Heat Transfer Fundamentals

$$Rate\,(Heat\ IN) - (Heat\ OUT) \pm Rate\ of\ \ Generation = Rate\ of\ \ Accumulation$$

Fourier's Law $\quad q_x = -k\,A\,\dfrac{dT}{dx}$ $\quad$ Newton's Law of Cooling $q = h\,A\,(T_s - T_\infty)$

$$Cartesian\ coordinates\,(x, y.z)$$

$$\rho\,\hat{C}_p\left(\frac{\partial T}{\partial t} + v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y} + v_z\frac{\partial T}{\partial z}\right) = k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] + \mu\phi_v + \dot{Q}$$

z

For a system experiencing only conduction and no viscous heat dissipation

$$\rho \widehat{c_p} \left(\frac{\partial T}{\partial t} \right) = k \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right] + \dot{Q}$$

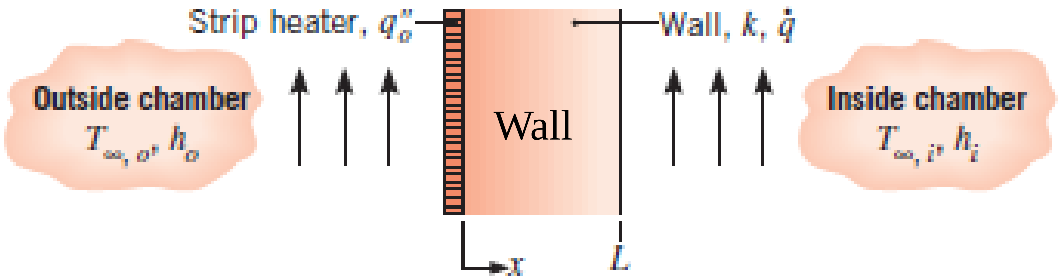
Possible boundary conditions:

1. Temperature specified at a point T (x) is known
2. Equality of temperature at the interface $Ts_1 = Ts_2$
3. Perfect insulation and/or adiabatic surface dT/dx = 0 at specified location
4. Conductive flux to the solid-fluid interface is equal to the convective flux away from the interface, $-k \left. \frac{dT}{dx} \right|_{x = L} = h \left(T_s - T_\infty \right)$

Wall Heat Generation with Strip Heater Experiencing Convection at the Boundaries

The air *inside* a chamber at $T_{\infty,i}$ = 50C is heated convectively with *hi* = 20 W/m^2 . K by a 200-mm-thick wall having a thermal conductivity of 4 W/m.K and a uniform heat generation of 1000 W/m^3.

To prevent any heat generated within the wall from being lost to the *outside* of the chamber at $T_{\infty,o}$ = 25C with *ho* = 5 W/m^2 .K, a very thin electrical strip heater is placed on the outer wall to provide a uniform heat flux, q''_0

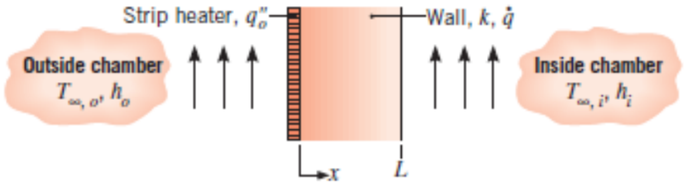


(a) Sketch the T-distribution in the wall on T - x coordinates for the condition where no heat generated within the wall is lost to the *outside* of the chamber.

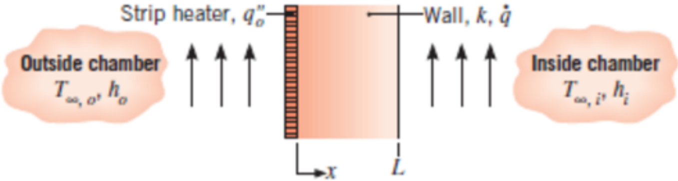
(b) What are the temperatures, $T(0)$ and $T(L)$, for the conditions of part (a)?

(c) Determine the value of q''_0 that must be supplied by the strip heater so that all heat generated within the wall is transferred to the *inside* of the chamber.

(d) If the heat generation in the wall were switched off while the heat flux to the strip heater remained constant, what would be the steady-state temperature, $T(0)$, of the outer wall surface?

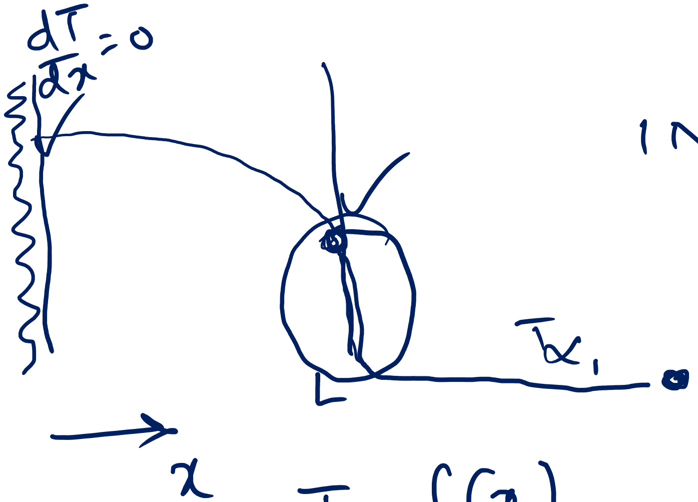


Sketch the T-distribution in the wall on T - x coordinates for the condition where no heat generated within the wall is lost to the *outside* of the chamber.



$T_{\infty 2}$

OUT



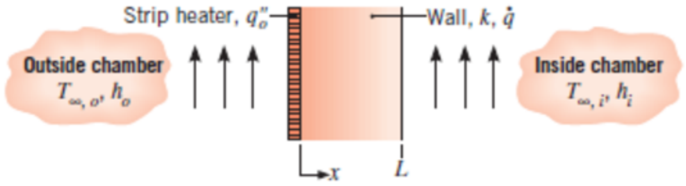
IN

$$k\frac{d^2T}{dx^2} + \dot{q} = 0$$

$$T = f(c_1 x^2 + c_2)$$

$$T = f(x) \rightarrow T_0 , T_L$$

What are the temperatures, $T(0)$ and $T(L)$, for the conditions of part (a) i.e. for the condition where no heat generated within the wall is lost to the *outside* of the chamber?



Cartesian coordinates $(x, y.z)$

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = k \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right] + \mu \phi_v + \dot{Q}$$

$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0$$

$$T = -\frac{\dot{q}}{2k} x^2 + C_1 x + C_2$$

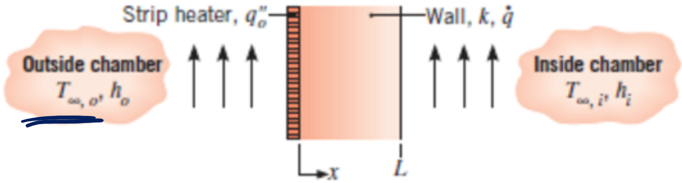
$x = 0$ $\quad \frac{dT}{dx} = 0$, 1 $\quad x = L$,

$T_L = 60^\circ$ $\quad T_{x=0} = 65$

COND. = CONV.

$$-k \left.\frac{dT}{dx}\right|_{x=L} = h_i (T_L - T_{\infty,i})$$

Determine the value of q''_0 that must be supplied by the strip heater so that all heat generated within the wall is transferred to the *inside* of the chamber.



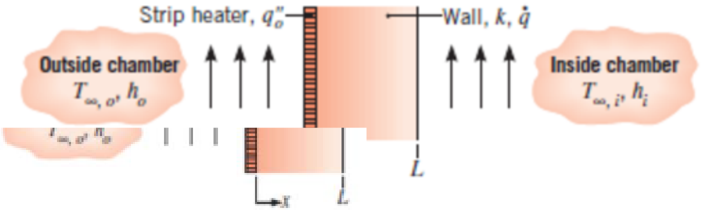
$h_o(65 - 25)$ ✓ $= q''_0$

$$q''_0 = h_o(T_L - T_{\infty o})$$

$T_{\infty o}$

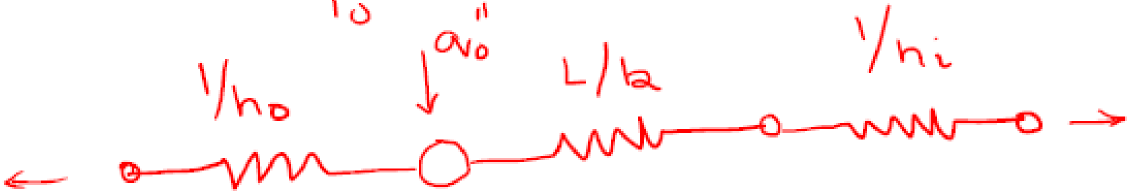
$T_L = T_s$

If the heat generation in the wall were switched off while the heat flux to the strip heater remained constant, what would be the steady-state temperature, $T(0)$, of the outer wall surface?



q_o''

T_o



$T_{x=0} = 55C$

$\frac{1}{hA}$

$\frac{1}{h}$

L/kA

L/k

Problem 2 – Heat Generation in a Plane Wall

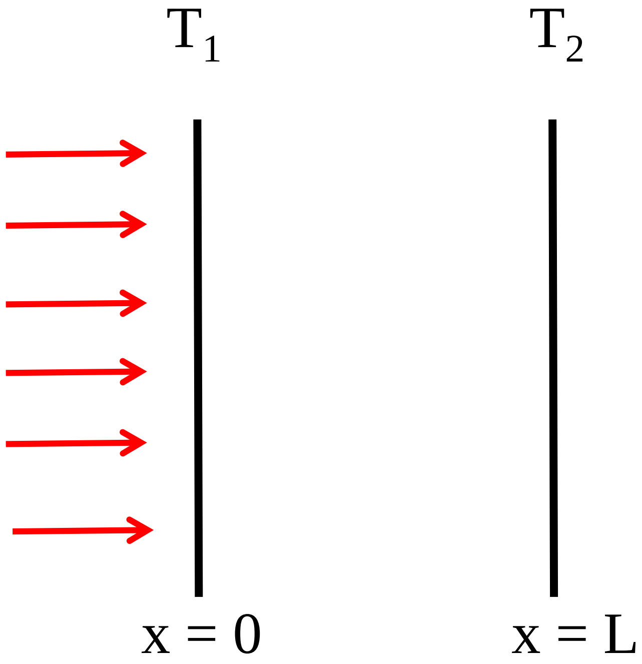
Consider a plane wall of thickness L, which is to act as shielding for a nuclear reactor. The inner surface receives gamma radiation that is partially absorbed within the shielding and has the effect of an internally distributed heat source. In particular heat is generated per unit volume within the shielding according to the relation

$$q\,(x) = \; q_o''\,\alpha\,e^{-\,\alpha\,x}$$

Where q_o'' is the incident radiation flux and α is the absorption coefficient of the shielding.

