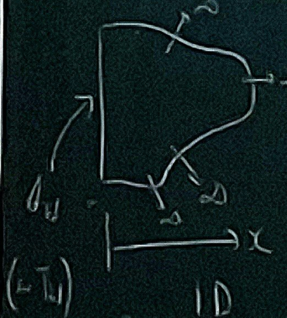
Integral Method

linear / nonlinear

Unsteady-State heat conduction in semi-infinite solid



$T(x, t=0) = T_i$

$T(x=0, t) = T_w$

$T(x,t)$?

Define $\theta = T(x,t) - T_i$

$$\frac{1}{\alpha}\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2}$$

$$\theta(x,0) = 0$$

$$\theta(0,t) = T_w - T_i \equiv \theta_w$$

$$\theta(x \to \infty, t) = 0$$

Define $\delta(t)$ penetration depth

$$x \geq \delta(t)$$

$$\theta(\delta, t) = 0$$

$$\left.\frac{\partial \theta}{\partial x}\right|_{x=\delta} = 0$$

$$\int_0^{\delta(t)} \frac{\partial^2 \theta}{\partial x^2}\,dx = \frac{1}{\alpha}\int_0^{\delta(t)} \frac{\partial \theta}{\partial t}\,dx$$

$$\Rightarrow \left.\frac{\partial \theta}{\partial x}\right|_0^{\delta}$$

$$\cancel{\left.\frac{\partial \theta}{\partial x}\right|_{\delta}} - \left.\frac{\partial \theta}{\partial x}\right|_0$$

Leibnitz Rule

$$\frac{1}{\alpha}\left[\frac{d}{dt}\int_0^{\delta(t)} \theta(x,t)\,dx - \cancel{\theta(\delta,t)}\frac{d\delta}{dt}\right]$$

$$= \frac{1}{\alpha}\frac{d}{dt}\int_0^{\delta(t)} \theta(x,t)\,dx$$

$$\frac{d}{dt}\int_0^{\delta(t)} \theta(x,t)\,dx = -\alpha \left.\frac{\partial \theta}{\partial x}\right|_{x=0}$$

Heat Balance Integral or Energy Integral

Assume $\theta(x,t) = a + bx + cx^2$

$$a = \theta_w, \quad b = -2\frac{\theta_w}{\delta}, \quad c = \frac{\theta_w}{\delta^2}$$

Solution

$$\frac{\theta(x,t)}{\theta_w} = \frac{T(x,t) - T_i}{T_w - T_i} = 1 - 2\left(\frac{x}{\delta}\right) + \left(\frac{x}{\delta}\right)^2$$

Substitute in Energy balance integral, to get $\delta \frac{d\delta}{dt} = 6\alpha$, $\delta(0) = 0$,

$$\delta = \sqrt{12\alpha t}$$

Integral Method

Surface Heat Flux, $q''_w = -k \left.\frac{\partial T}{\partial x}\right|_{x=0}$

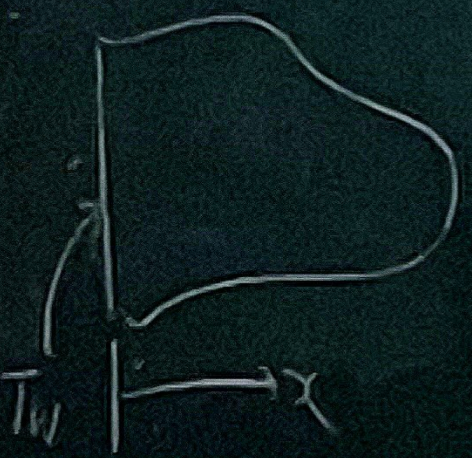
Exact Solution by Laplace Transformation

$$\frac{\theta(x,t)}{\theta_w} = \frac{T(x,t) - T_i}{T_w - T_i} = \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

$$= \frac{1}{\sqrt{3}} \frac{k(T_w - T_i)}{\sqrt{\alpha t}}$$

(Using Integral method)

$$q''_w = \frac{1}{\sqrt{\pi}} \frac{k(T_w - T_i)}{\sqrt{\alpha t}}$$



Try 3rd order polynomial

$$\theta(x,t) = a + bx + cx^2 + dx^3$$

$$\left.\begin{aligned} &\theta(0,t) = \theta_w \\ &\theta(\delta,t) = 0 \\ &\left.\frac{\partial\theta}{\partial x}\right|_\delta = 0 \end{aligned}\right\} \text{Natural conditions}$$

$$\left.\begin{aligned} &\left.\frac{\partial^2\theta}{\partial x^2}\right|_{x=\delta} = 0 \\ &\left.\frac{\partial^2\theta}{\partial x^2}\right|_{x=0} = 0 \end{aligned}\right\} \text{Derived condition}$$

