problem 1.

1. $\nabla f = [4x_1^3 - x_2^3 - 2x_1x_2^2 + 3x_1^2,\ 4x_2^3 - 3x_1x_2^2 - 2x_1^2x_2 + 3x_2^2]$

$$\nabla^2 f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_1 x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 12x_1 - 4x_1x_2 + 6x_1, & 3x_2^2 - 4x_2 \\ 3x_2^2 - 4x_2, & 12x_2 - 6x_1x_2 - 2x_1^2 + 6x_2 \end{bmatrix}$$

$$\nabla^2 f(1,1) = \begin{bmatrix} 12-4+6, & 3-4 \\ 3-4, & 12-6-2+6 \end{bmatrix} = \begin{bmatrix} 14, & -1 \\ -1, & 10 \end{bmatrix}$$

$\because |\nabla^2 f(1,1)| = 140+1 = 141 > 0 \quad \therefore$ it's positive definite.

problem 2.

$$\nabla f = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}\right] = \left[4x_1^3x_2 + 10e^{x_3},\ x_1^4 + \frac{2x_3}{(1+x_2)^3},\ \frac{-1}{(1+x_2)^2} + 10x_1e^{x_3},\ 3x_4^2\right]$$

$$\nabla^2 f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} & \frac{\partial^2 f}{\partial x_1 \partial x_4} \\ \vdots & \frac{\partial^2 f}{\partial x_2^2} & & \\ \frac{\partial^2 f}{\partial x_4 \partial x_1} & \cdots & \cdots & \frac{\partial^2 f}{\partial x_4^2} \end{bmatrix}$$

$$= \begin{bmatrix} 12x_1, & 4x_1^3, & 10e^{x_3}, & 0 \\ 4x_1^3, & \frac{-6x_3}{(x_2+1)^4}, & \frac{2}{(1+x_2)^3}, & 0 \\ 10e^{x_3}, & \frac{2}{(1+x_2)^3}, & 10x_1e^{x_3}, & 0 \\ 0 & 0 & 0 & 6x_4 \end{bmatrix}$$

$$\nabla^2 f(1,1,5,0) = \begin{bmatrix} 12, & 4, & 10e^5, & 0 \\ 4, & -\frac{15}{8}, & \frac{1}{4}, & 0 \\ 10e^5, & \frac{1}{4}, & 10e^5, & 0 \\ 0 & 0 & 0, & 0 \end{bmatrix} \not\succ 0$$

$$\nabla^2 f(1,1,5,2) = \begin{bmatrix} 12, & 4, & 10e^5, & 0 \\ 4, & -\frac{15}{8}, & \frac{1}{4}, & 0 \\ 10e^5, & \frac{1}{4}, & 10e^5, & 0 \\ 0, & 0, & 0, & 12 \end{bmatrix} \not\succ 0$$

$$\nabla^2 f(1,1,1,2) = \begin{bmatrix} 12, & 4, & 10e, & 0 \\ 4, & -\frac{3}{4}, & \frac{1}{4}, & 0 \\ 10e, & \frac{1}{4}, & 10e, & 0 \\ 0, & 0, & 0, & 12 \end{bmatrix} \not\succ 0$$

Problem 3. $f(x_1,x_2)=\frac{1}{3}x_1^3-4x_1+\frac{1}{3}x_2^3-16x_2$

$\nabla f=\left(\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2}\right)=(x_1^2-4,\ x_2^2-16)$ let $\nabla f=0$ $\quad x_1=\pm2$, $x_2=\pm4$

$\nabla^2 f=\begin{pmatrix}2x_1, & 0\\ 0 & 2x_2\end{pmatrix}$ $\quad (2,4)\ (2,-4)\ (-2,-4)\ (-2,4)$ are critical points

$\nabla^2 f(2,4)\succ0$ $\therefore$ $(2,4)$ is local minimum

$\because -\nabla^2 f(-2,-4)\succ0$ $\therefore$ $(-2,-4)$ is local maximum

$(-2,4)$ and $(2,-4)$ are saddle point.

