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2) Binomial distribution ~ Bin (n, π)

$$f(y) = \binom{n}{y}\left(\frac{\pi}{1-\pi}\right)^{y} (1-\pi)^{n}$$

Taking log on both sides

$$f(y) = \exp\left\{\log\binom{n}{y} + y\log\left(\frac{\pi}{1-\pi}\right) + n\log(1-\pi)\right\} \quad -①$$

let $\log\left(\frac{\pi}{1-\pi}\right) = \theta \Rightarrow \frac{\pi}{1-\pi} = e^{\theta} \Rightarrow \pi = \frac{e^{\theta}}{1+e^{\theta}}$.

$$1-\pi = 1 - \frac{e^{\theta}}{1+e^{\theta}} = \frac{1}{1+e^{\theta}}$$

~~comparing~~

$$\Rightarrow f(y) = \exp\left\{y\theta - n\log(1+e^{\theta}) + \log\binom{n}{y}\right\} \quad -②$$

comparing ② &

$$f_Y(y;\theta,\phi) = \exp\left\{\frac{y\theta - b(\theta)}{\phi} + c(y,\phi)\right\} \quad -③$$

we get, $\theta = \log\left(\frac{\pi}{1-\pi}\right)$, $\phi = 1$,

$$b(\theta) = n\log(1+e^{\theta})$$

$E(Y) = \mu = b'(\theta)$

$$= \frac{n e^{\theta}}{1+e^{\theta}} = n\pi \quad \text{~~as expected~~}$$

$var(Y) = \phi b''(\theta)$

$$= 1 \frac{\partial}{\partial \theta} \frac{n e^{\theta}}{1+e^{\theta}} = \frac{n e^{\theta}}{(1+e^{\theta})^2} = n\left(\frac{e^{\theta}}{1+e^{\theta}}\right)\left(\frac{1}{1+e^{\theta}}\right) = n\pi(1-\pi)$$

~~To show~~ canonical link :

To show: $g(\mu) = \theta$.

$$\frac{n e^{\theta}}{1+e^{\theta}} = \mu \Rightarrow \frac{\mu}{n} = \frac{e^{\theta}}{1+e^{\theta}} = \pi$$

$$\Rightarrow \theta = \log\left(\frac{\pi}{1-\pi}\right) = \log\left(\frac{\mu/n}{1-\mu/n}\right)$$

$\Rightarrow$ Logit fn.

$\therefore$ Logit function is the canonical link for binomial distribution.

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3) Poisson distribution $Pois(\lambda)$

$$f(y) = \frac{1}{y!}\lambda^{y} e^{-\lambda}$$

Taking log o.b.s

$$f(y) = \exp\{y \log \lambda - \lambda - \log(y!)\} \quad -①$$

To show:
~~Comparing~~

$$f(y;\theta,\phi) = \exp\left\{\frac{y\theta - b(\theta)}{\phi} + c(y,\phi)\right\} \quad -②$$

Comparing ① & ②

let $\log\lambda = \theta \Rightarrow \lambda = e^{\theta}$ $\quad \phi = 1 \quad c(y,\phi) = -\log(y!)$
$b(\theta) = e^{\theta}$.

$$E(Y) = b'(\theta) = \frac{d}{d\theta}e^{\theta} = \lambda = \mu$$

$$Var(Y) = \phi b''(\theta) = \frac{d}{d\theta}e^{\theta} = e^{\theta} = \lambda.$$

Canonical link : $g(\mu) = \theta \Rightarrow$ To show.
$\lambda = e^{\theta} \Rightarrow \log\lambda = \theta$
$\therefore$ log is the canonical link for Poisson distribution.

5) $f(y, \beta) = \binom{y+m-1}{m-1} \beta^m (1-\beta)^y$.

$$= f(y) = \exp\left[\log\binom{y+m-1}{m-1} + m\log\beta + y\log(1-\beta)\right]$$

$\cancel{= f(y) = \exp}$ let $\theta = \log(1-\beta)$

$$1-\beta = e^\theta \Rightarrow \beta = 1-e^\theta$$

$$\Rightarrow \log\beta = \log(1-e^\theta)$$

$$= \exp\left[\cancel{y\log}\ \theta y \cancel{+ m} - (-m\log(1-e^\theta)) + \log\binom{y+m-1}{m-1}\right]$$

$\phi = 1$ $\quad b(\theta) = -m\log(1-e^\theta)$

$$c(y, \phi) = \log\binom{y+m-1}{m-1}$$

$\cancel{E[y}$ $Ey = \dfrac{+me^\theta}{1-e^\theta}\ \cancel{\phi} = \dfrac{m(1-\beta)}{\beta}$.

