

Braids and the Jones polynomial

Thesis presentation

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IISERB

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Outline

Braids

- Geometric definition

- Generators and relations

- Algebraic definition

Closure

- Closure of a braid

- Equivalence of closures of braids

Orientation

Section 1

Outline

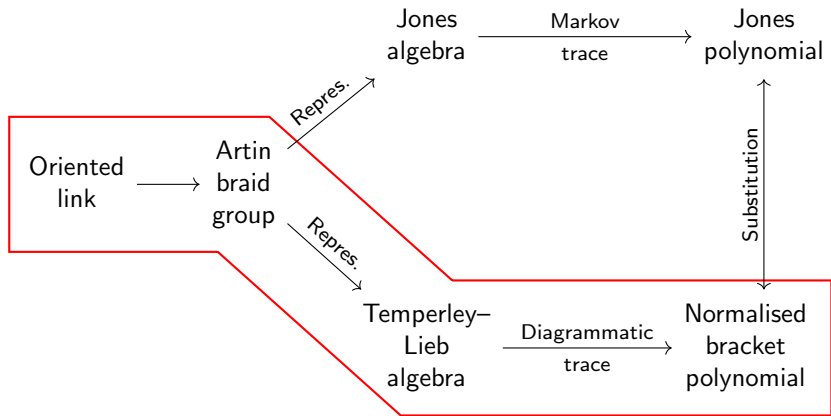


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Section 2

Braids

Subsection 1

Geometric definition

Three dimensional representation

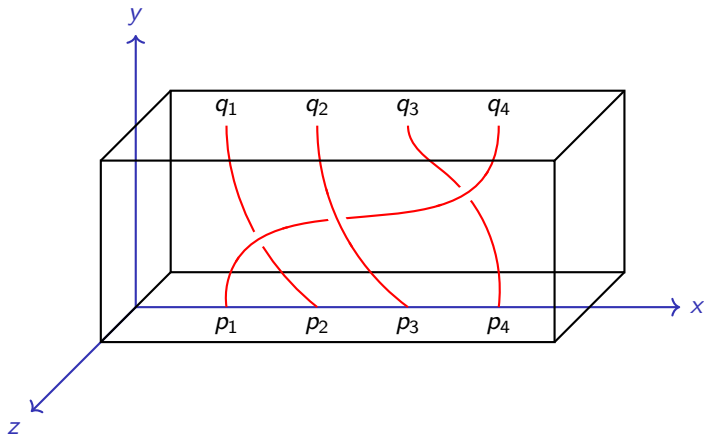


Figure: Three dimensional geometric representation of a braid

Two dimensional representation

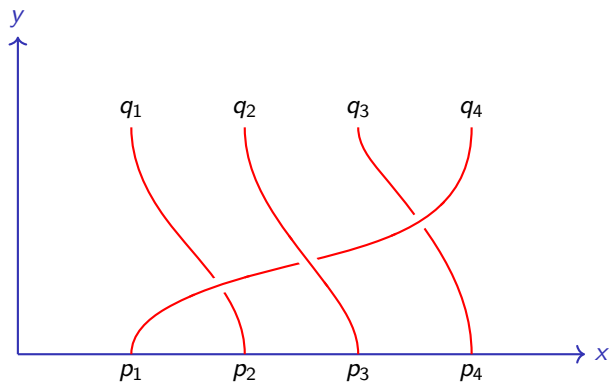


Figure: A projection of the braid

Multiplication of braids

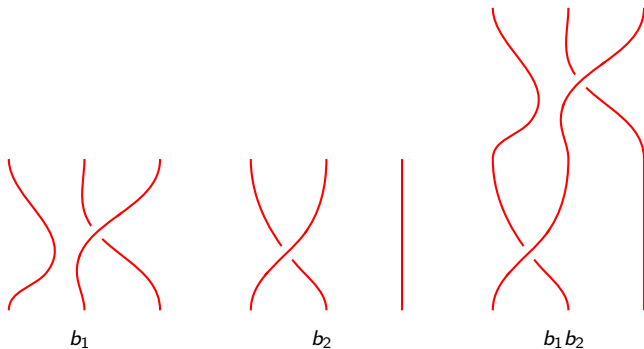


Figure: Multiplication of two braids

The identity braid \mathbb{I}_n

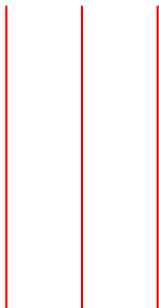


