Problem 1 | 2014 BC (Problem 1 | 2014 AB)

a)

$A(t) = 6.687 * (0.931)^t$

$AvgRate_{[0,30]} = \dfrac{A(30) - A(0)}{30 - 0} = -0.196802 \approx -0.197$ pounds per day.

The average rate of change of $A(t)$ on [0, 30] is -0.197 pounds per day.

b)

$A(t) = 6.687 * (0.931)^t$

$A'(15) \approx -0.163591 \approx -0.164$ pounds per day.

At $t = 15$ days, the amount of grass clippings remaining in the bin is decreasing at a rate of 0.164 pounds per day. The rate is negative because the grass clippings decompose.

c)

$A(t) = 6.687 * (0.931)^t$

Average Amount = $\dfrac{1}{30 - 0}\int_0^{30} A(t)\,dt = 2.75264$ pounds [TI-84]

Solve:

$A(t) = 2.75264$

$t = 12.4147$ days

The amount of grass clippings in the bin is equal to the average amount of grass clippings over the interval [0, 30] when $t = 12.415$ days.

d)

$A(t) = 6.687 * (0.931)^t$

$L(t) = A(30) + A'(30)(t - 30) = 0.782928 - 0.0559762(t - 30)$

$Solve:$

$L(t) = \dfrac{1}{2}$

$t = 35.0544$ days [TI-84]

According to the linear approximation, we predict that at approximately $t = 35.054$ days there will be half a pound of grass left in the bin. Note that this time value is an underestimate, since we are using a tangent line to a graph that is concave up (exponential decay).

Problem 2 | 2014 BC

a)

$$Area = \frac{\pi * 3^2}{4} + \int_0^{\pi/2} \frac{1}{2}\left(3 - 2\sin(2\theta)\right)^2 d\theta \approx 9.70796$$

The area of the shaded region R is approximately 9.708 square units.

b)

$$x(\theta) = r(\theta)\cos\theta = \left(3 - 2\sin(2\theta)\right)\cos\theta$$

$$\left.\frac{dx}{d\theta}\right|_{\theta=\frac{\pi}{6}} \approx -2.36603 \text{ [TI-84]}$$

The rate of change of x with respect to the angle θ is approximately -2.366 when $\theta = \pi / 6$.

c)

$$D(\theta) = 3 - (3 - 2\sin(2\theta)) = 2\sin(2\theta)$$

$$D'(\pi / 3) = 4\cos(2\theta)_{\theta=\pi/3} = -2$$

The distance between the two polar curves is decreasing at a rate of 2.000 units per radian, when the angle is $\pi / 3$.

d)

$$r(\theta) = 3 - 2\sin(2\theta)$$

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt}$$

$$\left.\frac{dr}{dt}\right|_{\theta=\frac{\pi}{6}} = (-4\cos(2\theta))_{\theta=\frac{\pi}{6}} * 3 = (-2) * 3 = -6$$

When $\theta = \pi / 6$, the distance from the curve $r(\theta)$ to the pole is decreasing at a rate of 6.000 units of length per unit of time.

Problem 3 | 2014 BC (Problem 3 | 2014 AB)

a)

$$g(3)=\int_{-3}^{3} f(t)\,dt=\frac{3*4}{2}+\frac{2*4}{2}+\frac{1*(-2)}{2}=6+4-1=9$$

b)

By the Fundamental Theorem of Calculus, the first and second derivatives of $g(x)$ are:

$$g'(x)=f(x)>0$$
$$g''(x)=f'(x)<0$$
$$(-5,-3) \text{ and } (0,2)$$

These are the only intervals for which f is both positive and decreasing (or where g is both increasing and concave down.)

c)

$$h(x)=\frac{g(x)}{5x}$$
$$h'(x)=\frac{5x\,g'(x)-5g(x)}{25x^2}$$
$$h'(3)=\frac{5*3*g'(3)-5g(3)}{25*9}=\frac{15*f(3)-5*9}{225}=\frac{15*(-2)-45}{225}=\frac{-3}{9}=-\frac{1}{3}$$

d)

$$p(x)=f(x^2-x)$$
$$p'(x)=f'(x^2-x)*(2x-1)$$
$$p'(-1)=f'(2)*(-3)=(-2)(-3)=6$$

Problem 4 | 2014 BC (Problem 4 | 2014 AB)

a)

$$\text{Average Acceleration}_{TRAIN\ A\,[2,8]} = \frac{v(8)-v(2)}{8-2} = \frac{-120-100}{8-2} = \frac{-220}{6} = \frac{-110}{3}\frac{meters}{\min^2}$$

b)

