SOME TOPICS IN KNOT THEORY

A THESIS

submitted in partial fulfillment of the requirements

for the award of the dual degree of

Bachelor of Science-Master of Science

in

MATHEMATICS

by

APOORV POTNIS

(18343)

DEPARTMENT OF MATHEMATICS,
INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH BHOPAL,
BHOPAL - 462 066

April 2023

भारतीय विज्ञान शिक्षा एवं अनुसंधान संस्थान भोपाल

Indian Institute of Science Education and Research Bhopal

(Estb. By MHRD, Govt. of India)

CERTIFICATE

This is to certify that **Apoorv Potnis**, BS-MS (Mathematics), has worked on the project entitled **'Some topics in knot theory'** under my supervision and guidance.

April 2023
IISER Bhopal

Dr. Dheeraj Kulkarni
Prof. Subhash Chaturvedi

Committee Member	Signature	Date

ACADEMIC INTEGRITY AND COPYRIGHT DISCLAIMER

I hereby declare that this project is my own work and, to the best of my knowledge, it contains no materials previously published or written by another person, or substantial proportions of material which have been accepted for the award of any other degree or diploma at IISER Bhopal or any other educational institution, except where due acknowledgement is made in the document.

I certify that all copyrighted material incorporated into this document is in compliance with the Indian Copyright Act (1957) and that I have received written permission from the copyright owners for my use of their work, which is beyond the scope of the law. I agree to indemnify and save harmless IISER Bhopal from any and all claims that may be asserted or that may arise from any copyright violation.

April 2023 **Apoorv Potnis**
IISER Bhopal

ACKNOWLEDGEMENT

Acknowledgement is here.

Apoorv Potnis

ABSTRACT

We shall study some selected topics in knot theory from the book *Knots and Physics*, 4th ed. by Louis Kauffman [1]. The basics of knot theory have been studied from the book *Knots and Links* by Peter Cromwell [2].

A knot is an embedding of the circle in three-dimensional space. We discuss the state model for bracket polynomial, Jones polynomial and some of its generalisations. One can apply these knot invariants to show that various knots are not ambient isotopic, which is the notion we shall use to distinguish knots. We shall see a resolution of some of the old conjectures in knot theory using the bracket polynomial. Vaughn Jones found the polynomial named after him while he was studying towers of von Neumann algebras. We shall follow a different route following Kauffman to understand the relation amongst braids, links, Temperley–Lieb algebras, which arose in statistical physics, and the Jones algebras.

Contents

Chapter 1

Introduction to Knot Theory

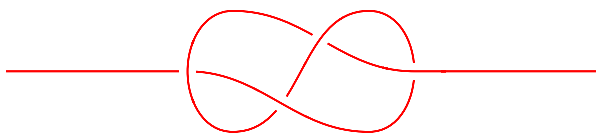
1.1 Definition of a knot

We normally conceive of knots as open strings with 'knotted' parts in between. Given a knotted string with two 'open' ends, we can simply pull one end of the string to untie it, in the usual way we untie a knot, by inserting an open end into the knotted region is a strategic manner and pulling it on the other side[1]. This way, any knot with two open ends can be untied. But if have a knot in a closed loop, we won't be able to untie it unless we cut it. We want our notion of a knot to be invariant under 'pulling'. Thus, we model our mathematical definition on closed loops instead of open. Refer to fig. 1.1.

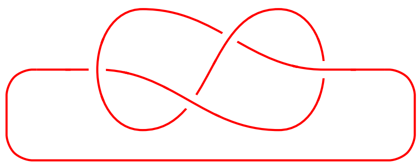
Definition 1 (Knot)**.** A knot K is the image a topological embedding of $\mathbb{S}^1$ in $\mathbb{R}^3$.

Although one can talk about embeddings of higher dimensional 'circles' in higher dimensional spaces, we restrict ourselves to knots in the three dimensional space. The above definition turns out out to be too general for our purposes. It takes into consideration certain pathological knots

[1]There exists another way of unknotting an open knot, often used in magic tricks. One creates another knot using the unknotted portion of the string on one side of the knotted portion such that this new knot 'cancels' the original knot. In mathematical terms, the new knot is constructed such that the *connected sum* of the original knot and the new knot gives an unknotted circle, or the *unknot*. We shall not deal with connected sums of knots in this thesis.



(a) A knot with open ends



(b) A knot with closed ends

Figure 1.1: Projections of a knot on a plane

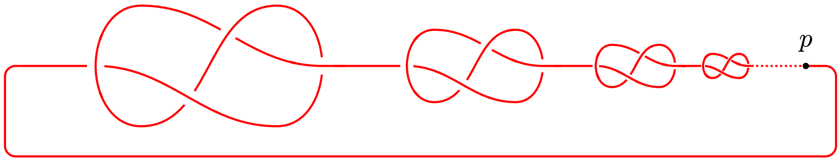


Figure 1.2: A wild knot. The knot is wild only at the point p.

known as *wild knots*. Although wild knots are an object of study, we shall not be dealing with them in this thesis due to their pathological nature. Figure 1.2 shows an example of a wild knot. A section of the knot is scaled down by a factor and then joined on one side of the previous section. If one repeats this process infinitely, the resulting curve formed by appending sections eventually converges to a point, provided that the scaling down is fast enough. We can join the convergence (limit) point to a an end-section on the other side to get wild knot. This knot is continuous everywhere, including the limit point. The behaviour of the knot at the limit point is different from the other points. Three common ways exist to exclude such behaviour by demanding extra conditions.

1. Differentiability.

   We can demand all knots to be differentiable at all points. We can verify visually that all the points, except the limit point of the wild knot are differentiable, or can be made (isotoped) differentiable. The derivative must necessarily change within a section. As the size of the section decreases, the derivative changes more rapidly. In a scaled down section, all the values which the derivative took in the previous section. As we approach the sections near the limit point, the derivative function must attain all the values it did before, but it must do so in a more rapid manner. At the limit point, the derivative shall cease to exist by the virtue of 'changing too rapidly'. Demanding differentiability forcibly removes the offending limit point. The wildness is due to the limit point. Along with differentiability, we can demand C^r smoothness or C^∞ smoothness as well.

   But this condition comes with a problem as well, namely we cannot use polygons for describing knots.

2. Piecewise linearity.

   We can demand all the edges of a knot to be piecewise linear with finitely many edges. Our knot shall be a polynomial (with finitely many edges) in that case. Infinitely (countably) many sections in a wild knot shall mean infinitely many edges (of decreasing length), which is not allowed. Thus, this condition excludes wild knots.

3. Local flatness.

In this thesis, we shall take the third route following Cromwell [2, chp. 1]. Local flatness is a topological condition unlike other other two. If we consider a local neighbourhood around each point of the knot, except the limit point, then we see visually see that we can always find a small enough local neighbourhood around each such point such that the strand is not 'knotted' in that neighbourhood. At the limit point, no matter how small a neighbourhood we take, the strand shall always be 'knotted'. We enforce this 'local unknottedness' condition by demanding local flatness. Let p be a point in a knot K, $\mathrm{B}(\mathrm{O}, 1)$ be the unit ball centered at origin O and d be a diameter of $\mathrm{B}(\mathrm{O}, 1)$.

Definition 2 (Local flatness)**.** The point p is said to be locally flat if there exists a neighbourhood $U \ni p$ such that the pair $(U, U \cap K)$ is homeomorphic to $(\mathrm{B}(\mathrm{O}, 1), d)$.

A knot is said to be locally flat if each point in that knot is locally flat. A point that is not locally flat is called wild, and a knot is wild if any of its points are wild.

Consider a spherical neighbourhood around a locally flat point. There exists a radius such that for all neighbourhoods less than this radius, the boundary of the neighbourhood intersects the strand in exactly two points. This is not possible at the limit point in wild knot figure.

Definition 3 (Tame knots)**.** A knot is said to be tame if all its points are locally flat.

1.2 Distinguishing knots

Any two homeomorphisms of the circle are homeomorphic to the circle and to each other, since being homeomorphic is an equivalence relation. But this means that all knots are homeomorphic to each other. Clearly, homeomorphism is not the correct notion to distinguish knots. When we mean that two knots are distinct, we mean that if we create a physical model of those knots, we cannot 'physically deform' one knot into another. *Cutting a knot is not allowed.* One might think that homotopy or isotopy are what we need, but it turns out that the notion of *ambient isotopy* is the correct one.

Definition 4 (Homotopy)**.** A homotopy of a space $X \subset \mathbb{R}^3$ is a continuous map $h\colon X \times [0,1] \to \mathbb{R}^3$. The restriction of h to level $t \in [0,1]$ is $h_t\colon X \times \{t\} \to \mathbb{R}^3$. h_0 must be the identity map.

Note that the continuity of h implies the continuity of h_t for all $t \in [0,1]$. The converse is not true though. Homotopy allows a curve to pass through itself. All knots are thus homotopic to the unknot, also referred to as the trivial knot. If we do not allow a curve to pass through itself, i.e. if we demand injectivity for each h_t, then we get what is known as an isotopy. But isotopy is not useful for distinguishing knots as well, due to bachelors' unknotting. All tame knots, or more generally, all knots with a tame arc turn out to be isotopic to the trivial knot. It is an open problem if all knots are isotopic to the trivial knot [3, 4]. In addition, it is not known as well if a knot known as the Bing sling, which is wild *at all points* is isotopic to the trivial knot [3, 4].

Proposition 5 (Bachelors' unknotting)**.** Every knot with a tame arc is isotopic to the trivial knot.

Proof. Refer to fig. 1.3. Let $K \subset \mathbb{R}^3$ be a knot with a tame arc.

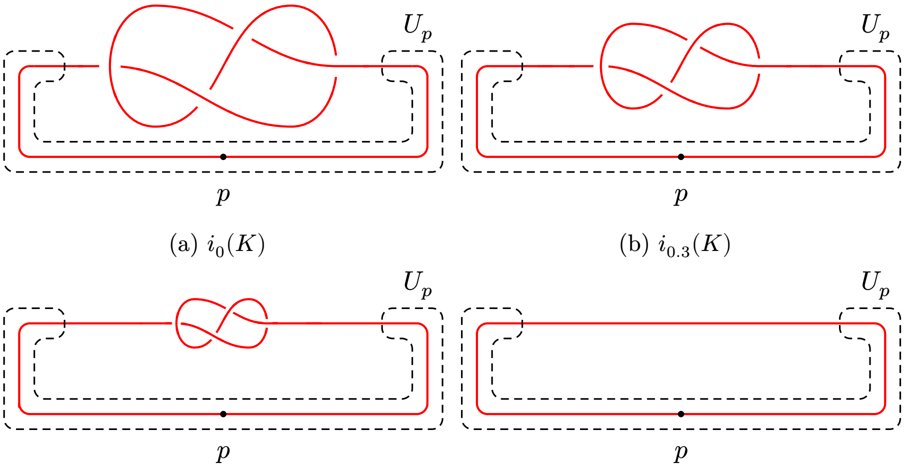
Let $p \in K$ in a tame arc of the knot. Since the knot is locally flat on the arc, by the definition of tameness, we take a ball $U_p \subset \mathbb{R}^3$ of radius ε around the point p such that the pair $U_p, U_p \cap K$ is homeomorphic to (B, d), where B is the unit ball in $\mathbb{R}^3$ centered at the origin and d is the diameter of B along the x-axis. We choose a parametrization $f\colon [0, 2\pi) \to K$ of the knot such that $f([a,b]) = K - (U_p \cap K)$, where $[a,b] \subset [0, 2\pi)$.

Let $r \in \mathbb{R}^3$ be a point outside U_p. Now consider the function $i_t\colon K \to \mathbb{R}^3$, defined for each $t \in [0,1]$ as follows. Let

$$\phi_t(a) := t\left(\frac{a+b}{2}\right) + (1-t)a$$

be a family of functions for all $t \in [0,1]$.

1. If $f(x) \in U_p$, then
$$i_t(f(x)) = f(x).$$
2. If $x \in [a, \phi_t(a))$, then
$$i_t(f(x)) = f(a) + \frac{\|f(a) - f(\phi_t(a))\|}{\|a - \phi_t(a)\|}(x-a)(r - f(a)).$$



(a) $i_0(K)$ (b) $i_{0.3}(K)$

(c) $i_{0.6}(K)$ (d) $i_1(K)$

Figure 1.3: Bachelors' unknotting demonstrating the isotopy equivalence of a knot with tame arc to the trivial knot. Here we have chosen the point r (not shown in the figure) to be the midpoint of the knotted region. $f(a)$ and $f(b)$ are the intersections of the knot K, represented by the thick red line, with U_p, shown using densely dashed lines. Note that $f(a)$, r and $f(b)$ are collinear in this case. The entirety of the knot is tame in this case, but we only require tameness in the chosen region of $U_p \cap K$.

3. If $x \in [\phi_t(a), \phi_t(b)]$, then
$$i_t(f(x)) = tr + (1-t)f(x).$$

4. If $x \in (\phi_t(b), b]$, then
$$i_t(f(x)) = f(b) + \frac{\|f(b) - f(\phi_t(b))\|}{\|b - \phi_t(b)\|}(b-x)(r - f(b)).$$

Let $i\colon [0,1] \times K \to \mathbb{R}^3$ be a function defined by $i(t, f(x)) := i_t(f(x))$.

i is defined such that the part inside U_p is kept unchanged for all t. For $t = 0$, i does not deform the knot at all. For $t \in (0,1)$, the knotted part (in $\mathbb{R}^3$) shrinks and the interval in the domain $[a,b]$ which maps to the knotted part also shrinks to $[\phi_t(a), \phi_t(b)]$. This shrinkage of the domain happens linearly with t. Refer to fig. 1.4. All points of the knotted part trace a straight line as they travel under isotopy from their original position to r. Eventually, the knotted part ceases to exist at $t = 1$ and a single point of the domain, namely, $(a+b)/2$ maps to r.

In the end, we get a figure consisting of two straight lines meeting at r, and $U_p \cap K$, the original part of the knot inside U_p. The other endpoints of these lines are $f(a)$ and $f(b)$. $U_p \cap K$ is isotopic to the line joining $f(a)$ and $f(b)$. Thus, we get a triangle with points r, $f(a)$ and $f(b)$ and we any that any triangle in $\mathbb{R}^3$ is isotopic to $\mathbb{S}^1$.

We now prove that i is continuous. We know that both i_t and i_x are continuous and injective for all $t \in [0,1]$ and $x \in [0, 2\pi)$ respectively, where i_x is defined to be the restriction of i for a particular x. We also see that i_t is linear in x. A function continuous in one argument and linear in the other is continuous in the product topology. We thus have an isotopy which sends a knot with a tame arc to $\mathbb{S}^1$. $\square$

Remark 6. It should be noted that the isotopy that we have constructed does not shrink $K - (U_p \cap K)$ uniformly. Parts of the strand closer to the point r are shrunk more than the parts

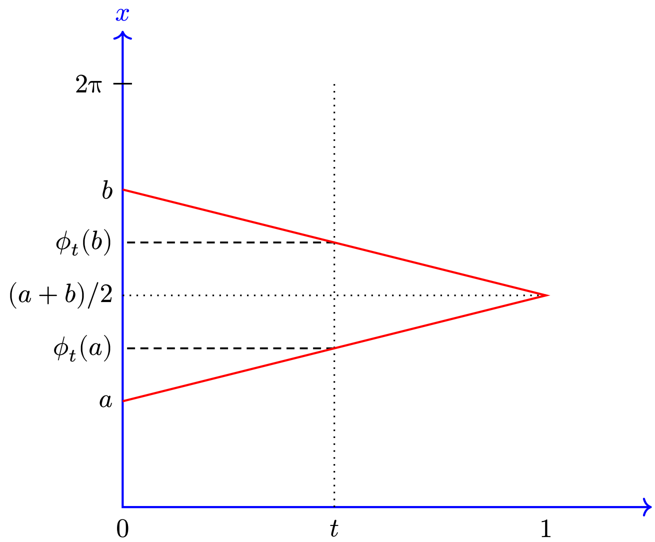


Figure 1.4: Illustration of shrinkage of domain with respect to t. For all $t \in [0,1]$, the interval $[\phi_t(a), \phi_t(b)] \in [0, 2\pi)$ maps to the shrunk 'knotted' part. The images of $[a, \phi_t(a)]$ and $[\phi_t(b), b]$ map to the straight lines which connect $f(a)$ to $i_t(f(\phi_t(a)))$ and $i_t(f(\phi_t(b)))$ to $f(b)$ respectively.

further away. Also, not all parts move at a uniform rate towards r. Parts closer to r move slower than the parts further.

In the above considered isotopy, we isotoped the set $X = K$. Instead, we take X to be the entire space $\mathbb{R}^3$, or a bounded set which completely covers the knot, then we get the notion of ambient isotopy. This modification ensures that the surrounding space is isotoped as well as we isotope the knot. The knot is a curve which has no volume. If we try bachelors' unknotting on the surrounding space as well, we observe that the surrounding space, which has a finite, non-zero volume cannot shrink to a set of zero volume under isotopy. This finally leads us to the equivalence relation induced by ambient isotopy.

