NAVIER-STOKES EQUATIONS FOR FLUID DYNAMICS

LONG CHEN

CONTENTS

1. BASIC EQUATIONS FOR FLUID DYNAMICS

In this section, we derive the Navier-Stokes equations for the incompressible fluid.

1.1. **Eulerian and Lagrangian coordinates.** Let us begin with Eulerian and Lagrangian coordinates. The Eulerian coordinate $(\boldsymbol{x}, t)$ is the physical space plus time. The Eulerian description of the flow is to describe the flow using quantities as a function of a spatial location $\boldsymbol{x}$ and time t, e.g. the flow velocity $\boldsymbol{u}(\boldsymbol{x}, t)$. This can be visualized by sitting on the bank of a river and watching the water pass a fixed location. The equations governing the flow will be mainly written and solved in the Eulerian coordinate.

In contrast, Lagrangian description can be visualized as sitting in a boat and drifting down a river. To make it precise, let us introduce the reference coordinate $\boldsymbol{\xi}$ which can be thought as a (very fine) uniform grid. Each cell is a fluid parcel which is a very small amount of the fluid consisting of reasonable microscopic particles (molecules and atoms). The size of fluid parcels is large enough such that the averaged quantities remains meaningful. Meanwhile it is small compared to the macro length scales of the flow under consideration such that it can be thought as a point.

We assume that the flow starts from a reference configuration and is mapped into its deformed configuration. Mathematically it can be described as a map, called *flow map*, from the reference coordinate to the Eulerian coordinate

$$\boldsymbol{X}(\cdot, t) : \mathbb{R}^n \to \mathbb{R}^n, \quad \text{and } \boldsymbol{X}(\cdot, 0) = \text{identity map.}$$

Namely $\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{\xi}, t)$ is the spatial location of the parcel $\boldsymbol{\xi}$ in the Eulerian coordinate at time t. This map is one-to-one and assumed to be smooth (at least continuous and differentiable).

The Lagrangian coordinate is $(\boldsymbol{X}(\boldsymbol{\xi}, t), t)$. One parcel, labeled by $\boldsymbol{\xi}$, can be thought as a boat. $\boldsymbol{X}(\boldsymbol{\xi}, t)$ is the position of the boat at time t drifting down a river. Describing the flow sitting in the boat is equivalent to using the moving coordinate $(\boldsymbol{X}(\boldsymbol{\xi}, t), t)$.

While the computation is easier in the Eulerian coordinate, the physical law is easier to be described in the reference coordinates. For example, since $\boldsymbol{X}(\boldsymbol{\xi}, t)$ denotes the location of the parcel in space, its derivative respect to t will represent the velocity of the parcel $\boldsymbol{\xi}$. We denote by

$$(1) \qquad\qquad \tilde{\boldsymbol{u}}(\boldsymbol{\xi}, t) = \frac{\partial \boldsymbol{X}}{\partial t}(\boldsymbol{\xi}, t).$$

We now push forward the function to the Eulerian coordinate. Recall that the mapping $\boldsymbol{X}$ is one-to-one. So we can write $\boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{x}, t) := \boldsymbol{X}^{-1}(\boldsymbol{x}, t)$. Here $\boldsymbol{X}^{-1}(\cdot, t) : \mathbb{R}^n \to \mathbb{R}^n$ is the map from Eulerian coordinate back to the reference coordinate. Therefore

$$\boldsymbol{u}(\boldsymbol{x}, t) := \tilde{\boldsymbol{u}}(\boldsymbol{\xi}(\boldsymbol{x}, t), t), \quad \text{or} \quad \tilde{\boldsymbol{u}}(\boldsymbol{\xi}, t) = \boldsymbol{u}(\boldsymbol{x}(\boldsymbol{\xi}, t), t).$$

With slight abuse of notation, the $\tilde{}$ is usually skipped, i.e., we use $\boldsymbol{u}(\boldsymbol{x}, t)$ and $\boldsymbol{u}(\boldsymbol{\xi}, t)$ to represent the velocity in different coordinates. The same convention is applied to most functions.

On the other hand, given a smooth vector field $\boldsymbol{u}(\boldsymbol{x}, t)$ in Eulerian coordinate, we can solve the ODE systems

$$\frac{d}{dt} \boldsymbol{X}(\boldsymbol{\xi}, t) = \boldsymbol{u}(\boldsymbol{X}(\boldsymbol{\xi}, t), t), t > 0, \quad \boldsymbol{X}(\boldsymbol{\xi}, 0) = \boldsymbol{\xi}$$

to find $\boldsymbol{X}(\boldsymbol{\xi}, t)$ the location of each parcel $\boldsymbol{\xi}$ at any time t. Thus it suffices to derive equations for $\boldsymbol{u}$ and solve $\boldsymbol{u}$ in Eulerian coordinate.

