# Solutions to the 1999 AP Calculus AB Exam Free Response Questions

Louis A. Talman
Department of Mathematical & Computer Sciences
Metropolitan State College of Denver

## Problem 1.

### ■ a.

If $v[t] = t \sin t^2$ for $t \geq 0$, then when $t = 1.5$, we have

```
v[t_] = t Sin[t^2]

t Sin[t^2]
```

```
v[1.5]

1.16711
```

$v[1.5] > 0$. Velocity is positive, so motion is in the positive direction, or upward.

### ■ b.

Accleration at $t = 1.5$ is

```
v'[t]

2 t^2 Cos[t^2] + Sin[t^2]
```

```
v'[1.5]

-2.04871
```

Acceleration at $t = 1.5$ is negative, so velocity is decreasing.

### c.

By the Fundamental Theorem of Calculus, $y[t] = y[0] + \int_0^t v[t]\, dt = y[0] + \int_0^t \tau \sin[\tau^2]\, d\tau = 3 + \sin^2\left[\frac{t^2}{2}\right]$. Thus, $y[2] = 3 + \sin^2[2]$.

```
3.0 + Sin[2.0]^2

3.82682
```

### d.

Total distance traveled when $0 \le t \le 2$ is $\int_0^2 | v[t] |\, d\, dt = \int_0^2 | \tau \sin[\tau^2] |\, d\tau = \int_0^{\sqrt{\pi}} \tau \sin[\tau^2]\, d\tau - \int_{\sqrt{\pi}}^2 \tau \sin[\tau^2]\, d\tau$. Integrating numerically, we obtain

```
NIntegrate[τ Sin[τ^2], {τ, 0, √π}] -
 NIntegrate[τ Sin[τ^2], {τ, √π, 2}]

1.17318
```

## Problem 2

### a.

The area of the pictured region is $\int_{-2}^2 (4 - x^2)\, dx = (4x - \frac{1}{3}x^3) \Big|_{-2}^{2} = \frac{32}{3}$.

### b.

Revolving the pictured region about the $x$-axis produces a solid whos volume is $\pi \int_{-2}^2 (16 - x^4)\, dx = \frac{256}{5}\pi$.

## ■ c.

$\pi \int_{-2}^{2} \left( (k - x^2)^2 - (k - 4)^2 \right) dx = \frac{256}{5} \pi.$

For the curious:

```
Solve[∫_{-2}^{2} ((k - x^2)^2 - (k - 4)^2) dx == 256/5, k]

{{k → 24/5}}
```

# Problem 3.

First, let us assign appropriate values to the function $R$:

```
Map[Apply[(R[#1] = #2) &, #] &,
 {{0, 9.6}, {3, 10.4}, {6, 10.8}, {9, 11.2}, {12, 11.4},
  {15, 11.3}, {18, 10.7}, {21, 10.2}, {24, 9.6}}]

{9.6, 10.4, 10.8, 11.2, 11.4, 11.3, 10.7, 10.2, 9.6}
```

## ■ a.

```
Sum[R[k] 6, {k, 3, 21, 6}]

258.6
```

Approximately 258.6 gallons of water flows out of the pipe in the period $0 \le t \le 24$.

## ■ b.

The function $R$ is given differentiable on $[0, 24]$, and it is also given that $R[0] = 9.6 = R[24]$. By Rolle's Theorem, there must be a $t \in (0, 24)$ such that $R'[t] = 0$.

### ■ c.

The average rate of flow is approximately $\frac{1}{24}\int_0^{24}\frac{1}{79}(768+23t-t^2)\,dt=\frac{852}{79}$ gallons per hour.

## Problem 4

### ■ a.

An equation for the line tangent to the graph of $f$ at the point where $x=0$ is $y = 2-3x$.

### ■ b.

We don't have enough information to decide whether or not $f$ has an inflection point at $x=0$. If $f(x) = 2-2x-\sin x$, then $f$ meets the given conditions and $f''(x)=\sin x$, so that $f''(x)$ undergoes a sign change at $x=0$ and $f$ has an inflection point there. However, if $f(x)=3-3x-\frac{x^2}{2}-\cos x$, then $f$ meets the given conditions, but $f''(x)=\cos x-1$, which does not undergo a sign change at $x=0$—meaning that this $f$ has no inflection point at $x=0$.

### ■ c.

An equation for the line tangent to the graph of $f$ at the point where $x=0$ is $y = g[0]+g'[0](x-0) = 4+[e^0(3\cdot 2+2(-3))]x$, or $y=4$.

