TRANSPORT PHENOMENA

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TRANSPORT PHENOMENA

Basic Overview

Over the last few decades, the subject has revolutionized the way Chemical Engineering science is taught. This course deals with the unified treatment of the different transport processes, ubiquitous in industry as well as in nature. Momentum, heat and mass transfer are taught together due to the underlying similarities of the mathematics and molecular mechanisms describing such processes.

Books

1. **Transport Phenomena by Bird, Stewart and Lightfoot (Wiley)**
2. Transport Phenomena Fundamentals by J. L. Plawsky (CRC Press)
3. Heat and Mass Transfer – A Transport Phenomena Approach by K. S. Gandhi (New Age)
4. Fluid Mechanics by R. W. Fox, P. J. Pritchard and A. T. McDonald (Wiley)
5. Introduction to Heat and Mass Transfer by F. P. Incropera & D. P. DeWitt (Wiley)

Grading: 3 tests of 1 hour duration, on Moodle

Course Plan

Formulation and solution of momentum transfer in laminar flow.
Navier-Stokes equation and its applications
Boundary Layer concepts, boundary layer thicknesses (disturbance, displacement and momentum), Blasius solution for flow over a flat plate
Use of momentum integral equation, turbulent boundary layers, fluid flow about immersed bodies, drag
Formulation and solution of heat transfer in laminar flow
Development and use of energy equation
Transient conduction - lumped capacitance, analytical solutions and other methods.
Formulation and solution of mass transfer in laminar flow. Development and use of species balance equation
Introduction to convective flow, natural convection, relevant examples from heat and mass transfer
Mathematical treatment of the similarities between heat, mass and momentum transfer, similarity parameters, and relevant analogies.
Solution of coupled heat, mass and momentum transfer problems based on analogy.

$$Pr = \frac{C_p \mu}{k} \quad , \quad Sc = \frac{\mu \cancel{\rho}}{\rho D_{AB}}$$

$$Nu = \frac{hL}{k} \quad , \quad Sh = \frac{h_m L}{D_{AB}}$$

Heat | Mass

Re

$$Nu = f(Re, Pr) \quad \Big| \quad Sh = f_2(Re, Sc\ \cancel{Pr})$$

TRANSPORT PHENOMENA

Application of Navier Stokes Equations

Continuity Equation

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

Equation of Motion – Navier Stokes equation

$$\rho D\bar{v} / Dt = -\nabla p - [\nabla . \bar{\tau}] + \rho g$$

Cartesian Coordinate – z component

$$\rho \left(\frac{\partial\, v_z}{\partial t} + v_x \frac{\partial\, v_z}{\partial x} + v_y \frac{\partial\, v_z}{\partial y} + v_z \frac{\partial\, v_z}{\partial z} \right) = -\frac{\partial\, P}{\partial z} + \mu \left[\frac{\partial^2\, v_z}{\partial x^2} + \frac{\partial^2\, v_z}{\partial y^2} + \frac{\partial^2\, v_z}{\partial z^2} \right] + \rho\, g_z$$

Relevant Boundary Conditions

No slip at the liquid-solid interface

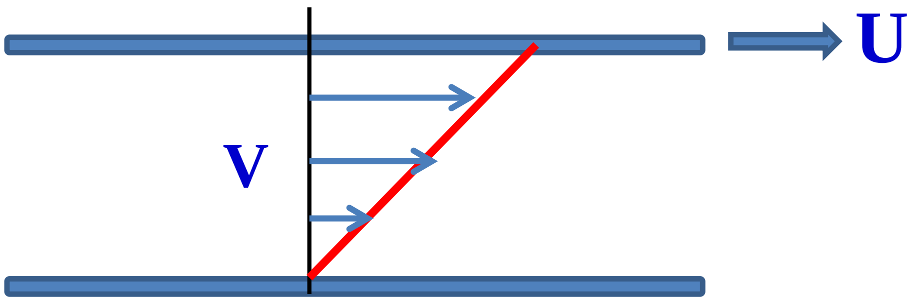
No shear at the liquid-vapor interface

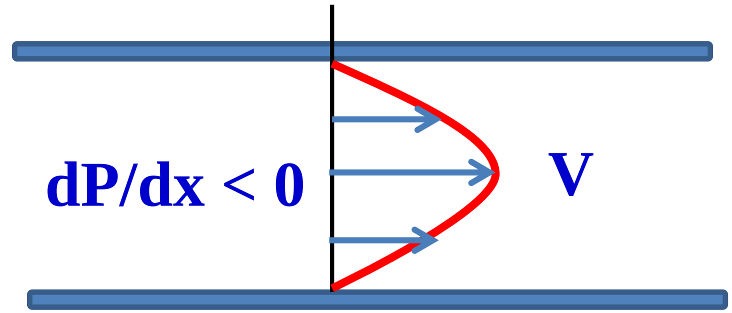
Transport Phenomena - Momentum Transfer

The record-write head for a computer disk storage system floats above the spinning disk on a very thin film of air (the film thickness is 0.5micron). The head location is 150mm from the disk centre line, the disk spins at 3600rpm. The record-write head is 10mm x 10mm square. Determine for standard air (density=1.23kg/m^3, viscosity=1.78x10^{-5} kg/(m.s)) in the gap between the head and the disk -
(a) the Reynold's number of the flow, (b) the viscous shear stress and (c) the power required to overcome viscous shear stress.

For such a small gap the flow can be considered as flow between parallel plates.

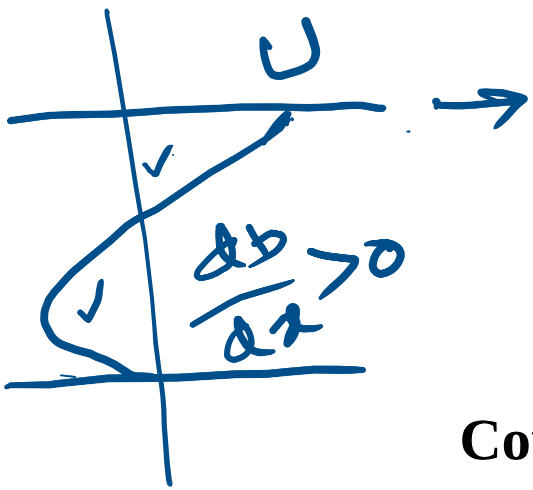
Flow between parallel plates

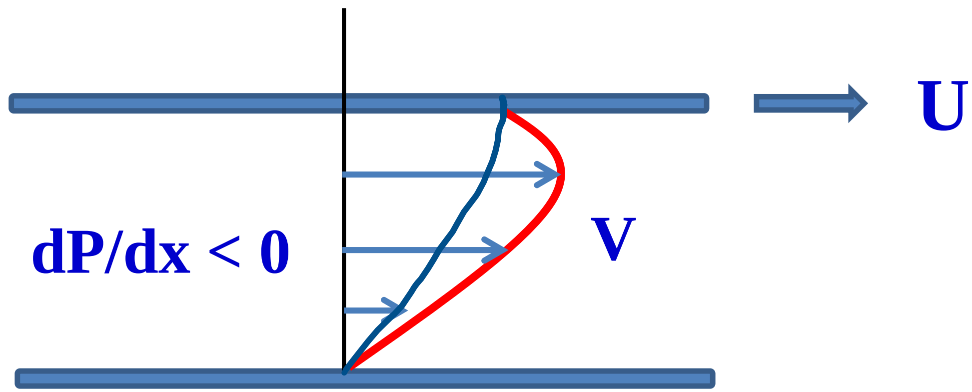




Couette Flow

Pressure gradient Driven Flow





Couette and Pressure gradient Driven Flow

Solution

Since the gap is too small, the situation can be approximated by the case of flow between two parallel plates with one plate moving and the other stationary with no applied pressure gradient.

Here,

$V = R\omega = 0.15 \text{ m} \times 3600 \text{ rev/min} \times 2\pi \text{ rad/rev} \times 1/60 \text{ min/s} = 56.5 \text{ m/s}$

$Re = (\rho V a)/\mu = (V a)/(\mu/\rho) = 56.5 \text{ m/s} \times 0.5 \times 10^{-6} \text{ m}/(1.45 \times 10^{-5} \text{ m}^2/\text{s}) = 1.95$ (Laminar Flow)

For Couette flow between two parallel plates, with no applied pressure gradient, the velocity profile will be linear with no-slip boundary conditions at the bottom (stationary) and top (moving) plates.