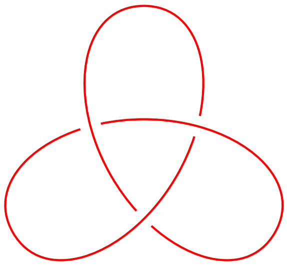
Remark 7. Unless mentioned otherwise, we shall always consider our knots to be tame from now on.

Definition 8 (Knot equivalence)**.** Two knots K_1 and K_2 are said to be ambient isotopic if there exists an isotopy $I\colon \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ such that $I(K_1, 0) = I_0(K_1) = K_1$ and $I(K_1, 1) = I_1(K_1) = K_2$.

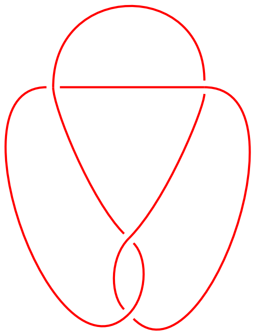
Knot equivalence is an equivalence relation as it satisfies the properties of reflexivity, symmetry and transitivity.

Remark 9. Each equivalence class of knots is called a *knot type*. We would often forget the distinction between a knot and its knot type. The intended meaning can be inferred from the context.

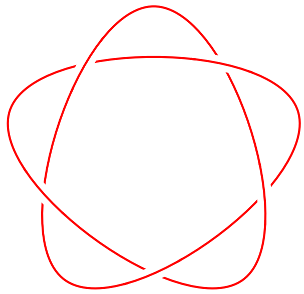
Remark 10. Note that we distinguish between *ambient isotopy* and *isotopy*. Many treatments of knot theory use the word isotopy for ambient isotopy as ambient isotopy is the useful construct in knot theory. Ambient isotopy is an isotopy of the whole space containing the knot, not just the knot.



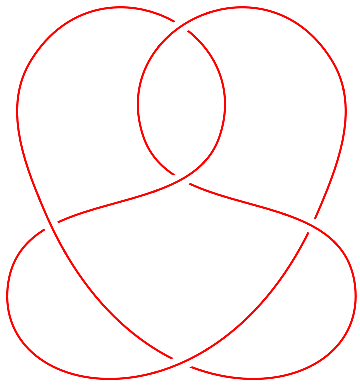
(a) A trefoil



(b) The figure-eight knot



(c) The cinquefoil knot



(d) The three-twist knot (5_2)

Figure 1.5: Various distinct knots.

Remark 11. All three approaches for excluding pathological behaviour, including C^r for all $r \geq 1$, generate the same knot isotopy classes. The knot isotopy classes corresponding to C^1 curves, piecewise-linear curves with finitely many components, smooth C^∞ curves and locally flat everywhere curves are equal [2, § 1.11].

Example 12. The knots depicted in fig. 1.5 are non-trivial and not ambient isotopic to each other. We shall prove this fact by the use of Kauffman's bracket polynomial in later chapters.

Remark 13. In this thesis, we shall look at knot theory in $\mathbb{R}^3$. One can compactify $\mathbb{R}^3$ to $\mathbb{S}^3$ and do knot theory in $\mathbb{S}^3$, as many treatments do. This does not result in a different knot theory.

1.3 Links

So far, we have looked at embeddings of a single circle. We can embed more than one circles to get links.

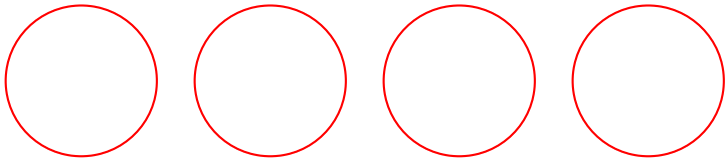


Figure 1.6: A trivial link with 4 components

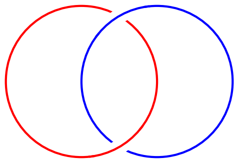


Figure 1.7: A Hopf link

Definition 14. A collection of disjoint topological embeddings of $\mathbb{S}^1$ in a three dimensional space is called a link.

Each knot (an embedding of a circle) belonging to a link is called a *component* of the link. The number of components of a link is called the *multiplicity* of the link. We shall denote the multiplicity of a link L by $\mu(L)$. A knot is thus a link with one component.

A link such that when projected onto a plane gives us n disjoint circles is referred to as a trivial link with $\mu(L) = n$ components. Refer to fig. 1.6 for a link with $n = 4$. A link is trivial iff all its components are trivial knots and they are 'unlinked'. Equivalently, a link is trivial if its components bound disjoint discs.

Since a link is a union of knots, we might try to describe a link L as $L = K_1 \cup \cdots \cup K_n$ such that images of K_i's are disjoint. But this does not completely describe a link as this only lists the component parts, and does not indicate how they are put together in space.

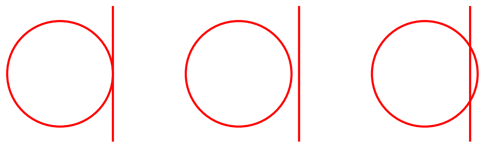
We can define two links to be equivalent if they are ambient isotopic, i.e. there exists an ambient isotopy which takes one link to another. This equivalence relation is *weak* in the sense that one has the freedom to match different components of the two links. We have not labelled the links.

1.4 Link diagrams

So far, we have depicted knots as curves in a plane (of paper) with the implicit understanding that if a strand goes under another strand, then we break the strand which goes underneath. We defined a knot as an object in three dimensions, but we can project the knot onto a plane to represent it, as we have done so far. We do not lose any topological information if we choose a 'nice enough' projection. What we have been doing implicitly can be formalized; we shall see how. We shall introduce some preliminary topological notions which shall be required.

1.4.1 Topological manifolds

Definition 15 (Topological manifold)**.** A topological n-manifold is a Hausdorff and second-countable topological space such that each point has a neighbourhood homeomorphic to $\mathbb{R}^n$ or $\mathbb{R}^n_{\geq 0}$.



(a) (b) (c)

Figure 1.8: Intersection of a line and a circle

Remark 16. Different sources define a topological manifold in slightly differing ways. Some require just paracompactness instead of second-countability. The above definition includes manifolds with boundary as well. Boundary points shall be the points which map to the boundary of $\mathbb{R}^n_{\geq 0}$.

Definition 17. A curve is a compact 1-manifold and a surface is a compact 2-manifold.

There are only four distinct connected 1-manifolds, namely the real line, the real half-line, the compact interval and the circle u p to homeomorphism.

1.4.2 Transversality

We shall need to consider the ways surfaces and curves intersect each other. We use the principle of general position to classify these intersections. We shall follow the treatment of Cromwell in this section [2, chp. 2, § 2.10].

A structure or property is said to be *stable* if it maintains its essential characteristics when it is perturbed a little. In our context, this means that the original set of curves and surfaces must be homeomorphic to the perturbed set of curves and surfaces. We desire our intersections to be stable. A collection of curves in a plane is said to be *stable* if the collection maintains its essential characteristics when perturbed a little. For example, consider the intersection of circle and a line as shown in fig. 1.8.

The intersection as shown in fig. 1.8a is not stable as any translation or rotation of the straight line shall lead to a figure homeomorphic to fig. 1.8b or fig. 1.8c. The figure does not represent a 1-manifold as no neighbourhood around the intersection point is homeomorphic to $\mathbb{R}^n$. Figure 1.8b is a union of two connected 1-manifold components. Figure 1.8c is again not a 1-manifold. Since the number of intersection points is different in each case, all three figures are topologically distinct (not homeomorphic). The intersections in fig. 1.8b and fig. 1.8c are stable though.

Definition 18. An intersection is said to be *transverse* if a neighbourhood of it is homeomorphic to one of the following four cases.

1. The union of x-axis and y-axis in $\mathbb{R}^2$.
2. The union of z-axis and xy-plane in $\mathbb{R}^3$.
3. The union of xz-plane and yz-plane in $\mathbb{R}^3$.
4. The union of xy-plane, xz-plane and yz-plane in $\mathbb{R}^3$.

Theorem 19. The above four cases the only stable arrangements of curves and surfaces in two and three dimensions.

The above theorem states that if we have a stable set of curves and surfaces in two and three dimensions, then the set is homeomorphic to one the above four cases. A set of objects is said to be in *general position* if all their intersections are transverse.

Theorem 20. Every finite set of curves and surfaces embedded in $\mathbb{R}^3$ is arbitrarily close to an ambient isotopic set in general position.

Thus, we can always eliminate points of tangency such as saddle points, maxima and minima. A detailed treatment about transversality of manifolds can be found in books of differential topology, such as the book of Guillemin and Pollack [5].

1.4.3 Projections and diagrams

Let L be a link and let $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ be a projection map. A point $p \in \pi(L)$ is called *regular* if $\pi^{-1}(p)$ is a single point, and *singular* otherwise. If $|\pi^{-1}(p)| = 2$, then p is called a *double* point.

We wish to get 'nice enough' projections. There can be infinitely many singular points that occur in a projection. We demand that a projection contains only a finite number of singular points. Singular points shall be isolated in such a case. This shall exclude cases such as two line segments of link projecting onto the same line segment on the plane. The singular points can have multiple pre-images as well. We thus demand that $|\pi^{-1}(x)| \leq 2$. We also demand our projections to be stable so that if we change the projection direction by a small amount, the projection is essentially unchanged: no singular points are created or destroyed.

Definition 21. If $\pi(L)$ has a finite number of singular points and they are all transverse double points, then the projection is said to be a *regular* projection.

Theorem 22. Every tame link admits a regular projection.

A link projection with the added information about the relative heights at crossings is called a link diagram. We adopt the convention that if a strand passes underneath another strand, then we break the underneath strand at the crossing in the diagram. A regular projection has only finitely many crossings.

Theorem 23. Every tame link admits a diagram.

Proofs of the above two theorems can be found in Cromwell [2, chp. 3]. From now on, unless mentioned otherwise, we shall always assume that our link diagrams correspond to regular projections.

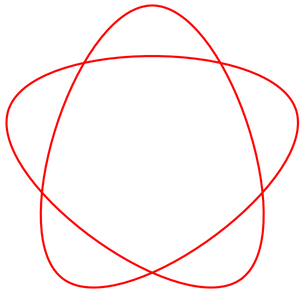
1.5 Reidemeister theorem

We can now encode all the topological information about a link in a two dimensional diagrammatic representation. If two links are ambient isotopic, then a natural question arises whether there exists relation between their diagrams. The famous and important theorem of Reidemeister answers this question in affirmative. Given a link diagram, we define three moves on local sections of a diagram, called as Reidemeister moves (fig. 1.10). These moves are local, i.e. we always perform these moves, or the ambient isotopies corresponding to these moves in a small enough neighbourhood such that everything is constant outside this neighbourhood. Such a neighbourhood exists as the knots are tame and the projections regular. moves preserve the link type.

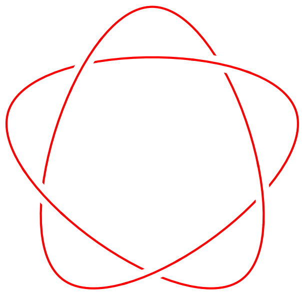
We now state the Reidemeister theorem. It was proved by Kurt Reidemeister in 1927 [6], and by James Alexander and Garland Briggs in 1926 [7].

Theorem 24 (Reidemeister)**.** Two diagrams of a link are related by a finite sequence of Reidemeister moves.

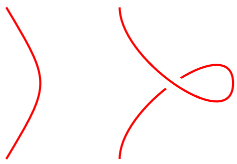
A proof of the above theorem can be found in the book of Murasugi on knots and links [8, chp. 4].



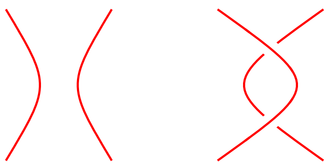
(a) A knot projection of the cinquefoil knot



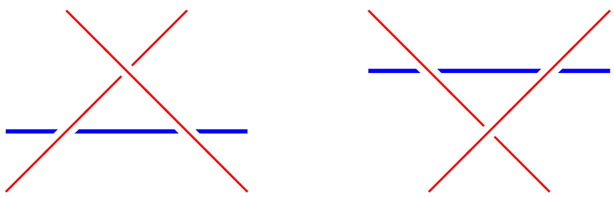
(b) A knot diagram of the cinquefoil knot



(a) A type I move

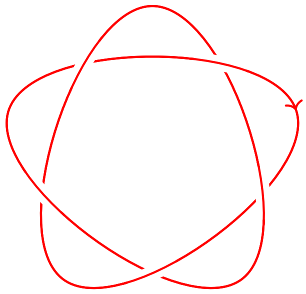


(b) A type II move

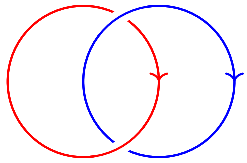


(c) A type III move

Figure 1.10: The Reidemeister moves.

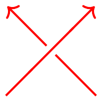


(a) The oriented cinquefoil knot

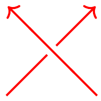


(b) An oriented Hopf link

Figure 1.11: Oriented links



(a) Positive crossing $+1$



(b) Negative crossing -1

Figure 1.12: Assignment of crossing values

1.5.1 Crossing number

Let $c(D)$ denote the number of crossings in a diagram D of a link L. The crossing number $c(L)$ is is the minimum number of crossings in any diagram of the link.

1.6 Orientation

We can assign a direction or orientation to links. Take three points a, b and c on a circle. We can define an orientation using a path such that one encounters the point a at the start, b in the middle and c at the end. If we transpose the points once, then we see that the orientation reverses. Even permutations of the set $\{a, b, c\}$ can be identified with one orientation and the odd permutations with the other. Figure 1.11 shows a trefoil knot and a Hopf link with orientations.

It can be seen that with orientations added, each crossing can be categorised in to two types, positive and negative. Refer to fig. 1.12. We can distinguish between the types of these crossings in the following manner. Hold your right hand thumb along the oriented direction of the upper strand. If the fingers curl along the oriented direction, then we call it a positive crossing and assign it the value $+1$. If the fingers curl in the opposite direction, then we call it a negative crossing at assign it -1.

1.6.1 Writhe

We define *writhe* w of a diagram D as the sum of all $\epsilon(c)$ for all $c \in D$.

$$\mathrm{w}(D) = \sum_{c \in D} \epsilon(c).$$

Note that writhe is a property of a diagram of a link, not the link itself. A link can have diagrams corresponding to all the possible writhes (the integers). We see this fact by applying the Reidemeister type I move repeatedly to generate a diagram with the desired writhe. A type I move can increase or decrease the writhe of a diagram depending upon the direction of twist we apply.

For example, the oriented cinquefoil knot in fig. 1.11 has a writhe of -5 as all the crossings are negative. Changing the orientation to another direction does not change the writhe of a knot as under an orientation change, the crossings preserve their signs. However, this is not true for links with more than one components, i.e. links which are not knots. The writhe of the oriented Hopf link as shown in fig. 1.11 is -2, but changing the orientations of the individual components gives us different writhes.

It was conjectured for many years that minimal diagrams of a link, i.e. diagrams with the minimal crossing number, had the same writhe. It was shown by Kenneth Perko in 1974 that two diagrams in the Rolfsen catalogue represented the same knot [9]. They were thought to represent different knots as both were minimal crossing diagrams with different writhes. These knot diagrams go by the famous name of 'Perko pair'.

1.6.2 Linking number

Let D be an oriented diagram of a 2-component link $K_1 \cup K_2$, and let D_i denote the component of D corresponding to K_i. The crossings of D are of three types: D_1 with itself, D_2 with itself, and D_1 with D_2. We shall denote the last group by $D_1 \cap D_2$. We define the *linking number* of D_1 with D_2 as

$$\mathrm{lk}(D_1, D_2) = \frac{1}{2} \sum_{c \in D_1 \cap D_2} \epsilon(c).$$

As the name suggests, this quantity captures how many times one component 'winds around' or 'goes around' an other component. The linking number for the Hopf link in fig. 1.11 is -1. If we change the orientation of any of the components, keeping the other constant, we shall get a linking number of $+1$. Changing the orientation of a component and keeping the others constant shall change only the sign of the linking number, and not the absolute value.

Linking number is an invariant for oriented two component links as it does not change under the Reidemeister moves. This shall be a general strategy for proving that a function is an invariant; in order to prove that a function is a link invariant, we prove its invariance under the Reidemeister moves.

Chapter 2

The Bracket Polynomial

Given two links or knots, one desires a way to tell if the two links are distinct. Since we can represent knots faithfully on a paper using knot diagrams, one can ask for a way to distinguish knots based on their diagrams. A *link invariant* is a function of a link such that if the evaluation of the function on two links yields different outputs, then the links are distinct, i.e. they are not ambient isotopic to each other. A link invariant is said to be *complete* if it always gives different outputs for distinct links.

