \times 8.314$

=(342.23 + 3.68) KJ

= 345.9 KJ

$\therefore \triangle H = 345.9$ KJ

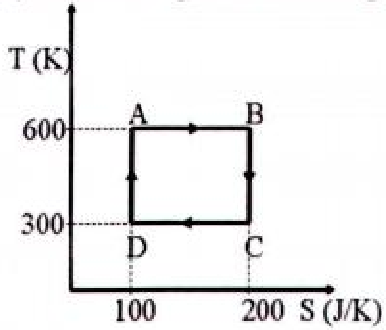
q = 0

W = 342.23 KJ

$\triangle U = 342.23$ KJ

$\triangle H = 345.9$ KJ

Q3. (a) From the given T-S diagram (ABCD) of a reversible Carnot engine shown below, find the



(i) Net work delivered by the engine in each cycle
(ii) Heat taken from the source in each cycle
(iii) ΔS_{sink} in each cycle

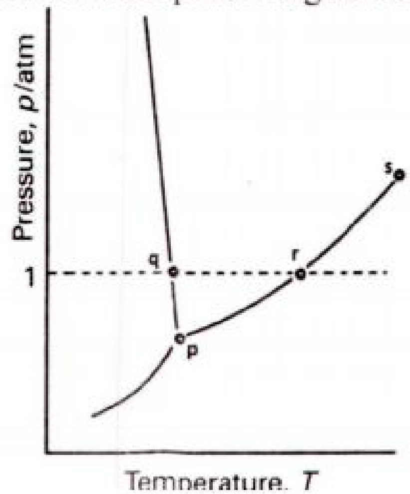
....[1 + 1 + 2 = 4]

(b) Calculate the following for a liquid whose vapour pressure (in ***Torr***) between 15 °C and 35 °C fits the expression: $\log(p_l) = 8.750 - \frac{1625}{T}$.

(i) The enthalpy of vaporization (in **kJ mol^{-1}**) of the liquid; **(ii)** The normal boiling point (**in K**)

....[2+ 2 = 4]

(c) Label the points (p, q, r, s) shown in the phase diagram of a one component system



And 3a): $Q = \int TdS$
For AB, $\triangle S = 100$ J/K
$Q = 600 \times 100 = 60$ KJ
For BC and DA, $\triangle S = -100$ J/K
$Q = -300 \times 100 = -30$ KJ
Net heat delivered by source = 60 KJ
Net heat taken by the engine = 60 − 30 KJ = 30 KJ
$\triangle U = q + W$
$\Rightarrow -W = q$
$\Rightarrow W = -q$
$\Rightarrow W = -30$ KJ
(i) Work delivered by the engine = 30 KJ
(ii) Heat taken from the source in each cycle = 60 KJ
(iii) $\triangle S_{system} + \triangle S_{surrounding} = 0$ [for reversible process]
In the process CD, surrounding = sink
$\therefore \triangle S_{carnot-engine} + \triangle S_{sink} = 0$
or -100 J/K + $\triangle S_{sink} = 0$
or $\triangle S_{sink} = 100$ J/K

Ans 3b): $\log(P_1) = 3.750 - \dfrac{1625}{T}$ . . . . . . (1)

(i) According to Clausius-Clapeyron equation,
$\frac{1}{P}\frac{\partial P}{\partial T} = \frac{\triangle H_{vap}}{RT^2}$ . . . . (2)
differentiating eq (1),
$\frac{1}{P_1}\frac{\partial P_1}{\partial T} = \frac{1625}{T^2}$ . . . . (3)
Comparing (2) and (3)
$\frac{1625}{T^2} = \frac{\triangle H_{vap}}{RT^2}$
$\Rightarrow \triangle H_{vap} = 1625R$
$\Rightarrow \triangle H_{vap} = 13.5$ KJ/mol
(ii) at normal boiling point,
vapour pressure $(P_i) = 1$ atm $= 760$ torr
$\log(P_i) = 8.750 - \frac{1625}{T}$
at T = 15^{o}C = 288 K
$\log(P_i) = 8.750 - \frac{1625}{288}$
$\Rightarrow \log(P_i) = 3.11 \quad \rightarrow \quad$ at 15^{o}C
Clausius-Clapeyron equation,
$\ln\left(\frac{P_1}{P_2}\right) = \frac{\triangle}{H_{vap}} R \left(\frac{1}{T_2} - \frac{1}{T_1}\right)$
$\Rightarrow \ln(760) - \ln(P_2) = \frac{\triangle H_{vap}}{R}\left(\frac{1}{288} - \frac{1}{T}\right)$
$\Rightarrow 6.63 - 3.11 = \frac{1625R}{R}\left(\frac{1}{288} - \frac{1}{T}\right)$
$\Rightarrow 2.16 \times 10^{-3} = \frac{1}{288} - \frac{1}{T}$
On solving,
$\Rightarrow T = 762.1$ K (The answer however seems to be unrealistic. Please consult your professor.)

Ans 3c): p $\rightarrow$ Triple point
q $\rightarrow$ melting/freezing point
r $\rightarrow$ boiling point
s $\rightarrow$ critical point

Q4. (a) For the following reaction at **25 ^{o}C** and **1 atm**, the Gibbs energy change is **+ 2.90 kJ mol^{-1}**.

C (*s*, graphite) $\rightarrow$ C (*s*, diamond)

Densities of graphite and diamond at **25 ^{o}C** are **2.25** and **3.51 g cm^{-3}** respectively.

(i) Will increase in pressure favour the conversion of graphite to diamond? Justify your answer.
(ii) If your answer in part (i) is yes, then calculate the maximum pressure necessary to make this reaction spontaneous at **25 ^{o}C**. ....[2 + 2 = 4]

(b) For the galvanic cell: **Pt | H_2 (*g*, P_{H2}) | HCl (*aq*, a_{H+}) | O_2 (*g*, P_{O2}) | Pt**

(i) Write the half-cell as well as complete cell reactions.
(ii) If the standard state Gibbs energy of formation ($\Delta_f G^o$) of **$H_2O(l)$** is **− 237.13 kJ mol^{-1}**. Calculate the EMF of the cell in the standard state.
(iii) Calculate the equilibrium constant of the overall cell reaction at **298** K. ....[2 + 2 + 2= 6]

Ans 4a): (i) Increase in pressure will result in increase in density because same amount of mass will be confined within smaller amount of volume. Thus density will increase.

Thus increase in pressure will favour conversion of graphite into diamond.
(ii) $dG = VdP - SdT$
At $T = 298$ K, $dT = 0$
$\therefore dG = VdP$ Now 1 mol = 149
$\therefore d(G_{diamond} - G_{graphite}) = (V_{graphite} - Vdiamond)dP$
At equilibrium ,
$G_{graphite} = G_{diamond}$
$\therefore dG_{diamond} = G_{diamond} - G^o{}_{diamond}$
Similarly for graphite.
$\therefore G^o_{diamond} - G^o_{graphite} = (V_{graphite} - V_{diamond})dP$
or $2900 = 14\left(\frac{1}{2.25} - \frac{1}{3.51}\right) \times 10^{-6}(P_f - P_i)$
or $2900 = \frac{14 \times 1.26}{2.25 \times 3.51} \times 10^{-6}(P_f - P_i)$
or $P_f - P_i = 12983$ bar
$P_i = 1$ bar
$\therefore P_f = 12984$ bar

Ans 4b): (i) $2e^- + 2H^+ \longrightarrow H^2$: anode
$2OH^- \longrightarrow H_2O + \frac{1}{2}O_2 + 2e^-$: cathode
Cell reaction : $2H^+ + 2OH^- \longrightarrow H_2O + \frac{1}{2}O_2 + H_2$
$\therefore \triangle_f G^o = -nFE$
or $-237.13 \times 10^3 = -1 \times 96500 \times E$
or $E = 2.46$ V
$\therefore -nFE = -RT \ln K$
or $\ln K = \frac{nFE}{RT} = \frac{\triangle G^o}{RT}$
or $\ln K = \frac{237130}{8.314 \times 298}$
or $K = 3.686 \times 10^{41}$

CHEMISTRY

END-SPRING SEMESTER **2016-17**

Part A: Inorganic Chemistry: ***Answer all the questions***

[Given data: Atomic No.: N:7, O:8, F:9, Fe:26, Co: 27, Cu:29, Ru:44, Rh:45, Sn: 50, Ir: 77, Pb:82. Velocity of light: 3.0×10^8 m/sec; Mass of electron: 9.1095×10^{-28} g; Gas constant: 8.314 J/mol/K; Faraday's constant: 9.648×10^3 emu; Joule's constant: 4.18 J/cal; Avogadro constant; 6.022×10^{23} mole^{-1}; Planck's constant: 6.626×10^{-34} J s; 1 eV = 1.602×10^{-19} J = 96.485 kJ/mol]

1. (a) Write the Hamiltonian for H_2^+ species? **[2]**

(b) Draw a figure to show the overall potential energy variation of a simple molecule with bond length. **[2]**

(c) The work function of a metallic cesium is 1.14 eV. If we shine it with a light of 700 nm wavelength, whether ejection of electrons will occur? **[2]**

Ans 1a):

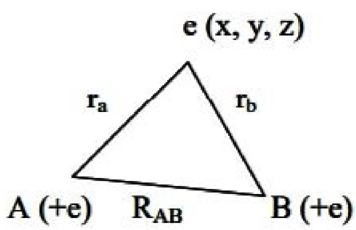


Figure 3

The potential energy is given by

$V = -e^2/r_a - e^2/r_b + e^2/R_{ab}$

Therefore Hamiltonian $H = -h^2/8\pi^2 m \, \nabla^2 - e^2/r_a - e^2/r_b + e^2/R_{ab}$

The Wave equation is

$$\nabla^2 \Psi + 8\pi^2 m/h^2 (E + e^2/r_a + e^2/r_b - e^2/R_{ab}) \Psi = 0$$

Ans 1c): Work Function of metallic Cesium $(w_o) = 1.14$ eV

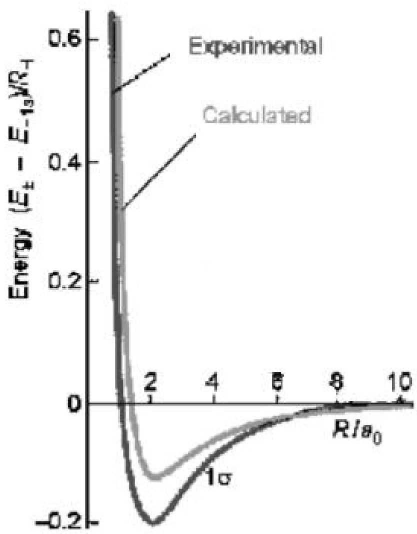
Wavelength of light $(\lambda) = 700$ nm $= 7 \times 10^{-7}$ m

Energy associated with light $= \dfrac{hc}{\lambda} = 2.84 \times 10^{-19} = 1.77$ eV

Energy associated with light $> w_o$

The light of wavelength 700 nm can eject electrons from metallic Cesium of 1.14 work function.

Ans 1b):

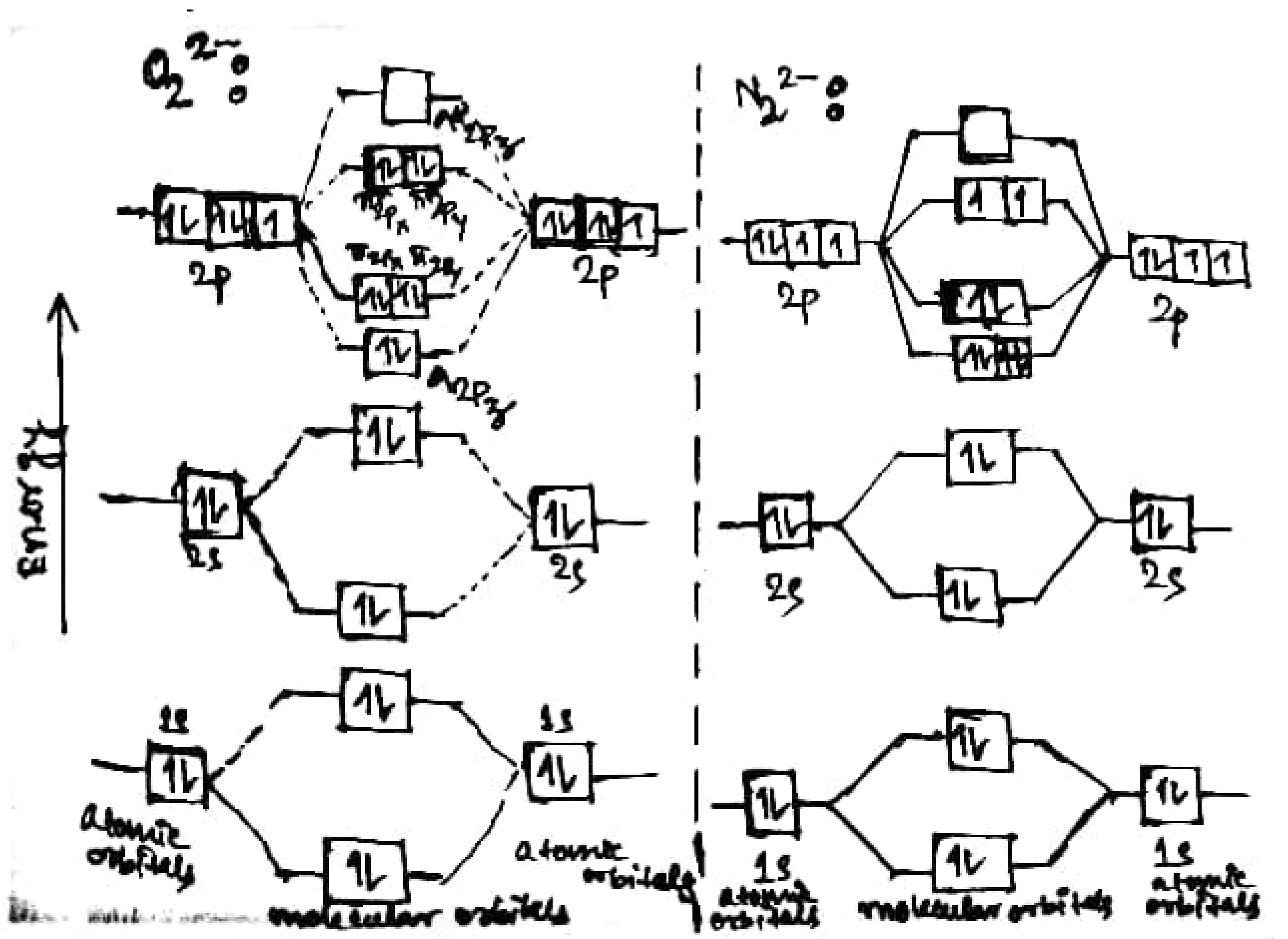


2. Draw molecurlar obrital diagrams for O_2^{2-} and N_2^{2-} ions and compare their bond order and bond stability with respect to their parent molecules. **[3+3]**

Ans 2):
Bond Order of $O_2^{2-} = \dfrac{10-8}{2} = 1$
Bond Order of $N_2^{2-} = \dfrac{10-6}{2} = 2$
Bond Order of $O_2 = \dfrac{10-6}{2} = 2$
Bond Order of $N_2 = \dfrac{10-4}{2} = 3$
$B.O_{O_2^{2-}} < B.O_{O_2}$
Stability of O_2^{2-} < Stability of O_2

$B.O_{N_2^{2-}} < B.O_{N_2}$
Stability of N_2^{2-} < Stability of N_2
Ans 3a): Bridging ligands :
Chelating Ligands :

Ans 3b): Complex I :
$\triangle G = -RT \ln \beta = -2.01RT$
Complex II:
$\triangle G = -RT \ln \beta = -2.36RT$
$\triangle G = \triangle H - T \triangle S$
Although complex I and Complex II have comparable $\triangle H$ there is a substantial difference in their formation constant (β) because of difference in their $\triangle S^o$. Complex II with higher value of lnβ most probably exhibit the phenomenon of chelation.(higher value of $\triangle S$ thus higher value of lnβ)
Ans 4a): Two types of Jahn-Teller distortions are Z-in and Z-out.



Z-out distortion is most preffered.

Ans 4b): Splitting of $[Co(H_2O)_6]^{2+}$:-

C.F.S.E. $= 2 \times \frac{3}{5} \triangle_o - \frac{2}{5} \triangle_o \times 5 + 2P \quad = \frac{6}{5} \triangle_o - 2 \triangle_o + 2P \quad = \quad -\frac{4}{5} \triangle_o + 2P$

$[Co(H_2O)_6]^{+2}$ + HCl(*conc.*) $\longrightarrow$ $[Co(Cl)_4]^{-2}$

Water is a comparatively stronger ligand as compared to Cl^-. So the splitting energy of $[Co(H_2O)_6]^{+2}$ is higher than that of $[Co(Cl)_4]^{-2}$ and thus e^- in $[Co(H_2O)_6]^{+2}$ need more energy to get excited so it will absorb higher energy light waves and reflect lower energy light waves to look pinkish, whereas for $[Co(Cl)_4]^{-2}$ the splitting energy gap between eg and t_{2g} orbitals is smaller so tit will absorb lower energy light waves to excite electrons and emit the higher energy light waves and thus appear bluish.

3. **(a)** Identify bridging and chelating ligand from the following list: **[2]**

Ehylenediaminetetraacetic acid, NH_3, OH^-, Cl^-, N_2H_4, ethylenediamine, PH_3.

(b) Following are the data given for two complexes (complex-I and complex-II): **[3+3]**

	ΔH° (kJ mol^{-1})	log β
Complex- I	**-53.14**	**7.44**
Complex- II	**-56.48**	**10.62**

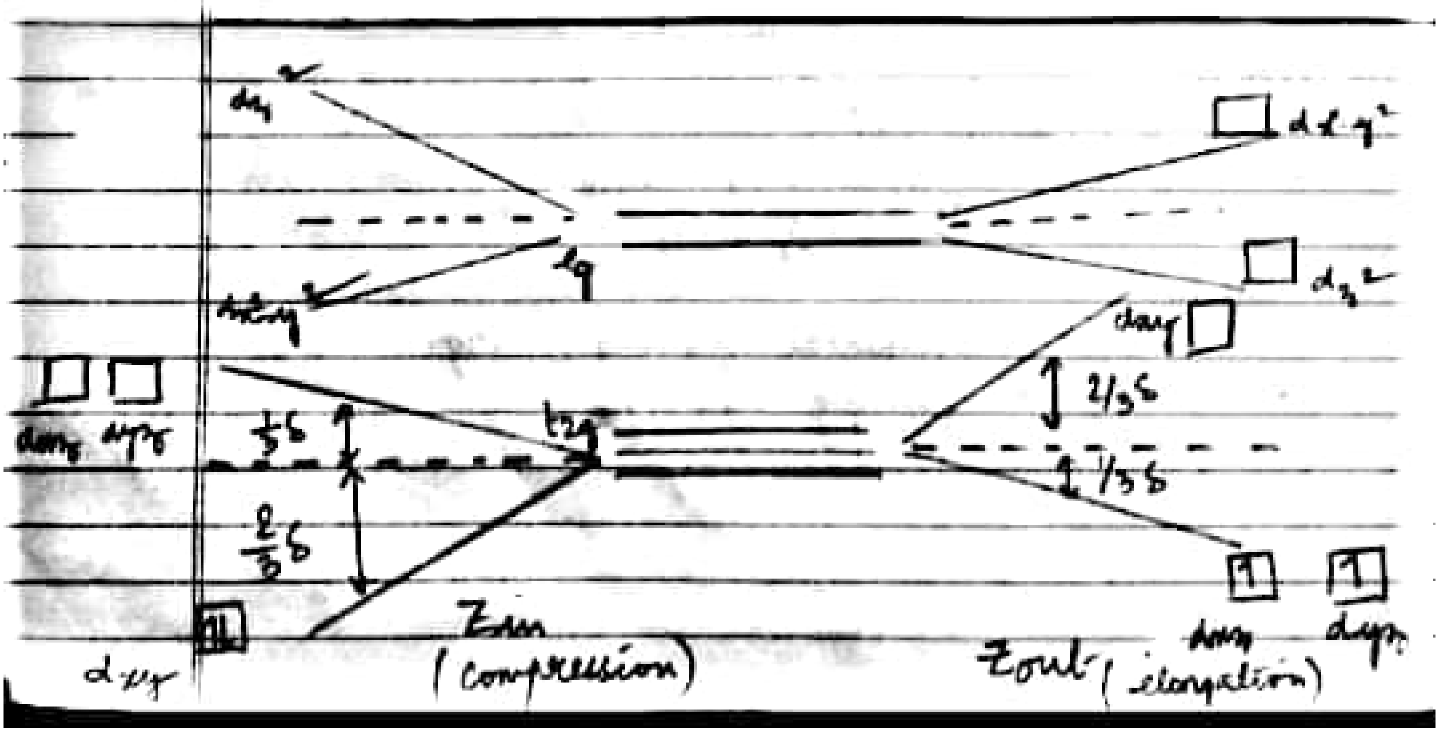
Calculate the thermodynamic parameters responsible for their formation at temperature 298 K and discuss your observations using ΔS° values.

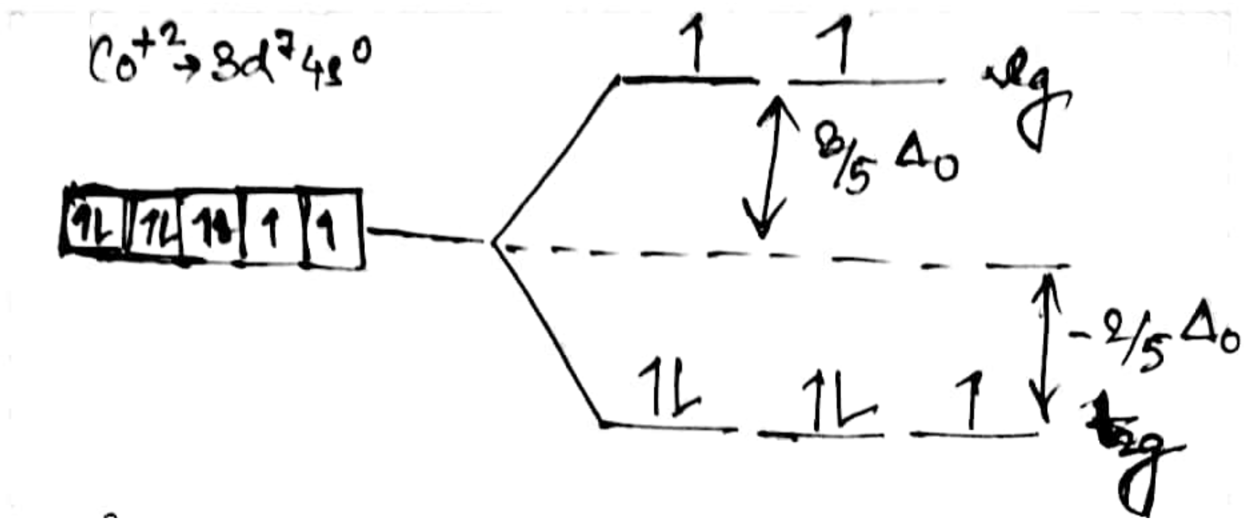
4. **(a)** With the help of diagram, show both types of J-T distortion for d^2 octahedral complex and choose which type of distortion is preferred. **[5]**

(b) Calculate the crystal field stabilization energy (CFSE) of $[Co(H_2O)_6]^{2+}$? **[1+2]**

Give the reason for colour change of pink solution of $[Co(H_2O)_6]^{+2}$ in water to deep blue in conc. HCl.

Ans 5a):



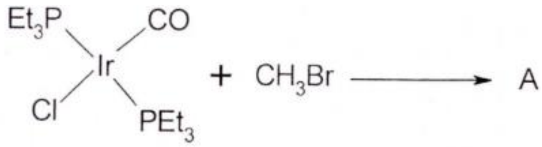


Ans 5b) Pre-catalyst of Monsanto acetic acid process is: $[Rh(CO)_2I_2]^-$
(repeat question from 2015 endsem question paper)
Ans 6a): (a) $2ClO^-(aq) + 2H_2O(l) + 2e^- \longrightarrow Cl_2(g) + 4OH^-(aq)$ $\qquad E^o = 0.4$ V
$Cl_2 + 2e^- \longrightarrow 2Cl^-$ $\qquad E^o = 1.35$ V
$$E^o = A = \frac{1 \times 0.04 + 1 \times 1.35}{2} = 0.875 \text{ V}$$
A = 0.875 V

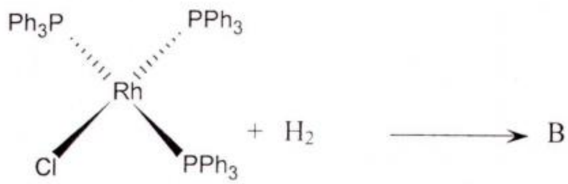
(b) $ClO^- \xrightarrow{0.40} Cl_2 \xrightarrow{1.35} Cl^-$

The potential on the left of Cl_2 is less positive than that on the right; therefore Cl_2 can oxidize and reduce itself; thus undergoes disproportionation.
Half cell reaction :
$Cl_2(g) + 4OH^-(aq) \longrightarrow 2ClO^-(aq) + 2H_2O + 2e^-$ $\qquad E^o = 0.40$ V
$CL_2 + 2e^- \longrightarrow 2Cl^-$ $\qquad E^o = 1.35$ V
Full cell reactions :
$2Cl_2 + 4OH^- \longrightarrow 2ClO^- + 2H_2O + 2Cl^-$
$Cl_2 + 4OH^- \longrightarrow ClO^- + H_2O + Cl^-$ $\qquad E^o = 0.95$ V

5. **(a)** Write down the products of the following reactions (A and B) and calculate the EAN for the complexes (I and II). **[4]**



(I)



(II)

(b) Draw the structure of pre-catalyst used in Monsanto acetic acid process.

(c) $ClO_4^- \xrightarrow{E^o} Cl_2$

$E^o = \dfrac{0.35 \times 2 + 0.3 \times 2 + 0.70 \times 2 + 0.4}{7}$

$nE = 3.1$

$n = 7$

Thus ClO_4^- is the rightmost element on the frost diagram with highest nE value, so is the strongest oxidizing agent.

Ans 6b): $\left(Fe^{3+} + e^- \longrightarrow Fe^{+2} \right)_{\rightarrow(1)}$ $\quad E^o_{Fe^{+3}/Fe^{+2}} = 0.77$ V

$\left(Sn^{+4} + 2e^- \longrightarrow Sn^{+2} \right)_{\rightarrow(2)}$ $\quad E^o_{Sn^{+4}/Sn^{+2}} = 0.15$ V

(1)$\times$ 2 - (2) :

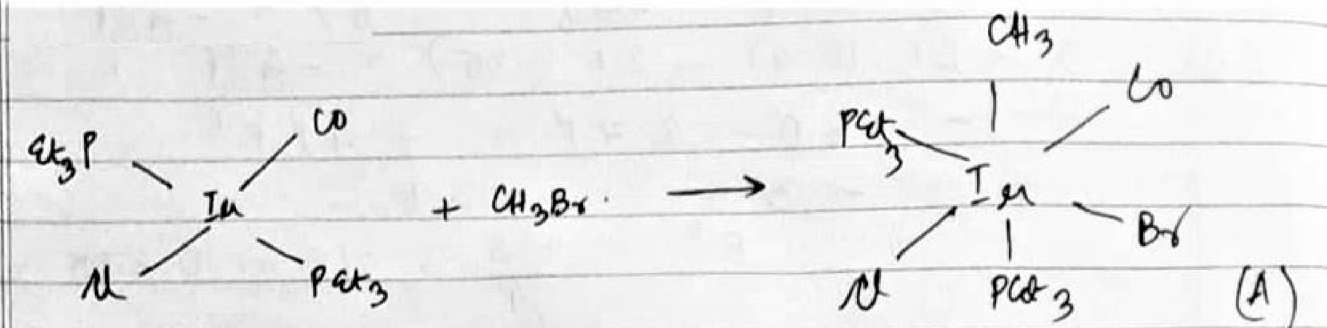
$\left(2Fe^{3+} + Sn^{+2} \longrightarrow 2Fe^{+2} + Sn^{+4} \right)$ $\quad E^o_{Fe^{+3}/Fe^{+2}} = 0.62$ V

E^o is positive so $\triangle G$ is negative so the above reaction is spontaneous. So Sn^{+2} will be oxidized by Fe^{+3} to Sn^{+4}.

Question 7 is out of syllabus.

(5) (a)

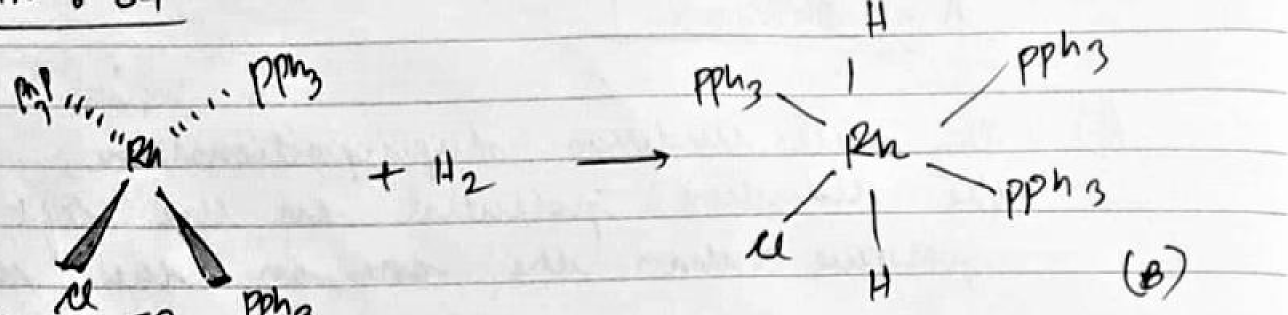
(I)



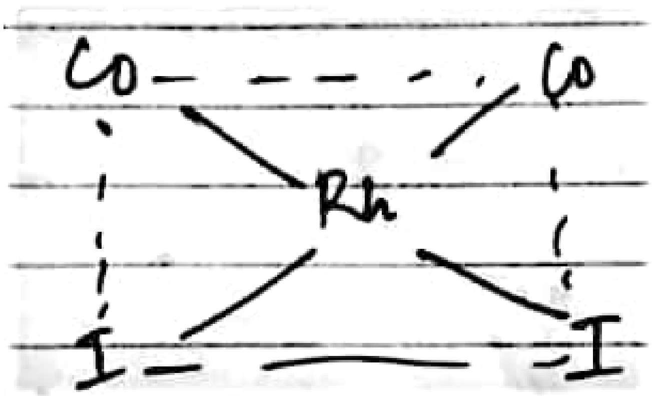
Ir^{+1}

EAN : 84

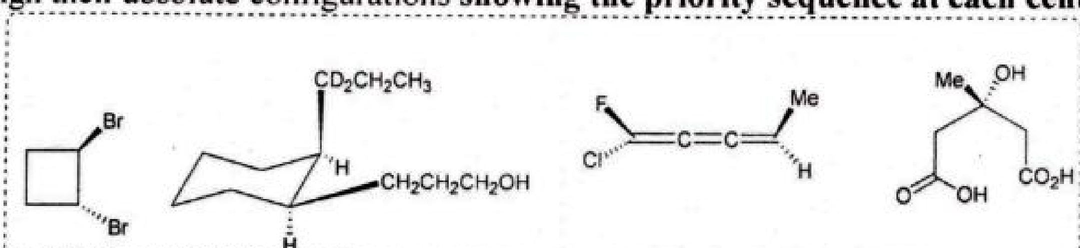
(II)

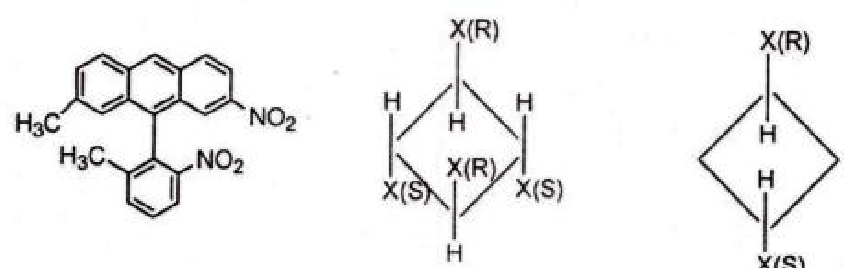


EAN : 52.



1. (a)	How many stereogenic center (s) is (are) present in each of the following molecules? Assign their absolute configurations **showing the priority sequence at each centre.**	[8]
(b)	Which of the following molecules are chiral? If not then what type(s) of symmetry is (are present)?	[4]





X contains a stereogenic centre and exist in R or S forms)

(c) Write the Fischer Projection Formulae of all possible stereoisomers of CA_4 (A is a [3]

6. (a) Answer the following questions based on the Latimer diagram for chlorine in basic solution as given below: [2+3 +1]

$$ClO_4^- \xrightarrow{0.35} ClO_3^- \xrightarrow{0.30} ClO_2^- \xrightarrow{0.70} ClO^- \xrightarrow{0.40} Cl_2 \xrightarrow{1.35} Cl^-$$

$$ClO^- \xrightarrow{E^0 = A} Cl^-$$

(a) Find out the value of A (show all half reactions).

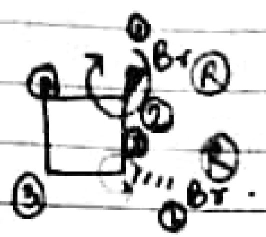
(b) Predict whether Cl_2 will undergo disproportionation reaction? Show all the half reactions and full reaction along with E^0 of the cell.

(c) Predict the strongest oxidizing agent from above.

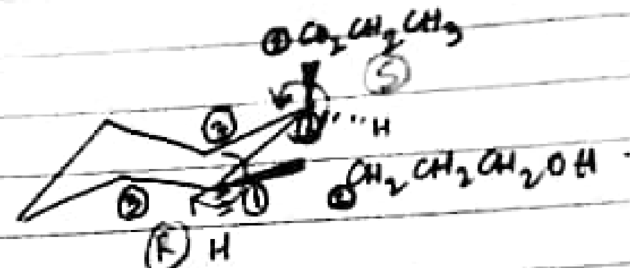
(b) From the following standard reduction potential data predict whether Sn^{2+} will be oxidised by Fe^{3+} (with complete cell reaction)? [2]

$$E^0_{Fe^{3+}/Fe^{2+}} = 0.77 \text{ V} \quad \& \quad E^0_{Sn^{4+}/Sn^{2+}} = 0.15 \text{ V}$$

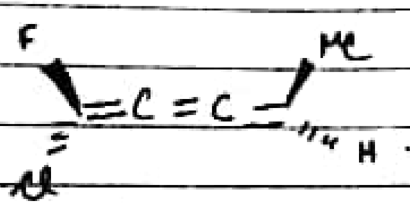
Ans 1a):



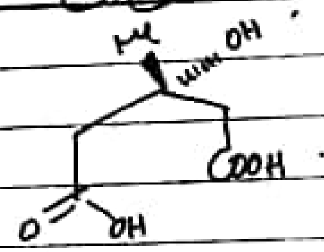
2 stereogenic centres
Both (R)



2 stereogenic centres
(R),(S)

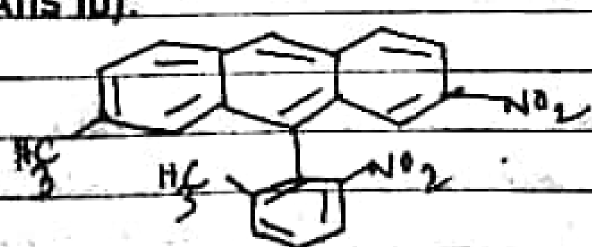


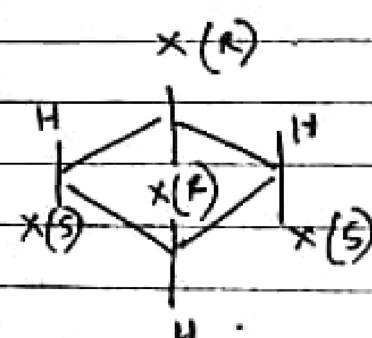
2 stereogenic centres
(E)

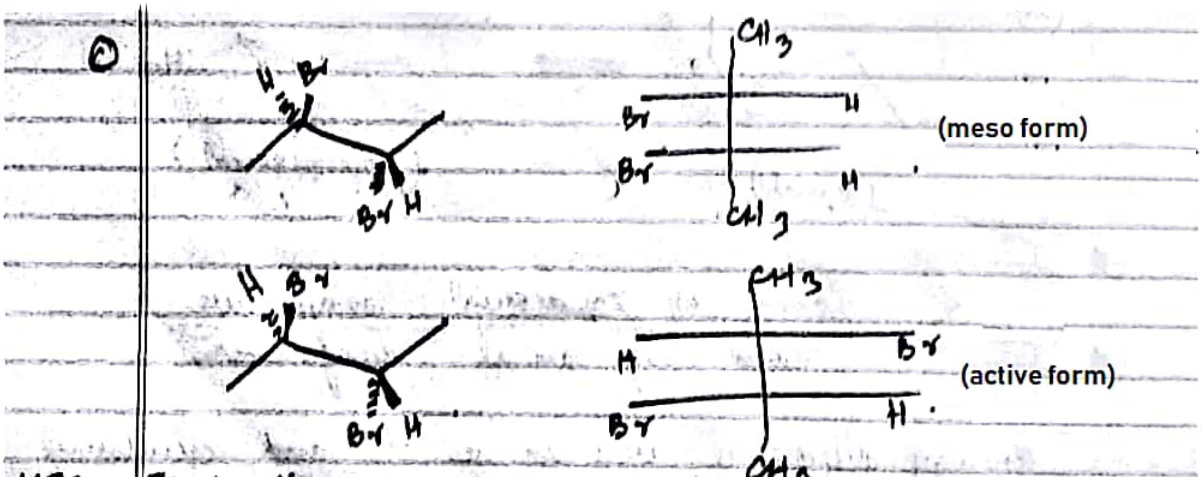


0 stereogenic centres.

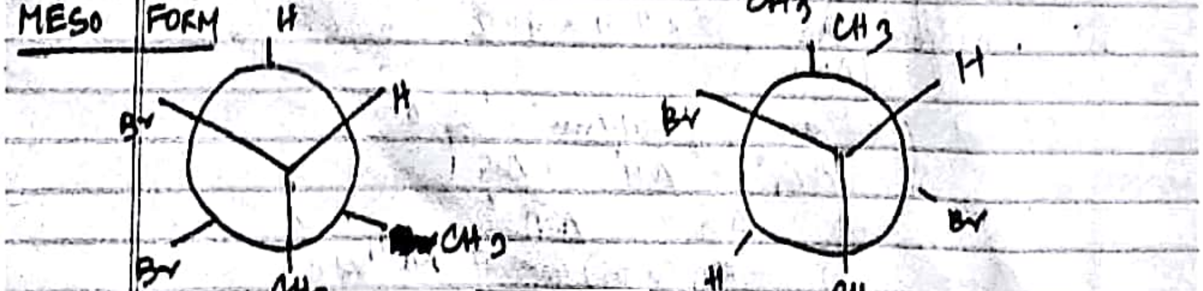
Ans 1b):







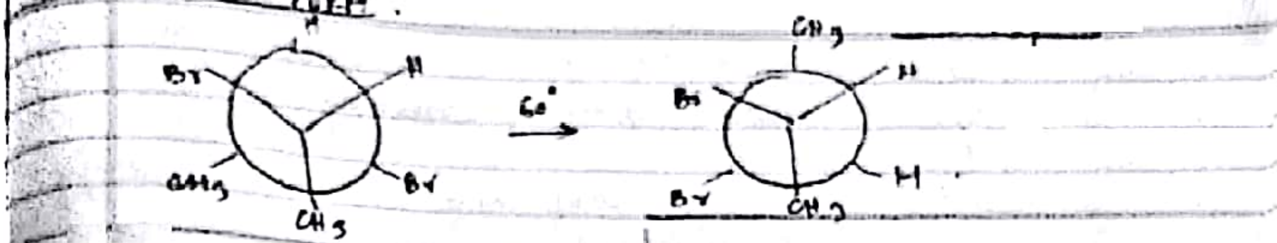
MESO FORM



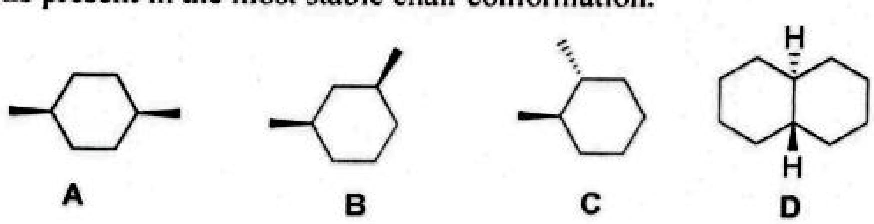
$\Delta H = 3 + 0.8 + 3.3 = \underline{7.1\ Kj/mol}$ $\Delta H = 2 \times 0.8 = \underline{1.6}$

$\Delta H = \underline{3.3 + 1.6 = 4.9\ KJ/mol}$ $\Delta H = \underline{3 + 1.6 = 4.6\ KJ/mol}$

ACTIVE FORM.