Therefore, the shear stress, τ, can be expressed as

$\tau = \mu du/dy = \mu V/a$ (linear velocity profile)

$\tau = 1.78 \times 10^{-5}$ kg /(m.s) x 56.5 m/s x 1/(0.5x10^{-6} m) = 2.01 KN/m^2

Therefore,

Force, $F = \tau A = \tau \times (W \times L)$, Torque, $T = F \times R = \tau \times (WL) \times R$

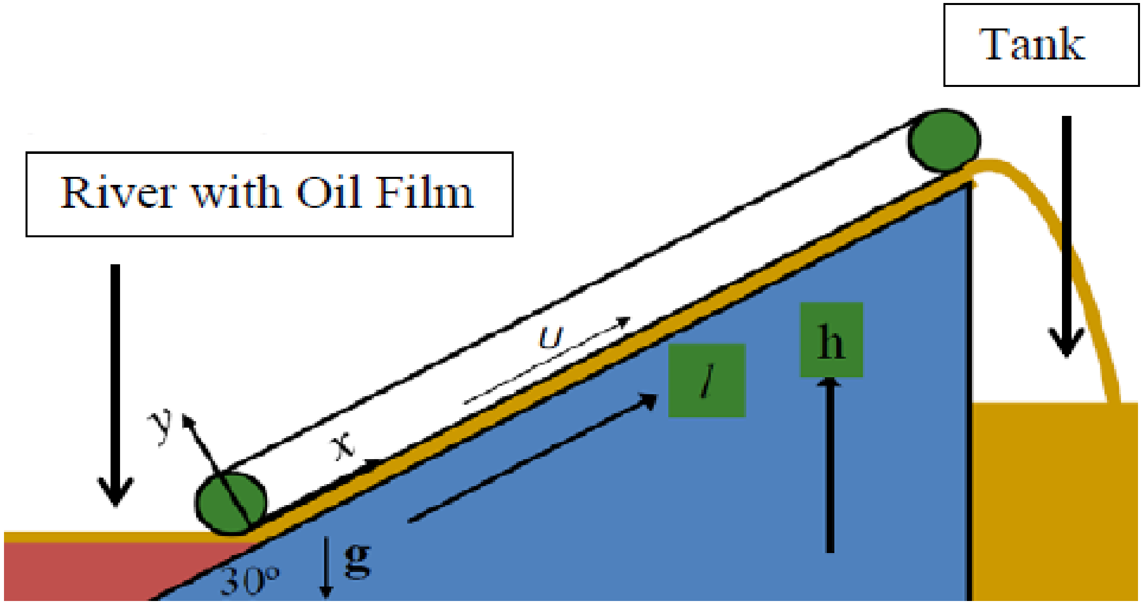
Power dissipation rate. $P = T \times \omega = \tau\, WL\, R\, \omega$

$P = 2.01 \times 10^{-3} \times 0.01 \times 0.01 \times 0.15 \times 3600 \times 2\pi \times 1/60 = 11.4$ W

Thus the power required to overcome viscous shear stress is 11.4 W

Transport Phenomena - Momentum Transfer
Problem 2

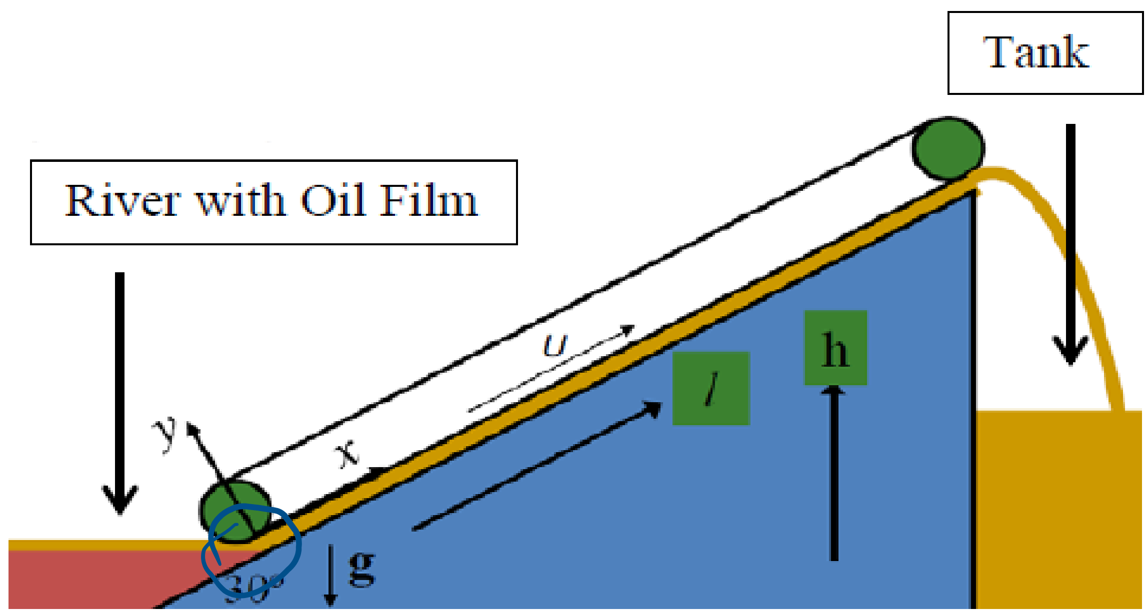
An oil skimmer uses a 5 m wide x 6 m long moving belt above a fixed platform (θ = 30º) to skim oil off of rivers (T = 10ºC). The belt travels at 3 m/s. The distance between the belt and the fixed platform is 2 mm.



The belt discharges into an open tank on the ship. The fluid is actually a mixture of oil and water. To simplify the analysis, assume crude oil dominates. Find the discharge of oil into the tank on the ship, the force acting on the belt and the power required (kW) to move the belt. For oil: ρ = 860 kg/m^3, viscosity, $\mu = 1 \times 10^{-2}$ N.s/m^2

The x-component of the equation of motion is given as

$$\rho\left(\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{v}\,\frac{\partial v}{\partial y} + \cancel{w}\,\frac{\partial w}{\partial z}\right) = -\cancel{\frac{\partial P}{\partial x}} + \mu\left[\cancel{\frac{\partial^2 u}{\partial x^2}} + \underline{\frac{\partial^2 u}{\partial y^2}} + \cancel{\frac{\partial^2 u}{\partial z^2}}\right] + \rho\,\underline{g_x} \checkmark$$



Considering one-dimensional flow (v = 0 = w), no applied pressure gradient, u is a function of only y, and a steady state process, the above equation reduces to the following governing equation

$$\frac{d^2u}{dy^2} = \frac{\rho g}{\mu} \sin\theta$$

The relevant boundary conditions (no slip) are

BC 1 At $y = 0$ $u = 0$
BC 2 At $y = h$, $u = U$

Therefore,

$$\frac{du}{dy} = \frac{\rho g sin\theta}{\mu}\, y + A \qquad and \qquad u = \left(\frac{\rho g sin\theta}{\mu}\right)\frac{y^2}{2} + Ay + B$$

Applying the BC s ➔ B = 0 and $A = \left(\frac{\rho g sin\theta}{\mu}\right)\frac{h}{2} + \frac{u}{h}$

Thus

$$u = -\left(\frac{\rho g sin\theta}{\mu}\right)\left(\frac{hy}{2} - \frac{y^2}{2}\right) + \frac{Uy}{h}$$

The volumetric flow rate per unit width of the film is given by

$$Q = \int_0^h u dy = -\int_0^h \left[\left(\frac{\rho g sin\theta}{\mu}\right)\left(\frac{hy}{2} - \frac{y^2}{2}\right) + \frac{Uy}{h}\right] dy = -\frac{\rho g h^3}{12\mu}\, sin\theta + \frac{Uh}{2}$$

$$Q = -\frac{860\ x\ 9{,}81\ x\ 0.002}{12\ x\ 10^{-2}} \sin 30 + \frac{3\ x\ 0.002}{2} = 0.0027 \frac{m^2}{s}\ (per\ unit\ width)$$

For the width of 5 m, the volumetric flow rate will be equal to 0.0135 m^3/s.

Evaluate $\tau = \mu\ du/dy$ at the moving belt from the expression for velocity as

$$\frac{du}{dy} = -\left(\frac{\rho g sin\theta}{\mu}\right)\left(\frac{h}{2} - y\right) + \frac{U}{h}$$

And at the moving belt,

$$\tau = \mu\left(\frac{du}{dy}\right)_{y\ =\ h} = \left(\frac{\rho g sin\theta}{2}\right) h + \frac{\mu U}{h}$$

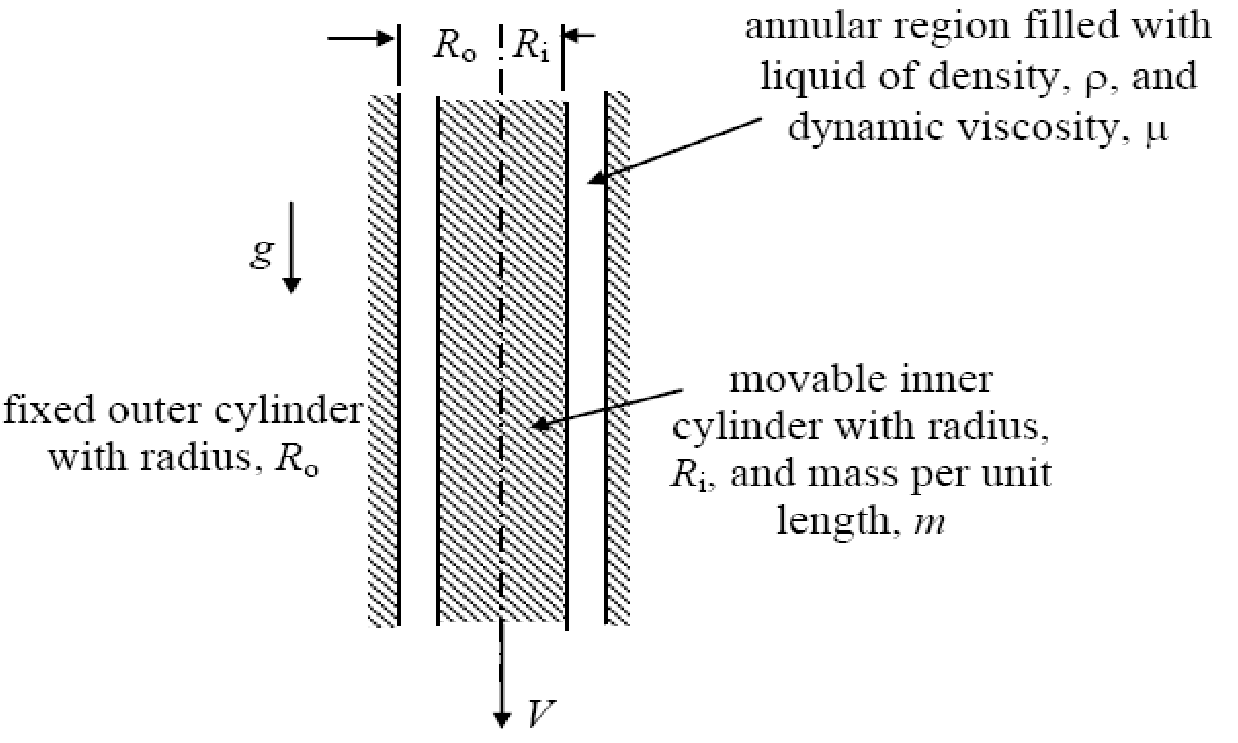
Using the values h = 0.002m, $\mu = 10^{-2}$ N.s/m^2 and U = 3 m/s