(a) If the inner ($x = 0$) and outer ($x = L$) surfaces of the shielding are maintained at constant temperatures of T_1 and T_2 respectively, find the temperature distribution in the shielding?

(b) Find an expression for the x location in the shield at which the temperature is a maximum.



Distributed heat source

$$q\,(x) = q_o''\,\alpha\,e^{-\alpha x}$$

q (x) = heat generation per unit volume

1 – Dimensional SS conduction

Governing equation

$$0 = k\left[\frac{d^2T}{dx^2}\right] + \dot{q}$$

Cartesian coordinates $(x, y.z)$

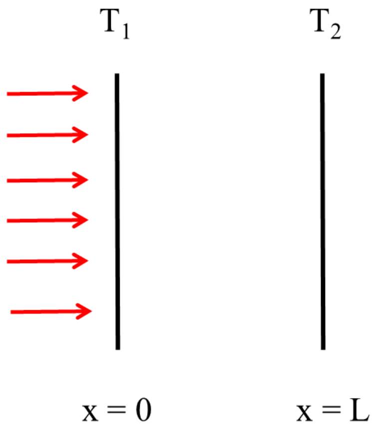
$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = k \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right] + \mu \phi_v + \dot{Q}$$

Gov. Eqn, $$0 = k\left[\frac{d^2T}{dx^2}\right] + \dot{q}$$

$$k\,\frac{d^2T}{dx^2} = -\,q_o''\,\alpha\,e^{-\alpha x} \quad \Longrightarrow \quad \frac{dT}{dx} = \frac{A}{-\alpha}e^{-\alpha x} + C_1 \qquad A \equiv -\,\frac{q''_o\,\alpha}{k}$$

Therefore

$$T = -\frac{q''_o}{\alpha\,k}e^{-\alpha x} + C_1\,x + C_2$$

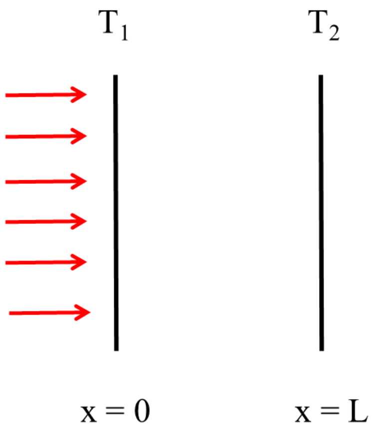


Using BC 1 at x = 0, $T = T_1$ $\Longrightarrow$ $$C_2 = T_1 + \frac{q''_o}{\alpha\,k}$$

$$T - T_1 = \frac{q''_o}{\alpha\, k}\left(1 - e^{-\alpha x}\right) + C_1 x$$

Using BC 2 at x = L $T = T_2$

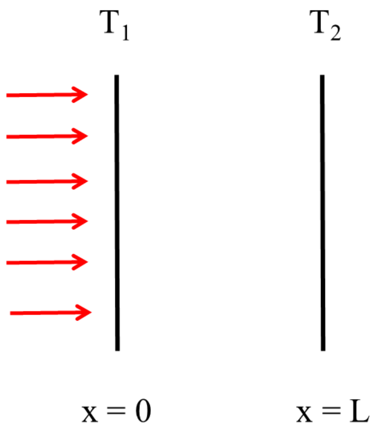
$$C_1 = \frac{1}{L}\left[(T_2 - T_1) - \frac{q''_o}{\alpha\, k}\left(1 - e^{-\alpha L}\right)\right]$$



Therefore

$$T = T_1 - (T_1 - T_2)\frac{x}{L} + \frac{q''_o}{\alpha\, k}\left(1 - \frac{x}{L}\right) + \frac{q''_o}{\alpha\, k}\left(\frac{x}{L}\, e^{-\alpha L} - e^{-\alpha x}\right)$$

Location at which T is maximum



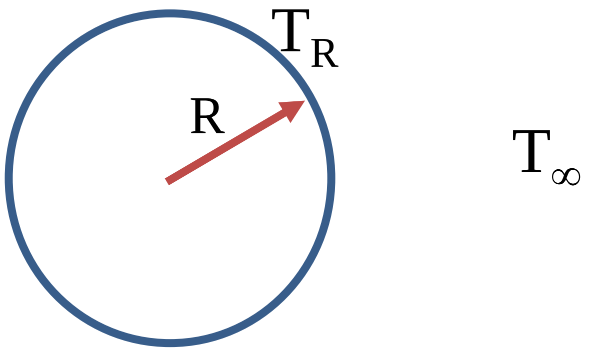
$$\frac{dT}{dx} = (T_2 - T_1)\frac{1}{L} + \frac{q''_o}{k} e^{-\alpha x} - \frac{q''_o}{\alpha kL}(1 - e^{-\alpha L})$$

$$\frac{dT}{dx} = 0 \; gives$$

$$e^{-\alpha x} = (T_2 - T_1)\frac{k}{Lq''_o} + \frac{q''_o}{\alpha kL}(1 - e^{-\alpha L})$$

Problem 3 - Heater Sphere in Quiescent Fluid

A heated sphere of radius R is suspended in a large, motionless body of fluid. It is desired to study the heat conduction in the fluid surrounding the sphere in the absence of convection. The surface temperature of the sphere is maintained at T_R and at a point far from the sphere, the temperature is T_∞ . Obtain the functional relationship of the temperature of the surrounding fluid as a function of r, the distance from the centre of the sphere. The thermal conductivity of the fluid, k, is constant. Using the above relation and the Newton's law of cooling evaluate an expression of Nusselt number for this case.



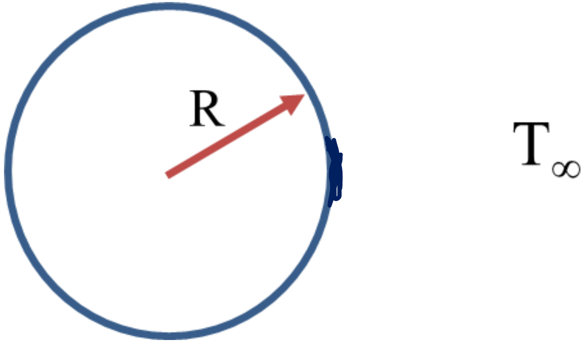
Spherical coordinates (r,θ,ϕ)

$$\rho \hat{C}_p\left(\frac{\partial T}{\partial t}+v_r\frac{\partial T}{\partial r}+\frac{v_\theta}{r}\frac{\partial T}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial T}{\partial \phi}\right)=k\left[\underline{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}\right]+\mu\phi_v+\dot{Q}$$

Use the above relation and the Newton's law of cooling to evaluate an expression of Nusselt number for this case.

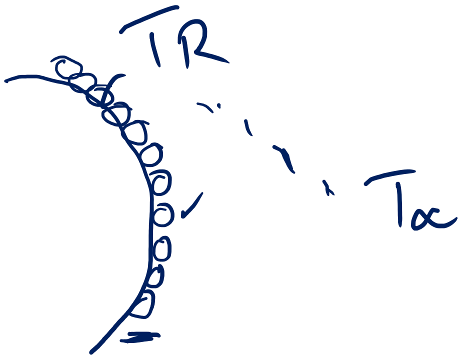
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{dT}{dr}\right)=0$$

$$T=-\frac{C_1}{r}+C_2$$



$r = R \quad T = T_R \qquad T = T_\infty$ as $r \to \infty$

$$\boxed{\frac{T - T_\infty}{T_R - T_\infty} = \frac{R}{r}}$$



$\underline{\underline{\dfrac{hD}{k}}}$

$$-k \left.\frac{dT}{dr}\right|_{r=R} = \underline{\underline{h}}\,(T_R - T_\infty)$$