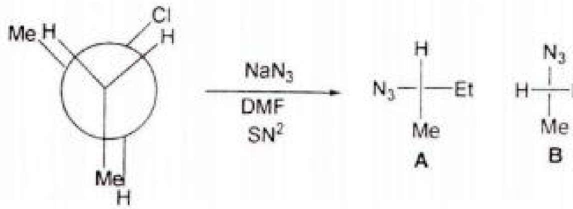


2. (a)	Draw trans-1,3-dimethylcyclohexane in its two chair conformations, and determine whether the two chairs are identical or enantiomers. Then do the same for the cis isomer.	[3]
(b)	Calculate the ratio of axial to equatorial methylcyclohexane that is present at room temperature (300 K). Assume $\Delta s = 0$; ($R = 2$ cal $mol^{-1}K^{-1}$; antilog(1.2) = 15.85; antilog(1.3) = 19.95, antilog(1.4) = 25.1; Boltzmann's constant = 1.38×10^{-23} JK^{-1}; Avogadro's number = 6.023×10^{23} mol^{-1}; 1 cal ≈ 4.2 J	[3]
(c)	For each of the compounds **A** through **D** indicate the number of **gauche butane interactions** present in the most stable chair conformation.	[2]

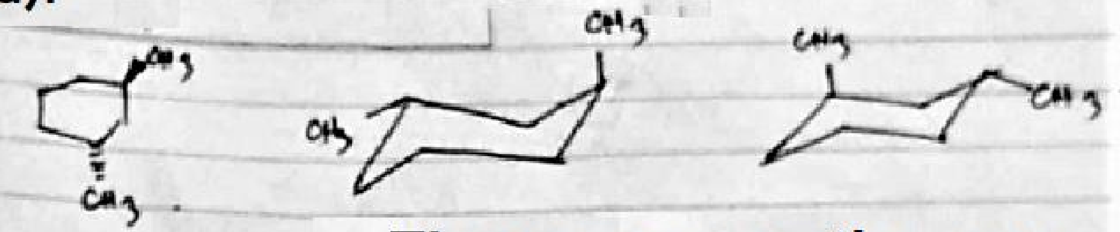


(d) Draw the most stable conformations of the following compounds. Predict with justification the rates of saponifications of the following acetates. **[4]**

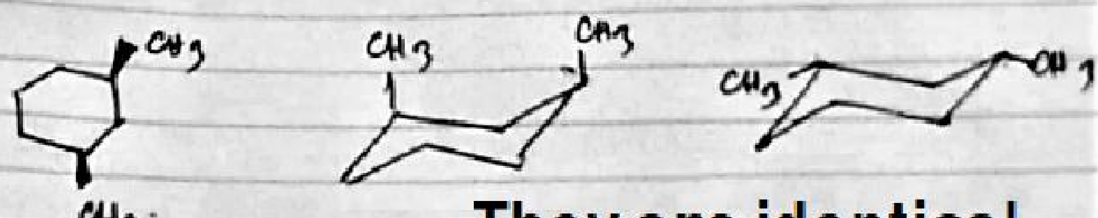
(e) Which one **A** or **B** is the correct product in the following reaction? Justify your answer. **[2]**



Ans 2a):

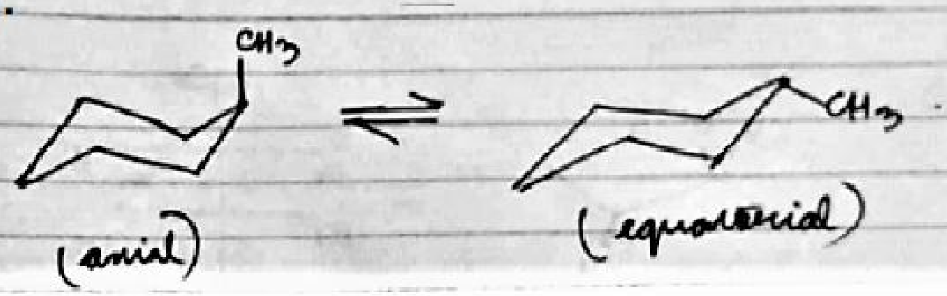


They are enantiomers



They are identical

Ans 2b):



$$K = \frac{\text{conc. of equatorial conformer}}{\text{conc. of axial conformer}}$$

Energy difference between axial and equatorial conformers ($\triangle$H) $= 2 \times 0.9 \times 4.2 = 7.56$ KJ/mol

$\because$ it is at equilibrium,

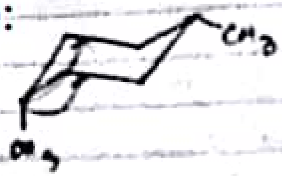
$\triangle G^o = \triangle H^o - T \triangle S^o$ [but $T \triangle S^o = 0$]

$\Rightarrow \triangle G = \triangle H$

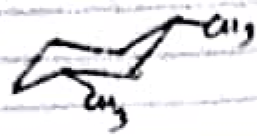
$\Rightarrow -RT\ln(K) = -7.56 \times 10^3$

$K = 20.7$ Ans: $\frac{1}{20.7} = 0.0483$

Ans 2c):

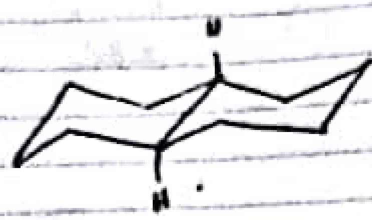


2 GB interactions

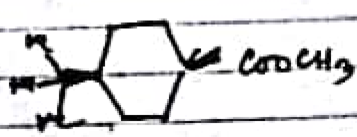


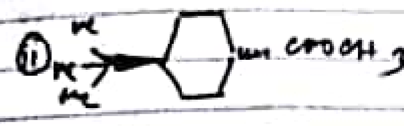
0 GB interactions

1 GB interactions

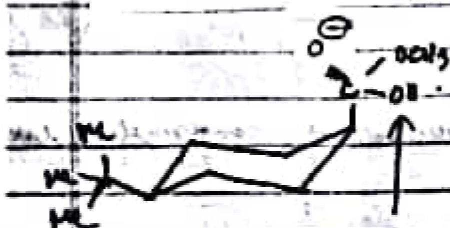


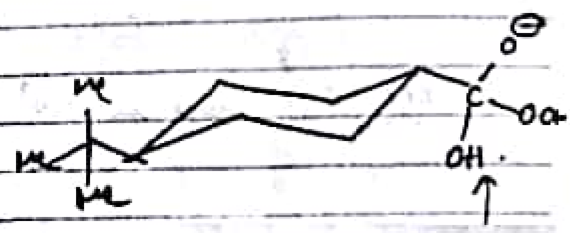
0 GB interactions





The rates of saponification is directly proportional to the stability of the intermediate formed.

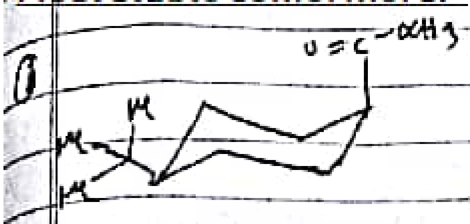


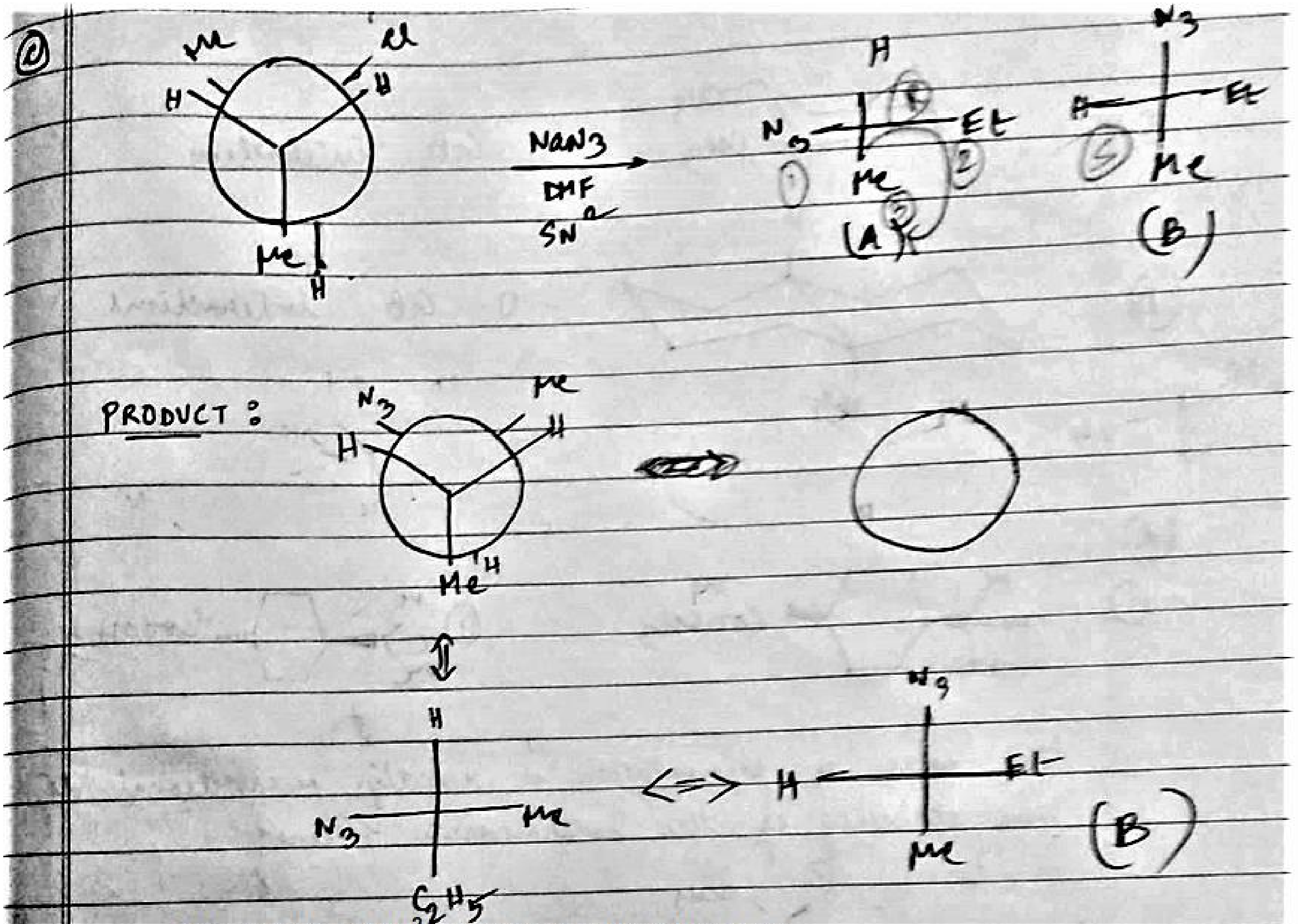


tetrahedarl bulky group present in the axial position, thus unstable

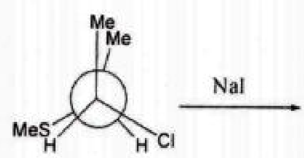
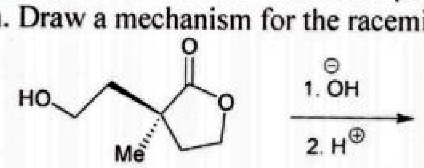
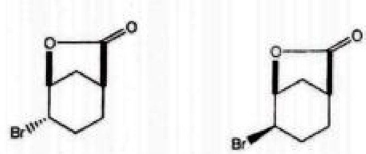
tetrahedral bulky group present in equatorial position, thus stable

Most stable conformers:-

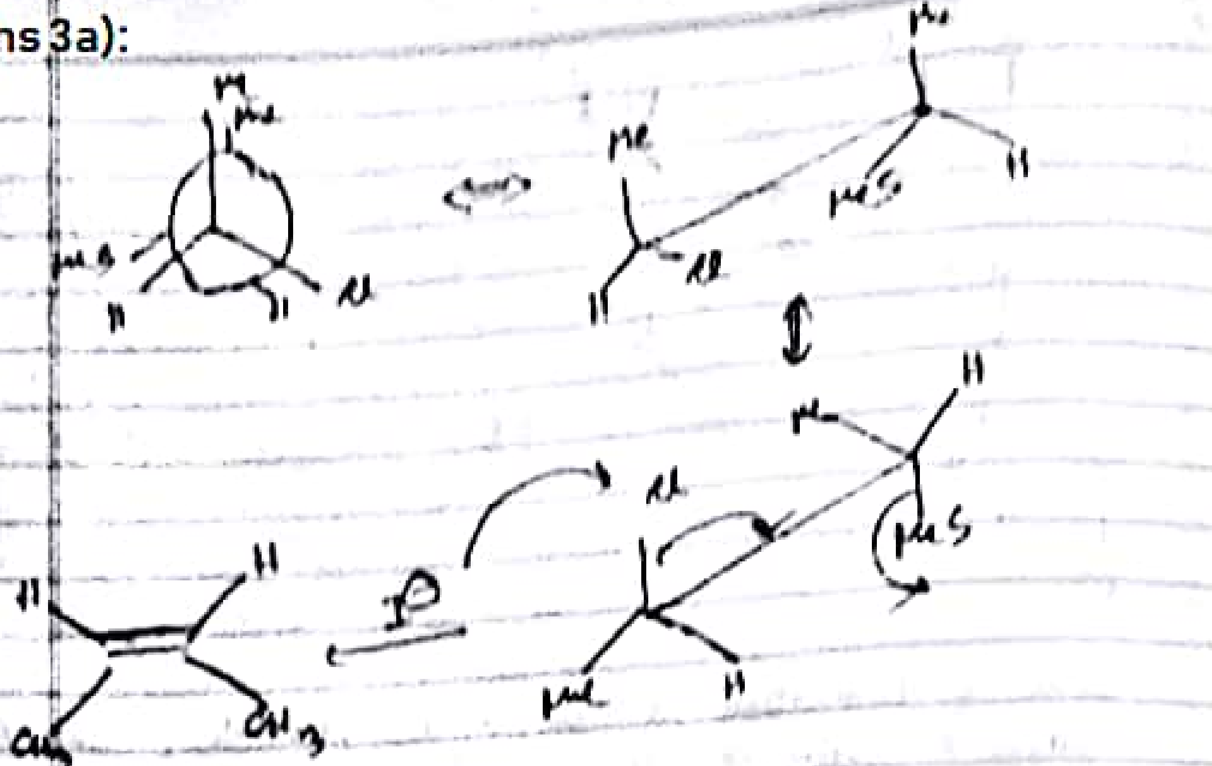




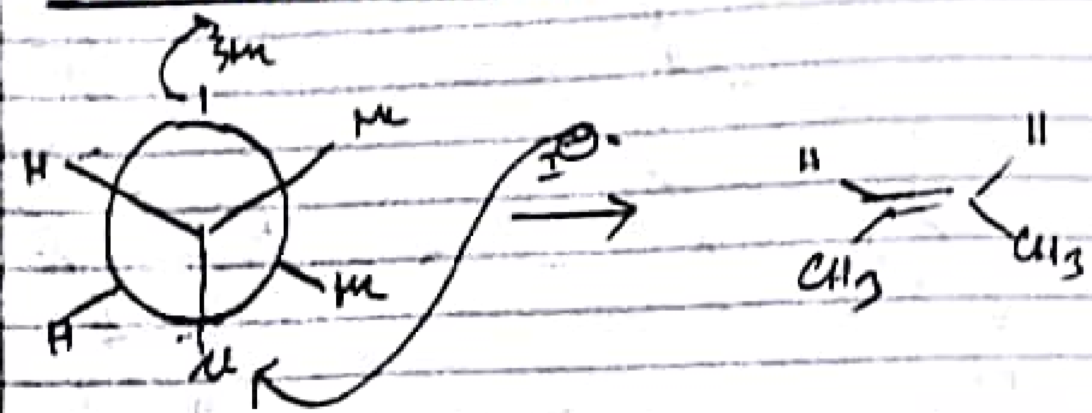
Reason: Since in SN^2, inversion of configuration occurs

3. (a)	Draw the correct Fisher Projection of the product in the following reaction: **Explain the steps to arrive at the solution using Newman projection.** 	[3]
(b)	The following compound gives a racemic mixture upon treatment with a base followed by acidification. Draw a mechanism for the racemisation. 	[4]
(c)	Among the following diastereomeric bromides, only one undergoes elimination upon treatment with a base. Write the structure of the elimination product and **also justify your answer.** 	[3]

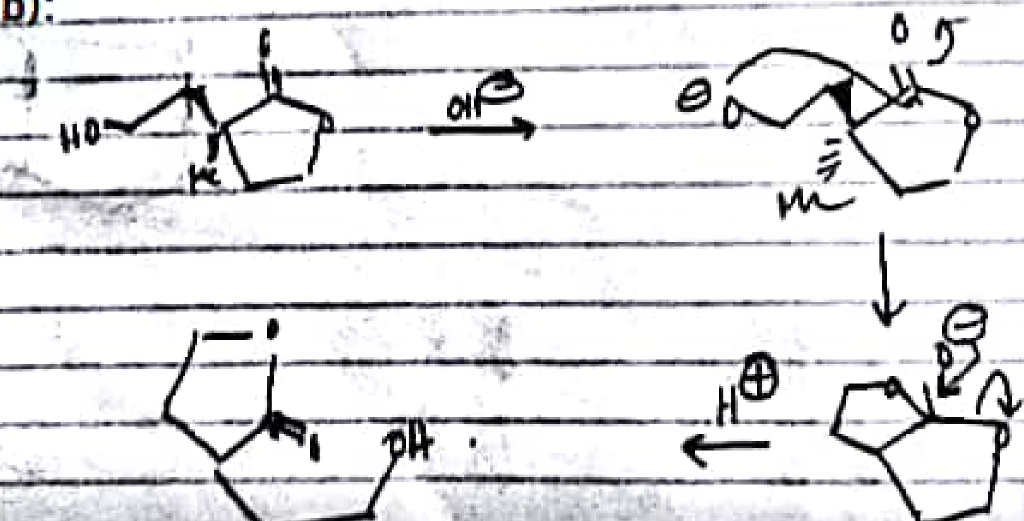
Ans 3a):



Using Newman Projection

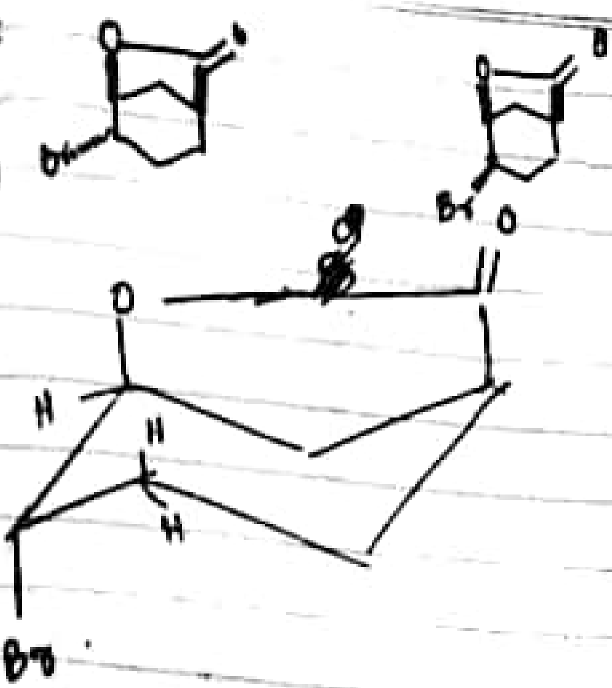


Ans 3b):

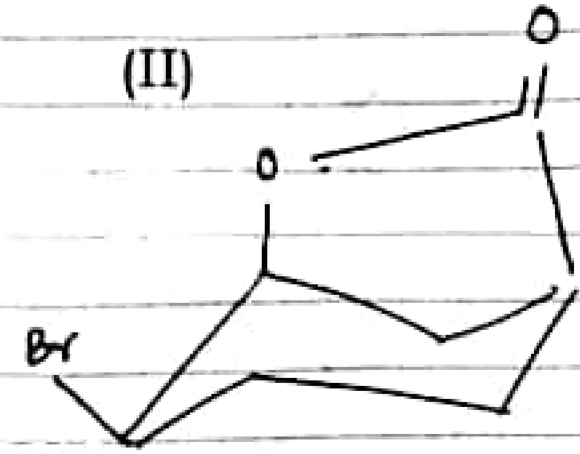


3c):

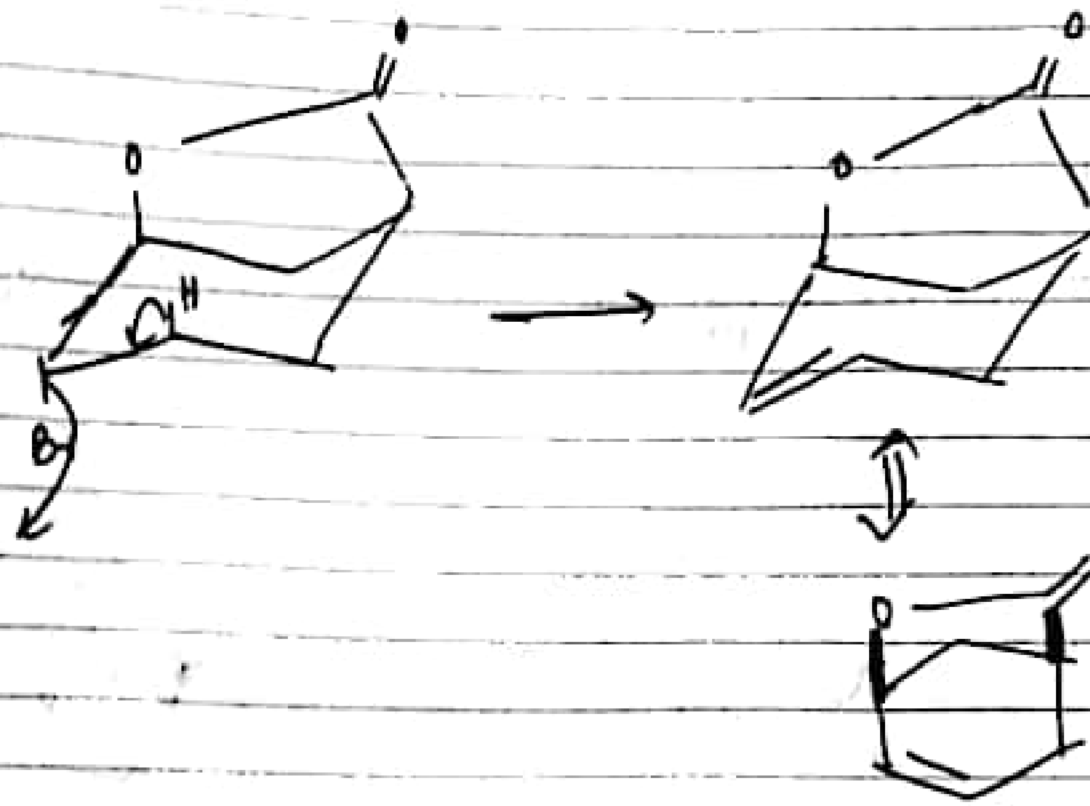
(I)



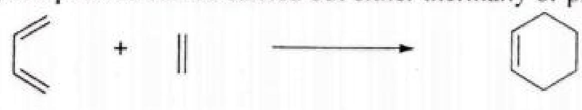
(II)



Since E_2 elimination is an anti process, the Br and H have to be anti to each otherwhich is persent in (I)

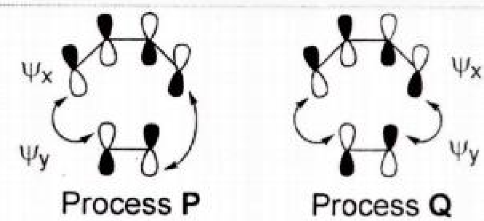


4. (a)	The following pericyclic process can be carried out either thermally or photochemically.	[4]



K **L** **M**

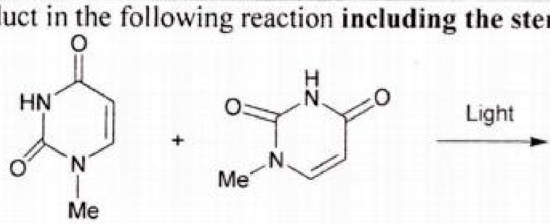
The combination of two molecular orbitals for two components **K** and **L** during two different cyclization processes **P** and **Q** are shown:



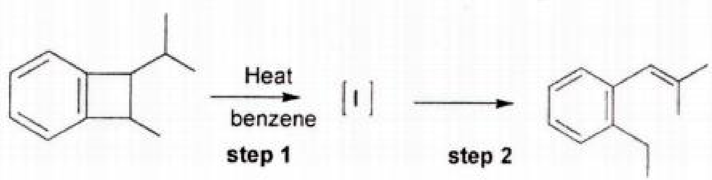
Consider the above MO picture and answer the following

(i) Identify the HOMO and LUMO considering molecular orbitals Ψ_x and Ψ_y.
(ii) Which one of process **P** or **Q** denotes supra-supra ring closure?
(iii) Which one of process **P** or **Q** denotes supra-antara ring closure?
(iv) Which process **P** or **Q** will be allowed thermally?

(b)	Draw the major product in the following reaction **including the stereochemistry**:	[2]

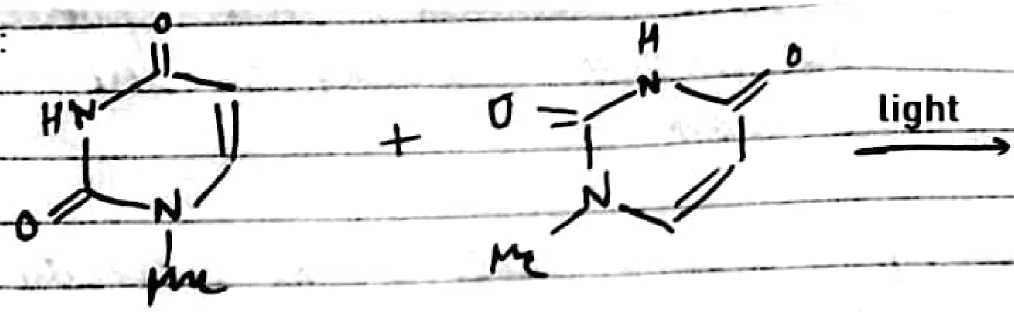


(c)	For the following pericyclic reactions, draw the structures of the intermediate **I**. Name the types of pericyclic reactions involved in the **steps 1** and **2**.	[3]

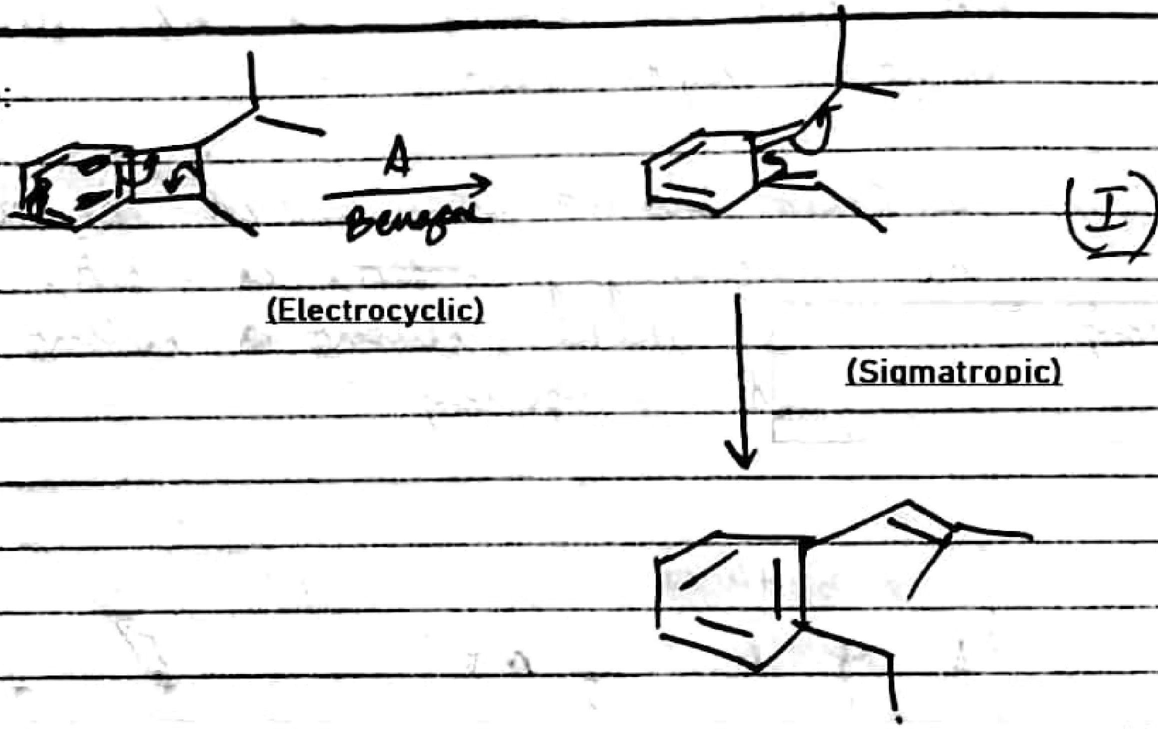


Ans 4a): (i) ψ_x– HOMO ψ_y– LUMO
(ii) Q denotes supra-supra ring closure.
(iii) P denotes supra-antara ring closure.
(iv) Process Q will be allowed thermally.

Ans 4b):



Ans 4c):



CHEMISTRY

END-AUTUMN SEMESTER **2015**

INORGANIC

1. (a) The work function of sodium is 2.5 eV. Predict whether the photon with wavelength of 6800 Å is suitable to eject photoelectron or not? 2

(b) Write the Hamiltonian and Schrödinger equation for the H_2^+ species. 3

Ans 1a): Work function $(w_o) = 2.5eV$
Wavelength of photon $(\lambda) = 6800\text{Å} \simeq 6.8 \times 10^{-7}\text{m}$
Energy of the photon $= \dfrac{hc}{\lambda}$
$= 2.923 \times 10^{-19}\text{J}$
$= 1.827\text{eV}$
Energy of the photon $<$ Work Function (w_o)
Photon with wavelength 6800Å is not suitable to eject photoelectrons from sodium of work function 2.5eV.
Ans 1b):

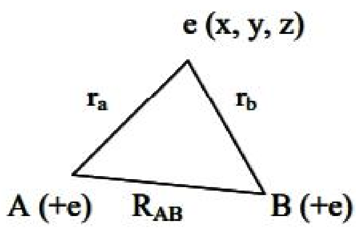


Figure 3

The potential energy is given by
$V = -e^2/r_a - e^2/r_b + e^2/R_{ab}$
Therefore Hamiltonian $H = -h^2/8\pi^2 m\, \nabla^2 - e^2/r_a - e^2/r_b + e^2/R_{ab}$
The Wave equation is

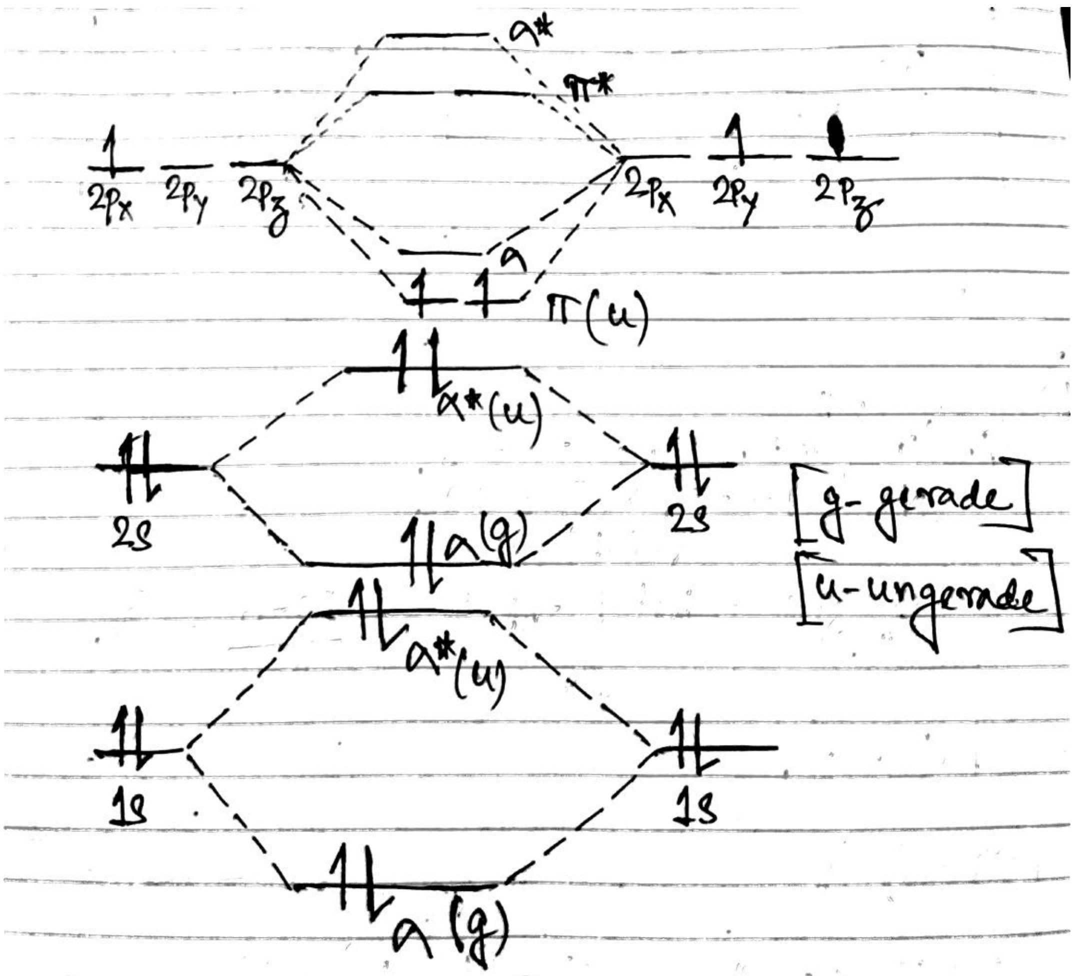
$$\nabla^2 \Psi + 8\pi^2 m/h^2 (E + e^2/r_a + e^2/r_b - e^2/R_{ab})\Psi = 0$$

2. (a) Draw the MO diagram of B_2 and show reason(s) for showing paramagnetism. Label the molecular orbitals with σ, π, g and u notations. Identify the molecular orbitals (if any) which undergo orbital mixing.

(b) Between O_2 and O_2^-, which one has longer bond length and why?

Ans 2a): B_2's molecular arrangement :

$$\sigma_{1s^2}\sigma^*_{1s^2}\sigma_{2s^2}\sigma^*_{2s^2}\sigma_{2p_z}\pi^*_{2p_x^1} \begin{matrix} \pi_{2p_x^1} \\ \\ \pi_{2p_y^1} \end{matrix} \quad \sigma_{2p_z} \quad \begin{matrix} \pi^*_{2p_x} \\ \\ \pi^*_{2p_y} \end{matrix} \quad \sigma^*_{2p_z}$$



MO diagram of B_2

There are 2 unpaired electrons and therefore this explains B_2's paramagnetism.
The suffix g and u are use in case of symmetrical and antisymmetrical with respect to its center, respectively.
$\Psi_A(1s) + \Psi_B(1s)$ . . . . g
$\Psi_A(2p_z) + \Psi_B(2p_z)$ . . . . u
$\sigma \longrightarrow$gerade $\qquad \pi \longrightarrow$ungerade
$\sigma^* \longrightarrow$ungerade $\qquad \pi^* \longrightarrow$gerade

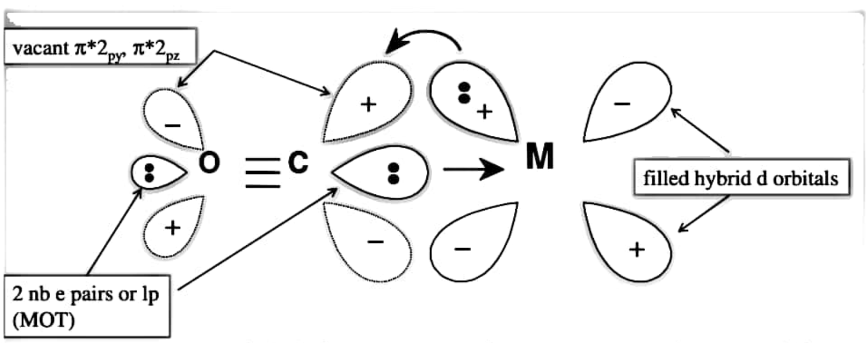
Ans 2b): For O_2:
No. of anti-bonding e^- (A.B.E)= 6
No. of bonding e^- (B.E) = 10
bond-order $= \dfrac{BE - ABE}{2} = \dfrac{10-6}{2} = 2$
Bond Order = 2
For O_2^-:
No. of anti-bonding e^- (A.B.E)= 7
No. of bonding e^- (B.E) = 10
bond-order $= \dfrac{BE - ABE}{2} = \dfrac{10-6}{2} = 1.5$
Bond Order = 1.5

$bond - length \propto \dfrac{1}{bond - order}$
as Bond-Order$_{O_2}$ > Bond-Order$_{O_2^-}$
So, Bond-Length$_{O_2}$ < Bond-Length$_{O_2^-}$
Thus O_2^- has longer bond length as compared O_2.

3. (a) Why does metallic copper show good electrical conductivity? 2
(b) What is π-acid ligand? Give an example. 1
(c) How does one π-acid ligand stabilize metal ions with low oxidation state forming stable organometallic complexes? Explain by showing orbital overlap responsible for forming the corresponding metal-ligand bond with proper labelling of the orbitals. 2

Ans 3a): Metallic copper is a good conductor due to the following reasons :-
- It has free electrons.
- The conduction and valence band overlap. There is no band gap so electrons are free to move.

Ans 3b): Ligands which can donate electrons to ligands (sigma donor) and also accept electron from metal are called pi acid ligand. the latter is called back bonding and the effect is called synergic effect.
Example: $Ni(CO)_4$
Ans 3c):



σ orbital (nb) serves as a donor to a metal atom (M)
&
M to CO pi backbonding

Carbonyl ligand is considered as a weak 2-electron σ-donor and very strong π-acceptor. Two types interactions are involved int he complexation of carbonyl with transition metal ion. The carbonyl donates its lone pair to the vacant metal d(σ) orbital and back bonding occurs from metal dπ to C-O π^* orbital. Thus via the phenomenon of back π bonding and synergic effect π-acid ligands like CO stabilizes metal ions with low oxidation state forming stable organometallic complexes. To stabilize low oxidation state we require ligands which can simultaneously bind the metal center and also withdraw electron density from the metal ion which is very high in electron density.

4. (a) Construct the Frost diagram of Mn from the following Latimer diagram (acidic medium). Show the required calculation and diagram with properly labelling of the axes. 4

$$MnO_4^- \xrightarrow{0.90} HMnO_4^- \xrightarrow{1.28} H_3MnO_4 \xrightarrow{2.9} MnO_2 \xrightarrow{0.95} Mn^{3+} \xrightarrow{1.5} Mn^{2+} \xrightarrow{-1.18} Mn$$

(b) From the Frost diagram, explain why Mn^{3+} is unstable in acidic solution. 1

Ans 4a): $MnO_4^- \xrightarrow{0.90} HMnO_4^- \xrightarrow{1.28} H_3MnO_4 \xrightarrow{2.9} MnO_2 \xrightarrow{0.95} Mn^{+3} \xrightarrow{1.5} Mn^{+2} \xrightarrow{-1.18} Mn$
(I) $Mn^{+2} \xrightarrow{-1.18} Mn$
n=2

nE^o = -2.36 V

(II) $Mn^{+3} \xrightarrow{E} Mn$

$$E^o = \frac{2 \times (-1.18) + 1 \times 1.5}{3} = -0.287V$$

$nE^o = -0.86V$

$n = 3$

(III) $\overset{(+4)}{Mn}\, O_2 \xrightarrow{E} \overset{(0)}{Mn}$

$$E^o = \frac{2 \times (-1.18) + 1.5 \times 1 + 1 \times 0.95}{4}$$

$nE^o = 0.09V$

$n = 4$

(IV) $H_3\, \overset{(+5)}{Mn}\, O_4 \xrightarrow{E} \overset{(0)}{Mn}$

$$E^o = \frac{0.09 + 2.9 \times 1}{5}$$

$nE^o = 2.99V$

$n = 5$

(V) $H\, \overset{(+6)}{Mn}\, O_4^- \xrightarrow{E} \overset{(0)}{Mn}$

$$E^o = \frac{2.99 + 1.28 \times 1}{6}$$

$nE^o = 4.27V$

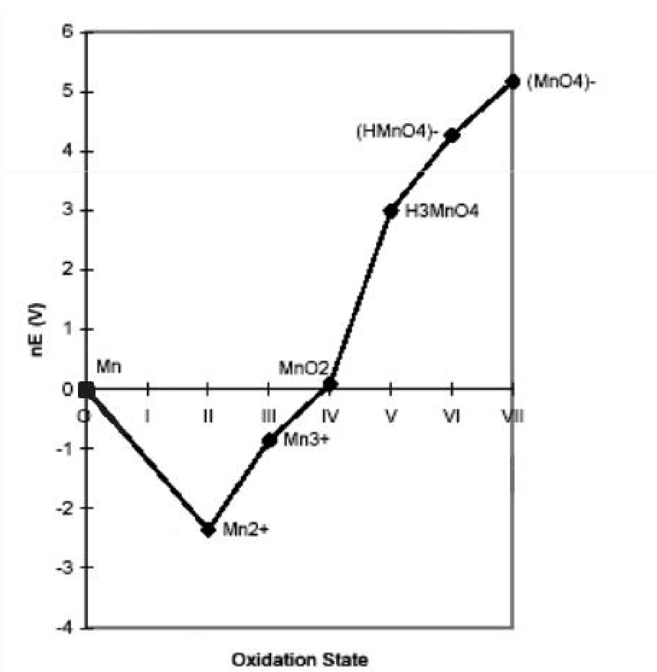
$n = 6$

(VI) $\overset{(+7)}{Mn}\, O_4^- \xrightarrow{E} \overset{(0)}{Mn}$

$$E^o = \frac{4.27 + 1 \times 0.09}{7}$$

$nE^o = 5.17V$

$n = 7$



Ans 4b): From the Frost diagram it is clear that Mn^{+3} lies above the line joining Mn^{+2} and MnO_2. So Mn^{+3} in acidic medium is unstable and has the tendency to disproportionate into Mn^{+2} and MnO_2.

