APOORVA ARYA
2020032

Assignment - 1

Ans 1 $P(w_1) = 1/4$, $P(w_2) = 3/4$

$P(x|w_1) = N(2,1)$ $P(x|w_2) = N(5,1)$

$$P(x|w_1) = \frac{1}{\sqrt{2\pi\sigma^2}} e\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$

$(\mu = 2, \sigma^2 = 1)$
$$= \frac{1}{\sqrt{2\pi}} e\left\{-\frac{1}{2}(x-2)^2\right\}$$

Similarly,
$$p(x|w_2) = \frac{1}{\sqrt{2\pi}} e\left\{-\frac{1}{2}(x-5)^2\right\}$$
$(\mu = 5, \sigma^2 = 1)$

(i) zero-one loss

$\lambda_{11} = \lambda_{22} = 0$, $\lambda_{12} = \lambda_{21} = 1$

Minimizing Risk would minimize error too.

$\therefore$ we need $1-P(w_1|x)$ & $1-P(w_2|x)$ to be max. and for Decision Boundary:

$$1-P(w_1|x) = 1-P(w_2|x)$$
$$P(w_1|x) = P(w_2|x)$$
$$\frac{P(x|w_1)P(w_1)}{\cancel{P(x)}} = \frac{P(x|w_2)P(w_2)}{\cancel{P(x)}}$$
$$\frac{1}{\cancel{4}} \times \frac{1}{\cancel{\sqrt{2\pi}}} e\left\{-\frac{1}{2}(x-2)^2\right\} = \frac{3}{\cancel{4}} \times \frac{1}{\cancel{\sqrt{2\pi}}} e\left\{-\frac{1}{2}(x-5)^2\right\}$$
$$e\left\{\frac{1}{2}(x-5)^2 - \frac{1}{2}(x-2)^2\right\} = 3$$
$$\frac{1}{2}\left[(x-5)^2 - (x-2)^2\right] = \ln 3$$
$$-3(2x-7) = 2\ln 3$$

Solving: $x = 3.13379$

1. (ii) $\lambda_{12} = 2$, $\lambda_{21} = 3$, $\lambda_{11} = 0$, $\lambda_{22} = 0$

Now, ~~likelihood ratio~~

for decision boundary:

$$(\lambda_{21} - \lambda_{11}) \, P(w_1|x) = (\lambda_{12} - \lambda_{22}) \, P(w_2|x)$$

$$(3-0) \frac{P(x|w_1)P(w_1)}{\cancel{P(x)}} = (2-0) \frac{P(x|w_2)\,P(w_2)}{\cancel{P(x)}}$$

$$\cancel{3} \times \frac{1}{\cancel{\sqrt{2\pi}}} e\left\{-\frac{1}{2}(x-2)^2\right\} \times \frac{1}{\cancel{4}} = 2 \times \frac{1}{\cancel{\sqrt{2\pi}}} \times e\left\{-\frac{1}{2}(x-5)^2\right\} \frac{\cancel{\times 3}}{\cancel{4}}$$

$$e\left\{-\frac{1}{2}(x-2)^2\right\} = 2 \times e\left\{-\frac{1}{2}(x-5)^2\right\}$$

$$e\left\{-\frac{1}{2}(x-2)^2 + \frac{1}{2}(x-5)^2\right\} = 2$$

$$\frac{1}{2}\left\{(x-5)^2 - (x-2)^2\right\} = \ln 2$$

$$-3(2x-7) = 2\ln 2$$

$$x = \frac{7}{2} - \frac{\ln 3}{3}$$

$$x = 3.26895 \text{ (approx)}$$

No, it'd be impractical and possibly dangerous to use zero-one loss for a task like cancer prediction on a real world dataset because it's too ideal and a false negative would be more costly than a false positive, since that'd lead to undetected case.