$$\frac{hD}{k} = \underline{\underline{2}}$$

$$\frac{T-T_\infty}{T_R-T_\infty} = \frac{R}{r}$$

AT THE INTERFACE

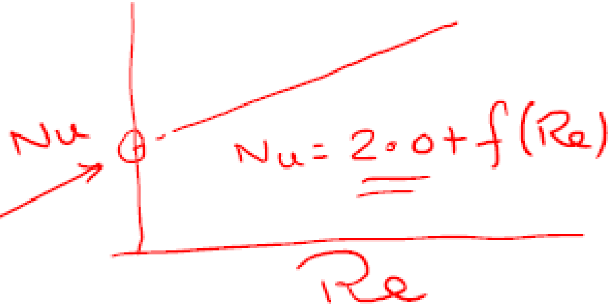
$$\boxed{Nu = 2}$$

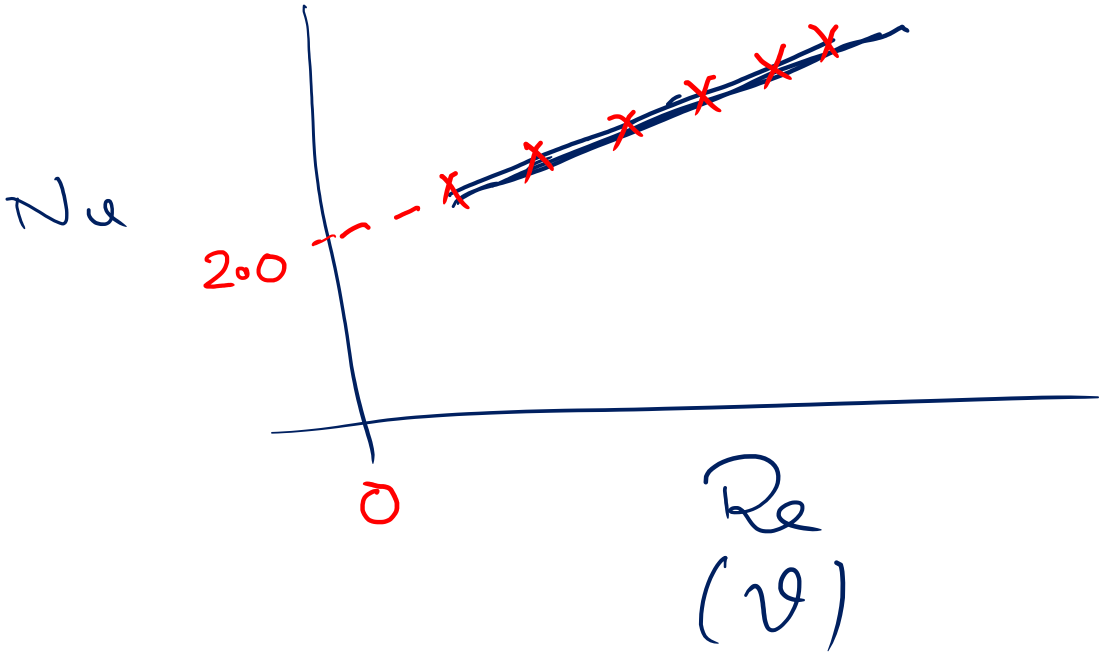
$$q_{cond.} = q_{conv.}$$

$$-k\left.\frac{dT}{dr}\right|_{r=R} = h\left(T_R - T_\infty\right)$$

$$\frac{hR}{k} = 1$$

$$\boxed{Nu = \frac{hD}{k} = 2}$$





Nu
2.0
0
Re
(v)

Transient Conduction

Lumped Capacitance Method

Major assumption: Temperature is space-wise isothermal inside the solid at any instant of time

Valid if the conduction resistance inside the solid is considerably less than the convective resistance at the periphery

Biot Number (= hL/k) << 1

$$\dot{In} - \dot{Out} + \dot{Generation} = \dot{Stored}$$

For a hot solid being cooled by a convective process, e.g. by putting it in a cold liquid and no heat generation

$$-\dot{E}_{OUT} = \dot{E}_{STORED}$$

$$-h\,A_s\,(T - T_\infty) = \rho\,V\,C\,\frac{dT}{dt}$$

$$\frac{\rho\,V\,C}{h\,A_s}\,\frac{d\theta}{dt} = -\,\theta$$

$$\frac{\rho\, V\, C}{h\, A_s} \ln \frac{\theta_i}{\theta} = \mathrm{t}$$

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp\left[-\left(\frac{h\, A_s}{\rho\, V\, C}\right) t\right]$$

Alternative to LC Method → Analytical, Numerical, Graphical (Heisler Charts)

Problem on Transient Conduction

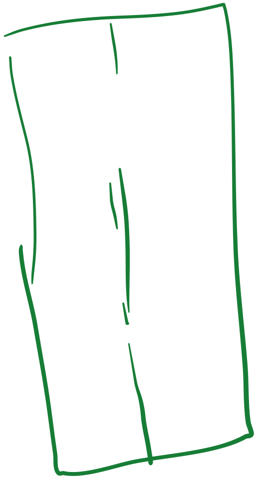
A long wire of diameter D = 1 mm is submerged in an oil bath of temperature T_∞ = 25C. The wire has an electrical resistance per unit length of $Re^{/}$ = 0.01 ohm/m. A current of I = 100 A flows through the wire and the convection coefficient is h = 500 $W/m^2.K$.

What is the steady state temperature of the wire?

From the time the current is applied, how long does it take for the wire to reach a temperature within 1^0 C of the steady state value?

For the wire are ρ = 8000 kg/m^3, c = 500 J/kg.K and k = 20 W/m.K

L



The first step is to calculate Biot number.

$$Bi \equiv \frac{h(r_o)}{k} = \frac{500 \frac{W}{m^2 K} \times 5\times10^{-4}\, m}{20 \ W/mK} = 0.012 < 0.1$$

As $Bi < 0.1$, LC method is applicable

($T \neq f(r)$, $T = f(t)$ only)

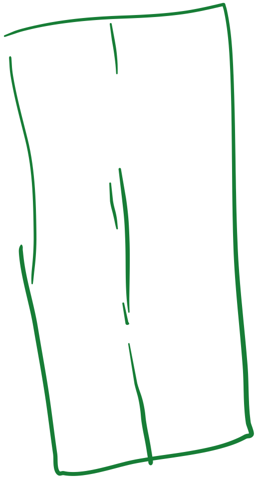
$L/2$

$$\boldsymbol{In - Out + Generation = Stored}$$

$$\frac{hL/2}{k}$$

$$In - Out + Generation = Stored$$

L



$L/2$

$\frac{hL/2}{h2}$

At steady state

$$-\dot{OUT} + \dot{GEN} = 0$$

$$\pi(hDL)(T - T_\infty) = I^2 R_e' L$$

R_e' = Resistance/length

$$T = T_\infty + \frac{I^2 R_e'}{\pi D h}$$

$$= 25 + \frac{(100)^2 \times 0.01}{\pi \times 0.001 \times 500} = 88.7\ C$$

∴ Steady state temperature = 88.7 C

For the transient case,

$$\overset{0}{IN} - \overset{0}{OUT} + \overset{0}{GEN} = \overset{0}{STORED}$$

$$0 - h\pi D(T - T_\infty) + I^2 R_e' = \rho c \frac{\pi D^2}{4} \frac{dT}{dt}$$

$$\frac{dT}{dt} = \frac{I^2 R_e'}{\rho c\, \pi D^2/4} - \frac{4h}{\rho c D}(T - T_\infty) \quad \dots \text{(1)}$$

$$\frac{d\theta}{dt} + A\theta = B \quad \left[\begin{array}{l} \theta \equiv T - T_\infty \\ A = \dfrac{4h}{\rho c D}, \; B = \dfrac{I^2 R_e'}{\rho c\, \pi D^2/4} \end{array}\right]$$

$$\frac{d}{dt}\left(\theta e^{At}\right) = B e^{At}$$

$$\theta e^{At} = \frac{B}{A} e^{At} + c$$

$$\theta = T - T_\infty = \frac{B}{A} + c e^{-At}$$

$$T - T_\infty - B/A = c e^{-At} \quad \text{— (2)}$$

At $t=0$, $T = T_i$

$$T_i - T_\infty - \frac{B}{A} = c \quad \text{—— (3)}$$

(2) ÷ (3)

$$\frac{T - T_\infty - B/A}{T_i - T_\infty - B/A} = \exp\left(-\frac{4h}{\rho C D}\, t\right).$$

$T = 87.7\,C$ (within 1C of ss temp.), $B/A = 63.7$

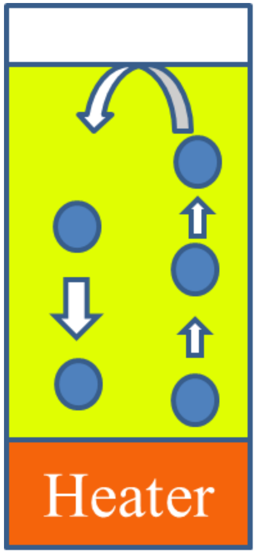
$h = 500\ W/m^2K$, $C = 500\ J/kgK$, $D = 0.001\,m$

$$\Rightarrow \quad t = 8.3\ s.$$

Problem 3 – Transient Conduction in a Rising Droplet

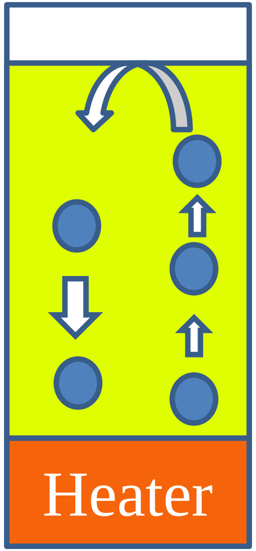
Consider the operation of the "lava" lamp. Polymer is heated at the base and due to buoyant forces begins to rise through a colored liquid in the form of "small" spheres (r_o = 1.5 cm). Assume that the colored liquid has a density of ρ_∞ = 900 kg/m^3 and a uniform temperature of T_∞ = 325 K. The polymer sphere has a temperature dependent density given by:

$\rho = \rho_o - \beta\,(T - T_\infty)$, ρ_o = 910 kg/m^3 , β = 2 kg/m^{3}K



If the polymer liquid begins its ascent at a temperature of 350 K, how long before it beings to fall back to be reheated?

Assume: h = 20 W/m^{2}K, Cp (sphere) = 400 J/kg K, k (sphere) = 10 W/m K and that the volume and area of the sphere are constant.



The first check should be the validity of LC model - the droplet temperature varies with time (position) but space-wise constant.

The heated droplet will rise and get cooled, increasing its density and therefore at some point it will start its downward journey again.

It is **assumed** that the volume and area of the droplet during this movement remain constant.

The colored liquid temperature is constant.