$\tau \big|_{\text{at the belt}} = 19.21\ N/m^2$

Power = (τ x L x W) U = 19.21 x 6 x 5 x 3 = 1.73 KW

Transport Phenomena - Momentum Transfer
Problem 3

Consider two concentric cylinders with a Newtonian liquid of constant density, ρ, and viscosity, μ, contained between them. The outer pipe, with radius, *R*o, is fixed while the inner pipe, with radius, *R*i, and mass per unit length, *m*, falls under the action of



gravity at a constant speed. There is no pressure gradient within the flow and no swirl velocity component. Determine the vertical speed, *V*, of the inner cylinder as a function of the following parameters: g, R_o, R_i, m, ρ, and μ. The space between the two cylinders is **not** ‘too small’ compared to the radii of the cylinders

Solution

For the inner cylinder moving at constant velocity, the downward force is exactly balanced by the viscous force as

$$\left(\tau_w\, A_w\right)\Big|_{Inner\ Cylinder} = m\, L\, g$$

The z component of the equation of motion in cylindrical coordinate is

$$\rho\left(\cancel{\frac{\partial v_z}{\partial t}} + \cancel{v_r}\frac{\partial v_z}{\partial r} + \cancel{\frac{v_\theta}{r}}\frac{\partial v_z}{\partial \theta} + v_z\cancel{\frac{\partial v_z}{\partial z}}\right)$$
$$= -\cancel{\frac{\partial p}{\partial z}} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_z}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}\right] + \rho g_z$$

Cancelling the terms with the observations i) steady state, ii) v_z is a function of r only, not of z or θ, iii) no applied pressure gradient

The following simplified form of the NS equation can be obtained.

Governing equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = -\frac{\rho g}{\mu}$$

Upon integration (with C_1 and C_2 being the constants of integration)

$$v_z = -\frac{\rho g r^2}{4\mu} + C_1 \ln r + C_2$$

Boundary Conditions

v_z = V at r = R_i => $$V = -\frac{\rho g}{4\mu} Ri^2 + C_1 \ln R_i + C_2$$

v_z = 0 at r = R_o => $$0 = -\frac{\rho g}{4\mu} Ro^2 + C_1 \ln R_o + C_2$$

Therefore $$V = \frac{\rho g}{4\mu}\left(Ro^2 \; - \; Ri^2\right) + \; C_1 \ln \frac{R_i}{R_o}$$

$$C_1 = \frac{1}{\ln \frac{R_i}{R_o}} \left[V \; - \frac{\rho g}{4\mu}\left(Ro^2 \; - \; Ri^2\right) \right]$$

$$\frac{dv_z}{dr} = - \; \frac{\rho g r}{2\mu} + \; \frac{C_1}{r} \qquad \tau_{rz} = \mu \frac{dv_z}{dr} = - \; \frac{\rho g r}{2} + \; \frac{C_1 \mu}{r}$$

Since the force on the inner cylinder = force due to gravity

$$\tau \Big|_{r=Ri} 2\pi R_i L = m \; L \; g$$

Upon substitution for the expression of the shear stress

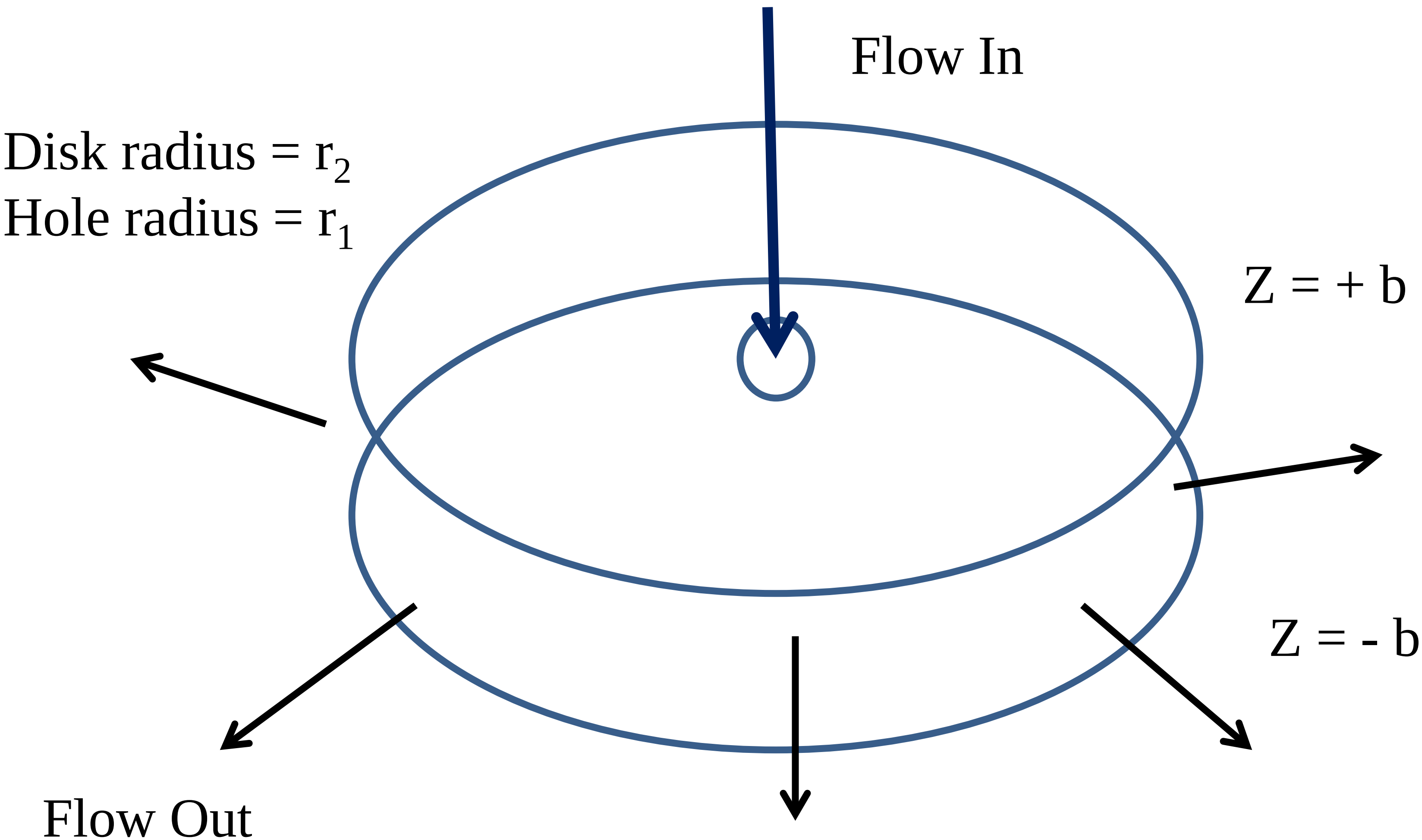
$$V = \; R_i \ln \frac{R_i}{R_o} \left(\frac{\rho g R_i}{2\mu} - \; \frac{mg}{2\pi R_i \mu} \right) - \; \frac{\rho g}{4\mu}\left({R_i}^2 \; - \; {R_0}^2\right)$$

Order of magnitude analysis of NS Equation

Is it possible to identify the relative magnitudes of the different terms (even approximately)?

It may then be possible to neglect the term(s) that may not play a crucial role in the transport process thereby simplifying NS equations.

Flow between two parallel disks with liquid entry through a small hole at the centre of the top plate



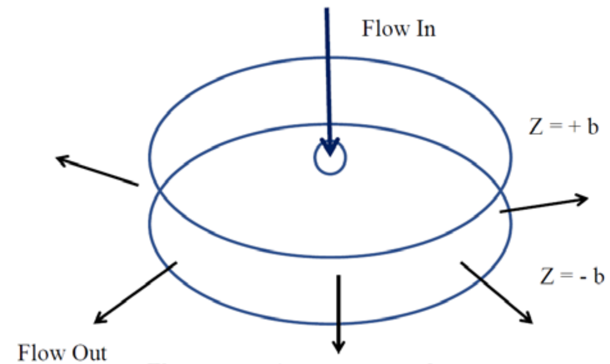
The pressure is p_1 at $r = r_1$ and p_2 at $r = r_2$

$$\cancel{\frac{\partial \rho}{\partial t}} + \frac{1}{r}\frac{\partial}{\partial r}\left(\rho r v_r\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\rho \cancel{v_\theta}\right) + \frac{\partial}{\partial z}\left(\rho \cancel{v_z}\right) = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\rho r v_r\right) = 0 \Rightarrow v_r = \frac{\phi}{r}$$

$$v_r = f(r, z)$$

$$\phi = f(z) \text{ ONLY}$$



The pressure is p_1 at $r = r_1$ and p_2 at $r = r_2$

$$\rho\left(\cancel{\frac{\partial v_r}{\partial t}} + v_r\frac{\partial v_r}{\partial r} + \cancel{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}} - \cancel{\frac{v_\theta^2}{r}} + \cancel{v_z\frac{\partial v_r}{\partial z}}\right)$$

$$= -\frac{\partial P}{\partial r} + \mu\left[\cancel{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r v_r\right)\right)} + \cancel{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}} - \cancel{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}} + \frac{\partial^2 v_r}{\partial z^2}\right] + \cancel{\rho g_r}$$

The governing equation

$$\rho\, v_r \cancel{\frac{\partial v_r}{\partial r}} = -\frac{\partial P}{\partial r} + \mu \frac{\partial^2 v_r}{\partial z^2} \qquad v_r = \frac{\phi(z)}{r}$$