$$Var(y) = \phi\frac{\partial^2}{\partial\theta^2}b(\theta) = \frac{me^\theta}{(1-e^\theta)^2} = \frac{r(1-\beta)}{\beta^2}$$

$$g(\mu) = \eta = \theta$$

$$E(y) = \frac{m e^{\theta}}{1-e^{\theta}} = \frac{m}{e^{-\theta}-1}$$

$$g^{-1}(\theta) = \frac{e^{-\theta}-1}{m}$$

$$m g^{-1}(\theta) + 1 = e^{-\theta}$$

$$\log(m g^{-1}(\theta) + 1) = e^{-\theta}$$

$$\log\left(\frac{1}{m g^{-1}(\theta)+1}\right) = +\theta$$

$$\log\left(\frac{1}{\frac{m}{Ey}+1}\right) = \theta$$

$$\log\left[\frac{Ey}{m+Ey}\right] = \theta$$

$\therefore$ canonical link is log

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(c) ~~(α,β)~~ $\text{Gamma}(\alpha,\beta) = \dfrac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y}$
$f(y;\beta)$ where $\alpha, \beta > 0$.

$$f(y) = \exp\{\alpha \log\beta - \log(\Gamma(\alpha)) + (\alpha-1)\log y - \beta y\} \quad -(1)$$

To show:

$$f(y;\phi;\theta) = \exp\left\{\frac{y\theta - b(\theta)}{\phi} + c(y,\phi)\right\} \quad -(2)$$

$$= \exp\left\{\frac{y(-\beta/\alpha) - (-\log\beta)}{1/\alpha} + (\alpha-1)\log y - \log(\Gamma(\alpha))\right\}$$

let $\theta = -\beta/\alpha \qquad \phi = 1/\alpha \qquad$ ~~$\theta = -\beta/\phi$~~

$-\beta = \alpha\theta \Rightarrow \beta = -\theta/\phi$

~~$\log\beta = -\log\theta - \log\phi$~~

~~$b(\theta) = -\log$~~

$$= \exp\left\{\frac{y\theta - [-\log(-\theta)]}{\phi} - \log(\phi)/\phi + \left(\frac{1}{\phi} - 1\right)\log y - \log\Gamma(1/\phi)\right\}$$

$-(3)$

Comparing ② + ③.

$b(\theta) = -\log(-\theta)$ $\phi = \frac{1}{\alpha}$ $\theta = -\beta/\alpha$.

~~Canonical link $g(\mu) = \theta$~~

$EY = b'(\theta) = -\frac{1}{\theta} = \mu$ — ④.

$\operatorname{Var} Y = \phi b''(\theta) = \phi / \theta^2$

To show:

Canonical link: $g(\mu) = \theta$.

From ④, we get $\theta = -\frac{1}{\mu} = g(\mu)$.

∴ The canonical link function is negative inverse function.

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1) Exp(λ)

$f(y;\lambda) = \lambda e^{-\lambda y}$

$f(y) = \exp\{\log\lambda - \lambda y\}$ — (1)

Comparing with

$f(y;\theta,\phi) = \exp\left\{\frac{y\theta - b(\theta)}{\phi} + c(y,\phi)\right\}$ (2)

$\theta = -\lambda$ — (3) $\Rightarrow \lambda = -\theta$

$f(y) = \exp\left\{\frac{\theta y - (-\log(-\theta))}{1}\right\}$

we get $b(\theta) = -\log(-\theta)$. $\phi = 1$. $\theta = -\lambda$.

$EY = b'(\theta) = \frac{-1}{\theta} = \frac{1}{\lambda} = \mu$ — (4)

$\operatorname{Var} Y = \phi b''(\theta) = \frac{1}{\theta^2} = \frac{1}{\lambda^2}$

To show: $g(\mu) = \theta$

From (4) $\mu = \frac{-1}{\theta} \Rightarrow \theta = \frac{-1}{\mu}$

$\therefore$ It is an inverse function

4) Chi-squared distribution

$$f(y;k) = \frac{1}{\Gamma(\frac{k}{2})\, 2^{k/2}}\, y^{k/2-1}\, e^{-y/2}$$

$$f(y) = \exp\left\{ -\log(\Gamma(\tfrac{k}{2})) + \tfrac{k}{2}\log\tfrac{1}{2} + (\tfrac{k}{2}-1)\log y - y/2 \right\}.$$

$$f(y) = \exp\left\{ -y/2 + \tfrac{k}{2}\log\tfrac{1}{2} + (\tfrac{k}{2}-1)\log y - \log(\Gamma(\tfrac{k}{2})) \right\}$$

$$= \exp\left\{ \frac{\frac{-y/2}{k/2} + \log\frac{1}{2}}{\frac{k}{2}} + (\tfrac{k}{2}-1)\log y - \log(\Gamma(\tfrac{k}{2})) \right\}$$

$$= \exp\left\{ \frac{-\frac{y}{k} - \log 2}{\frac{k}{2}} + (\tfrac{k}{2}-1)\log y - \log(\Gamma(\tfrac{k}{2})) \right\}$$

