CVX101 Homework 4 solutions

A4.1 *Numerical perturbation analysis example.* Consider the quadratic program

$$\begin{array}{ll} \text{minimize} & x_1^2 + 2x_2^2 - x_1x_2 - x_1 \\ \text{subject to} & x_1 + 2x_2 \leq u_1 \\ & x_1 - 4x_2 \leq u_2, \\ & 5x_1 + 76x_2 \leq 1, \end{array}$$

with variables x_1, x_2, and parameters u_1, u_2.

(a) Solve this QP, for parameter values $u_1 = -2$, $u_2 = -3$, to find optimal primal variable values $x_1^\star$ and $x_2^\star$, and optimal dual variable values $\lambda_1^\star$, $\lambda_2^\star$ and $\lambda_3^\star$. Let $p^\star$ denote the optimal objective value. Verify that the KKT conditions hold for the optimal primal and dual variables you found (within reasonable numerical accuracy).

Hint: See §3.7 of the CVX users' guide to find out how to retrieve optimal dual variables. To specify the quadratic objective, use `quad_form()`.

(b) We will now solve some perturbed versions of the QP, with

$$u_1 = -2 + \delta_1, \qquad u_2 = -3 + \delta_2,$$

where δ_1 and δ_2 each take values from $\{-0.1, 0, 0.1\}$. (There are a total of nine such combinations, including the original problem with $\delta_1 = \delta_2 = 0$.) For each combination of δ_1 and δ_2, make a prediction $p^\star_{\text{pred}}$ of the optimal value of the perturbed QP, and compare it to $p^\star_{\text{exact}}$, the exact optimal value of the perturbed QP (obtained by solving the perturbed QP). Put your results in the two righthand columns in a table with the form shown below. Check that the inequality $p^\star_{\text{pred}} \leq p^\star_{\text{exact}}$ holds.

δ_1	δ_2	$p^\star_{\text{pred}}$	$p^\star_{\text{exact}}$
0	0		
0	-0.1		
0	0.1		
-0.1	0		
-0.1	-0.1		
-0.1	0.1		
0.1	0		
0.1	-0.1		
0.1	0.1		

Solution.

(a) The following Matlab code sets up the simple QP and solves it using CVX:

```
Q = [1 -1/2; -1/2 2];
f = [-1 0]';
A = [1 2; 1 -4; 5 76];
b = [-2 -3 1]';

cvx_begin
    variable x(2)
    dual variable lambda
    minimize(quad_form(x,Q)+f'*x)
    subject to
        lambda: A*x <= b
cvx_end
p_star = cvx_optval
```

When we run this, we find the optimal objective value is $p^\star = 8.22$ and the optimal point is $x_1^\star = -2.33$, $x_2^\star = 0.17$. (This optimal point is unique since the objective is strictly convex.) A set of optimal dual variables is $\lambda_1^\star = 2.13$, $\lambda_2^\star = 3.31$ and $\lambda_3^\star = 0.08$. (The dual optimal point is unique too, but it's harder to show this, and it doesn't matter anyway.)

The KKT conditions are

$$\begin{array}{lll} x_1^\star + 2x_2^\star \leq u_1, & x_1^\star - 4x_2^\star \leq u_2, & 5x_1^\star + 76x_2^\star \leq 1 \\ \lambda_1^\star \geq 0, & \lambda_2^\star \geq 0, & \lambda_3^\star \geq 0 \\ \lambda_1^\star(x_1^\star + 2x_2^\star - u_1) = 0, & \lambda_2^\star(x_1^\star - 4x_2^\star - u_2) = 0, & \lambda_3^\star(5x_1^\star + 76x_2^\star - 1) = 0, \\ 2x_1^\star - x_2^\star - 1 + \lambda_1^\star + \lambda_2^\star + 5\lambda_3^\star = 0, & & \\ 4x_2^\star - x_1^\star + 2\lambda_1^\star - 4\lambda_2^\star + 76\lambda_3^\star = 0. & & \end{array}$$

We check these numerically. The dual variable $\lambda_1^\star$, $\lambda_2^\star$ and $\lambda_3^\star$ are all greater than zero and the quantities

```
A*x-b
2*Q*x+f+A'*lambda
```

are found to be very small. Thus the KKT conditions are verified.