Major assumption on the nature of the flow

NEGLIGIBLE CONVECTIVE ~~TRANSPORT~~

$\Rightarrow$ CREEPING FLOW

$$-\frac{dp}{dr} = -\mu \frac{d^2 v_r}{dz^2}$$

$\Rightarrow$ $p = \cancel{p_0 \text{ at } r} = r_0$ $\quad p = p_1$

$r = r_1$

Assume a constant applied pressure difference

$$\phi = -\frac{\Delta p z^2}{2\mu \ln r_2/r_1} + C_1 z + C_2$$

$$v_r(r, z) = \frac{\phi}{r} =$$

$z = +b$ $\quad v_r = 0$ $\quad$ NO SLIP

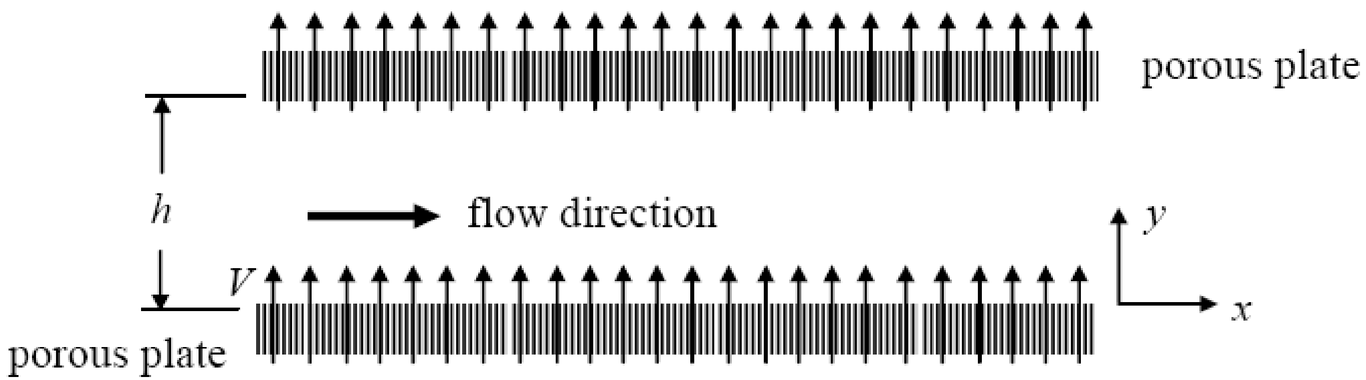
$z = -b$

Boundary Conditions

$$v_r(r,z) = \frac{\Delta P b^2}{2\mu r \ln \frac{r_2}{r_1}} \left[1 - \left(\frac{z}{b} \right)^2 \right]$$

$$Q = 2\pi \int_{-b}^{+b} r\, v_r \, dz = 2\pi \int_{-b}^{+b} \phi(z)\, dz = \frac{4\pi \Delta P\, b^3}{3\mu \ln \dfrac{r_2}{r_1}}$$

An incompressible fluid flows between two porous, parallel flat plates as shown in the figure. An identical fluid is injected at a constant speed V through the bottom plate and simultaneously extracted from the upper plate at the same velocity. Assume the flow to be steady, fully-developed, the pressure gradient in the x-direction is a constant, and neglect body forces.



Determine expressions for the y component of velocity.
Show that the x component of velocity can be expressed as

$$u_x = \frac{h}{\rho V}\left[\frac{\partial p}{\partial x}\right]\left[\left\{\frac{1-\exp\left(\frac{\rho\, V\, y}{\mu}\right)}{1-\exp\left(\frac{\rho\, V\, h}{\mu}\right)}\right\} - \frac{y}{h}\right]$$

The equation of continuity for fully developed, steady flow in x-direction

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

FD ⇒ 0

$$\frac{\partial v}{\partial y} = 0, \quad v = \text{const. for all } y$$

$v = V$ ✓

The x component of the NS equation

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}\right)$$

$$= -\frac{\partial P}{\partial x} + \mu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 x}{\partial z^2}\right] + \rho g_x$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \rho g_x$$

$$\rho V \frac{du}{dy} = \mu \frac{d^2 u}{dy^2} - A$$

The governing equation

$$\rho \, v_y \frac{\partial v_x}{\partial y} = -\frac{\partial P}{\partial x} + \mu \frac{\partial^2 v_x}{\partial y^2}$$

Assumptions:

Constant pressure gradient in the x direction

Boundary Conditions NO SLIP

A thrust bearing as shown in the figure is lubricated by pumping oil at a high pressure of p_0. The angular velocity is equal to ω. Note that under laminar conditions, both V_r and V_θ in the thin gap will be non-zero and p is a function of r only. Neglect convective and body force terms in equations of motion. You may also assume that the pressure at r = R is equal to p_{atm}, whereas the pressure from r = 0 till r = R_0 is equal to p_0.

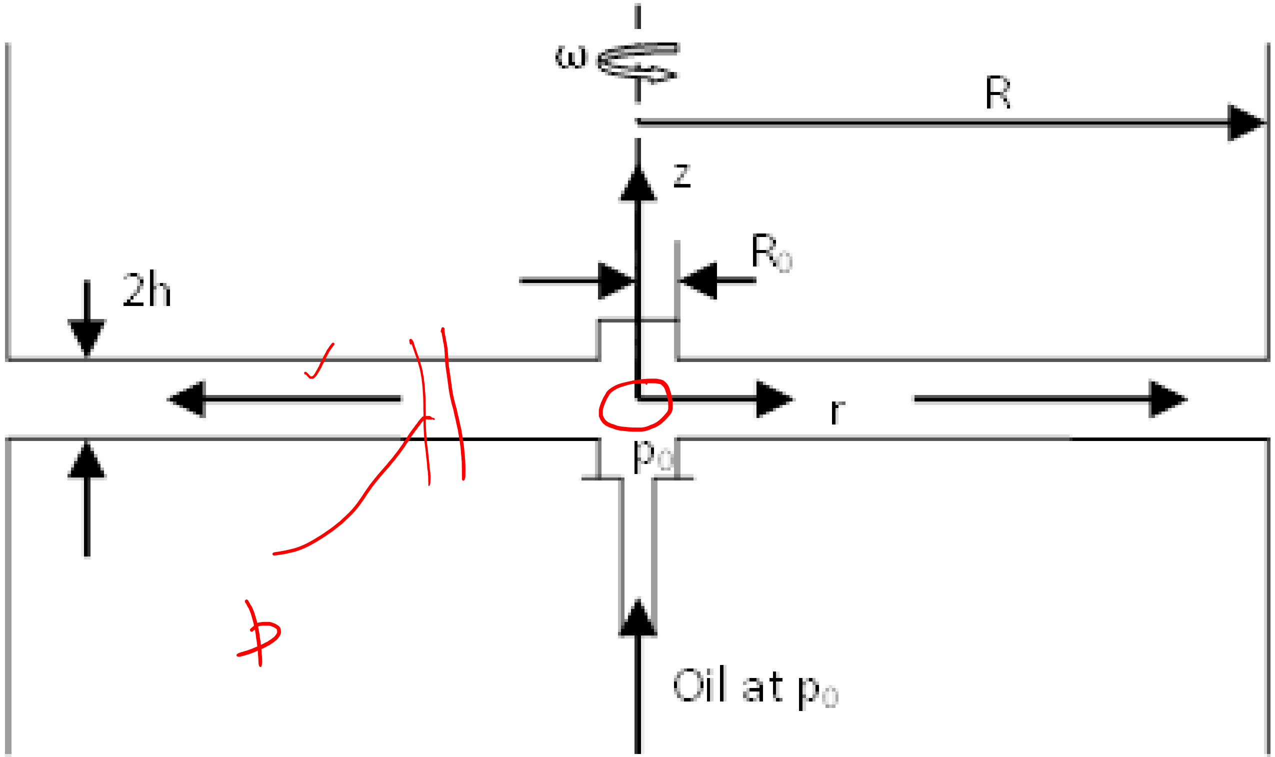
(i) Start with the equation of continuity to obtain the functional form of V_r.

(ii) Show that $V_\theta = \frac{\omega r}{2}\left(1+\frac{z}{h}\right)$ satisfies the θ component of the NS equation.

(iii) Write the r component of NS equation to show that $r\frac{\partial p}{\partial r}$ is a constant.

(iv) Evaluate the pressure distribution using the boundary conditions

(v) Find the vertical load the bearing can support and the flow rate of oil required.



ω
R
z
R_0
2h
r
p_0
Oil at p_0

(i) Start with the equation of continuity to obtain the functional form of V_r.

