Braids and the Jones polynomial

Thesis presentation

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IISERB

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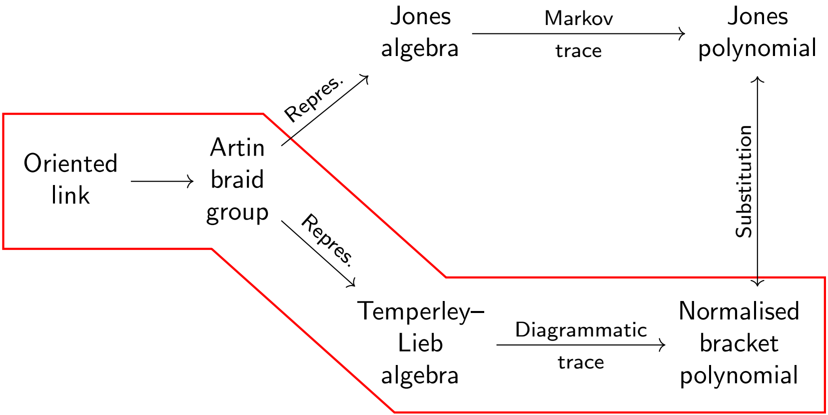
Outline



Jones algebra
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Repres.
Substitution
Oriented link
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Normalised bracket polynomial

