LIMITS

The limit of a function $f(x)$ as x approaches a finite or infinite amount $x = a$ is the answer to the question: what happens to the $y-$values as x tends towards a? If the answer is an amount L, finite or infinite, we write:

$$\lim_{x \to a} f(x) = L$$

Or equivalently:

$$y \to L \text{ as } x \to a$$

Note that $x \to a$ means that x need not equal but gets infinitely close to a.

One-Sided Limits

Left-Sided Limit $\lim_{x \to a^-} f(x)$

Geometrically, we trace along the portion of the graph to the left of $x = a$ and observe the trend in $y-$values. For beginners, it helps to draw tiny arrows on the graph.

Numerically, we construct a list (L1) of numbers getting close to $x = a$ from the left, and then populate a second list (L2) using the function rule. For example, if x is to approach 2 from the left, we compute the $y-$values at numbers such as: 1.9, 1.99, 1.999, 1.9999 etc.

Right-Sided Limit $\lim_{x \to a^+} f(x)$

Geometrically, we trace along the portion of the graph to the right of $x = a$ and observe the trend in $y-$values.

Numerically, we can apply the same graphing calculator steps as above. In this case we would compute the $y-$values at numbers such as: 2.1, 2.01, 2.001, 2.0001 etc.

Overall Limit – A definition

$$\lim_{x \to a} f(x) \text{ exists if an only if } \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

CONTINUITY

<u>Definition:</u> A function $f(x)$ is continuous at a point $x = a$ if and only if $\lim_{x \to a} f(x) = f(a)$.

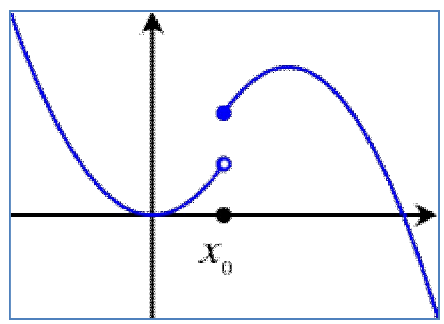
The above definition can be broken down into three true statements:

- ✓ $f(x)$ exists at $x = a$, so let us set $f(a) = M$
- ✓ The one sided limits must be equal to some number L, so that the overall limit exists.
- ✓ To satisfy continuity, we must have $M = L$.

Remarks:

- ✓ A function cannot be continuous at a point if the $x-$value is not in the domain. (There would be no $f(a)$.)
- ✓ A function cannot be continuous at a point if the overall limit fails to exist at that point.

The function is discontinuous at $x = x_0$ because the overall limit does not exist. The one-sided limits exist but do not coincide. We say that the function is continuous from the right side only.



One-Sided Continuities

<u>Definition:</u> A function $f(x)$ is continuous from the left at $x = a$ if and only if $\lim_{x \to a^-} f(x) = f(a)$.

<u>Definition:</u> A function $f(x)$ is continuous from the right at $x = a$ if and only if $\lim_{x \to a^+} f(x) = f(a)$.

<u>Remark:</u> If a function is continuous from both sides of $x = a$, then it is continuous at $x = a$. (We cannot satisfy both one-sided continuities without the two sides of the graph connecting, as it would result in two distinct $y-$values at $x = a$, hence failing to be a function.)

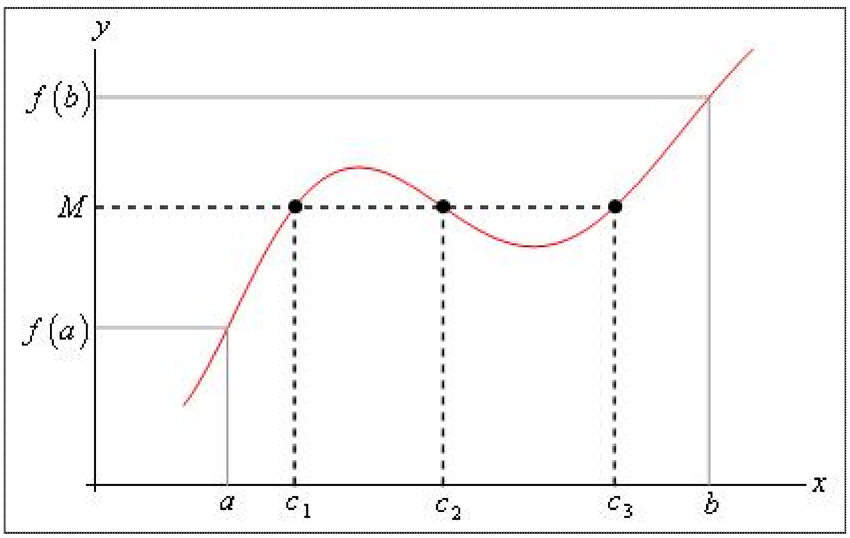
Definition: A function $f(x)$ is called continuous if and only if it is continuous at every point in its domain. Examples: polynomial, trigonometric, and rational functions are continuous throughout their domains.

Important Limits

$$\lim_{x\to 0}\frac{\cos x - 1}{x} = 0 \quad \lim_{x\to 0}\frac{\sin x}{x} = 1 \quad \lim_{x\to\infty}(1+\tfrac{1}{x})^x = e = 2.71828...$$

Intermediate Value Theorem

Let $f(x)$ be a continuous function on $[a,b]$ always defined on a closed interval $[a,b]$, and let $[L,N]$ be the image of $[a,b]$ under f. For every number $M \in (L,N)$, there exists at least one number $c \in (a,b)$ such that $f(c) = M$.



Continuity guarantees the existence of at least one value $x = c$ such that $f(c) = M$.