


# A Singularity Analysis Method for Stewart Parallel Mechanism with Planar Platforms

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**Abstract.** Singularity analysis is an important problem in the field of parallel mechanism, how to obtain a concise and analytical expression of the singularity locus has been the focus of the study for a long time. This paper presents a new method for the singularity analysis of the Stewart platform parallel mechanism. Firstly, the rotation matrix is described by quaternion; utilizing length constraint equations of the extensible limbs and properties of the quaternion, seven equivalent equations are obtained, and the variables of position and orientation are decoupled preliminarily. Secondly, by taking the derivative of seven the equivalent equations with respect to time, a new kind of Jacobian matrix is obtained, which reflects the mapping relationship between the change rate of the lengths of extensible limbs and the change rate of variables of position and orientation of the moving platform. Finally, the analytical expression of the singularity locus is derived from calculating the determinant of the new Jacobian matrix; when the quaternion are transformed into Rodriguez parameters, there are only 258 items in the fully expanded analytical expression of the singularity locus. Not only can this method be used to study the singularity problem of Stewart platform parallel mechanism, but it can also be used to study the singularity free workspace of the mechanism.

**Keywords:** Stewart platform · Parallel mechanism · Singularity analysis · Quaternion · Jacobian matrix

## 1 Introduction

It is well known that Stewart Platform parallel mechanism (also known as Gough platform), the typical representative of the six degrees of freedom spatial mechanism, is composed of a fixed base and a moving platform driven by six extensible limbs. Every limb is connected to the fixed base by spherical joint and to the moving platform by a universal joint. While working, the moving platform obtains three translational degrees of freedom and three rotational degrees of freedom by changing the lengths of the six extensible limbs, while the fixed base remains static [1]. Compared with the traditional serial mechanism, parallel mechanism possesses many advantages, such as higher precision of mobility, lower inertia, higher stiffness, larger payload capacity and better dynamic performance etc. It has been widely applied in the field of airplane simulators, parallel kinematic machines, parallel robots, micro displacement positioning devices, and medical and entertainment equipment, etc. [2].

Since the mid-20th century, parallel mechanism has become a hot research topic in the field of mechanism. Many scholars at home and abroad have been studying the Stewart Platform parallel mechanism from different aspects, such as kinematics analysis, singularity analysis, workspace and dexterity, dynamics and control. In both theoretical research and engineering applications, great progress has been made in Stewart Platform parallel mechanism. However, there are still a lot of problems having not been solved to this day, especially forward kinematics, singularity and workspace, which have been named as the three basic problems of parallel mechanism by Merlet [1]. When singularity occurs, the moving platform would like to get or lose some extra degrees of freedom in a certain direction. As a result of the moving platform getting some extra degrees of freedom, the position and orientation of the moving platform would be out of control; the driving forces of joint, which is used to balance the load effects on the moving platform, will tend to infinity, parallel mechanism would not work normally, it can even be damaged seriously [3]. Therefore, the singularity analysis of Stewart parallel mechanism cannot be avoided, neither in theoretical research nor engineering application.

There are some methods can be used to analyze the singularity of parallel mechanism, such as Jacobian matrix analysis, Grassman geometry and screw theory, etc. Gosselin has divided the singularity of parallel mechanism into three categories by using Jacobian matrix [4]. In addition, Gosselin studied the singularity representation and the maximal singularity-free zones in the six-dimensional workspace of the general Gough–Stewart platform [5, 6]. Huang obtained the analytical expression of singularity locus, which can be used to analyze both position singularity locus and orientation singularity locus of the 6-SPS parallel mechanism [7, 8]. Coste studied rational parameterization of the singularity locus of Gough–Stewart platform which the fixed base and the moving platform are both general planar hexagons [9]. Karimi studied singularity-free workspace analysis of general 6-UPS parallel mechanisms via Jacobian matrix and convex optimization [10]. Cao studied the position singularity characterization of a special class of the Stewart parallel mechanisms based on the Jacobian matrix [11]. Hunt studied the singularity of parallel mechanism by using the screw theory [12]. Huang analyzed the singularity of parallel mechanism by the method of general linear bundles [13]. Merlet proposed the method of singularity analysis based on Grassmann geometry [14]. Caro analyzed the singularity of a six-dof parallel manipulator using grassmann-cayley algebra and Gröbner bases [15]. Doyon studied the Gough–Stewart Platform with constant-orientation and obtained the singularity locus by vector expression [16]. Based on screw theory, Liu studied two types of singularity of 6-UCU parallel manipulator which are caused by both the active joints and passive universal joints [17]. Shanker obtained singular manifold of the Stewart platform [18]. Kaloorazi combined the study of maximal singularity-free sphere to the workspace analysis [19]. Some scholars pointed out that singularity can be avoided by kinematic redundancy [20], or by means of suitable control scheme [21], or by means of reconfigurable mass parameters [22], or by means of trajectory optimization and multi-model control law [23]. Some other scholars studied the singularity of different kinds of lower mobility parallel mechanisms by using different methods [24–26].