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Braids

Subsection 1

Geometric definition

Three dimensional representation

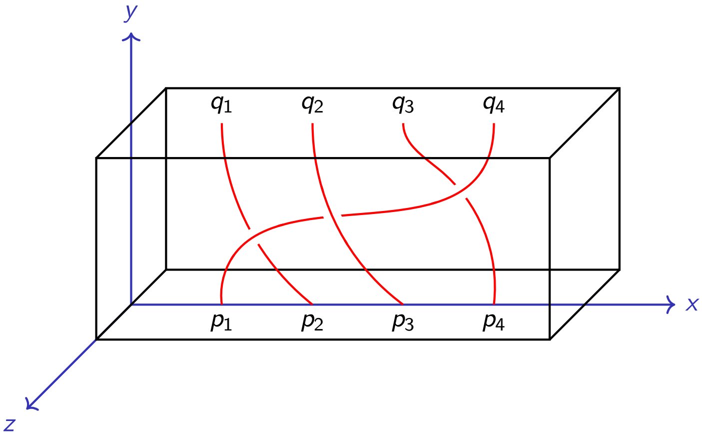


Figure: Three dimensional geometric representation of a braid

Two dimensional representation

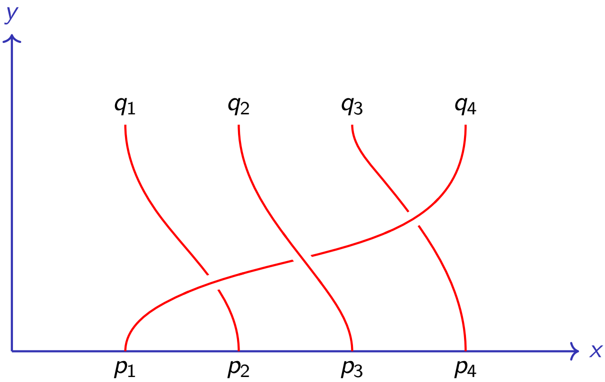


Figure: A projection of the braid

Multiplication of braids

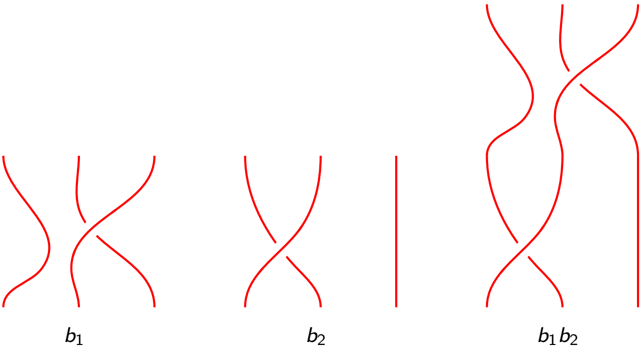


Figure: Multiplication of two braids

The identity braid $\mathbb{1}_n$

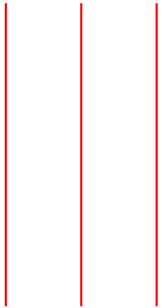
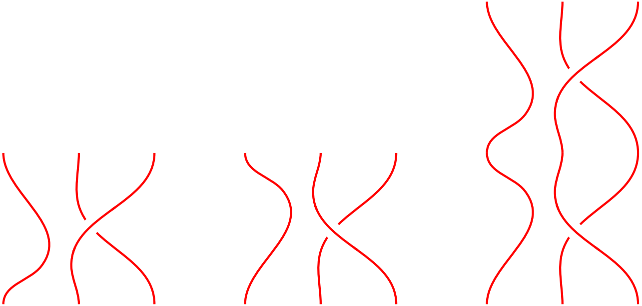


Figure: The identity $\mathbb{1}_3$

Inverse of braids



b b^{-1} $bb^{-1} = \mathbb{1}_3$

Figure: Inverse of a braid

Thus, braids form a group, known as the Artin braid group B_n.

Subsection 2

Generators and relations

Generators of the braid group

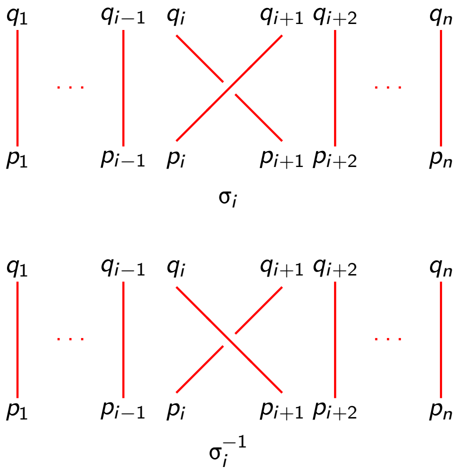


Figure: Generators σ_i and σ_i^{-1}

Type II move: $\sigma_i \sigma_i^{-1} = \mathbb{1}_n$

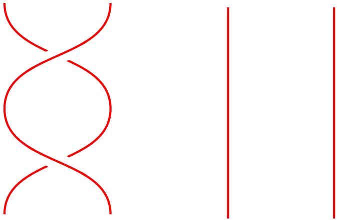
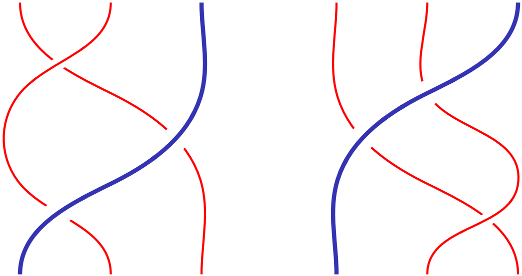


Figure: A type II move illustrating $\sigma_i \sigma_i^{-1} = \mathbb{1}_n$

Type III move: $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$



$\sigma_i \sigma_{i+1} \sigma_i$ $\sigma_{i+1} \sigma_i \sigma_{i+1}$

Figure: A type III move illustrating $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

Sliding of crossings: $\sigma_i \sigma_j = \sigma_j \sigma_i$

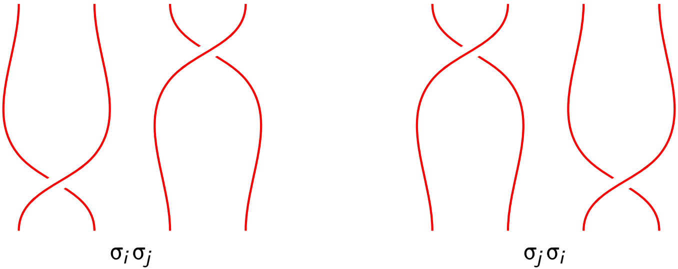


Figure: Sliding of crossings illustrating $\sigma_i \sigma_j = \sigma_j \sigma_i$

Subsection 3

Algebraic definition

Presentation of the braid group

The Artin braid group B_n admits the following presentation on the generators σ_i, for $1 \leq i \leq n-1$.

