Linear Approximation

The Linear Approximation of a function $f(x)$ is a common use/application of the derivative. Formally, the linear approximation of $f(x)$ near $x = a$ is given by the equation of the tangent line at $(a, f(a))$. The slope of the tangent line is $f'(a)$, hence the point-slope formula gives the linear approximation equation:

$$L(x) - f(a) = f'(x)(x-a)$$
$$\Rightarrow L(x) = f(a) + f'(a)(x-a)$$

For some value $x = b$ located near $x = a$, we use the linear approximation equation to *estimate* the true function value:

$$f(b) \approx L(b) = f(a) + f'(a)(b-a)$$

The error of approximation is given by:

$$E = L(b) - f(b)$$
$$E = f(a) + f'(a)(b-a) - f(b)$$

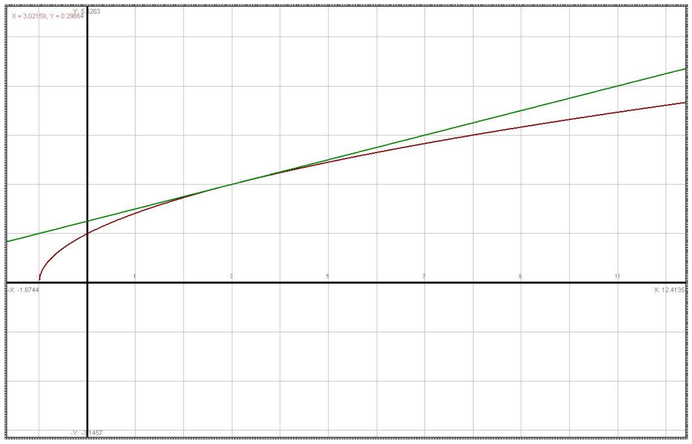
Geometrically, the error is the difference between the $y-$value on the tangent line and the $y-$value on the graph of the function at $x = b$. Whenever the tangent is above the graph of the function, the approximation is an overestimate. Whenever below, it is an underestimate.

Example: Use the tangent to $f(x) = \sqrt{x+1}$ to approximate the square root of 4.1.

Note that $\sqrt{4.1} = \sqrt{3.1+1}$, so we can use the linear approximation near $x = a = 3$ with $b = 3.1$, a number not too far from the point where the tangent is drawn ($x = 3$).

$$L(x) = f(3) + f'(3)(x-3) = 2 + \frac{1}{4}(x-3)$$
$$L(3.1) = 2 + \frac{1}{4}(3.1-3) = 2 + \frac{1}{40} = \frac{81}{40} = 2.025$$

We compare the approximation to the “true” value given by Wolfram Alpha: 2.02485 (also an approximation!). The numbers are very close. We conclude that the linear approximation yields an overestimate, as verified by the graph. The tangent line lies above the graph of $f(x) = \sqrt{x+1}$ at $x = 3.1$.



The graph of $f(x) = \sqrt{x+1}$ is concave down, so the linear approximations will be overestimates regardless of where the tangent line is drawn.