Math.SE: A Record

Andrew Norton

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Introduction

For the past several years, I was an avid contributer to Math.StackExchange; one of the largest math Q&A websites. At the time I am writing this, I have a reputation of 13,099. This places me in the top 3% of users; I have reached over 353k people with my answers, cast over 8k votes, and raised over 150 flags.

Due to various reasons, I believe it is time for my activity there to draw to a close. However, it was a large part of my life for the past four years. As such, I am creating this book of my posts (and the associated posts by other users) to serve as a justification for the time I spent on that site.

I have removed some answers to very low-quality questions that would not be helpful for reference; with other questions, I have altered the content slightly to better communicate the problem at hand. However, much of the content remains original; thus, the quality of language used by other users (typically in the "question" section) may be below that which I would typically use. To view the original question, the Question ID can be used on Math.SE.

Furthermore, much of this text was automatically converted from HTML/MathJax formatting to LaTeX automatically. I have manually corrected several of the problems this generated, but there still may be some computer-specific terms in the text. For instance, it will be somewhat common to read "which may be found here" in the text, since I normally would turn *here* into a link–of course, the link won't work in printed copy. Another common mistake will be the quotation marks; several quotations will be formated "like this" when they should be "like this." The difference is due to the way LaTeX handles a double-quote character, and search-and-replace does not work well to fix this issue.

Because I have only asked 16 questions, I have placed those in a chapter of their own along with the answer I "accepted." Many of these questions recieved multiple excellent answers, but the two perpetually limited resources (space and time) have prevented me from including all of them. I recommend that the interested reader look up the question online to see the other answers I received.

The remainder of the book consists of my *answers* to other users' questions. These are divided into chapters and sections based on subject matter. For every post, I have included both the question and answer scores; these are the net vote totals from other users of the site.

I hope that you find this book a useful reference and that it serves as a glimpse into my mind.

Andrew Norton
July, 2015

Chapter 1

Elementary Math

1.1 Puzzles

Can you find the formula of this function?

Question

I have been given the following table which describes part of a function, and need to find a formula matching this pattern:

x	y
1	1
2	2
3	4
4	8

I tried with the form y=ax+b and

$$y = ax^2 + bx + c$$

but these forms weren't appropriate for it.

Answer

Do you notice anything that the numbers in the y column have in common?

They're all powers of two!

With that information, see if you can play around with the general form for those numbers and get your desired function. (Hint: you might need to apply a horizontal shift.)

The answer, clearly, is

$$y = 2^{x-1}$$

Question Score: 0 Answer Score: 5
Question ID: 1100018

Patterns of no formula.

Question

How do I find the next number if the given pattern is

$$1, 2, 3, 2, 3, 4, 1, 2, 6, 23, 14, 19, 64, 69, 12, 78, 152, 93, 108, ?$$

(Find the question mark)

Answer

Let $p(n)$ be the nth term in the sequence. Clearly, this sequence follows the formula:

$$\begin{aligned}
p(x) = {} & \frac{600631x^{19}}{121645100408832000} - \frac{791723x^{18}}{800296713216000} + \frac{196988587x^{17}}{2134124568576000} && (1.1)\\
& - \frac{41785811x^{16}}{7846046208000} + \frac{8219611x^{15}}{38626689024} - \frac{49026370303x^{14}}{7846046208000} + \frac{26296057821373x^{13}}{188305108992000} && (1.2)\\
& - \frac{1098593289863x^{12}}{452656512000} + \frac{320897391017407x^{11}}{9656672256000} - \frac{39606777445183x^{10}}{109734912000} && (1.3)\\
& + \frac{30088961291838131x^{9}}{9656672256000} - \frac{(12871880314235441x^{8})}{603542016000} + \frac{(5410873671286319827x^{7})}{47076277248000} && (1.4)\\
& - \frac{(70887567483998273 3x^{6})}{1471133664000} + \frac{40339476695909643 73x^{5}}{2615348736000} && (1.5)\\
& - \frac{(599274486262658993x^{4})}{163459296000} + \frac{(49104110859304547x^{3})}{7916832000} - \frac{(2153634755170519x^{2})}{308756448} && (1.6)\\
& + \frac{(3188726258687x)}{692835} - 1320490 && (1.7)
\end{aligned}$$

Thus, $p(20) = 42$.

Ok, so that was a joke. However, this illustrates an important point–you can find some formula for $p(n)$ such that $p(20)$ is *any* value you wish. Without other context or information, coming up with the 20th term in this sequence is not a well-defined question. This is true for *any* "guess the next number" question.

Question Score: 1 Answer Score: 12
Question ID: 830412

Interview riddle

Question

On the Mathematics chat we were recently talking about the following problem @Chris'ssis had to solve during an interview :

$$3 \times 4 = 8$$
$$4 \times 5 = 50$$
$$5 \times 6 = 30$$
$$6 \times 7 = 49$$
$$7 \times 8 = ?$$

We have not managed to solve it so far, all we know is the solution (which was given **after** we had given up) :

224

How do we find this solution ?

Answer

Spoiler Alert: *(I use the answer given above in the response below. If you don't want to see it, you may want to skip this answer...)*

I'm replacing $\times$ by $\circ$, as the latter is more commonly used with unknown operations. I hate it when people redefine a common symbol, then "=" to describe a relationship.

Note that

$$3 \circ 4 = 4 \cdot 2 \tag{1.8}$$
$$4 \circ 5 = 5 \cdot 10 \tag{1.9}$$
$$5 \circ 6 = 6 \cdot 5 \tag{1.10}$$
$$6 \circ 7 = 7 \cdot 7 \tag{1.11}$$
$$7 \circ 8 = 8 \cdot 28 \tag{1.12}$$
$$\tag{1.13}$$

Thus, we can define:

$$a \circ b \overset{\text{def}}{=} b \cdot x_a$$

Where x_n is some sequence. OEIS yields three possible sequences:

$$x_n = \frac{\binom{n+2}{2} \gcd(n, 3)}{3}, \quad n \geq 0$$

(A234041)

$$x_n = \text{denominatorOf}\left(\frac{(n-2)(n+3)}{(n)(n+1)}\right) \quad n \geq 3$$

(A027626: GCD of n-th and $(n+1)$st tetrahedral numbers, offset by me for this problem)

The last sequence from OEIS is A145911 which is not promising *at all.* (It's a combination of, what appears to be, 3 other sequences.)

Question Score: 41 Answer Score: 28
Question ID: 866921

Bookshelf problem

Question

There is this thought problem I've been trying to solve, it goes as follows

Imagine a bookshelf with a finite number of books in it, to which a finite number of people have access. Each person has a paper and pen where they can keep track of what books they have already read. the books have no covers, so in order to find out if you have read it you must take it out and open it.

What is the most effective method each person can use to find a book they haven't read before?

conditions:

- You cannot change the order of the books in the bookshelf (you can however order the bookshelf into blocks/sections)
- The bookshelf - and its order - is the same for all people

edit:

- The bookshelf can also have new books added, so the system must be able to adapt to that,
- books can get stolen too. so a missing index can appear.

For example we can let each person take a book at random, then check on their list if they have read this book before, this will work. but once you reach the 900th/1000 book, you need to take 10 books in order to be able to read one.

Another example is where everyone writes down what they read, but after a few 100 books this gets rather slow. since you need to keep checking a list of 500 before you can pick one.

Answer

All jokes aside (see comment on OP)...

Let the people be numbered $P_1, P_2, P_3, \ldots$ and the books $B_1, B_2, B_3, \ldots$

If we have P_n start on book B_n, the next book they have not yet read is book B_{n+1}. If this book does not exist, wrap around to the start of the shelf.

Essentially, have each person start with a different book, and move down the shelf one-by-one.

No unnecessary books are opened, so this is optimal.

Question Score: 0 Answer Score: 2
Question ID: 304771

Puzzle: Dropping balls along the way

Question

A man has some balls in his pocket. Let the number of balls in his pocket be n.(Consider n as an integer. If any decimal value occurs, consider its floor value. For example, if $n = 2.6$ then take it as 2). For every mile that he runs, he is left with half the number of balls.

For instance, if initially he has 10 balls.

After running the first mile he'll have 5 balls.

After running the second mile, he'll have 2 balls.

After running the third mile, he'll have 1 ball.

After running the fourth mile, he'll have 0 balls.

So, it takes 4 miles to lose all the balls.

So how many miles does he have to run, in order to lose n balls.

Answer

Let a_k be the number of balls in the man's pocket after k miles. Thus, for the example:

$$a_0 = 10$$

$$a_1 = 5$$

$$a_2 = 2$$

$$a_3 = 1$$

$$a_4 = 0$$

The general form for a_k is:

$$a_k = \left\lfloor \frac{a_{k-1}}{2} \right\rfloor$$

Thus, the puzzle can easily be solved in O(n) time (for programming)...

To prove it in constant time, you need to find the number of times you can perform integer division by 2 before reaching 0. This makes me think it could be something similar to $log_2(n)$.

A quick check with a program shows that computing

$$\lceil \log_2(n) \rceil$$

returns the correct answer.

EDITED: As pointed out in the comments, the above does **not** return the correct answer for powers of 2. The correct answer is:

$$\lfloor \log_2(n) \rfloor + 1$$

Question Score: 5 Answer Score: 1
Question ID: 245656

1.2 Algebra

How do I remove a number from the numerator of a fraction so that I am left with the variable in the denominator in this equation?

Question

The question is this:

$$\frac{1}{R^p} = \frac{1}{4.5 \times 10^2} + \frac{1}{9.4 \times 10^2}$$

I calculated the equation so that it simplifyed to:

$$\frac{1}{R^p} = 0.003286$$

But now I am stuck...
Thanks for any help!!

Answer

You can reciprocate both sides of the equation. That is:

$$\frac{1}{a} = b \implies a = \frac{1}{b}$$

(assuming $b \neq 0$)
So, in your case:

$$\frac{1}{R^p} \approx 0.003286 \implies R^p \approx \frac{1}{0.003286} \approx 304.3$$

EDIT:
Another approach that is sometimes helpful depending on what p is equal to:

$$\frac{1}{a^b} = a^{-b}$$

So, in your case:

$$\frac{1}{R^p} = R^{-p} \approx 0.003286 \implies R \approx 0.003286^{-1/p}$$

Question Score: 1 Answer Score: 0
Question ID: 433091

Express the following formula in terms of n

Question

Express
$$T(2^k) = \frac{k(k+1)}{2}.$$
In terms of n, where $n = 2^k$.

I'm not sure how to go about with the conversion. Can someone concisely explain?

Thank you.

Answer

Alex's answer is correct; I thought I'd work the problem in "slow motion:"
$$n = 2^k$$
We have an exponential. We want to solve for k, which essentially begs for a log function. $\log_2$ is an ideal choice, as it cancels out the 2^k in the definition of T.

So, taking the $\log_2$ of both sides:
$$\log_2 n = \log_2(2^k)$$
$$\log_2 n = k \log_2(2)$$
$$\log_2 n = k$$
Substituting:
$$T(2^k) = \frac{k(k+1)}{2}$$
$$T(2^{\log_2 n}) = \frac{(\log_2 n)(\log_2 n + 1)}{2}$$
Simplify:
$$T(n) = \frac{(\log_2 n)(\log_2 n + 1)}{2}$$

Question Score: 1 Answer Score: 2
Question ID: 296018

Finding the root, domain, and limit to infinity of $f(x) = xe^{-x}$

Question

I had a calculus exam yesterday and the teacher asked the following question:

> Find the root and domain of:
> $$f(x) = xe^{-x}$$
> Also, find $\lim_{x\to\infty} f(x)$.

But from what I have researched, I didn't find that for e^x have roots, if someone know how to find the roots, thank you very much.

Answer

I'm going to assume you mean your question to be the following:

> Find the root and domain of:
> $$f(x) = xe^{-x}$$
> Also, find $\lim_{x\to\infty} f(x)$.

First, we find the root by setting $f(x) = 0$:

$$xe^{-x} = 0$$
$$\frac{xe^{-x}}{e^{-x}} = \frac{0}{e^{-x}}$$
$$x = 0$$

This is the only root. (We are justified dividing by e^{-x} because $e^{-x} \neq 0$ for all $x \in \mathbb{R}$.)

The domain is $\mathbb{R}$. I don't know exactly how you'd want to "prove" this, as it's kinda self-evident (there's no way to have a not-defined value...)

To find the limit:

$$\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x}$$

Here, we can "eyeball" this and note that e^x beats the *(insert favorite word (that fits) here)* out of x, in terms of how fast it grows. So, as x gets bigger and bigger, the denominator will get bigger much faster than the numerator. Thus the limit is 0:

$$\lim_{x\to\infty} \frac{x}{e^x} = 0$$

Question Score: 1 Answer Score: 4
Question ID: 430447

having trouble finding the inverse.

Question

Let $f(x) = \frac{1}{4}x^3 + x - 1$ What is the value of $f^{-1}(x)$ when $x = 3$?

First, check to make sure the fuction is stricly monotonic.
$f'(x) = \frac{3}{4}x^2 + 1$
$f''(x) = \frac{3}{2}x$ Which is a linear function thus $f(x)$ is strictly monotic
Now, this is where I am having troubles finding the inverse. I plugged the formula into wolfram alpha, but the result was not understood by me, thus, I don't understand how the result came about.
I have tried the switch x and y, but that is not very helpful.
$y = (1/4)x^3 + x - 1$
$y + 1 = (1/4)x^3 + x$
$y + 1 - x = (1/4)x^3$
$\sqrt[3]{4(y + 1 - x)} = x$
switch x and y, then solve for y
$\sqrt[3]{4(x + 1 - y)} = y$
$4(x + 1 - y) = y^3$
$x + 1 = \frac{y^3}{4} + y$
$x = \frac{y^3}{4} + y - 1$
$3 = \frac{y^3}{4} + y - 1$
$4 = (1/4)y^3 + y$
$0 = (1/4)y^3 + y - 4$
I have found equation to solve cubic polynomials
I think that is a bit much.
There has to be a better way, is there?

Answer

Let $f^{-1}(3) = a$. Then, $3 = f(a)$.
So, this problem simplifies to solving the equation:

$$\frac{1}{4}a^3 + a - 1 = 3$$

Rearranging:

$$\frac{1}{4}a^3 + a - 4 = 0$$

$$a^3 + 4a - 16 = 0$$

This is a polynomial with rational coefficients. The only possible (rational) roots are ±1, ±2, ±4, ±8, ±16. With this, it is probably easiest to plug each one in, and see which is a root.

We can easily see that $a = 2$ is a root to that equation. As a is the value we seek, we have our answer.

Question Score: 0 Answer Score: 3
Question ID: 534003

Evaluating a Function at a Point

Question

I am doing a course in calculus and I was given this problem :

Given that $f(x) = 3x^4 6x^3 + 4x^2 7x + 3$, evaluate $f(2)$.

The answer is meant to be 129 according to the tutor, but no matter how many times I try and work it out I can't see how they got that answer.

Answer

It appears your problem is with order of operations.

Let's look at an easier function:

$$g(x) = 6x^4$$

What is $g(-2)$?

$$g(-2) = 6(-2)^4$$

Remember–you evaluate exponents *before* multiplication:

$$g(-2) = 6(16)$$

$$g(-2) = 96$$

If the positive/negative thing is still hard, just turn the exponent into multiplication:

$$g(-2) = 6[(-2)(-2)(-2)(-2)]$$

$$g(-2) = 6[(-2)(-2)(4)]$$

$$g(-2) = 6[(-2)(-8)]$$

$$g(-2) = 6(16)$$

$$g(-2) = 96$$

From comments, it appears the error wasn't with raising the negative to an even power, but rather with the second term. The answer has been dealt with in other responses, but I'll include it here for archive purposes:

$$f(x) = 3x^4 6x^3 + 4x^2 7x + 3$$

$$f(-2) = 3(-2)^4 6(-2)^3 + 4(-2)^2 7(-2) + 3$$

$$f(-2) = 3(16)6(-8) + 4(4)7(-2) + 3$$

(Note the minus signs in front of the six and seven. I now multiply out those negatives, which makes them positive.)

$$f(-2) = 3(16) + 6(8) + 4(4) + 7(2) + 3$$

$$f(-2) = 48 + 48 + 16 + 14 + 3$$

$$f(-2) = 129$$

Question Score: 1 Answer Score: 3
Question ID: 275171

Algebra simplification in mathematical induction .

Question

I was proving some mathematical induction problems and came through an algebra expression that shows as follows:

$$\frac{k(k+1)(2k+1)}{6}+(k+1)^2$$

The final answer is supposed to be:

$$\frac{(k+1)(k+2)(2k+3)}{6}$$

I walked through every possible expansion; I combine like terms, simplify, factor, but never arrived at the answer.

Could someone explain the steps?

Answer

A good idea for this sort of thing is to use Wolfram Alpha to ensure that the two things are, indeed, equal. In this case, they are, so we can spend some time looking to factor.

$$\frac{k(k+1)(2k+1)}{6}+(k+1)^2=\frac{k(k+1)(2k+1)+6(k+1)^2}{6} \tag{1.14}$$

$$=\frac{(k+1)(k(2k+1)+6(k+1))}{6} \tag{1.15}$$

$$=\frac{(k+1)(2k^2+7k+6)}{6} \tag{1.16}$$

$$=\frac{(k+1)(k+2)(2k+3)}{6} \tag{1.17}$$

EDIT: A helpful trick is the one I did in the step from line 1 to line 2: I factored out the $k+1$ immediately, rather than expanding the whole expression. This makes it easier to deal with–most people have more practice factoring quadratics than cubics.

EDIT in response to comment:

$$\frac{k(k+1)(2k+1)+6(k+1)^2}{6}=\frac{k(k+1)(2k+1)+6(k+1)(k+1)}{6} \tag{1.18}$$

$$=\frac{(k+1)\Big(k(2k+1)+6(k+1)\Big)}{6} \tag{1.19}$$

Question Score: 5 Answer Score: 1
Question ID: 414184

Understanding underlying algebra behind simplified expression

Question

The solution to a linear algebra problem I'm working on reads:

$$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & -1 & 1-\lambda \end{pmatrix} = -\lambda(-\lambda(1-\lambda)+1)+1$$

This may be written as $\lambda^2(1-\lambda)+(1-\lambda)=(\lambda^2+1)(1-\lambda)$.

I understand how the determinant is calculated, but am struggling to understand the algebraic manipulation at the end. By my math, $-\lambda(-\lambda(1-\lambda)+1))+1$ simplifies to $-\lambda(-\lambda+\lambda^2+1)+1$, which simplifies to $(\lambda^2-\lambda^3-\lambda)+1$

What am I misunderstanding here?

Answer

Particularly, the algebra used to factor the expression is:

$$\begin{aligned} -\lambda(-\lambda(1-\lambda)+1)+1 &= \lambda^2(1-\lambda)-\lambda+1 && (1.20) \\ &= \lambda^2(1-\lambda)+1(1-\lambda) && (1.21) \\ &= (1-\lambda)(\lambda^2+1) && (1.22) \end{aligned}$$

Question Score: 2 Answer Score: 7
Question ID: 772227

Simplify algebra exponent

Question

I know I have asked a similar question in the past I am stuck on this question.
How would I simplify the following:

$$\left(\frac{xy^{-4}}{2^{-1}x^{-2}}\right)^3\left(\frac{8x^{-2}y^0}{3^{-1}xy^{-3}}\right)^{-2}$$

I have done

$$\frac{x^3y^{-12}}{2^{-3}x^{-6}}\left(\frac{3^{-1}xy^{-3}}{8x^{-2}\cdot 1}\right)^2$$

$$\frac{x^9y^{-12}}{2^{-3}}\frac{3^{-2}x^2y^{-6}}{64x^{-4}}$$

Unfortunately I am not sure how to proceed.

Answer

$$\frac{x^9y^{-12}}{2^{-3}}\frac{3^{-2}x^2y^{-6}}{64x^{-4}}$$

Remember that $a^{-n} = \frac{1}{a^n}$. Thus, we have:

$$\frac{2^3x^9}{y^{12}}\cdot\frac{x^2x^4}{64\cdot 3^2y^6}$$

At this point, we have $a^na^m = a^{n+m}$. I've also changed $64 = 2^6$.

$$\frac{2^3x^{15}}{2^6\cdot 3^2y^{18}}$$

Note that we can cancel some of the twos:

$$\frac{x^{15}}{2^3\cdot 3^2y^{18}}$$

These numbers are easier to work with:

$$\frac{x^{15}}{8\cdot 9y^{18}}$$

$$\frac{x^{15}}{72y^{18}}$$

Question Score: 1 Answer Score: 2
Question ID: 279749

Overall percentage difference

Question

In a corpus of text the expected letter frequency might be:

- e = 30%
- t = 30%
- a = 20%
- o = 20%

Actual recorded frequency:

- e = 90%
- t = 10%
- a = 0%
- o = 0%

I want to know the OVERALL percentage difference (not the average difference) between the expected frequencies and the actual frequencies, how would this be done? e.g: "Overall there is a 35% difference between what we expected our frequency count to be and our actual frequencies".

90% of the letters are e, but we expected only 30% of them to be e. So there is a 60% difference.

But this doesn't seem right to me, if it was expected to be 30% but we've got 90%, then that is 200% difference isn't it?!

Answer

It really depends on your definition of "percent difference." Some people use this to refer to the *point difference* between two percentages. (That is, $90\% - 30\% = 60\%$ percent difference).

However, other people refer to percent difference as a specific fraction:

$$\Delta\% = \frac{x - x_0}{x_0} \times 100\%$$

(where x is a given reading, and x_0 is the reference reading).

It depends on your specific application which definition you should use.

Question Score: 0 Answer Score: 1
Question ID: 420333

$$\sqrt[5]{x^3} = (\sqrt[5]{x})^3?$$

Question

$\sqrt[5]{x^3} = (\sqrt[5]{x})^3$?

I would suppose so given that $x^{3/5} = x^{3(1/5)} = \sqrt[5]{x^3}$ or $x^{3/5} = x^{(1/5)3} = (\sqrt[5]{x})^3$

Answer

The rule that $\sqrt[n]{a^m} = (\sqrt[n]{a})^m$ is true when $a \geq 0$ and n is even, or for any a when n is odd.

You need to be careful when dealing with even-index radicals and negative radicands. For example:

$$\sqrt[2]{-1}^2 = -1$$

But

$$\sqrt[2]{(-1)^2} = 1$$

Question Score: 0 Answer Score: 3
Question ID: 925850

Homework Help: Evaluate an expression with a "new" operation

Question

Problem Statement:

Imagine that a new operational symbol for mathematics has have been developed. This symbol is $\sim$ and is represented by

$$a \sim b = \frac{ab}{a-b}$$

Using this new symbol, find the value of $(2 \sim 3) \sim 4$

Any help?

Answer

We want to evaluate:

$$(2 \sim 3) \sim 4$$

First, simplify the parenthesis. Note that, for the part in the parenthesis, $a = 2, b = 3$.

$$a \sim b = \frac{ab}{a-b}$$

$$2 \sim 3 = \frac{(2)(3)}{2-3}$$

$$2 \sim 3 = \frac{6}{-1} = -6$$

Now, your expression is:

$$(-6) \sim 4$$

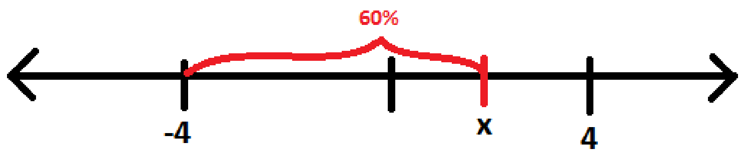
I'll leave the rest for you to do.

Question Score: 0 Answer Score: 2
Question ID: 290277

Calculate Percent of Range of Numbers

Question

If I have a range from -4 to 4, how can I figure out what 60% of that range is? Pictorially:



I'm looking for x such that the distance from -4 to x is 60% of the distance from -4 to 4. How can I find such an x?

Answer

First, you want to know what the *entire* range is. That is simply:

$$\text{range} = \text{big number} - \text{small number} = 4 - (-4) = 8$$

Then, we want sixty percent ***of*** the range. In many word problems, one may replace the word *of* with multiplication. That is, we're first looking for $(60\%) \times (\text{range})$. Recall that $60\% = 0.60$, and the rest should be straightforward.

The above tells us the distance from -4 to x. To find x, then, we simply add that distance to -4.

Question Score: -2 Answer Score: 2
Question ID: 1081784

Problem solving question with average

Question

Johnny had to take a test a day late. His 96 raised the class average from 71 to 72. How many students, including Johnny, took the test?

I tried to do trial and error to see how many students there were but I couldn't figure it out.

Answer

Hint: Letting n be the number of students excluding Johnny...

$$\text{Old average} = \frac{71n}{n}$$

$$\text{New average} = \frac{71n + 96}{n + 1}$$

Note that New average − Old average = 1.

Question Score: 0 Answer Score: 1
Question ID: 668869

Determine scale for random x and y so that aspect ratio is maintained and product is < 256

Question

It's entirely possible that this question already exists on here, I just simply don't have the vocabulary to search for it. That fact probably shows in the question title as well. Here's the question:

The largest size output by the third-party renderer I'm working with is 256in^2. There are no proscriptions on width and height. So, what I need to calculate is, for all area > 256, the scale that would need to be applied to width and height, retaining the aspect ratio between the two and reducing the area to ~ 256.

Thanks.

Answer

Let x_1 and y_1 be the width and height of the original size, and x_2 and y_2 be the width and height of the final size.

For aspect ratio to be the same, we want $x_2 = c \cdot x_1$ and $y_2 = c \cdot y_1$. We also want the area to be fixed:

$$x_2 \cdot y_2 = 256$$

Thus:

$$(c \cdot x_1)(c \cdot y_1) = 256$$

Rearranging:

$$c^2 = \frac{256}{x_1 y_1}$$

So, the scaling factor to resize the area is given by:

$$c = \sqrt{\frac{256}{x_1 y_1}}$$

Question Score: 3 Answer Score: 3
Question ID: 803167

What is the process of solving an equation with the square and the 4th power of an unknown?

Question

What is the process of solving an equation with the square and the 4th power of an unknown, for example $x^4 - 6x^2 - 27 = 0$?

Comment: It's not like I'm just waiting for the answer. I'm asking for the whole method in order to properly learn how to go about the problem.

Answer

This is a quadratic in terms of x^2. I know solutions have already been posted, but this is the way I like to solve these problems:

$$x^4 - 6x^2 - 27 = (x^2 - 9)(x^2 + 3)$$

If we are solving: $(x^2 - 9)(x^2 + 3) = 0$, this gives us two equations:

$$x^2 - 9 = 0 \qquad \text{or} \qquad x^2 + 3 = 0$$

The one on the left yields:

$$x^2 = 9 \implies x = \pm 3$$

The one on the right yields:(where i is the imaginary unit)

$$x^2 = -3 \implies x = \pm\sqrt{3}i$$

These are the four roots promised us by the Fundamental Theorem of Algebra. And, as expected, the complex roots came in conjugate pairs.

Question Score: -3 Answer Score: 1
Question ID: 598371

Ratios/Proportions of numbers: problem solving

Question

In a bag, there are red and blue cubes in the ratio 4 : 7. (red : blue) 4 : 7.
I add 10 more red cubes to the bag.
Now there are red and blue cubes in the ratio of (red : blue) 6 : 7.

How many blue cubes are in the bag?

I'm not sure, how would you work it out if you haven't got the amount of cubes altogether?

Answer

Let r be the number of red cubes (at the start), and b be the number of blue cubes. From the first statement, we know that:

$$\frac{r}{b} = \frac{4}{7}$$

From the second statement, we know that

$$\frac{r+10}{b} = \frac{6}{7}$$

Cross multiplying both of these, we obtain:

$$7r = 4b \tag{1}$$

$$7(r+10) = 6b \tag{2}$$

Distribute the 7 through in equation (2):

$$7r + 70 = 6b \tag{3}$$

Now plug the $7r$ from equation (1) into equation (3):

$$4b + 70 = 6b \tag{4}$$

Collect like terms:

$$70 = 2b$$

Dividing the two over:

$$b = 35$$

Question Score: 2 Answer Score: 0
Question ID: 385721

Proof of $\sqrt{2^{2^k}} = 2^{2^{k-1}}$?

Question

It's quite easy to observe that for $k \geq 0$:

$$2^{2^k} = 4, 16, 256, 65536, \ldots \tag{1.23}$$

$$\sqrt{2^{2^k}} = 2, 4, 16, 256, \ldots \tag{1.24}$$

More in general:

$$\sqrt{2^{2^k}} = 2^{2^{k-1}}$$

How can I prove this identity?

Answer

$$\begin{aligned}
\left(2^{2^{k-1}}\right)^2 &= \left(2^{2^{k-1}}\right)\left(2^{2^{k-1}}\right) && (1)\\
&= \left(2^{2^{k-1}+2^{k-1}}\right) && (2)\\
&= \left(2^{2(2^{k-1})}\right) && (3)\\
&= 2^{2^k} && (4)
\end{aligned}$$

To go from (1) to (2), we apply the rule that $a^b a^c = a^{b+c}$.
Going from (2) to (3), we note that $a + a = 2a$.
Going from (3) to (4), we again apply the rule that $a^b a^c = a^{b+c}$.

Question Score: 3 Answer Score: 2
Question ID: 351079

Simplifying fractions with exponents

Question

I'm revising for an exam, which I have the solutions to. One of the questions asks me to prove that a sequence is a Cauchy sequence, sequence is written as:

$a_n = \frac{2^{n+2}+1}{2^n}$,

and then in the solutions, it has:

$|a_n - a_m| = \left|\frac{2^{n+2}+1}{2^n} - \frac{2^{m+2}+1}{2^m}\right| = \left|\frac{1}{2^n} - \frac{1}{2^m}\right|$

I am extremely rusty on my math skills, but could someone explain how the 2^{nd} and 3^{rd} absolute values are equal?

Answer

The important equality is:

$$\frac{2^{n+2}+1}{2^n} = \frac{2^{n+2}}{2^n} + \frac{1}{2^n} = 2^2 + \frac{1}{2^n}$$

The same process is performed on both fractions and the 2^2 drops out. (One fraction contributes a $+4$ while the other fraction contributes a -4.)

Question Score: 1 Answer Score: 2
Question ID: 1096711

Find $g(y-1)$ given $g(y) = y^2 + 2y + 1$

Question

If $g(y) = y^2 + 2y + 1$, then $g(y-1) = \dots$
$g(y-1) = (y-1)^2 + 2(y-1) + 1$.
This is where I get stuck.

Answer

Starting from where you're at:

$$\begin{aligned}
g(y-1) &= (y-1)^2 + 2(y-1) + 1 && (1.25)\\
&= \underbrace{\left(y^2 - 2y + 1\right)}_{(y-1)^2} + \underbrace{2y - 2}_{2(y-1)} + 1 && (1.26)\\
&= y^2 - 2y + 2y + 1 + 1 - 2 && (1.27)\\
&= y^2 && (1.28)
\end{aligned}$$

EDIT: Whenever we write a^2, that means $a \cdot a$. So, in this particular instance, we can think of $y-1$ as our "a."

$$\begin{aligned}
(y-1)^2 &= (y-1)(y-1) && (1.29)\\
&= y(y-1) - 1(y-1) \quad \text{(distribute first factor)} && (1.30)\\
&= (y \cdot y - y \cdot 1) + (-y + 1) \quad \text{(distribute again)} && (1.31)\\
&= y^2 - y - y + 1 && (1.32)\\
&= y^2 - 2y + 1 && (1.33)
\end{aligned}$$

Question Score: 0 Answer Score: 2
Question ID: 1184575

Adding simultaneous rates

Question

If I have 3 rates like units/second produced by 3 separate devices and I want to get the total rate, is it ok to add the 3 rates up? This seems basic and I'd say it's correct but I thought I'd ask.

Thanks

Answer

Yes, you are correct. (In normal conditions.)

That is, if you're given a problem like:

> Worker A makes $5\frac{\text{widgets}}{\text{hour}}$, worker B makes $2\frac{\text{widgets}}{\text{hour}}$, and worker C makes $3\frac{\text{widgets}}{\text{hour}}$. What is the total rate?

The total rate is the sum of the other rates; in this case $10\frac{\text{widgets}}{\text{hour}}$.

However, if the rates were dependent on each other (e.g. only one worker can use a certain tool at a time), then the problem becomes a bit more difficult, and no "one-size-fits-all" approach exists.

Question Score: 1 Answer Score: 0
Question ID: 429912

Solving a quartic equation

Question

I am trying to be able to find the radius of a cone combined with a cylinder. see my other question (Solving for radius of a combined shape of a cone and a cylinder where the cone is base is concentric with the cylinder? part2)

I have a volume calculation that Has been reduced as far as I know how to.

Know values:

$$v = 65712.4$$
$$x = 3$$
$$y = 2$$
$$\theta = 30$$
$$r = unknown$$

$$v = \pi r^3 \left(2y - \frac{2}{3}\tan\theta - \frac{x}{r}\right)$$

Since I haven't solved a Quadratic equation in a while.

I would appreciate it explained in steps.

Thank You For Your Time.

Answer

I think your best bet is Newton's method for approximation.

When I type this into Wolfram Alpha, I get three nasty solutions, each that look something like this:

$$r = \frac{1}{2\left(\sqrt{3}\,\pi - 18\,\pi\right)} 3\left(2\,\pi^2 \sqrt{-2616\sqrt{3}\,v^2 + 12\,313 v^2 - 324\sqrt{3}\,\pi\,v + 2943\,\pi\,v} + 24\sqrt{3}\,\pi^2\,v - 218\,\pi^2\,v - 27\,\pi^3\right) \hat{}\,(1/3) + \left(27\,\pi^2\right) \Big/ \left(2\left(\sqrt{3}\,\pi - 18\,\pi\right)\left(2\,\pi^2\sqrt{-2616\sqrt{3}\,v^2 + 12\,313\,v^2 - 324\sqrt{3}\,\pi\,v + 2943\,\pi\,v} + 24\sqrt{3}\,\pi^2\,v - 218\,\pi^2\,v - 27\,\pi^3\right) \hat{}\,(1/3)\right) - \frac{9\,\pi}{2\left(\sqrt{3}\,\pi - 18\,\pi\right)} \approx$$

$$0.276618 - 0.02935\sqrt[3]{19.7392\sqrt{7781.96\,v^2 + 7482.69\,v} - 1741.3\,v - 837.169} - \frac{2.60706}{\sqrt[3]{19.7392\sqrt{7781.96\,v^2 + 7482.69\,v} - 1741.3\,v - 837.169}}$$

Question Score: 1 Answer Score: 1
Question ID: 452306

If A divided by B is equal to C and C is larger than D ...

Question

If A divided by B is equal to C and C is larger than D, and all these number are positive whole numbers other than one, it follows that

1. A is always larger than D
2. B is always smaller than C
3. B is always smaller than D
4. C is always larger than B
5. D is always larger than B

I got this from reasoning skill section. How would I choose it ?

Answer

$$\frac{A}{B} = C$$
$$\frac{A}{B} \geq D$$
$$A \geq BD$$

As B is a natural number greater than 1:

1. A is always larger than D.

Question Score: 1 Answer Score: 0
Question ID: 244354

Expansion of polynomial raised to high power

Question

Is there an easy way to expand something like $(x+x^2+x^3)^6$?
Thanks in advance!

Answer

My approach avoids multinomials:
Let's play with it a bit:

$$\begin{aligned}(x+x^2+x^3)^6 &= x^6(1+x+x^2)^6 && (1.34)\\ &= x^6\frac{((1-x)(1+x+x^2))^6}{(1-x)^6} && (1.35)\\ &= \frac{x^6(1-x^3)^6}{(1-x)^6} && (1.36)\\ & && (1.37)\end{aligned}$$

The numerator and denominator can now be expanded with the binomial theorem, and then long division will simplify the expression. (This is the way I would do it.)

Another approach:

$$\begin{aligned}(x+x^2+x^3)^6 &= (x+(x^2+x^3))^6 && (1.38)\\ &= x^6+6x^5(x^2+x^3)+15x^4(x^2+x^3)^2 && (1.39)\\ &+\ 20x^3(x^2+x^3)^3+15x^2(x^2+x^3)^4+6x(x^2+x^3)^5+(x^2+x^3)^6 && (1.40)\end{aligned}$$

Now, each of the binomials can be expanded as above.

Question Score: 1 Answer Score: 1
Question ID: 600883

Find the interest at the end of 30 years

Question

A sum of rupees 100 is invested at 8% per annum Simple Interset. Calculate the interest at the end of $1_{st}, 2_{nd}, 3_{rd}$.... years. Is the sequence of interest an Arithmetic Progression. Find the interest at the end of 30 years.

Hint: Arithmetic Progression follows the rule of $a_n = a + (n-1)d$ where n the term of the AP

My Problem: I am having problem in creating the Arithmetic Progression in this Question, because In this I am totally confused about concept.

Answer

Let's say you have 100 rupees at 8% a year:

- At the end of the first year, the total interest is $100 \cdot 8\%$.
- At the end of the second year, the interest is $100 \cdot 8\% + 100 \cdot 8\%$

So, after n years, you'll have a total interest owed as $n \cdot (100 \cdot 8\%)$. Now, think about how much you would need to pay each year.

- At the end of the first year, you'd pay $100 + 100 \cdot 8\%$ rupees.
- At the end of the second year, you'd pay $100 + 100 \cdot 8\% + 100 \cdot 8\%$

Thus, at the end of the nth year, you'd pay $100 + 100 \cdot 8\%$ rupees.

a is the starting value of the sequence, so $a = 100$

d is the "common difference" of the sequence–that's how much each term is greater than the last term. Thus, $d = 100 \cdot 8\% = 8$.

Thus,

$$a_n = a + (n-1) \cdot d$$

However, there is an offset–the sequence starts at 100, then 108, then 116, etc. We want it to count 108, 116, 124, etc. So, our altered series is

$$a_n = a + n \cdot d$$

$$a_n = 100 + n \cdot 8$$

Question Score: 0 Answer Score: 1
Question ID: 223927

1.3 Exponents and Logarithms

Filling in 'x' in a log function

Question

if $3^5 = x$ (exponential equation) converts to log form gives $\log_3 x = 5$
that makes sense.

$$3^5 = 243 \Rightarrow x = 243$$

So if I take the log form again: $\log_3 x = 5$ and replace x with 243. I then take the log of 243, expecting to get 5?? But instead, I get 2.3856??

Can anyone explain?

Answer

The `log` button on a calculator takes the base-10 log of the number. You want the base-3 log, so you either have to use the *change of base formula* or use an fancier or online calculator like Wolfram Alpha.

To input a log of a certain base in Wolfram alpha, use the underscore `_`. So, typing: `log_3(243)` gives the result `5`. (Or, as pointed out below, the underscore isn't really needed. I still like to use it for readability purposes.)

Question Score: 1 Answer Score: 2
Question ID: 431333

Variable with an exponent variable

Question

I'm actually dealing with an economics problem, but it seems like the math is always what messes me up. Ignoring what the variables mean, I'm trying to understand how to get from step 1 to step 2.

$$(d+g)k_e^* = sk_e^{*\alpha} \tag{1.41}$$

$$k_e^* = \left(\frac{s}{d+g}\right)^{\frac{1}{1-\alpha}} \tag{1.42}$$

Any input is appreciated

Answer

First, divide both sides by $(d+g)$:

$$(d+g)k_e^* = sk_e^{*\alpha} \implies k_e^* = \frac{s}{(d+g)}k_e^{*\alpha}$$

Now, divide both sides by k_e^*, and use exponent rules to simplify:

$$k_e^* = \frac{s}{(d+g)}k_e^{*\alpha} \implies \frac{k_e^*}{k_e^{*\alpha}} = \frac{s}{(d+g)} \implies k_e^{*1-\alpha} = \frac{s}{d+g}$$

Now, raise both sides to the $1/(1-\alpha)$ power:

$$k_e^{*1-\alpha} = \frac{s}{d+g} \implies \left(k_e^{*1-\alpha}\right)^{1/(1-\alpha)} = \left(\frac{s}{d+g}\right)^{1/(1-\alpha)}$$

Using exponent rules to simplify the right hand side, we have:

$$k_e^* = \left(\frac{s}{d+g}\right)^{\frac{1}{1-\alpha}}$$

Question Score: 0 Answer Score: 0
Question ID: 767287

Using Logarithms

Question

$$-2^{n-1}\ln 2 = -100\ln 10$$

$$-100\ln 10 = -230$$

$$\frac{-230}{\ln(2)} = -333$$

$$-2^{n-1} > -333$$

$$(n-1)\ln(-2) > \ln(-333)$$

Here is where I am stuck.
I am not sure if this part is correct: $-n-1=8$. Then solving we would get -9.

Answer

Just make sure that we take into account negative numbers:

$$-(2^{n-1})\ln(2) < -100\ln(10)$$

$$(2^{n-1})\ln(2) > 100\ln(10)$$

'note switch in inequality because of multiplying by a negative number...

$$(2^{n-1}) < \frac{100\ln(10)}{\ln(2)}$$

'Same thing: $\ln(2) < 0$.

$$(2^{n-1}) < \frac{100\ln(10)}{\ln(2)}$$

$$(n-1)\ln(2) < \ln\left(\frac{100\ln(10)}{\ln(2)}\right)$$

$$(n-1) > \frac{\ln\left(\frac{100\ln(10)}{\ln(2)}\right)}{\ln(2)}$$

$$n > \frac{\ln\left(\frac{100\ln(10)}{\ln(2)}\right)}{\ln(2)} + 1$$

For an integer n, then simply round up.

Question Score: 0 Answer Score: 0
Question ID: 520888

Equation involving logarithm, solvable without calculator?

Question

I'd like to know if I can solve the following equation without calculator:

$(0.4)^t = 5t$

I don't think it's possible, cause I always get stuck on formulas of the form $e^t = t$ or $t = \ln t$

I've also put the equation into wolframalpha, which was of no use to me unfortunately.

I'm not interested in the answer containing a W-function. Just want to know whether I can find the real solution or not!

Thanks!

Answer

No, it is not possible to solve in terms of elementary functions. That is why WolframAlpha returns an answer using the W function.

When you find the formula boils down to $t = \ln t$, you've come across a form of *Lambert's Transcendental Equation*.

Question Score: 1 Answer Score: 2
Question ID: 280438

log transformation for dummies

Question

We are given a formula $y = -\log(x)$.
Now, we're given a new expression $y = -\log(x^{1.5})$. How can we write up y?

Answer

We can make use of the following property of log:

$$\log(a^b) = b\log(a)$$

So, in this case:

$$y = -\log(x)$$

$$y = -\log(x^{1.5}) = -1.5\log(x)$$

Note: I treated the above as if y were a function of x, and we applied the transformation $x \mapsto x^{1.5}$. If this isn't a transform problem:

If $y = -\log(x)$, then:

$$-\log(x^{1.5}) = -1.5\log(x) = 1.5(-\log(x)) = 1.5(y)$$

Thus:

$$1.5(y) = -\log(x^{1.5})$$

Question Score: 3 Answer Score: 4
Question ID: 442029

proving that the differences of squares of hyperbolic sin/cos is an integer.

Question

The hyperbolic sine and cosine are defined as following:

$\sinh(x) = \dfrac{e^x - e^{-x}}{2}$

$\cosh(x) = \dfrac{e^x + e^{-x}}{2}$

How do I show that their differences of squares are always an integer for all real numbers x?

hint appreciated!

Answer

Of course, there is the identity listed in the comments. But, for fun, let's see what we can do without that:

$$\cosh^2(x) = \frac{(e^x + e^{-x})^2}{4} = \frac{e^{2x} + e^{-2x} + 2e^0}{4}$$

$$\sinh^2(x) = \frac{(e^x - e^{-x})^2}{4} = \frac{e^{2x} + e^{-2x} - 2e^0}{4}$$

So:

$$\begin{aligned}
\sinh^2(x) - \cosh^2(x) &= \frac{e^{2x} + e^{-2x} - 2}{4} - \frac{e^{2x} + e^{-2x} + 2}{4} && (1.43)\\
&= \frac{\cancel{e^{2x}} + e^{-2x} - 2 - \cancel{e^{2x}} - e^{-2x} - 2}{4} && (1.44)\\
&= \frac{\cancel{e^{-2x}} - 2 - \cancel{e^{-2x}} - 2}{4} && (1.45)\\
&= \frac{-2-2}{4} && (1.46)\\
&= \frac{-4}{4} && (1.47)\\
&= -1 && (1.48)\\
& && (1.49)
\end{aligned}$$

To prove for the other difference, we don't have to do near as much algebra:

$$\begin{aligned}
\cosh^2(x) - \sinh^2(x) &= -(\sinh^2(x) - \cosh^2(x)) && (1.50)\\
&= -(-1) && (1.51)\\
&= 1 && (1.52)
\end{aligned}$$

As 1 and -1 are integers, we have proven what we wanted.

Question Score: 2 Answer Score: 2
Question ID: 446715

Exponential Approximation

Question

This is taken from an example given in Gilbert Strings Linear Algebra. The topic is not relevant, but I don't understand the following:

$$\left(1+\frac{0.06}{N}\right)^{5N}=e^{0.30}$$

How is this derived?

Answer

Based on the definition of $\exp(x)$, we have:

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n}=e^{x}$$

Thus, a useful approximation for some situations is (for large N):

$$\left(1+\frac{x}{N}\right)^{N}\approx e^{x}$$

In your case (assuming N is very large):

$$\left(1+\frac{0.06}{N}\right)^{5N} = \left(1+\frac{5\cdot 0.06}{5N}\right)^{5N} \tag{1.53}$$

$$= \left(1+\frac{0.3}{5N}\right)^{5N} \tag{1.54}$$

$$\approx e^{0.3} \tag{1.55}$$

Question Score: 0 Answer Score: 6
Question ID: 1077093

Solve for x: question on logarithms.

Question

The question:

$$\log_3 x \cdot \log_4 x \cdot \log_5 x = \log_3 x \cdot \log_4 x \cdot \log_5 x \cdot \log_5 x \cdot \log_4 x \cdot \log_3 x$$

My mother who's a math teacher was asked this by one of her students, and she can't quite figure it out. Anyone got any ideas?

Answer

Following up on Jaeyong Chung's answer, and working it out:

$$1 = \log_3 x \log_4 x \log_5 x$$
$$1 = \frac{(\ln x)^3}{\ln 3 \ln 4 \ln 5}$$
$$(\ln x)^3 = \ln 3 \ln 4 \ln 5$$
$$(\ln x) = \sqrt[3]{\ln 3 \ln 4 \ln 5}$$
$$x = \exp\left(\sqrt[3]{\ln 3 \ln 4 \ln 5}\right) \approx 3.85093$$

And, of course, the obvious answer that everyone will overlook: $x = 1$ makes both sides of the equation zero.

Question Score: 3 Answer Score: 7
Question ID: 379486

Is there any "superlogarithm" or something to solve x^x?

Question

Is there any "superlogarithm" or something to solve an equation like this:

$$x^x = 10?$$

Answer

What you're looking for is the Lambert W function. This is the function such that:

$$W(xe^x) = x$$

It does not have a "simple" or explicit form.
To solve your equation, we follow this process:

$$x^x = 10$$

$$x \ln x = \ln 10$$
$$e^{\ln x} \ln x = \ln 10$$
$$W(e^{\ln x} \ln x) = W(\ln 10)$$
$$\ln x = W(\ln 10)$$
$$x = e^{W(\ln 10)}$$

Question Score: 13 Answer Score: 20
Question ID: 417657

Determine Equality: $(\frac{1}{n\sqrt{n}})^{\frac{1}{n}} = (n^{-\frac{\sqrt{n}}{n}})$

Question

I'm a little bit confused by this one. Is this correct?

$$\left(\frac{1}{n\sqrt{n}}\right)^{\frac{1}{n}} = \left(n^{-\frac{\sqrt{n}}{n}}\right) = \sqrt{n^{-\frac{1}{n}}}$$

Answer

Recall:

$$\left(\frac{1}{b^n}\right) = b^{-n} \tag{1}$$

$$(a^n)^m = a^{nm} \tag{2}$$

Now we have:

$$\left(\frac{1}{n\sqrt{n}}\right)^{\frac{1}{n}} = \left(n^{-\sqrt{n}}\right)^{\frac{1}{n}} \qquad \text{(applying equation (1))}$$

$$\left(n^{-\sqrt{n}}\right)^{\frac{1}{n}} = n^{-\sqrt{n}\cdot\frac{1}{n}} \qquad \text{(applying equation (2))}$$

Simplifying:

$$n^{-\sqrt{n}\cdot\frac{1}{n}} = n^{\frac{-\sqrt{n}}{n}}$$

Therefore, yes, the statement in your question is correct.

Question Score: 3 Answer Score: 1
Question ID: 304921

Getting stuck on simple logarithmic equation

Question

$$x \times \ln(x) = 1$$

I am trying to solve that equation. I used the theory $\ln(a) = \ln(b)$ being equivalent to $a = b$ and got stuck at

$$x = e^{\frac{1}{x}}$$

That's as far as I went and I know there's a solution (around 1.8 or 1.9), since I used my calculator, but I'd like to know how to do this by hand.

Answer

There is no solution using only algebraic manipulation. We have to use the "product-log" or Lambert-W function to solve this, and this function doesn't fall in the "simple functions" category. :)

Basically, the Lambert-W function is the inverse function of:

$$f(x) = xe^x$$

Equivalently:

$$x = W(xe^x)$$

So, using your expression:

$$x = e^{\frac{1}{x}}$$
$$1 = \frac{1}{x}e^{\frac{1}{x}}$$

Taking the product-log of both sides:

$$W(1) = W\left(\frac{1}{x}e^{\frac{1}{x}}\right)$$

$$W(1) = \frac{1}{x}$$

$$x = \frac{1}{W(1)}$$

Question Score: 1 Answer Score: 1
Question ID: 355268

How do you explain the concept of logarithm to a five year old?

Question

Okay I understand that it cannot be explained to a 5 year old. But, how do you explain the logarithm to primary school students?

Answer

The way I learned was pretty simple... First, know exponents (repeated multiplication).

So, to what power do you have to raise the base to get the number?

This is the logarithm function.

Question Score: 76 Answer Score: 2
Question ID: 129013

Units of a log of a physical quantity

Question

So I have never actually found a good answer or even a good resource which discusses this so I appeal to experts here at stack exchange because this problem came up again today. What happens to the units of a physical quantity after I take its (natural) logarithm. Suppose I am working with some measured data and the units are Volts. Then I want to plot the time series on a log-scale, only the ordinate is on the log scale, not the abscissa. So the x-axis is definitely in time (seconds let's say) but what are the units on the y-axis? Will it be Volts or log(Volts) or something? If I square the quantities, then the units are squared too so what if I take the log? A rationale in addition to the answer will be appreciated as well.

I guess whatever the answer, the same goes for taking the exponential or sine of the data too, right?

Answer

In the expression $\ln x$, x must be unitless.

This is because the log function is a series with x raised to differing powers. For instance:

$$-\sum_{k=1}^{\infty} \frac{(-1)^k(-1+x)^k}{k}$$

for

$$|-1+x| < 1$$

Let's say x had units of meters (for example). Then the first term in the series would have units of meters, the second term units of square meters, 3rd term in cubic meters, etc. You can't add quantities with differing powers of units, thus x must be unit-less.

The same argument applies for $|-1+x| \not< 1$.

Question Score: 5 Answer Score: 3
Question ID: 238390

1.4 Geometry and Trig

Is it possible to find the area of a shape from its perimeter?

Question

Is it possible to find the area of a free form shape knowing the perimeter? An example would be a clover leaf shape. If the perimeter is 96 how would I know what the area would be?

Answer

This is impossible. We can prove this by constructing two shapes with the same perimeter but different areas.

Consider, for instance, the unit equilateral triangle, with perimeter 3 and area $\sqrt{3}/4$, and the square with side lengths $3/4$ and area $9/16$. Since these two shapes have the same perimeter, but different areas, one cannot uniquely determine area from perimeter.

Question Score: 2 Answer Score: 1
Question ID: 1124507

Finding circumference without using π

Question

If the area of a circle is $254.34\ldots$ cm^2 it has a diameter of 18 cm, is it possible to find the circumference without using or making the irrational constant Pi ($\pi = 3,1415926535\ldots$) in any way at all, and if it is possible, how?

To find the area of 254.34 I used π as 3.14 in shorthand. The formula you give should allow me to find the Circumference as 56.52 cm (as this also uses π as 3.14)

Answer

Recall:

$$A = \pi \left(\frac{D}{2}\right)^2$$

$$C = \pi D$$

Where C is circumference, A is area, and D is diameter.

Thus:

$$\frac{A}{C} = \frac{\pi \left(\frac{D}{2}\right)^2}{\pi D}$$

The π's cancel out, as well as one of the D's:

$$\frac{A}{C} = \frac{D}{4}$$

Solving for C:

$$C = \frac{4A}{D}$$

Question Score: 13 Answer Score: 20
Question ID: 280402

PreCalc problem about high tide?

Question

The depth of water, y, at time x is governed by the following equation:

$$y = 3 + 4\cos\left(\frac{\pi}{5.7}(x-2)\right)$$

x is the time in hours after midnight
What time did the first high tide occur today?
How deep was the water at the time?
When will the second high tide occur today?

Answer

High tide occurs when the value of the cos function is equal to 1, which makes the equation:

$$y_{max} = 3 + 4(1) = 7$$

This is the maximum depth of the water.

What value of x makes the cos function equal to 1? Well, $\cos(2k\pi) = 1$ for $k \in \mathbb{Z}$. So, we want to solve:

$$\frac{\pi}{5.7}(x-2) = 2k\pi$$
$$(x-2) = 11.4k$$
$$x = 11.4k + 2$$

We pick the value of k that minimizes this expression (we want the soonest time after midnight), so we pick $k = 0$. Thus, we have:

$$x = 2$$

Or, 2:00am.

For the second high tide, we increment k by 1:

$$x = 11.4 + 2 = 13.4$$

Or, 1:24pm.

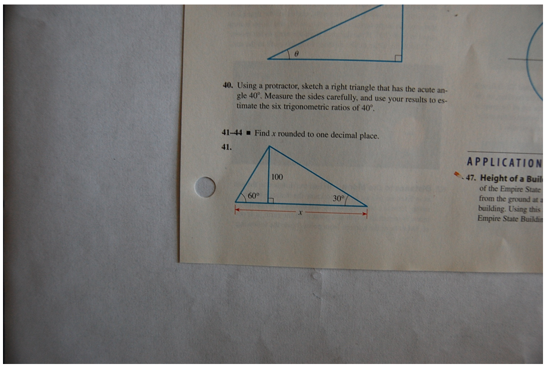
Question Score: 0 Answer Score: 1
Question ID: 389719

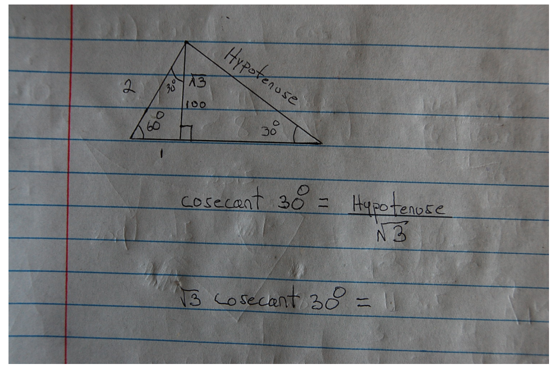
Solve The Triangle

Question

I am having a tough time trying to solve this problem. I have utilized the 30, 60, 90 triangle measures for the length of sides. However, I am stuck since the side that would be $\sqrt{3}$ has 100 as its length. How do I solve then?

This is what I have done so far:

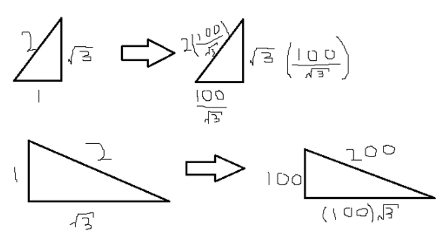




Answer

Both of the current answers utilize trig functions. I thought I would add an answer that doesn't use trig functions, but rather uses scaling of two $(30, 60, 90)$ triangles.

We are used to seeing the sides of a $(30, 60, 90)$ triangle being written as $(1, \sqrt{3}, 2)$ (where 1 is the length opposite to the 30 degree angle, $\sqrt{3}$ is the length opposite to the 60 degree angle, etc). We can then scale the side lengths by whatever multiplier we wish. The diagram below shows how to scale the side lengths appropriately for this problem.



For the left-hand triangle (top), we choose the multiplier $\frac{100}{\sqrt{3}}$ (because the side opposite the 60 degree angle is 100 units long). For the right-hand triangle (bottom), we choose the multiplier 100 (because the side opposite the 30 degree angle is 100 units long).

Thus, the length x can be found easily:

$$x = 100\sqrt{3} + \frac{100}{\sqrt{3}} = \frac{400\sqrt{3}}{3}$$

Question Score: 2 Answer Score: 0
Question ID: 865149

Find the value of the given trig function

Question

Find the value of the trigonometric function $\sec(v-u)$ given that

$$\sin u = \frac{20}{29} \quad , \quad \cos v = \frac{3}{5}$$

(Both u and v are in Quadrant III.)
Thanks!

Answer

Use the ID:

$$\cos(v-u) = \cos(v)\cos(v) + \sin(u)\sin(u)$$
$$\frac{1}{\cos(v-u)} = \frac{1}{\cos(v)\cos(v) + \sin(u)\sin(u)}$$
$$\sec(v-u) = \frac{1}{\cos(v)\cos(v) + \sin(u)\sin(u)}$$

Now, use the Pythagorean thm and quadrant number to find $\cos(u)$ and $\sin(v)$:

$$\sin(u) = \frac{y}{r}$$
$$x = -\sqrt{r^2 - y^2}$$
$$\cos(u) = \frac{-\sqrt{r^2 - y^2}}{r}$$
$$\cos(u) = \frac{-\sqrt{29^2 - 20^2}}{29}$$

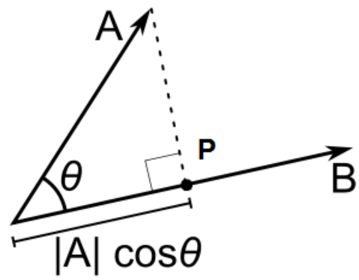
$\sin(v)$ is similar, and so omitted.
Now plug in to the secant formula.

Question Score: 0 Answer Score: 0
Question ID: 223200

Find the scalar projection point with dot product

Question

Take a look at the image bellow:



How can I find the point P, without knowing the vector B (or its end point)? Known is:

- The vector A (start point, end point and its length)
- Angle theta

Answer

Since you know A, you can find the angle A makes with the horizontal axis. Call this angle φ.

Then $\varphi - \theta$ is the angle that B makes with the x-axis. You can now find the x and y coordinates using right-triangle trig. (The hypotenuse of the triangle of interest is $\|A\| \cos\theta$.)

Another approach (that's more work than it's worth, but still an option) is to use the rotation matrix to make A parallel to B. Then compute the projection of A onto this rotated vector A. (This would be how I'd do it if I had a computer algebra system and only needed to solve the problem once.)

Question Score: 0 Answer Score: 0
Question ID: 1102826

How do I get the slope on a circle?

Question

I have drawn a circle by doing this in Matlab:

```
syms x;
ezplot(cos(x),sin(x))
```

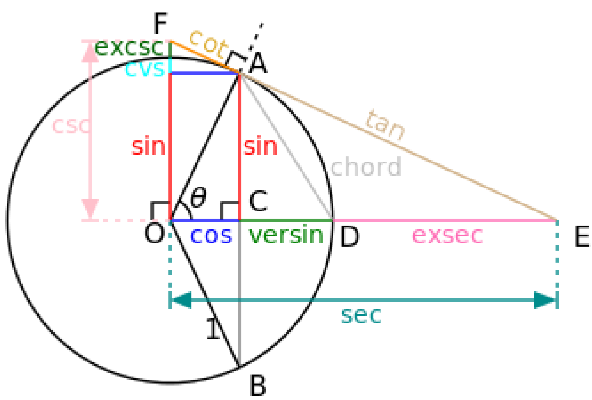
I get the tangent point at which I want my tangent to be by taking $x = \cos(3.1415)$, $y = \sin(3.1415)$, but now I am confused because normally I would differentiate the function to get the slope.

Now I have two functions that created my curve, which one do I differentiate to get the slope?

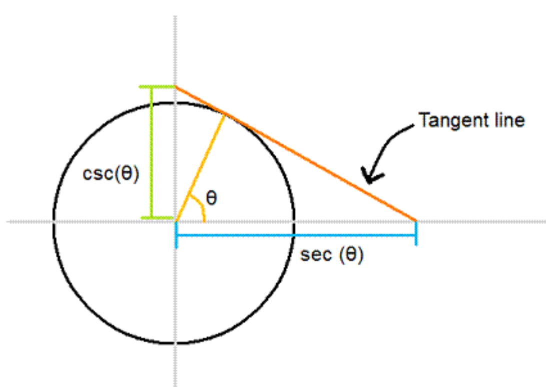
Answer

The prior answers have all used calculus. I'm going to post an answer using only trig.

The following diagram from Wikipedia's Trig Page is helpful.



However, that diagram also has a fault–the picture is very cluttered. :) Thus, I've redrawn it for you, labeling the components important for this problem:



Note that $\csc\theta$ returns the distance from the origin to the y-intercept of the tangent line, and $\sec\theta$ returns the distance from the origin to the x-intercept of the tangent line.

Let m represent slope:

$$m = \frac{\Delta y}{\Delta x} = \frac{\csc\theta}{\sec\theta}$$

Simplifying, this gives:

$$m = \cot\theta$$

Now, for your particular point, you will run into some difficulty:

$$m = \cot\pi = undefined$$

To interpret this, note that $\cot\pi$ is undefined because you are dividing by zero. This means that you had some change in the y direction, but none in the x direction. The only type of line like this is a vertical one. :)

Let me know if this needs more explaining–I'll edit the answer to make it clearer.

Question Score: 0 Answer Score: 1
Question ID: 232571

What's wrong with the calculation with polar coordinates here?

Question

Suppose $x = r\cos t$ and $y = r\sin t$. I did the following calculation:

$$x^4 + y^4 = (r\cos t)^4 + (r\sin t)^4 \tag{1.56}$$
$$= r^4(\sin^4 t + \cos^4 t) \tag{1.57}$$
$$= r^4[(\sin^2 t + \cos^2 t)^2 - 2\sin^2 t\cos^2 t] \tag{1.58}$$
$$= r^4(1 - 2\sin^2 t\cos^2 t) = r^4 - 2r^4\sin^2 t\cos^2 t \tag{1.59}$$

On the other hand

$$x^4 + y^4 = (x^2 + y^2)^2 - 2x^2y^2 \tag{1.60}$$
$$= r^4 - 2r^2\sin^2 t\cos^2 t \tag{1.61}$$

What is wrong with the calculation?

Answer

Notice that $2x^2y^2 = 2(r\sin t)^2(r\cos t)^2 = 2r^4\sin^2 t\cos^2 t$.

Question Score: 3 Answer Score: 2
Question ID: 1007903

Azimuth vs Yaw?

Question

What is the difference between these two terms, or are they completely synonymous? I have frequently seen either used in connection with pitch and roll.

Answer

There are others probably more qualified to speak on this subject than I, but from what I've gleaned from the Wikipedia pages on Azimuth and Yaw:

Azimuth seems to be used to represent an absolute heading, while yaw is a measurement of how much a craft turns from its current orientation.

Thus, if your azimuth went from $90°$ to $184°$, your yaw was $94°$, while your current azimuth is $184°$.

Question Score: 1 Answer Score: 2
Question ID: 296799

Volume of Shapes

Question

I'm doing some maths revision at home from a GSCE book and I'm a bit stuck. I'm not sure how to start off. (I know the formulae for a volume of cone and a formulae for a volume of a sphere but I don't know how that will help me).

QUESTION:

> If a cone with perpendicular height $6h$ and radius $2h$ has the same volume as a sphere of radius r, show that $r = \sqrt[3]{6h}$.

Answer

From the formulas for the area of a circle and cone:

$$V_{\text{circ}} = \frac{4}{3}\pi r^3$$

$$V_{\text{cone}} = \frac{1}{3}\pi \underbrace{(2h)^2}_{\text{cone radius}} \cdot \underbrace{(6h)}_{\text{cone height}}$$

We are told that *the volumes are the same.* Thus, we can set:

$$V_{\text{circ}} = V_{\text{cone}}$$

Thus:

$$\frac{4}{3}\pi r^3 = \frac{1}{3}\pi(2h)^2 \cdot (6h)$$

Simplifying the $(2h)^2$:

$$\frac{4}{3}\pi r^3 = \frac{1}{3}\pi(4r^2)(6h)$$

Now, multiply by $\frac{3}{4}$:

$$\pi r^3 = \pi(r^2)(6h)$$

Can you work from here? If you need more help, just leave a comment.

Question Score: 0 Answer Score: 0
Question ID: 430980

find minimum of given function

Question

today my relative asked a problem,which had strange solution and i am curious, how this solution is get from such kind of equations. let say function has form

$f(x) = a\sin(x) + b\cos(x)$

we should find it's minimum,we have not any constraints or something like this,as i know to find minimum,we should find point where it reaches minimum and then put this point into first equation,so in our case we have

$f'(x) = a\cos(x) - b\sin(x)$ or when we set this to zero and also convert in tangent form ,we get

$\tan(x) = a/b$

; or $x = tan^{-1}(a/b)$

now if we put this into first equation,it would be difficult without calculator to calculate minimum,let say $a = 3$ and $b = 2$, but my relative told me there exist such kind of formula that minimum is directly $\sqrt{a^2 + b^2}$, in our case $\sqrt{13}$, is it right? first of all i think that we can get value 3, if $\alpha = 0$;

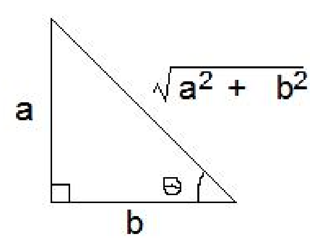
please help me

Answer

It appears that the main confusion here is how to find

$$\sin(\tan^{-1}(a/b)) \quad \text{or} \quad \cos(\tan^{-1}(a/b))$$

Well, we have $\tan(\theta) = \frac{a}{b}$, for some θ. So, let's create a right triangle like that, where the opposite side has length a, and the adjacent side has length b.



We know that $\sin(\theta) = \frac{\text{opp}}{\text{hyp}}$, so then:

$$\sin(\theta) = \frac{a}{\sqrt{a^2 + b^2}}$$

We also know that $\tan^{-1}(a/b) = \theta$, so then:

$$\sin(\tan^{-1}(a/b)) = \frac{a}{\sqrt{a^2 + b^2}}$$

The procedure for cosine is similar.

Question Score: 2 Answer Score: 3

Question ID: 437040

What does the secant value represent?

Question

What does the secant value represent?

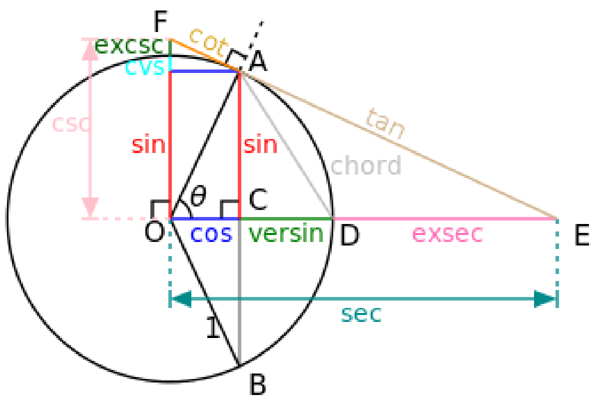
I know that

$$\sec = 1/\cos(\theta)$$

but really I do not know what this value represents, so I need your help. A clear example with images would be appreciated.

Answer

The numeric values of the different trig functions are the lengths of the line segments in the diagram below, if the circle has radius 1. This helped me a lot when trying to visualize what the numeric value of each trig function meant.



(Picture taken from Wikipedia.)

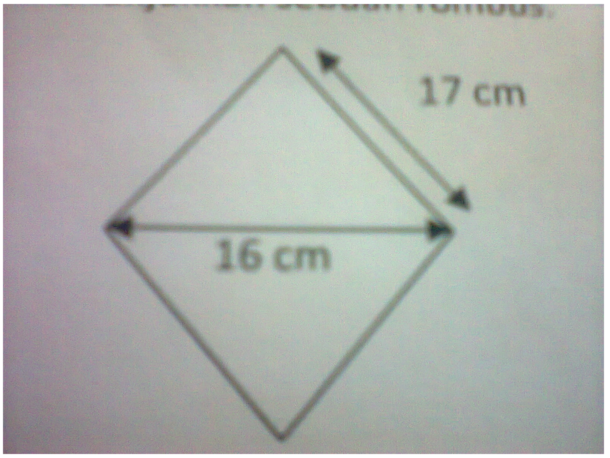
For Example, $\sec(\theta) = 2$ means that:

> If one draws a tangent line from the point on the unit circle at θ, the distance from the center of the unit circle to the intersection of the tangent line and the horizontal is 2 units.

Question Score: 4 Answer Score: 2
Question ID: 851736

how to calculate the area of a rhombus is in cm^2

Question



How do I calculate the area of a rhombus is in cm^2?

Is the formula $\frac{1}{2} \times 17 \times 16$? Anyone can help me to solved this? I don't know the rhombus formula.

Based on my exercise book, the answer is $240cm^2$

Answer

Your edited attempt is close, but not quite. After drawing the vertical diagonal, you have a right triangle with hypotenuse 17 cm and base 8 cm.

To find the height, we must use the Pythagorean Theorem:

$$b^2 + h^2 = c^2$$

Where b is the base of the triangle, h is the height, and c is the hypotenuse.

See if you can find the height of the triangle... Let me know in a comment if you need more help.

Question Score: 0 Answer Score: 1
Question ID: 452377

1.5 Complex Numbers

How to solve $z^6 = -15625$?

Question

$z^6 = -15625$ has six solutions.
$z^6 + 15625 = (z^2 + 25)(z^4 - 25z^2 + 625)$
$z^2 + 25 = 0$
$\Rightarrow x_1 = -5i$
$\Rightarrow x_2 = 5i$
That's easy, but I just don't find a way to get the other 4 solutions.
Thanks in advance

Answer

Thomas' method is better in principal, but if you insist on using factorization:

$$z^6 + 15625 = (z^2 + 25)(z^4 - 25z^2 + 625)$$

The second term is a quadratic in terms of z^2. We can use the quadratic formula to find the zeros generated by the second term:

$$z^2 = \frac{-(-25) \pm \sqrt{625 - 4(625)}}{2}$$

(etc)

Question Score: 1 Answer Score: 0
Question ID: 598733

Developing a Complex Number

Question

Develop 3^i.

I don't understand how to even begin approaching this problem.

The best way to for me to understand this, I think, would be to see a complete solution or a kick starter at least.

Answer

Do you know Euler's formula? $e^{\theta i} = \cos(\theta) + i\sin(\theta)$ (This will be helpful to memorize, fyi.)

Also, a helpful simplification is $a^b = e^{(\ln a)b}$.

With those two formulas:

$$3^i = e^{i\ln 3}$$

Thus, for Euler's formula, $\theta = \ln 3$. I think you can take it from here..

Question Score: 0 Answer Score: 0
Question ID: 322638

Solve $\cos(z) = \frac{3}{4} + \frac{i}{4}$

Question

I tried solving this using the definition of $cos(z) = \frac{e^{iz}+e^{-iz}}{2}$ and equating it to $\frac{3}{4} + \frac{i}{4}$ and converting it to a complex quadratic equation through a substitution $t = e^{iz}$ and finding roots via the complex quadratic formula but it didn't seem to work. I would prefer solutions via elementary methods.