Check for lumped capacitance validity

$Bi = h\, r_o / k = (20 \times 0.015)/10 < 0.1$, therefore LC model is valid

IN - OUT + GEN = ACCUM

$\rho = \rho_o - \beta\, (T - T_\infty)$

$$0 - h\, A\, (T - T_\infty) + 0 = \frac{d}{dt}\left(\rho\, C_p\, V\, T\right)$$

ρ is a function of T and therefore so are V and A of the sphere. However, as mentioned in the problem statement, V and A are assumed to be constant herein.

$$Let \quad \theta \equiv T - T_\infty$$

$$\theta \equiv T - T_\infty$$

$$-\frac{h\,A}{C_p\,V}\,\theta = \frac{d}{dt}\left[(\rho_0 - \beta\,T_\infty)\,\theta - \beta\,\theta^2 + \rho_0\,T_\infty\right]$$

$$-\frac{h\,A}{C_p\,V}\,\theta = \left[(\rho_0 - \beta\,T_\infty)\frac{d\theta}{dt} - 2\,\beta\,\theta\,\frac{d\theta}{dt}\right]$$

$$-\frac{h\,A}{C_p\,V}\int_0^t dt = \left[(\rho_0 - \beta\,T_\infty)\int_{\theta_0}^{\theta}\frac{d\theta}{\theta} - 2\,\beta\int_{\theta_0}^{\theta} d\theta\right]$$

Therefore

$$\boxed{t = \frac{C_p\,V}{h\,A}\left[(\rho_0 - \beta\,T_\infty)\ln\frac{\theta_0}{\theta} + 2\,\beta\,(\theta - \theta_0)\right]}$$

The polymer will begin to fall when its density reached that of the coloured liquid.
The temperature at which it occurs is

$\rho = \rho_o - \beta\,(T - T_{\infty})$

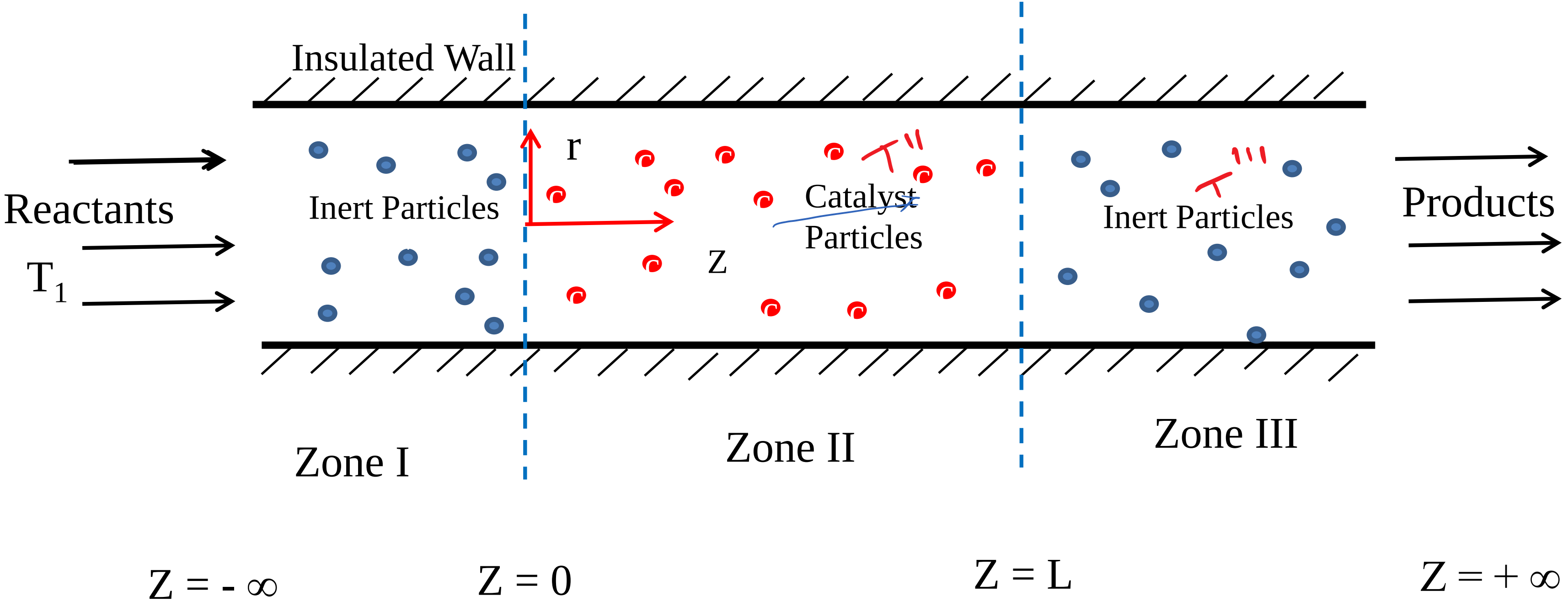
$$900 = 910 - 2\,(T - 325) \Rightarrow T = 330\ K$$

$$\theta_0 = 25\ C;\ \theta_{Reversal} = 5\ C$$

$$t = \frac{400\ x\ 0.015}{3\ x\ 20}\left[(910\ - 2\ x\ 325)\ \ln\frac{25}{5} + 2\ x\ 2\ (25\ - 5)\right]$$

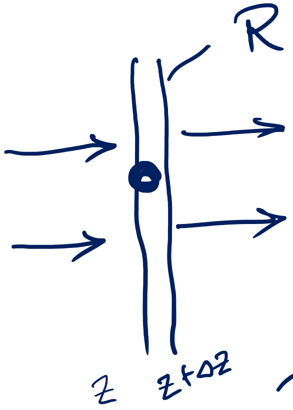
$$\boxed{t = 498\ s}$$

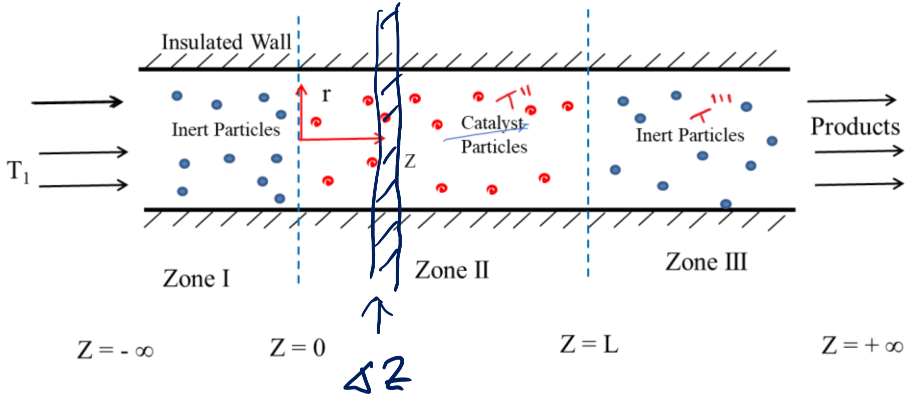
Heat Conduction with Chemical Heat Source



Superficial Velocity $v_1 = \dfrac{W}{\pi R^2 \rho_1}$ Volumetric Heat Generation S_C

IN - OUT + GEN = 0





CON CONV

$$\pi R^2 \left[q_z \Big|_z - q_z \Big|_{z+\Delta z} \right] + \pi R^2 \rho_1 v_1 C_p (T - T_0) \Big|_{z=z}$$

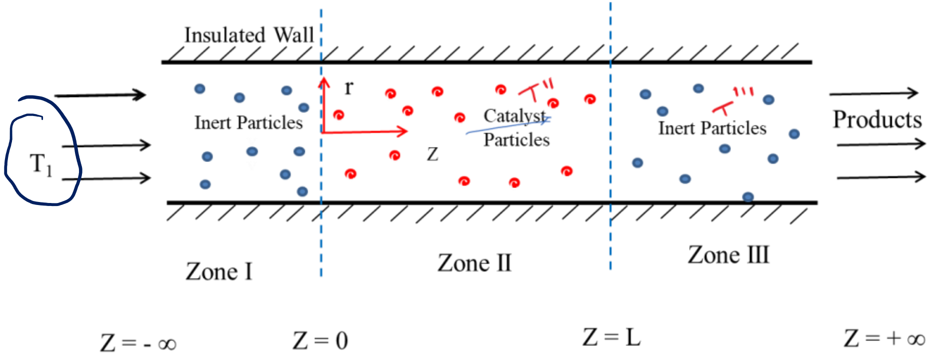
$$- \pi R^2 (\quad) \Big|_{z=z+\Delta z} + \pi R^2 \Delta z \, S_c = 0$$

$T = f(z)$ only

$\neq f(r)$

$\lim_{\Delta z \to 0}$

$$\frac{d v_z}{dz} + \rho\, \vartheta_1 C_p \frac{dT}{dz} = S_c$$



$$\boxed{-k_{z,eff} \frac{d^2T}{dz^2} + \rho_1 \vartheta_1 C_p \frac{dT}{dz} = S_c}$$

