Braids and the bracket polynomial

Thesis presentation

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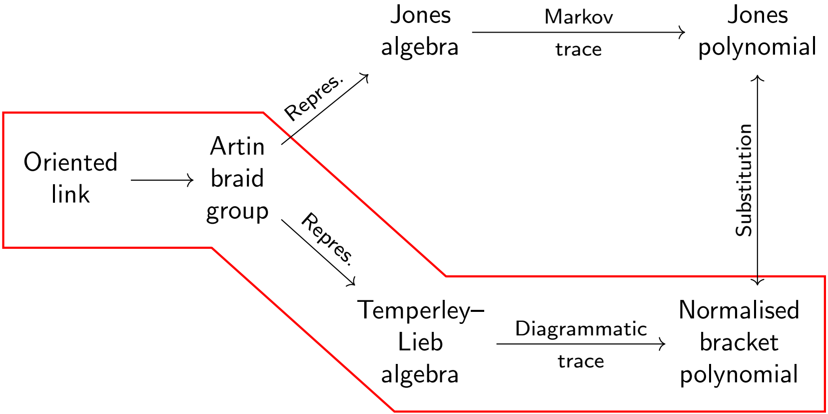
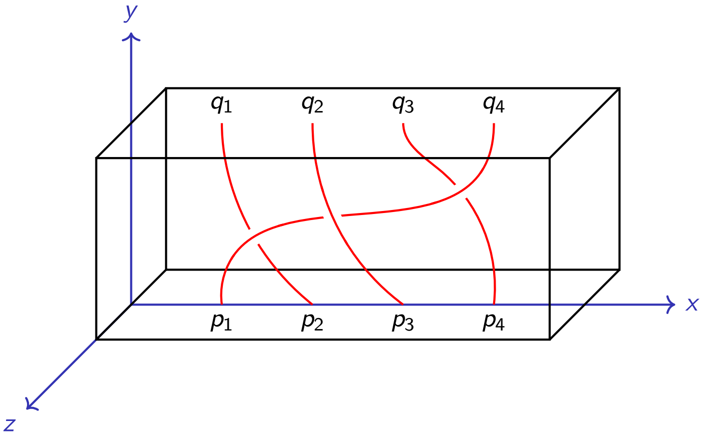
IISERB

April 17, 2023

Outline

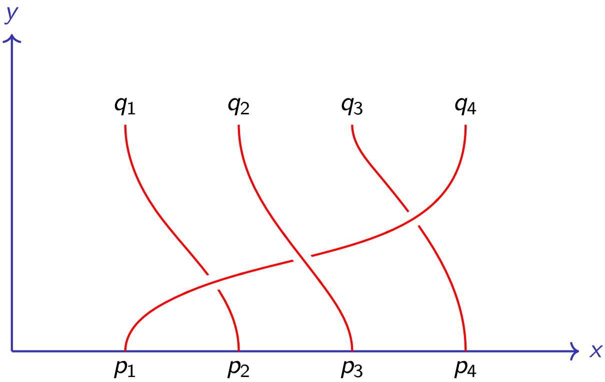

Jones algebra
Markov trace
Jones polynomial
Repres.
Substitution
Oriented link
Artin braid group
Repres.
Temperley–Lieb algebra
Diagrammatic trace
Normalised bracket polynomial

Three dimensional representation



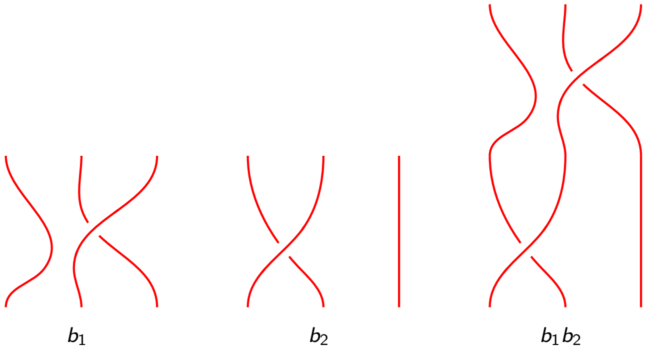
Three dimensional geometric representation of a braid