$$
B_n = \left\langle \; \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{rcll} \sigma_i \sigma_i^{-1} & = & \mathbb{1}_n & \\ \sigma_i \sigma_{i+1} \sigma_i & = & \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{if } i+1 \leq n-1 \\ \sigma_i \sigma_j & = & \sigma_j \sigma_i & \text{if } |i-j| \geq 2 \end{array} \; \right\rangle
$$

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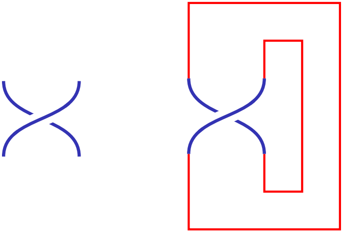
Section 3

Closure

Subsection 1

Closure of a braid

Closure of a braid $\overline{b}$



b $\qquad$ $\overline{b}$

Figure: Closure of a braid

Braids and links

Every closure of a braid is a link.

Theorem (Alexander)

Every link is ambient isotopic to a closure of a braid.

Subsection 2

Equivalence of closures of braids

Conjugation

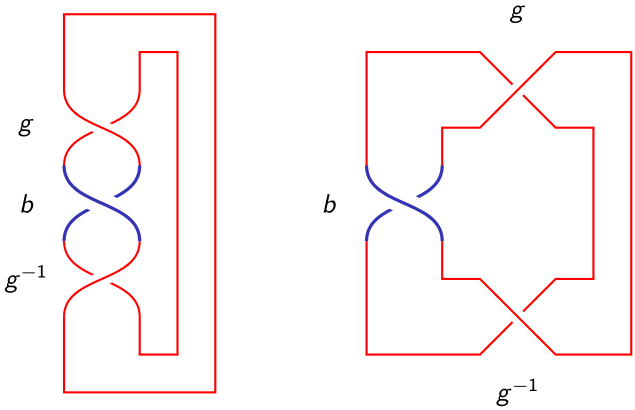
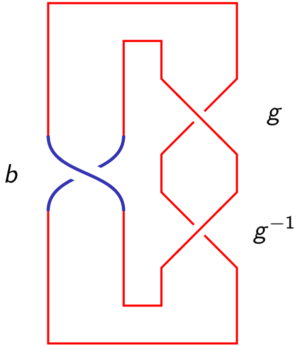


Figure: Conjugation process illustrating the link equivalence of $\overline{gbg^{-1}}$ and $\overline{b}$ (part 1)

Conjugation (contd.)



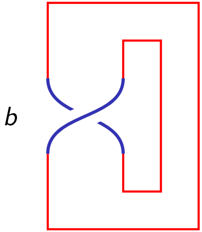


Figure: Conjugation process illustrating the link equivalence of $\overline{gbg^{-1}}$ and $\overline{b}$ (part 2)

Markov theorem

Theorem (Markov)

Two braids whose closures are ambient isotopic to each other are related by a finite sequence of the following operations.

1. Braid equivalences, i.e. equivalences resulting due to the braid relations.
2. Conjugation.
3. Markov moves.

Markov move

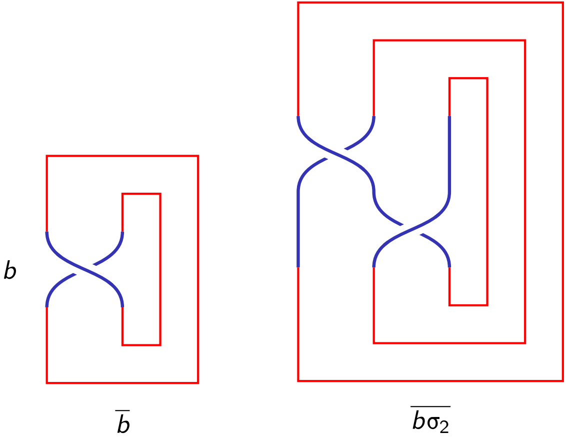


Figure: Markov move with $b = \sigma_1^{-1}$.