The Equation of Continuity

Cylindrical coordinates (r, θ, z):

$$\cancel{\frac{\partial \rho}{\partial t}} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \cancel{\frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta)} + \cancel{\frac{\partial}{\partial z}(\rho v_z)} = 0$$

$$v_r = \frac{f(z)}{r}$$

(ii) Show that $V_\theta = \dfrac{\omega r}{2}\left(1+\dfrac{z}{h}\right)$ satisfies the θ component of the NS equation.

$$\cancel{\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right)} = \cancel{-\frac{1}{r}\frac{\partial p}{\partial \theta}} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}} + \frac{\partial^2 v_\theta}{\partial z^2} + \cancel{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}\right] + \cancel{\rho g_\theta}$$

$$\frac{\partial^2 v_\theta}{\partial z^2} = -\cancel{\mu}\,\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right)$$

$\frac{\partial^2 v_\theta}{\partial z^2} = 0$ $\qquad$ $\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) = 0$

IT SATISFIES

(iii) Write the r component of NS equation to show that $r\dfrac{\partial p}{\partial r}$ is a constant

$$\cancel{\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right)} = -\frac{\partial p}{\partial r} + \mu\left[\cancel{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)} + \cancel{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + \cancel{\rho g_r}$$

$$\boxed{0 = -\frac{\partial p}{\partial r} + \mu\frac{\partial^2 v_r}{\partial z^2}}$$

$$r\frac{\partial p}{\partial r} = \mu\frac{\partial^2 r v_r}{\partial z^2}$$

$$\frac{\partial p}{\partial z} = 0 \qquad p = f(r) \text{ ONLY} \qquad P = P(r)$$

$$r\frac{\partial p}{\partial r} = \mu\frac{\partial^2 r v_r}{\partial z^2} = K$$

(iv) Evaluate the pressure distribution using the boundary conditions

$$r\frac{dp}{dr} = K$$

$r = R_o \quad p = p_o$, $\quad r = R = p_{atm}$

$$\boxed{K = \frac{p_o - p_{atm}}{\ln R_o/R}}$$

~~$p = K \ln r + C_1$~~

$p = K \ln r + c_1$

PR. DISTR. KNOWN

(v) Find the vertical load the bearing can support and the flow rate of oil required.

$$\frac{\partial^2 (r v_r)}{\partial z^2} = \frac{K}{\mu}$$

BC

$$z = h, \; v_r = 0, \quad z = -h, \; v_r = 0$$

$$v_r = \frac{K}{2\mu r}\left(z^2 - h^2\right)$$

$$Q = \int_{-h}^{h} v_r \, 2\pi r \, dz = \frac{p_0 - p_{atm}}{\ln R/R_0} \frac{4}{3} \frac{\pi h^3}{3\mu}$$

$$\text{AXIAL LOAD} = \int_{R_0}^{R} (p - p_{atm})\, 2\pi r\, dr$$

$$+ \; (p_o - p_{atm})\, \pi R_o^2$$

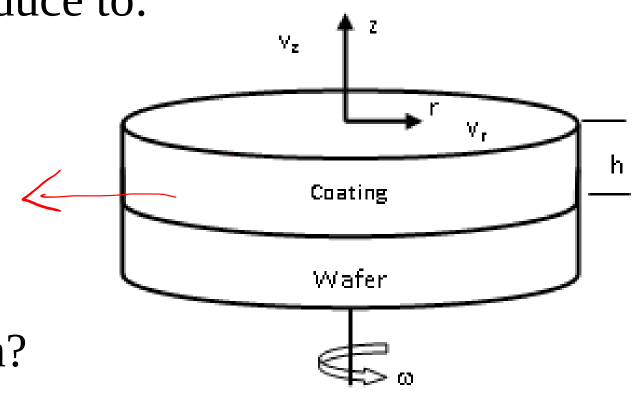
$$K \ln r/R$$

$$\text{Load} = \frac{\pi\, (p_o - p_{atm})}{2 \ln R/R_o} \left(R^2 - R_o^2\right)$$

Consider the spin-coating process used to coat silicon wafers with photoresist. The process is designed to produce a very thin, uniform coating by spinning a viscous, Newtonian, liquid onto a substrate (wafer). The process has angular symmetry, the rotation rate is constant, and since the film is thin, there are no real pressure gradients or fluid accelerations and body forces to speak of. The thin film also moves with the substrate as if it were a rigid body, $v_\theta \neq f(z)$

a) Show that the continuity and momentum equations reduce to:

$$-\rho\frac{v_\theta^2}{r}=\mu\frac{d^2 v_r}{dz^2}\ ; \qquad \frac{1}{r}\frac{\partial}{\partial r}(r v_r)+\frac{\partial v_z}{\partial z}=0$$



b) What are the boundary conditions for this problem?

c) Solve the equations for v_r and v_z.

d) The velocity, v_z, at the film/air interface is just the change in film thickness with time. Use this to obtain a differential equation for h, and integrate this equation to obtain the solution as:

$$\frac{1}{2}\left(\frac{1}{h^2}+\frac{1}{h_o^2}\right)=\left(\frac{2\rho\omega^2}{3\mu}\right)t$$

Where ω is the rate of rotation ($v_\theta = 2\pi r\omega$) and h_0 is the initial height of the film.

In this case, the following conditions will hold:

$$v_\theta = Cons\tan t,\ g_r = g_\theta = g_z = 0,\ \rho, \mu\, constant$$

$$\frac{\partial}{\partial \theta} = 0, \frac{\partial}{\partial t} = 0, \frac{\partial P}{\partial r} = \frac{\partial P}{\partial \theta} = \frac{\partial P}{\partial z} = 0$$

The continuity and momentum equations in cylindrical coordinates become

$$\cancel{\frac{\partial \rho}{\partial t}} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \cancel{\frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta)} + \frac{\partial}{\partial z}(\rho v_z) = 0 \qquad \frac{1}{r}\frac{\partial}{\partial r}\left(r\, v_r\right) + \frac{\partial v_z}{\partial z} = 0$$

Since the fluid moves essentially as a rigid body, $v_r = f(r, z)$. This film is very thin with respect to the radial or angular dimensions so that

$$v_\theta, v_r \gg v_z$$

Thus since the angular velocity is constant and v_z is small, we only need to deal with the momentum equation for v_r

$$\rho\left(\cancel{\frac{\partial v_r}{\partial t}} + v_r \frac{\partial v_r}{\partial r} + \cancel{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) =$$

$$\cancel{-\frac{\partial p}{\partial r}} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}} + \frac{\partial^2 v_r}{\partial z^2} - \cancel{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}\right] + \cancel{\rho g_r}$$

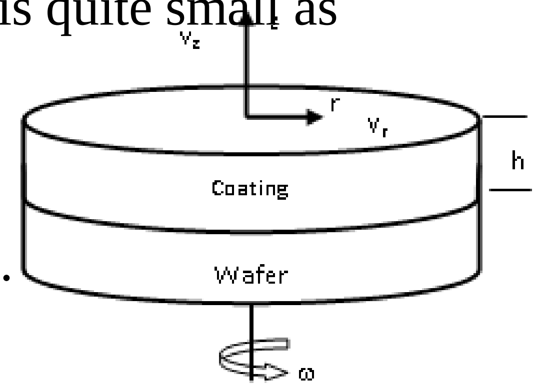
Neglecting all the obvious terms,

$$\rho\left(v_r \frac{\partial v_r}{\partial r} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z}\right) = \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{\partial^2 v_r}{\partial z^2}\right]$$

$$\mu\left[\cancel{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right)} + \frac{\partial^2 v_r}{\partial z^2}\right] = \rho\left(\cancel{v_r\frac{\partial v_r}{\partial r}} - \frac{v_\theta^2}{r} + \cancel{v_z\frac{\partial v_r}{\partial z}}\right)$$

$\frac{\partial}{\partial r}(v_r) \approx O(v_r/R)$ $\quad v_r = f(r, z)$ $\quad \frac{\partial v_r}{\partial z} \approx \frac{v_r}{z}$

Since the film is very thin, v_r changes rapidly with the film thickness, much more rapidly than it changes with the radial position. v_z is quite small as compared to the other components



Thus 1st term on LHS is neglected w.r.t. the second term.

For the same reason, 1st and 3rd terms on the rhs can be neglected.

In one case, v_z is very small and in the second $\frac{\partial v_r}{\partial r}$ is also very small.

v_θ is quite large

Thus

$$\mu \frac{d^2 v_r}{d z^2} + \frac{\rho v_\theta^2}{r} = 0$$

The boundary conditions are

$$z = 0; \; v_r = v_\theta = 0$$

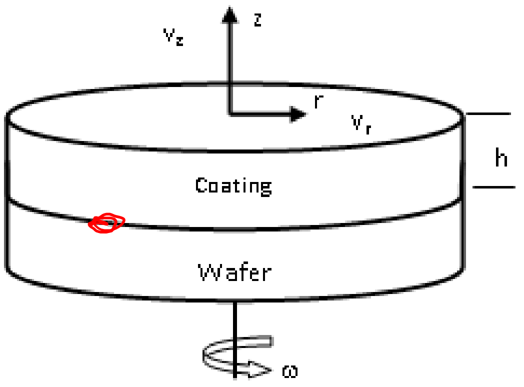
$$z = h; \; \frac{\partial v_r}{\partial z} = 0$$

Noting that $v_\theta = 2\pi r \omega$, the above equation is solved

$$v_r = \frac{4\pi^2\omega^2}{\gamma} r \left(hz - \frac{z^2}{2} \right)$$

Using the continuity equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r v_r \right) + \frac{\partial v_z}{\partial z} = 0$$



$$v_z = \frac{4\pi^2\omega^2}{\gamma} \left(\frac{z^3}{3} - h z^2 \right)$$

BC: $v_z = 0$ at $z = 0$

$$\langle v_z \rangle = \frac{1}{h}\int_0^h v_z\, dz = -\frac{\pi^2 \omega^2 h^3}{\gamma} = \frac{dh}{dt}$$

$$At \quad t = 0 \quad h = h_0$$

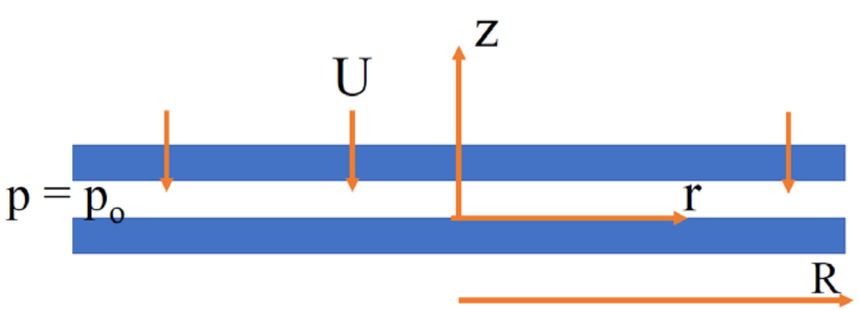
$$\frac{1}{h^2} - \frac{1}{h_0^2} = \frac{2\pi^2\omega^2}{\gamma} t$$