5. The spin only magnetic moment values of two complexes, $[Fe(L_A)_6]^{3-}$ and $[Fe(L_B)_6]^{3-}$, are 1.73 B.M and 5.92 B.M., respectively. L_A and L_B are mono-anionic ligands.

(a) Between L_A and L_B, which one is a strong-field ligand? 1

(b) Show the splitting of d-orbitals and electron distribution to account for the magnetic moment of each complex. 2

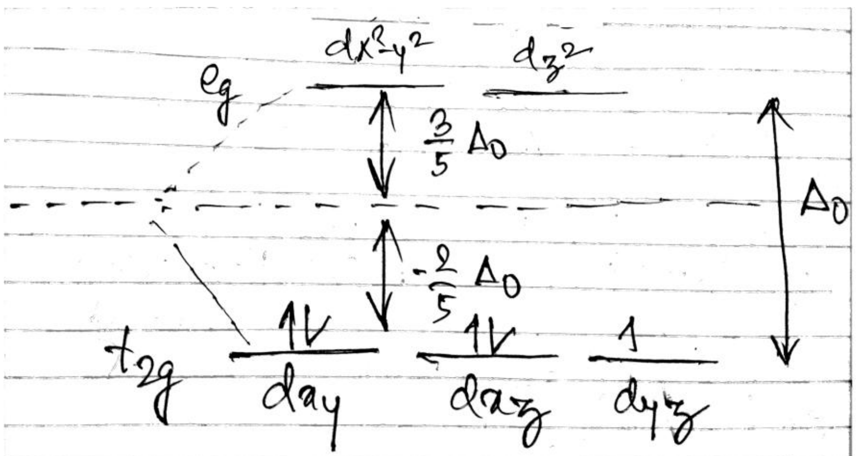
(c) Calculate the CFSE values for both the complexes. 2

Ans 5a): L_A is the strong field ligand.

Reason: Causes pairing, so less unpaired electrons and thus $[Fe(L_A)_6]^{-3}$ has comparatively lesser magnetic moment.

[see part (b) for detailed diagramatic analysis]

Ans 5b): $[Fe(L_A)_6]^{-3} \longrightarrow Fe^{+3} \longrightarrow 3d^5 4s^0$

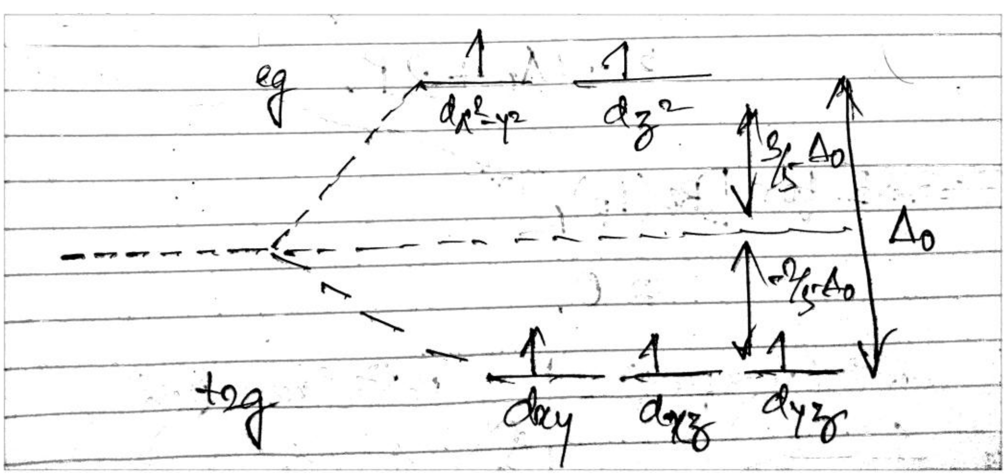


(Pairing due to strong field ligand L_A)

unpaired electrons(n) = 1

$\mu = \sqrt{n(n+2)} = 1.73$ BM

$[Fe(L_B)_6]^{-3}$



(no pairing as L_B is weak field ligand)

unpaired electrons(n) = 5

$\mu = \sqrt{5(7)} = \sqrt{35} = 5.92$ BM

Ans 5c): CFSE for $[Fe(L_A)_6]^{-3}$:

$\frac{3}{5} \triangle_o \times 0 - \frac{2}{5} \triangle_o \times 5 + 2P = -2 \triangle_o + 2P$ where P = pairing energy

CFSE for $[Fe(L_B)_6]^{-3}$:

$\frac{3}{5} \triangle_o \times 2 - \frac{2}{5} \triangle_o \times 3 = 0$

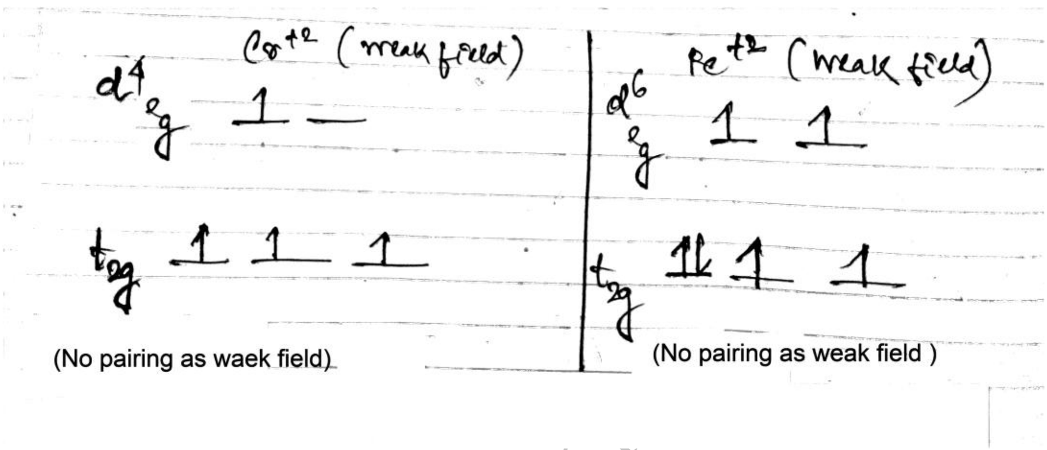
CFSE for $[Fe(L_A)_6]^{-3} = -2 \triangle_o + 2P$

CFSE for $[Fe(L_B)_6]^{-3} = 0$

6. (a) Between weak field Cr^{2+} and Fe^{2+} octahedral complexes, in which case do you expect strong Jahn-Teller distortion and why? 2

(b) Write down the expected product if excess CN^- is treated with an aqueous solution of Ni^{2+}. Show the splitting of d-orbitals with electron distribution for the resulting complex. 3

Ans 6a):



In Cr^{+2}, there is degeneracy (assymetry) in eg orbitals, so it will show strong Jahn-Teller distortions.

In Fe^{+2} there is degeneracy (assymetry) in t_{2g} orbitals so it will show weak Jahn-Teller distortions.

Ans 6b): When CN^- is treated with Ni^{+2}(aq)

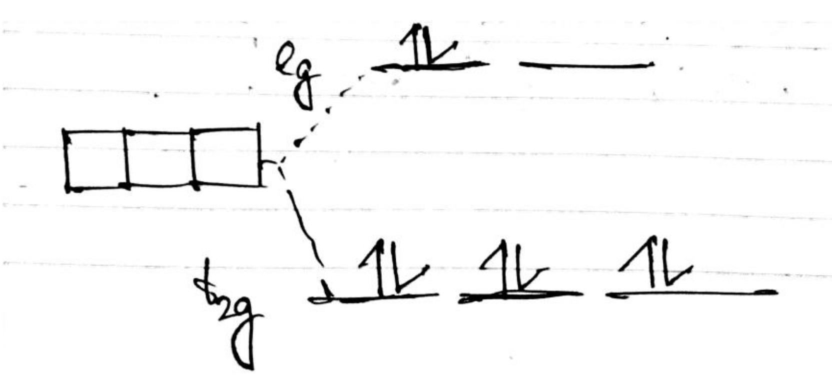
$Ni^{+2} + 2CN^- + nH_2O \longrightarrow Ni(CN)_2.nH_2O$ (green colour)

Now if excess CN^- is added,

$Ni(CN)_2 + 4CN^- \longrightarrow [Ni(CN)_6]^{4-}$

Configuration of Ni^{2+} in $[Ni(CN)_6]^{4-}$ is $3d^8 4s^0$

Splitting of d-orbital and electron distribution for $[Ni(CN)_6]^{4-}$

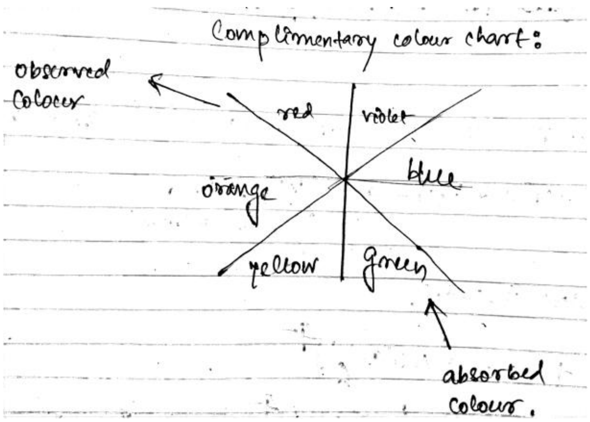


7. (a) Solutions of the complexes $[Co(NH_3)_6]^{2+}$, $[Co(H_2O)_6]^{2+}$ and $[CoCl_4]^{2-}$ are all colored. One is blue, another is yellow and third one is pink. Considering the spectrochemical series of ligands and relative magnitude of Δ_T and Δ_O, assign color for each of the complexes. 3

(b) Between $V(CO)_6$ and $Cr(CO)_6$, which one can show oxidizing power and why? 2

Ans 7a): For the given complexes $[Co(NH_3)_6]^{+2}$, $[Co(H_2O)_6]^{+2}$ and $[CoCl_4]^{-2}$ the order of ligand strength is $Cl^- < H_2O < NH_3$

Thus $\triangle_{o[Co(NH_3)_6]^{+2}} > \triangle_{o[Co(H_2O)_6]^{+2}} > \triangle_{o[CoCl_4]^{-2}}$



Now, since $\triangle_{o[Co(NH_3)_6]^{+2}}$ is highest among the given three complexes, so to excite its electrons more energy is required (more $\triangle_o$ means more gap between eg and t_{2g}) therefore it must absorb high energy (high frequency) light wave such as violet so from complementary colour chart if it absorbs violet then observed colour must be **yellow**.

Similarly, $\triangle_{o[Co(H_2O)_6]^{+2}}$ is second highest and so it must absorb high energy light wave and thus its observed colour must be lesser in energy. Among pink and blue, pink is lesser energy so $[Co(H_2O)_6]^{+2}$ must be pink and $[CoCl_4]^{-2}$ must be blue.

Remember: Higher the splitting energy ($\triangle_o$ or $\triangle_T$) higher is the gap between eg and t_{2g} and higher is the energy required to excite electrons, so a complex with higher splitting energy will absorb high energy light wave but will have a lower energy colour in accordance with complementary colour chart.

$$\text{Splitting energy} \propto \frac{1}{\text{Energy of observed colour}} \propto \text{Energy of absorbed colour}$$

8. (a) The $\log\beta$ for $[Cd(en)_2]^{2+}$ (en = ethylenediamine) formation is greater than that of $[Cd(MeNH_2)_4]^{2+}$ although ΔH values for both the reactions are very similar. Determine the thermodynamic parameters (at T = 298 K) responsible for the following reactions, and account for the marked difference in $\log\beta$. 3

$$[Cd(en)_2]^{2+} \xleftarrow[\substack{\log\beta = 10.6;\\ \Delta H = -56\ kJmol^{-1}}]{2\ \text{"en"}} \text{Aqueous } Cd^{2+} \xrightarrow[\substack{\log\beta = 6.5;\\ \Delta H = -57\ kJmol^{-1}}]{4\ MeNH_2} [Cd(MeNH_2)_4]^{2+}$$

(b) $EDTA^{4-}$ (ethylenediaminetetraacetate) is used to remove heavy toxic metal cations such as Pb^{2+} from human body through *chelation therapy*. Explain the mode of action of $EDTA^{4-}$ and how the Pb^{2+} is eliminated from body. 2

Ans 8a): $[Cd(en)_2]^{+2} \xleftarrow[\log\beta=10.6;\triangle H=-56KJ/mol]{2\text{"en"}} \text{Aqueous } Cd^{2+} \xrightarrow[\mathbf{log}\beta=6.5;\triangle H=-57KJ/mol]{\mathbf{4MeNH_2}} [Cd(MeNH_2)_4]^{+2}$

(I) (II)

$\triangle G = \triangle H - T\triangle S$

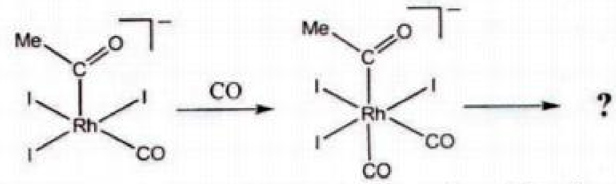
$\Rightarrow$ -RT ln(β) = $\triangle$H - T$\triangle$S $\rightarrow$ (i)
now, for (I)
G = -RTln(β)
I = -10.6RT
for (II),
G = -RTln(β)
II = -6.5RT
now, $G_{(I)} < G_{(II)}$
So (I) is more spontaneous than reaction (II).
from (i),
$$\ln(\beta) = \frac{-\triangle H}{RT} + \frac{\triangle S}{R}$$
for both (I) and (II) reactions $\frac{-\triangle H}{T}$ is almost same but the marked differences in ln(β) is because of difference in $\triangle$S.

In (I) 'en' is a chelating ligand , so $\triangle$S is very high for the reaction where as for Cd^{2+} to $[Cd(MeNH_2)_4]^{4+}$ the change in S is not so high so $\triangle$S is low.

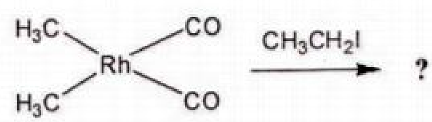
Ans 8b): The $EDTA^{4-}$ when injected in human body readily forms complexes with Pb^{+2} which due to chelation have very high stability constant. Therefore the complexes can be easily flushed out of human body and thus removing poisonous metals like Pb^{2+}.

9. (a) What is the active catalyst used in Monsanto acetic acid process? 1

(b) Draw the structures of the missing products of this reaction. Mention the type of reaction and change in oxidation state of "Rh". 2

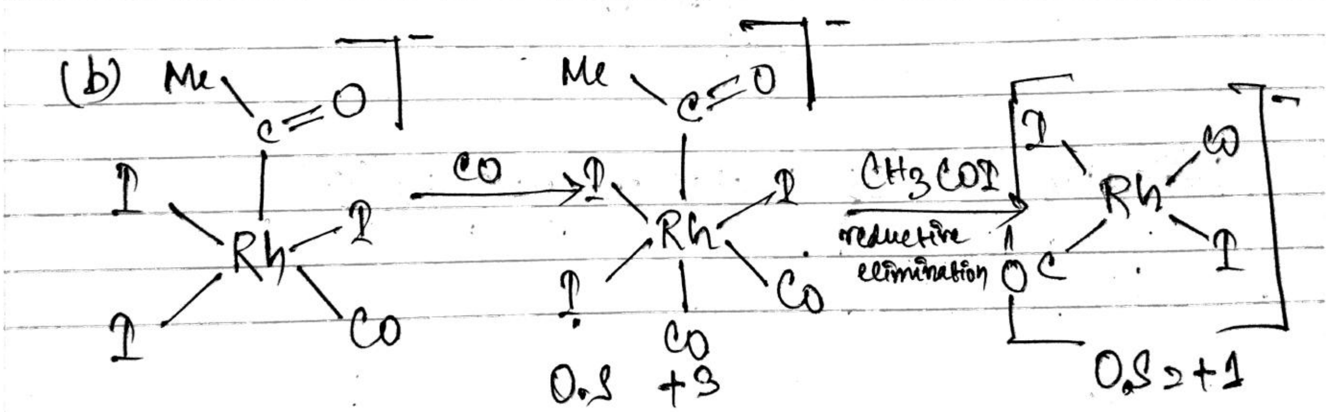


(c) Write down the expected organometallic compound in the following reaction. Mention the type of reaction and give reason why does the reaction undergo? 2

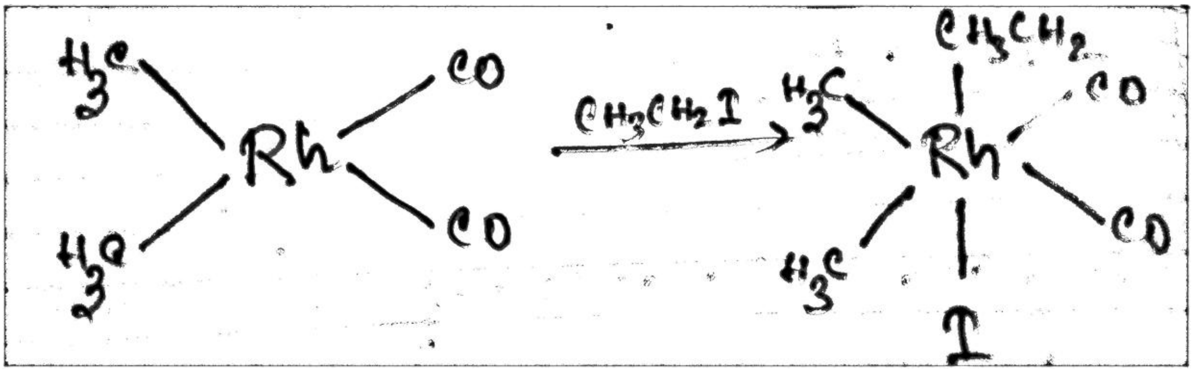


Ans 9a): The active catalyst used in Monsanto acetic acid process is rhodium carbonyl iodide catalyst $[Rh(CO)_2I_2]^-$.

Ans 9b):



Ans 9c):



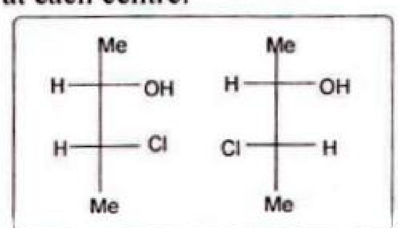
Type of reaction: Oxidative addition of CH_3CH_2I.

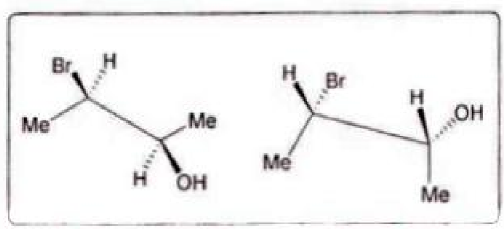
Question (10) is out of syllabus.

Part B: Organic Chemistry

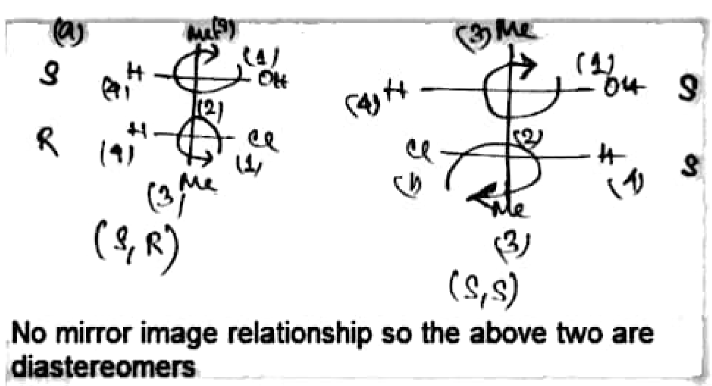
(Answer all questions in a separate answer sheet with your name, roll number and section)

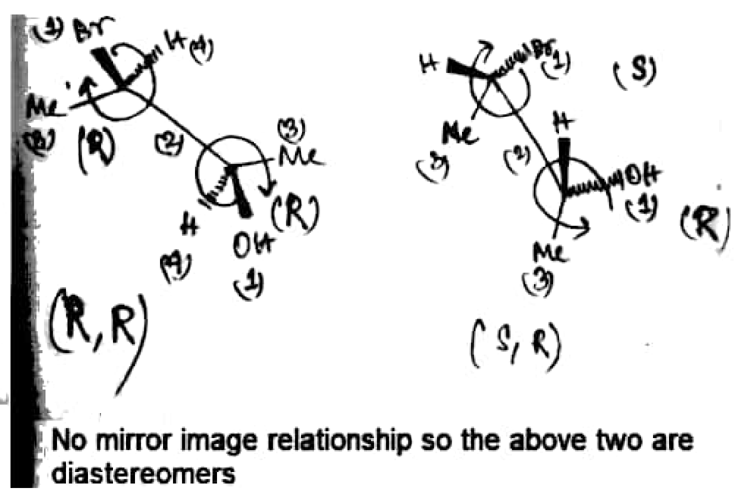
1. a) Indicate the relationship between the compounds of the following pairs as identical or enantiomers or diastereomers. Also indicate configurations of the stereogenic centers as 'R' or 'S' showing the priority sequence at each centre. **[2+2 = 4]**





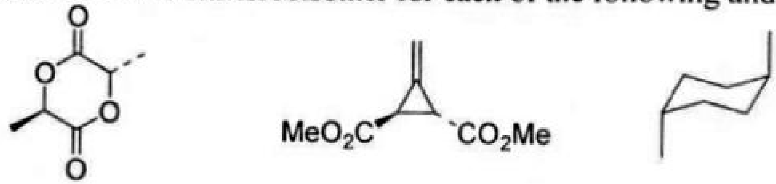
Ans 1a):



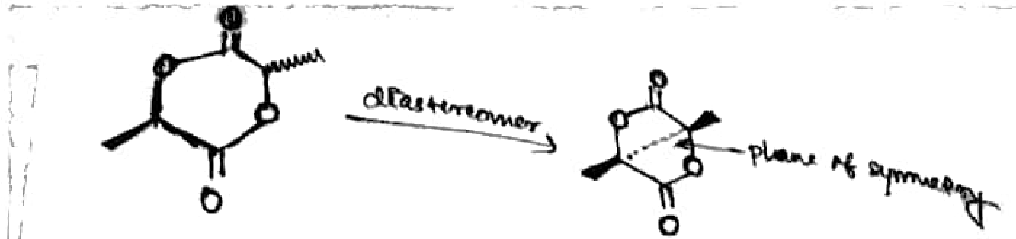


b) Draw the stereostructure of a diastereoisomer for each of the following and label them as chiral/achiral.

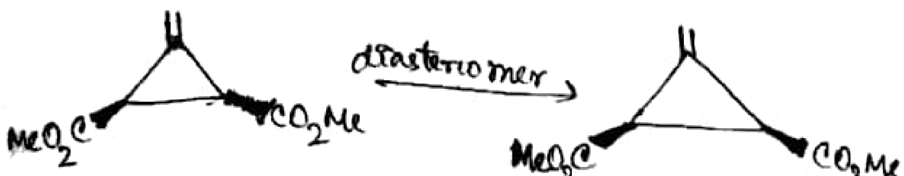
[1+1+1 = 3]



Ans 1b):

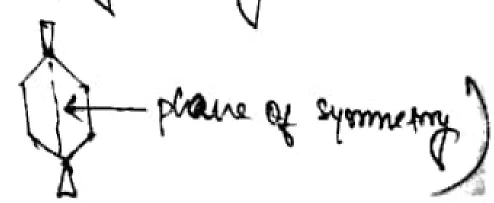


achiral
(plane of symmetry exists)

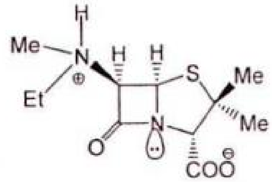


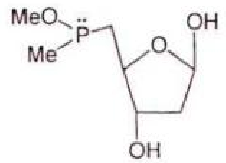
(chiral)

achiral
(plane of symmetry exists

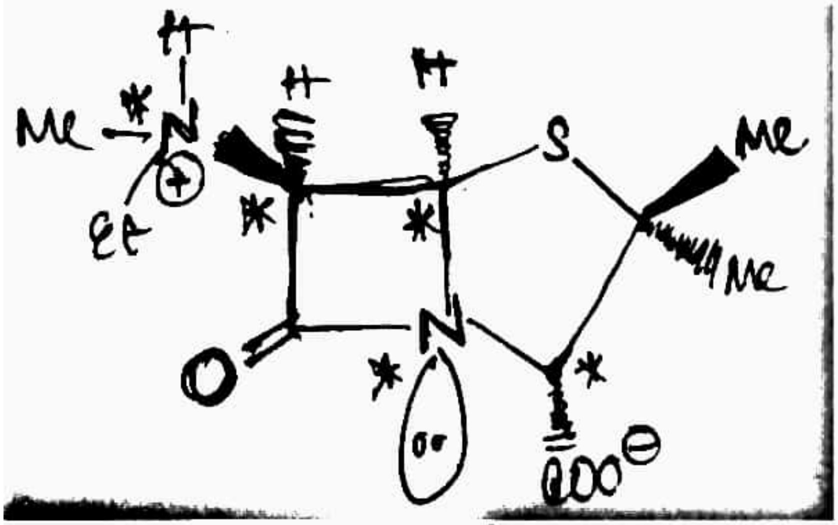


c) How many stereogenic center(s) is(are) present in each of the following molecules? Indicate them in the structures. **[1+1=2]**

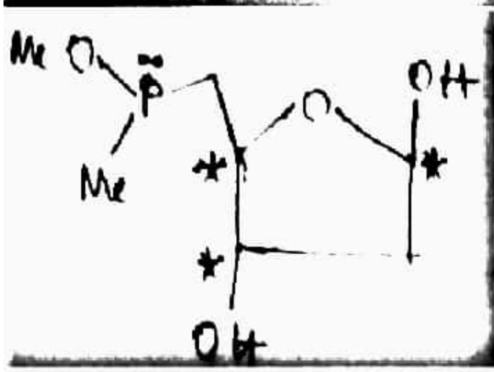




Ans 1c): No. of stereogenic centers (compound with N)= 5
No. of stereogenic centers (compound with P) = 3



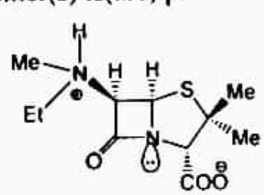
(✱marked are the stereogenic centres)

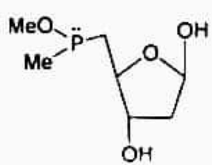


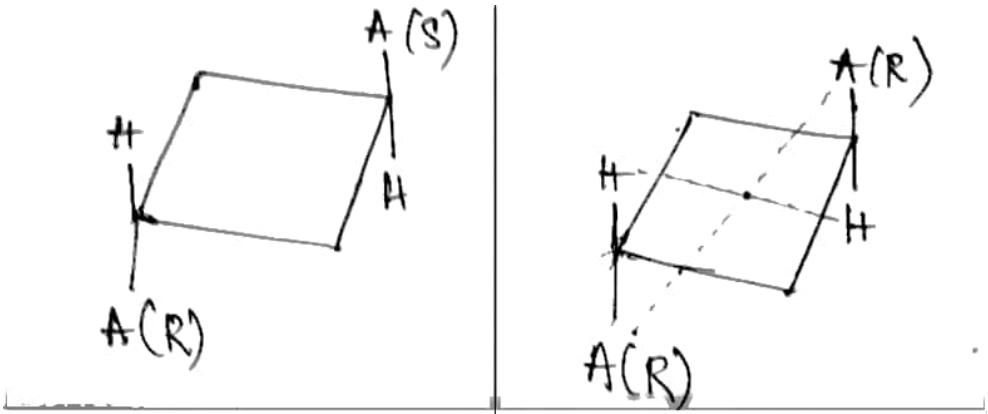
(✱marked are the stereogenic centers)

OR

c) How many stereogenic center(s) is(are) present in each of the following molecules? Indicate them in the structures. **[1+1=2]**

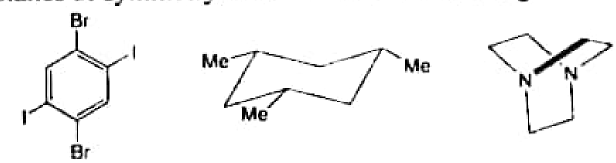


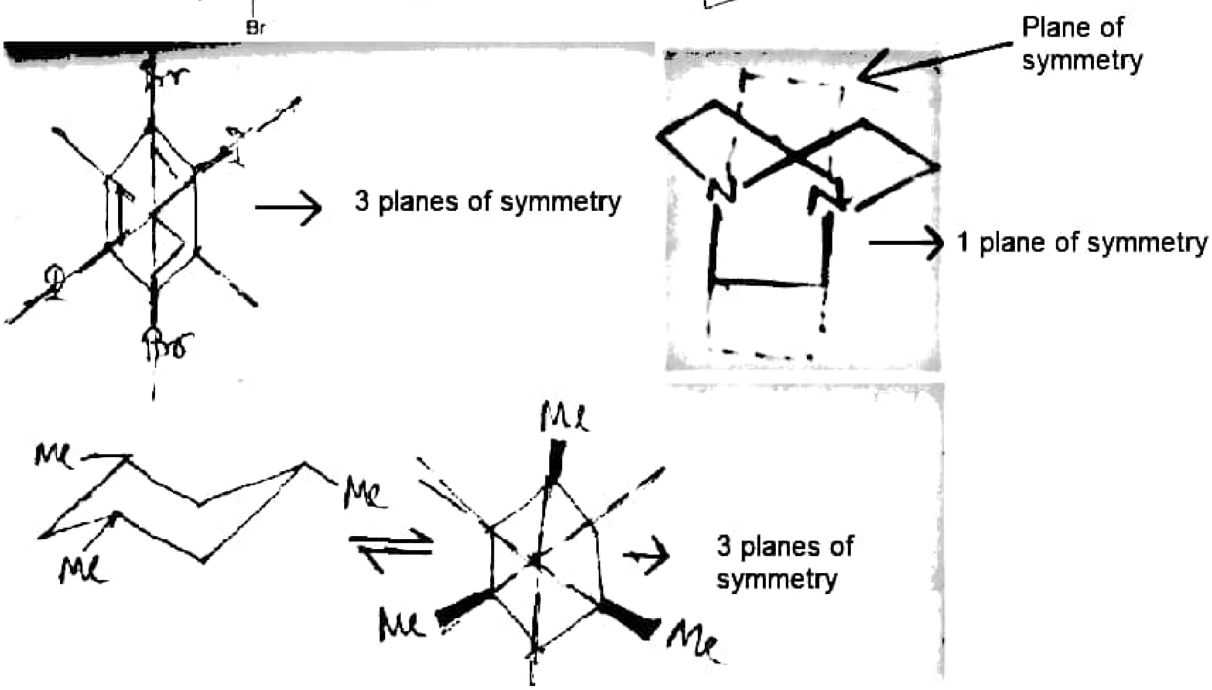




No center of symetry (as the absolute configuration of A are different across the center)	Center of Symmetry

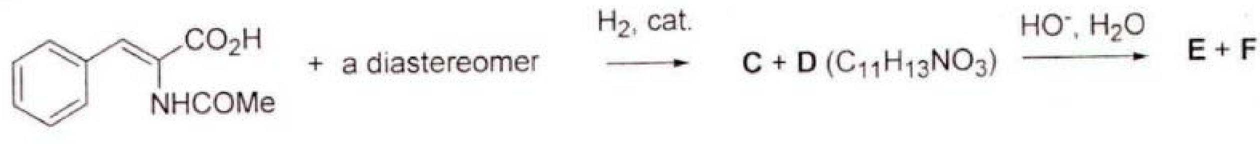
d) How many planes of symmetry does each of the following molecules possess?





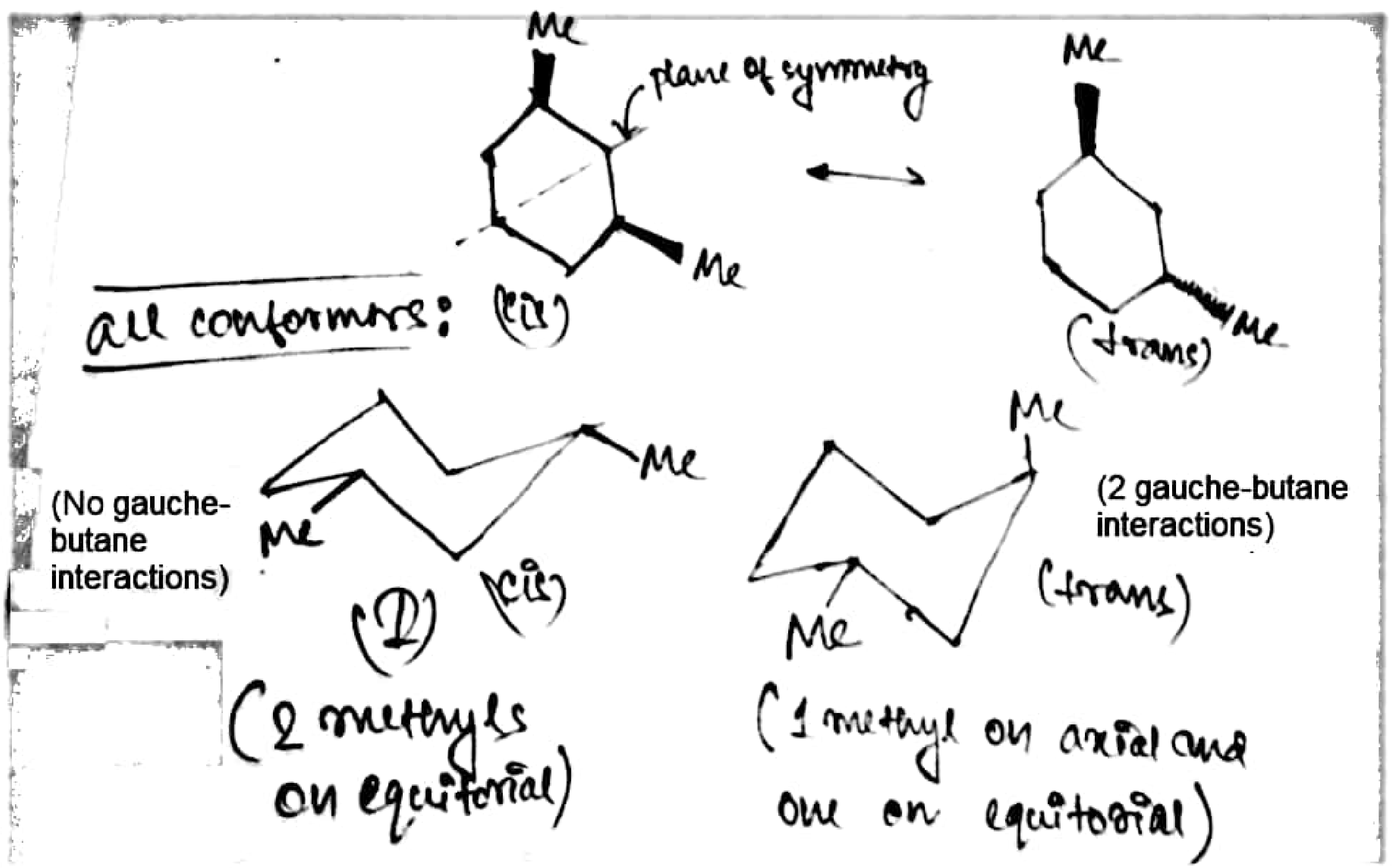
The **OR** part of the question is not in syllabus.

e) A mixture of two diastereoisomeric acids A and B, on catalytic hydrogenation, gives a racemic mixture of C and D ($C_{11}H_{13}NO_3$). The mixture, on alkaline hydrolysis, gives E and F. Write the structures of B, C, D, E and F. Comment on the optical activity of final mixture of products. Also assign E/Z configuration to A and B. **[4]**



A B

2. Draw all preferred conformations of the stereoisomers of 1,3-dimethylcyclohexane. Calculate their energy difference and comment on their optical activity. **[6]**



cis is optically inactive due to plane of symmetry, but due to two methyl groups being in equatorial position cis 1,3-dimethyl cyclohexane is more stable compared to its trans version which by the way is optically active. cis-1,3 dimethyl cyclohexane have no gauche-butane interaction while trans-1,3 dimethyl cyclohexane have 2 gauche butane interaction owing to which it is comparatively less stable.

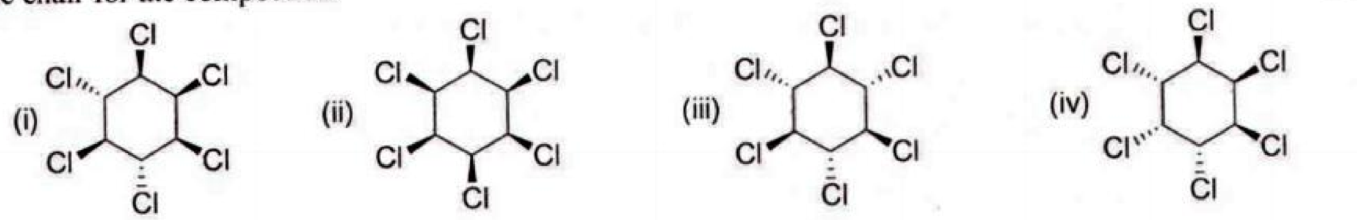
1 gauche-butane interaction aguments the energy of cyclohexane by 0.9 Kcal/mol.

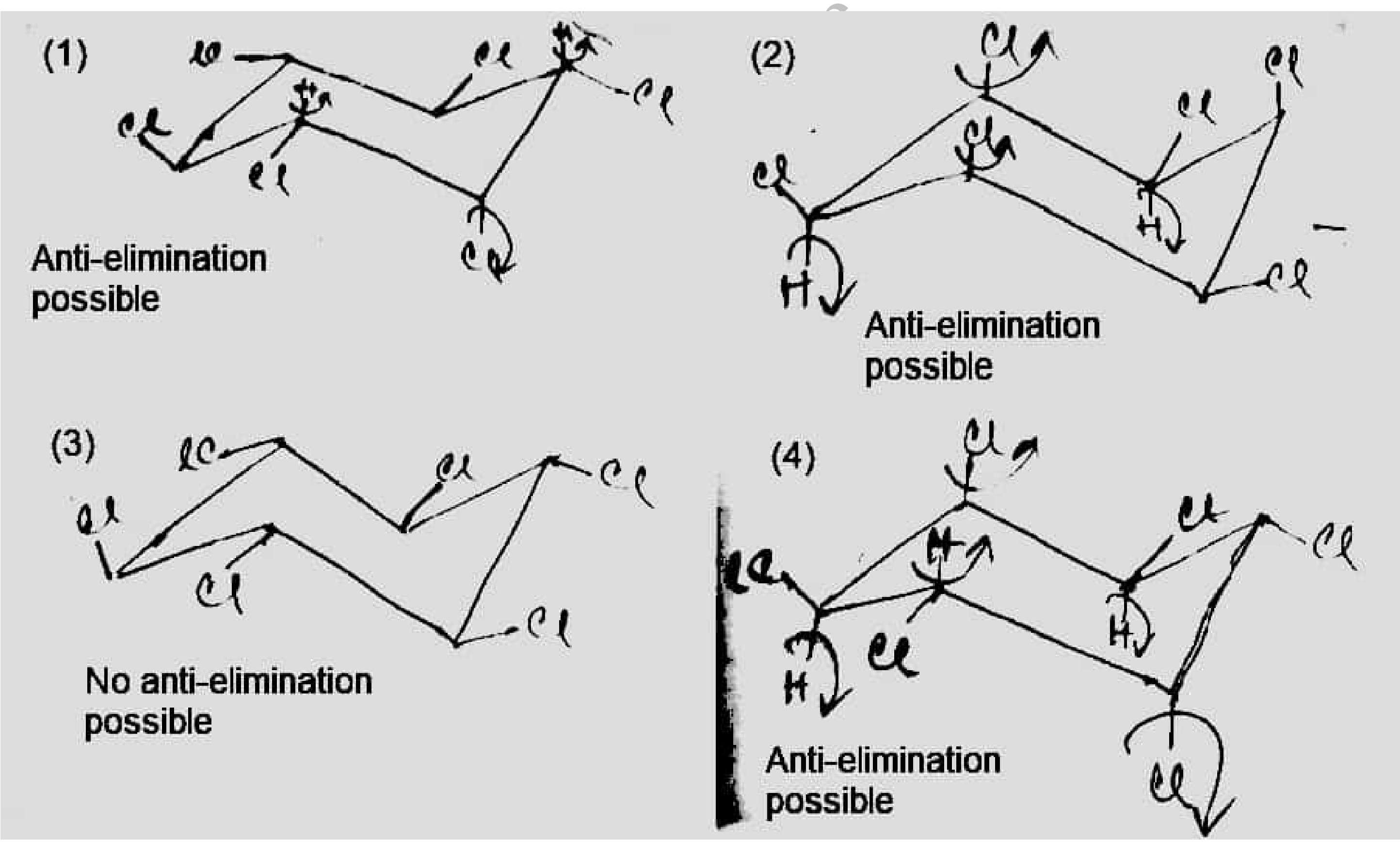
So 2 gauche-butane interactions are responsible for 1.8 Kcal/mol more energy for the trans counterpart thus making it coparatively less stable.

