SOLUTIONS / KEY D. SHUBLEKA

Part I. Multiple-Choice Questions (5 points each; please circle the correct answer.)

1. The series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n e^n}{n^2}$ diverges because

   I. The terms do not tend to 0 as n tends to ∞.

   II. The terms are not all positive.

   III. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$.

   (A) I only

   (B) II only

   (C) III only

   (D) I and II only

   (E) I and III only

$\lim_{n\to\infty} \frac{(-1)^n \cdot e^n}{n^2}$ DNE ($e^n \to \infty$ [faster], $n^2 \to \infty$) TRUE.

II. The terms don't need to all be positive for a series to converge, so false.

III. This is the Ratio Test.

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{e^{n+1}}{(n+1)^2} \cdot \frac{n^2}{e^n}\right| \longrightarrow e > 1$ ✓

2. The interval of convergence for the series $\sum_{n=1}^{\infty} \frac{(3x+2)^{n+1}}{n^{5/2}}$ is

   (A) $-1 \le x < -\frac{1}{3}$

   (B) $-1 < x \le -\frac{1}{3}$

   (C) $-1 \le x \le -\frac{1}{3}$

   (D) $\frac{1}{3} \le x \le 1$

   (E) $-1 < x < \frac{1}{3}$

Ratio Test:

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(3x+2)^{n+2}}{(n+1)^{5/2}} \cdot \frac{n^{5/2}}{(3x+2)^{n+1}}\right| = |3x+2| \cdot \left(\frac{n}{n+1}\right)^{5/2} \longrightarrow |3x+2| < 1$

$-1 < 3x+2 < 1$

$-3 < 3x < -1$

$-1 < x < -\frac{1}{3}$

Endpoints:

$x = -1: \sum \frac{(-1)^{n+1}}{n^{5/2}}$

$x = -\frac{1}{3}: \sum \frac{1}{n^{5/2}}$

Converges absolutely, hence converges at both endpoints.

3. Given that $f(x) = \sum_{n=0}^{\infty} \frac{n(x-a)^n}{2^n}$ on the interval of convergence of the Taylor series, $f^{(4)}(a) =$

   (A) 0

   (B) 6

   (C) 9

   (D) $\frac{1}{4}$

   (E) $\frac{1}{4!}$

$\frac{f^{(4)}(a)}{4!} = \frac{n}{2^n}\Big|_{n=4} = \frac{4}{2^4}$

$\frac{f^{(4)}(a)}{24} = \frac{4}{16} \Rightarrow f^{(4)}(a) = 6.$

Recall: $C_n = \frac{f^{(n)}(a)}{n!}$!

4. Which of the following series converge?

I. $\sum_{n=1}^{\infty}\left(\frac{n^2-n+5}{n^{7/2}+1}\right)$. ✓

II. $\sum_{n=1}^{\infty}\frac{(-1)^n 3}{n}$ ✓

III. $\sum_{n=1}^{\infty}\left(\frac{\cos 2n\pi}{n^2}\right)$. ✓

(A) I and II only
(B) I and III only
(C) II and III only
(D) They all do!
(E) None of them do!

I. Limit Comparison Test with $\sum_{n=1}^{\infty}\frac{n^2}{n^{7/2}}=\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$

By p-series test, the reference series converges. By limit comparison test, the given series converges too.

II $= 3\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ Alternating Harmonic Series. It converges.

III $= \sum_{n=1}^{\infty}\frac{1}{n^2}$ (p-series test ✓) (Converges)

5. $\underbrace{1-\frac{\pi^2}{2!}+\frac{\pi^4}{4!}+\cdots+(-1)^n\frac{\pi^{2n}}{(2n)!}+\cdots}$ =

(A) 0
(B) -1
(C) π
(D) 1
(E) $-\pi$

$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+(-1)^n\cdot\frac{x^{2n}}{(2n)!}+\cdots$

$\rightarrow = \cos\pi = -1$ ✓

Part II. Free-Response Questions

1. A function f is defined by

$$f(x) = \frac{1}{4} + \frac{2}{4^2}x + \frac{3}{4^3}x^2 + \cdots + \frac{n+1}{4^{n+1}}x^n + \cdots$$

for all x in the interval of convergence of the given power series.