Here is my attempt:

By definition we have $\frac{e^{iz}+e^{-iz}}{2} = \frac{3}{4} + \frac{i}{4} \implies e^{iz} + e^{-iz} = \frac{3}{2} + \frac{i}{2}$. Let $t = e^{iz}$ so then we have $t + \frac{1}{t} = \frac{3}{2} + \frac{i}{2}$ and if we multiply both sides by t we have $t^2 + 1 = (\frac{3}{2} + \frac{i}{2})t$ and hence $t^2 + (\frac{3}{2} + \frac{i}{2})t + 1 = 0$ By the quadratic formula for complex numbers we have, $a = 1, b = \frac{3}{2} + \frac{i}{2}, c = 1 \implies z = \frac{-(\frac{3}{2}+\frac{i}{2}) \pm \sqrt{(\frac{3}{2}+\frac{i}{2})^2 - 4(1)(1)}}{2(1)}$. Simplyifing we have $z = \frac{-\frac{3}{2}-(\frac{1}{2})i \pm \sqrt{-2+(\frac{3}{2})i}}{2}$ We wish to express $-2 + (\frac{3}{3})i$ in polar form so we have $|-2 + (\frac{3}{2})i| = \frac{5}{2}$. Now equating the real and imaginary parts we have $\frac{5}{2}\cos(\theta) = -2 \implies \cos(\theta) = -\frac{4}{5}$ and $\frac{5}{2}\sin(\theta) = \frac{3}{2} \implies \sin(\theta) = \frac{3}{5}$. From this we have $\tan(\theta) = -\frac{3}{4} \implies \theta = \arctan(-\frac{3}{4}) \approx -.6435$ rad. So we have $w = -2 + (\frac{3}{2})i = \frac{5}{4}(\cos(\theta) + i\sin(\theta)) = \frac{5}{4}e^{i\theta}$. By Proposition 1.3.12 we have $\sqrt{w} = \sqrt{\frac{5}{4}}e^{\frac{i\theta}{2}} = \frac{\sqrt{5}}{2}e^{\frac{i\theta}{2}}$. Similarily for $-\frac{3}{2} - (\frac{1}{2}) = \frac{\sqrt{10}}{2}e^{i\varphi}$ Where $\varphi = \arctan(\frac{1}{3})$. So finally we have $z = \frac{-(\frac{3}{2}+\frac{i}{2}) \pm \sqrt{(\frac{3}{2}+\frac{i}{2})^2 - 4(1)(1)}}{2(1)} = \frac{\sqrt{10}e^{i\varphi} \pm \sqrt{5}e^{\frac{i\theta}{2}}}{4}$ as solutions to $\cos(z) = \frac{3}{4} + \frac{i}{4}$.

Answer

I agree with your approach for the most part, but I think you've messed up on calculating b in the quadratic formula:

$$e^{iz} + e^{-iz} - \frac{3}{2} - \frac{i}{2} = 0$$

Using $t = e^{iz}$:

$$t + \frac{1}{t} - \frac{3}{2} - \frac{i}{2} = 0$$

Multiplying by t:

$$t^2 - \left(\frac{3}{2} + \frac{i}{2}\right)t + 1 = 0$$

This is where your solution starts to go wrong. Those minus signs are just lying in wait for the innocent mathematician!

Thus:

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && (1.62)\\
&= \frac{\left(\frac{3}{2} + \frac{i}{2}\right) \pm \sqrt{\left(\frac{3}{2} + \frac{i}{2}\right)^2 - 4}}{2} && (1.63)\\
&= \frac{\left(\frac{3}{2} + \frac{i}{2}\right) \pm \sqrt{\left(\frac{9}{4} - \frac{1}{4} + \frac{3}{2}i\right) - 4}}{2} && (1.64)\\
&= \frac{\left(\frac{3}{2} + \frac{i}{2}\right) \pm \frac{1}{2}\sqrt{-8 + 6i}}{2} && (1.65)\\
&= \frac{3 + i \pm (3i + 1)}{4} && (1.66)\\
&= \cdots && (1.67)
\end{aligned}
$$

Going from line three to line four:

$$
\begin{aligned}
\frac{\left(\frac{3}{2} + \frac{i}{2}\right) \pm \sqrt{\left(\frac{9}{4} - \frac{1}{4} + \frac{3}{2}i\right) - 4}}{2} &= \frac{\left(\frac{3}{2} + \frac{i}{2}\right) \pm \sqrt{\left(\frac{8}{4} + \frac{3}{2}i\right) - 4}}{2} && (1.68)\\
&= \frac{\left(\frac{3}{2} + \frac{i}{2}\right) \pm \sqrt{\left(2 + \frac{3}{2}i\right) - 4}}{2} && (1.69)\\
&= \frac{\left(\frac{3}{2} + \frac{i}{2}\right) \pm \sqrt{-2 + \frac{3}{2}i}}{2} && (1.70)\\
&= \frac{\left(\frac{3}{2} + \frac{i}{2}\right) \pm \sqrt{-\frac{8}{4} + \frac{6}{4}i}}{2} && (1.71)\\
&= \frac{\left(\frac{3}{2} + \frac{i}{2}\right) \pm \frac{1}{2}\sqrt{-8 + 6i}}{2} && (1.72)
\end{aligned}
$$

Question Score: 4 Answer Score: 1
Question ID: 652439

Linear Algebra Complex Numbers

Question

The solutions to the equation $z^2 - 2z + 2 = 0$ are $(a + i)$ and $(b - i)$ where a and b are integers. What is $a + b$?

I simplified and got $(z+1)(z+1) = -1$ and now I'm not sure where to go from there. I did this but I'm not sure.

$(a + i)^2 = a^2 - 1$

$(b - i)^2 = b^2 + 1$

$a + b = (a + i) + (b - i) = (a^2 - 1) + (b^2 + 1) = a^2 + b^2$

Answer

Re: Your Work

If you're going to solve this equation by factoring, etc., you do *not* want to factor and set equal to a non-zero number. That is, doing something like:

$$(z + 1)(z + 1) = -1$$

does not point you in the right direction, because you can only find roots by factoring when you have the equation set equal to *zero.*

Also, please see Chris K's comments about the other parts of your algebra... in general, $(x + y)^2 \neq x^2 + y^2$. That is, $(a + i)^2 \neq a^2 + i^2$.

Re: The Problem Statement Because this is a quadratic with real coefficients, we know by the Fundamental Theorem of Algebra that it will have exactly 2 solutions, where complex solutions occur in *conjugate pairs.*

We are given that the two solutions are $a + i$ and $b - i$. Since these are conjugate pairs, then $a = b$.

So, our two solutions are $a + i$ and $a - i$. Having one variable makes things simpler.

We also know that

$$\begin{aligned}(z - (a + i))(z - (a - i)) &= z^2 + (-(a + i) - (a - i))\, z + (a + i)(a - i) &\quad (1.73)\\ &= z^2 - 2z + 2 &\quad (1.74)\end{aligned}$$

Equating coefficients, we find that $(-(a + i) - (a - i)) = -2$ and $(a + i)(a - i) = 2$.

Can you take it from there?

Question Score: 2 Answer Score: 2

Question ID: 657927

rewrite $2ie^{i\pi} + i^3$

Question

i am asked to rewrite $2ie^{i\pi} + i^3$ into $x + iy$ form. i just tried all what i know so far, but couldnot come to solution. i said: $2ie^{i\pi} + i^3 = 2ie^{i\pi} - i$ but further i am stuck really. i am really eager to learn how things like this work. i appreciate any help and attempt to help.

the more difficult problems i am facing, the more i am loving maths. this problem was the first problem in my exam today. it took me 20 minutes. no sign of success..

EDIT: sorry, i forgot e. now it is there

Answer

$$2ie^{i\pi} + i^3$$

Recall that:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Thus, $e^{i\pi} = -1$. Substituting:

$$-2i + i^3$$

Continuing using basic properties of i:

$$-2i - i$$

$$\boxed{-3i}$$

Question Score: 2 Answer Score: 2
Question ID: 295834

1.6 Plotting

Graphing a Circle that doesn't have two of each variable

Question

Graph the circle:

$$x^2 + y^2 - 2x - 15 = 0$$

I know how to approach this problem if there were two y and x variables. But there is only one y variable. How would I approach this?

Answer

We do this by "completing the square" for the x variable, and then noticing that $y^2 = (y-0)^2$:

$$x^2 + y^2 - 2x - 15 = 0 \tag{1.75}$$

$$x^2 - 2x + \underbrace{1 - 1}_{+0} + y^2 - 15 = 0 \tag{1.76}$$

$$(x^2 - 2x + 1) + y^2 - 16 = 0 \tag{1.77}$$

$$(x-1)^2 + (y-0)^2 = 4^2 \tag{1.78}$$

Thus, our circle is of radius 4, and is centered at $(-1, 0)$.

Question Score: 0 Answer Score: 0
Question ID: 782945

Plotting for solution for $y = x^2$ and $x^2 + y^2 = a$

Question

Consider the system

$$y = x^2$$

and

$$x^2 + y^2 = a$$

for $x > 0$, $y > 0$, $a > 0$.

Solving for equations give me $y + y^2 = a$, and ultimately

$$y = \frac{-1 + \sqrt{4a+1}}{2}$$

(rejected $\frac{-1-\sqrt{4a+1}}{2}$ since $y > 0$).

The next part is to plot on the $x - y$ plane for different values of a. Is plotting the graph of $y = x^2$ insufficient?

Answer

Yes, it is insufficient.

You should notice that this equation is "special:"

$$x^2 + y^2 = a$$

This is the graph of a circle, radius $\sqrt{a}$.

So, your graph should contain both the parabola and the part of the circle in the region in question.

Here's a link to a graph from Wolfram Alpha which may help give some intuition. The darkest shaded region that is there is the region of interest.

Question Score: 1 Answer Score: 1
Question ID: 291095

How to shift $y = -\ln(-x)$ so that the x intercept is 0 instead of -1?

Question

I would like this function shifted to the right. I'm not quite sure how. As positive values of x grow, I need increasing slope.

Answer

These two rules are good to know (f is the original function, g is the shifted one):

- To shift $f(x)$ **up** by c units, let $g(x) = f(x) + c$
- To shift $f(x)$ **to the right** by c units, let $g(x) = f(x - c)$.

In your situation, $f = -\ln(-x)$.

Aside: Note that your (shifted) function is not defined (in the real number system) for values of $x \geq 1$

Question Score: 0 Answer Score: 2
Question ID: 911044

1.7 Functions and Relations

Which of the following is not used to determine the slope of a function algebraically?

Question

I don't know the answer to the above question. I think it is the slope of a secant line but I'm not sure

1. slope of a secant line
2. graphing calculator
3. difference quotient
4. rationalizing numerator

Answer

All of the given options can actually be used to find the slope of a function:

1. The slope of the secant line is the difference quotient.
2. A good graphing calculator (e.g. TI-89 or HP-50G) can find it algebraically.
3. Taking the limit of the difference quotient gives the slope of the function.
4. Rationalizing the numerator could be done as part of taking a limit (e.g. an example from the comments.)

Question Score: -1 Answer Score: 1
Question ID: 438215

Double branch $\sqrt{x}$ or square function turned 90?

Question

I have this idea for a graph but don't know what function could describe it better.

The idea is something like the "squared" function turned 90 degrees to the right, so that possible values for x are always positive and y may be both positive and negative.

The graphs of $\sqrt{x}$ and $-\sqrt{x}$ combined look good too, but I don't know how to write that as a single function (eh, I'm so bad with these things).

Basically, anything that may look like this will do.

Looking forward to some solution.

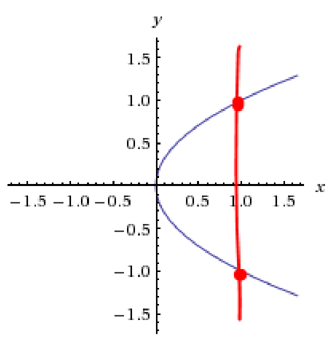
Answer

What you're looking for can be described as a *parabola opening towards the positive x axis.* I'm going to refer to it as a "sideways parabola," since the "standard" parabola that people learn opens towards the positive y axis.

The bad news: You cannot express a sideways parabola as a function of x. Why? Let's go back and look at a restriction on functions:

> **Vertical Line Test:** For a relation to be a function, it must (colloquially) pass the *vertical line test.* That is, you must be able to draw a vertical line anywhere on the relation's graph and the line must intersect the relation in at most one point.

In the picture below, I've marked the two intersections that a vertical line makes on a sideways parabola. This shows that the sideways parabola is not a function.



However, the good news is that one may still describe such a graph with mathematical notation. Two such ways are below, but keep in mind that they are **not** functions of x.

$$x = y^2$$
$$y = \pm\sqrt{x}$$

Question Score: 3 Answer Score: 6
Question ID: 1103580

Is this a isomorphism $(\mathbb{Z},+) \to (Z,+)$ where $\varphi(n) = 2n$? Why not?

Question

This is what I did :

a) $\varphi(x+y) = \varphi(x)\varphi(y)$: $2(x+y) = 2x+2y$ so φ is a homomorphism.

b) $\varphi(x) = \varphi(y)$: $2(1) = 2(1)$ so φ is one-to-one

c) $\varphi(x) = 2n$

$\varphi(1) = 2(1) = 2$ so is it onto? I say yes, but the answer says "$\varphi(n) \neq 1$ for all $n \in \mathbb{Z}$" so not isomorphic. Why I say yes: for every element in $\mathbb{Z}$, the first group, there is a element $2n$ in the second group. What am I missing here?

Answer

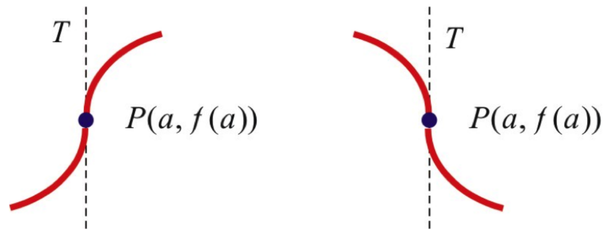
As pointed out in the comments, this question results from confusion over the definition of *onto*.

In order for ϕ to be onto, we need there to be an x such that, for every y, we have $y = \phi(x)$. A simple counterexample is $y = 1$; since y is odd, there is no integer x such that $y = 2x$.

Question Score: -1 Answer Score: 0
Question ID: 957086

Is a function that has a vertical tangent line a function?

Question



At, a, the function has a "infinite slope" or vertical tangent line. If the slope of the tangent line is considered to be the instantaneous rate of change, at that point, the function increases "straight up".

Since the function increases "straight up", the next point would be right above the previous point. Since a function can only have one value in the range per domain, and this function would at least two range values (the points right above each other) for the same domain value, wouldn't it violate the definition of a function?

Note that I understand that the function doesn't actually consist of discrete points, but is instead continuous. However, if the function is considered to consist of points infinitesimally close together, wouldn't the next "infinitesimally" far apart point be right above the previous point?

Answer

Having a vertical tangent line does not imply a graph does not represent a function. It may be easier to consider the example of a horizontal tangent line. For example:

$$y = x^3 \implies y' = 3x^2$$

At $x = 0$, we have a horizontal tangent line. Following the logic from the OP, the points just to the right and left of $x = 0$ would have the same y coordinate (0). However, we know from pre-calc that $y = x^3$ is a one-to-one function, which contradicts those other points having the same y coordinate.

The same intuition applies to the vertical example.

Why is this? Essentially, it's because you're taking a *limit*. The slope of the tangent line could be thought of as the slope at *one* point, not between 2 points...

Someone else may be able to give a more rigorous reasoning, but I hope this provides some intuition...

Question Score: 9 Answer Score: 10
Question ID: 309702

Relations and partial order

Question

I'm having some trouble answering a question and any help would be appreciated. The question is:

"Let $\mathbb{Z}$ be the set of integers and consider the set $X = \mathbb{Z} \times \mathbb{Z}$. Consider the relation R on X defined by $(x, y)R(z, w)$ if:

1. $x < z$ or
2. $x = z$ and $y \leq w$

Show that R is a partial order on X."

The whole set of integers is really throwing me off, how would you go about tackling this question?

Answer

You need 3 things to be a partial order relation. That is:

1. Antisymmetric property:
   $(x, y)R(w, z)$ and $(w, z)R(x, y) \implies (w, z) = (x, y)$
2. Reflexive property:
   $(x, y)R(x, y)$ for all (x, y)
3. Transitive property:
   $(x, y)R(w, z)$ and $(w, z)R(s, t) \implies (x, y)R(s, t)$

Antisymmetric Property

To show this, we pick two points in $\mathbb{Z}^2$. We call these points (a, b) and (c, d). If those two points have (what I'll call) a two-way relationship, then we have two options: $a < b$ **and** $b < a$ (impossible), or $a = b$ **and** ($c \leq d$ **and** $d \leq c$). Clearly, the only possible result is that $a = b$ and $c = d$.

See if that helps you do the other two options. (Pick arbitrary points, and show that they satisfy the requirements.) Comment if you need further assistance.

Question Score: 0 Answer Score: 0
Question ID: 786847

Can the inverse of a function be the same as the original function?

Question

I was wondering if the inverse of a function can be the same function.
For example when I try to invert
$g(x) = 2 - x$
The inverse seems to be the same function. Am I doing something wrong here?

Answer

That's perfectly fine, and your answer is correct. For another function that is its own inverse, see:

$$f(x) = \frac{1}{x} = f^{-1}(x)$$

Question Score: 13 Answer Score: 11
Question ID: 541978

Vertical line test

Question

> A vertical line crossing the x-axis at a point a will meet the set in exactly one point (a, b) if $f(a)$ is defined, and $f(a) = b$.

If the vertical line meets the set of points in two points then $f(a)$ is undefined?

Answer

(Short answer)
No. Rather, we conclude f is a relation, not a function.
Response to comment:
A *real function of one variable* is really saying three things:

1. It's a real mapping. This means that the object in question is a mapping from $\mathbb{R}^n \to \mathbb{R}^n$. In "normal talk," this means that you can put any real number in, and you get real numbers out. The object in question doesn't accept complex numbers.
2. It's a function. This means that every input has exactly one output. That is, you can't put in a number and be able to get out multiple numbers.
3. It's of one variable. This means that there is only one value used for each input. An example of a multivariable function is $f(x, y, z) = xyz$. This is a function of 3 variables.

Does this help?

Question Score: 2 Answer Score: 3
Question ID: 347816

1.8 Foundations of Math

Irrational roots don't exist

Question

I'm going through Apostol's Calculus vol. 1. It is excellent. But I was surprised to see him "prove" each number has a square root.

Inverse functions like divide usually introduce some invalid operations like divide by 0.

SQRT(x) (obviously an inverse function) where x is not a "square" number is just such an "invalid operation." It is impossible because there is no such number which satisfies this operation, and that is why it is irrational, you are trying to approximate something that doesn't exist.

I was really surprised such a clear, logical thinker as Apostol had this blind spot.

Answer

You seem to be confusing the concept of "not existing" with "non-terminating or non-repeating decimal expansion." Just because we cannot write something down in its entirety does not mean that it doesn't exist.

For example, we use the symbol $\sqrt{2}$ to denote the number such that $\sqrt{2} \cdot \sqrt{2} = 2$. This number most certainly exists.

Numbers that "do not exist" are fairly difficult (if not impossible) to come by–systems can be constructed to account for "impossibilities." Irrational numbers take care of those with non-terminating, non-repeating decimal expansions. Complex numbers take care of the "square root" of -1. It isn't a matter of existing, so much, but rather a matter of being able to construct a manner with which to work with the numbers.

Question Score: -4 Answer Score: 0
Question ID: 382808

Is there a way to prove that $2 + 2$ really equals 4?

Question

In elementary school, one learns that $2 + 2 = 4$ by experiment (putting two apples next to two other apples), and maybe also from some addition table to be memorized.

But is there any approach that *proves* $2 + 2 = 4$? If so, an example of such a proof would be good.

Answer

We assume the Peano axioms. Specifically:

1. Zero is a number.
2. If a is a number, the successor of a is a number.
   (We denote the successor of x as x'.)
3. Zero is not the successor of a number.
4. Two numbers of which the successors are equal are themselves equal.
5. (induction axiom.) If a set S of numbers contains zero and also the successor of every number in S, then every number is in S.

We then define addition recursively as follows:

$$a + 0 = a$$

$$a + b' = (a + b)'$$

Now, we will name some numbers. We will denote:

$$0' = 1 \tag{1.79}$$
$$1' = 2 \tag{1.80}$$
$$2' = 3 \tag{1.81}$$
$$3' = 4 \tag{1.82}$$

We could keep going, but we only need to be able to denote the numbers 0 through 4 (inclusive).

Now, showing that $2 + 2 = 4$ is a simple application of the recursive formula for addition:

$$2 + 2 = 2 + 1' \tag{1.83}$$
$$= (2 + 1)' \tag{1.84}$$
$$= (2 + 0')' \tag{1.85}$$
$$= ((2 + 0)')' \tag{1.86}$$
$$= ((2)')' \tag{1.87}$$
$$= 3' \tag{1.88}$$
$$= 4 \tag{1.89}$$

Question Score: 14 Answer Score: 5
Question ID: 896867

Constructing irrational numbers

Question

Which of the following numbers is constructible?

1) $3.14141414\ldots$

2) $\sqrt{3}$

3) $5^{\frac{1}{4}}$

4) $2^{\frac{1}{6}}$

Also,

Given a segment of length π, is it possible to construct, with a straight edge and compass, a segment of length 1?

I dont need full on proofs, just a little explanationif they could be constructed or not.

i have a feeling number 2 and 4 are constructible, just a educated guess because we could get $\sqrt{2}$ from a unit square and the diagonal is that and then we can just extend it I believe.

I just dont get it. Please help out

Answer

One is most certainly constructable, as all rationals are constructable, and $3.1414\ldots = \frac{311}{99}$.

Two is also constructable, as we can construct an equilateral triangle of side length 2, and its height is $\sqrt{3}$.

We can construct $\sqrt{5}$, so we should be able to construct $\sqrt{\sqrt{5}} = 5^{\frac{1}{4}}$.

I am not sure about $2^{\frac{1}{6}}$, but I do know $2^{\frac{1}{3}}$ is not constructable, so I doubt the former is. EDIT: as Andr Nicolas points out in the comment below, it is not.

As to the π question, I don't know.

Question Score: 0 Answer Score: 3
Question ID: 324143

1.9 Notation and Terms

We call “1/x” “inverse x;” what do we call “-1/x”?

Question

Just as $\sin(x)$ represents some $f(x)$, where the $f(x) =$ opposite/hypotenuse and it seems redundant to create another function $\csc(x)$ for the inverse of $\sin(x)$

As we call ”$1/f(x)$ the “inverse of f(x),” what do we name $-1/f(x)$?

Answer

TylerHG is correct, I just thought I would try to explain it with different phrasing.

“Inverse” is an ambiguous term *by itself.* We need to know *what* we’re doing the inverse of. An “inverse” is an object that, when applied to the original object, we get the “identity.” In layman’s terms, I like to think of the “inverse” as something that “undoes.”

However, just knowing we can “undo” something is *worthless* unless we know *what* we’re undoing! That’s why we must use the term “inverse” *with another term* that specifies *what* we’re “undoing.”

Thus, we call $\frac{1}{x}$ the **multiplicative inverse**, because multiplying by $\frac{1}{x}$ “undoes” multiplying by x.

Similarly, we call $(-x)$ the **additive inverse** of x because adding $(-x)$ “undoes” adding x.

This extends to functions as well–we call f^{-1} the **functional inverse** (or the **inverse function**) of f because applying f^{-1} “undoes” applying f. We can say the same about **inverse matrices** and **inverse operations** in general.

Inverses are a *really* interesting part of mathematics. If you want to learn more about them, consider reading a book on (or taking a course in) *Abstract Algebra.* You will learn a *lot* about what *exactly* an “inverse” is.

To address the particular scenario at hand, one could call $\frac{-1}{x}$ the “additive inverse of the multiplicative inverse of x.” A bit shorter is the “additive inverse of the reciprocal of x” (but that assumes you know that the reciprocal *is* the multiplicative inverse).

Shorter still is “the **negative reciprocal** of x,” because “negative” refers to the “additive inverse” and “reciprocal” refers to the multiplicative inverse. I use this term when discussing this quantity a lot.

Question Score: 0 Answer Score: 2
Question ID: 872210

How do you define the decimals indicator: $E-3$?

Question

How do you define E-3, which is used in the calculators to indicate that the first 3 decimal place-holders in the number are not displayed?

$$0.000563 = 5.63E-3$$

I need to write in my thesis something like: “Please, pay attention that the values are shifted by 3 decimals”.

Answer

It depends on what you’re writing about. It’s probably easier to just write 5.63×10^{-3} in the place of 0.00563 in a math text. If you *really* don’t want to write the $\times 10^{-3}$ (for example, in a table), just say “please note that the values in this table represent the error(or whatever they represent) divided by 1000.”

If you’re writing about something physical that has units (for example, distance) write: “Measurement is in millimeters” or “Measured in mm.”

Question Score: -1 Answer Score: 2
Question ID: 45115

“Nice” functions

Question

I see the statement of “nice functions” in textbooks and the authors usually don’t need to give the definition of “nice functions”. For example in a book which I read now the authors write

> Morrey spaces is not separable. A version of Morrey space where it is possible to approximate by “nice functions” is vanishing Morrey space.

But, the authors don’t give the definition of “nice functions” anywhere in the book.
I wonder in here what is the meaning of “nice functions”?
and
Is there a fixed definition of “nice functions”?

Answer

The custom is that “nice” or “well-behaved” is to be read as “satisfying all requirements for the appropriate theorems I am using.”

Question Score: 2 Answer Score: 4
Question ID: 764354

In the expression sqrt("cat"), what is the formal name of "cat"?

Question

This is more of a semantic question than an actual math question but I couldn't see where better to put it.

If I would write a silly equation like the aforementioned $\sqrt{\text{“}cat\text{”}}$, what is the proper name for the "cat" portion?

Can one say it's not in the domain? Is it a type error?

Answer

In the expression $\sqrt{x}$, x is referred to as the radicand.

If you literally mean that you pass the string "*cat*" as an argument to the square root function (that is, if $f(x) = \sqrt{x}$ and you wish to compute $f(cat)$): It doesn't make sense to pass a string value to a numeric function. This is because strings are out of the domain of math... (essentially)

Question Score: 3 Answer Score: 1
Question ID: 286861

Metrics on the plane

Question

Dene metrics ρ and d on the plane $\mathbb{R}^2$ as follows: for $x = (x_1, x_2)$ and $y = (y_1, y_2)$,

$$\rho(x, y) = |x_1 y_1| + |x_2 y_2| d(x, y) = max\{|x_1 y_1|, |x_2 y_2|\}$$

Draw accurate pictures in the x-y plane of the unit neighborhoods about the origin $O = (0, 0)$ in these two metrics; ie, draw pictures of $N_1(O, \rho)$ and $N_1(O, d)$.

I am confused on the definition of the metrics. Why are there two values for x and y? I'm really just struggling with grasping the definition of $\rho(x, y)$ and $d(x, y)$. Are they squares?

Answer

You can think of a metric as a way of measuring distance between two objects in a way that is consistent with the "normal" way of measuring distance between points.

What this definition is saying is that, given the two points (x_1, x_2) and $(y_1, y2)$, we can measure the "distance" between them in two ways. These ways to measure distance are described as the functions $\rho(x, y) = |x_1 - y_1| + |x_2 - y_2|$, and $d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.

We have two values for x and y because x and y are points.

We define the "unit neighborhood of the origin" to be the set of all points such that the distance (as defined by our metric) between the point and the origin is 1.

Question Score: 4 Answer Score: 1
Question ID: 556196

What is $\langle I \rangle$ in this text?

Question

I've read the following:

> It is easy to see that given any independent set I in V, the vertices of $V - I$ form a covering of G. Conversely, if $V - I$ forms a covering, then $\langle I \rangle$ must be empty; hence, it must be independent.

What is $\langle I \rangle$? I tried to search for it in the book I'm reading but couldn't. It doesn't have a notational index.

Answer

The comments by user66081 and Manuel Lafond are correct; $\langle I \rangle$ refers to the subgraph of G induced by I. In Gould's *Graph Theory*, the notation is introduced on page 6:

> Given a subset S of $V(G)$, the *subgraph induced by* S, denoted $\langle S \rangle$, is that graph with vertex set S and edge set consisting of those edges of G incident with two vertices of S.

Question Score: 1 Answer Score: 1
Question ID: 1125868

Meaning of math symbol $\sim$

Question

Segment of Example:

> We have:
> $$t \sim n\log(n)$$

Note: $\sim$ means "similarity" like in geometry, same shape but not same size. How is it interpreted here?

Answer

The answer by Daniel Littlewood is absolutely correct in this context. To extend this to the general definition (not tied to the example at hand):

$$f(n) \sim g(n) \iff \lim_{n\to\infty}\left(\frac{f}{g}\right)(n) = 1$$

Question Score: 7 Answer Score: 7
Question ID: 852742

Strange square brackets in recurrence equation

Question

I have the following recurrence given:

$$a_0 = 1$$

$$a_1 = 1$$

$$a_n = 3a_{n-2} + 3a_{n-1}$$

Why is that equal to something like this?:

$$a_n = 3a_{n-2} + 3a_{n-1} - 2[n = 1] + [n = 0]$$

What are those brackets?

Answer

These are most likely Iverson Brackets. If the Boolean expression inside the square bracket is true, then the bracket evaluates to 1. Otherwise, the bracket evaluates to 0.

Question Score: 2 Answer Score: 4
Question ID: 497884

What is this type of graph called

Question

I have two columns, User and Test Score.

Basically, I want to graph a sort of bell curve like chart.

The left side of the graph is the minimum score value, the right side is the maximum score value. X[] represents the number of slices between (linear).

Y(X[0]) would be the number of test scores that fall between X[0] and X[1]. It is sort of like a bell curve but it wouldn't be smooth unless there were a lot of points in between.

So two questions: 1) Is there a name of this graph? 2) Is it possible to do this with excel :-)?

Thanks in advance!

Answer

You are thinking of a histogram.

This functionality can be attained in Excel using the Column Chart type.

Question Score: 1 Answer Score: 3
Question ID: 231860

"Scoping" rules in mathematics

Question

I've been having a discussion with my friend about scoping rules in mathematical syntax. By scoping I mean a set of rules that define how variable and function names could be chosen so that at any given point no name used is ambiguous. As I'm a programmer, the concept of scoping feels very natural to me, so I started wondering whether there are some strict rules of such kind defined for mathematical syntax. If so, what are they?

The actual problem we've been arguing about with my friend was whether **arguments of a function should be considered "global" or "local"**. Is the following notation correct (unambiguous)?

$$f(x) = x + 3 \quad x \in \mathbb{R} \tag{1.90}$$

$$g(x) = 2x + 7 \quad x \in \mathbb{Z} \tag{1.91}$$

I believed it was, because x, as an argument, only has its meaning in the context of a function. My friend could not agree with that - he considered x "global" and therefore understood this snippet as some nasty and obviously incorrect attempt to double-define the domain of f and g. Do any standarized rules exist to support either claim?

Answer

I have never seen a list of scoping rules defined for mathematical expressions. However, I'd be willing to venture the following as generally accepted:

1. An argument of a function is local to the function. That is, it is considered clear to write:

$$f(x) = x^2, \text{ for } x \in \mathbb{R} \tag{1.92}$$

$$g(x) = x^4, \text{ for } x \in \mathbb{Z} \tag{1.93}$$

2. If one writes "Let x be..." this is a "global" definition with respect to the section of text; viz, x is defined that way until the end of the proof/lemma/chapter/whatever.

3. It is improper form to use a globally defined variable as an argument to a function. For example, it is poor form to write:

   > Let $x = 4$. And, let $g(x) = 4x^2$. (This causes confusion for the rest of the proof–to which x do you refer?)

   Rule of thumb: Try to make scoping not an issue. Trivial instances *where it is clear what the author means* (like in point 1) don't matter–a human is parsing the text, not a computer. If you have to think about it for more than a second or two, try to make it clearer.

4. Question Score: 2 Answer Score: 3
Question ID: 610967

Mathematical symbol for ”has”

Question

Just out of curiosity, I was wondering if there was a symbol for ”has” so intead of saying $x \in A$, we could say something like ”A has x”, they both mean the same thing but I was just wondering if there was another way to say it.

Thanks!

Answer

I believe I have seen $\ni$ used for this purpose. That is, $x \in A \iff A \ni x$

Question Score: 4 Answer Score: 3
Question ID: 417007

What does the notation $\min_x$ mean?

Question

I have a problem in which I need to find $\min_x(f(x))$. What does this notation mean?

Answer

In the contexts I've seen, $\min_x(f(x))$, means "the minimum of $f(x)$ over all x".

Question Score: 0 Answer Score: 1
Question ID: 1108322

Different of mapsto and right arrow

Question

Could someone please explain to me what is the difference in the two arrows

$$\to$$

and

$$\mapsto$$

For example in Probability wih Martingales (Willams)

$$\begin{aligned} \Omega &\to \text{ set of outcomes} \\ \omega &\mapsto \text{ outcome} \end{aligned}$$

Answer

We can say:

$$f : \{1,2,3\} \to \{4,5,6\}$$

This is a function from the set $\{1,2,3\}$ to the set $\{4,5,6\}$. Both things on the right and left of the arrow are *sets*.

But, if we say:

$$1 \mapsto 5$$

This says that when we put 1 into the function/relation/whatnot, we get out a 5. Both things on the left and right of the arrow are *elements* of the the domain/codomain.

This is not to say that we couldn't have a function like:*

$$f : \{\{1,2\},\{3,4\},\{5,6\}\} \to \{4,5,6\} \text{ such that:} \{1,2\} \mapsto 4 \vdots \text{etc}$$

In this case, one of the things on the right/left of the $\mapsto$ is a set. We could even make both of the things a set–it doesn't matter for $\mapsto$, but $\to$ must have sets on both sides. The $\mapsto$ is really just specifying what we get out when we put in something.

Question Score: 3 Answer Score: 1
Question ID: 473247

Symbol for such that (not in set)

Question

If A is a set, we can use the set notation

$$A = \{b \mid \text{property } p_1 \text{ of } b\}$$

But say A is an element like b,

$$A = b \mid \text{property } p_1 \text{ of } b$$

is this a usual notation? I am trying to say that A is a b that such that(|) it satisfies property p_1 of b, and assume that exactly one b satisfies property p_1.

Otherwise, is there a more usual convention to express this?

Answer

I had actually asked my prof about this a couple weeks ago... the symbol he gave is $\ni$. So, for an existential quantifier, we have:

$$\exists\, x \in \mathbb{R} \ni x^2 = x$$

He said we wouldn't use it in the class, as he thought it looked not so great...

This can also be seen here: http://www.physicsforums.com/showthread.php?t=195398

I, personally, like just abbreviating it "s.t." in my notes, as it's shorter, but more clear.

Question Score: 3 Answer Score: 0
Question ID: 309506

Concepts of Modern Mathematics (Ian Stewart) - 751=7.107+2?

Question

Concepts of Modern Mathematics by Ian Stewart (1995).

In Chapter 3 Ian Stewart talks about Short Cuts in the Higher Arithmetics, one section is on modular arithmetics. In one statement, he writes:

> "What is 751 days after Thursday?" We rephrase it as $4 + 751 =?$ and observe that $751 = 7.107 + 2$.

That is where I am getting lost. What is Ian Stewart doing to make this a true statement? I've attached the three pages on this topic, the statement in question is towards to bottom of page 3.

Answer

Rather than a decimal point, that should be a multiplication symbol. It should say:

> Now 751 isn't in our table, but we observe that
>
> $$751 = 7 \cdot 107 + 2$$
>
> etc.

After some Google-ing, I've found that some cultures use "." for multiplication and "·" for the decimal point. Thus, the expression could also be written:

$$751 = 7 \times 107 + 2 \quad \text{or} \quad 751 = (7)(107) + 2$$

Question Score: 4 Answer Score: 11
Question ID: 481370

Chapter 2

Calculus

2.1 Limits

Evaluating $\lim_{\theta\to 0} \frac{\sin 7\theta}{sin 3\theta}$

Question

$$\lim_{\theta\to 0} \frac{\sin 7\theta}{sin 3\theta}$$

My impression:

I don't see how I can change the variables in any way to get it in the form where I can simplify it to 1. I also looked up similar problems and they mention L'Hopital's rule to solve this, which I haven't learned yet. Do not give a straight answer. I already have that available to me.

Answer

If you're allowed to use Taylor series, note that $\sin(k\theta) \approx k\theta$ when θ is near zero:

Solution using this technique, now that I see you probably have solved this:

$$\lim_{\theta\to 0} \frac{\sin(7\theta)}{\sin(3\theta)} = \lim_{\theta\to 0} \frac{7\theta}{3\theta} = \frac{7}{3}$$

Question Score: 1 Answer Score: 4
Question ID: 1111513

rational limit problem

Question

I am teaching myself calculus and have run into a limit problem that I do not understand how to solve.

$$\lim_{x\to 0}\left(\frac{1}{x\sqrt{1+x}}-\frac{1}{x}\right)$$

All I am supposed to know at this point about calculus is the 11 limit laws. I think that I am just stuck on the algebra however. I do not know how to rid the denominators of the x's so it is not dividing by zero.

I have tried two ways to solve this problem as they are all I know.

1) obtaining a common denominator

$$\lim_{x\to 0}\left(\frac{x-x\sqrt{1+x}}{x^2\sqrt{1+x}}\right)$$
$$\lim_{x\to 0}\left(\frac{1-\sqrt{1+x}}{x\sqrt{1+x}}\right)$$

2) Multiplying to get rid of the denominators square root followed by a common denominator.

$$\lim_{x\to 0}\left(\frac{\sqrt{1+x}}{x(1+x)}-\frac{1}{x}\right)$$
$$\lim_{x\to 0}\left(\frac{\sqrt{1+x}-(1+x)}{x(1+x)}\right)$$

Please help me understand what I am missing. Any general principles for understanding/handling problems of this nature would be greatly appreciated.

p.p.s. I don't know if its a good idea to source where i got the problem from, but the book I am using is Single Variable Calculus Early Transcendentals 6E by James Stewart. The problem is in section 2.3, #29.

Answer

You've done very well thus far, but there's one other trick you might want to try...

Continuing from your first approach, multiply above and below by the conjugate of the numerator. That is:

$$\lim_{x\to 0}\left(\frac{1-\sqrt{1+x}}{x\sqrt{1+x}}\right)=\lim_{x\to 0}\left(\frac{1-\sqrt{1+x}}{x\sqrt{1+x}}\cdot\frac{1+\sqrt{1+x}}{1+\sqrt{1+x}}\right) \tag{2.1}$$
$$=\lim_{x\to 0}\left(\frac{1-(1+x)}{x\sqrt{1+x}(1+\sqrt{1+x})}\right) \tag{2.2}$$
$$\tag{2.3}$$

Hint: with just minor simplification, you should be able to cancel out that troublesome x.

Question Score: 1 Answer Score: 3

Question ID: 657969

Find simple limit with basic methods

Question

I need to find simple limit:

$$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n}$$

I can't use L'Hpital's rule, which I think is natural for such problems. How to calculate such limit using only basic methods (definition and simple transformations etc.)?

It's one problem (from 50 problems) I don't know how to solve from my homework.

Thanks for help.

Answer

Multiply by the radical conjugate:

$$\lim_{n\to\infty} \left(\frac{(\sqrt{n+1} - \sqrt{n}) \cdot (\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})} \right)$$

$$\lim_{n\to\infty} \left(\frac{n+1-n}{(\sqrt{n+1} + \sqrt{n})} \right)$$

Question Score: 1 Answer Score: 1
Question ID: 252673

Can someone explain this trigonometric limit?

Question

I have

$$\lim_{x\to 0} \frac{\tan(2x)}{\sin(x)}$$

and in my case the result is $\frac{2}{1}$ =2 not whether it is right.

This is my procedure.

$$\lim_{x\to 0} \frac{\frac{\sin(2x)}{\cos(2x)}}{\frac{\sin(x)}{1}} = \lim_{x\to 0} \frac{\sin(2x)}{(\cos(2x))(\sin(x))} = \frac{2x\frac{\sin(2x)}{2x}}{\cos(2x)\frac{x\sin(x)}{x}}$$

I separate the limit.

$$\frac{\left(\lim_{x\to 0} 2x\right)\cdot\left(\lim_{x\to 0}\frac{\sin(2x)}{2x}\right)}{\lim_{x\to 0}(\cos(2x))\cdot\left(\lim_{x\to 0}\frac{x\sin(x)}{x}\right)} = \lim_{x\to 0}\frac{2x}{x} = \frac{2}{1} = 2$$

Answer

That does give you the correct answer, but it takes a bit more work than necessary.

An alternative method is to use the *double-angle formula* for sin. That is:

$$\sin(2x) = 2\cos(x)\sin(x)$$

Thus:

$$\begin{aligned}
\lim_{x\to 0}\frac{\tan(2x)}{\sin(x)} &= \lim_{x\to 0}\frac{\sin(2x)}{\cos(2x)\sin(x)} && (2.4)\\
&= \lim_{x\to 0}\frac{2\cos(x)\cancel{\sin(x)}}{\cos(2x)\cancel{\sin(x)}} && (2.5)\\
&= \lim_{x\to 0}\frac{2\cos(x)}{\cos(2x)} && (2.6)\\
&= \frac{2\cdot 1}{1} && (2.7)\\
&= \boxed{2} && (2.8)
\end{aligned}$$

Question Score: 7 Answer Score: 14
Question ID: 467780

Comparison test for sequences?

Question

Let a_n, b_n such that for sufficiently large n: $a_n \leq b_n$.
Can we deduce that:

1. $\lim_{n\to\infty} a_n = \infty \implies \lim_{n\to\infty} b_n = \infty$
2. $\lim_{n\to\infty} b_n = L \implies \lim_{n\to\infty} a_n = L$, where $L < \infty$.

By the way,
I am aware of the squeeze theorem though I wondered if "one-sided-squeeze" is valid like mentioned above.

Answer

Your first implication is correct, but the second is not.

For a counterexample to assertion 2, consider $a_n = 1$ and $b_n = 2$. In this case, $\lim_{n\to\infty} b_n = 2$, but $\lim_{n\to\infty} a_n = 1 \neq 2$

Question Score: 4 Answer Score: 2
Question ID: 871837

Evaluating $\lim_{x\to 0}\frac{\sqrt{x+9}-3}{x}$

Question

The question is this.

In $h(x)=\dfrac{\sqrt{x+9}-3}{x}$, show that $\displaystyle\lim_{x\to 0} h(x)=\frac{1}{6}$, but that $h(0)$ is undefinied.

In my opinion if I use this expression $\displaystyle\lim_{x\to 0}\frac{\sqrt{x+9-3}}{x}$ above with the -3 inside the square root I got an undefinied expression, but if I put the -3 out of the square and I use this expression to calculate the limit $\displaystyle\lim_{x\to 0}\frac{\sqrt{x+9}-3}{x}$ I will get $\frac{1}{6}$.

Answer

You cannot pull the negative 3 out of the square root. For example:

$$\sqrt{1-3}=\sqrt{2}i\neq\sqrt{1}-3=-2$$

Question Score: 2 Answer Score: 2
Question ID: 345251

Find integral from 1 to infinity of $1/(1+x^2)$

Question

I am practicing for an exam and am having trouble with this problem. Find the integral from 1 to infinity of $\frac{1}{1+x^2}$. I believe the integral's anti-derivative is $\arctan(x)$ which would make this answer $\arctan(\infty) - \arctan(1)$ but from here I'm lost. I did find out that this comes out to $\pi/4$ but I don't know why.

Answer

Essentially, your question appears to be "Why does $\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$?"

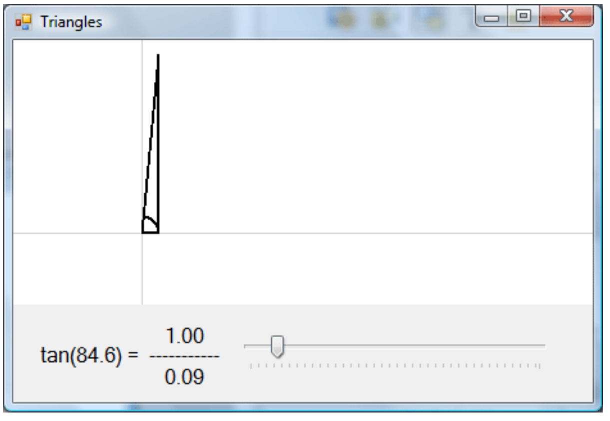
Remember that

$$\theta = \arctan\left(\frac{y}{x}\right)$$

When will $\frac{y}{x} \to \infty$? That's when $x \to 0$.

Think of a right triangle with height y and base x. θ is the angle between x and the hypotenuse. As x gets smaller and smaller, what does θ get close to? Well, θ approaches 90 degrees or $\frac{\pi}{2}$.

EDIT: As requested, here's a picture to illustrate this idea. The angle (in degrees) is displayed inside the tangent function, alongside the fraction $\frac{y}{x}$:



Question Score: 2 Answer Score: 6
Question ID: 253979

Explain this solution if it is correct.

Question

Link to solution. It is problem number 3.

$$\lim_{(x,y)\to(0,0)} \frac{y^2 \sin^2 x}{x^4 + y^4}$$

Solution: When x is small [close to 0], $\sin x$ has essentially the same growth as x. So the numerator is like x^2y^2, has the same degree as denominator. In this scenario, it's likely that the limit does not exist. To test this, we will try $y = mx$, where m is just any real number. Then, we get

$$\frac{y^2 \sin^2 x}{x^4 + y^4} = \frac{m^2x^2 \sin^2 x}{(1+m^4)x^4} = \frac{m^2 \sin^2 x}{(1+m^4)x^2}$$

Take the limit as x goes to 0, we will get $\frac{m^2}{1+m^4}$, which in particular means if we take $m = 2, m = 3$, the answers will not agree, and hence the limit does not exist.

I do not understand the last paragraph.

Answer

For a limit of a single-variable function, the limit exists if the right- *and* left-hand limits exist and are equal.

In higher dimensions, we don't just have a right-hand and left-hand limit. We also have limits coming in from parabolas, from various lines, etc. For the limit in a multi-variable function to exist, the limit must exist and be equal from all possible paths.

In this problem, the author picked two different paths: $y = 2x$ and $y = 3x$. Evaluate these limits:

Path $y = 2x$:

$$\lim_{(x,y)\to(0,0)} \frac{y^2 \sin^2 x}{x^4 + y^4}$$

Substitute in $2x$ for y:

$$\lim_{x\to 0} \frac{(2x)^2 \sin^2 x}{x^4 + (2x)^4}$$

Now this is just a Calc I limit. Skipping intermediate steps, we find the limit is equal to $\frac{4}{17}$.

Path $y = 3x$:

$$\lim_{(x,y)\to(0,0)} \frac{y^2 \sin^2 x}{x^4 + y^4}$$

Substitute in $3x$ for y:

$$\lim_{x\to 0} \frac{(3x)^2 \sin^2 x}{x^4 + (3x)^4}$$

Now this is just a Calc I limit. Skipping intermediate steps, we find the limit is equal to $\frac{9}{82}$.

Therefore...

We know that for the limit to exist, the limits along all paths must be identical. We have a counterexample-the limit along the path $y = 2x$ is not equal to the limit along the path $y = 3x$. Thus, the limit doesn't exist.

Question Score: 0 Answer Score: 1
Question ID: 254188

2.2 Derivatives

Find the open interval on which $x^{1/3} - 9$ is increasing/decreasing

Question

Consider the following function

$$f(x) = x^{\frac{1}{3}} - 9$$

How can I find the open interval on which the function is increasing or decreasing?

a. I know the critical points are $(0, -9)$

b. I know that the interval is not decreasing.

c. I know that by applying first derivative test to identify the relative extremum. (In this case no extrema).

All I cannot figure out is: in which interval the function is increasing?

Answer

To find where a function is increasing or decreasing, we look at the first derivative:

$$f(x) = x^{1/3} - 9 f'(x) = \frac{x^{-2/3}}{3}$$

We set the derivative equal to zero:

$$\frac{x^{-2/3}}{3} = 0$$

$$x^{-2/3} = 0$$
$$\frac{1}{x} = 0$$

Critical points come in two categories: *stationary* and *singular*. *Stationary* points are where the first derivative is equal to zero; *singular* points are where it is undefined. Note that $\frac{1}{x} \neq 0$ for any x; thus, there are *no stationary points*!

However, the function $f'(x)$ is undefined at $x = 0$, so there *is a singular point* at $x = 0$.

Now, we test $f'(x)$ to see if it is positive or negative on either side of the origin:

$$f(-1) = \frac{1}{3}, \qquad f(1) = \frac{1}{3}$$

Thus, the first derivative is positive for *all x values*, except $x = 0$ (where it is undefined). This tells us that the function is increasing on the interval $(-\infty, 0)$ and on $(0, \infty)$. But what about at $x = 0$?

As f is continuous at $x = 0$, f is increasing on all of $\mathbb{R}$. So, our final interval is $(-\infty, \infty)$.

Question Score: 3 Answer Score: 1
Question ID: 431001

Let $z(t) = (3x + 2y) \cdot exp(2x^2 - y^2)$, $x = \cos 3t$ and $y = \sin 3t$. Evaluate $z'(1)$.

Question

Let $z(t) = (3x + 2y) \cdot exp(2x^2 - y^2)$, $x = \cos 3t$ and $y = \sin 3t$. Evaluate $z'(1)$.

I know that

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

I found these, and then when evaluated at $t = 1$, $x = \cos(3)$ and $y = \sin(3)$, I obtained the answer -15.64 to 2 decimal places. This is wrong, but I'm not sure why.

Answer

A different approach: Unless you're required to do the problem using the chain rule, the easiest path is to just plug in the function values for x and y. So,

$$z(t) = (3x + 2y) \cdot exp(2x^2 - y^2) \implies z(t) = (3\cos 3t + 2\sin 3t) \cdot \exp(2\cos^2 3t - \sin^2 3t)$$

Now you can find z' the normal Calc 1 way. (You have a product rule and 4 chain rules, it looks like)

Question Score: 1 Answer Score: 0
Question ID: 378420

Find the derivative of y with respect to x,t,or theta, as appropriate

Question

$$y = \int_{\sqrt{x}}^{\sqrt{4x}} \ln(t^2)\, dt$$

I'm having trouble getting started with this, thanks for any help.

Answer

First Step

First, we need to **recognize** to which variables you are supposed to differentiate with respect. The important thing to realize here is that if you perform a definite integration with respect to one variable, that variable "goes away" after the computation. Symbolically:

$$\frac{d}{dt}\int_a^b f(t)\, dt = 0$$

Why? Because **the result of a definite integral is a constant, and the derivative of a constant is zero**! :)

So, it isn't appropriate here to differentiate with respect to t. With respect to θ doesn't make much sense, either–that's not even in the problem! **So, we are looking at differentiating with respect to x.**

Second Step

We now use a very fun theorem: the fundamental theorem of calculus! (bad pun, sorry)

The relevant part states that:

$$\frac{d}{dx}\int_a^x f(t)\, dt = f(x)$$

We now make your integral look like this:

$$y = \int_{\sqrt{x}}^{\sqrt{4x}} \ln(t^2)\, dt \tag{2.9}$$

$$= \int_{\sqrt{x}}^{a} \ln(t^2)\, dt + \int_a^{\sqrt{4x}} \ln(t^2)\, dt \tag{2.10}$$

$$= -\int_a^{\sqrt{x}} \ln(t^2)\, dt + \int_a^{\sqrt{4x}} \ln(t^2)\, dt \tag{2.11}$$

$$\tag{2.12}$$

Can you now find $\frac{dy}{dx}$? (Hint: Don't forget the chain rule!)

If you still want some more guidance, just leave a comment.

EDIT: Note that y is a sum of two integral functions, so you can differentiate both independently. I'll do one, and leave the other for you:

$$\frac{d}{dx}\left[\int_a^{\sqrt{4x}} \ln(t^2)\,dt\right] = \left[\ln\left(\sqrt{4x}^2\right)\right]\cdot\frac{d}{dx}\left(\sqrt{4x}\right) \tag{2.13}$$

$$= \left[\ln\left(4|x|\right)\right]\left(2\frac{x^{-1/2}}{1/2}\right) \tag{2.14}$$

$$= 4x^{-1/2}\ln\left(4|x|\right) \tag{2.15}$$

Question Score: 1 Answer Score: 2
Question ID: 376810

Question about Speeding and Calculus

Question

At 6.00 am, a driver picked up a fare card at the entrance of a tollway. At 10:30 am, the driver pulled up to a toll booth 250 miles away. After computing the fare, the toll booth operator issues a speeding ticket to the driver. (The posted speed limit was 65 mph). The driver said he was not speeding. Was he lying?

What I did to solve this problem is to find the average speed of the driver which came out to be 55 mph, using distance/time. Then by the mean value theorem, he/she apparently was not speeding.

Is there any other theorem that can be used to show that the driver was speeding?

Answer

There is no way to determine if the driver was speeding. Sure, they could go a constant 55mph the whole time (and did not speed), or they could have gone 400mph for a short time and slowed to a crawl until 10:30am. Since either approach would fit the two data points given, we cannot determine if they actually did exceed 55mph.

The way this problem *typically* goes, the driver's average speed exceeds the speed limit, and you can conclude they were speeding. However, this seems to be an odd case.

Question Score: 2 Answer Score: 0
Question ID: 1122970

Finding the derivative of a relational problem

Question

I am self studying some calculus and I have gotten really stuck! I thought I had the right idea but I keep getting the answer totally wrong. I am sure I am missing something important. Here is the problem:

For the equation $6x^{\frac{1}{2}} + 12y^{-\frac{1}{2}} = 3xy$, find an equation of the tangent line at the point $(1, 4)$.

Here is my work:

$$6x^{\frac{1}{2}} + 12y^{-\frac{1}{2}} = 3xy$$
$$6(x^{\frac{1}{2}} + 2y^{-\frac{1}{2}}) = 3xy$$
$$2(x^{\frac{1}{2}} + 2y^{-\frac{1}{2}}) = xy$$
$$2x^{\frac{1}{2}} + 4y^{-\frac{1}{2}} = xy$$
$$\ln(2x^{\frac{1}{2}}) + \ln(4y^{-\frac{1}{2}}) = \ln(xy)$$
$$\ln 2 + \ln x^{\frac{1}{2}} + \ln 4 + \ln y^{-\frac{1}{2}} = \ln x + \ln y$$

$$\ln 2 + \frac{1}{2}\ln x + \ln 4 - \frac{1}{2}\ln y = \ln x + \ln y$$
$$\ln 2 + \ln 4 + \frac{1}{2}\ln x - \ln x = \ln y + \frac{1}{2}\ln y$$
$$\frac{2}{3}\ln 2 + \frac{2}{3}\ln 4 - \frac{1}{3}\ln x = \ln y$$
$$e^{\frac{2}{3}\ln 2 + \frac{2}{3}\ln 4 - \frac{1}{3}\ln x} = y$$

$$\frac{2}{3}\ln 2 + \frac{2}{3}\ln 4 - \frac{1}{3}\ln x = \ln y$$
$$\frac{1}{3x} = y'\frac{1}{y}$$
$$y' = \frac{y}{3x}$$
$$y' = \frac{e^{\frac{2}{3}\ln 2 + \frac{2}{3}\ln 4 - \frac{1}{3}\ln x}}{3x}$$

$$y - 4 = \frac{4}{3}(x - 1)$$
$$y = \frac{4}{3}x - \frac{4}{3} + 4$$
$$y = \frac{4}{3}x + \frac{8}{3}$$

The Solution I found was: $y = \frac{4}{3}x + \frac{8}{3}$ but this is wrong! Can you tell me where I have gone wrong?

Answer

In general, you've just put yourself through a lot of unnecessary work. Just hit the thing immediately with implicit differentiation:

$$6x^{\frac{1}{2}} + 12y^{-\frac{1}{2}} = 3xy$$

Becomes:

$$3x^{-1/2} - 6y^{-3/2} = 3xy' + 3y$$

You're told to find the tangent line at $(1, 4)$, so plug in $x = 1,\ y = 4$:

$$3(1)^{-1/2} - 6(4)^{-3/2} = 3(1)y' + 3(4)$$

$$3 - 6\frac{1}{8} = 3y' + 12$$

Now solve for y':

$$y' = \frac{-9 - \frac{3}{4}}{3}$$

EDIT: Dang. I missed the "easy" mistake, and found a harder one instead. :o

Doing it your way:

$$\frac{2}{3}\ln 2 + \frac{2}{3}\ln 4 - \frac{1}{3}\ln x = \ln y$$

When you differentiated, you forgot a minus sign:

$$\frac{1}{3x} = \frac{y'}{y}$$

Question Score: 3 Answer Score: 2
Question ID: 423247

Tangent line of $y = A(x)$ at $x = \pi/2$, where $A(x)$ is an integral function

Question

Given the function A defined by:

$$A(x) = \int_{\pi/x}^{x} \frac{\sin t}{t}\, dt$$

Find the equation of the tangent line of $y = A(x)$ at $x = \frac{\pi}{2}$.

please try to explain in detail
Thank you in advance

Answer

We want to find a tangent line, and that involves finding the slope (derivative). So, let's find $A'(x)$.

We can apply the Fundamental Theorem of Calculus (FTC), which gives the result:

$$\frac{d}{dx} \int_{a}^{x} f(t) dt = f(x) \tag{1}$$

So, we need to break your $A(x)$ into a form that matches (1). Basically, we want a constant in the lower limit of integration, and a function of x to be in the upper limit of integration. So, we have:

$$\begin{aligned} A(x) &= \int_{\pi/x}^{x} \frac{\sin t}{t}\, dt && (2.16) \\ &= \int_{\pi/x}^{a} \frac{\sin t}{t}\, dt + \int_{a}^{x} \frac{\sin t}{t}\, dt && (2.17) \\ &= -\int_{a}^{\pi/x} \frac{\sin t}{t}\, dt + \int_{a}^{x} \frac{\sin t}{t}\, dt && (2.18) \\ & && (2.19) \end{aligned}$$

Now, we can apply the FTC to this to find $A'(x)$:

$$
\begin{align}
A'(x) &= \frac{d}{dx}\left[-\int_a^{\pi/x} \frac{\sin t}{t}\, dt\right] + \frac{d}{dx}\left[\int_a^{x} \frac{\sin t}{t}\, dt\right] \tag{2.20}\\
&= \frac{d}{dx}\left[-\int_a^{\pi/x} \frac{\sin t}{t}\, dt\right] + \left[\frac{\sin x}{x}\right] \tag{2.21}\\
&= \underbrace{\left[\frac{\sin(\pi/x)}{\pi/x} \cdot \frac{\pi}{x^2}\right]}_{\text{don't forget chain rule!}} + \left[\frac{\sin x}{x}\right] \tag{2.22}\\
& \tag{2.23}\\
&= \left[\frac{\sin(\pi/x)}{x}\right] + \left[\frac{\sin x}{x}\right] \tag{2.24}\\
&= \frac{\sin(\pi/x) + \sin x}{x} \tag{2}
\end{align}
$$

Now, just use (2) with julien's formula from the comments above, and you're basically done. Leave a comment if you have questions.

Question Score: 1 Answer Score: 1
Question ID: 384709

Manipulating differential expression

Question

How would you simplify something like

$$\frac{d(1+5x)}{d\ln x}$$

such that the denominator ends up just being dx. Presumably you can't just do

$$\frac{d(1+5e^x)}{dx}$$

Answer

Essentially, you want to differentiate the following w.r.t. $\ln x$:

$$(1+5x)$$

So, let $u = \ln x$.

$$x = e^u$$

$$1+5x = 1+5e^u$$

Now, differentiate the R.H.S. w.r.t. u.

$$\frac{d}{d\ln x}[1+5x] = \frac{d}{du}[1+5e^u]$$

$$\frac{d}{d\ln x}[1+5x] = 5e^u$$

$$\frac{d}{d\ln x}[1+5x] = 5x$$

Question Score: 1 Answer Score: 1
Question ID: 276005

Derivative of $e^{\ln(1/x)}$

Question

This question looks so simple, yet it confused me.

If $f(x) = e^{\ln(1/x)}$, then $f'(x) = ?$

I got $e^{\ln(1/x)} \cdot \ln(1/x) \cdot (-1/x^2)$.

And the correct answer is just the plain $-1/x^2$. But I don't know how I can cancel out the other two function.

Answer

The other hints provided will help you find the correct solution the easy way.

However, I'd like to point out that your approach has a flaw:

$$\frac{d}{dx}\left(e^{f(x)}\right) \neq e^{f(x)} \cdot f(x)$$

Rather, it is:

$$\frac{d}{dx}\left(e^{f(x)}\right) = e^{f(x)} \cdot f'(x)$$

Thus, we'd have (this is the hard way, applying the line of thought you tried to use):

$$\frac{d}{dx}\left(e^{\ln 1/x}\right) = e^{\ln 1/x} \cdot \left(\frac{d}{dx}(\ln 1/x)\right) \tag{2.25}$$

$$= e^{\ln 1/x} \cdot \left(\frac{1}{1/x} \cdot \frac{d}{dx}(1/x)\right) \tag{2.26}$$

$$= e^{\ln 1/x} \cdot \left(\frac{1}{1/x} \cdot \left(\frac{-1}{x^2}\right)\right) \tag{2.27}$$

$$= e^{\ln 1/x} \cdot \left(\frac{1}{1/x} \cdot \left(\frac{-1}{x^2}\right)\right) \tag{2.28}$$

$$\tag{2.29}$$

This is the correct answer, we don't simplify anything. Simplifying (using the same property as other have hinted that you should use:

$$e^{\ln 1/x} \cdot \left(\frac{1}{1/x} \cdot \left(\frac{-1}{x^2}\right)\right) = \frac{1}{x} \cdot \left(\frac{1}{1/x} \cdot \left(\frac{-1}{x^2}\right)\right) \tag{2.30}$$

$$= \frac{1}{x} \cdot \left(x \cdot \left(\frac{-1}{x^2}\right)\right) \tag{2.31}$$

$$= \frac{-1}{x^2} \tag{2.32}$$

Question Score: 3 Answer Score: 2
Question ID: 318881

Can you factor before finding derivative?

Question

Say the function is $y = \frac{x^2-1}{x-1}$
Can you factor functions before finding the derivative or does that not work?

Answer

Short answer: Yes.
Factoring an expression preserves the function it represents. That is, the functions

$$f(x) = (x+1)(x-1)$$

$$g(x) = x^2 - 1$$

are the *same function.*

Thus, taking the derivative of the factored form *or* the non-factored form is equivalent.

Question Score: 0 Answer Score: 1
Question ID: 878913

Taking the derivative from a graph?

Question

I have a problem that's graphed. It's linear from $(0,0)$ to $(1,1)$, then it's a horizontal line after that.

I have to find four derivatives from this, and I've never done a problem like this before. I have to find $f'(1/2)$, $\frac{d}{dx}f(e^x)$ at $x=0$, $\frac{d}{dx}f(e^{-x})$ at $x=1$, and $\frac{d}{dx}f(e^x)$ at $x=1$.

Any help is much appreciated.

Answer

This looks like an exercise in the chain rule.

First, recall that the derivative of a function is basically a fancy term for the slope at a given point. (At least, in a simplified form...) So, to find $f'(a)$, look at the slope of the graph at $x=a$. This should help with the first question.

For the rest, recall that $\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$. So, for example, $\frac{d}{dx}f(e^x) = f'(e^x) \cdot e^x$. You have the graph of the function, so you can find $f'(a)$ for a given value of $x=a$ as above. You also can evaluate e^x at any point. This will help with the rest of the questions. I will warn that not all points on that function are differentiable.

If you need more guidance, let me know via a comment. I'm changing the way I leave answers to a more open-ended method, so I'm still gauging how much answering to give. :)

Question Score: 0 Answer Score: 3
Question ID: 324288

Implicit Differentiation: Find dy/dx for $y = (1 + x\ln(x))^{\tan(2x)}$ when $x = 1$

Question

Find dy/dx for $y = (1 + x\ln(x))^{\tan(2x)}$ when $x = 1$

Now, I was thinking about taking the natural log of both sides to get:

$\ln(y) = \tan(2x)\ln(1 + x\ln(x))$;

However, differentiating we get:

$(dy/dx)/y = 2\sec 2(2x)\ln(1 + x\ln(x)) + \tan(2x)\ldots$ a lot of stuff.

My point is that there is a y term left over in the equation and we do not have a value for y At this point I am stuck.

Answer

First we differentiate. Taking ln of both sides is a good idea:

$$y = (1 + x\ln(x))^{\tan(2x)}$$

$$\ln y = \tan(2x)\ln(1 + x\ln(x))$$

Using implicit differentiation:

$$\frac{dy}{dx}\frac{1}{y} = \tan(2x)\left[\frac{d}{dx}\ln(1 + x\ln(x))\right] + \ln(1 + x\ln(x)) \cdot \frac{d}{dx}\tan(2x)$$

$$\frac{dy}{dx}\frac{1}{y} = \tan(2x)\left(\frac{1 + \ln(x)}{1 + x\ln(x)}\right) + \ln(1 + x\ln(x))\, 2\sec^2(2x)$$

Solving for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = y\left(\tan(2x)\left(\frac{1 + \ln(x)}{1 + x\ln(x)}\right) + \ln(1 + x\ln(x))\, 2\sec^2(2x)\right)$$

Oh no! We have a y variable, but we don't have its value! What do we do? Not a problem. We know that:

$$y = (1 + x\ln(x))^{\tan(2x)}$$

So, when $x = 1$:

$$y = (1 + 1\ln(1))^{\tan(2\cdot 1)} \tag{2.33}$$

$$= 1^{\tan(2)} \tag{2.34}$$

$$= 1 \tag{2.35}$$

Thus, we plug in $x = 1, \quad y = 1$:

$$\frac{dy}{dx} = 1\left(\tan(2\cdot 1)\left(\frac{1 + \ln(1)}{1 + 1\ln(1)}\right) + \ln(1 + 1\ln(1))\, 2\sec^2(2\cdot 1)\right) \tag{2.36}$$

$$= 1\left(\tan(2)\left(\frac{1}{1}\right) + \ln(1) \cdot 2\sec^2(2)\right) \tag{2.37}$$

$$= \tan(2) + 0 \tag{2.38}$$

$$= \boxed{\tan(2)} \tag{2.39}$$

Question Score: 0 Answer Score: 2

Question ID: 472481

Help with differentiation of natural logarithm

Question

Find $\dfrac{dy}{dx}$ given $y = \frac{\ln(8x)}{8x}$.
The answer is $\dfrac{1-\ln(8x)}{8x^2}$.
Can you show the process of how this is worked?
Thanks.

Answer

There are two approaches here. The most straightforward way is to apply the quotient rule. That is, given $f(x) = \frac{g(x)}{h(x)}$:

$$f'(x) = \frac{g'(x) \cdot h(x) - g(x) \cdot h'(x)}{(h(x))^2}$$

Or, in words: "bottom times the derivative of the top, minus the top times the derivative of the bottom; all over the bottom squared."

So, we have that the "top" is $\ln 8x$. The derivative of that is $\frac{1}{x}$, applying the chain rule and the derivative of natural log.

The bottom is $8x$. The derivative of this is 8. (just a power rule)

We plug into the quotient rule:

$$f'(x) = \frac{(8x)\frac{1}{x} - \Big(\ln(8x)\Big)8}{(8x)^2}$$

Simplifying:

$$f'(x) = \frac{8 - 8\ln(8x)}{(8x)^2}$$

$$f'(x) = \frac{1 - \ln(8x)}{8x^2}$$

This is the correct answer.

EDIT: I forgot I mentioned that there were two approaches! The other approach is to re-write the problem using properties of exponents, and apply the product rule:

$$f(x) = \frac{1}{8}x^{-1}\ln(8x)$$

(Differentiate with the product rule–it's a good practice. You should end up with the same function as above.)

Question Score: 2 Answer Score: 2
Question ID: 423257

Show the derivative is bounded

Question

Let $f:(0,\infty)$ such that:

$$f'(x)=\frac{1}{2\sqrt{x}}\left(\sin\frac{1}{x}+1\right)+\sqrt{x}\left(-\frac{1}{x^2}\right)\cos\frac{1}{x}$$

Now, We can bound it by:

$$\left|f'(x)\right|\leq\frac{1}{2}\cdot 2+\frac{1}{1}\cdot 1=2$$

This part of proof is from a book. Now, it's a bit unclear to me how it was done. Just be observing the derivative and familiarity with trigonometry?

Answer

I'm pretty sure that the bounds require $x\geq 1$.

$\sin(t)$ and $\cos(t)$ are bounded above by 1 and below by -1. That accounts for the trig terms.

$\frac{1}{2\sqrt{x}}$ is bounded above by $\frac{1}{2}$ for $x\geq 1$

Finally, $\frac{\sqrt{x}}{x^2}=x^{-3/2}$ is bounded above by 1 for $x\geq 1$.

Question Score: 1 Answer Score: 3
Question ID: 652554

Differential problem, find the maximum and minimum value

Question

Find the maximum, minimum value and inflection/saddle point of the following function

1. $f(x) = 12x^5 - 45x^4 + 40x^3 + 6$
2. $f(x) = x + \frac{1}{x}$
3. $f(x) = (2x+4)(x^2 - 1)$

Give a little explanation or procedural details if possible

Answer

I'll do the second one, and try to solve the other two based on how I did this one. If you need more explanation, let me know.

(1) Set the first derivative equal to zero to find critical points:

$$f(x) = x + \frac{1}{x}$$

$$f'(x) = 1 - \frac{1}{x^2}$$

$$0 = 1 - \frac{1}{x^2}$$

Solving, we find that we have critical points at $x = \pm 1$.

(2) Check the second derivative to determine max/min/unknown:

$$f''(x) = \frac{2}{x^3}$$

At $x = +1$, $f''(x) > 0$. Thus we have a minimum.

At $x = -1$, $f''(x) < 0$. Thus, we have a maximum.

Question Score: 2 Answer Score: 2
Question ID: 321819

Find y" using implicit differentiation

Question

Pretty sure I messed up

$$x^3 - 3xy + y^3 = 1$$
$$3x^2 - 3xy' - 3y + 3y^2y' = 0$$
$$3y^2y' - 3xy' = 3y - 3x^2$$
$$(3y^2 - 3x)y' = 3y - 3x^2$$
$$y' = \frac{y - x^2}{y^2 - x}$$
$$y'' = \frac{(y^2 - x)(y' - 2x) - (y - x^2)(2yy' - 1)}{(y^2 - x)^2}$$
$$y'' = \frac{y^2y' - 2xy^2 - xy' + 2x^2 - 2y^2y' + y + 2x^2yy' - x^2}{(y^2 - x)^2}$$
$$y'' = \frac{x^2 - 2xy^2 + y + (2x^2y - x - y^2)y'}{(y^2 - x)^2}$$

After subbing in y', expanding and simplifying...

$$y'' = \frac{x^2 - 2xy^2 + y + 3x^2y^2 - 2x^4y - xy + x^3 - y^3}{(y^2 - x)^3}$$

The answer should be

$$y'' = -\frac{4xy}{(y^2 - x)^3}$$

Answer

As pointed out in the comments, the error occurred when plugging back in the value for y'. The part that reads:

$$x^2 - 2xy^2 + y$$

Should read:

$$(x^2 - 2xy^2 + y)(y^2 - x)$$

This multiplies out and helps simplify further.

Question Score: 4 Answer Score: 2
Question ID: 445448

2.3 Optimization and Related Rates

Minimizing the distance between two boats.

Question

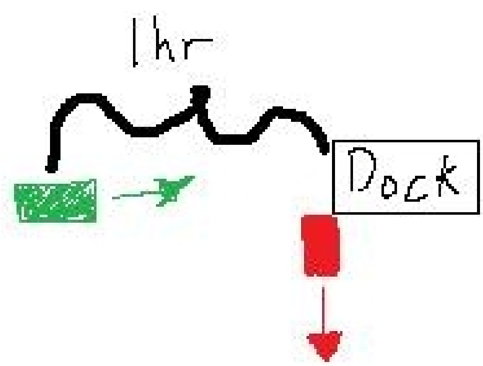
the source of this problem is Stewart's Essential Calculus (Early Transcendentals) 2nd ed.

> A boat leaves a dock at 2:00PM and travels due south at a speed of 20 km/h. Another boat has been heading due east at 15 km/h and reaches the same dock at 3:00 PM. I want to find at what time the two boats were closest together.

My plan of approach is to find the positions of the boats at any time t, and minimize the square of the distance between them. For the boat traveling due east, its position could be written as $(f(t), 0)$ and for the one due south, $(0, g(t))$. My difficulty lies in determining what $f(t)$ and $g(t)$ are. I'd appreciate it if someone could assist. Thanks.

Answer

First, draw a picture describing the starting position of the boats and the dock. We know it takes 1hr for the green boat to reach the dock, so I've marked that distance as 1hr.



We want the units of $f(t)$ and $g(t)$ to both be in kilometers, so let's start with writing down a function that gives the starting position for each when they're at $t = 0$.

I'm using f for the vertical boat, g for the horizontal boat.

$$f(t) = 0 + \text{something to be added}$$

$$g(t) = -(1\text{ hr})\left(15\frac{\text{km}}{\text{hr}}\right) + \text{something to be added}$$

The constants give me my offset because they don't start at the same place.

See if you can figure out the rest. If you can't, see (mouse-over) below:

> Moving on, we need to figure out what that "something" is. Well, we know distance = rate $\times$ time, so let's try that:

$$f(t) = 0 - \left(20\frac{\text{km}}{\text{hr}}\right)t$$

$$g(t) = -(1\text{ hr})\left(15\frac{\text{km}}{\text{hr}}\right) + \left(15\frac{\text{km}}{\text{hr}}\right)t$$

Why do we need the minus sign in f? Well, it's because we're going south, which is in the negative y direction.