To save notation we still use $\boldsymbol{x}$ for the mapping $\boldsymbol{X}$ (avoid typing capital letters and assume the function $f(\boldsymbol{x})$ is better received than $f(\boldsymbol{X})$). This could be a source of confusion when taking derivatives. In Eulerian coordinate, $\boldsymbol{x}$ and t are independent variables while in Lagrangian coordinate $(\boldsymbol{x}(\boldsymbol{\xi}, t), t) = (\boldsymbol{X}(\boldsymbol{\xi}, t), t)$ the spatial variable is a function of t. To avoid such confusion, let us treat the change from Eulerian coordinate to Lagrangian coordinate as a change of variable

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{\xi}, \tau), \quad t = \tau.$$

Here we use a different time variable τ to indicate that in Lagrangian coordinate $(\boldsymbol{x}(\boldsymbol{\xi}, \tau), \tau)$, the space and time variables are dependent. That is $\boldsymbol{x}$ is independent of t (in Eulerian coordinate) but dependent of τ (in Lagrangian coordinate).

We summarize different coordinates below:

- Eulerian coordinate $(\boldsymbol{x}, t)$;
- Lagrangian coordinate $(\boldsymbol{x}(\boldsymbol{\xi}, \tau), \tau) = (\boldsymbol{X}(\boldsymbol{\xi}, \tau), \tau)$;
- Reference coordinate (i.e. $\tau = 0$ in Lagrangian coordinate) $\boldsymbol{\xi}$;

1.2. **Material derivatives.** Let $f(\boldsymbol{x}, t)$ be a function in Eulerian coordinate. The partial derivative in the coordinate $(\boldsymbol{x}, t)$ is understood in the canonical sense. To describe the physical law, we need to change to Lagrangian coordinate $(\boldsymbol{x}(\boldsymbol{\xi}, \tau), \tau)$. By the chain rule, we have

$$\frac{\partial}{\partial \tau}(f(\boldsymbol{x}(\boldsymbol{\xi}, \tau), t(\tau))) = \nabla f \cdot \frac{\partial \boldsymbol{x}}{\partial \tau} + f_t = \boldsymbol{u} \cdot \nabla f + f_t.$$

The derivative with respect to τ variable, i.e.,

$$(2) \qquad\qquad D_\tau := \frac{\partial}{\partial \tau} = (\partial_t + \boldsymbol{u} \cdot \nabla)$$

is called *material derivative*. Here the *material* refer to fluid parcels. (The Lagrangian coordinate is also called material coordinate.)

Physically, this means the amount of change of f (in time) in Lagrangian coordinate consists of two parts: a temporal change and a spatial change. The temporal change f_t is observed in Eulerian coordinate (sitting still and track the rate of change of f in time). The other part is due to spatial change in the moving coordinate: we are in different locations at different time!

In the literature usually it is denoted by D_t or d/dt. We use D_τ to emphasize all physical laws are now described in Lagrangian coordinate. For example, since $\boldsymbol{u}$ is the velocity, then $D_\tau \boldsymbol{u} = \boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$ is the acceleration of parcel $\boldsymbol{\xi}$.

Let us take an arbitrary domain Ω_0 in the reference domain and denote $\Omega_\tau := \boldsymbol{X}(\Omega_0, \tau) = \{\boldsymbol{x} : \boldsymbol{x} = \boldsymbol{x}(\boldsymbol{\xi}, \tau), \boldsymbol{\xi} \in \Omega_0\}$ as the deformed domain of Ω_0 at time τ.

We now derive a formula for the material derivative of an integral $\int_{\Omega_\tau} f(\boldsymbol{x}, t) \, d\boldsymbol{x}$ in the Eulerian coordinate. Besides the derivative for f, we have to consider the change of the domain Ω_τ with respect to τ.