If the velocity function of train A is differentiable, then it is also continuous in the given interval. We can hence apply the Intermediate Value Theorem on [5, 8] to conclude that the velocity function must take on every value between 40 and -120 at least once; this interval also includes the value of -100 meters per minute, so the train's velocity must have been -100 meters/minute at least once on the interval $(5,8)$.

c)

$$x(t)-x(2) = \int_{w=2}^{w=t} v_A(w)\,dw$$

$$x(t)-300 = \int_{w=2}^{w=t} v_A(w)\,dw$$

$$x(t) = 300 + \int_{w=2}^{w=t} v_A(w)\,dw$$

$$\int_{w=2}^{w=t} v_A(w)\,dw \approx \frac{(5-2)(100+40)}{2} + \frac{(8-5)(-120+40)}{2} + \frac{(12-8)(-150-120)}{2} =$$

$$\int_{w=2}^{w=t} v_A(w)\,dw \approx 210-120-540 = -450\ meters.$$

$$x(12) \approx 300-450 = -150\ meters.$$

d)

Pythagorean Theorem:

$$z^2(t) = x^2(t) + y^2(t)$$

$$(x,y,z) = (300,400,500)$$

$$2z(t)z'(t) = 2x(t)x'(t) + 2y(t)y'(t)$$

$$y'(2) = v_B(2) = 125$$

$$x'(2) = v_A(2) = 100$$

$$z'(2) = \frac{x(2)x'(2)+y(2)y'(2)}{z(2)} = \frac{300*100+400*125}{500} = 160\ meters/\min$$

The rate at which the distance between train A and train B is changing at time $t = 2$ minutes is 160 meters per minute.

Problem 5 | 2014 BC

a)

$$Area = \int_0^1 y_{above} - y_{below}\, dx = \int_0^1 xe^{x^2} - (-2x)\, dx = \left(\frac{1}{2}e^{x^2} + x^2\right)_{[0,1]} = \frac{e}{2} + 1 - \frac{1}{2} - 0 = \frac{e+1}{2}$$

b)
Washer Method:

$$R(x) = 2 + xe^{x^2}$$

$$r(x) = 2 + (-2x) = 2 - 2x$$

$$V = \pi\int_0^1 \left(2 + xe^{x^2}\right)^2 - (2-2x)^2 dx$$

c)

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx + \sqrt{1^2 + (-2)^2} + (1 * e^{1^2} - (-2*1))$$

$$L = \int_0^1 \sqrt{1 + \left(e^{x^2} + 2x^2 e^{x^2}\right)^2}\, dx + \sqrt{5} + e + 2$$

Problem 6 | 2014 BC

a)
We apply the Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}(x-1)^{n+1}}{n+1} * \frac{n}{2^n(x-1)^n}\right| = \left|\frac{2(x-1)}{1} * \frac{n}{n+1}\right| \to 2|x-1| < 1$$

$$|x-1| < \frac{1}{2}$$

$$R = \frac{1}{2}$$

b)

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{2^n}{n}(x-1)^n = \frac{2}{1}(x-1) - \frac{2^2}{2}(x-1)^2 + \frac{2^3}{3}(x-1)^3 - \frac{2^4}{4}(x-1)^4 + \ldots$$

$$f'(x) = \sum_{n=1}^{\infty} (-1)^{n+1} 2^n (x-1)^{n-1}$$

$$f'(x) = 2 - 2^2(x-1) + 2^3(x-1)^2 - 2^4(x-1)^3 + \ldots + (-1)^{n+1}2^n(x-1)^{n-1} + \ldots$$

$$f'(x) = 2 - 4(x-1) + 8(x-1)^2 - 16(x-1)^3 + \ldots + (-1)^{n+1}2^n(x-1)^{n-1} + \ldots$$

c)

$$a = 2$$

$$r = -2(x-1)$$

$$f'(x) = \frac{a}{1-r} = \frac{2}{1-(-2)(x-1)} = \frac{2}{1+2x-2} = \frac{2}{2x-1}$$

$$f(x) = \int \frac{2}{2x-1}\,dx = C + \ln|2x-1|$$

$$f(1) = 0$$

$$C = 0$$

$$f(x) = \ln|2x-1|$$