Question Score: 1 Answer Score: 1
Question ID: 418852

Questions about derivative (related rates)

Question

1. An airplane, flying horizontally at an altitude of 1 mile, passes directly over an observer. If the constant speed of the airplane is 400 miles per hour, how fast is its distance from the observer increasing 45 seconds later? Hint: Note that in 45 seconds, the airplane goes 5 miles.

2. A metal disk expands during heating. If its radius increases at the rate of 0.02 inch per second, how fast is the area of one of its faces increasing when its radius is 8.1 inches?

Answer

The first problem is best accompanied by a picture. The second doesn't really need one, as the picture is trivial. I'll address the second problem first, as it is somewhat easier.

For the second problem, the rate of change of the radius is $0.02\frac{in}{s}$. Thus:

$$\frac{dr}{dt} = 0.02$$

Area of circle is easy:

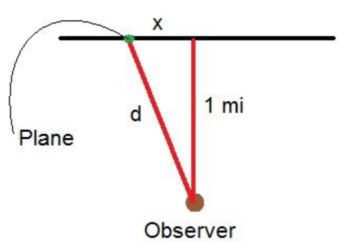
$$A = \pi r^2$$

Differentiate area:

$$\frac{dA}{dt} = \pi((2r)\frac{dr}{dt})$$

All that remains is to plug in the radius and rate of change of radius.

Now, we tackle the first problem.



x is the horizontal distance the plane has travelled since directly passing over the observer. d is the distance you are seeking. (oops–just realized d won't work as a symbol very well for obvious reasons. I'm going to substitute r.)

$$\frac{dx}{dt} = velocity = 400$$

What is the distance?

$$r = \sqrt{x^2 + 1^2}$$

$$\frac{dr}{dt} = \frac{(2x)\frac{dx}{dt}}{2\sqrt{x^2+1}}$$

Now you just have to plug in the values for x and $\frac{dx}{dt}$.

Question Score: 0 Answer Score: 0
Question ID: 244358

Find the volume of the largest right circular cone that can be inscribed in a sphere of radius r?

Question

I checked this question but didn't fully understand it.

I know that the volue of a right circular cone is $V = \frac{1}{3}\pi x^2 h$

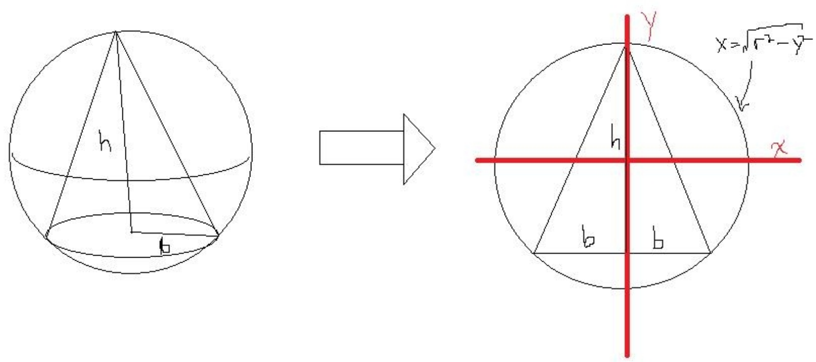
I know that I must take the first derivative and set it equal to zero. Which will find the maxium.

My problem is how to deal with the variable h, the height?

How can I rewrite that in terms of x or r.

I have tried to implicitly differentiate this equation, but that wasn't helpful for me.

Answer



Ok. In the image above, there are two sketches. The one on the left is in 3d, but is kinda hard to refer to. The one on the right is in 2d and easier to see what's going on, so I'm going to use that (just imagine it's a cross-section of the 3d pic).

The secret of the whole problem is to relate h and b. This is done using the equation for the right half of the circle:

$$x = \sqrt{r^2 - y^2}$$

b is the x value when y is offset from the top of the sphere by h. Thus, the equation in terms of b, h, and r is:

$$b = \sqrt{r^2 - (r-h)^2}$$

Now to relate to the volume of a cone:

$$V_{cone} = \frac{\pi}{3} \cdot b^2 h$$

$$V_{cone} = \frac{\pi}{3} \cdot (r^2 - (r-h)^2)h$$

Simplifying:

$$V_{cone} = \frac{\pi}{3} \cdot (2h^2 r - h^3)$$

Differentiate:

$$\frac{dV_{cone}}{dh} = \frac{\pi}{3} \cdot (4hr - 3h)h$$

Solve:

$$h = \left\{0, \frac{4r}{3}\right\}$$

We want the maximum, which obviously will occur at the second value of h. Finding the maximal volume is a simple algebra problem now.

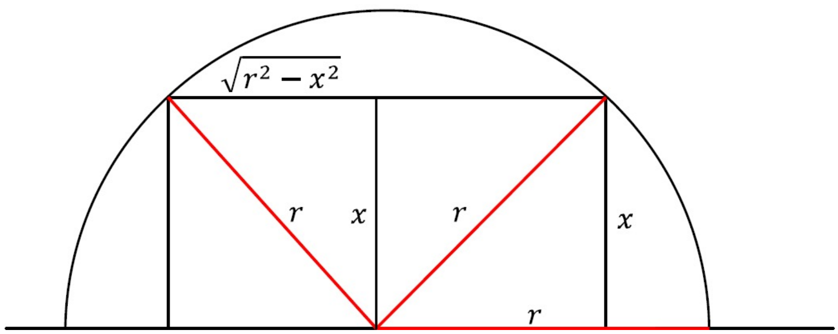
Question Score: 3 Answer Score: 3
Question ID: 225718

maximum area of a rectangle inscribed in a semi - circle with radius r.

Question

> A rectangle is inscribed in a semi circle with radius r with one of its sides at the diameter of the semi circle. Find the dimensions of the rectangle so that its area is a maximum.

My Try:
Let length of the side be x, Then the length of the other side is $2\sqrt{r^2 - x^2}$, as shown in the image.



Then the area function is

$$A(x) = 2x\sqrt{r^2 - x^2}$$

$$A'(x) = 2\sqrt{r^2 - x^2} - \frac{4x}{\sqrt{r^2 - x^2}} \tag{2.40}$$

$$= \frac{2}{\sqrt{r^2 - x^2}}(r^2 - 2x - x^2) \tag{2.41}$$

setting $A'(x) = 0$,

$$\implies x^2 + 2x - r^2 = 0$$

Solving, I obtained:

$$x = -1 \pm \sqrt{1 + r^2}$$

That however is not the correct answer, I cannot see where I've gone wrong? Can someone point out any errors and guide me the correct direction. I have a feeling that I have erred in the differentiation.
Also how do I show that area obtained is a maximum, because the double derivative test here is long and tedious.
Thanks!

Answer

You have dropped an x in calculating your derivative. By applying the product rule:

$$A'(x) = 2x\left(\frac{1}{2}(r^2 - x^2)^{-1/2}(-2x)\right) + 2\sqrt{r^2 - x^2} \tag{2.42}$$

$$= \frac{-2x^2}{\sqrt{r^2 - x^2}} + 2\sqrt{r^2 - x^2} \tag{2.43}$$

Question Score: 1 Answer Score: 1
Question ID: 872223

Related Rates: Calculus

Question

An airplane flies over an airport at an altitude of 10000 meters and at a speed of 900 km/hr. Find the rate at which the actual distance from the airport is increasing 2 minutes after the airplane was directly over the airport.

I'm getting a weird answer. Someone please help. The correct answer is 854 km/hr.

Answer

Let x be the horizontal distance from the airport to the plane. Let y be the vertical distance, and let r be the diagonal (that is, $r = \sqrt{x^2 + y^2}$).

Thus, we have:

$$\frac{dr}{dt} = \frac{d}{dt}(x^2 + y^2)^{1/2} \tag{2.44}$$
$$= \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot \frac{d}{dt}(x^2 + y^2) \tag{2.45}$$
$$= \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot (2x\frac{dx}{dt} + 2y\frac{dy}{dt}) \tag{2.46}$$

Based on the speed of the plane, we know that:

$$\frac{dx}{dt} = 900\frac{\text{km}}{\text{hr}}$$

Because the height of the plane, y is constant, we know that $\frac{dy}{dt} = 0$.

At 2 minutes after the flyover, we know that the horizontal distance x is described by:

$$x = \underbrace{900\frac{\text{km}}{\text{hr}}}_{\text{rate}} \cdot \underbrace{\frac{1}{30}\text{hr}}_{\text{time}} = 30\text{km}$$

Also, $y = 10\text{km}$.

Thus:

$$\frac{dr}{dt} = \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot (2x\frac{dx}{dt} + 2y\frac{dy}{dt}) \tag{2.47}$$
$$= \frac{1}{2}(30^2 + 10^2)^{-1/2} \cdot (2(30)(900) + 2(10)(0)) \tag{2.48}$$
$$= \frac{1}{2}(1000)^{-1/2} \cdot (54000) \tag{2.49}$$
$$\approx 854\frac{\text{km}}{\text{hr}} \tag{2.50}$$

Question Score: 0 Answer Score: 2

Question ID: 680969

2.4 Simple Integrals

Integrate by partial fraction decomposition

Question

$$\int \frac{5x^2+9x+16}{(x+1)(x^2+2x+5)}dx$$

Here's what I have so far...

$$\frac{5x^2+9x+16}{(x+1)(x^2+2x+5)} = \frac{\mathrm{A}}{x+1} + \frac{\mathrm{B}x+\mathrm{C}}{x^2+2x+5}$$

$$5x^2+9x+16 = \mathrm{A}(x^2+2x+5) + (\mathrm{B}x+\mathrm{C})(x+1) =$$

$$\mathrm{A}(x^2+2x+5) + \mathrm{B}(x^2+x) + \mathrm{C}(x+1) =$$

$$(\mathrm{A}+\mathrm{B})x^2 + (2\mathrm{A}+\mathrm{B}+\mathrm{C})x + (5\mathrm{A}+\mathrm{C})$$

$$\mathrm{A} = -3,\ \mathrm{B} = 8,\ \mathrm{C} = 31$$

$$\int \frac{5x^2+9x+16}{(x+1)(x^2+2x+5)}dx = \int \left(-\frac{3}{x+1} + \frac{8x+31}{x^2+2x+5} \right) dx \Rightarrow$$

$$\int -\frac{3}{x+1}dx + \int \frac{8}{x^2+2x+5}dx + \int \frac{31}{x^2+2x+5}dx$$

Hopefully I've got it correct until this point (if not, someone point it out please!). I can do the first integration by moving the -3 out and using $u = x+1$ to get

$$-3\ln(x+1)$$

but I'm stuck on the next two.

Answer

(Just working on the last integral, but not checking the partial fractions.) Complete the square:

$$\int \frac{dx}{x^2+2x+5} = \int \frac{dx}{(x+1)^2+4}$$

Now, substitute $x+1 = 2\tan(t)$. So, $dx = 2\sec^2 t\ dt$. Thus:

$$\int \frac{dx}{(x+1)^2+4} = \int \frac{2\sec^2(t)dt}{4(\tan^2 t+1)} \tag{2.51}$$

$$= \frac{1}{2}\int \frac{\sec^2 t\ dt}{\sec^2 t} \tag{2.52}$$

$$= \frac{1}{2}\int 1\ dt \tag{2.53}$$

$$= \cdots \tag{2.54}$$

Question Score: 4 Answer Score: 2
Question ID: 950462

Compute the integrals between zero and x

Question

I probably should be embarrassed to ask this question, but I am totally confused on how to compute these integrals. Could someone please help.

$$\int_0^x (3t^2+2t)\sin(t^3+t^2)\,dt$$
$$\int_0^x (3t^2+2t)\sin(x^3+x^2)\,dt$$

Does the first remain the same but replace t with x, and with the second one, do I replace the first part with x and find the antiderivative of the second part?

Answer

The first integral is nothing special:

$$\int_0^x (3t^2+2t)\sin(t^3+t^2)\,dt$$

Let $u = t^3 + t^2 \implies du = 3t^2 + 2tdt$

Thus the integral becomes:

$$\int_0^{x^3+x^2} \sin(u)\,du$$
$$\cos u\Big|_0^{x^3+x^2} = \cos(x^3+x^2) - 1$$

For the second integral, note that x is constant with respect to the integrand, and can be "pulled out". Thus, we have:

$$\int_0^x (3t^2+2t)\sin(x^3+x^2)\,dt = \sin(x^3+x^2)\int_0^x 3t^2+2t\,dt$$
$$\sin(x^3+x^2)\left[t^3+t^2\Big|_0^x\right]$$
$$\sin(x^3+x^2)\left[x^3+x^2\right]$$

Question Score: 1 Answer Score: 4
Question ID: 311714

Need help with integration by parts

Question

I absolutely despise integration by parts, because it never seems to work for me. Here's an example:

$$\int 4x \sin(2x)\, \mathrm{d}x$$

What I did:

$$\int 4x \sin(2x)\, \mathrm{d}x = -2x\cos(2x) - \int -4\cos(2x)\, \mathrm{d}x$$

Here I'm already stuck. I know about the ILATE/LIATE/whatever but it didn't work out for me either. What should I do at this juncture?

Answer

You've done the hard part of the integration by parts right; you're just off by a constant multiple. If we let $u = 4x,\ dv = \sin(2x)dx \implies du = 4\,dx,\ v = \frac{-\cos(2x)}{2}$.

When we put this together:

$$uv - \int v\,du = -2x\cos(2x) + \int \left(\frac{\cos(2x)}{2}\right) 4\,dx \tag{2.55}$$

$$= -2x\cos(2x) + \int 2\cos(2x)\,dx \tag{2.56}$$

So, that's basically where you're at (except for the 2 instead of the 4). At this point, we can use u substitution.

Hint:

Let $u = 2x$.

Question Score: 0 Answer Score: 1
Question ID: 423752

What is the definite integral of...

Question

$$\int_{-L}^{L} x\sin(\frac{\pi n x}{L})$$

I've seen something like this in Fourier theory, but I'm still not sure how to approach this integral. Wolfram Alpha gives me the answer, but no method. Integrate by parts? Substitution?

Answer

Whenever you have $p(x)f(x)$, where $p(x)$ is a polynomial, and $f(x)$ is a function like $\sin(x)$ or e^x, such that differentiating doesn't increase its "order," integration by parts is always a good technique. Differentiate the polynomial so it becomes "less complicated," and integrate the other function.

$$\int_{-L}^{L} x\sin\left(\frac{n\pi x}{L}\right)\,dx$$

Let $u = x, dv = \sin\left(\frac{n\pi x}{L}\right)dx$. Then $du = dx$ and $v = \frac{-L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)$.

Thus,

$$\int_{-L}^{L} x\sin\left(\frac{n\pi x}{L}\right)\,dx = \frac{-xL}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\bigg|_{-L}^{L} - \int_{-L}^{L}\frac{-L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\,dx$$

Question Score: 3 Answer Score: 2
Question ID: 394489

U-Substitution Integral Problem

Question

I am trying to solve the following homework problem.
$\int_{26}^{27} x\sqrt{x-26}\ dx$
This is what I have atttempted so far:
$u = x - 26; du = 1$
$\int_{26}^{27} \sqrt{u}\ du$
$x = 26 \rightarrow u = 0$
$x = 27 \rightarrow u = 1$
$\int_{0}^{1} \sqrt{u}\ du$
$= (\frac{2}{3} \times \sqrt{u^3})$ Evaluated at x = 1 - 0
This seems to give me a completely incorrect answer though, and I have no idea why.
Any help would be appreciated.
Edit: I believe I am using the wrong substitution, any hints?

Answer

You have the right idea for what to pick for u, but your error is because you have completely ignored the x outside of the square root. What you should have is this:

If you have $u = x - 26$, this means $x = u + 26$. Your limit changes are right, so this is:

$$\int_{26}^{27} x\sqrt{x-26}\,dx = \int_0^1 (u+26)\sqrt{u}\,du \tag{2.57}$$
$$= \int_0^1 u^{3/2} + 26u^{1/2}\,du \tag{2.58}$$
$$= \cdots \tag{2.59}$$

Does that help?

Question Score: 1 Answer Score: 1
Question ID: 655262

Easiest way to calculate this indefinite integral

Question

What's the easiest way to calculate the following indefinite integral:

$$\int \frac{\cos(x)}{\sqrt{2\sin(x)+3}}\mathrm{d}x$$

Answer

First, make the substitution $u = 2\sin(x) + 3$. Then, $du = 2\cos(x)\,dx$. Thus:

$$\int \frac{\cos(x)}{\sqrt{2\sin(x)+3}}dx = \int \frac{du}{2\sqrt{u}} \tag{2.60}$$

$$= \sqrt{u} + C \tag{2.61}$$

$$= \dots \tag{2.62}$$

Question Score: 3 Answer Score: 1
Question ID: 627416

Integral of $\frac{t}{t^4+2}dt$

Question

The answer in the back of the calculus book is

$$\frac{1}{2\sqrt{2}}\arctan\left(\frac{t^2}{\sqrt{2}}\right)+C$$

and I have no idea how they reached this answer. My first guess was to try partial fractions but I don't think I can in this case. I then tried u substitution using $u=t^2$ and $du=2t$, giving me

$$\frac{1}{2}\int\frac{du}{u^2+2}$$

I thought that I'd be able integrate this to reach an answer like $\frac{1}{2}\ln|t^2+1|+C$ but that's of course not the case and I'm not sure why. How should I approach this?

Answer

Note that:

$$\int\frac{1}{u^2+2}du=\frac{1}{2}\int\frac{du}{\left(\frac{u}{\sqrt{2}}\right)^2+1}=\frac{1}{\sqrt{2}}\arctan\left(\frac{u}{\sqrt{2}}\right)+C$$

Question Score: 5 Answer Score: 3
Question ID: 821134

How to take an integral when values aren't included or aren't part of my function?

Question

My question is rather dumb, but it's making my head hurt. I have to calculate a continuous probability on some interval in the following way:

$$p(0 < x < 0.5) = \int_0^{.5} f(x)dx$$

The problem I have is:

My function $f(x)$ is piecewise and takes a value of zero at x=0. And what I'm asked for doesn't include 0.5 or 0, so why is it appropriate to integrate this way (that is to say, why can I include these numbers as limits)? What's the criteria that determines how to take the integral when a point isn't included? My intuition is it has to do with limit rules, but I'd like to know the reason for it.

Answer

If $f(x)$ is defined everywhere on the interval $a < x < b$, then:

$$\int_a^b f(x)\ dx = \lim_{t \to a} \lim_{s \to b} \int_t^s f(x)\ dx$$

(if the limits exist)

Thus, it doesn't matter what the values *at* the endpoints are; just the behavior of the function as it approaches the endpoints.

Question Score: 0 Answer Score: 1
Question ID: 961544

Evaluate $\int x\cos(6x)\mathrm{d}x$?

Question

$$\int x\cos(6x)\,\mathrm{d}x$$

I have many similar problems to do, but I keep getting stumped on what to do with what resides inside the parenthesis as opposed to an exponent or something in front of the problem say either $\cos^6(x)$ or $6\cos(x)$. What should I be doing differently to solve this integral which has the $6x$ evaluated within cosine?

Answer

Whenever you have the product of:

1. Something you know how to differentiate (e.g. x), and...
2. Something you know how to integrate (e.g. $\cos(6x)$)

...you should use *integration by parts*.

For evaluating $\int \cos(6x)\,dx$, we use a u-substitution; let $u = 6x$. This means $du = 6\,dx$. Now we have:

$$\int \frac{\cos(u)}{6} du$$

Question Score: 1 Answer Score: 3
Question ID: 480923

Constant of integration use question.

Question

I get mixed up with what I can and can't do with the constant of integration C.

After integrating I have

$$-10\ln(y) = t + C.$$

What I did was:

$$\ln(y) = \frac{t}{-10} + C \implies y = e^{-\frac{t}{10}} + C \implies y = Ce^{-\frac{t}{10}},$$

and using an initial condition solved for C at $t = 0$. which was $y(0) = 30$, so $C = 30$.

However there should be an equilibrium value at 10 which I'm not getting in this graph, so I must of mixed up some rule with the C.

Any help?

Answer

I'm assuming you mean ln when you wrote *nl*.

You can basically do anything with the C, so long as it represents an *arbitrary constant value.*

So, you can write:

$$-10\ln y = t + C$$
$$\ln y = \frac{t}{-10} + \frac{C}{-10}$$

Now, note that some arbitrary constant divided by -10 is *still* an arbitrary constant. Thus, we have:

$$\ln y = \frac{t}{-10} + C$$
$$y = e^{\frac{t}{-10}+C}$$
$$y = e^{\frac{t}{-10}}e^{C}$$

But, e raised to an arbitrary constant is *also* an arbitrary constant. Thus, you have:

$$y = Ce^{t/(-10)}$$

Using your initial conditions, we have:

$$y = 30e^{t/(-10)}$$

So, what you did *is* basically right. However, the equation you gave *does not* have an equilibrium solution (ever), as the derivative of an exponential as above is not equal to 0 for any value of t. So, my guess is you did something wrong before that...

Question Score: 2 Answer Score: 1
Question ID: 333028

$$\int \tan^5(x)\ \mathrm{d}x$$

Question

I would like guidance on evaluating

$$\int \tan^5(x)\ \mathrm{d}x.$$

I have attempted using the Pythagorean identities to get

$$\int \tan^5(x)\ \mathrm{d}x = \int \tan(x)\left[\sec^2(x) - 1\right]^2\ \mathrm{d}x.$$

This doesn't look helpful. So I thought, why not turn only ONE of the $\tan^2$ terms to the $\sec^2 - 1$ form? This gives

$$\int \tan^5(x)\ \mathrm{d}x = \int \tan^3(x)\sec^2(x)\ \mathrm{d}x - \int \tan^3(x)\ \mathrm{d}x.$$

Clearly the second term is $\dfrac{\tan^4(x)}{4}$ (ignoring the constant term for now). Using a similar trick,

$$\int \tan^3(x)\ \mathrm{d}x = \int \tan(x)\sec^2(x)\ \mathrm{d}x - \int \tan(x)\ \mathrm{d}x \tag{2.63}$$

$$= \frac{\tan^2(x)}{2} - (-1)\ln|\cos(x)| \tag{2.64}$$

$$= \frac{\tan^2(x)}{2} + \ln|\cos(x)|. \tag{2.65}$$

So this suggests to me that

$$\int \tan^5(x)\ \mathrm{d}x = \frac{\tan^4(x)}{4} - \frac{\tan^2(x)}{2} - \ln|\cos(x)| + C.$$

But the answer in Stewart (section 7.2., #31) is

$$\frac{1}{4}\sec^4(x) - \tan^2(x) + \ln|\sec(x)| + C.$$

It's very clear where the $\ln|\sec(x)|$ term is coming from - and I tried to take the difference of my answer and Stewart's answer using Wolfram Alpha and unfortunately, the difference is not a constant.

Answer

The answers are equivalent, you have nothing to worry about:

$$\frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} = \frac{\tan^4 x + 2\tan^2 x}{4} - \tan^2 x \tag{2.66}$$

$$= \frac{\sec^4 x - 1}{4} - \tan^2 x \tag{2.67}$$

$$= \frac{\sec^4 x}{4} - \tan^2 x + C \tag{2.68}$$

Question Score: 8 Answer Score: 6
Question ID: 979056

2.5 Area and Volume

Finding area in a bounded region

Question

How would I solve the following problem?

> Sketch the region bounded by the graphs of $f(x) = x^2 - 4x$ and $-3x^2$ and then find the area of the region.

I have made a sketch for this question but what I am not sure how to do is find the area.
I have found the intersections $x = 0$ and $x = 1$
$x^2 - 4x = -3x^2$
$4x^2 - 4x = 0$
$x = 0, x = 1$

$$f(x) = \int_0^1 x^2 - 4x - (-3x^2)(dx)$$

$$f(x) = \int_0^1 4x^2 - 4x(dx)$$

$$G(X) = \frac{4x^3}{3} - 2x^2$$

So I did

$$A = \frac{4}{3}(1) - 2(1)^2 - 0$$

$$A = \frac{-2}{3}$$

But how can an Area be negative?

Answer

When finding the area bounded by the two curves, make sure you're always integrating **top** minus **bottom**. In this case, the $-3x^2$ is actually greater than $x^2 - 4x$. Thus, your integral should be:

$$\int_0^1 -3x^2 - \left(x^2 - 4x\right)\ dx$$

That was the only error you made.

Question Score: 3 Answer Score: 3
Question ID: 360717

Volume of liquid needed to fill sphere to height h

Question

Find the volume of liquid needed to fill a sphere of radius R to height h.

The picture shows h up to maybe a quarter, I am not sure it seems pretty ambiguous.

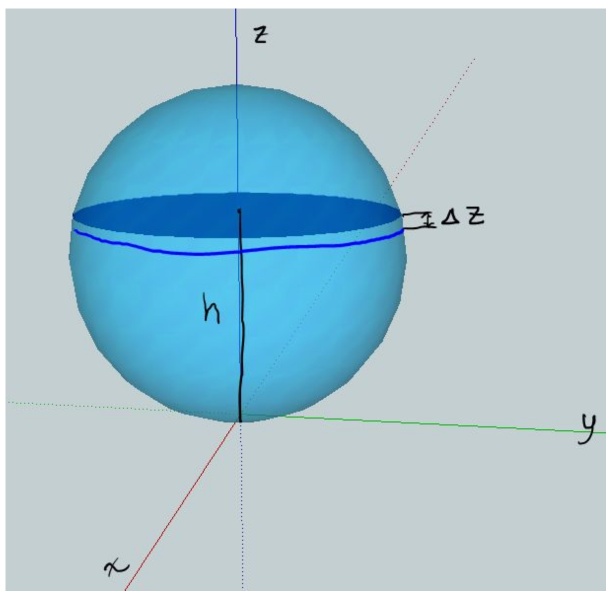
No clue what to do here. I just know that the formula I am suppose to memorize is

$$V = \int \pi(R^2 - x^2)$$

Answer

In this answer, I'm not really going to explain how to fit the formula; rather, I'm giving a process that will work for nearly any type of Calc II "application" problem.

The first step in any word problem is to **draw a picture** (even if you're given one–it still helps for you to draw your own.):



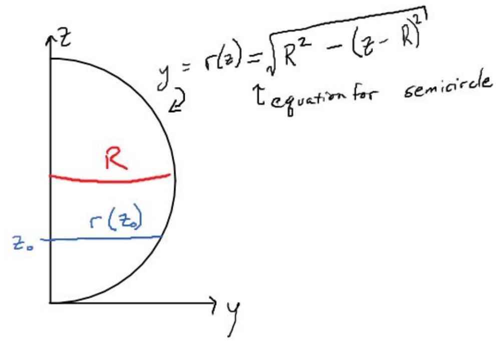
Now, we can turn the drawing into an integral. If we were to use discrete "slices" of the sphere to compute the volume, we'd have the following: (Let $r(z)$ be the radius of a slice at a given height z, and Δz be the height of a given slice.)

$$\sum \left(\pi \cdot \big(r(z)\big)^2 \cdot \Delta z\right)$$

This is just summing up the volume of a bunch of really short cylinders. As $\Delta z \to 0$, we have an integral:

$$\int_0^h \pi \cdot \big(r(z)\big)^2 \, \mathrm{d}z$$

Now, what is $r(z)$? Well, let's look at a circle of radius R, and see what its radius is as a function of z:



So, we see that $y = r(z) = \sqrt{R^2 - (z - R)^2}$. Let's plug this into our integral we have above:

$$\int_0^h \pi \left(\sqrt{R^2 - (z - R)^2} \right)^2 \, \mathrm{d}z$$

$$\int_0^h \pi(R^2 - (z - R)^2) \, \mathrm{d}z$$

The above integral gives the volume of the water in a sphere, filled to height h.

Note: my answer differs from the other answers because I positioned my sphere with the bottom at the origin, rather than centered at the origin. In hindsight, the other way is simpler, but I didn't want to have to redo all my graphics. :) To show they're the same, we perform the subsitution:

$$u = z - R \implies \mathrm{d}u = \mathrm{d}z$$

This implies:

$$\int_{-R}^{h-R} \pi(R^2 - u^2) \, \mathrm{d}u$$

Question Score: 3 Answer Score: 2
Question ID: 429481

Area Between Curves

Question

The problem I am working on is, "In Exercises 17 and 18, find the area of the region by integrating (a) with respect to and x (b) with respect to y."

The two functions: $g(y) = 4 - y^2$, and $f(y) = y - 2$

I was able to solve part (b), but (a) proved to be a bit more complicated.

Here is my work, for part (a).

So, the region bounded by the two functions is found by determining where the intersect: $(0, 2), (-5, -3)$

Rewriting the the functions in terms of x being the independent variable:

$g(x) = \pm\sqrt{4 - x}$ and $f(x) = x + 2$

The interval of integration being, $[-5, 0]$; but I have a feeling this is wrong. Another part I felt insecure about is what to do with the $\pm$.

I would appreciate the help.

Answer

The first step in any problem like this is to draw a picture. Then you can find the limits a bit easier... Here's a plot: Wolfram Alpha link.

(Note: You rewrote $f(y)$ incorrectly... it should change into $f(x) = x + 2$)

This problem is really just trying to emphasize how integrating w.r.t. y or x can make a problem a lot harder or easier.

Basically, when integrating w.r.t. x, you're going to have to split the region in two: Region one (R_1) is from x=-5 to x=0, and region two (R_2) from x=0 to x=4.

This is because your "top" curve changes at x=0. On R_1, your area is bounded above by y=x+2, below by $y = -\sqrt{4 - x}$. On R_2, you are bounded above by $y = \sqrt{4 - x}$, below by $y = -\sqrt{4 - x}$.

Thus, your area is:

$$\int_{-5}^{0} [(x + 2) - (-\sqrt{4 - x})]dx + \int_{0}^{4} [(\sqrt{4 - x}) - (-\sqrt{4 - x})]dx$$

Simplifies to:

$$\int_{-5}^{0} [(x + 2) + \sqrt{4 - x})]dx + \int_{0}^{4} 2\sqrt{4 - x}dx$$

EDIT: To determine analytically, without graph.

It is possible, but I wouldn't recommend it, as it could (and typically does) involve more work. There is also more of a chance of making a mistake.

First, find all intersections of the functions. In this case, there are three functions: $f_1(x) = x + 2$, $f_2(x) = \sqrt{4 - x}$, $f_3(x) = -\sqrt{4 - x}$. Now find the intersections of these graphs, which are:

$$(-5, -3), (0, 2), (4, 0)$$

This splits the x axis into five regions:

$$(-\infty, -3), (-3, 0), (0, 4), (4, \infty)$$

(Note: the above are not points, but rather ranges of x. I have made the endpoints excluded really because it doesn't matter so much...)

Now, determine the top and bottom curves for each region. This means you must determine, for example, the function with the greatest y value for the region $(-\infty, -3)$. Do this for all ranges. This basically involves a lot of manipulating inequalities.

Now you know your upper and lower limits for each of those regions. Now you need to know what regions you need to find the area of. Do this by a similar process as above for the y-axis.

Now you know your limits of integration as well as your integrands.

I think you will agree that drawing a graph is a better approach... :P

Question Score: 1 Answer Score: 3
Question ID: 244566

Volume by rotation?

Question

Find the volume of the solid that results when the regions bounded by $y = x^3, x = 2$, and the x-axis is revolved around the line $x = 2$?

I don't really understand what it meant by "x-axis is revolved around the line $x = 2$"

Answer

There will be a region between the lines $y = x^3$, $x = 2$ and $y = 0$ (the last one is the x axis). If you revolved that region around the line $x = 2$, you would have a 3D region. You are being asked to find the volume of that region.

The region looks like the almost-triangle in this chart. you're going to revolve it around the line $x = 2$.

Question Score: 2 Answer Score: 1
Question ID: 336312

2.6 Unusual Integrals

Evaluating $\int \frac{\sqrt{x^2-1}}{x} \mathrm{d}x$

Question

How can one evaluate the integral

$$\int \frac{\sqrt{x^2-1}}{x} \mathrm{d}x$$

?

I tried substituting $x = \cosh t$ but got stuck at

$$\int \frac{\sinh^2 t}{\cosh t} \mathrm{d}t$$

Any hints?

Answer

I'll use the hyperbolic substitution you made. (Why not?) Of importance is the hyperbolic dual of the Pythagorean identity, $\cosh^2 x - \sinh^2 x = 1$. Then, one can see that:

$$\frac{\sinh^2 t}{\cosh t} = \frac{\cosh^2 t - 1}{\cosh t}$$

This makes your integral:

$$\int \cosh t - \operatorname{sech} t \, dt$$

If you know your hyperbolic trig integrals as well as most people know their "normal" trig integrals, you're home free.

Hint:

$$\int \operatorname{sech} t \, dt = 2 \arctan\left(\tanh\left(\frac{t}{2}\right)\right) + C$$

(According to Wolfram.)

Question Score: 7 Answer Score: 3
Question ID: 1104550

Evaluating the improper integral $\int_0^\infty \frac{\sin x}{x+x^2}\,dx$

Question

Evaluating the improper integral

$$\int_0^\infty \frac{\sin x}{x+x^2}\,dx$$

I'm trying to determine if the integral exists.

I can't seem to deal with

$$\lim_{a\to 0^+}\int_a^\infty \frac{\sin x}{x+x^2}\,dx,$$

could someone help with the limit above?

Edit: doing it by parts does seem to work, but wondering if there is a neater way to evaluate the limit at 0

Answer

We only seek to prove convergence, not to find the actual value.

Let $f(x) = \frac{\sin(x)}{x(x+1)}$. Note that:

$$\int_0^\infty f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^\infty f(x)\,dx$$

(if the rhs converges)

Note that:

$$\int_0^1 f(x)\,dx \le \int_0^1 \frac{x}{x(1+x)}\,dx$$

(via Taylor series for $\sin(x)$)

Thus:

$$\int_0^1 f(x)\,dx \le \int_0^1 \frac{dx}{1+x} = \ln(2)$$

So, the first integral converges. Moving on to the second integral:

$$\int_1^\infty \frac{\sin(x)}{x(x+1)}\,dx \le \int_1^\infty \frac{1}{x(x+1)}\,dx \le \int_1^\infty \frac{1}{x^2}\,dx = 1$$

So, the second integral converges. Thus:

$$\int_0^\infty f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^\infty f(x)\,dx \tag{2.69}$$

$$\le \ln(2) + 1 \tag{2.70}$$

Thus, the integral converges.

Question Score: 2 Answer Score: 3

Question ID: 764831

Why does $\int_{-\infty}^{\infty} e^{-x^2} \sin x \, dx = 0$?

Question

I can't get my head around something...

Why does $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} \sin x \, dx = 0$ but $\displaystyle\int_{-\infty}^{\infty} \sin x \, dx$ or $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x^{2n}} \, dx$ doesn't converge?

I thought maybe the first equality can be justified by saying the integrand is odd, but since this is also the case for the others, I don't understand why they aren't 0. Does this have something to do with the exponential function "dominating" the sine?

Answer

Why does any improper integral converge? Its really based on how quickly the integrand "goes to zero."

The exponential is negative in the first integral, which means the value of the integrand will decrease very rapidly as x increases or decreases.

Sine just bounces up and down between -1 and 1, so it doesn't ever converge to a given value.

In the third integral, as x gets bigger, the oscillations will get bigger, which certainly doesn't converge.

EDIT: For the new third integral–it is kinda curious that it doesn't converge. (That is, you can't tell just by looking without practice.) However, evidently, the x^{-2n} term doesn't decay fast enough to make the integral converge.

EDIT V2: see @sos440's comment as to why the third integral doesn't converge.

Question Score: 5 Answer Score: 2
Question ID: 319591

Formula for $\Gamma(\frac{1}{2}+it)$

Question

I have been working on the following problem for my complex analysis class involving Euler's Gamma function: For

$$\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}\,dt\ ,\ Re(s) > 0$$

Show that

$$\left|\Gamma\left(\frac{1}{2}+it\right)\right|^2 = \frac{2\pi}{e^{\pi t}+e^{-\pi t}}$$

for $t \in \mathbb{R}$. I am most of the way there, but have gotten hung up. So far, I have used the reflection forumla:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

which initially holds only for $Re(z) > 0$ but is shown to hold for $z \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0}$ by analytic continuation. It is clear that for any $t \in \mathbb{R}$, $\frac{1}{2}+it \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0}$, so I apply the reflection formula with $z = \frac{1}{2}+it$. A computation using the complex sine function shows that the desired quantity is obtained on the right hand side; namely,

$$\Gamma\left(\frac{1}{2}+it\right)\Gamma\left(1-\left(\frac{1}{2}+it\right)\right) = \frac{2\pi}{e^{\pi t}+e^{-\pi t}}$$

What I am having difficulty with is showing that

$$\Gamma\left(\frac{1}{2}+it\right)\Gamma\left(1-\left(\frac{1}{2}+it\right)\right) = \left|\Gamma\left(\frac{1}{2}+it\right)\right|^2$$

Any guidance would be much appreciated, as always!

Answer

Taking the hint from the comment, we can easily verify that $\Gamma(\bar{s}) = \overline{\Gamma(s)}$. I'll do that at the end.

Then:

$$\begin{aligned}
\frac{2\pi}{e^{\pi t}+e^{-\pi t}} &= \Gamma\left(\frac{1}{2}+it\right)\Gamma\left(1-\left(\frac{1}{2}+it\right)\right) && (2.71)\\
&= \Gamma\left(\frac{1}{2}+it\right)\Gamma\left(\frac{1}{2}-it\right) && (2.72)\\
&= \Gamma\left(\frac{1}{2}+it\right)\Gamma\left(\overline{\frac{1}{2}+it}\right) && (2.73)\\
&= \Gamma\left(\frac{1}{2}+it\right)\overline{\Gamma\left(\frac{1}{2}+it\right)} && (2.74)\\
&= \left|\Gamma\left(\frac{1}{2}+it\right)\right|^2 && (2.75)\\
& && (2.76)
\end{aligned}$$

As desired.

Now to show that $\Gamma(\bar{s}) = \overline{\Gamma(s)}$.

$$\begin{aligned}
\Gamma\left(\overline{a+bi}\right) &= \Gamma\left(a-bi\right) && (2.77)\\
&= \int_0^\infty t^{(a-bi)-1} e^{-t}\,dt && (2.78)\\
&= \int_0^\infty e^{\ln(t)((a-bi)-1)} e^{-t}\,dt && (2.79)\\
&= \int_0^\infty e^{\ln(t)(a-1)} e^{-\ln(t)bi} e^{-t}\,dt && (2.80)\\
&= \int_0^\infty t^{(a-1)} e^{-t}\left(\cos(-\ln(t)b) + i\sin(-\ln(t)b)\right)\,dt && (2.81)\\
&= \int_0^\infty t^{(a-1)} e^{-t}\left(\cos(\ln(t)b) - i\sin(\ln(t)b)\right)\,dt && (2.82)\\
&= \int_0^\infty t^{(a-1)} e^{-t}\cos(\ln(t)b)\,dt - i\int_0^\infty t^{(a-1)} e^{-t}\sin(\ln(t)b)\,dt && (2.83)\\
&= \left(\overline{\int_0^\infty t^{(a-1)} e^{-t}\cos(\ln(t)b)\,dt + i\int_0^\infty t^{(a-1)} e^{-t}\sin(\ln(t)b)\,dt}\right) && (2.84)\\
&\vdots && (2.85)\\
&= \overline{\Gamma(a+bi)} && (2.86)
\end{aligned}$$

(simply follow the same steps backward to complete the conjugate demonstration)

Question Score: 1 Answer Score: 2
Question ID: 564997

$\int_0^1 \int_0^1 e^{\max(x^2,y^2)} dxdy$

Question

I am currently having difficulty trying to evaluate a certain double integral.

$$\int_0^1 \int_0^1 e^{\max(x^2,y^2)} dxdy$$

I've broken it down into two cases,
(i) $\max(x^2, y^2) = y^2$
then

$$\int_0^1 \int_0^1 e^{\max(x^2,y^2)} dxdy = \int_0^1 e^{y^2} dy$$

(ii) $\max(x^2, y^2) = yx^2$
then

$$\int_0^1 \int_0^1 e^{\max(x^2,y^2)} dxdy = \int_0^1 e^{x^2} dx$$

since the inner integral does not depend on y

I believe that this is correct. It is at this point that I cannot evaluate. From my understanding this integral cannot be solved by any traditional means.

By using series, I can write

$$\int_0^1 e^{x^2} dx = \int_0^1 \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} dx = \sum_{n=0}^{\infty} \frac{x^{(2n+1)}}{(2n+1)(n!)} |_0^1 = \sum_{n=0}^{\infty} \frac{1}{(2n+1)(n!)}$$

Would this be the correct way of doing this? Also, if it is the correct method, is that series the appropriate answer? I started to have second thoughts here so I have not shown convergence (or not), I suspect this would be the next step.

Answer

Series is not the way you want to solve this problem, although it is possible. When you examine $\max(x^2, y^2)$ over the region $[0, 1] \times [0, 1]$, you find that:

$$\max(x^2, y^2) = \begin{cases} y^2, & \text{for } y \geq x \\ x^2, & \text{for } y < x \end{cases}$$

Thus, by symmetry:

$$\int_0^1 \int_0^1 e^{\max(x^2,y^2)} dy\, dx = 2 \int_0^1 \int_0^x e^{x^2} dy\, dx \tag{2.87}$$

$$= 2 \int_0^1 x e^{x^2} dx \tag{2.88}$$

$$= \ldots \tag{2.89}$$

(Which is certainly integrable with Calc II methods)

Question Score: 6 Answer Score: 3
Question ID: 715653

Legendre Polynomials!

Question

I need some help to show that $\int_{-1}^{1} P_n^2(x)\ dx = \frac{2}{2n+1}$ where P_n is the n^{th} Legendre's Polynomial. I've already done some calculations, so I would be very grateful whether someone just showed me why:

$\int_0^1 s^n(1-s)^n ds = \frac{(n!)^2}{(2n+1)!}$

It would be enough :-)

P.S. The "reduction formula" can be used... (?)

Answer

To start, note that the identity you are trying to prove is related to:

$$\mathrm{B}(a,b) = \int_0^1 s^{a-1}(1-s)^{b-1}\,ds = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

(Your particular scenario is $\mathrm{B}(n,n)$.)

I'm sure there are many proofs of this, but one that I know relates to the convolution integral and Laplace transforms.

> **Declarations:** Let $f(t) = t^{a-1}$, $g(t) = t^{b-1}$. Also, let $h(t) = (f * g)(t)$, where $*$ is convolution. Finally, designate the Laplace transform of $f(t)$, $g(t)$, $h(t)$ as $F(s)$, $G(s)\, H(s)$, respectively.

Note that $H(s) = F(s)G(s)$. (This is a property of convolution and the Laplace transform.)

By deriving or looking up the Laplace transforms of f and g, we see that:

$$H(s) = F(s)G(s) \tag{2.90}$$

$$= \frac{\Gamma(a)}{s^a}\frac{\Gamma(b)}{s^b} \tag{2.91}$$

$$= \frac{\Gamma(a)\Gamma(b)}{s^{a+b}} \tag{2.92}$$

Taking the inverse transform:

$$h(t) = \mathcal{L}^{-1}\left\{\frac{\Gamma(a)\Gamma(b)}{s^{a+b}}\right\} \tag{2.93}$$

$$= \Gamma(a)\Gamma(b)\mathcal{L}^{-1}\left\{\frac{1}{s^{a+b}}\right\} \tag{2.94}$$

$$= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\left\{\frac{\Gamma(a+b)}{s^{a+b}}\right\} \tag{2.95}$$

$$= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}t^{a+b-1} \tag{2.96}$$

By definition of convolution, we have that:

$$h(t) = (f*g)(t) = \int_0^t r^{a-1}(t-r)^{b-1}\,dr$$

But, we know $h(t)$ from the inverse transform. This leads us to the identity:

$$\int_0^t r^{a-1}(t-r)^{b-1}\,dr = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}t^{a+b-1}$$

For the Beta function, we substitute $t = 1$, obtaining:

$$\mathrm{B}(a,b) = \int_0^1 r^{a-1}(1-r)^{b-1}\,dr = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Thus, for your case:

$$\int_0^1 r^n(1-r)^n\,dr = \frac{\Gamma(n+1)\Gamma(n+1)}{\Gamma((n+1)+(n+1))} \tag{2.97}$$

$$= \frac{n!n!}{(2n+1)!} \tag{2.98}$$

$$= \frac{(n!)^2}{(2n+1)!} \tag{2.99}$$

This is a "guided" exercise in Boyce and DiPrima's Ordinary Differential Equations and Boundary Value Problems, in the Convolution Integral section of the Laplace Transforms chapter. (I'm mentioning this to give credit to where I first saw this.)

Question Score: 1 Answer Score: 4
Question ID: 947516

How does one calculate the integral of the sum of two absolute values?

Question

I know how to find the integral of just one absolute value, but this problem presents the integral of the sum of two absolute values. Help!

I want to evaluate:

$$\int_a^b (|x-1|+|x+1|)dx$$

Answer

Note that $f(x) = |x-1| + |x+1|$ can be broken down into multiple cases:

$$f(x) = \begin{cases} (x-1)+(x+1) & \text{if } x \geq 1 \\ -(x-1)+(x+1) & \text{if } -1 \leq x \leq 1 \\ -(x-1)+-(x+1) & \text{if } x \leq -1 \end{cases}$$

Simplifying:

$$f(x) = \begin{cases} 2x & \text{if } x \geq 1 \\ 2 & \text{if } -1 \leq x \leq 1 \\ -2x & \text{if } x \leq -1 \end{cases}$$

This is easily visualized with a graph (see Wolfram—Alpha).

So, to integrate, break up the interval $[a, b]$ with -1 and 1.

An example will probably help:

$$\begin{aligned} \int_{-3}^{3} |x-1| + |x+1| \, dx &= \int_{-3}^{-1} -2x \, dx + \int_{-1}^{1} 2 \, dx + \int_{1}^{3} 2x \, dx && (2.100) \\ &= -x^2\Big|_{-3}^{-1} + 2x\Big|_{-1}^{1} + x^2\Big|_{1}^{3} && (2.101) \\ &= 8+4+8 && (2.102) \\ &= 20 && (2.103) \end{aligned}$$

This area can be seen shaded in this Wolfram—Alpha link.

Question Score: 3 Answer Score: 1
Question ID: 314504

Help on Proof involving integrals

Question

Good night.

I'm starting to learn proofs and I'm facing the following question.

Given the linear function $f(x)$, prove that $[\int_0^1 f(x)\,dx]^2 < \int_0^1 [f(x)]^2 dx$

As $f(x)$ is a linear function, I represented it as $f(x) = ax + b$. How to proceed with the proof now ?

Answer

Let f be an arbitrary linear function, namely $f(x) = ax + b$. It follows:

$$\left[\int_0^1 f(x)\,dx\right]^2 = \left[\int_0^1 ax + b\,dx\right]^2 \tag{2.104}$$

$$= \left[\frac{ax^2}{2} + bx\Big|_0^1\right]^2 \tag{2.105}$$

$$= \left[\frac{a}{2} + b\right]^2 \tag{2.106}$$

$$= \frac{a^2}{4} + ab + b^2 \tag{2.107}$$

It also follows:

$$\int_0^1 [f(x)]^2\,dx = \int_0^1 [(ax + b)]^2\,dx \tag{2.108}$$

$$= \int_0^1 [(ax + b)]^2\,dx \tag{2.109}$$

$$= \frac{a^2x^3}{3} + \frac{2abx^2}{2} + b^2x\Big|_0^1 \tag{2.110}$$

$$= \frac{a^2}{3} + ab + b^2 \tag{2.111}$$

$$\tag{2.112}$$

From here, it should be pretty obvious...

Question Score: 0 Answer Score: 1

Question ID: 429685

Please integral question help?

Question

So,once I asked to know the integral

$$\int \frac{\sqrt{x^2+1}}{x} dx$$

... and the advice I got was to subsitute $x = \tan u$.

How about substituting $\sqrt{x^2+1} = u$? Will it work that way?

Answer

Yes, it works. It's not pretty, but it works.

$$\int \frac{\sqrt{x^2+1}}{x} dx$$

$$u = \sqrt{x^2+1} \implies x = \pm\sqrt{u^2-1}$$

$$du = -(x^2+1)^{-\frac{1}{2}} 2x dx \implies dx = -(\sqrt{x^2+1}) \frac{1}{2x} du$$

$$dx = -\left(\sqrt{u^2-1+1}\right) \frac{1}{\pm 2\sqrt{u^2-1}} du = \mp \frac{u}{2\sqrt{u^2-1}} du$$

Substitute into the integral:

$$\int \frac{u}{\pm\sqrt{u^2-1}} \left(\mp \frac{u}{2\sqrt{u^2-1}}\right) du$$

$$\int \frac{u^2}{u^2-1} du$$

$$\int 1 + \frac{1}{(u-1)(u+1)} du$$

Note that, by partial fractions,

$$\frac{1}{(u-1)(u+1)} = \frac{1}{2(u-1)} - \frac{1}{2(u+1)}$$

Thus, you have the integral:

$$\int 1 + \frac{1}{2(u-1)} - \frac{1}{2(u+1)} du$$

Evaluating:

$$u + \frac{1}{2}\ln(u-1) - \frac{1}{2}\ln(u+1) + C$$

Simplify:

$$u + \frac{1}{2}\ln\left(\frac{u-1}{u+1}\right) + C$$

Now substitute back in:

$$\sqrt{x^2+1}+\frac{1}{2}\ln\left(\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}\right)+C$$

$$\sqrt{x^2+1}+\frac{1}{2}\ln\left(\frac{x^2+1-1}{\left(\sqrt{x^2+1}+1\right)^2}\right)+C$$

$$\sqrt{x^2+1}+\frac{1}{2}\ln(x^2)-\frac{1}{2}\ln\left(\left(\sqrt{x^2+1}+1\right)^2\right)+C$$

Yielding the solution:

$$\sqrt{x^2+1}+\ln(x)-\ln\left(\sqrt{x^2+1}+1\right)+C$$

Verification from W—A.

Question Score: 0 Answer Score: 0
Question ID: 290769

Evaluating the primitive $\int \frac{\mathrm{d}x}{e^{2x}+e^x+1}$

Question

Could someone help me evaluate this?

$$\int \frac{\mathrm{d}x}{e^{2x}+e^x+1}$$

I tried to solve it for hours with no success. I tried Wolframalpha but it's giving a step by step solution that is too long that in an exam I won't even have the time to write the solution.

Thanks in advance.

Answer

Let $u = e^x$. Then, $du = e^x\,dx$, or, equivalently, $dx = \frac{1}{u}du$. Thus, the integral becomes:

$$\int \frac{dx}{e^{2x}+e^x+1} = \int \frac{du}{(u^2+u+1)u} \tag{2.113}$$

$$= \int \frac{du}{(u^2+u+1)u} \tag{2.114}$$

Now, hit it with partial fractions:

$$\frac{1}{(u^2+u+1)u} = \frac{Au+B}{u^2+u+1} + \frac{D}{u} \tag{2.115}$$

$$= \frac{Au^2+Bu+Du^2+Du+D}{(u^2+u+1)u} \tag{2.116}$$

$$= \frac{(A+D)u^2+(B+D)u+D}{(u^2+u+1)u} \tag{2.117}$$

Thus:

$$A+D = 0B+D = 0D = 1$$

So, the integral is:

$$\int \frac{du}{(u^2+u+1)u} = \underbrace{\int \frac{-u-1}{u^2+u+1}du}_{\text{integral 1}} + \underbrace{\int \frac{1}{u}du}_{\text{integral 2}}$$

Integral 2 is trivial, so I won't write it out. For integral 1, we apply the substitution $w = u^2+u+1$, so $dw = (2u+1)du$.

$$\int \frac{-u-1}{u^2+u+1}du = \frac{-1}{2}\int \frac{2u+1}{u^2+u+1}du + \frac{-1}{2}\int \frac{1}{u^2+u+1}du \tag{2.118}$$

$$= \frac{-1}{2}\int \frac{1}{w}dw - \frac{1}{2}\int \frac{1}{u^2+u+1}du \tag{2.119}$$

$$= \frac{-1}{2}\ln|w| - \frac{1}{2}\underbrace{\int \frac{1}{u^2+u+1}}_{\text{integral 3}} \tag{2.120}$$

For Integral 3, we use our knowledge of the derivative of arctangent. We have:

$$\int \frac{1}{u^2+u+1}du = \int \frac{1}{\left(u+\frac{1}{2}\right)^2+\frac{3}{4}}du \tag{2.121}$$

$$= \int \frac{1}{\left(\frac{u+\frac{1}{2}}{\sqrt{3}/2}\right)^2+1}du \tag{2.122}$$

$$= \int \frac{1}{\left(\frac{2u+1}{\sqrt{3}}\right)^2+1}du \tag{2.123}$$

$$= \frac{2}{\sqrt{3}}\arctan\left(\frac{2u+1}{\sqrt{3}}\right)+C \tag{2.124}$$

Thus, our overall answer is:

$$\int \frac{dx}{e^{2x}+e^x+1} = I_1 + \ln|e^x| \tag{2.125}$$

$$= \frac{-1}{2}\ln|e^{2x}+e^x+1| - \frac{1}{\sqrt{3}}\arctan\left(\frac{2e^x+1}{\sqrt{3}}\right)+x+C \tag{2.126}$$

Question Score: 5 Answer Score: 10
Question ID: 627383

Derivative of an integral wrt a different variable

Question

I have a small problem which I just have no idea on how to calculate, I am trying to find the derivative of p of an integral that is with respect to y (and ultimately solve the equation).

$$\frac{d}{dp}[2\int_0^1 (p-y)^2 y^{x-1}(1-y)^{-x} dy] = 0$$

Answer

Your tool is differentiation under the integral. Essentially:

$$\frac{d}{dp}\int_a^b f(y,p)\,dy = \int_a^b \frac{\partial}{\partial p} f(y,p)\,dy$$

So:

$$\frac{d}{dp}[2\int_0^1 (p-y)^2 y^{x-1}(1-y)^{-x} dy] = 2\int_0^1 \frac{\partial}{\partial p}(p-y)^2 y^{x-1}(1-y)^{-x} dy \tag{2.127}$$

$$= 2\int_0^1 2(p-y)y^{x-1}(1-y)^{-x} dy \tag{2.128}$$

$$\tag{2.129}$$

To go further, we'd need to know for what variable you are trying to solve, and what in the world the x's are doing there. :)

Question Score: 0 Answer Score: 0
Question ID: 554273

2.7 Multivariable Calculus

Computing $\iint \sin(4x^2+2y^2)\,\mathrm{d}A$ over an elliptical region

Question

$$\iint \sin(4x^2+2y^2)dA,$$

where the region is bounded by ellipse $4x^2+2y^2=\pi$ and the lines $y=0$ and $y=\sqrt{2}x$.

This looks like a change of variables integral. Need some hints. You guys are right. **say region is also bounded by first quadrant.**

Here is what I did:

ok you must make change of variables u= 2x and v = root(2)* y which makes ellipse transform to circle with radius root(pi), compute jacobian d(x,y)/d(u,v) which is equal to 1/(2*root(2)). y=0 implies v=0. other line implies v=u.

thus using polar coordinates, region is a circle with radius rootpi. angle from 0 to pi/4 then compute integral. $sin(4x^2+2y^2)$ implies $sin(u^2+v^2)=sin(r^2)$. theta is from 0 to pi/4 and r from 0 to root(pi).

Answer

If you ever see the requirement to integrate over an ellipse, it is oftentimes best to use the Jacobian to change this to a circle, and use the adjusted polar coordinates.

Your ellipse is:

$$\frac{x^2}{\left(\frac{\sqrt{\pi}}{2}\right)^2}+\frac{y^2}{\left(\frac{\sqrt{\pi}}{\sqrt{2}}\right)^2}=1$$

Now, we let:

$$x=\left(\frac{\sqrt{\pi}}{2}\right)r\cos\theta$$

$$y=\left(\frac{\sqrt{\pi}}{\sqrt{2}}\right)r\sin\theta$$

The Jacobian is:

$$J=\begin{bmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{bmatrix}=\begin{bmatrix}\left(\frac{\sqrt{\pi}}{2}\right)\cos\theta & -\left(\frac{\sqrt{\pi}}{2}\right)r\sin\theta\\ \left(\frac{\sqrt{\pi}}{\sqrt{2}}\right)\sin\theta & \left(\frac{\sqrt{\pi}}{\sqrt{2}}\right)r\cos\theta\end{bmatrix}$$

The determinant is:

$$\det J=\left(\left(\frac{\sqrt{\pi}}{2}\right)\cos\theta\cdot\left(\frac{\sqrt{\pi}}{\sqrt{2}}\right)r\cos\theta\right)-\left(-\left(\frac{\sqrt{\pi}}{2}\right)r\sin\theta\cdot\left(\frac{\sqrt{\pi}}{\sqrt{2}}\right)\sin\theta\right) \tag{2.130}$$

$$=r\frac{\sqrt{2}\pi}{4}\left(\sin^2\theta+\cos^2\theta\right) \tag{2.131}$$

$$=r\frac{\sqrt{2}\pi}{4} \tag{2.132}$$

To find the "difficult" bound:

$$y = \sqrt{2}x$$

$$\left(\frac{\sqrt{\pi}}{\sqrt{2}}\right) r \sin\theta = \sqrt{2}\left(\frac{\sqrt{\pi}}{2}\right) r \cos\theta$$

$$\tan\theta = 1 \implies \theta = \frac{\pi}{4}$$

So, in our "new coordinate system," our bounds are $0 \leq \theta \leq \frac{\pi}{4}$, $0 \leq r \leq 1$.

Thus, after the substitution, we have:

$$\int_0^{\frac{\pi}{4}} \int_0^1 \sin(\pi r^2) \det J \, dr \, d\theta$$

$$\int_0^{\frac{\pi}{4}} \int_0^1 \left(\sin(\pi r^2)\right) \left(r \frac{\sqrt{2}\pi}{4}\right) \, dr \, d\theta$$

And I'm sure you can solve the above (hint: u-substitution).

EDIT: I only set this integral up for the region in the first quadrant. The work done for finding the Jacobian is the same for both regions (the first quadrant and the third quadrant), so all the OP needs to do to solve his/her problem is to set up a similar integral (using the same substitutions) over the second part of the region.

It should be noted that I checked absolutely none of the above after calculating the Jacobian, and I was taking some pretty far mental leaps at the very end. Thus, I highly suggest that you take this just as some guidance, rather than me working the whole problem out for you, as there might be some mistakes in it.

Question Score: 0 Answer Score: 4
Question ID: 325102

Determining existence of limit with multiple variables

Question

Given the following limit:

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^3+y^3}$$

And the instrucion to "Determine whether the limit exists, give a complete argument", would the following be a "complete argument"?

Approaching the limit from the line y=0, gives:

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^3+y^3} = \lim_{(x,y)\to(0,0)} \frac{0}{x^3+y^3} = 0$$

Approaching the limit from the line y=x, gives:

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^3+y^3} = \lim_{(x,y)\to(0,0)} \frac{x^3}{2x^3} = \frac{1}{2}$$

These limits do not agree, thus the original limit

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^3+y^3}$$

does not exist.

Or should another method besides approaching from different lines be used to give a "complete argument" whether this limit exists be given?

Answer

This is exactly the way you should approach the problem. Showing that the limit takes on different values depending on what path you use to approach is a *sufficient condition* for a multi-variable limit to not exist.

Question Score: 2 Answer Score: 3
Question ID: 1185295

Vector functions and motion along a curve

Question

> A particle moves along the curve $x = \ln y$ with a constant speed of 4 units per second. Find the normal scalar component of acceleration as a function of x.

Honestly, what I don't understand is how to make it into a vector function. I know how to proceed if I had it all in terms of t. I would derive the acceleration and use the formula for the normal scalar. Essentially what I have is the positions of the particle, how do I take it and make $\vec{r}(t)$? Is this how?

$$\vec{r}(t) = x\hat{\imath} + \ln y\hat{\jmath}$$

Answer

The easiest way to to convert between a curve that is given explicitly is to let x or y be t, and then continue from there. So, if we have the equation:

$$x = \ln y$$

This is the same as:

$$y = e^x$$

Now, we can re-write this as a system of equations:

$$\begin{cases} x = t \\ y = e^t \end{cases}$$

This system can be turned into a vector-valued-function:

$$\vec{r} = (t, e^t)$$

Or, in $\hat{\imath}\hat{\jmath}\hat{k}$ form:

$$\vec{r} = t\hat{\imath} + e^t\hat{\jmath}$$

If I've skipped over a step somewhere, let me know with a comment below.

Question Score: 0 Answer Score: 0
Question ID: 600823

Double Integral Confusion

Question

A buddy was asking me for help with one of his MV Calc problems, and I ended up getting the same answer as him so I figured I'd ask it here...

Question Find

$$\iint_R (x-1)\,dA$$

where R is the region enclosed by $y=x$ and $y=x^3$ in the first quadrant. So naturally I told him to compute

$$\int_0^1 \int_{x^3}^x (x-1)\,dy\,dx$$

as x varies from 0 to 1 and y from x^3 to x. Using this integral, we got $\frac{-7}{60}$. His textbook gives an answer of $\frac{-1}{2}$.

Answer

As mentioned in the comments, your answer is correct. I'll provide a solution here for future readers.

First, draw a picture. I'm going to use Wolfram Alpha to show one online–it should be simple enough to graph by hand. We can see the intersections are at $(0,0)$ and a $(1,1)$. Thus, we have our limits of integration. Evaluation is as follows:

$$\int_0^1 \int_{x^3}^x (x-1)\,dy\,dx = \int_0^1 y(x-1)\Big|_{x^3}^{x}\,dx \tag{2.133}$$

$$= \int_0^1 (x-x^3)(x-1)\,dx \tag{2.134}$$

$$= \int_0^1 (-x^4+x^3+x^2-x)\,dx \tag{2.135}$$

$$= -\frac{x^5}{5}+\frac{x^4}{4}+\frac{x^3}{3}-\frac{x^2}{2}\Big|_0^1 \tag{2.136}$$

$$= \frac{-1}{5}+\frac{1}{4}+\frac{1}{3}-\frac{1}{2} \tag{2.137}$$

$$= \frac{-7}{60} \tag{2.138}$$

Question Score: 2 Answer Score: 2
Question ID: 652798

how to find this type of definite double integral?

Question

could any one tell me how to find this type of definite double integral?

$$\int_0^\infty \int_x^\infty \frac{e^{-y/2}}{y} dydx$$

Thank you.

Answer

Gah! Don beat me with the same idea while I was writing this, but I'm posting anyway. :) We can re-arrange the order of integration:

$$\int_0^\infty \int_0^y \frac{e^{-y/2}}{y}\, dx\, dy = \int_0^\infty x\frac{e^{-y/2}}{y}\bigg|_{x=0}^{x=y} dy \tag{2.139}$$

$$= \int_0^\infty y\frac{e^{-y/2}}{y}\, dy \tag{2.140}$$

$$= \int_0^\infty e^{-y/2}\, dy \tag{2.141}$$

$$\tag{2.142}$$

This integral can now be computed in a standard Calc II way.

Question Score: 0 Answer Score: 2
Question ID: 458066

Paramaterizing a path C along a parabola $y = 2x^2$

Question

I am doing a line integral where the path C is defined as the arc of the parabola $y = 2x^2$ from the points $(-1, 2)$ to $(2, 8)$.

Is there a "catch all" approach or method that can be applied here? Or is the only way to parametrize this is to think of an expression in terms of t that works for a particular interval of t?

Answer

The approach of $x = t, y = f(t)$ works for all curves that are functions of x.

That is, the approach fails for $y^2 = x$ (although a nearly identical approach may be used).

A worse case is when it completely fails for something that is not a function of y *or* x. For example, a circle of radius 1: $x^2 + y^2 = 1$. For arbitrary shapes like this, a little more thinking is required; however, the independent variable substitution works pretty well in most cases.

Question Score: 0 Answer Score: 0
Question ID: 785805

Line Integral and potential Fields

Question

Field :

$$\int_\ell \frac{-1}{1+(y-x)^2}\,dx + \frac{1}{1+(y-x)^2}\,dy$$

Find the path from point $(0,0)$ to $(1,2)$ along the ellipse $(x-1)^2+(y/2)^2=1$.

I thought of checking the green formula because there are no undefined points. I get the answer zero, which mean (on a closed loop, with solid inner area) that the path doesnt matter.

I pick an Easy road to $(1,2)$ with $y=2x$. I get the answer $13/3$ But the answer is $\pi/4$.

Regards Oskar

Answer

The vector field $\vec{F} = \left(\frac{-1}{1+(y-x)^2}, \frac{1}{1+(y-x)^2}\right)$ is conservative, so the value of the integral is independent of path.

By following the path $y=2x$, we get the integral:

$$\int_0^1 \frac{1}{1+x^2} dx = \arctan(1) - \arctan(0) = \frac{\pi}{4}$$

Your mistake is probably found where you substituted the path into the integral.

Question Score: 1 Answer Score: 2
Question ID: 809097

How to integrate over polar coordinates

Question

Evaluate the following double integral by rewriting it in polar coordinates:

$$\iint_D xy\,dA, \text{ where } D \text{ is the disc with center at the origin and radius 5}$$

I have very little understanding about how to do this. The most I know right now is the following:

1. $x = r\cos(\theta)$
2. $y = r\sin(\theta)$
3. $dA = r\,dr\,d\theta$
4. $D = \{(x, y) \mid x^2 + y^2 \leq 25\}$ or $D = \{(r, \theta) \mid r \leq 5\}$

It's given in the problem that $r = 5$, so that's a start. I'm assuming then that my limits for r is $0 \leq r \leq 5$. But I have no idea how to define the limits for θ. My guess would be $0 \leq \theta \leq 2\pi$, but several examples with different regions seem to use $0 \leq \theta \leq \pi$.

So here's part of the integral with missing limits on θ:

$$\int_\alpha^\beta \int_0^5 r^3 \sin\theta\cos\theta\,dr\,d\theta$$

Is my limited understanding correct so far? How do I fill in the holes of this problem? I know how to integrate after I have the proper limits; I just don't know how to define the limits given the information I have.

Answer

That's exactly right! As far as the limits on θ:

Imagine the radius of a circle "sweeping" out the upper limit on θ. (The starting place for most problems is $\theta = 0$.) I'll call this upper limit β. See these pictures below for two different limits:

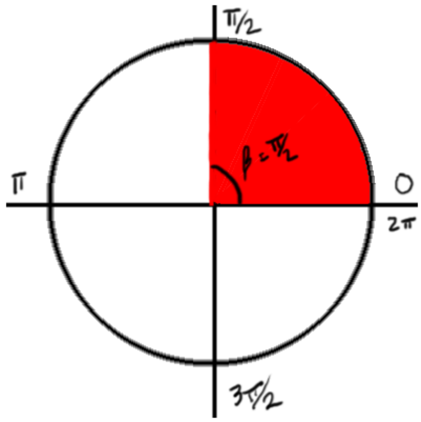


Figure 2.1: $\beta = \frac{\pi}{2}$

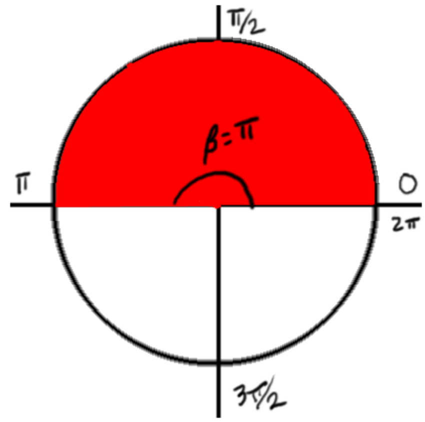


Figure 2.2: $\beta = \pi$

Since you want the *whole* disk of a given radius, we want β to be 2π. That is, we want to "sweep out" the whole circle.

Question Score: 2 Answer Score: 2
Question ID: 456916

Continuity and limit value of function at $(0,0)$

Question

> Does the function $f(x,y) = (x^2+y^2)/(x^2+y^2)$ have a limit of 1 as (x,y) approaches $(0,0)$ and is it continuous at $(0,0)$?

I tried approaching the origin along the x-axis (y = 0), the y-axis (x = 0), and the line y = x. All three answers gave me 1 but I do not know if that is enough or if I am missing a path that gives a value other than 1. Would an existing limit of 1 confirm whether or not the function is continuous at (0,0)

I would appreciate any feedback!

Answer

As pointed out in the comments, $f(x,y) = 1$ everywhere except $(0,0)$. Since the limit doesn't depend on the actual point, but only on the points "around" the "target" point, it is clear that the limit is 1.

If you can't see the above intuitively, you can use the substitution $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Then,

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2+y^2} = \lim_{(r,\theta)\to(0,0)} \frac{(r\sin(\theta))^2 + (r\cos(\theta))^2}{(r\sin(\theta))^2 + (r\cos(\theta))^2} \tag{2.143}$$

$$= \lim_{(r,\theta)\to(0,0)} \frac{r^2(\sin^2(\theta)+\cos^2(\theta))}{r^2(\sin^2(\theta)+\cos^2(\theta))} \tag{2.144}$$

$$= \lim_{r\to 0} \frac{r^2}{r^2} \tag{2.145}$$

$$= 1 \tag{2.146}$$

Question Score: 0 Answer Score: 2
Question ID: 844162

Finding volume using triple integrals.

Question

Use a triple integral to find the volume of the solid: The solid enclosed by the cylinder

$$x^2 + y^2 = 9$$

and the planes

$$y + z = 5$$

and

$$z = 1$$

This is how I started solving the problem, but the way I was solving it lead me to 0, which is incorrect.

$$\int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_{1}^{5-y} dzdxdy = \int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} (4-y)\, dxdy \tag{2.147}$$

$$= \int_{-3}^{3} [4x - xy]_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\, dy \tag{2.148}$$

$$= 8\int_{-3}^{3} \sqrt{9-y^2}dy - 2\int_{-3}^{3} y\sqrt{9-y^2}dy \tag{2.149}$$

If this is wrong, then that would explain why I'm stuck. If this is correct so far, that's good news, but the bad news is that I'm still stuck. If someone could help me out, that would be wonderful, thanks!

Answer

Ok. So you have the triple integral:

$$\int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_{1}^{5-y} dz\; dx\; dy = \int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} 4 - y\; dx\; dy \tag{2.150}$$

$$= \int_{-3}^{3} 4x - xy\Big|_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\; dy \tag{2.151}$$

$$= \int_{-3}^{3} 8\sqrt{9-y^2} - 2y\sqrt{9-y^2}\; dy \tag{2.152}$$

$$= 8\int_{-3}^{3} 3\sqrt{1-\left(\frac{y}{3}\right)^2}\; dy - 2\int_{-3}^{3} y\sqrt{9-y^2}\; dy \tag{2.153}$$

Now, I'm going to break this up. For the left-hand integral, we must use trig-substitution. Let $\cos(t) = \frac{y}{3}$. This implies that $dy = -3\sin(t)\; dt$. The limits of integration change as well, to $t = \arccos\left(\frac{-3}{3}\right) = \pi$ to $t = \arccos\left(\frac{3}{3}\right) = 0$.

So, the integral becomes:

$$\begin{aligned}
24\int_{\pi}^{0}\sqrt{1-\cos^2(t)}(-3\sin t)\,dt &= -72\int_{\pi}^{0}\sin^2(t)\,dt && (2.154)\\
&= 72\int_{0}^{\pi}\frac{1}{2}-\frac{\cos(2t)}{2}\,dt && (2.155)\\
&= 36\int_{0}^{\pi}1-\cos(2t)\,dt && (2.156)\\
&= 36\left(t-\frac{\sin(2t)}{2}\right)\Bigg|_{0}^{\pi} && (2.157)\\
&= \boxed{36\pi} && (2.158)
\end{aligned}$$

Now, for the left-hand integral, we apply u-substitution. If we set $u = 9 - y^2$, then $du = -2y\,dy$. The limits are transformed to $u = 9 - (-3)^2 = 0$ to $u = 9 - (3)^2 = 0$

So, the integral becomes:

$$\begin{aligned}
-2\int_{-3}^{3}y\sqrt{9-y^2}\,dy &= \int_{0}^{0}\sqrt{u}\,du && (2.159)\\
&= \boxed{0} && (2.160)
\end{aligned}$$

Well, that wasn't exciting. :)

So, putting it all together, we end up with:

$$V = 36\pi + 0 = \boxed{36\pi}$$

Question Score: 4 Answer Score: 5
Question ID: 457557

Evaluate the double integral using substitution:

Question

Evaluate:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} e^{-(x^2+y^2)} \, \mathrm{d}y \, \mathrm{d}x$$

I am trying to use substitution on this problem by making $x^2 + y^2 = u$ and $\sqrt{1-x^2} = v$. I then tried to add u and v to get something like $v + u = x^2 + y^2 + \sqrt{1-x^2}$ and I'm trying to set it equal to x, and then y, but I'm not sure if I'm doing this correctly.

Any suggestions as to how to make the substitution process less tedious?

Thank you so much.

