Solutions to the 1998 AP Calculus AB Exam Free Response Questions

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Problem 1.

a.

The area of the region R is

$$\int_0^4 \sqrt{x}\, dx$$

$$\frac{16}{3}$$

b.

$$\text{Solve}\left[\int_0^h \sqrt{x}\, dx == \frac{8}{3},\ h\right]$$

$$\{\{h \to 2\, 2^{1/3}\}\}$$

We need $h = 2^{4/3}$.

c.

The volume generated by revolving R about the x-axis is

$$\pi \int_0^4 x \, dx$$

8π

d.

```
Solve[π ∫_0^k x dx == 4 π, k]
```

$\{\{k \to -2\sqrt{2}\}, \{k \to 2\sqrt{2}\}\}$

We need $k = 2\sqrt{2}$.

Problem 2

a.

$\lim_{x\to-\infty} 2\,x\,e^{2x} = \lim_{x\to-\infty} \frac{2x}{e^{-2x}}$. L'Hôpital's Rule is applicable to the latter expression. Thus, $\lim_{x\to-\infty} \frac{2x}{e^{-2x}} = \lim_{x\to-\infty} \frac{2}{-2\,e^{-2x}} = 0$.

b.

If $f[x] = 2\,x\,e^{2x}$, then $f'[x] = (2 + 4\,x)\,e^{2x}$. Consequently, $f'[x] = 0$ only when $x = -\frac{1}{2}$. Now (by part a, above) $\lim_{x\to-\infty} f[x] = 0$, while $\lim_{x\to\infty} f[x] = \infty$. Consequently there are numbers x_1 and x_2, $x_1 < \frac{-1}{2} < x_2$, such that $x \le x_1$ implies that $f[x] \ge -\frac{1}{100} > f[-1/2] = -e^{-1}$ and $x_2 \le x$ implies that $f[x] \ge -\frac{1}{100} > f[-1/2]$. But f must have an absolute minimum in the interva l $[x_1, x_2]$, and it cannot be located at either x_1 or x_2. Because $x = -1/2$ is the only critical point in this interval, it must give the absolute minimum for $f[x]$ when $x_1 \le x \le x_2$, and therefore for $-\infty < x < \infty$.

c.

By the observations we have made in part b. above, the range of f is $[-e^{-1}, \infty)$.

■ d.

Let us assume, for the moment, that $b > 0$. Then, arguing as we have in parts a. and b. above, we find that $f[x] = b\,x\,e^{b\,x}$ has an absolute minimum at $x = -\frac{1}{b}$. This minimum value is $f[\frac{-1}{b}] = -e^{-1}$, which is independent of b. If $b < 0$, we obtain the same result after the change of variables $u = -x$, which amounts to a reflection about the y-axis.

Problem 3.

The following unpleasant syntax assigns the values given in the table to the function v.

```
Map[Apply[(v[#1] = #2) &, #] &,
 {{0, 0}, {5, 12}, {10, 20}, {15, 30}, {20, 55}, {25, 70},
  {30, 78}, {35, 81}, {40, 75}, {45, 60}, {50, 72}}]

{0, 12, 20, 30, 55, 70, 78, 81, 75, 60, 72}
```

■ a.

Acceleration is the derivative of velocity, so acceleration is positive at each point where the tangent line to the graph of the velocity function has positive slope. From the picture given, we find that acceleration is positive on the intervals [0, 35) and (45, 50].

■ b.

Taking $a[t]$, as the acceleration, average acceleration is $\frac{1}{50}\int_0^{50} a[t]\,dt = \frac{1}{50}(v[50] - v[0]) = \frac{1}{50}(72-0) = \frac{36}{25}$ ft/sec^2.

■ c.

The acceleration $a[40]$ is approximately $\frac{v[45]-v[35]}{10}$, or

```
v[45] - v[35]
-------------
     10

 21
-——
 10
```

(This is in feet/sec^2.)

■ d.

The required approximation is:

```
Sum[10 v[t], {t, 5, 45, 10}]

2530
```

The integral measures, in feet, distance traveled during the time interval $0 \le t \le 50$.

Problem 4

■ a.

At the point $(1,\ f[1])$, the slope is $\frac{3(1)^2+1}{2 f[1]} = \frac{1}{2}$.

■ b.

The line tangent to $y = f[x]$ at $(1,\ f[1])$ has equation $y = 4 + \frac{1}{2}(x - 1)$. Consequently, $f[1.2]$ is approximately $4 + \frac{1}{2}(0.2) = \frac{41}{10}$.

■ c.

From $f'[x] = \frac{3x^2+1}{2 f[x]}$, together with $f[1] = 4$, we have $2\int_1^x f[t]\, f'[t]\, dt = \int_1^x (3t^2 + 1)\, dt$, or $2\int_4^{f[x]} u\, du = \int_1^x (3t^2 + 1)\, dt$. Thus, $f[x]^2 - 16 = x^3 + x - 2$, or $f[x] = \sqrt{x^3 + x + 14}$. (We have chosen the positive square root because $f[1] = 4 > 0$.)

