Thesis draft

Apoorv Potnis

March 6, 2023

Contents

Chapter 1

Braids and the Jones polynomial

1.1 Motivation

As remarked earlier, Jones arrived at his polynomial indirectly while working on the theory of operator algebras. In his course of investigations, he constructed a tower of algebras nested in one another with the property that each of these algebras is generated by a set of generators satisfying a particular set of relations. A degree of similarity between these relations and the relations among the generators of the *Artin braid group* was pointed out to Jones by a student during a seminar, which led to the investigations of Jones into knot theory. Jones had defined a notion of a *trace* on his algebras; more specifically a trace function obeying the *Markov property*. As we shall see, one can express every link in terms of a (non-unique) *braid*. Jones then defined a representation of such a braid into his algebras. The trace of an algebra representation of a braid, which is in turn obtained from the link can be calculated. The Jones polynomial can be realized as such a trace.

insert reference

insert reference

insert figure reference

In this chapter, we shall not travel the original route of Jones to reach his polynomial as it requires the knowledge of the theory of von Neumann algebras. Instead, we shall follow the approach of Kauffman to construct a representation of the Artin braid group into the *Temperley–Lieb algebra*. These algebras admit a diagrammatic intrepretation and our definition of a trace on these algebras shall be diagrammatic in nature as well. We shall eventually reach the bracket polynomial, which we already know to be equivalent to the Jones polynomial. The Jones algebra can be recovered from the Temperley–Lieb algebra by a choice of substitutions. The Temperley–Lieb algebra arose during the study of certain statistical models in physics. The Temperley–Lieb algebra can be viewed as a part of a broader framework of Partition algebras.

insert reference

insert reference

insert Partition algebras wiki link

insert figure reference and construct one

1.2 Geometric representation of braids

1.2.1 Definition

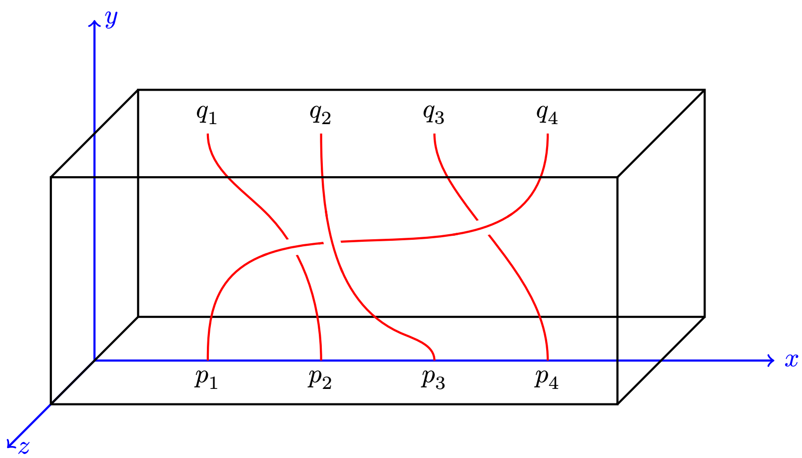


Figure 1.1: Three dimensional geometric representation of braids.

An n-braid is an element of the Artin braid group B_n, defined via the following presentation on the generators σ_i, for $1 \leq i \leq n-1$.

$$\mathrm{B}_n := \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{rcll} \sigma_i \sigma_i^{-1} & = & \mathbb{I}_n & \\ \sigma_i \sigma_{i+1} \sigma_i & = & \sigma_{i+1} \sigma_i \sigma_{i+1} & \\ \sigma_i \sigma_j & = & \sigma_j \sigma_i & \text{if } |i-j| \geq 2 \end{array} \right\rangle.$$

We shall now see a geometric construction in the three dimensional Euclidean space to represent the Artin braid group. This shall make clear the geometric intrepretation of the relations as well.