Comparing with

$$f(y;\phi,\theta) = \exp\left\{ \frac{y\theta - b(\theta)}{\phi} + c(y,\phi) \right\}$$

we get $\phi = \frac{k}{2}$, $b(\theta) = \log 2$, $\theta = \frac{-1}{2}$,

$Y \sim$ Bernoulli

Problem 3) $f(y_i, \pi) = \pi^{y_i}(1-\pi)^{1-y_i}$

$\cancel{L(Y,\pi) = \prod_{i=1}^{n} f(y_i;\pi) = \prod_{i=1}^{n} \pi^{y_i}(1-\pi)^{1-y_i}}$

log-likelihood:

$$l(Y,\pi) = \sum_{i=1}^{n}\left[y_i \log\pi + (1-y_i)\log(1-\pi)\right] \quad -①$$

$$S(\pi) = \frac{\partial(l(Y,\pi))}{\partial\pi} = \sum_{i=1}^{n}\left[\frac{y_i}{\pi} - \frac{1-y_i}{1-\pi}\right]$$

$$= \frac{\sum_{i=1}^{n} y_i}{\pi} - \frac{\sum_{i=1}^{n}(1-y_i)}{1-\pi}$$

$$= \frac{\sum_{i=1}^{n} y_i}{\pi} - \frac{n - \sum_{i=1}^{n} y_i}{1-\pi}$$

Assume $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ $= \frac{n\bar{y}}{\pi} - \frac{n-n\bar{y}}{1-\pi}$

$$= \frac{n\bar{y}(1-\pi) - n(1-\bar{y})\pi}{\pi(1-\pi)} = \frac{n\bar{y}-n\pi}{\pi(1-\pi)} \quad -②$$

For MLE, put ② $=0$

$$n\bar{y} = n\pi \Rightarrow \hat{\pi}_{MLE} = \bar{y}$$

$$I(\pi) = E\left(-\frac{\partial^2 l(y,\pi)}{\partial \pi^2}\right)$$

$$= E\left(-\frac{-n\pi(1-\pi) - n(\bar{y}-\pi)(1-2\pi)}{\pi^2(1-\pi^2)}\right]$$

$$= E\left[\frac{n\pi(1-\pi) + n(\bar{y}-\pi)(1-2\pi)}{\pi^2(1-\pi^2)}\right]$$

$$= \frac{n\pi(1-\pi) + n(1-2\pi)E(\bar{y}-\pi)}{\pi^2(1-\pi)^2} = \frac{n\pi(1-\pi)}{\pi^2(1-\pi)^2}$$

$$= \frac{n}{\pi(1-\pi)} \quad -③$$

Deriving statistics :-

Wald : $TS_W = (\hat{\pi}_{MLE} - \pi_0) . I(\hat{\pi}_{MLE}) (\hat{\pi}_{MLE} - \pi_0)$

$$= (\bar{y} - \pi_0)^2 \frac{n}{\bar{y}(1-\bar{y})} = \frac{n(\bar{y}-\pi_0)^2}{\bar{y}(1-\bar{y})}$$

Score : $TS_S = S(\pi_0) . I^{-1}(\pi_0) . S(\pi_0)$

$$= \left[\frac{n\bar{y} - n\pi_0}{\pi_0(1-\pi_0)}\right]^2 \frac{\pi_0(1-\pi_0)}{n}$$

$$= n(\bar{y}-\pi_0)^2 / \pi_0(1-\pi_0)$$

$LR : TS_{LR} = 2\left[\ell(y, \hat{\pi}_{MLE}) - \ell(y, \pi_0) \right]$

$$\ell(y, \bar{y}) = \sum_{i=1}^{n} y_i \log \bar{y} + (1-y_i) \log 1-\bar{y} \,]$$

$$\ell(y, \pi_0) = \sum_{i=1}^{n} \left[y_i \log \frac{\bar{y}}{\pi_0} + (1-y_i) \log \frac{1-\bar{y}}{1-\pi_0} \right]$$

$$TS_{LR} = 2 \sum_{i=1}^{n} \left[y_i \log \frac{\bar{y}}{\pi_0} + (1-y_i) \log \frac{1-\bar{y}}{1-\pi_0} \right]$$

$$= 2\left[n\bar{y} \log \frac{\bar{y}}{\pi_0} + (n - n\bar{y}) \log \frac{1-\bar{y}}{1-\pi_0} \right]$$

