Volumes of Solids with Known Cross Sections

If the cross sections are perpendicular to the horizontal axis, the volume of a solid of known cross-section is given by:

$$V = \int_{x=a}^{x=b} A(x)dx$$

, where $A(x)$ represents the area of the cross section at level some level x between a and b. When the cross sections are perpendicular to the vertical axis, the volume of the solid is given by:

$$V = \int_{y=c}^{y=d} A(y)\,dy .$$

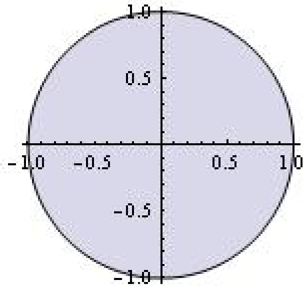
To set up the volume formula correctly, we need to identify the following:

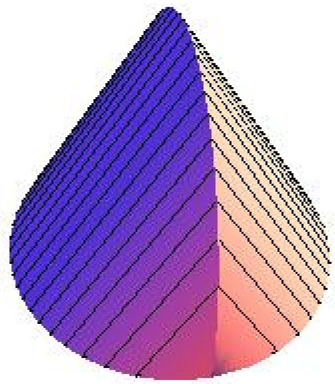
- ✓ The base region of the solid supports the cross sections. The base can be a square, triangle, ellipse, semi-circle, circle etc.
- ✓ The geometric shape of the cross sections. Examples include: equilateral triangles, semi-circles, squares, rectangles, etc.
- ✓ The axis the cross sections are perpendicular to can be identified by words such as ‘parallel’ and ‘perpendicular’.
- ✓ The bounds of integration are easily determined using the base region and the direction in which the cross sections are stacked.

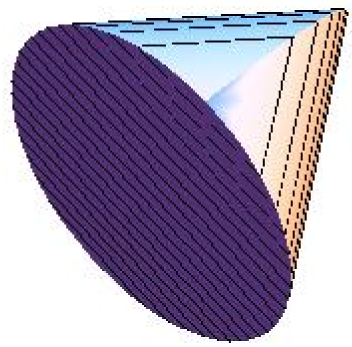
Example: The unit circle is the base of a solid whose known cross-sections are equilateral triangles perpendicular to the $x-$axis. Find the volume.

- ✓ Base region: unit circle, whose equation is $x^2 + y^2 = 1$
- ✓ Cross section: equilateral triangle. The area in terms of its side m is $A = \frac{\sqrt{3}m^2}{4}$.
- ✓ The cross sections are stacked horizontally, so the volume is: $V(x) = \int_{x=a}^{x=b} A(x)\,dx$.
- ✓ The integration bounds are $x = -1$ and $x = 1$.

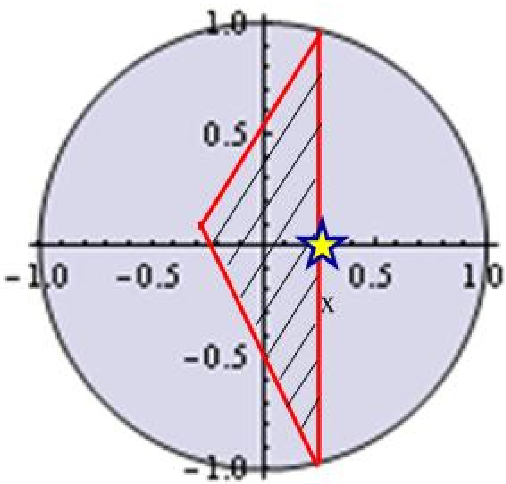
The base region and 3D impressions of the solid are shown:

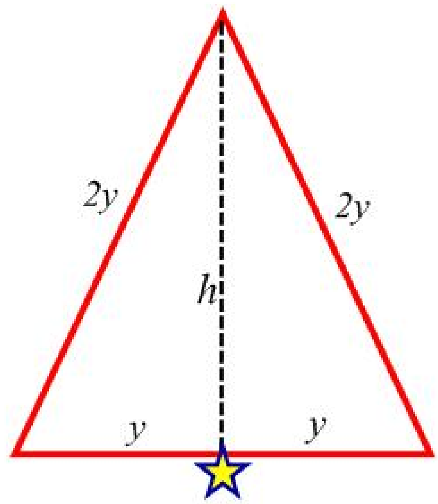






To find the area $A(x)$ of a typical cross section, we draw its 3D impression on the base region and a 2D representation to help express the area in terms of x.