II

$-k$ - - - - - - - $= 0$

I & III

6 BC.

1) $Z = -\infty \quad T^{I} = T_1$

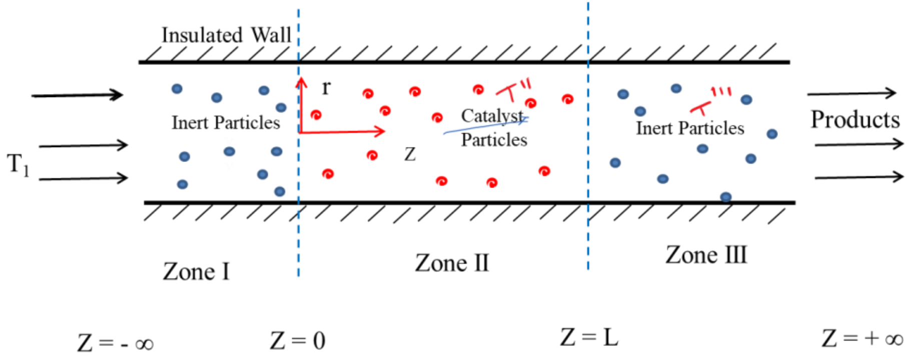
2) $Z = 0 \quad T^{I} = T^{II}$

3) $Z = L \quad T^{II} = T^{III}$ — T is a cont. of Z

4) $Z = 0 \quad -k_{eff} \dfrac{dT^{I}}{dz} = -k \dfrac{dT^{II}}{dz}$

5) $Z = L \quad -k \dfrac{dT^{II}}{dz} = -k \dfrac{dT^{III}}{dz}$

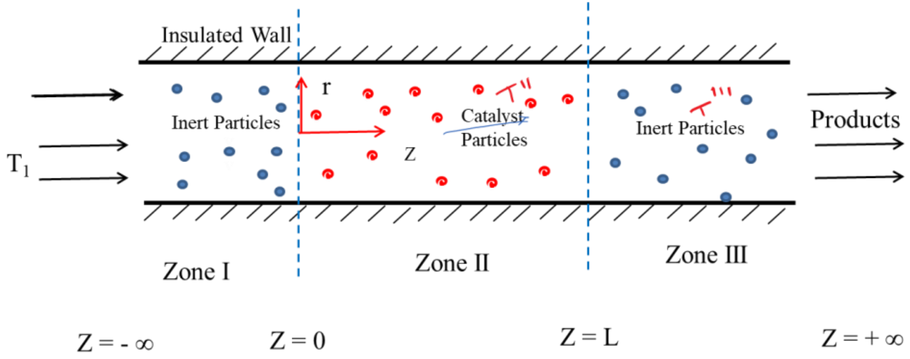
6)



$$Z = \frac{z}{L} \qquad \theta = \frac{T - T^0}{T_1 - T^0}$$

$$\boxed{\begin{aligned}\theta^{I} &= C_1 + C_2 e^{BZ}\\ \theta^{II} &= C_3 e^{m_3 Z} + C_4 e^{m_4 Z}\\ \theta^{III} &= C_5 + \cancel{C_6} e^{BZ}\end{aligned}}$$

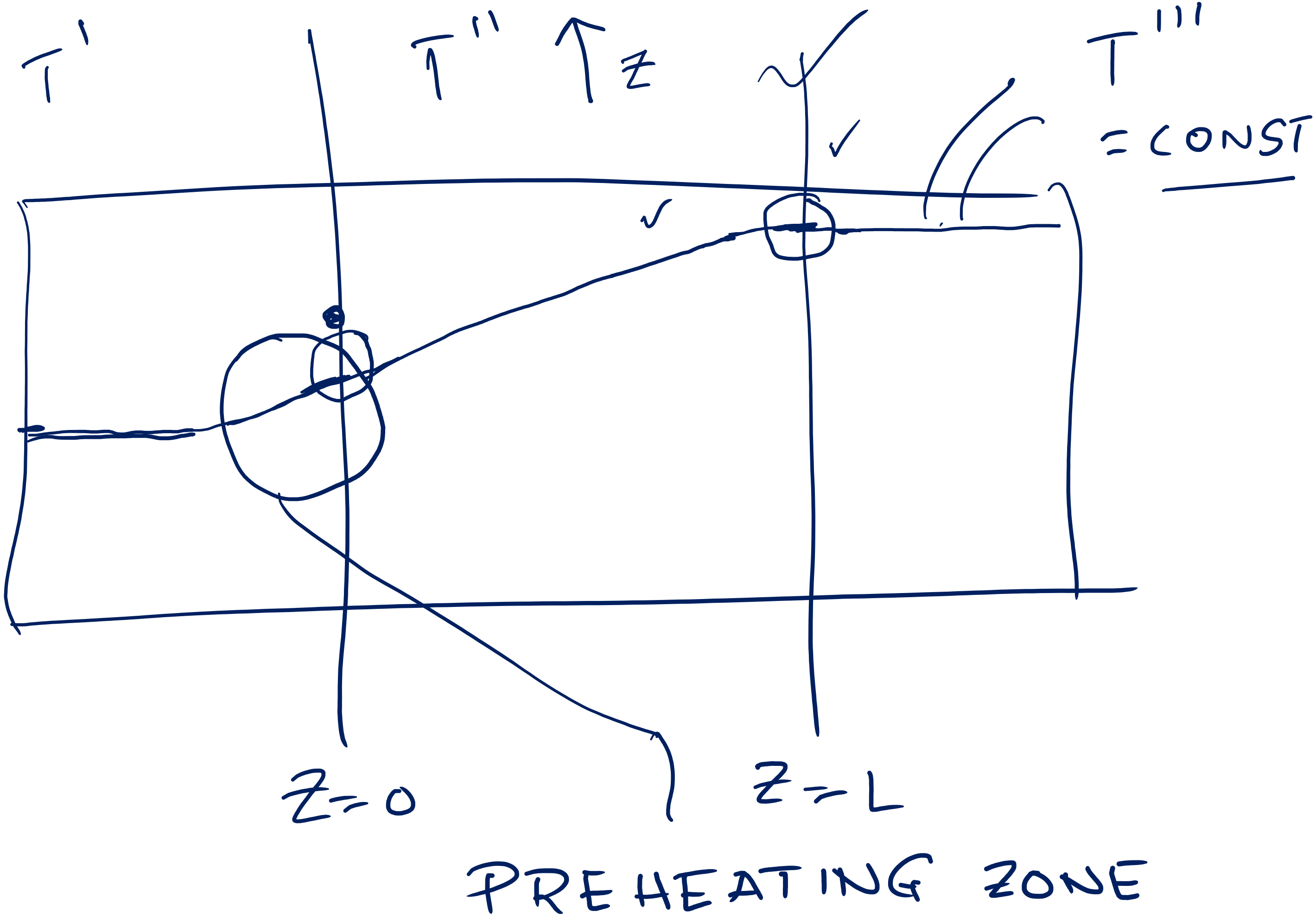
$= 0$



$Z = +\infty$ $Z = \frac{z}{L}$, $\theta^{III} = \frac{T - T^0}{T_1 - T^0}$ FINITE

$$B \equiv \frac{\rho v_1 C_p L}{k_{eff}}$$

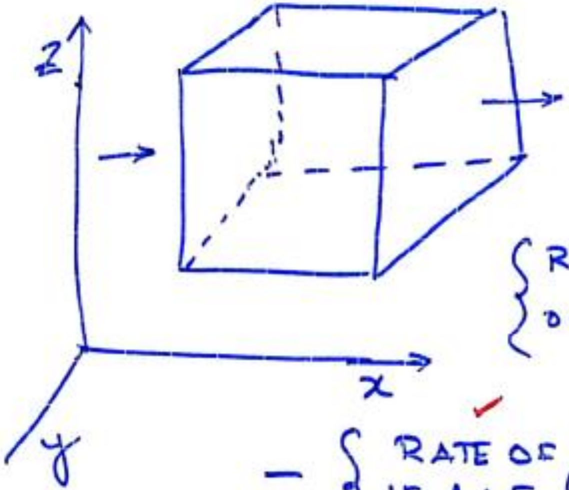
$$\Theta = \frac{T - T^0}{T_1 - T^0}$$



Heat Transfer Fundamentals

Energy Equation

EQ^N OF CHANGE FOR A NON ISOTHERMAL SYSTEM



FLUID IN THRU ~~AT~~ x, y, z FACES
$\Delta y \Delta z$, $\Delta x \Delta z$, $\Delta x \Delta y$

FLUID OUT " $x+\Delta x$, $y+\Delta y$, $z+\Delta z$

CONSERVATION OF ~~ENG~~ ENERGY.

{RATE OF ACCUMULATION OF INTERNAL & KINETIC ENERGY} = {RATE OF IE & KE IN BY CONV.}

− {RATE OF IE & KE OUT BY CONV.} + {NET RATE OF HEAT ADDITION BY COND.} − {NET RATE OF WORKDONE BY THE SYSTEM ON THE SURROUNDINGS}

−

RATE OF ACCUM OF IE + KE $= \Delta x \Delta y \Delta z \dfrac{\partial}{\partial t}\left(\rho U + \frac{1}{2}\rho v^2\right)$.

U = IE PER UNIT MASS OF THE FLUID

v = MAG. OF LOCAL FLUID VEL.

RATE OF CONVECTION OF IE & KE INTO THE ELEMENT

(x FACE)

$$\Delta y \Delta z \left\{ \underline{v_x}\left(\rho U + \frac{1}{2}\rho v^2\right)\Big|_x^{\text{IN}} - v_x\left(\rho U + \frac{1}{2}\rho v^2\right)\Big|_{x+\Delta x}^{\text{OUT}} \right\}$$

y FACE

$$+ \Delta x \Delta z \left\{ v_y\left(\rho U + \frac{1}{2}\rho v^2\right)\Big|_y - v_y(\text{-- -- -- --})\Big|_{y+\Delta y} \right\}$$

$$+ \Delta x \Delta y \left\{ v_z(\qquad)\Big|_z - v_z(\qquad)\Big|_{z+\Delta z} \right\}$$

COND.

NET RATE OF ENERGY IN BY COND.

$$\underline{\Delta y \Delta z}\left\{ q_x\big|_x - q_x\big|_{x+\Delta x} \right\} + \Delta x \Delta z \left\{ q_y\big|_y - q_y\big|_{y+\Delta y} \right\}$$
$$+ \Delta x \Delta y \left\{ q_z\big|_z - q_z\big|_{z+\Delta z} \right\}$$

WORK DONE → AGAINST VOLUMETRIC FORCE (BODY) (GRAVITY) ✓

WORK DONE → AGAINST SURFACE FORCES

SURFACE FORCES ↓ (PRESSURE ✓, VISCOUS ✓)

RATE OF WORK DONE = FORCE X VEL. IN THE DIRECT^N OF FORCE

RATE OF WORK DONE GRAVITATIONAL FORCE

$$\boxed{-\rho(\Delta x\Delta y\Delta z)(v_x g_x + v_y g_y + v_z g_z)}$$

RATE OF WORK DONE AG. STATIC PR.

$$\Delta y\Delta z\left\{p v_x\big|_{x+\Delta x} - p v_x\big|_x\right\} + \Delta x\Delta z\left\{p v_y\big|_{y+\Delta y} - p v_y\big|_y\right\} + \Delta x\Delta y\left\{p v_z\big|_{z+\Delta z} - p v_z\big|_z\right\}.$$

AGAINST VISCOUS FORCES.