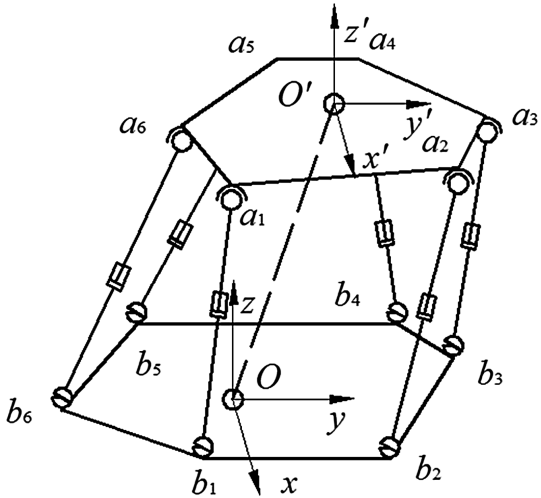
The existing research results cannot express singular locus concisely or yield higher computational efficiency, especially they cannot distinguish the singularity of given configurations quickly and easily.

The remainder of this paper is organized as follows: in Sect. 2, coordinate parameters of the hinge points and rotation matrix used in this paper are introduced; in Sect. 3, seven equivalent equations are obtained by utilizing length constraint equations of the extensible limbs and properties of the quaternion; in Sect. 4, a new kind of Jacobian matrix is derived from the equivalent equations, and the analytical expression of the singularity locus is derived from calculating the determinant of the new Jacobian matrix; in Sect. 5, a numerical example is introduced to verify the method presented in this paper. Finally, Sect. 6 draws the conclusions.

2 Coordinate Parameters and Rotation Matrix

As shown in Fig. 1, a spatial 6-dof mechanism known as the Stewart parallel mechanism. The hinge points  $a_i$  for  $i = 1$  to 6 are sketched on a circle symmetrically as the moving platform; while the hinge points  $b_i$  for  $i = 1$  to 6 are sketched on a circle symmetrically as the fixed base. Due to the symmetry, coordinates of hinge points, neither on the fixed base nor on the moving platform can be described by four parameters, namely,  $r_1, r_2, \theta_1, \theta_2$ , as shown in Table 1.

该结构由4个参数确定  $r, r, \theta, \theta$



**Fig. 1.** Stewart Platform parallel mechanism. This mechanism consists of a moving platform, a fixed base and six extendable links. The moving platform is driven by six extendable links  $a_i b_i (i = 1 \sim 6)$ , and every one of them is connected to the moving platform by a spherical joint and connected to the base by a universal joint. Static coordinate system  $O\text{-}xyz$  and moving coordinate system  $O'\text{-}x'y'z'$  are fixed to the base and the moving platform respectively.

**Table 1.** Coordinate parameters of hinge points.

	$a_{x,k}$	$a_{y,k}$	$b_{x,k}$	$b_{y,k}$
1	$r_2 \cos[-\pi/6 - \theta_2]$	$r_2 \sin[-\pi/6 - \theta_2]$	$r_1 \cos[-\pi/6 - \theta_1]$	$r_1 \sin[-\pi/6 - \theta_1]$
2	$r_2 \cos[-\pi/6 + \theta_2]$	$r_2 \sin[-\pi/6 + \theta_2]$	$r_1 \cos[-\pi/6 + \theta_1]$	$r_1 \sin[-\pi/6 + \theta_1]$
3	$r_2 \cos[-\pi/2 - \theta_2]$	$r_2 \sin[-\pi/2 - \theta_2]$	$r_1 \cos[\pi/2 - \theta_1]$	$r_1 \sin[\pi/2 - \theta_1]$
4	$r_2 \cos[-\pi/2 + \theta_2]$	$r_2 \sin[-\pi/2 + \theta_2]$	$r_1 \cos[\pi/2 + \theta_1]$	$r_1 \sin[\pi/2 + \theta_1]$
5	$r_2 \cos[-7\pi/6 - \theta_2]$	$r_2 \sin[-7\pi/6 - \theta_2]$	$r_1 \cos[7\pi/6 - \theta_1]$	$r_1 \sin[7\pi/6 - \theta_1]$
6	$r_2 \cos[-7\pi/6 + \theta_2]$	$r_2 \sin[-7\pi/6 + \theta_2]$	$r_1 \cos[7\pi/6 + \theta_1]$	$r_1 \sin[7\pi/6 + \theta_1]$