Answer

Whenever you see an integral with x^2+y^2 and a $\sqrt{1-x^2}$ or a $\sqrt{1-y^2}$, the best substitution is to go into polar coordinates. Don't make a custom substitution unless you absolutely must. (A custom substitution would require computing the Jacobian, which isn't a trivial thing.)

Now, recall that, for polar, $r = x^2 + y^2$. We can now make the substitution:

$$\iint e^{-r^2} r \, \mathrm{d}r \, \mathrm{d}\theta$$

But, what are our limits of integration? To answer this, draw the region given in the question. It is the part of the unit circle in the first quadrant. Thus, our new integral is:

$$\int_0^{\pi/2} \int_0^1 e^{-r^2} r \, \mathrm{d}r \, \mathrm{d}\theta$$

I'm sure you can proceed from here.

Question Score: 2 Answer Score: 1

Question ID: 431643

Volume of a slanted cylinder

Question

I have a cylinder of radius 4 and height 10 that is at a 30 degree angle. I need to find the volume.

I have no clue how to do this, I have spent quite a while on it and went through many ideas but I think my best idea was this.

I know that the radius is 4 so if I cut the cylinder in half from corner to corner I will have two side lengths giving me a third side length. So this gives

$$\sqrt{116} = height$$

Or the length of the tall sides.

Now I just plug this into my formula

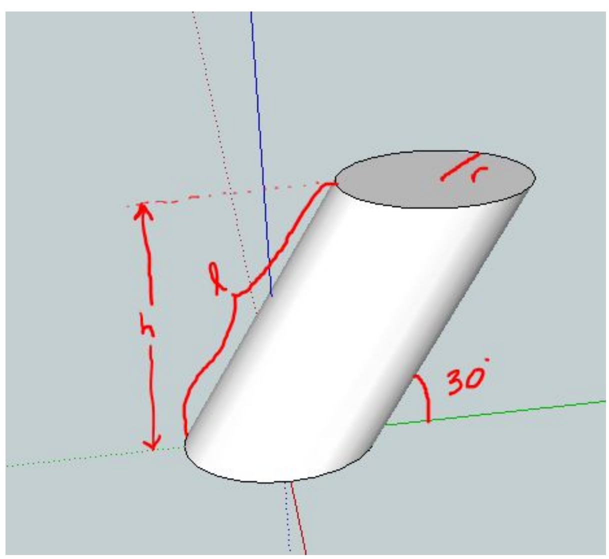
$$\pi r^2 h$$

$$\pi * 16 * \sqrt{116}$$

This is about 34π which is way off. What did I do wrong?

Answer

Picture for reference:



Let's get our terms straight here. h is the height of the cylinder; ℓ is the side length, and r is the radius. This cylinder is tilted at 30°.

The volume of a cylinder like this is given by the formula:

$$V = \pi r^2 h$$

For your problem, when you say "height of 10," I'm assuming you actually mean $\ell = 10$. From some trig, we see that:

$$h = \ell \sin 30^\circ = \frac{\ell}{2} = 5$$

Thus, our volume is:

$$V = \pi(4)^2(5) = 80\pi = 251.3 \text{ cubic units}$$

EDIT:

In the comments, it was mentioned that the vertical length is known. Thus, the solution is much simpler:

$$V = \pi r^2 h = \pi(4)^2(10) = 160\pi = 502.6$$

Question Score: 0 Answer Score: 2
Question ID: 431236

derivative of multivariable function, FTC

Question

Let $f : \mathbb{R}^n \to \mathbb{R}$ be some function. Is there something analogous to the statement: $f(x) - f(0) = \int_0^1 f'(x)dx$? Of course, this statement on it's own doesn't make sense, but could one say something similar?

I have learn't Calculus, but it's been a while, and apparently, I did not learn it well.

Answer

The closest thing I know of that relates to multiple-variable functions is the **Fundamental Theorem Of Line Integrals**:

> Let C be a smooth curve joining the point A to the point B in the plane or in space and parametrized by $\vec{r}(t)$. Let f be a differentiable function with a continuous gradient vector $\vec{F} = \nabla f$ on a domain D containing C. Then:
>
> $$\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A)$$

Source: *University Calculus, Early Transcendentals (2nd Edition)*, Hass, Weir, Thomas. Page 849.

Essentially, this says that if a vector field is conservative, the line integral over any path is equal to the difference of the potential function evaluated at each point. Applied to the physical world, this is why you can calculate the work done by gravity without needing to know the path taken.

Question Score: 0 Answer Score: 1
Question ID: 297731

Finding a volume of a solid by using integration. Calc III level.

Question

Find the volume of the solid with the plane $z = 0$ as the bottom, the cylinder $x^2 + y^2 = 4$ as the side, and the plane $z = 3xy$ as the top.

So I set

$$0 < z < 3 - x - y,\ 0 < x < 2,\ 0 < y < 2$$

Is that a correct way to set up an integration? or do i have to use different coordinate system?

it's simple but i just want to clarify!

Answer

That is not correct–the region you have denoted is a 2×2 unit square, not a cylinder. For the correct volume, compute:

$$\int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{0}^{3-x-y} dz\ dx\ dy$$

This may be simplified by going to a cylindrical coordinate system, by using the substitutions $x = r\cos\theta,\ y = r\sin\theta$:

$$\int_{0}^{2\pi} \int_{0}^{2} \int_{0}^{3-r(\cos\theta-\sin\theta)} r\ dz\ dr\ d\theta$$

Question Score: 1 Answer Score: 2

Question ID: 393420

Triple integral - finding boundaries

Question

Evaluate the triple integral

$$\iiint_E y \, dV$$

where E is bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $2x + 2y + z = 4$.

How we can find other boundaries for x, y, z? So far, we have $0 \leq x, 0 \leq y, 0 \leq z$. Is there a way to find those boundaries without sketching the graph ?

Answer

I am sure there are ways of finding bounds without sketching the graph, but they involve a lot of inequalities. Just trying to do so for a 2D region with a single integral takes quite a bit of computing. (I covered the basic idea for another user through that in this post (didn't compute, but just gave concept): Area Between Curves)

I highly recommend NOT trying to find bounds without graphing. The graph in question is a simple tetrahedron–it's not hard at all. Difficult bounds to sketch aren't (typically) asked of students unless they have graphing calculators. In the "real world" you have access to computer algebra systems, so you can use that to find the bounds.

tl;dr: Don't. It opens up too much opportunity for error.

Question Score: 1 Answer Score: 1
Question ID: 246187

Double Integral Over a General Region

Question

Evaluate the $\iint_R \sin\left(\frac{(x+y)}{2}\right)\cos\left(\frac{(x-y)}{2}\right) dA$ where R is a triangle with vertices $(0,0),(2,0),(1,1)$ using $u = \frac{(x+y)}{2}$ and $v = \frac{(x-y)}{2}$

I had trouble performing the substitution because I was not sure how to find du or dv or if I even needed to to perform the substitution.

Sorry for my poor formatting and any help would be appreciated

Answer

The technique you need to use isn't as directly related to u-substitution as you may be thinking. Instead, you are supposed to calculate the Jacobian of the change of variables.

Specifically, you want to compute:

$$J = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

Then, your integral becomes

$$\iint_G \sin(u)\cos(v)\,|J| dA$$

(Where G is the appropriate transformation of R under the new variables – see $8\pi r$'s answer for more information regarding that.)

Let me know if you need more guidance.

Question Score: 1 Answer Score: 1
Question ID: 866506

Derivative of a vector with Respect to scalar?

Question

I have a function $f = w\begin{bmatrix} a \\ b \\ c \end{bmatrix}$. So what is the $\frac{\partial f}{\partial w}$? I have never seen this. Thank you!

Answer

Note that $\vec{f} = \begin{bmatrix} aw \\ bw \\ cw \end{bmatrix}$.

Now, differentiate each component of the vector with respect to w. That is:

$$\frac{\partial \vec{f}}{\partial w} = \begin{bmatrix} \frac{\partial}{\partial w} aw \\ \frac{\partial}{\partial w} bw \\ \frac{\partial}{\partial w} cw \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

This assumes a, b, c could be constants *or* functions (but not of w).

Question Score: 5 Answer Score: 2
Question ID: 419496

$\iiint_R (x^2+y^2+z^2)^{\frac{3}{2}} e^{-x^2-y^2-z^2} dV_{xyz}$

Question

im having trouble evaluating this triple integral
$\iiint_R (x^2+y^2+z^2)^{\frac{3}{2}} e^{-x^2-y^2-z^2} dV_{xyz}$
$R = (x, y, z) : 1 \leq x^2+y^2+z^2 \leq 9$
any help is appreciated as always.

Answer

Hint:

Whenever you see $x^2+y^2+z^2$, this is begging to be transformed to $\rho^2 = x^2+y^2+z^2$ (spherical coordinates).

So, applying that substitution (and including the Jacobian):

$$\iiint_G (\rho^3 e^{-\rho^2}) \rho^2 \sin(\varphi) dV_{\rho,\varphi,\theta}$$

EDIT: For converting the region (as seems to be your questions from the comments), we can see that it's a sphere of radius 3 with a hole of radius 1 (as Ron Gordon noted). To model this, we let ρ range from 1 to 3, φ from 0 to π and θ from 0 to 2π.

Question Score: 1 Answer Score: 3
Question ID: 387100

Integrating a function over a domain

Question

How could you integrate the function $f(x, y) = x^2 + y^2$ over the triangle with vertices $(0, 0)$, $(0, 1)$ and $(1, 0)$?

I define the set $D = \{(x, y) \mid 0 \leq x \leq 1 \text{ and } 0 \leq y \leq x\}$ and then calculate

$$\int_0^1 \int_0^x x^2 + y^2 \, \mathrm{d}y \, \mathrm{d}x = \frac{1}{3},$$

but apparantly the answer is $\frac{1}{6}$.

Answer

Look at the graph of the region. You have described this region
However, the region in question is really this one.
Thus, your domain is

$$D = \{(x, y) | 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 - x\}$$

I believe you can take it from here.

Question Score: 2 Answer Score: 0
Question ID: 256514

2.8 Differential Equations

Nonlinear differential equation

Question

Question: Solve the following first-order equation.
$(1 + e^x)y' + e^y = 0$

Answer

It's a separable equation. It is first order, but certainly not linear (e^y term...)
Note that we can make it:

$$(1 + e^x)y' + e^y = 0$$

$$(1 + e^x)y' = -e^y$$

$$\frac{(1 + e^x)y'}{-e^y} = 1$$

$$-e^{-y}y' = \frac{1}{1 + e^x}$$

Now, integrate both sides. Let me know if you need further help.

Question Score: 2 Answer Score: 2
Question ID: 326695

Solving Differential equation with partial fraction decomposition

Question

I am a little rusty with some calculus and need some help with the follow equation:

$$\int \frac{f'(x)}{f(x)} dx = \int \frac{1}{-x + x^2 y} dx \tag{2.161}$$

Where y is a constant. My idea is to use some kind of U substitution as I know $\int \frac{dv}{v} = \ln(v)$. This gives:

$$\ln(f(x)) = \int \frac{1}{-x + x^2 y} dx \tag{2.162}$$
$$\tag{2.163}$$

Then I see to solve for $f(x)$ I can exponentiate. To solve the right integrand I would first do partial fraction decomposition which gives: $A = -1$ and $B = y$. Then I have:

$$\ln(f(x)) = \int \frac{-1}{x} dx + \int \frac{y}{-1 + xy} dx \tag{2.164}$$
$$\tag{2.165}$$

Then I get:

$$f(x) = x + -1 + xy = -1 + (y + 1)x \tag{2.166}$$

Is this correct?

Answer

Everything you've done looks correct, except from the second-to-last to the last lines. After the integration you get:

$$\ln(f(x)) = -\ln(x) + \ln(-1 + xy) + C$$

(Don't forget the +C)

Exponentiating both sides, we get ($\exp(x)$ is shorthand for e^x):

$$f(x) = \exp(-\ln(x) + \ln(-1 + xy) + C) \tag{2.167}$$
$$= \exp(-\ln(x)) \cdot \exp(\ln(-1 + xy)) \cdot \exp(C) \tag{2.168}$$
$$= \left(\frac{1}{x}\right)(xy - 1) \cdot (C) \tag{2.169}$$
$$= \left(y - \frac{1}{x}\right) \cdot C \tag{2.170}$$

Note that $\exp(a + b) \neq \exp(a) + \exp(b)$.

EDIT:

To show that this satisfies the original equation:

$$\frac{df}{dx} = \frac{C}{x^2}$$

$$\frac{f'(x)}{f(x)} = \frac{\frac{C}{x^2}}{\left(y - \frac{1}{x}\right) \cdot C} \tag{2.171}$$

$$= \frac{1}{x^2 \left(y - \frac{1}{x}\right)} \tag{2.172}$$

$$= \frac{1}{x^2 y - x} \tag{2.173}$$

Thus:

$$\int \frac{f'(x)}{f(x)}\, dx = \int \frac{1}{x^2 y - x}\, dx$$

Question Score: 3 Answer Score: 2
Question ID: 434146

Stability and Homogeneous Equations?

Question

Im having some trouble understanding the concept of stability in the context of a homogenous system $x = Ax$. According to my textbook, a system is stable if it has a fundamental matrix whose entries all remain bounded as $t \to +\infty$. Can someone please explain what this actually means.

Also why is it that if A has distinct real eigenvalues, then $x(t) = Ax(t)$ is stable iff all eigenvalues are negative. This is a theorem that I dont quite understand either. Please keep explanations simple if at all possible.

Answer

This is for the second part of your question...

A good way to think about this to gain intuition (NOT to be mathematically rigorous) is to forget about the matrix nature of the system for a second, but just think about it as if it's a scalar equation, with constant coefficients. Then we have:

$$x' = Ax$$

...for some constant A. Solving, using separation of variables, we have:

$$x = ce^{At}$$

Now, what happens if $A > 0$? Well, as $t \to \infty$, x gets larger and larger! If $A < 0$, we know the equation is asymptotically stable.

An "intuitive" way to think of the eigenvalues of a matrix is that they are *scalars* that *act like* the matrix. So, for the scalar case above, replace the conditionals on A with eig$A$.

Question Score: 1 Answer Score: 1
Question ID: 333353

How can i solve this System of first-order differential Equations?

Question

My Problem is this given System of differential Equations:

$$\dot{x} = 8x + 18y$$
$$\dot{y} = -3x - 7y$$

I am looking for a gerenal solution.

My Approach was: i can see this is a System of linear and ordinary differential equations. Both are of first-order, because the highest derivative is the first. But now i am stuck, i have no idea how to solve it. A Transformation into a Matrix should lead to this expression:

$$\overrightarrow{y} = \begin{pmatrix} 8 & 18 \\ -3 & -7 \end{pmatrix} \cdot x$$

or is this correct:

$$\overrightarrow{x} = \begin{pmatrix} 8 & 18 \\ -3 & -7 \end{pmatrix} \cdot y \text{ ?}$$

But i don't know how to determine the solution, from this point on.

Answer

I'm going to rename your variables. Instead of x and y, I will use x_1 and x_2 (respectively). Now, let's look at the system:

$$\begin{cases} \dot{x}_1 = 8x_1 + 18x_2 \\ \dot{x}_2 = -3x_1 - 7x_2 \end{cases}$$

To change this into matrix form, we rewrite as $\dot{\vec{x}} = \mathbf{A}\vec{x}$, where $\mathbf{A}$ is a matrix.

This looks like:

$$\underbrace{\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}}_{\dot{\vec{x}}} = \underbrace{\begin{pmatrix} 8 & 18 \\ -3 & -7 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\vec{x}}$$

To solve the system, we find the eigenvalues of the matrix. These are $r_1 = 2$ and $r_2 = -1$. Two corresponding eigenvectors are $\vec{\xi}_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $\vec{\xi}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, respectively.

We now plug these into the equation:

$$\vec{x} = c_1 e^{r_1 t} \vec{\xi}_1 + c_2 e^{r_2 t} \vec{\xi}_2$$

This yields:

$$\vec{x} = c_1 e^{2t} \begin{pmatrix} 3 \\ -1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

So, your individual solutions are:

$$x_1 = 3c_1 e^{2t} + 2c_2 e^{-t} x_2 = -c_1 e^{2t} - c_2 e^{-t}$$

Question Score: 5 Answer Score: 6
Question ID: 433098

how to solve, $(x^2 + y^2 + 2x)dx + 2ydy = 0$?

Question

$(x^2 + y^2 + 2x)dx + 2ydy = 0$
$f(x, y) = (x^2 + 2x) + y^2$
$g(y) = 2y$
differentiating $f(x, y)$, w.r.t y, we have, $f_y'(x, y) = 2y$
differentiating $g(y)$, w.r.t x, we have, $g_x'(y) = 0$

Answer

We have the differential equation:

$$(x^2 + y^2 + 2x) + (2y)y' = 0 \tag{1}$$

If we let:

$$f(x, y) = x^2 + y^2 + 2x$$
$$g(x, y) = 2y$$

Then:

$$f_y = 2y$$
$$g_x = 0$$

"Oh No! It's not exact, as we hoped!" However, there is still a chance to save the equation. Let's try the integrating factor method.

If we let:

$$\frac{d\mu}{dx} = \frac{f_y - g_x}{g}\mu = \frac{2y}{2y}\mu$$
$$\frac{d\mu}{dx} = \mu$$

Solving, $\mu = e^x$.
Multiply (1) by the integrating factor μ:

$$e^x(x^2 + y^2 + 2x) + (2ye^x)y' = 0 \tag{2}$$

Now we have two new functions, let's call them M and N:

$$M(x, y) = e^x(x^2 + y^2 + 2x) \tag{2.174}$$
$$N(x, y) = 2ye^x \tag{2.175}$$

Thus:

$$M_y = 2ye^x = N_x$$

Therefore, equation (2) is exact. Solve using the typical method. (If you need guidance past this point, please let me know via comment.)

Question Score: 1 Answer Score: 2
Question ID: 304517

Differential Equations: Population Problem $dp/dt = 0.5p - 380$

Question

I just want to make sure this is right because I'm doing the homework online and I'm on my last attempt and I'm pretty sure I got the other two right yet the computer program said no.

First at I have to find the time when the population becomes extinct when $p(0) = 710$. My answer was $2\ln\left(\frac{760}{50}\right)$ and its in months.

Then I have to find the initial population if they become extinct after 1 year. Now does that mean I use $t = 12$ since the first answer is in months?

EDIT: When I edited it for the first time I deleted, unintentionally, some information of the title that I've already corrected, sorry.

Answer

The DE:

$$\frac{dp}{dt} = \frac{1}{2}p - 380$$

$$\int \frac{dp}{\frac{1}{2}p - 380} = \int dt$$

$$2\int \frac{dp}{p - 760} = \int dt$$

$$2\ln|p - 760| = t + C$$

$$\ln|p - 760| = \frac{t}{2} + C$$

$$p - 760 = Ce^{\frac{t}{2}}$$

Solving for C:

$$C = 710 - 760 = -50$$

Solution to IVP:

$$p = -50e^{\frac{t}{2}} + 760$$

Ok, I've got the same thing you do here... Solving for t when $p = 0$:

$$0 = -50e^{\frac{t}{2}} + 760$$

$$0 = -50e^{\frac{t}{2}} + 760$$

$$e^{\frac{t}{2}} = \frac{76}{5}$$

$$t = 2\ln\frac{76}{5} \approx 5.44$$

That's odd... I'm getting your same answer. This probably means its a problem with the entry into the online homework system.

As to the second part of the question, yes, keep your units consistent across the problem.

Question Score: 2 Answer Score: 0

Question ID: 288414

General solution of $\frac{dx}{dt} - 2tx = t^3$

Question

I'm trying to find the general solution of the following first order differential equation (using the integrating factor method):

$$\frac{dx}{dt} - 2tx = t^3$$

I found the integrating factor to be e^{-t^2} meaning I would have to integrate

$$\frac{t^3}{\mathrm{e}^{-t^2}}$$

but I don't know how to integrate this. can anyone help?

Answer

$$\int \frac{t^3}{e^{-t^2}} dt = \int t^3 e^{t^2} dt$$

Let $u = t^2 \implies du = 2tdt$

Now we have:

$$\frac{1}{2} \int t^2 e^u du = \frac{1}{2} \int u e^u du$$

Now integrate by parts.

If you need further help, just leave a comment.

Question Score: 0 Answer Score: 1

Question ID: 309195

How do you determine the order of a differential equation?

Question

$$t^5y^{(4)} - t^3y'' + 6y = 0$$

the answer is fourth order, but I don't understand why exactly is it because of $y^{(4)}$? If so, is $y^{(4)}$ equivalent to y''''?

also, it says the equation is linear, but how is that possible if the exponent of t is 5 (and not 1)? Shouldn't a linear equation be of the form $ax + c$?

Answer

You're on the right track. Basically, $y^{(4)}$ is shorthand for y''''. This notation arises because counting little tickmarks is bothersome, to say the least. For example, imagine if I wanted $\frac{d^{104}y}{dx^{104}}$. You'd have to draw 104 little tickmarks, or you could simply say $y^{(104)}$.

And in general, the order of a differential equation is the order of the greatest derivative in the problem. So, $y^{(4)} = y$ is a fourth-order, and $y' = y$ is a first order.

Regarding linear differential equations: In determining order of differential equations, we don't care about what t does. The same is true for determining if a DE is linear or not. We want a linear combination of the derivatives of the function. So, the first two examples below are linear, the second two are not:

$$y' + y'' + y = t$$

$$t^4y'' + \cos t + y = 1$$

$$\cos(y'') + y = 3$$

$$(y')^2 + y = t$$

(Note that for the last one, I'm denoting exponentiation, not the order of the derivative. The difference is the lack of parenthesis in the superscript.)

Question Score: 1 Answer Score: 2
Question ID: 378880

Differential Equation for improper integrals

Question

How do I use the definiton of the improper integral to find the Laplace transform $F(s)$ for the function $f(t) = e^{(t-1)^2}$

Answer

This Laplace transform does not exist. Why?

From Boyce and DiPrima's Elementary Differential Equations:

Theorem 6.1.2: Suppose that:

1. f is piecewise continuous on the interval $0 \leq t \leq A$ for any positive A
2. $|f(t)| \leq Ke^{at}$ when $t \geq M$. In this inequality, K, a, and M are real constants, K, M necessarily positive.

Then the Laplace Transform $\mathcal{L}\{f(t)\} = F(s)$, defined by Eq. (4), exists for $s\,a$.

The problem we have is that $f(t) = e^{(t-1)^2}$ is not of exponential order–the a and K values above don't exist for this function.

Question Score: -1 Answer Score: 1
Question ID: 379766

Algebra help, logarithms, calculus

Question

I'm seriously out of my depth. I have very basic understanding of logarithms and calculus. Please could someone walk me through how to get from:

$$\frac{dp}{p} = -\frac{g}{R} \cdot \frac{dh}{T_0 - \lambda h}$$

To:

$$\log p = \frac{g}{\lambda R} \log(T_0 - \lambda h) + constant$$

I'd appreciate terminology of any manipulations done so I can Google them for more information. Thank you!

Answer

This is a first order, non-linear, ordinary differential equation. (For terminology) The dependent variable is p, and the independent variable is h.

The technique used to solve is called "separation of variables." (more terminology)

Starting with your expression:

$$\frac{dp}{p} = -\frac{g}{R} \cdot \frac{dh}{T_0 - \lambda h}$$

$$\int \frac{dp}{p} = \int -\frac{g}{R} \cdot \frac{dh}{T_0 - \lambda h}$$

Recall that $\int \frac{1}{x} dx = \ln|x| + C$. Thus we have:

$$\ln|p| + C = \int -\frac{g}{R} \cdot \frac{dh}{T_0 - \lambda h}$$

Now we perform a u-substitution on the RHS, letting $u = T_0 - \lambda h$. This implies that $du = -\lambda dh$

$$\ln|p| + C = \int -\frac{g}{R(-\lambda)} \cdot \frac{du}{u}$$

Integrating (using the same rule as above):

$$\ln|p| + C = \frac{g}{R(\lambda)} \cdot \ln|u|$$

Rearranging, and substituting u back in:

$$\ln|p| = \frac{g}{R\lambda} \cdot \ln|T_0 - \lambda h| + C$$

Leave a comment if you need more explanation on any step...

Question Score: 2 Answer Score: 2
Question ID: 317449

separable equation3

Question

Is this solution correct?

Equation:

$x \sin y \; dx + (x^2+1)\cos y \; dy = 0$

My solution:

$$x \sin y \, dx = -(x^2+1)\cos y \, dy \tag{2.176}$$
$$\frac{x}{-(x^2+1)} dx = \frac{\cos y}{\sin y} \, dy \tag{2.177}$$
$$\frac{x}{-(x^2+1)} dx = \frac{\cos y}{\sin y} \, dy \tag{2.178}$$
$$\frac{x}{-(x^2+1)} dx = \frac{\cos y}{\sin y} \, dy \tag{2.179}$$
$$\frac{x}{-(x^2+1)} dx = \cot y \, dy \tag{2.180}$$
$$\implies -\frac{1}{2}\ln(|x^2+1|) = -\cot^{-1} y \, dy \tag{2.181}$$

Answer

You almost had it, but not quite.

From your second-to-last line:

$$\frac{-x}{(x^2+1} \, dx = \cot(y) \, dy$$

$$\int \frac{-x}{(x^2+1)} \, dx = \int \cot(y) \, dy$$

$$-\frac{1}{2}\ln(x^2+1) + C = \int \frac{\cos(y)}{\sin(y)} \, dy$$

For the left integral, let $u = \sin(y) \implies du = \cos(y) \, dy$.

$$-\frac{1}{2}\ln(x^2+1) + C = \int \frac{du}{u} \, dy$$

$$-\frac{1}{2}\ln(x^2+1) + C = \ln|\sin y|$$

This can be solved for y, but it's not pretty (and may introduce some domain error):

$$C \exp\left(-\frac{1}{2}\ln(x^2+1)\right) = \sin y$$

$$y = \arcsin\left(C \exp\left(-\frac{1}{2}\ln(x^2+1)\right)\right)$$

Question Score: 1 Answer Score: 3

Question ID: 319683

Find Laplace Transform of the following function

Question

How do I find the Laplace transform for the function:
$f(t) = t, 0 \leq t \leq 1$ and $2 - t, t \geq 1$
I tried looking up the process online, but it remains unclear to me.
Thanks in advance!

Answer

You must break the integral up on the discontinuities to take the Laplace Transform of a discontinuous function:

$$\mathcal{L}\{(s) = \int_0^\infty f(t)e^{-st}dt \tag{2.182}$$

$$= \int_0^1 \underbrace{t}_{\text{f(t) for t ¡ 1}} e^{-st}dt + \int_1^\infty \underbrace{(2-t)}_{\text{f(t) for t ¿ 1}} e^{-st}\,dt \tag{2.183}$$

$$= \cdots \tag{2.184}$$

Please let me know if this needs more explaining.

More explanation:
The function f is defined by:

$$f(t) = \begin{cases} t & \text{if } 0 \leq t \leq 1 \\ 2-t & \text{if } 1 \leq t \end{cases}$$

To compute the Laplace transform, we compute the integral:

$$\int_0^\infty f(t)e^{-st}dt$$

Based on the property of integrals that says $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$, we can write:

$$\int_0^\infty f(t)e^{-st}dt = \int_0^1 f(t)e^{-st}\,dt + \int_1^\infty f(t)e^{-st}\,dt$$

We now replace $f(t)$ based on the definition:

$$\int_0^1 f(t)e^{-st}\,dt + \int_1^\infty f(t)e^{-st}\,dt = \int_0^1 te^{-st}\,dt + \int_1^\infty (2-t)e^{-st}\,dt$$

The rest is simple integration. (The first you do by-parts, the second you distribute, then a u-sub and by-parts again.)

Question Score: 0 Answer Score: 1
Question ID: 764002

Differential Equation - Word Problem

Question

Need help with this problem. I want to write it in a differential equation of the form:

$$p(t)' + f(t)p(t) = g(t)$$

Because of restoration of an island habitat, the maximum population of birds it can support at a time t is given by $1100 * \exp(t/80)$. The growth rate of the population $p(t)$ is equal to 1/20 of the difference between the maximum population and the current population. Initially the island has a population of 200 birds.

Answer

When you get a differential equations word problem, key phrases to look for include "growth rate" and "current ." "Growth rate" corresponds to the first derivative, and the "current [quantity]" corresponds to the actual function.

So, as we read this problem, we can note "The growth rate... is equal to," so we write down:

$$p'(t) = \text{(to be filled in later)}$$

Continuing, we see "1/20 of." "Of" in math word problems often refers to multiplication. So, we write:

$$p'(t) = \frac{1}{20} \cdot \text{(to be filled in)}$$

Continuing again: "difference of maximum population and the current population." Well, we're given the maximum population: $1100 \exp(t/80)$. What is the current population? That's $p(t)$. So, filling in our equation again:

$$p'(t) = \frac{1}{20} \cdot \left(1100 \exp\left(\frac{t}{80}\right) - p(t)\right)$$

Finishing the problem statement, we find that the original population is 200 birds. As you wrote in the comments, yes, this is the initial condition: $p(0) = 200$. So, we're done writing the differential equation. To make the equation fit the form you want, we can do some algebra:

$$p'(t) = \left(\frac{1}{20}\right) 1100 \exp\left(\frac{t}{80}\right) - \frac{1}{20} p(t), \qquad p(0) = 200 \tag{2.185}$$

$$p'(t) + \frac{1}{20} p(t) = 55 \exp\left(\frac{t}{80}\right), \qquad p(0) = 200 \tag{2.186}$$

Question Score: 3 Answer Score: 1
Question ID: 385669

Existence and Uniqueness Theorem

Question

I had a question about how to do one of these problems. So here's the question:

Given this equation $y' = \frac{-\cos(t)y(t)}{(t+2)(t-1)} + t$, find if the initial conditions $y(0) = 10$, $y(2) = -1$, $y(-10) = 5$ exist.

So I think the first step is just to take the partial derivative with respect to y which gives me:

$$y'' = \frac{-\cos(t)y'(t)}{(t+2)(t-1)}$$

So the 1'st equation doesn't exist at $t = -2, 1$ and the partial derivative doesn't exist at $t = -2, 1$ ....so do I conclude that all the initial values exists since none of them are $y(-2)$ or $y(1)$.

Don't really know how to do this whole existence and uniqueness thing....so am I right or completely off track?

Answer

What you've done looks perfectly fine.

Here's a general outline of what you do when you're looking to find where solutions exist for the first-order differential equation $y' + p(t)y = g(t)$.

1. Write the differential equation in the form: $y' = f(y, t)$
2. Find $f_y = \frac{\partial}{\partial y} f$.
3. Determine points of discontinuities of both f_y and f.

At this point, if you're just looking to see if a particular initial condition (t_0) has a solution, just check if t_0 is one of the points of discontinuity.

If you're looking for *where* the solution exists:

1. Draw a number line denoting where the discontinuities are (if possible).
2. Find where the initial condition falls on the number line.
3. If the discontinuity to the left of t_0 is a, and the discontinuity to the right of t_0 is b, then the solution exists on the interval (a, b).

EDIT Based on requests from comments below, here's a statement of the existence and uniqueness theorem:

> Let the functions f and $\frac{\partial f}{\partial y}$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the initial value problem:
>
> $$y' = f(t, y) \quad y(t_0) = y_0.$$

Source: *Elementary Differential Equations and Boundary Value Problems*, Boyce and DiPrima, 10th Edition, pg 70.

Question Score: 1 Answer Score: 2
Question ID: 296796

Solving the differential equation $\frac{dy}{dt} = e^{t-y}$

Question

I am working on the equation

$$\frac{dy}{dt} = e^{t-y}, \qquad y(0) = 1$$

This is what I have tried to get it to its exact solution:

$$\frac{dy}{dt} = e^t e^{-y}$$
$$\frac{1}{e^{-y}} dy = e^t dt$$
$$e^y dy = e^t dt$$
$$\int e^y dy = \int e^t dt$$
$$e^y = e^t + C$$
$$e^1 - e^0 = C < y = 1, t = 0 >$$
$$e^1 - 1 = C$$

going back:

$$e^y = e^t + \ln(e^1 - 1)$$
$$\ln(e^y) = \ln(e^t + e - 1)$$
$$y = \ln(e^t + e - 1)$$

Latex made me see my problem, Thank you for helping if you answered!

Answer

Your mistake was where you went:

$$e^y - e^t = C \implies \ln(e^y) - \ln(e^t) = C$$

Instead, you should have:

$$e^y - e^t = C \implies \ln(e^y - e^t) = C$$

(I would plug in the values for y and t in the RHS of the implication above. No need to solve further for C.)

Question Score: 5 Answer Score: 3
Question ID: 894625

Satisfying a Differential Equation

Question

For which values of k does the function $y = 2\cos(kt)$ satisfy the differential equation:

$$4\frac{d^2y}{dt^2} = -16y$$

So, $y = 2\cos(kt)$ Therefore, $y' = -2\sin(kt)k$ Thus, $y'' = -2\cos(kt)k^2$
Plugging this into $4y'' = -16y$ ...
$4(-2\cos(kt)k^2) = -16(2\cos(kt))$
I've simplified it this far:
$(\cos(k)k^2 + 4\cos(k)) = 0$
What next ?

Answer

What's next? Just some trig/algebra.
Factor out $\cos(kt)$:

$$(\cos(kt))\left(k^2 - 4\right) = 0$$

Now, the expression will be zero whenever either of the two factors are zero.
I think you should be able to take it from here...

Question Score: 1 Answer Score: 1
Question ID: 295877

Differential homogenous equation

Question

We have this:

$$y' = \frac{y}{x} + e^{-\frac{y}{x}}$$

The way to solve this equations (as I have learned) is to make the var. change $y = xu$, so you get a linear equation:

$$\frac{du}{f(1,u) - u} = \frac{dx}{x}$$

Being $f(x, y) = y'$

Now, when you have an equation with a no homogenous term $b(x)$, you can remove it, find a solution for the new equation, make the constant of that solution a function of x, substitute that solution in the initial equation and find what that function is. You couldn't do it with the above equation because the $e^{-\frac{y}{x}}$ is not a function of x only. But if you do it, you will get to the solution

$$y = x \log|\log|x| + k|, \quad k \in \mathbb{R}$$

And this equation is a solution of the above, so the method works, why? Is it correct? I did it in an exam and I don't know if I did it right. Obviously the final solution is correct, I didn't bother to recheck the problem because I pluged it into the equation and it worked, but I don't know why this works.

Thanks in advance.

ADDED Example asked by Babak Sorouh $y' + 2y = 3x$ We remove the $3x$ and solve, the solution is ce^{-2x}, we make the c a function of x and substitute in the first one: $c'e^{-2x} = 3x \Longrightarrow c(x) = 3/2xe^{2x} - 3/4e^{2x} + k; \quad k \in \mathbb{R}$ Substitute the $c(x)$ in the solution for the equation without the $3x$, and you got it: $3/2x - 3/4 + ke^{-2x}$

Answer

I don't think this will work for all cases. What you have described is essentially the method for exact equations, but with the addition of integrating factors. (If you have Boyce/DiPrima, it's on pg 99.) There are condition(s) for the method of exact equations to work.

An exact equation is of the form:

$$M(x, y) + N(x, y)y' = 0$$

...where $M_y = N_x$.

So, if you have the DE (using your example):

$$y' + 2y - 3x = 0$$

This implies:

$$M(x, y) = 2y - 3x$$

$$N(x, y) = 1$$

So, this is not an exact equation ($0 \neq 2$). However, if $\frac{M_y - N_x}{N}$ is a function of x only, there exists a factor to make the equation exact. (I am leaving out the derivation of this.) For this DE, $\frac{M_y - N_x}{N} = \frac{2-0}{1} = 2$, which satisfies the criteria.

Thus, we find our integrating factor as follows:

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu$$

$$\frac{d\mu}{dx} = 2\mu$$

$$\int \frac{d\mu}{\mu} = \int 2dx$$

$$\ln|\mu| = 2x$$

$$\mu = e^{2x}$$

So, multiplying through the original DE:

$$e^{2x}y' + (2y - 3x)e^{2x} = 0$$

Now, $M(x,y) = (2y - 3x)e^{2x}$, and $N(x,y) = e^{2x}$. This equation is exact, as $M_y = N_x = 2e^{2x}$

Now solve using method for exact equations as usual. (If needed I can demonstrate. I don't think it adds much to the answer, though...)

Question Score: 2 Answer Score: 1
Question ID: 298087

2.9 Theorems and Analysis

Is there an algorithm to read this mathematical definition?

Question

I'm reading Gemignami's *Calculus and Statistics.*

I'm a little stuck in this definition:

> **Definition 7**: Suppose f is a function from a set T of real numbers into R. Then f is said to be **continuous at a point** a of T if, given any positive number p, there is a positive number q such that whenever x is a point of T and $|x - a| < q$, then $|f(x) - f(a)| < p$. The function f is said to be **continuous** if it is continuous at every point of T.

The question may seem naive, but I really miss an *algorithm* to read this definition. I usually need to look at examples, analogies, wikipedia and other sources to form a intuitive conception of the concept, and I guess that definition should be self-contained (right?). The problem for me is that there are a lot of premises that are assumed at the same time and I feel confused with that. I am unable to *figure out what's happening* in such definitions and to make things a little worse, I'm learning alone at the moment. So, Is there an algorithm/way to read this mathematical definition?

Answer

I like Cameron's answer, but I have one thing to add:

If you know about limits already, there's a nice way of phrasing this definition: The function is continuous at point $(a, f(a))$ iff:

$$\lim_{x \to a} f(x) = f(a)$$

The function is continuous if it satisfies the above for all x in the domain.

Of course, there's some qualifiers, etc. (e.g. "Let f be a function from blaa blaa blaa...") for this definition, but you can think of continuity this way and be pretty safe.

Question Score: 1 Answer Score: 1
Question ID: 424429

Given certain conditions for δ, how do I show that an inequality relating delta to x is true?

Question

This is a problem out of a textbook (though there's no answer to this one in the back).

If $0 < \delta < 1$ and $|x-4| < \delta$

show: $|\sqrt{x}-2| < \frac{\delta}{\sqrt{3}+2}$

Hint: Rationalize $|\sqrt{x}-2|$.

I rationalize $|\sqrt{x}-2|$ by multiplying by a form of 1, $\frac{\sqrt{x}+2}{\sqrt{x}+2}$.

This yields $|\frac{x-4}{\sqrt{x}+2}| < \frac{\delta}{\sqrt{3}+2}$. (1)

Given the conditions from the beginning of the problem, I think that:

$0 < \delta < 1, |x-4| < \delta$ can be rewritten as $-\delta < x-4 < \delta$...

...which can then be rewritten as $4-\delta < x < 4+\delta$.

Given this and (1):

$$4-\left|\frac{(x)-4}{\sqrt{x}-2}\right|(\sqrt{3}+2) < x < 4+\left|\frac{(x)-4}{\sqrt{x}-2}\right|(\sqrt{3}+2)$$

Further, because I know that $0 < \delta < 1, |x-4| < \delta$ means that $|x-4| < 1$ must be true.

The absolute value inequality can be rewritten as $-1 < x-4 < 1$...

...or $3 < x < 5$.

By plugging possible values for x in, I get:

$$4-\left|\frac{(3)-4}{\sqrt{3}-2}\right|(\sqrt{3}+2) < x < 4+\left|\frac{(3)-4}{\sqrt{3}-2}\right|(\sqrt{3}+2)$$

and

$$4-\left|\frac{(5)-4}{\sqrt{5}-2}\right|(\sqrt{3}+2) < x < 4+\left|\frac{(5)-4}{\sqrt{5}-2}\right|(\sqrt{3}+2)$$

Simplifying, I find that:

$$4+\left|\frac{\sqrt{3}+2}{\sqrt{3}-2}\right| < x < 4-\left|\frac{\sqrt{3}+2}{\sqrt{3}-2}\right|$$

and

$$4-\left|\frac{\sqrt{5}+2}{\sqrt{5}-2}\right| < x < 4+\left|\frac{\sqrt{5}+2}{\sqrt{5}-2}\right|$$

What I'd like to know is: *am I even barking up the right tree?*

If so, how might I proceed?

If not, where did I go astray (and how do I go about understanding this problem and other problems like it?)

I feel like I don't really understand the question being asked, and I'm just manipulating the values I've been given because busy work *feels* like progress.

I will admit that (even after a semester of calculus) I'm still at a bit of a loss as to what a $\epsilon - \delta$ proof (or the "precise definition of the limit [...]", which I can only do because I've practiced a few problems while working from reference material) is useful for (and what it means beyond mapping some arbitrary values to some other arbitrary values).

I appreciate the time y'all have spent reading this - and especially any help you can offer.

Answer

You look as if you're trying to start with $|\sqrt{x} - 2| < \frac{\delta}{\sqrt{3}+2}$ and end up with $|x - 4| < \delta$. But, the problem is asking for it the other way around. That is, we must start with:

$$|x - 4| < \delta$$

Thus (assuming $x \geq 0$, which is valid because x is in 4 ± 1):

$$\left|(\sqrt{x} - 2)(\sqrt{x} + 2)\right| < \delta$$

As $\sqrt{x} + 2 \geq 0$ for all $x \geq 0$:

$$\left|(\sqrt{x} - 2)\right| (\sqrt{x} + 2) < \delta$$

Dividing over:

$$\left|(\sqrt{x} - 2)\right| < \frac{\delta}{\sqrt{x} + 2}$$

Since x is within 4 ± 1, we have:

$$\frac{\delta}{\sqrt{x} + 2} \leq \frac{\delta}{\sqrt{4 - 1} + 2}$$

This implies:

$$\left|\sqrt{x} - 2\right| < \frac{\delta}{\sqrt{3} + 2}$$

Question Score: 2 Answer Score: 2
Question ID: 457106

Identity and relationship between integrals and antiderivatives

Question

If I have that

$$G(ab) = \int_1^{ab} \frac{dt}{t} = \int_1^{a} \frac{dt}{t} + \int_a^{ab} \frac{dt}{t}.$$

$$G(ab) = G(a) + \int_a^{ab} \frac{dt}{t}.$$

How is that true? Specifically this line is reasoning (link to discussion):

$$G(\text{banana}) = \int_1^{\text{banana}} \frac{dt}{t}$$

As clever as banana man is he forgot to explain why that is true and I can't find it in any of my books or wikipedia entries. What does it mean?

Does this mean that the area under the graph of 2 from 1 to 2 is given as

$$\int_1^2 1dx$$

?

Answer

It seems that you are confused as to how they are defining $G(x)$. That is, $G(x)$ is defined as follows:

$$G(x) = \int_1^x \frac{1}{t}\, dt$$

Thus, by definition of G, you have that:

$$G(\text{banana}) = \int_1^{\text{banana}} \frac{1}{t}\, dt$$

Now, for the if/then statement, we make use of the theorem that:

$$\int_a^b f(x)\, dx + \int_b^c f(x)\, dx = \int_a^c f(x)\, dx$$

(If it helps, $G(t)$ is actually just the natural log, $\ln t$.)

EDIT: I'm going to try to further explain this, as it appears that your question wasn't really about the if/then statement, but rather about a sort of function called an "integral function."

In an integral function, the variable is not in the integrand, but rather in one of the limits. That is, we can define an integral function $G(x)$ as:

$$G(x) = \int_a^x f(t)\, dt$$

Note that the variable of the function (x) is different than the variable of integration (t).

Thus, $G(1) = \int_a^1 f(t)\, dt$.

It is *very important* to note that this **does not imply** that $f(1) = \int_a^1 f(t)\, dt$.

Question Score: 2 Answer Score: 1
Question ID: 433136

Why doesn't "$\lim_{x\to 1}\frac{1}{x-1}=\frac{1}{0}=\infty$" prove division by zero is possible?

Question

A friend of mine asked me one day: is a possible division by zero? - I answer you that, division by zero is not possible, the said ok, I want to solve this this example

$$\lim_{x\to 1}\frac{1}{x-1}$$

I did not think much and said that the result is ∞, because

$$\lim_{x\to 1}\frac{1}{x-1}=\frac{1}{1-1}=\frac{1}{0}=\infty$$

He then asked, amazed, as in this case are then division by zero is possible, therefore, $\lim_{x\to 1}\frac{1}{x-1}=\infty$, when the first bit of thae that division by zero is not possible.

My question is: is my answer correct, that is a $\lim_{x\to 1}\frac{1}{x-1}=\infty$, and how is it possible then that division by zero is not possible.

Please help to clarify this example.

Previously, thank you for your answers.

Answer

The reason that the existence of the limit above doesn't imply that division by 0 is possible is simply because

taking the limit $\neq$ actually dividing

Division by 0 is ambiguous, *unless defined* otherwise. (I believe there are some instances in which mathematicians do define it, but these are few and far between.)

A side note:

The example limit you have doesn't actually tend to ∞, rather it does not exist (the left- and right-hand limits are not the same).

An example that demonstrates what you want is:

$$\lim_{x\to 1}\frac{1}{(x-1)^2}\to\infty$$

Question Score: 0 Answer Score: 6
Question ID: 457365

Leibniz notation - how to get dx out of a derivative $v = \frac{dx}{dt}$

Question

I know that velocity equals $v = \frac{dx}{dt}$ which is writen in Leibniz notation. How can i get dx out of it in a proper way? I don't like it when people say that i should just multiply denominator dt with v like this:

$$v = \frac{dx}{dt}$$
$$v\,dt = dx \longrightarrow \boxed{dx = v\,dt}$$

I think that there must be some rules behind this which aren't mentioned most of the time. So what are these rules? How can i get same result in a proper mathematical way? Could you recommend any good book which focuses only on Leibniz notation and its tricks.

Answer

Not one of my best answers, but it is what it is...

It is notation, just like much of math. Using prime notation, we can define this quantity called the *differential*. For the below example, the differential is dx:

$$dx = f'(t)dt$$

This is why you will see confusion when performing separation of variables. Really, the dx, dt, or whatever other variable at the end of an integral is a differential. That is why we can simplify:

$$\frac{dx}{dt} = xt$$
$$\frac{1}{x}\frac{dx}{dt} = t$$
$$\int \frac{1}{x}\frac{dx}{dt}dt = \int t\ dt$$

Note that we're realy integrating both sides with respect to t! However, $\frac{dx}{dt}dt = dx$, based on the definition of the differential.

So, this becomes:

$$\int \frac{1}{x}dx = \int t\ dt$$

Question Score: 0 Answer Score: 1
Question ID: 337300

Extreme Value Theorem Proof (Spivak)

Question

Them: If f is continuous on $[a, b]$, then there is a y in $[a, b]$ such that $f(y) \geq f(x)$ for each $x \in [a, b]$

Proof. We already know that f is bounded on $[a, b]$, which means that the set

$$\{f(x) : x \text{ in } [a, b]\}$$

is bounded. This set is obviously not $\varnothing$, so it has a least upper bound α. Since $\alpha \geqslant f(x)$ for x in $[a, b]$ it suffices to show that $\alpha = f(y)$ for some y in $[a, b]$.

Suppose instead that $\alpha \neq f(y)$ for all $y$in $[a, b]$. Then the function g defined by

$$g(x) = \frac{1}{\alpha - f(x)}, \quad x \text{ in } [a, b]$$

is continuous on $[a, b]$, since the denominator of the right side is never 0. On the other hand, α is the least upper bound of $\{f(x) : x \text{ in } [a, b]\}$; this means that

$$\text{for every } \epsilon > 0 \text{ there is } x \text{ in } [a, b] \text{ with } \alpha - f(x) < \epsilon.$$

This, in turn, means that

$$\text{for every } \epsilon > 0 \text{ there is } x \text{ in } [a, b] \text{ with } g(x) > 1/\epsilon.$$

But *this* means that g is not bounded on $[a, b]$, contradicting the previous theorem.

OKay first of all how on earth does one come up with $g(x)$? It just feels like it comes out of nowhere. On the other hand what does

This means that: for every $\epsilon > 0$, there is x in $[a, b]$ with $\alpha - f(x) < \epsilon$

even mean? This statement is true, I agree, but I have absolutely no feeling for it. I also do not understand the last line with g. He could have chosen $\epsilon = \alpha - f(x)$ and be done it no?

Answer

Where did g come from?

The author needed a function that would be continuous on $[a, b]$ if α was not a value of $f(x), x \in [a, b]$. Basically, he chose this function because it is easy to see that it becomes discontinuous only if the least upper bound for the set is also in the set.

He needed it to become discontinuous to provide the contradiction found in the last step.

What does that quoted statement mean?

$$\text{for every } \varepsilon > 0, \text{ there is } x \in [a, b] \text{ with } \alpha f(x) < \varepsilon$$

In non-math talk, this basically means that, if you pick as small an ε as you want, then there is an x value in the range such that $\alpha - f(x)$ is less that that value you picked.

Even further in non-math talk: Pick any small value, and the function gets closer to the least upper bound than the magnitude of that small value.

(I know this is somewhat late, but I drafted it before leaving for a meeting... I didn't want it to go to waste. ;))

Question Score: 6 Answer Score: 3
Question ID: 384948

Two variable function continuity

Question

$$f(x,y)=\begin{cases}\frac{x^2y}{x^3+y} & \text{if } x^3+y\neq 0\\ 0 & \text{if } x^3+y=0\end{cases}$$

So how do I find out whether this function is continuous? My problem was using the condition, because it isn't continuity at a point.

Thanks in advance!

Answer

As pointed out in the comments, all that is needed is a single counter-example. A convenient point for the counter example is $(1,1)$.

Question Score: 2 Answer Score: 1
Question ID: 418716

Simple question about the Fundamental Theorem of Calculus

Question

We learned and proved the Fundamental Theorem of Calculus, and spoke about functions defined using the theorem. I still feel I haven't quite understood the relation:

Let's say $f(x)$ is differentiable on $\mathbb{R}$ and $f'(x)$ is continuous on $\mathbb{R}$.

Then can we say:

$$f(x) = \int_a^x f'(t)dt \quad \text{for some } a \in \mathbb{R}?$$

If so, is there any significance to $a \in \mathbb{R}$?

Answer

As mentioned in the comments, we should pick a value of a such that $f(a) = 0$. Then the statement is true.

This is important because, for some functions, picking a value of a such that $f(a) = 0$ is impossible. For example: $f(x) = x^2 + 1$.

Question Score: 0 Answer Score: 2
Question ID: 386768

Looking for a differentiable function which behaves somewhat like $\min(x, 1)$

Question

Is there a differentiable function $f : [0, 2] \to [0, 1]$ such that $f(x) = 0$ iff $x = 0$ and $f(x) = 1$ iff $x \in [1, 2]$? What about n times differentiable for any n, or infinitely differentiable? Thank you!

Answer

This just addresses the "does there exist such a function" with only one differentiation.

We want $f'(x) = 0$ for $x \in [1, 2]$ so that the function is constantly 1. We could define such a function piecewise:

$$f(x) = \begin{cases} \sin^2 \frac{\pi x}{2} & 0 \leq x < 1 \\ 1 & 1 \leq x \leq 2 \end{cases}$$

Question Score: 4 Answer Score: 2
Question ID: 385058

Chapter 3

Sequences and Summation

3.1 Sequences

Does $\lim_{n\to\infty} \frac{\exp(H_n)}{n+1}$ exist?

Question

Question:
Does $\lim_{n\to\infty} \frac{\exp(H_n)}{n+1}$ exist? If so, what is its value?

I know that the answer to the second part is e^γ, where γ is the Euler-Mascheroni constant, but I don't know how to get there.

The solution probably involves the definition of γ:

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right)$$

Answer

$$\begin{aligned}
\lim_{n\to\infty} \frac{\exp(H_n)}{n+1} &= \lim_{n\to\infty} \frac{\exp(H_n)}{\exp\ln(n+1)} && (3.1)\\
&= \lim_{n\to\infty} \left(\exp(H_n - \ln(n+1)\right) && (3.2)\\
&= \exp\left(\lim_{n\to\infty} \left(H_n - \ln(n+1)\right)\right) && (\star)\\
&= \exp(\gamma) && (3.3)
\end{aligned}$$

The move to line $(\star)$ is justified because $\exp(\cdot)$ is a continuous function.

Question Score: 2 Answer Score: 4
Question ID: 1081813

Making a sequence alternating

Question

So I have the sequence $\{x_j = k + \frac{1+2(j-\sum_{i=0}^{k} 2^{n-i})}{2^{n-k}}\}_{j\in\mathbb{N}}$, where $n = \lfloor log_2 j \rfloor$ and k= $\{l \in \mathbb{N}$:maximum l s.t. $\sum_{i=0}^{l} 2^{n-i} \leq j\}$, and I want to get the sequence :

$y_j = x_{\frac{j+1}{2}}$ if j odd and $y_j = -x_{\frac{j}{2}}$ if j even

into a single formula eg. $y_j = (-1)^{j+1} x_{f(j)}$.

What would be a way to do that?

Answer

You can use the ceiling function:

$$f(j) = \left\lceil \frac{j}{2} \right\rceil$$

(With your use of $f(j)$.)

Question Score: 1 Answer Score: 1
Question ID: 902198

How to create alternating series with happening every two terms

Question

I'm looking for a technique for creating alternating negatives and positives in a series. Specifically: when n=1, the answer is +, n=2 is +, n=3 is -, n=4 is -... etc.

I have every other part of the series written but I can't figure out that last piece... here's what I have now:

$$\sum_{1}^{\infty} 2^{n-1}(1^n + (-1)^n)/(3^{n-1}n!) * x^n$$

Technically, every other term is 0 so there doesn't really need to be two negatives in a row, it just has to sync up where I need them–I'm just guessing that I'd need it to work that way. Thanks for your assistance!

Answer

Try:

$$a_n = (-1)^{\frac{\left(2n+1+(-1)^{n+1}\right)}{4}+1}$$

Then,

$$\langle a_n \rangle = 1, 1, -1, -1, 1, 1, \ldots$$

Derivation:

"Any sufficiently advanced technology is indistinguishable from magic." - Clark's Third Law

I don't really know how to describe how I got that...

Question Score: 0 Answer Score: 1
Question ID: 768198

Working on sequence, possibly recursive

Question

I am working on this problem which asks to find if the sequence converges or not and if so the value it converges to. I am not sure how to deal with this type of question, but I feel like it may be a recursive relation. It is $a_n = \frac{1^2}{n^3} + \frac{2^2}{n^3} + \cdots + \frac{n^2}{n^3}$

What I have tried was calculating the first couple terms, $a_1 = 1$, $a_2 = 0.625$, $a_3 = 0.5185$ etc.I also tried writing it as $\frac{1^2 + 2^2 + .. + n^2}{n^3}$, but I am just really lost on where to go with this.

Thanks all

Answer

First, note that $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$. Then:

$$\frac{1^2}{n^3} + \cdots + \frac{n^2}{n^3} = \frac{\frac{n(n+1)(2n+1)}{6}}{n^3} \tag{3.4}$$

$$\underset{n\to\infty}{=} \frac{2n^3}{6n^3} \tag{3.5}$$

$$= \frac{1}{3} \tag{3.6}$$

Question Score: 1 Answer Score: 1

Question ID: 1103394

3.2 Finite Sums

Prove that $1+4+7+\cdots+3n-2=\frac{n(3n-1)}{2}$

Question

Prove that
$$1+4+7+\cdots+3n-2=\frac{n(3n-1)}{2}$$
for all positive integers n.

Proof:
$$1+4+7+\ldots+3(k+1)-2=\frac{(k+1)[3(k+1)+1]}{2}$$
$$\frac{(k+1)[3(k+1)+1]}{2}+3(k+1)-2$$
Along my proof I am stuck at the above section where it would be shown that:
$$\frac{(k+1)[3(k+1)+1]}{2}+3(k+1)-2=\frac{(k+1)[3(k+1)+1]}{2}$$
Any assistance would be appreciated.

Answer

Non-inductive derivation:

$$\sum_{k=1}^{n}(3k-2)=\sum_{k=1}^{n}3k-\sum_{k=1}^{n}2 \tag{3.7}$$
$$=3\left(\sum_{k=1}^{n}k\right)-2n \tag{3.8}$$
$$=\frac{3(n)(n+1)}{2}-\frac{4n}{2} \tag{3.9}$$
$$=\frac{3n^2-n}{2} \tag{3.10}$$
$$=\frac{n(3n-1)}{2} \tag{3.11}$$
$$\tag{3.12}$$

This, of course, relies on one knowing the sum of the first n natural numbers, but that's a well-known identity.

Question Score: 13 Answer Score: 33
Question ID: 1050814

Dealing with summations over elements $i \neq j$

Question

I have spent a quite a while trying to manipulate $\sum_{i,j=1:i\neq j}$ into series or product of sums that includes a $\sum_{i,j=1:j\neq 0}$ term. i.e.

$$\sum_{i,j=1:i\neq j} = \_\_\_\_\_\_ \text{ (times) } \sum_{i,j=1:j\neq 0}$$

or

$$\sum_{i,j=1:i\neq j} = \_\_\_\_\_\_ \text{ (plus) } \sum_{i,j=1:j\neq 0}$$

I considered multiplying the elements being summed over by

$$1 = \frac{\sum_{i,j=1:j\neq 0}}{\sum_{i,j=1:j\neq 0}}$$

but I figured this is not a valid way to introduce the desired $\sum_{i,j=1:j\neq 0}$ summation because it would leave another $\sum_{i,j=1:j\neq 0}$ that would not target the elements being summed over.

By this I mean:

$$A = \sum_{i,j=1:i\neq j} x_i x_j y_i y_j$$

where

$$\sum_{i,j=1:j\neq 0} x_i x_j = C$$

would result in (after multiplying by 1)

$$A = \frac{\sum_{i,j=1:i\neq j}}{\sum_{i,j=1:j\neq 0}} C y_i y_j$$

Which would either not be equivalent, or would raise the issue of dividing the two sums.

Thanks for the help, any advice is much appreciated.

Answer

You can break the sum up into two sums...

$$A = \sum_{i\neq j} x_i x_j y_i y_j \tag{3.13}$$

$$= \sum_{i\neq j,\ \text{s.t. } j\neq 0} x_i x_j y_i y_j + \sum_{i\neq j,\ \text{s.t. } j=0} x_i x_j y_i y_j \tag{3.14}$$

I don't know if this is what you're looking for, though.

Question Score: 2 Answer Score: 0

Question ID: 668647

Arithmetic sum question help

Question

An arithmetic sum has first term 5 and the tenth term is equal to 26. Find the common difference hence find the least value of n for which the sum of the first n terms of the series exceeds 1000.

Answer

Recall that the explicit formula for an arithmetic series is:

$$a_n = a_0 + nd \tag{1}$$

... where d is the common difference.

So, you're given that $a_9 = 26$ and that $a_0 = 5$. With that information, can you solve equation (1) for d?

For the sum question, what you want is the first n such that:

$$\sum_{k=0}^{n} a_k > 1000 \tag{2}$$

This can be done brute-force with a calculator (or pencil and paper with a lot of patience), or with formulas.

We know that $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$, and that $\sum_{k=0}^{n} 1 = n + 1$. Thus, plugging equation (1) in to equation (2):

$$\sum_{k=0}^{n} a_0 + kd > 1000$$

$$\sum_{k=0}^{n} a_0 + \sum_{k=0}^{n} kd > 1000$$

$$a_0 \sum_{k=0}^{n} 1 + d \sum_{k=0}^{n} k > 1000$$

$$a_0(n+1) + d\frac{n(n+1)}{2} > 1000 \tag{3}$$

So, all you need to do is solve (3) for the smallest integer n for which the equality holds.

Question Score: 0 Answer Score: 2
Question ID: 349372

How to evaluate the following summation

Question

I am trying to find the definite integral of a^x between b and c as the limit of a Riemann sum (where $a > 0$):

$$I = \int_b^c a^x \, \mathrm{d}x$$

However, I'm currently stuck in the following part, in order to find S:

$$S = \sum_{i=1}^{n} a^{(i(c-b))/n}$$

Is there a formula for this kind of expression? Thank you for your help.

Answer

Note that:

$$S = \sum_{i=1}^{n} a^{\dfrac{i(c-b)}{n}} \tag{3.15}$$

$$= \sum_{i=1}^{n} \left(a^{\left(\dfrac{(c-b)}{n}\right)} \right)^i \tag{3.16}$$

Now, we can use the finite form of the geometric series formula:

$$S = \frac{\left(a^{\left(\dfrac{(c-b)}{n}\right)} \right) - \left(a^{\left(\dfrac{(c-b)}{n}\right)} \right)^{n+1}}{1 - a^{\left(\dfrac{(c-b)}{n}\right)}}$$

...and that limit will be pretty nasty, but do-able. (I think.)

Question Score: 4 Answer Score: 2
Question ID: 633260

Triple-Nested Summation

Question

I have the following nested sum :

$$\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} x = x + 1$$

I don't have a clue how to solve this one, can somebody help me?
Thanks in advance.

Answer

I will use the rules found on Wikipedia's summation page.
First, note that x is not in the base or limit of any of the sums, so it can be factored out:

$$x\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1 = x + 1$$

Work from inside to outside, simplifying the sum:

$$\sum_{k=1}^{j} 1 = j$$

$$\sum_{j=1}^{i} j = \frac{i(i+1)}{2} = \frac{1}{2}\cdot\left(i^2 + i\right)$$

$$\sum_{i=1}^{n}\frac{1}{2}\cdot\left(i^2+i\right) = \frac{1}{2}\left(\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i\right) = \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right)$$

Now we're done, and can plug back in:

$$\frac{x}{2}\left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right) = x + 1$$

Now, solving for x:

$$x\left(\frac{n(n+1)(2n+1)}{12} + \frac{n(n+1)}{4} - 1\right) = 1$$

$$\begin{aligned} x &= \frac{1}{\left(\frac{n(n+1)(2n+1)}{12} + \frac{n(n+1)}{4} - 1\right)} \\ &= \frac{6}{n^3 + 3n^2 + 2n - 6} \end{aligned}$$

Question Score: 0 Answer Score: 5
Question ID: 312859

How to convert a sequence of numbers in the formula?

Question

I'm trying to understand different sorting algorithms and their BigO notation. Suppose, I'm using insertion sort and I have the worst case:

```
6 | 5 | 4 | 3 | 2 | 1
```

The number of comparisons will be $1 + 2 + 3 + 4 + 5 = 15$ or we can write it like:

$$1 + 2 + 3 + \cdots + (n - 1)$$

Also we can use formula to calculate number of comparisons for this sequence:

$$n(n - 1)/2$$

But it's not clear for me how the following equality holds:

$$1 + 2 + 3 + ... + (n - 1) = n(n - 1)/2$$

Answer

Write the sequence twice, once the "right" way, and once in reverse, then add vertically:

$$\begin{array}{ccccccccc} 1 & + & 2 & + & 3 & + & \dots & + & (n-1) \\ (n-1) & + & (n-2) & + & (n-3) & + & \dots & + & 1 \\ \hline n & + & n & + & n & + & \dots & + & n \end{array}$$

Now, what does the new sum add up to? Well, there are $n - 1$ "n"'s to add up. Thus we have $n(n - 1)$ as the new sum.

BUT, we know the new sum is twice the old one (after all, we added it twice). So, divide the new sum by two to get the old sum:

$$\frac{n(n-1)}{2}$$

Question Score: 1 Answer Score: 2
Question ID: 291313

Telescoping Sum with Discrete Calculus

Question

I'm trying to solve the following exercises, but I have a difficulty:

Find the sum of the telescoping series :

$$\sum_{k=1}^{n} k^2(k+1)(k+2)$$

Answer

As a variation, let's try discrete calculus. We can write the above as:

$$\sum_{k=1}^{n} k^2(k+1)(k+2) = \sum_{1}^{n+1} x(x+2)^{\underline{3}}\,\delta x$$

Now, we apply the summation-by-parts formula (that is, $\sum u\ \Delta v = uv - \sum \mathrm{E}v\ \Delta u$). We let $u = x$ and $\Delta v = (x+2)^{\underline{3}}$. Thus, $\Delta u = 1$ and $v = \frac{(x+2)^{\underline{4}}}{4}$. This means:

$$\sum_{1}^{n+1} x(x+2)^{\underline{3}}\,\delta x = x\frac{(x+2)^{\underline{4}}}{4}\Bigg|_{1}^{n+1} - \sum_{1}^{n+1} \frac{(x+3)^{\underline{4}}}{4}\,\delta x \tag{3.17}$$

$$= x\frac{(x+2)^{\underline{4}}}{4}\Bigg|_{1}^{n+1} - \left(\frac{(x+3)^{\underline{5}}}{20}\right)\Bigg|_{1}^{n+1} \tag{3.18}$$

$$= \left((n+1)\frac{(n+3)^{\underline{4}}}{4} - \frac{3^{\underline{4}}}{4}\right) - \left(\frac{(n+4)^{\underline{5}}}{20} - \frac{4^{\underline{5}}}{20}\right) \tag{3.19}$$

$$= \frac{(n+1)\cdot(n+3)^{\underline{4}}}{4} - \frac{(n+4)^{\underline{5}}}{20} \tag{3.20}$$

$$= \frac{n(n+1)^2(n+2)(n+3)}{4} - \frac{n(n+1)(n+2)(n+3)(n+4)}{20} \tag{3.21}$$

$$= \frac{n(n+1)(n+2)(n+3)(4n+1)}{20} \tag{3.22}$$

Simplification of the last two lines performed with Wolfram Alpha.

Question Score: 0 Answer Score: 0
Question ID: 910088

Formula for the $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n \cdot (n+1)$ sum

Question

Is there a formula for the following sum?
$S_n = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \ldots + n \cdot (n+1)$

Answer

We can use discrete calculus! Let $x^{\overline{k}}$ denote the kth rising factorial power of x. That is:

$$x^{\overline{k}} = \underbrace{x(x+1)(x+2)\cdots(x+k-1)}_{k \text{ factors}}$$

Then, $S_n = \sum_{k=0}^{n} k(k+1) = \sum_{0}^{n+1} x^{\overline{2}}\,\delta x$. (Notice that I started the summation at $k=0$; this makes it easier to plug in the lower limit, but does not affect the value of the summation since the first term is 0.)

Using the power rule for summation, we have:

$$S_n = \sum_{0}^{n+1} x^{\overline{2}}\,\delta x \tag{3.23}$$

$$= \left.\frac{(x-1)^{\overline{3}}}{3}\right|_{0}^{n+1} \tag{3.24}$$

$$= \frac{n^{\overline{3}}}{3} \tag{3.25}$$

$$= \frac{n(n+1)(n+2)}{3} \tag{3.26}$$

Question Score: 5 Answer Score: 2
Question ID: 347585

3.3 Infinite Series

Does this series violate the decreasing condition of the Integral Test for Convergence?

Question

I'm working on the section involving the Integral Test for Convergence in my calculus II class right now, and I've run into a seeming conflict between the definition of the Integral Test, and the solutions to some of the homework exercises as given by both my professor and the textbook.

According to the definition, my research, and my understanding of the integral test, the integral test can only be used for the series a_n where $a_n = f(n)$ and $f(x)$is positive, continuous and decreasing for all $x \geq N$, where N is the index of n. However, there are several problems where $f(x)$ is only decreasing if we add a condition, such as $f'(x) < 0 \iff x > 3$, where $N < 3$. It seems to me that the Integral Test cannot be used to determine convergence of a series when the function is only decreasing when $x > k$ and $k < N$, yet the book and my professor apply the test anyway.

For example, with the series:

$$\sum_{n=1}^{\infty} = \frac{n}{(4n+5)^{\frac{3}{2}}}$$

If we let $a_n = f(n)$, then for $f(x)$:

- $f(x) > 0$ for all x in the domain
- $f(x)$ is continuous for all $x > -\frac{5}{4}$

But, $f'(x) < 0$ only when $x > \frac{5}{2}$, as shown when testing the critical points with the derivative:

$$f'(x) = \frac{5-2x}{(4x+5)^{\frac{5}{2}}}$$

The professor notes this in her solution, but instead of ending with that and writing, "The Integral Test cannot be applied because $f(x)$ fails to satisfy the required conditions," she applies the test using the original index for n:

$$\int_1^{\infty} \frac{x}{(4x+5)^{\frac{3}{2}}} \to \infty \Rightarrow a_n \text{ Diverges}$$

The textbook reaches the same conclusion. Also, the problems in question are listed under a section where the instructions state, "Confirm that the Integral test can be applied to the series. Then use the Integral Test to determine the convergence or divergence of the series," implying the test can be used on the subsequent exercises.

Is there a reason the Integral Test for Convergence can be used to test for convergence in these problems, where $N < k$ and $f'(x) < 0 \iff x > k$? Am I missing something, or are the book and my professor wrongly using the Integral Test for these series?

Other exercises with same result:

$$\bullet \sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$

$$\bullet \frac{\ln 2}{2} + \frac{\ln 3}{3} + \frac{\ln 4}{4} + \frac{\ln 5}{5} + \frac{\ln 6}{6}$$

Answer

I do not believe your definition of the test is correct. Citing the ever-accurate (tongue-in-cheek) Wikipedia, we find the definition to be:

> Consider an integer N and a non-negative function f defined on the unbounded interval $[N, \infty)$, on which it is monotone decreasing. Then the infinite series
>
> $$\sum_{n=N}^{\infty} f(n)$$
>
> converges to a real number if and only if the improper integral
>
> $$\int_N^{\infty} f(x)\, dx$$
>
> is finite. In other words, if the integral diverges, then the series diverges as well.

(Source: http://en.wikipedia.org/wiki/Integral test for convergence)

It is important to note that various authors of textbooks may define a test or theorem in the way that it best fits their course outline. This is particularly true of elementary texts (e.g. first or second year calculus). Remember that mathematicians also like to generalize–if you can take a specific formulation of a theorem and generalize it (e.g. "f must be decreasing everywhere" to "f only has to have decreasing *end-behavior*"), that's perfectly acceptable.

As a further note in response to your comment on the main question: "decreasing a majority of the time" is different than "decreasing beyond $x = N$." The latter is the option you want.

Question Score: 2 Answer Score: 0
Question ID: 866430

Series convergence test of $\sum_{n=1}^{\infty} \frac{1}{n10^{n-1}}$

Question

Given the following series I have to test the convergence.

$$\sum_{n=1}^{\infty} \frac{1}{n10^{n-1}}$$

Then applying d'Alembert method I get:

$$\lim_{n\to\infty} \frac{1}{(n+1)10^{(n+1)-1}} \frac{n10^{n-1}}{1} = \lim_{n\to\infty} \frac{n10^{n-1}}{(n+1)10^{(n+1)-1}} = \frac{1}{10}$$

Thus, $R = 10$ so $x \in (-10; 10)$. So next I investigate the two boundary cases.
First, when $x = 10$, I substitute it back and get:

$$\sum_{n=1}^{\infty} \frac{10^n}{n10^{n-1}} = \sum_{n=1}^{\infty} \frac{10}{n} = 10 \sum_{n=1}^{\infty} \frac{1}{n}$$

Second, when x=-10, I calculate:

$$\sum_{n=1}^{\infty} \frac{(-1)^n 10^n}{n10^{n-1}} = 10 \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

The question is by what criteria and how do I know which of the two cases the series converges or diverges.

Answer

I will assume that you intend to determine convergence of the series:

$$\sum_{n=1}^{\infty} \frac{1}{n10^{n-1}}$$

One way to solve this is to note that $\sum_{n=N}^{\infty} \frac{1}{n10^{n-1}} < \sum_{n=N}^{\infty} \frac{1}{n^2}$ for large enough N. Thus, this series converges by comparison.

You seem to be confusing "finding radius of convergence" (which you do for power series) and "determining convergence" (which you can do for just regular series). This problem does not require d'Alembert's test (also called the Ratio test).

Question Score: 0 Answer Score: 0
Question ID: 839972

Fibonacci sequences and related series

Question

Let $\{a_n\}$ be a sequence such that $a_1 = a_2 = 1$ and $a_{n+1} = a_n + a_{n-1}$ for $n \geq 2$. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges.

My work:

Let $b_n = \frac{1}{a_n}$. Then I proved that $|\frac{b_{n+1}}{b_n}| < 1$. I was going to use ratio test. But how can I show that limsup $|\frac{b_{n+1}}{b_n}| \neq 1$? Can anyone please help me?

Answer

Bazooka-to-kill-a-fly approach:

Note that $a_n = f_n$, where f_n is the nth Fibonacci number. Thus, $a_n = [\phi^n/\sqrt{5}]$, where $\phi = \frac{1+\sqrt{5}}{2}$ and $[\cdot]$ is the nearest integer function.

Then:

$$\sum_{n=1}^{\infty} \frac{1}{a_n} = \sum_{n=1}^{\infty} \frac{1}{[\phi^n/\sqrt{5}]} \tag{3.27}$$

$$\leq \sum_{n=1}^{\infty} \frac{\sqrt{5}}{\phi^n} \tag{3.28}$$

$$= \frac{\sqrt{5}}{1-\frac{1}{\phi}} \tag{3.29}$$

Question Score: 1 Answer Score: 1
Question ID: 1103402

For which of the following choice of a_k is $\sum a_k$ convergent?