$$\frac{\theta(x,t)}{\theta_w} = \frac{T - T_i}{T_w - T_i} = \left(1 - \frac{x}{\delta}\right)^3$$

$$\delta = \sqrt{24\alpha t}$$

$$q_w''(t) = \sqrt{\frac{3}{8}}\,\frac{k(T_w - T_i)}{\sqrt{\alpha t}}$$

Error = 8%

$$= \frac{\partial^2\theta}{\partial x^2}$$

$$\frac{\theta(x,t)}{\theta_w} = \frac{T(x,t) - T_i}{T_w - T_i} = 1 - \frac{3}{2}\left(\frac{x}{\delta}\right) + \frac{1}{2}\left(\frac{x}{\delta}\right)^3$$

$$\delta = \sqrt{8\alpha t}$$

$$q_w'' = \frac{3}{4\sqrt{2}}\,\frac{k(T_w - T_i)}{\sqrt{\alpha t}}$$

Error = 6%

Cylindrical Coordinate System

$$\theta = (\text{Polynomial in } r) \ln r$$

$$\theta = (a + br + cr^2) \ln r$$

$$\begin{pmatrix} \text{1D Cylindrical} \\ \text{C S} \end{pmatrix} \xrightarrow{U = rT} \begin{pmatrix} \text{1D Cartesian} \\ \text{C S} \end{pmatrix}$$

Spherical Coordinate System

$$T = \frac{(\text{Polynomial in } r)}{r}$$

$$\Rightarrow U = Tr = \text{Polynomial in } r$$

with heat source

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{k}$$

W/m^3

Integral Method

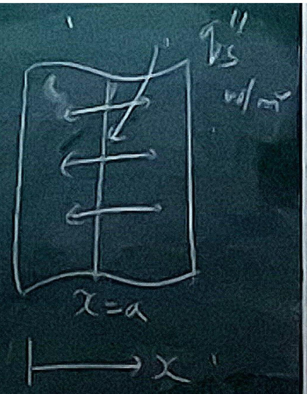
Additional Step

$$\int_0^{\delta(t)} \frac{\dot{q}}{k}\, dx$$

$\dot{q} \rightarrow$ distributed heat source

Planar Heat Source of strength $q_s''(t)$ W/m^2

$$\dot{q}(t) = q_s''(t)\, \delta(x-a)$$



$$\int_0^{\delta(t)} \frac{\dot q}{k}\, dx$$

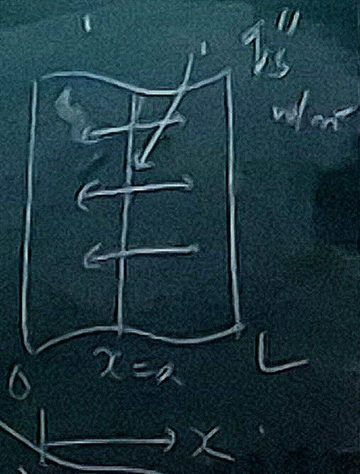
$\dot q \to$ distributed heat source

Planar Heat Source of strength $q_s''(t)$ W/m²

$$\dot q(t) = q_s''(t)\ \delta(x-a)$$

NOTE $f(x)\,\delta(x-a) = f(a)$

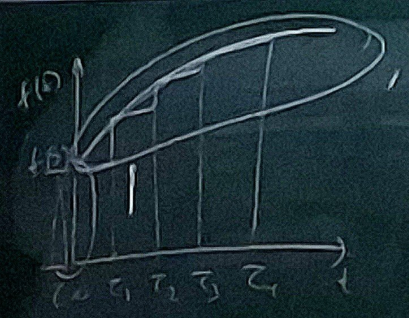
$$\int f(x)\,\delta(x-a)\,dx = f(a)$$

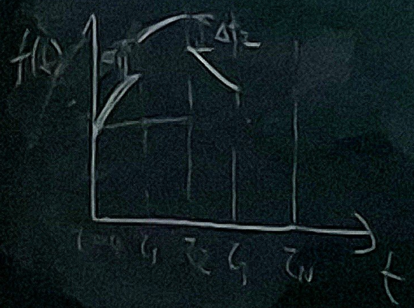


$$\int_0^L \delta(x-a)\,\sin\lambda_n x$$

$$= \sin\lambda_n a$$

$$\lambda_n = \frac{n\pi}{L}$$





Duhamel Method

Original problem:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$T(0,t) = f(t)$

$T(L,t) = 0$

IC $T(x,0) = 0$

Auxiliary Problem

Can be solved by SOV

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \phi}{\partial t}$$

$\phi(0,t) = 1$

$\phi(L,t) = 0$

IC $\phi(x,0) = 0$

$$T(x,t) = f(0)\,\phi(x,t) + \int_{\tau=0}^{t} \phi(x, t-\tau)\,\frac{d f(\tau)}{d\tau}\,d\tau$$

