1. Big O(n)

$f(n) \Rightarrow O(g(n))$

if $f(n) \leq g(n) \times c \quad \forall n \geq n_0$

for some constant, $c > 0$

$g(n) \rightarrow$ tight upper bound of $f(x)$

eg $f(n) = n^2 + n$

$g(n) = n^3$

$n^2 + n \leq c * n^3$

$n^2 + n = O(n^3)$

(ii) Big Omega (Ω)

when $f(n) = \Omega(g(n))$

$g(n) \rightarrow$ tight upperbound of $f(n)$

i.e $f(x) = \Omega g(x)$

if and only if

$f(x) \geq c \cdot g(x)$

$\forall n_2 > n_0$ and c = constant

eg. $f(n) = n^3 + 4n^2$

$g(n) = n^2$

ie $f(n) \geq c * g(n)$

$n^3 + 4n^2 = \Omega(n^2)$

(iii) Big Theta (Θ)

when $f(n) = \Theta(g(n))$ gives tight upperbound and lowerbound both.

i.e $f(n) = \Theta(g(n))$

if and only if

$$c_1 * g(n_1) \leq f(n) \leq c_2 * g(n_2)$$

for all $n \geq \max(n_1, n_2)$ some constant

$c_1 > 0$ & $c_2 > 0$

i.e $f(n)$ can never go beyond $c_2 g(n)$ and will never come down of $c_1 g(n)$.

eg. $3n + 2 = \Theta(n)$ as $3n + 2 \geq 3n$

$3n + 2 \leq 4n$ for n, $c_1 = 3$ $c_2 = 4$ & $n_0 = 2$

(iv) Small o (o)

When $f(n) = o(g(n))$ gives the upper bound

i.e $f(n) = o(g(n))$

If and only if

$f(n) < c * g(n)$

$\forall$ $n > n_0$ & $n > 0$

eg. $f(n) = n^2$; $g(n) = n^3$

$f(n) < c * g(n)$

$n^2 = o(n^3)$

(v) Small Omega (ω)

It gives lower bound i.e $f(n) = \omega(g(n))$ where $g(n) \rightarrow$ lower bound of $f(n)$ if and only if $f(n) > c * g(n)$

$\forall$ $n > n_0$ & some constant, $c > 0$

Ans 2:- for $i = 1, 2, 4, 6, 8, \ldots$ n times

i.e series is a GP

so $a = 1$ $r = 2$

k^{th} value of GP :

$$t_k = ar^{k-1}$$

$$t_k = 1(2)^{k-1}$$

$$2n = 2^k$$

$$\log_2(2n) = k(\log 2)$$

$$\log_2 2 + \log_2 n = k$$

$$\log_2 n + 1 = k \quad \text{(Neglecting 1)}$$

So TC $T(n) \Rightarrow O(\log_2 n)$

Ans 3:- $T(n) = 3T(n-1)$ — (i)

$T(n) = 1$

put $n \Rightarrow n-1$ in (i)

$T(n-1) = 3T(n-2)$ — (ii)

put (ii) in (i)

$T(n) = 3(3T(n-2))$

$T(n) = 9T(n-2)$ — (iii)

put $n = n-2$ in (i)

$T(n-2) = 3T(n-3)$

put in (iii)

$T(n) = 27T(n-3) \longrightarrow$ (iv)

Generalizing series,

$$T(k) = 3^k T(n-k) \quad \text{--- (V)}$$

for k^{th} terms, let $n-k = 1$ (Base Case)

$$k = n-1$$

put in (V)

$$T(n) = 3^{n-1} T(1)$$

$$T(n) = 3^{n-1}$$

$$T(n) = O(3^n)$$

Ans 4:-

$$T(n) = 2T(n-1) - 1 \longrightarrow ①$$

put $n = n-1$

$$T(n-1) = 2T(n-2) - 1 \longrightarrow ②$$

put in ①

$$T(n) = 2(2T(n-2) - 1) - 1$$

$$T(n) = 4T(n-2) - 2 - 1 \text{ --- } ③$$

put $n = n-2$ in ①

$$T(n-2) = 2T(n-3) - 1$$

put in ③

$$T(n) = 4(2T(n-3) - 1) - 2 - 1$$

$$= 8T(n-3) - 4 - 2 - 1 \text{ --- } ④$$

Generalizing series,

$$T(n) = 2^k T(n-k) - 2^{k-1} - 2^{k-2} \ldots 2^0$$

K^{th} term

Let $n-k=1$

$k=n-1$

$$T(n) = 2^{n-1}\, T(1) - 2^{k}\left(\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^k}\right)$$