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Section 4

Orientation

Orientation

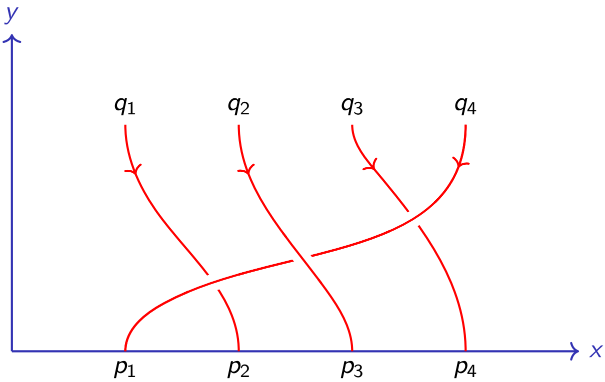
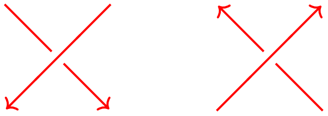
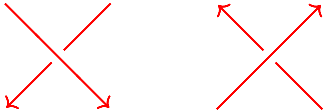


Figure: A projection of the braid with downward orientation

For the consistency of the orientation, it must be either upwards or downwards for all strands.



σ_i gets $+1$ assigned



σ_i^{-1} gets -1 assigned

Writhe is the sum of the assigned numbers.

Bracket polynomial of a braid:

$$\langle\cdot\rangle \colon \mathrm{B}_n \to \mathbb{Z}[A, A^{-1}]$$
$$\langle\cdot\rangle \colon b \mapsto \langle\overline{b}\rangle$$

$\langle\cdot\rangle$ is well defined and invariant under conjugation.

Normalisation using writhe:

$$L(K) \coloneqq (-A^3)^{-w(b)}\langle\overline{b}\rangle$$

 $= \mathbb{1}_n$ and $\mathsf{U}_i :=$ 

$$\langle \sigma_i^{-1} \rangle = A\langle \mathsf{U}_i \rangle + A^{-1}\langle \mathbb{1}_n \rangle$$

$$\langle \sigma_i \rangle = A\langle \mathbb{1}_n \rangle + A^{-1}\langle \mathsf{U}_i \rangle$$

We refer to U_i's as "hooks" or "input-output" forms.

They don't belong to the Artin braid group.

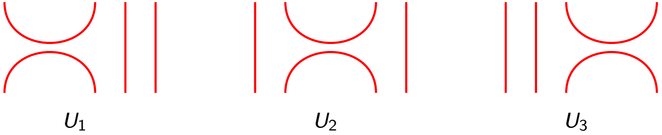
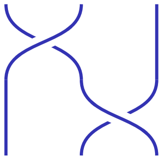
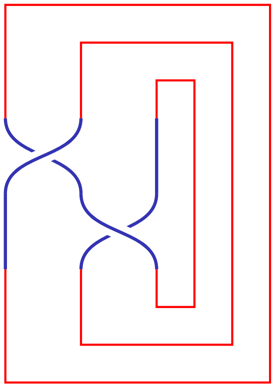


Figure: Input-output forms or hooks for 4 strands

Example

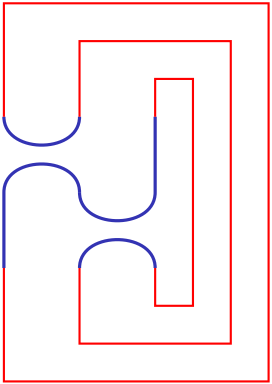


b

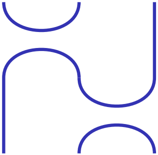


$\overline{b}$

Example (contd.)



$s = \overline{U_1 U_2}$



$U_1 U_2$

Figure: Writing a state of a braid closure in terms of input-output forms

$$\langle b \rangle = \langle S(b) \rangle = \sum_s \langle b|s \rangle \langle P_s \rangle = \sum_s \langle b|s \rangle \delta^{\|s\|}$$

$\langle S(b) \rangle$: Substituting $\langle \sigma_i \rangle = A \langle \mathbb{1}_n \rangle + A^{-1} \langle \mathrm{U}_i \rangle$ and $\left\langle \sigma_i^{-1} \right\rangle = A \langle \mathrm{U}_i \rangle + A^{-1} \langle \mathbb{1}_n \rangle$

s : A state in the expansion

P_s : Product of U_i's

$\langle b|s \rangle$: Product of A's and A^{-1}'s

δ : $-A^2 - A^{-2}$

$\|s\|$: Number of loops in s minus one