(b) The predicted optimal value is given by

$$p^\star_{\text{pred}} = p^\star - \lambda_1^\star\delta_1 - \lambda_2^\star\delta_2.$$

The following Matlab code fills in the table

```
arr_i = [0 -1 1];
delta = 0.1;
pa_table = [];
```

```
for i = arr_i
    for j = arr_i
        p_pred = p_star - [lambda(1) lambda(2)]*[i; j]*delta;
        cvx_begin
            variable x(2)
            minimize(quad_form(x,Q)+f'*x)
            subject to
                A*x <= b+[i;j;0]*delta
        cvx_end
        p_exact = cvx_optval;

        pa_table = [pa_table; i*delta j*delta p_pred p_exact]
    end
end
```

The values obtained are

δ_1	δ_2	$p^\star_{\text{pred}}$	$p^\star_{\text{exact}}$
0	0	8.22	8.22
0	-0.1	8.55	8.70
0	0.1	7.89	7.98
-0.1	0	8.44	8.57
-0.1	-0.1	8.77	8.82
-0.1	0.1	8.10	8.32
0.1	0	8.01	8.22
0.1	-0.1	8.34	8.71
0.1	0.1	7.68	7.75

The inequality $p^\star_{\text{pred}} \leq p^\star_{\text{exact}}$ is verified to be true in all cases.

5.1 *A simple example.* Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & x^2 + 1 \\ \text{subject to} & (x-2)(x-4) \leq 0, \end{array}$$

with variable $x \in \mathbf{R}$.

(a) *Analysis of primal problem.* Give the feasible set, the optimal value, and the optimal solution.

(b) *Lagrangian and dual function.* Plot the objective $x^2 + 1$ versus x. On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x, \lambda)$ versus x for a few positive values of λ. Verify the lower bound property ($p^\star \geq \inf_x L(x, \lambda)$ for $\lambda \geq 0$). Derive and sketch the Lagrange dual function g.

(c) *Lagrange dual problem.* State the dual problem, and verify that it is a concave maximization problem. Find the dual optimal value and dual optimal solution $\lambda^\star$. Does strong duality hold?

(d) *Sensitivity analysis.* Let $p^\star(u)$ denote the optimal value of the problem

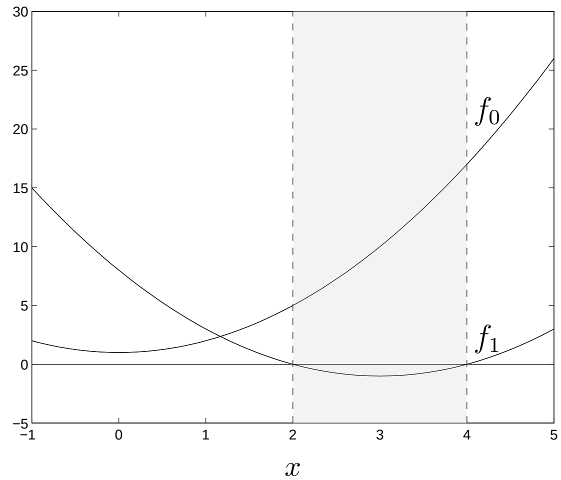
$$\begin{array}{ll} \text{minimize} & x^2+1 \\ \text{subject to} & (x-2)(x-4) \le u, \end{array}$$

as a function of the parameter u. Plot $p^\star(u)$. Verify that $dp^\star(0)/du = -\lambda^\star$.

Solution.

(a) The feasible set is the interval $[2,4]$. The (unique) optimal point is $x^\star = 2$, and the optimal value is $p^\star = 5$.

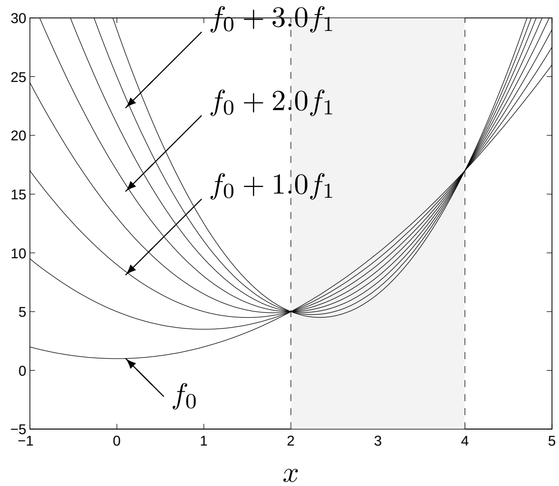
The plot shows f_0 and f_1.