In this chapter, we shall a link invariant called the normalised bracket polynomial. It attaches each link a polynomial with coefficients from $\mathbb{Z}[A, A^{-1}]$, where A and A^{-1} are some commuting variables. This invariant is not complete. It was discovered by Louis Kauffman in 1987 [10, 11]. The normalised version of this polynomial is equivalent to the Jones polynomial, which was discovered by Vaughn Jones in 1985 while working on the theory of operator algebras [12]. The discovery of the Jones polynomial created a flurry of activity as relations between knot theory and mathematical physics were found. Several generalisations of the Jones/bracket polynomial were immediately found and some long standing problems in knot theory, such as the Tait conjectures were proved. The approaches of Jones and Kauffman are very different while defining their polynomials. While understanding the original route taken by Jones to define his polynomial requires the knowledge of the theory of von Neumann algebras, Kauffman takes an elementary, but powerful diagrammatic approach while defining his polynomial. In this chapter, we shall look at Kauffman's bracket polynomial using the so called *state model*, as expounded in his book [1].

2.1 State model

Consider a crossing of an unoriented link as shown in fig. 2.1. We designate local regions around a crossing a label A or B based on the following scheme. Walk along the underpass towards a crossing. The area on the left is assigned the label A and the area on the right the label B. This assignment is unambiguous. We then 'resolve' or 'smoothen' or 'split' the crossing in two ways, one way which connects the A-regions and another way which connects the B-regions. If we resolve a crossing such that A-regions are connected, then we attach the label A to the resolved diagram. The same holds for B-regions. Note that the resolution of the crossing happens only locally, i.e. in an open ball around the crossing which does not intersect other crossings. Such an open ball exists due to the Hausdorff property of the plane.

We shall do this process recursively for all crossings to get a sets of Jordan curves in a plane with labels attached to them. Refer to fig. 2.2 where we have carried this process for a trefoil.

Each of the individual diagrams shown in curly brackets in fig. 2.2 is referred to as a state. To each state are the labels A and B attached to it. We can construct the original link unambiguously

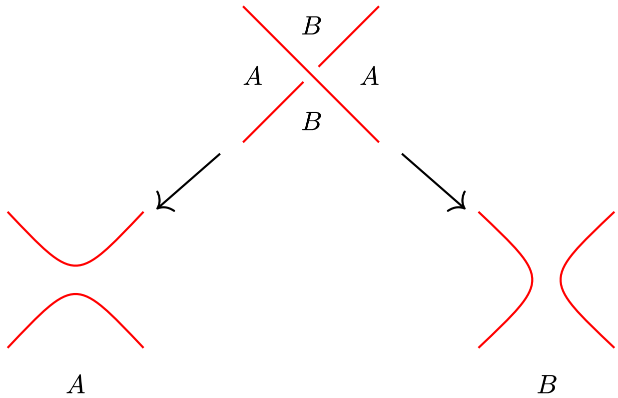
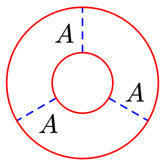


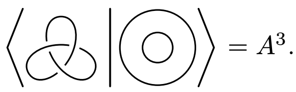
Figure 2.1: Resolution of a crossing with labels

using the states (and labels attached to them). We shall construct invariants of links by ‘averaging’ over these states.

By averaging, we mean the following. Let L be a link and σ denote a particular state obtained after resolving all the crossings recursively. We denote the commutative product of the labels associated to that state by $\langle K|\sigma\rangle$. For example, let K denote a right-handed trefoil, as shown in fig. 2.2 and σ denote the following state.



We then have



Let $\|\sigma\|$ denote one less than the number of loops in σ. Thus, for the above state, we have $\|\sigma\| = 1 - 1 = 0$.

We define the bracket polynomial $\langle L\rangle$ of a link L as follows.

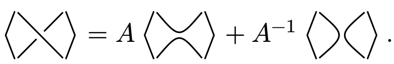
$$\langle L\rangle = \sum_{\sigma} \langle L|\sigma\rangle d^{\|\sigma\|},$$

where A, B and d are commuting variables. The bracket polynomial is thus a function of A, B and d. σ runs over all the states of K.

Thus, we calculate $\langle L|s\rangle$ for each state for the trefoil to get

$$\begin{aligned}\langle L\rangle &= A^3d^{2-1} + A^2Bd^{1-1} + A^2Bd^{2-1} + A^2Bd^{1-1}\\ &\quad + AB^2d^{2-1} + AB^2d^{2-1} + B^3d^{3-1}\\ \langle L\rangle &= A^3d + 3A^2Bd^0 + 3AB^2d^1 + B^3d^2.\end{aligned}$$

Theorem 25 (Skein relation)**.**



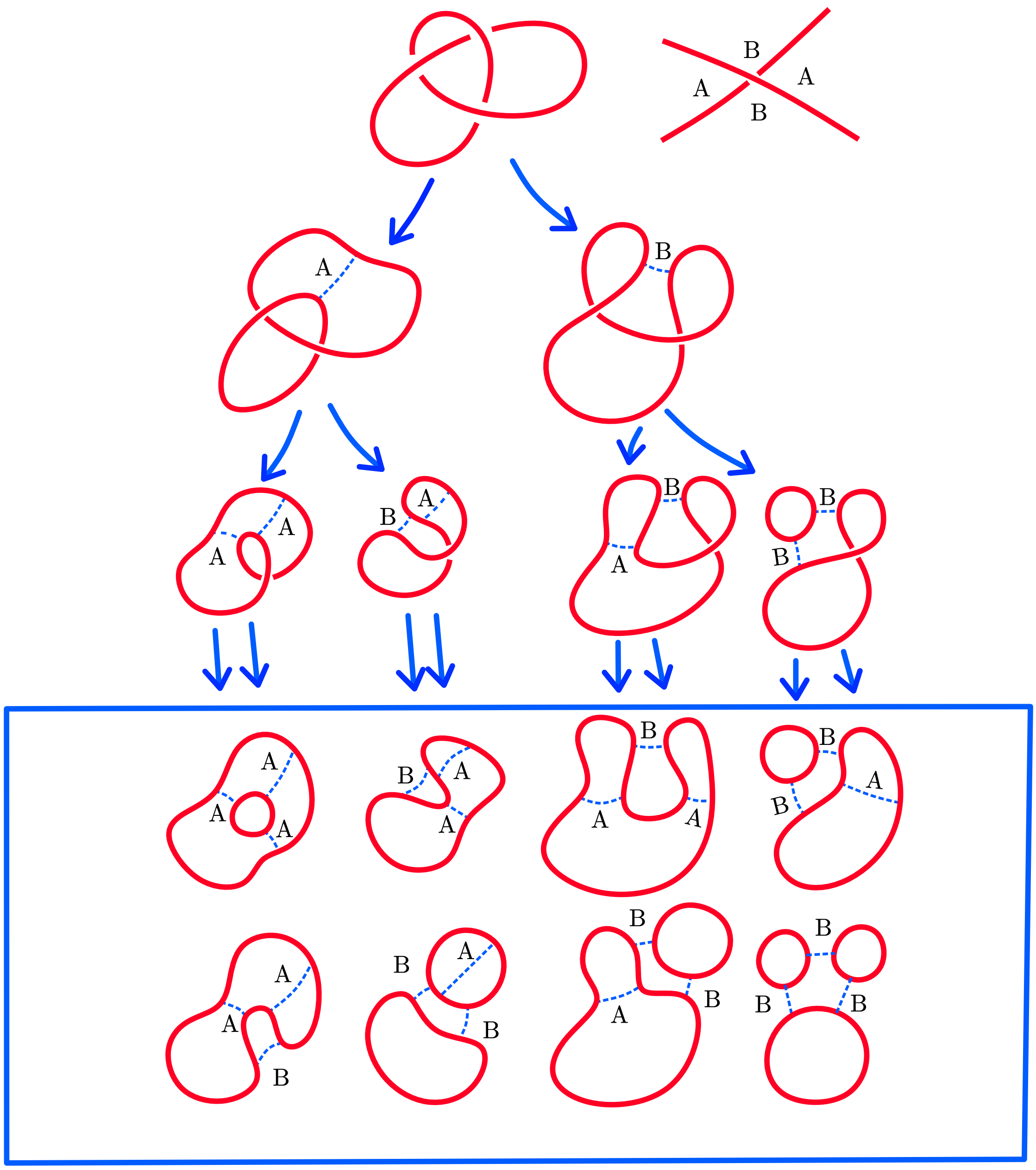


Figure 2.2: Resolution of all crossings for a trefoil.

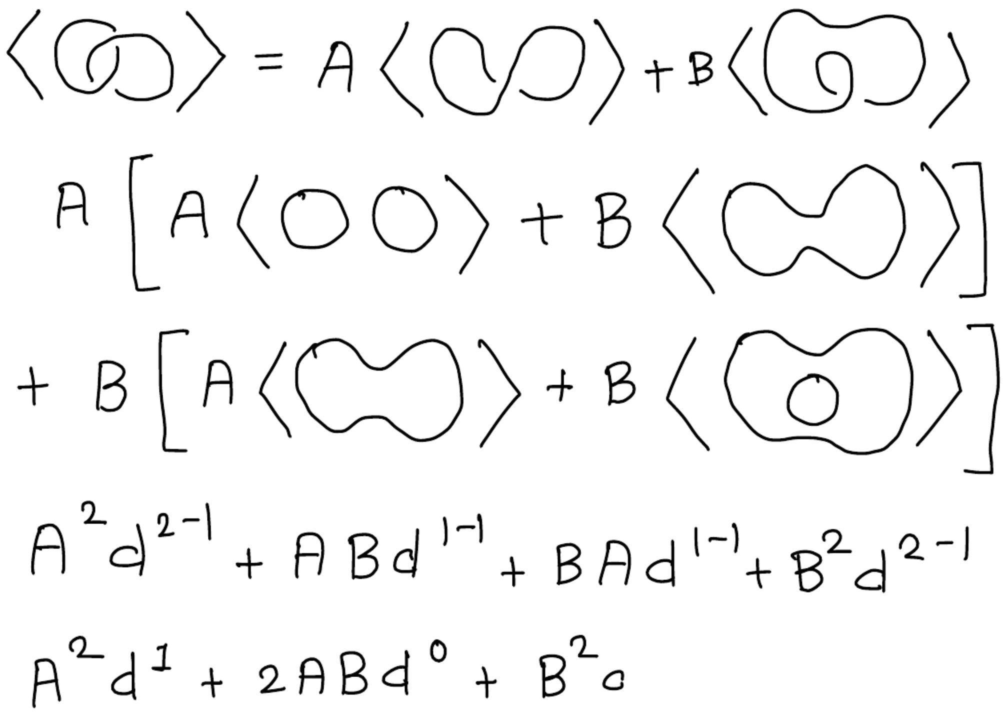
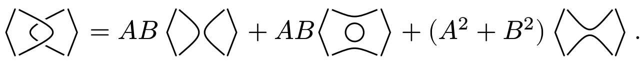


Figure 2.3: Bracket polynomial calculation for the Hopf link

The diagrams in the above theorem should be regarded as local diagrams. We make changes to a diagram locally near the crossing. There exists an open topological ball around each crossing such that the ball does not intersect any other crossing and the diagram remains identical outside this ball.

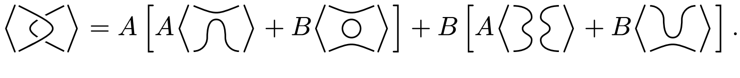
The proof of the above theorem follows from the definition of $\langle L \rangle$ and realizing that the states of a diagram are in one-to-one correspondence with the union of the diagrams with resolved crossings. The above identity is very important and we shall use it very often. For an example of a calculation of the bracket polynomial of a link, please refer to fig. 2.3.

Theorem 26.

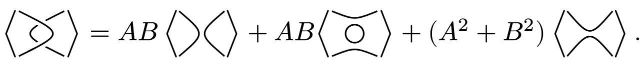


Proof.

This splits into

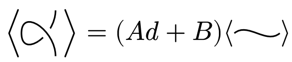


Finally we have

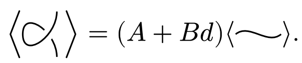


□

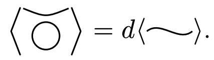
It is also clear that



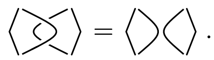
and



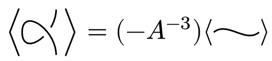
We also see that



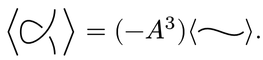
Let $B = A^{-1}$ and $d = -A^2 - A^{-2}$. We shall always use these substitutions from now on. We see that



This illustrates the invariance under a type II move. But note that



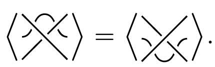
and



This follows from substituting $B = A^{-1}$ and $d = -A^2 - A^{-2}$ in $Ad + B$ to get $-A^3$. This illustrates that the bracket polynomial is *not* invariant under a type I move. Adding a twist multiplies the bracket by a factor of $-A^3$ or $-A^{-3}$, depending upon the type of the twist. This suggests that if we multiply by a compensatory factor, then we can possibly make it invariant under a type I move as well. We shall do this later using a normalisation factor.

Type II invariance of the bracket and the substitutions ensure that the bracket polynomial is invariant under a type III move as well.

Theorem 27.



Proof. We have

Using the type II moves, we have

□

We have thus found a regular isotopy invariant. A general principle is that once we have a regular isotopy invariant, we can suitably normalise it to make it ambient isotopy invariant. We know that writhe is an invariant of regular isotopy. Adding a positive twist increases the writhe by unit value and adding a negative twist decreases the writhe by unit value. As remarked earlier, we multiply the bracket by compensatory factor $-A^3)^{-\mathrm{w}(K)}$ to define the normalised bracket polynomial.

$$\mathrm{L}(A, K) := (-A^3)^{-\mathrm{w}(K)}\langle K\rangle.$$

Theorem 28. The normalised bracket polynomial is an invariant of ambient isotopy.

Proof. Since both the writhe and the bracket are regular isotopy invariant, it follows at once that the normalised bracket is regular isotopy invariant as well. We need to check type I move invariance. Recall that

$\Big\langle$  $\Big\rangle = (-A^{-3})\langle$  $\rangle$

and

$\Big\langle$  $\Big\rangle = (-A^{3})\langle$ $\rangle.$

These identities combined with the behaviour of writhe under adding twists ensures type I invariance. □

Theorem 29. Let K^* denote the mirror image of the oriented link K. Then we have that $\langle K^*\rangle(A) = \langle K\rangle(A^{-1})$ and $\mathrm{L}(A, K^*) = \mathrm{L}(A^{-1}, K)$.

Proof. Note that the link diagram of the mirror image of a link is obtained by switching all the crossings. This means that all the A's and the A^{-1}'s get exchanged as well. □

We calculate the bracket polynomial for various knots and links in the appendix. In particular, we calculate the normalised bracket polynomial for a trefoil and and its mirror image. For the trefoil shown in fig. 1.5, which is the left-handed trefoil knot, the normalised bracket polynomial turns out to be $A^4 + A^{12} - A^{16}$ and the normalised bracket polynomial for its mirror image turns out to be $A^{-4} + A^{-12} - A^{-16}$, as expected. But since the polynomials are different, we can conclude that a trefoil is *not* ambient isotopic to its mirror image. The normalised bracket polynomial for the Whitehead figure eight knot turns out to be $A^8 - A^4 + 1 - A^{-4} + A^{-6}$, which is symmetric in the powers of A, indicating that the mirror image of the figure eight knot is identical to the original one. While this proof of the chirality of trefoil is elementary, this fact was originally proved by Max Dehn in 1914 using the newly discovered fundamental group [13, p. 200].

Chapter 3

The Jones polynomial and its Generalisations

In the previous chapters, we defined Kauffman's bracket polynomial using the state model and then normalised it to get a link invariant. Instead of using the state model, one can define the bracket polynomial in an axiomatic way as well. One defines the polynomial as a function on link diagrams which satisfies certain properties. We would need to check well-definedness and invariance of such a function under the Reidemeister moves. In this chapter, we shall take a cursory look at the Jones polynomial and some of its generalisations. We shall not check well-definedness and invariance under the Reidemeister moves.

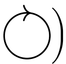
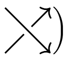
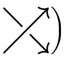
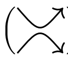
3.1 Jones polynomial

Definition 30 (Jones polynomial)**.** The Jones polynomial invariant $\mathrm{V}(t, K)$ is defined as

$$\mathrm{V}(t, K) := \mathrm{L}\left(\frac{1}{t^{1/4}}, K\right)$$

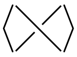
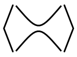
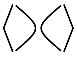
for an oriented link K.

Here, $t = \dfrac{1}{A^4}$. $\mathrm{V}(t, K)$ is a Laurent polynomial in $t^{1/2}$; $\mathrm{V}(K) \in \mathbb{Z}\left[t^{1/2}, \dfrac{1}{t^{1/2}}\right]$. With the above definition, we have the following two properties of the Jones polynomial.

