CSE 250B: Assignment #5

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1 Singular values versus eigenvalues

1.1 $Mv_i = U\Lambda V^T v_i = \sigma_i u_i$

1.2 $M^T u_i = V\Lambda U^T u_i = \sigma_i v_i$

1.3 $M^T M v_i = M^T \sigma_i u_i = \sigma_i M^T u_i = \sigma_i^2 v_i$, $MM^T u_i = M\sigma_i v_i = \sigma_i M v_i = \sigma_i^2 u_i$

1.4 EVs

From result of prev question for matrix MM^T,

$$MM^T u_i = \sigma_i^2 u_i$$

Therefore Eigen values of MM^T are σ_i^2 for $i \in [1, p]$ and its eigen vectors are vectors u_i for $i \in [1, p]$

1.5 EV relations in $MM^T$and $M^T M$

Similar to previous question, $M^T M v_i = \sigma_i^2 v_i$. The eigen values of MM^T and $M^T M$ are same (i.e. σ_i^2) and eigen vectors v_i.

1.6 Rank-k

If matrix M has rank-k,

$$\sigma_i = \begin{cases} \neq 0 & i \leq k \\ = 0 & i > k \end{cases}$$

2 Rank-1 matrices

2.1 Rank-1 Approximation

$\hat{M} = \sigma_1 u_1 v_1^T = \begin{bmatrix} 1.57 & 2.08 & 2.59 \\ 3.76 & 4.97 & 6.17 \end{bmatrix}$

2.2 Decomposition is not unique

$$M = uv^T$$

We can always scale down $u$by a constant c and scale up v by the same constant c to get the same matrix M. For eg. let

$$a = (1/c)u, b = (c)v$$
$$\implies M = uv^T = \frac{1}{c}.uv^T.c = ab^T$$

We can see that for different values of c, we get different values for vectors a and b which are just rescaled versions of original vectors $u$and v.

2.3 Sum of rank-1 matrices

Assuming $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_p$, the best rank-$k$ approximation will be given by k highest singular values and their corresponding vectors $u_i, v_i \mid i \in [1, k]$. Therefore,

$$M(rank - k) = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + ... + \sigma_k u_k v_k^T$$

3 The Gram matrix

3.1 Gram Matrix:

$$GramMatrix = B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 3 \end{pmatrix}$$

3.2 Uniqueness

Let Z be the matrix of points given i.e.

$$Z = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Then gram matrix is obtained by $B = Z^T Z$. Consider any orthogonal basis $O \in R^{3\times3}$. Therefore, $OO^T = I$.

$$B = Z^T Z = Z^T I Z = Z^T O O^T Z$$

Or in other words:

$$B = W^T W \qquad \text{(where } W = O^T Z\text{)}$$

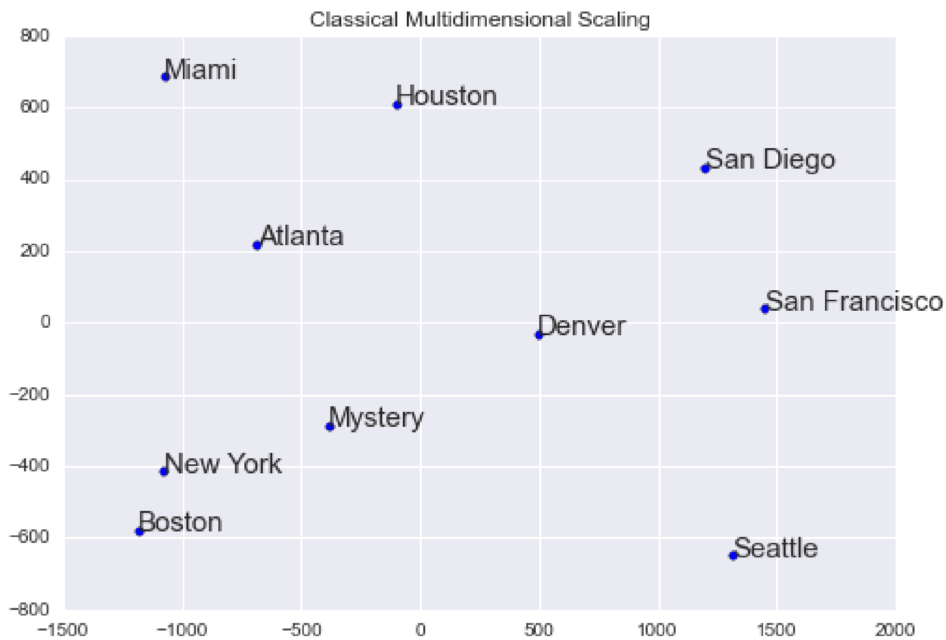
Here W represents any orthogonal tranformation of Z in any orthogonal basis O. Thus B can be achieved by any orthogonal transformation W of the original points Z and hence the points which give B are not unique.

4 Classical multidimensional scaling

4.1 Original Graph

This graph looks incorrect as the map seems to be rotated by 180 degrees. (North Down). The reason for this is SVD gives two orthogonal sets of directions ($U$and$V$) whereas the valid set of directions can be both (U and V) as well as ($-U$ and $-V$) resulting in same decomposition of matrix. Plotting the originally obtained graph below:

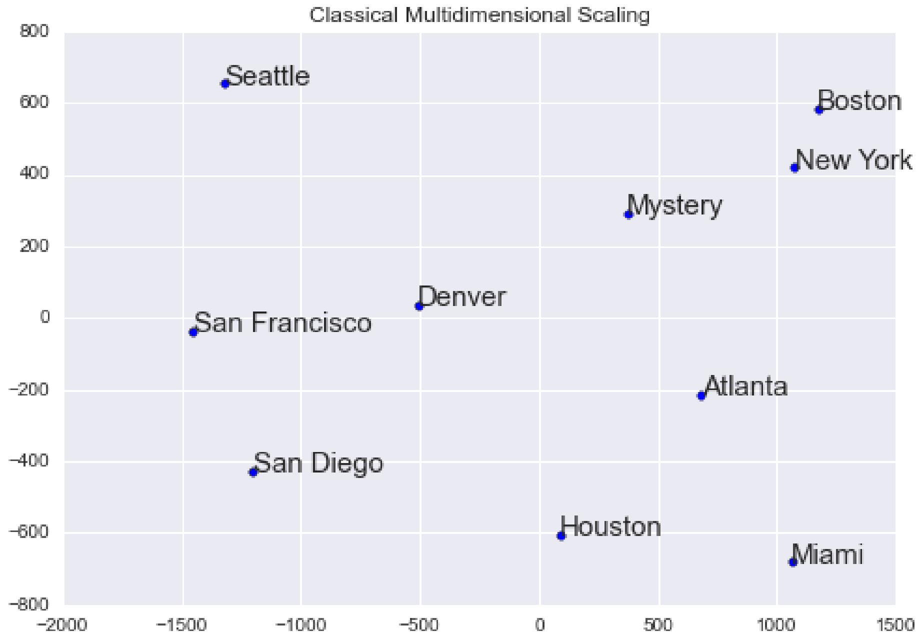
Figure 1: Original Graph



4.2 Correction

This looked like the graph was inverted. In other words, the directions $U$and V have been negated. After negating back all the data points the new graph looks like the following, which seems reasonably correct.

Figure 2: After Correction



4.3 Mystery City is Chicago