

# SOME TOPICS IN KNOT THEORY

A REPORT

*submitted in partial fulfillment of the requirements*

*for the award of the dual degree of*

Bachelor of Science-Master of Science

*in*

MATHEMATICS

*by*

APOORV POTNIS

(18343)

DEPARTMENT OF MATHEMATICS,  
INDIAN INSTITUTE OF SCIENCE EDUCATION AND  
RESEARCH BHOPAL,  
BHOPAL - 462 066

April 2023

# CERTIFICATE

This is to certify that **Apoorv Potnis**, BS-MS (Mathematics), has worked on the project entitled ‘**Some topics in knot theory**’ under my supervision and guidance. The content of this report is original and has not been submitted elsewhere for the award of any academic or professional degree.

April 2023  
IISER Bhopal

Dr. Dheeraj Kulkarni  
Prof. Subhash Chaturvedi

Committee Member	Signature	Date
_____	_____	_____
_____	_____	_____
_____	_____	_____
_____	_____	_____

# ACADEMIC INTEGRITY AND COPYRIGHT DISCLAIMER

I hereby declare that this project is my own work and, to the best of my knowledge, it contains no materials previously published or written by another person, or substantial proportions of material which have been accepted for the award of any other degree or diploma at IISER Bhopal or any other educational institution, except where due acknowledgement is made in the document.

I certify that all copyrighted material incorporated into this document is in compliance with the Indian Copyright Act (1957) and that I have received written permission from the copyright owners for my use of their work, which is beyond the scope of the law. I agree to indemnify and save harmless IISER Bhopal from any and all claims that may be asserted or that may arise from any copyright violation.

April 2023  
IISER Bhopal

Apoorv Potnis

# ACKNOWLEDGEMENT

Acknowledgement is here.

Apoorv Potnis

# ABSTRACT

We shall study some selected topics in knot theory from the book *Knots and Physics*, 4th ed. by Louis Kauffman [1]. The basics of knot theory have been studied from the book *Knots and Links* by Peter Cromwell [2].

A knot is an embedding of the circle in three-dimensional space. We discuss the state model for bracket polynomial, Jones polynomial and some of its generalisations. One can apply these knot invariants to show that various knots are not ambient isotopic, which is the notion we shall use to distinguish knots. We shall see a resolution of some of the old conjectures in knot theory using the bracket polynomial. Vaughn Jones found the polynomial named after him while he was studying towers of von Neumann algebras. We shall follow a different route following Kauffman to understand the relation amongst braids, links, Temperley–Lieb algebras, which arose in statistical physics, and the Jones algebras.

# Contents

<b>Certificate</b>	<b>i</b>
<b>Academic Integrity and Copyright Disclaimer</b>	<b>ii</b>
<b>Acknowledgement</b>	<b>iii</b>
<b>Abstract</b>	<b>iv</b>
<b>1 Introduction to knot theory</b>	<b>2</b>
1.1 Definition of a knot . . . . .	2
1.2 Distinguishing knots . . . . .	4
1.3 Knot diagrams . . . . .	7
1.3.1 Topological manifolds . . . . .	9
<b>2 Braids and the Jones polynomial</b>	<b>10</b>
2.1 Motivation . . . . .	10
2.2 Geometric representation of braids . . . . .	11
2.2.1 Definition . . . . .	11
2.2.2 Standard projection . . . . .	12
2.2.3 Group structure . . . . .	13
2.2.4 Generators . . . . .	13
2.3 Closure of braids . . . . .	16
2.3.1 Alexander Theorem . . . . .	17
2.3.2 Conjugation . . . . .	18
2.3.3 Markov move . . . . .	18
2.3.4 Markov Theorem . . . . .	18
2.4 Orientation . . . . .	20
2.5 Markov trace . . . . .	20
2.6 Temperley–Lieb algebra . . . . .	24
2.7 Jones algebra . . . . .	26

# Chapter 1

## Introduction to knot theory

### 1.1 Definition of a knot

We normally conceive of knots as open strings with ‘knotted’ parts in between. Given a knotted string with two ‘open’ ends, we can simply pull one end of the string to untie it, in the usual way we untie a knot, by inserting an open end into the knotted region in a strategic manner and pulling it on the other side<sup>1</sup>. This way, any knot with two open ends can be untied. But if we have a knot in a closed loop, we won’t be able to untie it unless we cut it. We want our notion of a knot to be invariant under ‘pulling’. Thus, we model our mathematical definition on closed loops instead of open. Refer to [fig. 1.1](#).

**Definition 1 (Knot).** A knot  $K$  is an embedding of  $S^1$  in  $\mathbb{R}^3$ .

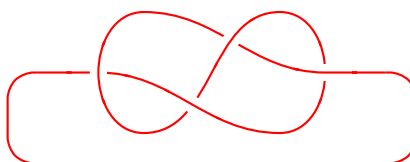
Although one can talk about embeddings of higher dimensional ‘circles’ in higher dimensional spaces, we restrict ourselves to knots in the three dimensional

---

<sup>1</sup>There exists another way of unknotting an open knot, often used in magic tricks. One creates another knot using the unknotted portion of the string on one side of the knotted portion such that this new knot ‘cancels’ the original knot. In mathematical terms, the new knot is constructed such that the *connected sum* of the original knot and the new knot gives an unknot. We shall not deal with connected sums of knots in this thesis.



(a) A knot with open ends



(b) A knot with closed ends

Figure 1.1: Projections of a knot on a plane

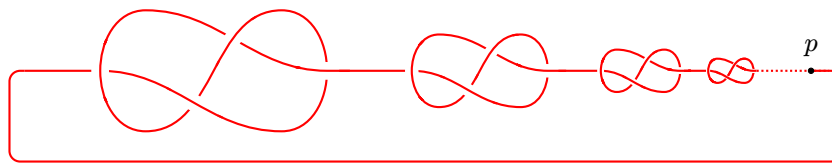


Figure 1.2: A wild knot. The knot is wild only at the point  $p$ .

space. The above definition turns out to be too general for our purposes. It takes into consideration certain pathological knots known as *wild knots*. Although wild knots are an object of study, we shall not be dealing with them in this thesis due to their pathological nature. Figure 1.2 shows an example of a wild knot. A section of the knot is scaled down by a factor and then joined on one side of the previous section. If one repeats this process infinitely, the resulting curve formed by appending sections eventually converges to a point, provided that the scaling down is fast enough. We can join the convergence (limit) point to an end-section on the other side to get wild knot. This knot is continuous everywhere, including the limit point. The behaviour of the knot at the limit point is different from the other points. Three common ways exist to exclude such behaviour by demanding extra conditions.

#### 1. Differentiability.

We can demand all knots to be differentiable at all points. We can verify visually that all the points, except the limit point of the wild knot are differentiable, or can be made (isotoped) differentiable. The derivative must necessarily change within a section. As the size of the section decreases, the derivative changes more rapidly. In a scaled down section, all the values which the derivative took in the previous section. As we approach the sections near the limit point, the derivative function must attain all the values it did before, but it must do so in a more rapid manner. At the limit point, the derivative shall cease to exist by the virtue of ‘changing too rapidly’. Demanding differentiability forcibly removes the offending limit point. The wildness is due to the limit point. Along with differentiability, we can demand  $C^r$  smoothness or  $C^\infty$  smoothness as well.

But this condition comes with a problem as well, namely we cannot use polygons for describing knots.

#### 2. Piecewise linearity.

We can demand all the edges of a knot to be piecewise linear with finitely many edges. Our knot shall be a polynomial (with finitely many edges) in that case. Infinitely (countably) many sections in a wild knot shall mean infinitely many edges (of decreasing length), which is not allowed. Thus, this condition excludes wild knots.

#### 3. Local flatness.

In this thesis, we shall take the third route following Cromwell [2, chp. 1]. If we consider a local neighbourhood around each point of the knot, except the limit point, then we see visually see that we can always find a small enough local neighbourhood around each such point such that the strand is not ‘knotted’ in



that neighbourhood. At the limit point, no matter how small a neighbourhood we take, the strand shall always be ‘knotted’. We enforce this ‘local unknottedness’ condition by demanding local flatness. Let  $p$  be a point in a knot  $K$ ,  $B(O, 1)$  be the unit ball centered at origin  $O$  and  $d$  be a diameter of  $B(O, 1)$ .

**Definition 2** (Local flatness). The point  $p$  is said to be locally flat if there exists a neighbourhood  $U \ni p$  such that the pair  $(U, U \cap K)$  is homeomorphic to  $(B(O, 1), d)$ .

A knot is said to be locally flat if each point in that knot is locally flat. A point that is not locally flat is called wild, and a knot is wild if any of its points are wild.

Consider a spherical neighbourhood around a locally flat point. There exists a radius such that for all neighbourhoods less than this radius, the boundary of the neighbourhood intersects the strand in exactly two points. This is not possible at the limit point in wild knot figure.