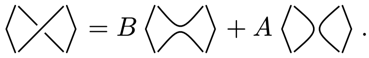
1. $\mathrm{V}\Big(t,$  $\Big) = 1.$
2. $\dfrac{1}{t}\mathrm{V}\Big(t,$  $\Big) - t\,\mathrm{V}\Big(t,$  $\Big) = \left(t^{1/2} - \dfrac{1}{t^{1/2}}\right)\mathrm{V}\Big($  $\Big).$

We can define the Jones polynomial axiomatically as a Laurent polynomial in $t^{1/2}$ satisfying the above two properties as well. But then we would have to check well-definedness and invariance under the Reidemeister moves. We derive these properties using the definition. The first property follows evaluating the bracket polynomial on the trivial knot. Orientation of a knot does not matter in this case. Just as with the normalised bracket polynomial, we shall drop the variable in which the polynomial is based if it is clear from the context.

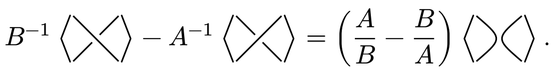
Proof of the second property. We know that

$\Big\langle$  $\Big\rangle = A\Big\langle$  $\Big\rangle + B\Big\langle$  $\Big\rangle$

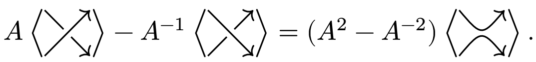
and

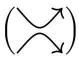
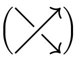
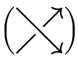


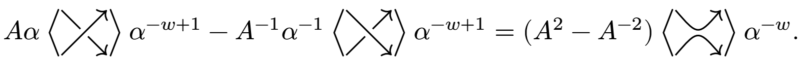
Hence,



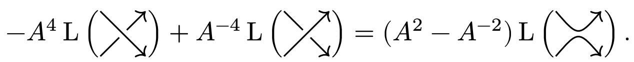
Thus,



Let $w = \mathrm{w}$  so that w  $= w - 1$ and w  $= w + 1$. Also, let $\alpha = -A^3$. Then



Thus,



Substituting $A = t^{-1/4}$ yields the desired result. $\square$

The Jones polynomial follows a reversing property.

Theorem 31 (Reversing property)**.** Let K be an oriented link. Let K' be such that K' is obtained by reversing the orientation of a component $K_i \subset K$.

Let $\lambda = \mathrm{lk}(K_i, K - K_i)$ denote the total linking number of K_i with the remaining components of K. Then

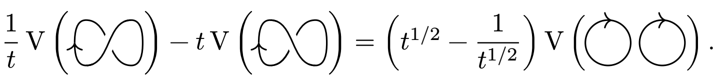
$$\mathrm{V}(t, K') = t^{-3\lambda} \cdot \mathrm{V}(t, K).$$

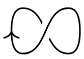
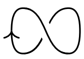
Proof. We see that $\mathrm{w}(K') = \mathrm{w}(K) - 4\lambda$.

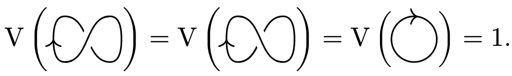
$$\begin{aligned} \mathrm{L}(A, K') &= (-A^3)^{-\mathrm{w}(K')} \langle K' \rangle \\ &= (-A^3)^{-\mathrm{w}(K) + 4\lambda} \langle K \rangle. \end{aligned}$$

Therefore $\mathrm{L}(A, K') = (-A^3)^{4\lambda}\, \mathrm{L}(A, K)$. Substituting $A = t^{-1/4}$ yields the desired result. $\square$

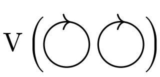
Example 32. Using the skein relation, we have that



But since both  and  are trivial knots,



Thus,

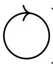
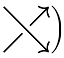
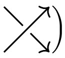
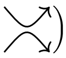


$$= \frac{t^{-1} - t}{t^{1/2} - 1/t^{1/2}} = \frac{(t^{-1} - t)(t^{1/2} + 1/t^{1/2})}{t - t^{-1}} = -(t^{1/2} + 1/t^{1/2}) = \delta.$$

The third property is surprising when viewed with respect to the skein relation [14, 15]. Skein relations of this sort form a general defining feature for various polynomials.

3.2 Alexander–Conway polynomial

Definition 33 (Alexander–Conway polynomial)**.** Let K be an oriented link diagram. Then the Alexander–Conway polynomial invariant $\nabla(z, K) \in \mathbb{Z}[z]$ is defined by the rules:

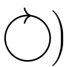
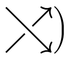
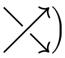
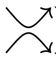
1. $\nabla\Big(z,$  $\Big) = 1.$
2. $\nabla\Big(z,$  $\Big) - \nabla\Big(z,$  $\Big) = \nabla\Big(z,$  $\Big).$

This polynomial is a generalisation and reformulation of the original Alexander polynomial.

3.3 HOMFLYPT polynomial

The HOMFLYPT polynomial was discovered independently by two groups, one group consisting of Jim Hoste, Adrian Ocneanu, Kenneth Millett, Peter Freyd, William Lickorish, and David Yetter and the other group consisting of Polish mathematicians Józef Przytycki and Paweł Traczyk.

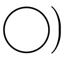
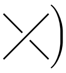
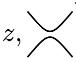
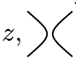
Definition 34 (HOMFLYPT polynomial)**.** Let K be an oriented link diagram. Then the HOMFLYPT polynomial invariant $\mathrm{P}(\alpha, z, K)$ is defined by the rules:

1. $\mathrm{P}\Big(\alpha, z,$  $\Big) = 1.$
2. $\alpha\,\mathrm{P}\Big(\alpha, z,$  $\Big) - \frac{1}{\alpha}\mathrm{P}\Big(\alpha, z,$  $\Big) = z\,\mathrm{P}\Big(\alpha, z,$  $\Big).$

Note that this polynomial is a two variable polynomial. If we take $\alpha = t^{-1}$ and $z = t^{1/2} - 1/t^{1/2}$, we retrieve the Jones polynomial. If we take $\alpha = 1$, we retrieve the Alexander–Conway polynomial.

3.4 Kauffman polynomial

The Kauffman polynomial $\mathrm{F}(\alpha, z, K)$ is a semi-oriented two variable polynomial invariant which generalises Kauffman's bracket and the Jones polynomial. This polynomial is a normalisation of a polynomial, $\overline{\mathrm{L}}(\alpha, z, K)$, defined for unoriented links which satisfies the following properties.

1. $\overline{\mathrm{L}}\Big(\alpha, z,$  $\Big) = 1.$
2. $\overline{\mathrm{L}}\Big(\alpha, z,$  $\Big) + \overline{\mathrm{L}}\Big(\alpha, z,$  $\Big) = z\Big[\overline{\mathrm{L}}\Big(\alpha, z,$  $\Big) + \overline{\mathrm{L}}\Big(\alpha, z,$  $\Big)\Big].$
3. $\overline{\mathrm{L}}\Big(\alpha, z,$  $\Big) = \alpha\,\overline{\mathrm{L}}\,(\alpha, z,$  $)$
4. $\overline{\mathrm{L}}\Big(\alpha, z,$  $\Big) = \frac{1}{\alpha}\,\overline{\mathrm{L}}\,(\alpha, z,$  $).$

The Kauffman polynomial is defined by the formula

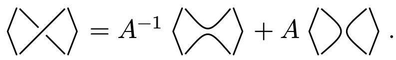
$$\mathrm{F}(\alpha, z, K) = \alpha^{-w(K)}\,\overline{\mathrm{L}}(\alpha, z, K).$$

The bracket and the Jones polynomials are special cases of the Kauffman polynomial.

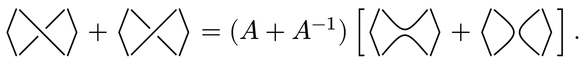
Theorem 35. $\langle K\rangle(A) = \overline{\mathrm{L}}(-A^3, A + A^{-1}, K).$

Proof. We recall the bracket identities

and



Adding them, we have



Thus, $\langle K\rangle(A) = \overline{\mathrm{L}}(-A^3, A + A^{-1}, K)$. $\square$

Theorem 36. $\mathrm{V}(t, K) = \mathrm{F}(-t^{-3/4}, t^{-1/4} + t^{1/4}, K)$.

Proof. Since

$$\mathrm{V}(t, K) = \mathrm{L}(t^{-1/4}, K)$$

and

$$\mathrm{L}(A, K) = (-A^3)^{-\mathrm{w}(K)}\langle K\rangle(A),$$

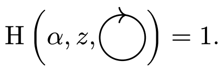
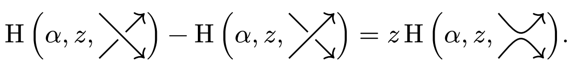
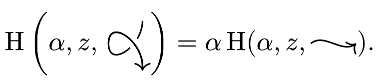
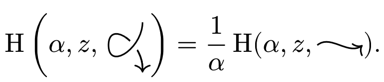
we have that

$$\mathrm{V}(t, K) = (-t^{-3/4})^{-\mathrm{w}(K)}\,\mathrm{L}(-t^{-3/4}, t^{-1/4} + t^{1/4}, K).$$

Hence, $\mathrm{V}(t, K) = \mathrm{F}(-t^{-3/4}, t^{-1/4} + t^{1/4}, K)$. $\square$

3.5 Regular isotopy HOMFLYPT polynomial

The regular isotopy HOMFLYPT polynomial $\mathrm{H}(\alpha, z, K)$, is defined by the following properties.

1. 
2. 
3. 
4. 

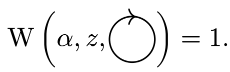
We take the existence of this invariant for granted.

Theorem 37. $\mathrm{P}(\alpha, z, K) = \alpha^{-\mathrm{w}(K)}\,\mathrm{H}(\alpha, z, K)$.

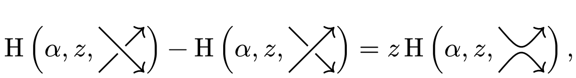
Proof. Let

$$\mathrm{W}(\alpha, z, K) = \alpha^{-\mathrm{w}(K)}\,\mathrm{H}(\alpha, z, K).$$

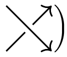
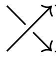
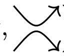
We see that $\mathrm{W}(\alpha, z, K)$ is invariant under ambient isotopy, and

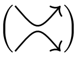


We thus check the behaviour under the exchange identity. We have



and thus,

$\alpha^{+1}\alpha^{-(w+1)}\,\mathrm{H}\left(\alpha, z,\right.$  $\left.\right) - \alpha^{-1}\alpha^{-(w-1)}\,\mathrm{H}\left(\alpha, z,\right.$  $\left.\right) = z\alpha^{-w}\,\mathrm{H}\left(\alpha, z,\right.$  $\left.\right)$,

where $w = w\left(\right.$  $\left.\right)$. $\square$

Chapter 4

Braids and the Jones polynomial

4.1 Motivation

$$\begin{tikzcd}
 & & \text{Jones algebra} \arrow[r, "\text{Markov trace}"] & \text{Jones polynomial} \arrow[dd, leftrightarrow, "\text{Substitution}"] \\
\text{Oriented link} \arrow[r] & \text{Artin braid group} \arrow[ur, "\text{Repres.}"] \arrow[dr, "\text{Repres.}"] & & \\
 & & \text{Temperley–Lieb algebra} \arrow[r, "\text{Diagrammatic trace}"] & \text{Normalised bracket polynomial}
\end{tikzcd}$$

As remarked earlier, Jones arrived at his polynomial indirectly while working on the theory of operator algebras [12]. In his course of investigations, he constructed a tower of algebras nested in one another with the property that each of these algebras is generated by a set of generators satisfying a particular set of relations. A degree of similarity between these relations and the relations among the generators of the *Artin braid group* was pointed out to Jones by a student during a seminar [2, p. 216], which led to the investigations of Jones into knot theory. Jones had defined a notion of a *trace* on his algebras; more specifically a trace function obeying the *Markov property*. As we shall see, one can express every link in terms of a (non-unique) *braid*. Jones then defined a representation of such a braid into his algebras. The trace of an algebra representation of a braid, which is in turn obtained from the link, can be calculated. The Jones polynomial was realized as such a trace.

In this chapter, we shall not travel the original route of Jones to reach his polynomial as it requires the knowledge of the theory of von Neumann algebras. Instead, we shall follow the approach described by Kauffman in his book to construct a representation of the Artin braid group into the *Temperley–Lieb algebra* [1, chp. 8]. These algebras admit a diagrammatic intrepretation and our definition of a trace on these algebras shall be diagrammatic in nature as well. Via this trace, we eventually reach the bracket polynomial, which we already know to be equivalent to the Jones polynomial as demonstrated earlier. The Jones algebra can be recovered from the Temperley–Lieb algebra by a choice of substitutions. The Temperley–Lieb algebra arose during the study of certain statistical models in physics [16]. This algebra can be viewed as a sub-algebra

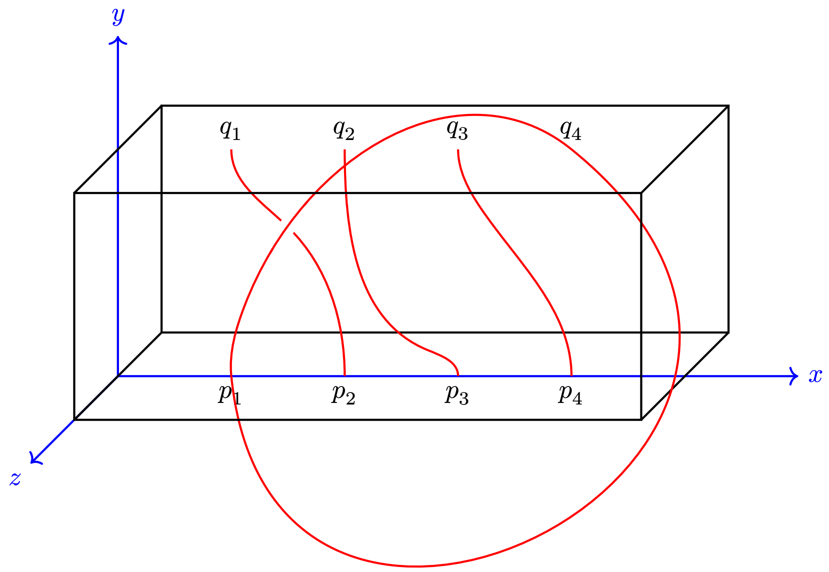


Figure 4.1: Three dimensional geometric representation of braids.

of a broader framework of the *partition algebra* [17].

4.2 Geometric representation of braids

4.2.1 Definition

We shall now understand some basics of braid theory. Emil Artin introduced the Artin braid group explicitly [18, 19, 20].

An n-braid is an element of the Artin braid group B_n, defined via the following presentation on the generators σ_i, for $1 \leq i \leq n-1$.

$$\mathrm{B}_n := \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{rcll} \sigma_i \sigma_i^{-1} & = & \mathbb{I}_n^{\mathrm{a}} & \\ \sigma_i \sigma_{i+1} \sigma_i & = & \sigma_{i+1} \sigma_i \sigma_{i+1} & \\ \sigma_i \sigma_j & = & \sigma_j \sigma_i & \text{if } |i-j| \geq 2 \end{array} \right\rangle,$$

where $\mathbb{I}_n^{\mathrm{a}}$ is the identity of B_n. Thus, B_n is the quotient of the free group of $n-1$ generators with the smallest normal subgroup of the free group containing the elements $\sigma_i \sigma_i^{-1}$, $\sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1}$ and $\sigma_i \sigma_j \sigma_i^{-1} \sigma_j^{-1}$ with appropriately restricted i and j. We want to recover this algebraic definition using the intuitive understanding of braids that we have. For, that we shall now see a geometric construction in the three dimensional Euclidean space to represent the Artin braid group. This shall make clear the geometric interpretation of the relations as well.

Consider two *ordered* sets of points $L_1 := \{p_1 := (1,0,0), \dots, p_n := (n,0,0)\}$ and $L_2 := \{q_1 := (1,1,0), \dots, q_n := (n,1,0)\}$ as shown in fig. 4.1 for $n = 4$. Elements of L_1 are called bottom points and elements of L_2 are called the top points. For $1 \leq i \leq n$, consider a family of non-intersecting continuous curves $\gamma_i \colon [0,1] \to \mathbb{R}^3$ such that

1. $\gamma_i(0) = p_i$ and $\gamma_i(1) = q_j$ for $1 \leq i, j \leq n$.
2. Any plane perpendicular to the xy-plane and parallel to the x-axis intersects each of the curves either exactly once or not at all.
3. All the curves strictly lie in the cube determined by the vertices $(0,0,1)$, $(0,0,-1)$, $(0,1,1)$, $(0,1,-1)$, $(n+1,0,1)$, $(n+1,0,-1)$, $(n+1,1,1)$, $(n+1,1,-1)$.

Such a labelled curve is called a strand in standard position and a family of such labelled n stands is called an n-strand set in a standard position. We can ambient isotope or rigidly move an n-strand set to get another n-strand set, possibly not in a standard position. Two n-strand sets are said to be equivalent if they are related by a finite sequence of the following operations.