Figure: The identity \mathbb{I}_3

Inverse of braids

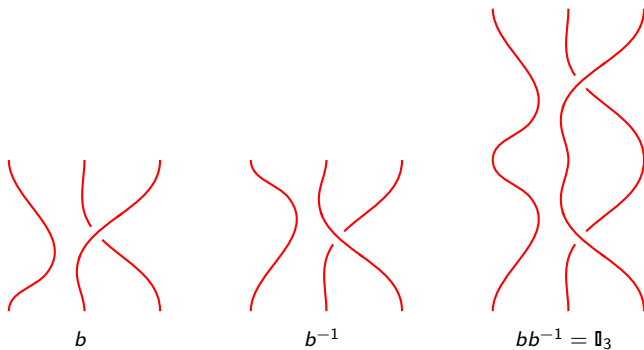


Figure: Inverse of a braid

Thus, braids form a group, known as the Artin braid group B_n .

Subsection 2

Generators and relations

Generators of the braid group

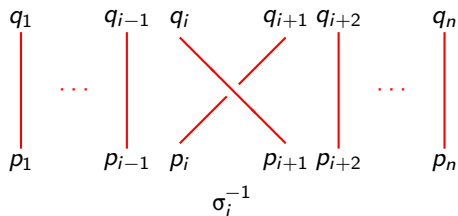
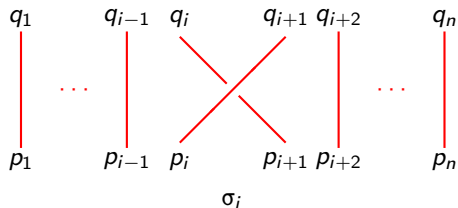


Figure: Generators σ_i and σ_i^{-1}

Type II move: $\sigma_i \sigma_i^{-1} = \mathbb{I}_n$

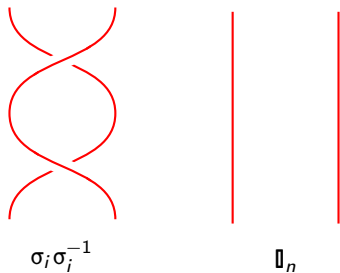


Figure: A type II move illustrating $\sigma_i \sigma_i^{-1} = \mathbb{I}_n$

Type III move: $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

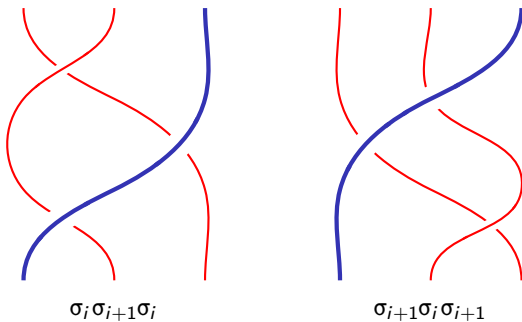


Figure: A type III move illustrating $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

Sliding of crossings: $\sigma_i \sigma_j = \sigma_j \sigma_i$

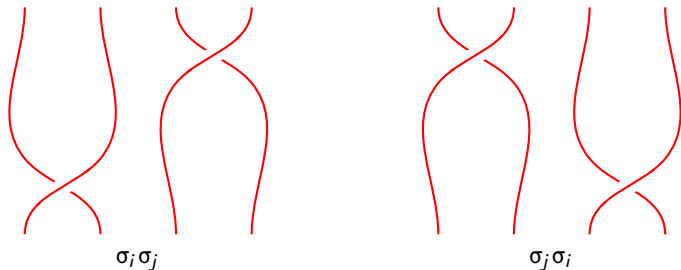


Figure: Sliding of crossings illustrating $\sigma_i \sigma_j = \sigma_j \sigma_i$

