Braids and the bracket polynomial

Thesis presentation

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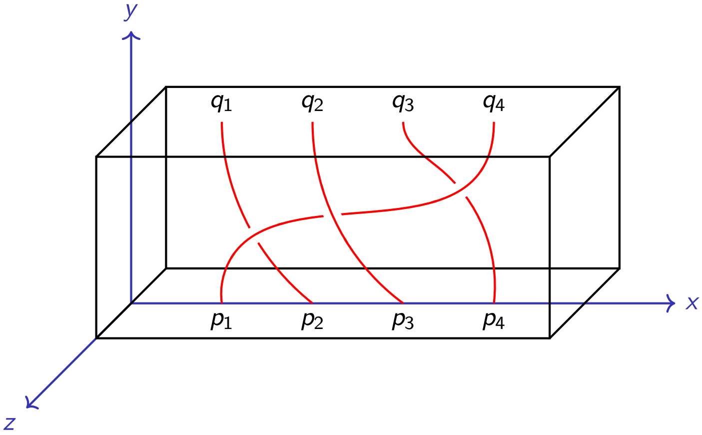
Outline

$$
\begin{tikzcd}
 & & \text{Jones algebra} \arrow[r, "\text{Markov}", "\text{trace}"'] & \text{Jones polynomial} \arrow[dd, leftrightarrow, "\text{Substitution}"'] \\
\text{Oriented link} \arrow[r] & \text{Artin braid group} \arrow[ur, "\text{Repres.}"] \arrow[dr, "\text{Repres.}"] & & \\
 & & \text{Temperley–Lieb algebra} \arrow[r, "\text{Diagrammatic}", "\text{trace}"'] & \text{Normalised bracket polynomial}
\end{tikzcd}
$$

Three dimensional representation



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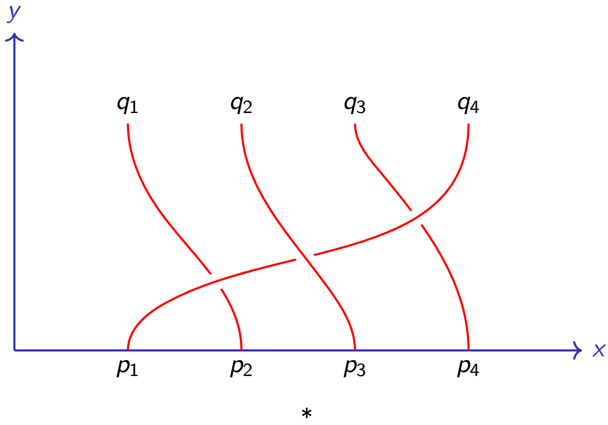
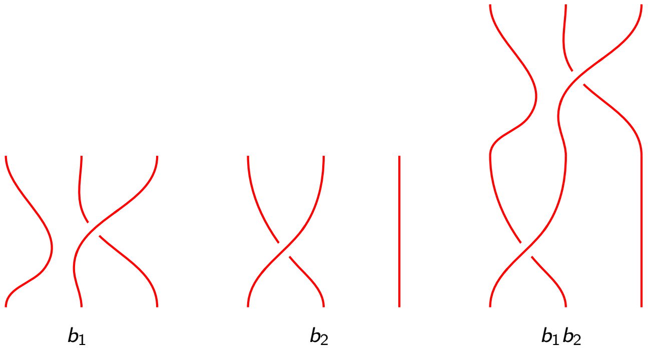
Three dimensional geometric representation of a braid