Therefore, coordinates of the hinge points on the moving platform can be expressed in the moving coordinate system  $\mathbf{O}'\text{-}\mathbf{x}'\mathbf{y}'\mathbf{z}'$  as:

$$\mathbf{a}_k = (a_{x,k} \quad a_{y,k} \quad 0)^T \quad (1)$$

Coordinates of the hinge point on the fixed base can be expressed in the static coordinate system  $\mathbf{O}\text{-}\mathbf{xyz}$  as:

$$\mathbf{b}_k = (b_{x,k} \quad b_{y,k} \quad 0)^T \quad (2)$$

The rotation matrix, which is described by quaternion [27], is shown as:

$$\mathbf{R} = \begin{bmatrix} \varepsilon_0^2 + \varepsilon_1^2 - \varepsilon_2^2 - \varepsilon_3^2 & 2\varepsilon_1\varepsilon_2 - 2\varepsilon_0\varepsilon_3 & 2\varepsilon_1\varepsilon_3 + 2\varepsilon_0\varepsilon_2 \\ 2\varepsilon_1\varepsilon_2 + 2\varepsilon_0\varepsilon_3 & \varepsilon_0^2 - \varepsilon_1^2 + \varepsilon_2^2 - \varepsilon_3^2 & 2\varepsilon_2\varepsilon_3 - 2\varepsilon_0\varepsilon_1 \\ 2\varepsilon_1\varepsilon_3 - 2\varepsilon_0\varepsilon_2 & 2\varepsilon_2\varepsilon_3 + 2\varepsilon_0\varepsilon_1 & \varepsilon_0^2 - \varepsilon_1^2 - \varepsilon_2^2 + \varepsilon_3^2 \end{bmatrix} \quad (3)$$

where,  $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_0 \in \mathbb{R}$ ,  $i^2 = j^2 = k^2 = -1$ , and  $ij = -ji = k$ ,  $jk = -kj = i$ ,  $ki = -ik = j$ . Similar to unit vectors,  $\boldsymbol{\varepsilon}^T \boldsymbol{\varepsilon} = 1$ .

The rotation matrix, which is described by the Rodriguez parameters, can be expressed as follows:

四元数和欧拉法的转角表示式的关系

$$\mathbf{R} = \frac{1}{\Delta} \cdot \begin{bmatrix} U^2 - V^2 - W^2 + 1 & 2UV - 2W & 2UW + 2V \\ 2UV + 2W & -U^2 + V^2 - W^2 + 1 & 2VW - 2U \\ 2UW - 2V & 2VW + 2U & -U^2 - V^2 + W^2 + 1 \end{bmatrix} \quad (4)$$

where,  $\Delta = U^2 + V^2 + W^2 + 1$ ,  $U, V, W$  are the Rodriguez parameters. It can be proved that the transformation relationship between quaternion  $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_0)^T$  and the Rodriguez parameters  $(U, V, W)$  can be expressed as follows:

$$\varepsilon_0 = \frac{1}{\sqrt{\Delta}}; \varepsilon_1 = \frac{U}{\sqrt{\Delta}}; \varepsilon_2 = \frac{V}{\sqrt{\Delta}}; \varepsilon_3 = \frac{W}{\sqrt{\Delta}} \quad (5)$$

The position of the origin of the moving coordinate system respect to the static coordinate system can be described by a vector  $\mathbf{P} = (P_x \quad P_y \quad P_z)^T$ . Then linkage vector between a pair of hinge points is written as:

$$l_k \mathbf{e}_k = \mathbf{P} + \mathbf{R} \cdot \mathbf{a}_k - \mathbf{b}_k \quad (k = 1 \sim 6) \quad (6)$$

where,  $l_k$  is the length of  $k^{\text{th}}$  extensible limb,  $\mathbf{e}_k$  is the unit vector along the axis of  $k^{\text{th}}$  extensible limb,  $\mathbf{a}_k$  is the position vector of the hinge point of the moving platform in the moving coordinate system,  $\mathbf{b}_k$  is the position vector of the hinge point of the fixed base in the static coordinate system,  $\mathbf{P}$  is the position vector of the reference point of the moving platform in the static coordinate system,  $\mathbf{R}$  is the rotation matrix.

### 3 Construction of Equivalent Equations

In order for the elements of the Jacobian matrix to have a simple form, the rotation matrix is described by the quaternion as shown in Eq. (3). Substituting the position vectors  $\mathbf{a}_k$ ,  $\mathbf{b}_k$  and  $\mathbf{P}$ , the length of  $k^{\text{th}}$  extensible limb can be obtained by taking the dot product of  $l_k \mathbf{e}_k$  with itself. After reorganizing, because the z component of  $\mathbf{a}_k$  and  $\mathbf{b}_k$  is zero, square linkage length equation, it can be expressed as (for conciseness of expression, omit the subscripts  $k$ ):

$$l^2 - r_1^2 - r_2^2 = \underline{P_P} - 2b_x \underline{P_x} - 2b_y \underline{P_y} + 2a_x \underline{W_x} + 2a_y \underline{W_y} - 2(\varepsilon_1^2 - \varepsilon_2^2)(a_x b_x - a_y b_y) - 2(\varepsilon_0^2 - \varepsilon_3^2)(a_x b_x + a_y b_y) + 2(\underline{2\varepsilon_0\varepsilon_3})(a_y b_x - a_x b_y) - 2(\underline{2\varepsilon_1\varepsilon_2})(a_y b_x + a_x b_y) \quad (7)$$

