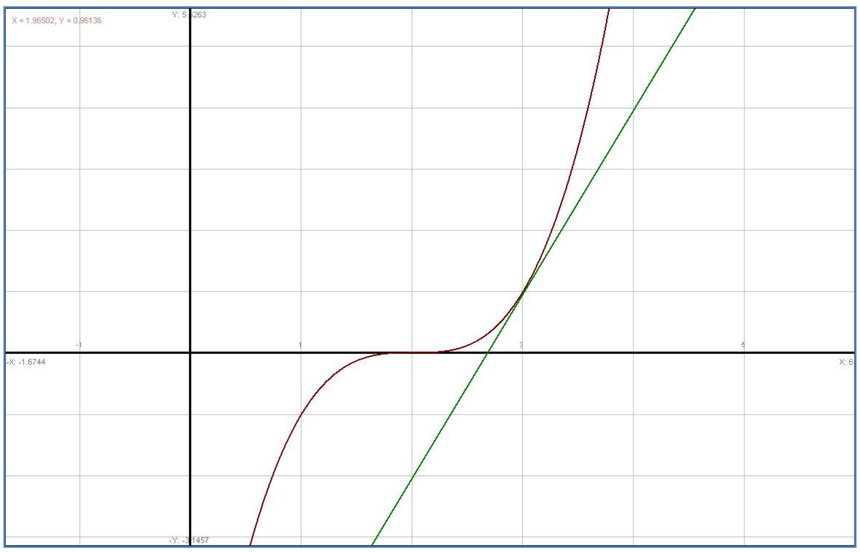
Differentiability at a Point

Definition: $f(x)$ is differentiable at a point $x=a$ if and only if $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ exists.

We write:

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a} \text{ or } f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Example: The graph of $f(x)=(x-2)^3$ is differentiable $x=3$ since a non-vertical tangent line to the graph can be drawn at this point.



Algebraically, assuming we live in a "pre-shortcuts to differentiation" world, we can compute the limit of the quotient to find the precise value of the slope:

$$f'(3)=\lim_{x\to 3}\frac{(x-2)^3-(3-2)^3}{x-3}$$
$$f'(3)=\lim_{x\to 3}\frac{x^3-6x^2+12x-9}{x-3}$$
$$f'(3)=\lim_{x\to 3}\frac{(x^2-3x+3)(x-3)}{x-3}$$
$$f'(3)=\lim_{x\to 3}(x^2-3x+3)=3$$

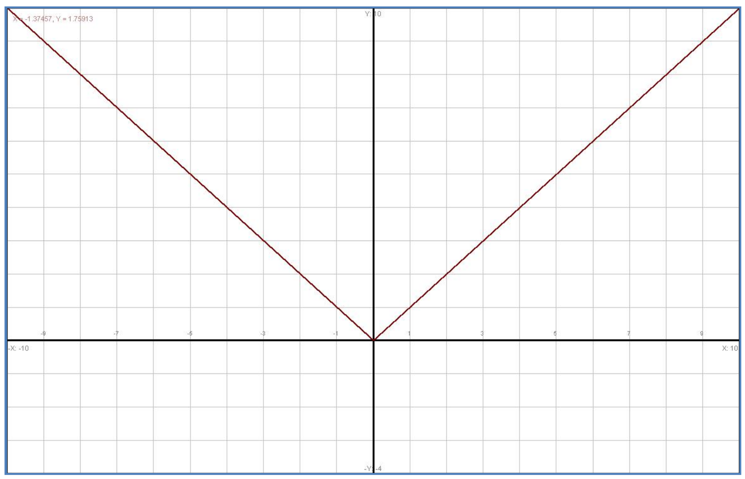
Fact: All polynomial functions are continuous and differentiable everywhere.

One-Sided Derivatives

Definition: $f(x)$ is differentiable from the right at $x=a$ if and only if $\lim\limits_{x\to a^+}\frac{f(x)-f(a)}{x-a}$ exists.

Definition: $f(x)$ is differentiable from the left at $x=a$ if and only if $\lim\limits_{x\to a^-}\frac{f(x)-f(a)}{x-a}$ exists.

Example: $f(x)=|x|$ fails to be differentiable at $x=0$, but both one-sided derivatives exist. From the left, the slopes from the left and right are -1 and 1, respectively.



Algebraically, we verify the one-sided derivatives. Note that $x\to 0^+$ implies that $x>0$, so we can write $|x|=x$. Also, $x\to 0^-$ implies that $x<0$, so we can write $|x|=-x$.

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{|x|}{x}=\lim_{x\to 0^+}\frac{x}{x}=1$$
$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{|x|}{x}=\lim_{x\to 0^-}\frac{-x}{x}=-1$$

We write:

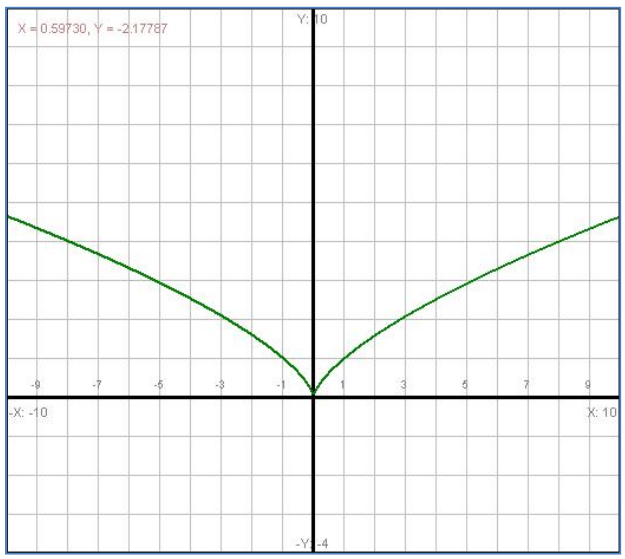
$$f'_+(0)=1$$
$$f'_-(0)=-1$$

Theorem: If $f(x)$ is differentiable at a point $x = a$, then $f(x)$ is continuous at $x = a$.