The area of the typical cross section (equilateral triangle) is expressed in terms of x by using the fact that the point (x, y) is on the circle, and hence it satisfies $x^2 + y^2 = 1$:

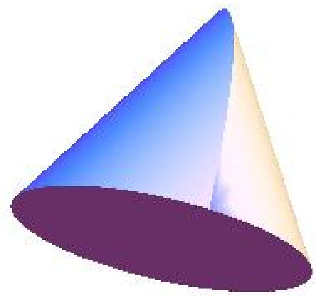
$$A = \frac{1}{2}bh = \frac{1}{2}(2y)\sqrt{3}y = \sqrt{3}y^2$$

$$y^2 = 1 - x^2 \Rightarrow A(x) = \sqrt{3}(1 - x^2)$$

Now we can easily compute the volume:

$$V = \int_{-1}^{1} A(x)\,dx = \int_{-1}^{1} \sqrt{3}(1-x^2)\,dx = \sqrt{3}\int_{-1}^{1}(1-x^2)\,dx =$$

$$2\sqrt{3}\int_{0}^{1}(1-x^2)\,dx == 2\sqrt{3}\left(x-\frac{x^3}{3}\right)\Bigg|_0^1 = \frac{4\sqrt{3}}{3} \text{ cubic units}$$

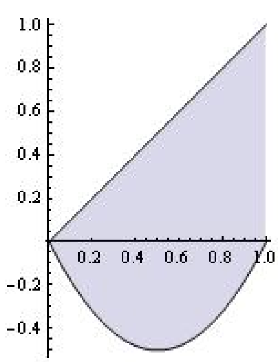


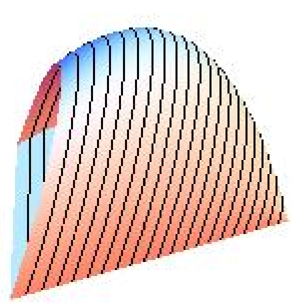
Tips when computing the volume of a solid of known cross sections:

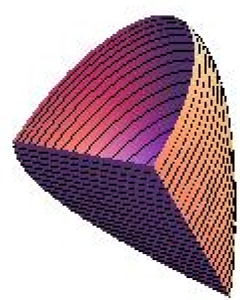
1. The number π should not feature in the final answer unless the typical cross section is part of a circle (semi-circle or less often, quarter of a circle). Remember that π appears in the volumes of solids of revolution (disk, washer, shell)
2. Most of the known cross sections exhibit "even" symmetry, hence we should feel free to use the property of even functions:

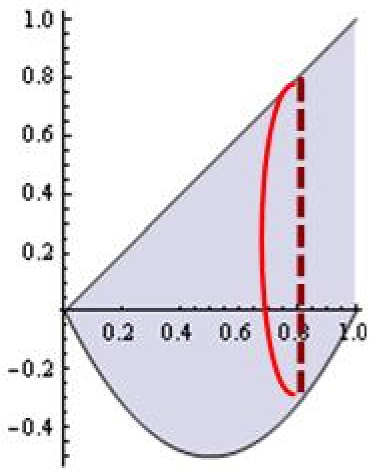
$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

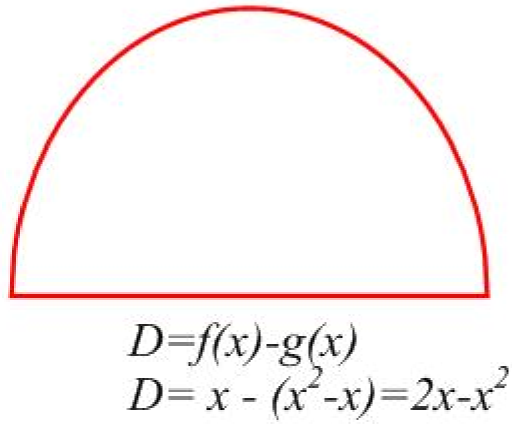
Example: The base region is bounded by $y = x$ and $y = x^2 - x$ on the interval [0, 1]. The cross sections are semi-circles parallel to the $y-$axis.











The area of a typical cross section is given by:

$$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left(\frac{D}{2}\right)^2 = \frac{1}{8}\pi D^2 = \frac{1}{8}\pi(2x - x^2)^2 = \frac{\pi}{8}(x^4 - 4x^3 + 4x^2)$$

The volume is:

$$V(x) = \int_{x=0}^{x=1} A(x)\,dx = \int_{x=0}^{x=1} \frac{\pi}{8}(x^4 - 4x^3 + 4x^2)\,dx =$$

$$= \frac{\pi}{8}\int_{x=0}^{x=1} (x^4 - 4x^3 + 4x^2)\,dx = \frac{\pi}{8}\left(\frac{x^5}{5} - x^4 + \frac{4x^3}{3}\right)\Bigg|_0^1 = \frac{\pi}{8}\frac{8}{15} = \frac{\pi}{15} \text{ cubic units}$$

Note that the number π appeared in this example because the typical cross section is a semi-circle.