**Definition 3** (Tame knots). A knot is said to be tame if all its points are locally flat.

## 1.2 Distinguishing knots

Any two homeomorphisms of the circle are homeomorphic to the circle and to each other, since being homeomorphic is an equivalence relation. But this means that all knots are homeomorphic to each other. Clearly, homeomorphism is not the correct notion to distinguish knots. When we mean that two knots are distinct, we mean that if we create a physical model of those knots, we cannot ‘physically deform’ one knot into another. *Cutting a knot is not allowed.* One might think that homotopy or isotopy are what we need, but it turns out that the notion of *ambient isotopy* is the correct one.

**Definition 4** (Homotopy). A homotopy of a space  $X \subset \mathbb{R}^3$  is a continuous map  $h: X \times [0, 1] \rightarrow \mathbb{R}^3$ .

The restriction of  $h$  to level  $t \in [0, 1]$  is  $h_t: X \times \{t\} \rightarrow \mathbb{R}^3$ .  $h_0$  must be the identity map.

Note that the continuity of  $h$  implies the continuity of  $h_t$  for all  $t \in [0, 1]$ . The converse is not true. Insert example here. Homotopy allows a curve to pass through itself. All knots are thus homotopic to the trivial knot. If we do not allow a curve to pass through itself, i.e. if we demand injectivity for each  $h_t$ , then we get what is known as an isotopy. But isotopy is not useful for distinguishing knots as well, due to bachelors’ unknotting. All tame knots, or more generally, all knots with a tame arc turn out to be isotopic to the trivial knot. It is an open problem if all knots are isotopic to the trivial knot [3, 4]. In addition, it is not known as well if a knot known as the Bing sling, which is wild *at all points* is isotopic to the trivial knot [3, 4].

**Proposition 5** (Bachelors’ unknotting). Every knot with a tame arc is isotopic to the unknot.

*Proof.* Refer to [fig. 1.3](#). Let  $K \subset \mathbb{R}^3$  be a knot with a tame arc.

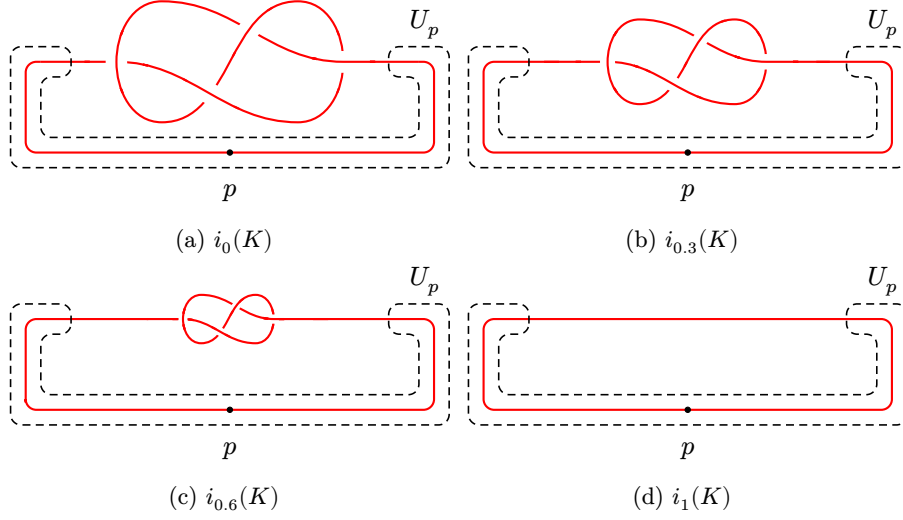


Figure 1.3: Bachelors' unknotting demonstrating the isotopy equivalence of a knot with tame arc to the trivial knot. Here we have chosen the point  $r$  (not shown in the figure) to be the midpoint of the knotted region.  $f(a)$  and  $f(b)$  are the intersections of the knot  $K$ , represented by the thick red line, with  $U_p$ , shown using densely dashed lines. Note that  $f(a)$ ,  $r$  and  $f(b)$  are collinear in this case. The entirety of the knot is tame in this case, but we only require tameness in the chosen region of  $U_p \cap K$ .

Let  $p \in K$  in a tame arc of the knot. Since the knot is locally flat on the arc, by the definition of tameness, we take a ball  $U_p \subset \mathbb{R}^3$  of radius  $\varepsilon$  around the point  $p$  such that the pair  $U_p, U_p \cap K$  is homeomorphic to  $(B, d)$ , where  $B$  is the unit ball in  $\mathbb{R}^3$  centered at the origin and  $d$  is the diameter of  $B$  along the  $x$ -axis. We choose a parametrization  $f: [0, 2\pi] \rightarrow K$  of the knot such that  $f([a, b]) = K - (U_p \cap K)$ , where  $[a, b] \subset [0, 2\pi]$ .

Let  $r \in \mathbb{R}^3$  be a point outside  $U_p$ . Now consider the function  $i_t: K \rightarrow \mathbb{R}^3$ , defined for each  $t \in [0, 1]$  as follows. Let

$$\phi_t(a) := t \left( \frac{a+b}{2} \right) + (1-t)a$$

be a family of functions for all  $t \in [0, 1]$ .

1. If  $f(x) \in U_p$ , then

$$i_t(f(x)) = f(x).$$

2. If  $x \in [a, \phi_t(a))$ , then

$$i_t(f(x)) = f(a) + \frac{\|f(a) - f(\phi_t(a))\|}{\|a - \phi_t(a)\|} (x - a)(r - f(a)).$$

3. If  $x \in [\phi_t(a), \phi_t(b)]$ , then

$$i_t(f(x)) = tr + (1-t)f(x).$$

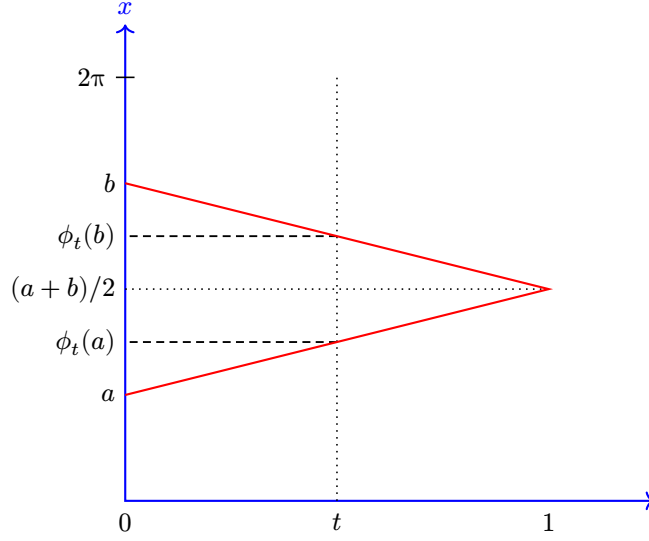


Figure 1.4: Illustration of shrinkage of domain with respect to  $t$ . For all  $t \in [0, 1]$ , the interval  $[\phi_t(a), \phi_t(b)] \in [0, 2\pi)$  maps to the shrunk ‘knotted’ part. The images of  $[a, \phi_t(a)]$  and  $[\phi_t(b), b]$  map to the straight lines which connect  $f(a)$  to  $i_t(f(\phi_t(a)))$  and  $i_t(f(\phi_t(b)))$  to  $f(b)$  respectively.

4. If  $x \in (\phi_t(b), b]$ , then

$$i_t(f(x)) = f(b) + \frac{\|f(b) - f(\phi_t(b))\|}{\|b - \phi_t(b)\|}(b - x)(r - f(b)).$$

Let  $i: [0, 1] \times K \rightarrow \mathbb{R}^3$  be a function defined by  $i(t, f(x)) := i_t(f(x))$ .

$i$  is defined such that the part inside  $U_p$  is kept unchanged for all  $t$ . For  $t = 0$ ,  $i$  does not deform the knot at all. For  $t \in (0, 1)$ , the knotted part (in  $\mathbb{R}^3$ ) shrinks and the interval in the domain  $[a, b]$  which maps to the knotted part also shrinks to  $[\phi_t(a), \phi_t(b)]$ . This shrinkage of the domain happens linearly with  $t$ . Refer to [fig. 1.4](#). All points of the knotted part trace a straight line as they travel under isotopy from their original position to  $r$ . Eventually, the knotted part ceases to exist at  $t = 1$  and a single point of the domain, namely,  $(a + b)/2$  maps to  $r$ .

In the end, we get a figure consisting of two straight lines meeting at  $r$ , and  $U_p \cap K$ , the original part of the knot inside  $U_p$ . The other endpoints of these lines are  $f(a)$  and  $f(b)$ .  $U_p \cap K$  is isotopic to the line joining  $f(a)$  and  $f(b)$ . Thus, we get a triangle with points  $r$ ,  $f(a)$  and  $f(b)$  and we any that any triangle in  $\mathbb{R}^3$  is isotopic to  $\mathbb{S}^1$ .

We now prove that  $i$  is continuous. We know that both  $i_t$  and  $i_x$  are continuous and injective for all  $t \in [0, 1]$  and  $x \in [0, 2\pi)$  respectively, where  $i_x$  is defined to be the restriction of  $i$  for a particular  $x$ . We also see that  $i_t$  is linear in  $x$ . A function continuous in one argument and linear in the other is continuous in the product topology. We thus have an isotopy which sends a knot with a tame arc to  $\mathbb{S}^1$ .  $\square$

**Remark 6.** It should be noted that the isotopy that we have constructed does not shrink  $K - (U_p \cap K)$  uniformly. Parts of the strand closer to the point  $r$  are

shrunk more than the parts further away. Also, not all parts move at a uniform rate towards  $r$ . Parts closer to  $r$  move slower than the parts further.

