Problem 1

a)
$f'(x) = 8x - 3x^2$
$f'(3) = 24 - 27 = -3 = m$

b)
The $x-$intercept of the tangent line is: $18 - 3x = 0 \rightarrow x = 6$
The $x-$intercepts of $f(x)$ are: $4x^2 - x^3 = 0 \rightarrow x = 0, x = 4$

$$S = \int_{x=3}^{x=6} 18 - 3x\,dx - \int_{x=3}^{x=4} f(x)\,dx = \frac{27}{2} - \frac{67}{12} = \frac{95}{12} = 7.916...$$

c)
We use the disk method here:
$$S = \pi \int_{x=0}^{x=4} [f(x)]^2 dx = \pi \int_{x=0}^{x=4} [4x^2 - x^3]^2 dx = \frac{16384\pi}{105} = 490.208$$

Problem 2

a)
$\int_{t=0}^{t=12} H(t)\,dt \approx 70.571$ gallons of oil are pumped into the tank during the 12 hour interval.

b)
We compare the two rates:

$$H'(6) = 2 + \frac{10}{1 + \ln(6+1)} = 5.394536660...$$
$$R'(6) = 12 \sin\left(\frac{6^2}{47}\right) = 8.31872839...$$

The removal rate is greater, therefore the overall volume of oil in the tank is falling at this point in time.

c)

$$V(12)-V(0)=\int_{t=0}^{t=12} RateIn(t)-RateOut(t)\,dt=\int_{t=0}^{t=12} H(t)-R(t)\,dt$$
$V(12)=125+\int_{t=0}^{t=12} H(t)-R(t)\,dt=122.02571$ gallons of oil are present in the tank.

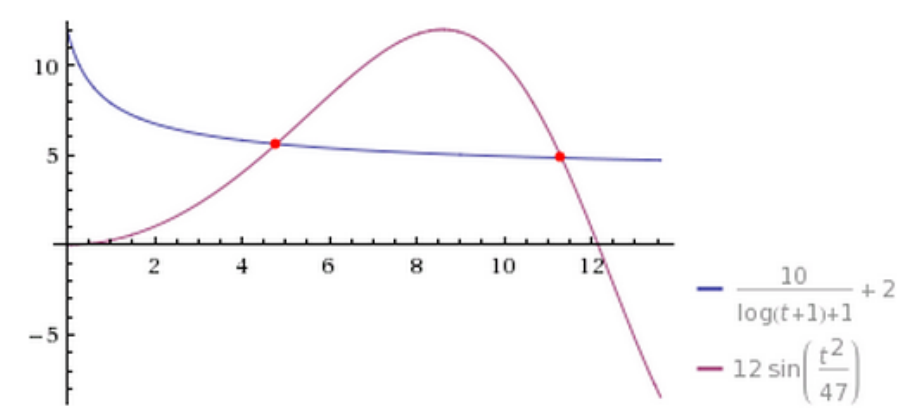
d)

Goal: minimize volume.

$$V(x)-V(0)=\int_{t=0}^{t=x} H(t)-R(t)\,dt$$
$$V(x)=125+\int_{t=0}^{t=x} H(t)-R(t)\,dt$$
$V'(x)=H(x)-R(x)$ by FTC
$V''(x)=H'(x)-R'(x)$ by FTC

Therefore, for critical number(s), we set the two rates equal to each other.

Plot:



The two solutions are t = 4.79005 and t=11.3185.

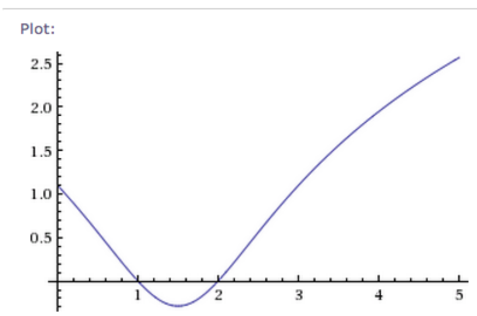
For a minimum, applying the second derivative test, we need $V''(x^*)>0 \rightarrow H'(x^*)-R'(x^*)>0 \rightarrow H'(x^*)>R'(x^*)$. Looking at the slopes of the two rate functions at the intersection above, this condition is satisfied at t=11.3185; therefore, at the point in time the volume in the tank is a minimum. [Note: At the other point, the volume is a maximum.]