3. (a) Among 1-bromo-3-methylbutane, 2-bromo-3-methylbutane and 2-bromo-2-methylbutane, identify the one i) that undergoes faster S_N2 with NaOMe ii) undergoes faster S_N1 in EtOH; iii) that is optically active and produces an optically active compound when treated with NaN_3 in S_N2 mechanism. **[6]**

Ans 3a): (i) Faster $S_N{}^2$ with NaOMe :
1-bromo-3-methylbutane ; reason: alkylhalides favours SN^2
(ii) Faster $S_N{}^1$ with EtOH :
2-bromo-2-methylbutane ; reason: 3^o alkylhalides favour $S_N{}^1$
(iii) optically active product when reacted with NaN_3 via $S_N{}^2$ mechanism:
2-bromo-3-methylbutane ; reason: the resultant product will be optically active

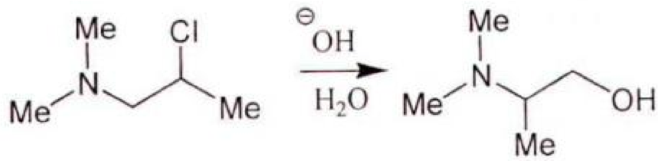
b) Which of the following hexachlorocyclohexanes is the least reactive in an E2 reaction. Explain by writing the chair for the compounds. **[4]**



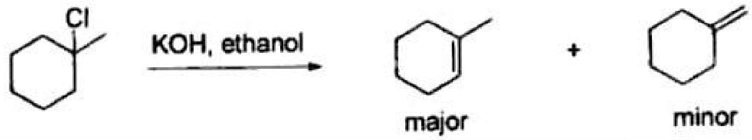


In (3) we clearly see that no chlorine is there in axial position, so no anti-elimination is possible with it. E_2 is favoured by anti-elimination, so (3) is least favoured to undergo E_2 mechanism.

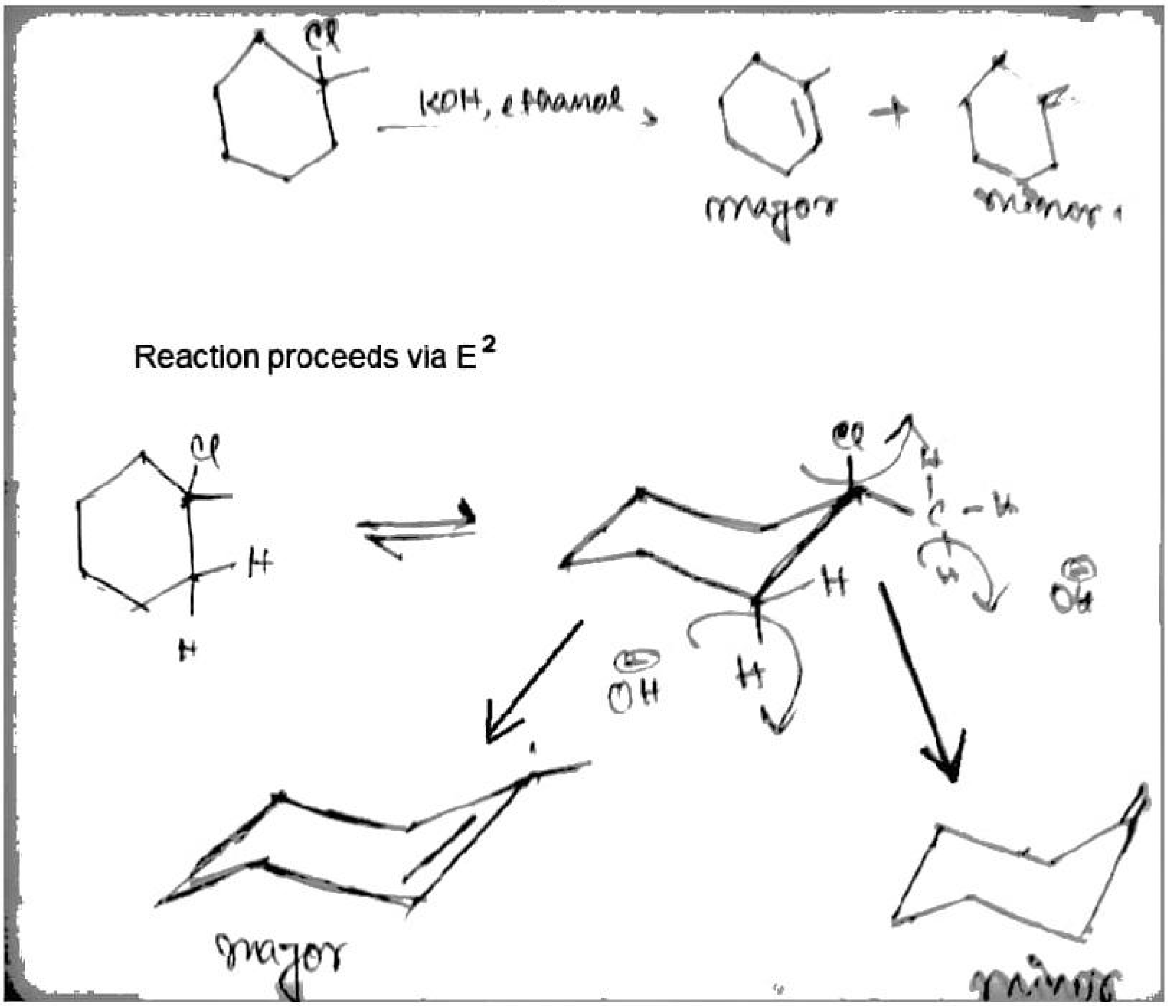
c) Draw a mechanism for the following transformation: [3+1=4]

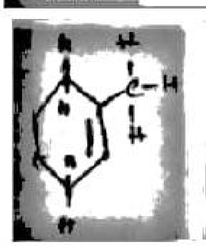


d) For the following E2 elimination reaction, draw the structures of transition states for both products and explain why 1-methylcyclohexene is formed as the major product. **[4]**



Ans:



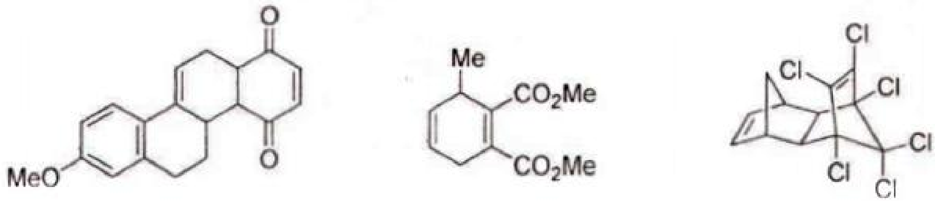


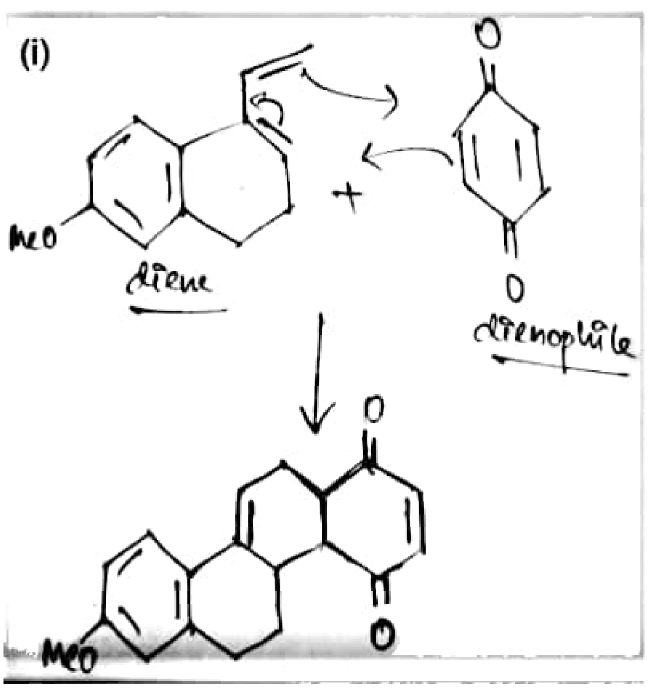
7 alpha H present for hyperconjugation

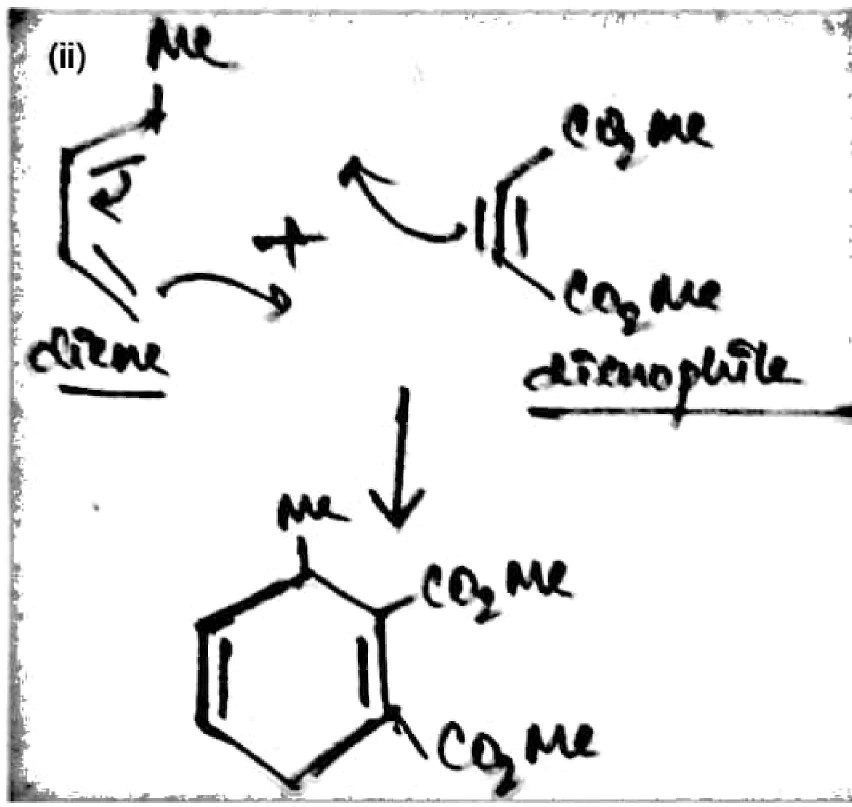
Only 4 alpha H present for hyperconjugation

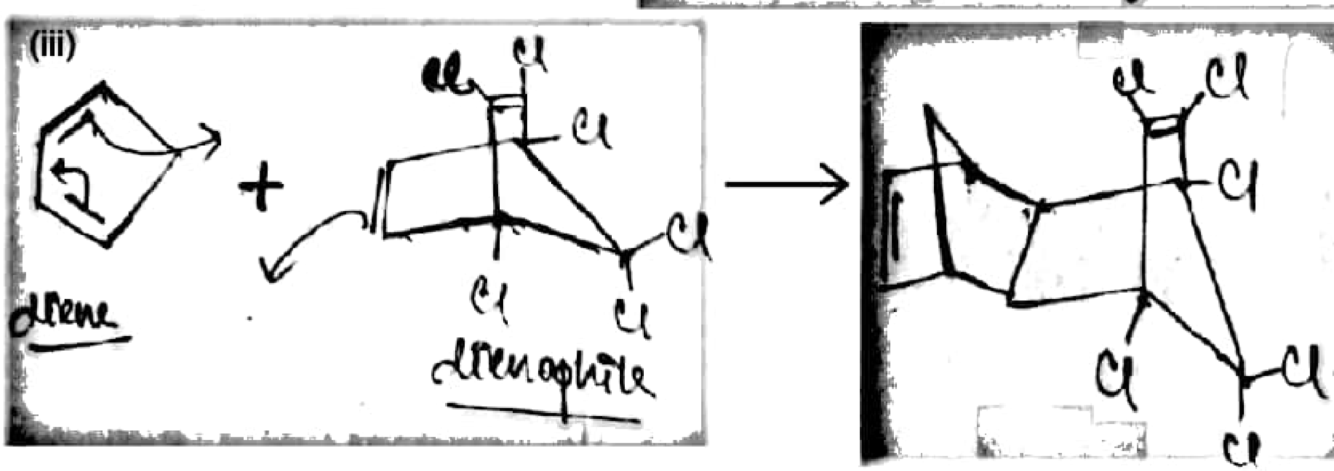
∴ This is more suitable and major product

4. (a) Draw the structures of the dienophiles that are needed to synthesize the following compounds by Diels-Alder reactions. **[3]**

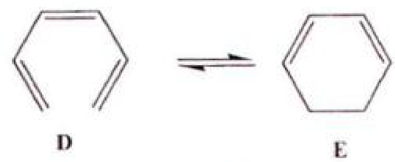




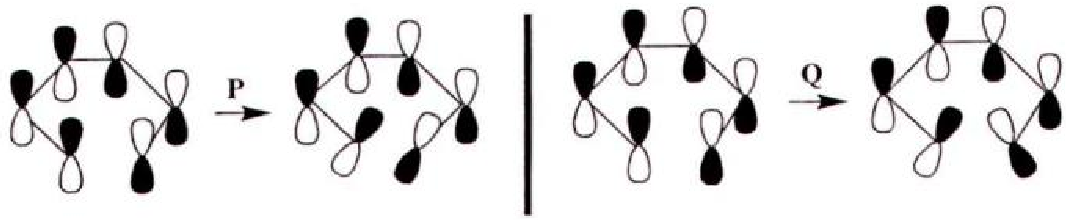




c) The following electrocyclization process can be carried out either thermally or photochemically. [3]



One of the π-molecular orbitals for **D** along with their orientation during cyclization processes **P** and **Q** are shown:



Consider the above MO picture and answer the following:

(i) Which of process P or Q denotes conrotatory ring closure?
(ii) Which process P or Q denotes disrotatory ring closure?
(iii) Based on the aromatic transition state theory, find out which process P or Q is allowed thermally?

Ans 4c):

(i) Process P denotes conrotatory ring closure.

(ii) Process Q denotes disrotatory ring closure.

(iii) According to aromatic transition state theory, for 6n electron system, thermally disrotatory ring c
allowed. So process Q is allowed thermally.

CHEMISTRY

END-AUTUMN SEMESTER **2012**

Part A - Inorganic (Answer in separate script and label as PART A)

1. (a) Calculate the E^0 value for the reduction of HClO to Cl^- in aqueous acidic solution

$$\underset{+7}{ClO_4^-} \xrightarrow{+1.2} \underset{+5}{ClO_3^-} \xrightarrow{+1.18} \underset{+3}{HClO_2} \xrightarrow{+1.65} \underset{+1}{HClO} \xrightarrow{+1.63} \underset{0}{Cl_2} \xrightarrow{+1.36} \underset{-1}{Cl^-}$$

using the Latimer diagram.

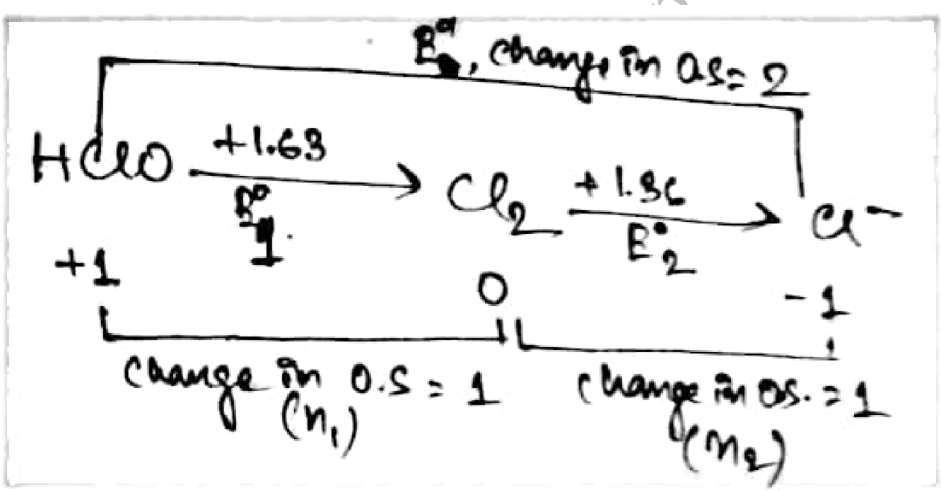
(b) Draw Frost diagram form the given Latimar diagram (acidic medium).

$$O_2 \xrightarrow{+0.70} H_2O_2 \xrightarrow{+1.76} H_2O$$

(c) Calculate the de Broglie wavelength of a He atom travelling at 1000 m s^{-1}.

(d) The work function of certain metal is 3.44×10^{-18} J. The absorption of a photon of unknown wavelength ejects electrons with velocity 1.03×10^6 m s^{-1}. What is the wavelength of the incident radiation? 2 + 2 + 2 + 2 = 8

Ans 1a):



$$E^o = \frac{n_1E_1^o + n_2E_2^o}{(n_1+n_2)}$$
$$= \frac{1\times 1.63 + 1.36 \times 1}{2}$$
$$= 1.495\text{V}$$

The E^o value for reduction of HClO to Cl^- is 1.495 V.

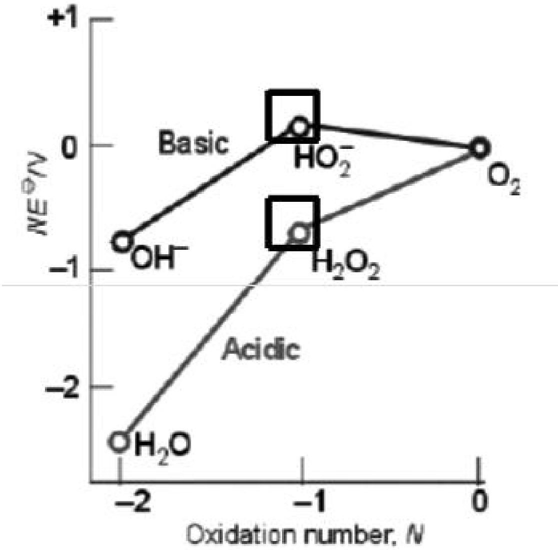
Ans 1b): $O_2 \xrightarrow{+0.70} H_2O_2 \xrightarrow{+1.76} H_2O$

$\therefore O_2 \xrightarrow{E^o} H_2O$

(0) (-2)

$\therefore n = -2$

$$E^o = \frac{0.07 \times 1 + 1.76 \times 1}{2} = 1.23\text{V}$$
$nE^o = -2.46\text{V}$
$\therefore O_2 \xrightarrow{+0.70} H_2O_2$
$\therefore n = -1$
$nE^o = -0.70\text{V}$



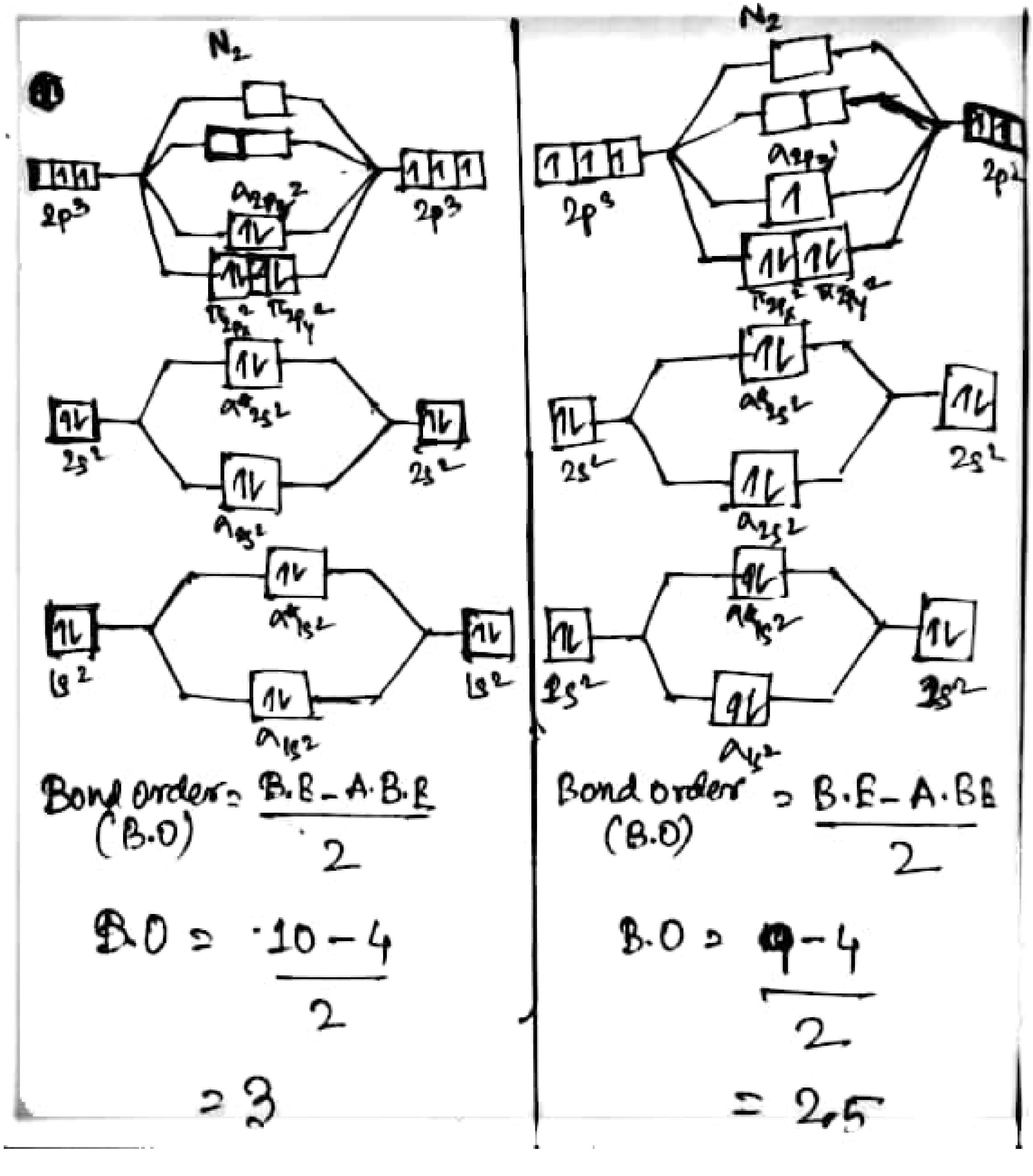
Ans 1c): Velocity of the atom $(v) = 1000\text{ms}^{-1}$
De-Broglie wavelength $(\lambda) = \dfrac{h}{mv}$
mass of the atom = $2\times$ mass of proton + $2\times$mass of neutron (mass of e^- can be neglected)
$= 6.63 \times 10^{-27}$ Kg
$mv = 6.63 \times 10^{-24}$ Kgms^{-1}
$$\lambda = \frac{h}{mv} = \frac{6.626 \times 10^{-34}}{6.63 \times 10^{-24}} = 1 \times 10^{-10} = 1\text{Å}$$

Ans 1d): Work function of metal $(w_o) = 3.44 \times 10^{-19}$ J
velocity of ejected photon $= 1.03 \times 10^6$ ms^{-1}
From Einstein's photoelectric equation,
$$\frac{1}{2}mv = \frac{hc}{\lambda} - w_o$$
$$\Rightarrow \frac{hc}{\lambda} = \frac{1}{2} \times 9.1 \times 10^{-31} \times (1.03 \times 10^6)^2 + 3.44 \times 10^{-18}$$
$$\Rightarrow \frac{hc}{\lambda} = 4.83 \times 10^{-19} + 34.4 \times 10^{-19} = 39.23 \times 10^{-19}$$
$$\Rightarrow \lambda = \frac{hc}{39.23 \times 10^{-19}} = \frac{6.626 \times 10^{-34} \times 2.997 \times 10^8}{39.23 \times 10^{-19}} = 0.51 \times 10^{-8}\text{m} = 51\text{Å}$$

2. (a) "The experimental dissociation energy of N_2 is 945 kJ mol^{-1}, whereas that of N_2^+ is 842 kJ mol^{-1}." – Explain the difference in terms of MOT.
(c) Arrange the following molecules in increasing order of dissociation energy: O_2, N_2, B_2. Which molecule will be stabilized when an electron is removed form it?

$2 + 2 + 2 = 6$

Ans 2a): Bond Order $\propto$ Bond Strength



Since, Bond Order of N_2 > Bond Order of N_2^+
so Bond Strength N_2 > Bond Strength of N_2^+
$\therefore$ more energy is required to break the bond of N_2 than N_2^+
The orbitol diagram of N_2 and N_2^+ are given in the next page,
$\therefore$ dissociation energy of N_2 is greater than N_2^+.

Ans 2c): Molecular orbital configuration of O_2 , N_2 and B_2

$$O_2: \quad \sigma_{1s^2} \quad \sigma^*_{1s^2} \quad \sigma_{2s^2} \quad \sigma^*_{2s^2} \quad \sigma_{2p_x^2} \quad \begin{matrix} \pi_{2p_y^2} & \pi^*_{2p_y^1} \\ \pi_{2p_z^2} & \pi^*_{2p_z^1} \end{matrix}$$

$$\text{B.O.} = \frac{10-6}{2} = 2$$

$$N_2: \quad \sigma_{1s^2} \quad \sigma^*_{1s^2} \quad \sigma_{2s^2} \quad \sigma^*_{2s^2} \quad \begin{matrix} \pi_{2p_y^2} \\ \pi_{2p_z^2} \end{matrix} \quad \sigma_{2p_x^2}$$

$$\text{B.O.} = \frac{10-4}{2} = 3$$

$$B_2: \quad \sigma_{1s^2} \quad \sigma^*_{1s^2} \quad \sigma_{2s^2} \quad \sigma^*_{2s^2} \quad \begin{matrix} \pi_{2p_x^1} \\ \pi_{2p_y^1} \end{matrix}$$

$$\text{B.O.} = \frac{6-4}{2} = 1$$

We know B.O $\propto$ Bond dissociation energy
Since, B.O.$_{N_2}$ > B.O.$_{O_2}$ > B.O.$_{B_2}$
N_2 , O_2 and B_2 in decreasing order of bond dissociation energy is:
$N_2 > O_2 > B_2$

On removing an electron , the electron will be removed from Bonding orbitals in case of N_2 and B_2 , but for O_2 it will be removed from an anti-bonding orbital. (see MO configuration)

Thus on removing an electron bond order of O_2 will increase so O_2^+ will be more stable than O_2.

But for N_2 and B_2 , N_2^+ and B_2^+ will have lesser bond order respective to their parent element and so will be less stable with respect to them.

3. (a) Arrange the following in increasing order of Δ_O value: $[Fe(CN)_6]^{3-}$, $[Ru(CN)_6]^{3-}$, $[Os(CN)_6]^{3-}$.

(b) Calculate the ΔS value of the following reaction. $Cd^{2+} + 2en = [Cd(en)_2]^{2+}$. The ΔG of the reaction is -60 kJ mol^{-1} and ΔH of the reaction is -55 kJ mol^{-1}. Temperature is 25 °C.

(c) Find out the CFSE and calculate the expected magnetic moment for Co^{3+} in strong and weak octahedral fields. Assume pairing energy to be P and crystal field splitting energy to be Δo. $2 + 2 + 2 = 6$

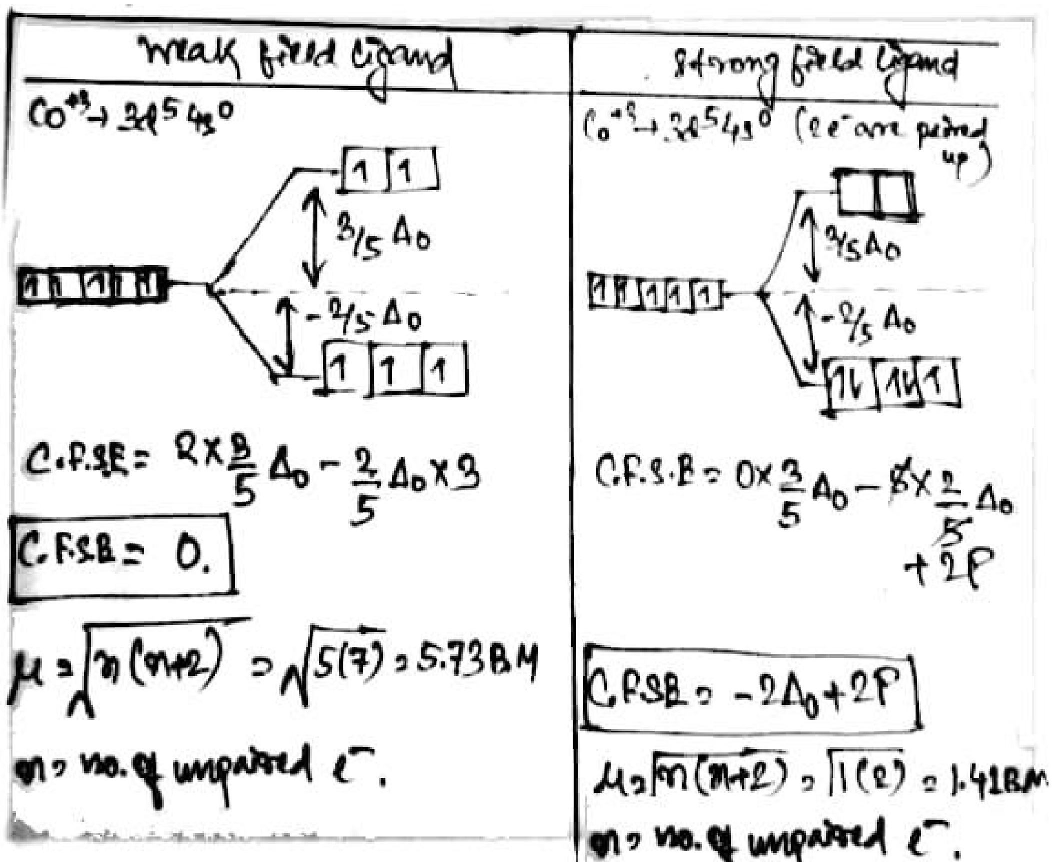
Ans 3a): Down the group the metal orbitals are more diffused so the overlap with the ligand orbital is better and therefore down the group $\triangle_o$ increases.

So for $[Fe(CN)_6]^{-3}$, $[Ru(CN)_6]^{-3}$ and $[Os(CN)_6]^{-3}$
the value of $\triangle_o$ in increasing order would be
$[Fe(CN)_6]^{-3} > [Ru(CN)_6]^{-3} > [Os(CN)_6]^{-3}$

Ans 3b): $Cd^{+2} + 2en = [Cd(en)_2]^{+2}$
$\triangle G = -60$ KJ/mol , $\triangle H = -55$ KJ/mol , T= 25^{o}C= 298 K
$\triangle G = \triangle H - T \triangle S$

$$\Rightarrow \triangle S = \frac{\triangle H - \triangle G}{T} = \frac{-55+60}{298} = 0.168 \text{ KJ mol}^{-1}\text{K}^{-1} \approx 16.8 \text{ J/mol/K}$$

Ans 3c): Orbital splitting of Co^{+3} under the influence of weak and strong field ligands is shown tn the next page:



4. (a) "CO is a π acid ligand." – Explain.

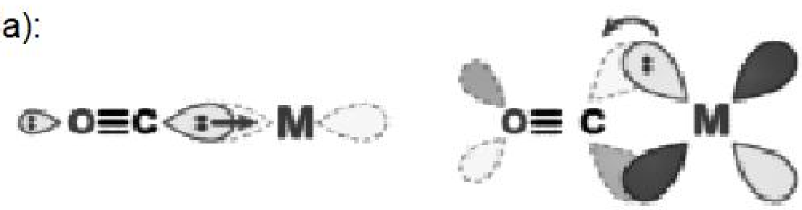
(b) "O^{2-} stabilizes metal in high formal oxidation states." – Explain.

(c) Explain the observed difference in equilibrium constants of the following two equlibria (i) $[Cu(H_2O)_6]^{2+} + en = [Cu(H_2O)_4(en)]^{2+} + 2H_2O$. $\log K_1 = 10.6$

(ii) $[Cu(H_2O)_6]^{2+} + 2NH_3 = [Cu(H_2O)_4(NH_3)_2]^{2+} + 2H_2O$. $\log \beta_2 = 5.0$

2 + 1 + 2 = 5

ANS 4a):



CO-M sigma bond M to CO pi backbonding

Carbonyl is cosidered a weak 2 electron σ donor adn π acceptor. Two types of interactions are involved in the complexation of carbonyl with transition metal ion. The carbonyl donates its lone pair to vacant metal d(σ) and back donation occurs from metal dπ orbital to C-O π^* orbital.

Ans 4c): (i) $[Cu(H_2O)_6]^{+2} + en = [Cu(H_2O)_4(en)]^{+2} + 2H_2O$

$$\log \beta_1 = 10.6$$

(ii) $[Cu(H_2O)]^{+2} + 2NH_3 = [Cu(H_2O)_4(NH_3)_2]^{+2} + 2H_2O$

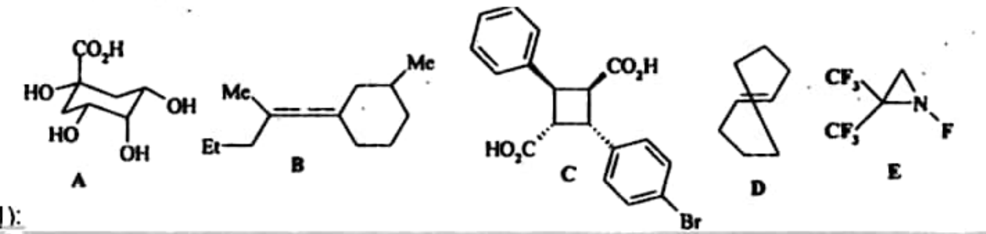
$$\log \beta_2 = 5.0$$

the formation constant β_1 in reaction (i) is higher compared to reaction (ii) because in reaction (i) the product formed has chelating ligand "en". Thus $\triangle S$ of reaction (i)$\log \beta_1$ is higher compared to $\log \beta_2$.

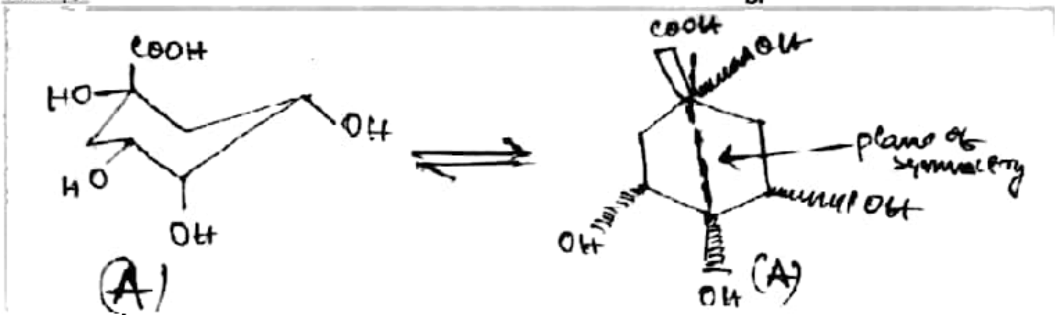
Part: B (Organic)

1. Whether the following compounds will be chiral/achiral/meso or exist as a racemic mixture (No explanation is required)?

   [1x5]



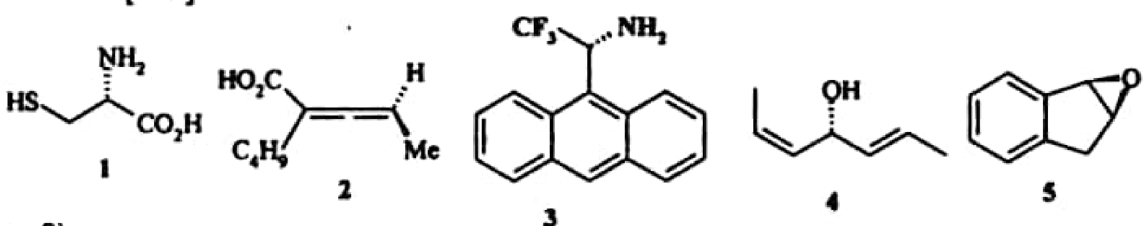
Ans 1):



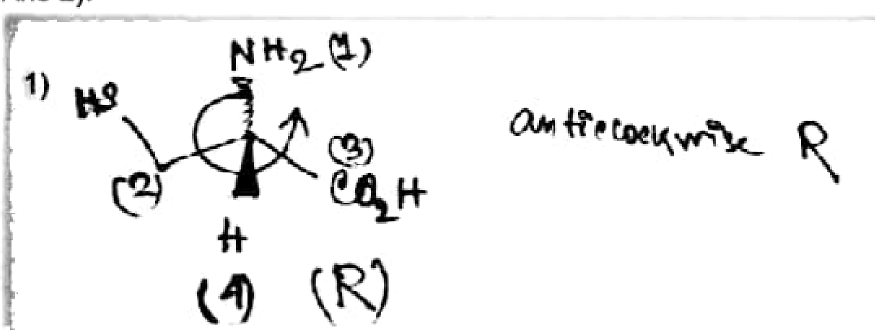
A: meso ; B: chiral (no plane of symmetry) ; C: chiral (no symmetry) ; D: chiral ; E: chiral

2. Write down the absolute configurations for the following molecules (as *R,S* notations of the asymmetric centers).

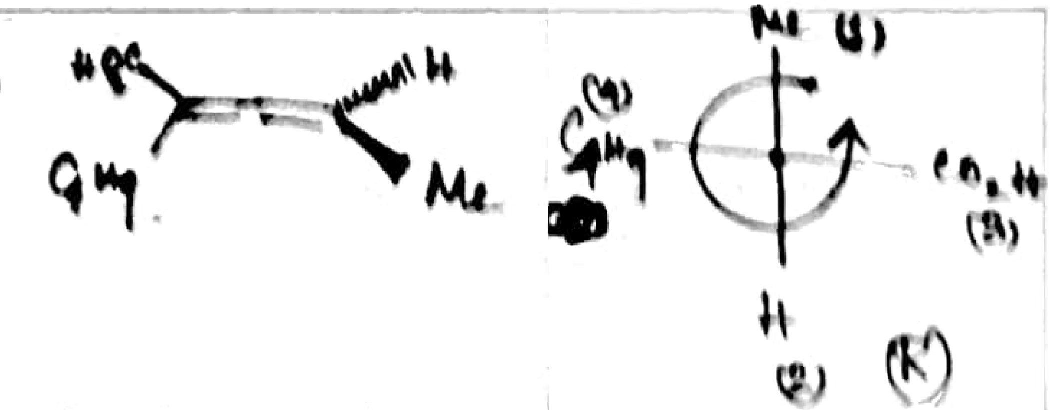
   [2x5]



Ans 2):

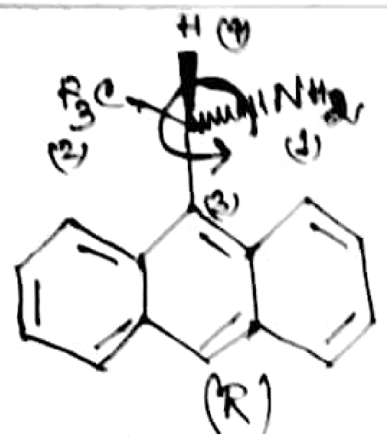


2)



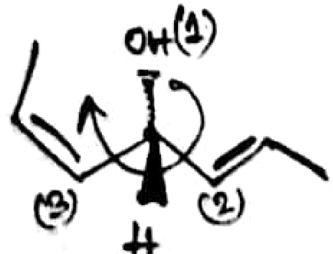
anticlockwise with least priority group in horizontal, so R

3)



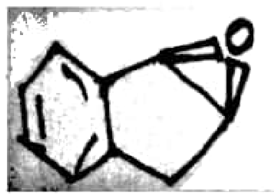
anticlockwise so, **R**

4)

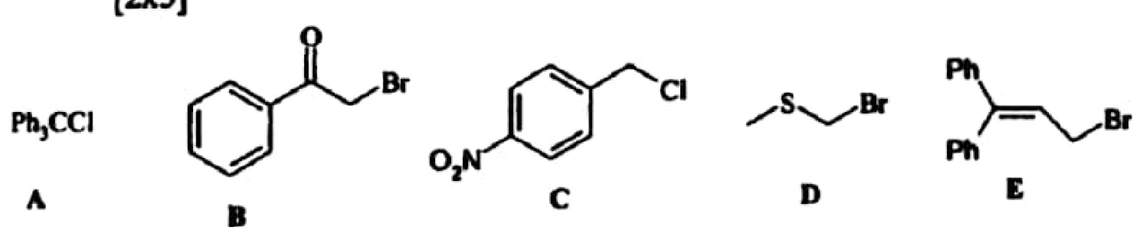


clockwise with
least priority group
in wedge so, **S**

5)



3. **Predict with concise explanation (one sentence) whether the following substrates will react under S_N1, S_N2 or both conditions?**
   [2x5]



Ans 3) (A) Ph_3CCl_3 will react under S_N1 mechanism because via the mechanism Ph_3C^+ (tri-phenyl cabocation) will be formed which is highly stable intermediate.