Date

Ans 2 $X = [X1, X2, X3]$, $Y = A^T X + B$

Mean vector $u = [5, -5, 6]$

$A = (2, -1, 2)^T$

$\bar{Y} = A^T \bar{X} + B$

Then, $A^T \bar{X} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} [5, -5, 6]$

(3×1) (1×3) (For mean)

$= 10 + 5 + 12 = 27$

$\bar{Y} = A^T \bar{X} + B = 27 + 5 = 32$

Ans 3 (A) $P(x|w_1) = \frac{1}{\pi b} \cdot \frac{1}{1 + \left(\frac{x - a_1}{b}\right)^2}$

$P(x|w_2) = \frac{1}{\pi b} \cdot \frac{1}{1 + \left(\frac{x - a_2}{b}\right)^2}$

for zero - one loss :

$(\lambda_{21} - \lambda_{11}) P(w_1|x) = (\lambda_{12} - \lambda_{22}) P(w_2|x)$

$1 \cdot \frac{P(x|w_1) \cancel{P(w_1)}}{\cancel{P(x)}} = 1 \cdot \frac{P(x|w_2) \cancel{P(w_2)}}{\cancel{P(x)}}$

$[P(w_1) = P(w_2)]$

$\frac{1}{\cancel{\pi b}} \cdot \frac{1}{1 + \left(\frac{x - a_1}{b}\right)^2} = \frac{1}{\cancel{\pi b}} \cdot \frac{1}{1 + \left(\frac{x - a_2}{b}\right)^2}$

$\left(\frac{x - a_1}{b}\right)^2 = \left(\frac{x - a_2}{b}\right)^2$

$\left[\frac{x - a_1}{b} + \frac{x - a_2}{b}\right] \left[\frac{x - a_1}{b} - \left(\frac{x - a_2}{b}\right)\right] = 0$

$\left(\frac{2x - (a_1 + a_2)}{b}\right) \left(\frac{a_2 - a_1}{b}\right) = 0$

Date

$$2x = a_1 + a_2 \Rightarrow x = \left(\frac{a_1+a_2}{2}\right)$$

(B) Now, we know that

$$P(x) = P(x|w_1)\,P(w_1) + P(x|w_2)\,P(w_2)$$

$$= \frac{1}{\pi b} \cdot \frac{1}{1+\left(\frac{x-a_1}{b}\right)^2} \cdot \frac{1}{2} + \frac{1}{\pi b}\,\frac{1}{1+\left(\frac{x-a_2}{b}\right)^2} \cdot \frac{1}{2}$$

Now, $b = 1$, $a1 = 3$, $a2 = 5$

$$= \frac{1}{2\pi}\left[\frac{1}{1+(x-3)^2} + \frac{1}{1+(x-5)^2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{2+(x-3)^2+(x-5)^2}{\left(1+(x-3)^2\right)\left(1+(x-5)^2\right)}\right]$$

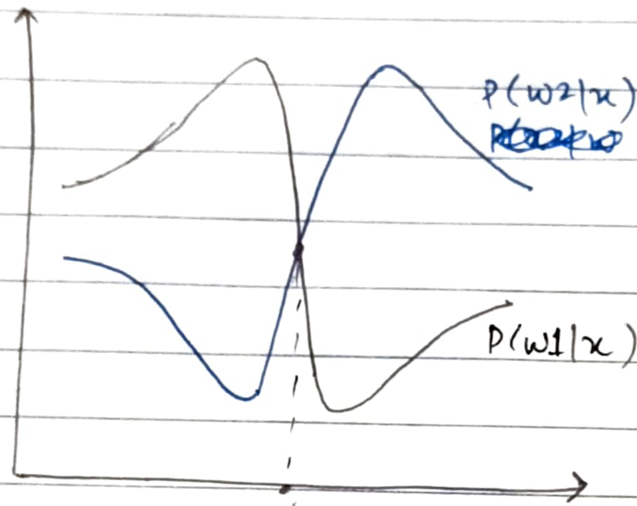
$$P(w1|x) = \frac{p(x|w_1)\;p(w_1)}{P(x)}$$

$$= \frac{\frac{1}{\cancel{2}} \cdot \frac{1}{\cancel{\pi}}\left(\frac{1}{\cancel{1+(x-3)^2}}\right)}{\frac{1}{\cancel{2\pi}}\left(\frac{2+(x-3)^2+(x-5)^2}{\cancel{(1+(x-3)^2)}\,(1+(x-5)^2)}\right)}$$

$$\therefore\; p(w1|x) = \frac{1+(x-5)^2}{2+(x-3)^2+(x-5)^2}$$

Now, we plot this on graph
(code in .py file)

