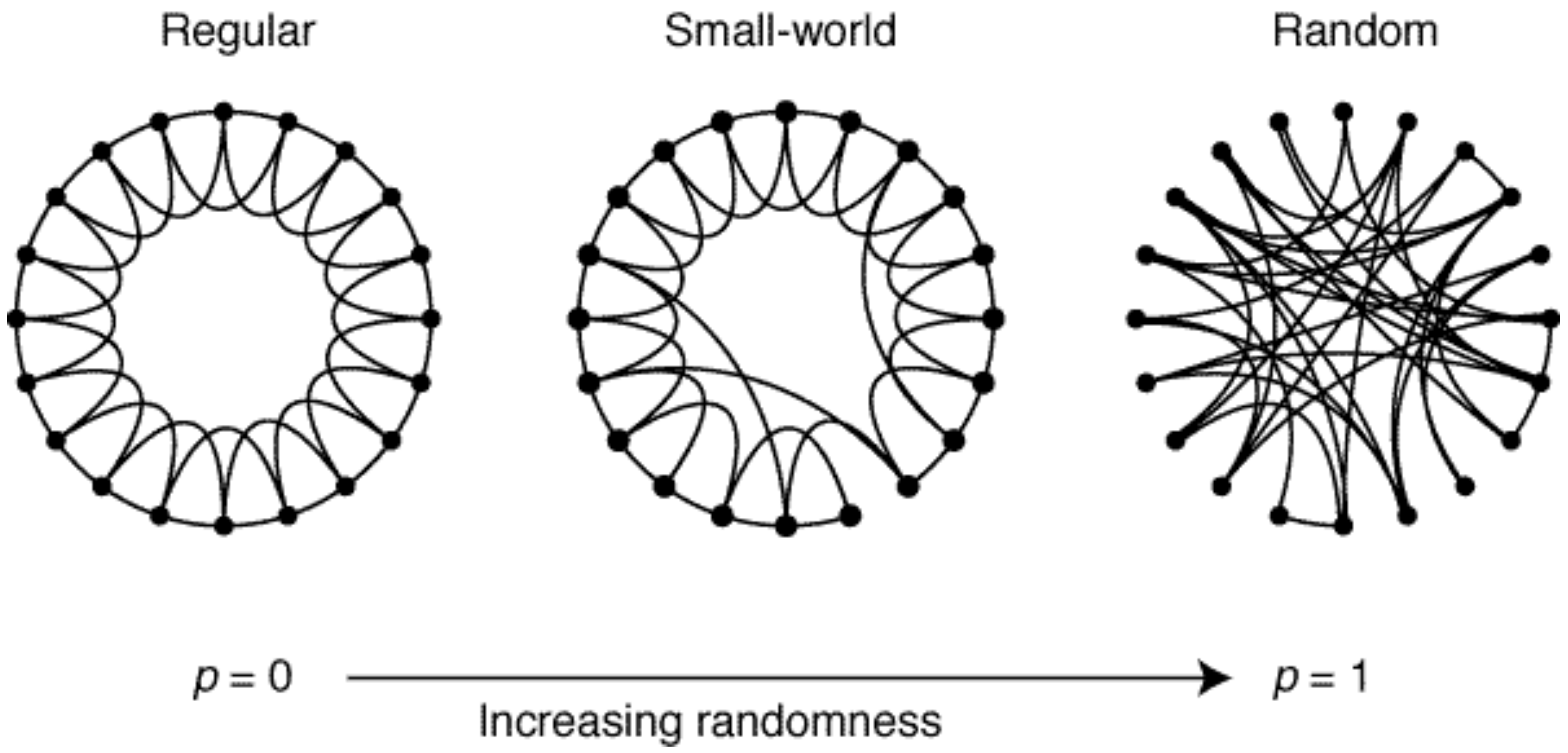
Navigable Small Worlds – 2

Alexander Ponomarenko

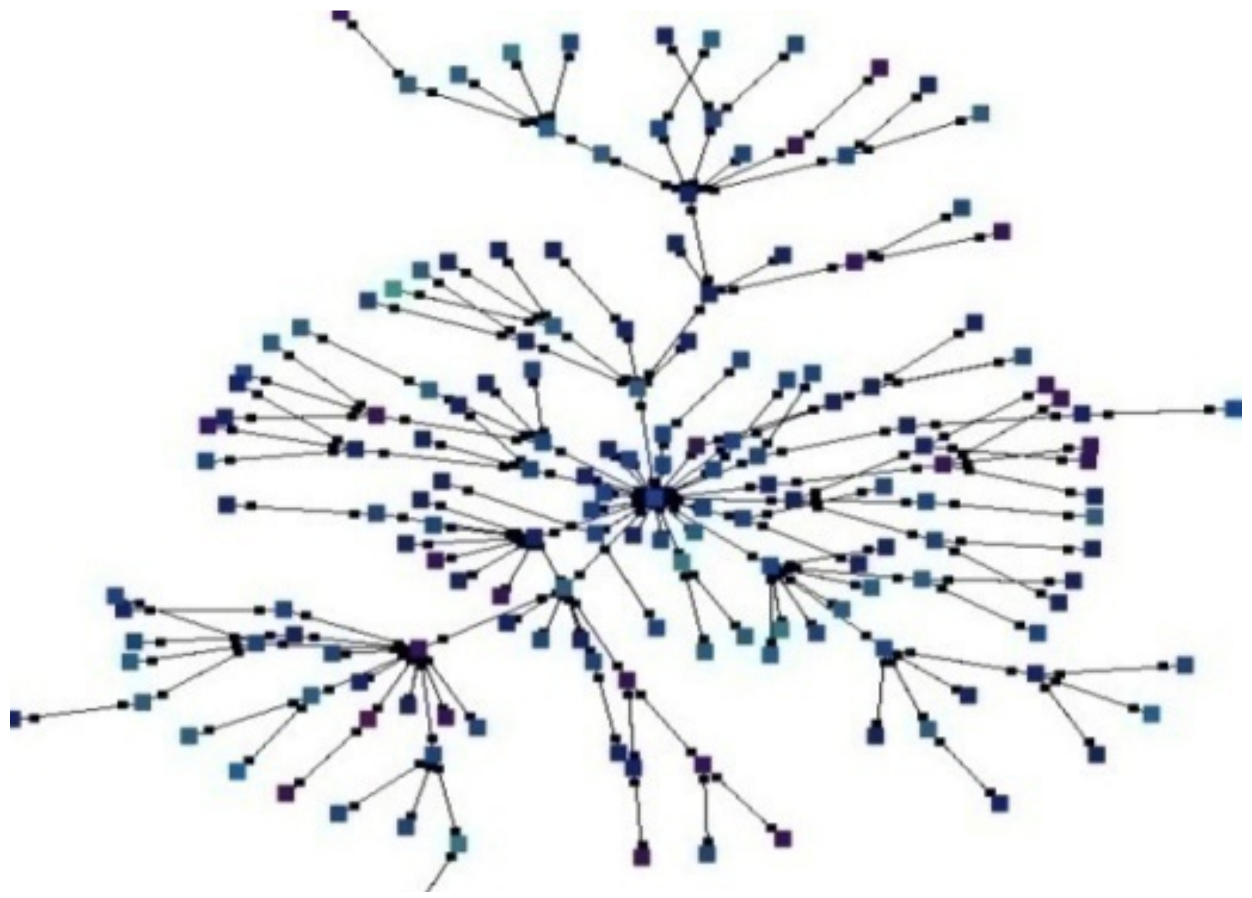
1. Small diameter ~ log(n)
2. Hight Clustering Coefficient
3. Power law degree distribution
4. Navigable