Consider two *ordered* sets of points $L_1 := \{p_1 := (1,0,0), \dots, p_n := (n,0,0)\}$ and $L_2 := \{q_1 := (1,1,0), \dots, q_n := (n,1,0)\}$ as shown in fig. 1.1 for $n = 4$. Elements of L_1 are called bottom points and elements of L_2 are called the top points. For $1 \leq i \leq n$, consider a family of non-intersecting continuous curves $\gamma_i \colon [0,1] \to \mathbb{R}^3$ such that

1. $\gamma_i(0) = p_i$ and $\gamma_i(1) = q_j$ for $1 \leq i, j \leq n$.

2. Any plane perpendicular to the xy-plane and parallel to the x-axis intersects each of the curves either exactly once or not at all.

3. All the curves lie in the cube determined by the vertices $(0,0,1)$, $(0,0,-1)$, $(0,1,1)$, $(0,1,-1)$, $(n+1,0,1)$, $(n+1,0,-1)$, $(n+1,1,1)$, $(n+1,1,-1)$.

Such a labelled curve is called a strand in standard position and a family of such labelled n stands is called an n-strand set in a standard position. We can ambient isotope or rigidly move an n-strand set to get another n-strand set, possibly not in a standard position. Two n-strand sets are said to be equivalent if they are related by a sequence of rigid motions of the strand sets, and ambient isotopies of the strand sets such that the space outside the cube, along with the endpoints, remains fixed. We shall refer to an equivalence class of such n-strands as a geometric n-braid. Thus, a geometric n-braid is well-defined.

Remark 1. Even though we have restricted our strands to the a bounded cube in the standard position, we can in principle change the bounds of our cube in x and z directions to any value and get the same theory. We shall not pursue this approach here.

1.2.2 Standard projection

We call the projection of a standard position n-strand set onto the xy-plane to be a two dimensional representation of a braid. Such a projection is drawn in fig. 1.2. It should be noted that a standard position n-strand set is unique only upto ambient isotopy, thus correspondingly the two dimensional representation of such a set is also unique only upto ambient isotopy, namely the ambient isotopies of the projection of the cube and the ambient isotopies such that the projection is a two dimensional representation of a braid for all times.

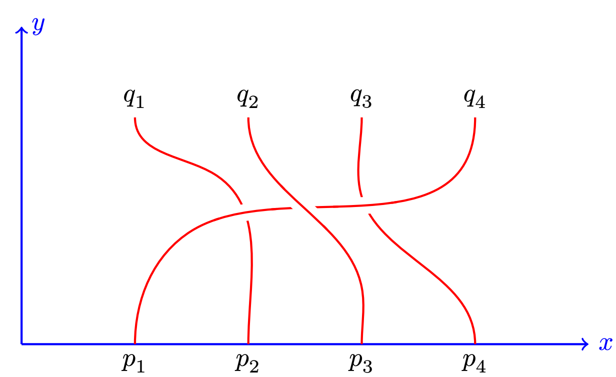


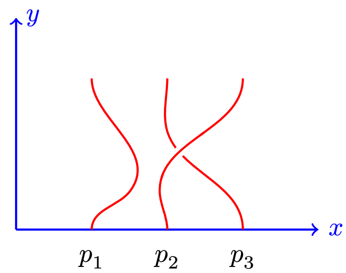
Figure 1.2: Two dimensional geometric representation of braids.

Now onwards, we shall always visually represent geometric n-braids using their standard two dimensional projections.

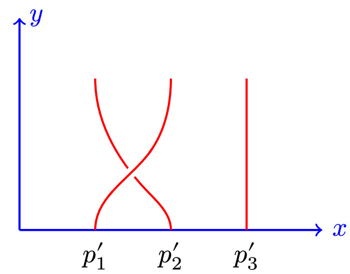
1.2.3 Group structure

Multiplication of any two n-geometric braids b_1 and b_2, denoted by $b_1 b_2$ is defined as follows (fig. 1.3). Ambient isotope and then rigidly move b_1 and b_2 *separately* in the standard position. Now translate only b_1 in the $+y$ direction by unit distance. The bottom points of b_1 and the top points of b_2 now coincide. Concatenate their strands and shrink the concatenated strands in the y