(a) **(4 points)** Find the interval of convergence for this power series. Show the work that leads to your answer.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{[(n+1)+1]x^{n+1}}{4^{(n+1)+1}} \cdot \frac{4^{n+1}}{(n+1)x^n}\right| = \left|\frac{n+2}{n+1} \cdot \frac{4^{n+1}}{4^{n+2}} \cdot x\right|$$

$$= \left(\frac{n+2}{n+1}\right) \cdot \frac{1}{4}|x| \longrightarrow \frac{1}{4}|x| < 1 \iff |x| < 4$$

$$-4 < x < 4$$

Endpoints:

$x = -4$: $\sum_{n=0}^{\infty} \frac{n+1}{4^{n+1}} \cdot (-4)^n = \sum_{n=0}^{\infty} \frac{(n+1)(-1)^n 4^n}{4^n \cdot 4} = \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)}{4}$

$\lim_{n\to\infty} a_n$ DNE. Hence it diverges.

$x = 4$: $\sum_{n=0}^{\infty} \frac{(n+1)}{4^{n+1}} \cdot 4^n$ By the Divergence Test, it diverges.

Interval: $(-4, 4)$ $R = 4$.

(b) **(3 points)** Find $\lim_{x\to 0} \frac{f(x) - \frac{1}{4}}{x}$.

$$\lim_{x\to 0} \frac{\left[\frac{1}{4} + \frac{2}{4^2}x + \frac{3}{4^3}x^2 + \cdots\right] - \frac{1}{4}}{x} = \lim_{x\to 0} \frac{\left[\frac{2}{4^2}x + \frac{3}{4^3}\cdot x^2 + \cdots\right]}{x}$$

$$= \lim_{x\to 0}\left(\frac{2}{4^2} + \frac{3}{4^3}x + \cdots\right) = \frac{2}{4^2} = \boxed{\frac{1}{8}}$$

#1, continued; $f(x) = \frac{1}{4} + \frac{2}{4^2}x + \frac{3}{4^3}x^2 + \cdots + \frac{n+1}{4^{n+1}}x^n + \cdots$

(c) (**3 points**) Write the first three nonzero terms and the general term for an infinite series that represents $\int_0^2 f(x)\,dx$.

$$\int_0^2 \frac{1}{4} + \frac{2}{4^2}x + \frac{3}{4^3}x^2 + \ldots + \frac{(n+1)}{4^{n+1}}x^n + \ldots \, dx$$

$$= \left(\frac{1}{4}x + \frac{2}{4^2}\cdot\frac{x^2}{2} + \frac{3}{4^3}\frac{x^3}{3} + \ldots + \frac{n+1}{4^{n+1}}\frac{x^{n+1}}{n+1} + \ldots\right)\Big|_0^2$$

$$= \left(\frac{x}{4} + \frac{x^2}{4^2} + \frac{x^3}{4^3} + \ldots + \frac{x^{n+1}}{4^{n+1}} + \ldots\right)\Big|_0^2$$

$$= \frac{2}{4} + \frac{2^2}{4^2} + \frac{2^3}{4^3} + \ldots + \frac{2^{n+1}}{4^{n+1}} + \ldots$$

$$= \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots + \left(\frac{1}{2}\right)^{n+1} + \ldots$$

(d) (**4 points**) Find the sum of the series determined in part (c).

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n+1} \text{ or } \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots$$

$a = \frac{1}{2}$

$r = \frac{1}{2}$ $\quad |r| < 1$

$$= \frac{a}{1-r} = \frac{1/2}{1-1/2} = \frac{1/2}{1/2} = \boxed{1}$$

2. Let f be a function with derivatives of all orders and for which $f(2) = 7$. When n is odd, the nth derivative of f at $x = 2$ is 0. When n is even and $n \geq 2$, the nth derivative of f at $x = 2$ is given by $f^{(n)}(2) = \dfrac{(n-1)!}{3^n}$.

