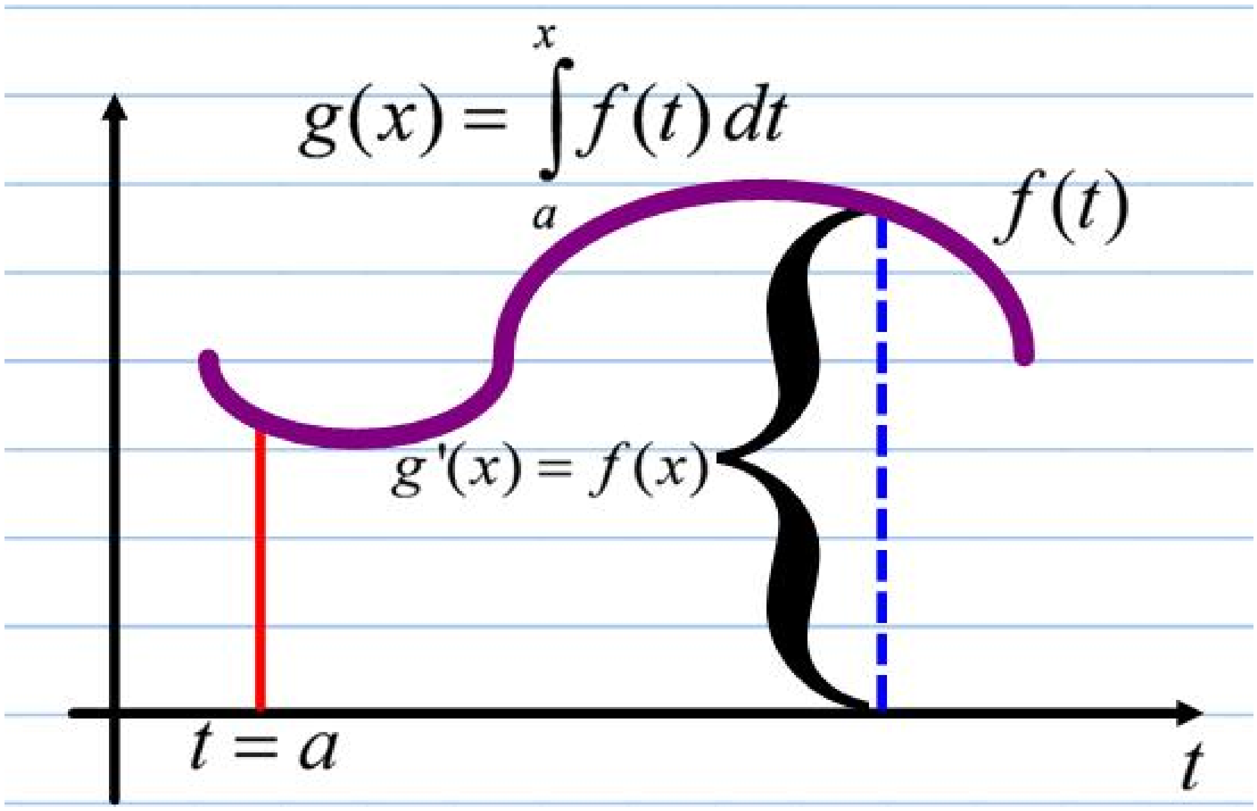
Fundamental Theorem of Calculus – Part One

Let $f(t)$ be a continuous function. If we define an accumulation function g by $g(x)=\int_a^x f(t)\,dt$, then $g(x)$ is continuous and differentiable, and $g'(x)=f(x)$.