In the above considered isotopy, we isotoped the set  $X = K$ . Instead, we take  $X$  to be the entire space  $\mathbb{R}^3$ , or a bounded set which completely covers the knot, then we get the notion of ambient isotopy. This modification ensures that the surrounding space is isotoped as well as we isotope the knot. The knot is a curve which has no volume. If we try bachelors' unknotting on the surrounding space as well, we observe that the surrounding space, which has a finite, non-zero volume cannot shrink to a set of zero volume under isotopy. This finally leads us to the equivalence relation induced by ambient isotopy.

**Remark 7.** Unless mentioned otherwise, we shall always consider our knots to be tame from now on.

**Definition 8** (Knot equivalence). Two knots  $K_1$  and  $K_2$  are said to be ambient isotopic if there exists an isotopy  $I: \mathbb{R}^3 \times [0, 1] \rightarrow \mathbb{R}^3$  such that  $I(K_1, 0) = I_0(K_1) = K_1$  and  $I(K_1, 1) = I_1(K_1) = K_2$ .

**Proposition 9.** Knot equivalence is indeed an equivalence relation.

*Proof.* Reflexivity. Symmetry. Transitivity. □

**Remark 10.** Each equivalence class of knots is called a *knot type*. We would often forget the distinction between a knot and its knot type. The intended meaning can be inferred from the context.

**Remark 11.** Note that we distinguish between *ambient isotopy* and *isotopy*. Many treatments of knot theory use the word isotopy for ambient isotopy as ambient isotopy is the useful construct in knot theory. Ambient isotopy is an isotopy of the whole space containing the knot, not just the knot.

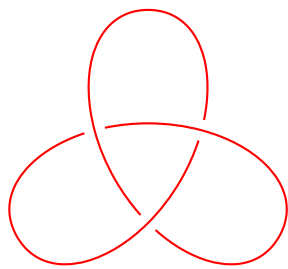
**Remark 12.** All three approaches for excluding pathological behaviour, including  $C^r$  for all  $r \geq 1$ , generate the same knot isotopy classes (insert citations). The knot isotopy classes corresponding to  $C^1$  curves, piecewise-linear curves with finitely many components, smooth  $C^\infty$  curves and locally flat everywhere curves are equal [2, chp. 1, § 1.11].

**Example 13.** The knots depicted in [fig. 1.5](#) are non-trivial and not ambient isotopic to each other. We shall prove this fact by the use of Kauffman's bracket polynomial in later chapters.

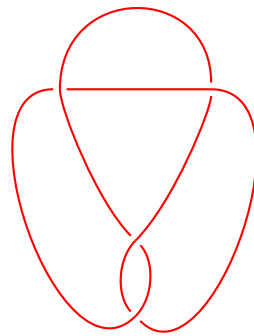
**Remark 14.** In this thesis, we shall look at knot theory in  $\mathbb{R}^3$ . One can compactify  $\mathbb{R}^3$  to  $\mathbb{S}^3$  and do knot theory in  $\mathbb{S}^3$ , as many treatments do. This does not result in a different knot theory.

## 1.3 Knot diagrams

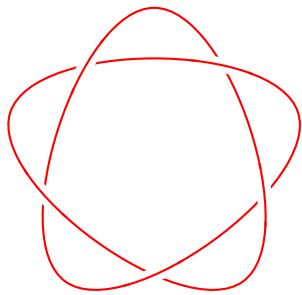
So far, we have depicted knots as curves in a plane (of paper) with the implicit understanding that if a strand goes under another strand, then we break the strand which goes underneath. We defined a knot as an object in three dimensions, but we can project the knot onto a plane to represent it, as we have done so far. We do not lose any topological information if we choose a 'nice enough'



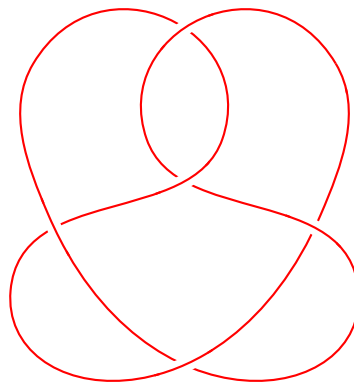
(a) A trefoil



(b) The figure-eight knot



(c) The cinquefoil knot



(d) The three-twist knot ( $5_2$ )

Figure 1.5: Various distinct knots.

projection. What we have been doing implicitly can be formalized; we shall see how. We shall introduce some preliminary topological notions which shall be required.

### 1.3.1 Topological manifolds

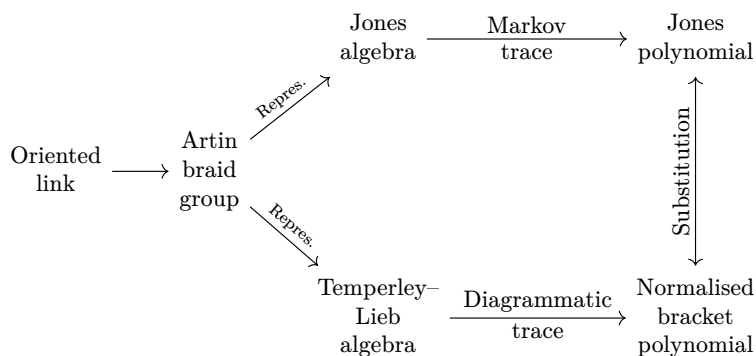
**Definition 15** (Topological manifold). A topological  $n$ -manifold is a Hausdorff and second-countable topological space such that each point has a neighbourhood homeomorphic to  $\mathbb{R}^n$  or  $\mathbb{R}_{\geq 0}^n$ .

Let  $K$  be a knot and let  $\pi: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be a projection map. A point  $p \in \pi(K)$  is called *regular* if  $\pi^{-1}(p)$  is a single point, and *singular* otherwise. If  $|\pi^{-1}(p)| = 2$ , then  $p$  is called a *double* point.

## Chapter 2

# Braids and the Jones polynomial

### 2.1 Motivation



As remarked earlier, Jones arrived at his polynomial indirectly while working on the theory of operator algebras [5]. In his course of investigations, he constructed a tower of algebras nested in one another with the property that each of these algebras is generated by a set of generators satisfying a particular set of relations. A degree of similarity between these relations and the relations among the generators of the *Artin braid group* was pointed out to Jones by a student during a seminar [2, p. 216], which led to the investigations of Jones into knot theory. Jones had defined a notion of a *trace* on his algebras; more specifically a trace function obeying the *Markov property*. As we shall see, one can express every link in terms of a (non-unique) *braid*. Jones then defined a representation of such a braid into his algebras. The trace of an algebra representation of a braid, which is in turn obtained from the link, can be calculated. The Jones polynomial was realized as such a trace.

In this chapter, we shall not travel the original route of Jones to reach his polynomial as it requires the knowledge of the theory of von Neumann algebras. Instead, we shall follow the approach described by Kauffman in his book to construct a representation of the Artin braid group into the *Temperley-Lieb algebra* [1, chp. 8]. These algebras admit a diagrammatic interpretation and our

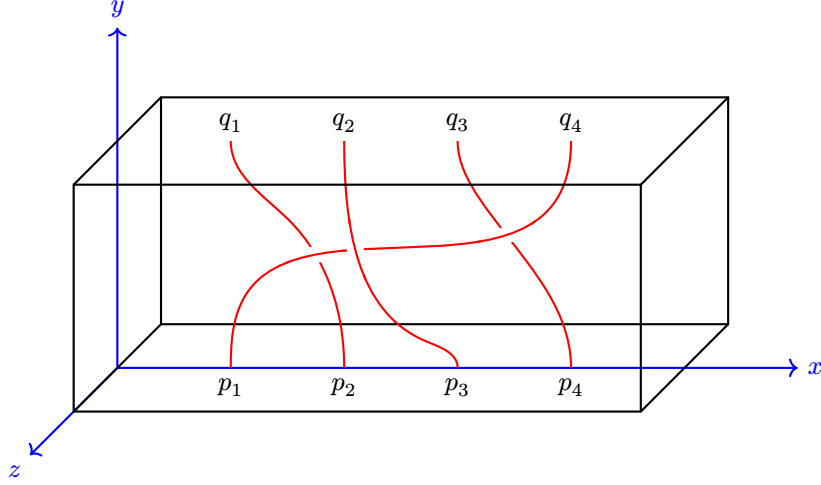


Figure 2.1: Three dimensional geometric representation of braids.

definition of a trace on these algebras shall be diagrammatic in nature as well. Via this trace, we eventually reach the bracket polynomial, which we already know to be equivalent to the Jones polynomial as demonstrated earlier. The Jones algebra can be recovered from the Temperley–Lieb algebra by a choice of substitutions. The Temperley–Lieb algebra arose during the study of certain statistical models in physics [6]. This algebra can be viewed as a sub-algebra of a broader framework of the *partition algebra* [7].