Two dimensional representation



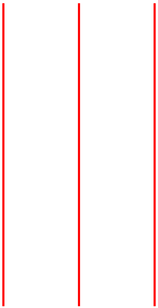
A projection of the braid

Multiplication of braids



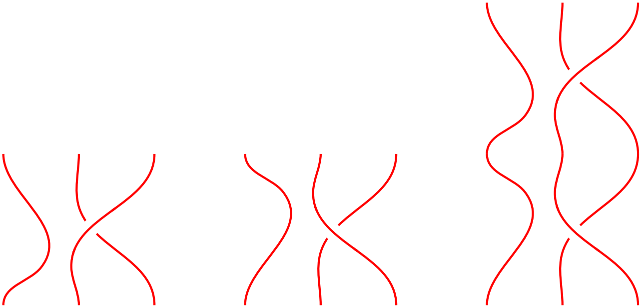
Multiplication of two braids

The identity braid $\square_n$



The identity $\square_3$

Inverse of braids

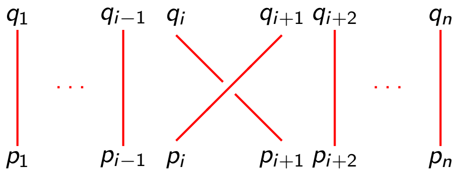


b $\quad$ b^{-1} $\quad$ $bb^{-1} = \mathbb{1}_3$

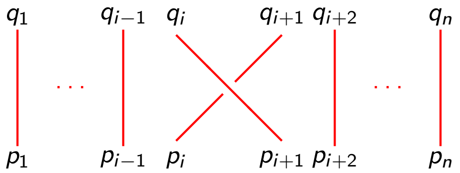
Inverse of a braid

Thus, braids form a group, known as the Artin braid group B_n.

Generators of the braid group



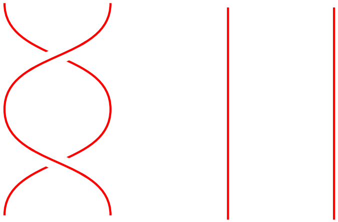
σ_i



σ_i^{-1}

Generators σ_i and σ_i^{-1}

Type II move: $\sigma_i \sigma_i^{-1} = ▯_n$

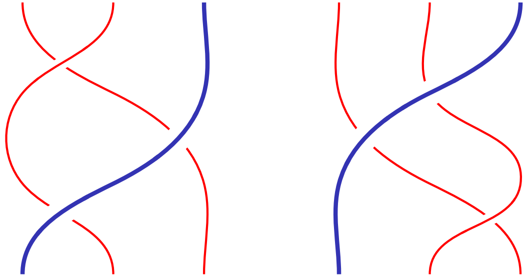


$\sigma_i \sigma_i^{-1}$ $▯_n$

o

ffigureA type II move illustrating $\sigma_i \sigma_i^{-1} = ▯_n$

Type III move: $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

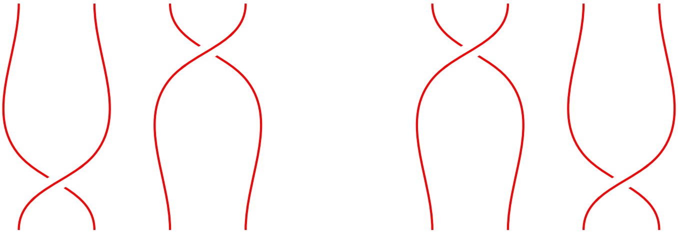


$\sigma_i \sigma_{i+1} \sigma_i$ $\sigma_{i+1} \sigma_i \sigma_{i+1}$

o

ffigureA type III move illustrating $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

Sliding of crossings: $\sigma_i \sigma_j = \sigma_j \sigma_i$



$\sigma_i \sigma_j$ $\sigma_j \sigma_i$

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ffigureSliding of crossings illustrating $\sigma_i \sigma_j = \sigma_j \sigma_i$

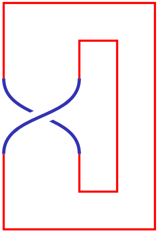
Presentation of the braid group

The Artin braid group B_n admits the following presentation on the generators σ_i, for $1 \leq i \leq n-1$.

$$\mathrm{B}_n = \left\langle \ \sigma_1, \ldots, \sigma_{n-1} \ \middle| \ \begin{array}{rcll} \sigma_i \sigma_i^{-1} & = & \mathbb{1}_n & \\ \sigma_i \sigma_{i+1} \sigma_i & = & \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{if } i+1 \leq n-1 \\ \sigma_i \sigma_j & = & \sigma_j \sigma_i & \text{if } |i-j| \geq 2 \end{array} \ \right\rangle$$