(B) **B** will react under S_N2 mechanism as the carbon attached to Bromine is 1^o and S_N2 is favoured in 1^o carbon. (also the carbocation intemediate for S_N1 would be highly unstable due to the **O** attached to the adjacent carbon)

(C) **C** will react under S_N2 mechanism. Although the carbocation intermediate for S_N1 is benzyl carbocation but due to the presence of strong -M.E. 'N_2O' group the carbocation would become unstable. So S_N2 will be favoured more.

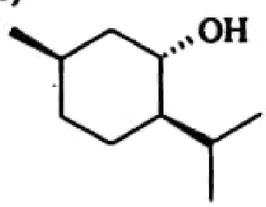
(D) **D** can react via both S_N1 and S_N2 because 1^o carbon but the carbocation can be stabilized byconfugation with lone pair of Sulphur .

(E) **E** can react via both S_N1 and S_N2. the carbocation will be stabilized fue to conjugation with the double bond (allyl carbocation) but the nature of alkyl halide is 1^o.

4. Draw the structures of most stable conformers for the given molecules? [2x5]

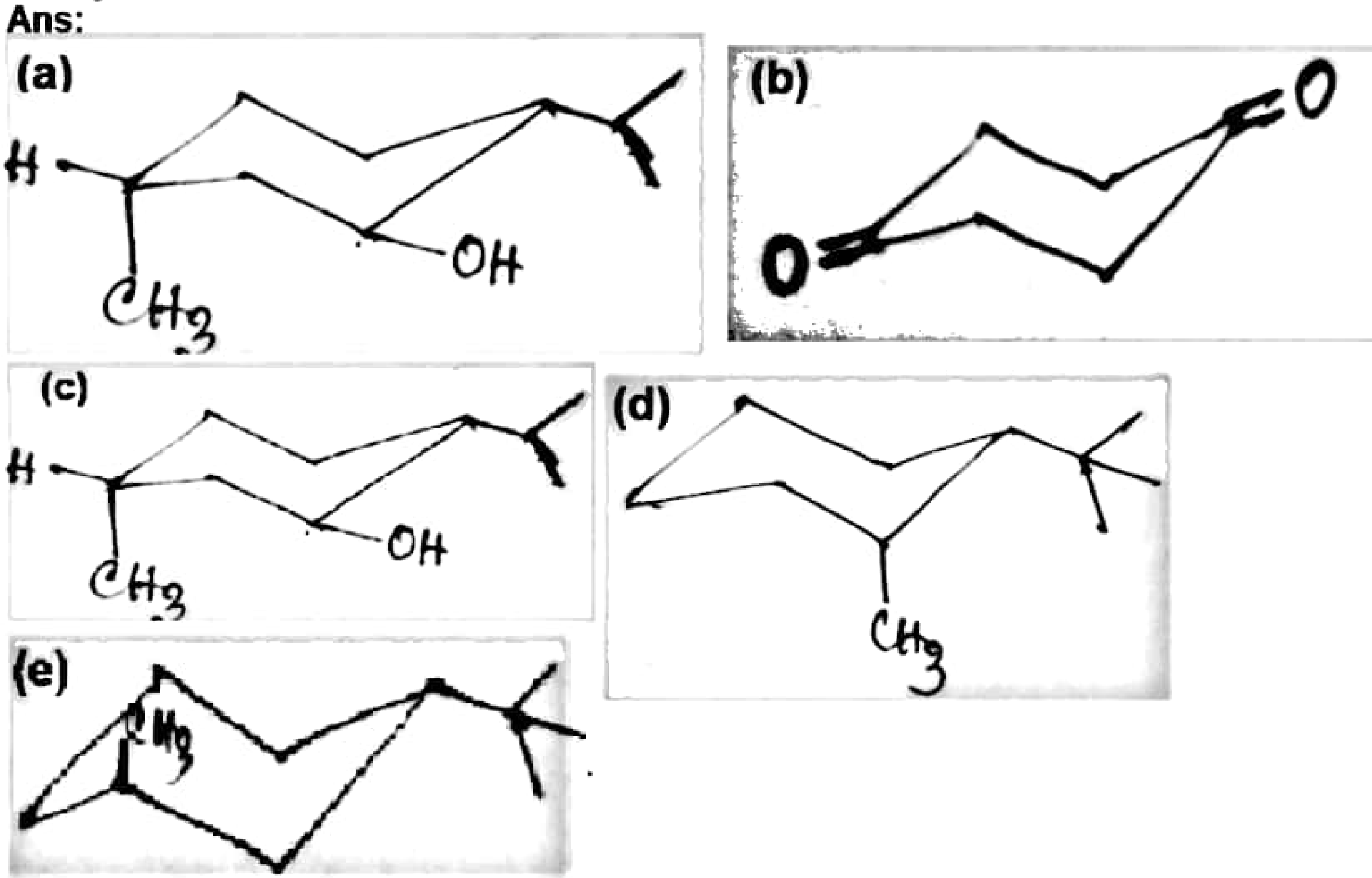
a) Meso-2,3-butane-diol; b) cyclohexane-1,4-dione;

c)

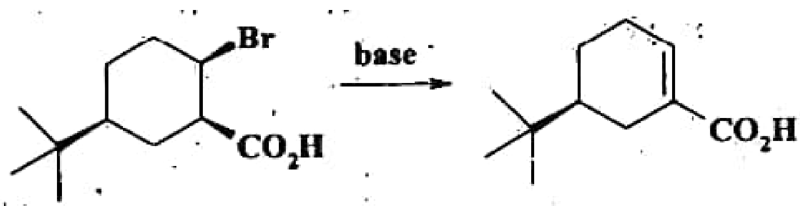


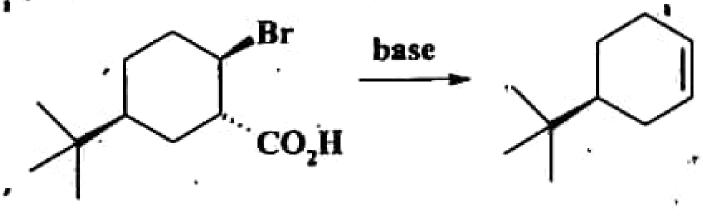
d) *cis*-1-tertbutyl-2-methyl cyclohexane, e) *trans*-1-tertbutyl-3-methyl cyclohexane

Ans:

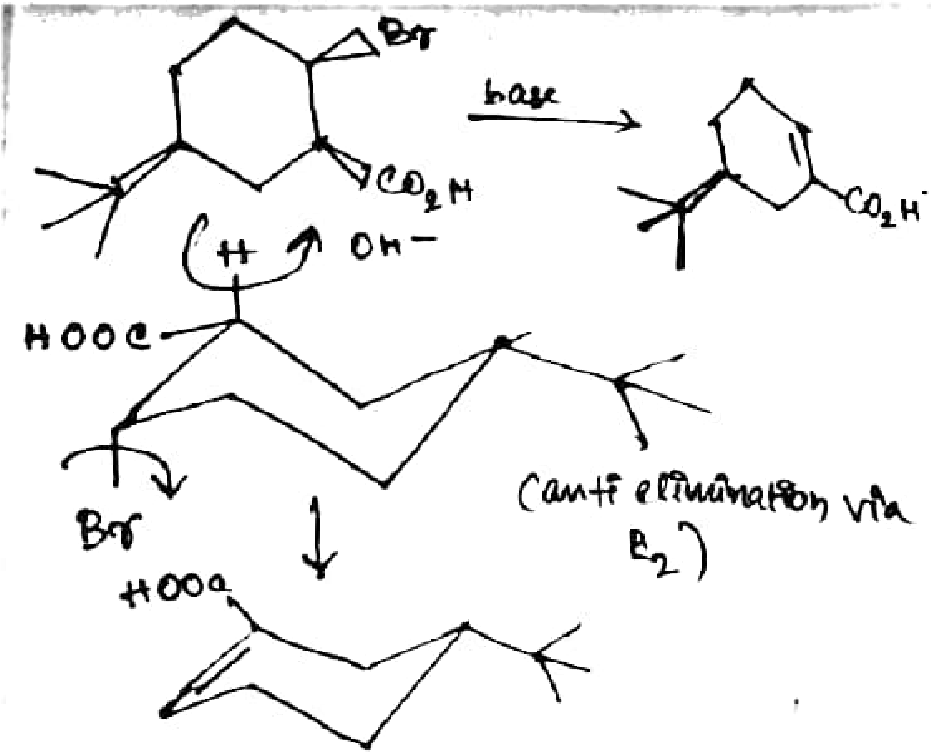


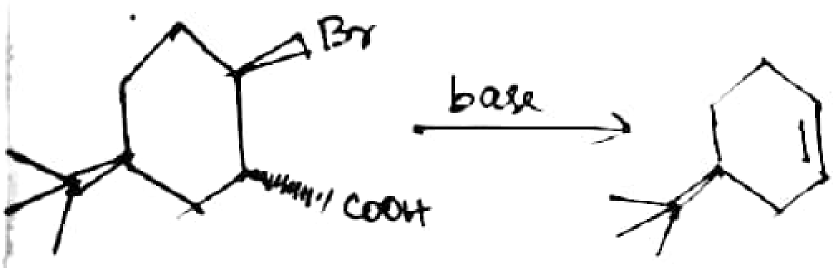
5. Account for the contrasting results in these two reactions (explain in terms of conformational analysis)?

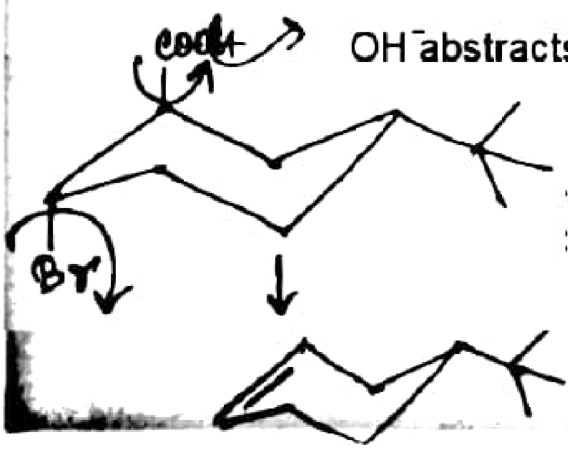




Ans:

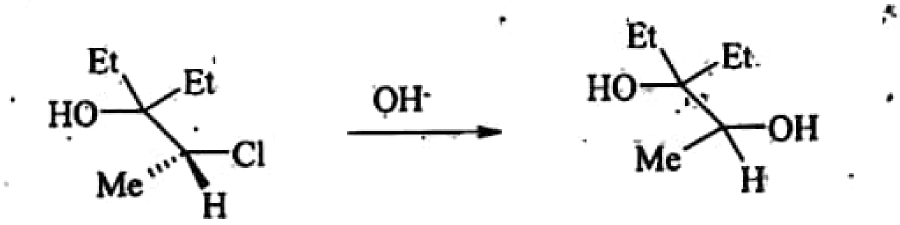






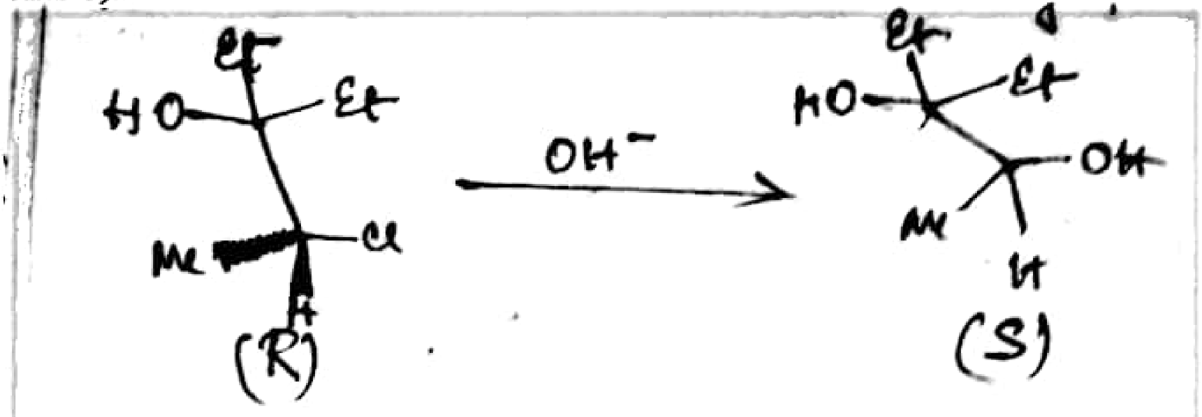
OH⁻ abstracts proton from -COOH and then Br breaks out and CO_2 comes out to form the result

6. **Predict the stereochemistry of the final product with proper reasoning?**

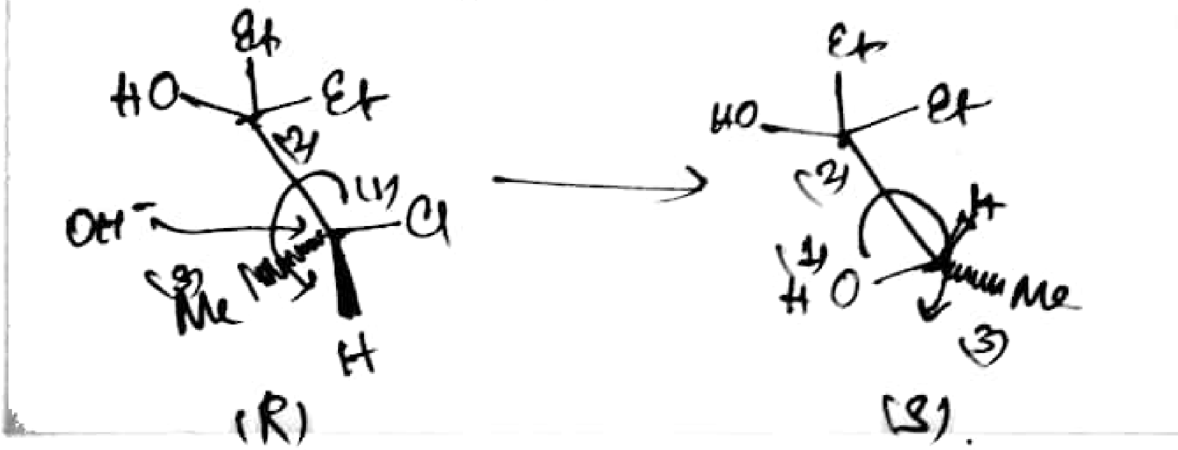


(R)

ANS 6):



The reaction will proceed via SN^2 reaction so an inversion of configuration will take place



Therefore the product will be 'S' configured.

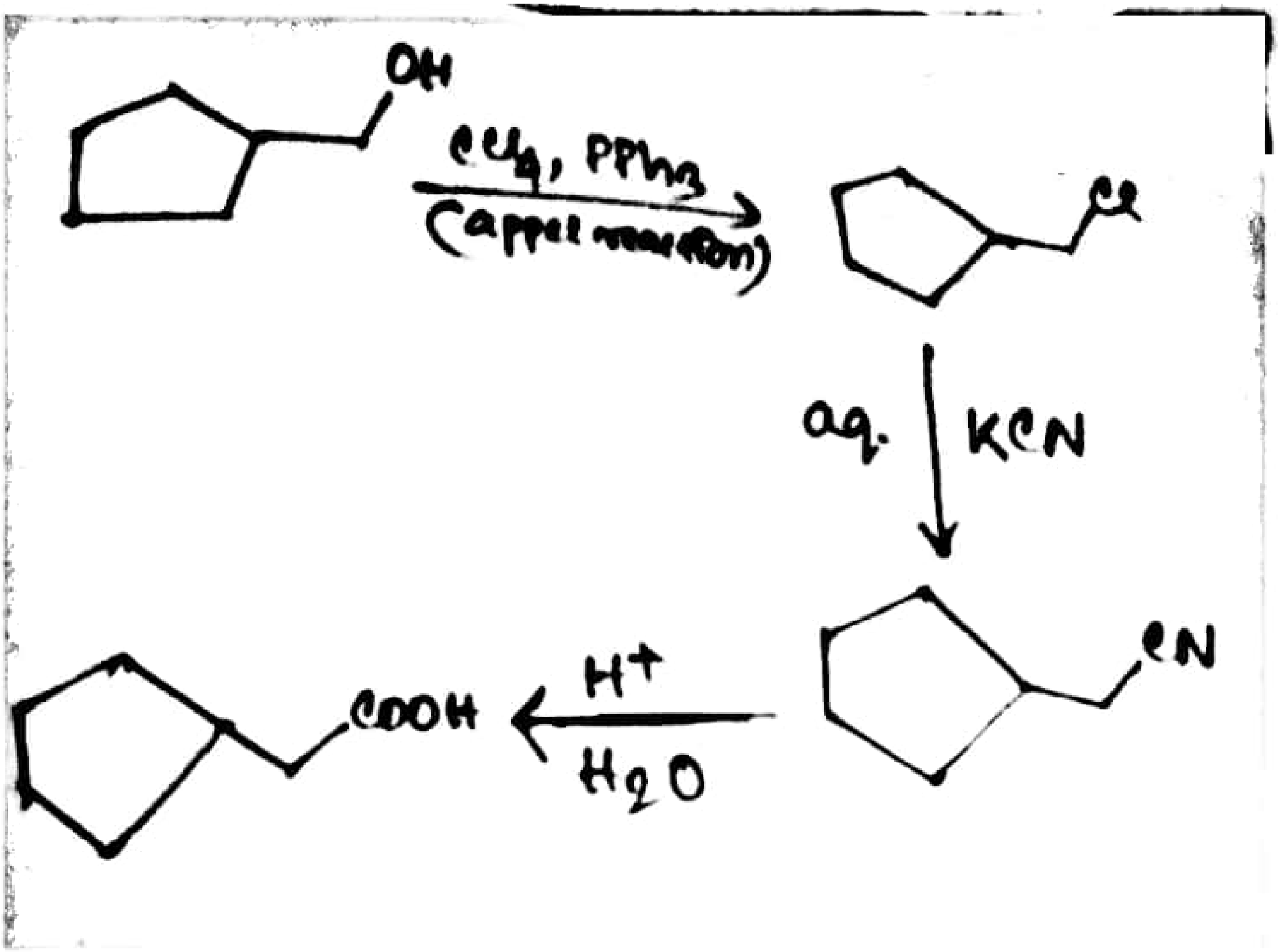
Note: S_N2 does not guarantee an inversion of absolute configuration. You will have to consider the arrangement and priority order of species around the chiral carbon in the product and assign 'R' or 'S' accordingly.

8. How do you carry out the following transformation as indicated?

[3]

in 3 steps (Use Appel reaction)

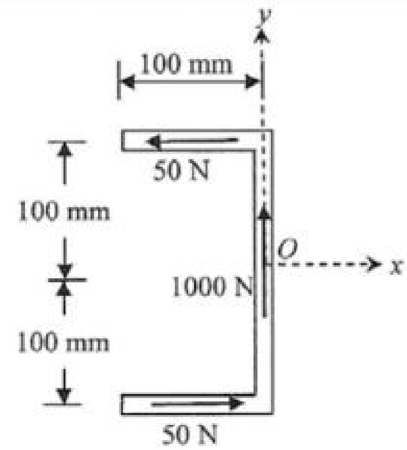
Ans 8):



MECHANICS

MID-SPRING SEMESTER EXAMINATION **2017-2018**

Question 1: A channel section is subjected to three forces as shown in the figure. If the force-system is to be replaced by a single force, $\overrightarrow{\mathbf{R}}$ then find (a) the magnitude and the direction of $\overrightarrow{\mathbf{R}}$ and (b) the equation of the line of action of $\overrightarrow{\mathbf{R}}$ in the given co-ordinate system. [3+3 = 6]



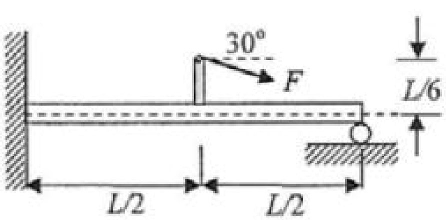
Ans: The given force system consists of a force $\overrightarrow{F} = 1000\hat{j}$ N passing through (0,0) and a couple of moment $\overrightarrow{C} = 0.2 \times 50\hat{k}$ (N-m)$= 10\hat{k}$ (N-m)

Let us find the equivalent force-couple system at (x,0) where x is expressed in meter. The equivalent force-couple system is a force $\hat{R} = 1000\hat{j}$ (N) and a couple of moment $\hat{C_{eq}} = \hat{C} - (1000x)\hat{k} = \Big(10 - 1000x\Big)\hat{k}$ (N-m)

If $x = \dfrac{10}{1000} = 0.01$ m, then the equivalent force system consists of a single free $\hat{R}$, only. Hence,

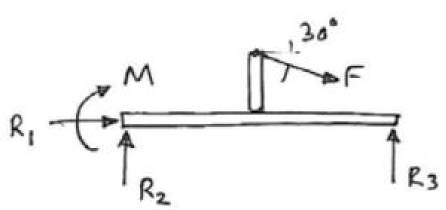
(a) The magnitude of $\hat{R}$ **is 1000(N)** and it's direction is **along positive y-axis i.e** $\hat{j}$

(b) the equation of the line of action is $\mathbf{x = 10}$ (where x is in mm)



Question 2: A propped cantilever beam is loaded as shown in the figure. (a) Draw a neat Free Body Diagram (FBD) of the beam. (b) State with proper justification whether it is possible to find the reaction force at the roller by considering the equilibrium equations alone. [3+2 = 5]

Ans a): The FBD if the cantilever beam is drawn below



b) Only two equations of equilibrium are obtained from the above free body diagram that relate the vertical forces and the moment.

They are,

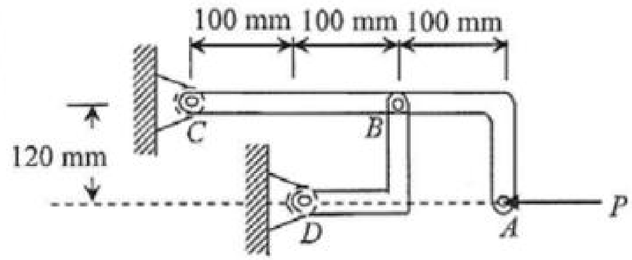
$$R_2 + R_3 = Fsin(30^o)$$

$$R_2 \times (L/2) = R_2 \times (L/2) + M$$

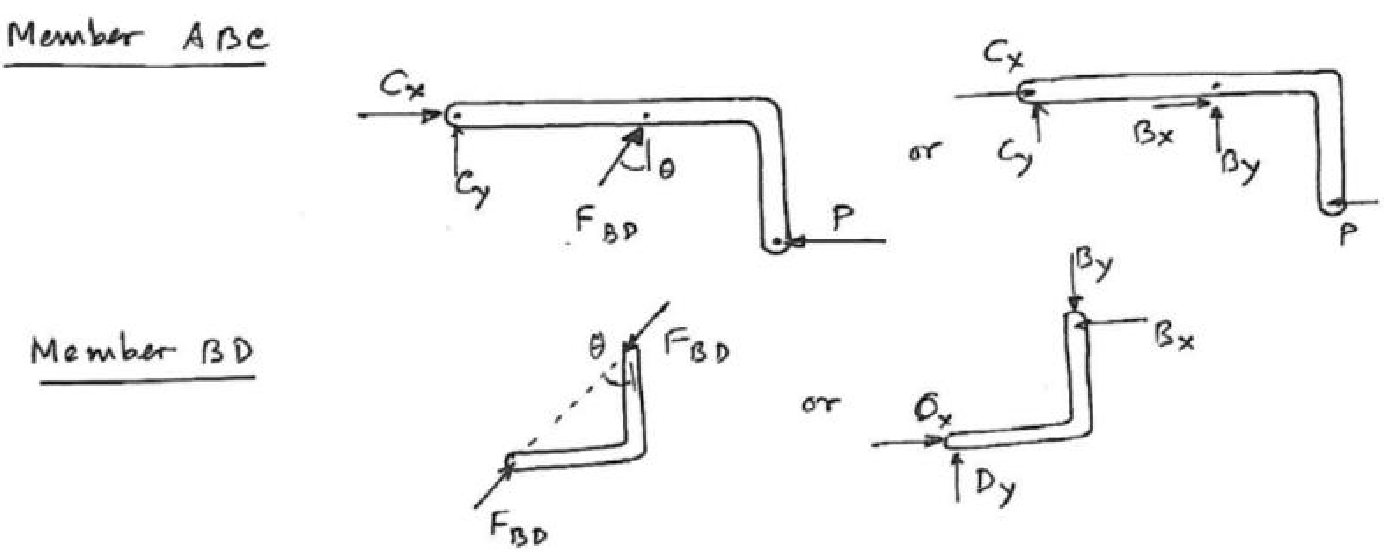
Thus it is not possible to get R_3, the reaction force at the roller.

Question 3: A frame consisting of two members *ABC* and *BD* is designed to carry a force *P* as shown in the figure. It is known that the reaction force at *B* is 5 kN. (a) Draw the FBD of members *ABC* and *BD*, (b) Calculate the value of *P* when its direction is as shown in the figure and (c) Calculate the magnitude and the direction of the reaction force on the frame at *C*.

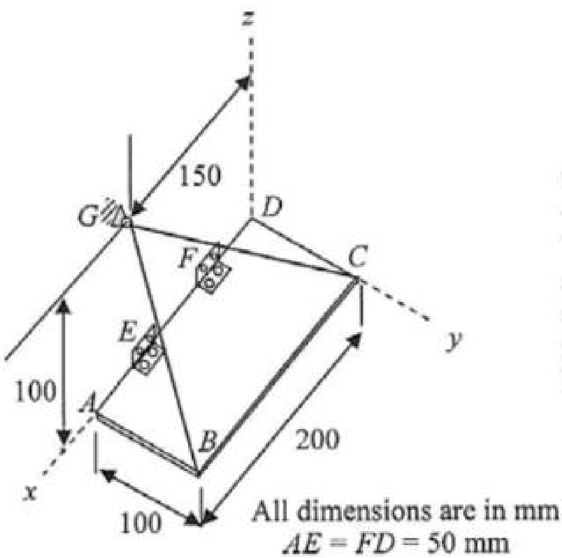
[4+4+4 = 12]



Ans a): The FBDs are as shown below



b): Given $|F_{BD}| = 5$kN
Considering FBD of member ABC and using the following equilibrium equations
$M_c = 0 \qquad \Rightarrow F_{BD}cos(\theta) \times 0.2 - P \times 0.12 \ldots\ldots\ldots\ldots\ldots (1)$
We get $\quad P = F_{BD} \times \dfrac{0.2 \times cos(\theta)}{0.12} = 1.2804\ F_{BD}$
Since P¿0 (for the direction is given), we take $F_{BD} = 5$kN and accordingly $\mathbf{P = 6.402}$ **kN**
c): Consider the FBD of member ABC. The following equations of equilibrium are written
$C_x + F_{BD}sin(\theta) = P \ldots\ldots\ldots (2)$
$C_y + F_{BD}cos(\theta) = P \ldots\ldots\ldots (3)$
From equations (2) and (3) we get
$C_x = 3.2$kN
$C_y = 3.84$kN
The reaction force at C has magnitude $\sqrt{C_x^2 + C_y^2} = 5$**kN** and **direction along** -50.2^o **from positive x-axis.**

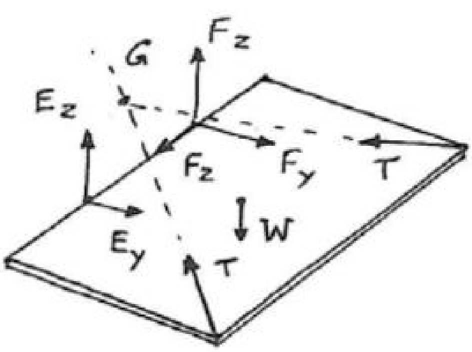


Question 4: A 100 mm × 200 mm uniform wooden panel weighing 400 N is supported by two frictionless hinges as shown in the figure. The panel is kept horizontal with the help of a single inextensible string that passes over a frictionless ring at *G*. The reaction moments at the hinges are negligible. Further the hinge at *E* allows sliding motion along its axis (i.e., *AD*). (a) Draw the FBD of the panel (b) Find the string tension.

[4+6=10]

1

Ans a): The FBD of the panel is drawn below



b): Considering moment balance equation about the axis AD, we get

$$M_x = 0 \qquad \Rightarrow T\big(\overrightarrow{AB} \times \hat{n}_{BG}\big).\hat{i} + T\big(\overrightarrow{DC} \times \hat{n}_{CG}\big).\hat{i} - \frac{0.1}{2} \times 400 = 0 \ . \ . \ . \ . \ . \ . \ (1)$$

where T is in Newton, $\overrightarrow{AB} = 0.1\hat{j}$ (m), $\overrightarrow{DC} = 0.1\hat{j}$ (m)

$$\hat{n}_{BG} = \frac{\overrightarrow{r_G} - \overrightarrow{r_B}}{|\overrightarrow{r_G} - \overrightarrow{r_B}|} = \frac{-0.05\hat{i} - 0.1\hat{j} + 0.1\hat{k}}{\sqrt{(0.05)^2 + (0.1)^2 + (0.1)^2}}$$

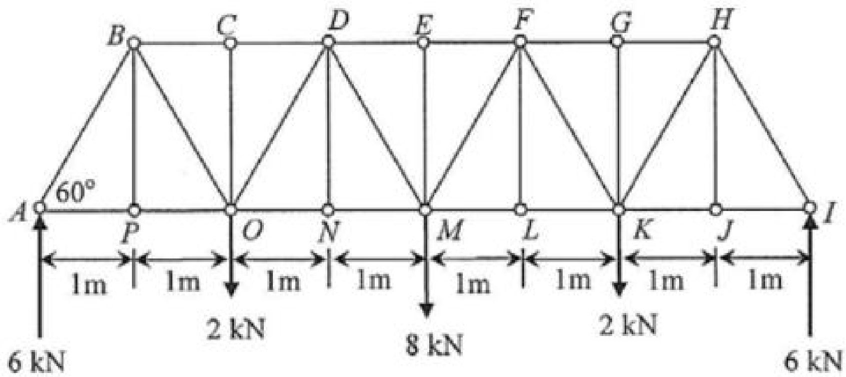
$$\hat{n}_{CG} = \frac{\overrightarrow{r_G} - \overrightarrow{r_C}}{|\overrightarrow{r_G} - \overrightarrow{r_C}|} = \frac{0.15\hat{i} - 0.1\hat{j} + 0.1\hat{k}}{\sqrt{(0.15)^2 + (0.1)^2 + (0.1)^2}}$$

From equation (1) we get

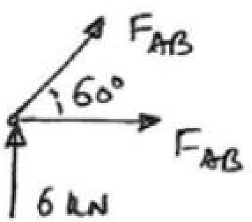
$$T\Big(\frac{0.1 \times 0.1}{\sqrt{(0.05)^2 + (0.1)^2 + (0.1)^2}} + \frac{0.1 \times 0.1}{\sqrt{(0.15)^2 + (0.1)^2 + (0.1)^2}}\Big) = 20$$

i.e. $T = 173.65$(N)

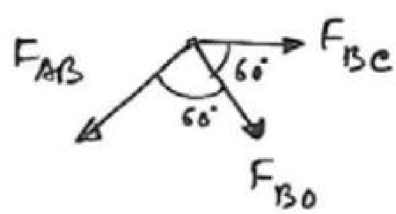
Question 5: A plane truss is subjected to loads as shown in the figure. (a) Calculate the force in member *BO* by using *method of joints*, (b) Calculate the force in member *FM* by *method of section* and (c) Identify the zero force members in the truss (*wrong identification carries penalty*). [4+4+4 = 12]



Ans a): Consider joint-A whose FBD is shown below



Joint A

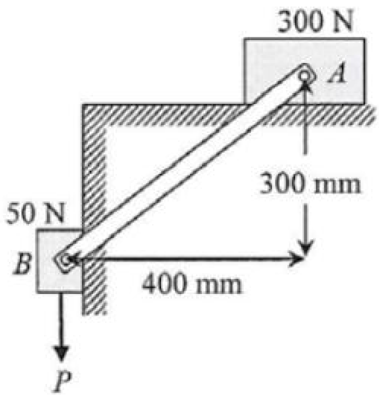


Joint B

From equilibrium equations ($\sum F_y = 0$) we get
$F_{AB} sin(60^o) + 6000 = 0$ . . . . (1)
i.e. $F_{AB} = -6000 \times \frac{2}{\sqrt{3}}$ (N)

We further note that member BP is a zero-force member. Next consider joint B. The FBD is shown above. The following equations of equilibrium are obtained
$F_{AB} cos(30^o) + F_{BD} cos(30^o) = 0$ . . . . (2)
$F_{AB} sin(30^o) = F_{BC} + F_{BD} sin(30^o)$ . . . . (3)
We thus get $F_{BD} = -F_{AB} = \frac{1200}{\sqrt{3}}$(N) = 6.928kN,
The force in member BD is 6.928kN (T).

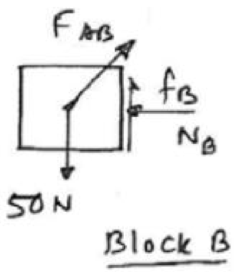
c): The zero force members are:
BP, CO, DN, EM, FL, GK, HJ

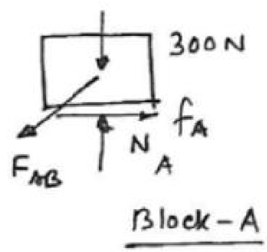


Question 6: Two rigid blocks A and B of weights 300 N and 50 N, connected by a massless rigid bar as shown in the figure, are in equilibrium. A vertical downward force P is then applied on block B to slide it down. The coefficient of static and kinetic friction between the blocks and the hatched surfaces are 0.2 and 0.18, respectively. (a) Draw the FBDs of both the blocks when $P = 0$, (b) Draw the FBDs of both the blocks when P is just enough to start motion. (c) Calculate the value of P required to start the motion.

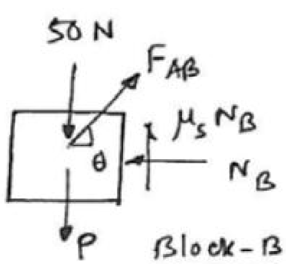
[3+3+4 = 10]

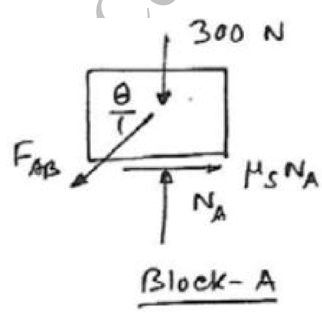
Ans a): FBDs of the blocks for P=0 are shown below (knowing that blocks are in equilibrium)





b): For impending downward motion of Block B the FBDs are shown below,





c): The following equations pf equilibrium are obtained from the free body diagrams shown in (b)

<u>Block-A:</u> $F_{AB}cos(\theta) = \mu_s N_A$ $(\sum F_x = 0)$ . . . . (1)

$F_{AB}sin(\theta) + 300 = N_A$ $(\sum F_y = 0)$ . . . .(2)

<u>Block-B:</u> $F_{AB}cos(\theta) = N_B$ $(\sum F_x = 0)$ . . . . (3)

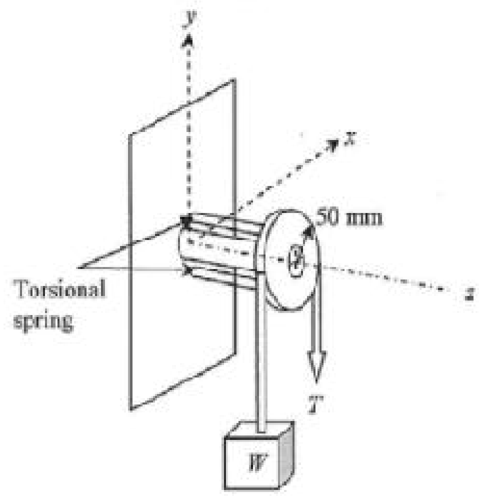
$F_{AB}sin(\theta) + \mu_s N_B = P + 50$ $(\sum F_y = 0)$ . . . .(4)

From equations (1)-(4) we get,

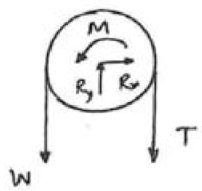
$$P = F_{AB}(sin(\theta) + \mu_s cos(\theta)) - 50 = 300\mu_s \frac{\big(sin(\theta) + \mu_s cos(\theta)\big)}{\big(cos(\theta) - \mu_s sin(\theta)\big)} - 50 = 17.06 \text{ N}$$

The required downward force is 17.06 N.

Question 7: A massless pulley required to lift a 1 kN weight, W, is able to rotate freely about the smooth axle. However its rotation is restricted by a torsional spring with spring constant 100 N.m/rad (i.e., 100 N.m torque is required to rotate the pulley by 1 rad about its axis) as shown in the figure. The coefficients of static and kinetic friction between the belt and the pulley are 0.2 and 0.15, respectively. (a) Draw the FBD of the pulley and the belt as a single body, (b) Find the range of belt tension, T, for which the belt does not slip over the pulley and (c) Find by how many degrees the pulley rotates when the tension is large enough to lift the weight up with uniform speed. [2+4+4 = 10]



Ans a): The FBD of the pulley + belt is shown below



, $M = K_T\theta$, where θ is measured positive in clockwise direction

b): When the weight is moving upward with uniform speed

$T = We^{\mu_x \pi} = 1.602$ kN

From the FBD shown in (a) we get the following equilibrium equation after taking moment about the center of the pulley.

$M = (T - W) \times r$

since $M = K_t\theta$, we get

$\theta = \dfrac{(T - W) \times r}{K_t} = \dfrac{W(e^{(}\mu_x\pi) - 1)r}{K_t} = 0.301$ rad

The pulley rotates by 17.2454^o in the clockwise direction.

MECHANICS

MID - AUTUMN SEMESTER EXAMINATION**2017-2018**

1. A woman supports an 80 kg homogeneous box on a horizontal rough ledge by providing only an upward vertical force at the corner B, as shown in the figure . We need to determine the range (F_{min}, F_{max}) within which the vertical force at box B must lie for keeping the box in equilibrium without tilting or moving it from the horizontal position shown.