Subsection 3

Algebraic definition

Presentation of the braid group

The Artin braid group B_n admits the following presentation on the generators σ_i , for $1 \leq i \leq n-1$.

$$B_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \left| \begin{array}{ll} \sigma_i \sigma_i^{-1} &= \mathbb{I}_n \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{if } i+1 \leq n-1 \\ \sigma_i \sigma_j &= \sigma_j \sigma_i \quad \text{if } |i-j| \geq 2 \end{array} \right. \right\rangle$$

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Closure

Subsection 1

Closure of a braid

Closure of a braid \bar{b}

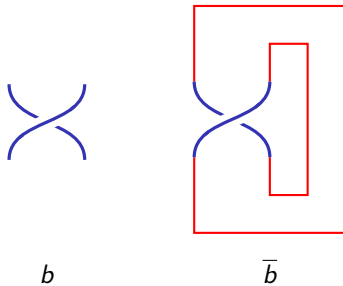


Figure: Closure of a braid

Braids and links

Every closure of a braid is a link.

Theorem (Alexander)

Every link is ambient isotopic to a closure of a braid.

Subsection 2

Equivalence of closures of braids

Conjugation

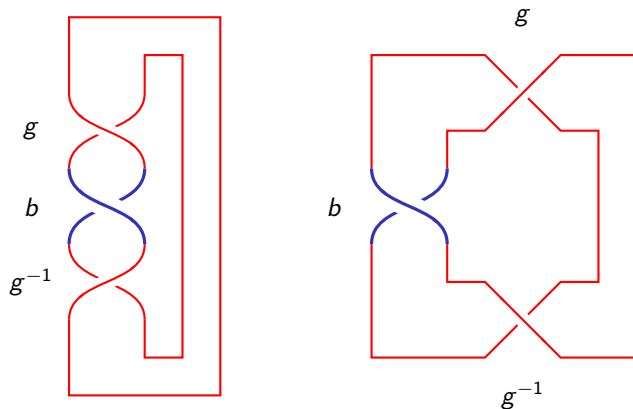


Figure: Conjugation process illustrating the link equivalence of $\overline{gbg^{-1}}$ and \overline{b} (part 1)

Conjugation (contd.)

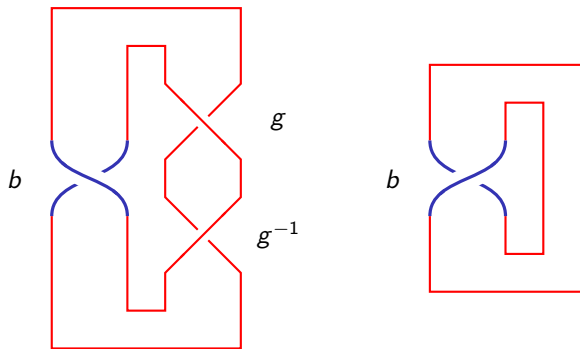


Figure: Conjugation process illustrating the link equivalence of $\overline{gbg^{-1}}$ and \overline{b} (part 2)

Markov theorem

Theorem (Markov)

Two braids whose closures are ambient isotopic to each other are related by a finite sequence of the following operations.

1. Braid equivalences, i.e. equivalences resulting due to the braid relations.
2. Conjugation.
3. Markov moves.

Markov move

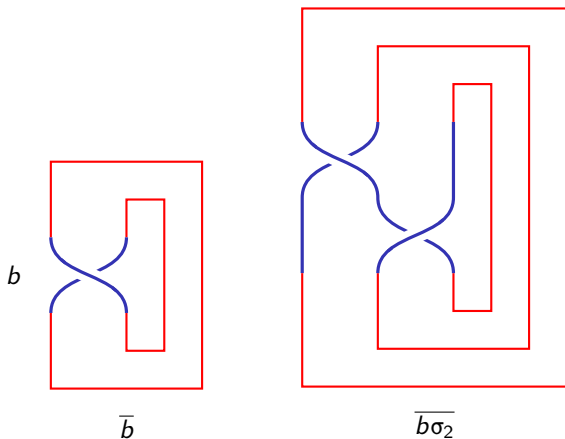


Figure: Markov move with $b = \sigma_1^{-1}$.