Two dimensional representation



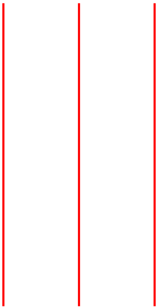
A projection of the braid

Multiplication of braids



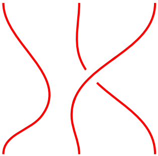
Multiplication of two braids

The identity braid $\square_n$

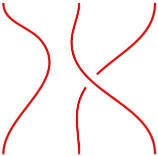


The identity $\square_3$

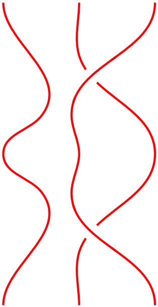
Inverse of braids



b



b^{-1}

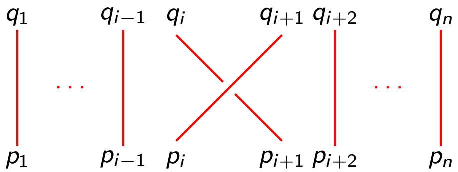


$bb^{-1} = \mathbb{1}_3$

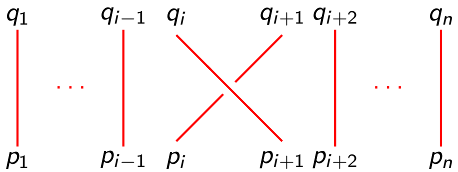
Inverse of a braid

Thus, braids form a group, known as the Artin braid group B_n.

Generators of the braid group



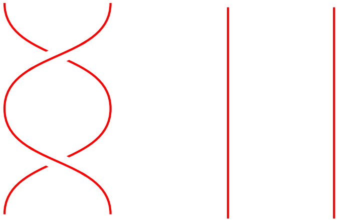
σ_i



σ_i^{-1}

Generators σ_i and σ_i^{-1}

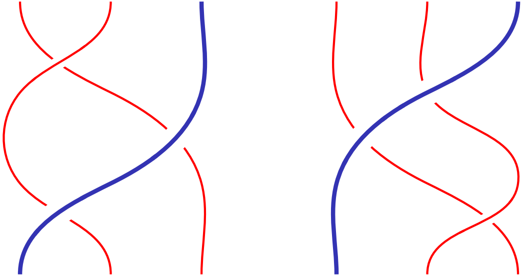
Type II move: $\sigma_i \sigma_i^{-1} = \mathbb{1}_n$



$\sigma_i \sigma_i^{-1}$ $\qquad\qquad$ $\mathbb{1}_n$

A type II move illustrating $\sigma_i \sigma_i^{-1} = \mathbb{1}_n$

Type III move: $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$



$\sigma_i\sigma_{i+1}\sigma_i$ $\sigma_{i+1}\sigma_i\sigma_{i+1}$

A type III move illustrating $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

Sliding of crossings: $\sigma_i\sigma_j = \sigma_j\sigma_i$

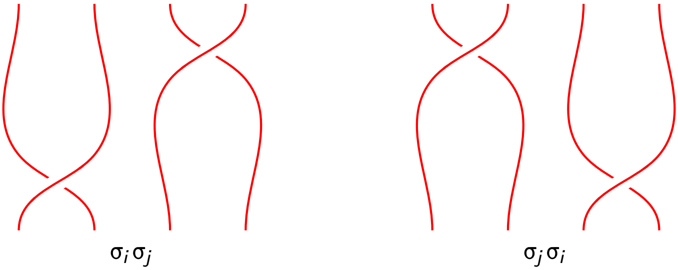


Figure: Sliding of crossings illustrating $\sigma_i\sigma_j = \sigma_j\sigma_i$

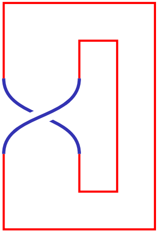
Presentation of the braid group

The Artin braid group B_n admits the following presentation on the generators σ_i, for $1 \leq i \leq n-1$.

$$B_n = \left\langle \ \sigma_1, \ldots, \sigma_{n-1} \ \middle| \ \begin{array}{rcll} \sigma_i \sigma_i^{-1} & = & \mathbb{1}_n & \\ \sigma_i \sigma_{i+1} \sigma_i & = & \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{if } i+1 \leq n-1 \\ \sigma_i \sigma_j & = & \sigma_j \sigma_i & \text{if } |i-j| \geq 2 \end{array} \ \right\rangle$$

Closure of a braid $\overline{b}$

b



$\overline{b}$

Closure of a braid

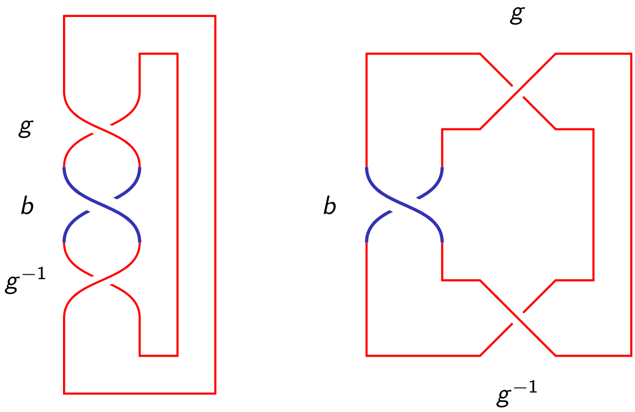
Braids and links

Every closure of a braid is a link.