$$= 2^{n-1} - 2^{n-1}\left(\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{k-1}}\right)$$

i.e Series in GP

$a=\frac{1}{2}$, $r=\frac{1}{2}$

So,

$$T(n) = 2^{n-1}\left(1-\left(\frac{1}{2}\,\frac{(1-(1/2)^{k-1})}{1-1/2}\right)\right)$$

$$= 2^{n-1}\left(1-1+\left(\frac{1}{2}\right)^{n-1}\right)$$

$$= \frac{2^{n-1}}{2^{n-1}}$$

$$T(n) = O(1)$$

Ans. As $i^2=n$

$i=\sqrt{n}$

$i = 1, 2, 3, 4, \ldots \sqrt{n}$

$\sum_{i=1} 1+2+3+4+\cdots+\sqrt{n}$

$$T(n) = \frac{\sqrt{n}+(\sqrt{n}+1)}{2}$$

$$T(n) = \frac{n+\sqrt{n}}{2}$$

$$T(n) = O(n)$$

Ans 7 :-

Since for $k = k^2$

$k = 1, 2, 4, 8, \ldots n$

∴ Series is in GP

so $a = 1$, $r = 2$

$$= \frac{a(r^n - 1)}{r - 1}$$

$$= \frac{1(2^k - 1)}{2 - 1}$$

$$n = 2^k - 1$$

$$n + 1 = 2^k$$

$$\log_2(n) = k$$

i	j	k
1	log(n)	log(n) * log(n)
2	log(n)	log(n) * log(n)
¦	¦	¦
n	log(n)	log(n) * log(n)

$TC \Rightarrow O(n * \log n * \log n)$

$\Rightarrow O(n \log^2(n))$

Ans 8 :-

for (i = 1 to n)

we get j = n times every turn

$\therefore$ $i * j = n^2$

k^{th},

Now,

$T(n) = n^2 + T(n-3)$;

$T(n-3) = (n^2 3)^2 + T(n-6)$;

$T(n-6) = (n^3 6)^2 + T(n-9)$;

and $T(1) = 1$;

Now Substitute each value in T (n)

$T(n) = n^2 + (n-3)^2 + (n-6)^2 + \text{---} + 1$

Let

$k^n - 3k = 1$

$k = (n-1)/3$ total terms = k+1

$T(n) = n^2 + (n-3)^2 + (n-6)^2 + \text{....} + 1$

$T(n) \simeq kn^2$

$T(n) \simeq (k-1)/3 \; n^2$

So $T(n) = O(n^3)$

Ans 9: -

for $i = 1$ $\quad j = 1 + 2 + \dots (n \geq j + i)$

$i = 2$ $\quad j = 1 + 3 + 5 + \dots (n \geq j + i)$

$i = 3$ $\quad j = 1 + 4 + 7 + \dots (n \geq j + i)$

n^{th} term of AP is

$$T(n) = a + d * m$$

$$T(m) = 1 + d \times m$$

$$(n-1)/d = m$$

for $i = 1$ $\quad (n-1)/1$ times

$i = 2$ $\quad (n-1)/2$ times

$i = n-1$

We get

$$T(n) = i_1 j_1 + i_2 j_2 + \dots + i_{n-1} j_{n-1}$$

$$= \frac{(n-1)}{1} + \frac{(n-2)}{2} + \frac{(n-3)}{3} \dots +$$

$$= n + \frac{n}{2} + \frac{n}{3} + \dots + \frac{n}{n-1} - n \times 1$$

$$= n\left[1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-1}\right] - n * 1$$

$$= n \times \log n - n + 1$$

Since $\int \frac{1}{x} = \log x$

$$T(n) = O(n \log n)$$

Ans10 :-

As given n^k and c^n

Relationship b/w n^k and c^n is

$$n^k = O(c^n)$$

$$n^k \leq a(c^n)$$

$\forall\ n \geq n_0$ & constant , $a > 0$

for $n_0 = 1$; $c = 2$

$\Rightarrow 1^k < a2$

$n_0 = 1$ & $c = 2$