FORCE X VEL.

$$\begin{array}{ccc} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{array}$$

$$\underline{\Delta y\Delta z}\left\{\begin{array}{c} \tau_{x\boxed{x}}\, v_x \\ \tau_{x\boxed{y}}\, v_y \\ \tau_{x\boxed{z}}\, v_z \end{array}\right. \quad \begin{array}{c} \underline{\Delta x\Delta z} \\ \boxed{\begin{array}{c} \tau_{y\boxed{x}}\, v_x \\ \tau_{yy}\, v_y \\ \tau_{yz}\, v_z \end{array}} \end{array} \quad \boxed{\begin{array}{c} \tau_{zx}\, v_x \\ \tau_{zy}\, v_y \\ \tau_{zz}\, v_z \end{array}} + \Delta x\Delta y$$

NET RATE

$$\Delta y \Delta z \left\{ \left(\tau_{xx} v_x + \tau_{xy} v_y + \tau_{xz} v_z \right\} \right|_{x+\Delta x}$$

$$- \left(\tau_{xx} v_x + \cdot - - - - \right) \Big|_x$$

$$+ 2 \text{ '}y\text{'} \quad + 2 \text{ '}z\text{'}$$

$$\boxed{\rho C \frac{DT}{Dt}} = k \nabla^2 T - T \left(\frac{\partial P}{\partial T} \right) \nabla v + \mu \phi_v$$

HEAT COND.

EXPANSION EFFECTS

VISCOUS DISSIPATION.

$$\rho C_p \left[\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} \cdots \right]$$

$$k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \cdots \right)$$

SIMPLIFICATIONS

$$\boxed{\rho \neq f(T) \quad , \text{NO } \phi}$$

$$\Rightarrow \rho C \cdot \frac{DT}{Dt} = k \nabla^2 T + \dot{q}$$

BIRD
STEWART
LIGHTFOOT

SOLID ρ CONST

$$\rho C_p \frac{DT}{Dt} = k \nabla^2 T + \dot{q}$$

$$\boxed{\rho C_p \frac{\partial T}{\partial t} = k \nabla^2 T + \dot{q}}$$

COND. EQN.

Heat Transfer Fundamentals

Fourier's Law $q_x = -k\,A\,\frac{dT}{dx}$ Newton's Law of Cooling $q = h\,A\,(T_s - T_\infty)$

Equation of Energy in Cartesian and Cylindrical Coordinate Systems

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = k \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right] + \mu \phi_v + \dot{Q}$$

Transient, Convection, Conduction, Viscous Dissipation, Generation

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right) = k \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right] + \mu \phi_v + \dot{Q}$$

Equations of Energy

Cartesian coordinates $(x, y.z)$

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = k \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right] + \mu \phi_v + \dot{Q}$$

Cylindrical coordinates (r, θ, z)

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right) = k \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right] + \mu \phi_v + \dot{Q}$$

Spherical coordinates (r, θ, ϕ)

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin\theta} \frac{\partial T}{\partial \phi} \right) = k \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 T}{\partial \phi^2} \right] + \mu \phi_v + \dot{Q}$$

The Dissipation Function

Cartesian coordinates (x, y, z)*:*

$$\Phi_v = 2\left[\left(\frac{\partial v_x}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial y}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right] + \left[\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right]^2 + \left[\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right]^2 + \left[\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right]^2 - \frac{2}{3}\left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right]^2$$

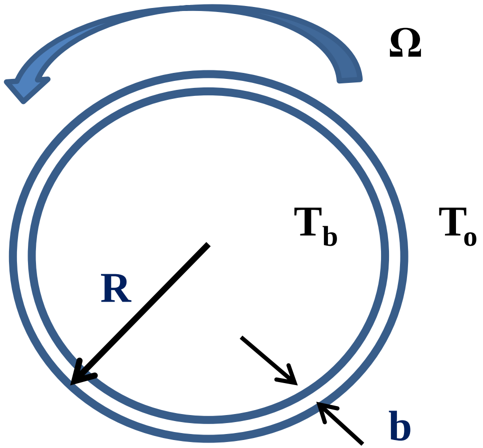
Cylindrical coordinates (r, θ, z)*:*

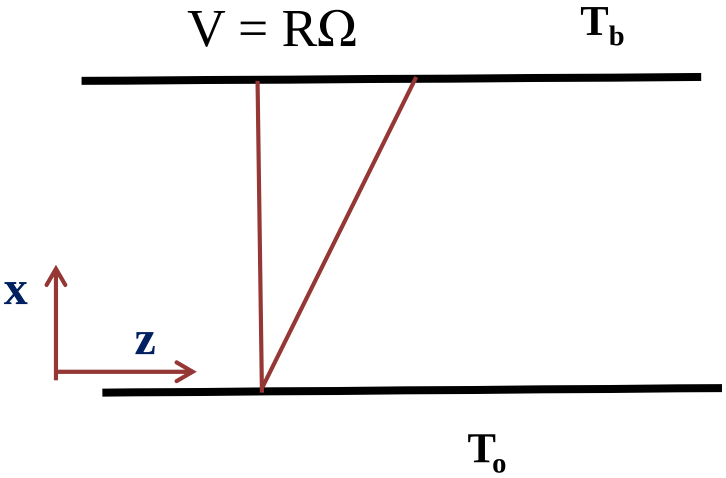
$$\Phi_v = 2\left[\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right] + \left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]^2 + \left[\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right]^2 + \left[\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right]^2$$

$$- \frac{2}{3}\left[\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}\right]^2$$

Viscous Heat Generation

Heat Conduction with Viscous Heat Source

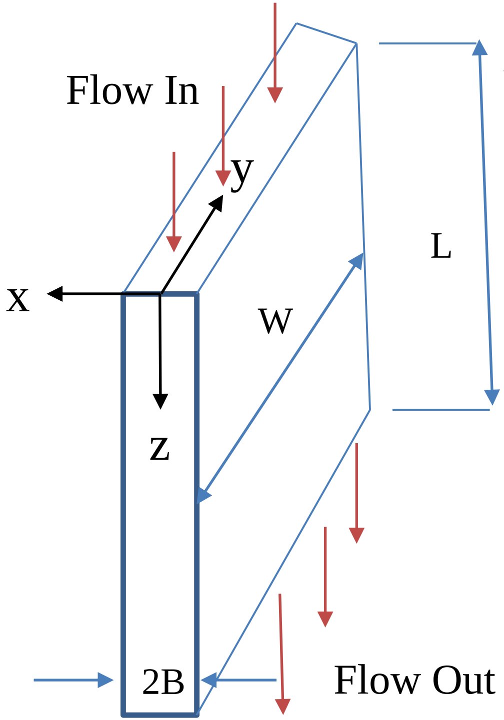




Cartesian coordinates (x, y, z):

$$\Phi_v = 2\left[\left(\frac{\partial v_x}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial y}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right] + \left[\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right]^2 + \left[\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right]^2 + \left[\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right]^2 - \frac{2}{3}\left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right]^2$$

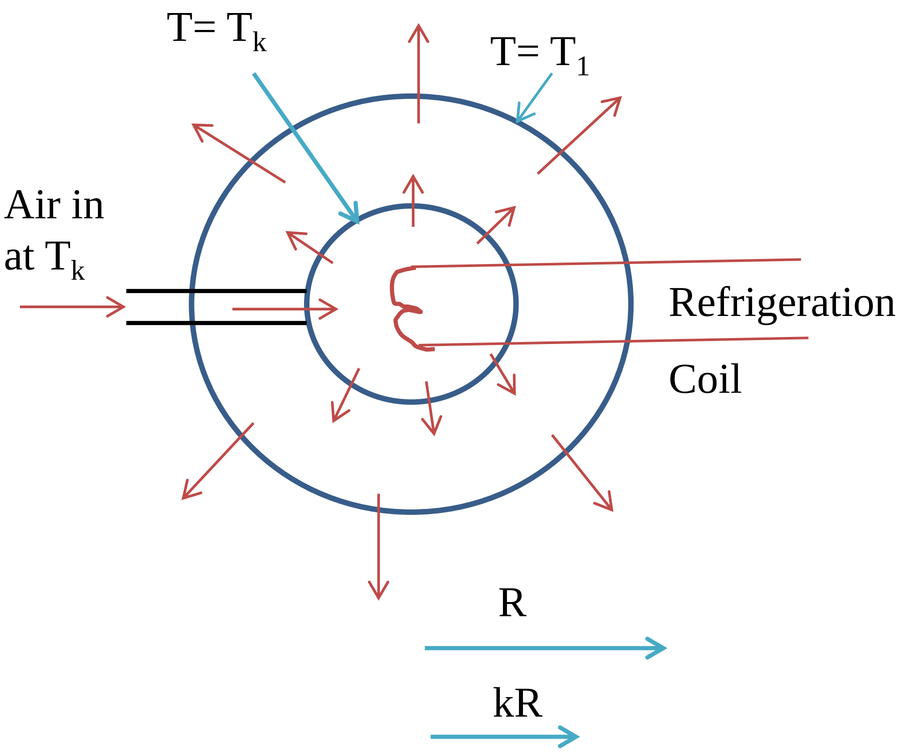
Laminar flow in a slit with viscous heat generation, both plates at T_o



$W \gg 2b$, $v_z = f(x)$ only

Derive an expression for the temperature profile

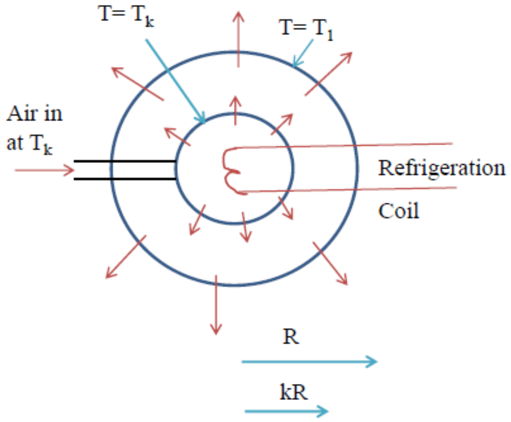
Transpiration Cooling



Two concentric porous spherical shells. T_k is to be kept lower than T_1 by passing dry air at Tk in a radially outward direction from the inner shell and through the outer one to the outside.