### ■ d.

$g'[x] = e^{-2x}(3f[x]+2f'[x])$, so
$g''[x] = [\frac{d}{dx}(e^{-2x})](3f[x]+2f'[x]) + e^{-2x}[\frac{d}{dx}(3f[x]+2f'[x])] =$
$-2e^{-2x}(3f[x]+2f'[x]) + e^{-2x}(3f'[x]+2f''[x]) =$
$-6e^{-2x}f[x]-4e^{-2x}f'[x]+3e^{-2x}f'[x]+2e^{-2x}f''[x] = e^{-2x}(-6f[x]-f'[x]+2f''[x]$ .

Hence $g''[0] = (-6)\cdot(2)-(-3)+2\cdot 0=-9$. By the Second Derivative Test, $g$ has a local maximum at $x=0$.

# Problem 5

## a.

On the interval [2, 4], the graph is symmetric about the point (3, 0), so the integral over that interval is zero. Consequently, $\int_1^4 f[t]\, dt = \int_1^2 f[t]\, dt$ is the area of the trapezoid whose corners are (1, 0), (2, 0), (2, 1), and (1, 4), or $\frac{(4+1)}{2} \cdot 1 = \frac{5}{2}$. Thus, $g[4] = \frac{5}{2}$. $g[-2] = \int_1^{-2} f[t]\, dt = -\int_{-2}^1 f[t]\, dt$ is the negative of the area of a triangle with base 3 and height 4, or $-6$.

## b.

By the Fundamental Theorem of Calculus, $g'[x] = \frac{d}{dx} \int_1^x f[t]\, dt = f[x]$. Hence $g'[1] = f[1] = 4$.

## c.

The absolute minimum of $g[x]$ for $-2 \le x \le 4$ lies either at a point where $g'[x] = 0$ or where $x = -2$ or where $x = 4$. We have already seen, in part a above, that $g[-2] = -6$ and that $g[4] = \frac{5}{2}$. If $g'[x] = 0$, then, by our first observation in part b above, $f[x] = 0$. This happens only at $x = -2$, which we have already computed, and at $x = 3$. But $f$, and therefore $g'$ undergoes a sign change from positive to negative as $x$ increases through 3, so $x = 3$ must give a local maximum for $g$. It follows that $g$ attains its local minimum of $-6$ at $x = -2$.

## d.

If $g$ is to have an inflection point somewhere, then $g'$ must change from increasing to decreasing or from decreasing to increasing at that point. This happens when $x = 1$, but not when $x = 2$. Hence $g$ has an inflection point at just one of the two points in question.

# Problem 6

## a.

If $y = x^{-2}$, then $y' = -2\, x^{-3}$. So the equation of a line tangent to the curve $y = \frac{1}{x^2}$ at the point $(w, \frac{1}{w^2})$ is $y = \frac{1}{w^2} - \frac{2}{w^3}(x - w) = -\frac{2}{w^3}\, x + \frac{3}{w^2}$. Hence $k = 3\, \frac{w}{2}$. When $w = 3$, this gives $k = \frac{9}{2}$.

## b.

We saw in part a. that, in general, $k = \frac{3\, w}{2}$.

## ■ c.

From part a or part b, we have $k = 3\frac{w}{2}$. Hence, by implicit differentiation with respect to $t$, $\frac{dk}{dt} = \frac{3}{2}\frac{dw}{dt}$. If $\frac{dw}{dt} = 7$, then when $w = 5$ we must have $\frac{dk}{dt} = \frac{21}{2}$ units per second.

## ■ d.

The tangent line at $(w, \frac{1}{w^2})$ crosses the $x$-axis when $0 = -\frac{2}{w^3}x + \frac{3}{w^2}$, or when $x = \frac{3}{2}w$. The area of the triangle PQR is therefore $A = \frac{1}{2}\cdot(\frac{3}{2}w - w)\cdot\frac{1}{w^2} = \frac{1}{4w} = \frac{1}{4}w^{-1}$. Consequently, $\frac{dA}{dt} = -\frac{1}{4}w^{-2}\frac{dw}{dt}$. When $w = 5$ and $\frac{dw}{dt} = 7$, this gives $\frac{dA}{dt} = -\frac{1}{4}\cdot\frac{1}{25}\cdot 7 = -\frac{7}{100}$. Because this is negative, area is decreasing at this instant. (Note: Because $w$ and $\frac{dw}{dt}$ are both positive at the critical instant, it suffices for solving this problem to notice that the differentiation produced a minus sign.)