Theorem (Alexander)

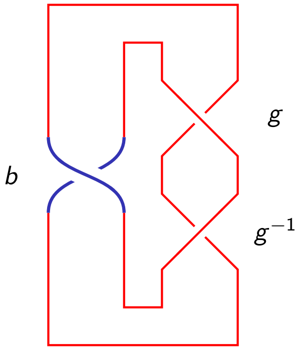
Every link is ambient isotopic to a closure of a braid.

Conjugation



Conjugation process illustrating the link equivalence of $\overline{gbg^{-1}}$ and $\overline{b}$ (part 1)

Conjugation (contd.)



Conjugation process illustrating the link equivalence of $\overline{gbg^{-1}}$ and $\overline{b}$ (part 2)

Markov theorem

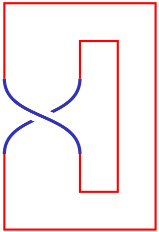
Theorem (Markov)

Two braids whose closures are ambient isotopic to each other are related by a finite sequence of the following operations.

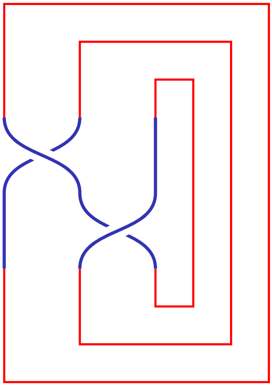
1. Braid equivalences, i.e. equivalences resulting due to the braid relations.
2. Conjugation.
3. Markov moves.

Markov move

b



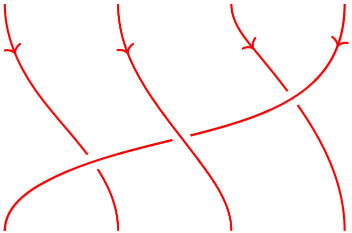
$\overline{b}$



$\overline{b\sigma_2}$

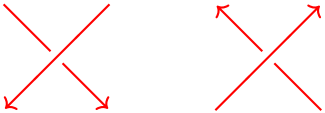
Markov move with $b = \sigma_1^{-1}$.

Orientation

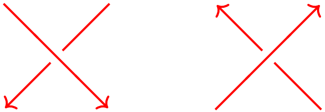


A braid with downward orientation

For the consistency of the orientation, it must be either upwards or downwards for all strands.



σ_i gets assigned $+1$



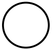
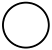
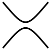
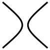
σ_i^{-1} gets assigned -1

Writhe is the sum of the assigned numbers.

Kauffman's bracket polynomial $\langle K \rangle$

Definition (Kauffman's bracket polynomial)

Let K an un-oriented link diagram. Then the bracket $\langle K \rangle \in \mathbb{Z}[A, A^{-1}]$ is defined by the rules:

1. $\Big\langle$  $\Big\rangle = 1.$
2. $\Big\langle$  $\cup K \Big\rangle = (-A^2 - A^{-2})\langle K \rangle.$
3. $\Big\langle$  $\Big\rangle = A \Big\langle$  $\Big\rangle + A^{-1} \Big\langle$  $\Big\rangle.$

$\langle K \rangle$ is invariant under the type II and type III moves.

Normalised bracket polynomial $L(K)$

We can normalise $\langle K \rangle$ by multiplying it with $(-A^3)^{-w(K)}$ gain type I move invariance.

$$L(K) := (-A^3)^{-w(K)} \langle K \rangle$$

Bracket polynomial of a braid

Bracket polynomial of a braid:

$$\langle\cdot\rangle\colon \mathrm{B}_n \to \mathbb{Z}[A, A^{-1}]$$
$$\langle\cdot\rangle\colon b \mapsto \langle\overline{b}\rangle$$

$\langle\cdot\rangle$ is well defined and invariant under conjugation.