Question

For which of the following choice of a_k is $\sum a_k$ convergent?

i) $\dfrac{\sinh(k)}{2^k}$

ii) $\left(1 - \dfrac{1}{k}\right)^{k^2}$

Honestly, I have no idea. Usually, when I see sin or cos, I use consider absolute convergence, since it is easier; however, clearly this will not work since $\sum \dfrac{1}{k}$ diverges.

I considered using the integral test, but am not sure actually how to properly use it.

For the second, was switching between partial sums and Comparison Test. I tried Ratio Test, but it didn't produce a result (i.e., $L = 1$).

Answer

For the first one:

$$\lim_{k\to\infty} \frac{\sinh(k)}{2^k} = \lim_{k\to\infty} \frac{e^k - e^{-k}}{2^{k+1}} \tag{3.30}$$

$$= \lim_{k\to\infty} \frac{e^k}{2^{k+1}} \to \infty \tag{3.31}$$

Since the kth term does not go to zero as $k \to \infty$, we know the first diverges.

Question Score: 0 Answer Score: 1
Question ID: 1046658

Maclaurin series expansion of $\frac{1}{(1+x)^n}$

Question

I am trying to figure out the Maclaurin Series expansion of the function, preferribly in a sneaky and clever way. Any ideas?

Thanks.

Answer

Simple, yet not straightforward, approach:

$$\begin{aligned}
\frac{1}{(1-x)^n} &= (1-x)^{-n} && (3.32)\\
&= \sum_{k=0}^{\infty} \binom{-n}{k} (-x)^k && (1)\\
&= \sum_{k=0}^{\infty} \left((-1)^k \binom{n+k-1}{k} \right) (-x)^k && (2)\\
&= \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k && (3.33)\\
& && (3.34)
\end{aligned}$$

We arrive at (1) using the binomial series, and at (2) using the relationship between the multiset coefficient and the binomial coefficient $\binom{-n}{k}$.

Question Score: 1 Answer Score: 2
Question ID: 972668

Taylor Polynomial for $x^{1/3}$

Question

a. Compute the Taylor polynomial $T_3(x)$ for the function $(x)^{1/3}$ around the point $x = 1$.

b. Compute an error bound for the above approximation at $x = 1.3$.

I'm having trouble figuring out what to do for this problem.
I have to find the first three derivatives and after that I'm not quite sure what to do.

Answer
Hints:
A Taylor polynomial centered at a has the form:

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!}$$

So, when you compute the first three derivatives, you plug them into that formula:

$$T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2} + \frac{f'''(a)(x-a)^3}{6}$$

In your case, $a = 1$.
For the error bound, we use the Taylor series remainder term:

$$R_n = \frac{f^{(n+1)}(a)(x-a)^{(n+1)}}{(n+1)!}$$

To find the error at $x = 1.3$, plug in that value.

Question Score: 1 Answer Score: 2
Question ID: 393636

Can I know the value of an infinite serie?

Question

$$\sum_{n=0}^{\infty} \frac{n}{e^n}$$

I have found through a software that the value is $\dfrac{e}{(e-1)^2}$.

I've been trying to do it manually but I am getting $\dfrac{\infty}{\infty}$, since I have in the

$$\sum_{0}^{\infty} x\, e^{-x} = \frac{1}{e} + \frac{2}{e^2} + \cdots + \frac{\infty}{e^{\infty}} = \frac{e^2 * \ldots * e^{\infty} + e * e^3 * \ldots * e^{\infty} + e * e^3 * \ldots * e^{\infty-1}}{e * e^2 * \ldots * e^{\infty}} = \frac{\infty}{\infty}$$

Answer

Consider, instead, the series:

$$\begin{aligned}
\sum_{n=0}^{\infty} nx^n &= x \sum_{n=0}^{\infty} nx^{n-1} && (3.35)\\
&= x \sum_{n=0}^{\infty} \frac{d}{dx} x^n && (3.36)\\
&= x \frac{d}{dx} \sum_{n=0}^{\infty} x^n && (3.37)\\
&= x \frac{d}{dx} \left(\frac{1}{1-x} \right) && (3.38)\\
&= x \left(\frac{1}{(1-x)^2} \right) && (3.39)\\
&= \frac{x}{(1-x)^2} && (3.40)
\end{aligned}$$

This converges whenever $|x| < 1$. By making a smart substitution, you have your result.

Question Score: 0 Answer Score: 3
Question ID: 952631

Is $\sum_{n=1}^{\infty} \frac{e^{\arctan(n)}}{n^2+1}$ convergent or divergent?

Question

I don't know how should I determine if this series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{e^{\arctan(n)}}{n^2+1}$$

Any help would be appreciated.

Answer

Since $\arctan(x) < \frac{\pi}{2}$ for $x \geq 1$, we have that $e^{\arctan(n)} < e^{\pi/2}$ for $n \geq 1$. Thus:

$$\sum_{n=1}^{\infty} \frac{e^{\arctan(n)}}{n^2+1} \leq \sum_{n=1}^{\infty} \frac{e^{\pi/2}}{n^2+1} = e^{\pi/2} \sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

Therefore, by comparison, the series...

Question Score: 1 Answer Score: 2
Question ID: 641799

3.4 Recurrence Relations

Reasoning about a recursive function

Question

First of all, I am a computer science student, not a maths student. So maybe this is a trivial question, I just would like to understand it :)

Suppose I have the following (pointless) recursive function:

$$f(x) = a + b \cdot f(x)$$

for some constants a and b. I suppose that it does not really compute any values, or maybe maps its argument to $\pm\infty$ (except for the cases $b = 0$ or $a = 0, b = 1$).

I would like to reason:

$$f(x) - b \cdot f(x) = a \tag{3.41}$$
$$\vdots \tag{3.42}$$
$$f(x) = \frac{a}{1-b} \tag{3.43}$$

In terms of computation, this seems like a completely different function. So what is wrong about this reasoning (am I not allowed to treat the = as equality? or $f(x)$ as a constant?).

Answer

As pointed out in the comments, the issue you're running into is that your $f(x)$ **isn't** defined recursively. When one says "f is defined recursively," they mean that the value of $f(x)$ is a function of values of f at points *other than* x. For instance, the following are recursive functions:

$$f(x) = a + b \cdot f(x-1), \quad x > 0 \tag{3.44}$$
$$g(n) = g(n-1) + g(n-2), \quad n \geq 2 \tag{3.45}$$

(Notice that we need to specify the values of x where the recurrence is valid; we also need some initial conditions or boundary values, but that strays from your current question.)

However, ***your*** function $f(x)$ does not depend on prior values of x. Rather, you've really just described a constant function *implicitly*, rather than explicitly:

$$f(x) = a + b \cdot f(x) \implies (1-b)f(x) = a \implies f(x) = \frac{a}{1-b}$$

The equation on the far left is an *implicit* definition because you have defined $f(x)$ *in terms of itself* (not for values of $f(a)$, $a \neq x$), while the equation on the right is a *explicit* definition.

Question Score: 1 Answer Score: 2
Question ID: 386972

Find $f(n)$ when $n = 2^k$ where f satisfies the recurrence relation $f(n) = f\left(\frac{n}{2}\right) + 1$ with $f(1) = 1$

Question

Given: $f(1) = 1$.

Answer:

$$f(2) = f(1) + 1 = 1 + 1$$

$$\dots$$

$$f(4) = f(2) + 1 = 1 + 1 + 1.$$

How do I find the value of $f(n)$ where n is an odd integer?

Let say $f(3) = f\left(\frac{3}{2}\right) + 1$ which is it by the way. Then the question comes up what would the value of $f\left(\frac{3}{2}\right)$ be?

Answer

As already mentioned, without more initial conditions, you cannot compute terms using the recurrence. However, if you note that

$$f(n) = \log_2(n) + 1$$

(Which should be proven by induction)

You can then extend the function to accept all real numbers.

Question Score: 0 Answer Score: 1

Question ID: 767378

Repertoire method for solving recursions

Question

I am trying to solve this four parameter recurrence from exercise 1.16 in Concrete Mathematics:

$$g(1) = \alpha$$

$$g(2n+j) = 3g(n) + \gamma n + \beta_j$$

$$\text{for } j = 0, 1 \text{ and } n \geq 1$$

I have assumed the closed form to be:

$$g(n) = A(n)\alpha + B(n)\gamma + C(n)\beta_0 + D(n)\beta_1$$

Next i plugged $g(n) = 1$ in the recurrence equations from which I obtained $\alpha = 0, \beta_0 = -2, \beta_1 = -2$ and $\gamma = 0$

Substituting these values back into the closed form, I got:

$$A(n) - 2C(n) - 2D(n) = 1$$

Similarly plugging $g(n) = n$, I got $\alpha = 1, \beta_0 = 0, \beta_1 = 1$ and $\gamma = -1$ and plugging this back into the closed form, we get:

$$A(n) - B(n) + D(n) = n$$

Also, from the text in chapter 1, a recursion of general form

$$f(j) = \alpha_j$$

$$f(dn+j) = cf(n) + \beta_j$$

has a radix representation of

$$f((b_m b_{m-1} ... b_1 b_0)_d) = (\alpha_{b_m} \beta_{b_{m-1}} ... \beta_{b_1} \beta_{b_0})_c$$

Applying the generalization to the current problem we have

$$A(n)\alpha + C(n)\beta_0 + D(n)\beta_1 = (\alpha \beta_{b_{m-1}} ... \beta_{b_1} \beta_{b_0})_3$$

where $n = (1 b_{m-1} ... b_1 b_0)_2$

I am unable to see how to proceed further from here. Any help will be appreciated :)

Answer

You don't need to substitute $g(n) = 1$. If you do, however, you should get $\alpha = 1$, not $\alpha = 0$.

We know that

$$g(n) = \alpha A(n) + \gamma B(n) + \beta_0 C(n) + \beta_1 D(n) \tag{1}$$

We also know that:

$$\alpha A(n) + \beta_0 C(n) + \beta_1 D(n) = (\alpha\beta_{b_{m-1}} \dots \beta_{b_0})_3 \tag{2}$$

Thus, all that remains is to determine $B(n)$, then we have solved the problem.

From substituting $g(n) = n$, we have that:

$$A(n) - B(n) + D(n) = n$$

Thus:

$$\begin{aligned} B(n) &= \underbrace{A(n) + C(n) + D(n)}_{\text{simplify using (2)}} - C(n) - n && (3.46) \\ &= \underbrace{(1\dots1)_3}_{m+1 \text{ digits}} - C(n) - n && (3.47) \\ &= \frac{3^{m+1}-1}{2} - \left(\sum_{k, \text{ where } b_k = 0} 3^k \right) - n && (3.48) \end{aligned}$$

This leads us to the solution:

$$g(n) = (\alpha\beta_{b_{m-1}} \dots \beta_{b_0})_3 + \gamma \left(\frac{3^{m+1}-1}{2} - \left(\sum_{k, \text{ where } b_k = 0} 3^k \right) - n \right)$$

...where $n = (b_m b_{m-1} \dots b_0)_2$.

I'd like to get the sum out of the solution, but I don't know of a good way to do so. (I doubt if it is possible.)

Question Score: 7 Answer Score: 3
Question ID: 374142

sequence $x_{n+1} = x_n + \sin x_n$

Question

There is a sequence which satisfies

$$x_1 = a$$

$$x_{n+1} = x_n + \sin x_n$$

where a = 1.

Why does $\lim_{n\to\infty} x_n = \frac{\pi}{2}$ hold?

Answer

If we assume that the limit exists, let $\lim_{n\to\infty} x_n = x$. Then:

$$\left(\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n + \sin(x_n)\right) \implies x = x + \sin(x)$$

Thus:

$$0 = \sin(x) \implies x = k\pi, \quad k \in \mathbb{Z}$$

Thus, your stated equality does not hold.

Question Score: 1 Answer Score: 3
Question ID: 864342

$T(n) = T(\log n) + \mathcal{O}(1)$ Recurrence Relation

Question

what is the solution of following recurrence relation?

$$T(1) = 1 \tag{3.49}$$
$$T(n) = T(\log n) + \mathcal{O}(1) \tag{3.50}$$

a) $O(logn)$
b) $O(log^*n)$
c) $O(log^2n)$
d) $O(n/logn)$

Answer

I like to think of recurrences of the form $T(n) = T(f(n)) + \Theta(1)$ † as "accumulator" recurrences. This is because they "add up" the number of iterations–or some multiple thereof–it takes for $f(n)$ to go beyond some limit.

In words, this recurrence is determining the number of times one must take a logarithm before reaching 1. This should sound familiar to a computer science student, as there's a special function to describe this number!

This special function is $\lg^* n$, which is the iterated logarithm. Since some may not be familiar with this function, it is defined as:

$$\lg^* n := \begin{cases} 0 & n \leq 1 \\ 1 + \lg^*(\lg(n)) & n > 1 \end{cases}$$

So, a good guess would be that $T(n) \in \Theta(\lg^* n)$. (In fact, if we had $+1$ instead of $+\Theta(1)$, the exact solution would be: $T(n) = \lg^* n + 1$.) The guess can be proven somewhat easily using strong induction.

---

† I must make an important distinction: if you are really talking about big-$\mathcal{O}$ notation, there isn't a unique "right answer" to this question. This is because $\mathcal{O}(\cdot)$ is *an upper bound* for the asymptotic behavior of the recurrence, rather than a tight bound. People often confuse $\mathcal{O}(\cdot)$ with $\Theta(\cdot)$. As such, I'm assuming that the you meant $\Theta(\cdot)$ wherever you wrote $\mathcal{O}(\cdot)$.

Question Score: 3 Answer Score: 1
Question ID: 1145030

Solve the recurrence relation: $2a_n = 7a_{n-1} - 3a_{n-2}; a_0 = a_1 = 1$

Question

$2a_n = 7a_{n-1} - 3a_{n-2}$
$a_0 = a_1 = 1$

My attempt:

$2t^2 - 7t + 3 = 0$
$t = -\frac{1}{2}, -3$

$U_n = b(-\frac{1}{2})^n + d(-3)^n$
$b + d = 1 = -\frac{1}{2}b - 3d$
$b = \frac{8}{5}, d = -\frac{3}{5}$
$a_n = \frac{8}{5}(-\frac{1}{2})^n - \frac{3}{5}(-3)^n$
$a_2 = -5 \neq 2 = \frac{1}{2}(7-3)$

Where did I go wrong?

Update: changing the t values to positive, new solution:

$a_n = \frac{4}{7}(\frac{1}{2})^n + \frac{3}{7}(3)^n$
$a_2 = 4$

But following the initial equation given, $2a_2 = 4$, so shouldn't a_2 be 2?

a_2 according to Wolfram Alpha is indeed 2.

Answer

The answer given by Wolfram Alpha **does** match your textbook. (Try multiplying the 2^{-n} through.)

Your mistake was when you changed the roots of the characteristic to positive; if

$$a_n = A\left(\frac{1}{2}\right)^n + B\,(3)^n$$

then the correct method of solving for A and B would give $A = \frac{4}{5}$ and $B = \frac{1}{5}$.

Question Score: 2 Answer Score: 0
Question ID: 1068309

Stirling numbers of the second kind – a series-expansion typo?

Question

In H. S. Wilf's *generatingfunctionology*, (1.6.8) describes:

$$A_n(y) = \sum_k \left\{ {n-1 \atop k-1} \right\} y^k + \sum \left\{ {n-1 \atop k} \right\} y^k$$
$$= yA_{n-1}(y) + \left(y\frac{d}{dy} \right) A_{n-1}(y)$$

where

$$A_n(y) = \sum_k \left\{ {n \atop k} \right\} y^k$$

but surely if

$$A_{n-1}(y) = \sum_k \left\{ {n-1 \atop k} \right\} y^k = \left\{ {n-1 \atop 1} \right\} y + \ldots + \left\{ {n-1 \atop k} \right\} y^k$$

then

$$A_n(y) = \left(yA_{n-1}(y) - \left\{ {n-1 \atop k} \right\} y^{k+1} \right) + y\frac{d}{dy}A_{n-1}(y)$$

since $A_n(y)$ only sums over k rather than $k+1$. So is this a typo or am I missing something?

Answer

If I understand correctly, it seems like you've confused what the author means by $\sum_k f(k)$. That is,

$$\sum_k f(k) \neq f(1) + f(2) + \cdots + f(k)$$

Rather:

$$\sum_k f(k) = \sum_{k\in\mathbb{Z}} f(k) = f(0) + f(1) + f(-1) + f(2) + f(-2) + \cdots$$

(See Convention 1.3.)

Taking into account that $\left\{ {n \atop k} \right\} = 0$ if $k > n$ or $n, k < 0$, we can see that:

$$A_n(y) = \sum_k \left\{ {n \atop k} \right\} y^k = \left\{ {n \atop 0} \right\} + \left\{ {n \atop 1} \right\} y + \cdots + \left\{ {n \atop n} \right\} y^n \tag{3.51}$$

$$\implies A_{n-1}(y) = \sum_k \left\{ {n-1 \atop k} \right\} y^k = \left\{ {n-1 \atop 0} \right\} + \left\{ {n-1 \atop 1} \right\} y + \cdots + \left\{ {n-1 \atop n-1} \right\} y^{n-1} \tag{3.52}$$

$$\tag{3.53}$$

With this, the formulas become readily apparent.

Question Score: 1 Answer Score: 1
Question ID: 1005158

non-homogeneous recurrence relations

Question

The question is: $a_n = 12a_{n-2} + 16a_{n-3} + 9 \cdot 4^n + 81n$with $a_0 = 0, a_1 = 1, a_2 = 98$ I tried to deal with the particular solution first by: $A4^n + Bn = 12[A4^{n-2} + B(n-2)] + 16[A4^{n-3} + B(n-3)] + 9 \cdot 4^n + 81n$. However, turns out the coefficient of 4^nbeing 0 which means 9=0 which is clearly wrong. Could anyone please show me where did I do wrong in the working? Many thanks.

Answer

I see user21820 has given a good answer while I went AFK, but I figured I'd share this anyway since I worked it out...

We can apply the method of annihilators to convert this into a 6th order, constant coefficient, homogeneous recurrence. For background on the method of annihilators, see these links.

First, we rewrite and reindex the recurrence:

$$a_n = 12a_{n-2} + 16_{n-3} + 9 \cdot 4^n + 81n \implies a_{n+3} - 12a_{n+1} - 16a_n = 9 \cdot 4^{n+3} + 81(n+3)$$

Next, we rewrite using the shift operator (and I've simplified some of the products):

$$(E^3 - 12E - 16)\langle a_n \rangle = \langle 576 \cdot 4^n \rangle + \langle 81n \rangle + \langle 243 \rangle$$

Note that the annihilator for 4^n is $(E-4)$, the annihilator for n is $(E-1)^2$, and the annihilator for 243 is $(E-1)$. We apply the annihilators to the whole equation:

$$(E-4)(E-1)^2(E^3 - 12E - 16)\langle a_n \rangle = (E-4)(E-1)^2\langle 576 \cdot 4^n \rangle \tag{3.54}$$
$$+ (E-4)(E-1)^2\langle 81n \rangle \tag{3.55}$$
$$+ (E-4)(E-1)^2\langle 243 \rangle \tag{3.56}$$

The RHS is annihilated, so:

$$(E-4)(E-1)^2(E^3 - 12E - 16)\langle a_n \rangle = \langle 0 \rangle$$

Factoring the annihilators (which gives the roots of the characteristic), we find this is equivalent to:

$$(E-4)^2(E-1)^2(E+2)^2\langle a_n \rangle = \langle 0 \rangle$$

We have 3 roots, each of of multiplicity 2. So:

$$a_n = c_1 4^n + c_2 n 4^n + c_3 1^n + c_4 n 1^n + c_5(-2)^n + c_6 n(-2)^n$$

Or, simplified:

$$a_n = 4^n(c_1 + nc_2) + (c_3 + nc_4) + (-2)^n(c_5 + nc_6)$$

Computing the first 6 terms of the sequence provides 6 linear equations in 6 unknowns, which can be (relatively) easily solved.

(And my answer matches the other one, which is a good sign...) :)

Question Score: 1 Answer Score: 1
Question ID: 765894

Generalized Josephus Recurrence

Question

$$T(n) = \begin{cases} 4 & n \le 2 \\ 3T(n/3) + 5 & n > 2 \end{cases}$$

I am trying to solve the above recurrence relation, but am unsure about my work:

$$T(n/3) = 3T(n/9) + 5 \tag{3.57}$$
$$\implies T(n) = 3(3T(n/9) + 5) + 5 \tag{3.58}$$
$$= 3^k T(n/3^k) + 5k \tag{3.59}$$

Letting $k = \log_3(n)$:

$$= 3^{\log_3(n)} T\left(n/3^{\log_3(n)}\right) + 5\log_3(n) \tag{3.60}$$
$$= nT(1) + 5\log_3(n) \tag{3.61}$$

Answer

This is a **generalized Josephus recurrence**, of which an in-depth solution is given in *Concrete Mathematics* (by Graham, Knuth, Patashnik).

If f is defined as:

$$f(j) = \alpha_j \qquad \text{for } 1 \le j < d \tag{3.62}$$
$$f(dn + j) = cf(n) + \beta_j \qquad \text{for } 0 \le j < d \text{ and } n \ge 1 \tag{3.63}$$

Then $f((b_m b_{m-1} \ldots b_1 b_0)_d) = (\alpha_{b_m} \beta_{b_{m-1}} \beta_{b_{m-2}} \ldots \beta_{b_1} \beta_{b_0})_c$ (where $(b_m b_{m-1} \ldots b_1 b_0)_d$ is the radix-d representation of n).

Applied to your problem, we have $d = 3$, $\alpha_j = 4$, $c = 3$, and $\beta_j = 5$.
Thus, your solution is:

$$f((b_m b_{m-1} \ldots b_1 b_0)_d) = (\alpha_{b_m} \beta_{b_{m-1}} \beta_{b_{m-2}} \ldots \beta_{b_1} \beta_{b_0})_c \tag{3.64}$$
$$= \underbrace{(4\ 5\ 5 \ldots 5\ 5)_3}_{m+1 \text{ digits}} \tag{3.65}$$
$$= \underbrace{(5\ 5\ 5 \ldots 5\ 5)_3}_{m+1 \text{ digits}} - 3^m \tag{3.66}$$
$$= 5 \cdot \underbrace{(1\ 1 \ldots 1\ 1)_3}_{m+1 \text{ digits}} - 3^m \tag{3.67}$$
$$= 5\left(\frac{3^{m+1} - 1}{2}\right) - 3^m \tag{3.68}$$

(Where $m + 1$ is the number of digits in the base-3 representation of n.)

Question Score: 0 Answer Score: 0
Question ID: 888687

General Solution for a non-Homogeneous Recurrence

Question

What is the general solution to the recurrence:

$$x(n+2) = x(n+1) + x(n) + n - 1$$

where $n \geq 1$ with $x(1) = 0, x(2) = 1$?

It's a question on a practice exam I'm reviewing and I'm not quite sure why my solution is incorrect.

So I went through the characteristic equation and found the constants, and the solution to the homogeneous part of this question is:

$$x(n) = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{\sqrt{5}}$$

This is the Fibonacci expression, right? So, what do I do with the non-homogeneous part of the expression? My professor mentioned something about solving them separately but I'm not sure what to do with it now.

Answer

Method of Annihilators Approach

Let E denote the "shift" operator, and let $\langle x_n \rangle$ denote the sequence $x_1, x_2, x_3, \ldots$

Then our recurrence can be re-written as:

$$x_{n+2} - x_{n+1} - x_n = n - 1 \iff (E^2 - E - 1)\langle x_n \rangle = \langle n - 1 \rangle$$

The annihilator for $n-1$ is $(E-1)^2$, we have:

$$(E-1)^2(E^2 - E - 1)\langle x_n \rangle = (E-1)^2 \langle n - 1 \rangle = \langle 0 \rangle$$

So, we've now converted our problem from a non-homogenous problem to a homogeneous problem; the only difficulty is that we have also changed the recurrence from a 2nd order recurrence to a 4th order recurrence. Thus, we need to compute two more initial conditions:

$$x_2 = 1 \tag{3.69}$$
$$x_3 = 3 \tag{3.70}$$

We can easily move from the annihilator operator form to the characteristic of this fourth-order recurrence:

$$(r-1)^2(r^2 - r - 1) \implies r = 1, 1, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$$

Thus, our general solution has the form:

$$x_n = A + Bn + C\left(\frac{1+\sqrt{5}}{2}\right)^n + D\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Using our four initial conditions, we find:

$$x_1 = 0 = A + B + \left(\frac{1+\sqrt{5}}{2}\right) C + \left(\frac{1-\sqrt{5}}{2}\right) D \tag{3.71}$$

$$x_2 = 1 = A + 2B + \left(\frac{1+\sqrt{5}}{2}\right)^2 C + \left(\frac{1-\sqrt{5}}{2}\right)^2 D \tag{3.72}$$

$$x_3 = 1 = A + 3B + \left(\frac{1+\sqrt{5}}{2}\right)^3 C + \left(\frac{1-\sqrt{5}}{2}\right)^3 D \tag{3.73}$$

$$x_4 = 3 = A + 4B + \left(\frac{1+\sqrt{5}}{2}\right)^4 C + \left(\frac{1-\sqrt{5}}{2}\right)^4 D \tag{3.74}$$

This is a system of linear equations; using any method, we arrive at the values:

$$A = 0, B = -1, C = 1, D = 1$$

Thus, the explicit solution is:

$$x_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n - n$$

Generating Function Approach

The more ways to solve, the merrier! The below recurrence is equivalent to the recurrence in the OP, but has been extended to allow $n = 0$:

$$a_{n+2} = a_{n+1} + a_n + n - 1 \tag{$\star$}$$

for $n \geq 2$, and $a_0 = 2, a_1 = 0$.

Define $A(x) = \sum_{n=0}^{\infty} a_n x^n$. Then, multiplying ($\star$) by x^n and summing over n, we have: (note: all sums are to be taken as $\sum_{n=0}^{\infty}$, but I'm leaving off the limits for ease of typing.)

$$\sum_n a_{n+2} x^n = \sum_n a_{n+1} x^n + \sum_n a_n x^n + \sum_n (n-1) x^n \tag{$\star\star$}$$

We can rewrite ($\star\star$) as:

$$\frac{A(x) - 2}{x^2} - \frac{A(x) - 2}{x} - A(x) = \sum_n (n-1) x^n = \frac{2x - 1}{(1-x)^2}$$

Rearranging the above, we have:

$$A(x) \frac{1 - x - x^2}{x^2} = \frac{2 - 2x}{x^2} + \frac{2x - 1}{(1-x)^2}$$

Thus,

$$A(x) = \frac{2(1-x)}{1-x-x^2} + \frac{2x^3 - x^2}{(1-x-x^2)(1-x)^2} \tag{3.75}$$

$$= \frac{5x^2 - 6x + 2}{(1-x-x^2)(1-x)^2} \tag{3.76}$$

$$= \frac{x-2}{x^2+x-1} - \frac{1}{(x-1)^2} - \frac{1}{x-1} \tag{3.77}$$

The Taylor Series for each of the above can be found (albeit the first term takes some work), then the coefficient of x^n is the nth term.

For what it's worth, the first several values of the sequence are:

n	$T(n)$
1	0
2	1
3	1
4	3
5	6
6	12
7	22
8	39
9	67

Question Score: 2 Answer Score: 0
Question ID: 1064364

Difference Equations

Question

Toms old car has a major oil leak, losing 25% of the oil in the engine every week. Tom adds a quart of oil weekly. The capacity of the engine is 6 quarts of oil. In the long run, what will the oil level (in quarts) be at the end of every week before each quart of oil is added?

Answer

Let's write out a few test cases and see if we can find a pattern:

At the end of week:	Capacity (quarts):	After filling (quarts):
1	4.5	5.5
2	4.125	5.125
3	3.84375	4.84375

Well, that wasn't too useful for pattern finding, but we can at least use it to check our work...

Let a_i represent the amount of oil in the tank at then end of a given week i (before filling the tank):

$$a_n = (a_{n-1} + 1) \cdot \frac{3}{4}$$

for $n >= 0$ where $a_0 = 5$

Expanding:

$$a_{n+2} = (((a_n + 1) \cdot \frac{3}{4}) + 1) \cdot \frac{3}{4}$$

$$a_{n+2} = (((a_n + 1) \cdot \frac{3^2}{4^2}) + \frac{3}{4})$$

$$a_{n+2} = (((\frac{3^2}{4^2} \cdot a_n + \frac{3^2}{4^2})) + \frac{3}{4})$$

I will generalize this to:

$$a_n = \left(\frac{3}{4}\right)^n \cdot a_0 + \sum_{k=1}^{n} \frac{3^k}{4^k}$$

By geometric series:

$$a_n = \left(\frac{3}{4}\right)^n \cdot a_0 + \left(\frac{3}{4} \cdot \frac{1 - \left(\frac{3}{4}\right)^n}{1 - \left(\frac{3}{4}\right)}\right)$$

$$a_n = \left(\frac{3}{4}\right)^n \cdot a_0 + \left(3\frac{2^{2n} - 3^n}{2^{2n}}\right)$$

Computing a_3 yields 3.84375, thus I assume the formula is correct.

EDIT: I now see you want what happens in the "long term." Taking the limit of the series as $n \to \infty$ evaluates to 3 quarts.

Question Score: 1 Answer Score: 1
Question ID: 253053

Solve the recurrence $T(n) = 2T(n-1) + n$

Question

Solve the recurrence $T(n) = 2T(n-1) + n$ where $T(1) = 1$ and $n \geq 2$.
The final answer is $2^{n+1} - n - 2$
Can anyone arrive at the solution?

Answer

Let's use the **method of annihilators** to turn this into a third-order, homogeneous recurrence, and then solve with a characteristic equation. First, we write the recurrence so n is the least index:

$$T(n) - 2T(n-1) = n \implies T(n+1) - 2T(n) = n+1$$

Then, we rewrite the recurrence in terms of the shift operator E:

$$(E-2)T(n) = n+1$$

Applying the $(E-1)$ operator to both sides of the equation, we have:

$$(E-1)^2(E-2)T(n) = (E-1)^2(n+1)$$

Now, since $(E-1)^2(n+1) = 0$ (it annihilates that term), we have:

$$(E-1)^2(E-2)T(n) = 0$$

The characteristic equation is, then $(r-1)^2(r-2) = 0$, so our roots are $r = 1$ (with multiplicity 2) and $r = 2$. So, the recurrence has the form:

$$\begin{aligned} T(n) &= c_1 \cdot 2^n + c_2 \cdot n1^n + c_3 \cdot 1^n \\ &= c_1 \cdot 2^n + nc_2 + c_3 \end{aligned}$$

Using the recurrence relation, we compute $T(1) = 1, T(2) = 4, T(3) = 11$ and can now solve for (c_1, c_2, c_3), which gives the solution $(2, -1, -2)$.

Question Score: 2 Answer Score: 0
Question ID: 239974

Chapter 4

Discrete Math

4.1 Logic

Biconditional Statement

Question

Upon reading my textbook it gives a definition for a biconditional statement as the following:

> Given statement variables p and q, the **biconditional of** p **and** q is "p if, and only if, q and is denoted $p \leftrightarrow q$.

It then mentions that:

> It is true if both p and q have the same truth values and is false if p and q have opposite truth values. The words...

My question is, from the second part of the definition, is it talking about if p is equal to *true* and q is equal to *true* then the outcome will be *true*? Likewise if both are *false*, the outcome will be *true*? Also, if that is the case, is it saying that the rest, i.e. p being *true* or *false* and q being its opposite, the outcome will always be *false*?

Answer

Yes, your interpretation is correct. Here is the truth table for the biconditional:

p	q	$p \iff q$
T	T	T
T	F	F
F	T	F
F	F	T

Question Score: 0 Answer Score: 1
Question ID: 643356

Express the statements using quantifiers example

Question

I'm having a little trouble understanding quantifiers and therefore doubting all my homework answers. Since there is no where to check if the answers are correct, I'm very very worried I am just practicing incorrectly. So I've set up two examples with what I think the answers are.

It would be brilliant if you could confirm if I am correct or not so I could use these answers as a base to check my other answers. If I am incorrect, it would be awesome if you could point me the right direction!

Express the statements using quantifiers. *note: $\neg$ = negation.

a) Everybody knows everybody.

my answer: $\forall x E(x)$, E(x) = knows everybody.

b) Somebody knows everybody.

my answer: $\exists x E(x)$

c) There is somebody whom no one knows.

my answer: $\neg\ \exists x E(x)$

Answer

It is probably more in the intent of the exercises to let $R(x, y)$ be "x knows y," and to use double-quantifiers... Also, you should technically have a domain (e.g. all people)

But, your answers for `a` and `b` are correct (`c` is wrong, see comments), given your definition of E and assuming you aren't required to list the domain.

For an example of double quantifiers: Let $R(x, y)$ be "x knows y," and let D be the set of all people. Then:

$$\text{"Everybody knows somebody"} \iff \forall x \in D, \exists y \in D \text{ such that } R(x, y)$$

Question Score: 2 Answer Score: 1
Question ID: 512634

Negating Quantified statements

Question

The problem I am working on is:

> Express each of these statements using quantifiers. Then form the negation of the statement, so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the phrase "It is not the case that.")
>
> 1. Some old dogs can learn new tricks.
> 2. No rabbit knows calculus.
> 3. Every bird can fly.
> 4. There is no dog that can talk.
> 5. There is no one in this class who knows French and Russian.

I am having trouble with only two parts–namely, d) and e)

For d): $P(x) = x$ cannot talk.

$\exists x P(x)$ Negating this, $\neg \exists x P(x) \rightarrow \forall x \neg P(x)$

This would read in English, "Every dog can talk". However, the answer is, "There is a dog that talks."

For e): $F(x) = x$ doesn't know French. $R(x) = x$ doesn't know Russian.

$\forall x (F(x) \wedge R(x)$ Negating this, $\neg \forall x (F(x) \wedge R(x) \rightarrow \exists x (\neg F(x) \vee \neg R(x))$ Translating this back into English: "There is a person in class that knows French or Russian."

However the answer is, "There is someone in this class who knows French and Russian"

Answer

Problem D

"There does not exist a dog that can talk."

Now, $P(x) =$ x can talk, your domain, D, is all dogs. Thus, the statement becomes:

$$\neg \exists x \in D \text{ such that } P(x)$$

I'm pretty sure you can negate the above.

Another approach is to read the statement as "All dogs cannot talk." Then, $Q(x) =$ "x cannot talk," and your domain, D, is all dogs. Thus, the statement becomes:

$$\forall x \in D, Q(x)$$

Problem E

This is similar. Let me know if you need help with this one, and I'll edit my answer to include it.

Question Score: 1 Answer Score: 0
Question ID: 298889

Quantifier Notation

Question

What's the difference between $\forall x \exists y$ and $\exists y \forall x$? I don't believe they mean the same thing even though the quantifiers are attached to the same variable, but I'm having a hard time understanding the difference. Any examples to make the distinction clear would be appreciated.

Answer

One reads "For every x, there exists a y...", and the other says "There exists a y, such that for every x..."

An example of the difference can be found by making the (totally non-mathy) statement:

$$\forall x\ \exists y \text{ s.t. } x \text{ loves } y$$

That is, everybody loves at least one other person.

On the other hand:

$$\exists y\ \forall x \text{ s.t. } x \text{ loves } y$$

That is, there is a person that everyone loves.

Does this make the difference a bit more clear?

Question Score: 12 Answer Score: 8
Question ID: 340196

How can I use a truth table to show that this is a tautology?

Question

How can I show that this is a tautology by using a truth table? $(pq)(pr) \to (qr)$

I know how to do it by logical equivalences, but now I have to use a truth table. Never done it before so I dont even know where to start.

Answer

The idea is that you create a truth table for the whole statement. If the "output" column is only "true" values, then it's a tautology (by definition). It helps to build this truth table slowly, adding only one binary operation at a time:

p	q	r	$\neg p$	$(p \vee q)$	$(\neg p \vee q)$	$(p \vee q) \wedge (\neg p \vee q)$	$(q \vee r)$	$(p \vee q) \wedge (\neg p \vee q) \to (q \vee r)$
T	T	T	F	T	T	T	T	T
T	T	F	F	T	T	T	T	T
T	F	T	F	T	F	F	T	T
T	F	F	F	T	F	F	F	T
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
F	F	F	T	F	T	F	F	T

I've left out some rows so there's still some fun for you to have... ;)

Question Score: 1 Answer Score: 5
Question ID: 379680

De-Morgan's theorem for 3 variables?

Question

The most relative that I found on Google for *de morgan's 3 variable* was: (ABC)' = A' + B' + C'.

I didn't find the answer for my question, therefore I'll ask here:

What is De-Morgan's theorem for $(A + B + C)'$?

Answer

This is one instance where introducing another variable provides some insight. Let $D = B \vee C$.

Then, we have:

$$\begin{aligned}
\neg(A \vee B \vee C) &= \neg(A \vee D) \quad (4.1)\\
&= \neg A \wedge \neg D \quad (4.2)\\
&= \neg A \wedge \neg(B \vee C) \quad (4.3)\\
&= \neg A \wedge \neg B \wedge \neg C \quad (4.4)
\end{aligned}$$

Thus:

$$\neg(A \vee B \vee C) = \neg A \wedge \neg B \wedge \neg C$$

The idea is effectively the same for even more terms. Thus we can have:

$$\neg(P_1 \vee P_2 \vee \cdots \vee P_n) = \neg P_1 \wedge \neg P_2 \wedge \cdots \wedge \neg P_n$$

...and...

$$\neg(P_1 \wedge P_2 \wedge \cdots \wedge P_n) = \neg P_1 \vee \neg P_2 \vee \cdots \vee \neg P_n$$

(Note: I'm more familiar with this notation for logic, so I'm using it. $\vee$ is or, $\wedge$ is and, and $\neg$ is not.)

Question Score: 6 Answer Score: 5
Question ID: 320678

Clarification on the definition of logical conjunction

Question

First of all, I have never studied Logic seriously before. I am reading this article on Wikipedia. The definition is the following:

> Logical conjunction is an operation on two logical values, typically the values of two propositions, that produces a value of true if and only if both of its operands are true.

I would like to know the motivation for this definition. For example, I do not understand why if A is false and B is true, then $A \wedge B$ is false.

Answer

Logical conjunction is more commonly (in some circles) referred to as "and." That is, $A \wedge B$ is true iff A and B are true. "And" implies *both* (not only one) are true.

Question Score: 2 Answer Score: 0
Question ID: 612305

Boolean Algebra (Help Needed)

Question

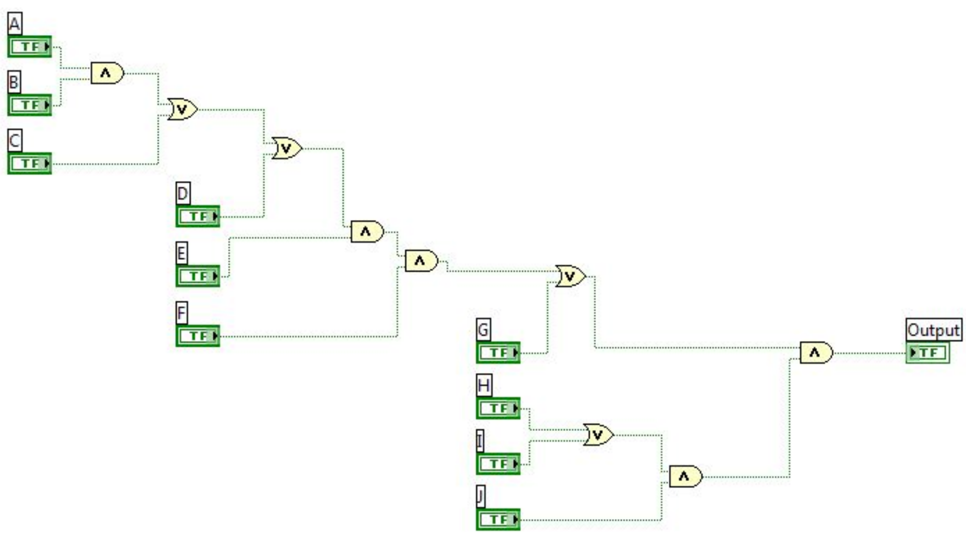
How would I draw the gate-level logic circuit of the following Boolean expression?

$$(((A \wedge B \vee C) \vee D \wedge E \wedge F) \vee G \wedge (H \vee I \wedge J)).$$

Then how would I implement this Boolean expression using NAND gates only?

Answer

Here's the non-simplified circuit (using order of operations from left-to-right unless otherwise noted by parenthesis):



Ok. So, to "simplify" to NAND gates, I'm just going to brute-force it using the conversions from this page: http://en.wikipedia.org/wiki/Logical NAND

(Good grief! This took longer than I thought it would. I can't imagine what this problem is good for other than punishing helpless students...)

Actually, what I'm going to do, is simplify the expression just to $\uparrow$ (NANDs) and $\neg$ (not), as $\neg P = P \uparrow P$, and is easy to do in the circuit.

The (Starting) expression:

$$((A \wedge B \vee C) \vee D \wedge E \wedge F) \vee G \wedge (H \vee I \wedge J)$$

Part 1

$$(A \wedge B \vee C)$$

$$(\neg[A \uparrow B] \vee C) \equiv \neg([A \uparrow B] \wedge \neg C) \equiv [A \uparrow B] \uparrow \neg C$$

Adding in the next term (D):

$$\begin{aligned}
([A \uparrow B] \uparrow \neg C) \vee D &\equiv \neg\left([A \uparrow B] \uparrow \neg C\right) \wedge \neg D && (4.5)\\
&\equiv ([A \uparrow B] \wedge \neg C) \wedge \neg D && (4.6)\\
&\equiv ([A \uparrow B]) \wedge (\neg C \wedge \neg D) && (4.7)
\end{aligned}$$

Now add in the next two terms (E, F):

$$\begin{aligned}
([A \uparrow B]) \wedge (\neg C \wedge \neg D) \wedge E \wedge F &\equiv ([A \uparrow B]) \wedge (\neg C \wedge \neg D) \wedge \neg[E \uparrow F] && (4.8)\\
&\equiv ([A \uparrow B]) \wedge \neg(\neg C \uparrow \neg D) \wedge \neg[E \uparrow F] && (4.9)\\
&\equiv ([A \uparrow B]) \wedge \neg\big(\neg(\neg C \uparrow \neg D) \uparrow \neg[E \uparrow F]\big) && (4.10)\\
&\equiv \neg\Big(([A \uparrow B]) \uparrow \neg\big(\neg(\neg C \uparrow \neg D) \uparrow \neg[E \uparrow F]\big)\Big) && (4.11)
\end{aligned}$$

Let's call that big old thing P, so I don't have to keep copy/pasting it...

Part 2

Now let's look at the last expression in parentheses:

$$\begin{aligned}
H \vee I \wedge J &\equiv \neg(\neg H \wedge \neg I) \wedge J && (4.12)\\
&\equiv (\neg H \uparrow \neg I) \wedge J && (4.13)\\
&\equiv \neg\Big((\neg H \uparrow \neg I) \uparrow J\Big) && (4.14)
\end{aligned}$$

Let's call that Q.

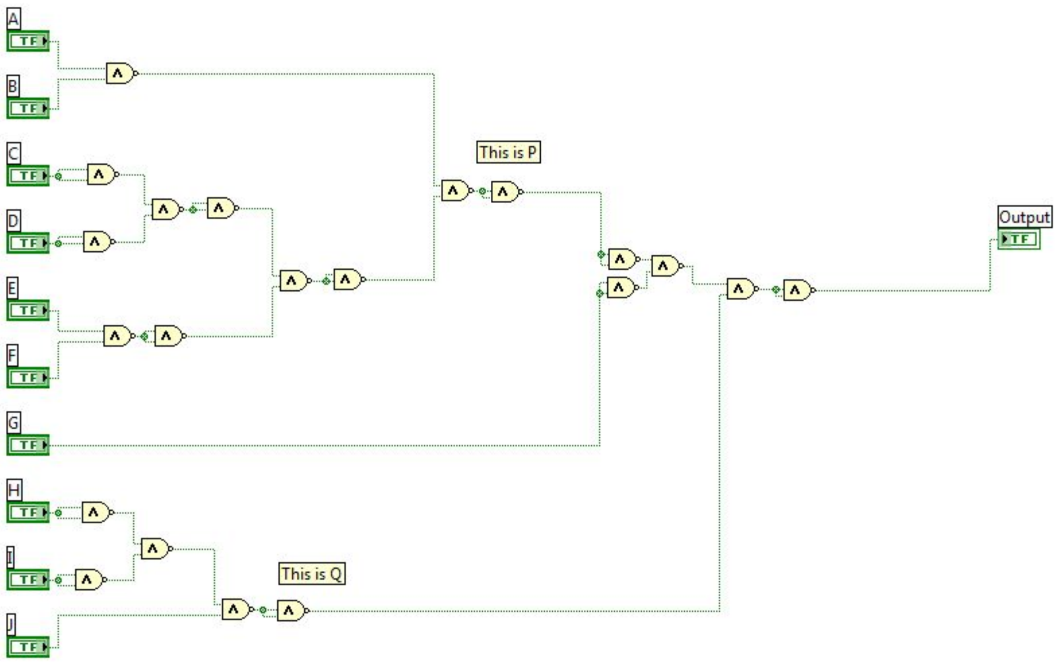
Part 3

Let's now put it together:

$$\begin{aligned}
P \vee G \wedge Q &\equiv (\neg P \uparrow \neg G) \wedge Q && (4.15)\\
&\equiv \neg\Big((\neg P \uparrow \neg G) \uparrow Q\Big) && (4.16)
\end{aligned}$$

The circuit

Oh–and did I mention we're not done? Here's the circuit:



Is this totally simplified? No–some double negations can be taken out. But, it is valid per the problem statement.

FYI: I created the graphics in LabVIEW, and tested both expressions output (to catch careless errors), and they match.

Question Score: -7 Answer Score: 9
Question ID: 309722

Knights and Knaves

Question

A very special island is inhabited only by knights and knaves. Knights always tell the truth, and knaves always lie. You meet four inhabitants: Bozo, Marge, Bart and Zed.

- Bozo says, "Bart and Zed are both knights".
- Marge tells you that both Bart is a knight and Zed is a knave
- Bart tells you, "Neither Marge nor Zed are knaves".
- Zed says that neither Bozo nor Marge are knaves.

Can you determine who is a knight and who is a knave?
I am having extreme difficulty with this can anyone help me? I assume is starts like this.
So
$Bo \equiv (Ba \wedge Ze)$
$Ma \equiv (Ba \wedge \neg Ze)$
$Ba \equiv (Ma \vee Ze)$
$Ze \equiv (Bo \vee Ma)$
Where
Bo= Bozo is a knight
Ma= Marge is a knight
Ba= Bart is a knight
Ze= Zed is a knight

Answer

> Bozo says, "Bart and Zed are both knights."
>
> Marge tells you that both Bart is a knight and Zed is a knave.
>
> Bart tells you, "Neither Marge nor Zed are knaves."
>
> Zed says that neither Bozo nor Marge are knaves.

Marge and Bart contradict each other. If Marge is telling the truth, Zed lies, and Bart tells the truth. Bart, however, Zed and Marge tell the truth. Thus, Marge and Bart are Knaves. Therefore, Zed is a knave (Because Marge said he was a knight). Bozo says Bart and Zed are knights, which has been established false. Thus **all are knaves**.

The way I typically solve these problems is to look for contradictions.

Question Score: 1 Answer Score: 0
Question ID: 250429

Logic Negation Symbols

Question

This is a rather simple question but I can't find an exact answer on it. In examples, I've seen $\sim$ and $\neg$. These fall under negation. If they both fall under negation, does that mean that I can substitute them for each other?

- p: Sam goes to work.
- q: Max goes to work.

So, if I were to write: Neither Sam goes to work nor Max goes to work, could I write it as $\neg(p \vee q)$ and also $\sim(p \vee q)$? Or, if I had It is not the case that if Sam goes to work then Max goes to work, could I write $\sim(\sim p \vee q)$ or $\neg(\neg p \vee q)$?

Answer

Yes, these symbols both represent the same thing, so all of your examples are valid.

However, I suggest using only one at a time for clarity's sake. That is, don't write: $\sim(p \vee q)$. Mixing the symbols may cause people to think that you're using one to represent something other than negation.

Question Score: 2 Answer Score: 4
Question ID: 385753

Why does one counterexample disprove a conjecture?

Question

Can't a conjecture be correct about most solutions except maybe a family of solutions? For example, a few centuries ago it was widely believed that $2^{2^n}+1$ is a prime number for any n . For $n=0$ we get 3 , for $n=1$ we get 5 , for $n=2$ we get 17 , for $n=3$ we get 257 , but for $n=4$ it was too difficult to find if this was a prime, until Euler was able to find a factor of it. It seems like this conjecture stopped after that. But what if this conjecture isn't true only when n satisfies a certain equation, or when n is a power of $2 \geq 4$, or something like that? Did anybody bother to check? I am not asking about this conjecture specifically, but as to why we consider one counterexample as proof that a conjecture is totally wrong.

P.S. Andre Nicolas pointed out that Euler found a factor when $n=5$, not 4 .

Answer

This is because, in general, a conjecture is typically worded "Such-and-such is true *for all* values of [some variable]." So, a single counter-example disproves the "for all" part of a conjecture.

However, if someone refined the conjecture to "Such-and-such is true *for all* values of [some variable] *except* those of the form [something]." Then, this revised conjecture must be examined again and then can be shown true or false (or undecidable–I think).

For many problems, finding one counter-example makes the conjecture not interesting anymore; for others, it is worthwhile to check the revised conjecture. It just depends on the problem.

Question Score: 12 Answer Score: 27
Question ID: 440859

Discrete Mathematics - Show that a conditional statement is a tautology.

Question

I am trying to show that the conditional statement:

$$[\sim p \wedge (p \vee q)] \rightarrow q$$

is a tautology without using truth tables. Could someone help me understand how to do this?

Answer

Distributing:

$$[(\neg p \wedge p) \vee (\neg p \wedge q)] \implies q$$

$$[c \vee (\neg p \wedge q)] \implies q$$

$$(\neg p \wedge q) \implies q$$

Now, we can convert the implication to disjunction/negation:

$$\neg(\neg p \wedge q) \vee q$$

Using DeMorgan's:

$$(p \vee \neg q) \vee q$$

Can you take it from here?

Question Score: 0 Answer Score: 1
Question ID: 655264

Boolean simplification, 5 variables

Question

I'm currently learning for my maths exam, and in the part about boolean algebra I came across an exercise that I can't seem to solve. I probably only need the first few steps to get started.

$$(xyz + uv)(x + \overline{y} + \overline{z} + uv)$$

Usually, if I get into trouble, I can fall back to a truth table or VK-diagram, but that's just too much work for 5 variables.

Thanks in advance!

Answer

Let's write it like this:

$$(xyz + uv)((x + \bar{y} + \bar{z}) + uv) \equiv ((xyz)(x + \bar{y} + \bar{z}) + (xyz)(uv)) + (uv(x + \bar{y} + \bar{z}) + uvuv) \tag{4.17}$$

$$\equiv ((xyz)(x + \bar{y} + \bar{z}) + xyzuv) + (uv(x + \bar{y} + \bar{z}) + uv) \tag{4.18}$$

$$\equiv ((xyz)(x + \bar{y} + \bar{z}) + xyzuv) + (uv) \tag{4.19}$$

$$\equiv ((xyzx + xyz\bar{y} + xyz\bar{z}) + xyzuv) + (uv) \tag{4.20}$$

$$\equiv ((xyz + 0 + 0) + xyzuv) + (uv) \tag{4.21}$$

$$\equiv (xyz + xyzuv) + (uv) \tag{4.22}$$

$$\equiv (xyz) + (uv) \tag{4.23}$$

From line 1 to line 2, we distributed. From 2 to 3, we noted that $ab + a \equiv a$. From 3 to 4, we distributed again. From 4 to 5, we noted that $a\bar{a} \equiv 0$ (contradiction), which we remove from 5 to 6. 6 to 7 we note $ab + a \equiv a$.

Question Score: 1 Answer Score: 1
Question ID: 412941

Functions for boolean operators, that return 1 or 0

Question

Are there any purely mathematical expressions that are equivalent to boolean operators and return 1 or 0?

For example: $a > b$

Is there any $f(a, b)$ for which if $a > b$, $f(a, b) = 1$ and if $a < b$, $f(a, b) = 0$? Can you express such a function with a simple expression and with no branches/multiple definitions ?

Answer

You could use an Iverson Bracket:

$$f(x, y) = [x > y]$$

Question Score: 1 Answer Score: 2
Question ID: 861619

4.2 Set Theory

Does a set include its elements or just its subsets?

Question

I am a little confused about the basic definition of inclusion.

I understand that, for example, $\{4\} \subset \{4\}$.

I also understand that $4 \in \{4\}$, and that it is false to say that $\{4\} \in \{4\}$.

However, is it possible to say that $4 \subset \{4\}$?

Answer

There are two symbols, here, and I think you may be getting them confused. I would suggest to *not* use the word "inclusion" (at least, not all the time) because that a different meaning in English than in math.

The $\in$ symbol is used to designate if something is *inside of* a set. That is, $4 \in \{4\}$, and $\{4\} \in \{\{4\}\}$, but $\{4\} \notin \{4\}$.

The $\subset$ symbol is used to show whether the elements of one set are inside another set. That is, if $A \subset B$, then $a \in B$ for every $a \in A$. Another way of looking at it: $\subset$ always has a set on both sides.

Question Score: 3 Answer Score: 1
Question ID: 633799

Prove by induction that $P_n < 2^{2^n}$, being P_n the n^{th} prime number

Question

Prove by induction that $P_n < 2^{2^n}$, been P_n the n^{th} prime number

The prime numberss set is defined as $\mathbb{P} := \{2, 3, 5, 7, 11, \ldots\}$

Let $P(n)$ be the proposition we want to prov, ie: $P(n) := P_n < 2^{2^n}$

For $P(1)$ we have the first prime number, ie, 2 so $2 < 2^{2^1} \implies 2 < 4$, so $P(1)$ is true.

For $P(2)$ we have the second prime number, ie, 3 so $2 < 2^{2^2} \implies 3 < 16$, so $P(2)$ is true.

Inductive Hypothesis: Let $n = k$ and we assume that

$$P(k) := P_n < 2^{2^n}$$

is true.

I dont know how to do the inductive step, how it should be done?

Answer

Hint: Use Bertrand's postulate. Applied to this situation, the postulate says that there is always a prime between P_n and $2P_n$. (Thus, you just have to show that $2^{2^{n+1}} > 2P_n$)

Edit: If need be, I can provide more detail to the proof. Just leave a comment.

Question Score: 2 Answer Score: 2
Question ID: 633258

Subtracting a set from a set of set

Question

Let say that I have a set of set X = a,b, a,c and I want to remove the element a,b from X. What is the proper way to write this subtraction operation.

X a,b

or

X a,b

I think that the first one is incorrect and that I should use the second one. Is this correct? Or is there other way to write this?

Answer

The second one is correct.

Note that $\{a, b\}$ is **not** a subset of $S = \{\{a, b\}, \{a, c\}\}$, but rather an **element** of S.

The set difference operator $(A \setminus B)$ is defined between two **sets**, not a set and an element thereof. Thus, you want to use $\{\{a, b\}, \{a, c\}\} \setminus \{\{a, b\}\}$.

Question Score: 1 Answer Score: 2

Question ID: 881787

Please check these proofs for sets

Question

I would appreciate the insight again for a couple of proofs since I'm learning. These are homework problems in so much as they are problems from the textbook. They are not required by my professor. I'm doing a little extra through spring break.

The objective was to prove the statement or provide counter examples. I'd like to have my work critiqued since I've got to refine my abilities for proofs.

The following applies to both problems: Let A, B, C be sets.

Problem 1

Prove: $A \oplus B = A \oplus C \Rightarrow B = C$.

My objective is to is to show that $B \subseteq C$ and $C \subseteq B$.

Proof:

Case 1:

$$\text{let}\, x \in B \mid x \in A \oplus B \tag{4.24}$$
$$\text{so}, x \in B \,\text{and}\, x \notin A, \text{ by definition} \tag{4.25}$$
$$\text{since}, A \oplus B = A \oplus C, x \in C \,\square \tag{4.26}$$

Case 2:

$$\text{let}\, x \in C \mid x \in A \oplus C \tag{4.27}$$
$$\text{so}, x \in C \,\text{and}\, x \notin A, \text{ by definition} \tag{4.28}$$
$$\text{since}, A \oplus C = A \oplus B, x \in B \,\square \tag{4.29}$$

Problem 2

Prove: $A \times B = A \times C \Rightarrow B = C$.

A similar problem and I must show that $B \subseteq C$ and $C \subseteq B$.

Proof:

Case 1:

$$\text{let}(a, b) \in A \times B \,\text{and}\, (a, c) \in A \times C \tag{4.30}$$
$$\text{since}\, A \times B = A \times C, \forall (a, b), (a, b) = (a, c) \tag{4.31}$$
$$\text{so}, a = a \,\text{and}\, b = c \tag{4.32}$$
$$\text{thus}\, \forall b \in B, b \in C \,\square \tag{4.33}$$

Case 2:

$$\text{let}(a, c) \in A \times C \,\text{and}\, (a, b) \in A \times B \tag{4.34}$$
$$\text{since}\, A \times C = A \times B, \forall (a, c), (a, c) = (a, b) \tag{4.35}$$
$$\text{so}, a = a \,\text{and}\, c = b \tag{4.36}$$
$$\text{thus}\, \forall c \in C, c \in B \,\square \tag{4.37}$$

Answer

As pointed out in the comments, your solution to Problem 2 is correct.

I believe that your solution to Problem 1 is correct, but (as Pedro Tamaroff pointed out) it's not the "easiest" proof. A more straightforward proof is to say:

Let A, B, C be sets, and $A \oplus B = A \oplus C$. Adding A on the left gives:

$$A \oplus (A \oplus B) = A \oplus (A \oplus C)$$

$$(A \oplus A) \oplus B = (A \oplus A) \oplus C \qquad (\oplus \text{ is associative})$$

$$\emptyset \oplus B = \emptyset \oplus C \qquad (A \oplus A = \emptyset \text{ for all } A)$$

$$B = C \qquad (\emptyset \oplus A = A \oplus \emptyset = A \text{ for all } A)$$

Question Score: 3 Answer Score: 1
Question ID: 731910

Proving by contradiction that if $a \in \mathbb{Q}, b \in \mathbb{R} \setminus \mathbb{Q}$ then $a + b \in \mathbb{R} \setminus \mathbb{Q}$

Question

I'm trying to prove by contradiction that if $a \in \mathbb{Q}, b \in \mathbb{R} \setminus \mathbb{Q}$ then $a + b \in \mathbb{R} \setminus \mathbb{Q}$, I already proved it with contra position and a direct proof seems impossible.

Suppose $a \in \mathbb{Q}, b \in \mathbb{R} \setminus \mathbb{Q}$ show that $a + b \in \mathbb{Q}$.

So $a = \frac{p}{q}, p, q \in \mathbb{Z}, q \neq 0$, so we have $\frac{p}{q} + b = \frac{p+bq}{q}$ and now I'm basically back to square one since I can't assume that a sum and product of integers and irrationals isn't an integer.

Answer

Your problem is a misunderstanding of the technique of proof by contradiction.

When performing a proof by contradiction, you make your usual assumptions (e.g., $a \in \mathbb{Q}, b \in \mathbb{R} \setminus \mathbb{Q}$) and you **assume the logical negation** of what you're trying to prove (e.g., $a + b \in \mathbb{Q}$).

Per your description, it looks like you made your usual assumptions, and then *tried to prove* the logical negation of what you were trying to prove; that is impossible to do.

Question Score: 1 Answer Score: 3
Question ID: 1122404

How can I model multiplication of two sets of integers

Question

First a disclaimer, I am a physician and nowhere near a mathematician. I am struggling with a problem in which I want to model a set of equations using a discrete system.

For example I want to calculate a list of totals when a prescription frequency can be $\{2, 4, 6\}$ and the amount can be $\{5, 10, 15\}$. This gives a set of possible totals of $\{10, 20, 30, 40, 60, 90\}$.

My first, naive, solution was to model the set like:

$$\text{Factor } n = a + bx \wedge x = \{x \in \mathbb{N} : x \leq (c - a) \div b \wedge (a + bx) \mid d\}$$

Where a is the minimum value in the set, b is the increment of the set, c is the maximum of the set and d is a dividend, of which the set is a divisor. Thus, I can describe the frequency as: Factor $n = 2 + 2x \wedge x = \{x \in \mathbb{N} : x \leq (6 - 2) \div 2 \wedge (2 + 2x) \mid \infty\} = \{2, 4, 6\}$.

But what happens when I multiply such a set definition with another set? If there is no maximum, i.e. $c = \infty$, no problem, the resulting set is just a multiple of the multiplication, i.e. $b_{result} = b_1 \times b_2$. And $a_{result} = a_1 \times a_2$ as is b_{result}. But, obviously, this is not correct when $c \neq \infty$, or $d \neq \infty$.

There are two possible solutions, either generate the set and use that in subsequent calculations (can get messy) or come up with some sort of equation that can generate the set (the prefered strategy).

What I want to accomplish is that a set of equations like: $total = frequency \times quantity$ and $quantity = runtime \times infusionrate$ can be solved, while disallowing entries that have no solution. For example if $total = 9$ and frequency is a multiple of 1 than quantity cannot be set at 5, thus the equation with the above values would set the dividend of quantity to 9, limiting quantity to divisors of 9.

Any, help will be greatly appreciated, I have been struggling with this for years and there are no practical existing systems that help doctors with prescription calculations like this.

Answer

From my understanding, you have an input of two sets, S_1 and S_2. For example:

$$S_1 = \{2, 5, 6\}$$

$$S_2 = \{5, 10, 15\}$$

My understanding is that you want to compute a set consisting of all unique products between an element of S_1 and S_2.

Well, we typically don't consider duplicate elements in sets anyway (we use a multiset for that), so if there's a duplicate value created by our definition, we just ignore it. Thus, we can define the product $S_1 * S_2$ as follows:

$$S_1 * S_2 = \{a \cdot b : a \in S_1, b \in S_2\}$$

Is this what you're looking for?

Question Score: 3 Answer Score: 1
Question ID: 335108

4.3 Counting and Probability

Probability arrangements

Question

> A sports committee of four is to be chosen from a group of nine students, made up of three boys and six girls. Calculate the expected number of boys on the sports committee of four.

I get in order to calculate expected value you have to find the probability of there being $0, 1, 2$, or 3 boys and then compute:

$$0 \cdot P(X = 0) + 1 \cdot P(X = 1) + 2 \cdot P(X = 2) + \cdots$$

But, how do I calculate the individual probabilities?

Answer

Hint: There are $\binom{3}{k} \cdot \binom{6}{4-k}$ ways of choosing a team with k boys.
How many ways are there of choosing a team of 4 people (without restrictions)?
Can you combine the two lines above to find $P(X = n)$? (The answer should be yes.)

Question Score: 0 Answer Score: 0
Question ID: 866535

Simple probability problem

Question

A bag contains $(2m+1)$ coins. It is known that m of these coins have a *head* on both sides and the remaining coins are fair. A coin is picked up at random from the bag and tossed.

If the probability that the toss results in a head is $14/19$, then m is equal to ?

How to go about this? What Sample space to consider?

Answer

HINT: Break it into two parts–first find the probability of getting heads with a "guaranteed" coin, then the probability of getting heads with a fair coin. You can do one *or* the other, so you add those two probabilities.

Answer: (mouse over grey spots to see equations)

I'm not entirely sure about the sample space to consider, but the way I would approach this problem is as follows:

You can pick any one of the m coins that are double-headed, and be guaranteed a head. So, the probability of getting heads this way is the number of double-headed coins over the total number of coins:

$$\frac{m}{2m+1}$$

You can pick any one of the remaining $m+1$ coins, and have $50:50$ chances of getting heads or tails. So, we first find the probability of picking a fair coin, then multiply it by the chances of getting heads with that coin:

$$\frac{m+1}{2m+1}\cdot\frac{1}{2}$$

So, the overall probability of getting a head is:

$$\underbrace{\frac{m}{2m+1}}_{\text{double headed}}+\underbrace{\frac{1}{2}\cdot\frac{m+1}{2m+1}}_{\text{flip a fair head}}=\frac{3m+1}{4m+2}$$

We can tell that $\frac{14}{19}$ won't fit this for, but perhaps try a non-reduced form of the fraction. For example: $\frac{28}{38}, \frac{42}{57}, \frac{56}{76}, \ldots$

Question Score: 1 Answer Score: 5
Question ID: 436033

probability - customer who bought this also bought this

Question

How to calculate probability that a customer bought this item also bought this other item? I tried using google to no avail. It seems very simple, but I still am confused.

Answer

A simple way to do this would be to count all the people who bought product X, then count all the people who bought product X and Y. This (somewhat) tells you the probability that someone who buys X will buy Y.

However, this isn't that great unless you have a list of a *lot* of purchases.

Question Score: 0 Answer Score: 1
Question ID: 258396

Why placing people in a circle has (n-1)! distinct arrangements?

Question

Why placing people in a circle has (n-1)! distinct arrangements?

I saw in books something about a placing a 'pivot',

But As i continued reading i saw something that really confused me. Once we can distinguish any particular position in the circle, the rotational symmetry is lost and there are n! distinct arrangements. Say if one of the chairs is broken (and we choose to take note of this fact) then there are n! distinct ways n people can sit pick one person to sit in the broken chair, then a person to sit to their right, and so on.

What's actually the difference between the 'broken chair' and placing a 'pivot'? And why is 'the broken chair' has n! while the other method has (n-1)!?

Answer

When they refer to placing a pivot, they just mean picking a starting place from which to count seating arrangements. You could think of it this way: We get to pick an arbitrary place for the pivot, but we don't get to pick an arbitrary place for the broken chair.

Question Score: 2 Answer Score: 2
Question ID: 427651

Number of 6-digit passwords, starting with even or ending with odd digit

Question

My problem is

> A password consists of six digits, each in $\{0,\ldots,9\}$ How many passwords start with an even digit or end with an odd digit?

the answer is $750,000$.
I would like to know how exactly do you get $750,000$ as the answer?

Answer

I want to say firstly that this is basically the same information as presented in the answers by Jorge and julien, and is based off of Steve Kass' comment, but I wanted to say it in a slightly different way/different details, in case someone needs this sort of explanation.

First, let's see how many possible passwords there are. Well, there are 6 "slots," and there are 10 possibilities for each slot. Thus, we have $10^6 = 1000000$ possible passwords.

Second, let's look at how many passwords are ***not*** acceptable. That is, they end in an even digit, *and* they start with an odd digit. This means there are 5 possibilities for both the starting and ending digit; the first digit can be in $\{1,3,5,7,9\}$, and the last digit can be in $\{0,2,4,6,8\}$. The rest of the digits have 10 possibilities. So, the number of not acceptable passwords is:

$$5\cdot 10\cdot 10\cdot 10\cdot 10\cdot 5 = 5^2\cdot 10^4 = 250000$$

Finally, we know that the number of acceptable passwords is the number of total passwords minus the number of unacceptable passwords. So, we have:

$$\text{acceptable passwords} = 1000000 - 250000 = 750000$$

Question Score: 7 Answer Score: 3
Question ID: 324241

What is the expression for putting n indistinguishable balls into k indistinguishable cells?

Question

I'm looking for the expressions for the number of ways in which n indistinguishable balls can be placed into k indistinguishable cells, with

- No cell being empty
- Some cells being empty

I knew I've read it (in *Applied Combinatorics* by Fred Roberts), but unfortunately, I don't happen to have access to it anymore. I would be grateful if someone could help me remember it!

Answer

According to the comments, it appears that what you're discussing are partitions of an integer.

There isn't a really "nice" way of explicitly computing these, but we have published a blog post that presents a somewhat simple technique using memoization; specifically, Euler's Pentagonal Formula. This was written by Paramanand Singh and may be found here.

If you want an asymptotic approximation, there is the Hardy-Ramanujan formula:

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(2\pi\sqrt{n/6}\right)$$

Question Score: 6 Answer Score: 2
Question ID: 92594

Probability - something with a small chance of occurring, but is repeated multiple times.

Question

For example, if you have a 1.5% chance of obtaining admission to any school and you apply to 15 schools what is the chance that you'll get into a least 1 school? Is this as simple as $1.5\% \times 15 = 22.5\%$ chance of getting into one school?

Phrased another way: if an event has a 1.5% chance of occurring every time it is performed, and you perform it 15 times what is the probability that the event occurs at least once?

Answer

If an event happens with probability p, then the probability of it not happening is $1-p$. Instead of looking at the question "does the event occur *at least once?*", consider its complement. That is, "what is the probability of the event *never* occurring?"

If we repeat the event n times, the probability of the event never happening is $(1-p)^n$. Thus, the probability of it happening at least once is $1-(1-p)^n$.

Applied to your scenario, we have $p = 0.015$, $n = 15$. Thus...

The reason that we can't just multiply the probability of *one* occurrence by the number of samples is best illustrated by an example: Suppose we had $p = 1.5\%$, as in your problem. If we applied to 1000 schools, what is the probability that we get in? Saying $P[\text{I get in}] = 1000 \cdot 1.5\% = 1500\%$ doesn't make since–probabilities must be between 0 and 1!

EDIT: I wrote this up, and then saw Andr's comment on the main post. The assumption of independence is key, here. If the probability of the event changes every time you perform an experiment, the answer is a bit more involved.

Question Score: 1 Answer Score: 2
Question ID: 1236513

How many combinations are there

Question

Given I have N fields. I want to produce X possible matches with a variable M for how many of the fields do not have to equal.

e.g values and results.

$N = 4$

$M = 1$

$X = 5$ (height of the matrix with N as the length)

Considering this could be represented as a matrix with zero representing a non-required match and 1 representing a match:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Another example:

$N = 4$

$M = 2$

$X = ?$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

What is the function for X?

Answer

Let me rephrase the question.

> Given a string a of length N, how many strings b are there such that a and b differ by (at maximum) M characters?

(The numeric answer to the question above is your X.)

Note that there is 1 way for the strings to differ by zero characters. There are N ways for the strings to differ by 1 character. There are $\binom{N}{2}$ ways for the strings to differ by 2 characters. In general:

$$X = \sum_{k=0}^{M} \binom{N}{k}$$

In response to comment:

We define the binomial coefficient to be:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Thus, our expression above becomes:

$$X = \sum_{k=0}^{M} \binom{N}{k} = \sum_{k=0}^{M} \frac{N!}{k!(N-k)!}$$

Where $a!$ is the factorial of a.

Question Score: 0 Answer Score: 2
Question ID: 670484

How many subsets of S are there that contain x but do not contain y?

Question

> Let S be a set of size 37, and let x and y be two distinct elements of S. How many subsets of S are there that contain x but do not contain y?

This question is on a practice exam that I am reviewing for tomorrow. The answer key says that the answer to this question is $2^{37} - 2^{35}$.

I fail to see how this is correct as 2^{37} would be the amount of all subsets and 2^{35} is the amount of subsets containing x and y (I believe), therefore the answer would be the # of subsets that do not contain x or y, which doesn't answer the question.

I thought the answer should be $2^{36} - 2^{35}$, where 2^{36} is the number of subsets containing x, taking away the number of subsets that contain both x and y, therefore being left with the number of subsets that contain x and not y.

Can someone please confirm that I am right or explain to me why I am wrong?

Answer

You are correct; the answer should be $2^{36} - 2^{35} = 2^{35}$.

Another way of looking at it: x must be in any such subset, and y must not be. Thus, we have 35 "free" elements that could either be *in* or *out* of the subset. So, there are 2^{35} ways of forming that subset.

Question Score: 2 Answer Score: 2
Question ID: 1237647

How many 4 digit numbers can be constructed ..

Question

How many 4 digit numbers can be constructed from the numbers 1,2,3 that are odd and less than 2000.

My answer: 1(less than 2 so it's less than 2000) * 3 * 3 * 5 (odd numbers)

Why is this wrong ? : answer is 18 can someone explain why

Answer

Your process seems to be right, but I think you picked the wrong values for your odd numbers.

A four digit number to answer your problem has the following form:

$$\square \quad \square \quad \square \quad \square$$

Where each $\square$ could be a number in the set $\{1, 2, 3\}$.