## 2.2 Geometric representation of braids

### 2.2.1 Definition

We shall now understand some basics of braid theory. Emil Artin introduced the Artin braid group explicitly [8, 9, 10].

An  $n$ -braid is an element of the Artin braid group  $B_n$ , defined via the following presentation on the generators  $\sigma_i$ , for  $1 \leq i \leq n - 1$ .

$$B_n := \left\langle \sigma_1, \dots, \sigma_{n-1} \left| \begin{array}{ll} \sigma_i \sigma_i^{-1} &= \mathbb{I}_n^a \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i \sigma_j &= \sigma_j \sigma_i \quad \text{if } |i - j| \geq 2 \end{array} \right. \right\rangle,$$

where  $\mathbb{I}_n^a$  is the identity of  $B_n$ . Thus,  $B_n$  is the quotient of the free group of  $n - 1$  generators with the smallest normal subgroup of the free group containing the elements  $\sigma_i \sigma_i^{-1}$ ,  $\sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1}$  and  $\sigma_i \sigma_j \sigma_i^{-1} \sigma_j^{-1}$  with appropriately restricted  $i$  and  $j$ . We want to recover this algebraic definition using the intuitive understanding of braids that we have. For, that we shall now see a geometric construction in the three dimensional Euclidean space to represent the Artin braid group. This shall make clear the geometric interpretation of the relations as well.

Consider two *ordered* sets of points  $L_1 := \{p_1 := (1, 0, 0), \dots, p_n := (n, 0, 0)\}$  and  $L_2 := \{q_1 := (1, 1, 0), \dots, q_n := (n, 1, 0)\}$  as shown in [fig. 2.1](#) for  $n = 4$ .



Elements of  $L_1$  are called bottom points and elements of  $L_2$  are called the top points. For  $1 \leq i \leq n$ , consider a family of non-intersecting continuous curves  $\gamma_i: [0, 1] \rightarrow \mathbb{R}^3$  such that

1.  $\gamma_i(0) = p_i$  and  $\gamma_i(1) = q_j$  for  $1 \leq i, j \leq n$ .
2. Any plane perpendicular to the  $xy$ -plane and parallel to the  $x$ -axis intersects each of the curves either exactly once or not at all.
3. All the curves strictly lie in the cube determined by the vertices  $(0, 0, 1)$ ,  $(0, 0, -1)$ ,  $(0, 1, 1)$ ,  $(0, 1, -1)$ ,  $(n+1, 0, 1)$ ,  $(n+1, 0, -1)$ ,  $(n+1, 1, 1)$ ,  $(n+1, 1, -1)$ .

Such a labelled curve is called a strand in standard position and a family of such labelled  $n$  strands is called an  $n$ -strand set in a standard position. We can ambient isotope or rigidly move an  $n$ -strand set to get another  $n$ -strand set, possibly not in a standard position. Two  $n$ -strand sets are said to be equivalent if they are related by a finite sequence of the following operations.

1. Rigid motions of  $\mathbb{R}^3$ . Note that the labels of the strands do not change.
2. Ambient isotopies of the strand sets such that the space outside the cube remains constant, and the ambient isotoped set satisfies the above mentioned first and second conditions for all times.

**Remark 16.** Though rigid motions of the Euclidean space include rotations and reflections, we carry the labels of the endpoints as well when we isotope. For example, suppose we have two strand sets such that if we reflect points of one strand set with respect to a plane, we the unmarked points of the other strand set. This reflection shall reflect the marked endpoints of the strands as well. The two strand sets in consideration shall be considered equivalent iff all the marked endpoints match as well.

**Remark 17.** Ambient isotopies which preserve the points outside the cube must preserve the points on the boundary as well, especially the endpoints. The curves cannot touch the boundary at all times.

We shall refer to an equivalence class of such  $n$ -strands as a geometric  $n$ -braid. A geometric  $n$ -braid is well-defined.

**Remark 18.** Even though we have restricted our strands to the a bounded cube in the standard position, we can in principle change the bounds of our cube in  $x$  and  $z$  directions to any value and get the same theory. We shall not pursue this approach here.

### 2.2.2 Standard projection

We call the projection of a standard position  $n$ -strand set onto the  $xy$ -plane to be a two dimensional representation of a braid. Such a projection is drawn in [fig. 2.2](#). It should be noted that a standard position  $n$ -strand set is unique only up to ambient isotopy, thus correspondingly the two dimensional representation of such a set is also unique only up to ambient isotopy, namely the ambient isotopies of the projection of the cube and the ambient isotopies such that the projection is a two dimensional representation of a braid for all times.

Now onwards, we shall always visually represent geometric  $n$ -braids using their standard two dimensional projections.

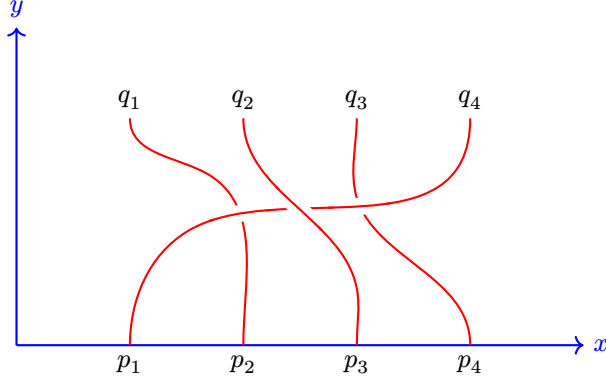


Figure 2.2: Two dimensional geometric representation of braids.

### 2.2.3 Group structure

Multiplication of any two  $n$ -geometric braids  $b_1$  and  $b_2$ , denoted by  $b_1 b_2$  is defined as follows (fig. 2.3). Ambient isotopy and then rigidly move  $b_1$  and  $b_2$  *separately* in the standard position. Now translate only  $b_1$  in the  $+y$  direction by unit distance. The bottom points of  $b_1$  and the top points of  $b_2$  now coincide. Concatenate their strands and shrink the concatenated strands in the  $y$  direction by half keeping fixed the bottom points of  $b_2$ . The result is another geometric  $n$ -braid  $b_1 b_2$  in the standard position. Multiplication defined this way is associative.

**Remark 19.** Note that braid multiplication is not commutative in general. The reader is advised to refer to remark 16.

We shall now drop the axes as well while representing two dimensional geometric  $n$ -braids.

An  $n$ -strand set such that each  $\gamma_i$  is a straight line segment connecting the  $i^{\text{th}}$  bottom point to the  $i^{\text{th}}$  top point is called the identity geometric  $n$ -braid and is denoted by  $\mathbb{I}_n$  (fig. 2.4).

A geometric  $n$ -braid  $a$  such that  $ab = ba = \mathbb{I}_n$  for some geometric  $n$ -braid  $b$  is called the inverse of  $a$  and denoted is by  $a^{-1}$ . We shall see that each element has an inverse.

With these operations, the set of geometric  $n$ -braids becomes a group, which we shall denote by  $\text{GB}_n$ .

### 2.2.4 Generators

By the virtue of ambient isotopy, we can move the crossings in a two dimensional representation of a geometric  $n$ -braid such that each crossing lies in a region bounded by two lines parallel to the  $x$ -axis. Moreover, we can arrange the crossings such that each such region contains only one crossing. Thus, if we give the information regarding the type of each crossing for each such region, we can faithfully reconstruct the two dimensional representation. To this end, we define the generators of a geometric  $n$ -braid.

Denote by  $\tau_i$  the geometric  $n$ -braid such that

1.  $\gamma_i(1) = q_{i+1}$ ,  $\gamma_{i+1}(1) = q_i$ , and  $\gamma_j(1) = q_j$  when  $j$  does not equal  $i$  or  $i + 1$ .

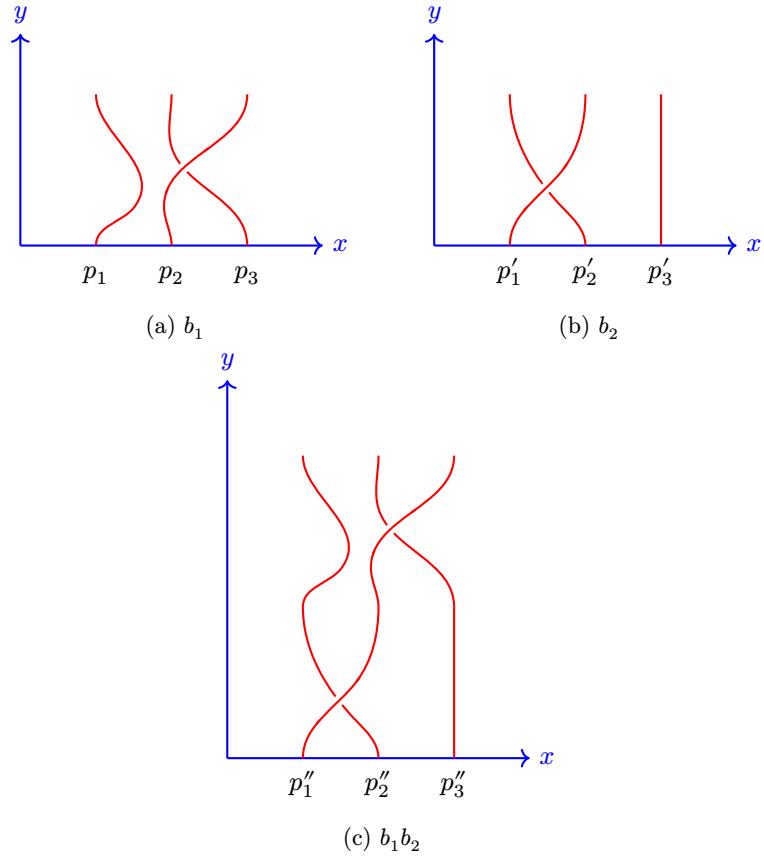


Figure 2.3: Multiplication of two braids (before shrinking).