This can also be solved by assuming v_z at z = h to be equal to dh/dt

Problem: One disk approaching another displacing a liquid in between

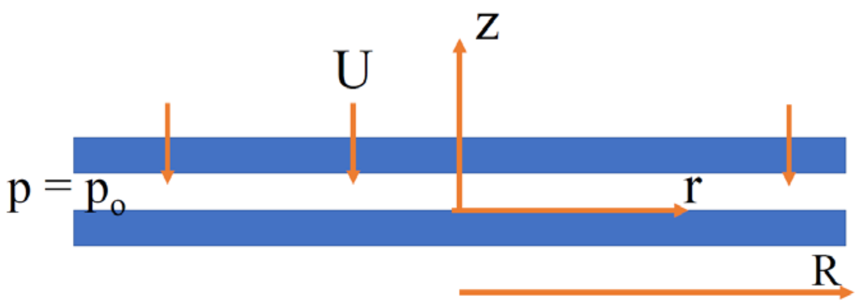
Two parallel, plane circular disks (of radius R) lie one above the other a small distance apart. The space between them is filled with a liquid. The upper disk approaches the lower at a constant (small) velocity U, displacing the liquid. The imposed pressure at $r = R$ is p_o and p_r is not a function of z. Simplify the basic equations (continuity and motion, r component) using an analysis based on your understanding and estimate of the magnitudes of the different terms in the original equations.

Solve the these equations with proper boundary conditions to evaluate a expression of the resistance to motion on the moving disk in terms of the separation between the two plates. Take the origin to be at the centre of the lower (stationary) plate.



Assumptions and special considerations

1. R is very large compared to h
2. Leakage rate is small (Small v_r)
3. $\partial p/\partial z$ is negligible
4. v_z is very small as compared to v_r
5. $\partial v_z/\partial z$ may not be small (as z is very small)
6. $\frac{\partial}{\partial z}\left(\frac{\partial v_z}{\partial z}\right)$ can be appreciable
7. v_r is not small as compared to v_z
8. $\frac{\partial v_r}{\partial r}$ is small compared to $\frac{\partial v_r}{\partial z}$
9. $V_\theta = 0$, no dependence of v_r , v_z



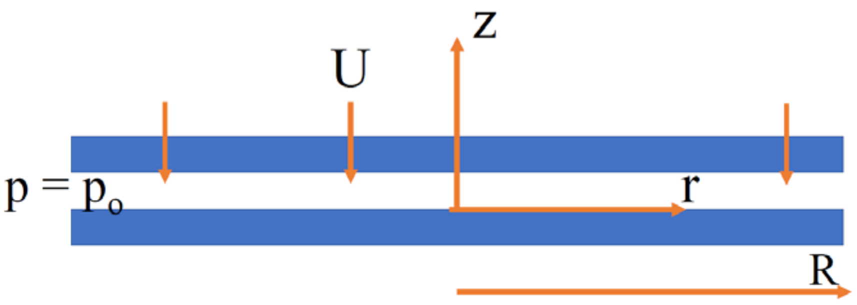
The equation of continuity

$$\cancel{\frac{\partial \rho}{\partial t}}+\frac{1}{r}\frac{\partial}{\partial r}\left(r v_r\right)+\cancel{\frac{1}{r}\frac{\partial}{\partial \theta}\left(\rho v_\theta\right)}+\frac{\partial v_z}{\partial z}=0 \quad (A)$$

$$\frac{v_r}{r}+\frac{\partial v_r}{\partial r}+\frac{\partial v_z}{\partial z}=0$$

The terms in the equation

Relative magnitudes of 1,2 w.r.t. 3



The equation of motion (r component)

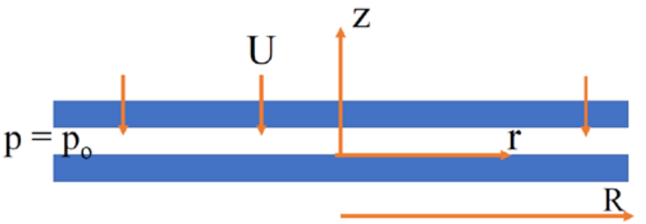
$$\rho\left(\cancel{\frac{\partial v_r}{\partial t}} + \cancel{v_r\frac{\partial v_r}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}} + \cancel{v_z\frac{\partial v_r}{\partial z}} - \cancel{\frac{v_\theta^2}{r}}\right) =$$

$$-\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}} + \frac{\partial^2 v_r}{\partial z^2} - \cancel{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}\right] + \cancel{\rho g_r}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \gamma\left[\cancel{\frac{1}{r}\frac{\partial v_r}{\partial r}} - \cancel{\frac{v_r}{r^2}} + \cancel{\frac{\partial^2 v_r}{\partial r^2}} + \frac{\partial^2 v_r}{\partial z^2}\right]$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \gamma\left[\frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{\partial^2 v_r}{\partial r^2} + \frac{\partial^2 v_r}{\partial z^2}\right]$$

Compare terms to arrive at the following equation



$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \gamma\frac{\partial^2 v_r}{\partial z^2} \quad (B)$$

$$v_r = \frac{1}{2\mu}\frac{\partial P}{\partial r}z^2 + C_1 z + C_2$$

$$BC: \quad z = 0;\ v_r = 0 \quad \Rightarrow \quad C_2 = 0$$

$$z = h;\ v_r = 0 \quad \Rightarrow \quad C_1 = -\frac{1}{2\mu}\frac{dp}{dr}h$$

$$v_r = \frac{1}{2\mu}\frac{\partial P}{\partial r}\left(z-h\right)z \quad (C)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,v_r\right) + \frac{\partial v_z}{\partial z} = 0 \qquad (A)$$

Putting the expression of v_r from (C) to (A) and integrating w.r.t. z and as

$$BC: \quad z = 0;\ v_r = 0 = v_z$$

$$z = h;\ v_r = 0;\ v_z = -U$$

$$U = \frac{1}{r}\frac{d}{dr}\int_0^h r\,v_r\,dz = \frac{1}{r}\frac{d}{dr}\left(r\frac{dp}{dr}\right)\frac{1}{2\mu}\int_0^h z\left(z-h\right)dz = -\frac{h^3}{12\mu}\frac{1}{r}\frac{d}{dr}\left(r\frac{dp}{dr}\right)$$

$$U = -\frac{h^3}{12\mu}\frac{1}{r}\frac{d}{dr}\left(r\frac{dp}{dr}\right)$$

Integrating w.r.t. r

$$-\frac{12\mu}{h^3}\frac{r^2}{2} = r\frac{dp}{dr} + C_1$$

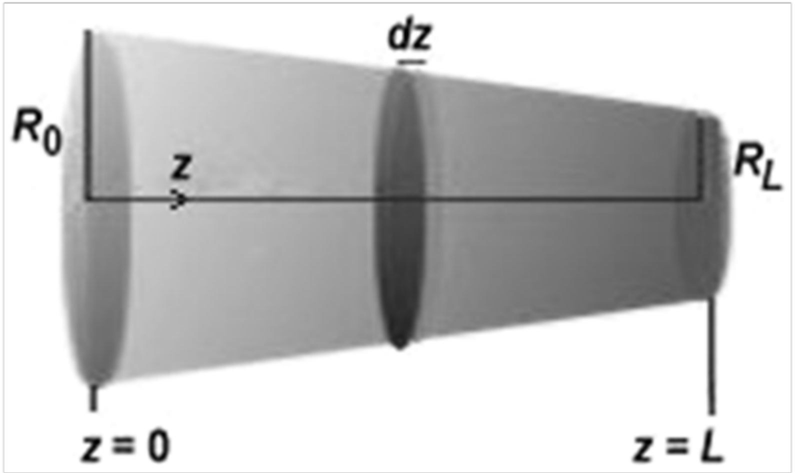
As dp/dr is finite at r = 0 $C_1 = 0$ and p = p_o at r = r

$$p - p_o = \frac{3\mu U}{h^3}\left(R^2 - r^2\right)$$

$$Force,\ F = \int_0^R 2\pi r\,dr\left(p - p_o\right) = 2\pi\int_0^R \frac{3\mu U}{h^3}\left(R^2 r - r^3\right)dr$$

$$Force,\ F = \frac{3\pi\mu U R^4}{2h^3}$$

A fluid (of constant density ρ) is in incompressible, laminar flow through a tube of length L. The radius of the tube of circular cross section changes linearly from R_0 at the tube entrance ($z = 0$) to a slightly smaller value R_L at the tube exit ($z = L$).



Using the lubrication approximation, determine the mass flow rate vs. pressure drop (ΔP) relationship for a Newtonian fluid (of constant viscosity μ).

The approximation where a flow between non-parallel surfaces is treated locally as a flow between parallel surfaces is commonly called the lubrication approximation because it is often employed in the theory of lubrication.