Watts-Strogatz

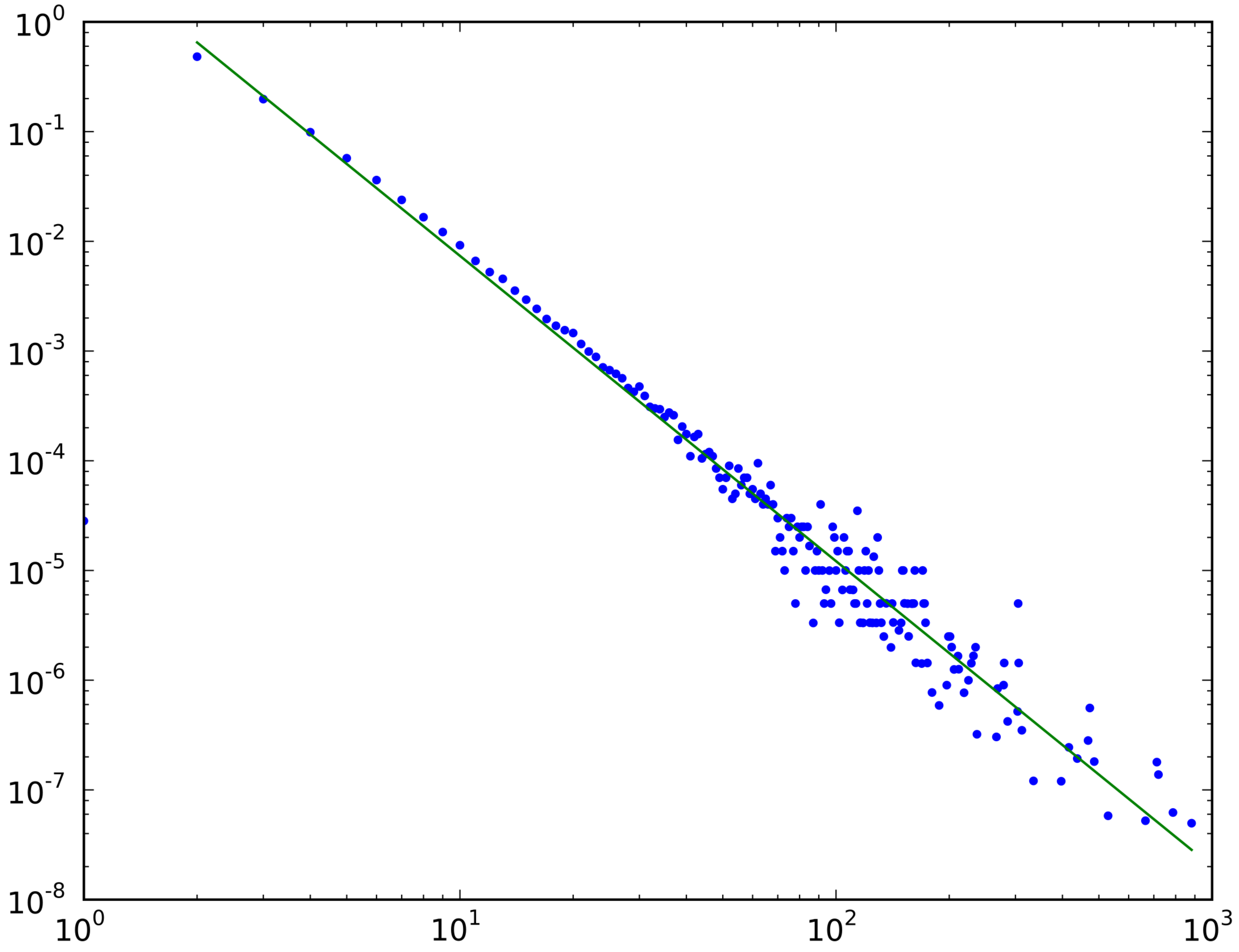


Barabási–Albert model

$$p_i = \frac{k_i}{\sum_j k_j}$$



$$P(k) \approx k^{-3}$$



The probability that the new node is connected to node ***i***

see also [Generalized preferential attachment by Ostroumova et.al."]