(a) Draw two separate free body diagrams corresponding to F_{Bmin} and F_{Bmax}. (4)

(b) Determine the range F_{Bmin}, F_{Bmax} . Take $g = 10m/s^2$ (12)

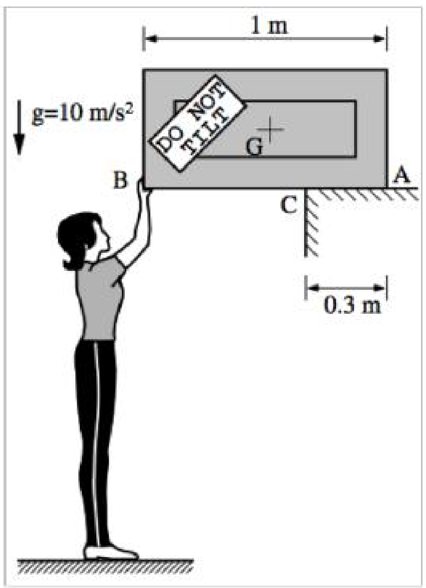
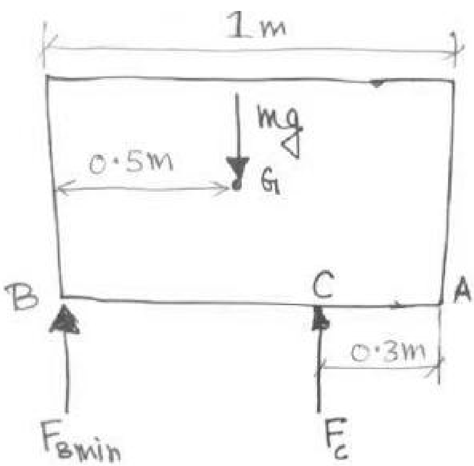


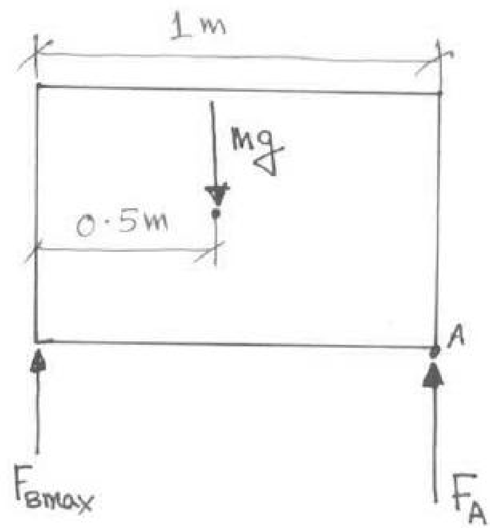
Figure 1

ANS:

(a) Only verticle forcwe applied at B. The box may tilt about the point C or A.

For F_{Bmin} the box will tilt about point C.	ForF_{Bmax} the box will tilt about point A.





(b)

F_{Bmin} :
$\sum M_c = 0$(left FBD)
$F_{Bmin} \times 0.7 - mg \times 0.2 = 0$
$\Rightarrow F_{Bmin} = \frac{2}{7}mg = \frac{2}{7} \times 80 \times 10N$
$\Rightarrow F_{Bmin} = 228.57N$

F_{Bmin}:
$\sum M_A = 0$(right FBD)
$F_{Bmax} \times 1 - mg \times 0.5 = 0$
$\Rightarrow F_{Bmax} = \frac{1}{2}mg = 40 \times 10N$
$\Rightarrow F_{Bmax} = 400N$

The range of F_B to maintain equilibrium is

$(F_{Bmmin}, F_{Bmax}) = (228.57N, 400N)$

2. The 7 m long massless beam AB is supported by a ball-and-socket joint at A and two inextensible cables BC and DE, as shown in Figure 2. The beam makes equal angles with x, y and z axes. The cable DE is parallel to y-axis. A vertically downward load of $W = 2$ kN is applied to the beam at the end B.

   (a) Draw a neat free body diagram of the beam AB. **(5)**

   (b) Determine the cable tensions T_{BC} and T_{DE}. **(10)**

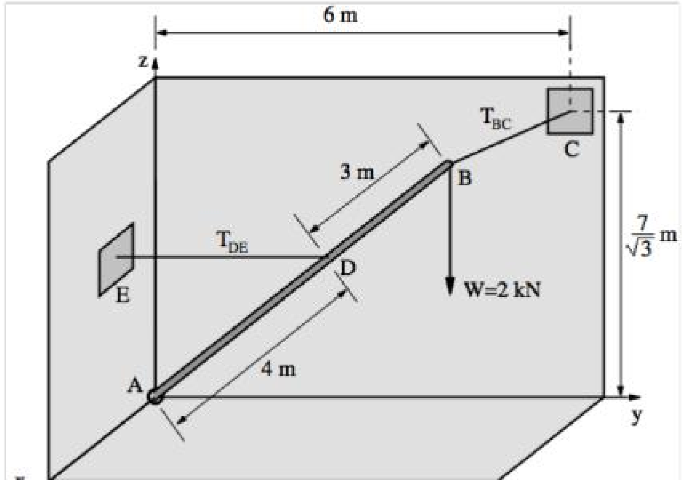


Figure 2

ANS:

(b)

$\overrightarrow{r_{AB}} = \frac{7}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})m$ (as AB makes equal angle with x, y and z axes)
$\overrightarrow{r_{AD}} = \frac{4}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})m$
$\overrightarrow{r_{AB}} = (6\hat{j} + \frac{7}{\sqrt{3}}\hat{k})m$
$\Rightarrow \overrightarrow{r_{BC}} = \overrightarrow{r_{AC}} - \overrightarrow{r_{AB}} = -\frac{7}{\sqrt{3}}\hat{i} + (6 - \frac{7}{\sqrt{3}}\hat{j}$
Forces: $\overrightarrow{T_{BC}} = T_{BC}.n\hat{_{BC}} = \frac{T_{BC}}{|\overrightarrow{r_{BC}}|} - \frac{7}{\sqrt{3}}\hat{i} + (6 - \frac{7}{\sqrt{3}}\hat{j}$
$\overrightarrow{T_{DE}} = -T_{DE}\hat{j}$ (parallel to y axis)
$\overrightarrow{F} = -(2KN)\hat{k}$ (vertically downward)

Take moment about A: $\sum M_A = 0 \Rightarrow \overrightarrow{r_{AD}} \times \overrightarrow{T_{DE}} + \overrightarrow{r_{AB}} \times \overrightarrow{T_{BC}} + \overrightarrow{r_{AB}} \times \overrightarrow{F} = 0$
$\Rightarrow \frac{4}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k}) \times (-T_{DE}\hat{j}) + \frac{7}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k}) \times \frac{T_{BC}}{|\overrightarrow{r_{BC}}|} - \frac{7}{\sqrt{3}}\hat{i} + (6 - \frac{7}{\sqrt{3}})\hat{j} + \frac{7}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k}) \times -2\hat{k} = 0$

$\Rightarrow (-4T_{DE}\hat{k} + 4T_{DE}\hat{i}) + \frac{7T_{BC}}{|\overrightarrow{r_{BC}}|}(\frac{7}{\sqrt{3}}\hat{k} - \frac{7}{\sqrt{3}}\hat{j} + (6 - \frac{7}{\sqrt{3}})\hat{k} - (6 - \frac{7}{\sqrt{3}}\hat{i})) + 14\hat{j} - 14\hat{i} = 0$
$\Rightarrow (4T_{DE} - \frac{7T_{BC}}{|\overrightarrow{r_{BC}}|}(6 - 7/\sqrt{3}) - 14)\hat{i} + (-\frac{7T_{BC}}{|\overrightarrow{r_{BC}}|} \times (7/\sqrt{3}) + 14)\hat{j} + (-4T_{DE} + \frac{7T_{BC}}{|\overrightarrow{r_{BC}}|} \times 6)\hat{k} = 0 = 0\hat{i} + 0\hat{j} + 0\hat{k}$

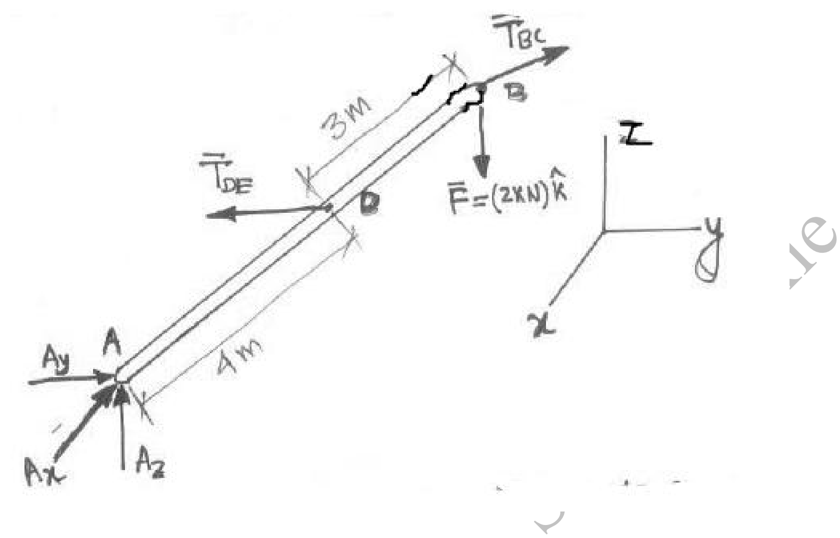
From the component along y we obtain:
$-\frac{7T_{BC}}{|\overrightarrow{r_{BC}}|} \times (7/\sqrt{3}) + 14 = 0 \Rightarrow T_{BC} = \frac{14\sqrt{3}}{49} \times |\overrightarrow{r_{BC}}|$ KN
Now :$|\overrightarrow{r_{BC}}| = ((6 - \frac{7}{\sqrt{3}})^2 + \frac{49}{3})^{0.5} = 4.491$

$\Rightarrow T_{BC} = 2.222$ KN

From the component along z axis we obtain:
$-4T_{DE} + 7\frac{T_{BC}}{|\overrightarrow{r_{BC}}|} \times 6 = 0$
$\Rightarrow T_{DE} = \frac{21}{2} \times \frac{T_{BC}}{|\overrightarrow{r_{BC}}|} = 21/2 \times 14\sqrt{3}/49$ KN
$\Rightarrow T_{DE} = 3\sqrt{3}$KN $= 5.196$ KN

Therefore the cable tensions are : $T_{BC} = 2.222$KN
$T_{DE} = 5.196$KN

(a). FBD of the beam AB:



3. The massless cantilever beam AB shown in Figure 3 is subject to a couple M at the midspan C and a force P at the free end B.

   (a) Draw a neat free body diagram of the beam AB. **(4)**

   (b) Calculate the reaction components at A. **(6)**

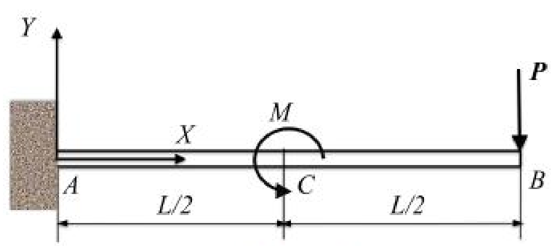
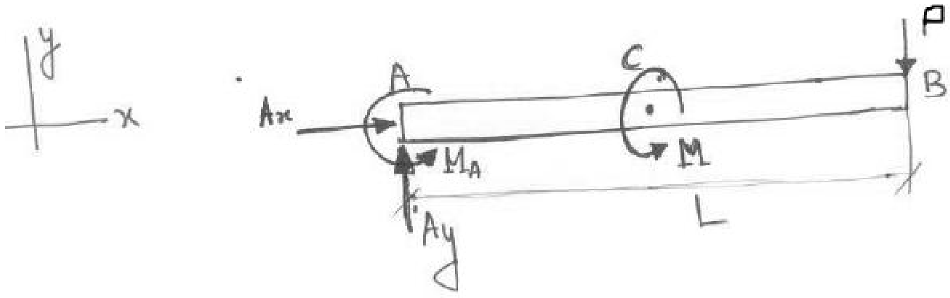


Figure 3

ANS: (a) FBD of the beam AB:



(b) Reaction Components at A: $\sum F_x = 0 : A_x = 0$ $\qquad \sum F_y = 0 : A_y - P = 0 \Rightarrow A_y = P$
$\sum M_{atA} = 0 : M_A + M - P.L = 0 \Rightarrow M_A = PL - M$

4. For the truss shown in Figure 4

(a) Identify the zero force members. **(6*)**
(* *Wrong identification carries penalty.*)

(b) Compute the forces in members CF and BC and state whether they are in tension or compression. **(6)**

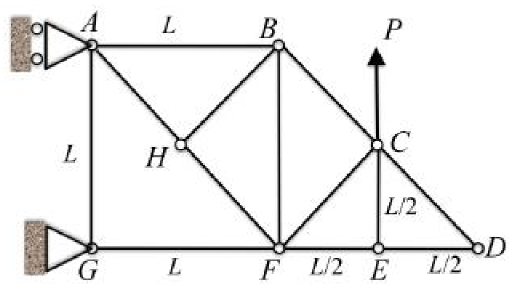
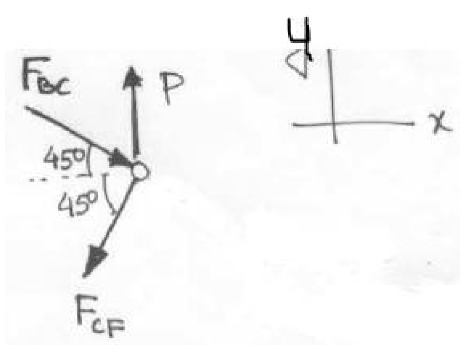


Figure 4

ANS:
(a) Zero force members:
1. CD and ED(from joint D)
2. EF and CE (from joint E)
3. BH (from joint H)
(b) Forces in members CF and BC:
FBD of joint C:



$\sum F_x = 0 : F_{BC} cos45 - F_{CF} cos45 = 0 \Rightarrow F_{BC} = F_{CF}$
$\sum F_y = 0 : P - F_{BC} sin45 - F_{CF} sin45 = 0$
$P - F_{BC} \times \frac{1}{\sqrt{2}} = 0$ (using $F_{CF} = F_{BC}, sin45 = \frac{1}{\sqrt{2}}$)
$\Rightarrow F_{BC} = P/\sqrt{2}$

$\Rightarrow F_{CF} = P/\sqrt{2}$
Therefore : $F_{BC} = P/\sqrt{2}$(compression)
$F_{CF} = P/\sqrt{2}$(tension)

5. The massless frame shown in Figure 5 is subject to a 6 kN load at end E. The pin at C is rigidly attached to member ABC and is supported by the frictionless slot in member DE.

(a) Draw free body diagrams of all the members. **(6)**

(b) Compute the components of forces at the pins A, B and C. **(8)**

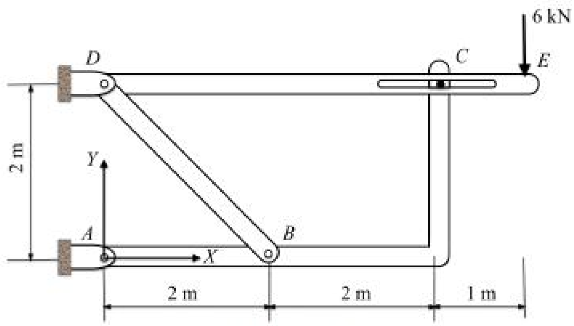
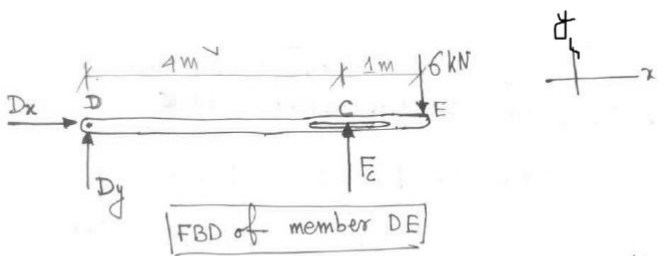
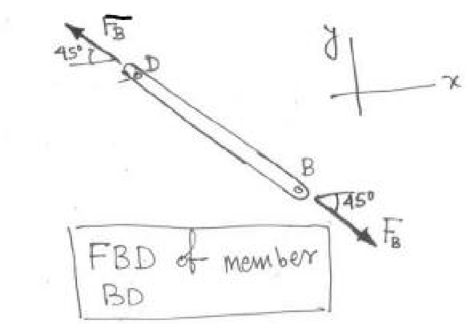


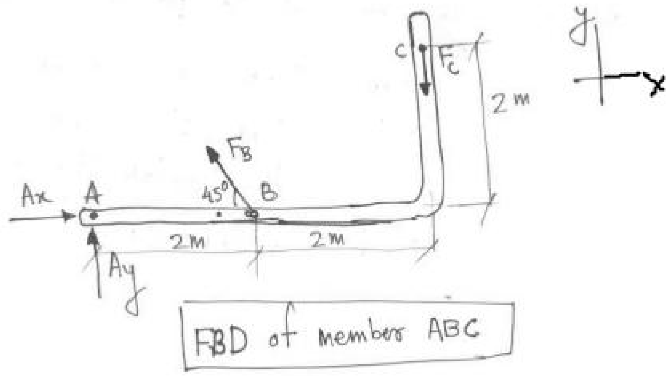
Figure 5

ANS:
(a) Free body diagrams:



Member BD has forces acting along at its ends through pin joints. Therefore BD is a two force member.





(b) Forces at pins B and C: From the fbd of member DE: $\sum M_D = 0 : F_C \times 4 - 6KN \times 5 = 0$
$\Rightarrow F_C = \frac{30}{4} = 7.5KN$ (along y direction)
From the fbd of member ABC
$\sum M_A = 0 : F_B sin45 \times 2 - F_C \times 4 = 0 \Rightarrow F_B = 2\sqrt{2}F_C = 15\sqrt{2}KN$ Component of F_B along x: $F_{Bx} = F_B cos45 = 15KN$
along y :$F_{By} = F_B sin45 = 15KN$
$\sum F_x = 0 : A_x - F_B cos45 = 0$
$\Rightarrow A_x = F_B cos45 = 15KN$

$\sum F_y = 0 : A_y + F_B sin45 - F_c = 0$
$\Rightarrow A_y = F_c - F_B/\sqrt{2} = -7.5KN$
Therefore the forces at the pins A, B, C are
$A_x = 15KN$ $A_y = -7.5KN$
$F_{Bx} = 15KN$ $F_{By} = 15KN$ or $F_B = 15\sqrt{2}KN$ (tension)
$F_c = 7.5KN$ (downward and along y axis)
ANS:

6. The circular cylinder A rests on two half-cylinders B and C as shown in Figure 6. All cylinders are homogeneous and have same radius r. The coefficient of friction between the half-cylinders and the horizontal surface is $\mu = 0.5$. The contact between the cylinders is frictionless. Determine the maximum distance d, between the half-cylinders, to maintain the arrangement in equilibrium. (14)

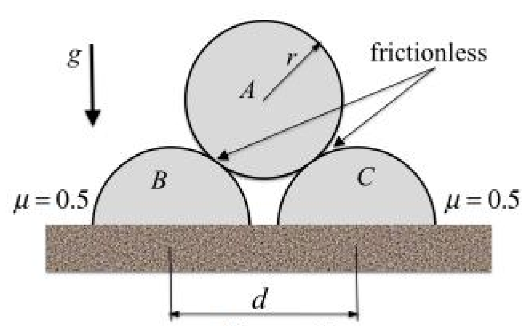
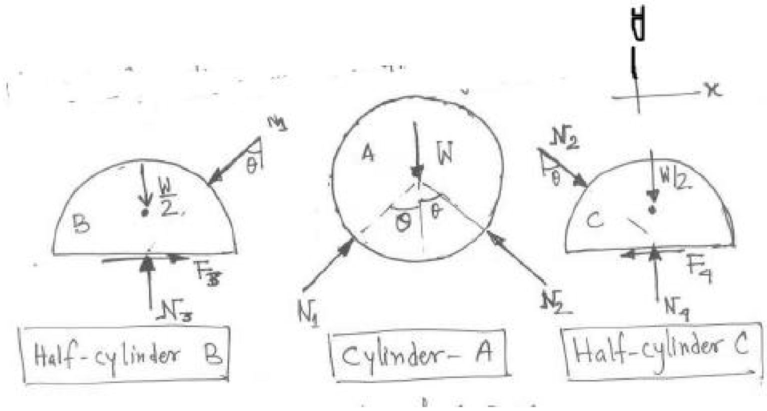
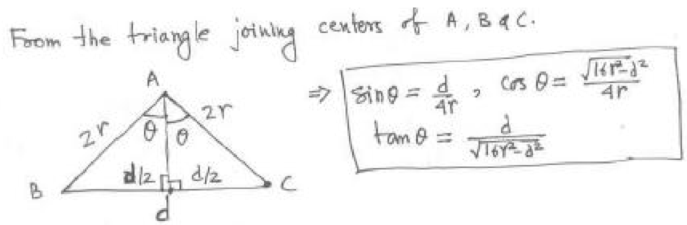


Figure 6

Free body diagrams of the cylinders:



From the triangle joining centers of A, B and C:



From the FBD of cylinder A: $\sum F_x = 0 : N_1 = N_2$
$\sum F_y = 0 : (N_1 + N_2)cos\theta = W$ or $2N_1 cos\theta = W - - - - - - -(1)$ Note : since $N_1 = N_2$ from fbd of half cylinders B and C , we get $F_3 = F_4$ and $N_3 = N_4$
From the FBD of cylinder-B:
$\sum F_y = 0 : N_3 - \frac{W}{2} - N_1 cos\theta = 0$
$N_3 = W$ (after using equation (1))
$\sum F_x = 0 : F_3 - N_1 sin\theta = 0$
or $F_3 = N_1 sin\theta = \frac{W}{2} tan\theta$ (using eq. (1))
To maintain equilibrium - need to avoid slip of half cylinders B and C.
$\Rightarrow F_3 \leq \mu N_3 = \mu W$
$\Rightarrow \frac{W}{2} tan\theta \leq \mu W \Rightarrow tan\theta \leq 2\mu$
$\Rightarrow \frac{d}{\sqrt{(16r^2-d^2)}} \leq 2\mu = 1$
$d \leq 2\sqrt{2}r$
Since , $F_3 = F_4$ and $N_3 = N_4$, we obtain the relation considering the fbd of cylinder C.
For equilibrium , minimum distance d is $d_{min} = 2\sqrt{2}r$

7. A massless belt-idler, shown in Figure 7, comprises of a wooden cylinder fixed rigidly to an arm which is hinged at O. An inextensible light belt passes over the cylinder and moves at a steady speed from right to left as shown. The coefficient of kinetic friction between the belt and the cylinder is $\mu_k = 0.5$ and $T_1 = 10$ N. If both T_1 and T_2 always remain vertical while moving, determine

(a) the steady angle θ that the arm makes with the vertical, and (12)

(b) the net force magnitude on the hinge at O. (2)

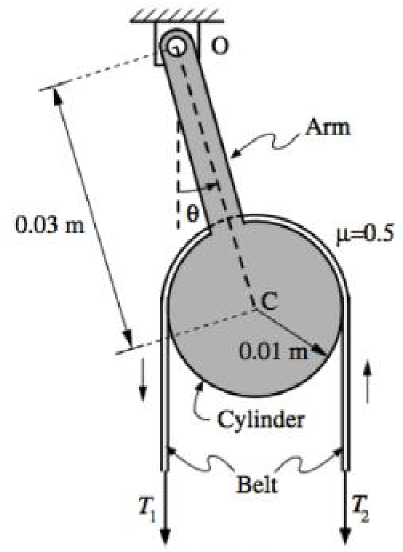
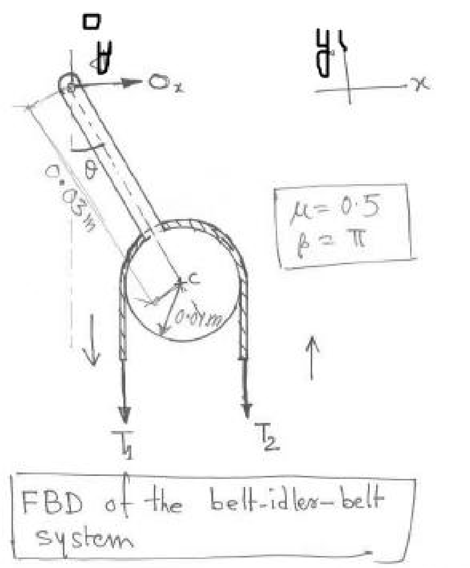


Figure 7

ANS:



$\sum F_x = 0 : O_x = 0 \qquad F_y = 0 \qquad O_y = T_1 + T_2 - - - - - - - - - - - (2)$
$\sum M_c = 0 : (T_1 - T_2) \times 0.01 - O_y \times 0.03 sin\theta - O_x \times 0.03 cos\theta = 0$
$\Rightarrow (T_1 - T_2) - (T_1 + T_2) \times 3 sin\theta = 0$
$\Rightarrow sin\theta = \frac{T_1 - T_2}{3(T_1 + T_2)} - - - - - - - - - - - - - (3)$
(a) Since belt moves from right to left : $T_1 > T_2$
$\Rightarrow T_1/T_2 = e^{\mu\beta} \Rightarrow T_2 = T_1 e^{-0.5\pi}$
$sin\theta = 0.2186 \Rightarrow \theta = 12.63\,\text{deg}$
(b)$O_y = T_1 + T_2 = 10(1 + e^{-0.5\pi})N = 12.08N, O_x = 0N$
$\Rightarrow$ Force magnitude at hinge O is 12.08N

MECHANICS

MID - AUTUMN SEMESTER EXAMINATION **2016-2017**

1. The vertical force F is decomposed into F_1 and F_2 as shown in Fig.1. Determine the magnitude of F_1 and the angle β. *(10)*

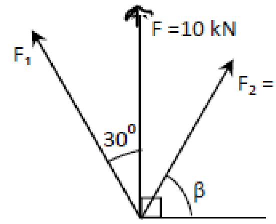
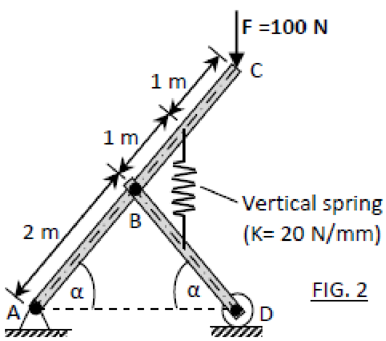


FIG. 1

ANSWER:
$F_1 + F_2 = F$
$\Rightarrow F_2 cos\beta - F_1 sin30 = 0$
and$F_2 sin\beta + F_1 cos30 = 10$
$\Rightarrow 5cos\beta = F_1 sin30$ and $5sin\beta = 10 - F_1 cos30$

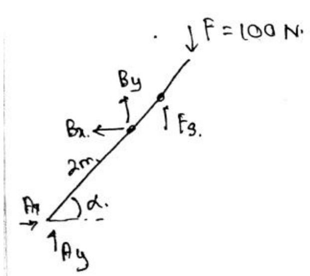
Square and add to get β and then find F_1
Therefore solving this we get $\beta = 30\,\text{deg}$ and $F_1 = 8.67KN$



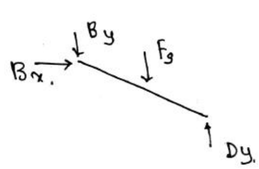
2. The massless structure shown in Fig.2 is in equilibrium, when the vertical force F=100 N is applied at point C. Determine the value of α, if the un-stretched length of the vertical spring was 1.0 m. *(12)*

ANSWER:

For rod ABC

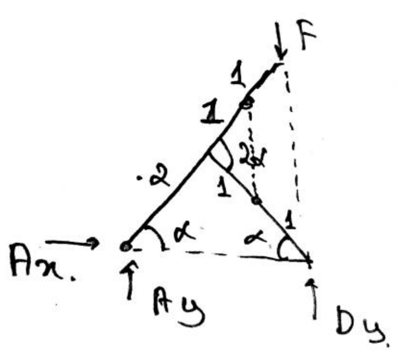


For rod BC



At point D there is a roller, so horizontal force at point D $\Rightarrow D_x = 0$

Looking at the FBD of the whole body.



$\Rightarrow \sum M_A = 0$ or $F = D_y = 100 \Rightarrow A_y = 0[\sum F_y = 0]$ also $A_x = 0(\sum A_x = 0)$
Unstretched length of spring = 1m
Compressed length $= 1 - 2 \times 1 sin\alpha$
$A_x = A_y = 0$[proved]
$\Rightarrow \sum M_B = 0$ or $F_3 \times 1 cos\alpha = F \times 2cos\alpha$
or $F_3 = 2F$ or $kx = 2F$ or $x = \frac{2 \times 100}{20} = 10mm$
$\Rightarrow x = 1 - 2sin\alpha$
or 0.001 $=1 - 2sin\alpha$ or $2sin\alpha = 0.99$ or $\alpha = 29.67\,\text{deg}$

3. For the truss shown in Fig.3, **(a)** determine the support reactions, **(b)** identify the zero force members, and **(c)** determine the forces in the members CD, GF and BH. State if they are in tension or in compression. *(15)*

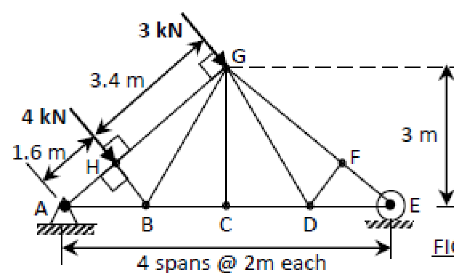
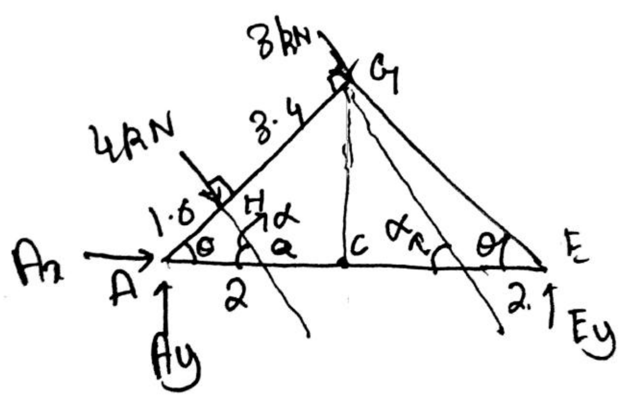


FIG. 3

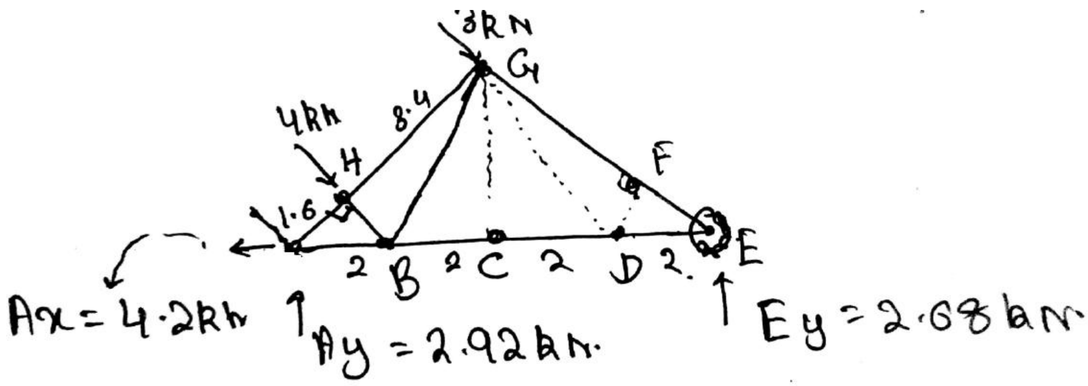
ANSWER: At point E ,$E_x = 0$
$\rightarrow E_y(8) = (4KN)(1.6) + (3KN)(3.4)$ [Taking FBD of the whole body] $\sum M_A = 0$
or $E_y = 2.68KN$



$\Rightarrow cos\theta = AC/AG = 0.8 \rightarrow sin\theta = 0.6$ From $\Delta AHO, \alpha = 90 - \theta$
$\sum F_x = 0:$

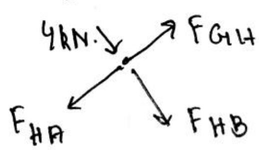
or $A_x + 4sin\theta + 3sin\theta = 0$
or $A_x = -4.2KN$
$\sum F_y = 0 : A_y + F_y - 4cos\theta - 3cos\theta = 0$ or $A_y = 2.92KN$

(b) At point F , there is no truss to balance forces $\perp$ to EG.
$\rightarrow$ DF is zero member truss , if $F_{DF} = 0$
$\rightarrow$ At point D , there is no member to balance sorces$\perp$ to CE.
$\rightarrow$ DG is zero member truss
Similar logic for GC $\rightarrow$ GC,GD,FD.
[3KN and 4KN are both parallel]



(c) $\Rightarrow F_{ED} + F_{EF}cos\theta = 0$

At joint H



and $E_y + F_{EF}sin\theta = 0$
solving $F_{EF} = -4.47KN = F_{GF}$ and $F_{ED} = 3.57KN = F_{CD}; F_{HB} = -4KN$
$\rightarrow F_{HB} = 4KN(C)$
$F_{GF} = 4.47KN(C)$
$F_{CD} = 3.57KN(T)$

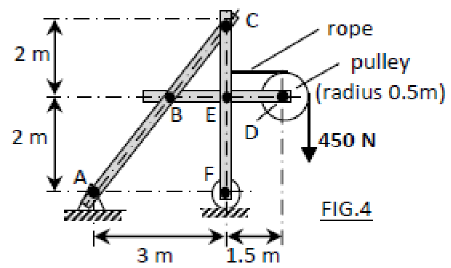


FIG.4

4. The members of the frame shown in Fig.4 are assembled with frictionless pin joints. Neglecting weight of the members, determine the horizontal and vertical components of the force at C exerted on the member CEF. *(16)*

ANSWER:

$tan\alpha = 4/3$

$\Rightarrow sin\alpha = 4/5 = 0.8$

$cos\alpha = 3/5 = 0.6$

$\Rightarrow \sum M_A = 0$ [For whole body]

$T(0.5 + 1.5 + 3) = F_y(3)$

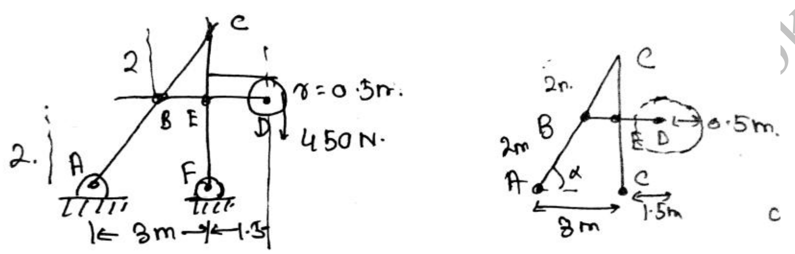
or $F_y = 450 \times (5/3) = 750N$

Also $\sum F_x = 0$ [for whole body]

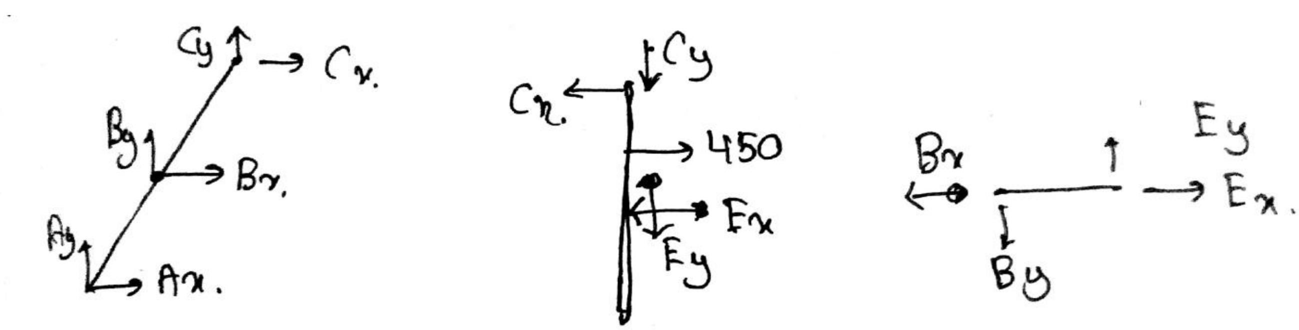
$\Rightarrow A_x = 0$

$\sum F_y = 0$ [for whole body]

$\Rightarrow A_y + F_y = 430$ or $A_y = -300N$



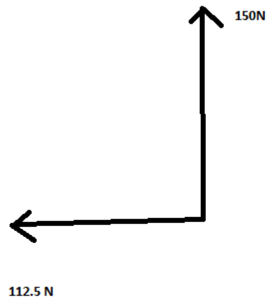
[diagram in reference to the above answer]



$\sum M_B = 0, \rightarrow A_y(1.5) + C_x(2) = C_y(1.3) \rightarrow C_y = -150N($ on member AC)

$\sum M_E = 0 \rightarrow 450(0.5) = C_x \times 2 \rightarrow C_x = 112.5$

ThereforeC_y on member CF $= 150$ N and $C_x = 112.5N$



5. In Fig. 5, a right angle bent rigid bar, hinged at C, is pressed against a rigid wedge at A by a 20 kN force at B. Neglecting weight of the members, determine the horizontal force P required to initiate motion of the wedge to the left. The coefficient of friction between the wedge and the bar is 0.25 and the coefficient of friction between the wedge and the ground is 0.15. *(16)*

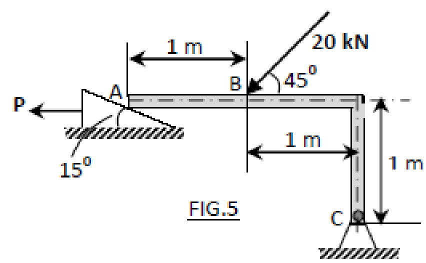
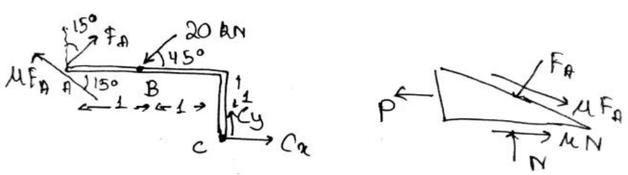
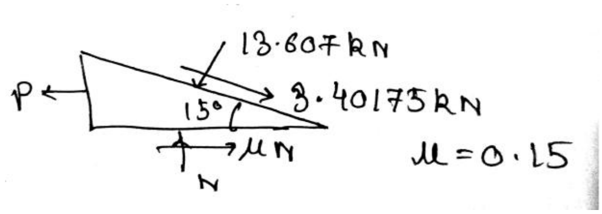


FIG.5

ANSWER:



$\mu_A = 0.25$
$\sum M_C = 0$
$F_A cos15 \times$
$2 + F sin15 \times$
$1 - 20cos45 \times$
$1 - 20sin45 \times 1 - \mu F_A cos15 \times 1 + \mu_A F_A sin15 \times 2 = 0$
Solving we get $F_A = 13.607KN$



on wedge : $\sum F_x = 0, P - \mu_A F_A cos15 - F_A sin15 - 0.15N = 0$
or $P = 0.15N - 0.236$
$\sum F_y = 0.N - F_A cos15 - 0.25 F_A sin15 = 0$
or $N = 14.024KN \Rightarrow P = 1.867KN$

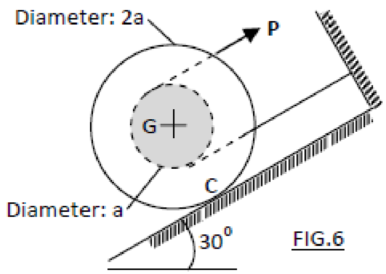
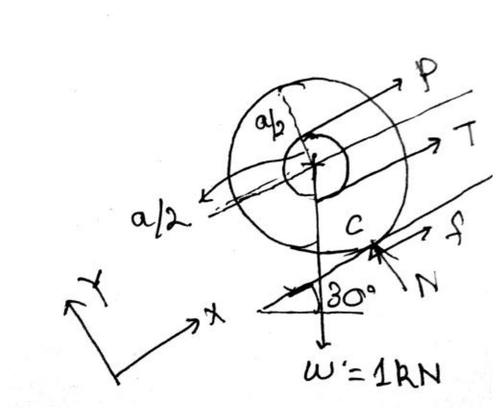


FIG.6

6. A spool of weight W = 1000 N, outer diameter 2a and inner diameter a, is rolled up an inclined plane of 30^0 using an inextensible belt, as shown in Fig. 6. Assuming that the spool has pure rolling at C and sliding between the belt and the inner diameter of the spool, determine **(a)** the minimum force P required for rolling up the spool, and **(b)** the minimum coefficient of friction required at C for rolling up without slipping at C. The coefficient of friction between the belt and the spool is 0.6. *(16)*

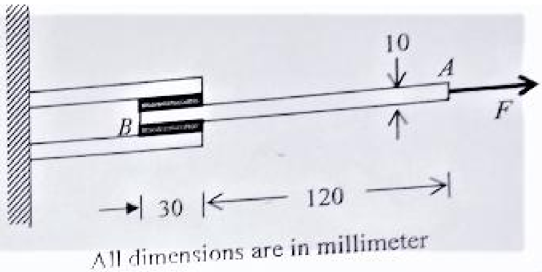


N = Wcos30 [$\sum F_y = 0$] − − − − − − −(1)
P + T + f = Wsin30 [$\sum F_x = 0$]———(2)
$\sum M_C = 0$
or P(3a/2) + T(a/2) = W sin$\theta(a)$———-(3)
Also by law of friction
$\frac{P}{T} = e^{0.6\pi} \mu = 0.6$(At the spool)
or T = 0.152P ——(4)
Solving we get P = 317.3 N
From (2) f = 134.5N ⇒ Minimum coefficient of friction = f/(Wcos30) = 0.155

MECHANICS

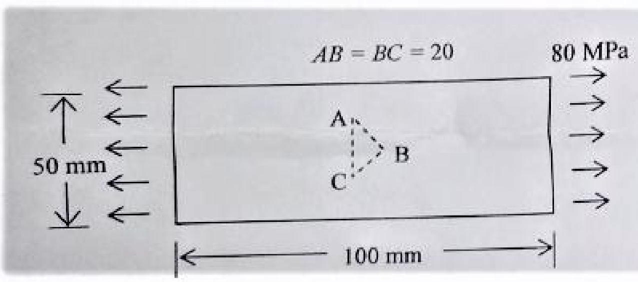
END - SPRING SEMESTER EXAMINATION **2017-2018**

Question 1: A rigid 150 mm × 25 mm × 10 mm-bar, which is connected to two shorter rigid bars, is pulled by an axial force, F, as shown in the figure. A 5 mm-thick rubber block of width 25 mm and length 30 mm is glued to the long bar and each of the shorter bars. If the joint fails when the shear stress in the glue exceeds a limit of 800 kPa, then determine (a) the maximum value of F (b) displacement of the end point, A, of the bar from its initial unloaded position. Given data: G_{rubber} = 20 MPa. [4+6 = **10 Marks**]



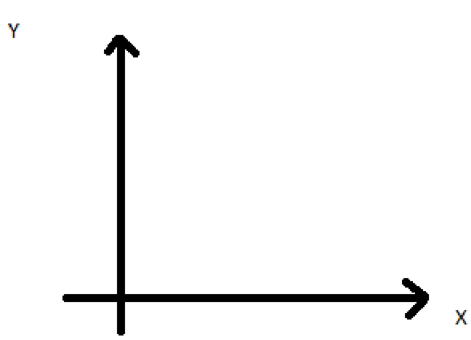
1.a) Shear stress $= \frac{F/2}{A}$
$= \frac{F}{2\times(25\times30)\times10^{-6}}$
$\therefore \frac{F}{2\times(25\times30)\times10^{-6}} = 800\times10^3$
or F =1.2kN

b) $\frac{\delta}{l} = \frac{shearstress}{G}$
or $\delta = 0.2mm$



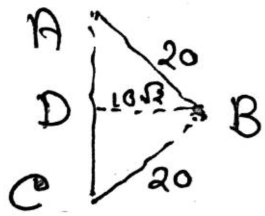
Question 2: A right-angled triangle ABC with lengths $AB = BC$ = 20 mm is inscribed on a 100 mm × 50 mm × 10 mm aluminium plate (Young's modulus E = 70 GPa and Poisson's ratio ν = 0.33) with the side AC vertical to the length axis. The plate is subjected to plane stress as shown in the figure. Find (a) the change in the angle ABC, (b) the change in plate thickness (c) change in plate volume after deformation.
[8+3+4=**15 Marks**]

2)

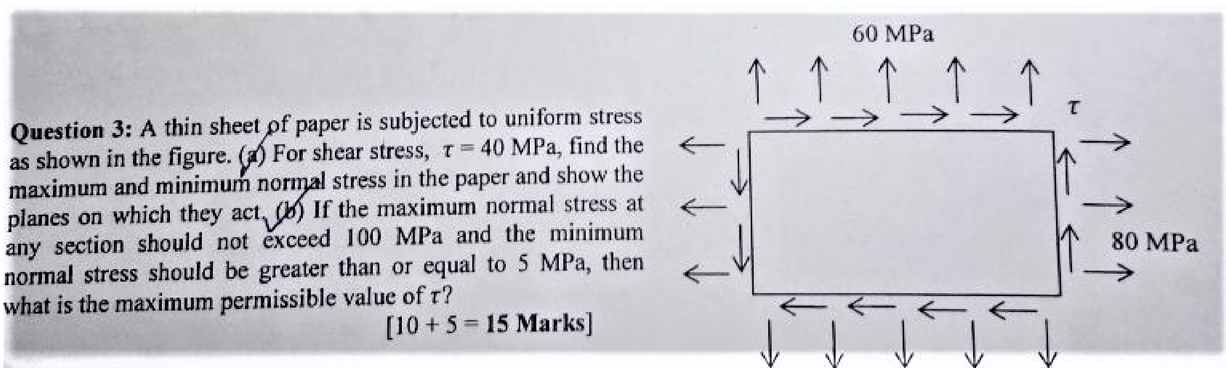


$r_x = \frac{1}{E}(\sigma_x - \gamma\sigma_y) = \frac{\sigma_x}{E}$

$DB^{'} = (1 + r_x)DB$



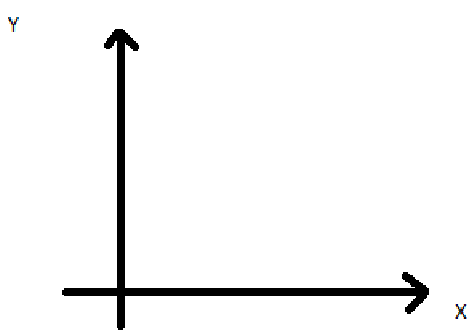
$= (1 + \frac{80 \times 10^6}{70 \times 10^9})10\sqrt{2}$
$= 14.15829806mm$
$r_y = \frac{-\gamma \sigma_x}{E}$
$\therefore \phi_{new} = 2tan^{-1}(\frac{AD^{'}}{DB^{'}} = 2tan^{-1}(14.1368202/14.15829806) \approx 89.913°$



Question 3: A thin sheet of paper is subjected to uniform stress as shown in the figure. (a) For shear stress, $\tau = 40$ MPa, find the maximum and minimum normal stress in the paper and show the planes on which they act. (b) If the maximum normal stress at any section should not exceed 100 MPa and the minimum normal stress should be greater than or equal to 5 MPa, then what is the maximum permissible value of τ?