Problem 4. $f(x)=\frac{x^TAx}{x^Tx}$ $\qquad \nabla f(x)=\frac{x^T(A+A^T)\cdot x^Tx-2x\,x^TAx}{(x^Tx)^T\cdot(x^T\cdot x)}$

$\because$ A is symetric $A^T=A$ $\qquad =\frac{2x^TAx^Tx-2x\,x^TAx}{xx^T\cdot x^T\cdot x}=\frac{2Axx^Tx-2Ax^Txx^T}{xx^T\cdot x^T\cdot x}$

let $\nabla f(x)=0$ $\quad x=x^T$ $\qquad =\frac{2A(x-x^T)}{xx^Tx^Tx}$

$\because$ A is symmetric matrix $\quad x^T=x^{-1}$ $\quad x^Tx=I$

so $\nabla f(x)\triangleq(x^TAx)=2Ax=0$

iff x are eigenvectors of A. $\qquad Ax=\lambda x \quad (A-\lambda)x=0$

problem 5. $\min_{x \in \mathbb{R}^n} \sum_{i=1}^{m} w_i (a_i^T x - b_i)^2$ w is diagonal matrix $\begin{bmatrix} w_1 & & \\ & \ddots & \\ & & w_n \end{bmatrix}$

$$= (Ax-b)^T w (Ax-b)$$

$$= x^T A^T w A x - x^T A^T w b - b^T w A x + b^T w b$$

$$\nabla_x = A^T w^T A x - A^T w b - A^T w^T b$$

Let $\nabla_x = 0$ $A^T w^T A x = A^T (w + w^T b)$ $x = (A^T w^T A)^{-1} \cdot A^T (w + w^T b)$

$$\nabla_x^2 = A^T w A.$$

(1) if $\nabla_x^2 \preceq 0$ the minimum is $-\infty$

(2) if $\nabla_x^2 \succ 0$ W is PD then minimizer exits and is unique.

problem 6.

$$\min_{x \in \mathbb{R}^n} \frac{1}{k} \sum_{i=1}^{k} \|x - y_i\|_2^2$$

$$= \frac{1}{k} (X - Y)^T (X - Y)$$

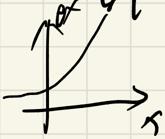
$$= \frac{1}{k} (X^T - Y^T)(X - Y) = \frac{1}{k} (X^T X - 2X^T Y - Y^T Y)$$

$\nabla_x = \frac{1}{k}(2X - 2Y)$ $X = Y$ is minimum.

$$\nabla_x^2 = \frac{2}{k} \succeq 0$$

problem 7:

(1) $f(x) = e^x - 1$ $\quad \nabla f(x) = e^x \quad \nabla^2 f(x) = e^x$

from the plot  It's convex

(2) $f(x,y) = x \cdot y \quad \nabla f(x) = [\, y, x \,] \quad \nabla^2 f(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

It's neither

(3) $f(x,y) = \frac{x_1^2}{x_2}$ It's convex beave the quandratic over linear is convex

(4) $f(x,y) = -\log(e^{-x} + e^{-y})$ $\quad -\log(x)$ is convex

$$\nabla f(x,y) = \begin{bmatrix} \frac{e^{-x}}{e^{-x}+e^{-y}} , & \frac{e^{-y}}{e^{-x}+e^{-y}} \end{bmatrix}$$

$$\nabla^2 f(x,y) = \begin{bmatrix} \frac{-e^{-x}(e^{-x}+e^{-y}) + e^{-x}\cdot e^{-x}}{(e^{-x}+e^{-y})^2} & \frac{-e^{-x}\cdot e^{-y}}{(e^{-x}+e^{-y})^2} \\ \frac{-e^{-x}\cdot e^{-y}}{(e^{-x}+e^{-y})^2} & \frac{-e^{-y}(e^{-x}+e^{-y}) + e^{-y}\cdot e^{-y}}{(e^{-x}+e^{-y})^2} \end{bmatrix}$$

for the quadatic term is ≥ 0

So $mat(1) = e^{-x}\cdot(-e^{-y}) = -e^{-x}e^{-y} < 0$

$mat(1,4) = e^{-x}(-e^{-y}) \cdot e^{-y}(-e^{-x}) + e^{-2x} e^{-2y}$

$= 2 e^{-2x} e^{-2y} > 0$

It's neither,