Closure of a braid $\overline{b}$

b



$\overline{b}$

Closure of a braid

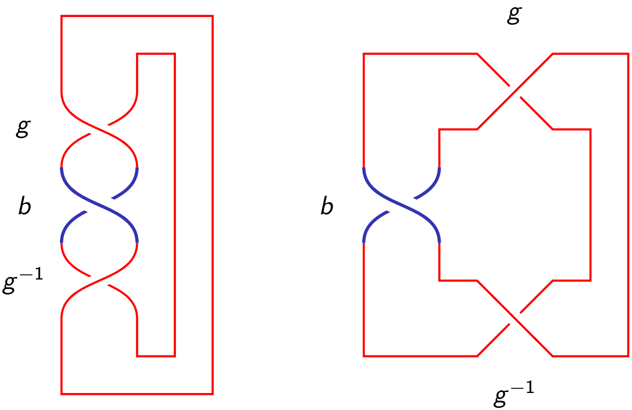
Braids and links

Every closure of a braid is a link.

Theorem (Alexander)

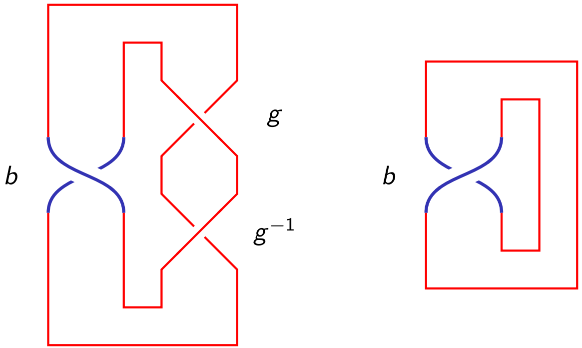
Every link is ambient isotopic to a closure of a braid.

Conjugation



Conjugation process illustrating the link equivalence of $\overline{gbg^{-1}}$ and $\overline{b}$ (part 1)

Conjugation (contd.)



Conjugation process illustrating the link equivalence of $\overline{gbg^{-1}}$ and $\overline{b}$ (part 2)

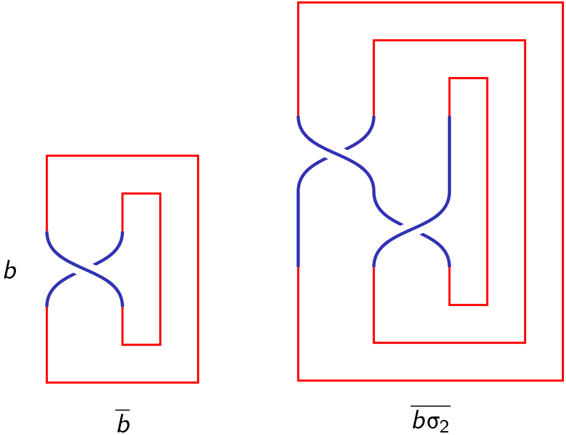
Markov theorem

Theorem (Markov)

Two braids whose closures are ambient isotopic to each other are related by a finite sequence of the following operations.

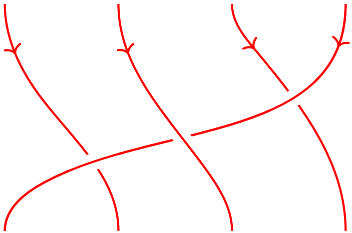
1. Braid equivalences, i.e. equivalences resulting due to the braid relations.
2. Conjugation.
3. Markov moves.

Markov move



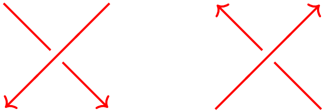
Markov move with $b = \sigma_1^{-1}$.

Orientation



A braid with downward orientation

For the consistency of the orientation, it must be either upwards or downwards for all strands.