(b) The Lagrangian is

$$L(x,\lambda) = (1+\lambda)x^2 - 6\lambda x + (1+8\lambda).$$

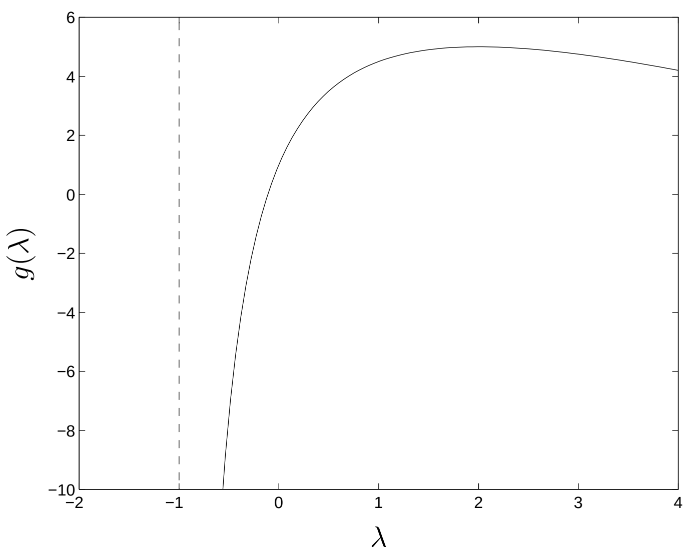
The plot shows the Lagrangian $L(x,\lambda) = f_0 + \lambda f_1$ as a function of x for different values of $\lambda \ge 0$. Note that the minimum value of $L(x,\lambda)$ over x (*i.e.*, $g(\lambda)$) is always less than $p^\star$. It increases as λ varies from 0 toward 2, reaches its maximum at $\lambda = 2$, and then decreases again as λ increases above 2. We have equality $p^\star = g(\lambda)$ for $\lambda = 2$.



For $\lambda > -1$, the Lagrangian reaches its minimum at $\tilde{x} = 3\lambda/(1+\lambda)$. For $\lambda \leq -1$ it is unbounded below. Thus

$$g(\lambda) = \begin{cases} -9\lambda^2/(1+\lambda) + 1 + 8\lambda & \lambda > -1 \\ -\infty & \lambda \leq -1 \end{cases}$$

which is plotted below.



We can verify that the dual function is concave, that its value is equal to $p^\star = 5$ for $\lambda = 2$, and less than $p^\star$ for other values of λ.

(c) The Lagrange dual problem is

$$\begin{array}{ll} \text{maximize} & -9\lambda^2/(1+\lambda) + 1 + 8\lambda \\ \text{subject to} & \lambda \geq 0. \end{array}$$

The dual optimum occurs at $\lambda = 2$, with $d^\star = 5$. So for this example we can directly observe that strong duality holds (as it must — Slater's constraint qualification is satisfied).

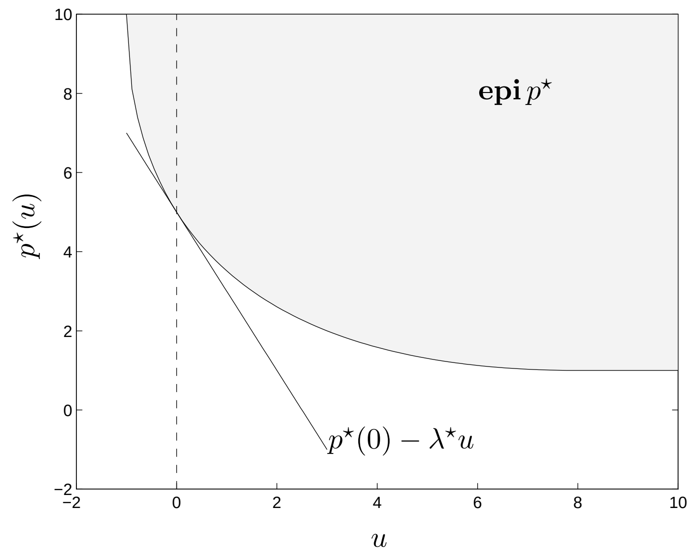
(d) The perturbed problem is infeasible for $u < -1$, since $\inf_x(x^2 - 6x + 8) = -1$. For $u \geq -1$, the feasible set is the interval

$$[3 - \sqrt{1+u}, 3 + \sqrt{1+u}],$$

given by the two roots of $x^2 - 6x + 8 = u$. For $-1 \leq u \leq 8$ the optimum is $x^\star(u) = 3 - \sqrt{1+u}$. For $u \geq 8$, the optimum is the unconstrained minimum of f_0, *i.e.*, $x^\star(u) = 0$. In summary,

$$p^\star(u) = \begin{cases} \infty & u < -1 \\ 11 + u - 6\sqrt{1+u} & -1 \leq u \leq 8 \\ 1 & u \geq 8. \end{cases}$$

The figure shows the optimal value function $p^\star(u)$ and its epigraph.