(a) **(4 points)** Write the sixth-degree Taylor polynomial for f about $x = 2$.

$$T_6(x) = f(2) + \underbrace{f'(2)}_{\text{zero}}(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{\overbrace{f'''(2)}^{\text{zero}}}{3!}(x-2)^3$$

$$+ \cdots + \frac{f^{(6)}(2)}{6!}(x-2)^6 = f(2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f^{(4)}(2)}{4!}(x-2)^4$$

$$+ \frac{f^{(6)}(2)}{6!}(x-2)^6 = 7 + \left[\frac{1!}{3^2}\right]\Big/2!\,(x-2)^2 + \left[\frac{3!}{3^4}\right]\Big/4!\,(x-2)^4 + \left[\frac{5!}{3^6}\right]\Big/6!\,(x-2)^6.$$

(b) **(3 points)** In the Taylor series for f about $x = 2$, what is the coefficient of $(x-2)^{2n}$ for $n \geq 1$?

(a) continued: $T_6(x) = 7 + \frac{1}{2!\,3^2}(x-2)^2 + \frac{1}{4 \cdot 3^4}(x-2)^4 + \frac{1}{6 \cdot 3^6}(x-2)^6.$

b) $$C_{2n} = \frac{f^{(2n)}(2)}{(2n)!} = \frac{\left[\frac{(2n-1)!}{3^{2n}}\right]}{(2n)!} = \frac{1}{2n \cdot 3^{2n}}$$ ✓

(c) **(4 points)** Find the inteval of convergence of the Taylor series for f about $x = 2$. Show the work that leads to your answer.

$$a_n = \frac{f^{(n)}(2)}{n!}(x-2)^n = \frac{(n-1)!}{n! \cdot 3^n} \cdot (x-2)^n = \frac{(x-2)^n}{n \cdot 3^n}$$

$$a_{n+1} = \frac{(x-2)^{n+1}}{(n+1)3^{n+1}} \Longrightarrow \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-2)^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n \cdot 3^n}{(x-2)^n}\right|$$

$$= \frac{|x-2|}{3} \cdot \left(\frac{n}{n+1}\right) \longrightarrow \frac{|x-2|}{3} < 1$$

$$|x-2| < 3$$
$$-3 < x-2 < 3$$
$$-1 < x < 5$$

$x = -1$: $\displaystyle\sum_{n=2}^{\infty} \frac{(-3)^n}{n \cdot 3^n} = \sum_{n=2} \frac{(-1)^n}{n}$ ✓ converges

$x = 5$: $\displaystyle\sum_{n=2}^{\infty} \frac{3^n}{n \cdot 3^n} = \sum_{n=2}^{\infty} \frac{1}{n}$ Harmonic -It diverges.

Interval: $[-1, 5)$ $R = 3$.

Part III. Multiple-Choice Questions (5 points each; please circle the correct answer.)

1. The series $\sum_{n=1}^{\infty} \frac{\sqrt{n^p - 1}}{n^{p+2} + 1}$ will converge, provided that

(A) $p > 1$
(B) $p > 2$
(C) $p > -1$
(D) $p > -2$
(E) $p > 0$

$n^p - 1 \geq 0 \iff n^p \geq 1 \iff p \geq 0$ *

Also: $\sqrt{n^p - 1} \overset{vs.}{\sim} n^{p/2}$

$n^{p+2} + 1 \overset{vs.}{\sim} n^{p+2}$

Consider: $\sum_{n=1}^{\infty} \frac{n^{p/2}}{n^{p+2}} = \sum_{n=1}^{\infty} \frac{1}{n^{p/2+2}}$

** $p > -2 \Leftarrow \frac{p}{2} > -1 \iff \frac{p}{2} + 2 > 1$

* and **

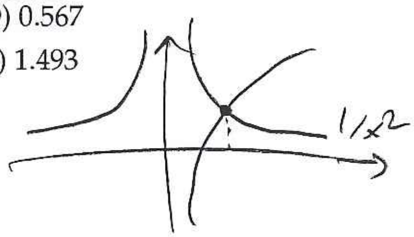
2. The graph of the function represented by the Taylor series $\sum_{n=0}^{\infty} n(x+1)^{n-1}$ intersects the graph of $y = \ln x$ at $x \approx$

(A) 1.763
(B) 0.703
(C) 1.532
(D) 0.567
(E) 1.493

$\sum_{n=0}^{\infty} n(x+1)^{n-1} = \frac{d}{dx}\left(\sum_{n=0}^{\infty} (x+1)^n\right) = \frac{d}{dx}\left(\frac{1}{1-(x+1)}\right)$

↑ Geometric

$= \frac{d}{dx}\left(\frac{-1}{x}\right) = \frac{1}{x^2}$



$\begin{cases} y_1 = 1/x^2 \\ y_2 = \ln x \end{cases} \rightarrow$ TI-84 ?