Lemma 1.1. *Let $J(\boldsymbol{\xi}, \tau) = (\partial \boldsymbol{x}/\partial \boldsymbol{\xi}) = (\partial x_i/\partial \xi_j)_{i,j=1,n}$ be the Jacobi matrix of the flow map, and let $|J| = \det J$ be the Jacobian. Then*

$$(3) \qquad \partial_\tau |J| = |J| \operatorname{div} \boldsymbol{u}.$$

Proof. By changing of variables, the volume of Ω_τ is

$$|\Omega_\tau| = \int_{\Omega_\tau} 1 \, d\boldsymbol{x} = \int_{\Omega_0} |J| \, d\boldsymbol{\xi}.$$

We then compute

$$\partial_\tau |\Omega_\tau| = \lim_{\Delta\tau \to 0} \frac{|\Omega_{\tau+\Delta\tau}| - |\Omega_\tau|}{\Delta\tau}.$$

The key observation is that the change of the volume is equal to the volume of the band

$$|\Omega_{\tau+\Delta\tau}| - |\Omega_\tau| = \int_{\partial\Omega_\tau} (\Delta\tau)\boldsymbol{u} \cdot \boldsymbol{n} \, dS,$$

since the boundary particles moving in the normal direction with velocity $\boldsymbol{u} \cdot \boldsymbol{n}$ in $\Delta\tau$ time step. We then have

$$\int_{\Omega_0} \partial_\tau |J| \, d\boldsymbol{\xi} = D_\tau |\Omega_\tau| = \int_{\partial\Omega_\tau} \boldsymbol{u} \cdot \boldsymbol{n} \, dS = \int_{\Omega_\tau} \operatorname{div} \boldsymbol{u} \, d\boldsymbol{x} = \int_{\Omega_0} \operatorname{div} \boldsymbol{u} |J| \, d\boldsymbol{\xi},$$

which implies (3) since Ω_0 is arbitrary. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

An algebraic version of (3) is summarized in the following exercise.

Exercise 1.2. *Let $A(t)$ be invertible matrices and differentiable with respect to t. Prove*

$$(4) \qquad \frac{d}{dt} \det A(t) = (\det A)\operatorname{tr}\left(A^{-1}\frac{d}{dt}A(t)\right).$$

Derive (3) *from* (4)*.*

Theorem 1.3 (Transport Theorem I)**.** *For a smooth function $f(\boldsymbol{x}, t)$, we have:*

$$(5) \qquad \frac{d}{d\tau}\left(\int_{\Omega_\tau} f(\boldsymbol{x}, \tau) \, d\boldsymbol{x}\right) = \int_{\Omega_\tau} (D_\tau f + f \operatorname{div} \boldsymbol{u}) \, d\boldsymbol{x} = \int_{\Omega_\tau} (f_t + \nabla \cdot (f\boldsymbol{u})) \, d\boldsymbol{x}.$$

Proof. Since the domain is changing with respect to τ, we go back to the reference domain Ω_0 and apply the formula (3) to get

$$
\begin{aligned}
\frac{d}{d\tau}\left(\int_{\Omega_\tau} f(\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{x}\right) &= \frac{d}{d\tau}\left(\int_{\Omega_0} f(\boldsymbol{x}(\boldsymbol{\xi}, \tau), \tau)|J|\,\mathrm{d}\boldsymbol{\xi}\right) \\
&= \int_{\Omega_0} D_\tau f(\boldsymbol{x}(\boldsymbol{\xi}, \tau), \tau)\,|J|\,\mathrm{d}\boldsymbol{\xi} + \int_{\Omega_0} f(\boldsymbol{x}(\boldsymbol{\xi}, \tau), \tau)\partial_\tau|J|\,\mathrm{d}\boldsymbol{\xi} \\
&= \int_{\Omega_0} (f_t + \boldsymbol{u}\cdot\nabla f + f\operatorname{div}\boldsymbol{u})\,|J|\,\mathrm{d}\boldsymbol{\xi} \\
&= \int_{\Omega_\tau} (f_t + \nabla\cdot(f\boldsymbol{u}))\,\mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

$\square$

1.3. Conservation laws. We now derive fundamental equations for fluid. Let $(\rho, \boldsymbol{u})$ denotes the mass density and velocity of fluid. Equations are written in the Eulerian coordinate but the derivation is easier in Lagrangian coordinate. We assume all functions are smooth enough such that the differentiation and integration is interchangeable.

1.3.1. *Conservation of mass.* The fluid parcel will move with the fluid flow. Its shape could change due to the distortion by the flow while the mass of a fluid parcel remains constant. This is considered as a basic principle of classical mechanics.