Finally, we note that $p^\star(u)$ is a differentiable function of u, and that

$$\frac{dp^\star(0)}{du} = -2 = -\lambda^\star.$$

5.13 *Lagrangian relaxation of Boolean LP.* A *Boolean linear program* is an optimization problem of the form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & x_i \in \{0, 1\}, \quad i = 1, \ldots, n, \end{array}$$

and is, in general, very difficult to solve. In exercise 4.15 we studied the LP relaxation of this problem,

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & 0 \leq x_i \leq 1, \quad i = 1, \ldots, n, \end{array} \qquad (1)$$

which is far easier to solve, and gives a lower bound on the optimal value of the Boolean LP. In this problem we derive another lower bound for the Boolean LP, and work out the relation between the two lower bounds.

(a) *Lagrangian relaxation.* The Boolean LP can be reformulated as the problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & x_i(1-x_i) = 0, \quad i = 1, \ldots, n, \end{array}$$

which has quadratic equality constraints. Find the Lagrange dual of this problem. The optimal value of the dual problem (which is convex) gives a lower bound on the optimal value of the Boolean LP. This method of finding a lower bound on the optimal value is called *Lagrangian relaxation.*

(b) Show that the lower bound obtained via Lagrangian relaxation, and via the LP relaxation (1), are the same. *Hint.* Derive the dual of the LP relaxation (1).

Solution.

(a) The Lagrangian is

$$\begin{array}{rcl} L(x,\mu,\nu) & = & c^T x + \mu^T(Ax-b) - \nu^T x + x^T \mathbf{diag}(\nu) x \\ & = & x^T \mathbf{diag}(\nu) x + (c + A^T\mu - \nu)^T x - b^T \mu. \end{array}$$

Minimizing over x gives the dual function

$$g(\mu,\nu) = \left\{ \begin{array}{ll} -b^T\mu - (1/4)\sum_{i=1}^n (c_i + a_i^T\mu - \nu_i)^2/\nu_i & \nu \succeq 0 \\ -\infty & \text{otherwise} \end{array} \right.$$

where a_i is the ith column of A, and we adopt the convention that $a^2/0 = \infty$ if $a \neq 0$, and $a^2/0 = 0$ if $a = 0$.

The resulting dual problem is

$$\begin{array}{ll} \text{maximize} & -b^T\mu - (1/4)\sum_{i=1}^n (c_i + a_i^T\mu - \nu_i)^2/\nu_i \\ \text{subject to} & \mu \succeq 0, \end{array}$$

with implicit constraint $\nu \succeq 0$.

In order to simplify this dual, we optimize analytically over ν, by noting that

$$\begin{array}{rcl} \displaystyle\sup_{\nu_i \geq 0} \left(-\frac{(c_i + a_i^T\mu - \nu_i)^2}{\nu_i} \right) & = & \left\{ \begin{array}{ll} 4(c_i + a_i^T\mu) & c_i + a_i^T\mu \leq 0 \\ 0 & c_i + a_i^T\mu \geq 0 \end{array} \right. \\ & = & \min\{0, 4(c_i + a_i^T\mu)\}. \end{array}$$

This allows us to eliminate ν from the dual problem, and simplify it as

$$\begin{array}{ll} \text{maximize} & -b^T\mu + \sum_{i=1}^n \min\{0, c_i + a_i^T\mu\} \\ \text{subject to} & \mu \succeq 0. \end{array}$$

(b) We follow the hint. The Lagrangian and dual function of the LP relaxation are

$$\begin{aligned} L(x,u,v,w) &= c^T x + u^T(Ax-b) - v^T x + w^T(x-\mathbf{1}) \\ &= (c + A^T u - v + w)^T x - b^T u - \mathbf{1}^T w \\ g(u,v,w) &= \begin{cases} -b^T u - \mathbf{1}^T w & A^T u - v + w + c = 0 \\ -\infty & \text{otherwise.} \end{cases} \end{aligned}$$

The dual problem is

$$\begin{array}{ll} \text{maximize} & -b^T u - \mathbf{1}^T w \\ \text{subject to} & A^T u - v + w + c = 0 \\ & u \succeq 0, v \succeq 0, w \succeq 0, \end{array}$$

which is equivalent to the Lagrange relaxation problem derived above. We conclude that the two relaxations give the same value.