However, we can further refine this: We need the number to be less than 2000, so, as you correctly deduced, the first digit must be 1:

$$1 \quad \square \quad \square \quad \square$$

We can further refine this by noting that the number must be odd. This means that the last digit must be 1 or 3. So, we can say that our number will look like:

$$1 \quad \square \quad \square \quad \{1, 3\}$$

The center two digits have no restriction, so they can be any of $\{1, 2, 3\}$. So our final number looks like:

$$1 \quad \{1, 2, 3\} \quad \{1, 2, 3\} \quad \{1, 3\}$$

To determine how many numbers fit this form, we multiply the count of possibilities for each digit. That is:

$$\underbrace{1}_{\text{first digit}} \times \underbrace{3}_{\text{second digit}} \times \underbrace{3}_{\text{third digit}} \times \underbrace{2}_{\text{fourth digit}} = 18 \text{ possibilities}$$

Question Score: 2 Answer Score: 2
Question ID: 430428

Show that $0 \leq 1 - \frac{n!}{(n-k)!n^k} \leq 1$

Question

I would like to prove that

$$0 \leq 1 - \frac{n!}{(n-k)!n^k} \leq 1$$

for $n \geq k \in \mathbb{N}$

I tried to use the fact that $n^k \geq n$ and binomial coefficient but it doesn't look good, plus I have other ideas (like Stirling) but I think it's inappropriate.

Thank you in advance

Answer

Clearly, $\frac{n!}{(n-k)!n^k}$ 0, so the upper bound follows immediately.

What remains is to show that $n! \leq (n-k)!n^k$. This is also easily seen, given that:

$$n! = \underbrace{n(n-1)(n-2)\cdots}_{k \text{ numbers}}\underbrace{(n-k)(n-k-1)\cdots 2\cdot 1}_{(n-k)!} \tag{4.38}$$

$$\leq \underbrace{n\cdot n\cdot n\cdots}_{k \text{ numbers}}\underbrace{(n-k)(n-k-1)\cdots 2\cdot 1}_{(n-k)!} \tag{4.39}$$

$$\leq n^k \cdot (n-k)! \tag{4.40}$$

Since the denominator is greater than the numerator, the fraction is less than 1. This shows the lower bound.

Question Score: 1 Answer Score: 1
Question ID: 766737

How many ways can you fill a list with n slots with m letters, $m < n$, making sure to use each one once?

Question

I've been trying to solve this problem and haven't had any luck. I also haven't been able to find anything online. Clearly there are m^n ways if we do not need to use each one once, but what restriction gets placed if we do? Some must be duplicated, but not all.

EDIT: I'm very tired and massively miswrote this question at first. I explained this in the comments to a response, but it seems to have been deleted now. Sorry to anyone who answered the first question.

Answer

Permutations are exactly what you describe. For example:

```
"GFEDCBA" is a permutation of "ABCDEFG"
"ABC"     is NOT a permutation of "ABCDEFG"
```

Thus, the number of rearrangements possible (requiring every letter to be used once) is n!

Question Score: 1 Answer Score: 0
Question ID: 317454

How do I know when to use nCr button and when to use nPr button on my calculator?

Question

How do I know when to use nCr button and when to use nPr button on my calculator?
nCr= combinations I believe NPR= permutations
Is there a general rule I can use to figure out which one to use and when?

Answer

In permutations, order counts.

So, if I wanted to know the number of ways to *arrange* a set of four unique books out of a total supply of 20 books, I'd use 20 `nPr` 4. (Order counts in arranging books.)

In combinations, order doesn't count.

So, if I wanted to know the number of ways to make a team of 4 people out of a total group of 20 people, I'd use 20 `nCr` 4. This is because order doesn't matter in choosing a team.

Question Score: 2 Answer Score: 2
Question ID: 429480

Probabilities: k out of n

Question

I have 10 numbers. each of them represents a coordinate.

I think that by combining these ten numbers, 100 points can be generated:

$$10^2 = 100$$

Then by choosing k points out of those 100, there can be:

$$\frac{100!}{k!(100-k)!}$$

different combinations of k points.

Is this right?

For the different values of k

$$1 < k < 11$$

how many combinations can be generated?

Answer

Short answer: You are correct.

Longer-ish answer (for future readers):

You have a set of ten numbers. When you "combine" them to form coordinates, there are 10 ways to choose the first coordinate and 10 ways to choose the second coordinate. Thus, there are $10 \times 10 = 100$ possible coordinates.

To select k points, we use the binomial coefficient, with $n = 100$:

$$\text{Ways for k points} = \binom{n}{k} = \binom{100}{k} = \frac{100!}{k!(100-k!)}$$

Question Score: 2 Answer Score: 2
Question ID: 429018

What is the meaning of $(2n)!$

Question

I came across something that confused me

$$(2n)! =?$$

What does this mean:

$$2!n!, \quad 2(n!)$$

or

$$(2n)! = (2n)(2n-1)(2n-2)...n...(n-1)(n-2)...1$$

Which one is right?

The exercise is to show that

$$(n+1) \Bigg| \left(\begin{array}{c} 2n \\ n \end{array} \right)$$

Then I thought of using the combination formula $\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!(n-k)!}$ to decrease my expression, but then I came across

$$(2n)!$$

Answer

You are correct:

$$(2n)! = (2n)(2n-1)(2n-2)\cdots(3)(2)(1)$$

So, for example, if $n = 3$, then $(2n)! = 720$.

Question Score: 3 Answer Score: 2

Question ID: 451044

Binomial formula for $(x+1)^{1/3}$ (related to Newton's binomial theorem)

Question

I know that

$$\sqrt{1+x} = \sum_{j=0}^{\infty} \left(\frac{(-1)^{(j-1)}}{2^{2j-1} \cdot (2j-1)} \binom{2j-1}{j} x^j \right).$$

Now, I want to evaluate $\sqrt[3]{1+x}$ but stuck at some point:
To evaluate $\sqrt[3]{1+x}$, I first tried to find what $\binom{1/3}{j}$ is

$$\binom{1/3}{j} = \frac{(1/3 \cdot (1/3-1) \cdot (1/3-2) \cdots (1/3-j+1))}{j!} = \frac{(-1)^{j-1} \cdot 1 \cdot 2 \cdot 5 \cdots (3j-4)}{3^j \cdot j!}$$

However I could not continue from here. Is there a way to write $1 \cdot 2 \cdot 5 \cdots (3j-4)$ in another form? (Or is $3 \cdot 4 \cdot 6 \cdot 7 \cdots (3j-5)$ possible?) Could you please help me?

Regards

Answer

Note that $\binom{n}{r} = \frac{n!}{r!(n-r)!}$. We also have the identity $\Gamma(n) = (n-1)!$

Thus, we can generalize the binomial coefficient as:

$$\binom{n}{r} = \frac{\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)}$$

Therefore:

$$\binom{1/3}{j} = \frac{\Gamma(4/3)}{\Gamma(j+1)\Gamma(j+2/3)}$$

Note, though, that you may run into some issues, as r (as in, $\binom{n}{r}$) should be in the range $[0, n]$.

Question Score: 9 Answer Score: 1
Question ID: 334243

Number of possible outcomes in a license plate

Question

If a license plate consists of 3 letters followed by 3 digits and having at least one digit or letter repeated .. How many outcomes are there?

26 * 26 * 10* 10 * 10 .. Is that right?

Answer

That is incorrect.

First, let's find the number of license tags, total, that can be made with 3 letters and 3 digits. This is:

$$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 26^3 10^3$$

Second, look at how many tags can be made *without* repeating any character (letter or number):

$$26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8$$

Now, if we must repeat at least one character, then number of tags that satisfy are:

$$\text{Total} - \text{ThoseThatDon'tRepeat} = 26^3 10^3 - 26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8$$

Question Score: 0 Answer Score: 3
Question ID: 425530

An unusual combination lock problem

Question

Suppose there's a 4-digit combination padlock and you're asked to open it.

But this lock has an unique defect in a way that the four digits need not be in the correct order for it to open. For example, if the right key is `0-1-2-3`, combinations such as `3-2-1-0`, `2-0-1-3` or any other orderings of 0, 1, 2, 3 are sufficient to crack it open.

1. How many tries does it take to open this lock with brute force?
2. What is the best strategy to ensure the least number of tries?

(edit) Yes, let's assume repetition of digits is allowed.

Answer

Essentially, this is asking how many combinations are there of 4 numbers in range 0-9. This is equivalent to:

$$\binom{\text{count of possible inputs}}{\text{number in each group}} = \binom{10}{4} = 210$$

Thus, a maximum of 210 attempts are all that are necessary.

I'll leave the optimal strategy to someone else... `:)`

Question Score: 3 Answer Score: 0
Question ID: 313225

Discrete Mathematics problem

Question

Consider this equation:
$x_1 + x_2 + \cdots + x_k = n$
with these following constraints:
$x_i 0;\ i = 1, 2, \ldots, k$
How would I go about formulating each of the following problems as a variation of the above problem? I have the answer key, but it's one of those useless keys that just give an answer with no explanation or steps shown so I am completely stuck.

1. Determine the number of ways to select k objects with replacements from a set of n objects.

Answer: $x_1 + x_2 + \cdots + x_n = k;\ \ x_i 0; i = 1; 2; \ldots ; n.$
(b) Determine the number of ways to place n nondistinguisable balls in k boxes.
Answer: $x_1 + x_2 + \cdots + x_k = n;\ \ x_i 0; i = 1; 2; \ldots ; k.$
Can anyone explain to me how to get those answers?

Answer

For 1**:**
x_i is the number of times you selected object i.
For an example: Let's say we have 4 objects, and we want to select 3 (with replacement). We can:

- Select object 1 three times, and all other objects 0 times. $(x_1 = 3,\ x_2 = x_3 = x_4 = 0)$
- Select object 1 two times, object 2 once, and all others 0 times. $(x_1 = 2,\ x_2 = 1,\ x_3 = x_4 = 0)$
- Etc.

For 2**:**
x_i is the number of balls in the i-th box. Let's say we have 2 boxes, and 2 balls. Then, we can:

- Place both balls in box 1 $(x_1 = 2,\ x_2 = 0)$
- Place one ball in each box $(x_1 = x_2 = 1)$
- Place both balls in box 2 $(x_1 = 0,\ x_2 = 0)$

Question Score: 1 Answer Score: 0
Question ID: 809127

Is this question a pigeon hole question?

Question

> How many integers must you pick in order to be sure that at least two of them have the same remainder when divided by 15? Explain.

It seems like this is similar to the birthday pigeon hole example.

I really want to understand this question as I have been staring at it for quite some time. So please, just hints if possible to get me started.

Answer

This problem **does** use the pigeonhole principle. A major hint for this problem is: How many possible remainders are there when you divide by 15?

> There are 15 remainders when you divide by 15. If you draw 20 socks, are you guaranteed to have a duplicate? What about if you draw 13? What about 17?

Question Score: 2 Answer Score: 1
Question ID: 1066387

How many integers in the range [1,999] are divisible by exactly 1 of 7 and 11?

Question

This is a question in Kenneth Rosen's Discrete Mathematics textbook 6th edition. I haven't had trouble with any other counting problems regarding "how many numbers in range [x,y] have divisibility property Z?" My issue is I have no idea what Rosen is asking for, i.e. I don't understand the question because I don't know what he wants me to compute.

I have no idea why the number of integers divisible by 7 or 11 minus the number of integer divisible by 77 (11 and 7) is the answer to this question. Both of these values I've already computed correctly (in separate questions).

In context: 20. How many positive integers less than 1000 e) are divisible by exactly one of 7 and 11?

Thus my question is: what/which numbers am I supposed to count/compute?

Answer

Let's look at a smaller example: the positive integers less than 10 that are divisible by exactly one of 2 and 3.

Well, $2, 4, 6, 8$ are divisible by 2, and $3, 6, 9$ are divisible by 3. There are 6 positive integers that are less than 10 that are divisible by 2 or 3.

But, 6 is divisible by **both** 2 **and** 3. We only should consider integers that are divisible by 2 or 3 **but not both**. So, we have $6 - 1 = 5$ integers that are divisible by exactly one of 2 and 3.

Does that help?

Question Score: 5 Answer Score: 3
Question ID: 929373

Probability of balls in boxes

Question

If 12 balls are thrown at random into 20 boxes, what is the probability that no box will receive more than 1 ball?

So my book says the answer is: $\frac{20!}{8!20^{12}}$

However I am having some trouble understanding this result and was hoping for some clarification.

There's a formula in my book: $p = \frac{n!}{(n-k)!n^k}$ so it's apparent that $n = 20$ and $k = 12$, but I don't understand the reasoning behind it.

Answer

Outline:

1. Count the number of ways to throw 12 balls into 20 boxes
2. Count the number of ways to throw 12 balls into 20 boxes *with at most one ball per box.*
3. Divide the number of desired ways by the total number of ways.

Ways to throw 12 balls into 20 boxes

For each of the 12 balls, there are 20 options of where it can be thrown. (The first ball could be thrown into any of 20 boxes, the second could be thrown into any of 20 boxes, etc.) So, there are:

$$\underbrace{20 \cdot 20 \cdots 20}_{12 \text{ times}} = 20^{12} \text{ ways}$$

...that the balls could have been thrown.

Ways with *at most one ball per box*

- For the first ball, we have 20 options of where to throw it (because all the boxes are empty).
- For the second ball, we can throw it in any box *except the one taken by the first ball*, so there are 19 options for the second ball.
- For the third ball, we can't throw it in the boxes occupied by the first or second balls, so there are only 18 options.
- $\vdots$
- For the twelfth ball, we can throw it in any box *except those occupied by balls $1, 2, 3 \ldots, 11$. So, there are 9 options for the twelfth ball.

All together, then, there are:

$$20 \cdot 19 \cdot 18 \cdots 9 = \frac{(20 \cdot 19 \cdot 18 \cdots 9)\cancel{(8 \cdot 7 \cdots 2 \cdot 1)}}{\cancel{(8 \cdot 7 \cdots 2 \cdot 1)}}$$
$$= \frac{20!}{8!} \text{ ways}$$

Divide to get probability

The probability (when dealing with equally likely outcomes) is the number of desired outcomes divided by the total number of outcomes. Here, we desire *one ball per box*. So, the probability is:

$$\frac{\frac{20!}{8!}}{20^{12}} = \frac{20!}{8!20^{12}}$$

Question Score: 3 Answer Score: 1
Question ID: 1126981

sequences of six digits (0-9)

Question

How many sequences of six digits(0-9) contain at least one 3, at least one 5 , and at least one 8? Can someone please give me a hint?

Answer

First, notice that there 10^6 sequences of 6 digits.

Next, note that the question is equivalent to requesting the complement of the number of six digit sequences that contain no 3s, or no 5, or no 8's.

We recall that:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Now, let $A = \{$sequences with no 3's$\}$, $B = \{$sequences with no 5's$\}$, and $C = \{$sequences with no 8's$\}$.

Then:

$$|A| = 9^6 \tag{4.41}$$
$$|B| = 9^6 \tag{4.42}$$
$$|C| = 9^6 \tag{4.43}$$
$$|A \cap B| = 8^6 \tag{4.44}$$
$$|A \cap C| = 8^6 \tag{4.45}$$
$$|B \cap C| = 8^6 \tag{4.46}$$
$$|A \cap B \cap C| = 7^6 \tag{4.47}$$

Thus, the number of such sequences is:

$$10^6 - |A \cup B \cup C| = 10^6 - (|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|) \tag{4.48}$$
$$= 10^6 - 3(9^6) + 3(8^6) - 7^6 \tag{4.49}$$
$$= 74460 \tag{4.50}$$

Question Score: 0 Answer Score: 0
Question ID: 768266

4.4 Floor/Ceiling and Max/Min

Function that is 1 if x is smaller than or equal to 10 and 0 otherwise

Question

Is there any function that returns 1 if x is smaller than 10 or equal to 10? Otherwise 0 should be returned.

I need this value as a coefficient for another function. Something should be added or subtracted only if x is smaller than 11.

Thank you in advance.

Answer

The *unit step* is the function you're looking for. In Boyce and DiPrima's Differential Equations text, they refer to this as $u_t(x)$. It is defined as:

$$u_t(x) = \begin{cases} 1 & \text{if } x \geq t \\ 0 & \text{otherwise} \end{cases}$$

Then, you can write your function as:

$$f(x) = 1 - u_{10}(x)$$

Another option is:

$$f(x) = 1 - \left\lfloor \frac{\lceil x - 1 \rceil}{10} \right\rfloor$$

Where $\lfloor \cdot \rfloor$ is the floor function, and $\lceil \cdot \rceil$ is the ceiling function.

Yet another option is:

$$f(x) = [x \leq 10]$$

Where $[\cdot]$ are Iverson brackets.

Question Score: 0 Answer Score: 1

Question ID: 685329

Question about greatest integer function.

Question

$[x]$ denotes the greatest integer $\leq x$. Let $f(x) = [x]$ and let $g(x) = [2x]$.

I am having hard time understanding this. What is meant by "greatest integer?" Can anyone refer me to any visual/graphical explanation for $[x]$?

For example, how would we draw the graph for $f(x) = [\sqrt{x}]$?

Also I am a little confused by the including properties such as $[2x] = [x] + [x + 1/2]$ and with $[3x], [4x]$...

Lastly, how would you integrate it? What would be the evaluation of the integral of $\int_1^3 [x]dx$ and integral of $\int_0^9 [\sqrt{t}]dt$?

In general, I am completely confused by $[x]$ and its meaning and its functional ability. So any intuitive explanation would be much appreciated.

Answer

In your example, $[x]$ is also known as the ceiling function. You can think of this as the "rounding up" function. It's also sometimes denoted $\lceil x \rceil$. As far as graphs go, it looks like this.

For your graph of the $[\sqrt{x}]$, you can see that here.

Question Score: 5 Answer Score: 1
Question ID: 408953

How to calculate with $\lceil \cdot \rceil$

Question

I have a problem calculating with ceils.

If I have $\frac{\lceil \frac{n}{2} \rceil}{np}$, this is not the same as $\frac{\lceil \frac{1}{2} \rceil}{p}$. Are there some rules on how to calculate with these things in order to be able to divide?

Answer

Floors are generally nicer than ceilings. So, note that $\lceil x \rceil = -\lfloor -x \rfloor$.

Also note that $x = \lfloor x \rfloor + \{x\}$, where $\{x\}$ is the *fractional part* of x.

Then, your case is:

$$\frac{\lceil \frac{n}{2} \rceil}{np} = \frac{-\lfloor -\frac{n}{2} \rfloor}{np} \tag{4.51}$$

$$= \frac{-\left(-\frac{n}{2} - \left\{-\frac{n}{2}\right\}\right)}{np} \tag{4.52}$$

$$= \frac{n}{2np} + \frac{\left\{-\frac{n}{2}\right\})}{np} \tag{4.53}$$

$$= \frac{1}{2p} + \frac{\left\{-\frac{n}{2}\right\}}{np} \tag{4.54}$$

$$\tag{4.55}$$

If n is a real number, this is about as far as you can go.

Question Score: 0 Answer Score: 1

Question ID: 958065

Proving that for any odd integer: $\left\lfloor \frac{n^2}{4} \right\rfloor = \frac{(n-1)(n+1)}{4}$

Question

I'm trying to figure out how to prove that for any odd integer, the floor of:

$$\left\lfloor \frac{n^2}{4} \right\rfloor = \frac{(n-1)(n+1)}{4}$$

Any help is appreciated to construct this proof!
Thanks guys.

Answer

Let n be an odd integer.
Then there exists an integer k, such that:

$$n = 2k + 1$$

It follows that:

$$\left\lfloor \frac{n^2}{4} \right\rfloor = \left\lfloor \frac{(2k+1)^2}{4} \right\rfloor \tag{4.56}$$
$$= \left\lfloor \frac{(4k^2+4k+1)}{4} \right\rfloor \tag{4.57}$$
$$= \left\lfloor \frac{(4k^2+4k)}{4} + \frac{1}{4} \right\rfloor \tag{4.58}$$
$$= \left\lfloor (k^2+k) + \frac{1}{4} \right\rfloor \tag{4.59}$$

Because $k^2 + k$ is an integer, we can now say:

$$\left\lfloor \frac{n^2}{4} \right\rfloor = k^2 + k$$

It also follows that:

$$\frac{(n-1)(n+1)}{4} = \frac{n^2-1}{4} \tag{4.60}$$
$$= \frac{(2n+1)^2-1}{4} \tag{4.61}$$
$$= \frac{(4k^2+4k+1)-1}{4} \tag{4.62}$$
$$= \frac{4k^2+4k}{4} \tag{4.63}$$
$$= k^2 + k \tag{4.64}$$

Therefore:

$$\left\lfloor \frac{n^2}{4} \right\rfloor = \frac{(n-1)(n+1)}{4}$$

Q.E.D.

Question Score: 2 Answer Score: 4
Question ID: 300493

Can I really factor a constant into the min function?

Question

Say I have $\min(5x_1, x_2)$ and I multiply the whole function by 10, i.e. $10\min(5x_1, x_2)$. Does that simplify to $\min(50x_1, 10x_1)$? In one of my classes I think my professor did this but I'm not sure (he makes very hard to read and seemingly bad notes), and I'm just trying to put these notes together. Thanks!

Answer

Let's try it out. There's three cases:

Case 1:

$5x_1 < x_2$, therefore $10\min(5x_1, x_2) = 50x_1$

If $5x_1 < x_2$, then $50x_1 < 10x_2$. Therefore, $\min(50x_1, 10x_2) = 50x_1$.

Case 2:

$5x_1 > x_2$, therefore $10\min(5x_1, x_2) = 10x_2$

If $5x_1 > x_2$, then $50x_1 > 10x_2$.
Therefore, $\min(50x_1, 10x_2) = 10x_2$.

Case 3:

$5x_1 = x_2$, therefore $10\min(5x_1, x_2) = 50x_1 = 10x_2$

If $5x_1 = x_2$, then $50x_1 = 10x_2$.
Therefore, $\min(50x_1, 10x_2) = 50x_1 = 10x_2$.

Summary

We get the same value from both $10\min(5x_1, x_2)$ and $\min(50x_1, 10x_2)$, therefore the two expressions are equal. However, my intuition warns me against extending this to say $c\min(x_1, x_2) = \min(cx_1, cx_2)$. (My thought is that, if $c < 0$, we would have $c\min(x_1, x_2) = \max(cx_1, cx_2)$, but I haven't checked.)

Question Score: 3 Answer Score: 5
Question ID: 296867

4.5 Number Theory

Euler Fermat with double exponent

Question

I have to calculate

$$3^{2014^{2014}} \pmod{98}$$

What I already have is that the $\gcd(3, 98) = 1$ so I know that I can use the Euler Fermat formula. Then I know that $\varphi(m = 98) = 42$

Then I can say that

$$3^{2014^{2014} \pmod{42}} \pmod{98}$$

Answer

Well, I had written almost all of this up, and then @GohP.iHan posted a much better/shorter approach. :) I'll go ahead and post in case someone else finds it useful, but you should use his way.

Your problem, now, is that you want to compute $2014^{2014} \pmod{42}$. We can factor 2014, which should make it easier:

$$2014^{2014} = (2 \cdot 19 \cdot 53)^{2014}$$

So, we can use Euler/Fermat for 19^{2014} and 53^{2014}:

$$\begin{aligned}
19^{2014} &\equiv 19^{2014 \bmod \phi(42)} \pmod{42} && (4.65)\\
&\equiv 19^{2014 \bmod 12} && (4.66)\\
&\equiv 19^{10} && (4.67)\\
&\equiv \left(19^{-1}\right)^2 \equiv 31^2 && (4.68)\\
&\equiv 37 \pmod{42} && (4.69)
\end{aligned}$$

Note how I used that $9^{10} \equiv 19^{12} \cdot \left(19^{-1}\right)^2$; this saves me from having to do repeated-squares to evaluate the power. I use the same trick for 53^{2014} below:

$$\begin{aligned}
53^{2014} &\equiv 11^{10} \pmod{42} && (4.70)\\
&\equiv \left(11^{-1}\right)^2 \equiv 23^2 && (4.71)\\
&\equiv 25 \pmod{42} && (4.72)
\end{aligned}$$

Now we have that pesky $2^{2014} \pmod{42}$, where the exponent base isn't coprime to the modulus. At this point, I'll just say "use the CRT like @GohP.iHan did in his answer," because I wasn't sure how to approach that. You should have that $2^{2014} \equiv 16$.

Multiplying, we have an answer:

$$2014^{2014} \equiv 16 \cdot 37 \cdot 25 = 14800 \equiv 16 \pmod{4}2$$

So, the final answer is $3^{16} \equiv 25 \pmod{98}$

Question Score: 4 Answer Score: 2
Question ID: 1263584

Largest prime factor of 600851475143

Question

I'm trying to use a program to find the largest prime factor of 600851475143. This is for Project Euler here: http://projecteuler.net/problem=3

I first attempted this with the code that goes through every number up to 600851475143, tests its divisibility, and adds it to an array of prime factors, printing out the largest.

This is great for small numbers, but for large numbers it would take a VERY long time (and a lot of memory).

Now I took university calculus a while ago, but I'm pretty rusty and haven't kept up on my math since.

I don't want a straight up answer, but I'd like to be pointed toward resources or told what I need to learn to implement some of the algorithms I've seen around in my program.

Answer

Project Euler problems (at least the ones that I have done) tend to deal with a lot of number theory topics. So, reading an introductory number theory book could be helpful.

With regards to your particular situation, I suggest finding primes first, then testing the primes for divisibility. That is, to find prime factors of 25, don't test $1, 2, 3, 4, 5, 6, \ldots$–this would take forever, and you'd also be left with composite factors as well. Rather, find the primes under 25, and test those: $2, 3, 5, 7, \ldots$ This will be much faster.

Also, you may want to research prime-finding algorithms. If you start dealing with elliptic curves, that's probably overkill for Project Euler–there are some simple, yet effective, algorithms out there.

Question Score: 5 Answer Score: 1
Question ID: 389675

Counting Divisors Proof

Question

How can be proved that the number of positive divisors is equal to:

$$(e_1 + 1)(e_2 + 1)....(e_n + 1)$$

where e_i is the ith exponent of the prime factorization.

Answer

Idea of proof:

Let $p_0, \dots, p_i$ be the i prime factors of n. That is:

$$n = p_0^{e_0} p_1^{e_1} \cdots p_i^{e_i}$$

Then, the number of unique positive divisors is the same as the number of ways to select unique combinations of the e_ks.

There are $e_0 + 1$ ways to select exponent for p_0, $e_1 + 1$ ways to select the exponent for p_1, etc. The result follows immediately.

Question Score: 1 Answer Score: 1
Question ID: 611038

Using formula of sum of two squares ... create Pythagorean triplets from pairs of primitive Pythagorean triplets?

Question

Can you use a formula for writing products of two sums of squares as a sum of squares to create Pythagorean triplets from pairs of primitive Pythagorean triplets?

First, the formula for writing products of two sums of squares as a sum of squares is:

$$(a^2+b^2)(c^2+d^2) = (ad-bc)^2 + (ac+bd)^2$$

Attempt:

A Pythagorean triplet is: $a^2+b^2=c^2$.

Two examples of a Pythagorean triplet is $(3,4,5)$ and $(5,12,13)$.

Where $3^2+4^2=5^2$ and $5^2+12^2=13^2$.

Now, using the previously mentioned formula:

$$(3^2+4^2)(5^2+12^2) = (3\cdot 12 - 4\cdot 5)^2 + (3\cdot 5 + 4\cdot 12)^2 = 5^2\cdot 13^2 = 4225$$

So, $16^2+63^2=65^2$. Therefore the answer is yes. Is this sufficient? I can't help but feel I have overlooked something. The question thankfully doesn't say "prove", but maybe a "proof" is what I am looking for. Can anyone help me with this?

Answer

Let the following be your Pythagorean triplets.

$$a_1^2+b_1^2=c_1^2$$

$$a_2^2+b_2^2=c_2^2$$

We have (from the formula):

$$(a_1^2+b_1^2)\cdot(a_2^2+b_2^2) = (a_1\cdot b_2 - a_2\cdot b_1)^2 + (a_1\cdot a_2 - b_1\cdot b_2)^2$$

Obviously the RHS is the sum of two squares. To prove this is a Pythagorean triplet, we need to show the LHS is a perfect square. We know that the sum $a_1^2+b_1^2=c_1^2$, and similar for the other factor. Thus, we have:

$$(c_1^2)\cdot(c_2^2) = (a_1\cdot b_2 - a_2\cdot b_1)^2 + (a_1\cdot a_2 - b_1\cdot b_2)^2$$

$$(c_1\cdot c_2)^2 = (a_1\cdot b_2 - a_2\cdot b_1)^2 + (a_1\cdot a_2 - b_1\cdot b_2)^2$$

Which is a Pythagorean triple.

Question Score: 2 Answer Score: 1

Question ID: 255073

Existence of square root in $\mathbb{Z}_n$?

Question

I had this question on my final exam and I struggled with it.
It asks to prove or disprove the following:

$$\forall m \in Z,\ \forall\ [a] \in Z_m,\ \exists\ [b] \in Z_m, [a] = [b]^2$$

I claimed that it's true, and wrote that for an arbitrary m, and an arbitrary a, $[a] = [b]^2$ is equivalent to solving $x \equiv b^2 \pmod{m}$ which is possible since $\gcd(m, 1) \mid b^2$.

Answer

In words, this is asking if every element of $\mathbb{Z}_m$ is a perfect square. This is false, and can be shown by a counterexample.

Take $m = 3$, and consider the congruence class $a = [2]$. Since $[0]^2 = [0], [1]^2 = [1]$, and $[2]^2 = [1]$, there is no $b \in \mathbb{Z}_m$ such that $a = b^2$.

Question Score: 1 Answer Score: 1
Question ID: 1237636

The least value of $a+b+c$ where a, b, c are not perfect squares but their pairwise products are

Question

The positive integers a, b and c are all different. None of them is a square but all the products ab, ac and bc are squares. What is the least value that $a+b+c$ can take?

A 14 B 28 C 42 D 56 E 70

Is it C 42? $27+3+12$

Answer

WLOG, say $a < b < c$. Note that, for a minimum sum, a, b, and c would be of the form $n, n \cdot s_1, n \cdot s_2$, respectively, where n is a non-square positive integer, and s_1, s_2 are squares.

To minimize the sum $n + n \cdot s_1 + n \cdot s_2$, we minimize n and then the two squares. The smallest non-square n would be 2, which means $s_1 = 4$ and $s_2 = 9$ are the smallest squares we can choose.

Thus:

$$a + b + c = 2 + 2 \cdot 4 + 2 \cdot 9$$

Question Score: 0 Answer Score: 2
Question ID: 668765

Prove that if n is not divisible by 5, then $n^4 \equiv 1 \pmod 5$

Question

Suppose n is an integer which is not divisible by 5. Prove that $n^4 \equiv 1 \pmod 5$.

Answer

I would approach this with a proof by cases. There are 5 options for $n \pmod 5$:

Case $n \equiv 0 \pmod 5$**:** Not possible by assumption.

Case $n \equiv 1 \pmod 5$**:** In this case, note that $n^4 \equiv 1^4 \equiv 1 \pmod 5$

Case $n \equiv 2 \pmod 5$**:** (keep going–it's similar...)

Case ...

EDIT: this does assume that you meant "divisible," not "a divisor."

Question Score: 8 Answer Score: 8

Question ID: 871353

Is this procedure for 5^{300} mod 11 correct?

Question

I'm supposed to simplify $5^{300} \bmod 11$, and I'm new to modular exponentiation. Is this procedure correct?

$$5^1 \bmod 11 = 5 \tag{4.73}$$
$$5^2 \bmod 11 = 3 \tag{4.74}$$
$$5^4 \bmod 11 = 3^2 \bmod 11 = 9 \tag{4.75}$$
$$5^8 \bmod 11 = 9^2 \bmod 11 = 4 \tag{4.76}$$
$$5^{16} \bmod 11 = 4^2 \bmod 11 = 5 \tag{4.77}$$
$$5^{32} \bmod 11 = 5^2 \bmod 11 = 3 \tag{4.78}$$

Thus:

$$5^{300} = 3+3+3+3+3+3+3+3+3+4+9$$

Answer

You're *almost* there. Instead of your last line, you want:

$$5^{300} \equiv 5^4 5^8 5^{32} 5^{256} \pmod{11}$$

Now you replace each of those factors with the modular equivalent you found before (e.g. $5^4 \implies 9$)

Question Score: 5 Answer Score: 6
Question ID: 855657

Diophantus problem

Question

I was given following problem as an example of early mathematics with the solutions. But it seems i can't understand from where they are getting the $35z^2 = 5$ from in the solutions. Could someone please example what was the step done on that part? Thanks.

Solve Diophantus' problem IV-9. To add the same number to a cube and its side and make the second sum the cube of the first. (The equation is $x + y = (x^3 + y)^3$; he started by letting $x = 2z$ and $y = 27z^3 - 2z$.

We want to solve $x + y = (x^3 + y)^3$. We set $x = 2z$ and $y = 27z^3 - 2z$ (so that $x + y = (3z)^3$). Then $(x^3 + y)^3 = (35z^3 - 2z)^3 = (3z)^3$. It follows that $35z^2 = 5$. This is impossible for rational z. But now note that $35 = 27 + 8 = 3^3 + 2^3$ and $5 = 3 + 2$. In order that the equation in z be solvable in rationals, we need two numbers a and b (to replace the 3 and 2) so that $\frac{a^3+b^3}{a+b}$ is a square. So let $a + b = 2$ (where 2 is arbitrary). Then $b = 2 - a$ and $a^3 + b^3$ must equal 2 times a square. This implies that $a^3 + (2 - a)^3 = 8 - 12a + 6a^2 = 2(\text{square})$ or that $4 - 6a + 3a^2$ is a square. So set $4 - 6a + 3a^2 = (2 - 4a)^2$ and solve for a. We get $a = \frac{10}{13}$ and therefore $b = \frac{16}{13}$. Since it is only the ratio of a and b which is important, we can choose $a = 5$, $b = 8$ and therefore put $x = 5z$, $y = 512z^3 - 5z$ and repeat the initial calculation. We then get $637z^3 = 13z$ and $z^2 = \frac{1}{49}$, so $z = \frac{1}{7}$. Then $x = \frac{5}{7}$; $y = \frac{267}{343}$ is the desired solution.

Answer

Since $(35z^3 - 2z)^3 = (3z)^3$, we take cube roots.

This gives us $35z^3 - 2z = 3z$, at which point the result is immediate.

Question Score: 0 Answer Score: 1

Question ID: 960135

Let $a = 43120$ How many positive divisors does a have?

Question

I am doing a review assignment and I'm stuck on this problem.

a) How many positive divisors does a have? I got 60

b) How many positive integers less than a are relatively prime to a? I got 720

c) What is the smallest positive integer m such that a^2m is a cube?

d) list all positive divisors b of a for which a divides b^2 is also true.

Any help and advice would be greatly appreciated. Thank you for your time and help!

Answer

First, note that $43120 = 2^4 \cdot 5^1 \cdot 7^2 \cdot 11^1$.

Part A

The number of positive integer divisors is the product of one plus each exponent in the prime factorization. That is,

$$d(43120) = (4+1)(1+1)(2+1)(1+1) = (5)(2)(3)(2) = 60$$

Part B

The number of positive integers coprime to 43120 can be found using Euler's Totient function, $\phi(n)$. Since ϕ is multiplicative for coprime integers:

$$\phi(43120) = \phi(2^4)\phi(5)\phi(7^2)\phi(11)$$

Also, $\phi(p^n) = p^{n-1}(p-1)$ for prime integers p. Then:

$$\begin{aligned}
\phi(43120) &= \left[2^3(2-1)\right][5-1]\left[7^1(7-1)\right][11-1] && (4.79)\\
&= 8 \cdot 4 \cdot 42 \cdot 10 && (4.80)\\
&= 13440 && (4.81)
\end{aligned}$$

Part C We want all exponents of a^2 to be multiples of three, and we want the smallest such exponents.

$$\left(2^4 \cdot 5^1 \cdot 7^2 \cdot 11^1\right)^2 = 2^8 \cdot 5^2 \cdot 7^4 \cdot 11^2$$

Well, it is easy to see that we need to add 1 to the first exponent, 1 to the second, 2 to the third and 1 to the fourth. Thus, our integer m is:

$$m = 2^1 \cdot 5^1 \cdot 7^2 \cdot 11^1$$

Part D This is a similar exponent-related trick as in part c if I'm thinking it through correctly. I'll leave this one as an exercise to the reader. :)

Question Score: 8 Answer Score: 10
Question ID: 1061329

Proving $\binom{p}{k}$ is a multiple of p

Question

If p is a prime number greater than 2 and $k \in \mathbb{N}$ so that $k < p$, how can I prove that $\binom{p}{k}$ is congruent to $0 \mod p$. I'm sorry I know my formatting is rough but I don't know how to format it correctly.

Answer

So, you have:

$$\frac{p!}{k!(p-k)!}$$

and you want to prove this is a multiple of p.

Hint: $n! = n(n-1)!$

Question Score: 1 Answer Score: 1
Question ID: 251280

Solving infinite sums with primes.

Question

Let p_n denote the n'th prime number.

How would one go about proving that infinite products like:

$$\prod_{k=1}^{\infty} 1 - \frac{1}{(p_k)^2} = \frac{6}{\pi^2}$$

or

$$\prod_{k=1}^{\infty} \frac{{p_k}^2}{{p_k}^2 - 1} = \frac{\pi^2}{6}$$

are correct?

Is there any way to prove it except by exhaustion?

Answer

As @doppz mentioned in the comments, the way to prove these identities is to realize the requested product is requesting the value of $\zeta(2)$. Then, you can use any of the multitude of proofs that $\zeta(2) = \frac{\pi^2}{6}$.

For example proofs: http://math.cmu.edu/ bwsulliv/MathGradTalkZeta2.pdf

Question Score: 2 Answer Score: 1
Question ID: 621918

Prove that N is composite if and only if $p|N$ for some p prime, $p \leq \sqrt{N}$.

Question

Show that N is composite if and only if $p|N$ for some p prime, $p \leq \sqrt{N}$
I have absolutely no idea on how to start this, could you guys give me some tips?
I'll update if I can come up with something...

Answer

Hints

One direction of the proof should be just an application of the definition of composite, and nothing else.

For the other direction, prove this by contradiction. Assume that all primes dividing p were greater than $\sqrt{N}$. What happens when you multiply these primes together again?

Answer

First part: We aim to show that, if $p \mid N$ and $p \leq \sqrt{N}$, then N is composite. For sake of reaching a contradiction, assume N is prime. Then, N is only divisible by itself and 1. But, $1 < p \leq \sqrt{N}$ is a divisor of N by hypothesis–a contradiction! Hence, N is composite.

Now, going the other way: If N is composite, it is divisible by some primes (not necessarily unique) $p_1, p_2, \ldots, p_k$, $k \geq 2$. Suppose that all of p_i were greater than $\sqrt{N}$. Then:

$$N = p_1 p_2 \cdots p_k > \underbrace{\sqrt{N} \cdots \sqrt{N}}_{k \text{ times}} > N$$

Clearly, this is a contradiction. Thus, at least one of the p_i must be less than the square root of N.

Question Score: 2 Answer Score: 2
Question ID: 1236762

How often is $1 + \prod_{k=1}^{n} p_k$ not prime?

Question

How often is $1 + \prod_{k=1}^{n} p_k$ is not prime?,wkere p_k is k'th prime consider that $2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 59 \times 509 = 30031$ is this one off or, are there infinitely many composites of the form $1 + \prod_{k=1}^{n} p_k$?

Answer

Define $p_n\# = p_1 \cdots p_n$. This is called the *primorial* of p_n (like a factorial, but multiplying together only primes).

Prime numbers of the form $p_n\# \pm 1$ are called *primorial primes*. Along a similar vein, *Euclid Numbers* are of the form $p_n\# + 1$ (not necessarily prime).

Thus, your question is: "How many prime Euclid Numbers are there?" The Wolfram Mathworld article says:

> The largest known Euclid number is E_{13494}, and it is not known if there are an infinite number of prime Euclid numbers (Guy 1994, Ribenboim 1996).

Regarding composite Euclid numbers, this comment by Franois Brunault on MathOverflow says:

> According to Ribenboim (The little book of big primes, page 3), it is also not known whether there exist infinitely many composite Euclid numbers. Such questions are often very difficult to tackle.

Summary: There are many known prime Euclid Numbers (A014545 is a list of n such that E_n is prime), but it is not known if there are infinitely many such primes. There are also many known composite Euclid Numbers, but it is not known if there are infinitely many.

Question Score: 9 Answer Score: 6
Question ID: 1103357

How to do this discrete mathematics problem

Question

I'm working on this one for a homework assignment and I just can't come up with the solution:

"Let d be a positive integer and consider any set A of $d+1$ positive integers. Show that there exists two different numbers $x, y \in A$ so that $x \mod d \cong y \mod d$ and $x \neq y$."

I scanned through the forums but didn't see anything on this. Forgive me if I missed something.

It's probably painfully simple, I just can't think of it. Any help would be greatly appreciated, thanks!

Answer

Hint: There are only d unique residues when considered $\mod d$. But, we have $d+1$ elements in the set. Thus... (pigeonhole time!)

Question Score: 0 Answer Score: 0
Question ID: 776503

Method to solving this proof with a java app

Question

I'm writing a program to solve this proof, but I don't know how to go about solving it.

- For every odd integer n, $3 \leq n \leq 199$, there exists an integer $m \geq 0$, and a prime number p, such that $n = 2^m + p$.

Answer

Algorithm

Here's a "brute force" (but somewhat fast) approach for a generalized upper limit of N. (This case is $N = 199$)

1. Generate (or obtain) a sorted array of all primes less than N. (This is $\pi(N)$, where π is the prime counting function)
2. Generate a sorted array of all powers of two less than N. (There are $\lfloor \log_2(N) \rfloor$ of these.) This could be done dynamically (not stored in array) using bitshifts.
3. Iterate through all the odd numbers less than N but greater than 3 (there are about $N/2$ of these).
   (a) For each odd number, iterate through the powers of two
   (b) Subtract the power of two from the odd number, and do a binary search on the prime array to see if the difference is prime.
   (c) If you find a prime, the statement has been verified for the odd number. Continue to next odd.

Runtime:

Approximate $\pi(N) = \mathrm{Li}(N)$, where $\mathrm{Li}(x)$ is the logarithmic integral. Also assume we are given the prime list. Then, we have an upper bound of the runtime:

$$\mathcal{O}\left(\frac{N}{2} \lfloor \log_2(N) \rfloor \cdot \log_2(\mathrm{Li}(N))\right) \approx \mathcal{O}\left(N \lg(N) \cdot \lg(\mathrm{Li}(N))\right)$$

(The first factor from iterating through odd numbers, the second from the number of powers of two, and the third factor from the binary search on the prime array.)

Java Code

```
public class MathPost {
  static final int N = 199;

  //Assumed to have all primes under N
  static int[] primeArray = {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37,
      41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107,
      109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181,
       191, 193, 197, 199};

  public static void main(String[] args) {
```

```
    boolean isTrue = true;
    boolean trueForAPowerOfTwo = false;

    int powerOfTwo = 1;
    int supposedPrime = 0;
    int oddNumber;
    for (oddNumber = 3; oddNumber <= N; oddNumber += 2) {
      //We don't know if the proposition is true for this odd number with any power of two yet
      trueForAPowerOfTwo = false;

      for (int power = 0; (1 << power) < oddNumber; power++) {
        //Find the power of two we are considering
        powerOfTwo = 1 << power;

        //Determine the difference of the power of two and the odd number.
        supposedPrime = oddNumber - powerOfTwo;

        //The following sets the "wasTrueForAPowerOfTwo" variable to true if we find a prime difference
        trueForAPowerOfTwo |= java.util.Arrays.binarySearch(primeArray, supposedPrime) >= 0;
      }

      isTrue &= trueForAPowerOfTwo;

      if (!isTrue) break;
    }

    System.out.printf("The proposition is \%s\n", isTrue);
    if (!isTrue) {
      System.out.printf("Counterexample: The odd number \%d cannot be expressed as desired.\n", oddNumber);
    }

  }
}
```

The Result

The statement is false. Take the odd (prime) number 127. It cannot be expressed as the sum of a power of two and another prime:

$$127 - 1 = 126 \quad \text{(Not prime)} \tag{4.82}$$
$$127 - 2 = 125 \quad \text{(Not prime)} \tag{4.83}$$
$$127 - 4 = 123 \quad \text{(Not prime)} \tag{4.84}$$
$$127 - 8 = 119 \quad \text{(Not prime)} \tag{4.85}$$
$$127 - 16 = 110 \quad \text{(Not prime)} \tag{4.86}$$
$$127 - 32 = 95 \quad \text{(Not prime)} \tag{4.87}$$
$$127 - 64 = 63 \quad \text{(Not prime)} \tag{4.88}$$

Question Score: 0 Answer Score: 2
Question ID: 10910

Number theory proof from AoPS

Question

http://www.artofproblemsolving.com/Resources/articles.php?page=htw.readers
In the above link, he gives a problem, namely

> Let $S(n)$ be the sum of the digits of n. Find $S(S(S(4444^{4444})))$.

Then in the lemma he presents is where I get confused. He states

> Every integer n, written in decimal notation, is congruent to the sum of its digits modulo 9.

And he goes on to present the proof. But by the same argument he uses I can also show that it works for mod 8 as well. Also, how would I represent 18? 19?

Answer

Let's define a k-digit number n to have the decimal (base-10) digits $d_0, d_1, \dots d_{k-1}$. We can then write n as:

$$n = 10^0 d_0 + 10^1 d_1 + \cdots + 10^{k-1} d_{k-1}$$

Note that we can re-arrange this as:

$$\begin{aligned} n &= d_0 + 10^1 d_1 + \cdots + 10^{k-1} d_{k-1} && (4.89)\\ &= d_0 + (9d_1 + d_1) + \cdots + (10^{k-1} - 1)d_{k-1} + d_{k-1} && (4.90)\\ &= (d_0 + d_1 + \cdots + d_{k-1}) + \underbrace{(9d_1 + 99d_2 + \cdots (10^{k-1} - 1)d_{k-1})}_{\text{all this is a multiple of 9}} && (4.91) \end{aligned}$$

Because anything that is a multiple of 9 is congruent to $0 \pmod 9$, we have:

$$\begin{aligned} n &\equiv (d_0 + d_1 + \cdots + d_{k-1}) + 0 \pmod 9 && (4.92)\\ &\equiv d_0 + d_1 + \cdots + d_{k-1} \pmod 9 && (4.93) \end{aligned}$$

An example may help. Let's try $d_0 = 1, d_1 = 6, d_2 = 7, d_3 = 3, d_4 = 4$, which makes our 5-digit number 16734. Then, we have:

$$\begin{aligned} n &= 1 + 6 \cdot 10 + 7 \cdot 10^2 + 3 \cdot 10^3 + 4 \cdot 10^4 && (4.94)\\ &= 1 + (6 + 6 \cdot 9) + (7 + 7 \cdot 99) + (3 + 3 \cdot 999) + (4 + 4 \cdot 9999) && (4.95)\\ &= (1 + 6 + 7 + 3 + 4) + (6 \cdot 9 + 7 \cdot 99 + 3 \cdot 999 + 4 \cdot 9999) && (4.96)\\ &\equiv (1 + 6 + 7 + 3 + 4) + 0 \pmod 9 && (4.97)\\ &\equiv 11 \pmod 9 && (4.98) \end{aligned}$$

Question Score: 2 Answer Score: 1
Question ID: 501887

Let $a, b \in \mathbb{Z}$, $a \equiv 4 \pmod{13}$, and $b \equiv 9 \pmod{13}$. Find $c \equiv 9a \pmod{13}$.

Question

The Problem

I had my first exposure to number theory today. Trying to work on some problems in hope that it will start to make more sense. Here is the problem (part a) I'm stuck on right now.

13. Suppose that a and b are integers, $a \equiv 4 \pmod{13}$, and $b \equiv 9 \pmod{13}$. Find the integer c with $0 \leq c \leq 12$ such that

a) $c \equiv 9a \pmod{13}$.

b) $c \equiv 11b \pmod{13}$.

c) $c \equiv a + b \pmod{13}$.

d) $c \equiv 2a + 3b \pmod{13}$.

e) $c \equiv a^2 + b^2 \pmod{13}$.

f) $c \equiv a^3 - b^3 \pmod{13}$.

My Work

We know from the first congruence that $\frac{a-4}{13} = d$ where $d \in \mathbb{Z}$ and $\frac{b-9}{13} = e$ where $d \in \mathbb{Z}$. We also know $a = 4 + 13k$ where $k \in \mathbb{Z}$ and $b = 9 + 13j$ where $j \in \mathbb{Z}$. $\frac{c-9a}{13}$ = an integer.

My Question

I've derived what I can from what I was given. How can I use this information to help figure out this c? What is a good strategy for the rest of the questions?

Answer

We know $a \equiv 4 \pmod{13}$. We want to find an expression for c. Thus:

$$c \equiv 9a \pmod{13} \tag{4.99}$$
$$\equiv 9(4) \pmod{13} \tag{4.100}$$
$$\equiv 36 \pmod{13} \tag{4.101}$$
$$\equiv 10 \pmod{13} \tag{4.102}$$

Just like with normal equations, you can substitute values in congruences.

Question Score: 2 Answer Score: 1
Question ID: 1021111

Show that $9 \mid a^2$ if given that $6 \mid a$

Question

Does this prove I made seem correct to show that if 6 divides a then 9 divides a^2
If $6 \mid a$, then $a = 6k$ (k is some integer).
Then $a^2 = 36k^2 = 9(4k^2)$.
Which means that $9 \mid a^2$.
perhaps if not is there any other way?

Answer

What you've done is perfect.

An alternate method of proof for all those people who just love modular arithmetic:

Since $6 \mid a$, then $a \equiv 6,\ 3,\ \text{or}\ 0 \pmod 9$. Note that $6^2 \equiv 3^2 \equiv 0^2 \equiv 0 \pmod 9$. Thus, $9 \mid a^2$.

Question Score: 6 Answer Score: 4
Question ID: 749559

How do we know all numbers alternate between odd and even numbers?

Question

Basically, I tried proving that multiplying an odd and even number together gives you an even number to a friend of mine. This is what I said.

Let's first take a random number k. It doesn't matter if its odd or even. If we then multiply this by 2, we get an even number (from the definition of a even number, which is basically a number that can be divided by 2). Now we know that in the number line, it goes odd, even, odd, even,..., and so if we add 1 to this, then we get an odd number. We now have an odd number and an even number and so lets multiply them.

$$2k \cdot (2k+1) = 2k(2k+1) = 4k^2 + 2k) = 2(2k^2 + k)$$

We now have a random number $(2k^2 + k)$ multiplied by 2 and so this is an even number (by definition of an even number).

My friend then said that we make the assumption that $2k \pm 1$ is odd because the number line alternates between odd and even numbers, how do know that this is true?

I was thinking how to prove this. Would it be by some form of induction?

EDIT: I think I need to change "number" to "integer" in this post don't I?

Answer

Let m be an odd integer, and n be an even integer.

Thus, by definition of odd and even, we have integers k_1 and k_2 such that:

$$m = 2k_1 + 1$$

$$n = 2k_2$$

It follows that:

$$\begin{aligned} m \cdot n &= (2k_1 + 1) \cdot (2k_2) && (4.103) \\ &= 4k_1k_2 + 2k_2 && (4.104) \\ &= 2(2k_1k_2 + k_2) && (4.105) \end{aligned}$$

As $(2k_1k_2 + k_2)$ is an integer, we have that $m \cdot n$ is even.

Question Score: 10 Answer Score: 1
Question ID: 322911

Prove $\sqrt{k}$ is not a rational number.

Question

Suppose $k > 1$ is an integer, and k is not a square number, then $\sqrt{k}$ is not a rational number.
Proof:
Let $\sqrt{k} = \frac{p}{q}$, and $(p, q) = 1$,So $q^2|p^2$, $p \neq 1$, k is not an integer.When $q = p = 1$, and $k > 1$.
And $q = 1$, then $\sqrt{k} = p$, and k is a square number.
I do think it's so easy...Where is wrong?
How to do it.

Answer

In short: I think you started correctly, but made some odd jumps. I'm going to offer some critique of the proof–please don't take anything here personally. :)

A proof is more than just a string of symbols–rather, it must be a clearly (yet concisely) written work that allows the reader a peak inside your mind when you proved the proposition.

So, I would suggest:

- Whenever you start a proof by contradiction, make sure to denote it.
- Always "introduce" your variables. Are you assuming k is an integer? A complex number? What about p and q?
- Don't use notation unless it really helps. Sometimes it's easier to say (and nearly always easier to read) "p and q coprime," rather than "$(p, q) = 1$".
- Show your algebra, or at least mention that you're doing some. Don't make me think about why $\sqrt{k} = \frac{p}{q}$ implies $q^2|p^2$–show me from the definition of *divides*.

So, a start to the proof could be:

Proof:

Assume proposition is false. That is, there exists a $k \in \mathbb{Z}^+$ such that $k > 1$ is not a perfect square and $\sqrt{k}$ is rational.

As $\sqrt{k}$ is rational, there exist coprime integers p and q ($q \neq 0$) such that:

$$\sqrt{k} = \frac{p}{q}$$

Rearranging the above, we find that:

$$p = q\sqrt{k}$$

It follows:

$$p^2 = q^2 k$$

...

Question Score: 1 Answer Score: 2
Question ID: 449974

If a and b are two integers, and $a \mid b$, then $\gcd(a^2, b^2) =$

Question

If a and b are two integers, and $a \mid b$, then $\gcd(a^2, b^2) =$
I think the answer is a^2. Is it correct?

Answer

If $a \mid b$, then $b = ka$ for some integer k.

Then we aim to find $\gcd(a^2, (ka)^2) = \gcd(a^2, k^2a^2)$. So, yes, a^2 is the greatest common divisor.

Question Score: 2 Answer Score: 1
Question ID: 598689

Prove that for any integer $k \neq 0$, $gcd(k, k+1) = 1$

Question

I'm learning to do proofs, and I'm a bit stuck on this one. The question asks to prove for any positive integer $k \neq 0$, $gcd(k, k+1) = 1$.

First I tried: $gcd(k, k+1) = 1 = kx + (k+1)y$: But I couldn't get anywhere.

Then I tried assuming that $gcd(k, k+1) \neq 1$, therefore k and $k+1$ are not relatively prime, i.e. they have a common divisor d s.t. $d|k$ and $d|k+1 \implies d|2k+1$

Actually, it feels obvious that two integers next to each other, k and $k+1$, could not have a common divisor. I don't know, any help would be greatly appreciated.

Answer

Old John's answer in the comments is better than this, but I'm hoping to provide some intuition...

Look at $k!$ instead of k:

- 2 divides $k!$, thus it cannot divide $k! + 1$. (2 divides every other integer)
- 3 divides $k!$, thus it cannot divide $k! + 1$. (3 divides every third integer)

etc, up to and including "k divides $k!$...".

Now look back at k: Well, if 2 divides k, then it cannot divide $k+1$ (2 divides every other integer). If 3 divides k, then it cannot divide $k+1$. Etc, until you reach "does k divide k?".

Question Score: 0 Answer Score: 1
Question ID: 231887

Polynomial modulo n

Question

http://en.wikipedia.org/wiki/AKS primality test

$$(x-a)^n \equiv (x^n - a) \pmod{n} \qquad (1)$$

How can I interpret what the "mod n" means?

I have watched the Numberphile video on the AKS primality test, and based on that, I am assuming that "mod n" means the remainder when dividing by each of the coefficients.

$$(x-1)^5 \equiv (x^5-1) \pmod 5$$
$$x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1 \equiv x^5 - 1 \pmod 5$$
$$1x^5 - 0x^4 + 0x^3 - 0x^2 + 0x - 1 \equiv 1x^5 - 1$$
$$x^5 - 1 \equiv x^5 - 1$$

Is my assumption correct?

Answer

This is usually covered in more detail in an introductory number theory or abstract algebra textbook, and is referred to as "modular arithmetic."

Writing that two quantities are congruent/equivalent $\pmod n$ (this is said "modulo n") means that the two quantities produce the same remainder when divided by n. Thus, we can write $15 \equiv -1 \equiv 7 \pmod 8$.

We often want to work with polynomials, though; in this case, we can simplify expressions (as you have in the question) by reducing each of the coefficients of the polynomial to their "residues" (remainders) modulo n.

The big "watchout" in modular arithmetic is that, although we can multiply, add, and subtract freely, division isn't always guaranteed. Instead, we work with "inverses" of a number. Long story short, an integer a is the inverse of x modulo n if (and only if) $ax \equiv 1 \pmod n$. We write $a \equiv x^{-1} \pmod n$. It can be shown that the inverse always exists if x and n are coprime.

All that to say: yes, your approach in the question is correct.

Question Score: 0 Answer Score: 1
Question ID: 1066897

Determining gcd(94, 27)

Question

I want to determine $\gcd(94, 27)$. Using the Euclidean algorithm, I got

$$94 = 27(3) + 13 \tag{4.106}$$
$$\implies 27 = 13(2) + 1 \tag{4.107}$$
$$\implies 2 = 2(1) \tag{4.108}$$

Does this mean the GCD is 2? Clearly 2 doesn't divide 27, so what am I doing wrong?

Answer

As mentioned in the comments, the *last non-zero remainder* is the GCD, not the quotient corresponding to the expression with zero remainder. To highlight, that is the red/boxed number below:

$$94 = 27(3) + 13 \tag{4.109}$$
$$27 = 13(2) + \boxed{1} \tag{4.110}$$
$$13 = 13(1) + 0 \tag{4.111}$$

For another example, consider $\gcd(45, 81)$:

$$81 = (1)45 + 36 \tag{4.112}$$
$$45 = (1)36 + \boxed{9} \tag{4.113}$$
$$36 = (4)9 + 0 \tag{4.114}$$

Question Score: 2 Answer Score: 3
Question ID: 1114275

4.6 Asymptotics

how to proof this big-oh statement?

Question

I have a question on my homework which is:

Prove that if $f(x) = O(g(x))$, and $g(x) = O(h(x))$, then $f(x) = O(h(x))$

I am not to sure how to prove this. This is my attempt. Is it good enough or am i missing something important?

Thanks in advance!

Proof:

if you Assume $f(x) = n^2 + 1$

then $f(x) \leq C_1.g(x)$ where $g(x) = x^2$

if $g(x) = x^2$ then $g(x) \leq C_2.h(x)$ where $h(x) = n^2$

Therefore this shows that $f(x) \leq C_3 h(x)$ which proves $f(x) = O(h(x))$

Answer

A big warning for all proofs you do: ***Don't*** assume anything not given in the problem statement. For example, you don't know that the functions are quadratic. So that's a flaw in the proof.

Otherwise, you have the right idea. Just separate it from defining the function to be quadratic. Also, the last line should be an inequality, not an equality. That is, it should read $f(x) \leq C_3 h(x)$, not $f(x) = C_3 h(x)$. (This is because $\mathcal{O}$ represents an upper bound, not an exact bound.)

Question Score: 1 Answer Score: 0
Question ID: 346977

Determine whether $F(x) = 5x + 10$ is $O(x^2)$

Question

Please, can someone here help me to understand the Big-O notation in discrete mathematics?

> Determine whether $F(x) = 5x + 10$ is $O(x^2)$

Answer

Short answer:
Yes, it is $\mathcal{O}(x^2)$.
Long answer:
Let's look at the definition of $\mathcal{O}$:

> $f(x) = \mathcal{O}(g(x))$ if and only if there exists a positive real number M and a real number x_0 such that:
> $$|f(x)| \leq M|g(x)| \text{ for all } x\ x_0$$

In layman's terms, this is saying that $\mathcal{O}(g(x))$ is an *upper bound* to the function. Basically, at some point, $g(x)$ grows as fast or faster than $f(x)$.

Clearly, x^2 eventually beats out x in terms of growth over the long haul. In the "long haul," (as $x \to \infty$), it doesn't matter what is added to x or what x is multiplied by–x^2 will beat it. Thus, x^2 is an upper bound.

(This is a non-rigorous argument, of course. Rather than prove it for you, this is just meant to give a feel for how $\mathcal{O}$ works.)

Question Score: 4 Answer Score: 0
Question ID: 427971

Notation question: $\ll$

Question

I was perusing Wolfram's page on Highly Composite Numbers and saw the following at the end: "Nicholas proved that there exists a constant $c_2 > 0$ such that $Q(x) \ll (\ln x)^{c_2}$."

What does the $\ll$ mean in this context? (fyi, $Q(x)$ is the number of highly composite numbers less than or equal to x)

Answer

The symbols $\ll$ and $\gg$ denote "asymptotically very much less than" and "asymptotically very much greater than," respectively.

Question Score: 1 Answer Score: 4
Question ID: 1069343

Algorithm Analysis: Determining Order Class of a Sum

Question

Okay so I have a summation which goes:

$$\sum_{i=1}^{n^3} 3i^2 \cdot \log(i)$$

My goal is to find the **order** of the function, not the exact summation amount. I have found the order of it by writing out the sum, and noticing that my maximal term from the sum is $3(n^3)^2 \log(n^3)$.

Therefore, **I am looking at the entire summation as order** $(n^6)\log(n)$**.** Can it be simplified this way, or does the large "amount" of terms being summed (after all, it is going from $i = 1$ all the way to $i = n^3$) lead to the overall order of the function being higher than this? And if this is the case, does anyone giving pointers as to how to accomplish this?

Saying it another way, there are a ton of terms being summed up here. Does just taking the maximum term from the summation constitute the highest order of the summation, or does adding together so many such terms result in a higher order?

A common example is to view the series $1 + 2 + 3 + \cdots + n$. As we know, this is not order n. This is order n^2, because of the lemma which states that this series is equal to $n(n + 1)/2$. So this is where my concern is coming from.

Answer

In short/in general: The order of the last term in the summation is not necessarily the order of the sum. What you found with the Harmonic sum (and with this sum) exemplifies that.

For the problem at hand:

I'm going to approach this from a CS **competition** perspective. That is, I'm going to give some rough bounds for the algorithm, even though I know they're **not exact**. This is useful in ACM-style competitions when you want to know *quickly* if you need to re-structure your solution.

For an upper bound, we can note that $\log(i) < i$ (for large enough i). Then, our sum becomes:

$$\sum_{k=1}^{n^3} i^2 \log(i) \approx \sum_{k=1}^{n^3} i^3 = \left(\frac{n^3(n^3+1)}{2}\right)^2 \approx \mathcal{O}(n^{12})$$

A lower bound is (by eliminating the log term):

$$\sum_{k=1}^{n^3} i^2 \log(i) \approx \sum_{k=1}^{n^3} i^2 = \frac{n^3(n^3+1)(2n^3+1)}{6} \approx \Omega(n^9)$$

This method of upper/lower bounds is oftentimes helpful enough, so I thought I'd include it here for future readers.

Question Score: 3 Answer Score: 1
Question ID: 647098

Recursion and Time Complexity Concept

Question

The question prompt is as follows:

Consider the function $f(n)$ defined as:

$$f(n) = \begin{cases} n(n-1)f(n-2) & n > 1 \\ 1 & n = 0,\ n = 1 \end{cases}$$

How may be $g(n)$ be defined to make $f(n) \in \mathcal{O}(g(n))$? (I'm supposed to choose one response from 1 and a response from 2)

1. For $n > 0$, define $g(n)$ as:
   Option 1: $g(n) = g(n-1) + n$
   Option 2: $g(n) = ng(n-1)$.
2. Then, assign:
   Option 1: $g(0) = 0$
   Option 2: $g(0) = 100$.

Now, I solved the recursive equation and got the following:

$$f(n) = n!$$

If we define $g(n) = g(n-1) + n$:

$$g(n) = n(n+1)/2 + \{0 \text{ or } 100\}$$

If we define $g(n) = n \cdot g(n-1)$:

$$g(n) = n! \cdot \{0 \text{ or } 100\}$$

(The $\{0$ or $100\}$ because when $g(0)$ could be 0 or 100.)
However, I am not seeing how either definition of $g(n)$ yields $\lim_{n\to\infty}(f(n)/g(n)) = 0$.

Answer

Your problem is that you're seeking the wrong thing limit. In order for $f(n)$ to be $\mathcal{O}(g(n))$, you want:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c, \quad 0 \le c < \infty$$

Clearly, if we choose $g(n) = n! \cdot g(0)$ and $g(0) = 100$, the resulting value is $c = \frac{1}{100}$.
You probably were confused with the definition of *little-oh*:

$$f(n) \in o(g(n)) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

To conclude: you did everything else right, except you messed up the limit definition of $\mathcal{O}(\cdot)$.

Question Score: 2 Answer Score: 1
Question ID: 991351

Calculating run times of programs with asymptotic notation

Question

When calculating the run time of programs using asymptotic notation, I know how to set up the sums for things like for loops, but I'm getting stuck on summing them up.

Sorry if this is a dumb question but say we have something like

```
for i=4 to n^2
   for j = 5 to 3i log (i)
      s = s + ij
```

I would set this up with outer loop = sum from i=4 to n^2, inner loop sum from j = 5 to $3i \log i$, and finally a constant. I'm getting hung up on how to sum everything up. Can someone please explain this to me?

Answer

A simple way to determine runtime is to use summations. Each loop becomes a sum, where the lower index is the starting value of the loop and the upper index is the ending value of the loop. Because you're performing a constant time operation in the innermost loop, we are simply summing 1:

$$\sum_{i=4}^{n^2} \sum_{j=5}^{3i \lg i} 1 = \sum_{i=4}^{n^2} 3i \lg i - 5 + 1 \tag{4.115}$$

$$\approx \sum_{i=1}^{n^2} i \ln i \tag{4.116}$$

(I'm using $\approx$ to slide between expressions that are asymptotically equivalent throughout the problem.)

The last summation above is kind of difficult (its solution involves the hyperfactorial function). So, we switch to an approach that is accessible to more students: a summation may be approximated by an integral. So, we now compute:

$$\sum_{i=1}^{n^2} i \ln i \approx \int_1^{n^2} x \ln x \, dx \tag{4.117}$$

$$\approx x^2 \ln x + \mathcal{O}(x^2) \Big|_1^{n^2} \tag{4.118}$$

$$\approx n^4 \ln n \tag{4.119}$$

So, your program executes with $\mathcal{O}(n^4 \ln n)$ time complexity.

Question Score: 2 Answer Score: 3
Question ID: 701244

Proving Big-Θ if and only if Big-O and Big-Ω

Question

Given the definitions of Big-O and Big-Ω, I'd like to prove that $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

Here's what I've come up with, but I'm not sure it's a rigorous proof, or even correct. I'm looking for comments on both the structure and style of my proof as well as correctness.

Proof by contradiction.

Suppose $f(n) = \Theta(g(n))$ and either $f(n) \neq O(g(n))$ or $f(n) \neq \Omega(g(n))$.

By the definition of Θ,

$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$

For some positive constants c_1, c_2, and $n > n_0$.

But it was given that $f(n) \neq O(g(n))$, which means there exist no positive constants c, n_0 such that for all $n > n_0$,

$0 \leq f(n) \leq c \cdot g(n)$

Which is a contradiction. ■

Answer

To prove an *if and only if* statement, you must prove that $p \to q$ and $q \to p$. Applied to your statement, you must prove that:

$$f(n) = \Theta(g(n)) \implies f(n) = \mathcal{O}(g(n)) \wedge f(n) = \Omega(g(n)) \tag{1}$$

...and...

$$f(n) = \mathcal{O}(g(n)) \wedge f(n) = \Omega(g(n)) \implies f(n) = \Theta(g(n)) \tag{2}$$

Your proof only deals with the second case, and is therefore incomplete.

Also, I must ask, why did you go with a proof by contradiction for part 1? It appears that a direct proof would suffice:

> Let f and g be functions, such that $f(n) = \Theta(g(n))$.
>
> By definition of Θ, there exist positive constants k_1 and k_2 such that, for sufficiently large n:
>
> $$k_1 \cdot |g(n)| \leq |f(n)| \leq k_2 \cdot |g(n)|$$
>
> Thus, for sufficiently large n:
>
> $$|f(n)| \leq k_2 |g(n)|$$
>
> Therefore $f(n) = \mathcal{O}(g(n))$ by definition.
>
> And, for sufficiently large n:
>
> $$k_1 |g(n)| \leq |f(n)|$$
>
> Therefore $f(n) = \Omega(g(n))$ by definition.
>
> ■

Still, part 2 needs to be proven for it to be an if and only if.

Question Score: 0 Answer Score: 2

Question ID: 301196

Big-O notation Basics, is it related to derivatives?

Question

I am having the hardest time with Big-O notation (I am using this Rosen book for the class I am in).

On the surface, Big-O reminds me of derivatives, rate of change and what not; is this proper thinking? If $f(n)$ is $O(g(n))$, would the derivatives have any affect on this?

Essentially is there a good resource for learning Big-O for the first time?

If I missunderstand this forum and need a specific question, then:

Prove that if $f(n) \leq g(n)$ for all n, then $f(n) + g(n)$ is $O(g(n))$. (I'd rather gain an understanding of how to do this than to have an answer to a problem).

EDIT:

My attempt at the answer to my specific question using l'Hopital:

```
lim(x) f'(x)/(f'(x) + g'(x)) = lim(x)1/(g'(x)
```

Answer

I found this question (and the first answer) helpful: Big-O Notation and Asymptotics

For example, f(n) is O(g(n)). Then, f(n) may diverge (increase without bound). However, (f(n))/(g(n)) does not, as g(n) is always greater than f(n) beyond some number N.

So, really, it has more to do with the limit of the ratio of two functions than derivatives.

Question Score: 7 Answer Score: 2
Question ID: 221720

Understanding O-notation and the meaning of Ω

Question

I am studying algorithms, and I have problems on the concepts from an exercise. Thank you so much!

> Which of the following equations lie in $O(n)$, $\Omega(n)$, $\Theta(n)$ and why.
>
> a. $3n + 2n^2$
>
> b. $\log n + 4n^3 + 6n$
>
> c. $5n + 6$

My answer:
$O(n)$: c only
$\Omega(n)$: a only
$\Theta(n)$: c only
But the answers are saying:
$O(n)$: c only
$\Omega(n)$: a, b, c
$\Theta(n)$: c only
I am confused. Doesn't Ω mean a lower bound on the equation? Then the Ω of a, b, c should be:
a: $\Omega(n)$
b: $\Omega(\log n) \leftarrow \Omega(\log n)$ should be lower than $\Omega(n)$
c: $\Omega(1) \leftarrow$ much lower than $\Omega(n)$

Answer

Plot the functions... It will make it a lot clearer. $\Omega(n)$ means that the function $f(n)$ (for large enough n) will take on values such that $f(n) > cn$, where c is some constant.

For plot examples, look here:

> Problem A (Obviously here, $f(n) \gg n$)
>
> Problem B (Again, for large enough n, $f(n) \gg n$)
>
> Problem C (For all n, $f(n)\ n$)

EDIT: Note that Ω notation is not specifying the term in $f(n)$ that grows the slowest, but rather *any function* that grows slower than the *entirety* of $f(n)$.

Question Score: 2 Answer Score: 1
Question ID: 310820

Proving with Big O Notations

Question

Is there a way I can prove that $O(3^{2n})$ does NOT equal 10^n? How would that be done? Also, is it okay to simplify $O(3^{2n})$ to $O(9^n)$ to do so?

Answer

To answer your second question first: yes, it is allowable to simplify 3^{2n} to 9^n.

Recall that $f \in \mathcal{O}(g)$ iff:

$$\limsup_{x\to\infty} \frac{f(x)}{g(x)} = c, \quad 0 \le c < \infty$$

Letting $f(x) = 10^x$ and $g(x) = 9^x$, and taking the limit:

$$\begin{aligned}
\limsup_{x\to\infty} \frac{f(x)}{g(x)} &= \limsup_{x\to\infty} \frac{10^x}{9^x} && (4.120)\\
&= \limsup_{x\to\infty} \left(\frac{10}{9}\right)^x && (4.121)\\
&\to \infty && (4.122)
\end{aligned}$$

(The last simplification is because $10/9 > 1$.)

Therefore, $f \notin \mathcal{O}(g)$.

Question Score: 2 Answer Score: 1
Question ID: 1114284

Chapter 5

Abstract Algebra

Prove that φ is automorphism

Question

G is commutative group. $|G| = n$.

$m \in \mathbb{N}$ and $\gcd(m, n) = 1$.

I need to prove that $\varphi : G \to G$, $\varphi(x) = x^m$ is automorphism of G.

My try:

I assume that $a \in \ker(G)$, so $a \in G$ and in one hand: $a^m = e$ (because $\varphi(a) = e$), and at the other hand $a^n = e$ because $a \in G \implies$ because $\gcd(m, n) = 1$, a must be e, so $\ker(\varphi) = \{e\}$.

And that's mean that φ is Aut.

I'm right? my proof is OK?

Thank you!

Answer

As pointed out in the comments, the proof given above is nearly correct–it just needs to show that ϕ is a morphism. (This was done in the comments, but since comments are supposed to be ephemeral, I'll include the outline below.)

If $a, b \in G$, then $\phi(a)\phi(b) = a^m b^m = (ab)^m = \phi(ab)$. Thus, ϕ is a homomorphism. (This requires G to be abelian.)

Question Score: 4 Answer Score: 2
Question ID: 600321

Let p be prime in $\mathbb{Z}$. Find all roots of $x^p - 1$ in $\mathbb{Z}_p$.