Let us take an arbitrary domain Ω_0 in the reference coordinate and consider its deformation Ω_τ at time τ. The mass of the fluid inside the domain Ω_τ is expressed as

$$
\text{mass} = \int_{\Omega_\tau} \rho(\boldsymbol{x}, \tau)\,\mathrm{d}\boldsymbol{x}.
$$

Conservation of mass implies that

$$
\frac{d}{d\tau}(\text{mass}) = 0 \quad \longrightarrow \quad \int_{\Omega_\tau} D_\tau\rho + \rho\operatorname{div}\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} = \int_{\Omega_\tau} \rho_t + \nabla\cdot(\rho\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} = 0.
$$

Since the domain is arbitrary, we get the mass conservation (also known as continuity) equation

$$
(6) \qquad\qquad\qquad \rho_t + \nabla\cdot(\rho\boldsymbol{u}) = 0.
$$

To further illustrate the difference between Eulerian and Lagrangian coordinates, we derive (6) in Eulerian coordinate. Now take an arbitrary domain Ω in Eulerian coordinate, which does not change in time. We consider the change of the mass in Ω:

$$
\frac{d}{dt}\int_\Omega \rho(\boldsymbol{x}, t)\,d\boldsymbol{x} = \int_\Omega \partial_t\rho(\boldsymbol{x}, t)\,d\boldsymbol{x}.
$$

The change of mass is due to the fact that some fluid parcels flow into or out of Ω which is characterized by the surface integral of flux

$$
-\int_{\partial\Omega} \rho\boldsymbol{u}\cdot\boldsymbol{n}\,dS.
$$

The negative sign is chosen since $\boldsymbol{n}$ is the outwards normal direction and $\boldsymbol{u}\cdot\boldsymbol{n} > 0$ means the loss of mass.

Mass conservation is then described by

$$
\int_\Omega \partial_t\rho(\boldsymbol{x}, t)\,d\boldsymbol{x} = -\int_{\partial\Omega} \rho\boldsymbol{u}\cdot\boldsymbol{n}\,dS.
$$

Apply divergence theorem and let Ω shrink to a point, we obtain equation (6) again.

1.3.2. *Incompressible fluid.* The fluid is called *incompressible* if $D_\tau \rho = 0$, i.e., the mass density of fluid parcels does not change during the moving.

For incompressible flow, the continuity equation is simplified to

$$\text{(7)} \qquad\qquad \operatorname{div} \boldsymbol{u} = 0.$$

Physically, or mathematically by (3), the volume of fluid parcels does not change. Note that due to the continuity equation $D_\tau \rho + \rho \operatorname{div} \boldsymbol{u} = 0$, we have

$$D_\tau \rho = 0 \iff \operatorname{div} \boldsymbol{u} = 0$$

By the formulae on Jacobian (3), we conclude for incompressible fluid, the Jacobian is constant and since $|J(0)| = 1$, this constant is one. As a consequence, the volume of Ω_τ does not change.

For incompressible fluid, since $\operatorname{div} \boldsymbol{u} = 0$, we then have

$$\text{(8)} \qquad \frac{d}{d\tau} \left(\int_{\Omega_\tau} f(\boldsymbol{x}, \tau) \, \mathrm{d}\boldsymbol{x} \right) = \int_{\Omega_\tau} D_\tau f(\boldsymbol{x}, \tau) \, \mathrm{d}\boldsymbol{x} = \int_{\Omega_\tau} (f_t + \boldsymbol{u} \cdot \nabla f) \, \mathrm{d}\boldsymbol{x}.$$

Formally we can exchange the integration and the derivative and safely skip the effect from the changing of the domain.

1.3.3. *Conservation of momentum.* The momentum is given by

$$\text{momentum} = \int_{\Omega_\tau} \rho \boldsymbol{u} \, \mathrm{d}\boldsymbol{x}.$$

By Newton's second law, we have

$$\frac{d}{d\tau}(\text{momentum}) = \text{force}(\Omega_\tau),$$

where $\text{force}(\Omega_t)$ is the force acting on the domain Ω_τ.

We first figure out the formula for the change of momentum.

Theorem 1.4 (Transport theorem II). *For a smooth function $f(\boldsymbol{x}, t)$, we have:*

$$\frac{d}{d\tau} \left(\int_{\Omega_\tau} \rho f(\boldsymbol{x}, \tau) \, \mathrm{d}\boldsymbol{x} \right) = \int_{\Omega_\tau} \rho D_\tau f(\boldsymbol{x}, \tau) \, \mathrm{d}\boldsymbol{x} = \int_{\Omega_\tau} \rho(f_t + \boldsymbol{u} \cdot \nabla f) \, \mathrm{d}\boldsymbol{x}.$$

Proof. By transport theorem I (Theorem 1.3), and the chain rule, we have

$$\begin{aligned}
\frac{d}{d\tau} \left(\int_{\Omega_\tau} \rho f \, \mathrm{d}\boldsymbol{x} \right) &= \int_{\Omega_\tau} D_\tau(\rho f) \, \mathrm{d}\boldsymbol{x} + \int_{\Omega_\tau} \rho f \operatorname{div} \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} \\
&= \int_{\Omega_\tau} \rho D_\tau f \, \mathrm{d}\boldsymbol{x} + \int_{\Omega_\tau} f D_\tau \rho \, \mathrm{d}\boldsymbol{x} + \int_{\Omega_\tau} f \rho \operatorname{div} \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} \\
&= \int_{\Omega_\tau} \rho D_\tau f \, \mathrm{d}\boldsymbol{x}.
\end{aligned}$$

The last two terms canceled because of the conservation of mass, c.f. (6). $\qquad \square$

This is a generalization of (8) if we treat $\rho \, \mathrm{d}\boldsymbol{x}$ as a new measure. Formally it is the exchange of integral and material derivative.