σ_i gets assigned $+1$

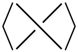
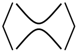
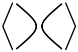
σ_i^{-1} gets assigned -1

Writhe is the sum of the assigned numbers.

Kauffman's bracket polynomial $\langle K \rangle$

Definition (Kauffman's bracket polynomial)

Let K an un-oriented link diagram. Then the bracket $\langle K \rangle \in \mathbb{Z}[A, A^{-1}]$ is defined by the rules:

1. $\left\langle \bigcirc \right\rangle = 1.$
2. $\left\langle \bigcirc \cup K \right\rangle = (-A^2 - A^{-2})\langle K \rangle.$
3.  $= A$  $+ A^{-1}$ .

$\langle K \rangle$ is invariant under the type II and type III moves.

Normalised bracket polynomial $L(K)$

We can normalise $\langle K \rangle$ by multiplying it with $(-A^3)^{-w(K)}$ gain type I move invariance.

$$L(K) := (-A^3)^{-w(K)} \langle K \rangle$$

Bracket polynomial of a braid

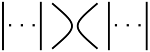
Bracket polynomial of a braid:

$$\langle\cdot\rangle\colon \mathrm{B}_n \to \mathbb{Z}[A, A^{-1}]$$
$$\langle\cdot\rangle\colon b \mapsto \langle\overline{b}\rangle$$

$\langle\cdot\rangle$ is well defined and invariant under conjugation.

Normalisation using writhe:

$$L(K) \coloneqq (-A^3)^{-w(b)}\langle\overline{b}\rangle$$

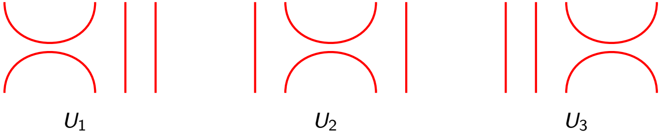
 $= \mathbb{1}_n$ and $\mathsf{U}_i :=$ 

$$\langle \sigma_i^{-1} \rangle = A\langle \mathsf{U}_i \rangle + A^{-1}\langle \mathbb{1}_n \rangle$$

$$\langle \sigma_i \rangle = A\langle \mathbb{1}_n \rangle + A^{-1}\langle \mathsf{U}_i \rangle$$

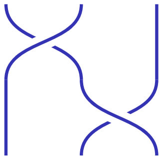
We refer to U_i's as "hooks" or "input-output" forms.

They don't belong to the Artin braid group.

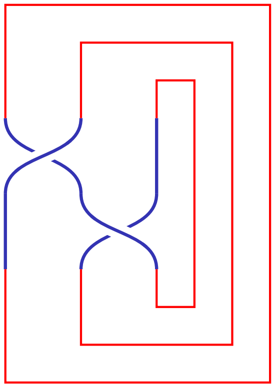


Input-output forms or hooks for 4 strands

Example

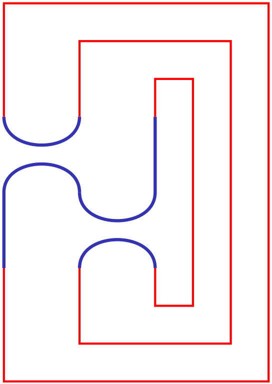


b

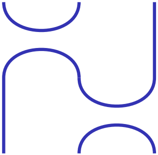


$\overline{b}$

Example (contd.)