Kleinberg Model

$$d((i,j),(k,l)) = |k-i| + |l-j|$$

$$P((u,v) \in E)^{-r} = \sum_{v} d(u,v)^{-r}$$

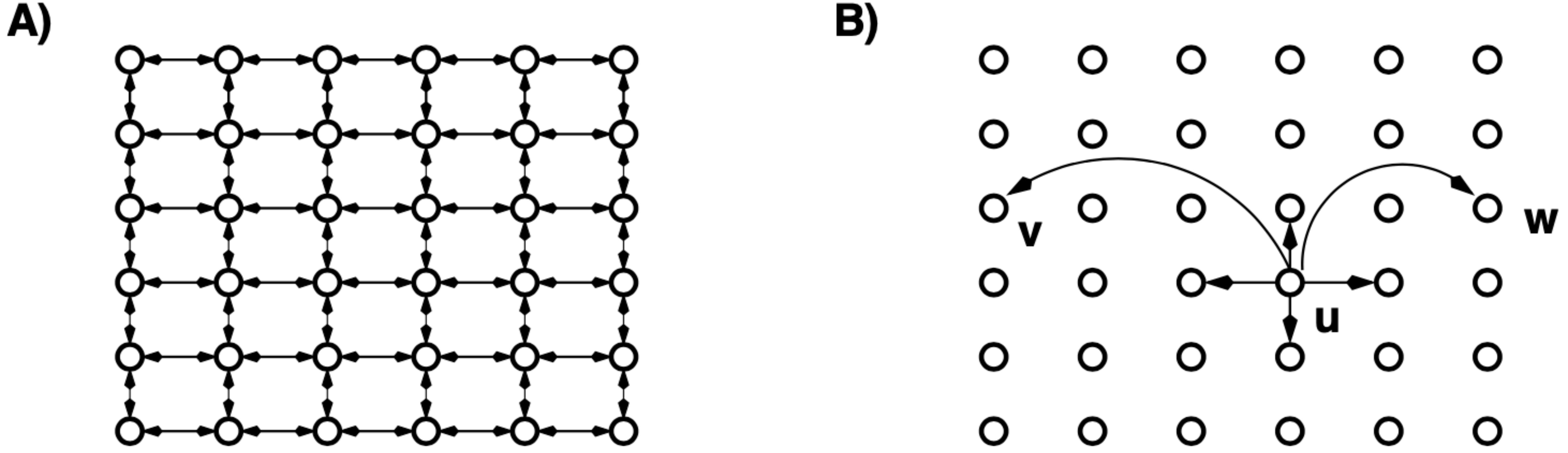


Figure 1: (A) A two-dimensional grid network with $n = 6$, $p = 1$, and $q = 0$. (B) The contacts of a node u with $p = 1$ and $q = 2$. v and w are the two long-range contacts.

[Kleinberg J. The small-world phenomenon: An algorithmic perspective //Proceedings of the thirty-second annual ACM symposium on Theory of computing. – ACM, 2000. – C. 163-170.]

Theorem 1 *There is a constant α_0, depending on p and q but independent of n, so that when $r = 0$, the expected delivery time of any decentralized algorithm is at least $\alpha_0 n^{2/3}$. (Hence exponential in the expected minimum path length.)*

Theoreme 2

There is a decentralized algorithm ***A*** , so that when r = 2 and p = q = 1, the expected delivery time of **A** is $O\big((\log n)^2\big)$

Theorem 3 *(a) Let $0 \le r < 2$. There is a constant α_r, depending on p, q, r, but independent of n, so that the expected delivery time of any decentralized algorithm is at least $\alpha_r n^{(2-r)/3}$.*

(b) Let $r > 2$. There is a constant α_r, depending on p, q, r, but independent of n, so that the expected delivery time of any decentralized algorithm is at least $\alpha_r n^{(r-2)/(r-1)}$.