A13.16 *Option price bounds.* In this problem we use the methods and results of example 5.10 to give bounds on the arbitrage-free price of an option. (See exercise 5.38 for a simple version of option pricing.) We will use all the notation and definitions from Example 5.10.

We consider here options on an underlying asset (such as a stock); these have a payoff or value that depends on S, the value of the underlying asset at the end of the investment period. We will assume that the underying asset can only take on m different values, $S^{(1)}, \ldots, S^{(m)}$. These correspond to the m possible scenarios or outcomes described in Example 5.10.

A risk-free asset has value $r > 1$ in every scenario.

A *put option* at *strike price* K gives the owner the right to sell one unit of the underlying stock at price K. At the end of the investment period, if the stock is trading at a price S, then the put option has payoff $(K-S)_+ = \max\{0, K-S\}$ (since the option is exercised only if $K > S$). Similarly a *call option* at strike price K gives the buyer the right to buy a unit of stock at price K. A call option has payoff $(S-K)_+ = \max\{0, S-K\}$.

A *collar* is an option with payoff

$$\begin{cases} C - S_0 & S > C \\ S - S_0 & F \le S \le C \\ F - S_0 & S < F \end{cases}$$

where F is the *floor*, C is the *cap* and S_0 is the price of the underlying at the start of the investment period. This option limits both the upside and downside of payoff.

Now we consider a specific problem. The price of the risk-free asset, with $r = 1.05$, is 1. The price of the underlying asset is $S_0 = 1$. We will use $m = 200$ scenarios, with $S^{(i)}$ uniformly spaced from $S^{(1)} = 0.5$ to $S^{(200)} = 2$. The following options are traded on an exchange, with prices listed below.

Type	Strike	Price
Call	1.1	0.06
Call	1.2	0.03
Put	0.8	0.02
Put	0.7	0.01.

A collar with floor $F = 0.9$ and cap $C = 1.15$ is not traded on an exchange. Find the range of prices for this collar, consistent with the absence of arbitrage and the prices given above.

Solution. This problem is exactly the one described at the end of Example 5.10. Let's label the risk-free asset as asset 1, the underlying asset as asset 2, the four options traded on the exchange as assets 3–6, and the collar as asset 7. A set of 7 prices for these, denoted p, is consistent with the no-arbitrage assumption if and only if there is a $y \succeq 0$ with $V^T y = p$, where V is the value matrix as defined in Example 5.10. We are given the first 6 entries of p, and need to bound the last entry. The lower bound is found by solving the LP

$$\begin{array}{ll} \text{minimize} & p_n \\ \text{subject to} & V^T y = p \\ & y \succeq 0, \end{array}$$

with variables $p_n \in \mathbf{R}$ and $y \in \mathbf{R}^m$. To find the upper bound we maximize p_n instead. The following code solves the problem:

```
% ee364a option price bounds
% additional exercise solution

r=1.05; % risk-free rate
m=200; % scenarios
n=7; % assets
V=zeros(m,n); % value/payoff matrix
V(:,1) = r; % risk-free asset
V(:,2) = linspace(0.5,2,m); % underlying
% the four exchange traded options:
V(:,3) = pos(V(:,2) - 1.1);
V(:,4) = pos(V(:,2) - 1.2);
V(:,5) = pos(0.8-V(:,2));
V(:,6) = pos(0.7-V(:,2));
% collar option:
F=0.9;C=1.15;
V(:,7) = min(max(V(:,2)-1,F-1),C-1);

p = [1; 1; 0.06; 0.03; 0.02; 0.01];  % asset prices (from exchange)
```

```
cvx_begin
variables p_collar y(m)
minimize p_collar
%maximize p_collar
y>=0
V'*y== [p; p_collar]
cvx_end
```

The bounds were as follows:

```
[lb ub]=
0.033 0.065
```

We didn't ask you to carry out the following check, but had this been a real pricing exercise, you should have done it. We vary both the range of final underlying asset prices (which we arbitrarily took to be 0.5 and 2, representing a halving or doubling of value), and m, the number of samples of the final underlying asset value (which arbitrarily took to be the round number 200). Sure enough, varying these arbitrary parameters doesn't change the call option price bounds very much.