The mass flow rate versus the pressure drop (W versus ΔP) relationship for a Newtonian fluid in a circular tube is given by (where $\Delta P = P_0 - P_L$)

$$W = \frac{\pi \Delta P R^4 \rho}{8 \mu L} \quad or \quad \frac{\Delta P}{L} = \frac{8 \mu W}{\pi \rho \underline{\underline{R}}^4} \quad — 1$$

$$-\frac{dP}{dz} = \frac{8 \mu W}{\pi \rho} \frac{1}{[R(z)]^4} \quad — 2$$

$\underline{R(z)}$

$$R(z) = R_0 + (R_L - R_0) \frac{z}{L}$$

$$\frac{dR}{dz} = \frac{R_L - R_0}{L} \quad — 3$$

$\underline{P}$ as a f^n of R

$$-dP = \frac{8\mu W}{\rho \pi} \frac{L}{R_L - R_0} \frac{dR}{R^4}$$

INTEGRATE $\quad z = 0, \quad z = L$

$(R_0) \qquad (R_L)$

$$\int_{P_0}^{P_L} (-dP) = \frac{8\mu W}{\rho \cancel{4} \pi} \frac{L}{R - R_0} \int_{R_0}^{R_L} \frac{dR}{R^4}$$

$$W = \frac{\cancel{\pi \Delta P R}_0^4 \, \rho}{8 \mu L} \left[\frac{3(\lambda - 1)}{1 - \lambda^{-3}} \right]$$

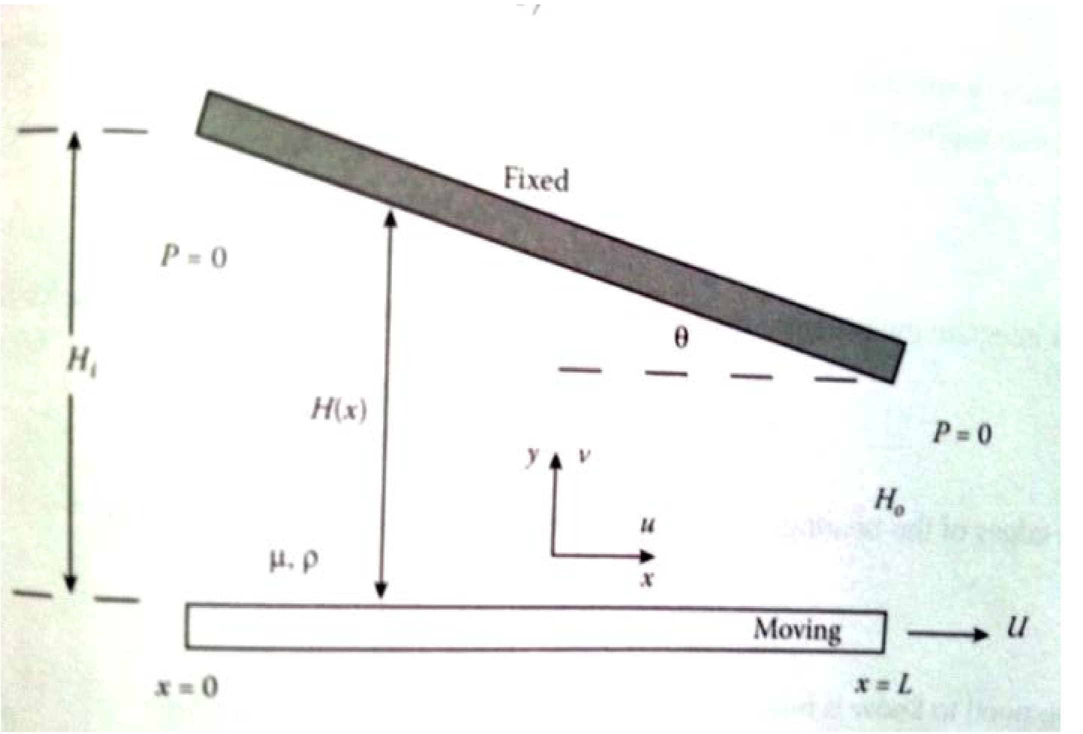
$$\lambda = R_L / R_0$$

$$W = \frac{\pi \, \Delta P \, R_o^4 \, \rho}{8 \, \mu \, L} \left[\frac{3(\lambda - 1)}{1 - \lambda^{-3}} \right] \quad ; \quad \lambda = R_L \big/ R_o$$

Taper correction to Hagen-Poiseuille equation

Lubrication flows are characterized by incompressible fluids to thin gaps e.g., the layer of water between the ice skate and the ice, the oil that lubricates the moving parts of an internal combustion engine etc. The fluid flows at relatively small velocities and so the inertial terms in the Navier-Stokes equations are insignificant compared to the diffusive terms. Since the gaps are so thin, lubrication flows can be treated as two-dimensional. Finally, if the gap is very thin, there is little effect of gravity.

Consider the lubrication flow occurring between the fixed and moving components of a bearing as shown in the figure in presence of a pressure gradient with one of the plate moving with a constant velocity Uo. Assume that there is a constant volumetric flow rate V of the lubricant in the system as a result of this motion of the plate and the applied pressure gradient.



Solve the governing equations with appropriate boundary conditions to obtain the x-component of the velocity, u, as a function of the pressure gradient and H(x).

Use relevant conditions to obtain an expression for dP/dx and local pressure P(x) in terms of H(x), Uo and V.

Finally obtain expressions for the (i) volumetric flow rate V and (ii) the load the bearing can support in terms of the system parameters (Hi, Ho, θ, Uo) and μ.

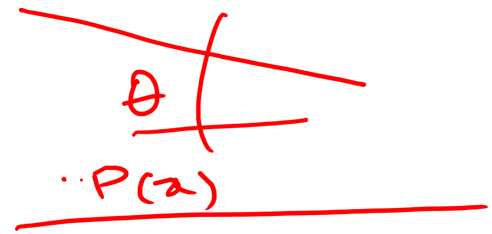
- IDENTIFY $P = f(x)$ only
- WRITE NS EQN, CANCEL LHS
- $$\frac{dP}{dx} = \mu\left(\cancel{\frac{d^2u}{dx^2}} + \underbrace{\frac{d^2u}{dy^2}}_{<<}\right) \checkmark$$
- INTEGRATE

FIND $u = f\left(U_0, \frac{dp}{dx}, H(x), y\right]$

- $$V = \int_0^{H(x)} \underline{\underline{u}}\, dy = f\left(U_0, H(x), \frac{dp}{dx}, \mu\right)$$

- $P(x)$
- H VARIES LINEARLY WITH x

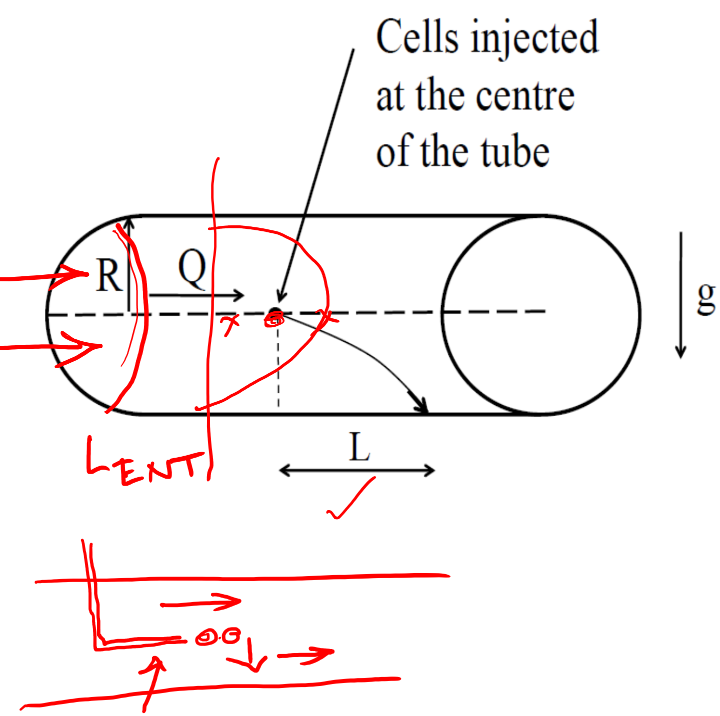
$$\text{LOAD} = \int_0^L P(x)\cos\theta \, dx$$



$$\text{LOAD} = 6\mu U_0 \left(\frac{\cos\theta}{\tan^2\theta}\right)\left[\ln\left(\frac{H_i}{H_o}\right) - 2\left(\frac{H_i - H_o}{H_i + H_o}\right)\right]$$

A cell separation (fractionation) system is based on cell density. Cells are injected at the centre of a tube of radius R, and are carried by fluid flowing at a flow rate of Q. Dense cells fall quickly under the action of gravity, adhere to the tube wall and hence do not pass out of the tube. Assume that the concentration of the cell is low enough that the laminar flow in the tube is not perturbed by the presence of the cells. Let the cells be spherical, with radius ‘a’ and let them have a density $\rho + \Delta\rho$, where ρ is the density of the flowing fluid.

Find the axial distance (L) that a cell travels before it hits the bottom wall. You need to use the fact that, in fully developed flow in the presence of gravity, the pressure distribution in the vertical distance is hydrostatic and the axial velocity profile is the familiar parabolic shape. You may assume that the cell is spherical, that it reaches its terminal (falling) velocity nearly immediately after injection into the tube, and that it gets carried axially at the local fluid velocity in the tube. State any other assumptions you make.



We need to find the axial distance L that the cell travels before it hits the bottom wall.