Find expression for the rate of heat removal from the inner sphere as a function of the mass flow rate of the air at SS condition.

Here, $v_\theta = 0$, $v_\varphi = 0$ and the equation of continuity becomes -



$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \rho v_r\right) = 0 \quad \Rightarrow \quad r^2 \rho v_r = C = \frac{W_r}{4\pi}$$

$$v_r = \frac{W_r}{4\pi r^2 \rho}$$

W_r is the radial mass flow rate of the gas

Equation of energy

$$\text{Spherical coordinates}\,(r,\theta,\phi)$$

$$\rho \hat{C}_p \left(\cancel{\frac{\partial T}{\partial t}} + v_r \frac{\partial T}{\partial r} + \cancel{\frac{v_\theta}{r}} \frac{\partial T}{\partial \theta} + \cancel{\frac{v_\phi}{r \sin\theta}} \frac{\partial T}{\partial \phi} \right) = k \left[\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \cancel{\frac{\partial T}{\partial \theta}} \right) + \frac{1}{r^2 \sin^2\theta} \cancel{\frac{\partial^2 T}{\partial \phi^2}} \right] + \cancel{\mu \phi_v} + \cancel{\dot{Q}}$$

Governing Equation

$$\rho \hat{C}_p \left(v_r \frac{dT}{dr} \right) = k \left[\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dT}{dr} \right) \right]$$

Using Equation of Continuity

$$\frac{dT}{dr} = \frac{4\pi k}{W_r \hat{C}_p} \frac{d}{dr} \left(r^2 \frac{dT}{dr} \right)$$

Integrating

$$\frac{T - T_1}{T_k - T_1} = \frac{e^{-R_o/r} - e^{-R_o/R}}{e^{-R_o/kR} - e^{-R_o/R}} \quad ; \quad R_o = \frac{W_r \hat{C}_p}{4\pi k}$$

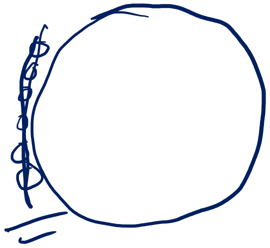
For small R_o

$$\frac{T - T_1}{T_k - T_1} = \frac{1/r - 1/R}{1/kR - 1/R}$$

$$R_o = \frac{W_r \hat{C}_p}{4\pi k}$$

If R is infinite and kR is finite

$$\frac{T - T_\infty}{T_k - T_\infty} = \frac{kR}{r}$$



The heat flow to the inner sphere in the –r direction

$$Q = -4\pi(\kappa R)^2 q_r \Big|_{r=kR} = 4\pi k (\kappa R)^2 \frac{dT}{dr} \Big|_{r=kR}$$

$$Q = 4\pi k R \frac{(T_1 - T_\kappa)}{\left[e^{\left(R_o/\kappa R\right)(1-\kappa)} - 1 \right]}$$

When $R_o = 0$, the rate of heat removal at zero gas rate

$$Q_o = 4\pi k R \frac{(T_1 - T_\kappa)}{(1-\kappa)}$$

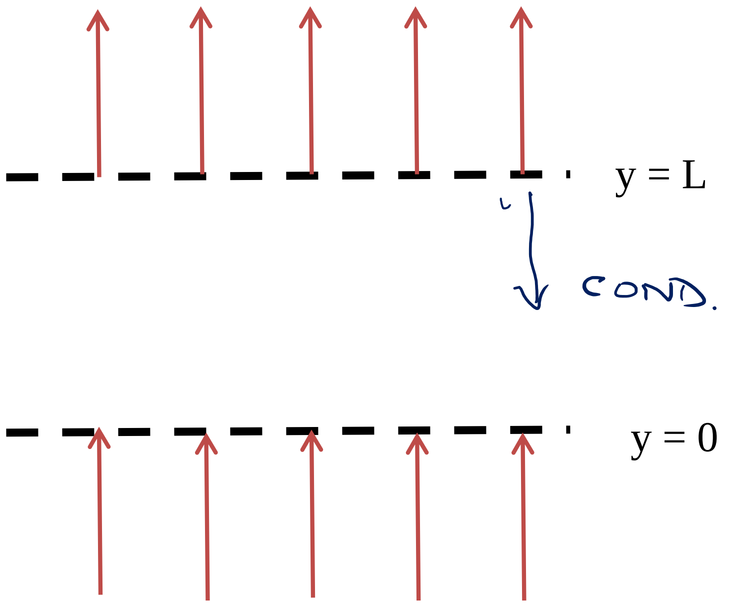
The effectiveness ϵ of the transpiration is given by $\varepsilon = \frac{Q_o - Q}{Q_o} = 1 - \frac{\phi}{e^\phi - 1}$; $\phi = \frac{R_o(1-\kappa)}{\kappa R}$

For low values of φ, ϵ tends to $\varphi/2$

Transpiration can be a very effective method for reducing the heat transfer rates. Especially important where heat and mass transfer occur simultaneously.

Transpiration Cooling

Two large porous horizontal plates are separated by a small distance L. The upper plate is at a temperature of $T = T_L$ while the lower plate is to b maintained at a lower temperature T_0. To reduce the amount of heat that must be removed from the lower plate, an ideal gas at T_0 is blown upward through both plates at a steady rate. Develop an expression of temperature distribution and the amount of heat q_0 that must be removed from the cold plate per unit area as a function of the fluid properties and the flow rate of the ideal gas.



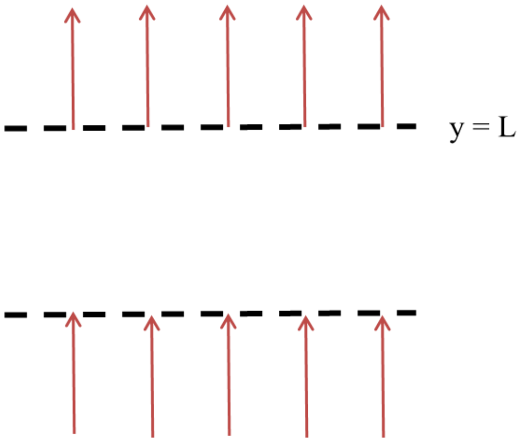
Ideal gas at T_0

$$\rho \hat{C}_p \left(\cancel{\frac{\partial T}{\partial t}} + \cancel{v_x \frac{\partial T}{\partial x}} + v_y \frac{\partial T}{\partial y} + \cancel{v_z \frac{\partial T}{\partial z}} \right) = k \left[\cancel{\frac{\partial^2 T}{\partial x^2}} + \frac{\partial^2 T}{\partial y^2} + \cancel{\frac{\partial^2 T}{\partial z^2}} \right] + \cancel{\mu \phi_v} + \cancel{\dot{Q}}$$

SS, 1D conduction, convection, no viscous dissipation, no heat generation

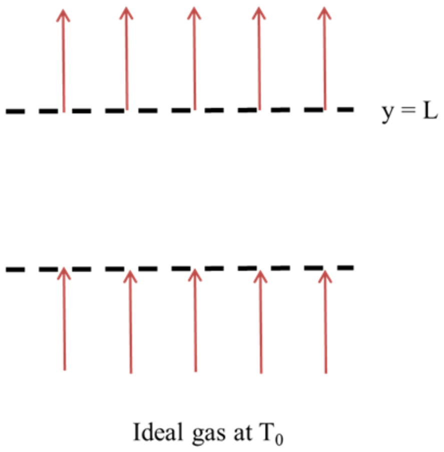
$$\rho \hat{C}_p \left(v_y \frac{dT}{dy} \right) = k \left[\frac{d^2 T}{dy^2} \right]$$

$$\frac{d^2 T}{d y^2} - \frac{1}{Y_o} \frac{dT}{dY} = 0 \, ; \quad Y_o = \frac{k}{\rho C_p v_y}$$



Ideal gas at T_0

To evaluate v_y equation of continuity may be used.



$$\frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z} = 0$$

$$\rho v_y = Constant = \frac{\dot{m}}{A}$$

$\dot{m}$ = mass flow rate, A = area

The governing equation can then be solved.

$$\frac{d^2T}{dy^2} - \frac{1}{Y_0}\frac{dT}{dY} = 0; \quad Y_0 = \frac{k}{\rho C_p v_y}$$

With the boundary conditions

$$T = T_o \quad at \quad y = 0$$
$$T = T_L \quad at \quad y = L$$

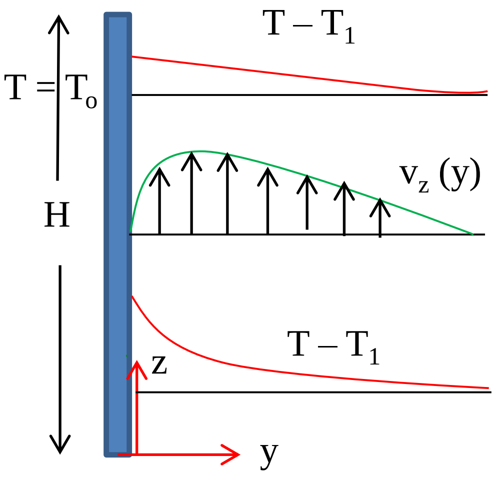
To obtain

$$\frac{T - T_L}{T_o - T_L} = \frac{e^{\left(y\phi/L\right)} - e^{\phi}}{1 - e^{\phi}}$$

Rate of heat removal by Fourier's law

$$q_o = k\left.\frac{dT}{dy}\right|_{y=0} = \frac{k\left(T_L - T_O\right)}{L}\frac{\phi}{e^{\phi} - 1}$$

Free Convection from a Vertical Plate



Flat plate heated to a temperature of T_o and is suspended in a large body of fluid. Which is at temperature T_1. Fluid near the plate rises due to buoyancy.