1. Rigid motions of $\mathbb{R}^3$. Note that the labels of the strands do not change.
2. Ambient isotopies of the strand sets such that all the space outside the cube (and consequently the boundary, with the end-points) remains constant.

Remark 38. Though rigid motions of the Euclidean space include rotations and reflections, we carry the labels of the endpoints as well when we isotope. For example, suppose we have two strand sets such that if we reflect points of one strand set with respect to a plane, we the get unmarked points of the other strand set. This reflection shall reflect the marked endpoints of the strands as well. The two strand sets in consideration shall be considered equivalent iff all the marked endpoints match as well.

We shall refer to an equivalence class of such n-strands as a geometric n-braid. A geometric n-braid is well-defined.

Remark 39. Even though we have restricted our strands to the a bounded cube in the standard position, we can in principle change the bounds of our cube in x and z directions to any value and get the same theory. We shall not pursue this approach here.

4.2.2 Standard projection

We call the projection of a standard position n-strand set onto the xy-plane to be a two dimensional representation of a braid. Such a projection is drawn in fig. 4.2. It should be noted that a standard position n-strand set is unique only up to ambient isotopy, thus correspondingly the two dimensional representation of such a set is also unique only up to ambient isotopy, namely the ambient isotopies of the projection of the cube and the ambient isotopies such that the projection is a two dimensional representation of a braid for all times.

Now onwards, we shall always visually represent geometric n-braids using their standard two dimensional projections.

4.2.3 Group structure

Multiplication of any two n-geometric braids b_1 and b_2, denoted by $b_1 b_2$ is defined as follows (fig. 4.3). Ambient isotope and then rigidly move b_1 and b_2 *separately* in the standard position. Now translate only b_1 in the $+y$ direction by unit distance. The bottom points of b_1 and the top points of b_2 now coincide. Concatenate their strands and shrink the concatenated strands in the y direction by half keeping fixed the bottom points of b_2. The result is another geometric n-braid $b_1 b_2$ in the standard position. Multiplication defined this way is associative.

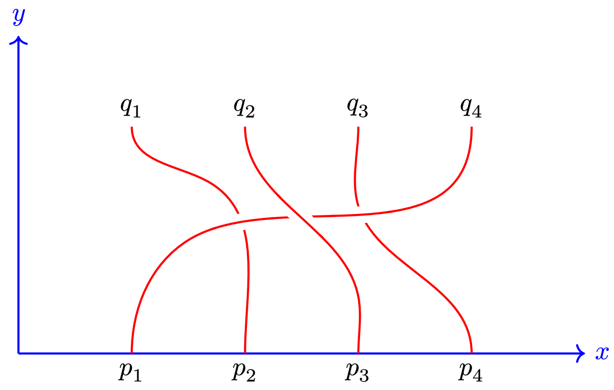


Figure 4.2: Two dimensional geometric representation of braids.

Remark 40. Braid multiplication is not commutative in general. The reader is requested to refer to remark 38.

We shall now drop the axes as well while representing two dimensional geometric n-braids.

An n-strand set such that each γ_i is a straight line segment connecting the i^{th} bottom point to the i^{th} top point is called the identity geometric n-braid and is denoted by $\mathbb{I}_n$ (fig. 4.4).

A geometric n-braid a such that $ab = ba = \mathbb{I}_n$ for some geometric n-braid b is called the inverse of b and denoted is by b^{-1}. We shall see that each element has an inverse.

With these operations, the set of geometric n-braids becomes a group, which we shall denote by GB_n.

4.2.4 Generators

By the virtue of ambient isotopy, we can move the crossings in a two dimensional representation of a geometric n-braid such that each crossing lies in a region bounded by two lines parallel to the x-axis. Moreover, we can arrange the crossings such that each such region contains only one crossing. Thus, if we give the information regarding the type of each crossing for each such region, we can faithfully reconstruct the two dimensional representation. To this end, we define the generators of a geometric n-braid.

Denote by τ_i the geometric n-braid such that

1. $\gamma_i(1) = q_{i+1}$, $\gamma_{i+1}(1) = q_i$, and $\gamma_j(1) = q_j$ when j does not equal i or $i+1$.
2. $\pi_{xy}(\gamma_i(t)) \geq 0$ and $\pi_{xy}(\gamma_{i+1}(t)) \leq 0$ for all $t \in [0, 1]$.

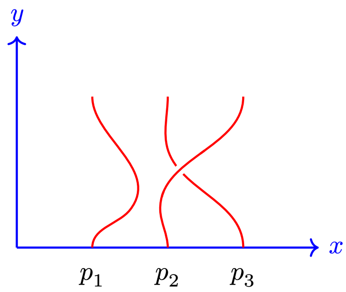
π_{xy} is the projection maps onto to xy-plane. $\tau_1, \dots, \tau_{n-1}$ are the generators of GB_n (fig. 4.5). For example, in fig. 4.3 we have $b_1 = \tau_2 \in \mathrm{GB}_3$, $b_2 = \tau_1 \in \mathrm{GB}_3$ and $b_1 b_2 = \tau_2\tau_1 \in \mathrm{GB}_3$.

If we multiply τ_i and τ_i^{-1} to form $\tau_i\tau_i^{-1}$, we observe that $\tau_i\tau_i^{-1} = \mathbb{I}_n$, where $\tau_i, \tau_i^{-1} \in \mathrm{B}_n$ for all $n \geq 2$ (fig. 4.6).

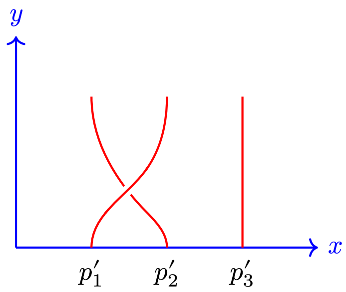
We can perform a move equivalent to the type III move to see that $\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$ (fig. 4.7).

We can slide two crossings vertically across each other if this does not change the ambient isotopy type. This is possible if the two crossings we wish to slide do not share a strand. This gives us the relation $\tau_i\tau_j = \tau_j\tau_i$ if $|i - j| \geq 2$ (fig. 4.8).

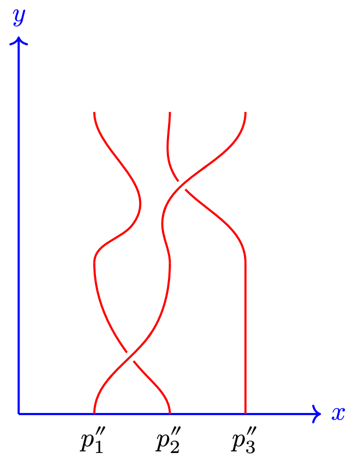
Let w be a word of length m in GB_n; $w = \prod_{j=1}^{m} \tau_{\alpha_j}^{\pm 1}$ where $1 \leq \alpha_j \leq n - 1$. Every element of GB_n and B_n can be expressed as a product of its generators, albeit non-uniquely. We define a



(a) b_1



(b) b_2



(c) b_1b_2

Figure 4.3: Multiplication of two braids (before shrinking).

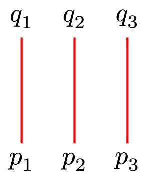
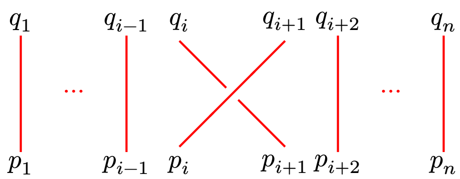
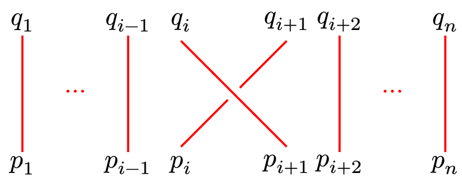


Figure 4.4: The identity $\mathbb{I}_3$



(a) τ_i



(b) τ_i^{-1}

Figure 4.5: Generators τ_i and τ_i^{-1}

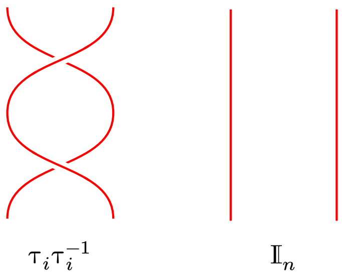


Figure 4.6: A type II move illustrating $\tau_i\tau_i^{-1} = \mathbb{I}_n$.

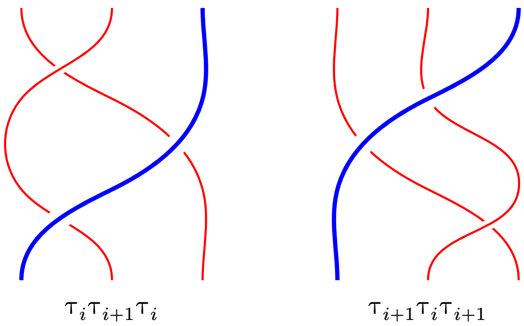


Figure 4.7: A type III move illustrating $\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$.

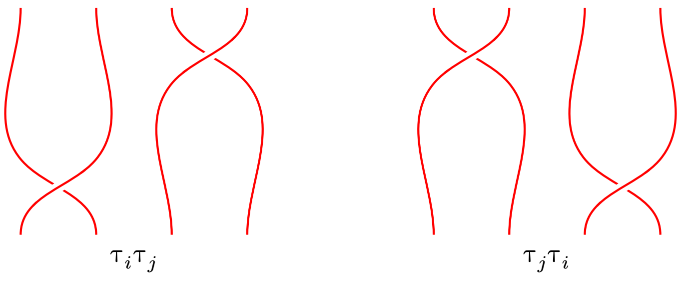
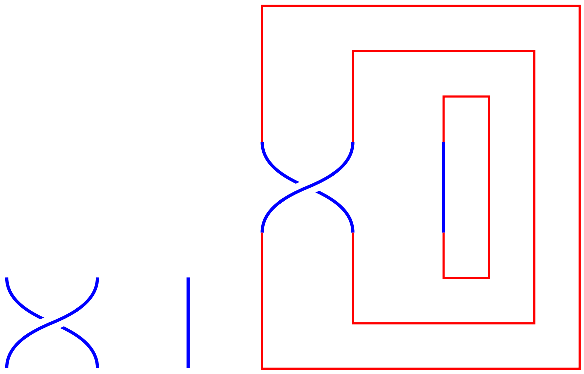


Figure 4.8: Sliding of crossings illustrating $\tau_i\tau_j = \tau_j\tau_i$.



(a) b (b) $\bar{b}$

Figure 4.9: Closure of a braid with $b = \tau_1 \in \mathrm{GB}_2$.

homomorphism

$$\Phi\colon \mathrm{GB}_n \to \mathrm{B}_n,$$
$$\Phi\colon \prod_{j=1}^{m} \tau_{\alpha_j}^{\pm 1} \mapsto \prod_{j=1}^{m} \sigma_{\alpha_j}^{\pm 1} \text{ for all } m \in \mathbb{N}.$$

We can see that Φ is a surjection as follows. Take an element $\prod_{j=1}^{m} \sigma_{\alpha_j}^{\pm 1} \in \mathrm{B}_n$, Φ maps $\prod_{j=1}^{m} \tau_{\alpha_j}^{\pm 1}$ to $\prod_{j=1}^{m} \sigma_{\alpha_j}^{\pm 1}$. Proving that Φ is an injection is harder and a proof can be found in [21, chp. 2], and [22].

Theorem 41. Φ is an isomorphism, i.e. B_n and GB_n are isomorphic.

This allows us to forget the distinction between B_n and GB_n.

4.3 Closure of braids

We define the closure of a geometric n-braid as follows. Consider a geometric n-braid in the standard position. For each $1 \leq i \leq n$, we construct the following sequence of line connected line segments. Join $(i, 1, 0)$, $(i, i, 0)$, $(i, i, 0)$, $(i, -i, 0)$, $(i, -i, 0)$, $(i, 0, 0)$ consecutively. We then join γ_i to the constructed line segments. Repeating this process for all i gives the closure of a braid (fig. 4.9). We denote the closure of a geometric n-braid b by $\bar{b}$. Closure of a braid is unique up to ambient isotopy. Two equivalent braid words have the same closures, thus making the closure well-defined.

Proposition 42. Every closure of a geometric n-braid is a link.

Proof. The outer line segments are locally flat by virtue of being piecewise linear. Due to the second condition regarding the intersection of a plane in the standard representation of geometric n-braid, we can project the strand onto the y-axis and this projection would be a homeomorphism. We take an ϵ neighbourhood, $N(\epsilon)$ around the strand. The pair $(N(\epsilon), \gamma_i)$ would then be homeomorphic to $(\mathrm{B}, \mathrm{B} \cap x\text{-axis})$, where B is the three-dimensional unit ball around the origin. We have local flatness at the end points as well due to the union of two locally flat curves. $\square$

Remark 43. Suppose $\tau_i \in \mathrm{GB}_n$ and $\tau'_i \in \mathrm{GB}_m$ are generators where $n < m$. Then the closures of these two generators are not ambient isotopic. The closure of the latter contains one more non-linking loop.

4.3.1 Alexander Theorem

The theorems of James Alexander [23] and Andrei Markov Jr. [24] relate braids to knots.

Theorem 44 (Alexander)**.** Every link is ambient isotopic to the closure of a geometric braid, for some $n \in \mathbb{N}$.

Proof. Consider a piecewise linear, regular projection $\pi(L)$ of a link L on a plane. We choose a point O in the projection plane which is not collinear with any of the line segments. This can be done since a the link has only finitely many line segments. Let $P \in \pi(L)$. The vector OP can move either clockwise or anti-clockwise as P moves along the link projection. We wish to modify the line segments such that OP moves in only one sense, say anti-clockwise, as P moves along the entire length of the link projection. We now fix our attention on a line segment corresponding to a clockwise rotation. We divide the segment into sub-parts such that each part shares at-most one crossing point with other line segments. If A and B are end-points of such a line segment, then we may replace this line segment with two another line segments AC and CB, such that C is another point not on belonging to $\pi(L)$ and the triangle ABC encloses O. If AB originally passed under (or over) a line segment of $\pi(L)$, then the modified line segments AC and CB must pass under (or over) of the line segments of $\pi(L)$ as well. This move shall not change the link type as it shall be a combination of sliding, type 2 and type 3 moves. In the resulting triangle, we have two orientations possible, one path which travels via C and the other path which does not. The vector OP shall move in the opposite, anti-clockwise sense while traversing from A to B via C, instead of AB. We can repeat this process for all of the (finitely many) line segments which turn clockwise. In the end, we obtain a projection such that OP moves in only the anti-clockwise sense, as P moves along the entire length of the link projection. We can ambient isotope the projection such that all the crossings lie in the projection of a cube, more precisely the cube constructed while defining a geometric braid. The end-points can be made to match as well. The above procedure of triangular moves shall guarantee the monotonicity that is required. ☐

Henceforth, unless specified otherwise, we shall always work with the projections of the standard representation of a geometric n-braid and its closure and refer to these projections simply as a braid and its closure.

4.3.2 Conjugation

If $b, g \in \mathrm{GB}_n$, we observe that $\overline{gbg^{-1}}$ is ambient isotopic to $\bar{b}$ (fig. 4.10). The upper strands in g and g^{-1} are connected via the closure strand. We can slide g and g^{-1} via the closure strands to the 'other side' of b to annihilate each other. This can be achieved by a type II move.

Remark 45. The closures of conjugate braids are ambient isotopic as links. The braids themselves are not.

4.3.3 Markov move

We observe that if $b \in \mathrm{GB}_n$, then $b\tau_n \in \mathrm{GB}_{n+1}$, $b\tau_n^{-1} \in \mathrm{GB}_{n+1}$ and b have ambient isotopic closures (fig. 4.11), although b, $b\tau_n$ and $b\tau_n^{-1}$ are not equivalent as braids. That is, we can add a strand and a crossing of that strand with another strand without changing the link type (of the closure). We can also remove a strand and a crossing if that strand does not cross any other

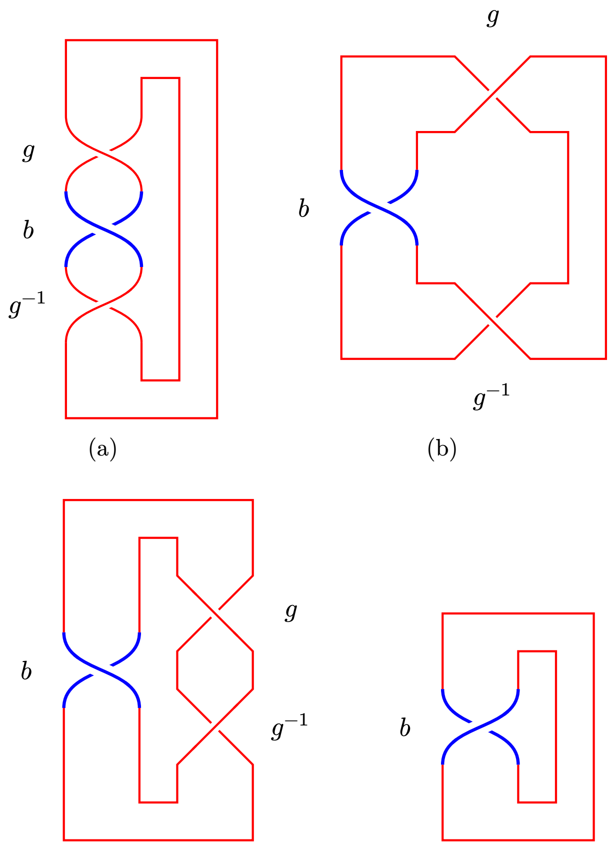
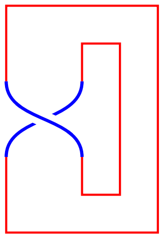
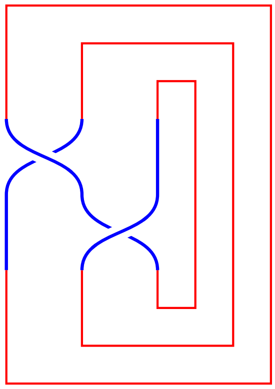


Figure 4.10: Conjugation process illustrating the link equivalence of $\overline{gbg^{-1}}$ and $\overline{b}$, with $b = \tau_1^{-1} \in$ GB_1 and $g = \tau_1^{-1} \in \mathrm{GB}_1$.