Alternative Form:

$$T(x,t) = \int_{\tau=0}^{t} f(\tau)\,\frac{\partial \phi(x,t-\tau)}{\partial t}\,d\tau$$

Where we have used

$\phi(x, t=0) = 0$

$$\frac{\partial \phi(x,t-\tau)}{\partial t} = -\frac{\partial \phi(x,t-\tau)}{\partial \tau}$$

... Method

$$\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$\ldots = f(t)$$

$$\ldots = 0$$

$$\ldots = 0$$

$$T(x,t) = f(0)\,\phi(x,t) + \int_{\tau=0}^{t} \phi(x,t-\tau)\,\frac{df(\tau)}{d\tau}\,d\tau$$

$$\int_0^t \phi(x,t-\tau)\frac{df(\tau)}{d\tau}\,d\tau = \int_0^{\tau_1} \phi(x,t-\tau)\frac{df(\tau)}{d\tau}\,d\tau + \int_{\tau_1}^{\tau_2} \phi(x,t-\tau)\frac{df(\tau)}{d\tau}\,d\tau + \cdots + \int_{\tau_N}^{t} \phi(x,t-\tau)\frac{df(\tau)}{d\tau}\,d\tau$$

$$\ldots\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \phi}{\partial t}$$

$$\phi(0,t) = 1$$

$$\phi(L,t) = 0$$

$$\phi(x,0) = 0$$

$$T(x,t) = \sum_{i=0}^{N} \Delta f_i\,\phi(x,t-\tau_i) + \int_{\tau=0}^{t} \phi(x,t-\tau)\,\frac{df(\tau)}{d\tau}\,d\tau$$

Stieltjes Integral

Example

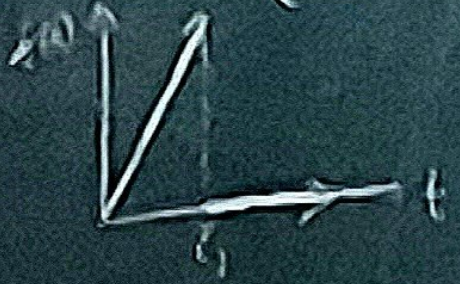
$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$T(x=0,t)=0$

$T(x=L,t)=f(t)$

$T(x,t=0)=0$

$$f(t) = \begin{cases} bt, & 0<t<\tau_1 \\ 0, & t>\tau_1 \end{cases}$$



$T(x,t) =$

$\int_0^t \phi(x,t-\tau$

Auxiliary Problem

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \phi}{\partial t}$$

$\phi(x=0,t)=0$

$\phi(x=L,t)=1$

$\phi(x,t=0)=0$

Solve by SOV

PDE BC?

ODE

Start $\phi(x,t) = \phi_{trs}(x,t) + \phi_{ss}(x)$

$$= \frac{x}{L} + \frac{2}{L}\sum_{n=1}^{\infty} \frac{(-1)^n}{\beta_n}\sin\beta_n x\, e^{-\alpha\beta_n^2 t}$$

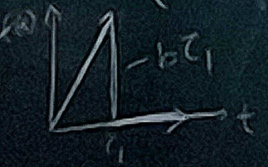
$\phi(x,t-\tau) = ??$

$\beta_n = \frac{n\pi}{L}$

$T(x,t)$

Sti

$$f(t) = \begin{cases} bt, & 0 < t < \tau_1 \\ 0, & t > \tau_1 \end{cases}$$



For $t < \tau_1$

$$T(x,t) = \cancel{f(0)}^{\,0}\,\phi(x,t) + \int_{\tau=0}^{t} \phi(x,t-\tau)\,\underbrace{\frac{df(\tau)}{d\tau}}_{=b}\,d\tau$$

For $t > \tau_1$

$$T(x,t) = \cancel{f(0)}^{\,0}\,\phi(x,t) + \phi(x,t-\tau_1)\,\Delta f_1 + \int_{\tau=0}^{\tau_1} \phi(x,t-\tau)\,\underbrace{\frac{df(\tau)}{d\tau}}_{=b}\,d\tau + \int_{\tau_1}^{t} \phi(x,t-\tau)\,\cancel{\frac{df(\tau)}{d\tau}}^{\,0}\,d\tau$$

Ozisik

$$\Delta f_1 = -b\tau_1$$

SLN $\phi(x,t) = \underbrace{\phi_{tr}(x,t)}_{\text{PDE soln}} + \underbrace{\phi_{ss}(x)}_{\text{ODE}}$

solve by SOV $\Rightarrow \phi(x,t) = \frac{x}{L} + \frac{2}{L}\sum_{n=1}^{\infty} \frac{(-1)^n}{\beta_n} \sin\beta_n x\, e^{-\alpha\beta_n^2 t}$

$\phi(x,t-\tau) = ??$ $\quad \beta_n = \frac{n\pi}{L}$