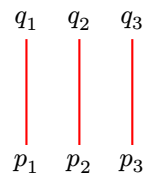


Figure 2.4: The identity  $I_3$

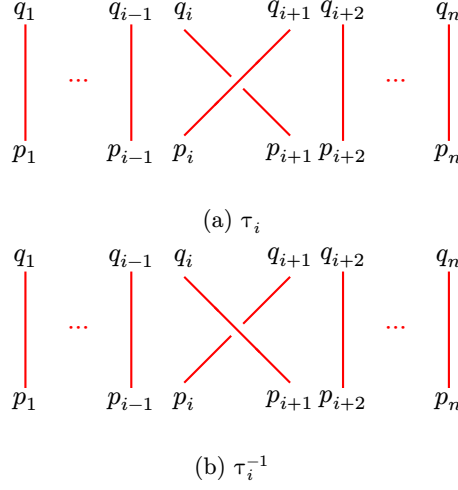


Figure 2.5: Generators  $\tau_i$  and  $\tau_i^{-1}$

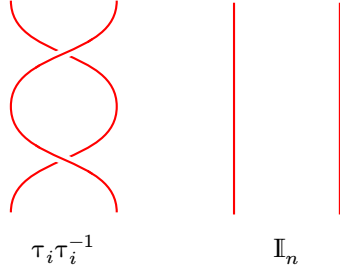


Figure 2.6: A type II move illustrating  $\tau_i \tau_i^{-1} = \mathbb{I}_n$ .

2.  $\pi_{xy}(\gamma_i(t)) \geq 0$  and  $\pi_{xy}(\gamma_{i+1}(t)) \leq 0$  for all  $t \in [0, 1]$ .

$\pi_{xy}$  is the projection maps onto to  $xy$ -plane.  $\tau_1, \dots, \tau_{n-1}$  are the generators of  $\text{GB}_n$  (fig. 2.5).

For example, in fig. 2.3 we have  $b_1 = \tau_2 \in \text{GB}_3$ ,  $b_2 = \tau_1 \in \text{GB}_3$  and  $b_1 b_2 = \tau_2 \tau_1 \in \text{GB}_3$ .

If we multiply  $\tau_i$  and  $\tau_i^{-1}$  to form  $\tau_i \tau_i^{-1}$ , we observe that  $\tau_i \tau_i^{-1} = \mathbb{I}_n$ , where  $\tau_i, \tau_i^{-1} \in \text{B}_n$  for all  $n \geq 2$  (fig. 2.6).

We can perform a move equivalent to the type III move to see that  $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$  (fig. 2.7).

We can slide two crossings vertically across each other if this does not change the ambient isotopy type. This is possible if the two crossings we wish to slide do not share a strand. This gives us the relation  $\tau_i \tau_j = \tau_j \tau_i$  if  $|i - j| \geq 2$  (fig. 2.8).

Let  $w$  be a word of length  $m$  in  $\text{GB}_n$ ;  $w = \prod_{j=1}^m \tau_{\alpha_j}^{\pm 1}$  where  $1 \leq \alpha_j \leq n - 1$ . Every element of  $\text{GB}_n$  and  $\text{B}_n$  can be expressed as a product of its generators,

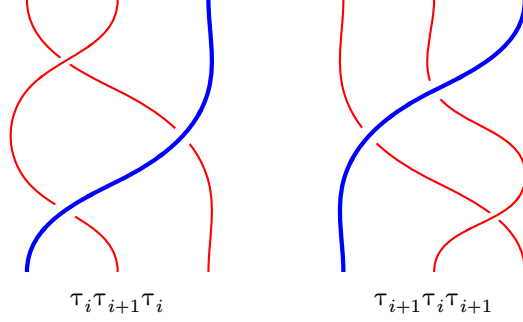


Figure 2.7: A type III move illustrating  $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$ .

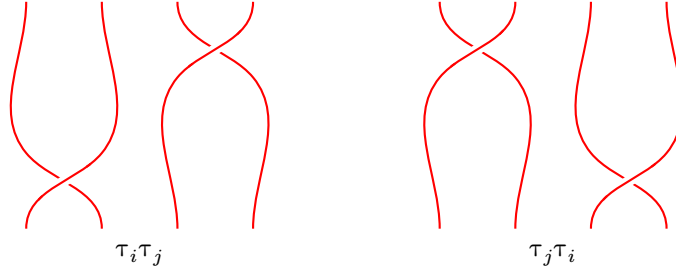


Figure 2.8: Sliding of crossings illustrating  $\tau_i \tau_j = \tau_j \tau_i$ .

albeit non-uniquely. We define a homomorphism

$$\begin{aligned} \Phi: \text{GB}_n &\rightarrow \text{B}_n, \\ \Phi: \prod_{j=1}^m \tau_{\alpha_j}^{\pm 1} &\mapsto \prod_{j=1}^m \sigma_{\alpha_j}^{\pm 1} \text{ for all } m \in \mathbb{N}. \end{aligned}$$

We can see that  $\Phi$  is a surjection as follows. Take an element  $\prod_{j=1}^m \sigma_{\alpha_j}^{\pm 1} \in \text{B}_n$ ,  $\Phi$  maps  $\prod_{j=1}^m \tau_{\alpha_j}^{\pm 1}$  to  $\prod_{j=1}^m \sigma_{\alpha_j}^{\pm 1}$ . Proving that  $\Phi$  is an injection is harder and a proof can be found in [11, chp. 2], and [12].

**Theorem 20.**  $\Phi$  is an isomorphism, i.e.  $\text{B}_n$  and  $\text{GB}_n$  are isomorphic.

This allows us to forget the distinction between  $\text{B}_n$  and  $\text{GB}_n$ .

## 2.3 Closure of braids

We define the closure of a geometric  $n$ -braid as follows. Consider a geometric  $n$ -braid in the standard position. For each  $1 \leq i \leq n$ , we construct the following sequence of line connected line segments. Join  $(i, 1, 0)$ ,  $(i, i, 0)$ ,  $(i, i, 0)$ ,  $(i, -i, 0)$ ,  $(i, -i, 0)$ ,  $(i, 0, 0)$  consecutively. We then join  $\gamma_i$  to the constructed line segments. Repeating this process for all  $i$  gives the closure of a braid (fig. 2.9). We denote the closure of a geometric  $n$ -braid  $b$  by  $\bar{b}$ . Closure of a braid is unique up to ambient isotopy. Two equivalent braid words have the same closures, thus making the closure well-defined.

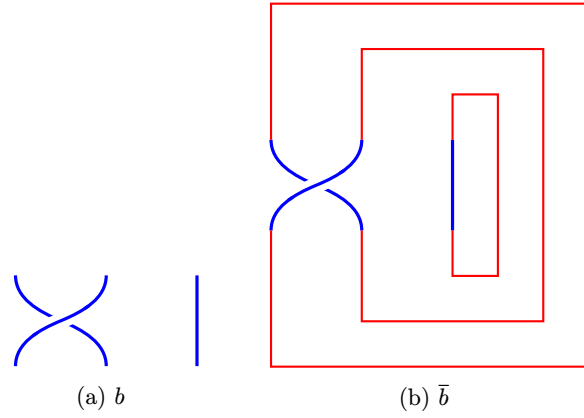


Figure 2.9: Closure of a braid with  $b = \tau_1 \in \text{GB}_2$ .

**Proposition 21.** Every closure of a geometric  $n$ -braid is a link.

*Proof.* The outer line segments are locally flat by virtue of being piecewise linear. Due to the second condition regarding the intersection of a plane in the standard representation of geometric  $n$ -braid, we can project the strand onto the  $y$ -axis and this projection would be a homeomorphism. We take an  $\epsilon$  neighbourhood,  $N(\epsilon)$  around the strand. The pair  $(N(\epsilon), \gamma_i)$  would then be homeomorphic to  $(B, B \cap x\text{-axis})$ , where  $B$  is the three-dimensional unit ball around the origin. We have local flatness at the end points as well due to the union of two locally flat curves.  $\square$

**Remark 22.** Suppose  $\tau_i \in \text{GB}_n$  and  $\tau'_i \in \text{GB}_m$  are generators where  $n < m$ . Then the closures of these two generators are not ambient isotopic. The closure of the latter contains one more non-linking loop.