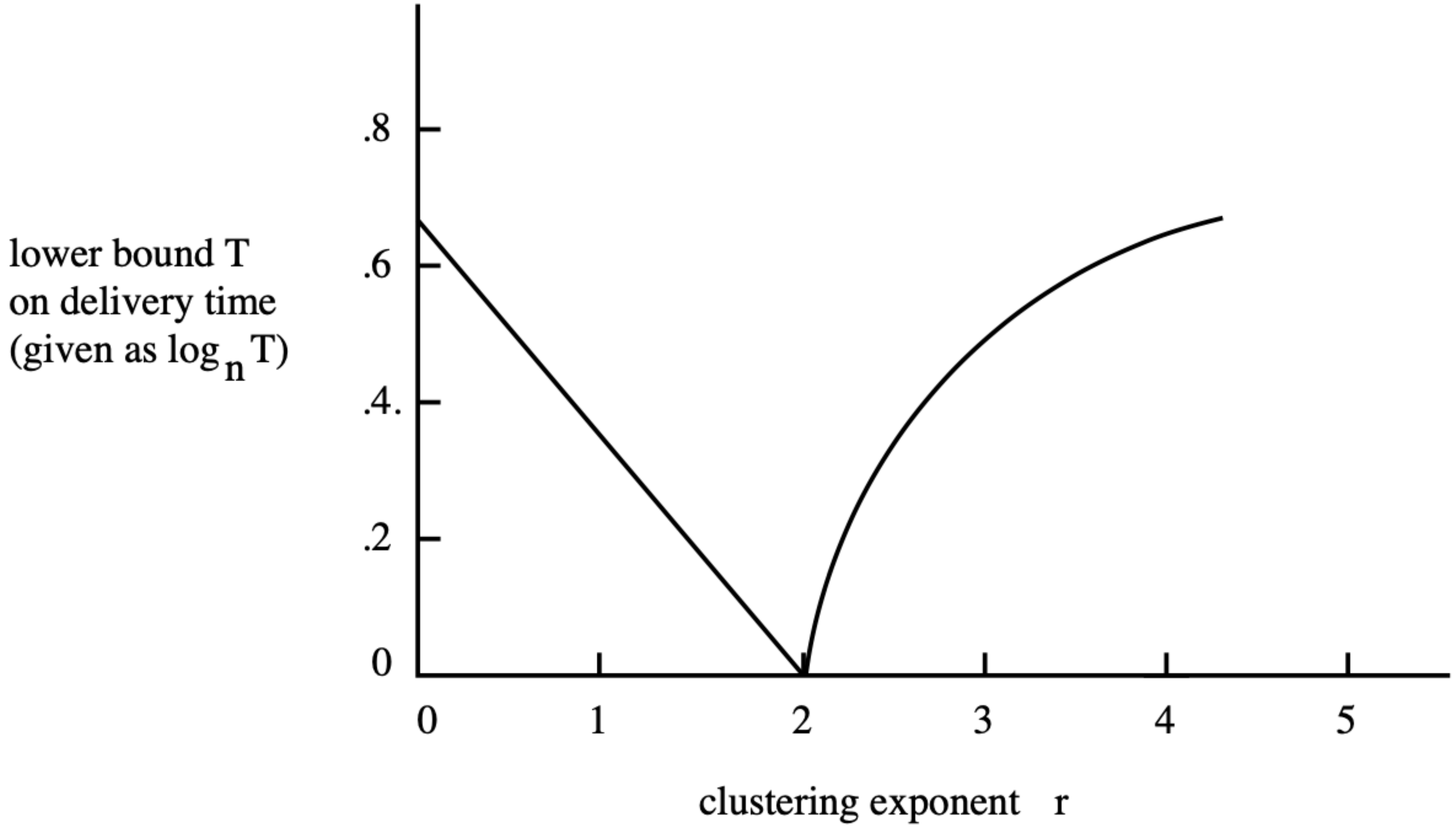


Figure 2: The lower bound implied by Theorem 3. The x-axis is the value of r; the y-axis is the resulting exponent on n.