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Orientation

Orientation

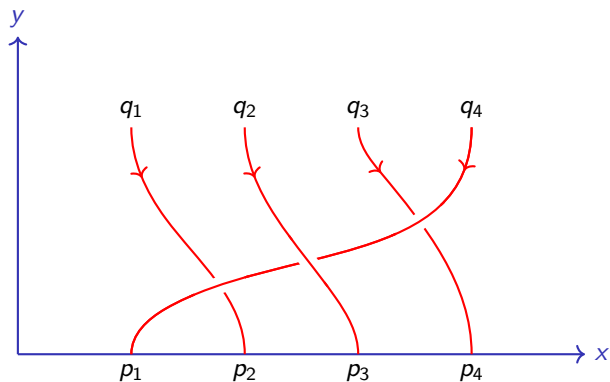


Figure: A projection of the braid with downward orientation

For the consistency of the orientation, it must be either upwards or downwards for all strands.



σ_i gets +1 assigned



σ_i^{-1} gets -1 assigned

Writhe is the sum of the assigned numbers.

Bracket polynomial of a braid:

$$\langle \cdot \rangle : B_n \rightarrow \mathbb{Z}[A, A^{-1}]$$

$$\langle \cdot \rangle : b \mapsto \langle \bar{b} \rangle$$

$\langle \cdot \rangle$ is well defined and invariant under conjugation.

Normalisation using writhe:

$$L(K) := (-A^3)^{-w(b)} \langle \bar{b} \rangle$$

$$\langle |\cdots| \times |\cdots| \rangle = A \langle |\cdots| \bowtie |\cdots| \rangle + A^{-1} \langle |\cdots| \rangle \langle |\cdots| \rangle$$

$$|\cdots\rangle \langle \cdots| = \mathbb{I}_n \quad \text{and} \quad U_i := |\cdots| \bowtie |\cdots|$$

$$\langle \sigma_i^{-1} \rangle = A \langle U_i \rangle + A^{-1} \langle \mathbb{I}_n \rangle$$

$$\langle \sigma_i \rangle = A \langle \mathbb{I}_n \rangle + A^{-1} \langle U_i \rangle$$

We refer to U_i 's as “hooks” or “input-output” forms.

They don't belong to the Artin braid group.

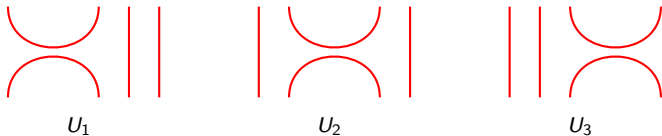
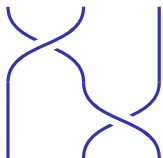
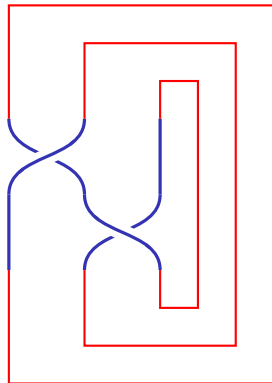


Figure: Input-output forms or hooks for 4 strands

Example

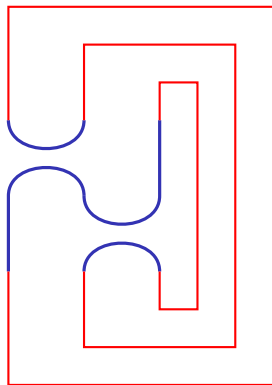


b

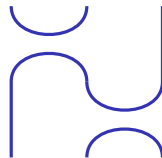


\bar{b}

Example (contd.)



$$s = \overline{U_1 U_2}$$



$$U_1 U_2$$

Figure: Writing a state of a braid closure in terms of input-output forms.