### 2.3.1 Alexander Theorem

The theorems of James Alexander [13] and Andrei Markov Jr. [14] relate braids to knots.

**Theorem 23** (Alexander). Every link is ambient isotopic to the closure of a geometric braid, for some  $n \in \mathbb{N}$ .

*Proof.* Consider a piecewise linear, regular projection  $\pi(L)$  of a link  $L$  on a plane. We choose a point  $O$  in the projection plane which is not collinear with any of the line segments. This can be done since a the link has only finitely many line segments. Let  $P \in \pi(L)$ . The vector  $OP$  can move either clockwise or anti-clockwise as  $P$  moves along the link projection. We wish to modify the line segments such that  $OP$  moves in only one sense, say anti-clockwise, as  $P$  moves along the entire length of the link projection. We now fix our attention on a line segment corresponding to a clockwise rotation. We divide the segment into sub-parts such that each part shares at-most one crossing point with other line segments. If  $A$  and  $B$  are end-points of such a line segment, then we may replace this line segment with two another line segments  $AC$  and  $CB$ , such that  $C$  is another point not on belonging to  $\pi(L)$  and the triangle  $ABC$  encloses  $O$ . If  $AB$

originally passed under (or over) a line segment of  $\pi(L)$ , then the modified line segments  $AC$  and  $CB$  must pass under (or over) of the line segments of  $\pi(L)$  as well. This move shall not change the link type as it shall be a combination of sliding, type 2 and type 3 moves. In the resulting triangle, we have two orientations possible, one path which travels via  $C$  and the other path which does not. The vector  $OP$  shall move in the opposite, anti-clockwise sense while traversing from  $A$  to  $B$  via  $C$ , instead of  $AB$ . We can repeat this process for all of the (finitely many) line segments which turn clockwise. In the end, we obtain a projection such that  $OP$  moves in only the anti-clockwise sense, as  $P$  moves along the entire length of the link projection. We can ambient isotope the projection such that all the crossings lie in the projection of a cube, more precisely the cube constructed while defining a geometric braid. The end-points can be made to match as well. The above procedure of triangular moves shall guarantee the monotonicity that is required.  $\square$

Henceforth, unless specified otherwise, we shall always work with the projections of the standard representation of a geometric  $n$ -braid and its closure and refer to these projections simply as a braid and its closure.

### 2.3.2 Conjugation

If  $b, g \in \text{GB}_n$ , we observe that  $\overline{gbg^{-1}}$  is ambient isotopic to  $\bar{b}$  (fig. 2.10). The upper strands in  $g$  and  $g^{-1}$  are connected via the closure strand. We can slide  $g$  and  $g^{-1}$  via the closure strands to the ‘other side’ of  $b$  to annihilate each other. This can be achieved by a type II move.

**Remark 24.** The closures of conjugate braids are ambient isotopic as links. The braids themselves are not.

### 2.3.3 Markov move

We observe that if  $b \in \text{GB}_n$ , then  $b\tau_n \in \text{GB}_{n+1}$ ,  $b\tau_n^{-1} \in \text{GB}_{n+1}$  and  $b$  have ambient isotopic closures (fig. 2.11), although  $b$ ,  $b\tau_n$  and  $b\tau_n^{-1}$  are not equivalent as braids. That is, we can add a strand and a crossing of that strand with another strand without changing the link type (of the closure). We can also remove a strand and a crossing if that strand does not cross any other strand. We visually see that adding the above mentioned strands anywhere between the existing strands is equivalent to adding the strands on the right.

### 2.3.4 Markov Theorem

**Theorem 25** (Markov). Two braids whose closures are ambient isotopic to each other are related by a finite sequence of the following operations.

1. Braid equivalences, i.e. equivalences resulting due to the braid relations.
2. Conjugation.
3. Markov moves.

A proof of the above theorem, based on the notes of J. H. Roberts and the seminar of an unknown speaker at Princeton University, can be found in the

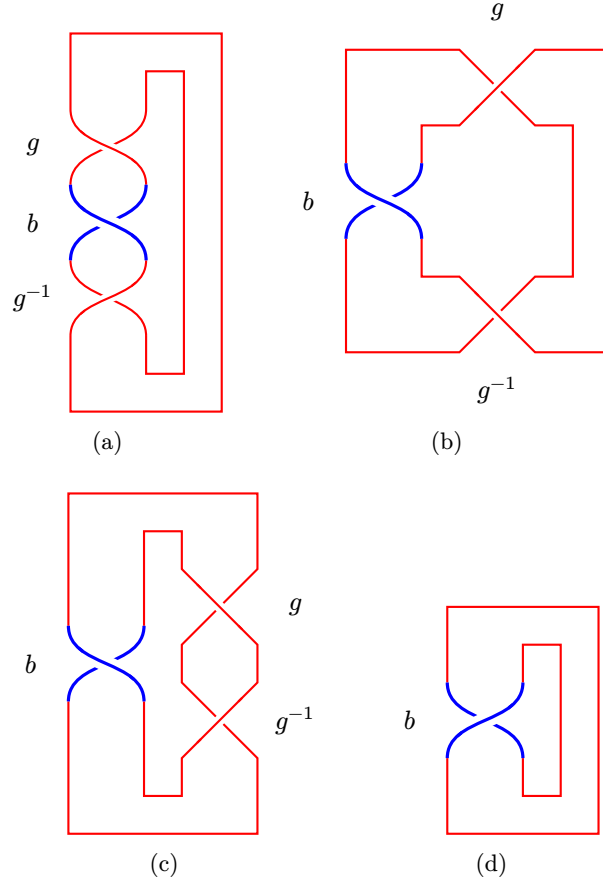


Figure 2.10: Conjugation process illustrating the link equivalence of  $\overline{gbg^{-1}}$  and  $\overline{b}$ , with  $b = \tau_1^{-1} \in \text{GB}_1$  and  $g = \tau_1^{-1} \in \text{GB}_1$ .

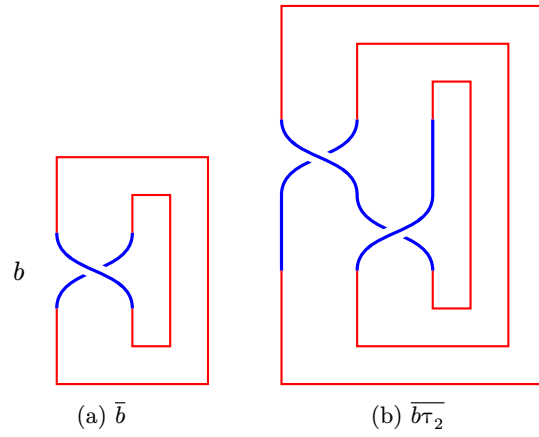


Figure 2.11: Markov move with  $b = \tau_1^{-1}$ .



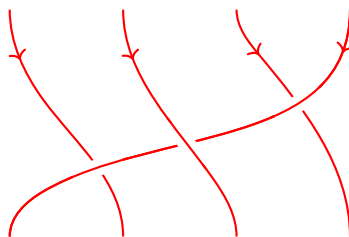


Figure 2.12: A braid with downward orientation

book of Joan Birman [15, chp. 2]. Markov gave a sketch of the proof in 1936 [14]. A shorter proof may be found in Morton's paper [16].

## 2.4 Orientation

We see visually that the result of a Markov move on a braid is equivalent to performing a type I move on the braid closure. We know that a type I move increases or decreases the writhe of a link by a unit value. Since all the crossings in a braid closure occur only in the cube containing the braid strands, we can define the writhe of a braid equal to the writhe of the braid closure by assigning each braid crossing a value, either  $+1$  or  $-1$ . Our assignment must be consistent with our earlier assignment for knots. But for this procedure, we need to assign an orientation to the braid. We assign all the strands (inside the braid cube) a downward orientation (fig. 2.12).

Doing so, we see that  $\tau_i$  inherits  $+1$  value while  $\tau_i^{-1}$  inherits  $-1$ . We could instead have assigned all the strands an upward orientation as well. This would not have changed the values of  $\tau_i$  and  $\tau_i^{-1}$  (fig. 2.13). What is not allowed is assigning arbitrary orientation to strands. If we assign the orientation arbitrarily, then the well-definedness of the orientation cannot be guaranteed. Two distinct strands in a braid could be connected via the closure strands and one would need to check the whole connected link component of the braid closure for a well-defined closure.

Thus, the writhe  $w$  of the braid  $b$  with a word representation  $\prod_{j=1}^m \tau_{\alpha_j}^{\beta_j}$  of length  $m$ , where  $\beta_j \in \{+1, -1\}$  is

$$w(b) = \sum_{j=1}^m \beta_j.$$

Note that the writhe of a braid is dependent on its word representation. We also see that  $w(b) = w(\bar{b})$ .

## 2.5 Markov trace

We won't distinguish between  $B_n$  and  $GB_n$  from now on. We can now move finally towards the Jones/bracket polynomial with the information we have. If we have a function  $J_n: B_n \rightarrow R$ , where  $R$  is a commutative ring, then using the Markov Theorem we can construct link invariants from the family of functions  $\{J_n\}$  if the following conditions are satisfied.