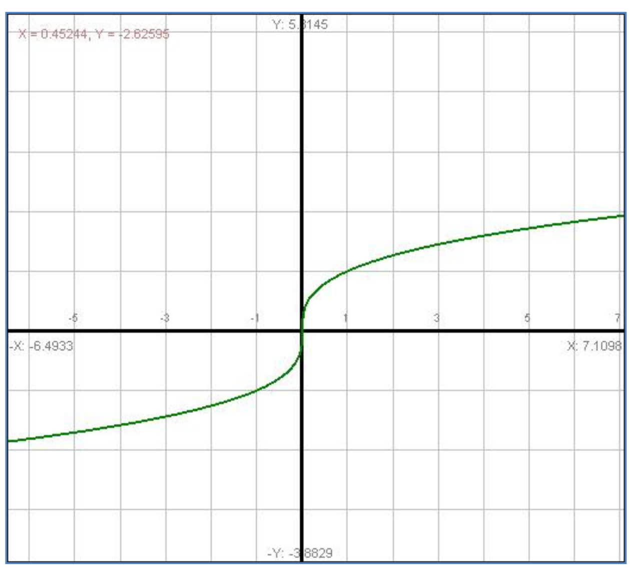
Differentiability $\Rightarrow$ Continuity

The converse is false. Continuity does not necessarily imply differentiability. The absolute value function in the example above helps illustrate. There are three cases of continuous functions that fail to be differentiable at a point: a sharp corner (absolute value function at the origin), a cusp, or a vertical tangent line.

Example: $y = x^{2/3}$ has a cusp at the origin.



Example: $y = x^{1/3}$ has a vertical tangent line at the origin.



FACTS, THEOREMS, and DEFINITIONS

A Critical Number

$x = c$ is a critical number of $f(x)$ if and only if $f(x)$ is continuous there and $f'(c) = 0$ or does not exist.

A Local (Relative) Maximum

$f(x)$ attains a local maximum at $(c, f(c))$ if and only $f(c) \geq f(x)$ for all $x-$values *near* $x = c$. ("near $x = c$" means "in a small open interval containing $x = c$")

A Local (Relative) Minimum

$f(x)$ attains a local minimum at $(c, f(c))$ if and only $f(c) \leq f(x)$ for all $x-$values *near* $x = c$.

A Global (Absolute) Maximum

$f(x)$ attains a global maximum at $(c, f(c))$ if and only $f(c) \geq f(x)$ for all $x-$values in the domain of $f(x)$.

A Global (Absolute) Minimum

$f(x)$ attains a global minimum at $(c, f(c))$ if and only $f(c) \leq f(x)$ for all $x-$values in the domain of $f(x)$.

Fermat's Theorem

If $f'(c)$ exists and $f(x)$ has a local max or min at $(c, f(c))$, then $f'(c) = 0$.

The converse of Fermat's Theorem is false. Consider $f(x) = x^3$ at $x = 0$.

Extreme Value Theorem

If $f(x)$ is continuous an a closed interval $[a,b]$, then $f(x)$ attains an absolute maximum and an absolute minimum in $[a,b]$.

The absolute extrema could occur anywhere in $[a,b]$, in the interior or at the endpoints. EVT does not show how to determine these points; it simply guarantees their existence.

Mean Value Theorem

If a function $f(x)$ is differentiable (and therefore continuous) on an interval $[a,b]$, then there exists at least one number $x=c\in(a,b)$ such that $f'(c)=\frac{f(b)-f(a)}{b-a}$.

In plain English, the instantaneous rate of change must equal the average rate of change on the given interval at least once.

Rolle's Theorem [Special Case of MVT]

Suppose $f(x)$ satisfies the following conditions:

i. $f(x)$ is continuous on $[a,b]$
ii. $f(x)$ is differentiable on (a,b)
iii. $f(a)=f(b)$

Then there exists at least one number $x=c\in\left(a,b\right)$ such that $f'(c)=0$.

Corollaries

I. If $f'(x)=0$ for all x, then $f(x)$ is a constant function: $f(x)=k$.
II. If $f'(x)=g'(x)$ for all x, then $f(x)=g(x)+c$.

Closed Interval Method

Goal: Determine the absolute extrema (max and min) of a *continuous* function whose domain is restricted to a closed interval $[a,b]$.

- ✓ Check that $f(x)$ is continuous on the interval $[a,b]$
- ✓ Find all critical numbers $c_1,c_2,\ldots$ in (a,b)
- ✓ Evaluate $f(a),f(b),f(c_1),f(c_2),\ldots$
- ✓ The largest and the least $y-$values from the previous step are the global (absolute) extrema.