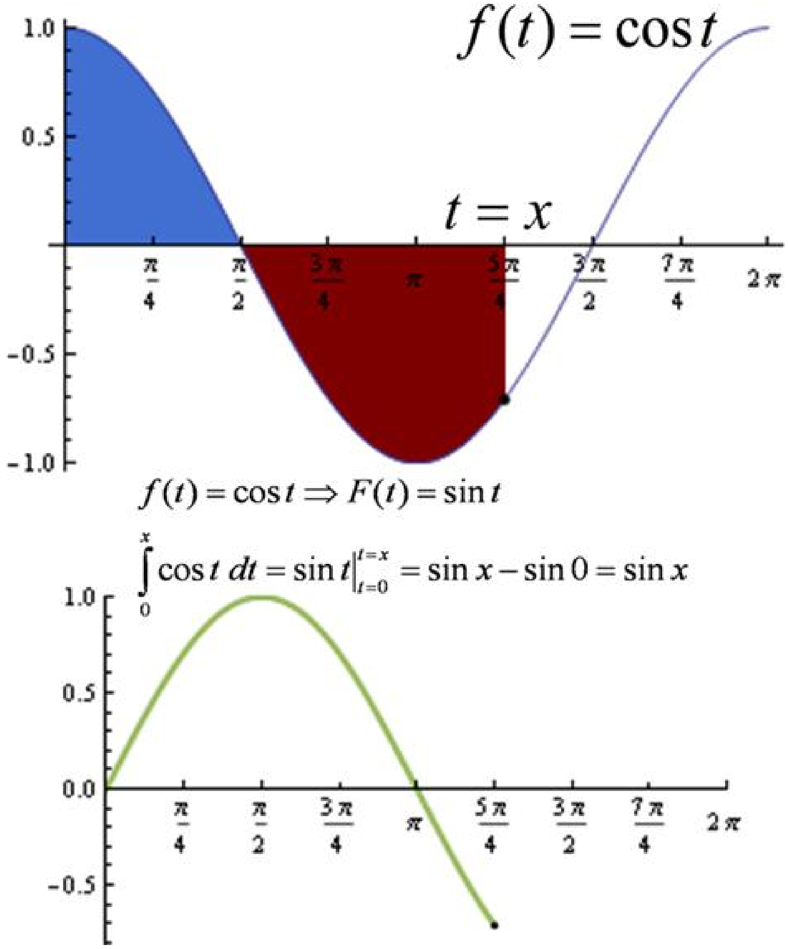
In plain English, the rate at which the net area changes is equal to the $y-$value of the rate function $f(t)$, whose net area is measured by $g(x)$. The rate does not depend on where we start to accumulate net areas.

When the variable bound of the integral (typically, the upper bound) is a function of x, let's say $h(x)$, we must account for the chain rule when differentiating:

$$\left[g(x)=\int_a^{h(x)} f(t)\,dt\right]\Rightarrow \tfrac{d}{dx}\left(\int_a^{h(x)} f(t)\,dt\right)=f(h(x))\bullet h'(x)$$

Fundamental Theorem of Calculus – Part Two

If $f(t)$ is continuous on $[a,b]$, then $\int_a^b f(t)\,dt = F(b) - F(a)$, where F is any anti-derivative of f.



The second part of the fundamental theorem of calculus can be also written as:

$$\int_a^b F'(t)dt = F(b) - F(a)$$

In the case where one of the bounds is seen as a changing amount, FTC says:

$$\tfrac{d}{dx}\int_a^x F'(t)dt = \tfrac{d}{dx}\left[F(x) - F(a)\right] = F'(x) - 0 = F'(x)$$

, since $F(a)$ is a constant amount.

When both bounds are variable, FTC says:

$$\tfrac{d}{dx}\int_{l(x)}^{h(x)} F'(t)dt = \tfrac{d}{dx}\left[F(h(x)) - F(l(x))\right] = F'(h(x))h'(x) - F'(l(x))l'(x)$$

Note that whenever we are given a rate function $f(t)$ and an accumulation function of the form $g(x) = \int_a^x f(t)\,dt$, FTC implies that the latter is simply an anti-derivative of f.

$$g(x) = F(x) - F(a) = F(x) + K$$

Hence, whenever asked to sketch a qualitatively correct anti-derivative of a given graph of a function, we use the concept of net area (FTC Part Two) and its rate of change (FTC Part One). The precise position of the graph will depend on the accumulation's starting point $t = a$. If a is stated as an arbitrary constant, we choose a starting point ourselves, usually the origin.

Here are a few tips on sketching $g(x) = \int_a^x f(t)\,dt$ when $f(t)$ and $t = a$ are given:

$f(t) > 0 \Rightarrow g(x) \uparrow (note : g'(x) = f(x))$
$f(t) < 0 \Rightarrow g(x) \downarrow (note : g'(x) = f(x))$
$f'(t) > 0 \Rightarrow g(x)\ is\ concave\ up\ (note : g''(x) = f'(x))$
$f'(t) < 0 \Rightarrow g(x)\ is\ concave\ down\ (note : g''(x) = f'(x))$
$f(t)\ changes\ sign \Rightarrow g(x)\ attains\ a\ local\ extremum$