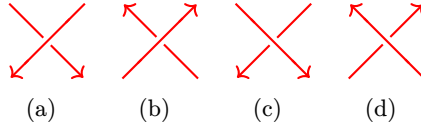


Figure 2.13: Assignment of crossing values. (a) Downward orientation for  $\tau_i$  corresponding to +1. (b) Upward orientation for  $\tau_i$  corresponding to +1. (c) Downward orientation for  $\tau_i^{-1}$  corresponding to -1. (d) Upward orientation for  $\tau_i^{-1}$  corresponding to -1.

1.  $J_n$  is well defined.  $J_n(b) = J_n(b')$  if  $b = b'$ .
2.  $J_n(b) = J_n(gbg^{-1})$  if  $g, b \in B_n$ .
3. If  $b \in B_n$ , then there exists a constant  $\alpha \in R$ , independent of  $n$ , such that

$$J_{n+1}(b\sigma_n) = \alpha^{+1} J_n(b)$$

and

$$J_{n+1}(b\sigma_n^{-1}) = \alpha^{-1} J_n(b).$$

The last condition reminds us of the normalisation needed in order to make the bracket polynomial invariant under the type I move. Its purpose here is the same.

A family of functions  $\{J_n\}$  satisfying the above given three conditions is called a Markov trace on  $\{B_n\}$ . For any link  $L$  which is ambient isotopic to  $\bar{b}$ , where  $b \in B_n$ , we define  $J(L) \in R$  as follows.

$$J(L) := \alpha^{-w(b)} J_n(b).$$

We call  $J(L)$  the link invariant for the Markov trace  $\{J_n\}$ .

**Theorem 26.**  $J$  is an invariant of ambient isotopy for oriented links.

*Proof.* Suppose  $L \sim \bar{b}$  and  $L'' \sim \bar{b}'$ , where  $\sim$  denotes the ambient isotopy relation. By the Markov Theorem, we can obtain  $\bar{b}'$  via an application of a finite sequence of the moves mentioned in the Markov Theorem on  $\bar{b}'$ . Each such move leaves  $J$  invariant.  $J_n$  is already invariant under braid equivalences and conjugation by definition. The  $\alpha^{-w(b)}$  factor cancels the effect of a type I move.  $\square$

We can define the bracket polynomial for braids in the same way as we did for links. Define

$$\begin{aligned} \langle \cdot \rangle : B_n &\rightarrow \mathbb{Z}[A, A^{-1}] \\ \langle \cdot \rangle : b &\mapsto \langle \bar{b} \rangle. \end{aligned}$$

We simply evaluate the bracket on the closure of the braid. We observe that this function is a Markov trace with  $\alpha = -A^3$ . We know that

$$\langle \text{crossing} \rangle = A \langle \text{positive crossing} \rangle + A^{-1} \langle \text{negative crossing} \rangle.$$

In terms of braids, we have

$$\langle \left| \cdots \right| \text{crossing} \left| \cdots \right| \rangle = A \langle \left| \cdots \right| \text{positive crossing} \left| \cdots \right| \rangle + A^{-1} \langle \left| \cdots \right| \text{negative crossing} \left| \cdots \right| \rangle.$$

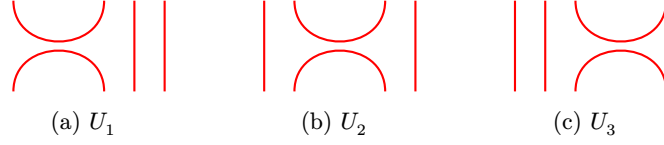


Figure 2.14: Input-output forms or hooks for 4 strands

But  $\left| \cdots \right\rangle \left\langle \cdots \right|$  is the identity braid. Thus, if we denote  $\left| \cdots \right\rangle \left\langle \cdots \right|$  by  $U_i$ , we can write

$$\langle \sigma_i^{-1} \rangle = A \langle U_i \rangle + A^{-1} \langle \mathbb{I}_n \rangle.$$

Similarly, one could write

$$\langle \sigma_i \rangle = A \langle \mathbb{I}_n \rangle + A^{-1} \langle U_i \rangle.$$

Note that  $U_i$  does not belong to the braid group and is a new object. We refer to them as “input-output forms” or as “hooks”. They are cup and cap combinations involving the  $i$ -th and  $(i+1)$ -th strands (fig. 2.14). We can thus use consider  $\sigma_i$  to be equivalent to  $A + A^{-1}U_i$ ,  $\sigma_i^{-1}$  to be equivalent to  $A^{-1} + AU_i$ , and create a formalism based on this. Given a braid word representation of a braid, we can simply substitute the above mentioned equivalences to get a product of  $U_i$ ’s, with  $A$  and  $A^{-1}$  as coefficients. Note that this product is dependent on the specific braid representation of a knot. It is not invariant under a type I move, as is the case with the un-normalised bracket polynomial.

We see that each bracket polynomial evaluation state of the closure of a braid can be written in terms of the closure of a product of the input-output forms. Given a braid word  $b$ , we can consider it equivalent to  $S(b)$ , where  $S(b)$  is the sum of products of the  $U_i$ ’s obtained by substituting  $\sigma_i$  by  $A + A^{-1}U_i$  and  $\sigma_i^{-1}$  by  $A^{-1} + AU_i$ . Closure of each term (a product in  $U_i$ ’s) in the sum corresponds to a state obtained while evaluating the bracket polynomial as it gives a collection of loops. If  $P$  is such a product, then  $\langle P \rangle = \langle \bar{P} \rangle = \delta^{\|P\|}$ , where  $\|P\|$  is the number of loops in  $\bar{P}$  minus 1, and  $\delta = -A^{-2} - A^2$ . Thus,

$$S(b) = \sum_s \langle b|s \rangle P_s,$$

where  $s$  denotes a state and indexes all the terms in the product, and  $\langle b|s \rangle$  is the product of  $A$ ’s and  $A^{-1}$ ’s multiplying each  $P$ -product  $P_s$ . We have

$$\langle b \rangle = \langle S(b) \rangle = \sum_s \langle b|s \rangle \langle P_s \rangle = \sum_s \langle b|s \rangle \delta^{\|s\|}.$$

**Example 27.** Let  $b = \sigma_1 \sigma_2^{-1}$ . We can resolve  $\bar{b}$  in many states. One of the states  $s$  and its corresponding product  $U_1 U_2$  in terms of the input-output forms is shown in fig. 2.15.

**Example 28.** Consider the same braid  $b = \sigma_1 \sigma_2^{-1}$ . We have

$$\begin{aligned} P(b) &= (A + A^{-1}U_1)(AU_2 + A^{-1}) \\ P(b) &= A^2U_2 + \mathbb{I}_3 + U_1U_2 + A^{-2}U_1 \\ \langle b \rangle &= \langle \sigma_1 \sigma_2^{-1} \rangle = \langle P(b) \rangle = A^2 \langle U_2 \rangle + \langle \mathbb{I}_3 \rangle + \langle U_1U_2 \rangle + A^{-2} \langle U_1 \rangle. \end{aligned}$$

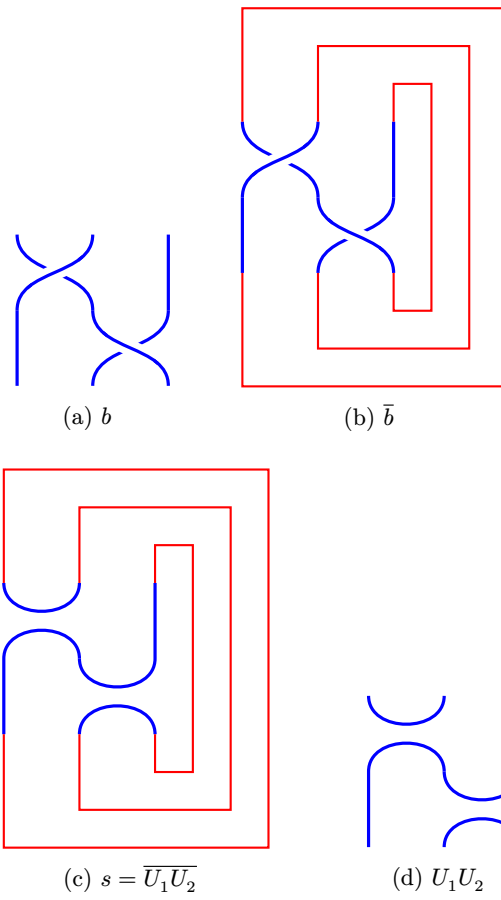


Figure 2.15: Writing a state of a braid closure in terms of input-output forms.