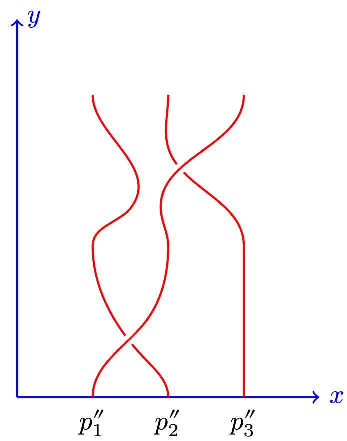
direction by half keeping fixed the bottom points of b_2. The result is another geometric n-braid $b_1 b_2$ in the standard position. Multiplication defined this way is associative.



(a) b_1



(b) b_2



(c) $b_1 b_2$

Figure 1.3: Multiplication of two braids (before shrinking).

We shall now drop the axes as well while representing two dimensional geometric n-braids.

An n-strand set such that each γ_i is a straight line segment connecting the i^{th} bottom point to the i^{th} top point is called the identity geometric n-braid and is denoted by $\mathbb{I}_n$ (fig. 1.4). Here we have used the same identity symbol of the Artin braid group. This abuse of notation shall not matter in the long run because we shall see that our n-strand set is isomorphic to the Artin braid group.

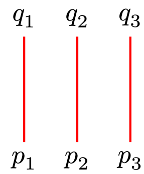


Figure 1.4: The identity $\mathbb{I}_3$

A geometric n-braid a such that $ab = ba = \mathbb{I}_n$ for some geometric n-braid b is called the inverse of b and denoted is by b^{-1}. We shall see that each element has an inverse.

With these operations, the set of geometric n-braids becomes a group, which we shall denote by GB_n.

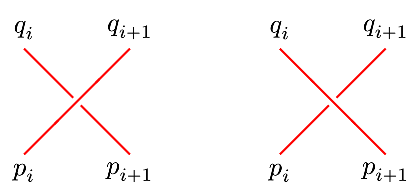
1.2.4 Generators

By the virtue of ambient isotopy, we can move the crossings in a two dimensional representation of a geometric n-braid such that each crossing lies in a region bounded by two lines parallel to the x-axis. Moreover, we can arrange the crossings such that each such region contains only one crossing. Thus, if we give the information regarding the type of each crossing for each such region, we can faithfully reconstruct the two dimensional representation. To this end, we define the generators of a geometric n-braid.

Denote by τ_i the geometric n-braid such that

1. $\gamma_i(1) = q_{i+1}$, $\gamma_{i+1}(1) = q_i$, and $\gamma_j(1) = q_j$ when j does not equal i or $i+1$.
2. $\pi_{xy}(\gamma_i(t)) \geq 0$ and $\pi_{xy}(\gamma_{i+1}(t)) \leq 0$ for all $t \in [0, 1]$.

π_{xy} is the projection maps onto to xy-plane. $\tau_1, \ldots, \tau_{n-1}$ are the generators of GB_n (fig. 1.5).

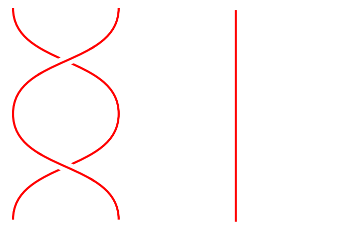


(a) τ_i (b) τ_i^{-1}

Figure 1.5: Generators τ_i and τ_i^{-1}. We have ommited the other straight strands.

For example, in fig. 1.3 we have $b_1 = \tau_2 \in \mathrm{B}_3$, $b_2 = \tau_1 \in \mathrm{B}_3$ and $b_1 b_2 = \tau_2 \tau_1 \in \mathrm{B}_3$.