Question

Let p be prime in $\mathbb{Z}$. Find all roots of $x^p - 1$ in $\mathbb{Z}_p$.
I don't know where to start.
Any help/hints would be greatly appreciated.

Answer

Fermat's Little Theorem says:

> If p is a prime number, then
> $$a^p = a \pmod{p}$$

You are trying to solve:

$$x^p - 1 \equiv 0 \pmod{p}$$

Then:

$$x^p \equiv 1 \pmod{p}$$

By the FLT:...

Does that help?

Question Score: 0 Answer Score: 2
Question ID: 591751

Question

Suppose $a, b, p \in \mathbb{Z}$ with p prime. Prove that if $p \mid a$ and $p \mid a^2 + b^2$, then $p \mid b$.

I am starting with the fact that $a = pt$ with $t \in \mathbb{Z}$ and $p = (a^2 + b^2) \cdot x$ with $x \in \mathbb{Z}$. I set them equal to each other in hopes that is would give me something that could give me $p \mid b$.

Answer

Saying "p divides a" (or, symbolically, $p|a$), means $a = pt$, for some integer t. (Not the other way around, like you have it.)

There might be a more obvious approach, but I would first show that $p|x \iff p|x^2$. This is a direct proof one way, and a trivial proof by contradiction the other way.

Second, I'd show that $p|(x - y) \wedge p|x \implies p|y$.

Substituting $a^2 + b^2$ for x and a^2 for y in the above should yield the result you want.

Question Score: 1 Answer Score: 0
Question ID: 670475

How to find left and right cosets of a subgroup

Question

Q:" G group, H subgroup. G=D6, D diedras group, and H=. Find the cosets of H."

I don't understand the method to find the cosets, I've searched for answers but somehow it stays confusing... Being aH the left coset, do I give an arbitrary value from D_6 to a? And how does it multiply by $H = \{e, r, r^2, r^3, r^4, r^5\}$?

Answer

When we write aH, it means that we multiply *each element* of H by a on the left. That is:

$$aH = \{ae, ar, ar^2, ar^3, ar^4, ar^5\}$$

To find *all* the cosets of H, you need to do the above computation for every possible value of $a \in G$. (Note that two different values of a may give the same coset.)

Question Score: 0 Answer Score: 0
Question ID: 1102809

Why Rational Numbers do not include pairs (a, b) with $b = 0$?

Question

Let $X = Z \times Z$

If we have the relation R on X defined by $(a, b)R(c, d)$ if and only if $ad = bc$. Then, what is the problem if $b = 0$?

Obviously, I'm not looking for the answer that we cannot divide by 0, but rather something more fundamental. I thought that perhaps it violates the reflexivity of the equivalence relation.

Can I have a hint?

Answer

$(1, 0)R(0, 0)$ and $(0, 0)R(0, 1)$, but $(1, 0) \not R(0, 1)$. Thus, R fails to be an equivalence relation without that restriction (not transitive).

Question Score: 1 Answer Score: 6
Question ID: 1058488

Inverse of a composite permutation

Question

In a homework assignment, I am asked to find $(P_3 \circ P_1)^{-1}$ knowing:

Let $P_1 = (3\ 4\ 1\ 2\ 5), P_2 = (3\ 5\ 1\ 2\ 4)$ and $P_3 = (5\ 1\ 4\ 2\ 3)$ be three permutations.

I am second-guesing the final guessing the final step (where the composition is inverted), since the inversion seems to be the same value as the non-inverted set.

Is my method correct?

$(P_3 \circ P_1)^{-1}$ can be written as $(P_3(P_1(x)))^{-1}$ for each $x \in P_3$. So:
$(P_3 \circ P_1(1)) = (P_3(P_1(1))) = (P_3(3)) = 4$
$(P_3 \circ P_1(2)) = (P_3(P_1(2))) = (P_3(4)) = 2$
$(P_3 \circ P_1(3)) = (P_3(P_1(3))) = (P_3(1)) = 5$
$(P_3 \circ P_1(4)) = (P_3(P_1(4))) = (P_3(2)) = 1$
$(P_3 \circ P_1(5)) = (P_3(P_1(5))) = (P_3(5)) = 3$
So, $(P_3 \circ P_1) = (4\ 2\ 5\ 1\ 3)$ and so $(P_3 \circ P_1)^{-1} = (4\ 2\ 5\ 1\ 3)$

Answer

You are right to second-guess the last line. $(P_3 \circ P_1)^{-1}$ should have the cycle of $(P_3 \circ P_1)$, yet in reverse. That is:

$$(P_3 \circ P_1)^{-1} = (3\ 1\ 5\ 2\ 4)$$

More examples of finding the inverse of a cycle can be found here.

Question Score: 2 Answer Score: 1
Question ID: 830000

Question

If we have to prove that the multiplicative group of integers modulo 8, $U(8)$, is isomorphic to a set of matrices, are we allowed to define the isomorphism by saying:

$$[1] \mapsto \text{(matrix)} \tag{5.1}$$
$$[2] \mapsto \text{(matrix)} \tag{5.2}$$
$$\vdots \tag{5.3}$$

Or, can we only define the isomorphism by a rule (for example, $f(x) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ is a "rule") that works for each element in $U(8)$?

Answer

I'm assuming that by U_8 you are referring to the group of units modulo 8. (Also denoted $\mathbb{Z}_8^*$, and a whole bunch of other ways.)

It is perfectly acceptable to define a function by enumerating the individual mappings. For example, I could define a function $f : U_8 \to \{1, i, -1, -i\}$ by stating:

$$f([1]) = 1 \tag{5.4}$$
$$f([3]) = i \tag{5.5}$$
$$f([5]) = -1 \tag{5.6}$$
$$f([7]) = -i \tag{5.7}$$

This defines such a function f. (Note that I'm not checking to see if this particular f is an isomorphism, but it is certainly possible to do so.)

Question Score: 0 Answer Score: 2
Question ID: 952785

if $m, n \in \mathbb{N}$, $m < n$, then S_m isomorphic to subgroup of S_n

Question

How do show that if $m, n \in \mathbb{N}$ and $m < n$, then S_m is isomorphic to a subgroup of S_n, without using any "overpowered" results?

Answer

Consider S_3 and S_5. When we write in cycle notation, it's clear that $(123) \in S_3$ **and** $(123) \in S_5$. The same can be said for all other cycles in S_3 (e.g. $(12) \in S_3$ and S_5). So, take the subgroup of S_5 that consists of only the cycles that are also in S_3.

The same idea extends to any S_n, S_m.

Of course, this abuses the notation a bit, because writing in cycle notation does not completely define a mapping–the domain is lacking from the cycle notation. However, thinking in these "loose" terms helped me visualize the isomorphism.

Question Score: 3 Answer Score: 1
Question ID: 1064342

What are some interesting coding projects (doable in Java) that relates to group theory?

Question

I would like some ideas of possible programs I can write in Java that involves some computational aspects of group theory. My only ideas so far is to write a program that computes the product of two elements of S_n, but this is too easy. Any suggestions/ideas will be greatly appreciated.

Answer

One option (off the top of my head) would be to determine the order of an element in a cyclic group.

I suggest looking at this book (free): http://abstract.pugetsound.edu/ It has some programming problems (designed for a different language, but should be close enough) that relate to whatever topic the chapter discusses. Obviously, group theory is included.

Question Score: 5 Answer Score: 1
Question ID: 299917

Chapter 6

Linear Algebra

6.1 Vectors and Matrices

Diagonalizing using a matrix P

Question

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix witth eigenvalue λ.

(a) Show that unless it is zero, the vector $\begin{pmatrix} b \\ \lambda - a \end{pmatrix}$ is an eigenvector (I solved this)

(b) Find a matrix P such that $P^{-1}AP$ is diagonal, assuming that $b \neq 0$ and that A has distinct eigenvalues.

I need help with (b). Here is what I tried:

I know that if I consider the canonical basis of $\mathcal{M}_2(\mathbb{R})$, then one of the two colums of P has the components of the eigenvector we considered in question (a). But how do I find the other column?

I tried looking for the eigenvalues of A from scratch but I get a messy expression.

Answer

Let's call the two eigenvalues λ_1 and λ_2.

Note that the sum of the eigenvalues is equal to the trace of the matrix. That is:

$$\lambda_1 + \lambda_2 = a + d$$

Then

$$\lambda_2 = a + d - \lambda_1$$

You showed that for *any* eigenvalue r, $\begin{pmatrix} b \\ r - a \end{pmatrix}$ is an eigenvector. Thus, for μ, $\begin{pmatrix} b \\ \mu - a \end{pmatrix}$ is an eigenvector. Thus, our two eigenvectors are:

$$\vec{v}_1 = \begin{pmatrix} b \\ \lambda_1 - a \end{pmatrix} \tag{6.1}$$

$$\vec{v}_2 = \begin{pmatrix} b \\ \lambda_2 - a \end{pmatrix} = \begin{pmatrix} b \\ -\lambda_1 + d \end{pmatrix} \tag{6.2}$$

We know that to find the eigendecomposition of a matrix A, we let P be the matrix with columns corresponding to the eigenvectors and

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Then, $A = P\Lambda P^{-1}$.
So, for this case

$$P = \begin{pmatrix} b & b \\ \lambda_1 - a & -\lambda_1 + d \end{pmatrix}$$

Question Score: 3 Answer Score: 2
Question ID: 505086

Showing Orthogonality

Question

How would I do this question..... I'm familiar with Gram-Schmidt and the basics but I have no idea how to do a and b in this question.

> Suppose $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ is an orthonormal set of vectors.
>
> a) Show that $\|\vec{x}_1 + \vec{x}_2 + \vec{x}_3\| = \sqrt{3}$.
>
> b) Suppose that a vector $\vec{y}$ is orthogonal to each of the vectors $\vec{x}_1, \vec{x}_2, \vec{x}_3$. Show that $\vec{y}$ is also orthogonal to $66\vec{x}_1 - 17\vec{x}_2 + \vec{x}_3$.

Answer

Problem A:

By definition of the norm:

$$\|x_2 + x_2 + x_3\| = \sqrt{\langle x_2 + x_2 + x_3, x_2 + x_2 + x_3 \rangle}$$

So, we compute the inner product:

$$\begin{aligned}
\langle x_2 + x_2 + x_3, x_2 + x_2 + x_3 \rangle &= 2(\langle x_2, x_2 \rangle + \langle x_2, x_2 \rangle + \langle x_2, x_3 \rangle) && (6.3)\\
&+ \langle x_3, x_2 \rangle + \langle x_3, x_2 \rangle + \langle x_3, x_3 \rangle && (6.4)\\
&= 2(1 + 1 + 0) + 0 + 0 + 1 && (6.5)\\
&= 5 && (6.6)
\end{aligned}$$

Note that the inner product of two identical members (e.g. $\langle x_2, x_2 \rangle$) is 1 due to the "normal" part of the orthonormal collection. Also note that the inner product of two different members (e.g. $\langle x_3, x_2 \rangle$ is 0 due to the "orthogonal" part of the orthonormal collection.

Our result above implies that the requested norm is $\sqrt{5}$. Note that this is different than our desired answer. This is because the problem has a typo; it should request $\|x_1 + x_2 + x_3\|$, not $\|x_2 + x_2 + x_3\|$

Problem B: I'm just giving a hint here. Take the inner product of y and each of the vectors, and show that it is 0 for all three. Recall that the inner product of two normal vectors is 0.

Question Score: 2 Answer Score: 4
Question ID: 766763

three dimensional cross product

Question

Why do two three dimensional vectors x and y such that $x \cdot y$ does not equal $x \times y$ do not not exist? They do not exist right? Please help me kinda lost in this.

Answer

The dot product is a scalar quantity. The cross product is a vector quantity. To compare a scalar and a vector is like comparing apples and oranges–it *really* doesn't make sense to do so.

For example:

$$(1, 0, 0) \times (0, 0, 1) = (0, 1, 0)(1, 0, 0) \cdot (0, 0, 1) = 0$$

You can't really compare $(0, 1, 0)$ and 0–one is a vector, the other is "just a number."

Question Score: 0 Answer Score: 4
Question ID: 440956

Inverting all values in matrix

Question

Lets say I have a matrix:

$$\begin{bmatrix} 2 & 4 \\ 3 & 7 \end{bmatrix}$$

And my maximum range value is 10, how would I go about creating another matrix that inverts those values? So that the matrix would end up looking like:

$$\begin{bmatrix} 8 & 6 \\ 7 & 3 \end{bmatrix}$$

In algebraic form?

Answer

Look at the individual entries in the matrix, and see what relationship you can find between those entries, and the "maximum range value".

From your example, in the $(1,1)$ cell in the first matrix, you have the entry 2. In the second matrix, you have the entry 8. Your maximum value is 10. Can you see some relationship between the numbers 2, 8, and 10? Does this relationship hold for the other values? (for example, 3, 7, and 10)

You should find that the appropriate formula is:

$$Y = m \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} - X$$

…where m is your "maximum range value" and X is your input matrix.

Question Score: 1 Answer Score: 0
Question ID: 324255

Finding vector of co-ordinates

Question

How do we find a co-ordinate vector in Algebra?

For example, given:

$$\begin{pmatrix} 2 & -3 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

How do we calculate v_1 and v_2?

Answer

The hard way

Whenever you have a linear system of the form:

$$A\vec{x} = \vec{b}$$

where A is a square matrix and $\vec{x}$ and $\vec{b}$ are column vectors, you can solve this by finding A^{-1}:

$$\vec{x} = A^{-1}\vec{b}$$

To find A^{-1}, we augment A with the identity, and row reduce.

So, for your example:

$$A = \begin{pmatrix} 2 & -3 \\ 0 & -4 \end{pmatrix}$$

$$\vec{b} = \vec{0}$$

$$\vec{x} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

So, to find A^{-1}

$$\left(\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 0 & -4 & 0 & 1 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 0 & 1 & 0 & \frac{-1}{4} \end{array}\right) \Longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{-3}{8} \\ 0 & 1 & 0 & \frac{-1}{4} \end{array}\right)$$

So, $A^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{-3}{8} \\ 0 & \frac{-1}{4} \end{pmatrix}$.

Now, we simply plug into our formula:

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{-3}{8} \\ 0 & \frac{-1}{4} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Multiplying matrices, we have:

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The easy way

We can also note that this is a homogeneous linear system with linearly independent rows. We can conclude, therefore, that the only solution of this system is the trivial one; that is, $v_1 = 0, v_2 = 0$.

Question Score: 0 Answer Score: 2
Question ID: 369619

Vector calculation question

Question

the points a b c d are concordantly (1,2,-3) , (-1,2,1) , (0,1,-2) , (2,-1,1)
find formula of the plane going thorugh d and which is pararlel to plane abc
calculate the volume of pyramid abcd.
attemp to solve:
the vectors ab ac ad are concordantly -204 -1-11 1-34
using the determinant: $\begin{vmatrix} x-1 & y-2 & z+3 \\ -2 & 0 & 4 \\ -1 & -1 & 1 \end{vmatrix}$ whihc yields: 2x+y +z-1=0 - formula for plane abc

normal is hence 2 1 1 and combining that with the info on point d we get that d (free argument) for desired plane is -4 and therfore its formula is 2x+y+z-4=0

for the volume we calulate that $ab \cdot ac$ (dot product) is 6, and the scalar product of their sizes is sqrt of 60. arccos of ratio of the two is 39.23 degrees. the volume would be sixth of the scalr prosuct of sizes of ab and ac and sin of said angle (39.23), yielding 0.816. I got other results in the textbook so I would like to ask help on this one.

Answer

Assuming your determinant calculation is correct, your work for the plane looks fine to me. I'm not seeing how to do the pyramid problem in my mind, so I can't check that for you.

As a side note: the dot product ($\cdot$) is the *scalar product*. Thus, I don't really understand what you're saying in your last paragraph ("the dot product is 6, and the scalar product is...") The scalar product and dot product are the same thing.

Question Score: 2 Answer Score: 0
Question ID: 854257

Vector Addition and Subtraction - interpretation

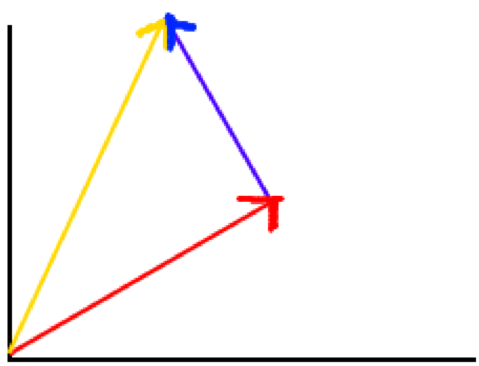
Question

If we have two vectors a and b, both in $\mathbb{R}^n$, is it correct to think of

1. $a - b$ as how similar the two vectors are?
2. $a + b$ as moving the vector a in the direction of vector b?

Answer

We can think of vector addition, $\vec{a} + \vec{b}$, as describing the point at the end of $\vec{b}$ if $\vec{b}$ started at the tip of $\vec{a}$. In the picture below, the red vector is $\vec{a}$, the blue vector is $\vec{b}$, and the yellow vector is $\vec{a} + \vec{b}$:



For vector subtraction, it is best to think of it with regards to addition. First, realize that $-\vec{a}$ is just $\vec{a}$ reflected about the origin. Then, treat $\vec{a} - \vec{b}$ as $\vec{a} + (-\vec{b})$ and visualize it as with addition.

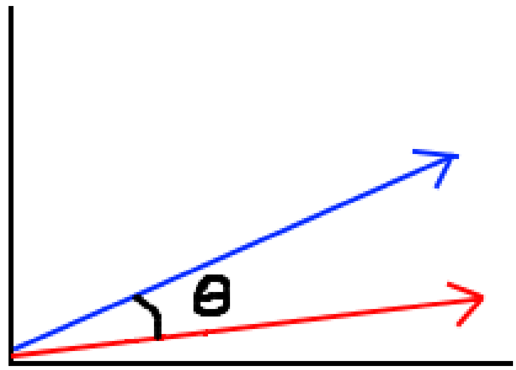
If you're looking for a measure of vector similarity, we typically use the *inner product* or *dot product*: $\vec{a} \cdot \vec{b}$. This makes since, because we'd like to think of "similarity" as a scalar quantity, rather than a whole vector. If you've learned the relationship between the dot product and cosine, this makes a bit more sense:

$$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos \theta$$

...where θ is the angle between the two vectors, as in the image below:

So, vectors that point in the same direction are "similar," while vectors that point in opposite directions are "dissimilar." Normalization is helpful here, because then $+1$ means "identical vectors," and -1 means "reflected vectors." A value of 0 means perpendicular.

Question Score: 3 Answer Score: 3
Question ID: 1226282



how to manipulte two vectors to result in a vector which elements are products of corresponding elements of the input vectors?

Question

so I have 2 vectors. How do I manipulate them so I end up with the vector of the same size which elements are a product of the corresponding elements of the two vectors. How would I do that for an arbitrary number of vectors? Thanks!

Answer

My interpretation of the question:

> If you have vectors:
> $$\vec{a} = \langle a_1, a_2, \ldots, a_n \rangle$$
> $$\vec{b} = \langle b_1, b_2, \ldots, b_n \rangle$$
> How do you create the vector below?
> $$\vec{c} = \langle a_1 b_1, a_2 b_2, \ldots, a_n b_n \rangle$$

This is the exact definition of element-wise multiplication, which is actually a defined operation. See this question (and those it is linked to) for more information: Symbol for elementwise multiplication of vectors

EDIT: It appears this is also called the Hadamard product.

Question Score: 2 Answer Score: 1
Question ID: 310150

Find an orthogonal vector to 2 vector

Question

I have the following problem:

> A B C D are the 4 consecutive vertices of a parallelogram with the following coordinates
>
> $$A = (1, -1, 1)$$
> $$B = (3, 0, 2)$$
> $$C = (2, 3, 4)$$
> $$D = (0, 2, 3)$$

I must find a vector that is orthogonal to both CB and CD.
How can I do this? Is there some kind of formula?
Thanks,

Answer

The cross product of two vectors is orthogonal to both, and has magnitude equal to the area of the parallelogram bounded on two sides by those vectors. Thus, if you have:

$$\overrightarrow{CB} = \langle 3-2, 0-3, 2-4\rangle = \langle 1, -3, -2\rangle$$

$$\overrightarrow{CD} = \langle 0-2, 2-3, 3-4\rangle = \langle -2, -1, -1\rangle$$

Compute the following, which is an answer to your question:

$$\langle 1, -3, -2\rangle \times \langle -2, -1, -1\rangle = \langle 1, 5, -7\rangle$$

Note, though, that there are infinitely many vectors that are orthogonal to $\overrightarrow{CB}$ and $\overrightarrow{CD}$. However, these are all non-zero scalar multiples of the cross product. So, you can multiply your cross product by any (non-zero) scalar.

Question Score: 1 Answer Score: 2
Question ID: 357746

Incremental Cartensian Coordinates Betwwen Two Known Coordinates

Question

I've done a lot of searches and haven't found exactly what I'm looking for. I'm looking for an algorithm that will provide me the cartesian coordinates (xyz) every 100ft between two known cartesian coordinates (xyz).

Any assistance would be great.

Answer

You have two points, $\vec{p_1} = (x_1, y_1, z_1)$ and $\vec{p_2} = (x_2, y_2, z_2)$.

The line between these two points is given by:

$$(x, y, z) = \hat{v}t + \vec{p_1}$$

where $\hat{v}$ is the unit vector from $\vec{p_1}$ to $\vec{p_2}$, and t ranges from 0 to $\|\vec{p_2} - \vec{p_1}\|$.

To find the points you're looking for, substitute multiples of 100 in for t: $0, 100, 200, \ldots$

Question Score: 0 Answer Score: 0

Question ID: 871082

Parallel Vectors

Question

My book defines "parallel vectors" as follows:

> Two nonzero vectors x and y are called **parallel** if $y = tx$ for some nonzero real number t. (Thus, nonzero vectors having the same or opposite directions are parallel.)

Would this alternate definition work?

> If we let $z = (p_0, p_1, p_2)$ and $\zeta = (q_0, q_1, q_2)$, and if we say $z \diamond \zeta$ defines the operation $(\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2})$, then if each $\frac{p_i}{q_i}$ is equal for all $i = 0, 1, 2$, then x is parallel to y.

Answer

The definition provided in the book is different than the definition you have created. How can we show this? Simply provide a counter-example:

Let $\vec{a} = \langle 1, 0, 4 \rangle$, and $\vec{b} = \langle 2, 0, 8 \rangle$.

These vectors are parallel by the book's definition, as $\vec{a} = 2\vec{b}$.

However, using your $\diamond$ operator:

$$\vec{a} \diamond \vec{b} = \left\langle \frac{1}{2}, \frac{0}{0}, \frac{4}{8} \right\rangle$$

We cannot conclude that $\frac{1}{2} = \frac{0}{0} = \frac{4}{8}$, as $\frac{0}{0}$ is undefined.

Thus, your definition is not equivalent.

Question Score: 0 Answer Score: 0

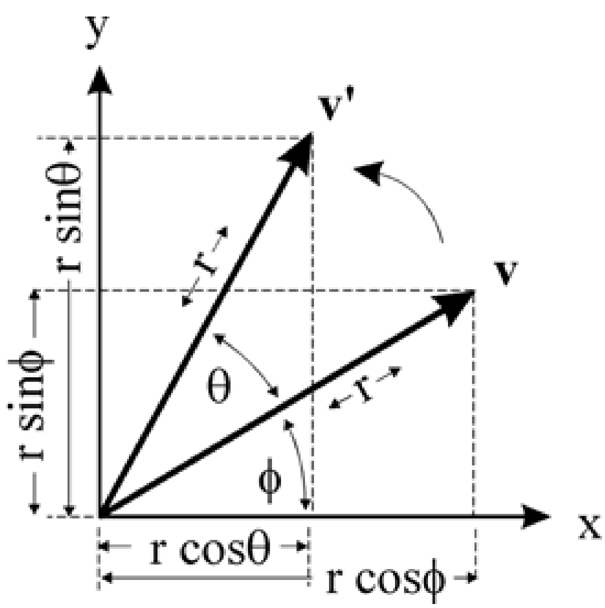
Question ID: 310795

6.2 The Rotation Matrix

2D rotation of point about origin

Question

I'm in the process of learning game development and have a question regarding a simple rotation. So far, I'm visualizing the rotation as such:



I've read a similar question, but I'm struggling to understand how to apply this given formula:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Given a point of `(2,3)`, what would the point be after a rotation in the `xy` plane about the origin through an angle of `-180` degrees?

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(-180°) & -\sin(-180°) \\ \sin(-180°) & \cos(-180°) \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Is this correct?

Answer

The difficulty you seem to be having is with matrix multiplication. I suggest that you watch the Khan Academy videos on this, as he does a great job of explaining it.

For your example, we apply this as follows:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos 180 & -\sin 180 \\ \sin 180 & \cos 180 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} (-1)(2) + (0)(3) \\ (0)(2) + (-1)(3) \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

So, a rotation of 180 degrees results in a new point of $(-2, -3)$.

Question Score: 2 Answer Score: 2
Question ID: 346672

How do I transform $f(x) = \log(1 + e^x)$ such that graph rotates 90° on the x-y axis

Question

I am looking for a function $f(x)$ that is of a specific shape on the x-y axis.
I have a function $f(x) = \log(1+e^x)$ that has right shape. I want it rotated 90° on x-y axis.
How can I get an $f(x)$ that is essentially a 90° rotation of $f(x) = \log(1 + e^x)$?

Answer

This approach is for an arbitrary angle change, and then applied to a 90° rotation.
We use the formula for the rotation matrix:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The vector on the LHS is the new, rotated point; the vector on the RHS is the input point.
For your equation, we have $x = x$, $y = \ln(1 + e^x)$. So, we put that in the RHS vector:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ \ln(1+e^x) \end{pmatrix}$$

Performing matrix multiplication:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x\cos\theta - \ln(1+e^x)\sin\theta \\ x\sin\theta + \ln(1+e^x)\cos\theta \end{pmatrix}$$

Now, we insert the desired value for θ, and simplify:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x\cos(90°) - \log(1+e^x)\sin(90°) \\ x\sin(90°) + \ln(1+e^x)\cos(90°) \end{pmatrix} \tag{6.7}$$

$$= \begin{pmatrix} x \cdot 0 - \log(1+e^x) \cdot 1 \\ x \cdot 1 + \ln(1+e^x) \cdot 0 \end{pmatrix} \tag{6.8}$$

$$= \begin{pmatrix} -\ln(1+e^x) \\ x \end{pmatrix} \tag{6.9}$$

This gives us a set of simultaneous equations to solve:

$$\begin{cases} x' = -\ln(1+e^x) \\ y' = x \end{cases}$$

We want to solve for y' in terms of x':

$$\begin{aligned} x' &= -\ln(1+e^{y'}) \\ -x' &= \ln(1+e^{y'}) \\ e^{-x'} &= 1 + e^{y'} \\ e^{y'} &= e^{-x'} - 1 \\ y' &= \ln(e^{-x'} - 1) \end{aligned}$$

This process gets the same answer as just switching y and x in the original equation, but it lets us perform other rotations (for example, a 45° rotation) instead of just a 90° rotation.

Question Score: 5 Answer Score: 0
Question ID: 435219

How to multiply matrices?

Question

I've written a java-programm(JOGL) and have my vertices from the object I want to draw on screen. I've read, when I want to rotate my object I must multiply the matrix of my object with the so-called "rotation-matrix". Link -¿ rotationsmatrix . I don't know what I've done wrong by multiplying the two matrices. So please, can anyone help me?

Answer

I cannot read German, but I used Google translate. Please forgive me if I've misunderstood your goal.

Your mistake was in your definition of the rotation matrix. Because you are rotating the object around the positive z axis, the z coordinate should **not** change after the rotation. The rotation matrix should be a 3×3 matrix, not a 4×4; the result should be a 3×1 column vector that describes a point, not a 4×4 matrix.

I will call $\vec{M}$ the original point vector and $\vec{M}'$ the rotated point vector.

Thus:

$$\vec{M}' = \mathbf{R}\vec{M}$$

We let $\vec{M} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Because we are rotating *around the positive z-axis*, the rotation matrix is:

$$R = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus:

$$\vec{M}' = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{6.10}$$

$$= \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ z \end{pmatrix} \tag{6.11}$$

Question Score: 0 Answer Score: 2
Question ID: 460447

6.3 Linear Systems

Assignment: Find the number of parameters in the general solution to a system of linear equations

---

Question

This is a question given in an assignment I'm working on:

> If the coefficient matrix A in a homogeneous system of 33 equations with 28 unknowns is known to have rank 12, how many parameters are there in the general solution?

I've deduced that, since this is a homogeneous system with fewer variables than equations, the only solution is the trivial solution; I'm unsure, however, how to find the number of parameters in this solution.

How would I go about finding the number of parameters in this kind of abstract situation?

Answer

You add a parameter for every column without a pivot in the REF form of the coefficient matrix. That is, there are the same number of parameters as the dimension of the null space of A. What is the relationship between $\dim NS(A)$ and $\mathrm{rk}(A)$? It is:

$$\dim NS(A) = \text{columns} - \mathrm{rk}(A)$$

Question Score: 0 Answer Score: 2
Question ID: 544822

What is the names of $A\vec{x} = \vec{b}$ linear equation system components?

Question

Having $A\vec{x} = \vec{b}$.

What is the names of $A\vec{x} = \vec{b}$ linear equation system components?

Answer

According to Elementary Linear Algebra by Venit and Bishop, we have:

$$A\vec{x} = \vec{b}$$

A is the coefficient matrix of the system

$\vec{x}$ is the (column) vector of unknowns

$\vec{b}$ is the (column) of constants.

Granted, its probably not the most well-known book, but it's the one I have ;)

Question Score: 2 Answer Score: 3

Question ID: 405226

Linear Algebra Complex values

Question

The cube roots of $-3+2i$ are $x_1 = (1.0106+1.1532i),\ x_2 = (0.4934-1.4519i),$ and $x_3 = (-1.5040 + bi)$

What is b?

So

$$-3 + 2i = (x_1)(x_2)(x_3) = -3.268 + 2.172bi + 1.351i + 0.898b$$

$$-3 + 2i - 1.351i = -3.268 + 2.172bi + 0.898b$$

$$-3 + 0.649i + 3.268 = 2.172bi + 0.898b$$

$$0.268 + 0.649i = 2.172bi + 0.898b$$

Now I'm stuck.

Answer

You're nearly there! :)

Since two complex numbers are equal iff their real and imaginary parts are equal, we know that:

$$0.268 = 0.898b \qquad 0.649 = 2.172b$$

But, we are dealing with rounded-off values. This means it is reasonable to expect that the two equations above may give slightly different answers.

We find that the first equation gives ≈ 0.298, and the second gives ≈ 0.298.

Question Score: 2 Answer Score: 2
Question ID: 657989

Linear Algebra Rank

Question

> If A is an $n \times n$ matrix and $A\vec{x} = \vec{b}$ has exactly one solution for each $\vec{b} \in \mathbb{R}^n$, then $\text{rk}(A) = n$.

Can you help me get started on this question? So far, I'm leaning towards the path of $A\vec{x} = 0$. And 0 falls under any $\mathbb{R}^n$. But if the rank is 0, then it must be a null vector. So I'm guessing it's false.

Is this true or false?

Answer

This statement is true. Why? Here's an outline:

If there is one solution to the system $A\vec{x} = \vec{b}$, then the columns of A are independent. (I believe this is typically shown by contradiction.)

$\text{rk}(A)$ is defined as the dimension of the row space of A, which is equivalent to the dimension of the column space of A. (This is typically given as a theorem... I forget how to prove the two dimensions are equal, but that should be a fairly well-known proof.)

Since all n columns are independent, the basis for $RS(A)$ has n elements. Thus, $\text{rk}(A) = n$.

Question Score: 2 Answer Score: 0
Question ID: 654120

Why would a system of equations be solved this way?

Question

If my instructions are to solve a system of equations using Gauss-Jordan elimination, and the matrix below is one of my final steps, why would I not be expected to fully row-reduce it to the second matrix shown here?

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 1/2 \\ 0 & 1 & -2 & 1 & 1 \\ 0 & 0 & -6 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

In other words, why would you write the solution (w,x,y,z) as (1/2+y,1+2y-z,y,z) which is what my book and calculator gave, rather than (1/2,1-z,0,z) when y apparently must be equal to 0? My only guess is for generality, but it seems to me that y must equal zero for the system to have a solution. I'm new to this.

Answer

You have just encountered the difference between Row-Echelon Form (REF) and Reduced Row-Echelon Form (RREF). The reasons that most people (myself included) don't perform the row operations to reach your second matrix from the first matrix are twofold:

1. The number of solutions to the system can already be ascertained from REF form. I can tell (at a glance) how the remaining steps to reach RREF will change the matrix, so I can save time. Often, one only cares *if* there's a solution, not necessarily what it is.

2. It is faster to backsubstitute than to continue row reducing. That is, once you reach your first matrix, simply turn it back into a set of equations and use standard high school algebra techniques (substitution) for solving.

Question Score: 1 Answer Score: 1
Question ID: 1083110

Least square solution of a matrix

Question

Determine the least squares solution to the system $A\mathbf{x} = B$ below:

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 5 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \\ -2 \end{bmatrix}$$

To find the least square regression we need to solve:

$$(A^\top A)^{-1} A^\top B = x$$

The important matrices are:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 5 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

$$A^\top = \begin{bmatrix} 1 & 1 & 2 & 2 & 3 \\ 2 & 3 & 5 & 0 & 1 \\ 1 & 2 & 3 & 1 & 1 \end{bmatrix}$$

$$A^\top A = \begin{bmatrix} 6 & 9 & 15 & 3 & 6 \\ 9 & 14 & 23 & 4 & 8 \\ 15 & 23 & 38 & 7 & 14 \\ 3 & 4 & 7 & 5 & 7 \\ 6 & 8 & 14 & 7 & 11 \end{bmatrix}$$

But, my problem is that $(A^\top A)^{-1}$ is singular.

Answer

You have computed AA^T, not $A^T A$. The correct $A^T A$ should be:

$$\begin{pmatrix} 19 & 18 & 14 \\ 18 & 39 & 24 \\ 13 & 24 & 16 \end{pmatrix}$$

Question Score: 1 Answer Score: 1
Question ID: 355486

6.4 Vector Spaces

Need to check the meaning of a Transition matrix

Question

Is the transition matrix just the change of basis matrix from a **non-standard** basis to the **standard basis**?

Answer

No. A transition matrix can convert between any two bases B and C.

Source: *Elementary Linear Algebra with Applications, 3rd ed.* by Richard Hill. Page 220.

Question Score: 0 Answer Score: 0

Question ID: 781349

Prove $e^x, e^{2x}..., e^{nx}$ is linear independent on the vector space of $\mathbb{R} \to \mathbb{R}$

Question

Prove $e^x, e^{2x}..., e^{nx}$ is linear independent on the vector space of $\mathbb{R} \to \mathbb{R}$

isn't it suffice to say that e^y for any $y \in \mathbb{R}$ is in $\mathbb{R}^+$

Therefore, there aren't $\gamma_1, ...\gamma_n$ such that $\gamma_1 e^x + \gamma_2 e^{2x} ... + \gamma_n e^{nx} = 0$.

Therefore, they're not linear dependent.

I've seen a proof goes as follow:

take $(n-1)$ derivatives of the equation. then, you got n equations with n variables. Arranging it in a matrix (which is found out to be Van-Der-Monde matrix).

calculate the determinant which is $\neq 0$. Therefore, only the trivial solution exist. Therefore, no linear dependency.

Is all that necessary?

Answer

Since the γ_i (using your notation) can be negative, it does not suffice to state that $e^x > 0 \; \forall x \in \mathbb{R}$.

You can either use the matrix/determinant method, or, I believe, you can look at the power series of e^x to do so.

Question Score: 6 Answer Score: 1
Question ID: 633724

Subspace of vector space.

Question

I have task sound like:

Examine that W is a subspace of the vector space M_{3x3}

$W = (A : A^t = -A)$

To check it from definition I have to check two conditions.

1) $\vec{u} + \vec{v} \in W$

2) $\alpha \cdot \vec{u} \in W$

1) B, C are matrix 3x3 and $\in W$

$B^t + C^t = (B + C)^t = -A$ and i don't know what i can do next. Could anyone tell me how i can do this task?

Answer

First Condition:

Two matrices in the subspace are B and C. These satisfy:

$$B^t = -B$$

$$C^t = -C$$

You must show that the sum of the two matrices is also in the subspace:

$$B^t + C^t = -B - C$$

$$(B + C)^t = -(B + C)$$

Which is in the subspace.

Don't compare the matrices to A; A is an arbitrary matrix in the subspace used to explain the definition of the subspace.

EDIT:

Second condition:

Let B be in the subspace (that is, $B^t = -B$), and a be an arbitrary scalar.

$$(a \cdot B)^t = a \cdot B^t$$

Because $B^t = -B$:

$$(a \cdot B)^t = a \cdot (-B)$$

$$(a \cdot B)^t = -(aB)$$

Thus $a \cdot B$ is in the subspace.

Question Score: 0 Answer Score: 0
Question ID: 249188

Is $S = \{(1,t) \mid t \in \mathbb{R}\}$ a subspace of $\mathbb{R}^2$?

Question

My professor introduced subspaces of $\mathbb{R}^n$ today and I don't think I understand them very well.

He posed this question as an example:

> Is the set $S = \{(1,t) \mid t \in \mathbb{R}\}$ a subspace of $\mathbb{R}^2$?

He said that it wasn't. Could anyone elaborate as to why it isn't?

Can I simply say that the zero vector, $\vec{0} = (0,0)$ can never equal $(1,t)$ and be done with it?

Answer

To show that some set is a subspace, you must show two things:

1. You can pick any two vectors in the set, add them together, and the result is in the set.
2. You can pick any vector in the set, multiply it by any scalar, and the result is in the set.

The easiest check is for the $\vec{0}$ to be in the set. Why? If it isn't in the set, then multiplying by zero (a scalar) results in a vector *not* in the set! (Violation of condition 2.)

So, in short, you're perfectly right. ;)

Question Score: 4 Answer Score: 3
Question ID: 417994

Properties of Derivative function on $\mathbb{R}[x]$

Question

> Let $\mathbb{R}[x]$ denote the vector space of all real polynomials. Let $D : \mathbb{R}[x] \to \mathbb{R}[x]$denote the map $Df = \frac{df}{dx}$, for all f. Is there $E : \mathbb{R}[x] \to \mathbb{R}[x]$ such that $E(D(f)) = f, \forall f$?

I agree with Olivier Oola's comment that E appears to act as an antiderivative. Here's what I've tried:

First, let a function $C : \mathbb{R}[x]\mathbb{R}[x]$ such that $C(a_0 + a_1x + \cdots + a_nx^n) = a_0$. Then, define $E(a_0 + a_1x + \cdots + a_nx^n) = \int(a_0 + a_1x + \cdots + a_nx^n) + C(a_0 + a_1x + \cdots + a_nx^n)$. But, I realize that this not right

Answer

Hint: Consider the polynomials $p, p' \in \mathbb{R}$, where $p = 5$ and $p' = 1$. Then, let $\alpha = D(p) = D(p')$. What is $E(\alpha)$? Can E be a map?

Question Score: 2 Answer Score: 1
Question ID: 1079858

How is b not a linear combination of these vectors?

Question

Determine if $\vec{b}$ is a linear combination of $\vec{a}_1, \vec{a}_2, \vec{a}_3$.

$\vec{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}, \vec{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$

Okay, so I made my constants x_1, x_2, x_3 for $\vec{a}_1, \vec{a}_2, \vec{a}_3$, respectively. I end up getting the following consistent system:

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which has the general solution:

$$\begin{cases} x_1 = 2 - 5x_3 \\ x_2 = 3 - 4x_3 \\ x_3 = \text{free} \end{cases}$$

So $\vec{b}$ is equal to infinitely many linear combinations of $\vec{a}_1, \vec{a}_2, \vec{a}_3$, right? Why does my book say that $\vec{b}$ is *not* a linear combination of these three vectors? Must the constants be a unique solution?

Answer

The book is wrong. $\vec{b}$ is a linear combination of those three vectors. (e.g. $\vec{b} = 2\vec{a_1} + 3\vec{a_2}$, among other combinations.)

Question Score: 6 Answer Score: 6

Question ID: 651283

Linear Algebra Question: Prove that no proper subset spans

Question

I have to prove the following:

> **S** is a basis for linear space **L** if and only if it is a minimal spanning set for **L**. In other words **S** is a basis for **L** if and only if **S** spans **L** and no proper subset of **S** spans **L**.

I've got the general setup for this proof, but I can't for the life of me figure out how to prove that no proper subset of S spans L. Any help would be greatly appreciated.

Answer

The key here is that all elements of the basis must be linearly independent. Proof outline:

1. Assume that S is a basis for L, and $S' \subsetneq S$ also spans L.
2. Then, since the elements of $A = (S' \setminus S)$ are in L, the elements of A can be written as linear combinations of the elements in S'.
3. This contradicts the independence of S (the basis).

Question Score: 0 Answer Score: 0
Question ID: 766713

What are elements of a field called

Question

From Linear Algebra by Serge Lang we have

> "Let K be a field. Elements of K will also be called **numbers** (without specification) if the reference to K is made clear by the context, or they will be called **scalars**."

What is the meaning of "if the reference to K is made clear by the context"? Is in this context the term numbers equivalent to scalars? Are there any difference between numbers and scalars in general?

Answer

In Linear Algebra, we can sometimes use vector spaces whose elements "look" just like scalars. For example:

> Define the vector space $V \subseteq \mathbb{R}^+$ with:
>
> - Addition: $x + y = xy$
> - Scalar multiplication: $rx = x^r$

Clearly, we could call x, y, and r "numbers." Yet, in this context, it is unclear which numbers are *vectors* and which are *scalars*. So, we call x and y **vectors** and r a **scalar**. This is what is meant by "when it cannot be determined by context."

Question Score: 2 Answer Score: 1
Question ID: 749073

Chapter 7

Computer Math

7.1 Computer Fundamentals

Ways of defining a recursive function that counts right-parenthesis in a string

Question

I'm trying to find a more elegant way of defining a recursive function on $\{(,)\}$ that counts right-parenthesis in a string.

Let r be a function on $\{(,)\}$ defined recursively so that:

- $r(\lambda) = 0$, and
- $r(sx) = r(s) + 1$, if x is symbol)
- $r(sx) = r(s)$, if x is symbol (

I have also found another way of saying the same thing:

Leg r be a function $\{(,)\}$ defined recursively so that:

- $r(\lambda) = 0$, $r\left(()\right. = 0$, $r\left.())\right. = 1$, and
- $r(sx) = r(s) + r(x)$, where x is a symbol

Is the last example correct? Is there anybody who can recommend a better way of defining this?

NB! I am studying mathematical logic in a different language, sorry in advance if I used some terms incorrectly.

Answer

The two definitions are equivalent. I agree with the commenter that the first form is more intuitive.

The only change I could suggest would be a typesetting/presentation change to the following:

Define $r(s)$ recursively by $r(\lambda) = 0$, and

$$r(sx) = \begin{cases} r(s) + 1 & \text{if } x = \,) \\ r(s) & \text{otherwise} \end{cases}$$

(where x is a character/symbol)

By using "otherwise" instead of "if $x = ($", it is abundantly clear that) is the only character we care about. This also extends the definition to work for larger alphabets than $\{(,)\}$.

Question Score: 2 Answer Score: 2
Question ID: 1012226

Decimal representation of the binary number 10001011

Question

So 10001011 is an 8-bit twos complement. Now what is the Decimal representation of the number x represented by 10001011? My steps:

1. $10001011 - 1$ and I get 01110110
2. Flip the digits and you get 10001001
3. Now I'm supposed to convert 1000 and 1001 into digits (0 to 9) but am not sure how to do it efficiently, it'll take a lot of time to just start calculating. Any suggestions ?

Answer

As a tip for the final binary to decimal conversion–what many programmers will do is convert first to hexadecimal, then convert to binary. The hexadecimal step makes the conversion process shorter, as one can easily memorize binary to hex conversions.

So, $0111\ 0101_2$ is the same as 75_{16}. (For the conversion to hex, note that $0111_2 = 7_{16}$, and $0101_2 = 5_{16}$.) To convert to decimal, now we just have to compute

$$7 \cdot 16 + 5 \cdot 1 = 112 + 5 = 117_{10}$$

As pointed out in other answers, this is the negative of the original number, so we have the final result of -117.

Question Score: 1 Answer Score: 0
Question ID: 612079

Factorial of 1,e+80

Question

Recently I started being very fascinated in logistics, and out of the blue came the question into my head, what is the factorial of the amount of atoms in the observeable universe, which is said to be between `1,e+78` and `1,e+82` but the amount of ways you can arrange these atoms is unimaginably larger. So I played with this number and I thought it might be interesting to see how far I could get on calculating the factorial of `1,e+80`, so I went ahead and created a simple java program:

```java
import java.awt.Color;
import java.math.BigDecimal;
import java.math.RoundingMode;
import java.text.DecimalFormat;
import java.text.NumberFormat;

import javax.swing.JFrame;
import javax.swing.JLabel;

public class Factorial {

    public static void main(String[] args) {
        JFrame frame = new JFrame("Project Factorial");
        frame.setAlwaysOnTop(true);
        frame.setDefaultCloseOperation(JFrame.
            DO_NOTHING_ON_CLOSE);
        frame.setLocation(0, 0);
        frame.setUndecorated(true);
        JLabel label = new JLabel();
        label.setForeground(Color.white);
        frame.getContentPane().setBackground(Color.BLUE);
        frame.getContentPane().add(label);
        frame.setVisible(true);
        BigDecimal atoms = new BigDecimal("1E+80");
        BigDecimal total = new BigDecimal("1");
        double increment = new BigDecimal("100.0").divide(atoms)
            .doubleValue();
        double percentage = increment;
        for (BigDecimal num = new BigDecimal("2"); num.compareTo
            (atoms) <= 0; num.add(BigDecimal.ONE)) {
            total = total.multiply(num);
            percentage += increment;
            label.setText("[" + String.format("%-10.3f%%",
                percentage) + "]" + format(total, total.scale()))
                ;
            frame.pack();
            Thread.yield();
        }
```

```
    }

    private static String format(BigDecimal x, int scale) {
        NumberFormat formatter = new DecimalFormat("0.0E0");
        formatter.setRoundingMode(RoundingMode.HALF_UP);
        formatter.setMinimumFractionDigits(scale);
        return formatter.format(x);
    }

}
```

Quickly after running this program for about an hour I realized running this kind of calculation on any computer today would absolutely take ages, I was hopping to reach 0,001 % on the calculation but it is certainly too large to calculate, but then the question came up, exactly how long would it take? That question is not so easy to answer considering there is alot of factors involved, but really what I am trying to solve is the factorial of `1,e+80`.

$$(1 \times 10^{80})!$$

The number can be very hard to understand just how big it is, so I'd like to visualise how big that number is, for instance, if you could calculate how long it would take for a computer to calculate the factorial of `1,e+80` that would be a cool visualization.

(**EDIT:** Thanks for the great answers, however, I wish to implement a way of calculating the factorial of `1,e+80` in a application despite I would need to use some kind of approximation formula, so I decided to use Stirling's approximation based on derpy's answer.

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

So with Stirling's approximation and GMP library for C programming language it would be possible to make a quite accurate and efficient program to calculate the factorial of `1,e+80`)

Answer

For the computation time aspect of the question:

Let's assume that a single multiplication takes "constant" time (and that we can perform 10^9 such operations in a second–a reasonable approximation with today's processor).

To compute the factorial of n, you must perform n multiplications. Recall that we can perform 10^9 multiplications per second. So, the runtime of that factorial is:

$$\frac{10^{80} \text{ multiplications}}{10^9 \text{ multiplications per second}} = 10^{71} \text{ seconds} \approx 3 \times 10^{63} \text{ years}$$

Basically, you'll never see that factorial program terminate.

Question Score: 6 Answer Score: 14
Question ID: 774303

binary subtraction

Question

I am trying to solve binary subtraction:

$$11000_2 - 1011_2 = 1001_2$$

I know that it should be 1001_2, when checking with answer key however I am not sure how it was calculated as I get different result : 1011_2, as $0 - 1$ should give 1.

Thank you for your help in advance!

Answer

Don't forget to "borrow" when subtracting 1 from 0.

Here it is in "slow motion:"

$$\begin{array}{rccccc} & 1 & \cancel{1}^0 & \cancel{0}^1 & \cancel{0}^1 & {}^1 0 \\ - & & 1 & 0 & 1 & 1 \\ \hline & . & . & . & . & 1 \end{array}$$

$$\begin{array}{rccccc} & \cancel{1}^0 & {}^1 0 & 1 & 1 & {}^1 0 \\ - & & 1 & 0 & 1 & 1 \\ \hline & & 1 & 1 & 0 & 1 \end{array}$$

Note: If the answer key has listed 1001 *as the correct answer, they're wrong... :)*

Question Score: 0 Answer Score: 7

Question ID: 1081731

Is showing a graph is non-Hamiltonian NP-Complete?

Question

> Show that graph is not Hamiltonian. Is this an NP-complete problem?

My guess is that this is not an NP-complete problem, because we can run DFS and check it. Like, if we have a Hamiltonian cycle than there is a path that can be traced with DFS that covers all vertices and comes back to the original source. Right?

I am not exactly sure though. Maybe my assumption is stupid! I just started to read on NP-complete problems. Need your help here...

Answer

You probably know that showing a graph *is* Hamiltonian is an NP-complete problem. Thus, showing that a graph is *not* Hamiltonian is a co-NP-complete problem. It is an open problem as to whether or not a co-NP-complete problem can also be NP-complete.

So, the short answer is, “we don’t know yet!”

Question Score: 1 Answer Score: 2
Question ID: 1263681

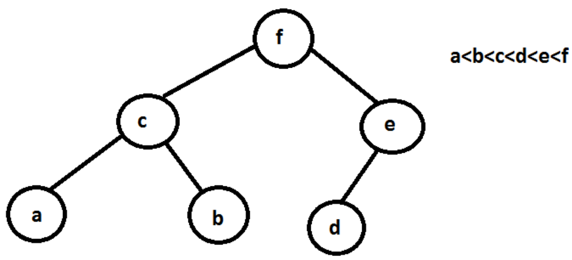
7.2 Graphs and Trees

Determine if there is a node in a binary postorder anti-sorted tree with key k

Question

A binary postorder anti-sorted tree is a binary tree for which the post-order traversal gives the keys that are saved at the nodes of the tree in descending order. Present a pseudocode for the most effective algorithm that you can think of that will find if there is a node in a binary postorder anti-sorted tree with key k. Which is the asymptotically most exact bound for the time execution of the above algorithm in the worst case?

I thought that the tree will be of the following form. Am I right?



I wrote the following algorithm, that returns 1 if there is a node with key k and 0 if there isn't such a node.

```
Alg(int k, pointer R){
    if (R==NULL) return 0;
    if (R->key==k) return 1;
    else if (R->key<k){
          if (R->lc!=NULL){
             if (k>R->lc->key){
                 if (R->rc!=NULL)   return Alg(k,R->rc);
             }
             else return Alg(k,R->lc);
           }
           else if (R->rc!=NULL) return Alg(k,R->rc);
    }
    else return 0;

  }
```

Could you tell me if my idea is right?

Answer

Firstly, the diagram you drew is a valid example of such a tree.

Secondly, the problem statement says "pseudocode;" everyone has a different understanding of what this means. I'd ask your instructor about how particular s/he is if this is for a class.

Since I'm not too familiar with C++, your pseudocode is a bit too code-like for me (but that's personal preference, not something that's right/wrong).

I'll convert your pseudocode to even more abstract pseudocode, so we're not worrying with pointers, etc:

```
Alg(Tree T, key):
  if (T is empty):  return "not found"

  Node currNode = T.root
  if (currNode.value = key):
    return "found"

  else if (currNode.value < key):
    if (currNode.hasLeftChild):
      if (currNode.leftChild.value < key):
        if (currNode.hasRightChild): // (1)
          return Alg(currNode.rightSubtree, key)
        else:
          return Alg(currNode.leftSubtree, key) // (2)

    else if (currNode.hasRightChild):
      return Alg(currNode.rightSubtree, key)

  else:
    return "not found"
```

I'm a bit confused on why you have the if/else constructs that you do between the lines that I've marked (1) and (2) above. It seems like you're doing some unnecessary checking. However, your algorithm looks as if it will return the right answer even so.

It seems to me that the a better way to decide which path to take is to check "is the key less than or equal to the value at my left node? If yes, follow the left branch. If no, follow my right branch."

That is, my pseudocode would be:

```
Alg(Tree T, key):
  if (T is empty):  return "not found"

  Node currNode = T.root
  if (currNode.value = key):
    return "found"

  else if (currNode.value < key):
    if (currNode.isLeaf):
      return "not found" //no children

    if (currNode.hasLeftChild AND key <= currNode.leftChild.
        value):
      //key is in the first half of the list
```

```
      return Alg(currNode.leftSubtree, key)

    else if (currNode.hasRightChild):
      //key must be in the second half of the list, or absent.
      return Alg(currNode.rightSubtree, key)

  else:
    return "not found"
```

As to asymptotic performance: the worst case is that the node is not there, and you must traverse the entire height of the tree. Thus, the algorithm operates in time proportional to $\mathcal{O}(\lg n)$, where n is the number of nodes in the tree.

Question Score: 1 Answer Score: 1
Question ID: 1121009

Parent and childs of a full d-node tree

Question

i have a full d-node tree (by that mean a tree that each node has exactly d nodes as kids). My question is, if i get a random k node of this tree, in which position do i get his kids and his parent?

For example, if i have a full binary tree, the positions that i can find the parent,left and right kid of the k node are $\frac{k}{2}, 2k, 2k+1$ respectively.

Thanks in advance.

Answer

Start by labeling your k-ary tree with the root as 0, and continue in a breath-first-search pattern.

1. Given a node n, with index $n.\mathrm{i}$, n's children have indices:

$$\{(n.\mathrm{i})k+1,\ (n.\mathrm{i})k+1,\ \ldots,\ (n.\mathrm{i})k+k\}$$

2. We can also see that given a node n with index $n.\mathrm{i} > 0$, we can find the index of the parent by the formula:

$$p.\mathrm{i} = \left\lfloor \frac{n.\mathrm{i}-1}{k} \right\rfloor$$

I haven't proved these, but these formulas appear to be correct based on several different drawings.

Question Score: 1 Answer Score: 0
Question ID: 393771

Looking for counterexample in graph theory

Question

I have this problem from graph theory:

Given a graph $G = (V, E)$ (maybe with multiple edges) find if it's possible to delete some edges such that the new graph is 1-regular (ie. all of its vertices have degree 1).

So far I have only discovered that if number of vertices is odd then it's impossible, this is an easy consequence of the degree formula $\sum_{v \in V} \deg(v) = 2|E|$. Also I can consider there are no multiple edges or loops since I can delete them without affecting the result.

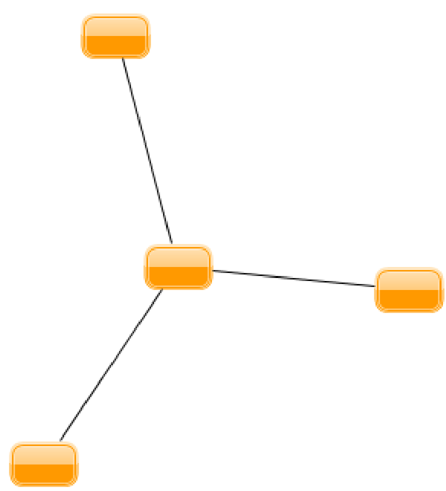
Now if I consider the adjacency matrix of the graph then the resulting adjacency matrix for the new graph must have determinant 1 or -1 because I can get it by permuting the rows/-columns of the identity matrix.

I'm guessing two things one bolder than the other. If the number of vertices is even and every vertex has degree at least one it is always possible to get a 1-regular graph and second, if the adjacency matrix of the original graph has determinant 1 or -1 it is possible to delete some edges to get a 1-regular graph. (Maybe deleting edges can be translated to using a pivot to delete some entry in the matrix?)

May someone give me some hints/help or a counterexample to my guessing?

Answer

For your first guess, consider the following graph:



I will continue to think about the second guess...

Question Score: 3 Answer Score: 2
Question ID: 634292

Recursive Sequence Tree Problem

Question

This question appears also on CSTheory.SE. I was told that cross-posting in this particular situation could be approved, since the question can be viewed from many angles.

I am a researcher in the field of computer science. In my research I have the following problem, which I have been thinking for quite a while now.

I think the problem is best explained through an example, so first assume this kind of a tree structure:

```
                    1, 2, 3, 4, 5, 6, 7, 8
                   /                      \
      6, 8, 10, 12                        -4, -4, -4, -4
     /            \                       /             \
   16, 20       -4, -4                -8, -8,           0, 0
   /    \       /    \                /     \          /    \
  36    -4    -8      0             -16      0        0      0
```

The root of the tree is always some sequence $s = (s_0, ..., s_{N-1})$ where $N = 2^p$ for some $p \in \mathbb{N}, p > 2$. **Please note that I am looking for a general solution to this, not just for sequences of the form** $1, 2, ..., 2^p$**.** As you can see, the tree is defined in a recursive manner: the left node is given by

$$left(k) = root(k) + root\left(\frac{N}{2} + k\right), \quad 0 \leq k \leq \frac{N}{2}$$

and the right node by

$$right(k) = root(k) - root\left(\frac{N}{2} + k\right), \quad 0 \leq k \leq \frac{N}{2}$$

So, for example, the following would give the second level of the tree:

$$(6 = 1 + 5, 8 = 2 + 6, 10 = 3 + 7, 12 = 4 + 8)$$
$$(-4 = 1 - 5, -4 = 2 - 6, -4 = 3 - 7, -4 = 4 - 7)$$

I am only interested in the lowest level of the tree, i.e., the sequence $(36, -4, -8, 0, -16, 0, 0, 0)$. If I compute the tree recursively, the computational complexity will be $O(NlogN)$. That is a little slow for the purpose of the algorithm. Is it possible to calculate the last level in linear time?

If a linear-time algorithm is possible, and you find it, I will add you as an author to the paper the algorithm will appear in. The problem constitutes about 1/10 of the idea/content in the paper.

If a linear-time algorithm is not possible, I will probably need to reconsider other parts of the paper, and leave this out entirely. In such a case I can still acknowledge your efforts in the acknowledgements. (Or, if the solution is a contribution from many people, I could credit the whole math SE community.)

Answer

An interesting note on Kaya's matrix $\mathbf{M}$: I believe that it can be defined recursively for any value of p. (I should note here that this is my belief. I have yet to prove it...)

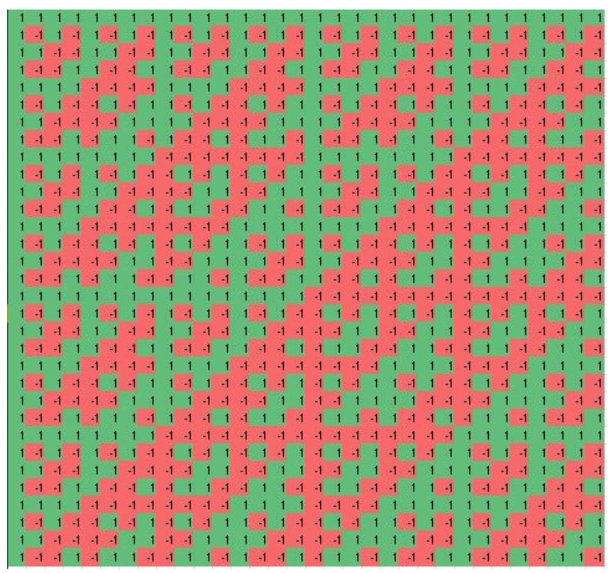
That is, let $\mathbf{M}_p$ be the matrix for the value of p (here, let's remove the bound on $p > 2$).

Let $\mathbf{M}_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

Then $\mathbf{M}_n = \begin{pmatrix} \mathbf{M}_{n-1} & \mathbf{M}_{n-1} \\ \mathbf{M}_{n-1} & -\mathbf{M}_{n-1} \end{pmatrix}$.

Thanks to some searches based off of Suresh Venkat's answer, I found that this matrix is called the Walsh Matrix. Multiplying this matrix by a column vector of your first sequence provides a column vector of the bottom sequence.

As a side note, this makes an almost fractal-like pattern when colored.



The above is for $p = 4$.

I'm almost sure I've seen a graphic similar to the one above before. If someone recognizes something similar, that would be great...

Question Score: 6 Answer Score: 1
Question ID: 415515

7.3 Cryptography

Toy cryptographic hash function for education purposes?

Question

I'm teaching some high school students about number theory and cryptography, and I'd like a hash function (or ideally, a family of hash functions) that I can use as simple demonstration for cryptographic hash functions. The students have learned everything involved in a simple implementation of RSA.

It will be used for inputs that are numbers in decimal format between 8 and 12 digits long.

- Can be worked out fairly quickly and easily with paper and a scientific calculator for decimal inputs of around 8-12 digits
- A short digest (say around 4 digits)
- Digest should change significantly with small modifications to the input
- It should *seem* difficult (at least on paper) to find an input that produces a given digest
- It should seem difficult to see a way to modify an input without modifying its digest
- It should seem difficult to find two inputs that will have the same digest

Not all of these properties are required but the more it has the better it will be for demonstration purposes. It's okay if it's reversible, so long as it isn't obvious at first glance how to do that.

Any ideas?

Answer

I'm looking in my algorithms textbook now for information on hash functions. For positive integers n, it says that the most common hash function is to simply take $n \pmod{p}$ for some prime p. This doesn't really fit the requirement of "large changes in output for small changes in input."

However, things get a bit more interesting if you don't use the hash function for numbers, but rather for text. For example, my textbook (by Sedgewick and Wayne) says an example process to hash a string is to:

```
hash = 0
for each character c in the string:
  hash = (R*hash + c) % p
```

...Where `R` is some number that is greater than the numeric value of any character, and `p` is a prime (e.g. 31 seems to be used often). You'd need some way of converting characters to numbers, which could be as simple as $A \mapsto 1, \ldots, Z \mapsto 26$

This doesn't fit your exact input/output requirements, but it would be a fairly straightforward example to teach high schoolers.

Question Score: 3 Answer Score: 1
Question ID: 961548

Using simple linear algebra for encryption?

Question

e.g. the character $a = 97$ (it's computer decimal format, commonly known)

and then using a pattern/key like $y = 31x + 5$ to get 3012 (substitute 97 into x, y is now the encrypted code).

1. How easy/hard would it be for someone to crack a code like this? 3012 3197 3135 (without them knowing the pattern)
2. If someone had these encrypted codes, are there any methods they could use besides trial and error to figure out the pattern and then the code?

Answer

What you have described is nothing more than a single-alphabet substitution cipher. This sort of cipher has been around for centuries, with variations on the type of mapping. It is very easy/simple to break if given enough ciphertext encoded with the method.

To break this cipher, first analyze the frequency of the resulting numbers. (e.g. 3012 occurs what percent of the time in the message?) Now, compare this to a frequency table of letters in the English alphabet. This will give you a starting place. You can also look at the frequency of pairs of letters. By that time, you should have a pretty good guess for the substitutions.

Question Score: 2 Answer Score: 5
Question ID: 439684

7.4 Applications to Programming

Function defined by rules

Question

I watched a Khan Academy video, and the speaker defined a function

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 2 \\ 1 & \text{if } x = 2. \end{cases}$$

My question is, is there a way to figure out the function's "body", just by making up rules like this?

For example, can I say that I have a function $f(x)$ that for

$$f(x) = \begin{cases} \sin(x) & \text{if } -1 < x < 1 \\ 0 & \text{if } x < -1 \text{ or } x > 1. \end{cases}$$

How would I figure out the function's body just by using those preset rules?

Just pointing me in the right direction would be more than enough, please use the catalogue at khanacademy.org and tell me what to watch. Thank you for your time!

P.S. This might not make a ton of sense at first, but it's interesting for me because I'm a front-end web developer and I'm doing lots of graphic or motion related programming, which involves math.

Answer

This is a *piecewise-defined function.* To write this in a programming language setting, you can take three routes:

If/Then: By far the simplest (here's pseudo-Python code)

```
def f(x):
    if (x > 1 || x < -1):
        return 0
    else:
        return math.sin(x)
```

Ternary Operator: Works in some languages, others it doesn't work. For Java:

```
public static double f(double x) {
    return (x > 1 || x < -1) ? 0 : Math.sin(x)
}
```

Cast Boolean to Numeric Also language dependent. In some languages, a false value is equivalent to 0, and a true value is equivalent to 1. This is true of Python:

```
def f(x):
    return (x > 1)*(x < -1)*math.sin(x)
```

Question Score: 2 Answer Score: 2
Question ID: 95921

How would you write the aspect ratio when height is greater than width?

Question

If you have an image that is sized 200x100 you would say the aspect ratio is ”2:1” correct? You can figure that out with width/height correct? What if the size was 100x200? Would you say the aspect ratio is ”1:2”? If so how do you do the math to get ”1:2” instead of ”.5:1”.

More information:

I have to figure this out in code and format it correctly. Here is as far as I got. Does the following look right?

```
width = 200
height = 100

if (width > height) {
    ratio = width/height + ":1" // results in "2:1"
}
else if (height > width) {
    ratio = "1:" + height/width // results in "1:2"
}
else if (width == height) {
    ratio = "1:1";
}
```

I'm not even sure I described what I'm trying to do correctly. Sorry!

Answer

A ratio is essentially a fraction. When you say the ratio of the width to the height is $2:1$, you mean:

$$\frac{width}{height} = \frac{2}{1}$$

If the ratio if $1:2$, this means:

$$\frac{width}{height} = \frac{1}{2}$$

Thus, your problem is equivalent to reducing a fraction.

Example: The height is 200, the width is 300. Thus:

$$\frac{width}{height} = \frac{300}{200} = \frac{3}{2}$$

Written as a ratio, this is $3:2$.

How do you reduce a fraction? Compute the greatest common divisor (GCD) of the numerator and denominator. Then divide the numerator and denominator by the the GCD. Information on computing the GCD can be found on Wikipedia; consider Euclid's Algorithm.

What would this look like in pseudocode? Assuming you have written a function to find the GCD of two numbers called gcd():

```
w = some input //this is the width variable
h = some input //this is the height variable

divisor = gcd(w, h)

w = w / divisor
h = h / divisor

print (w:h)
```

Question Score: 3 Answer Score: 3
Question ID: 246178

Given 4 corner points of a rectangle in 3d space, how to find its "plane" equation?

Question

Context:

> A BoundingPolytope defines a polyhedral bounding region using the intersection of four or more half spaces. The region defined by a BoundingPolytope is always convex and must be closed. Each **plane** in the BoundingPolytope specifies a half-space defined by the equation:
>
> $$Ax + By + Cz + D \leq 0$$

source: from a javadoc, this is for programming purpose

I need the value of A, B, C, and D for *a plane* as in that equation and all I have is corner points of a rectangle that lies in that plane.

Answer

You only need 3 of these corner points. Let these corner points be denoted $\vec{v_1}$, $\vec{v_2}$, and $\vec{v_3}$, treated as vectors from the origin to the point.

Then, the equation of your plane is:

$$((x, y, z) - \vec{v_1}) \cdot [(\vec{v_3} - \vec{v_1}) \times (\vec{v_2} - \vec{v_1})] = 0$$

Where $\times$ is the cross product, $\cdot$ is the dot product, and (x, y, z) are the free variables describing the plane.

Question Score: 1 Answer Score: 1
Question ID: 665579

Normalize data with large spread in values.

Question

I'm currently trying to rank a set of data. The issue is that my initial rank comes from a search on google and the returning result set.

The spread in values is ranging from 33 all the way up to 1580000000. This makes it very hard, at least as far as my skill set goes, to apply any sort of modifiers to these numbers.

What I'm wondering is if there is a way to normalize the data into a close range. I do NOT care about the difference between the numbers as long as the original order is kept.

I have no clue what tag to post this under so I apologize for that right off the bat.

Thank you Bruce

Answer

What tools are at your disposal, and how large is the dataset? One easy way is to simply sort the data by original rank and relabel each entry. I can show an easy way to do this in Excel, and I'll include some basic program code below that does the trick, too.

First, sort the data using the standard Excel sort menu and configure the sort appropriately (ascending order of rank, has headers):

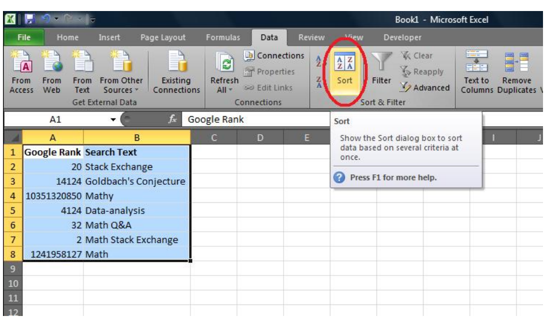


Figure 7.1: Sort data

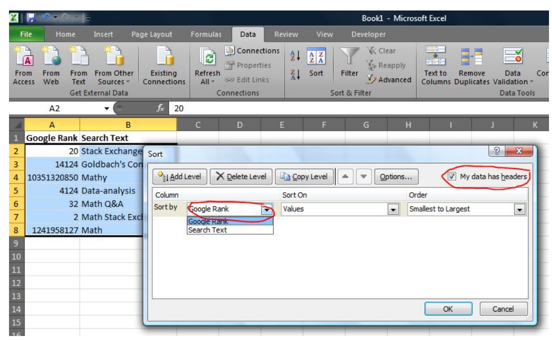
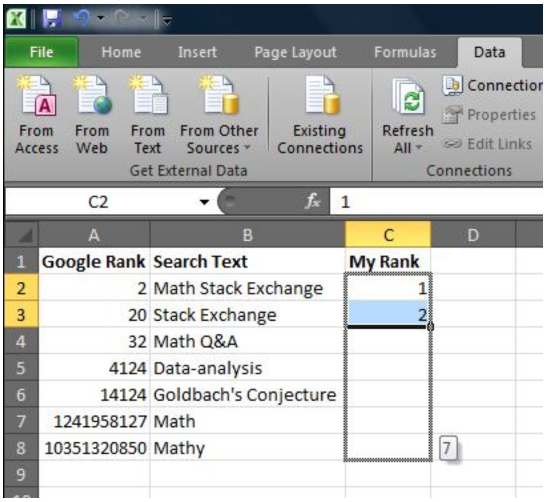


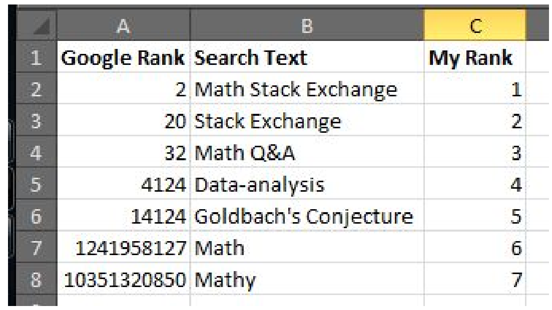
Figure 7.2: Configure sort

Finally, use Excel's auto-complete to relabel the data entries



	A	B	C	D
1	Google Rank	Search Text	My Rank	
2	2	Math Stack Exchange	1	
3	20	Stack Exchange	2	
4	32	Math Q&A		
5	4124	Data-analysis		
6	14124	Goldbach's Conjecture		
7	1241958127	Math		
8	10351320850	Mathy		
9				

Figure 7.3: Relabel



	A	B	C
1	Google Rank	Search Text	My Rank
2	2	Math Stack Exchange	1
3	20	Stack Exchange	2
4	32	Math Q&A	3
5	4124	Data-analysis	4
6	14124	Goldbach's Conjecture	5
7	1241958127	Math	6
8	10351320850	Mathy	7

Figure 7.4: Done!

If it's too hard to do this in Excel, here is some pseudocode for a programmatic attempt:

```
structure searchStruct {
   //Keep track of original rank, search text, and your new rank
       .
   int GoogleRank;
   String searchText;
   int myRank;
}

main{
  //Load in data
  searchStruct[] allData = <Some method of loading in data>;

  //Sort data
  sort allData by GoogleRank;

  //Relabel (the -1 for the array index is b/c of 0-based array)
  for (int i = 1; i <= allData.length; i++) {
    allData[i-1].myRank = i;
  }
}
```

Question Score: 1 Answer Score: 1
Question ID: 313198

intensity transformation 16-bit image to a 8-bit image

Question

Give an intensity transformation function T for converting a 16-bit image to a 8-bit image, i.e. T takes an integer from 0,1,2,...,65535 and returns an integer from 0,1,2,...,255.

Can we use $s = T(r) = \lfloor\sqrt{r}\rfloor$?