Problem 3

a)

$$a(t)=\frac{2t-3}{t^2-3t+3}\rightarrow a(4)=\frac{5}{7}$$

b)
A particle changes direction whenever the velocity changes sign. A graph of the velocity function shows that direction changes at t=1 and t=2. The particle travels to the left on the time interval (1, 2), since velocity is negative during this time.



c)

$$x(2)-x(0)=\int_0^2 v(t)\,dt$$

$$x(2)=x(0)+\int_0^2 \ln(t^2-3t+3)\,dt=8+0.368617=8.368617$$

d)
Average Speed is given by:

$$\frac{1}{2-0}\int_0^2 |v(t)|\,dt=\frac{1}{2}\int_0^2 |\ln(t^2-3t+3)|\,dt=0.370509$$

Problem 4

a)

$$g(-1) = \int_{-4}^{-1} f(t)\,dt = \frac{(-3+(-2))3}{2} = -7.500$$

$$g'(x) = f(x) \rightarrow g'(-1) = f(-1) = -2$$

$$g''(x) = f'(x) \rightarrow g''(-1) = f'(-1) = DNE \text{ (corner)}$$

b)

$g(x)$ has a point of inflection at $x = 2$ because at this point:

- its second derivative $g''(x) = f'(x)$ changes from positive to negative
- its first derivative $g'(x) = f(x)$ does not change sign (remains positive)
- the function $g(x)$ is continuous

This is the only point of inflection.

c)

$$h(x) = \int_{x}^{3} f(t)\,dt$$

By looking at the graph and thinking of the cancellation of the right triangles, conclude that:

$$h(1) = 0, h(-1) = 0$$

d)

Using FTC, $h'(x) = -f(x) < 0$. This is true on the interval (0, 2). The function $h(x)$ is decreasing on this interval. [Remark: Pay attention to the minus sign when applying the FTC. The left bound is the variable bound here.]

Problem 5

a)

Implicitly differentiate, then isolate the first derivative:

$$2y\frac{dy}{dx} = y + x\frac{dy}{dx} \rightarrow \frac{dy}{dx} = \frac{y}{2y-x}$$

b)
$$\frac{dy}{dx} = \frac{1}{2} = \frac{y}{2y-x} \rightarrow 2y - x = 2y \rightarrow x = 0$$
$$\rightarrow y^2 = 2$$
$$\rightarrow y = \pm\sqrt{2}$$
$$(0,-\sqrt{2}),(0,\sqrt{2})$$

c)
$$\frac{dy}{dx} = 0 = \frac{y}{2y-x} \rightarrow y = 0$$
$$\rightarrow 0 = 2$$

, therefore a horizontal tangent to this curve is impossible.

d)
Implicitly differentiate, but <u>with respect to time</u>:
$$y^2 = 2xy$$
$$t = 5 \rightarrow \frac{dy}{dt} = 6, y = 3$$
$$\rightarrow x = 1.5$$
$$2y\frac{dy}{dt} = 2(x\frac{dy}{dt} + \frac{dx}{dt}y)$$
$$y\frac{dy}{dt} = x\frac{dy}{dt} + \frac{dx}{dt}y$$
$$y\frac{dy}{dt} = x\frac{dy}{dt} + \frac{dx}{dt}y$$
$$3*6 = 1.5*6 + \frac{dx}{dt}3$$
$$\frac{dx}{dt} = 22/3$$

Problem 6

a)
$\lim_{x\to 3} f(x) = 2 = f(3)$, therefore the function is continuous. The limit there is equal to the function value.

b)
$$f_{avg[0,5]} = \frac{1}{5-0}\int_0^5 f(x)\,dx = \frac{1}{5}\int_0^3 \sqrt{x+1}\,dx + \frac{1}{5}\int_3^5 5-x\,dx =$$
$$f_{avg[0,5]} = \frac{1}{5}\left(\frac{14}{3}+2\right) = \frac{20}{15}$$

c)

The one sided limits must be equal to each other and coincide with the function value at $x = 3$, in order for the function to be continuous:
$2k = 3m + 2$

In order for the two pieces to connect in a smooth fashion, the one-sided derivatives must be equal:

$$g'_{-}(x) = \frac{k}{2\sqrt{x+1}} \to g'_{-}(3) = \frac{k}{4}$$
$$g'_{+}(x) = m \to g'_{+}(3) = m$$
$$\to \frac{k}{4} = m$$
$$\begin{cases} k = 4m \\ 2k = 3m+2 \end{cases} \to \begin{cases} 2k = 8m \\ 2k = 3m+2 \end{cases} \to \begin{cases} k = 4m \\ 5m-2=0 \end{cases} \to \begin{cases} k = \frac{8}{5} \\ m = \frac{2}{5} \end{cases}$$