Apply Theorem 1.4 to the momentum, we get

$$\frac{d}{d\tau}(\text{momentum}) = \int_{\Omega_\tau} \rho(\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u}) \, \mathrm{d}\boldsymbol{x}.$$

Recall that $\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = D_\tau \boldsymbol{u}$ is the acceleration. The momentum equation is the integral form of the Newton's second law $F = ma$.

We then look at the force. Given a parcel Ω_τ in the fluid, forces can be classified into two category: body (external) force or surface force of stress. Examples of body force are the gravity or the magnetism. The action can be written as $\int_{\Omega_\tau} f \, \mathrm{d}\boldsymbol{x}$. The stress can be further decomposed as pressure force and stress tensor. The pressure force on the boundary is given by $-\int_{\partial\Omega_\tau} p\boldsymbol{n} \, \mathrm{d}S = -\int_{\Omega_\tau} \nabla p \, \mathrm{d}\boldsymbol{x}$. Here the negative sign is due to the outward normal $\boldsymbol{n}$ is used. The pressure force is always in the normal direction. Another surface force is coming from the deformation of the material: $\int_{\partial\Omega_t} \boldsymbol{\sigma}\boldsymbol{n} dS$. Here $\boldsymbol{\sigma}(\boldsymbol{x}, t)$ is a matrix function and $\boldsymbol{\sigma}\boldsymbol{n}$ contains forces not only in the normal direction but also in the tangential direction. The stress can be unified as $-p\boldsymbol{I} + \boldsymbol{\sigma}$. If only pressure force exists, we can the fluid *idea fluid*.

The momentum equation becomes

$$\rho D_\tau \boldsymbol{u} = f - \nabla p + \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}) = f + \operatorname{div}(-p\boldsymbol{I} + \boldsymbol{\sigma}(\boldsymbol{u})).$$

We now focus on the formula for the stress tensor $\boldsymbol{\sigma}$. We assume that

(1) $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\nabla \boldsymbol{u})$ is linear in $\nabla \boldsymbol{u}$ which is called *Newtonian fluid*. If not, it is called *non-Newtonian fluid*.
(2) It is invariant with respect to the rotation R in the Eulerian coordinate. Namely $\boldsymbol{\sigma}(R\nabla \boldsymbol{u} R^{-1}) = R\boldsymbol{\sigma}(\nabla \boldsymbol{u})R^{-1}$.
(3) It is symmetric, i.e. $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$.

A commonly used formula for the viscosity $\boldsymbol{\sigma}$ satisfying the above assumptions is

$$(9) \qquad \boldsymbol{\sigma} = \lambda(\operatorname{div} \boldsymbol{u})\boldsymbol{I} + 2\mu\nabla^{(s)}\boldsymbol{u}, \quad \text{with } \nabla^{(s)}\boldsymbol{u} = \frac{\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T}{2},$$

where λ, μ are two positive parameters and $\nabla^{(s)}\boldsymbol{u}$ is known as the symmetric gradient or the deformation tensor.

Lemma 1.5. *For $\boldsymbol{\sigma}$ given by (9), we have*

$$\operatorname{div}\boldsymbol{\sigma} = \mu\Delta\boldsymbol{u} + (\lambda + \mu)\nabla(\operatorname{div} \boldsymbol{u}).$$

Proof.

$$\operatorname{div}\boldsymbol{\sigma} = (\operatorname{div}\boldsymbol{\sigma}_1, \cdots, \operatorname{div}\boldsymbol{\sigma}_n), \quad \text{where } \boldsymbol{\sigma}_k = (\boldsymbol{\sigma}_{1k}, \cdots, \boldsymbol{\sigma}_{nk})^T.$$

Let us compute

$$(\operatorname{div} 2\nabla^{(s)}\boldsymbol{u})_k = \sum_{i=1}^{n} \partial_i(\partial_i u_k + \partial_k u_i) = \Delta\boldsymbol{u}_k + \partial_k(\operatorname{div} \boldsymbol{u}).$$

Note that $\operatorname{div}(\operatorname{div} \boldsymbol{u})I = \nabla(\operatorname{div} \boldsymbol{u})$. The desired result then follows. $\square$

The divergence of the stress tensor for incompressible flow can be simplified to the Laplacian of the velocity since $\operatorname{div} \boldsymbol{u} = 0$. Then the momentum equation for the incompressible fluid is simplified as

$$(10) \qquad -\mu\Delta\boldsymbol{u} + \rho(\partial_t\boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}) + \nabla p = \boldsymbol{f}.$$

1.3.4. *Conservation of energy.* Now we have two equations: conservation of mass and conservation of momentum for three unknowns $(\rho, \boldsymbol{u}, p)$. To complete the system, we need one additional equation for the conservation of energy.

