Can use Laws of Logs

APMA Faculty
University of Virginia

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Introduction

Logarithmic functions are the inverses of exponential functions. They have several important properties that make them useful in mathematics, science, and engineering. This document will explore these properties in detail.

Definition of Logarithm

The logarithm of a number is the exponent to which the base must be raised to produce that number. Formally, for $b > 0$, $b \neq 1$,

$$\log_b(x) = y \quad \text{if and only if} \quad b^y = x$$

where: - b is the base, - x is the argument, - y is the logarithm.

Properties of Logarithms

Logarithmic functions have several key properties that are useful in simplifying expressions and solving equations. In the following 7 properties will be discussed.

Product Rule

The logarithm of a product is the sum of the logarithms of the factors.

$$\log_b(xy) = \log_b(x) + \log_b(y)$$

Example: Simplify $\log_2(8 \cdot 4)$:

$$\log_2(8 \cdot 4) = \log_2(8) + \log_2(4) = 3 + 2 = 5$$

Quotient Rule

The logarithm of a quotient is the difference of the logarithms.

$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$$

Example: Simplify $\log_{10}\left(\frac{100}{10}\right)$:

$$\log_{10}\left(\frac{100}{10}\right) = \log_{10}(100) - \log_{10}(10) = 2 - 1 = 1$$

some are not.

Power Rule

The logarithm of a power is the exponent times the logarithm of the base.

$$\log_b(x^y) = y\log_b(x)$$

Example: Simplify $\log_3(27)$:

$$\log_3(27) = \log_3(3^3) = 3\log_3(3) = 3\cdot 1 = 3$$

Change of Base Formula

The logarithm can be converted to a different base using the change of base formula.

$$\log_b(x) = \frac{\log_k(x)}{\log_k(b)}$$

Example: Convert $\log_2(8)$ to base 10:

$$\log_2(8) = \frac{\log_{10}(8)}{\log_{10}(2)} \approx \frac{0.903}{0.301} \approx 3$$

Logarithm of 1

The logarithm of 1 to any base is 0.

$$\log_b(1) = 0 \quad \text{for any base } b > 0$$

Example: $\log_5(1) = 0$

Logarithm of the Base

The logarithm of the base to itself is 1.

$$\log_b(b) = 1$$

Example: $\log_7(7) = 1$

Inverse Property

The logarithm function is the inverse of the exponential function.

$$\log_b(b^x) = x \quad \text{and} \quad b^{\log_b(x)} = x$$

Example: $\log_2(2^4) = 4$

Applications of Logarithmic Properties

These properties are widely used in simplifying logarithmic expressions, solving logarithmic equations, and in applications such as compound interest, population growth, and information theory.

Examples

1. **Simplifying Logarithmic Expressions:**

$$\log_3(27\cdot 9)-\log_3(3) = \log_3(243)-\log_3(3) = \log_3\left(\frac{243}{3}\right) = \log_3(81$$

2. **Solving Logarithmic Equations:**

$$\log_2(x) + \log_2(8) = 5$$

$$\log_2(8x) = 5 \quad \Rightarrow \quad 8x = 2^5 = 32 \quad \Rightarrow \quad x = 4$$

Natural Logarithm Function

The natural logarithm function, denoted as $\ln(x)$, is one of the most important functions in mathematics. It is the inverse of the exponential function with base e, where $e \approx 2.71828$. The natural logarithm has a wide range of applications in calculus, physics, engineering, and other fields.

Definition

The natural logarithm of a number x is defined as the power to which the base e must be raised to obtain x. Mathematically, this is expressed as:

$$\ln(x) = y \quad \text{if and only if} \quad e^y = x$$

where e is the base of the natural logarithm, approximately equal to 2.71828.

Domain and Range

- The domain of $\ln(x)$ is $x > 0$.
- The range of $\ln(x)$ is all real numbers, $-\infty < \ln(x) < \infty$.

Logarithm of e

The natural logarithm of e is 1:

$$\ln(e) = 1$$

This is because $e^1 = e$.

Properties

The logarithm of a product is the sum of the logarithms of the factors:

$$\ln(xy) = \ln(x) + \ln(y)$$

Example: Simplify $\ln(2 \times 3)$:

$$\ln(2 \times 3) = \ln(2) + \ln(3)$$

The logarithm of a quotient is the difference of the logarithms:

$$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$$

Example: Simplify $\ln\left(\frac{4}{2}\right)$:

$$\ln\left(\frac{4}{2}\right) = \ln(4) - \ln(2)$$

Properties

The logarithm of a power is the exponent times the logarithm of the base:

$$\ln(x^y) = y \ln(x)$$

Example: Simplify $\ln(3^2)$:

$$\ln(3^2) = 2\ln(3)$$

The logarithm of a reciprocal is the negative of the logarithm:

$$\ln\left(\frac{1}{x}\right) = -\ln(x)$$

Example: Simplify $\ln\left(\frac{1}{5}\right)$:

$$\ln\left(\frac{1}{5}\right) = -\ln(5)$$

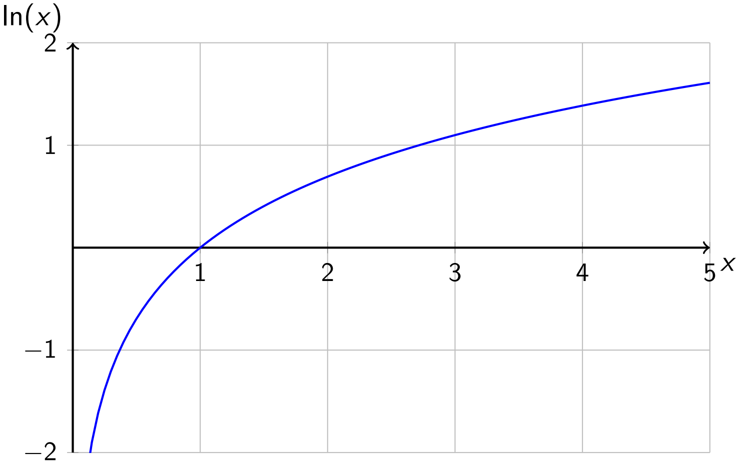
Applications of the Natural Logarithm

The natural logarithm function is used in many applications, including:

- Calculating compound interest in finance.
- Modeling exponential growth and decay in biology and chemistry.
- Solving differential equations in physics and engineering.
- Analyzing data in statistics and economics.

Plot of $\ln(x)$

Below is the plot of the natural logarithm function $\ln(x)$ for $x > 0$:



Properties of the Plot

- The natural logarithm $\ln(x)$ is only defined for $x > 0$.
- As x approaches 0 from the right, $\ln(x)$ approaches $-\infty$.
- As x increases, $\ln(x)$ increases, but at a decreasing rate.
- The function passes through the point $(1, 0)$, since $\ln(1) = 0$.