$l^2 - r_1^2 - r_2^2 = C_0 \vec{\eta}_1 + C_1 \vec{\eta}_2$

where,  $P_P = P_x^2 + P_y^2 + P_z^2$ ,  $W_x = P_x(\varepsilon_0^2 + \varepsilon_1^2 - \varepsilon_2^2 - \varepsilon_3^2) + P_y(2\varepsilon_1\varepsilon_2 + 2\varepsilon_0\varepsilon_3) + P_z(2\varepsilon_1\varepsilon_3 - 2\varepsilon_0\varepsilon_2)$ ,  $W_y = P_x(2\varepsilon_1\varepsilon_2 - 2\varepsilon_0\varepsilon_3) + P_y(\varepsilon_0^2 - \varepsilon_1^2 + \varepsilon_2^2 - \varepsilon_3^2) + P_z(2\varepsilon_2\varepsilon_3 + 2\varepsilon_0\varepsilon_1)$ . It can be seen from Eq. (7) that  $P_P$ ,  $P_x$ ,  $P_y$ ,  $W_x$ ,  $W_y$ ,  $\varepsilon_0^2 - \varepsilon_3^2$ ,  $\varepsilon_1^2 - \varepsilon_2^2$ ,  $2\varepsilon_0\varepsilon_3$  and  $2\varepsilon_1\varepsilon_2$  can be regarded as nine new unknown variables. These nine unknown variables can be arranged into two groups, namely  $\boldsymbol{\eta}_1 = (P_P \ P_x \ P_y \ W_x \ W_y \ 2\varepsilon_0\varepsilon_3)^T$  and  $\boldsymbol{\eta}_2 = (\varepsilon_0^2 - \varepsilon_3^2 \ \varepsilon_1^2 - \varepsilon_2^2 \ 2\varepsilon_1\varepsilon_2)^T$ . Equation (7) is equivalent to the expressions below:

$$\begin{aligned} P_P &= P_{P0} + k_0(\varepsilon_0^2 - \varepsilon_3^2) \\ P_x &= P_{x0} + 2k_1\varepsilon_1\varepsilon_2 \\ P_y &= P_{y0} + k_1(\varepsilon_1^2 - \varepsilon_2^2) \\ W_x &= W_{x0} + 2k_2\varepsilon_1\varepsilon_2 \\ W_y &= W_{y0} + k_2(\varepsilon_1^2 - \varepsilon_2^2) \\ 2\varepsilon_0\varepsilon_3 &= C_0 \end{aligned} \quad (8)$$

where parameters  $k_0$ ,  $k_1$ ,  $k_2$  are all constants, which are determined by parameters  $(r_1, r_2, \theta_1, \theta_2)$  of hinge points on both the fixed base and the moving platform; that is to say, once the structural parameters of the mechanism are given, parameters  $k_0$ ,  $k_1$ ,  $k_2$  are all constants.

$k_0, k_1, k_2$  是  $r_1, r_2, \theta_1, \theta_2$  的函数

式(7)  $\left\{ \begin{array}{l} \eta_1 \\ \eta_2 \end{array} \right\}$   $\frac{P_P \ P_x \ P_y \ W_x \ W_y \ 2\varepsilon_0\varepsilon_3}{\varepsilon_0^2 - \varepsilon_3^2 \ \varepsilon_1^2 - \varepsilon_2^2 \ 2\varepsilon_1\varepsilon_2} \rightarrow$  式(8)  $\left\{ \begin{array}{l} P_{P0} \\ P_{x0} \\ P_{y0} \\ W_{x0} \\ W_{y0} \\ C_0 \end{array} \right\}$

式(8)  $\rightarrow$   $\left\{ \begin{array}{l} k_0 \\ k_1 \\ k_2 \end{array} \right\} \rightarrow r_1, r_2, \theta_1, \theta_2$