Theoreme 2

There is a decentralized algorithm ***A*** , so that when r = 2 and p = q = 1, the expected delivery time of **A** is $O\left((\log n)^2\right)$

$$P((u,v) \in E) = \frac{d(u,v)^{-2}}{\boxed{\sum_{u \neq v} d(u,v)^{-2}}}$$

$$\boxed{\sum_{u \neq v} d(u,v)^{-2} \leq \sum_{j=1}^{2n-2} (4j)(j^{-2}) = 4 \sum_{j=1}^{2n-2} j^{-1} \leq 4 + 4\ln(2n-2) \leq 4\ln(6n)}$$

$$\Downarrow$$

$$\boxed{P((u,v) \in E) \geq \frac{1}{4\ln(6n) d(u,v)^2}}$$

We are at the stage j when $2^j < d(t, u) \leq 2^{j+1}$

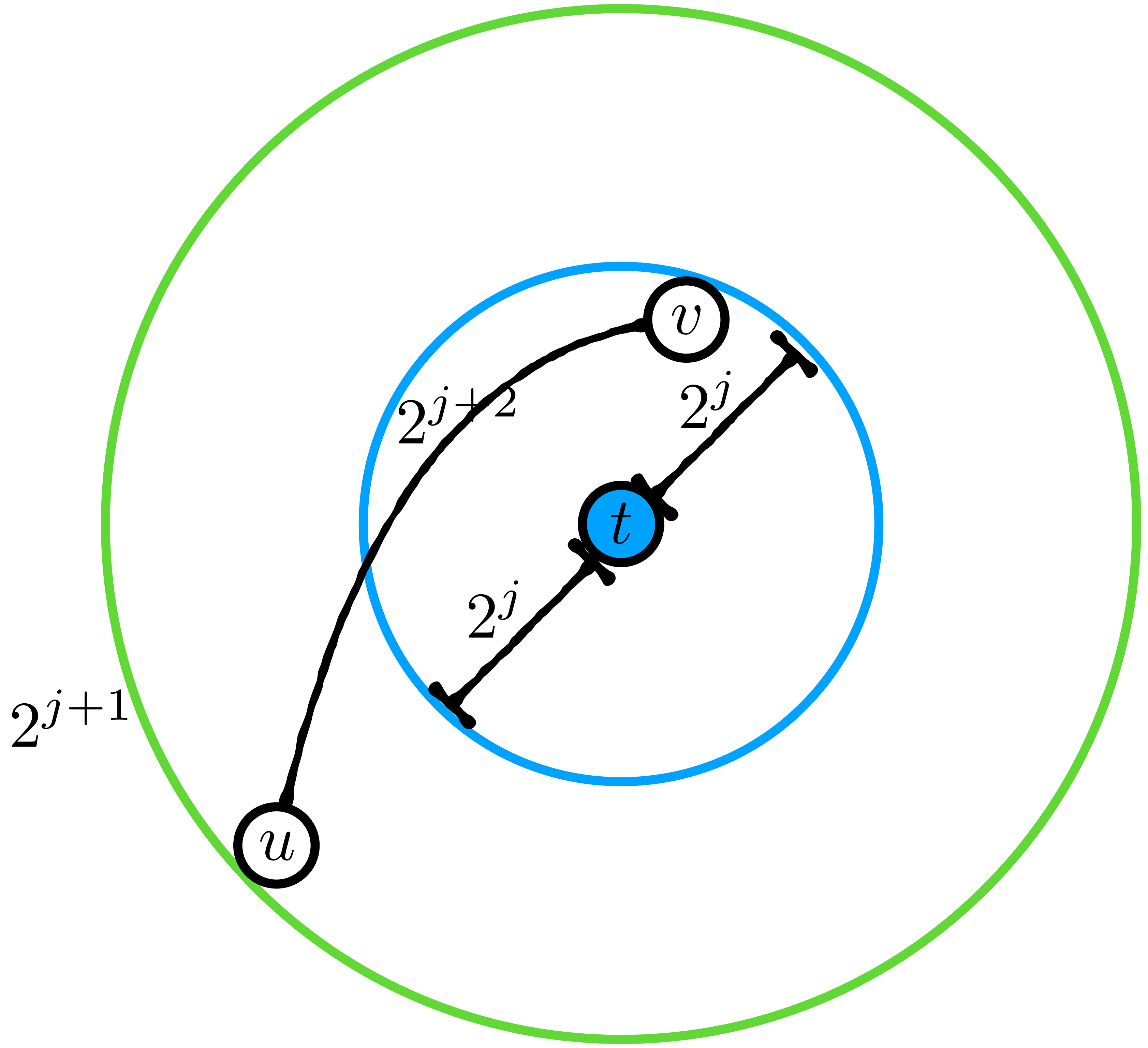
The initial value of *j* is at most log *n*.

