3 Constraint heat equation

Let $\Omega = [0,1] \times [0,1]$, then the heat equation with dirichlet and positivity constraint conditions is

$$\begin{aligned} u_t - \nabla.(\nabla u) &= f, \quad (x,t) \in \ \Omega \times (0,T] \\ u(x,t) &\geq 0, \quad (x,t) \in \ \Omega \times (0,T] \\ u(x,0) &= 0, \quad x \in \ \Omega \\ u(x,t) &= 0, \quad x \in \ \partial\Omega \times [0,T] \end{aligned} \tag{3.0.1}$$

After using Euler implicit scheme for time discretization in the time interval partition $0 =: t_0 < t_1 < ... < t_N := T$,
then $\forall n = 1, 2, ..., N$ we have

$$\begin{aligned} u(x,t^n) - (dt)\nabla.(\nabla u) = (dt)f + u(x,t^{n-1}), &\quad (x,t) \in \ \Omega \times (0,T] \\ u(x,t^n) \geq 0, &\quad x \in \ \Omega \\ u(x,t^n) = 0, &\quad x \in \ \partial\Omega \end{aligned} \tag{3.0.2}$$

Now we define energy functional as follows:

$$\mathcal{J}(v) = \frac{1}{2}\int_\Omega v^2 dx + \frac{dt}{2}\int_\Omega |\nabla v|^2 dx - dt\int_\Omega (f + u^{n-1})v dx, \quad v \in \mathcal{H} \tag{3.0.3}$$

where the space $\mathcal{H}$ should be chosen properly and $u^{n-1} = u(x, t^{n-1})$.

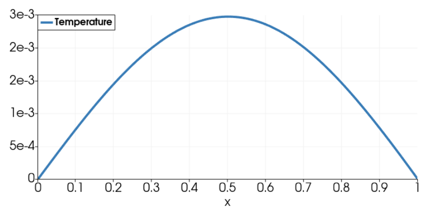
With some regularity assumption, it can be shown that if $u \in \mathcal{H}$ such that

$$\begin{aligned} u &= \arg\min_{v\in\mathcal{H}} \mathcal{J}(v) \\ u(x,t) &\geq 0, \quad (x,t) \in \ \Omega \times [0,T] \end{aligned} \tag{3.0.4}$$

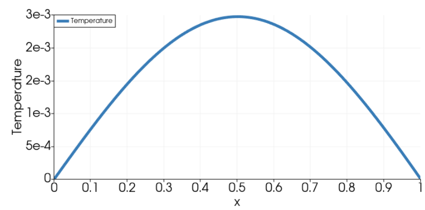
holds, then u is the solution of 3.0.2. Using lagrange multiplier functional $\mathcal{L}(v,\lambda) := \mathcal{J}(v) + \lambda v$ and first order optimality condition, the solution of the constraint optimization 3.0.4 is the solution of following variational equations : Find $(u,\lambda) \in \mathcal{H} \times \mathcal{S}$ such that

$$\begin{aligned} \int_\Omega uv dx + \int_\Omega \nabla u.\nabla v dx - \int_\Omega (f + u^{n-1} - \lambda)v dx = 0, &\quad \forall v \in \mathcal{H} \\ \int_\Omega (\lambda - \max 0,\ \lambda - u)w dx = 0, &\quad \forall w \in \mathcal{S} \end{aligned} \tag{3.0.5}$$

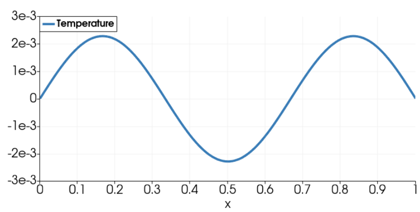
where $\mathcal{S}$ is the proper penalty space. The results of the equation 3.0.1 for $\Omega = [0,1] \times [0, 0.1]$, $T = 4$ and $f(x,t) = 2\sin(\pi\, t\, x)$ are shown in following figures



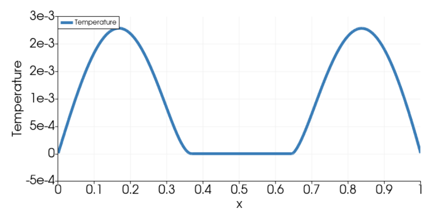
(a) Unconstraint heat equation $u(x, 1)$



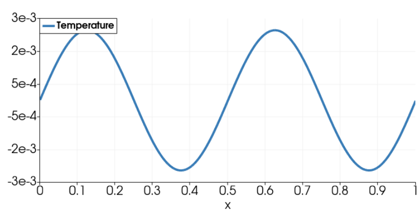
(b) Constraint heat equation $u(x, 1)$



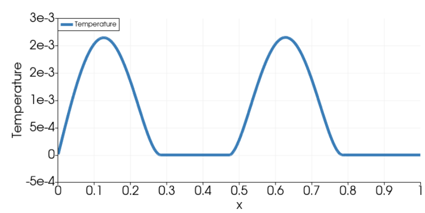
(a) Unconstraint heat equation $u(x, 3)$



(b) Constraint heat equation $u(x, 3)$



(a) Unconstraint heat equation $u(x, 4)$



(b) Constraint heat equation $u(x, 4)$