$s = \overline{U_1 U_2}$



$U_1 U_2$

Writing a state of a braid closure in terms of input-output forms

$$\langle b \rangle = \langle S(b) \rangle = \sum_s \langle b|s \rangle \langle P_s \rangle = \sum_s \langle b|s \rangle \delta^{\|s\|}$$

$\langle S(b) \rangle$	:	Substituting $\langle \sigma_i \rangle = A \langle \mathbb{1}_n \rangle + A^{-1} \langle \mathrm{U}_i \rangle$ and $\left\langle \sigma_i^{-1} \right\rangle = A \langle \mathrm{U}_i \rangle + A^{-1} \langle \mathbb{1}_n \rangle$
s	:	A state in the expansion
P_s	:	Product of U_i's
$\langle b\|s \rangle$	:	Product of A's and A^{-1}'s
δ	:	$-A^2 - A^{-2}$
$\|s\|$	:	Number of loops in s minus one

Temperley–Lieb algebra TL_n

We give U_i's a structure of their own by constructing

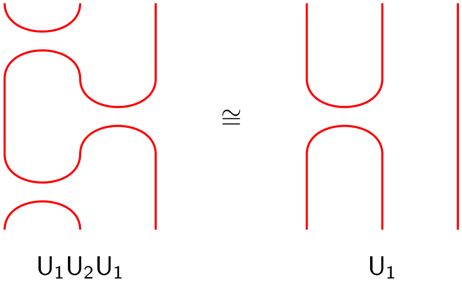
- over the ring $\mathbb{Z}[A, A^{-1}]$
- the free additive algebra TL_n
- with the generators $\mathrm{U}_1, \mathrm{U}_2, \ldots, \mathrm{U}_{n-1}$
- and the multiplicative relations coming from the interpretation of U_i's as input-output forms.

Multiplicative relations in TL_n

Multiplicative relations in TL_n:

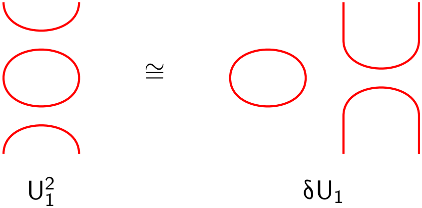
1. $\mathsf{U}_i \mathsf{U}_{i\pm 1} \mathsf{U}_i = \mathsf{U}_i$
2. $\mathsf{U}_i^2 = \delta \mathsf{U}_i$
3. $\mathsf{U}_i \mathsf{U}_j = \mathsf{U}_j \mathsf{U}_i$ if $|i-j| \geq 2$

Geometric interpretation of $U_1U_2U_1 = U_1$



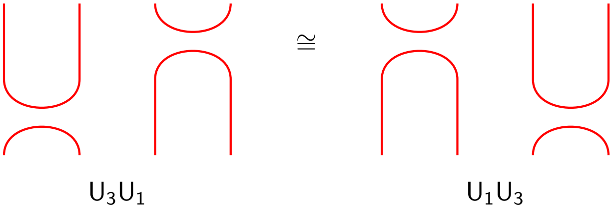
Geometric interpretation of $U_1U_2U_1 = U_1$

Geometric interpretation of $U_i^2 = \delta U_i$



Geometric interpretation of $U_i^2 = \delta U_i$

Geometric interpretation of $U_3 U_1 = U_1 U_3$



Geometric interpretation of $U_3 U_1 = U_1 U_3$

Representation of B_n in TL_n

We define a mapping

$$\rho\colon \mathsf{B}_n \to \mathsf{TL}_n$$

by

$$\begin{aligned} \rho(\sigma_i) &= A + A^{-1}\mathsf{U}_i \\ \rho(\sigma_i^{-1}) &= A^{-1} + A\mathsf{U}_i \end{aligned}$$

$\rho\colon \mathsf{B}_n \to \mathsf{TL}_n$ is a representation of the Artin braid group.

Trace

We define the diagrammatic trace

$$\mathrm{tr}\colon \mathrm{TL}_n \to \mathbb{Z}[A, A^{-1}]$$

by extending linearly

$$\mathrm{tr}(P) = \langle P \rangle.$$

This version of trace is diagrammatic in nature as we are counting loops in a state.

$$\langle b \rangle = \mathrm{tr}(\rho(b))$$

Whole procedure

So one can

- find a braid representation b of a link L by Alexander's theorem,
- calculate $\mathrm{tr}(\rho(b))$
- and normalise it to get the link invariant normalised bracket polynomial.

The substitution $A = t^{-1/4}$ yields us the Jones polynomial.