(a) $\bar{b}$



(b) $\overline{b\tau_2}$

Figure 4.11: Markov move with $b = \tau_1^{-1}$.

strand. We visually see that adding the above mentioned strands anywhere between the existing strands is equivalent to adding the strands on the right.

4.3.4 Markov Theorem

Theorem 46 (Markov)**.** Two braids whose closures are ambient isotopic to each other are related by a finite sequence of the following operations.

1. Braid equivalences, i.e. equivalences resulting due to the braid relations.
2. Conjugation.
3. Markov moves.

A proof of the above theorem, based on the notes of J. H. Roberts and the seminar of an unknown speaker at Princeton University, can be found in the book of Joan Birman [25, chp. 2]. Markov gave a sketch of the proof in 1936 [24]. A shorter proof may be found in Morton's paper [26].

4.4 Orientation

We see visually that the result of a Markov move on a braid is equivalent to performing a type I move on the braid closure. We know that a type I move increases or decreases the writhe of a link by a unit value. Since all the crossings in a braid closure occur only in the cube containing the braid strands, we can define the writhe of a braid equal to the writhe of the braid closure by assigning each braid crossing a value, either +1 or −1. Our assignment must be consistent with our earlier assignment for knots. But for this procedure, we need to assign an orientation to the braid. We assign all the strands (inside the braid cube) a downward orientation (fig. 4.12).

Doing so, we see that τ_i inherits +1 value while τ_i^{-1} inherits −1. We could instead have assigned all the strands an upward orientation as well. This would not have changed the values of τ_i and τ_i^{-1} (fig. 4.13). What is not allowed is assigning arbitrary orientation to strands. If we assign the orientation arbitrarily, then the well-definedness of the orientation cannot be guaranteed. Two distinct strands in a braid could be connected via the closure strands and one would need to check the whole connected link component of the braid closure for a well-defined closure.

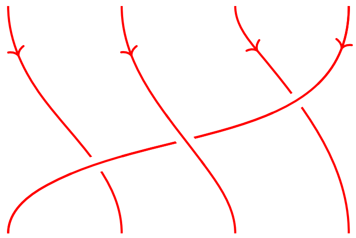


Figure 4.12: A braid with downward orientation

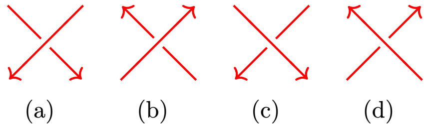


Figure 4.13: Assignment of crossing values. (a) Downward orientation for τ_i corresponding to $+1$. (b) Upward orientation for τ_i corresponding to $+1$. (c) Downward orientation for τ_i^{-1} corresponding to -1. (d) Upward orientation for τ_i^{-1} corresponding to -1.

Thus, the writhe w of the braid b with a word representation $\prod_{j=1}^{m} \tau_{\alpha_j}^{\beta_j}$ of length m, where $\beta_j \in \{+1, -1\}$ is

$$w(b) = \sum_{j=1}^{m} \beta_j.$$

Note that the writhe of a braid is dependent on its word representation. We also see that $w(b) = w(\bar{b})$.

4.5 Markov trace

We won't distinguish between B_n and GB_n from now on. We can now move finally towards the Jones/bracket polynomial with the information we have. If we have a function $J_n : \mathrm{B}_n \to R$, where R is a commutative ring, then using the Markov Theorem we can construct link invariants from the family of functions $\{J_n\}$ if the following conditions are satisfied.

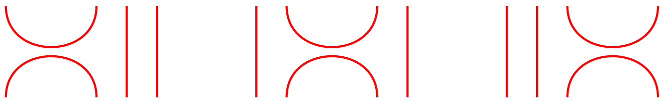
1. J_n is well defined. $J_n(b) = J_n(b')$ if $b = b'$.
2. $J_n(b) = J_n(gbg^{-1})$ if $g, b \in \mathrm{B}_n$.
3. If $b \in \mathrm{B}_n$, then there exists a constant $\alpha \in R$, independent of n, such that
$$J_{n+1}(b\sigma_n) = \alpha^{+1} J_n(b)$$
and
$$J_{n+1}(b\sigma_n^{-1}) = \alpha^{-1} J_n(b).$$

The last condition reminds us of the normalisation needed in order to make the bracket polynomial invariant under the type I move. Its purpose here is the same.

A family of functions $\{J_n\}$ satisfying the above given three conditions is called a Markov trace on $\{\mathrm{B}_n\}$. For any link L which is ambient isotopic to $\bar{b}$, where $b \in \mathrm{B}_n$, we define $J(L) \in R$ as follows.

$$J(L) := \alpha^{-w(b)} J_n(b).$$

We call $J(L)$ the link invariant for the Markov trace $\{J_n\}$.



(a) U_1 (b) U_2 (c) U_3

Figure 4.14: Input-output forms or hooks for 4 strands

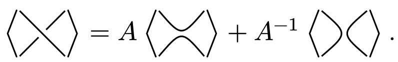
Theorem 47. J is an invariant of ambient isotopy for oriented links.

Proof. Suppose $L \sim \bar{b}$ and $L'' \sim \overline{b'}$, where $\sim$ denotes the ambient isotopy relation. By the Markov Theorem, we can obtain $\overline{b'}$ via an application of a finite sequence of the moves mentioned in the Markov Theorem on $\overline{b'}$. Each such move leaves J invariant. J_n is already invariant under braid equivalences and conjugation by definition. The $\alpha^{-w(b)}$ factor cancels the effect of a type I move. $\square$

We can define the bracket polynomial for braids in the same way as we did for links. Define

$$\begin{aligned} \langle\cdot\rangle &: \mathrm{B}_n \to \mathbb{Z}[A, A^{-1}] \\ \langle\cdot\rangle &: b \mapsto \langle\bar{b}\rangle. \end{aligned}$$

We simply evaluate the bracket on the closure of the braid. We observe that this function is a Markov trace with $\alpha = -A^3$. We know that



In terms of braids, we have

But  is the identity braid. Thus, if we denote  by U_i, we can write

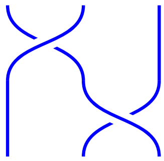
$$\langle\sigma_i^{-1}\rangle = A\langle\mathrm{U}_i\rangle + A^{-1}\langle\mathbb{I}_n\rangle.$$

Similarly, one could write

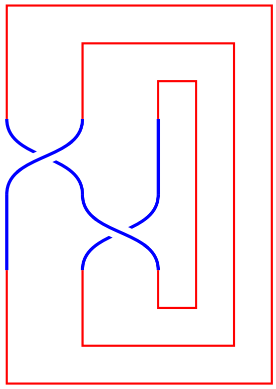
$$\langle\sigma_i\rangle = A\langle\mathbb{I}_n\rangle + A^{-1}\langle\mathrm{U}_i\rangle.$$

Note that U_i does not belong to the braid group and is a new object. We refer to them as "input-output forms" or as "hooks". They are cup and cap combinations involving the i-th and $(i+1)$-th strands (fig. 4.14). We can thus use consider σ_i to be equivalent to $A + A^{-1}\mathrm{U}_i$, σ_i^{-1} to be equivalent to $A^{-1} + A\mathrm{U}_i$, and create a formalism based on this. Given a braid word representation of a braid, we can simply substitute the above mentioned equivalences to get a product of U_i's, with A and A^{-1} as coefficients. Note that this product is dependent on the specific braid representation of a knot. It is not invariant under a type I move, as is the case with the un-normalised bracket polynomial.

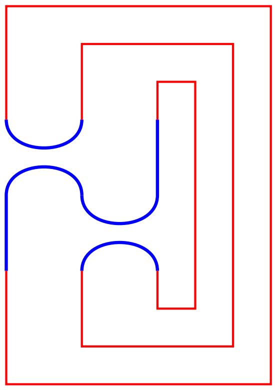
We see that each bracket polynomial evaluation state of the closure of a braid can be written in terms of the closure of a product of the input-output forms. Given a braid word b, we can consider it equivalent to $S(b)$, where $S(b)$ is the sum of products of the U_i's obtained by substituting σ_i by $A + A^{-1}\mathrm{U}_i$ and σ_i^{-1} by $A^{-1} + A\mathrm{U}_i$. Closure of each term (a product in U_i's) in the sum corresponds to a state obtained while evaluating the bracket polynomial as it gives a collection



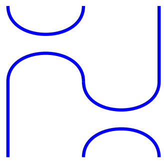
(a) b



(b) $\overline{b}$



(c) $s = \overline{U_1 U_2}$



(d) $U_1 U_2$

Figure 4.15: Writing a state of a braid closure in terms of input-output forms.

of loops. If P is such a product, then $\langle P \rangle = \langle \overline{P} \rangle = \delta^{\|P\|}$, where $\|P\|$ is the number of loops in $\overline{P}$ minus 1, and $\delta = -A^{-2} - A^2$. Thus,

$$S(b) = \sum_s \langle b|s \rangle P_s,$$

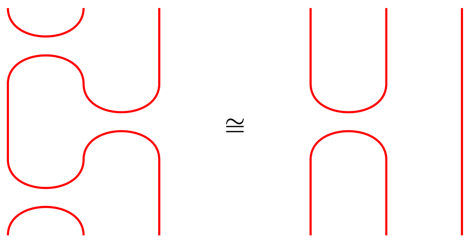
where s denotes a state and indexes all the terms in the product, and $\langle b|s \rangle$ is the product of A's and A^{-1}'s multiplying each P-product P_s. We have

$$\langle b \rangle = \langle S(b) \rangle = \sum_s \langle b|s \rangle \langle P_s \rangle = \sum_s \langle b|s \rangle \delta^{\|s\|}.$$

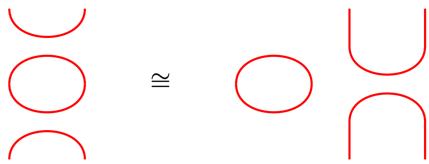
Example 48. Let $b = \sigma_1 \sigma_2^{-1}$. We can resolve $\overline{b}$ in many states. One of the states s and its corresponding product $\mathrm{U}_1 \mathrm{U}_2$ in terms of the input-output forms is shown in fig. 4.15.

Example 49. Consider the same braid $b = \sigma_1 \sigma_2^{-1}$. We have

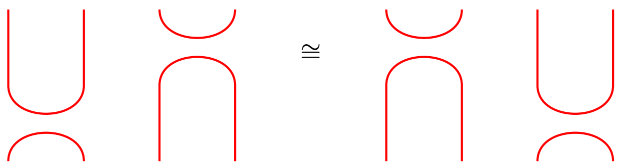
$$\begin{aligned} P(b) &= (A + A^{-1}\mathrm{U}_1)(A\mathrm{U}_2 + A^{-1}) \\ P(b) &= A^2\mathrm{U}_2 + \mathbb{I}_3 + \mathrm{U}_1\mathrm{U}_2 + A^{-2}\mathrm{U}_1 \\ \langle b \rangle = \langle \sigma_1 \sigma_2^{-1} \rangle = \langle P(b) \rangle &= A^2 \langle \mathrm{U}_2 \rangle + \langle \mathbb{I}_3 \rangle + \langle \mathrm{U}_1 \mathrm{U}_2 \rangle + A^{-2} \langle \mathrm{U}_1 \rangle. \end{aligned}$$



(a) $U_1U_2U_1 = U_1$



(b) $U_1^2 = \delta U_1$, we have a union of a loop(δ), and U_1 on the right side



(c) $U_3U_1 = U_1U_3$

Figure 4.16: Input-output form relations

Now, $\mathbb{I}_3$ corresponds to $|||$. Thus, $\langle I_3 \rangle = \delta^{3-1} = \delta^2$ as the closure of $\mathbb{I}_3$ shall give three loops. U_1 corresponds to . Thus, $\langle U_1 \rangle = \delta^{2-1} = \delta$ as the closure shall give two loops. Similarly, $\langle U_2 \rangle = \delta$ and $\langle U_1U_2 \rangle = \delta^{1-1} = 1$. For a visual representation of the state U_1U_2, see fig. 4.15, where the closure gives one single loop. In the end, we get

$$\langle \bar{b} \rangle = A^2\delta + \delta^2 + 1 + A^{-2}\delta.$$

4.6 Temperley–Lieb algebra

We can give the U_i's a structure of their own by constructing the free additive algebra TL_n with the generators $U_1, U_2, \dots, U_{n-1}$ and the multiplicative relations coming from the interpretation of U_i's as input-output forms. We can consider this algebra as module over the ring $\mathbb{Z}[A, A^{-1}]$ with $\delta = -A^{-2} - A^2 \in \mathbb{Z}[A, A^{-1}]$. We shall call TL_n the Temperley–Lieb algebra. The multiplicative relations in TL_n are as follows.

1. $U_iU_{i\pm1}U_i = U_i$.
2. $U_i^2 = \delta U_i$.
3. $U_iU_j = U_jU_i$ if $|i - j| \geq 2$.

These relations are a result of the geometric relations as illustrated in fig. 4.16.

Now that we have the Temperley–Lieb algebra, we can define a mapping

$$\rho\colon \mathrm{B}_n \to \mathrm{TL}_n$$

by the following formulae.

$$\begin{aligned}\rho(\sigma_i) &= A + A^{-1}\mathrm{U}_i,\\ \rho(\sigma_i^{-1}) &= A^{-1} + A\mathrm{U}_i.\end{aligned}$$

We see that ρ is indeed a representation by verifying that $\rho(\sigma_i)\,\rho(\sigma_i^{-1}) = 1$, $\rho(\sigma_i\sigma_{i+1}\sigma_i) = \rho(\sigma_{i+1}\sigma_i\sigma_{i+1})$ and if $|i-j| \geq 2$, $\rho(\sigma_i\sigma_j) = \rho(\sigma_j\sigma_i)$.

Proposition 50. $\rho\colon \mathrm{B}_n \to \mathrm{TL}_n$ is a representation of the Artin braid group.