$C_0 \rightarrow$  式(6)

$$\begin{aligned} k_0 &= 2r_1r_2\cos[\theta_1 - \theta_2]; k_1 = -r_2\csc[\theta_1 - \theta_2]\sin[\theta_1 + 2\theta_2]; \\ k_2 &= -r_1\csc[\theta_1 - \theta_2]\sin[2\theta_1 + \theta_2] \end{aligned} \quad (9)$$

where the six parameters  $P_{P0}$ ,  $P_{x0}$ ,  $P_{y0}$ ,  $W_{x0}$ ,  $W_{y0}$ ,  $C_0$ , are determined by parameters ( $r_1$ ,  $r_2$ ,  $\theta_1$ ,  $\theta_2$ ) of hinge points and the lengths of extensible limbs  $l_i$  (for  $i = 1$  to 6); their specific expressions are shown as follows:

$$\begin{aligned} P_{P0} &= \frac{-6r_1^2 - 6r_2^2 + l_1^2 + l_2^2 + l_3^2 + l_4^2 + l_5^2 + l_6^2}{6} \\ P_{x0} &= \frac{\csc[\theta_1 - \theta_2]((l_1^2 - l_2^2 - 2l_3^2 + 2l_4^2 + l_5^2 - l_6^2)\cos[\theta_2] + \sqrt{3}(l_1^2 + l_2^2 - l_5^2 - l_6^2)\sin[\theta_2])}{12r_1} \\ P_{y0} &= \frac{\csc[\theta_1 - \theta_2](\sqrt{3}(l_1^2 - l_2^2 - l_5^2 + l_6^2)\cos[\theta_2] - (l_1^2 + l_2^2 - 2l_3^2 - 2l_4^2 + l_5^2 + l_6^2)\sin[\theta_2])}{12r_1} \\ W_{x0} &= \frac{\csc[\theta_1 - \theta_2]((l_1^2 - l_2^2 - 2l_3^2 + 2l_4^2 + l_5^2 - l_6^2)\cos[\theta_1] + \sqrt{3}(l_1^2 + l_2^2 - l_5^2 - l_6^2)\sin[\theta_1])}{12r_2} \\ W_{y0} &= \frac{\csc[\theta_1 - \theta_2](\sqrt{3}(l_1^2 - l_2^2 - l_5^2 + l_6^2)\cos[\theta_1] - (l_1^2 + l_2^2 - 2l_3^2 - 2l_4^2 + l_5^2 + l_6^2)\sin[\theta_1])}{12r_2} \\ C_0 &= \frac{(l_1^2 - l_2^2 + l_3^2 - l_4^2 + l_5^2 - l_6^2)\csc[\theta_1 - \theta_2]}{12r_1r_2} \end{aligned} \quad (10)$$

#### 4 Establishment of Singular Locus Equation

When the Stewart platform parallel mechanism is working, the velocity and angular velocity of the moving platform are obtained by controlling the telescopic speeds of the six extensible limbs; the variables of input and output are connected by Jacobian matrix. The singularity of the Stewart platform parallel mechanism can be analyzed by studying the Jacobian matrix. The position and orientation of the moving platform is described by the position vector and quaternion respectively; so the output variables of the moving platform can be expressed with the derivative of the variables of position and orientation, it is  $(\dot{P}_X \ \dot{P}_Y \ \dot{P}_Z \ \dot{e}_0 \ \dot{e}_1 \ \dot{e}_2 \ \dot{e}_3)$ . That is to say, a mapping matrix can be established between the input variables  $\dot{l}_i (i = 1 \sim 6)$  and the output variables  $(\dot{P}_X \ \dot{P}_Y \ \dot{P}_Z \ \dot{e}_0 \ \dot{e}_1 \ \dot{e}_2 \ \dot{e}_3)$ . Actually, this mapping matrix is a new kind of Jacobian matrix.

For a specific Stewart platform parallel mechanism, there are only six variables in  $P_{P0}$ ,  $P_{x0}$ ,  $P_{y0}$ ,  $W_{x0}$ ,  $W_{y0}$ ,  $C_0$ , and the others are all constants; so Eq. (8) can be rewritten as:

$$\begin{aligned} P_{P0} &= P_x^2 + P_y^2 + P_z^2 - k_0(e_0^2 - e_3^2) \\ P_{x0} &= P_x - 2k_1e_1e_2 \\ P_{y0} &= P_y - k_1(e_1^2 - e_2^2) \\ W_{x0} &= P_x(e_0^2 + e_1^2 - e_2^2 - e_3^2) + P_y(2e_1e_2 + 2e_0e_3) + P_z(2e_1e_3 - 2e_0e_2) - 2k_2e_1e_2 \\ W_{y0} &= P_x(2e_1e_2 - 2e_0e_3) + P_y(e_0^2 - e_1^2 + e_2^2 - e_3^2) + P_z(2e_2e_3 + 2e_0e_1) - k_2(e_1^2 - e_2^2) \\ C_0 &= 2e_0e_3 \end{aligned} \quad (11)$$