3. Using the fourth-degree Maclauren polynomial of the function $f(x) = e^x$ to estimate e^{-2}, this estimate is

(A) 7.000
(B) 0.333
(C) 0.135
(D) 0.067
(E) 0.375

$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

$e^{-2} \approx 1 - 2 + \frac{(-2)^2}{2!} + \frac{(-2)^3}{3!} + \frac{(-2)^4}{4!}$

$= -1 + \frac{4}{2} - \frac{8}{6} + \frac{16}{24}$

$= -1 + 2 - \frac{4}{3} + \frac{2}{3}$

$= 1 - \frac{2}{3} = 1/3 \approx 0.333$

4. What is the approximation of the value of $\cos(2°)$ obtained by using the sixth-degree Taylor polynomial about $x = 0$ for $\cos x$?

(A) $1 - 2 + \frac{2}{3} - \frac{4}{45}$

(B) $1 + 2 + \frac{16}{24} + \frac{64}{720}$.

(C) $1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720}$

(D) $1 - \frac{\pi^2}{2 \cdot 90^2} + \frac{\pi^4}{4! \cdot 90^4} - \frac{\pi^6}{6! \cdot 90^6}$ ✓

(E) $1 + \frac{\pi^2}{2 \cdot 90^2} + \frac{\pi^4}{4! \cdot 90^4} + \frac{\pi^6}{6! \cdot 90^6}$

$$\cos(2^\circ) = \cos\left(2 \cdot \frac{\pi}{180}\right) = \cos\left(\frac{\pi}{90}\right)$$
$$= 1 - \frac{(\pi/90)^2}{2!} + \frac{(\pi/90)^4}{4!} - \frac{(\pi/90)^6}{6!}$$
$$+ \ldots$$

5. Which of the following gives a Taylor polynomial approximation about $x = 0$ for $\sin 0.5$, correct to four decimal places?

(A) $0.5 + \frac{(0.5)^3}{3!} + \frac{(0.5)^5}{5!}$

(B) $0.5 - \frac{(0.5)^3}{3!} + \frac{(0.5)^5}{5!}$

(C) $0.5 - \frac{(0.5)^3}{3} + \frac{(0.5)^5}{5}$

(D) $0.5 + \frac{(0.5)^2}{2!} + \frac{(0.5)^3}{3!} + \frac{(0.5)^4}{4!} + \frac{(0.5)^5}{5!}$

(E) $0.5 - \frac{(0.5)^2}{2!} + \frac{(0.5)^3}{3!} - \frac{(0.5)^4}{4!} + \frac{(0.5)^5}{5!}$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$
$$+ \ldots + \frac{(-1)^n}{(2n+1)} x^{(2n+1)} + \ldots$$

Part IV. Free-Response Questions

1. The function f has derivatives of all orders for all real numbers x. Assume $f(2) = -3$, $f'(2) = 5$, $f''(2) = 3$, and $f'''(2) = -8$.

   (a) **(3 points)** Write the third-degree Taylor polynomial for f about $x = 2$ and use it to approximate $f(1)$.

$$T_3(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3$$
$$= -3 + 5(x-2) + \frac{3}{2}(x-2)^2 + \frac{-8}{6}(x-2)^3$$
$$f(1) \approx T_3(x) = -3 + 5(-1) + \frac{3}{2}\cdot(-1)^2 - \frac{4}{3}(-1)^3 =$$
$$= -3 - 5 + \frac{3}{2} + \frac{4}{3} = -8 + 1.5 + 1.\overline{333} = \frac{-48}{6} + \frac{17}{6} = \boxed{\frac{-31}{6}}$$

   (b) **(4 points)** The fourth derivative of f satisfies the inequality $|f^{(4)}(x)| \leq 3$ for all x in the closed interval $[1, 2]$. Use the Lagrange error bound on the approximation to $f(1)$ found in part (a) to explain why $f(1) \neq -5$.