Proof. We first prove that $\rho(\sigma_i)\,\rho(\sigma_i^{-1}) = 1$.

$$\begin{aligned}\rho(\sigma_i)\,\rho(\sigma_i^{-1}) &= (A + A^{-1}\mathrm{U}_i)(A^{-1} + A\mathrm{U}_i)\\ &= 1 + (A^{-2} + A^2)\mathrm{U}_i + \mathrm{U}_i^2\end{aligned}$$

But since $\mathrm{U}_i^2 = \delta\mathrm{U}_i$ and $\delta = -A^{-2} - A^2$, we have

$$\begin{aligned}\rho(\sigma_i)\,\rho(\sigma_i^{-1}) &= 1 + (A^{-2} + A^2)\mathrm{U}_i + \delta\mathrm{U}_i\\ &= 1 + (A^{-2} + A^2)\mathrm{U}_i + (-A^{-2} - A^2)\mathrm{U}_i\\ &= 1.\end{aligned}$$

We now prove that $\rho(\sigma_i\sigma_{i+1}\sigma_i) = \rho(\sigma_{i+1}\sigma_i\sigma_{i+1})$.

$$\begin{aligned}\rho(\sigma_i\sigma_{i+1}\sigma_i) &= (A + A^{-1}\mathrm{U}_i)(A + A^{-1}\mathrm{U}_{i+1})(A + A^{-1}\mathrm{U}_i)\\ &= (A^2 + \mathrm{U}_{i+1} + \mathrm{U}_i + A^{-2}\mathrm{U}_i\mathrm{U}_{i+1})(A + A^{-1}\mathrm{U}_i)\\ &= A^3 + A\mathrm{U}_{i+1} + A\mathrm{U}_i + A^{-1}\mathrm{U}_i\mathrm{U}_{i+1} + A^{-2}\mathrm{U}_i^2 + A\mathrm{U}_i\\ &\quad + A^{-1}\mathrm{U}_{i+1}\mathrm{U}_i + A^{-3}\mathrm{U}_i\mathrm{U}_{i+1}\mathrm{U}_i\\ &= A^3 + A\mathrm{U}_{i+1} + (A^{-1}\delta + 2A)\mathrm{U}_i\\ &\quad + A^{-1}(\mathrm{U}_i\mathrm{U}_{i+1} + \mathrm{U}_{i+1}\mathrm{U}_i) + A^{-3}\mathrm{U}_i\\ &= A^3 + A\mathrm{U}_{i+1} + (A^{-1}(-A^2 - A^{-2}) + 2A + A^{-3})\mathrm{U}_i\\ &\quad + A^{-1}(\mathrm{U}_i\mathrm{U}_{i+1} + \mathrm{U}_{i+1}\mathrm{U}_i)\\ &= A^3 + A(\mathrm{U}_{i+1} + \mathrm{U}_i) + A^{-1}(\mathrm{U}_i\mathrm{U}_{i+1} + \mathrm{U}_{i+1}\mathrm{U}_i).\end{aligned}$$

Since symmetry of the above expression in i and $i+1$, we can conclude that $\rho(\sigma_i\sigma_{i+1}\sigma_i) = \rho(\sigma_{i+1}\sigma_i\sigma_{i+1})$. We now prove that if $|i-j| \geq 2$, then $\rho(\sigma_i\sigma_j) = \rho(\sigma_j\sigma_i)$.

$$\begin{aligned}\rho(\sigma_i\sigma_j) &= \rho(\sigma_i)(\sigma_j)\\ &= (A + A^{-1}\mathrm{U}_i)(A + A^{-1}\mathrm{U}_j)\end{aligned}$$

Now since $\mathrm{U}_i\mathrm{U}_j = \mathrm{U}_j\mathrm{U}_i$ if $|i-j| \geq 2$, we have $(A+A^{-1}\mathrm{U}_i)(A+A^{-1}\mathrm{U}_j) = (A+A^{-1}\mathrm{U}_j)(A+A^{-1}\mathrm{U}_i)$ which equals $\rho(\sigma_j\sigma_i)$. □

We now define the diagrammatic trace $\mathrm{tr}\colon \mathrm{TL}_n \to \mathbb{Z}[A, A^{-1}]$ by extending linearly $\mathrm{tr}(P) = \langle P\rangle$, where P is a product term in $S(b)$. This version of trace is diagrammatic in nature as we are counting loops in a state. We thus arrive at the formula $\langle b\rangle = \mathrm{tr}(\rho(b))$. With what we have learnt so far, one can now find a braid representation b of a link L by Alexander's theorem, calculate $\mathrm{tr}(\rho(b))$ and normalise it to get the link invariant normalised bracket polynomial. One can substitute $A = t^{-1/4}$ to arrive at our long sought destination of the Jones polynomial.

4.7 Jones algebra

Jones considered a sequence of algebras A_n for $n = 2, 3, ...$ with multiplicative generators $e_1, e_2, \dots, e_{n-1}$ and the following relations.

1. $e_i^2 = e_i$.
2. $e_i e_{i\pm1} e_i = c e_i$.
3. $e_i e_j = e_j e_i$ if $|i - j| \geq 2$.

Here, c is a scalar which commutes with all the other elements. We can consider A_n as the free additive algebra on these generators viewed as a module over the ring $\mathbb{C}[c, c^{-1}]$, i.e. we take the free ring in the generators e_1, e_2,..., e_{n-1} and quotient it with the smallest ideal containing the elements e_i, $c^{-1}e_i e_{i+1}$ and $e_i e_j e_i^{-1} e_j^{-1}$ with appropriately restricted i and j. While c is often taken to be a complex number, we can view it as another algebraic variable which commutes with the generators. A_n arose in the theory of classification of von Neumann algebras and one can realize A_n as a von Neumann algebra as well. The reader is requested to compare the above mentioned relations for Jones algebra with the relations for the Artin braid group and the Temperley–Lieb algebra.

As mentioned in the beginning of this chapter, it is natural to construct a nested tower of algebras

$$M_0 \hookrightarrow M_1 \hookrightarrow M_2 \hookrightarrow M_3 \hookrightarrow \cdots \hookrightarrow M_n \hookrightarrow M_{n+1} \hookrightarrow \dots$$

with the following properties.

1. M_0 and M_1 are given.
2. $e_i \colon M_i \to M_{i-1}$ is a 'projection'.
3. $e_i^2 = e_i$.
4. $M_{i+1} = \langle M_i, e_i \rangle$.

Jones constructed such a tower of algebras with the other two properties of generators $e_i e_{i\pm1} e_i = c e_i$ and $e_i e_j = e_j e_i$ if $|i - j| \geq 2$. He defined a notion of a trace $\mathrm{tr}\colon M_n \to \mathbb{C}$, i.e. a function which vanishes on the commutator of any two elements of M_n. This trace satisfied the so called Markov property: $\mathrm{tr}(we_i) = c\,\mathrm{tr}(w)$ for w in the algebra generated by M_0, e_i,..., e_{i-1}.

We now see a concrete example of such a tower of algebras. Consider

$$\mathbb{R} \hookrightarrow \mathbb{R}[x_1] \hookrightarrow \mathbb{R}[x_1, x_2] \hookrightarrow \mathbb{R}[x_1, x_2, x_3] \hookrightarrow \cdots \hookrightarrow \mathbb{R}[x_1, x_2, \dots, x_n] \hookrightarrow \dots .$$

We have a sequence of the set of all real numbers, the set of all real polynomials in one variable, the set of all real polynomials in two variables, the set of all real polynomials in three variables, and so on. Each variable x_i for $i \geq 2$, is a map from $\mathbb{R}[x_1, x_2, \dots, x_{i-1}]$ to $\mathbb{R}[x_1, x_2, \dots, x_{i-2}]$, defined as the coefficient of $x_{i-1}^0 \in \mathbb{R}[x_1, x_2, \dots, x_{i-1}]$.

$$\begin{aligned} x_i &\colon \mathbb{R}[x_1, x_2, \dots, x_{i-1}] \to \mathbb{R}[x_1, x_2, \dots, x_{i-2}] \\ x_i &\colon \sum_{j=0}^{m} p_j x_{i-1}^j \mapsto p_0, \end{aligned}$$

where $p_j \in \mathbb{R}[x_1, x_2, \dots, x_{i-2}]$ and $m \in \mathbb{N}$. As an example, if $\sum_{j=0}^{m} p_j x_1^j \in \mathbb{R}[x_1]$, then

$$\begin{aligned} x_2 &\colon \mathbb{R}[x_1] \to \mathbb{R} \\ x_2 &\colon \sum_{j=0}^{m} p_j x_1^j \mapsto p_0, \end{aligned}$$

where p_0 is the coefficient of x_1^0, i.e. the constant term. For example,

$$x_2(3x_1^4 + 6x_1^3 + x_1 + 5) = 5.$$

Similarly, x_3 is a map from $\mathbb{R}[x_1, x_2]$ to which maps a polynomial in two variables to the polynomial (in the single variable x_1) which does not contain any power of x_2. As an example,

$$x_3(4x_1x_2^2 + 3x_1^3 + x_2 + x_2^5 + 1) = 3x_1^3 + 1.$$

$(\mathbb{R}[x_1, x_2, \dots, x_n], +, \cdot)$ is an algebra with the $+$ operation defined as polynomial addition (component-wise addition) and the $\cdot$ operation defined as polynomial multiplication (Cauchy product). The generator relations for the Jones algebra, however, are not satisfied with the $\cdot$ operation; $x_1^2 \coloneqq x_1 \cdot x_1 \neq x_1$. But we can make sense of them if we interpret the operation amongst the generators as the function composition operator. x_1, x_2,..., x_n are the generators $(\mathbb{R}[x_1, x_2, \dots, x_n], +, \cdot)$. Since x_i's are functions as well, we can interpret x_i^2 as $x_i \circ x_i$. If we do that, then $x_i^2 \coloneqq x_i \circ x_i = x_i$ as x_i is a projection operator. Similarly, we see that $x_i \circ x_{i\pm1} \circ x_i = x_i$ and $x_i \circ x_j = x_j \circ x_i$ by restricting the domains appropriately.

We can define a Markov trace on the above tower as follows. For an element $w \in \mathbb{R}[x_1, x_2, \dots, x_n]$, define $\mathrm{tr}\colon \mathbb{R}[x_1, x_2, \dots, x_n] \to \mathbb{Z}$ by assigning w the degree of the polynomial. For an element of $\mathbb{R}[x_1, x_2, \dots, x_{n+1}]$, $\mathrm{tr}\colon \mathbb{R}[x_1, x_2, \dots, x_{n+1}] \to \mathbb{Z}$. Even though the domains of the trace are different in the above two examples, we can perform this abuse of notation since there is no risk of confusion here. We see that $\mathrm{tr}(wx_{n+1}) = \mathrm{tr}(w)$. Thus, tr is a Markov trace as it satisfies the Markov property with $c = 1$ for all $w \in \mathbb{R}[x_1, x_2, \dots, x_n]$ where $n \geq 1$.

We can retrieve the Jones algebra from the Temperley–Lieb algebra by substituting $e_i = \delta^{-1}\mathrm{U}_i$, $e_i^2 = e_i$ and $e_ie_{i\pm1}e_i = \delta^{-2}e_i$. We have taken $c = \delta^{-2}$. Note that the underlying rings are different.

Chapter 5

Alternating Knots

We shall see a remarkable application of the bracket polynomial in this chapter to resolve a conjecture of Peter Guntherie Tait, who was one of the early pioneers in knot theory. He had created one of the first knot tables. Based on his data, he had conjectured that the number of crossings in a reduced alternating diagram of a knot is an ambient isotopy invariant.

5.1 Shading link projections

We take a regular link projection as shown in fig. 5.1. We shall refer to hatched regions (connected components) as coloured black and regions which are not hatched as white. The regions have been coloured such that the opposite regions with respect to a crossing point share the same colour. We have thus two-coloured the diagram. One can ask whether it is always possible to two-colour a given (tame) link diagram. We shall now see that this is true.

We thicken the curve in the diagram to create a 'road' (fig. 5.2) with 'road crossings'. We find a route such that we travel every street only once, and turn at right angles at every crossing. We do not wish to create a route such that we transversely cross a road crossing (fig. 5.3). Such a route for the trefoil is shown in fig. 5.4.

We effectively want to resolve each crossing such the the result curve is one single continuous curve. Given a link projection, we at first randomly select a crossing and resolve such that connectivity is not lost. It is always possible to do such a resolution as if one resolution gives us two disconnected components, then the other resolution shall necessarily give us a single connected component. Refer to fig. 5.5 for this.

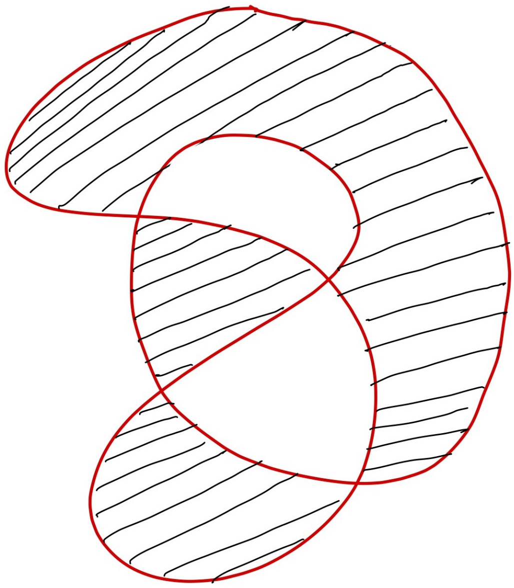


Figure 5.1: A two-coloured link projection.

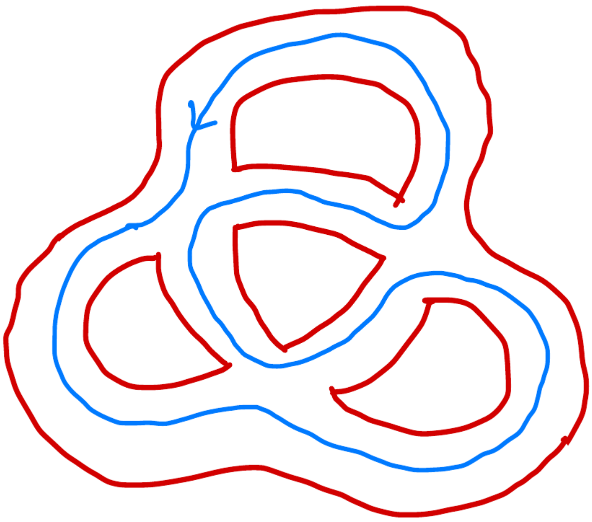


Figure 5.2: 'Thickened' trefoil projection to resemble a road.

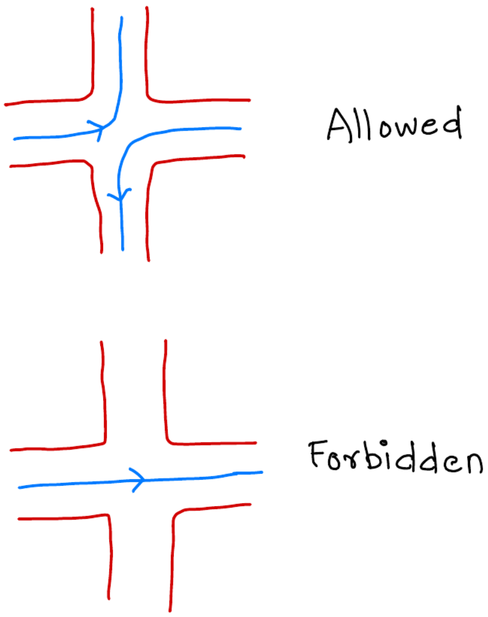


Figure 5.3: Allowed and forbidden paths at crossings.

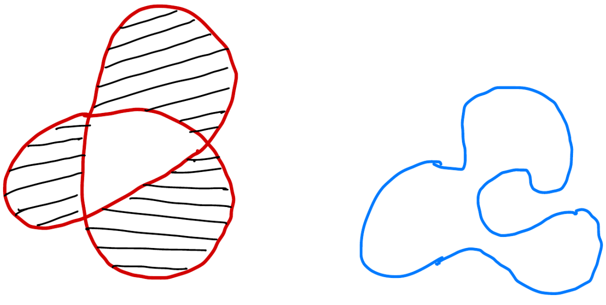


Figure 5.4: A route for trefoil.

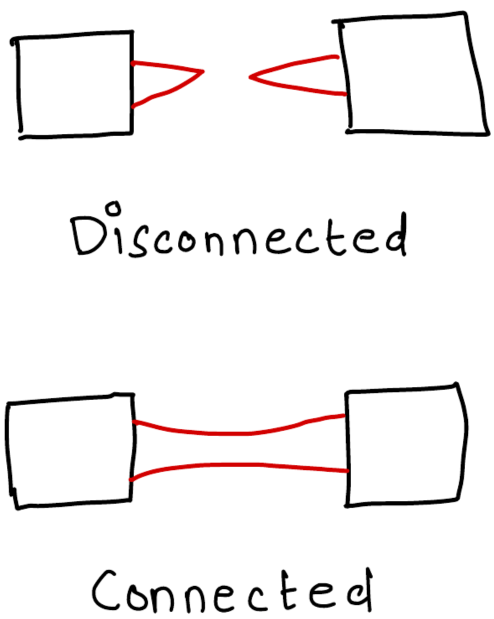


Figure 5.5: Resolutions resulting in connected and disconnected components.

We have applied the Jordan curve theorem here. Once we resolve one crossing we iteratively resolve other crossings as well, making sure that connectivity is not lost (fig. 5.6).

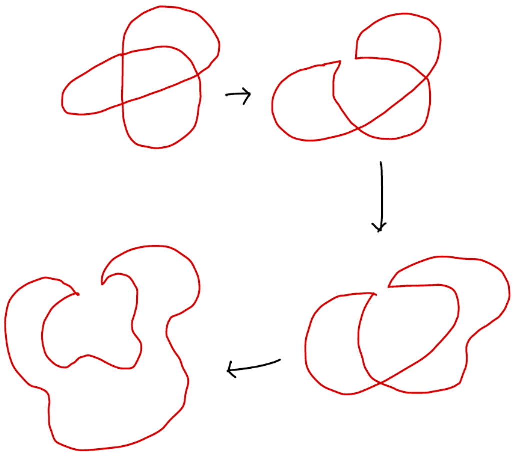


Figure 5.6: Iterative connectivity preserving resolutions.

In the end, since the crossings are finitely many, we get a single Jordan curve. This curve separates the plane into two connected components. We can shade the bounded component (fig. 5.7). This brings us to two-colouring a regular link projection. Being able to find a path for a regular link projection as shown in fig. 5.2 is thus equivalent to being able to two-colour a regular link projection.