The total energy consists of kinetic nervy and internal energy. The kinetic energy is

$$E_{\text{kinetic}} = \frac{1}{2} \int_{\Omega} \rho |\boldsymbol{u}|^2 \, \mathrm{d}\boldsymbol{x}.$$

The internal energy E_{internal} is more complicated and a detailed explanation requires knowledge on thermodynamics.

To simplify the discussion, we shall restrict to incompressible flow only. The following exercise will show that for incompressible flow with constant internal energy, the balance of energy is implied by the momentum equation and thus will not introduce new equations. In other words, for incompressible fluid, the system will be completed by the equation $\operatorname{div} \boldsymbol{u} = 0$.

Exercise 1.6. Let e_{internal} be the density of the internal energy. The total energy density will be $e_{\text{total}} = e_{\text{internal}} + |\boldsymbol{u}|^2/2$. The change of energy of Ω_{τ} of the fluid should be the same as the work done by body forces and surface forces times the distance. Thus the rate of change of the work is $F(\Omega_{\tau}) \cdot \boldsymbol{u}$. Simplify the conservation of energy

$$D_{\tau} \int_{\Omega_{\tau}} \rho e_{\text{total}} \, d\boldsymbol{x} = F(\Omega_{\tau}) \cdot \boldsymbol{u}$$

for incompressible flow with constant e_{internal}. And show it is equivalent to the momentum equation.

For compressible flow, we need an additional relation between pressure p and density ρ derived from the first law of thermodynamics. An important class of compressible flow is isentropic flow; see [1] (page 14).

1.4. **Naiver-Stokes equations.** For incompressible, viscous and Newtonian fluid, we then obtain the Naiver-Stokes equation (plus suitable boundary and initial conditions).

$$
\begin{aligned}
-\mu \Delta \boldsymbol{u} + \rho(\boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) + \nabla p &= \boldsymbol{f}, \\
\rho_t + \boldsymbol{u} \cdot \nabla \rho &= 0, \\
\operatorname{div} \boldsymbol{u} &= 0.
\end{aligned}
$$

In this note, we consider the constant mass density. Then the Navier-Stokes equations is simplified to

(11) $$-\mu \Delta \boldsymbol{u} + \boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \boldsymbol{f},$$

(12) $$\operatorname{div} \boldsymbol{u} = 0.$$

Note that in the momentum equation (11) the viscosity constant μ, the pressure, and the force is normalized by dividing the constant density ρ. The mass equation is equivalent to the incompressible equation.

Now let us consider the non-dimensionalization by the transformation

$$\bar{\boldsymbol{u}} = \boldsymbol{u}/U, \bar{p} = p/U^2, \bar{\boldsymbol{x}} = \boldsymbol{x}/L, \text{ and } \bar{t} = t/T.$$

Then the momentum equation (11) becomes

$$\bar{\boldsymbol{u}}_{\bar{t}} + \bar{\boldsymbol{u}} \cdot \nabla \bar{\boldsymbol{u}} - \frac{1}{Re} \Delta \bar{\boldsymbol{u}} + \nabla \bar{p} = \boldsymbol{f},$$

where

$$Re = LU/\mu.$$

Flow with the same Reynolds number (in domains with the same shape) will be similar. Therefore one can construct an experiment using practical size in lab to model flow in large scales. As Re increases, the equation becomes inviscid. From the definition of Re, for large scale problem (L or U big), the viscosity is tiny. In the limiting case, $Re = \infty, \mu = 0$, Navier-Stokes equation becomes the so-called Euler equation. The fluid is called ideal fluid. Note that N-S equation is second order while Euler is first order. The boundary conditions $\boldsymbol{u} = 0$ should be changed accordingly to $\boldsymbol{u} \cdot \boldsymbol{n} = 0$. If there is a mismatch in the boundary condition, it cause problems near the boundary, known as boundary layer effect; see [1] for details.