$$\text{Error Bound} = \left| \frac{f^{(4)}(z)}{4!}(x-2)^4 \right|$$
$$= \left| \frac{3\cdot(1-2)^4}{4!} \right| = \frac{3}{4!} = \frac{3}{24} = \frac{1}{8}.$$

$$\left| \underbrace{f(1)}_{\text{true}} - \underbrace{T_3(1)}_{\text{approx.}} \right| < \frac{1}{8}$$
$$-\frac{1}{8} < f(1) + \frac{31}{6} < \frac{1}{8} \Rightarrow \boxed{-5.291\overline{6} < f(1) < -5.041\overline{6}}$$

Hence $f(1) \neq -5$

   (c) **(4 points)** Write the fourth-degree Taylor polynomial, $P(x)$, for $g(x) = f(x^2 + 2)$ about $x = 0$. Use P to explain why g must have a relative minimum at $x = 0$.

$$g(x) = f(x^2+2) = -3 + 5(x^2+2-2) + \frac{3}{2}(x^2+2-2)^2 =$$
$$= -3 + 5x^2 + \frac{3}{2}x^4 = g(0) + g'(0)\cdot x + \frac{g''(0)}{2!}x^2 + \frac{g'''(0)}{3!}x^3 + \frac{g^{(4)}(0)}{4!}x^4$$

$u = x^2+2$

$$f(u) = -3 + 5(u-2) + \frac{3}{2}(u-2)^2 + \ldots$$

$$g'(0) = 0$$

$$\frac{g''(0)}{2!} = 5 \Rightarrow g''(0) = 10$$

By the Second Derivative Test $\rightarrow$ local min.

2. Let f be a function having derivatives of all orders for al real numbers. The third-degree Taylor polynomial for f about $x = 2$ is given by

$$T(x) = -5(x-2)^2 - 3(x-2)^3.$$

(a) **(2 points)** Find $f(2)$ and $f''(2)$.

$$T_3(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-3)^3$$

$$= 0 + 0(x-2) + (-5)(x-2)^2 + (-3)\cdot(x-2)^3$$

So: $f(2) = 0$ $\quad \frac{f''(2)}{2!} = -5 \Rightarrow f''(2) = -10$

(b) **(4 points)** Is there enough information given to determine whether f has a critical point at $x = 2$? If not, explain why not. If so, determine whether $f(2)$ is a relative maximum, a relative minimum, or neither, and justify your answer.

@ $x = 2$, $f'(2) = 0$

and $f''(2) = -10 < 0$

Yes, $x=2$ is a critical number and $f(2)$ is a local max, by the second derivative test.

#2, continued; $T(x) = -5(x-2)^2 - 3(x-2)^3$.

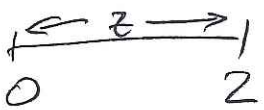
(c) **(4 points)** Use $T(x)$ to find an approximation for $f(0)$. Is there enough information given to determine whether f has a critical point at $x = 0$? If not, explain why not. If so, determine whether $f(0)$ is a relative maximum, a relative minimum, or neither, and justify your answer.

$$f(0) \approx T_3(0) = -5(-2)^2 - 3(0-2)^3 =$$
$$= -20 + 24 = 4.$$

The coefficients of $T(x)$ give information of the form $\frac{f^{(n)}(2)}{n!}$; hence we don't know the derivative values of $f(x)$ @ $x=0$.

$\longrightarrow$ Not enough info.

(d) **(4 points)** The fourth derivative of f satisfies the inequality $|f^{(4)}(x)| \leq 5$ for all x in the closed interval $[0, 2]$. Use the Lagrange error bound on the approximation to $f(0)$ found in part (c) to explain why $f(0)$ is positive.



$$\text{Error Bound} = \left| \frac{f^{(4)}(z)}{4!}(x-2)^4 \right|$$

$$= \left| \frac{5 \cdot (-2)^4}{4!} \right| = \frac{5 \cdot 16}{24} = \frac{10}{3}.$$

$$|f(0) - T_3(0)| < \frac{10}{3}$$

$$|f(0) - 4| < \frac{10}{3}$$

$$-\frac{10}{3} < f(0) - 4 < \frac{10}{3}$$

$$-\frac{10}{3} + \frac{12}{3} < f(0) < \frac{10}{3} + \frac{12}{3}$$

$$\frac{2}{3} < f(0) < \frac{22}{3}$$

So $f(0)$ must be positive