Answer

You can certainly use whatever you'd like. However, this won't produce an even-looking image. Why? Look at a table of what input maps to what output:

Input	Output
1	1
$\vdots$	$\vdots$
4	2
$\vdots$	$\vdots$
9	3
$\vdots$	$\vdots$
16	4
$\vdots$	$\vdots$
25	5

Ok. So that was interesting and trivial. What does this mean? Notice that the input maps 3 digits to 1, 5 digits to 2, 7 digits to 3, etc. This indicates that the image will be more weighted in the lighter/higher colors.

Another way to look at this is to say that the ideal image transformation would have the midpoint be the same on both. That is, if you plug $\frac{2^{16}}{2}$ into the transformation, you'd get back $\frac{2^8}{2}$. The same goes for all other denominators: $\frac{2^{16}}{4} \to \frac{2^8}{4}$, etc.

You may want to look at using the $\log_2(x)$ function... (not sure how this would pan out, but it could work).

Question Score: 0 Answer Score: 1
Question ID: 354816

JQuery placing elements X pixels by degree (Basic trig)

Question

Essentially I want to place an element X pixels from the current position towards the center. Here's my code:

```
var sitX = parseInt($('div.par[id='+my+']').css("left"))+43; // 43 is half the width
    var sitY = parseInt($('div.par[id='+my+']').css("top"))+43; //43 is half the height
var deltaY = sitY - 200; //200 is the vertical center of the screen
var deltaX = sitX - 380; //Same for the horizontal
ang1 = Math.atan(deltaY / deltaX) * 180 / Math.PI;
...
```

I later create it with `right=(Math.cos(ang1)*60)` and `top=(Math.sin(ang1)*60)` on an element with absolute positioning.

What am I missing?

Here is a fiddle: http://jsfiddle.net/mgfVR/1/

Answer

For future reference, you'd probably get a better answer on Stack Overflow.

Anyway, the error I'm seeing is that the `Math.cos()` and `Math.sin()` functions should take arguments in radians, so you don't need the `*180/Math.PI` in the definition of `ang1`.

Question Score: 0 Answer Score: 1
Question ID: 417959

Moving point along the vector

Question

I'm making a game. I have came across a problem. I have to move a point along a vector for some distance. Can anyone help me? Any ideas?

Answer

Let's say you have a vector $\vec{P} = [P_1, P_2, P_3]$ that represents the point's location in space. Another vector, $\vec{B} = [B_1, B_2, B_3]$ represents the ball's (or bullet's or whatever) position in space.

The vector $\vec{BP}$ between the two is:

$$\vec{BP} = \vec{P} - \vec{B} = [P_1 - B_1, P_2 - B_2, P_3 - B_3]$$

So, if you want the ball to move *all* the way to the player, you would say:

$$\vec{B}_{new} = \vec{B} + \vec{BP}$$

Or, if you want the ball to move only 1/100th of the way to the player, you would say:

$$\vec{B}_{new} = \vec{B} + \frac{1}{100}\vec{BP}$$

In general, to move some small ϵ to the player:

$$\vec{B}_{new} = \vec{B} + \epsilon\vec{BP}$$

Question Score: 1 Answer Score: 3
Question ID: 333350

7.5 Computer Algebra Systems

How to notate or write recurring decimals in Matlab?

Question

I'm new on Matlab, and I have a question. Can I represent recurring decimals? For example I want to operate 216.6666666 by another number.

I'll appreciate your answers.

Answer

If it's absolutely necessary, I suggest using a toolbox to extend to arbitrary precision, and then use a loop to extend this to as far as you want.

However, by default, MATLAB uses standard binary floating point representation. So, you're limited to either 32 or 64 bits of precision. (I forget which.) Thus, you won't really be able to represent a repeating decimal to infinite precision.

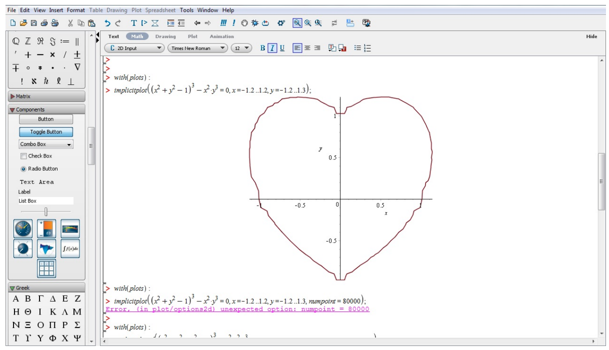
That said, MATLAB is pretty good with roundoff error–if I recall correctly, `(1/3)*3` still returns `1`.

Question Score: 2 Answer Score: 1
Question ID: 414857

How to draw heart shape using Maple

Question

I'm drawing a heart shape using Maple, but the heart-shaped curve is broken. Please hint to me how to draw it correctly.



Answer

The syntax is `numpoints=8000`, not `numpoint=8000`. :)

Another option (instead of `numpoints`) is to use `grid`.

You may also want to look at plotting Cardioids.

Question Score: 6 Answer Score: 3

Question ID: 346556

Does the integral of $\sqrt{1 - \sin t}$ exist?

Question

I have searched for answers everywhere, but none of them seems to be correct. Since the area under $\sqrt{1 - \sin t}$ is always positive, I imagine the integral to be a graph resembling an infinitely rising stairs, but the answers floating around the internet (including WolframAlpha) seem to be wrong.

Answer

You are correct. The integral should not evaluate to 0, if we are working with real numbers. Sometimes a CAS needs a little help. Some creatively placed absolute value signs worked for Wolfram.

Question Score: 3 Answer Score: 0
Question ID: 250697

Plot of $x^{1/3}$ has range of 0-inf in Mathematica and R

Question

Just doing a quick plot of the cuberoot of x, but both Mathematica 9 and R 2.15.32 are not plotting it in the negative space. However they both plot x cubed just fine:

```
Plot[{x^(1/3), x^3},
    {x, -2, 2}, PlotRange -> {-2, 2}, AspectRatio -> Automatic]
```

Wolfram Link

```
plot(function(x){x^(1/3)} , xlim=c(-2,2), ylim=c(-2,2))
```

Is this a bug in both software packages, or is there something about the cubed root that I don't understand?

```
In[19]:= {1^3, 1^(1/3), -1^3, -1^(1/3), 42^3, -42^3, 42^(1/3) //
    N, -42^(1/3) // N}
Out[19]= {1, 1, -1, -1, 74088, -74088, 3.47603, -3.47603}
```

Interestingly when passing -42 into the R function I get NaN, but when I multiply it directly I get -3.476027.

```
> f = function(x){x^(1/3)}
> f(c(42, -42))
[1] 3.476027        NaN
> -42^(1/3)
[1] -3.476027
```

Answer

My Calc professor talked about that last semester.

Many software packages plot the *principal root,* rather than the real root.

For example, $\sqrt[3]{3}$ has three values (See W—A.)

Mathematica uses the roots in the upper-left quadrant when plotting the cube root. Thus, it thinks it's complex, and therefore doesn't graph it.

Question Score: 1 Answer Score: 1
Question ID: 273149

Chapter 8

Physics

Projectile Motion - Finding initial velocity

Question

You need to send a bowling ball up exactly 205 meters so someone at the top of the Washington Monument can take a picture of it "hanging" in space. The bowling ball will be shot from a mortar tube from a platform 25 meters above the ground. You can determine what the initial velocity is because the technician will put in enough black powder to give the bowling ball whatever initial velocity you tell him to give you. So....

(1) What initial velocity (v_o) will cause the bowling ball to "freeze" in mid-air at exactly 205 meters above the ground?

(2) How fast will the bowling ball be moving right before it hits the ground?

Using The formula $s(t) = s_0 + v_0(t) + .5gt^2$

$s_0 = 25$ and $t = 18$ sec and $g = 10$ m/sec^2

Answer

This is probably a better fit for Physics.SE...

I'm going to assume you are ignoring air resistance, as there's no Differential Equations tag.

Part 1

We have from conservation of energy:

$$\Delta K + \Delta U_g = 0$$

$$\frac{1}{2}m\left(v_f^2 - v_0^2\right) + mg(y_f - y_0) = 0$$

Dividing by m and plugging in values:

$$\frac{1}{2}\left(-v_0^2\right) + g(205\text{m} - 25\text{m}) = 0$$

Simplify and solve for v_0:

$$\frac{1}{2}\left(v_0^2\right) = g(205\text{m} - 25\text{m})$$

$$v_0 = \sqrt{2g(205\text{m} - 25\text{m})}$$

$$v_0 = \sqrt{2(9.80)\text{m}/\text{s}^2(180\text{m})}$$

$$v_0 = \sqrt{3528}\frac{\text{m}}{\text{s}} \approx 59.39\frac{\text{m}}{\text{s}}$$

Part 2

The ball will fall 205 meters. Again, conservation of energy:

$$\Delta K + \Delta U_g = 0$$

$$\frac{1}{2}m\left(v_f^2 - v_0^2\right) + mg(y_f - y_0) = 0$$

Dividing by m and plugging in values:

$$\frac{1}{2}\left(v_f^2\right) + g(-205\text{m}) = 0$$

$$v_f^2 = 2g(205)\text{m}$$

$$v_f = \sqrt{9.80\frac{\text{m}}{\text{s}^2}(410)\text{m}}$$

$$v_f \approx 63.39\frac{\text{m}}{\text{s}}$$

With Kinematic Equations

I guess this is supposed to be solved with the kinematic equations:

$$s = s_0 + v_0 t - \frac{gt^2}{2}$$

$$v(t) = v_0 - gt$$

(With g being the *magnitude* of gravity ($9.80\frac{\text{m}}{\text{s}^2}$))

The first step is to solve for the time it takes to reach the top. The easiest way is to use the equation:

$$v(t) = v_0 - gt$$

Thus, we have:

$$0 = v_0 - gt_{top}$$

$$t_{top} = \frac{v_0}{g}$$

Now we plug this in to the other equation:

$$s = s_0 + v_0 t - \frac{gt^2}{2}$$

$$s_{top} = s_0 + v_0\left(\frac{v_0}{g}\right) - \frac{g\left(\frac{v_0}{g}\right)^2}{2}$$

$$205 = 25 + \frac{v_0^2}{g} - \frac{v_0^2}{2g}$$

$$180 = \frac{2v_0^2 - v_0^2}{2g}$$

$$180 = \frac{v_0^2}{2g}$$

$$v_0^2 = 180(2g)$$

$$v_0 = \sqrt{360 \cdot 9.80 \frac{\mathrm{m}^2}{\mathrm{s}^2}} \approx 59.4 \frac{\mathrm{m}}{\mathrm{s}}$$

The falling speed is very similar, so I will not work it out for you.

Question Score: 0 Answer Score: 0
Question ID: 299892

What Mathematics questions can be better solved with concepts from Physics?

Question

Over the years, I've seen several questions in mathematics that can be solved using concepts borrowed from Physics. Having seen these question, I'm interested to find out what other mathematics questions you've found that can be better solved with a concept from physics - or at least where the application of physics is interesting and perhaps illuminating.

Examples

One of these questions is on minimizing the time taken for a lifeguard to go out to a stationary distressed swimmer. In the scenario, the lifeguard runs faster than he swims in the water, and as such the straight line is not the fastest way for the lifeguard to reach the swimmer. Students will normally use calculus to solve this problem, and the answer can be obtained after some work - however, a much more convenient (and intuitive) way is to borrow from the idea of refractive indices in geometric optics. We recast the situation by replacing the beach and the sea with two materials with different refractive indices, choosing the appropriate refractive indices proportionate to the ratio between the lifeguard's velocities while running and swimming. The problem is then reduced to finding a beam of light that passes through both the swimmer and the lifeguard's position. (For a more complete explanation, you can visit this site: http://findingmoonshine.blogspot.sg/2012 05 01 archive.html)

Another of these questions requires one to prove that, in an acute-angled triangle, the angle subtended by any side of the triangle at the Toricelli point is 120. Again, instead of using trigonometry, one can use the concept of hanging equal weights from a (frictionless) string at each of the vertices of the triangle, and then tying each of the three strings together at one knot placed on the surface of the triangle. The equilibrium position of the knot is the Toricelli point, and one can then complete the proof by considering forces acting on the knot.

Looking forward to hearing from you!

Answer

The most interesting such result that I have seen is the connection between the moments of the Riemann zeta function and quantum energy shells.

The story goes that Brian Conrey challenged quantum physicists to use a proposed relationship between the Zeta function and quantum mechanics to continue a sequence relating to the moments of the zeta function (using solely mathematical methods, an explicit formula had not (at the time) been found). Jon Keating, a physicist, continued the sequence; in fact, he developed a formula for the terms of said sequence.

An article about this is found here.

Question Score: 18 Answer Score: 1
Question ID: 255464

I need to know the formula to compute time to reach a star

Question

I am working on a sci-fy short story and I need to use a somewhat realistic value for the time in transit for going to a star. Assume that energy on the space ship is not a problem. The space ship accelerates at some amount of G-force until reaching some fraction of the speed-of-light. It then coasts until reaching the midpoint and then reverses the procedure (including the coasting) until reaching the destination star.

So what would be the formula to compute the time to the coast segment and distance covered?

Assume G = .1, D = 50 light-years, Max-speed .5 of light-speed. How do I compute D(s) - distance covered until coasting T(s) - time taken to reach D(s)

Added bonus; would there be any time dilatation at this speed and if so how to I compute it?

Answer

Simplifications for Modeling

Let's assume that gravity is not an issue here. (Once you include the gravitational attraction from one planet, you're dealing with an improper integral, typically dea¡ with in second semester calculus. With 3 or more planets, you're dealing with a graduate level problem, that, IIRC, is still open.)

Now, you're dealing with a period of acceleration, followed by a a period of constant velocity, followed by a period of negative acceleration.

One problem with your setup is that you can't accelerate by some amount of force without knowing the mass of the spacecraft. It's better to say that you accelerate at, say, a rate of 10^{-6} of the speed of light every second (that is, $\sim 300m/s^2$).

Working the Problem

So, we accelerate at a constant rate of 300 meters per second per second (meaning, we advance around 1 Mach number every second) until we reach $1.499x10^8$ meters per second. We use this formula, which calculates velocity as a function of time, assuming constant acceleration:

$$v = at + v_0$$

We start at rest, so $v_0 = 0$. The acceleration is 300, and the final velocity (v) is 1.499×10^6 Thus, we have:

$$1.499 \times 10^6 = 300t$$

Solving for t yields 500,000 seconds.

So, it takes 500,000 seconds to accelerate to $0.5c$, where c is the speed of light.

It will also take 500,000 seconds to decelerate at the end, as our acceleration constant is the same.

So, we have $1,000,000$ seconds accounted for in the flight. We want to travel a distance of 50 light-years, denoted D.

How far will those $1,000,000$ seconds take us? Well, we have a position formula under constant acceleration:

$$x = \frac{a}{2}t^2 + v_0 t$$

So, we have:

$$x = 150t^2$$

$$x = 150(500,000)^2$$

$$x = 3.75 \times 10^{13}$$

So, on the first leg of the journey (we haven't coasted yet), we'll travel 3.75×10^{13} meters.

While we're coming to rest:

$$x = 150t^2 + 1.499 \times 10^6 t$$

Plugging in as above yields a distance traveled of approx 3.82×10^{13} meters.

So, we have to travel 50 light years-3.82×10^{13}meters-3.75×10^{13}meters $\approx$ 49.99light years. (calculation here)

This is essentially 50 light years, so the acceleration time could have been ignored.

Traveling 50 light years at one half the speed of light will take 100 years.

Analysis

Note that the mathematical model that we used was overly complex. Because of the distances and speeds involved, it doesn't really matter if you accelerate or instantly assume the target speed.

One could account for relativity because of the speed involved, but (per my physics teacher), it doesn't have much effect until you're hitting around 70%+ the speed of light.

Hope this helps, let me know if you have questions.

Question Score: 4 Answer Score: 1
Question ID: 280445

Show that the velocity of the particle is... (Newtons Law question)

Question

A particle of mass m is released from rest from height of ten metres above a body of viscous fluid. Show that the velocity of the particle at the moment of impact with the fluid is 14m s^{-1} and estimate the duration of the drop. (Use the approximation $g = 10$m s^{-2}).

So far I have worked out the initial conditions must be $x(0) = 10, \dot{x}(0) = 0$.

With this my motion for all time is $x(t) = -5t^2 + 10$

where do i go from here?

Answer

Hints/Solution Outline:

Set $x(t) = 0$ to find the time at which it hits the viscous fluid. (This should be $\sqrt{2}$s.) This is also called the duration of the drop.

Next, find the velocity equation $\dot{x}(t)$. Now, plug in the time that you found from before, and you should have your answer ($\dot{x}(\sqrt{2})$).

Question Score: 0 Answer Score: 0

Question ID: 774293

Can somebody help and explain me with angular and linear Velocity textproblems?

Question

So we got this as homework today, and I just don't understand it; how to start *or* how to do it. Explanations and/or setups would be great :)

> A wheel with radius of 5 feet is rotating at 100 rpm.
>
> a) Determine the angular speed of the wheel in radians per minute.
>
> b) Determine the linear speed of a point on the circumference of the wheel in feet per minute.

Thank you so much.

Answer

This question is probably a better fit for Physics.SE. (Don't worry about that, though–if it is a better fit there, it will be moved without you having to do anything.)

Anyway... The first part simply asks for you to find *angular speed* in *radians per minute.* This is a somewhat simple unit conversion, if you have learned about radians. Recall from trigonometry (and I guess precalc–not sure where they're introduced) that there are 2π radians in one revolution. That is, 2π rad = 1 rev.

Converting units:

$$\begin{array}{c|c} 100\text{ rev} & 2\pi\text{ rad} \\ \hline \text{min} & \text{rev} \end{array} = 200\pi\frac{\text{rad}}{\text{min}} \approx 628\frac{\text{rad}}{\text{min}}$$

Second part: This one is a *little* more tricky–that is, you need to know a physics formula:

$$v = r\omega$$

This is one you should memorize and hold dear–it will serve you well throughout the rest of your studies of rotary motion. Now that I've said it's important, *what does it mean*? Well, this equation relates *linear velocity* and *rotational velocity.* ω represents rotational velocity in $\frac{\text{rad}}{\text{some time unit}}$, r is the radius of the rotating body, and v is linear velocity.

Using that equation, and the previous result:

$$v = r\omega$$
$$v = (5\text{ feet})\left(628\frac{\text{rad}}{\text{min}}\right) \approx 3141\frac{\text{feet}}{\text{min}}$$

Addendum: Because I worked that problem, I'll create a somewhat new one for you that uses similar concepts:

> a) A wheel is rotating at $10\frac{\text{rad}}{\text{sec}}$. What is the speed in rpm?
>
> b) The same wheel (still rotating at $10\frac{\text{rad}}{\text{sec}}$) has a linear velocity of $v = 5\frac{\text{feet}}{\text{sec}}$. What is the wheel's radius?

Question Score: 0 Answer Score: 0
Question ID: 295016

Quadratic Baseball Question

Question

The height of a baseball is modeled by the function $h(x) = -0.005x^2 + 0.3x + 1.5$, would an outfielder which is modeled by the function $m(x) = -0.06x + 5.6$ where $50 \leq x \leq 90$, catch the ball?

Answer

Answer to your question:

Look at this graph of the equations. The blue represents the height of the baseball, and the red represents something about the outfielder. Notice one of the solutions is well outside the accepted range for x.

Some things that don't sit right with me:

Firstly, the height of a baseball is governed (without considering air resistance) by the equation $y = -\frac{g}{2}t^2 + v_0 t + y_0$, where g is the gravitational acceleration. So, this occurs on some planet (or with some units) where gravitational acceleration is -0.005. Whoever set the problem should have attached some units, or used an accurate gravitational acceleration.

Also, the process of setting the expressions equal to each other is very wrong. One function returns the elevation of the baseball, the other is the position of the outfielder. In order for setting them equal to make sense, we either need the function for the outfielder to return the outfielder's elevation with respect to time (which is very odd, to say the least). Or, we can have the function for the baseball return its distance in the horizontal plane. This makes more sense, but is against the problem definition.

Question Score: -1 Answer Score: 3
Question ID: 347803

Chapter 9

Soft Question

9.1 General Mathematics

Real numbers in math

Question

What are real numbers for a person who doesnt know ANYTHING about math, and had to explain them what real numbers are.

Are real numbers only rational and irrational? if so then do we have to say what are rational and irrational

and please it cant have a lot of math involved in it (even if it sounds silly). Like say I am in uni or maybe you as a graduate of a university you want to explain to your niece, your little brother that is what I want like I know the good hard theorems but he wont get it.

So i want to simply break it down for someone who wants to know what a real number actually is

so this is what i have so far

So basically say you are a non math person and this is my explanation would it be valid. So a real number is basically any number that you can think of, let it be decimal numbers, rational numbers which are just fractions, irrational numbers. Real numbers can range anywhere from ∞ to $-\infty$ where ∞ is the highest number you can think of in your imagination and $-\infty$ is the lowest number you can think of in your imagination. Like numbers really dont end say the biggest number is 1000000000000000000000 but there is a number bigger than that like 1000000000000000000000, or the lowest number you can think of is -1000000000000000000000 what about -2000000000000000000000, etc real numbers are never ending

Answer

If you're explaining to a non-math person, the easiest way to describe a real number is to say:

> *"Pick any number, and it's real." (And say after that there are some exceptions that you only come across only in "higher" math.)*

This is the way I explain it to people who aren't mathematically mature enough to understand what a complex number is (that is, they don't know square roots yet). Someone who can take a square root (or name a complex number) is probably math-literate enough to understand a more rigorous explanation.

Will this explanation make mathematicians cringe? Yes. Will it get the point across to a layperson? Also yes. Sometimes specificity needs to be sacrificed to explain a difficult concept to a beginner.

Question Score: 3 Answer Score: 4
Question ID: 299172

What is the definition of an extremum of a function?

Question

In calculus extremum is a common topic, but I don't understand what it is. What is the use of this extremum, especially in practical life?

Answer

EDIT: As to what an extrema is: Extrema are maximums and minimums–over its whole "lifetime" how high and how low does a function go?

Original post:

Many, many processes in "real life" can be represented by functions. These applications can range in topic from physics (for example, the velocity of an object), to financial (cost to produce a certain quantity of goods), to the mundane (how much water you use per day, on an average basis).

Finding the extreme values of the functions is done whenever knowing the max or min is necessary. For example–how high will my model rocket go? That's a calculus problem.

Another example would be to find the maximal profit you can earn with a certain production method. Or, what is the maximal volume of a box that uses a fixed amount of material (or even a minimal amount of material)? What is the minimum amount of fencing to be used for your cattle herd?

These are just some simple examples–there are thousands of applications out there, it is just a matter of finding a process and needing to maximize or minimize its output.

Question Score: 1 Answer Score: 3
Question ID: 456546

What was the first bit of mathematics that made you realize that math is beautiful? (For children's book)

Question

I'm a children's book writer and illustrator, and I want to to create a book for young readers that exposes the beauty of mathematics. I recently read Paul Lockhart's essay *"The Mathematician's Lament,"* and found that I, too, lament the uninspiring quality of my elementary math education.

I want to make a book that discredits the notion that math is merely a series of calculations, and inspires a sense of awe and genuine curiosity in young readers.

However, I myself am mathematically unsophisticated.

> **What was the first bit of mathematics that made you realize that math is beautiful?**

For the purposes of this children's book, accessible answers would be appreciated.

Answer

Two instances where I thought math was amazing:

1. In like 4th grade or whenever you learn areas of rectangles, one of the exercises in my book was to estimate the area of some squiggly shape overlaid on a rectangular grid. I thought this was pretty cool, and reasoned that if you could make the grid "smaller" (higher resolution), you could be more accurate. I mentioned this to my mom, who proceeded to tell me that was basically how Calculus 2 worked. :) That was very fun for me.
2. Deriving the quadratic formula in Algebra 1. That was fun–it showed that some totally un-intuitive formula could be easily found using other previously found results.

Question Score: 518 Answer Score: 11
Question ID: 323334

7 Drinks - 7 Flavors - Infinite variety?

Question

Me and three friends are trying to find the answer to a question I posed about a self-service drinks machine in our local Burger King:

There is a drinks machine that has 7 varieties of drinks (coke, lemonade, orange-aid etc) and each drink has seven varieties of flavor (vanilla coke, lime coke, vanilla lemonade, lime lemonde etc). If you can choose the ratios of the different drinks in your cup, could you make an infinite number of different flavored drinks?

My friends feel it is important to point out that the cup is 500ml (I don't think this makes any difference).

So far no-one believes me in saying it is infinite, even after we sent the same questions as stated above to a renowned text answering service, whose answer was:

> There is an infinite number of drinks combinations with 7 drinks varieties with 7 flavors if you can choose the ratios, as there are infinite numbers.

My friends response to this was that they (the text answering service) didn't understand the question and got it wrong.

I hope you can answer my question and, if I am right, could you please explain your answer so my friends won't think you have misunderstood the question as well.

Answer

The answer is no, because of limitations in the physical world. In a container of given size, a limited number of soda molecules will fit.

For example, assume you have a very small cup only capable of containing 100 molecules of soda. (To simplify the problem, assume all soda molecules are the same size.)

So, you have 100 "slots" for soda molecules. Each slot can have one of up to $7 \cdot 7 = 49$ possible flavors. Thus, you only have 49^{100} possible combinations of flavors–a large number, for sure, but still a long ways off from infinite.

EDIT:

The limited capacity of the cup is important. If we had no limit of capacity, we could form any ratio we so wished between any mixture of drinks. However, this is not possible with limited capacity. For example, let's say we wanted a ratio of 100 : 101 molecules of soda (in the previous example). We'd need a total capacity of 201 molecules to have this precision. Unfortunately, we only can fit 100 molecules in a cup.

Question Score: 10 Answer Score: 12
Question ID: 254797

Why do we still do symbolic math?

Question

I just read that most practical problems (algebraic equations, differential equations) do not have a symbolic solution, but only a numerical.

Numerical computations, to my understanding, never deal with irrational numbers, but only rational numbers.

Why do mathematicians construct the real numbers and go through the pain of using their complicated properties to develop real analysis, etc. to be able to solve just a few cases symbolically?

Why do non-mathematicians deal with real numbers and symbolic calculation if the majority of cases must use rationals and numerical computation only?

Answer

The major problem I have with your question is that the premise is false.

> I just read that most practical problems (algebraic equations, differential equations) do not have symbolic solution, but only numerical.

This is patently incorrect–it is very rare for me to find an exact, numerical solution to any of the problems I have encountered in physics, engineering, chemistry, or computer science. The real numbers are exceedingly useful in the sciences.

Question Score: 66 Answer Score: 17
Question ID: 881689

Baloney detection kit for Math

Question

There are folks who claim they proved Fermat's Last Theorem, Riemman Hypothesis offering no more than a half a dozen pages (sometimes one or two pages) of proof, very few if any citations of previous works (where are the giants whose shoulders they stepped in?), among other suspicious characteristics.

I am not suggesting the one can measure the quality of a work by their number of pages alone, but for problems like those I mentioned, common sense suggest that their proofs must be really very complex and probably very long because if they were so simple as to be solved in a few pages, then someone would have discovered how to solve them much before the genius of Wiles or Perelman.

I also have seen various things like functions, theorems, etc which carries the name of its discoverer and that are considered by some to be trivial or/and of dubious importance for Math.

It is easy for a professional mathematician to the faults in those works and sometimes it is not too difficult for someone like me, that have little knowledge of math to see some faults, for they are common outside math, like lack of references.

What worries me is that if some particular work deals with advanced Math, it is not obvious for almost everyone if the work has or has not value, more or less like the Bogdanov affair. The Bogdanovs write about something that has a jargon and concepts so complex/advanced that seemed esoteric or post-modern to almost anyone, so in the absence of another easily identifiable clues one cannot see the work's real value unless that one is very well-versed in whatever the work deals with and those are unfortunately few.

This being so, I can elaborate the question: has someone designed a baloney detection kit specific/optimized for math, that can help in identifying if not all, most of the Math works that have no value even if they seemed at first sight elaborate, serious and erudite?

I can risk some opinions: lack of formalism, non-trivial notions that are not properly explained/defined, incomplete reasoning which omits intermediate calculations/deductions that are also non-trivial. But that is not specific to Math but for other sciences too.

Answer

In addition to the "crank detector" listed in Sam's answer, we also have a couple of "crackpot indices:"

http://primes.utm.edu/notes/crackpot.html

http://math.ucr.edu/home/baez/crackpot.html (for physics, but still applies)

Question Score: 0 Answer Score: 2
Question ID: 659849

9.2 How To Learn

Mathematical notation for computer science

Question

Can anyone point me in the direction of good introductory material on the use of mathematical notation in the field of computer science? I often come across notation in research papers that I don't understand and I would like to at least have an overview of its meaning/intent, if not a thorough understanding. I don't have much of a mathematical background and don't expect to become an expert in this field overnight but any ideas would be appreciated.

UPDATE: The kind of notation I've been having trouble with can be seen on page three of this article. Note: this isn't an article I've been using for research, it's just the first publicly available example I came across!

Answer

It looks like the notation you've referenced is from set theory. I know the first chapters of this book discuss set theory somewhat. It's a free reference, so it doesn't cost anything but time to look at...

I haven't formally covered set theory yet in my curriculum, so I don't know of any other books to suggest... Perhaps look in your discrete math book?

Question Score: 2 Answer Score: 2
Question ID: 186299

Is there a correct order to learning maths properly?

Question

I am a high school student but I would like to self-learn higher level maths so is there a correct order to do that? I have learnt pre-calculus, calculus, algebra, series and sequences, combinatorics, complex numbers, polynomials and geometry all at high school level. Where should I go from here? Some people recommended that I learn how to prove things properly, is that a good idea? What textbooks do you recommend?

Answer

One general approach is to select a college, and start working through topics that an undergraduate mathematician would see.

It is a good idea to know proper techniques of proof, but that can also be picked up by reading lots of "good" proofs. If you feel comfortable with proving some basic things (e.g. the sum of two odd numbers is even) on your own, then I'd suggest just picking up proper methods of proof by reading other people's more advanced proofs.

From looking at what you've done, it seems that Linear Algebra could be a good next step, or perhaps a multivariable calculus course (if you haven't done that already).

Question Score: 7 Answer Score: 1
Question ID: 379819

High School Mathematics

Question

Could you recommend any high school mathematics books that are rigorous and present high school mathematics in a higher level.I just realized that I only memorized a lot of things that I was taught in school and want to revise and understand those concepts deeply.

Answer

The Forester books. (Amazon.com)

That's algebra 1, but there's others that cover trig, calculus, etc.

The concepts are presented very well and have interesting applications/challenge problems. I recommend these.

Question Score: 5 Answer Score: 1
Question ID: 309453

Do we need to formally teach the Greek Alphabet?

Question

This is a question that I am purely interested in because I think we never thought about this before in Mathematics education... or even so was not discussed.

When did we learn the Greek alphabets when we learnt mathematics? For example, I was pretty afraid when I saw Pressure = $h\rho g$ or even the idea where the variable for an angle was θ and that was when I was in 8th grade.

When I went to high school, I got even more confused when I saw letters like μ, λ, σ during statistics class. During calculus class, don't we remember the δ represnting small changes and our peers write it like a small d? Even only when I was in *4th year of* college that I realize Σ represents sum because Sigma and Sum start with 'S' and Π represents produce because Pi and Product start with 'P' (it's for my memory but I'm not sure if it was taught this way.

So the question is: do we need to formally teach the Greek alphabet (not all but slowly) and tell them that researchers use these letters frequently to represent certain variables before we teach them? Of course, as student of math, we hardly (if ever) say X follows a Poisson distribution with parameter m or other correct, but "weird" sounding statements. I remember doing it during high school statistics... because I strongly believe that these "seemingly scary" letters "turn off" math students in the pre-college level.

Answer

To me, the Greek alphabet is just another character set we use. (For a non-Greek example, the Hebrew letter *aleph*, $\aleph$, is used in discussions of cardinal numbers.)

I feel that it is a waste of time to "formally" cover it in a class–a student can easily "pick up" the alphabet just by using the letters in a class. I can recognize nearly all of the Greek letters (upper and lower case) just from casual browsing of math/science sites/textbooks. In one introductory engineering class, I was required to memorize the lowercase Greek alphabet, but I found I didn't need to from that casual browsing... While that was being discussed, I felt a little cheated that valuable class time was being spent discussing something so trivial.

If students didn't have access to Google or some other search engine, any good dictionary or encyclopedia should have a total list of the characters along with their names.

Question Score: 49 Answer Score: 3
Question ID: 701970

Relearn mathematics

Question

I believe that this question is well known, but since I'm not a US resident and do wish to learn math according to the U.S. school syllabus I wish that somebody can help me.

At my first eight years at school, I was good at math. Suddenly, I was involved in car accident and couldn't attend school for a long time. I've missed a lot of subjects and couldn't complete it until the final exams.

I didn't fail at final math exams but the average grade I got can't satisfy me since I wish to be able to learn Computer Science at the university (I'm working today as a software programmer but I can't touch many subjects because of math knowledge limitations). I've registered to it (Hebrew University of Jerusalem) and accepted but at the start of the year I've realized that I don't understand the subjects so I've stopped my studies and wish to rebuild my mathematics knowledge before re enrolling.

Could somebody points me to a book/website/syllabus that's talks about mathematics from scratch (Algebra, Geometry, Trigonometry etc.)? I've learned at Israel, but I do wish to follow the standard U.S. high-school syllabus.

Thanks you very much!

Answer

The website I would suggest is Khan Academy: http://www.khanacademy.org/ Literally hundreds of videos on math from counting to differential equations and linear algebra.

If you want books (specifically, high school text books), I highly recommend the Saxon Math series. I used these books from elementary math through precalculus (Saxon calls this Advanced Mathematics). (FWIW, I'm now in Vector Calculus and loving it...) Here's a link to the homeschool section–they will sell individual texts (rather than bulk orders) through this site: http://saxonhomeschool.hmhco.com/en/saxonhomeschool.htm

Khan Academy, though, is probably the best resource; plus, it's free! :)

Some helpful questions for syllabus information:

In what order should the following areas of mathematics be learned?

Good book to help me relearn Algebra 1...

Learning mathematics as if an absolute beginner?

Question Score: 3 Answer Score: 2
Question ID: 245314

Which calculus text should I use for self-study?

Question

I am 36 years old, and have forgotten a lot of math from high school, of which I only took up to Algebra 2. However I am teaching myself mathematics and am now, as an adult, completely fascinated with the logic and beauty of it, but pitiably I am only now beginning to end Algebra 1. I've a long way to go yet, and have the Saxon method at home (Alg.1, Alg.2, Advanced Mathematics (trig and precalc), and Saxon Calculus). So far I like it, as it is what I am accustomed to, having used it in high school. But my question is if a calc text like Spivak or Apostol would be better than the Saxon Calc. I've searched the web for any insight I might glean, but I have found that Saxon is not mentioned in the same circles as Apostol or Spivak. I throw those names out, by the way, not due to any knowledge of Calculus, as I am far away from that yet, but because I've seen good things written about them. Anyway, any helpful advice would be appreciated.

Answer

1. I love Saxon's books. I'm actually at (or near by 1-2pts) the top of my college CalcII class, and know that Saxon's books have prepared me incredibly well.

2. As far as a recommendation for learning Calc, I suggest using http://www.khanacademy.org/.

Question Score: 14 Answer Score: 2
Question ID: 128220

How to prevent doing the same mistakes over and over again?

Question

At the moment I am preparing for the GMAT. However, a phenomenon that occured and has occured in the past is that sometimes I always make the same mistakes at similar problems, especially when doing GMAT problems or collegue homework problems. How do YOU prevent yourself from the phenomenon?

I really appreciate your replys!!!

PS.: I would be thankful, if you do not immediately close my open question

Answer

Practice, practice, practice! :)

Here's one way that I've made use of in the past. Find a bunch of problems that involve the mistake you keep making. Then, when you come across that part of the problem, don't just solve it.

Instead, return to your textbook (not the solution guide to the problem) and look at other similar problems. Follow the process outlined in the book. Repeat until this is overly boring.

At some point, after doing this enough, you will have memorized the *correct* way of doing the problem. For the next problem, don't look it up in the book, but work from what you remember. Check your work (without looking in the answer book), and then look at the answer. If you messed up, look at the textbook again and try another problem.

Basically, you want to practice doing the right way a whole bunch of times. Whatever helps you to remember to do it correctly for one problem, do it over and over and over again.

Question Score: 2 Answer Score: 5
Question ID: 315987

Where to relearn algebra?

Question

I need a recommendation for a book for relearning basic algebra and re-understanding concepts like logarithms ans so on...any recommendation?In the past I've used Basic mathematics By serge lang ,I want something different.

Answer

For books, I highly recommend the Saxon math texts. You'd probably most benefit from their Algebra 1 or Algebra 2 textbook.

Also, I second Damien's recommendation of Khan academy. Those videos are really helpful.

Question Score: 1 Answer Score: 0
Question ID: 465832

Gap year to study math

Question

This is a plan in its earliest and thus least concise stage, so either bear with me or don't read the following babble (I bolded some of the important stuff):

I am a high school graduate who is about to enter a top 5 US college ("top 5" might sound arrogant and unnecessary, but I'm keeping it because I think it could somewhat affect your response), and I'm very interested in majoring in mathematics. I am not blind to other potential majors, as I am also very interested in philosophy, and to a lesser degree, certain natural sciences. However, over the past year or so I have fallen further and further in love with pure mathematics.

All that David Copperfield kind of crap: Last summer I self-studied much of basic calculus. This past year I complete AP **calc AB**, and for the past couple of months I have been going through **How to Prove it by Velleman** pretty thoroughly (really like it so far). **During the rest of the summer** I intend to finish HTPI and do as much of **Spivak's Calculus** as I can (I loved calc this year and want to try some more advanced/proof-based material).

You might be saying, "wow, this kid is basically a mathematical virgin." And you would be correct: Compared to some of the incredible people on my college's Facebook group who did calculus in 9th grade, I am highly inexperienced. But as Einstein said, "I am not a genius, I am only passionately curious," and I am the second one.

So the question: **Would it be a good idea for me to take a gap year to (continue to) self-study mathematics?**

On my hypothetical gap year, I would begin my self-study (as I think I learn better and possibly a little faster that way) by either **continuing with calculus or starting linear algebra**. Ideally I could devote, say, the first half of my year to one/both of those and the rest to **basic/introductory abstract algebra**, which I realize might be out of my league, but I just find it so damn intriguing.

The only reason I'm even considering this idea is because I have never really immersed myself in mathematics or had that much time to pursue it on my own, other than last summer/right now, and so I've always felt like there's a next level that I've never really experienced and probably wouldn't have time to experience in the first few years of college.

I have also spent a lot of time reading about great mathematicians of the past, and I've gotten the feeling that many of them (Grothendieck, Galois, Euler, even Newton, to some extent) learned the most in own independent studies. Now, I'm not trying to compare myself to these demi-gods, but **I feel like if there's any time I could get ahead and have a chance to learn how to think like a mathematician, it is now.**

So what do you think? Any personal experience in the matter? Do you think I should be worried about forgetting stuff (this is the usual concern with a gap year for mathematicians, so I thought I'd ask about it, although given that I would be doing and learning math on may gap year, it probably doesn't apply to me as much)?

It might seem kind of odd to take a gap year from learning just to learn, however, a whole year would give me a chance to, as I said before, immerse myself and learn more intensively than I'll be able to at college and beyond.

I realize that this might be better asked in the academia community, however, I would like a mathematics-specific answer before I consider this question from a university's perspective.

I also realize that this is ultimately my decision, so no need to tell me to just do what I want or what I think is best for me; the purpose of this is to help me determine what would be best for me. :)

I also realize that this question is way too long, but I'd ask you not to respond if you didn't read most of it, just to ensure that you are not misunderstanding my situation.

Answer

Some things to keep in mind:

- No matter what, **don't** lose your self-driven attitude. This is is probably your greatest strength, and is rare to come by. *Always* engage in some form of self-study.
- Will your offer of acceptance to your top-5 institution still be there next year if you don't go this fall? If not, are you willing to risk losing that acceptance? (This is just a question to consider.)
- Self-study is certainly the best way to learn math (IMHO). You will learn an incredible amount through the "struggle" (not the best word, but I think it approximates my meaning) to understand a topic.
- I am taking a non-traditional route to college education, by spending an extra year at a community college before continuing to a four-year school. There are *plenty* of people who take off a year between high school and college–so long as you are purposeful with your time, it won't cause many problems.
- One of my classmates who stands out as a "great math person," (who is actually majoring in math, whereas a lot of my other friends are engineering students) took precalculus for the first time (I believe) at the college level. You certainly don't have to finish Spivak before entering college to do well in mathematics.

Question Score: 18 Answer Score: 11
Question ID: 855557

Resources to help an 8yo struggling with math

Question

Friends of mine asked me for suggestion for one of their children (age 8) who had bad scores at the local Star test (the family is based in California).

Both parents work, so they have also limited time/energies to go through math exercise with the kid (or may have time only at the end of the day, when the student's energies are depleted, too).

This is not to say that anything requiring parent support should automatically disqualified - it's just to make clear that parent assistance could be a limited resource, so either something that can be done more or less alone by the student, or that gets maximum bang-for-the-buck for the parents time would be preferred.

Books (including exercise workbooks)? Software? Online videos? Games (boardgames, computer games)?

Answer

Online videos: Khan Academy has some great resources. Check out their Arithmetic section.

Online game: This is a fun "constructions" game. Maybe a bit advanced, but is pretty coo, nonetheless.

Some general ideas:

Give the kid a jar of pennies, and have him/her make patterns (e.g. rectangles, squares, etc). Start it off with supervision/guidance, and then let them play on their own. Have them count the pennies, first by ones, then by twos, etc. Show how grouping the pennies in stacks of 5 can make it easier to count without mistakes.

Put up a "hundred chart" in their bedroom (or somewhere they spend a lot of time). Show them patterns like how the multiples of 5 are in two columns, the multiples of 2 are in diagonals, the multiples of 9 are also on diagonals. Show them the Sieve of Eratosthenes to find "prime numbers" when they understand multiplication.

As soon as they know of a multiplication table, show how it can be used to find the number of pennies in a 8×12 rectangle.

Introduce them to "recreational math" problems There are plenty here under the recreational-mathematics ranging from super easy to super hard.

Sure: many of these involve some parent (or other adult) help. But that's just because **there's no substitute for a parent/significant adult who cares**. If the parent takes time out of their busy schedule to (patiently) work with the kid on math, it sends a message: "Math is important to me (the parent), and I care about helping you succeed." Kids (especially young ones) care about what their parents care about.

Question Score: 4 Answer Score: 2
Question ID: 457446

How to explain that proof is important

Question

I don't know if this is the right place to post this or not, but I will go ahead anyway (sorry if it ain't the right place)

Yesterday I was discussing a particular theorem of geometry with my brother which he just learnt in the school. I had asked him if he knew the proof for it, he replied saying his teacher has said that wouldn't be necessary.

Then, I asked him to sit and try to prove the theorem. He said that knowing the theorem counts for more than knowing the proof. How do I explain to him that knowing the proof is more important and how it can even help expand his thinking?

I know this question doesn't have a single pointed answer as is pre-requisite for questions posted here, but I would appreciate any replies

Answer

An idea on why learning proofs (not just theorems) is important: One could say that it is unimportant to know how to prepare food because there is a McDonald's down the street. But, if a person becomes strictly reliant on McDonald's for preparing food, then we can be assured that (s)he will never be able to produce a (worthwhile) dish of their own creation.

Likewise with proofs–one could say it is unimportant to know how to "prepare" the "food" of a theorem via proof because there is the "McDonalds" of the math book nearby. But, after years of just relying on memorizing theorems, a person will never be able to come up with a sound theorem of their own.

Being able to prove something makes it much more solidified in one's mind, and gives you a tool that is applicable to many circumstances, not just a single instance. For example, my double angle formula may not be useful when I need a triple angle formula, but I could use the proof/derivation of the double angle formula to *find my own* triple angle! This is where proof is much more powerful than memorization.

Question Score: 10 Answer Score: 3
Question ID: 600800

9.3 Problem Solving

Repeating u-substitution

Question

A question about the integration technique of u-substitution:

Is it allowed to apply u-substitution over and over again, to reduce the integral to a more manageable form?

Answer

Most certainly.

However, I do suggest that you do not keep using the letter u for your substitutions to avoid confusion; I typically use w for my second substitution, and t for a third, etc. Situations in which you need more than 3 substitutions are fairly rare... (in my experience)

Also, these situations oftentimes arise with applying u substitution, followed by some other technique, then another substitution, etc. Back-to-back substitutions can be handled by one (bigger) substitution.

Question Score: 2 Answer Score: 3
Question ID: 389671

When to give up on a hard math problem?

Question

I practice olympiad problems from books like *Putnam and Beyond.* Often I come across a problem that I simply can't solve. After ~ 30 minutes of deep thinking it feels like I'm ramming my head into a brick wall, since I've exhausted all avenues of thought I am aware of. What should I do in these situations? Move on to another problem? Give up and see the answer? Or spend more time on it?

Answer

The answer for *you* changes with time.

This is much like the overused joke about the math professor working for 45 minutes on a problem, looking up, and saying "it's trivial!" As you work more and more on math problems, you start to be able to work longer without becoming discouraged. For example, 30 minutes may seem like *forever* now, but after working on a ton of (hard/stretching) problems, you'll probably think it seems small.

Always stretch/push yourself, and never lose track of *why* you're solving the problem.

Of course, if you mean "When to give up on a hard math problem, *during competition for strategic reasons,*" that's a different problem entirely.

Question Score: 38 Answer Score: 4
Question ID: 820004

When to do u-substitution and when to integrate by parts

Question

I'm in my first semester of calculus, so the problems I'm facing are about as hard as those on KhanAcademy calculus playlist. I'm currently doing integration, a somewhat diffucult part of the course. Doing derivatives is mechanichs; finding the integral is an art.

The two main techniques showed are

- u-substitution
- integration by parts

My quesiton is: are there any rules of thumbs (preferably with a logical reason behind it) of when to use which?

Answer

Always do a u-sub if you can; if you cannot, consider integration by parts.

A u-sub can be done whenever you have something containing a function (we'll call this g), and that something is multiplied by the derivative of g. That is, if you have $\int f(g(x))g'(x)dx$, use a u-sub.

Integration by parts is whenever you have two functions multiplied together–one that you can integrate, one that you can differentiate.

My strategy is to try to "play it out" in my mind and try to see which one will work better. The best way to get better at these sorts of integrals is to practice large sets of each type. Then, you start to think "Oh–this looks like a u-sub!" or, "maybe by-parts is better for this." **Practice is really the best way to get better at recognizing each type.**

Question Score: 2 Answer Score: 1
Question ID: 538654

Chapter 10

My Questions

Question from Putnam '08: Given $F_n(x)$, find $\lim_{n\to\infty} \frac{n!F_n(1)}{\ln(n)}$

Question

Problem Statement: Let $F_0(x) = \ln(x)$. For $n \geq 0$ and $x > 0$, let $F_{n+1}(x) = \int_0^x F_n(t)dt$. Evaluate

$$\lim_{n\to\infty} \frac{n!F_n(1)}{\ln(n)}$$

Source: Putnam 2008, Problem B2.

My solution: First, show by induction that:

$$F_n(x) = \frac{x^n}{n!}\left(\ln(x) - H_n\right)$$

...where H_n is the nth harmonic number. (Induction proof omitted because it is trivial, and doesn't relate to my question)

Then, the limit becomes:

$$\begin{aligned}
\lim_{n\to\infty} \frac{n!F_n(1)}{\ln(n)} &= \lim_{n\to\infty} \frac{n!}{\ln(n)} \frac{1^n}{n!}\left(\ln(1) - H_n\right) \\
&= \lim_{n\to\infty} \frac{\left(\ln(1) - H_n\right)}{\ln(n)} \\
&= \lim_{n\to\infty} \frac{-H_n}{\ln(n)} \\
&= \lim_{n\to\infty} \frac{-H_n}{\ln(n)} && (1) \\
&= -\lim_{n\to\infty} \frac{\ln(n)+\gamma}{\ln(n)} && (2) \\
&\stackrel{\text{L'H}}{=} -\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n}} && (3) \\
&= -1 && (4)
\end{aligned}$$

(where γ is Euler's Constant)

My Questions

1. For Putnam-style grading, do I have to prove the limit exists by definition (e.g. $\epsilon - \delta$ proof)?

2. Do I have to show that I can apply limit rules (e.g. L'Hopital's from (2) to (3)) to limits of discrete variables?

3. In between lines (1) and (2), I made a jump based on the definition that $\lim_{n\to\infty}(H_n - \ln(n)) = \gamma$ So, as n becomes large, H_n is approximately $\ln(n)+\gamma$. I am most uneasy about this step, as it feels like a "back of the paper" sort of substitution and not a rigorous one. Is this a justified step?

And, of course, I'd be open to suggestions/comments on the style of my solution and general rigor, but the above three points are the main ones to which I'm looking for answers.

Answer

I feel like your perception of this problem may be backwards: to my mind, the induction to prove the form of $F_n(x)$ is the 'meat' of the problem, and once you've got that result it's the rest of the problem that's trivial. To answer your specific questions, though: you shouldn't need an epsilon-delta proof for limits on a Putnam; once you have an explicit form for F_n (and note that the first integral is improper so a little justification may help there), you can manipulate the limit quite a bit – as long as you don't do anything improper, you should be fine.

In this case, if you didn't know the form of the Harmonic series explicitly (and I would personally take $H_n = \ln n + O(1)$ as well-enough established that it didn't need independent justification, but I wouldn't fault someone for feeling otherwise) then you can use Riemann estimates for $\int_1^n \frac{1}{x}dx$ to bound it: just break it up as $\sum_{i=1}^{n-1}\left(\int_i^{i+1}\frac{1}{x}dx\right)$ and note that the integral in parentheses is bounded between $\frac{1}{i+1}$ and $\frac{1}{i}$. Summing, this gives $\ln n \leq H_n \leq \ln n + 1$, and that's more than enough to give the result: since $F_n(1) = -\frac{H_n}{n!}$ then $-\frac{\ln n+1}{n!} \leq F_n(1) \leq -\frac{\ln n}{n!}$ and so $-\left(1+\frac{1}{\ln n}\right) \leq \frac{n!F_n(1)}{\ln n} \leq -1$; the squeeze here is trivial, and you don't need L'Hopital's rule at all.

In general, rigor in contest problems is to be encouraged, but it should also be the last thing you work on; for an exam like the Putnams where you (almost certainly) won't be able to complete all the problems, putting effort into a new problem is IMHO more likely to bear fruit (and points) than the last few drops of rigor on a problem you've already gotten a result for.

Answer by: Steven Stadnicki
Question Score: 4
Question ID: 609913

Question

This webpage (as well as Wikipedia) explains how one can use the Miller-Rabin test to determine if a number in a particular range is prime. The size of the range determines the number of required bases to test. For example:

> If $n < 9,080,191$ is a base 31 and base 73 strong probable prime, then n is prime.
>
> If $n < 2,152,302,898,747$ is a base $2, 3, 5, 7$, and 11 strong probable prime, then n is prime.

How many bases must I test to deterministically tell if any integer $n < 2^{63}-1$ is prime?

As one may guess by the bound, I'm interested in using this approach for primality testing in programs. Assuming the truth of the extended Riemann hypothesis (which I'd be willing to do, given that I am using this code simply for competitions and/or "toy" problems, but not for high-risk applications), it is sufficient to test all bases a in the range $1 < a < 2\log^2(n)$.

Reducing the number of bases I need to check could dramatically improve the runtime of my code; thus, I was wondering if there was a smaller number of bases.

Answer

According to A014233,

> $a(12) > 2^{64}$. Hence the primality of numbers $< 2^{64}$ can be determined by asserting strong pseudoprimality to all prime bases $\leqslant 37 (= \text{prime}(12))$. Testing to prime bases $\leqslant 31$ does not suffice, as $a(11) < 2^{64}$ and $a(11)$ is a strong pseudoprime to all prime bases $\leqslant 31 (= \text{prime}(11))$. [Joerg Arndt, Jul 04 2012]

$$a(11) = 3825123056546413051 < 2^{62}$$

Answer by: Daniel Fischer
Question Score: 3
Question ID: 1004807

Iterations of modulus operation

Question While working on a completely unrelated task, I thought up the following problem:

> Consider the following process. Let a_0 and n be given, and determine $a_1, \ldots, a_k$ as follows:
>
> $$a_{j+1} = n \bmod a_j$$
>
> ... where the process terminates when $a_j \mid n$. That is, $a_k = 0$.
>
> Define $f(a_0, n) = k$.
>
> Aside from direct iteration, how might one compute or express $f(a_0, n)$?

Example: $f(124323, 938342) = 12$

```
   68081 = 938342 mod 124323
   53289 = 938342 mod 68081
   32429 = 938342 mod 53289
   30330 = 938342 mod 32429
   28442 = 938342 mod 30330
   28198 = 938342 mod 28442
    7808 = 938342 mod 28198
    1382 = 938342 mod 7808
    1346 = 938342 mod 1382
     180 = 938342 mod 1346
       2 = 938342 mod 180
       0 = 938342 mod 2
```

This process always terminates (as can be trivially proven).

Aside from the obvious $f(n \pm 1, n)$ case, I don't really know where to begin approaching this–problems I just think up tend to be harder than the math I know. ';)'

So, my questions are:

1. Obviously, if someone sees an obvious route to computing this, that would be great.
2. If this is something that is known to be impossible, that would also be good to know.
3. If someone has a hint, I'd appreciate that–I like trying to figure things out, but I know that good hints can be difficult to write.

Answer

This is what I call Gauss's algorithm for computing inverses, when $n = p$ prime. By replacing a by $a' := n \bmod a = n - qa$ it yields $a' \equiv -qa$ which is a *smaller* multiple of a. Iterated, it yields a *descent* on the elements of the ideal (a) of all multiples of a mod n. It terminates with a factor of n. This factor can only be 1 when $n = p$ is prime, so it shows that 1 is a multiple of a, by the product of the sequence of quotients q_i, so yielding $a^{-1} \bmod n$.

Answer by: Bill Dubuque
Question Score: 6
Question ID: 643047

Question

Yesterday, a question was posted that went along the lines of:

> Suppose we have a number of workers, N. These workers work at a constant rate of R widgets per unit time. ($R > 0$) We also have a fixed time interval, $I > 0$ (in units of time), such that after I time units, we add another worker.
>
> Is there a closed form for the number of widgets produced, W, after t time units pass? (t is not necessarily a multiple of I.)

I solved the problem given above after looking at the graph of output versus input. However, my solution relies on the time interval I being constant. (See my work below.)

My question is, if I were to vary by some rule, could a closed form be found? For example, could a solution be found if we were to add one worker every F_n time-units? (where F_n is the nth Fibonacci number)

My work for the old question: Essentially, at $t = kI$, $k \in \mathbb{N}$, we have:

$$W = \left(\frac{k(k+1)}{2}\right) IR = \left(\frac{\frac{t}{I}\left(\frac{t}{I}+1\right)}{2}\right) IR$$

This is because our "overall rate" (rate for all the workers) is R for the first I time-units, $2R$ for the next I time-units, etc. The total output for the first k intervals is the sum of these overall rates times I. This is equivalent to the sum of the first k natural numbers times IR

At t between multiples of I, we simply use the amount at the previous multiple of I and add to it the amount of widgets produced since then:

$$W = \left(\frac{\lfloor \frac{t}{I} \rfloor \left(\lfloor \frac{t}{I} \rfloor + 1\right)}{2}\right) IR + \left(\left\lfloor \frac{t}{I} \right\rfloor + 1\right)\left(t - \left\lfloor \frac{t}{I} \right\rfloor I\right) R$$

(I haven't paid attention to the units of the above expressions. They work out numerically, but not with respect to their units.)

Answer

Let $f(n)$ be the number of units of time that pass before the nth worker is added, and define $f(0) = 0$. That is, at time $f(0)$ there are N workers, at time $f(1)$ there are $N+1$ workers, and so on. Then the number of widgets produced up to time $f(n)$ is

$$\begin{aligned}
&\sum_{i=0}^{n-1} \#\{\text{Widgets produced between } f(i) \text{ and } f(i+1)\} \\
=&\sum_{i=0}^{n-1} [f(i+1) - f(i)] \cdot [N+i] \cdot R \\
=&\sum_{i=0}^{n-1} Ri[f(i+1) - f(i)] + \sum_{i=0}^{n-1} RN[f(i+1) - f(i)] \\
=&\sum_{j=1}^{n} R(j-1)f(j) - \sum_{j=1}^{n-1} Rjf(j) + \sum_{j=1}^{n} RNf(j) - \sum_{j=1}^{n-1} RNf(j) \\
=&(n-1)Rf(n) - R\sum_{j=1}^{n-1} f(j) + RNf(n) \\
=&\boxed{R\left[(N+n)f(n) - \sum_{j=1}^{n} f(j)\right]}.
\end{aligned}$$

As a check that this formula is right, note it satisfies certain edge cases: $n = 0$ correctly gives 0, and $f(n) = In$ starting with one worker at the start ($N = 1$) gives your expression $IR\frac{n(n+1)}{2}$.

Philosophically one may debate about whether a summation counts as a "closed form", but it is clear that in the case of a general f there is no simpler expression.

When $f(n)$ is the nth Fibonacci number, using Sum of the first n Fibonacci number we get that:

$$R[(N+n)F_n - F_{n+2} + 1]$$

widgets have been produced up to time F_n.

Answer by: 6005
Question Score: 2
Question ID: 288289

Question I was playing around with some integrals, and this question popped into my head:

> What functions exist such that the following is true?
>
> $$\int f(g(x))\ \mathrm{d}x = f\left(\int g(x)\ \mathrm{d}x\right)$$

There there's the obvious example of $f(x) = x,\ g(x) = e^x$, but I was wondering if others exist.

EDIT 1: As pointed out in the comments, this is true for any g if $f(x) = x$. But, this is sort of trivial–I'd really like to know about for other assignments of f...

My question is twofold:

1. Are there known functions that satisfy this equality?
2. What sort of topic in math would this fall under? (e.g. Abstract Algebra, Differential/Integral equations, etc.)

I'd also accept an answer to a similar, but slightly different question, as phrased in the comments by user1551; if it's easier/more feasible to answer:

> Find a pair of functions f and g such that $\int_a^b f(g(x))dx = f\left(\int_a^b g(x)dx\right)$ for any interval $[a, b]$

Answer

I can reformulate the problem in the following way. Suppose we have

$$\int_0^x f(g(y))dy = f\left(\int_0^x g(y)dy\right)$$

Take a derivative of each side to get

$$f(g(x)) = g(x)f'\left(\int_0^x g(y)dy\right).$$

Assume $g(x)$ is invertible. Then letting $z = g(x)$ we get

$$\frac{f(z)}{z} = f'\left(\int_0^{g^{-1}(z)} g(y)dy\right).$$

Let $h(z) = \int_0^{g^{-1}(z)} g(y)dy$ be some arbitrary function. Note that this is the same as $h(g(x)) = \int_0^x g(y)dy$. Then we look for solutions to

$$\frac{f(z)}{z} = f'(h(z)), \qquad f(0) = 0$$

$$h'(z) = z/g'(g^{-1}(z)), \qquad h(g(0)) = 0.$$

The boundary conditions come from plugging in $x = 0$ into the appropriate expressions. We see that regardless of what $h(z)$ is, $f(z) = z$ is always a solution, as mentioned in the comments.

I spent some time trying to find another solution, but I could not. However, if you modify the original problem to

$$\int_0^x f(g(y))dy = f\left(\int_{-\infty}^x g(y)dy\right),$$

then our new equations reduce to

$$\frac{f(z)}{z} = f'(h(z)), \qquad f(0) = 0$$

$$h'(z) = z/g'(g^{-1}(z)), \qquad h(g(-\infty)) = 0.$$

It turns out that $f(x) = x^p$ and $h(x) = x/p^{p-1}$ satisfy the first equation. Then if we choose $g(x) = e^{xp^{p-1}}$, we find the second equation is also satisfied.

If we really asked, for what f and g does there exist such that $\int_{-\infty}^x f(g(y))dy - f\left(\int_{-\infty}^x g(y)dy\right)$ is constant, then the above gives a new solution.

Answer by: nayrb
Question Score: 14
Question ID: 430420

Question

I'm not a math major, but would like to compete in the Putnam. As suggested in other questions here, I'm working some old contest problems. I'd like some input on this attempted proof–general input is gladly welcomed, but I do have some specific questions (see bottom of post.)

> **Prompt:**
>
> How many primes among the positive integers, written as usual in the base 10, are such that their digits are alternating 1's and 0's, beginning and ending with 1?
>
> **Source:** KSU Putnam

Answer: There is only one such prime, namely, 101.

Proof:

Consider the numbers of the form $101\ldots01$. Let a_n be the number of this form with n ones in its base-10 representation. That is:

$$a_n = \sum_{k=0}^{n-1} 10^{2k} \tag{1}$$

It follows:

$$\begin{aligned} a_n &= \sum_{k=0}^{n-1} \left(10^2\right)^k \\ &= \frac{\left(10^2\right)^n - 1}{10^2 - 1} \\ &= \frac{(10^n)^2 - 1}{99} \\ &= \frac{(10^n - 1)(10^n + 1)}{9 \cdot 11} \end{aligned}$$

We now note that $10^n - 1 = 9\sum_{k=0}^{n-1} 10^k$. Thus:

$$a_n = \frac{\left(\sum_{k=0}^{n-1} 10^k\right)(10^n + 1)}{11} \tag{2}$$

Let b_n be the numerator in (2); that is:

$$b_n = \left(\sum_{k=0}^{n-1} 10^k\right)(10^n + 1)$$

From formula (1), we can see that a_n is the sum of integers, thus, a_n is clearly an integer. For a_n to be a prime, b_n must satisfy the following:

$$b_n = 11 \cdot p, \text{ where } p \text{ is prime.}$$

Therefore, one of either $\left(\sum_{k=0}^{n-1} 10^k\right)$ or $(10^n + 1)$ must be 11, and the other is prime. For $n = 1$, we see:

$$\left(\sum_{k=0}^{n-1} 10^k\right) = 1; \qquad (10^n + 1) = 11$$

For $n = 2$, we see:

$$\left(\sum_{k=0}^{n-1} 10^k\right) = 11; \qquad (10^n + 1) = 101$$

As both terms are monotonically increasing, these two are the only ways to have either one equal to 11. Of these two, only one pair contains a prime (namely, 101).

My questions: First and foremost, I want to know if my proof is valid/on the right track. After that, I'd like to know if I'm covering the proof in enough detail. Specifically:

1. I assert that $10^n - 1 = 9\sum_{k=0}^{n-1} 10^k$. Do I need to prove this, or is it obvious enough? (Subtracting 1 from a power of 10 leaves a number with as many 9s as there are 0's in the power of 10.)

2. Do I need to show further why b_n must be 11 times a prime? Or is it clear that all other cases yield composite a_n?

3. Is my reasoning that there are only these two solutions (i.e. monotonically increasing terms) legitimate?

And, finally, is there some well-known theorem that I have totally overlooked that would make solving this problem easy?

Answer

You can actually cut your proof short a fair bit earlier – once you have $a_n = (10^n+1)(10^n-1)/99$, you can note that for $n > 2$, both of these factors are larger than 2×99 and so there must be a factorization of a_n into two numbers each larger than 2.

More formally, you can split into the case of even n and odd n; for even n then $99|10^n - 1$ so you have $a_n = (10^n + 1) \times ((10^n - 1)/99)$ with each factor > 1, while for odd n $11|10^n + 1$ and $9|10^n - 1$ so $a_n = ((10^n + 1)/11) \times ((10^n - 1)/9)$ with each factor > 1. (These divisibility properties are immediate from $10 \equiv 1 \pmod 9$ and $10 \equiv -1 \pmod{11}$). This also explains why the proof breaks down at $n = 2$; there, you still have $a_n = (10^n + 1) \times ((10^n - 1)/99)$, but it's no longer the case that the right factor is > 1.

Answer by: Steven Stadnicki
Question Score: 16
Question ID: 479500

Question

I've recently taken an interest in the Method of Annihilators for solving recurrence relations. However, I taught myself using nearly solely this Power Point presentation. Thus, my knowledge is lacking in some ways.

My question is, **how does one "come up" with an annihilator?** The presentation I saw showed how to prove/demonstrate an annihilator *works*, but now how to just pull one out of thin air.

An example of what I'm talking about: How would I find an annihilator for $\sin\left(\frac{\pi}{2}n\right)$? I could just start guessing/writing out the sequence, and stumble upon the annihilator $(E^2 - 1)$ But, is there a *systematic* way to find this annihilator?

Answer

The trick behind such terms is to consider them in the complex plane, I.e. $\sin\frac{\pi}{2}n = \frac{1}{2\mathrm{i}}\left(\exp\left(\frac{\pi\mathrm{i}}{2}n\right) - \exp\left(-\frac{\pi\mathrm{i}}{2}n\right)\right)$, and this is just two n-th powers

Answer by: vonbrand
Question Score: 1
Question ID: 766863

Notation: Why write the differential first?

Question From reading answers here, I've noticed that some people write integrals as $\int dx\ f(x)$, while other people write them as $\int f(x)\ dx$.

I realize that there is no mathematical difference between the two notation forms, but was wondering *why* some people choose the first method over the second. Is there some place in higher maths that it becomes beneficial to write the differential first?

(I, personally, have always used the second method, just because I was taught that way...)

Answer

When you have a lot of integrals, particularly with limits, it can be very helpful at times to be able to tell at a glance which integral is over which variable.

$$\int_0^1 \int_2^3 f(x,y)\ \mathrm{d}x\mathrm{d}y$$

This is not particularly readable or clear, especially when f is lengthy and there are more nested integrals etc. I could also imagine it being misinterpreted.

By contrast,

$$\int_0^1 \mathrm{d}y \int_2^3 \mathrm{d}x\ f(x,y)$$

makes it very clear what is going on. The only price you pay is possible ambiguity about where the integral ends, but this is easier to make clear with formatting and less of an issue anyway.

It also just occurred to me that the second notation ties in better with the syntax of an operator. That is, if one thinks of $\int_0^1 \mathrm{d}x$ as being an operator, taking a function to its integral, it's more natural to have the whole operator together in one lump. Think of how one changes

$$\frac{\partial f}{\partial x} \to \frac{\partial}{\partial x} f$$

Answer by: Sharkos
Question Score: 20
Question ID: 387572

Proving the Riemann Hypothesis and Impact on Cryptography

Question

I was talking with a friend last night, and she raised the topic of the Clay Millennium Prize problems. I mentioned that my "favorite" problem is the Riemann Hypothesis; I explained what it posits and mentioned that, if proven, it would have great impact on cryptography. Her immediate response was, "Why do we have to wait to prove it? We're pretty sure it's true, so why can't cryptographers assume so until shown otherwise?" As weird as it sounds, I was unable to give a good response.

In "popular mathematics" culture (i.e. not necessarily accurate but well-known), I've heard over and over that "proving the Riemann Hypothesis would cause a lot of problems in Cryptography." (See, for instance, the Numb3rs episode where a mathematician supposedly solves it and puts all of encryption at risk.) However, if a positive answer to the Riemann Hypothesis would result in us being able to break much of modern encryption, why *couldn't* we just assume that it has a positive answer and use that "maybe-valid" result to break codes *now*?

My best guess is that the idea that R.H. puts cryptography at risk is actually false; instead, either the process to reach the solution is the risky part, or it just produces a theoretical attack strategy that is still infeasible. That is, it would prove modern encryption insecure to *some* attack, but that attack technique is not possible at this time (much like creating a collision with SHA-1; theoretically, but not practically, possible).

tl;dr: I've been told that a positive answer to the Riemann Hypothesis is bad for cryptography. Why can't we assume a positive answer *now* to break codes, but just realize that what we're doing isn't guaranteed correct?

Answer

She is right - we already do generally assume the truth of the Riemann hypothesis, but knowing that RH is true does not provide any way to attack primes-based cryptography like RSA. Rather it's actually the opposite that is true, if anything. More on that in a second. But the implications of the Riemann Hypthesis (either way) for cryptography is greatly exaggerated and overhyped; the most likely truth is that a determination one way or the other won't affect security at all, and if it did have any effect it would likely be only theoretical.

Also see this question: Would a proof of the Riemann hypothesis affect security?

RH has numerous implications for regularity in the distribution of primes. Primes-based security is based on the belief that finding one of the two prime factors of an appropriately-generated semiprime is difficult. If there were a chink in the regularity of primes that made some part of them more predictable than they otherwise would be, then conceivably such a phenomenon could be exploited to detect prime factors better than we currently know how. Thus it would be a *counterexample* to RH that could actually pose a problem, not its truth.

But whether or not RH's negation could say something about primes that could be exploited for detecting prime factors is not clear. The most notable consequence of RH is the best error term for counting primes up to a given magnitude using the prime number theorem. Can you see how primes appearing a little more frequently than we currently believe could be transformed into a clever algorithm for prime factor detection? I don't think there is any known literature on how a nontrivial zero off the critical line could be utilized in cryptography.

The popular attacks and ideas on how to approach the Riemann hypothesis - noncommutative geometry and trace formulas, the field with one element, quasicrystals, quantum energy levels viz Hilbert-Polya and Berry-Keating, etc. - all seem to delve into territory far beyond and properly outside of analytic number theory, which is the subject dedicated to statistical and asymptotic properties of primes. (Although if anything does happen to stick in the future it will subsequently become subsumed into number theory.) But it is at least possible that the *ideas* that will be used in proving the Riemann Hypothesis (assuming it's true) will be strictly number-theoretic and provide direct insight into the structure of the primes that we did not previously have, that could conceivably be exploited to attack primes-based security. Once again, though, these are some pretty big ifs.

Answer by: whacka
Question Score: 7
Question ID: 1272296

Question

In a "math structures" class at the community college I'm attending (uses the book Discrete Math by Epp, and is basically a discrete math "light" edition), we've been covering some basic logic.

I've been reading some of the logic questions on here to get used to notation, etc. However, when I came across the question http://math.stackexchange.com/questions/279015/visualizing-concepts-in-mathematical-logic, I didn't understand what the $\vdash$ symbol means.

It's not in Discrete Math by Epp, nor is it in my mom's old logic book from when she went to college.

Wikipedia's Math Symbols page says it means "can be derived from" when used in a logic context. However, that doesn't make any sense in the above question, as there is nothing on the left of the $\vdash$.

So, what does $\vdash$ mean, especially in the context of the question linked above?

Answer

Let S be a set of (logical) formulae and ψ be a formula. Then $S \vdash \psi$ means that ψ can be derived from the formulae in S. Intuitively, S is a list of assumptions, and $S \vdash \psi$ if we can prove ψ from the assumptions in S.

$\vdash \psi$ is shorthand for $\varnothing \vdash \psi$. That is, ψ can be derived with no assumptions, so that in some sense, ψ is 'true').

More precisely, systems of logic consist of certain axioms and rules of inference (one such rule being "from ϕ and $\phi \to \psi$ we can infer ψ"). What it means for ψ to be 'derivable' from a set S of formulae is that in a finite number of steps you can work with (i) the formulae in S, (ii) the axioms of your logical system, and (iii) the rules of inference, and end up with ψ.

In particular, if $\vdash \psi$ then ψ can be derived solely from the axioms by using the rules of inference in your logical system.

Answer by: Clive Newstead
Question Score: 16
Question ID: 280384

Relationship between $\int_a^b f(x)\,dx$ and $\int_a^b xf(x)\,dx$

Question

I was working a problem, and came to a point where it would help greatly if there was a relationship between the following two expressions:

1. The numeric value of $\int_a^b f(x)\,dx$, and
2. The numeric value of $\int_a^b xf(x)\,dx$.

That is, if I know the numeric value for the first integral, but nothing about $f(x)$, is it possible to determine the numeric value for the second?

I've tried experimenting with integration by parts, but that always seems to need the indefinite integral of $F(x)$.

In response to Calvin Lin's comment: I'm looking to compute the second integral, so equalities are the type of relationship that I'm looking for.

In response to JoeHobbit's comment: The particular problem that I'm trying to solve is really at this link. However, this sprung off some other thoughts, not necessarily tied to specific problems.

Answer

It's impossible to determine the numerical value of $\int_a^b xf(x)dx$ exactly from $\int_a^b f(x)dx$ (for example, $\int_0^1 xdx = \int_0^1 (1-x)dx = \frac{1}{2}$, but $\int_0^1 x \cdot xdx = \frac{1}{3} \neq \int_0^1 x(1-x)dx = \frac{1}{6}$. However, you still know some things. As you mentioned, integration by parts tells you that $\int_a^b xf(x)dx = bF(b) - aF(a) + \int_a^b F(x)dx$, where $F(x)$ is an antiderivative of $f(x)$. If you could find bounds to the antiderivative, you could tell a bit about this integral. Another inequality was given in the comments, but you can't get an equality.

Answer by: William Ballinger
Question Score: 9
Question ID: 416959