Suppose we are at the stage *j*, and $\log(\log n) \leq j < \log(n)$

What is the probability that phase *j* will end in this step?

The **stage *j*** if the message at ***x***, and $2^j < d(t,x) \leq 2^{j+1}$

$$d(u,v) \leq 2^{j+2}$$



$$\boxed{P((u,v) \in E) \geq \frac{1}{4\ln(6n)d(u,v)^2}}$$

$$P\Big((u,v) \in E\Big) \geq \frac{1}{4\ln(6n)2^{(j+2)^2}} = \frac{1}{4\ln(6n)2^{2j+4}}$$

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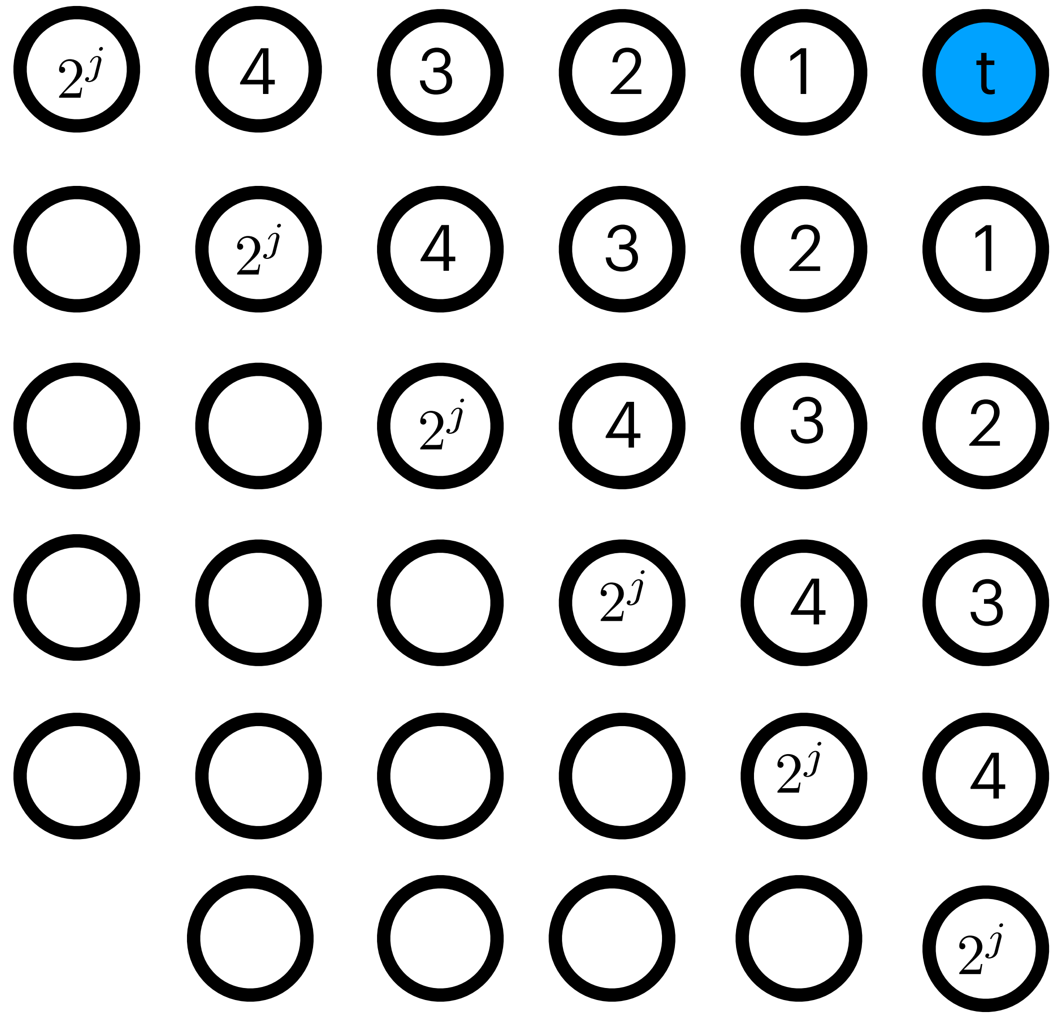