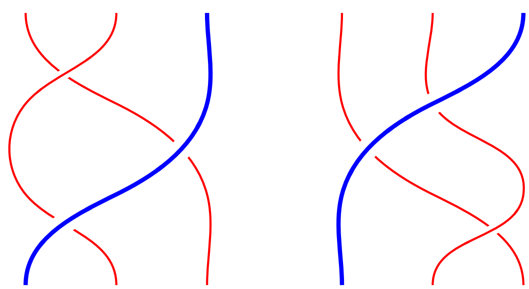
If we multiply τ_i and τ_i^{-1} to form $\tau_i\tau_i^{-1}$, we observe that $\tau_i\tau_i^{-1} = \mathbb{I}_n$, where $\tau_i, \tau_i^{-1} \in \mathrm{B}_n$ for all $n \geq 2$ (fig. 1.6).



$\tau_i\tau_i^{-1}$ $\mathbb{I}_n$

Figure 1.6: A type II move illustrating $\tau_i\tau_i^{-1} = \mathbb{I}_n$

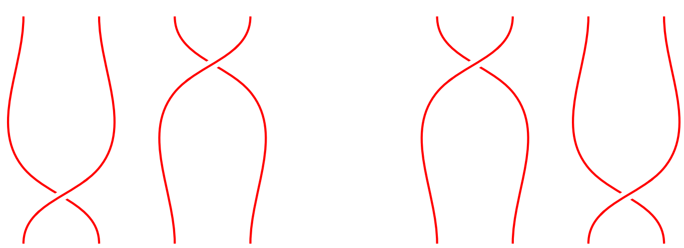
We can perform a move equivalent to the type III move to see that $\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$ (fig. 1.7).



$\tau_i\tau_{i+1}\tau_i$ $\tau_{i+1}\tau_i\tau_{i+1}$

Figure 1.7: A type III move illustrating $\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$.

We can slide two crossings vertically across each other if this does not change the ambient isotopy type. This is possible if the two crossings we wish to slide do not share a strand. This gives us the relation $\tau_i\tau_j = \tau_j\tau_i$ if $|i - j| \geq 2$ (fig. 1.8).



$\tau_i\tau_j$ $\tau_j\tau_i$

Figure 1.8: Sliding of crossings illustrating $\tau_i\tau_j = \tau_j\tau_i$.

Thus, we can define GB_n as the quotient of the free group F generated by $\{\tau_1, \dots, \tau_{n-1}\}$ with the smallest normal subgroup containing the elements $\tau_i\tau_i^{-1}$, $\tau_i\tau_{i+1}\tau_i\tau_{i+1}^{-1}\tau_i^{-1}\tau_{i+1}^{-1}$, and if $|i-j| \geq 2$, then $\tau_i\tau_j\tau_i^{-1}\tau_j^{-1}$.

$$\mathrm{GB}_n := \left\langle \tau_1, \dots, \tau_{n-1} \;\middle|\; \begin{array}{rcll} \tau_i\tau_i^{-1} & = & \mathbb{I}_n & \\ \tau_i\tau_{i+1}\tau_i & = & \tau_{i+1}\tau_i\tau_{i+1} & \\ \tau_i\tau_j & = & \tau_j\tau_i & \text{if } |i-j| \geq 2 \end{array} \right\rangle.$$

We thus define a map

$$\phi\colon \{\sigma_1, \dots, \sigma_{n-1}\} \to \{\tau_1, \dots, \tau_{n-1}\},$$
$$\phi\colon \sigma_i \mapsto \tau_i.$$

Let w be a word of length m in B_n; $w = \prod_{j=1}^{m} \sigma_{\alpha_j}^{\pm 1}$ where $1 \leq \alpha_j \leq n-1$. We can define a group isomorphism