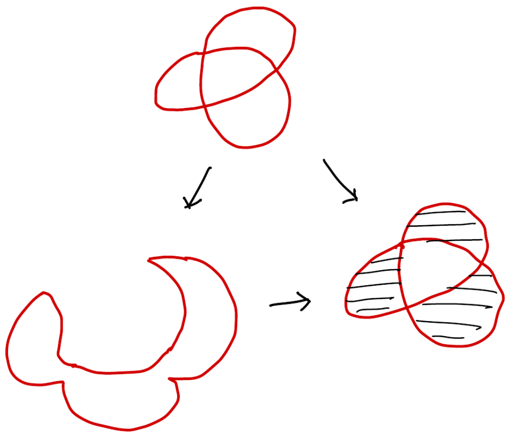


Figure 5.7: Equivalence of the road question and two-colouring a link projection.

Given a link projection, we have two choices for each crossing regarding the choice of the under-strand (or over-strand). By making a choice for each crossing, we can construct a link associated to that link projection. In fact, we always make choices such the diagram is *alternating*. By an alternating diagram, we mean that if a strand passes over another strand in a crossing, then it passes underneath an other strand in the subsequent crossing if we travel along the original strand. For example, all the knots in fig. 1.5 are alternating. The preceding fact can be understood by observing that each link projection admits a two colouring and making the choice for each crossing such that the A-regions are the shaded regions.

We shall say that a diagram is *reduced* if the diagram is not the form as shown in fig. 5.8. There should not be any circle in the diagram plane such that it meets the diagram at a single point which is a crossing for a diagram to be reduced.

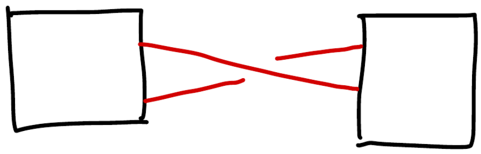


Figure 5.8: Diagram with a bridge (a nugatory crossing).

We now consider the bracket polynomial $\langle K \rangle$ for a reduced alternating diagram of a link K. If the diagram is two-coloured in such a way that every pair of A-regions is shaded, then the state S obtained by splicing each such shading shall contribute

$$A^{V(K)}(-A^2 - A^{-2})^{l(S)-1},$$

where $V(K)$ denotes the number of crossings in K and $l(S)$ the number of loops in the state S. The highest power term contributed by S is

$$(-1)^{l(S)+1} A^{V(K)+2l(S)-2}.$$

To see why this is the highest degree term in $\langle K \rangle$, we consider any other state S' of K. S' shall necessarily be obtained from S be switching the choices of some of the resolutions. We have already assumed that K is reduced. S is a state such that only one region admits a shading. There are loops inside the shaded region. Switching a resolution choice shall result in joining two previously disconnected loops. This means that $l(S') = l(S) - 1$. Substituting $l(S')$ in place of $l(S)$ in $A^{V(K)}(-A^2 - A^{-2})^{l(S)-1}$, we realise that the largest degree contribution from S' is four less than that of S. We request the reader to refer to fig. 5.9.

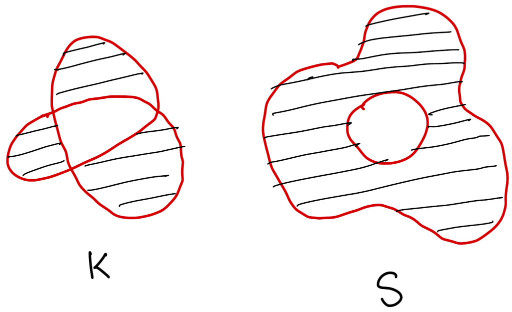


Figure 5.9: A state S of a trefoil K.

The number of white (or unshaded) regions in shaded state is equal to the number of loops in that state. Thus, the maximum degree of the bracket is given by

$$\max\deg\langle K\rangle = V(K) + 2W(K) - 2,$$

where $W(K)$ denotes the number of white regions. Similarly, the minimum degree is given by

$$\min\deg\langle K\rangle = -V(K) - 2B(K) + 2,$$

where $B(K)$ denotes the number of black regions.

For the trefoil in fig. 5.9, we have that $V = 3$, $W = 2$ and $B = 3$.

$$\max\deg = 3 + 4 - 2 = 5$$

and

$$\min\deg = -3 - 6 + 2 = -7.$$

This matches with bracket polynomial for the trefoil $-A^5 - A^{-3} + A^{-7}$.

5.2 Tait conjecture

Definition 51. The *span* of an unoriented is the difference between the highest and lowest degrees of the bracket polynomial.

$$\mathrm{span}(K) = \max\deg(K) - \min\deg(K).$$

Ambient isotopy invariance of the normalised bracket polynomial implies the ambient isotopy invariance of the span as well. For a reduced alternating diagram, $\mathrm{span}(K) = 2V + 2W + 2B - 4$. But since $W + B$ equals the total number of regions (connected components), which is equal to the number of white regions plus two, we have that $\mathrm{span}(K) = 4V$. Thus, the number of crossings in a reduced alternating diagram is an ambient isotopy invariant. Tait had conjectured this on the basis of knot tabular data in the 19$^{\text{th}}$ century. It was proved by Kauffman and Murasugi in 1987 [27, 28].

We know that $\mathrm{L}(A, K^*) = \mathrm{L}(A^{-1}, K)$. Thus, for an alternating link K,

$$\operatorname{span}(K) = 4V = \max\deg\langle K\rangle.$$

Thus,

$$\max\deg\langle K\rangle = \min\deg\langle K\rangle = 2V.$$

We know that $\mathrm{L}(A, K) := (-A^3)^{-\mathrm{w}(K)}\langle K\rangle$. This implies that

$$\max\deg(\mathrm{L}(K)) = -3\,\mathrm{w}(K) + \max\deg\langle K\rangle$$

and

$$\min\deg(\mathrm{L}(K)) = -3\,\mathrm{w}(K) + \min\deg\langle K\rangle.$$

Thus,

$$\max\deg\langle K\rangle = -\min\deg\langle K\rangle.$$

We have

$$6\,\mathrm{w}(K) = \max\deg\langle K\rangle + \min\deg\langle K\rangle,$$

and substituting $\max\deg\langle K\rangle = V(K) + 2W(K) - 2$ and $\min\deg\langle K\rangle = -V(K) - 2B(K) + 2$ gives us

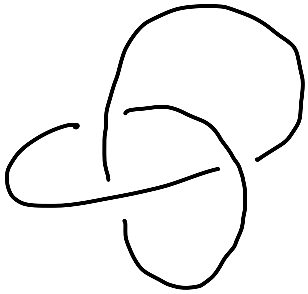
$$3\,\mathrm{w}(K) = W - B.$$

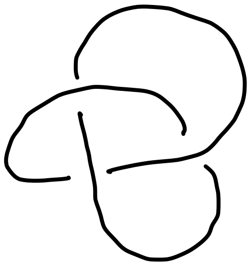
For a reduced and alternating achiral knot, $3\,\mathrm{w}(K) = W - B$.

Chapter 6

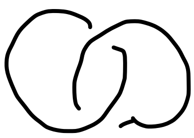
Appendix

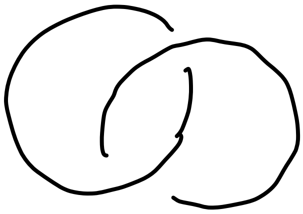
Bracket polynomial calculation for trefoil knot.

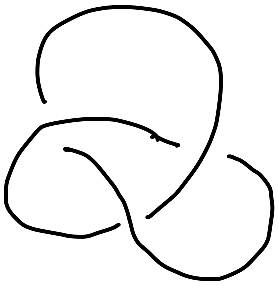
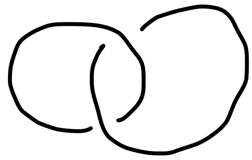
T^*:  Left-handed trefoil

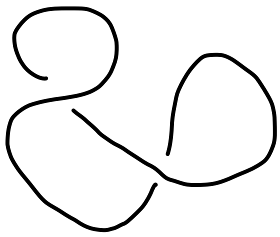
T:  Right handed trefoil

We know that

$\langle$  $\rangle = -A^4 - A^{-4}$

 $\leftarrow$ Hopf link

$\langle$  $\rangle = A \langle$  $\rangle$

$+ A^{-1} \langle$  $\rangle$

$$= A(-A^{4} - A^{-4}) + A^{-1}(-A^{-3})^{2}$$

$$\langle T \rangle = -A^{5} - A^{-3} + A^{-7}$$

$$w(T) = 3$$

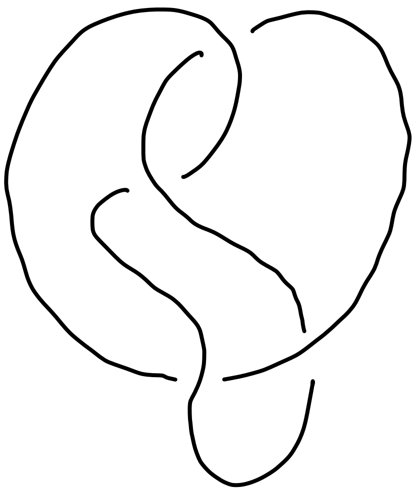
$$\therefore \; L_T = (-A^{3})^{-3} \langle T \rangle$$

$$L_T = A^{-4} + A^{-12} - A^{-16}$$

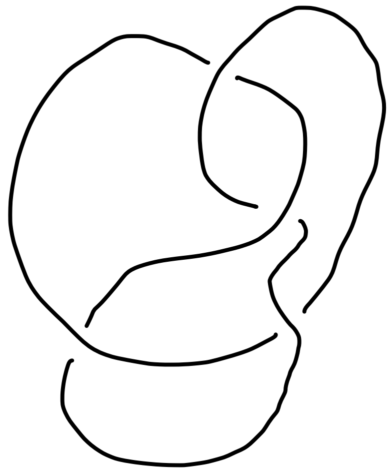
$$L_{T^*} = A^{4} + A^{12} - A^{16}$$

The above calculation shows that the left-handed trefoil and the right-handed trefoil, which are mirror images of each other, are not ambient isotopic. Trefoil knot is chiral.

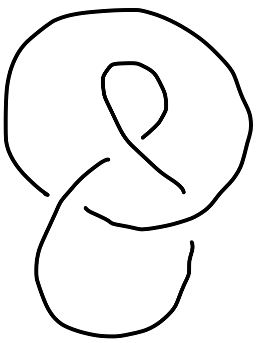
The Figure Eight knot

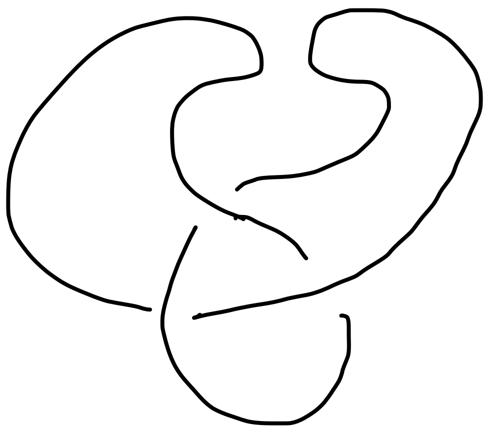


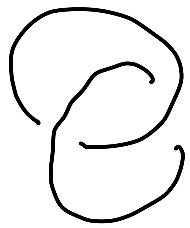
E

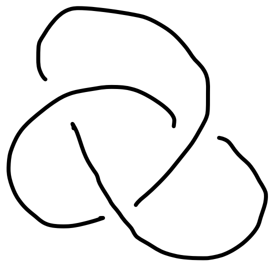


E^*

$\langle E \rangle = A \Big\langle$  $\Big\rangle$

$+ A^{-1} \Big\langle$  $\Big\rangle$

$= A(-A^3)\left\langle$  $\right\rangle$

$+ A^{-1}\left\langle$  $\right\rangle$

$$= -A^4\left[-A^4 - A^{-4}\right]$$

$$+ A^{-1}\left[-A^5 - A^{-3} + A^{-7}\right]$$

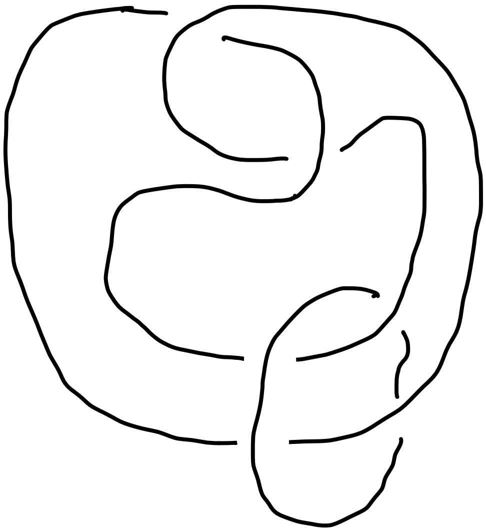
$$= A^8 + 1 - A^4 - A^{-4} + A^{-8}$$

$$\langle E \rangle = A^8 - A^4 + 1 - A^{-4} + A^{-8}$$

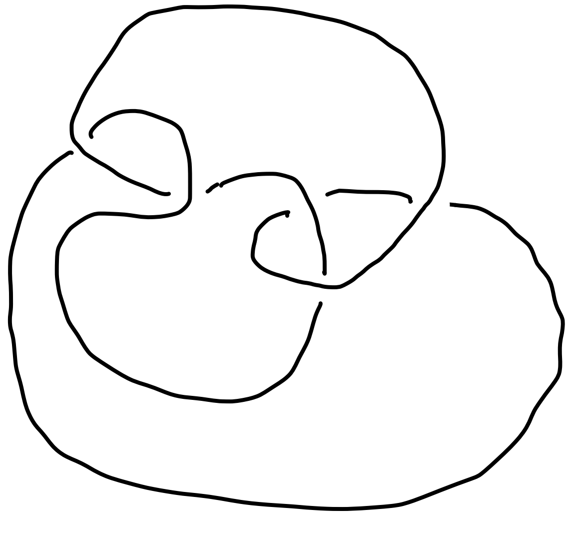
$w(E) = 0$. $\quad L(E) = \langle E \rangle = L(E^*)$

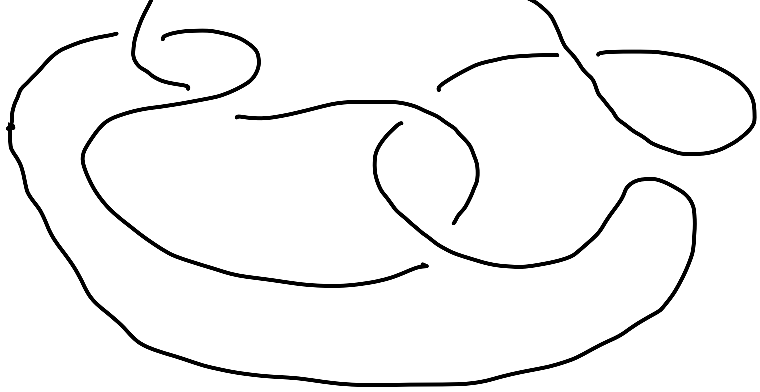
E and E^* are ambient isotopic as $L(E)$ is symmetric in powers of A.

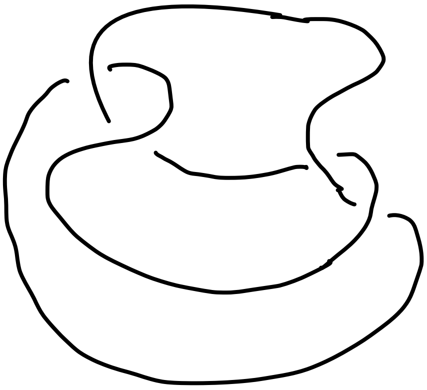
The Whitehead link

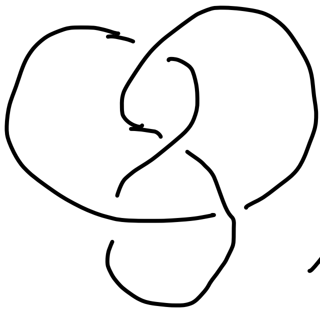


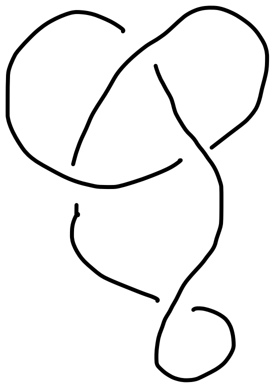
W

$\langle W \rangle = A \Big\langle$  $\Big\rangle$

$+ A^{-1} \Big\langle$  $\Big\rangle$

$= A(-A^3) \langle$  $\rangle$

$- A^{-4} \langle$  $\rangle$

$= (-A^4) \langle$  $\rangle - A^{-4} \langle E^* \rangle$

$$= (-A^4)(-A^{-3}) \langle T^* \rangle - A^{-4} \langle E^* \rangle$$

$$= A^8 + A^{-8} + 2A^4 - 2A^{-4} + 1 - A^{-12}$$