式(11)和式(10)共有7个方程, 包含7个未知量 $P_x, P_y, P_z, e_0, e_1, e_2, e_3$

建立 $(P_x, P_y, P_z, e_0, e_1, e_2, e_3)$ 与 $l_1, l_2, l_3, l_4, l_5, l_6$ 的函数关系, 即四阶非线性方程组

And according to the nature of the quaternion, in order to maintain a unified form with Eq. (11), that is:

$$1 = \varepsilon_0^2 + \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 \quad (12)$$

In both Eqs. (11) and (12), there are a total of seven equations about seven variables, that is  $P_X, P_Y, P_Z, \varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3$ , which are changing with time as well as the length of the extensible limbs  $l_i (i = 1 \sim 6)$ . Therefore, the mapping relationship between the variables of input and output can be obtained by taking the derivative of Eqs. (11) and (12) with respect to time yields:

$$\begin{pmatrix} M_l \\ 0 \end{pmatrix} (\dot{l}_1 \ \dot{l}_2 \ \dot{l}_3 \ \dot{l}_4 \ \dot{l}_5 \ \dot{l}_6)^T = M(P, \varepsilon) \cdot (\dot{P}_X \ \dot{P}_Y \ \dot{P}_Z \ \dot{\varepsilon}_0 \ \dot{\varepsilon}_1 \ \dot{\varepsilon}_2 \ \dot{\varepsilon}_3)^T \quad (13)$$

where  $\vec{0} \in R^{1 \times 6}$ ,  $M_l \in R^{6 \times 6}$ ,  $M_l(i, j) = f(l_j)$  is a constant when the length of the extensible limbs are given.

$$M(P, \varepsilon) = \begin{pmatrix} P_X & P_Y & P_Z & -k_0 \varepsilon_0 & 0 & 0 & k_0 \varepsilon_3 \\ 1 & 0 & 0 & 0 & -2k_1 \varepsilon_2 & -2k_1 \varepsilon_1 & 0 \\ 0 & 1 & 0 & 0 & -2k_1 \varepsilon_1 & 2k_1 \varepsilon_2 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} & m_{45} & m_{46} & m_{47} \\ m_{51} & m_{52} & m_{53} & m_{54} & m_{55} & m_{56} & m_{57} \\ 0 & 0 & 0 & \varepsilon_3 & 0 & 0 & \varepsilon_0 \\ 0 & 0 & 0 & \varepsilon_0 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \end{pmatrix} \quad (14)$$

where,

$$\begin{aligned} m_{41} &= \varepsilon_0^2 + \varepsilon_1^2 - \varepsilon_2^2 - \varepsilon_3^2 & m_{51} &= 2\varepsilon_1 \varepsilon_2 - 2\varepsilon_0 \varepsilon_3 \\ m_{42} &= 2\varepsilon_1 \varepsilon_2 + 2\varepsilon_0 \varepsilon_3 & m_{52} &= \varepsilon_0^2 - \varepsilon_1^2 + \varepsilon_2^2 - \varepsilon_3^2 \\ m_{43} &= -2\varepsilon_0 \varepsilon_2 + 2\varepsilon_1 \varepsilon_3 & m_{53} &= 2\varepsilon_0 \varepsilon_1 + 2\varepsilon_2 \varepsilon_3 \\ m_{44} &= 2P_X \varepsilon_0 - 2P_Z \varepsilon_2 + 2P_Y \varepsilon_3 & m_{54} &= 2P_Y \varepsilon_0 + 2P_Z \varepsilon_1 - 2P_X \varepsilon_3 \\ m_{45} &= 2P_X \varepsilon_1 + 2P_Y \varepsilon_2 - 2k_2 \varepsilon_2 + 2P_Z \varepsilon_3 & m_{55} &= 2P_Z \varepsilon_0 - 2P_Y \varepsilon_1 - 2k_2 \varepsilon_1 + 2P_X \varepsilon_2 \\ m_{46} &= -2P_Z \varepsilon_0 + 2P_Y \varepsilon_1 - 2k_2 \varepsilon_1 - 2P_X \varepsilon_2 & m_{56} &= 2P_X \varepsilon_1 + 2P_Y \varepsilon_2 + 2k_2 \varepsilon_2 + 2P_Z \varepsilon_3 \\ m_{47} &= 2P_Y \varepsilon_0 + 2P_Z \varepsilon_1 - 2P_X \varepsilon_3 & m_{57} &= -2P_X \varepsilon_0 + 2P_Z \varepsilon_2 - 2P_Y \varepsilon_3 \end{aligned}$$

Equation (13) states that  $M(P, \varepsilon)$  is the mapping transformation matrix between the driving speed and the change rate of position and orientation of the moving platform, namely the Jacobian matrix. When singularity occurs, the determinant of Jacobian matrix should be equal to zero, that is:

$$f = \text{Det}[M(P, \varepsilon)] = 0 \quad (15)$$

It can be seen from Eq. (13) that the elements of the Jacobian matrix are very simple monomials; Except for the elements of the 4<sup>th</sup> and 5<sup>th</sup> rows, in which rows every element is more complex. This feature of the new Jacobian matrix makes the

calculation of the determinant faster, compared to the calculation of determinant of the traditional Jacobian matrix; the terms of polynomial is much less after fully expanded.