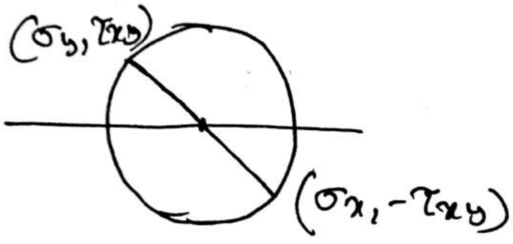
[10 + 5 = **15 Marks**]

3)



Using Mohr's circle
$\sigma_x = 80MPa$
$\sigma_y = 60MPa$
$\tau_{xy} = 40MPa$

$\therefore Radis = \sqrt{(10)^2 + (40)^2} = 41.23MPa$
$Center = (70MPa, 0)$
$\therefore (\sigma_N)_{max} = (70 + 41.23)MPa = 111.23MPa$
$(\sigma_N)_{min} = (70 - 41.23)MPa = 28.77MPa$
$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}cos2\theta + \tau_{xy}sin2\theta$
$\tau_{x'y'} = -(\frac{\sigma_x - \sigma_y}{2})sin2\theta + \tau_{xy}cos2\theta$

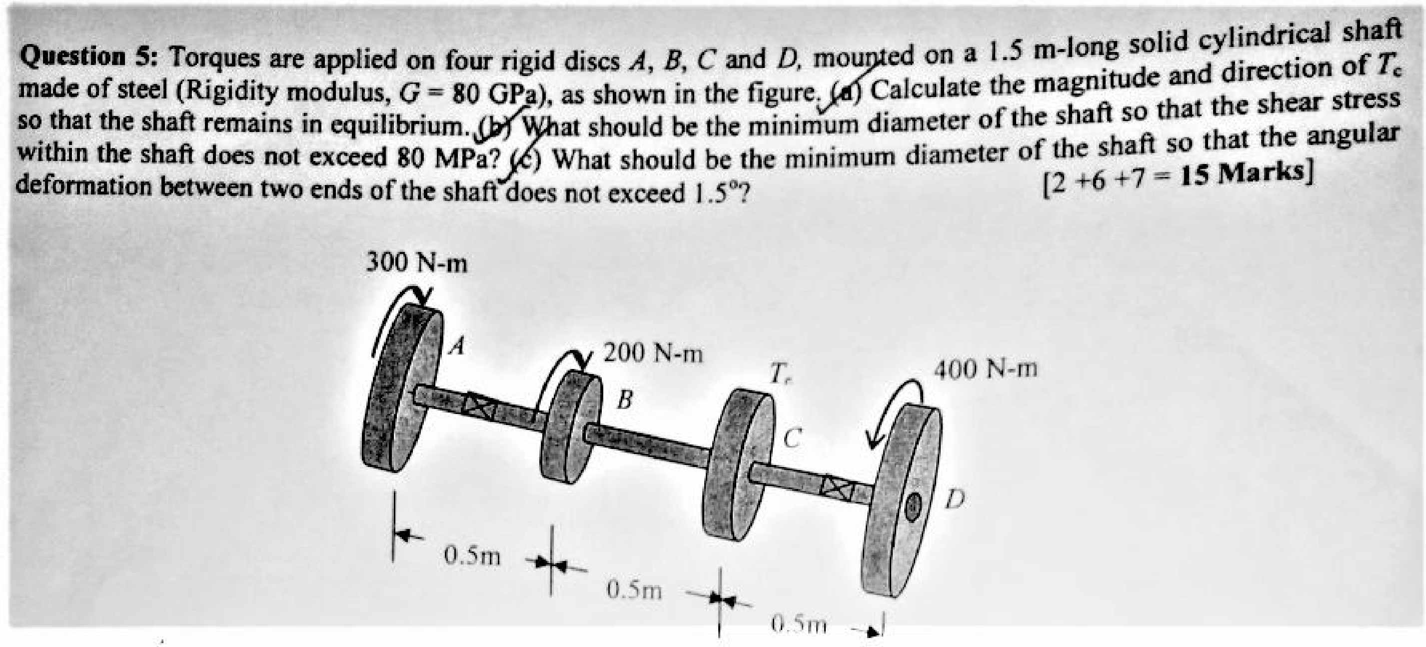


$(\sigma_{x'})_{max} = \frac{\sigma_x+\sigma_y}{2} + \sqrt{(\frac{\sigma_x-\sigma_y}{2})^2 + \tau_{xy}^2}$
$(\sigma_{x'})_{min} = \frac{\sigma_x+\sigma_y}{2} - \sqrt{(\frac{\sigma_x-\sigma_y}{2})^2 + \tau_{xy}^2}$
$\therefore \sigma_x = 80MPa, \sigma_\sigma y = 60MPa$
$\therefore 70 + \sqrt{100+\tau_{xy}^2} \leq 100$ and $70 - \sqrt{100+\tau_{xy}^2} \geq S$
or $\sqrt{100+\tau_{xy}^2} \leq 30$ or $65 \geq \sqrt{100+\tau_{xy}^2}$
or $100 + \tau_{xy}^2 \leq 900$ or $4225 \geq 100 + \tau_{xy}^2$
or $\tau_{xy} \leq 28.3MPa$ or $\tau_{xy} \leq 64.23MPa$
$\therefore \tau_{xy}$ maximum is 28.3MPa

Question 4: The maximum normal stress within the cylindrical portion of a thin-walled cylindrical pressure vessel, made of 12.5 mm thick plate (E = 200 GPa and ν = 0.25), reached its allowable limit 200 MPa when the internal pressure was 1.5 MPa (gauge). (a) By how much should the plate thickness be increased if the vessel is to be designed for an internal pressure of 2 MPa (gauge)? (b) Calculate the diameter of the vessel (c) Calculate the amount by which the diameter of the cylinder increases with respect to unpressurized state when 2 MPa (gauge) pressure is applied.
[2+2+6 = 10 Marks]

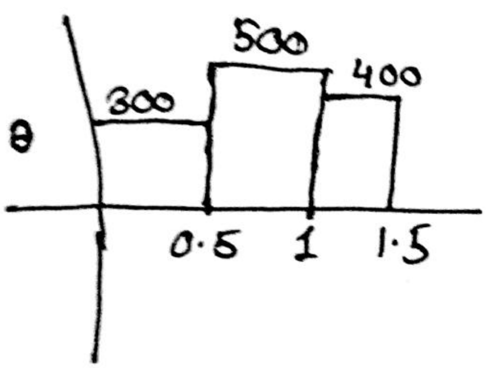
4) $Z = 12.5mm$
$E = 200GPa$
$\gamma = 0.25 \sigma_{\theta\theta} = 2\sigma_{xx} = \frac{Pr}{Z}$
$\therefore \sigma_{\theta\theta} = \frac{Pr}{t}$ [normal stress is maximum for $\sigma_{\theta\theta}$ among $\sigma_{\theta\theta}$ and σ_{xx}
a) To keep $\sigma_{\theta\theta}$ constant and r constant
P/t = constant
or $t = \frac{2.5\times2}{1.5}mm = 16.67mm$
$\therefore$ Increase in thickness = 4.17mm
b) $\sigma_{\theta\theta} = \frac{Pr}{t}$
or $200 \times 10^6 = \frac{1.5\times10^6\times r}{12.5}$
or d =2r =3.33m

c) $\frac{Dd}{d} = \frac{Dr}{r} = \frac{1}{E}(\sigma_{\theta\theta} - \gamma\ \sigma_{xx}) = (1-(\gamma/2)\frac{Pr}{Et}$
or $\frac{Dd}{d} = \frac{2\times10^6\times(5/3)}{200\times10^9\times12.5\times10^{-3}}(1-\frac{0.33}{2})$ [Here P = 2MPa]
or Dd = 3.71mm



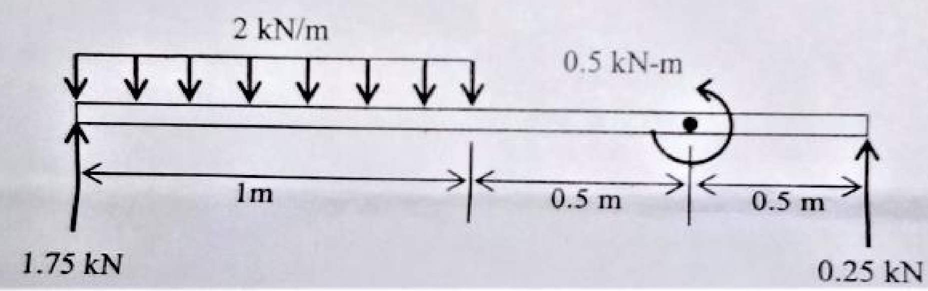
Question 5: Torques are applied on four rigid discs A, B, C and D, mounted on a 1.5 m-long solid cylindrical shaft made of steel (Rigidity modulus, G = 80 GPa), as shown in the figure. (a) Calculate the magnitude and direction of T_c so that the shaft remains in equilibrium. (b) What should be the minimum diameter of the shaft so that the shear stress within the shaft does not exceed 80 MPa? (c) What should be the minimum diameter of the shaft so that the angular deformation between two ends of the shaft does not exceed 1.5°? [2 +6 +7 = **15 Marks]**

5.a) total sum must be zero
$\therefore 300 + 200 + T_c - 400 = 0$
or $T_c = 100Nm$

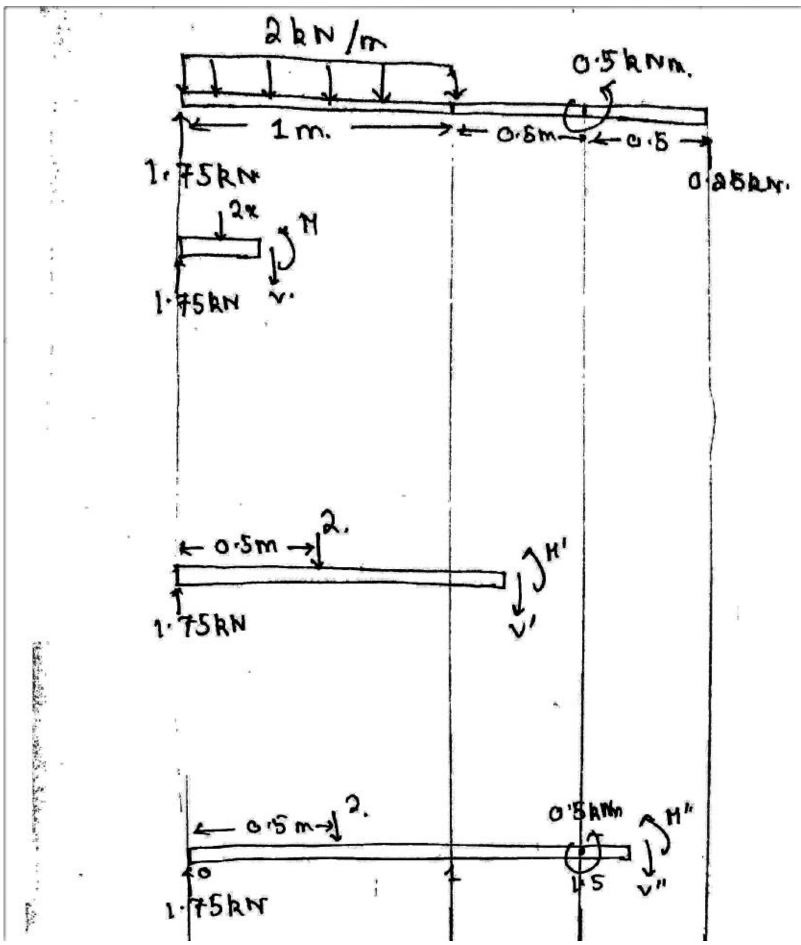


$\frac{T}{J} = \frac{G\phi}{L} = \frac{\tau}{P}$
$\tau = \frac{TP}{J}$
$\therefore \tau$ is maximum at the position BC
$T_{max} = 500Nm$
$\therefore \tau = \frac{(500)\times r}{(1/2)\pi r^4}$
or $r = 15.8mm$
$\therefore d = 31.69mm$
$\phi = \frac{TL}{GJ}$
$\therefore \phi_{max}$ is at BC $T_{max} = 500Nm$
$\therefore \phi_{max} = \frac{500\times 0.5}{80\times 10^9 \times \frac{\pi}{2} \times r^4} = \frac{1.5}{180} \times \pi$
or $\frac{10\times 0.5\times 180}{80\times 10^9\times \pi\times 1.5\times \pi} = r^4 \Rightarrow r \approx 16.6mm$ and $d = 33.21mm$ $\sigma_{\theta\theta}$

Question 6: Draw the (a) Shear Force Diagram (SFD) and (b) Bending Moment Diagram (BMD) of the beam sho
in the figure. Clearly indicate the sign convention that you adopt while drawing the diagrams. [10 + 10 = **20 Marks**

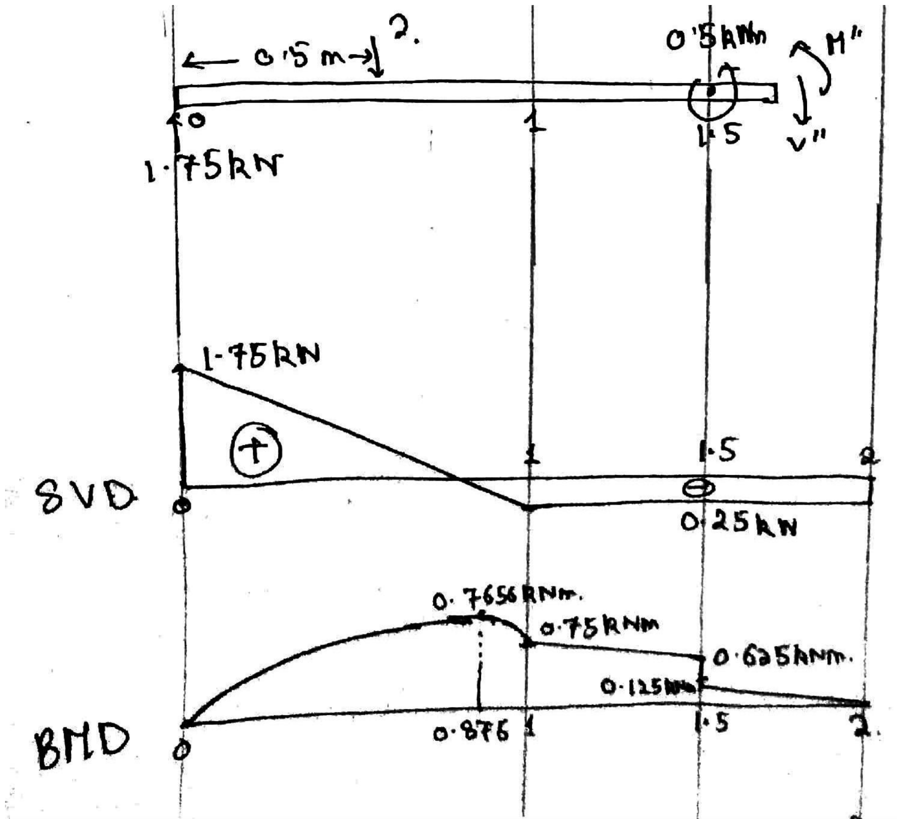


6)



[Be careful about the curavatre of the quadratic]
$V + 2x = 1.75kN$ or $V = (1.75 - 2x)$
$M + 2x(\frac{x}{2}) = 1.75x$
or $M = 1.75x - x^2$

$V' + 2 = 1.75$
or $V' = -0.25kN$

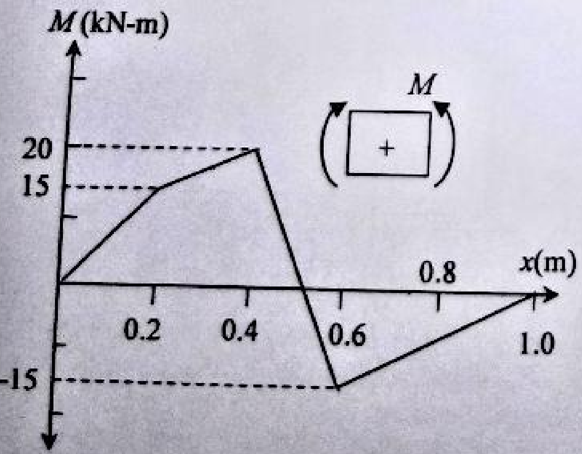


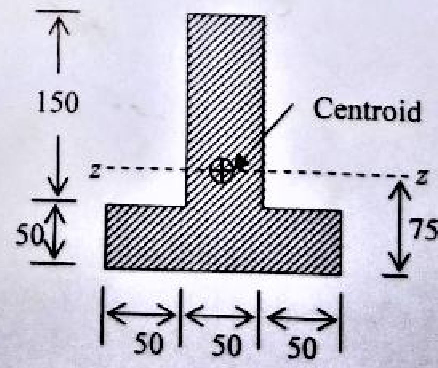
$M' + 2(x - (1/2)) = 1.75x$
$\Rightarrow M' = 1.75x - 2x + 1$
$= 1 - 0.25x$

$V'' + 2 = 1.75$
$\Rightarrow V'' = -0.25kN$
$M'' + 2(x - \frac{1}{2}) + \frac{1}{2} = 1.75x$
$\Rightarrow M'' = 1/2 - 0.25x$

Question 7: The bending moment diagram of a slender inverted T-beam is shown in the left figure. The cross section is shown in the right. The bending moment that causes compression in the top fibre of the beam is taken to be positive. (i) Determine, with justification, the section(s), x, where the beam experiences the maximum tensile stress due to bending, (ii) maximum compressive stress due to bending, (iii) Calculate the maximum tensile and compressive bending stress within the beam in terms of the second moment of area I_{zz}. (iv) Calculate I_{zz} about the centroidal axis Z-Z.

[3 + 3 + 8 + 6 = **20 Marks**]





All dimensions are in mm

7. $\sigma = \frac{My}{I}$

At the two maxima , x =0.4 , x=0.6

At x = 0.4

$\sigma_c = \frac{20\times10^3\times0.125}{I_{zz}} = \frac{2.5\times10^3}{I_{zz}}$

$\sigma_T = \frac{20\times10^3\times0.075}{I_{zz}} = \frac{1.5\times10^3}{I_{zz}}$

At x = 0.6

$\sigma_T = \frac{15\times10^3\times0.125}{I_{zz}}$

$= \frac{1.875\times10^3}{I_{zz}}$

$\sigma_c = \frac{15\times10^3\times0.075}{I_{zz}}$

$= \frac{1.125\times10^3}{I_{zz}}$

$\therefore$ Compressive stresse maximum at x = 0.4 with $\sigma_c = (2500/I_{zz})P_a$

Tensile stress is maximum at x =0.6

with $\sigma_T = (1875/I_{zz})P_a$

$I_{zz} = \frac{0.15\times(0.05)^2}{12} + (0.05)(0.15)(0.125)^2 + \frac{0.05(0.12)^3}{12} + (0.05)(0.15)(0.125)^2 \quad [i = \frac{bh^3}{12} + A_1d^2]$

$1.75\times10^{-4}m^4$

MECHANICS

END SEMESTER AUTUMN EXAMINATION **2017-18**

1. Two 40 mm wide and 15 mm thick flat plates are loaded in tension. They are joined using two rectangular splice plates of same width and thickness as the plates and two 10 mm diameter rivets as shown in Figure 1. The factor of safety against any of the ultimate load that can be carried is 2.5. The ultimate strength in tension for the plate and splice material is 400 MPa. The ultimate strength in shear of the rivet material is 170 MPa.

   (a) Calculate the tensile stress in the critical areas of the plate in terms of F. **(6)**

   (b) Calculate the shear stress in the critical rivet cross-section in terms of F. **(6)**

   (c) Find the allowable load F_{allow} considering the failure due to tension in plate and shearing of rivet. **(5)**

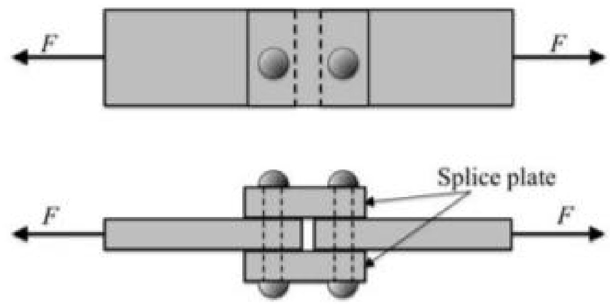
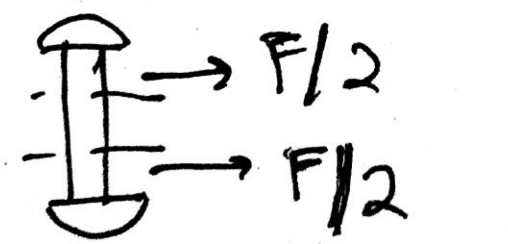


Figure 1

Ans 1a): Factor of safety = 2.5
Critical Area A = (40 - 10)×15 = 450mm^2
Critical stress, $\tau = F/A = \dfrac{F}{450}$ MPa, where F is in Newton.
Ans 1b): Critical shear stress $= \dfrac{F/2}{\pi d^2/4} = \dfrac{2F}{\pi d^2} = \dfrac{F}{50\pi}$ MPa



Ans 1c): Considering fail due to tension
$\therefore \dfrac{F_{max}}{450} = \dfrac{400}{2.5}$
or $F_{max} = 72$ kN
Considering fail due to stress,
$\Rightarrow \dfrac{F_{max}}{500} = \dfrac{170}{2.5}$ or $F_{max} = 10.68$ kN

2. A 3 m long hollow aluminum shaft used in building structures has inner and outer diameters $d_1 = 80$ mm and $d_2 = 100$ mm, respectively. The shear modulus of aluminum is $G = 30$ GPa and the tube is subjected to pure torsion at ends.

(a) Find the angle of twist in degrees when the maximum shear stress in the hollow shaft is 50 MPa. **(6)**

(b) Find the diameter of a solid shaft of same material and length resisting the same torque and has the same maximum shear stress. **(8)**

(c) Calculate that ratio of the weights of the hollow and solid shafts. **(4)**

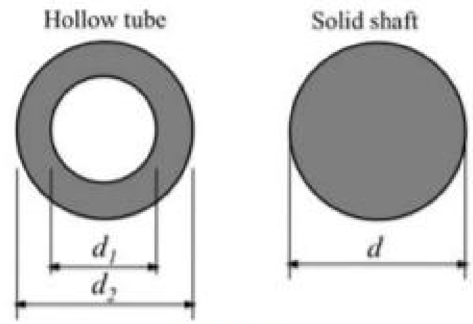


Figure 2

Ans 2a): $\dfrac{T}{J} = \dfrac{G\phi}{L} = \dfrac{\tau}{r}$ G=30 GPa r = 50mm

$\therefore \phi = \dfrac{\tau L}{Gr} = \dfrac{50 \times 10^6 \times 3}{30 \times 10^9 \times 50 \times 10^{-3}} = 0.1\text{rad} = 5.73^o$

Ans 2b): $\tau_{max} = \dfrac{Tr}{J}$ To get τ_{max} and T constant

we got $\dfrac{J}{r}$ =constant $J = \dfrac{1}{2}\pi\left({r_2}^4 - {r_1}^4\right)$

$\therefore \dfrac{1}{2}\pi\dfrac{(0.05^4 - 0.04^4)}{0.05} = \dfrac{1}{2}\pi r^3$

or diameter = 83.9mm

Ans 2c): Ratio of weights = ratio of areas

$\therefore$ ratio $= \dfrac{\pi(d_2^4 - d_1^4)}{\pi d^2/4} = \left(\dfrac{d_2^4 - d_1^4}{d^2}\right) \simeq 0.51$

3. A $L = 2$ m long cantilever beam of square cross section of side b is subject to a distributed load $q = 1$ kN/m and a moment $M_1 = 1$ kN-m at the mid span as shown in Figure 3.

(a) Draw shear force and bending moment diagrams of the beam mentioning the sign convention. **(8)**

(b) Identify and state the location at which the bending moment is maximum in magnitude. **(2)**

(c) Determine the minimum depth b of the beam such that the bending stress in the beam does not exceed 300 MPa. **(6)**

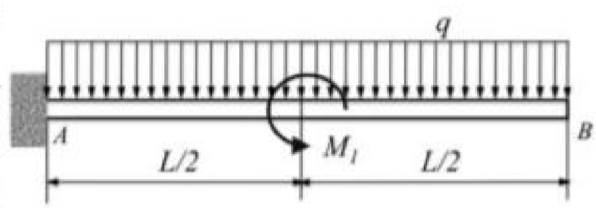
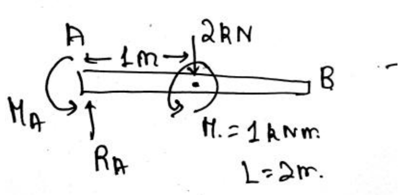
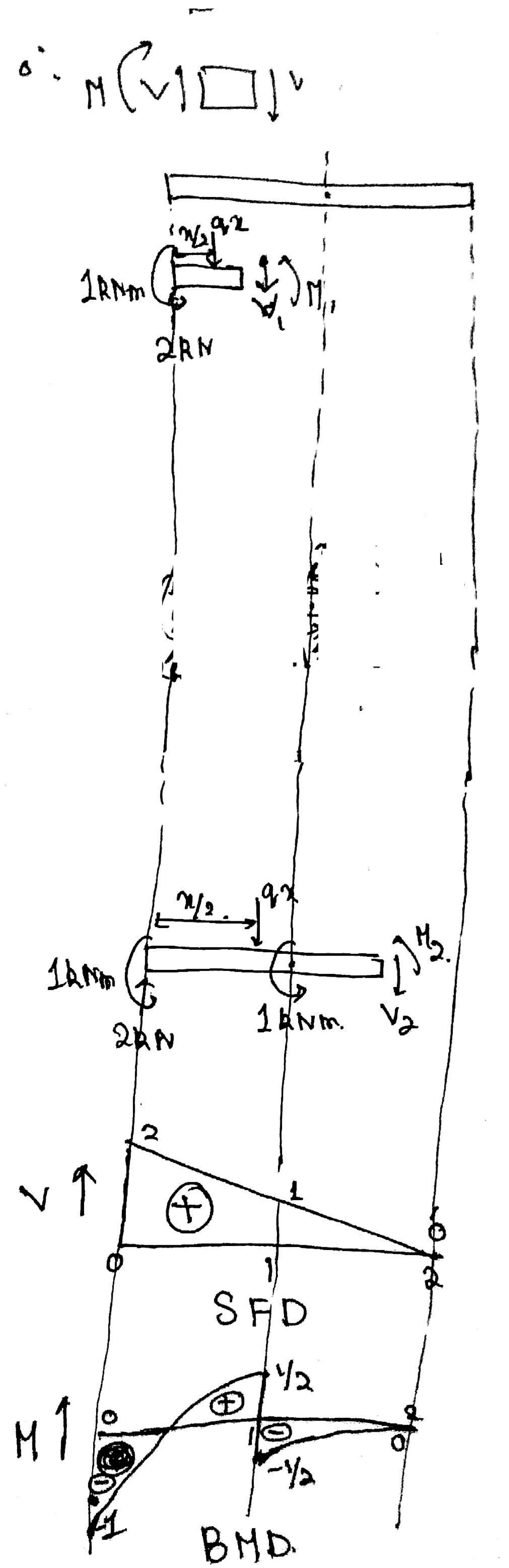


Figure 3

Ans 3): The 'q' force can be replaced by 1 single force of magnitude qL at the center.
q = 1 kN/m



$\therefore \Sigma M_A = 0$: 2×1 - 1 - $M_A = 0$ $M_A = 1$kNm
$\Sigma F = 0$: $R_A = 2$kN



Moments about the open end

$$\therefore M_1 + qx \times (x/2) + 1 = 2x$$

$$\text{or } M_1 = (2x - \frac{x^2}{2} - 1)\text{kNm}$$

$$\text{and } v_1 = \frac{dM_1}{dx} = (2 - x)\text{kN}$$

$$M_2 + 1 + 1 - 2x + x \times (x/2) = 0$$

$$\text{or } M_2 = 2x - 2 - \frac{x^2}{2}$$

$$\therefore v_2 = \frac{dM_2}{dx} = 2 - x$$

$$\therefore M_{max} = 1\text{kN at x=0}$$

4. A wooden sample shown in Figure 4 was subjected to an axial load P. The sample was found to fail (break) at an angle $\theta = 60°$ when the normal stress on the oblique plane at 60° reached 50 MPa. The cross-sectional area (section $a - a$) of the sample was 100 mm^2.

(a) Calculated the load P at failure. **(6)**

(b) Compute the shear stress on the 60° oblique plane at failure . **(4)**

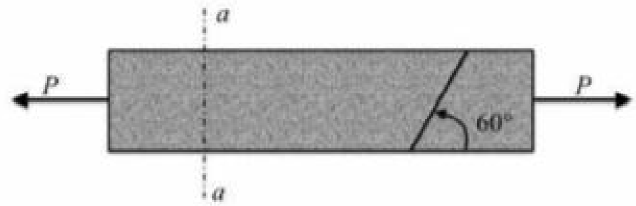
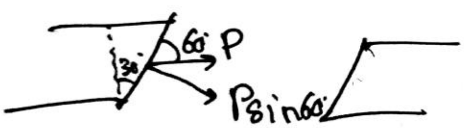


Figure 4

Ans 4a):



Also $A = A_N / \cos 30^o$

$$\therefore \sigma_N = \frac{P \sin 60^o}{A_N \sec 30^o} = \frac{P}{A} \cos^2 30^o$$

$$\therefore 50 \times 10^6 = \frac{3}{4} \times \frac{P}{100}$$

or $P = 6.67$ kN

Ans 4b): $$\tau = \frac{P \cos 60^o}{A_N \sec 30^o} = \frac{P}{2A_N} \sin 60^o$$

$$= \frac{1}{2} \times \left(\frac{50 \times 10^6 \times 4}{3\sqrt{3}} \right) \times \frac{\sqrt{3}}{2}$$

$$= 28.87\text{MPa}$$

5. A rigid block of weight 1000 N was initially resting on a rigid ground. It is being slowly pulled upwards using a 3 m long massless rope of cross sectional area 100 mm^2 as shown in Figure 5. The material of the rope has Young's modulus $E = 1$ GPa. Determine the displacement of the top end of the rope when the mass just looses contact with the ground. **(10)**

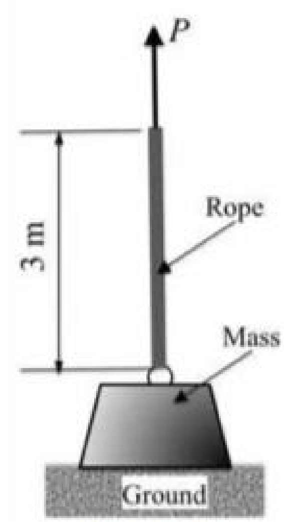


Figure 5

Ans 5): $\dfrac{F}{A} = E \times \dfrac{\triangle l}{l}$
force = weight = 1000 N, E = 1 GPa, A = 100mm^2, l = 3m
or, $\dfrac{1000}{100 \times 10^{-6}} = 10^9 \times \dfrac{\triangle l}{3}$
or, $\triangle l = 30 \times 10^{-3} = 80$mm

Figure 5

6. A strain gauge (strain measuring device) is installed on the surface of an aluminium beverage can along its longitudinal direction as shown in Figure 6. The internal radius to thickness ratio of the can is $r/t = 100$. When the lid of the can is popped open, the strain gauge indicates a change of axial/longitudinal strain by $\epsilon = 150 \times 10^{-6}$.

   Consider the can to be cylindrical and the material has $E = 70$ GPa and $\nu = 0.33$. Neglect the pressure due to weight of the fluid in the can.