There are mainly three difficulties associated to the Navier-Stokes equations:

(1) First it is time dependent. Stability in time could be an issue for both PDE and numerical methods. For example, we still do not know wether solutions to N-S equation will below up in finite time or not (for a reasonable large class of initial conditions).

(2) Second, it is nonlinear. Efficient numerical methods can be developed for this special quadratic nonlinearity. But the convection $\boldsymbol{u} \cdot \nabla$ derivative, especially when it is dominate ($\mu \ll 1$), will cause a serious trouble in the numerical computation.

(3) Third it is a coupled system of $(\boldsymbol{u}, p)$. The pressure p can be eliminate if we consider $\boldsymbol{u}$ in the exactly divergence free space. But the divergence free condition is hardly to impose in numerical methods.

We shall solve this tangle by focusing on one difficulty at a time. We first skip the time derivative and nonlinearity to get the steady-state Stokes equations

$$
\begin{align}
-\mu \Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f}, \tag{13} \\
\operatorname{div} \boldsymbol{u} &= 0. \tag{14}
\end{align}
$$

and will study Stokes equations in detail in the next Chapter.

1.5. **Different formulations.**

1.5.1. *Lagrangian formulation.* We use material derivative to write the equations as

$$
\begin{align}
\rho D_\tau \boldsymbol{u} + \nabla p &= \rho \boldsymbol{f}, \\
D_\tau \rho &= 0, \\
\operatorname{div} \boldsymbol{u} &= 0.
\end{align}
$$

This formulation is useful for numerical methods based on Lagrangian coordinate including semi-Lagrangian method and characteristic method.

1.5.2. *Vorticity-Velocity-Pressure formulation.* We introduce the vorticity $\boldsymbol{w} = \nabla \times \boldsymbol{u}$ which is a vector function in three dimensions and a scalar function in two dimensions. For a two dimensional velocity $\boldsymbol{u} = (u_1, u_2)$, the $\nabla \times$ is applied to $(u_1, u_2, 0)$ and only nonzero component is kept, i.e., $\nabla \times \boldsymbol{u} = \partial_1 u_2 - \partial_2 u_1$.

Using the identity

$$
(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = (\operatorname{curl} \boldsymbol{u}) \times \boldsymbol{u} + \frac{1}{2}\nabla(|\boldsymbol{u}|^2) \tag{15}
$$

we can obtain the rotation form of the Navier-Stokes equations:

$$
\begin{align}
-\mu \Delta \boldsymbol{u} + \partial_t \boldsymbol{u} + \boldsymbol{w} \times \boldsymbol{u} + \nabla P &= \boldsymbol{f}, \\
\operatorname{div} \boldsymbol{u} = 0, \quad \boldsymbol{w} &= \nabla \times \boldsymbol{u}.
\end{align}
$$

where the kinematic pressure is replaced by the Bernoulli (or dynamic) pressure $P = p + |\boldsymbol{u}|^2/2$. Note that the boundary condition is only imposed on the velocity $\boldsymbol{u}$ but no boundary condition for vorticity $\boldsymbol{w}$.

Using the identity

$$-\Delta\boldsymbol{u} = \nabla \times \nabla \times \boldsymbol{u} - \nabla\nabla \cdot \boldsymbol{u}$$

and $\operatorname{div}\boldsymbol{u} = 0$, the momentum equation can be rewritten as

$$-\mu\nabla \times \boldsymbol{w} + \partial_t\boldsymbol{u} + \boldsymbol{w} \times \boldsymbol{u} + \nabla P = \boldsymbol{f}.$$

1.5.3. *Vorticity-Velocity formulation.* It is advantageous to describe dynamics of a flow in terms of the evolution of the vorticity. In 3-D, taking curl of equations (11) and combining the equation (12), we obtain the vorticity-velocity formulation

$$-\mu\Delta\boldsymbol{w} + D_\tau\boldsymbol{w} - (\boldsymbol{w} \cdot \nabla)\boldsymbol{u} = \nabla \times \boldsymbol{f},$$
$$\operatorname{div}\boldsymbol{u} = 0, \quad \boldsymbol{w} = \nabla \times \boldsymbol{u}.$$

One of the main advantages of this formulation is the elimination of the pressure.

For two dimensional incompressible flow, the equation is

$$D_\tau w - \mu\Delta w = \nabla \times \boldsymbol{f},$$
$$\operatorname{div}\boldsymbol{u} = 0, \quad w = \nabla \times \boldsymbol{u}.$$

1.5.4. *Vorticity-Stream function formulation.* Suppose Ω is a simply connected ('no holes') domain in two dimensions. Since $\operatorname{div}\boldsymbol{u} = 0$, there exists a function ψ such that $\boldsymbol{u} = \nabla\times\psi$. Here the $\nabla\times$ applied to a scalar function ψ can be thought as $\nabla \times (0,0,\psi)$, i.e., $\nabla \times \psi = (\partial_2\psi, -\partial_1\psi)$. The function ψ is called stream function. Note that $\nabla \times \nabla \times \psi = \Delta\psi$.