2) i) $\pi_0 = 0.1$, $n = 10$, $\pi = 0.3$

$$TS_W = \frac{10(0.3 - 0.1)^2}{0.3(1-0.3)} = 1.9 \checkmark$$

$$TS_S = \frac{10(0.3 - 0.1)^2}{0.1(1-0.1} = 4.44 \times$$

$$TS_{LR} = 2\left[10 \cdot 0.3 \log^{0.3/0.1} + 10(1-0.3) \log \frac{1-0.3}{1-0.1} \right]$$

$$= 3.07 \checkmark$$

ii) $\pi_0 = 0.3$, $\pi = 0.3$, $n = 10$.

$$TS_W = 10\frac{(0.3-0.3)^2}{0.3(1-0.3} = 0.$$

$$TS_S = 10\frac{(0.3-0.3)^2}{0.3(1-0.3)} = 0.$$

$$TS_{LR} = 0 = 2\left[10.0.3\log\frac{0.3}{0.3} + 10(1-0.3)\log\frac{1-0.3}{1-0.3}\right] = 0.$$

iii) $\pi_0 = 0.5$

$$TS_W = 10\frac{(0.3-0.5)^2}{0.3(1-0.3)} = 1.9$$

$$TS_S = 10\frac{(0.3-0.5)^2}{0.5(1-0.5)} = 1.6$$

$$TS_{LR} = 2\left[10.0.3\log\frac{0.3}{0.5} + 10(1-0.3)\log\frac{1-0.3}{1-0.5}\right]$$

$$= 1.65.$$

3) Not all of them.

Problem2 : $y_i \sim Bin(m, \pi_i)$.

$$f(y) = \binom{m}{y_i} \log \pi_i^{y_i} (1-\pi_i)^{m-y_i}$$

$$l(\beta, \pi_i) = \sum_{i=1}^{m} \{ y_i \log \pi_i + (m-y_i) \log(1-\pi_i) + \log \binom{m}{y_i} \}$$

$$\Rightarrow \log \frac{\pi_i}{1-\pi_i} = x_i \beta = n_i$$

For a saturated model, all π_i's are different.

so $\beta = [\pi_1, \dots, \pi_N]^T$

MLE $\Rightarrow$ $\hat{\pi}_i = \frac{e^{x_i \hat{\beta}}}{1+e^{x_i \hat{\beta}}} = \frac{y_i}{m}$

$$l(b_{max}, y) = \sum \left[y_i \log\left(\frac{y_i}{m_i}\right) - y_i \log\left(\frac{m_i - y_i}{m_i}\right) + m \log\left(\frac{m_i - y_i}{m_i}\right) + \log \binom{m_i}{y_i} \right]$$

For any other model

let $\hat{y}_i = m_i \hat{\pi}_i$

$$l(b;y) = \sum \left[y_i \log\left(\frac{\hat{y}_i}{m}\right) - y_i \log\left(\frac{m_i - \hat{y}_i}{m}\right) + m_i \log\left(\frac{m_i - \hat{y}_i}{m}\right) + \log\binom{m_i}{y_i} \right]$$

$\therefore$

$$D = 2\left[l(b_{max}; y) - l(b; y) \right]$$

$$= 2 \sum_{i=1}^{n} \left[y_i \log\left(\frac{y_i}{\hat{y}_i}\right) + (m_i - y_i) \log\left(\frac{m_i - y_i}{m_i - \hat{y}_i}\right) \right]$$

Since $\hat{y}_i = m_i \hat{\pi}_i$

$$= 2 \sum_{i=1}^{n} \left[y_i \log\left(\frac{y_i}{m_i \hat{\pi}_i}\right) + (m_i - y_i) \log\left(\frac{m_i - y_i}{m_i - m\hat{\pi}_i}\right) \right]$$

where $\hat{\pi}_i = \dfrac{e^{x_i \hat{\beta}}}{1 + e^{x_i \hat{\beta}}}$

deviance residual

$$d_i = \text{sign}(y_i - m_i\hat{\pi}_i)\sqrt{2\left(y_i \log\left(\frac{y_i}{m\hat{\pi}_i}\right) + (m - y_i)\log\left(1 - \frac{y_i}{m_i}\right)/(1-\hat{\pi}_i)\right)}$$

where $\hat{\pi}_i = \dfrac{e^{x_i\beta}}{1+e^{x_i\beta}}$.

Pearson's statistic :

$$G = \sum_{i=1}^{n} (y_i - \hat{\mu}_i)^2 / V(\hat{\mu}_i)).$$

$$= \sum_{i=1}^{n} \left(\frac{y_i - m_i\hat{\pi}_i}{\sqrt{m_i\hat{\pi}_i(1-\hat{\pi}_i)}}\right)$$