Normalisation using writhe:

$$L(K) \coloneqq (-A^3)^{-w(b)}\langle\overline{b}\rangle$$

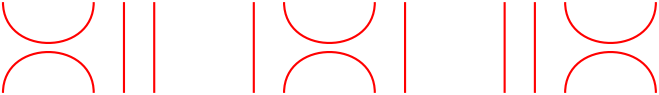
 $= \mathbb{1}_n$ and $\mathsf{U}_i :=$ 

$$\langle \sigma_i^{-1} \rangle = A\langle \mathsf{U}_i \rangle + A^{-1}\langle \mathbb{1}_n \rangle$$

$$\langle \sigma_i \rangle = A\langle \mathbb{1}_n \rangle + A^{-1}\langle \mathsf{U}_i \rangle$$

We refer to U_i's as "hooks" or "input-output" forms.

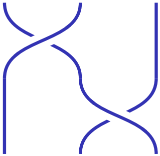
They don't belong to the Artin braid group.



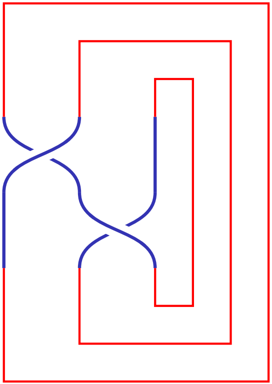
U_1 U_2 U_3

Input-output forms or hooks for 4 strands

Example

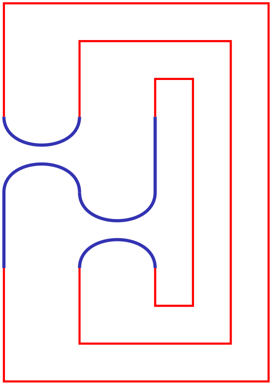


b

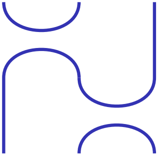


$\overline{b}$

Example (contd.)



$s = \overline{U_1 U_2}$



$U_1 U_2$

Writing a state of a braid closure in terms of input-output forms

$$\langle b \rangle = \langle S(b) \rangle = \sum_s \langle b|s \rangle \langle P_s \rangle = \sum_s \langle b|s \rangle \delta^{\|s\|}$$

$\langle S(b) \rangle$	:	Substituting $\langle \sigma_i \rangle = A \langle \mathbb{1}_n \rangle + A^{-1} \langle U_i \rangle$ and $\left\langle \sigma_i^{-1} \right\rangle = A \langle U_i \rangle + A^{-1} \langle \mathbb{1}_n \rangle$
s	:	A state in the expansion
P_s	:	Product of U_i's
$\langle b\|s \rangle$	:	Product of A's and A^{-1}'s
δ	:	$-A^2 - A^{-2}$
$\|s\|$	:	Number of loops in s minus one

Temperley–Lieb algebra TL_n

We give U_i's a structure of their own by constructing

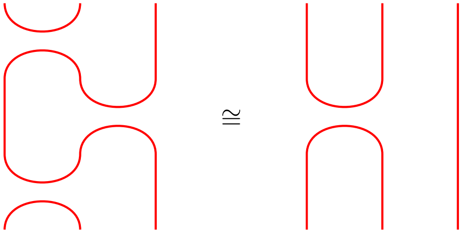
- over the ring $\mathbb{Z}[A, A^{-1}]$
- the free additive algebra TL_n (as a module)
- with the generators $\mathrm{U}_1, \mathrm{U}_2, \ldots, \mathrm{U}_{n-1}$
- and the multiplicative relations coming from the interpretation of U_i's as input-output forms.

Multiplicative relations in TL_n

Multiplicative relations in TL_n:

1. $\mathsf{U}_i \mathsf{U}_{i\pm 1} \mathsf{U}_i = \mathsf{U}_i$
2. $\mathsf{U}_i^2 = \delta \mathsf{U}_i$
3. $\mathsf{U}_i \mathsf{U}_j = \mathsf{U}_j \mathsf{U}_i$ if $|i-j| \geq 2$

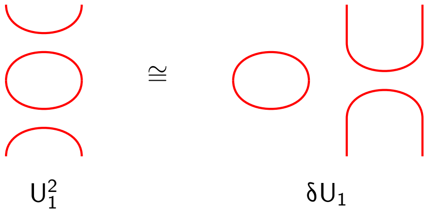
Geometric interpretation of $U_1 U_2 U_1 = U_1$



$U_1 U_2 U_1 \cong U_1$

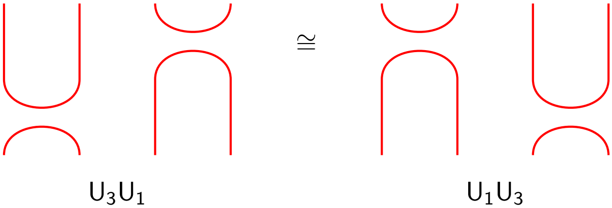
Geometric interpretation of $U_1 U_2 U_1 = U_1$

