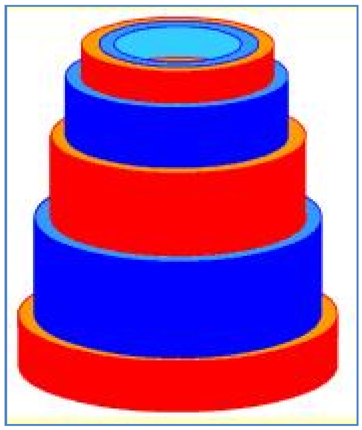
Shell Method

We use the Cylindrical Shells method whenever the Washer Method fails to apply because the inverse functions (outer and/or inner radius) are difficult or impossible to solve for algebraically.

The General Formula: $V = \int_a^b designated \bullet hitter\ dx = \int_a^b d(x) \bullet h(x)\ dx$



Steps:

- ✓ Carefully draw the region that is being revolved and the rotational axis.
- ✓ Draw a thin rectangle into the the region so that its longer dimension stretches parallel with the rotational axis and captures the extent of the region.
- ✓ The function $h(x)$ or $h(y)$ will measure the "height" of this rectangle at various levels.
- ✓ We define R as the distance from the rotational axis to the rectangle. Assume the rectangle has infinitely small (close to zero!) height so the nearest point from it to the rotational axis is also the farthest point.
- ✓ The function $d(x)$ or $d(y)$ measures the circumference of the typical rectangle as it makes a full revolution, hence creating a shell!

The discussion below summarizes the most general cases dealing with two functions enclosing a region. One can easily set one of the functions equal to zero when dealing with a region that borders one of the axes.

Horizontal Rotational Axis

- ✓ The equation of the function to the right (FAR) is of the form $x = f(y)$.
- ✓ The equation of the function to the left (NEAR) is of the form $x = g(y)$.
- ✓ The interval of $y-$values is $[c,d]$.
- ✓ The rotational axis has equation $y = m$, located at a distance above or below the enclosed region by $f(y)$ and $g(y)$.

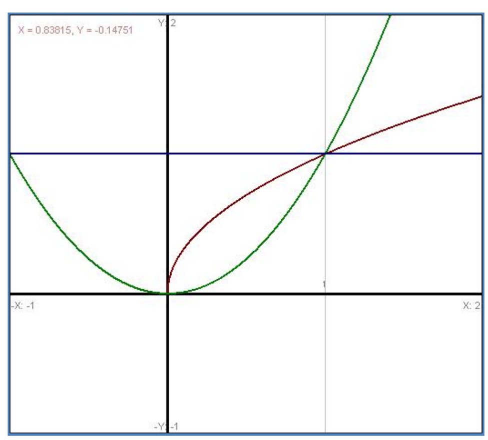
$$R = |y-m| = |m-y| \qquad d(y) = 2\pi R = 2\pi|y-m|$$
$$h(y) = x_{FAR} - x_{NEAR} = f(y) - g(y)$$
$$Volume = \int_c^d d(y)h(y)dy = \int_c^d 2\pi|y-m|[f(y)-g(y)]dy$$

- ✓ Special case: the rotational axis is $y = 0$.

$$Volume = \int_c^d 2\pi|y|[f(y)-g(y)]dy$$

Example: Use the Shell Method to set up, but do not evaluate the volume of the solid generated when the region enclosed by the graph of $x = y^2$ and $x = \sqrt{y}$ is rotated about $y = 1$.

$$x_{FAR} = \sqrt{y} \qquad x_{NEAR} = y^2$$
$$h(y) = x_{FAR} - x_{NEAR} = \sqrt{y} - y^2$$
$$R = |1-y| = 1-y \qquad (y<1)$$
$$V = 2\pi \int_{y=0}^{y=1} (1-y)\left(\sqrt{y}-y^2\right)dy$$



Vertical Rotational Axis

- ✓ The equation of the top function is of the form $y = f(x)$.
- ✓ The equation of the bottom function is of the form $y = g(x)$.
- ✓ The interval of x – values is $[a,b]$.
- ✓ The rotational axis has equation $x = n$, located at a distance to the left or right of the enclosed region by $f(x)$ and $g(x)$.

$$R = |x-n| = |n-x| \qquad d(x) = 2\pi R = 2\pi |x-n|$$
$$h(x) = y_{TOP} - y_{BOTTOM} = f(x) - g(x)$$
$$Volume = \int_a^b d(x)h(x)dx = \int_a^b 2\pi |x-n| [f(x) - g(x)] dx$$

- ✓ Special case: the rotational axis is $x = 0$.

$$Volume = \int_a^b 2\pi |x| [f(x) - g(x)] dx$$

Example: Use the Shell Method to set up, but do not evaluate the volume of the solid generated when the region enclosed by the graph of $y = x$ and $y = \sqrt{x}$ in the first quadrant is rotated about $x = 1$.

$$y_{TOP} = \sqrt{x} \qquad y_{BOTTOM} = x \qquad h(x) = \sqrt{x} - x$$
$$R = |1-x| \qquad d(x) = 2\pi |1-x|$$
$$x < 1 \Rightarrow d(x) = 2\pi(1-x)$$
$$V = 2\pi \int_{x=0}^{x=1} |1-x| \left(\sqrt{x} - x\right) dx$$