The expressions of the forces acting on the spherical cell are:

Gravitational force

$$F_g = m_{cell}\, g = \frac{4}{3}\pi a^3 \left(\rho + \Delta\rho\right) g$$

Buoyancy force

$$F_b = m_{fluid}\, g = \frac{4}{3}\pi a^3 \rho\, g$$

Drag force

$$F_d = 6\pi \mu a v_r$$

The cell is moving downward with a velocity v_r in the r direction

$$\frac{4}{3}\pi a^3 \left(\rho + \Delta\rho\right) g = \frac{4}{3}\pi a^3 \rho\, g + 6\pi \mu a v_r$$

$$v_r = \frac{2 a^2 g \rho}{9 \mu}$$

Find the radial location as a function of time as

$$r(t) = \int_0^t v_r \, dr = \frac{2 a^2 g \Delta\rho t}{9 \mu}$$

The time T, when the cell will hit the wall

$$T = \frac{R}{v_r} = \frac{9 \mu R}{2 a^2 g \Delta\rho}$$

$$v_x = \frac{\Delta P R^2}{4 \mu L}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

$$\langle v_x \rangle = \frac{\Delta P R^2}{8 \mu L}; \qquad Q = \pi R^2 \langle v_x \rangle = \frac{\Delta P \pi R^4}{8 \mu L}$$

$$\frac{2 Q}{\pi R^2} = \frac{\Delta P R^2}{4 \mu L}$$

$$v_x = \frac{2Q}{\pi R^2}\left[1-\left(\frac{r(t)}{R}\right)^2\right]$$

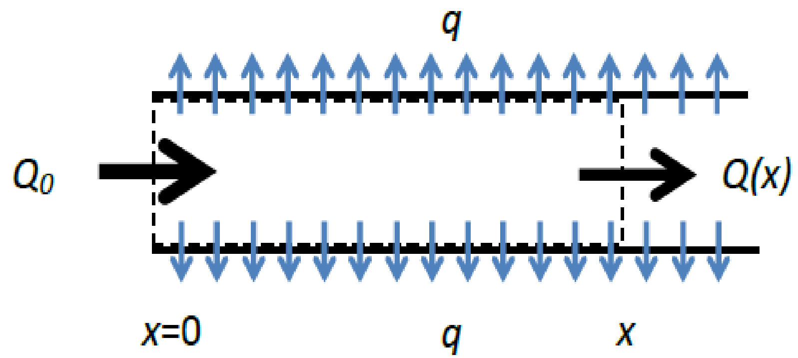
$$L = \int_0^T v_x\, dt = \frac{2Q}{\pi R^2}\int_0^T\left[1-\frac{[r(t)]^2}{R^2}\right]dt$$

$$L = \frac{2Q}{\pi R^2}\left[T-\left(\frac{2a^2 g\,\Delta\rho}{9\mu R^2}\right)^2\frac{T^3}{3}\right]$$

$$As \quad T = \frac{9\mu R}{2a^2 g\,\rho}$$

$$L = \frac{6\mu Q}{\pi R a^2 g\,\Delta\rho}$$

Epithelium is one basic type of animal tissue, which lines the cavities and surfaces of structures throughout the body. Epithelial layers contain no blood vessels, so they must receive nourishment via diffusion of substances from the underlying connective tissue, through the basement membrane.



An apparatus has been built for testing the effect of various drugs on the rate at which an epithelium can pump fluid from its luminal side (the side facing the fluid) to its basal side (which lies on the channel wall). The cells line the top and bottom surface of a flow channel that has a separation of h (from top plate to bottom plate; ignore the thickness of cells), a length L, and a depth into the page of W. Each of these walls is porous so that any fluid pumped by the cells can leave the channel. Let each cell layer (top and bottom) pump fluid at a rate of q per unit area of the channel walls (q has thus units of length/time). The height of the channel is much less than its length ($h \ll L$).

Fluid enters the channel at the left at a flow rate Q_0 and a gauge pressure of P_0. Because of the pumping action of the cells, the flow rate through the channel decreases as a function of x, the distance from the beginning of the channel. To determine the rate at which the cells are pumping fluid out of channel, the channel is instrumented with pressure transducers that can measure P(x). We would like to use this information to find the rate at which the cells pump fluid. The fluid in the channel has a density of ρ and a viscosity of μ. The flow is dominated by viscous effects and is steady.

(a) Find the pressure distribution, P(x) , in the flow channel if q=0.

(b) Find the pressure distribution P(x) in the channel for $q \neq 0$.

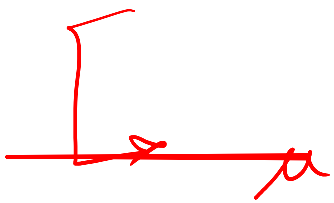
(c) Given that $P(x=L)=P_e$, find q .

(d) Find the criterion necessary for the assumption that viscous flow dominates to be valid.

All answers must be given in terms of the known quantities e.g., x,L,W,h,Q_0 , P_0, P_e, ρ and μ (not all of these parameters need necessarily be used).

Find the pressure distribution, P(x) , in the flow channel if q=0

$q=0 \Rightarrow dp/dx = \text{const.}$, $q \neq 0 \quad \dfrac{dp}{dx} \neq \text{const.}$



$$\frac{d^2 v_x}{dy^2} - \frac{dp}{dx} = 0$$

$\underline{BCs}$ $y=0, h \quad v_x = 0$

$v_x = ?$

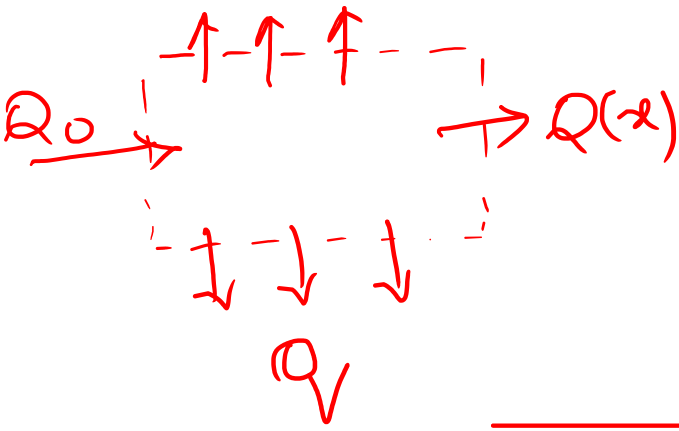
$v_x \longrightarrow \langle v_x \rangle \quad f\left[\left(\dfrac{dp}{dx}\right), h, \mu\right]$

$Q_0 = wh\langle v_x \rangle$

$$\frac{dp}{dx} = f^n(Q_0, \mu, w, h)$$

$$\cancel{P(x)} \quad P(x) = P_0 - \frac{12\mu Q_0}{wh^3}\, x$$

Find the pressure distribution P(x) in the channel for q≠0.



$$Q(x) = Q_0 - \cancel{2}qwx$$

$$\cancel{\frac{dp}{dx} = \mu \frac{d}{}}$$

$$\frac{dp}{dx} = \mu \left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right]$$

$$\left[\frac{\partial^2 v_x}{\partial x^2} \right] << \left[\frac{\partial^2 v_x}{\partial y^2} \right]$$

ASSUM. q IS SMALL

$$\frac{dp}{dx} = -\frac{12\mu Q(x)}{wh^3}$$

$$p = p_0 - \frac{12\mu x}{wh^3}\left[Q_0 - qwx \right]$$

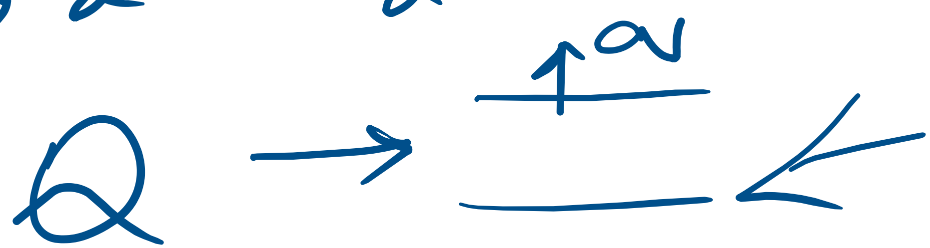
Given that P(x=L)=Pe , find q .

$$P_e = P_o - \frac{12\mu L}{wh^3}\left[Q_o - qwx\right]$$

Find the criterion necessary for the assumption that viscous flow dominates to be valid.

$$\frac{\rho\left(v_x \dfrac{\partial v_x}{\partial x} + v_y \dfrac{\partial v_x}{\partial y}\right)}{\mu\left(\dfrac{\partial^2 v_x}{\partial x^2} + \dfrac{\partial^2 v_x}{\partial y^2}\right)} \ll 1$$

$$\frac{\partial v_x}{\partial x} \approx \frac{d}{dx}\left(Q/A\right) = \frac{1}{wh}\frac{d}{dx}\left(Q_0 - 2qwx\right)$$



$v_y \approx a$ ✓

$$\frac{\rho\left(v_x \cdot \frac{a}{h} + a\frac{v_x}{h}\right)}{\mu\, v_x / h^2} \ll 1$$

$$\frac{\rho a h}{\mu} \ll 1$$

The Equation of Continuity

$$[\partial\rho/\partial t + (\nabla \cdot \rho\mathbf{v}) = 0]$$

Cartesian coordinates (x, y, z):

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0 \qquad \text{(B.4-1)}$$

Cylindrical coordinates (r, θ, z):

$$\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0 \qquad \text{(B.4-2)}$$

Spherical coordinates (r, θ, ϕ):

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\rho v_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(\rho v_\phi) = 0 \qquad \text{(B.4-3)}$$

[a] When the fluid is assumed to have constant mass density ρ, the equation simplifies to $(\nabla \cdot \mathbf{v}) = 0$.

The Equation of Motion for a Newtonian Fluid with constant μ, ρ

The Navier Stokes Equation

Cartesian coordinates (x, y, z):

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right] + \rho g_x$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right] + \rho g_y$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$

Cylindrical coordinates (r, θ, z):

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right] + \rho g_\theta$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$

Spherical coordinates (r, θ, ϕ):

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right) = -\frac{\partial p}{\partial r}$$

$$+ \mu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial v_r}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 v_r}{\partial \phi^2}\right] + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta - v_\phi^2 \cot\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta}$$

$$+ \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v_\theta \sin\theta)\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 v_\theta}{\partial \phi^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{2 \cot\theta}{r^2 \sin\theta}\frac{\partial v_\phi}{\partial \phi}\right] + \rho g_\theta$$

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r + v_\theta v_\phi \cot\theta}{r}\right) = -\frac{1}{r \sin\theta}\frac{\partial p}{\partial \phi}$$

$$+ \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v_\phi \sin\theta)\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{2}{r^2 \sin\theta}\frac{\partial v_r}{\partial \phi} + \frac{2 \cot\theta}{r^2 \sin\theta}\frac{\partial v_\theta}{\partial \phi}\right] + \rho g_\theta$$