There are four components in quaternion, without loss of generality, define  $\varepsilon_0 > 0$ , as a result, it can be obtained:

$$\varepsilon_0 = \sqrt{1 - \varepsilon_1^2 - \varepsilon_2^2 - \varepsilon_3^2} \quad (16)$$

A new singular trajectory equation can be obtained by substituting Eq. (16) into Eq. (15), it is shown as follows:

$$f(\varepsilon_1, \varepsilon_2, \varepsilon_3, P_X, P_Y, P_Z) = 0 \quad (17)$$

Alternatively, Eq. (5) shows the transformation relationship between the quaternion and the Rodriguez parameters. Another singular trajectory equation, which is expressed of the Rodriguez parameter, can be obtained by substituting Eq. (5) into Eq. (15), it is shown as follows:

四次方程求解比较难, 只有心算多项式中

$$f(U, V, W, P_X, P_Y, P_Z) = 0 \quad (18)$$

In the case of any three variables of position and orientation are given, the variation of singular locus with respect to the remaining three variables can be studied, through neither Eq. (17) nor Eq. (18). When the variables of orientation are known, the singular locus of position is solved, and vice versa. As for the set of specific parameters of position and orientation, it can be determined whether the mechanism is in the singular pose or not. The two singular locus equations are equivalent in nature. In addition to the variables of position and orientation, there are only three parameters  $k_0, k_1, k_2$  in them. The three parameters  $k_0, k_1, k_2$  are constants and depend on the mechanism parameters. There are only 258 items in Eq. (18) when the symbolic expression is expanded completely. If the singularity is analyzed in the case of the symbolic expression factorized, the calculation speed will be further improved.

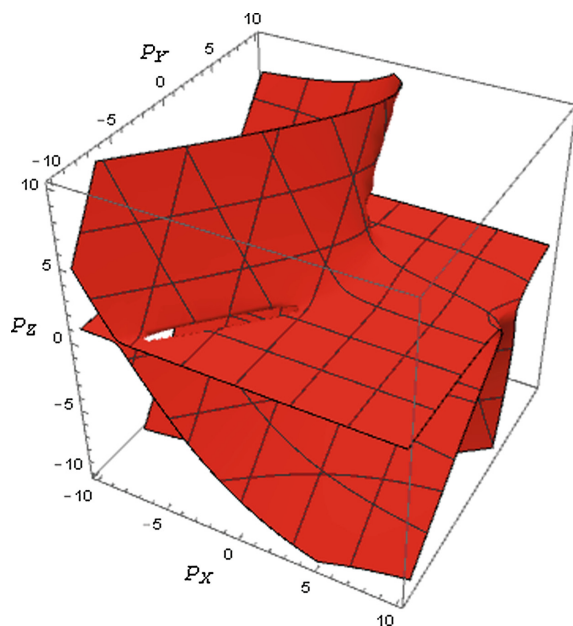
## 5 Numerical Example

As mentioned above, the structure parameters of the Stewart platform parallel mechanism can be described by  $\theta_1, \theta_2, r_1, r_2$ . The values of these four parameters are  $\pi/5, \pi/9, 1, 0.618$ . It can be obtained that  $k_0 = 1.18812, k_1 = -2.17548, k_2 = -3.62575$ .

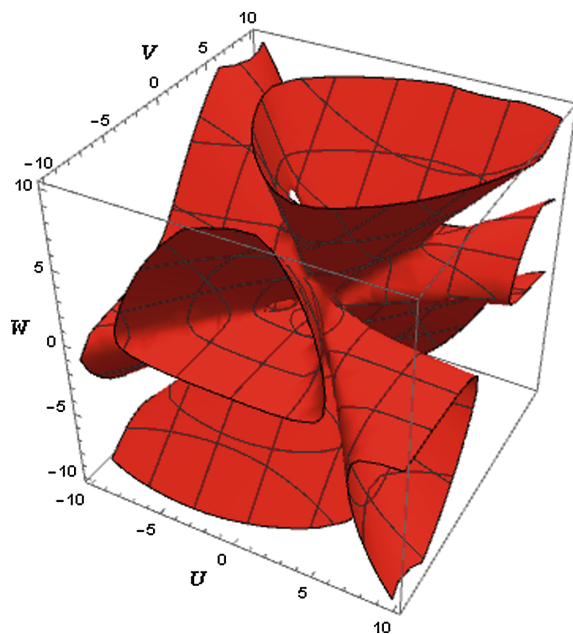
- (1) When the variables of orientation of the moving platform are determined, the change of the singular locus with respect to position variables can be studied by Eq. (17) or Eq. (18). The singular locus obtained by these two equations are identical in theory. For example: the Rodriguez parameters ( $U = 0.7, V = 0.3, W = 0.4$ ) and the quaternion ( $\varepsilon_0 = 0.758098, \varepsilon_1 = 0.530669, \varepsilon_2 = 0.227429, \varepsilon_3 = 0.303239$ ) can be used to describe the same given pose. The computer simulation shown that the singular locus equations are 3<sup>th</sup> polynomials with respect to the variables of position, and in these two cases, the singular locus can be expressed in in Fig. 2.