Let $B_j = \{x : d(t,x) \le 2^j\}$

$$P\Big((u,v) \in E\Big) \ge \frac{1}{4\ln(6n)2^{2j+4}}$$

$$|B_j| \ge \sum_{i=1}^{2^{j+1}} i > \sum_{i=1}^{2^j} i = 2^j(2^j+1)\frac{1}{2} = 2^{2j}\frac{1}{2} + 2^j\frac{1}{2} > 2^{2j-1}$$

$$\boxed{P\Big(\Big\{\{(u,v) : v \in B_j\} \cap E\Big\} \ne \emptyset\Big) \ge \frac{2^{2j-1}}{4ln(6n)2^{2j+4}} = \frac{1}{128\ln(6n)}}$$

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Let X_j — the number of steps at stage j

$$E[X_j] = \sum_{i=1}^{\infty} P(X_j \geq i)$$

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Если случайная величина X может принимать только натуральные значения $(0, 1, 2, \ldots)$, то имеется красивая формула для её математического ожидания:

$$\begin{aligned} \mathsf{M}[X] &= \sum_{i=0}^{\infty} i\mathsf{P}\{X = i\} \\ &= \sum_{i=0}^{\infty} i(\mathsf{P}\{X \geqslant i\} - \mathsf{P}\{X \geqslant i+1\}) \\ &= \sum_{i=1}^{\infty} \mathsf{P}\{X \geqslant i\}. \end{aligned} \tag{6.28}$$

В самом деле, каждый член $\mathsf{P}\{X \geqslant i\}$ присутствует в сумме i раз со знаком плюс и $i-1$ раз со знаком минус (исключение составляет член $\mathsf{P}\{X \geqslant 0\}$, вовсе отсутствующий в сумме).

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+	+	+
0*(p{X>=0} - p{X>=1})	0*p{X>=0}	
1*(p{X>=1} - p{X>=2})	1*p{X>=1} - 0*p{X>=1}	1*p{X>=1}
2*(p{X>=2} - p{X>=3})	-1*p{X>=2} + 2*p{X>=2}	1*p{X>=2}
3*(p{X>=3} - p{X>=4})	-2*p{X>=3} + 3*p{X>=3}	1*p{X>=3}
i-1*(p{X>=i-1} - p{X>=i})		
i*(p{X>=i} - p{X>=i+1})	-(i-1)*p{X>=i} + i*p{X>=i})	1*p{X>=i}
i+1*(p{X>=i+1} - p{X>=i+2})		1*p{X>=i+1}

Let X_j — the number of steps at stage j

$$E[X_j] = \sum_{i=1}^{\infty} P(X_j \geq i) \quad \leq \sum_{i=1}^{\infty} \left(1 - \frac{1}{128 \ln(6n)}\right)^{i-1}$$

$$\boxed{P\left(\left\{\{(u,v) : v \in B_j\} \cap E\right\} \neq \emptyset\right) \geq \frac{2^{2j-1}}{4 ln(6n) 2^{2j+4}} = \frac{1}{128 \ln(6n)}}$$

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$$S_n = b_1 + b_2 + ... + b_n = b_1 + b_1 q + b_1 q^2 + ... + b_1 q^{n-1}$$

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$$b_1 = 1$$

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$$X = \sum_{j=0}^{\log n} X_j$$ — total number of steps

$\blacksquare$

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