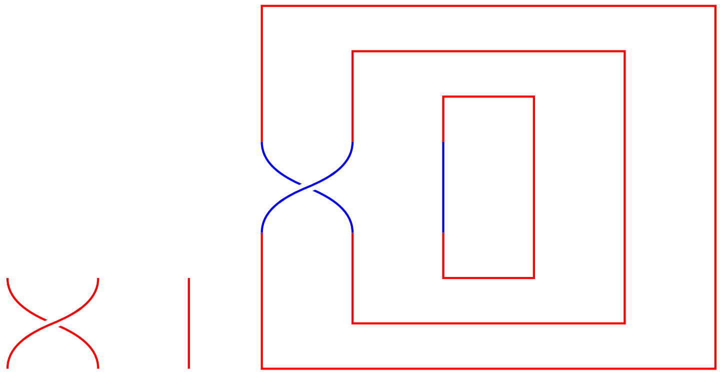
$$\Phi\colon \mathrm{B}_n \to \mathrm{GB}_n,$$
$$\Phi\colon \prod_{j=1}^{m} \sigma_{\alpha_j}^{\pm 1} \mapsto \prod_{j=1}^{m} \phi(\sigma_{\alpha_j}^{\pm 1}) \text{ for all } m \in \mathbb{N}.$$

We have finally succeeded in retrieving the algebraically defined Artin braid group from the geometric definition.

1.3 Closure of braids

1.3.1 Introduction

We define the closure of a geometric n-braid as follows. Consider a geometric n-braid in the standard position. For each $1 \leq i \leq n$, we construct the following sequence of line connected line segments. Join $(i, 1, 0)$, $(i, i, 0)$, $(i, i, 0)$, $(i, -i, 0)$, $(i, -i, 0)$, $(i, 0, 0)$ consecutively. We then join γ_i to the constructed line segments. Repeating this process for all i gives the closure of a braid (fig. 1.9). We denote the closure of a geometric n-braid b by $\bar{b}$. Closure of a braid is unique upto ambient isotopy.



(a) b (b) $\bar{b}$

Figure 1.9: Closure of a braid with $b = \tau_1 \in \mathrm{GB}_2$.

Proposition 2. Every closure of a geometric n-braid is a link.

Proof. The outer line segments are locally flat by virtue of being piecewise linear. Due to the second condition regarding the intersection of a plane in the standard representation of geometric n-braid, we can project the strand onto the y-axis and this projection would be a homeomorphism. We take an ϵ neighbourhood, N_ϵ around the strand. The pair (N_ϵ, γ_i) would then be homeomorphic to $(\mathrm{B}, \mathrm{B} \cap x\text{-axis})$, where B is the three-dimensional unit ball around the origin. We have local flatness at the end points as well due to the union of two locally flat curves. □

Remark 3. Suppose $\tau_i \in \mathrm{GB}_n$ and $\tau_i' \in \mathrm{GB}_m$ are generators where $n < m$. Then the closures of these two generators are not ambient isotopic. The closure of the latter contains one more non-linking loop.

1.3.2 Alexander's theorem

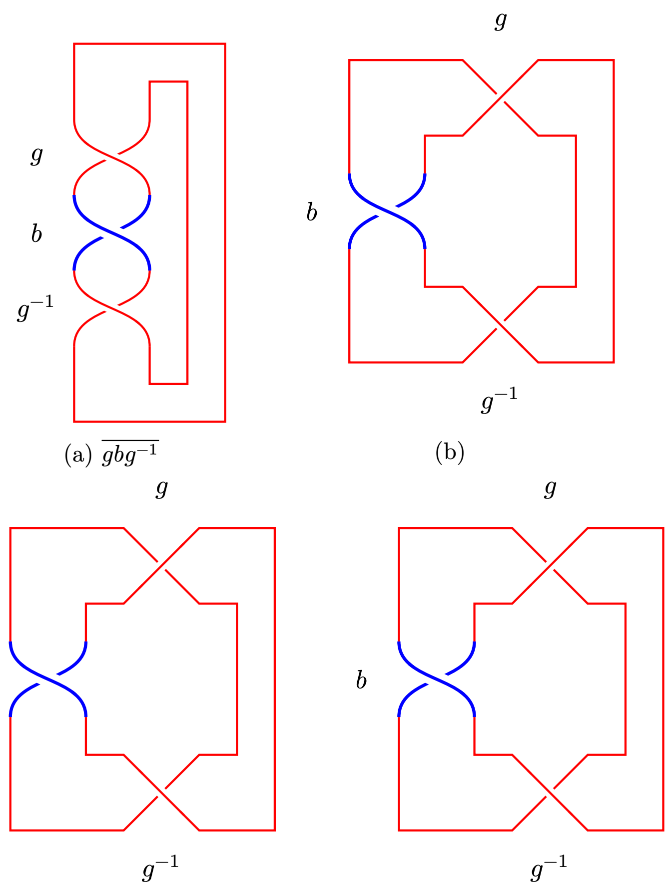
Theorem 4 (Alexander)**.** Every link is ambient isotopic to the closure of a geometric braid, for some $n \in \mathbb{N}$.