   (a) What was the internal pressure in the can before opening? **(8)**

   (b) Determine the in-plane normal and shear stresses (σ_{x1}, σ_{y1} and τ_{x1y1}) on an element rotated from longitudinal direction by $\theta = 30°$ as shown, before the can was opened. **(12)**

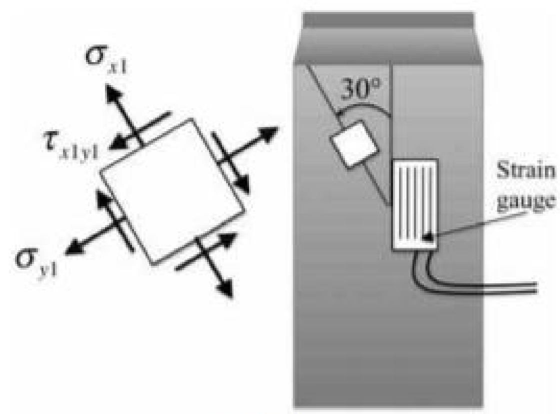
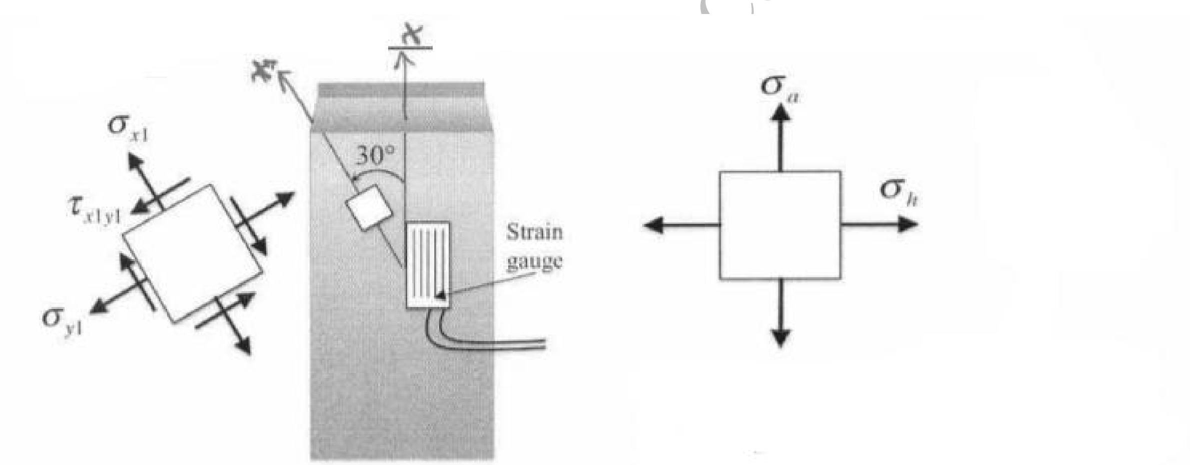


Figure 6

Ans 6a):



Given : $\dfrac{r}{t} = 100$
Hoop stress: $\sigma_n = \dfrac{Pr}{t} = 100$P
Longitudinal Stress: $\sigma_a = \dfrac{Pr}{2t} = 50$P
Assuming state of plane stress:
Axial strain: $\varepsilon_a = \dfrac{\sigma_a}{E} - \dfrac{\nu\sigma_n}{E} = \dfrac{1}{E}(50P - \nu \times 100P)$
$\Rightarrow \varepsilon_a = \dfrac{50P}{E}(1 - 2\nu) = 150 \times 10^{-6}$ (Given)
$\Rightarrow P = \dfrac{150 \times 10^{-6} E}{50(1-2\nu)} = \dfrac{150 \times 10^{-6} \times 70 \times 10^9}{50 \times (1 - 2 \times 0.33)}$ Pa
P = 617.65 $\times 10^3$ Pa
$\therefore$ pressure before can opening is = 617.65 kPa

Ans 6b): $\tau_{xy} = 0 \qquad \theta = 30^o$

$\sigma_x = \sigma_a = 50P = 30.88$ MPa

$\sigma_y = \sigma_n = 100P = 61.77$ MPa

$$\sigma_{x_1} = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$= 46.33 \text{ - } 15.45\cos 60^o = 38.61 \text{ MPa}$$

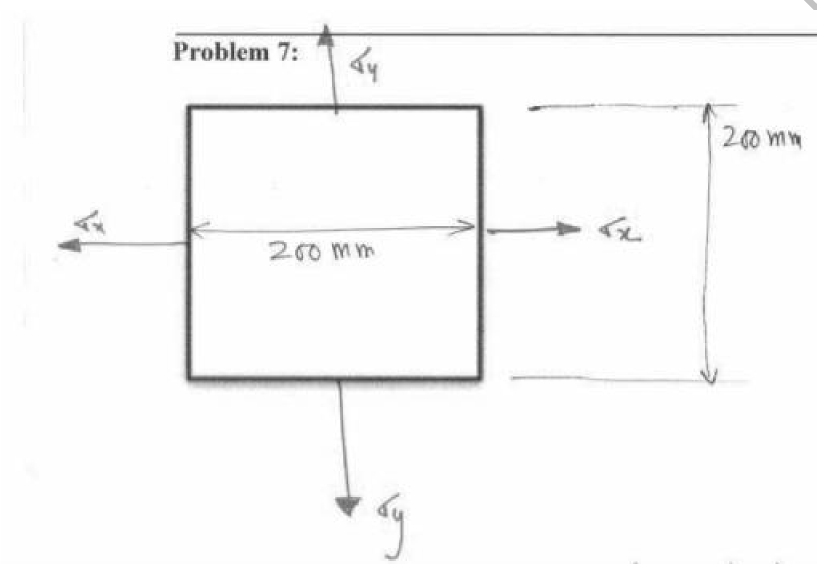
$$\sigma_{y_1} = \left(\frac{\sigma_x + \sigma_y}{2}\right) - \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta - \tau_{xy}\sin 2\theta = 54.04 \text{ MPa}$$

$$\tau_{x_1y_1} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta = 13.37 \text{ MPa}$$

$\sigma_{x_1} = 38.61$MPa , $\sigma_{y_1} = 54.04$MPa , $\tau_{x_1y_1} = 13.37$MPa

7. A 200 mm square plate is subjected to tensile stresses $\sigma_x = 10$ MPa and $\sigma_y = 20$ MPa. Find the percentage change in the volume of the plate if the thickness of the plate is 1 mm. The plate material has $E = 100$ GPa and $\nu = 0.3$. **(14)**

Ans 7):



$E = 100$GPa $= 100{\times}10^3$MPa $\qquad \nu = 0.3$

$\sigma_x = 10$MPa , $\sigma_y = 20$MPa

Strains: $\varepsilon_x = \dfrac{1}{E}(\sigma_x - \nu\sigma_y) = \dfrac{1}{100 \times 10^3}(10 - 0.3 \times 20) = 4 \times 10^{-5}$

$\varepsilon_y = \dfrac{1}{E}(\sigma_y - \nu\sigma_x) = \dfrac{1}{100 \times 10^3}(20 - 0.3 \times 10) = 17 \times 10^{-5}$

$\varepsilon_z = -\dfrac{\nu}{E}(\sigma_x + \sigma_y) = -9 \times 10^{-5}$

Elongation : $\delta_x = \varepsilon_x \times L = 4 \times 10^{-5} \times 200\text{mm} = 8 \times 10^{-3}\text{mm}$

$\delta_y = \varepsilon_y \times L = 17 \times 10^{-5} \times 200\text{mm} = 34 \times 10^{-3}\text{mm}$

$\delta_z = \varepsilon_z \times t = -9 \times 10^{-5} \times 1\text{mm} = -9 \times 10^{-5}\text{mm}$

New Volume : $(200 + \delta_x)(200 + \delta_y)(1 + \delta_z)\text{mm}^3$

$= (200.008)(200.034)(0.99991)\text{mm}^3$
$V_1 = 40004.8 \text{ mm}^3$

Original volume : $V = 200 \times 200 \times 1\text{mm}^3 = 40000 \text{ mm}^3$
$\Rightarrow$ Percentage change in volume :
$\frac{\triangle V}{V} = \frac{4.8}{40000} \times 100 = 1.2 \times 10^{-2}\%$

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MECHANICS

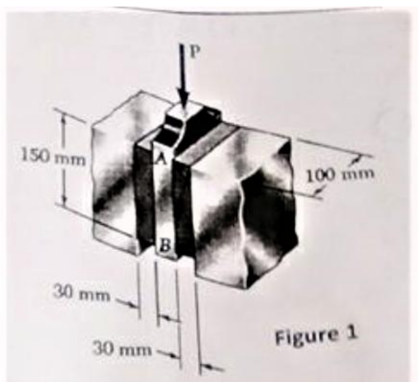
END SEMESTER SPRING EXAMINATION **2016-17**

1. A vibration isolation unit consists of two blocks of rubber bonded to a rigid metal plate AB and to rigid supports as shown in figure. The modulus of rigidity of the rubber is G=20 MPa, Poisson's ratio ν=0.5 and modulus of elasticity is E=60 MPa. If a force of magnitude P=40KN is applied to the plate AB as shown, such that the bonding remains intact at all surfaces, then find the

a) Deflection of plate AB in the downward direction.

b) Change in volume of the two rubber blocks.

c) Maximum tensile stress developed in the rubber blocks.



Ans 1a): Force = P/2 = 20 KN

G = 20 MPa

$$\therefore \tau = \frac{P}{2\pi} = \frac{20\times10^3}{150\times100\times10^{-6}}$$

$$\frac{\tau}{G} = \theta = \frac{20\times10^3}{150\times100\times10^{-6}} \times \frac{1}{20\times10^6} = \frac{1}{15}$$

$$\theta = \frac{\triangle\, l}{l} = \frac{\triangle\, l}{30mm}$$

$\therefore \triangle\, l$ = 2mm

Ans 1b): V = xyz

Thus, $\frac{\triangle\, V}{V} = \frac{\triangle\, x}{x} + \frac{\triangle\, y}{y} + \frac{\triangle\, z}{z}$

taking $\triangle\, x$ and $\triangle\, y$=0

$$\frac{\triangle\, V}{V} = \frac{\triangle\, z}{z} = \frac{\triangle\, l}{l} = \frac{1}{15}$$

Ans 1c)* There should not be any tensile stress. **[Clarification Required]**

2. The walls of a thin-walled cylindrical pressure vessel made of epoxy are strengthened with glass fibers oriented at $\beta = 45^o$ to the axis of the vessel as shown in figure 2. The pressure vessel is subjected to internal gague pressure of 1 MPa. The walls are 1 mm thick and the diameter D and length L are 500mm and 1000mm respectively. If a factor of safety of 2.0 is to be used them what must be the

a) Tensile strength of the glass fibers so that they do not rupture.

b) Shear strength of the interface between the glass fibers and the epoxy.

c) Change in the circumference of the pressure vessel if the material of the pressure vessel is taken to be isotropic with elastic properties E=20GPa and ν = 0.28 .

Ans 2a): $\sigma_{\theta\theta} = \frac{Pr}{z(\frac{2r}{L})+1}$

$$= \frac{1\times(0.5/2)}{0.001\times(0.5/1+1)}\text{MPa}$$

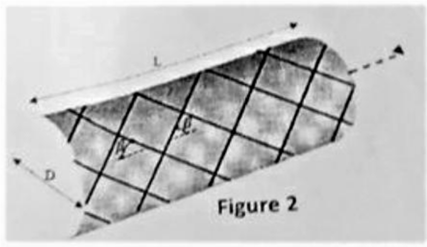
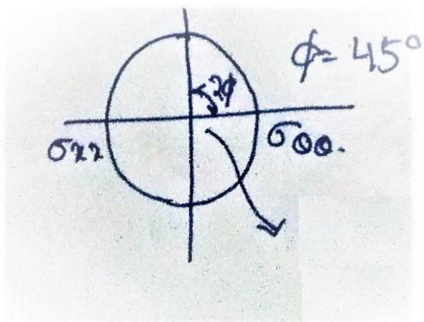


Figure 2

$$= \frac{0.5 \times 0.5}{0.001(1.5)}\text{MPa}$$
$$= \frac{500}{3}\text{MPa}$$

$\sigma_{xx} = \dfrac{Pr}{2t} = \dfrac{1 \times 0.5/2}{2 \times 0.001} = 125$ MPa
Using Mohr's Circle, Tensile stress at 45^o rotation



$\sigma = \left(\dfrac{\sigma_{\theta\theta} + \sigma_{xx}}{2}\right. \approx 145.8$ MPa
Using factor of safety 2, $\sigma' = 292$ MPa
Ans 2b): Sheer strength $= (\sigma - \sigma_{xx})$ Radius of Mohr's Circle
$= 145.83 - 125 = 20.83$MPa
Ans 2c): C $= 2\pi r$
or $\dfrac{\triangle\, C}{C} = \dfrac{\triangle\, r}{r} = \dfrac{\sigma_{\theta\theta} - \nu\sigma_{xx}}{E} = \dfrac{166.67 - 0.28 \times 125}{20 \times 10^3} = 6.5835 \times 10^{-3}$
$\therefore \triangle\, C = 6.5835 \times 10^{-3} \times \pi \times 0.5 \simeq 10.34$mm

3. A solid and a hollow cylindrical shaft made of the same material have the same length and volume but are subjected to torsion T_s and T_h respectively. Derive the relation between the ratios of applied torque (T_s/T_h) and n(as shown in figure 3) if the maximum shear stress in both the cases is exactly the same.

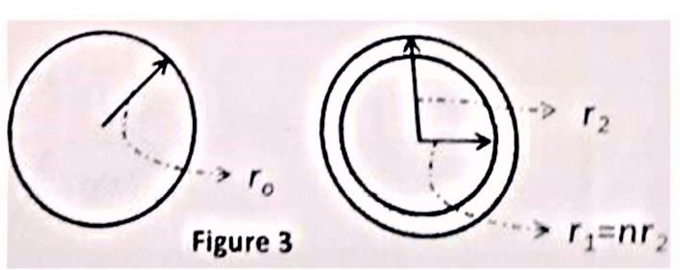
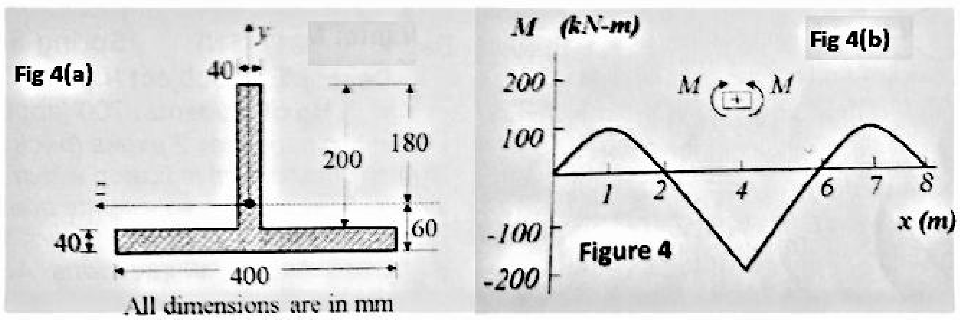


Figure 3

Ans 3): $\dfrac{\tau}{r} = \dfrac{T}{J}$
or $\tau = \dfrac{rT}{J}$ Given τ is same
$\therefore \dfrac{r_o T_s}{\frac{\pi}{2} r_o^4} = \dfrac{r_2 T_h}{\frac{\pi}{2}({r_2}^4 - n^4 {r_2}^4)}$

or $\dfrac{T_s}{T_h} = \dfrac{(r_o/r_2)^3}{(1-n^4)}$

4. A straight beam 8m long with inverted T section is transversely loaded. The cross-section of the beam along with its centroid is shown in the figure



a) I of the cross-section about the cetroidal axis.

b) Location of the section of the beam where the tensile and the compressive bending stresses are maximum.

c) Ratio between the maximum tensile bending stress and the maximum compressive bending stress within the beam.

d) Location of the section where average shearing stress (shear force divided by area) vanishes.

e) Points where the bending stress is zero.

Ans 4a): I $= \dfrac{bh^3}{12}$

$\therefore I_1 = \dfrac{(0.4)(0.04)^3}{12} = 2.133 \times 10^{-6}$ $\qquad I_2 = \dfrac{(0.04)(0.2)^3}{12} = 2.67 \times 10^{-5}$

$\therefore I_{net} = I_1 + 0.4 \times 0.04 \times (0.04)^2 + I_2 + 0.4 \times 0.04 \times (0.08)^2$
$= 1.568 \times 10^{-4}$

Ans 4b): $\sigma = \dfrac{My}{I}$ peaks at x=1, x=4, x=7

$\therefore$ Compressive stress is maximum at x=1, x=7

As I is constant, compare the products of My.

$\therefore$ Tensile stress is maximum at x=4.

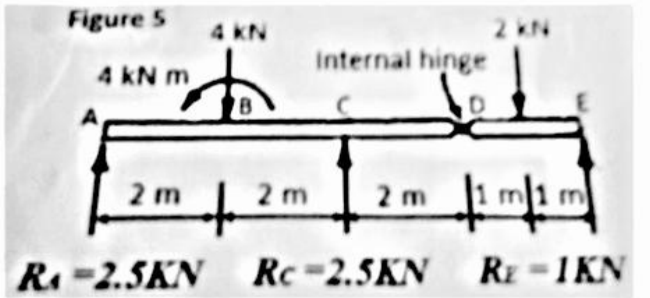
4c): $\sigma_{max\ tensile} = 200 \times 180$ $\qquad \sigma_{max\ compressive} = 100 \times 180$

Thus, ratio = 2

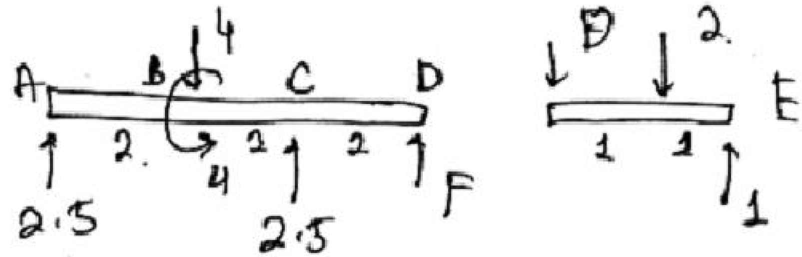
4d): Probably not in syllabus.

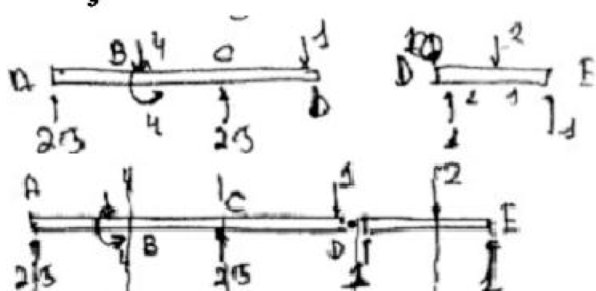
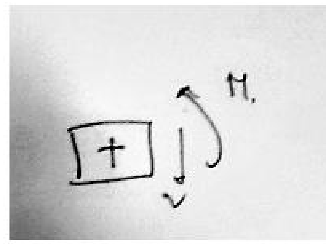
4e): x=2,6 as M=0

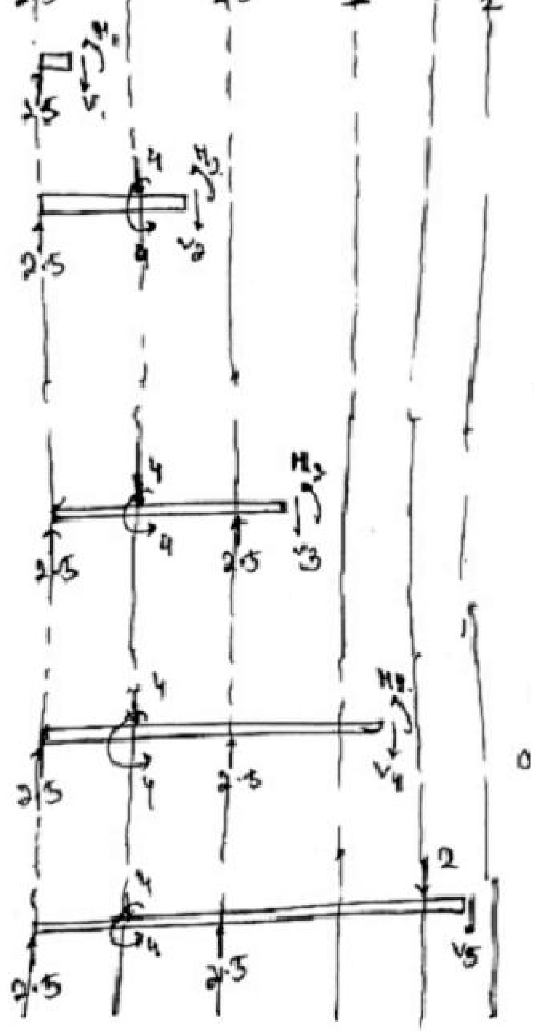
5. The compound beam ABCDE shown in the figure 5 consists of two beams AD and DE joined by a hinged connection at D. The loads on the beam consists of a 4KN force and 4KN-m moment at B and a 2KN force at the mid-point of DE. Reaction forces at supports A,B and E have magnitudes and direction as shown in figure 5. Draw the shear force diagram for ABCDE (with proper labeling). Calculate bending moment at C and D and the maximum bending moment.



Ans 5):



$\therefore \Sigma F_y = 0 \qquad \therefore F = 4 - 5 = 1$





$v_1 = 2.5\text{KN}$

$M = 2.5x$

$v_2 + 4 = 2.5$

or $v_2 = -1.5\text{KN}$

$M_2 + 4 + 4(x-2) = 2.5x$

or $M_2 - 4 = -1.5x$

or $M_2 = 4 - 1.5x$

$v_3 + 4 = 5$

$v_3 = 1\text{KN}$

$M_3 + 4(x-2) + 4 = 2.5(x + x - 4)$

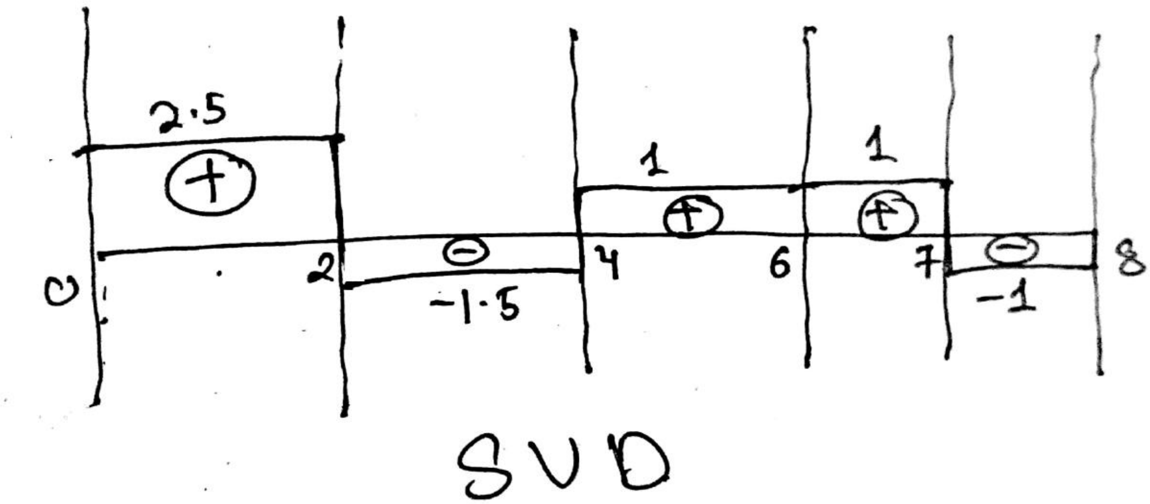
$M_3 = x - 6$

$v_4 = 1\text{KN}$

$v_5 = -1\text{KN}$

$\therefore$ From $M_2, M_C(x=4) =$ -2 KN

$\therefore$ From $M_3, M_D(x=6) = 0$ KN



Probably the maximum bending moment is 2KN-m.

Q 6. is probably not in syllabus.

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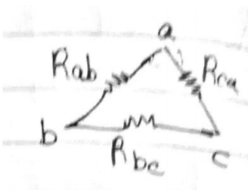
ELECTRICAL TECHNOLOGY

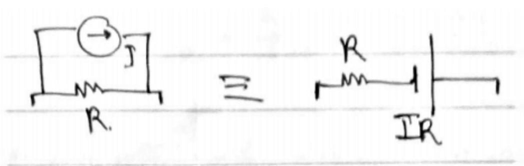
Formulae

1) $\triangle$ to Y conversion: $R_a = \dfrac{R_{ab} \times R_{ca}}{R_{ab} + R_{bc} + R_{ca}}$

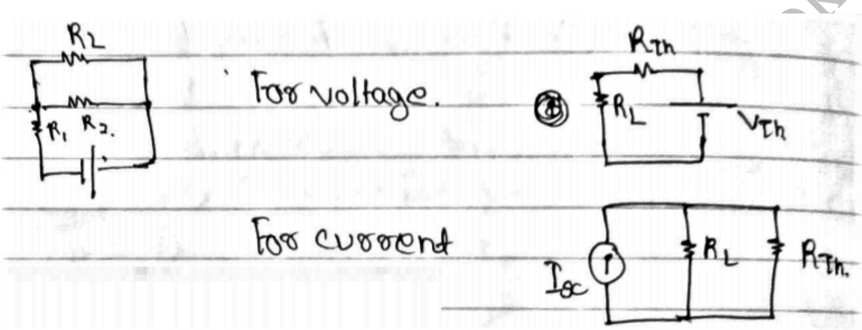


2) Y to $\triangle$ conversion: $\dfrac{R_aR_b + R_bR_c + R_cR_a}{R_c} = R_{ab}$

3) Source conversion:



4) For Thevenin theorem



I_{sc} is the short circuit current when R_L is shorted.
V_{Th} is the voltage across R_L terminals when R_L is open.

5) Maximum power transfer theorem is when internal resistance is equal to load resistance.

6) Form factor $= \dfrac{E_{rms}}{E_{ang}} = \dfrac{\pi}{2\sqrt{2}} \simeq 1.112$

$$E_{rms} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} V_o^2 \sin^2\theta . d\theta} = \frac{V_o}{\sqrt{2}}$$

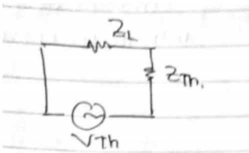
$$E_{avg} = \frac{1}{\pi}\int_0^{\pi} V_o \sin\theta . d\theta = \frac{2V_o}{\pi}$$

7)Peak factor $= \dfrac{E_{peak}E_{rms}}{=}\sqrt{2} = 1.414$

8) For composite excitations; $E = E_1\sin(\omega_1 t) + E_2\sin(\omega_2 t)$
$E_{rms} = \sqrt{E_{1rms}^2 + E_{2rms}^2}$

9) If $X_L + X_C > 0$ then its an inductive circuit and if $X_L + X_C < 0$ then its a capacitive circuit.

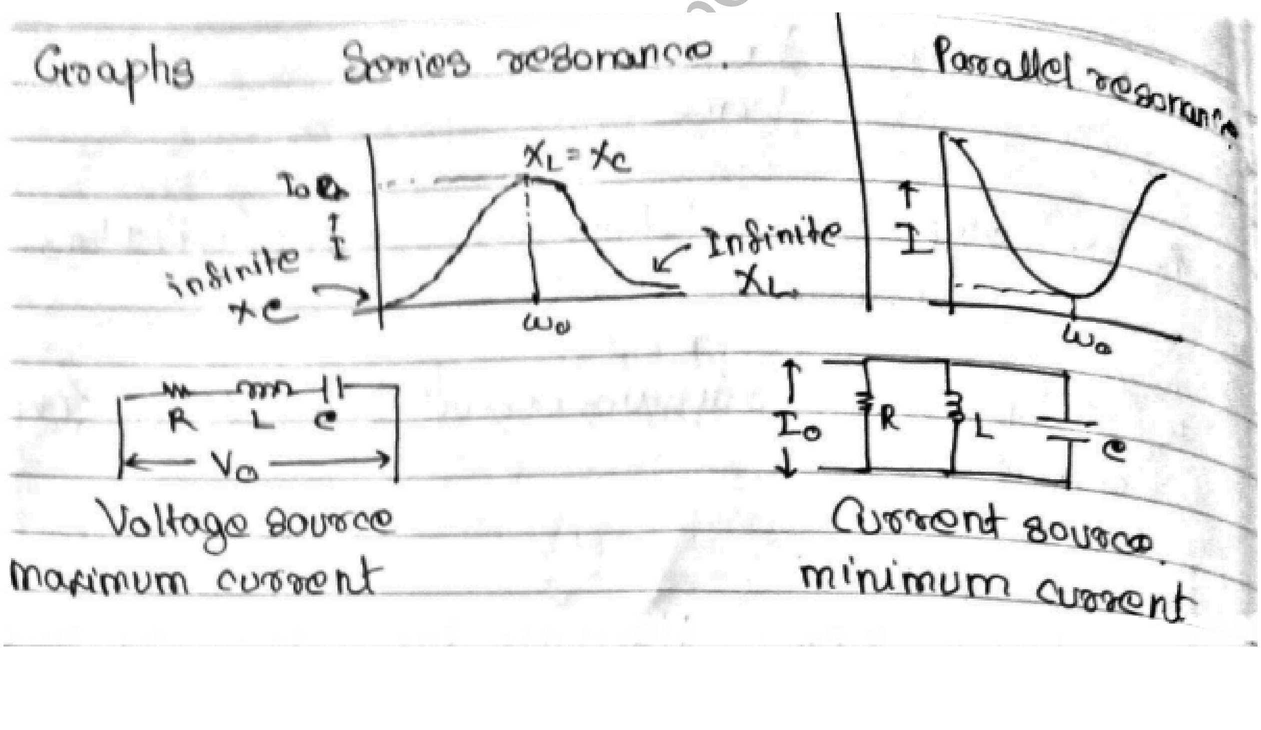
10) Maximum power transfer, when



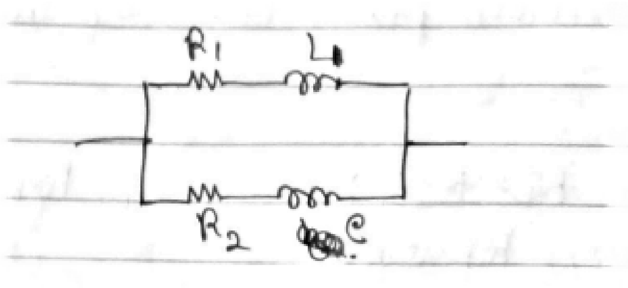
$Z_L = Z_{Th}^*$
Special cases: (i) For $X_{Th} = 0, R_L = R_{Th}, |Z_L| = R_{Th}$
(ii)For X_L constant (even 0) $R_L = [R_{Th}^2 + (X_{Th} + X_L)^2]^{1/2}$
(iii) R_L constant (explicitly), then, $X_L = -X_{Th}$
P $= \dfrac{(V_{Th})^2}{2} \dfrac{R_L}{(R_L + R_{Th})^2 + (X_L + X_{Th})^2}$

11) Half power bandwidth, $P = P_o/2$
$\therefore \omega_1 = -\alpha + \sqrt{\alpha^2 + \omega_o^2}$
$\omega_2 = \alpha + \sqrt{\alpha^2 + \omega_o^2}$
where, $\alpha = \dfrac{R}{2L}, \omega_o = \dfrac{1}{\sqrt{LC}}$
$\therefore \triangle\omega = \omega_1 - \omega_2 = 2\alpha = \dfrac{R}{L}$
Quality factor (for series circuits) $= \dfrac{\omega_o}{\triangle\omega} = \dfrac{1}{\sqrt{LC}} \times \dfrac{L}{R} = \dfrac{\omega_o L}{R} = \omega\tau$
Quality factor (for parallel circuits) $= \dfrac{\omega_o}{\triangle\omega} = \dfrac{\omega_o}{1/RC} =$
$omega_o RC = \omega_o\tau$

12)



13)

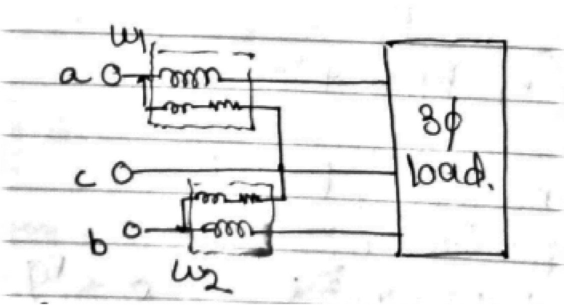


$\omega_o = \frac{1}{\sqrt{LC}}\left(\frac{R_1^2C-L}{R_2^2C-L}\right)^{1/2}$
See R_1 is with L and R_2 is with C.

14) Power $= V_LI_L\cos\phi = 3V_{ph}I_{ph}\cos\phi$ where $\cos\phi$ is power factor of load (balanced)

15 For balanced cases $Z_\triangle = 3Z_Y$

16)



ϕ(power factor angle)= $\tan^{-1}\left(\frac{\sqrt{3}(\omega_1-\omega_2)}{\omega_1+\omega_2}\right)$
If $\omega_1 > \omega_2$ inductive load.
If $\omega_1 < \omega_2$ capacitive load.

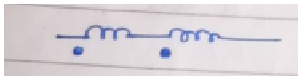
17) Hl = NI in AT(ampere turns)
B(flux density) $= \mu H$ $\phi = BA \longrightarrow$flux
R = l/$\mu A \longleftarrow$ Magnetic reluctance(AT/Wb)
$\oint Hdl = NI$

18 Magnetic pressure = Energy per unit volume $= \frac{B^2}{2\mu_o}$

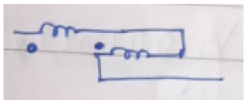
19) $L = \frac{N^2\mu \times Area}{length}$ N = no. of turns
$L = \frac{N^2}{R}$ R = magnetic reluctance
$M = (\sqrt{L_1L_2}k \longleftarrow$ coupling coefficient (k=1 means, no leakage condition)

20) L = Nϕ/I Energy $= \frac{1}{2}LI^2$

21)
$L_{eq} = L_1 + L_2 + 2M$ for,

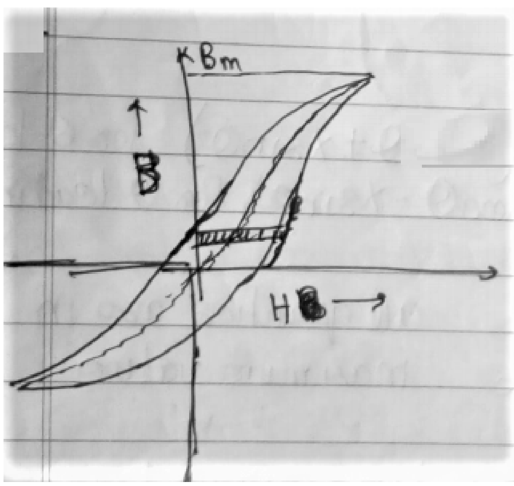


$L_{eq} = L_1 + L_2 - 2M$ for,



Energy $= \frac{1}{2} L_{eq} I^2$

22)



Energy stored $= \int H.dB$

23) Hysterisis loss $= R_n B_m^n f V$
where, $R - n$ = characteristic constant of core
$B_m = B_{max}$ n = steinmetz exponent
f = frequency
V = volume

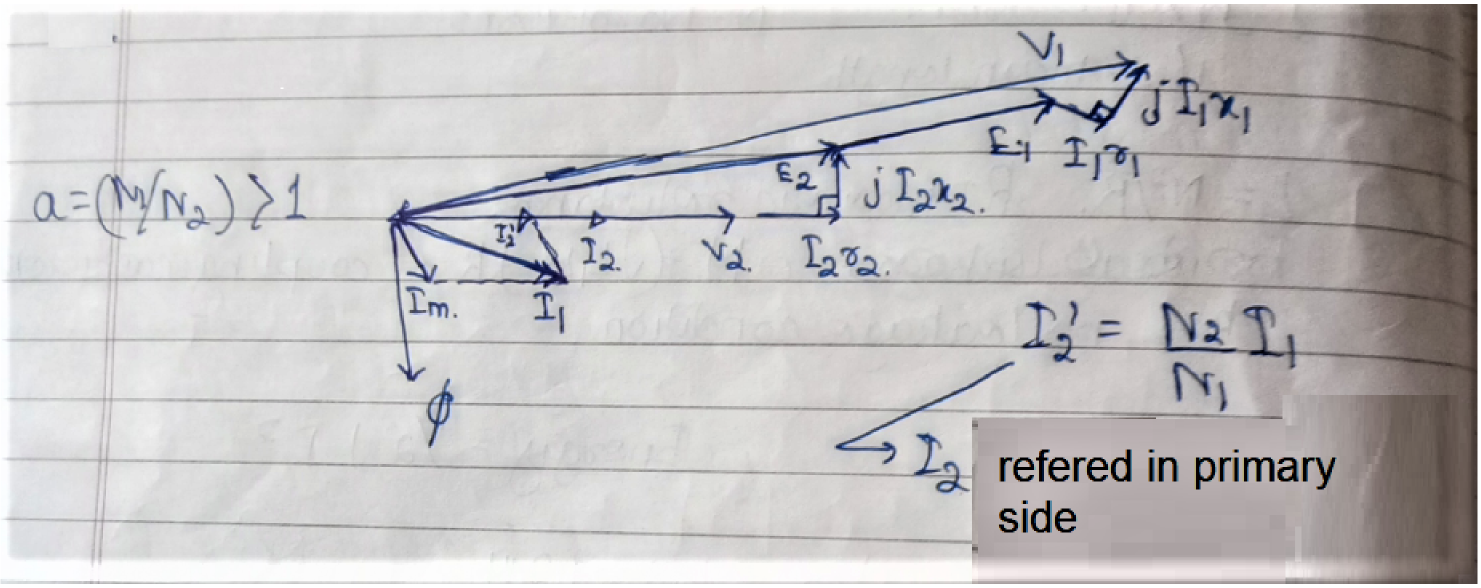
21) Eddy current loss $P_e = K_e f^2 B_m^2 V$ where $K_e = kt^2$
t = thickness, V = volume, f = frequency
$B_m = B_{max}$

22) Induced voltage(rms value), $e_1 = \sqrt{2}\pi f N_1 \phi_{max} = 4.44 f N_1 \phi_{max}$

23) $\frac{E_1}{N_1} = \frac{E_2}{N_2}$ $\frac{M}{N_2}$ = a = turn ratio of transformer

24) % voltage regulation $= \frac{V_{2(no-load)} - V_{2(full-load)}}{V_{2(no-load)}} \times 100\%$
Taking approximations ,
Voltage regulation $= I(R\cos\theta + X\sin\theta)$ for θ lagging
$= I(R\cos\theta - X\sin\theta)$ for θ leading

25)



26) Efficiency$(n) = \dfrac{xS\cos\theta}{xS\cos\theta + P_i + x^2P_c}$ all quantities are in maximum value
maximum efficiency when Iron loss = Copper loss
$\therefore x = \sqrt{P_i/P_c}$

27) $(V_a)_{Auto} = \dfrac{1}{1-1/a'}(V_A)_{Tw}$

where a' = auto transformer ratio $= \dfrac{V_1}{V_2}$

28) $\dfrac{\theta_{electrical}}{\theta_{mechanical}} = \dfrac{P}{2}$ where P = no. of poles

29) Air gap power $= \dfrac{3I_2^2r_2}{S} = P_G$
Rotor copper loss $= SP_G$
Mechanical load (including mechanical loss) $= (1-S)P_G$
Mechanical load $r_2(-1+\dfrac{1}{S})$

30) Torque, $T = \dfrac{P_m}{\omega_r} = \dfrac{(1-S)P_G}{(1-S)\omega_s} = \dfrac{P_G}{\omega_s}$

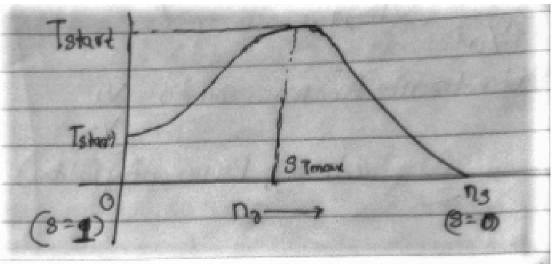
31) $T = \dfrac{3}{\omega_s} \times \dfrac{{V_{Th}}^2(r_2'/s)}{(r_{Th}+\frac{r_2'}{s})^2 + (X_{Th}+X_2')^2}$

$$s_{T_{max}} = \frac{r_2'}{\sqrt{R_{Th}^2 + (X_{Th}+X_2')^2}}$$

$$T_{max} = \frac{3V_{Th}^2}{2\omega_s} \times \frac{1}{R_{Th} + \sqrt{R_{Th} + (X_{Th}+X_2')^2}}$$

32) $T_c = \dfrac{2T_{emax}}{\dfrac{S}{S_{max}} + \dfrac{S_{max}}{s}}$ For neglecting stator impedence.

33)



Peak remains constant with changing r_2. Increasing r_2 shifts the peak towards left.

34) n_s (no. of rotations /minute) $= \dfrac{120 \times f}{P}$ where P=pole and slip rpm $= sn_s$

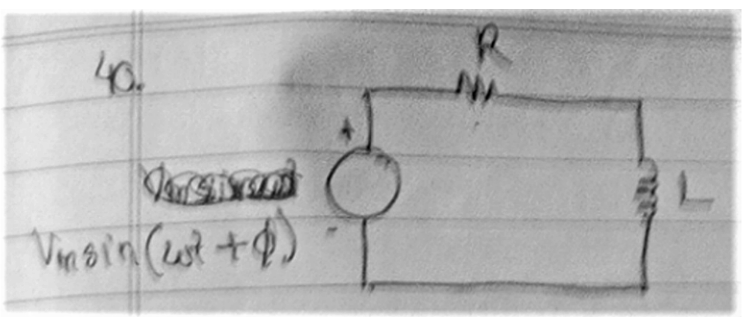
35) $\dfrac{I_s(line) - \triangle}{I_s(line) - Y} = 3$

36) $q(t) = q_\alpha + (q_o - q_\alpha)e^{-t/\tau}$ $\tau = RC$
$i(t) = i_\alpha + (i_o - i_\alpha)e^{-t/\tau}$ $\tau = L/R$

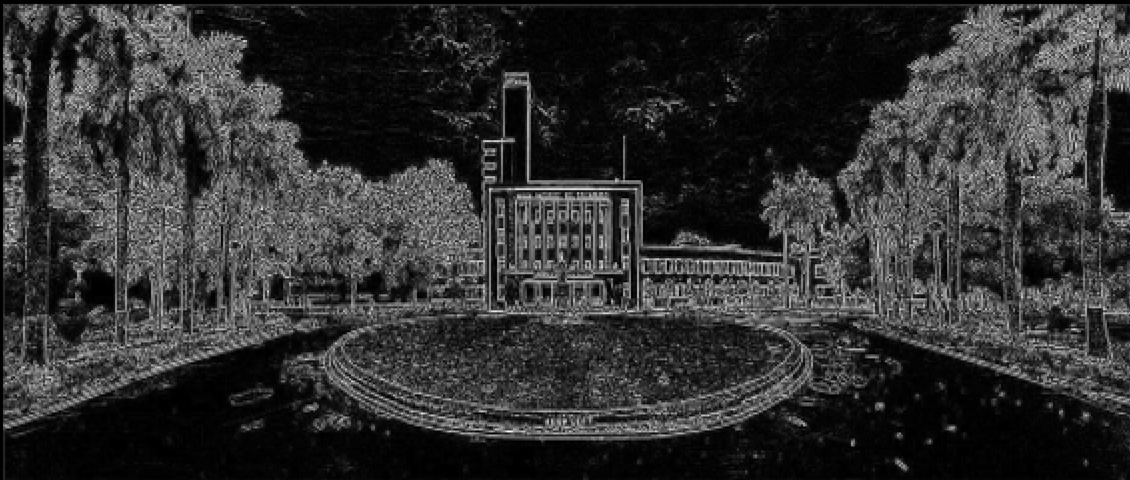
37) Induced emf in stator coil $= sE_2$

38) 200/400 V, 4000 VA $\longrightarrow$ 400V at the supply of higher side referenced equivalent circuit.
Full load higher side referenced equivalent circuit = 10A

39) 400V, 100Hp $\longrightarrow$ 400 volt line voltage supply ; 100 Hp is the output power.
40)



$I = \dfrac{V_m}{|Z|}\sin(\omega t + \theta - \phi) + ke^{-R/L}$
$\therefore k = \dfrac{V_m}{|Z|}\sin(\theta - \phi)$ where$\theta = \tan^{-1}\left(\dfrac{\omega L}{R}\right)$
$\therefore$ For transient free switching, $\phi = \tan^{-1}\left(\dfrac{\omega L}{R}\right)$
When you replace L with C, then $\phi = -\theta$
$\therefore \phi = -\tan^{-1}\left(\dfrac{1}{\omega RC}\right)$ for Capacitors.



About the book:

This is a first of its kind attempt by now sophomore students namely Aniruddha, Aadi, Kanishka and Parth. This book is a dedicated help for anyone aiming for a high CGPA or a department change. This book is a compilation of 42 previous years solved papers along with important formulae and key concepts. We sincerely believe that our efforts will bear fruitful results.