i) On plotting $P(w_1|x)$ & $P(w_2|x)$, we get:



$$x^* = \frac{a_1 + a_2}{2}$$

$$x^* = \frac{3+5}{2} = 4$$

$$\therefore \quad P(\text{error}) = \int_X P(\text{error}|x)\, P(x) \cdot dx$$

$$\Rightarrow \int_{-\infty}^{4} P(w_2|x)\, P(x) \cdot dx + \int_{4}^{\infty} P(w_1|x)\, P(x) \cdot dx$$

$$\frac{1}{2\pi}\left[\int_{-\infty}^{4} \frac{\cancel{1+(x-3)^2}}{\cancel{2+(x-3)^2+(x-5)^2}} \cdot \frac{\cancel{2+(x-3)^2+(x-5)^2}}{(\cancel{1+(x-3)^2}) \cdot (1+(x-5)^2)}\, dx + \int_{4}^{\infty} \frac{\cancel{1+(x-5)^2}}{\cancel{2+(\ldots)}} \cdot \frac{\cancel{(2+(x-3)^2+\ldots)}}{1+(x-3)^2 \cdot \cancel{(1+(x-5)^2)}}\, dx\right]$$

$$\Rightarrow \frac{1}{2\pi}\left[\int_{-\infty}^{4} \frac{1}{1+(x-5)^2}\, dx + \int_{4}^{\infty} \frac{1}{1+(x-3)^2}\, dx\right]$$

Date

$$\frac{1}{2\pi}\left\{ \tan^{-1}(x-5)\Big]_{-\infty}^{4} + \tan^{-1}(x-3)\Big]_{4}^{\infty} \right\}$$

$$\frac{1}{2\pi}\left\{ \left(-\frac{\pi}{4} - \left(-\frac{\pi}{2}\right)\right) + \left(\frac{\pi}{2} - \frac{\pi}{4}\right) \right\}$$

$$\frac{1}{2\pi}\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = \frac{1}{4}$$

$\therefore$ overall rate : $\frac{1}{4} = 0.25$

Ans 4 a) Observing the covariance matrix :

$a_{12} = a_{21} = 0$

Then, a and b are independent.

$x = [a \;\; b]$ a $\rightarrow$ Bernoulli

b $\rightarrow$ Gaussian

~~pdf~~ pdf of a : $\theta^{a}(1-\theta)^{1-a}$

of b : $\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(b-m)^2\right\}$

pdf of x = $\theta^{a}(1-\theta)^{1-a} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e\left\{\frac{-1}{2\sigma^2}(b-m)^2\right\}$

Ans 4b

$$q(x) = \prod_{i=1}^{N} \theta^{a_i} (1-\theta)^{1-a_i} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(b_i - m)^2\right\}$$

$$\ln(q(x)) = \sum_{i=1}^{N} a_i \ln\theta + (1-a_i)\ln(1-\theta) + \ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)$$

$$+ \left\{-\frac{1}{2\sigma^2}(b_i - m)^2\right\}$$

Differentiating w.r.t θ to maximise $q(x)$:

$$\frac{\partial}{\partial\theta} \ln(q(x)) = \sum_{i=1}^{N} \frac{a_i}{\theta} - \frac{(1-a_i)}{(1-\theta)} + 0$$

$$= \sum_{i=1}^{N} \frac{a_i(1-\theta) - (1-a_i)\theta}{\theta(1-\theta)}$$

$$= \sum_{i=1}^{N} \frac{a_i - \cancel{a_i\theta} - \theta + \cancel{a_i\theta}}{\theta(1-\theta)}$$

$$= \sum_{i=1}^{N} \frac{a_i - \theta}{\theta(1-\theta)}$$

$$\frac{\partial}{\partial\theta} \ln(q(x)) = 0 \text{ for max.}$$

$$\therefore \sum_{i=1}^{N} \frac{a_i - \theta}{\theta(1-\theta)} = 0$$

$$\therefore \frac{a_1 - \theta}{\theta(1-\theta)} + \ldots + \frac{a_N - \theta}{\theta(1-\theta)} = 0$$

$$\theta = \frac{a_1 + a_2 + \ldots a_N}{N}$$

$$\therefore \theta = \frac{1}{N}\sum_{i=1}^{N} a_i$$