Deduce the dependence of the heat loss on system variables. Free convection form of the equation of motion is to be used.

Assumptions:

1. Dimension in the x direction is very large; thus v_y and v_z are functions of y and z only
2. v_y is small; fluid almost moves directly upward and therefore, y component of the equation of motion can be neglected.

$$\beta = \frac{1}{V}\frac{\Delta V}{\Delta T} \quad Buoyant\ Force = \rho\, g\, \Delta V = \rho\, g\, \beta V \Delta T \quad ; \quad Buoyant\ Force \big/ Volume = \rho\, g\, \beta\, \Delta T$$

The Boussinesq Equation of Motion for Forced and Free Convection

In non-isothermal flow, the fluid ρ and μ depend on T as well as P. The variation in ρ is particularly important as it gives rises to buoyant forces, and thus to free convection.

The buoyant force appears automatically when an equation of state is inserted into the equation of motion. For example, one can use the simplified equation of state into the equation of motion (known as the Boussinesq approximation).

$$\rho\,(T) = \bar{\rho} - \bar{\rho}\,\bar{\beta}\,(T - \bar{T});$$ Where $\beta = -\left(\frac{1}{\rho}\right)(\partial\rho/\partial \mathrm{T})_{\mathrm{p}}$, evaluated at $\bar{T}$

When the above equation is substituted into the ρg term of the Eq. of motion, but not into the $\rho Dv/Dt$ term), one would get the ***Boussinesq equation***

$$\rho\frac{Dv}{Dt} = \left(\cancel{-\nabla P + \bar{\rho}\,g}\right) - [\nabla.\tau] - \cancel{\bar{\rho}\,g\,\bar{\beta}\left(T - \bar{T}\right)}$$

Neglected in Free Convection

Neglected in Forced Convection

Continuity Equation

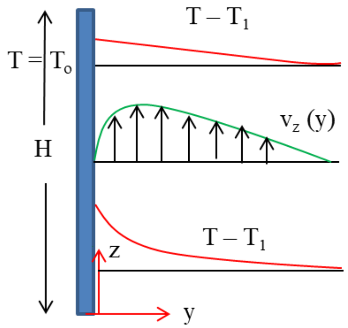
$$\frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

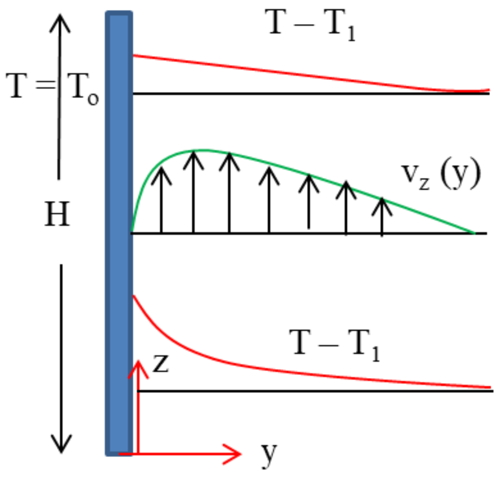
Equation of Motion

$$\rho \left(v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = \mu \left[\frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right] + \rho \, g \, \beta \, (T - T_1)$$

Equation of Energy

$$\rho \hat{C}_p \left(v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} \right) (T - T_1) = k \left[\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] (T - T_1)$$





Boundary Conditions

$$y = 0, \qquad v_y = v_z = 0, \quad T = T_o$$

$$y \to \infty, \qquad v_z = 0, \quad T = T_1$$

$$z \to -\infty, \quad v_y = v_z = 0, \quad T = T_1$$

$$\rho \left(v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = \mu \left[\frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right] + \rho\, g\, \beta\, (T - T_1)$$

$$\rho \hat{C}_p \left(v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} \right) (T - T_1) = k \left[\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] (T - T_1)$$

Dimensionless variables

$$\theta = \frac{T - T_1}{T_o - T_1} \qquad \zeta = {}^{z}/_{H}$$

$$\eta = \left(\frac{B}{\mu\ \alpha\ H}\right)^{1/4} y$$

$$\phi_z = \left(\frac{\mu}{B\ \alpha\ H}\right)^{1/4} v_z$$

$$\phi_y = \left(\frac{\mu\ H}{\alpha^3\ B}\right)^{1/4} v_y$$

Governing Equations

$$\frac{\partial \varphi_y}{\partial \eta} + \frac{\partial \varphi_z}{\partial \zeta} = 0$$

$$\frac{1}{\mathrm{Pr}}\left(\varphi_y \frac{\partial \phi_z}{\partial \eta} + \varphi_z \frac{\partial \phi_z}{\partial \zeta}\right) = \frac{\partial^2 \phi_z}{\partial \eta^2} + \phi$$

$$\left(\varphi_y \frac{\partial \theta}{\partial \eta} + \varphi_y \frac{\partial \theta}{\partial \zeta}\right) = \frac{\partial^2 \theta}{\partial \eta^2}$$

$$\phi_y \ \&\ \phi_z \quad = f\left(\eta, \zeta, \mathrm{Pr}\right)$$

$$\theta \quad = f\left(\eta, \zeta, \mathrm{Pr}\right)$$

$$q_{avg} = + \frac{k}{H} \int_0^H -\left.\frac{\partial T}{\partial y}\right|_{y=0} dz$$

$$q_{avg} = k(T_o - T_1)\left(\frac{B}{\mu \alpha H}\right) \int_0^1 -\left.\frac{\partial \theta}{\partial \eta}\right|_{\eta=0} d\zeta$$

$$\frac{\partial \varphi_y}{\partial \eta} + \frac{\partial \varphi_z}{\partial \zeta} = 0$$

$$\frac{1}{\Pr}\left(\varphi_y \frac{\partial \phi_z}{\partial \eta} + \varphi_z \frac{\partial \phi_z}{\partial \zeta}\right) = \frac{\partial^2 \phi_z}{\partial \eta^2} + \phi$$

$$\left(\varphi_y \frac{\partial \theta}{\partial \eta} + \varphi_y \frac{\partial \theta}{\partial \zeta}\right) = \frac{\partial^2 \theta}{\partial \eta^2}$$

$$\theta \quad = f\left(\eta, \zeta, \Pr\right)$$

$$q_{avg} = C \frac{k}{H} (T_o - T_1)\left(Gr. \Pr\right)^{1/4}$$

$$q_{avg} = k(T_o - T_1)\left(\frac{B}{\mu\alpha H}\right)\int_0^1 -\frac{\partial\theta}{\partial\eta}\bigg|_{\eta=0} d\zeta \qquad q_{avg} = C\,\frac{k}{H}(T_o - T_1)\left(Gr.\,\mathrm{Pr}\right)^{1/4}$$

$$\theta = f\left(\eta, \zeta, \mathrm{Pr}\right)$$

$$\frac{\partial\theta}{\partial\eta}\ \textit{is a function of}\ \eta, \xi, \mathrm{Pr}.$$

$$\frac{\partial\theta}{\partial\eta}\bigg|_{\eta=0}\ \textit{is a function of}\ \xi\ \textit{and}\ \mathrm{Pr} \qquad \int_0^1 -\frac{\partial\theta}{\partial\eta}\bigg|_{\eta=0} d\zeta\ \textit{is a function of only}\ \mathrm{Pr}$$

$$C = f\left(\mathrm{Pr}\right) \rightarrow \cancel{Strong}\,/\,Weak \quad Function$$

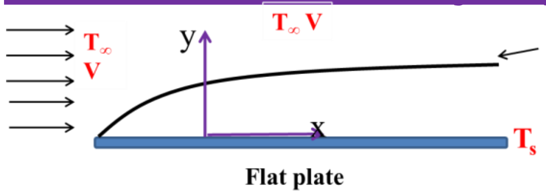
Experimental measurements provide, in laminar flow range ($GrPr < 10^9$), variation of C with Pr is indeed quite small.

Pr	0.73	10	10^2	10^3
C	0.517	0.612	0.652	0.653

Convection Boundary Layers

Velocity boundary layer - Fundamental importance to convection transport problems - Concept of C_f

Thermal boundary layer - BL thickness is the value for which the ratio



$$\boxed{\frac{T_s - T}{T_s - T_\infty} \approx 0.99}$$

Concept of convection coefficient

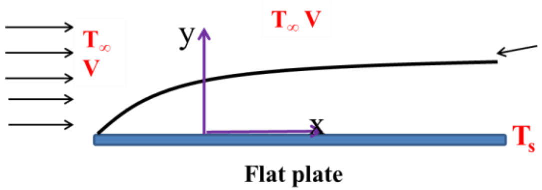
At any distance x from the leading edge, the local heat flux my be obtained by applying Fourier's law to the fluid at y = 0

$$q_s = -k \left.\frac{\partial T}{\partial y}\right|_{y=0} = h\left(T_s - T_\infty\right)$$

$$h = \frac{-k_f \left.\frac{\partial T}{\partial y}\right|_{y=0}}{\left(T_s - T_\infty\right)}$$

If $T^* \equiv \dfrac{T - T_s}{T_\infty - T_s}$ *and* $x^* \equiv x / L$

$$\frac{hL}{k_f} = \left.\frac{\partial T^*}{\partial y^*}\right|_{y=0}$$



- Nusselt number is the dimensionless temp gradient at the interface

- Conditions in the thermal BL strongly influences and determine the rate of heat transfer across the BL

- $(T_s - T_\infty)$ is a constant but δ_t increases with x, therefore, the gradient decreases with increasing x and q_s and h decrease with x

	Velocity B.L.	**Thermal B.L.**
Characterized by	Velocity Gradient Shear Stress	Temperature gradient Heat transfer
Principal manifestations	Surface friction	Convection heat transfer
Key parameters	Friction coefficient, C_f	Convective heat transfer coefficient, h