Geometric interpretation of $U_i^2 = \delta U_i$



Geometric interpretation of $U_i^2 = \delta U_i$

Geometric interpretation of $U_3U_1 = U_1U_3$



Geometric interpretation of $U_3U_1 = U_1U_3$

Representation of B_n in TL_n

We define a mapping

$$\rho\colon B_n \to TL_n$$

by

$$\begin{aligned} \rho(\sigma_i) &= A + A^{-1}U_i \\ \rho(\sigma_i^{-1}) &= A^{-1} + AU_i \end{aligned}$$

$\rho\colon B_n \to TL_n$ is a representation of the Artin braid group.

Trace

We define the diagrammatic trace

$$\mathrm{tr}\colon \mathrm{TL}_n \to \mathbb{Z}[A, A^{-1}]$$

by extending linearly

$$\mathrm{tr}(P) = \langle P \rangle.$$

This version of trace is diagrammatic in nature as we are counting loops in a state.

$$\langle b \rangle = \mathrm{tr}(\rho(b))$$

Whole procedure

So one can

- find a braid representation b of a link L by Alexander's theorem,
- calculate $\mathrm{tr}(\rho(b))$
- and normalise it to get the link invariant normalised bracket polynomial.

The substitution $A = t^{-1/4}$ yields us the Jones polynomial.

References

Louis Kauffman's book on which this presentation is based:

Louis H. Kauffman. *Knots and physics*. 4th ed. Knots and Everything 53. Singapore: World Scientific, 2013. ISBN: 978-981-4383-00-4

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Original papers by Emil Artin and Frederic Bohnenblust:

1. Emil Artin. "Theorie der zöpfe". In: *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg* 4.1 (Oct. 1925), pp. 47–72. ISSN: 1865-8784. DOI: 10.1007/bf02950718
   This paper is in German. I am not aware of an English translation. It contains some errors which have been corrected in the 1947 paper by Artin.
2. Emil Artin. "Theory of braids". In: *Annals of Mathematics* 1 (1947), pp. 101–126. ISSN: 0003486X. DOI: 10.2307/1969218
3. H. Frederic Bohnenblust. "The algebraical braid group". In: *The Annals of Mathematics* 48.1 (Jan. 1947), p. 127. ISSN: 0003-486X. DOI: 10.2307/1969219

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General reference books for knot theory and braids:

1. Kunio Murasugi and Bohdan I. Kurpita. *A study of braids*. Springer Science & Business Media, 1999. ISBN: 978-0-7923-5767-4
2. Peter R. Cromwell. *Knots and links*. Cambridge, UK: Cambridge University Press, 2004. ISBN: 0-521-83947-5
3. Joan S. Birman. *Braids, links, and mapping class groups*. Annals of Mathematics Studies 82. Princeton University Press, 1974. ISBN: 978-14-0088142-0
   This book contains the first proof of the Markov theorem, based on the notes of J. H. Roberts and an unknown speaker at Princeton University.

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1. Andrei A. Markov Jr. "Über die freie äquivalenz der geschlossenen zöpfe". German. In: *Recueil Mathématique. Nouvelle Série* 1 (1936), pp. 73–78. URL: https://www.mathnet.ru/php/archive.phtml?wshow=paper&jrnid=sm&paperid=5479&option_lang=eng
   This paper is in German and contains a sketch of a proof.
2. James W. Alexander. "A lemma on systems of knotted curves". In: *Proceedings of the National Academy of Sciences* 9.3 (Mar. 1923), pp. 93–95. DOI: 10.1073/pnas.9.3.93
   This paper introduced and proved the now known Alexander's theorem.
3. H. R. Morton. "Threading knot diagrams". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 99.2 (1986), pp. 247–260. DOI: 10.1017/S0305004100064161
   This paper contains a shorter proof of the Markov theorem.

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1. H. Neville V. Temperley and Eliott H. Lieb. "Relations between the 'percolation' and 'colouring' problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the 'percolation' problem". In: *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences* 322.1549 (Apr. 1971), pp. 251–280. DOI: 10.1098/rspa.1971.0067
   This paper introduces the Temperley–Lieb algebras in a statistical physics context.
2. F. R. Vaughan Jones. "A polynomial invariant for knots via von Neumann algebras". In: *Bulletin of the American Mathematical Society* 12.1 (1985), pp. 103–111. DOI: 10.1090/s0273-0979-1985-15304-2
   This paper introduces the Jones polynomial.