■ d.

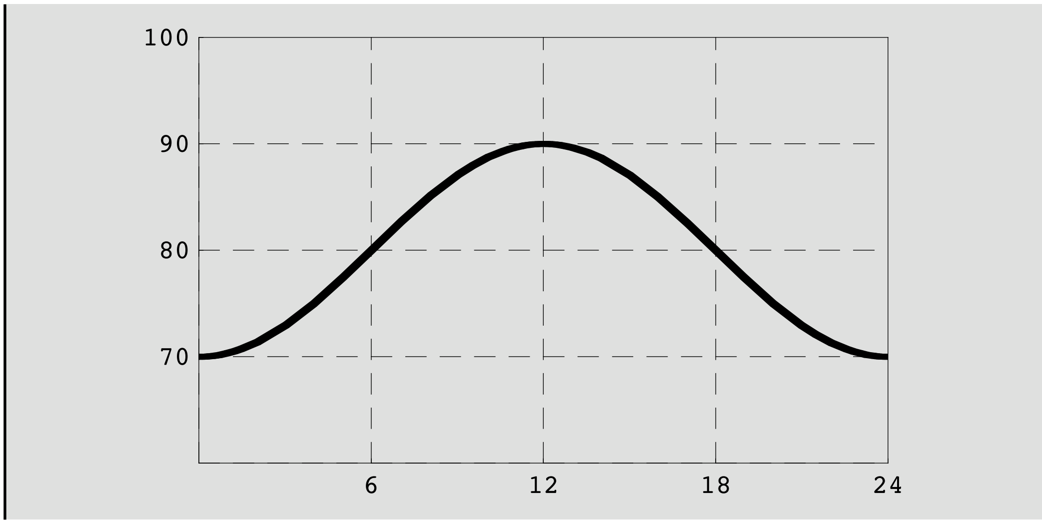
$f[1.2]$ is given by

```
√(x³ + x + 14) /. x → 1.2

4.11437
```

Problem 5

■ a.



■ b.

Average temperature is $\frac{1}{14-6}\int_6^{14} F[t]\,dt$:

$$\frac{1}{14-6}\int_6^{14}\left(80 - 10\,\mathrm{Cos}\left[\frac{\pi\,t}{12}\right]\right)dt$$

$$\frac{1}{8}\left(640+\frac{180}{\pi}\right)$$

```
N[%]

87.162
```

To the nearest degree, this is 87 degrees.

c.

```
FindRoot[80 - 10 Cos[π t/12] == 78, {t, 6}]
```

```
{t → 5.23087}
```

```
a = t /. %
```

```
5.23087
```

```
FindRoot[80 - 10 Cos[π t/12] == 78, {t, 18}]
```

```
{t → 18.7691}
```

```
b = t /. %
```

```
18.7691
```

The air conditioner ran when $5.231 \le t \le 18.769$.

d.

The approximate total cost is $0.05 \int_a^b (2 - 10\,\text{Cos}[\frac{\pi t}{12}])\, dt$, or

```
0.05 ∫_a^b (2 - 10 Cos[π t/12]) ⅆt
```

```
5.09637
```

To the nearest cent, this is $5 .10.

Problem 6

■ a.

If $2y^3 + 6x^2 y - 12x^2 + 6y = 1$, then, differentiating implicitly with respect to x while treating y as a function of x, we obtain $6y^2 y' + 12xy + 6x^2 y' - 24x + 6y' = 0$. Hence, $(6y^2 + 6x^2 + 6)y' = 24x - 12xy$, or $y' = (4x - 2xy)/(x^2 + y^2 + 1)$.

■ b.

Tangent lines are horizontal where $y' = 0$. This can happen, according to part a., above, only where $4x - 2xy = 0$, from which we conclude that $x = 0$ or that $y = 2$. If $x = 0$, then from $2y^3 + 6x^2 y - 12x^2 + 6y = 1$ we conclude that $2y^3 + 6y = 1$, and we thus have, approximately,

```
FindRoot[y^3 + 6 y == 1, {y, 0}]

{y → 0.165906}
```

An equation for the line tangent to the curve at the corresponding point is $y = 0.166$ (approximately).

If $y = 2$, then $16 + 12x^2 - 12x^2 + 12 = 1$, or $28 = 1$, so there are no horizontal tangent lines where $y = 2$.

■ c.

At the point where the line through the origin with slope -1 is tangent to the curve, we must have $y = -x$, because that is the equation of the line in question and the point of tangency lies on that line. Hence $-2x^3 - 6x^3 - 12x^2 - 6x = 1$, or $8x^3 + 12x^2 + 6x + 1 = 0$. Equivalently, $(2x + 1)^3 = 0$, so that the only root of this equation is $x = \frac{-1}{2}$. Because $y = -x$ at the point in question , $y = \frac{1}{2}$. The coordinates of the point are $(-\frac{1}{2}, \frac{1}{2})$.