Proof. Consider a piecewise linear, regular projection $\pi(L)$ of a link L on a plane. We choose a point O in the projection plane which is not collinear with any of the line segments. This can be done since a the link has only finitely many line segments. Let $P \in \pi(L)$. The vector OP can move either clockwise or anti-clockwise as P moves along the link projection. We wish to modify the line segments such that OP moves in only one sense, say anti-clockwise, as P moves along the entire length of the link projection. We now fix our attention on a line segment corresponding to a clockwise rotation. We divide the segment into sub-parts such that each part shares at-most one crossing point with other line segments. If A and B are end-points of such a line segment, then we may replace this line segment with two another line segments AC and CB, such that C is another point not on belonging to $\pi(L)$ and the triangle ABC encloses O. If AB originally passed under (or over) a line segment of $\pi(L)$, then the modified line segments AC and CB must pass under (or over) of the line segments of $\pi(L)$ as well. This move shall not change the link type as it shall be a combination of sliding, type 2 and type 3 moves. In the resulting triangle, the vector OP shall move in the opposite, anti-clockwise sense while traversing from A to B via C, instead of AB. We can repeat this process for all of the (finitely many) line segments which turn clockwise. In the end, we obtain a projection such that OP moves in only the anti-clockwise sense, as P moves along the entire length of the link projection. We can ambient isotope the projection such that all the crossings lie in the projection of a cube, more precisely the cube constructed while defining a geometric braid. The end-points can be made to match as well. The above procedure of triangluar moves shall guarantee the monotonicity that is required. $\square$

Henceforth, unless specified otherwise, we shall always work with the projections of the standard representation of a geometric n-braid and its closure and refer to these projections simply as a braid and its closure.

1.3.3 Conjugation

If $b, g \in \mathrm{GB}_n$, we observe that $\overline{gbg^{-1}}$ is ambient isotopic to $\overline{b}$ (fig. 1.10). The upper strands in g and g^{-1} are connected via the closure strand. We can slide g and g^{-1} via the closure strands to the 'other side' of b to annihilate each other. This can be achieved by a type II move.



(c) (d) $\overline{b}$

Figure 1.10: Conjugation process illustrating the link equivalence of $\overline{gbg^{-1}}$ and $\overline{b}$, with $b = \tau_1^{-1} \in \mathrm{GB}_1$ and $g = \tau_1^{-1} \in \mathrm{GB}_1$.

Remark 5. The closures of conjugate braids are ambient isotopic as links. The braids themselves are not.

1.3.4 Markov move

We observe that if $b \in \mathrm{GB}_n$, then b, $b\tau_n \in \mathrm{GB}_{n+1}$ and $b\tau_n^{-1} \in \mathrm{GB}_{n+1}$ have ambient isotopic closures. That is, we can add a strand and a crossing of that strand with another strand without changing the link type (of the closure). We can remove a strand and a crossing if that strand does not cross any other strand. It can be shown that adding the above mentioned strands anywhere between the existing strands is equivalent to adding the strands on the right.

1.3.5 Markov's theorem

Theorem 6 (Markov)**.** Two braids whose closures are ambient isotopic to each other are related by a finite sequence of the following operations.

1. Braid equivalences.
2. Conjugation.
3. Markov moves.

Proof. Please refer to insert Birman citation. $\square$