它奇异点曲面方程是二次多项式 - 图2 - 四次方程和螺旋线结果一致

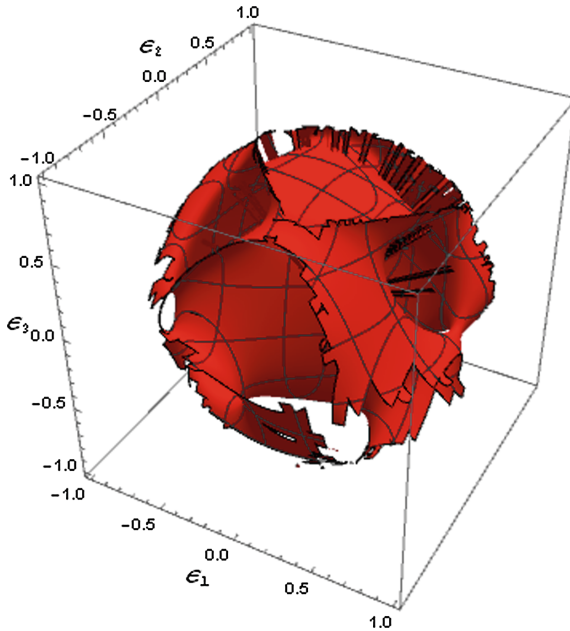




**Fig. 2.** Singular locus about variables of position. Whether Rodriguez parameters or quaternion can be used to describe the rotation matrix. The singular locus is identical.



**Fig. 3.** Singular locus about variables of orientation at the given position. The variables of orientation are described by Rodriguez parameters.



**Fig. 4.** Singular locus about variables of orientation at the given position. The variables of orientation are described by quaternions.

位置工作空间。UVW是6次多项式，因此是8次多项式 - fig 3. fig. 4

- (2) When the variables of position of the moving platform are determined, the change of the singular locus with respect to orientation variables can be researched by Eq. (17) or Eq. (18). For example, the position vector is defined as  $\mathbf{P} = (2 \ 2 \ 4)^T$ , the rotation matrix is described by Rodriguez parameters; The singular locus equations is 6 times polynomial about of Rodriguez parameters  $U, V, W$ , singular locus is shown in Fig. 3. When the rotation matrix is described by quaternion; The singular locus equations is 8 times polynomial about that of each component of quaternion ( $\varepsilon_0 = \sqrt{1 - \varepsilon_1^2 - \varepsilon_2^2 - \varepsilon_3^2}$ ) singular locus is shown in Fig. 4.

- (3) When the variables of position and orientation of the moving platform are determined, whether Stewart platform parallel mechanism is singular or not can be researched by Eq. (17) or Eq. (18). For example, the position vector is defined as  $\mathbf{P} = (0 \ 0 \ 5)^T$ ; Rodriguez parameters are  $U = 0, V = 0$  and  $W = 1$  respectively; singular locus equation is equal to zero, Stewart platform parallel mechanism occurs singular.

位置时确定。判断及点是奇异

## 6 Conclusions

- (1) In this paper, a new approach used to analyze the singularity of the Stewart Platform parallel mechanism is studied. There are only 258 items in analytical expression of the singular locus equation when it is expanded completely.

- (2) The rotation matrix is described by quaternion, and the kinematics equation of the Stewart Platform parallel mechanism is transformed into a new form in this paper. In addition, normalization of quaternion is used and 7 equivalent equations are obtained, which can be used to study the singularity and the forward kinematics of the Stewart Platform parallel mechanism.
- (3) Based on these equivalent equations, this paper presents a new Jacobian matrix, which states the mapping relationship between the driving speed and the change rate of position and orientation of the moving platform. Every component of the Jacobian matrix is a rather simple monomial.
- (4) Evaluating the determinant of Jacobian matrix, the analytical expression of the singular locus equation can be derived. These singular trajectory equations can not only be used to study the distribution of singular locus in the workspace, but can also be used to judge whether a set of parameters of position and orientation corresponds to a singularity pose. The singularity analysis approach presented in this paper provides an important theoretical basis for studying the singularity free workspace.

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