In two dimensions, the vorticity-stream function formulation consists of a convection-diffusion equation for the vorticity and a Poisson equation for the stream function:

$$\tag{16} \partial_t w + (\nabla \times \psi) \cdot \nabla w - \mu\Delta w = \nabla \times \boldsymbol{f},$$
$$\tag{17} \triangle\psi + w = 0.$$

The boundary condition $\boldsymbol{u} = 0$ on $\partial\Omega$ will be transformed to $\psi = 0$ and $\partial_n\psi = 0$. The Dirichlet boundary condition $\psi = 0$ is easy to impose but the Neumann boundary condition $\partial_n\psi = 0$ can be only imposed weakly through the weak formulation of (17).

1.5.5. *Vorticity-Velocity-Stream function formulation.* For three dimensional incompressible flow, the stream function is a little bit complicated. First of all from $\operatorname{div}\boldsymbol{u} = 0$ to write $\boldsymbol{u} = \nabla \times \boldsymbol{A}$, we require the domain contains no 'solid holes'. The vector function $\boldsymbol{A}$ is not unique as $\boldsymbol{A} + \operatorname{grad}\phi$ for any smooth enough ϕ will still give $\boldsymbol{u} = \nabla \times \boldsymbol{A}$. To eliminate such freedom, we can further ask $\operatorname{div}\boldsymbol{A} = 0$. Then the equation

$$\boldsymbol{w} = \operatorname{curl} \times \boldsymbol{u} = \operatorname{curl} \times \operatorname{curl} \times \boldsymbol{A} = -\Delta\boldsymbol{A}.$$

We obtain a vorticity-velocity-stream function formulation of Navier-Stokes equations

$$-\mu\Delta\boldsymbol{w} + D_\tau\boldsymbol{w} - (\boldsymbol{w} \cdot \nabla)\boldsymbol{u} = \nabla \times \boldsymbol{f},$$
$$\boldsymbol{w} + \Delta\boldsymbol{A} = 0, \quad \operatorname{div}\boldsymbol{A} = 0, \quad \boldsymbol{w} = \nabla \times \boldsymbol{u}.$$

Again the boundary condition for $\boldsymbol{A}$ is $\boldsymbol{A} \times \boldsymbol{n} = 0$ which implies $\boldsymbol{u} \cdot \boldsymbol{n} = 0$. The tangential boundary condition of $\boldsymbol{u}$ can be imposed weakly in the equation $\boldsymbol{w} = \nabla \times \boldsymbol{u}$.

1.6. **Types of fluid.**

- Incompressible fluid: $\operatorname{div} \boldsymbol{u} = 0$ or $D_\tau \rho = 0$.
- Idea fluid: surface force consists of only pressure force. No viscosity $\mu = 0$.
- Isentropic fluid: there exists a function w, called enthalpy, such that $\nabla w = \rho^{-1}\nabla p$. If $p = p(\rho)$, then it is isentropic and $w = \int p'/\rho$. Example: Gas $p = A\rho^\gamma$ with constant A and $\gamma \geq 1$.
- Newtonian fluid: $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\nabla \boldsymbol{u})$ is linear in $\nabla \boldsymbol{u}$.
- non-Newtonian fluid: $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\nabla \boldsymbol{u})$ is a non-linear function of $\nabla \boldsymbol{u}$, e.g., xxx.
- homogeneous fluid: $\rho = $ constant in space. For a homogeneous and incompressible fluid, ρ is also constant in time due to the conservation of mass.

1.7. **Further reading.** Wikipedia is always a good place to check first. Search Navier-Stokes equations in Wiki or click

   http://en.wikipedia.org/wiki/NavierStokes_equations.

Among plenty of material on fluid dynamics on the internet, I would like to recommend the lecture notes [2] (for a concise but clear introduction, especially the setting of right notation), the book [1] (for an elaborate discussion), and the book [3] (for nice pictures since a picture is worth a thousand words).

Explain streamline v.s. path line. Add physical/geometry meaning of vorticity.

REFERENCES

[1] A. Chorin and J. Marsden. *A mathematical introduction to fluid mechanics*, volume 4. Springer, 1993.
[2] Y. J. Kim. A mathematical introduction to fluid mechanics. *Lecture Notes*, 2008.
[3] P. Markowich. *Applied partial differential equations: a visual approach*. Springer Verlag, 2007.