Now,  $\mathbb{I}_3$  corresponds to  $\left| \begin{array}{c} \parallel \\ \parallel \\ \parallel \end{array} \right\rangle$ . Thus,  $\langle I_3 \rangle = \delta^{3-1} = \delta^2$  as the closure of  $\mathbb{I}_3$  shall give three loops.  $U_1$  corresponds to  $\left| \begin{array}{c} \times \\ \parallel \end{array} \right\rangle$ . Thus,  $\langle U_1 \rangle = \delta^{2-1} = \delta$  as the closure shall give two loops. Similarly,  $\langle U_2 \rangle = \delta$  and  $\langle U_1 U_2 \rangle = \delta^{1-1} = 1$ . For a visual representation of the state  $U_1 U_2$ , see [fig. 2.15](#), where the closure gives one single loop. In the end, we get

$$\langle \bar{b} \rangle = A^2 \delta + \delta^2 + 1 + A^{-2} \delta.$$

## 2.6 Temperley–Lieb algebra

We can give the  $U_i$ 's a structure of their own by constructing the free additive algebra  $TL_n$  with the generators  $U_1, U_2, \dots, U_{n-1}$  and the multiplicative relations coming from the interpretation of  $U_i$ 's as input-output forms. We can consider this algebra as module over the ring  $\mathbb{Z}[A, A^{-1}]$  with  $\delta = -A^{-2} - A^2 \in \mathbb{Z}[A, A^{-1}]$ . We shall call  $TL_n$  the Temperley–Lieb algebra. The multiplicative relations in  $TL_n$  are as follows.

1.  $U_i U_{i \pm 1} U_i = U_i$ .
2.  $U_i^2 = \delta U_i$ .
3.  $U_i U_j = U_j U_i$  if  $|i - j| \geq 2$ .

These relations are a result of the geometric relations as illustrated in [fig. 2.16](#).

Now that we have the Temperley–Lieb algebra, we can define a mapping

$$\rho: B_n \rightarrow TL_n$$

by the following formulae.

$$\begin{aligned} \rho(\sigma_i) &= A + A^{-1} U_i, \\ \rho(\sigma_i^{-1}) &= A^{-1} + A U_i. \end{aligned}$$

We see that  $\rho$  is indeed a representation by verifying that  $\rho(\sigma_i) \rho(\sigma_i^{-1}) = 1$ ,  $\rho(\sigma_i \sigma_{i+1} \sigma_i) = \rho(\sigma_{i+1} \sigma_i \sigma_{i+1})$  and if  $|i - j| \geq 2$ ,  $\rho(\sigma_i \sigma_j) = \rho(\sigma_j \sigma_i)$ .

**Proposition 29.**  $\rho: B_n \rightarrow TL_n$  is a representation of the Artin braid group.

*Proof.* We first prove that  $\rho(\sigma_i) \rho(\sigma_i^{-1}) = 1$ .

$$\begin{aligned} \rho(\sigma_i) \rho(\sigma_i^{-1}) &= (A + A^{-1} U_i)(A^{-1} + A U_i) \\ &= 1 + (A^{-2} + A^2) U_i + U_i^2 \end{aligned}$$

But since  $U_i^2 = \delta U_i$  and  $\delta = -A^{-2} - A^2$ , we have

$$\begin{aligned} \rho(\sigma_i) \rho(\sigma_i^{-1}) &= 1 + (A^{-2} + A^2) U_i + \delta U_i \\ &= 1 + (A^{-2} + A^2) U_i + (-A^{-2} - A^2) U_i \\ &= 1. \end{aligned}$$

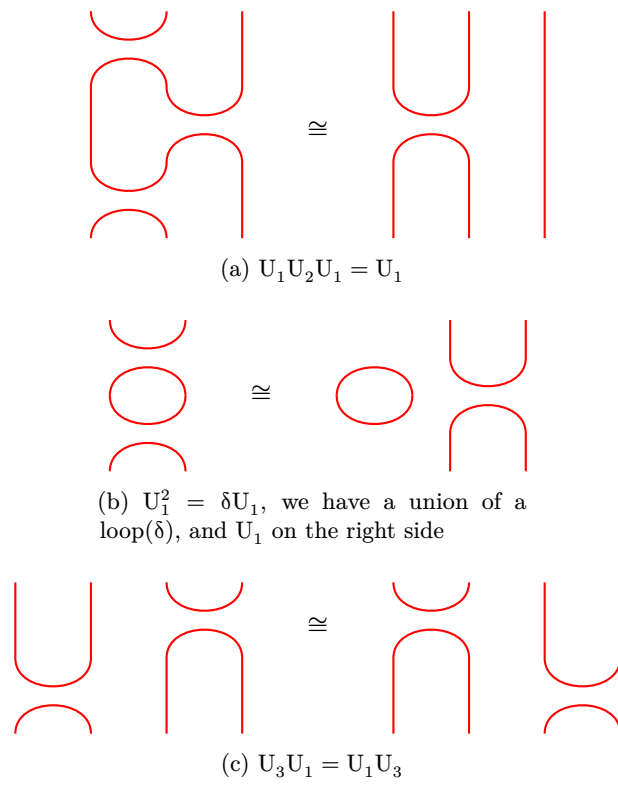


Figure 2.16: Input-output form relations

We now prove that  $\rho(\sigma_i \sigma_{i+1} \sigma_i) = \rho(\sigma_{i+1} \sigma_i \sigma_{i+1})$ .

$$\begin{aligned}
\rho(\sigma_i \sigma_{i+1} \sigma_i) &= (A + A^{-1}U_i)(A + A^{-1}U_{i+1})(A + A^{-1}U_i) \\
&= (A^2 + U_{i+1} + U_i + A^{-2}U_i U_{i+1})(A + A^{-1}U_i) \\
&= A^3 + AU_{i+1} + AU_i + A^{-1}U_i U_{i+1} + A^{-2}U_i^2 + AU_i \\
&\quad + A^{-1}U_{i+1}U_i + A^{-3}U_i U_{i+1}U_i \\
&= A^3 + AU_{i+1} + (A^{-1}\delta + 2A)U_i \\
&\quad + A^{-1}(U_i U_{i+1} + U_{i+1}U_i) + A^{-3}U_i \\
&= A^3 + AU_{i+1} + (A^{-1}(-A^2 - A^{-2}) + 2A + A^{-3})U_i \\
&\quad + A^{-1}(U_i U_{i+1} + U_{i+1}U_i) \\
&= A^3 + A(U_{i+1} + U_i) + A^{-1}(U_i U_{i+1} + U_{i+1}U_i).
\end{aligned}$$

Since symmetry of the above expression in  $i$  and  $i + 1$ , we can conclude that  $\rho(\sigma_i \sigma_{i+1} \sigma_i) = \rho(\sigma_{i+1} \sigma_i \sigma_{i+1})$ . We now prove that if  $|i - j| \geq 2$ , then  $\rho(\sigma_i \sigma_j) = \rho(\sigma_j \sigma_i)$ .

$$\begin{aligned}
\rho(\sigma_i \sigma_j) &= \rho(\sigma_i)(\sigma_j) \\
&= (A + A^{-1}U_i)(A + A^{-1}U_j)
\end{aligned}$$

Now since  $U_i U_j = U_j U_i$  if  $|i - j| \geq 2$ , we have  $(A + A^{-1}U_i)(A + A^{-1}U_j) = (A + A^{-1}U_j)(A + A^{-1}U_i)$  which equals  $\rho(\sigma_j \sigma_i)$ .  $\square$

We now define the diagrammatic trace  $\text{tr}: \text{TL}_n \rightarrow \mathbb{Z}[A, A^{-1}]$  by extending linearly  $\text{tr}(P) = \langle P \rangle$ , where  $P$  is a product term in  $S(b)$ . This version of trace is diagrammatic in nature as we are counting loops in a state. We thus arrive at the formula  $\langle b \rangle = \text{tr}(\rho(b))$ . With what we have learnt so far, one can now find a braid representation  $b$  of a link  $L$  by Alexander's theorem, calculate  $\text{tr}(\rho(b))$  and normalise it to get the link invariant normalised bracket polynomial. One can substitute  $A = t^{-1/4}$  to arrive at our long sought destination of the Jones polynomial.

## 2.7 Jones algebra

Jones considered a sequence of algebras  $A_n$  for  $n = 2, 3, \dots$  with multiplicative generators  $e_1, e_2, \dots, e_{n-1}$  and the following relations.

1.  $e_i^2 = e_i$ .
2.  $e_i e_{i \pm 1} e_i = c e_i$ .
3.  $e_i e_j = e_j e_i$  if  $|i - j| \geq 2$ .

Here,  $c$  is a scalar which commutes with all the other elements. We can consider  $A_n$  as the free additive algebra on these generators viewed as a module over the ring  $\mathbb{C}[c, c^{-1}]$ , i.e. we take the free ring in the generators  $e_1, e_2, \dots, e_{n-1}$  and quotient it with the smallest ideal containing the elements  $e_i$ ,  $c^{-1}e_i e_{i+1}$  and  $e_i e_j e_i^{-1} e_j^{-1}$  with appropriately restricted  $i$  and  $j$ . While  $c$  is often taken to be a complex number, we can view it as another algebraic variable which commutes with the generators.  $A_n$  arose in the theory of classification of von Neumann

algebras and one can realize  $A_n$  as a von Neumann algebra as well. The reader is requested to compare the above mentioned relations for Jones algebra with the relations for the Artin braid group and the Temperley–Lieb algebra.

As mentioned in the beginning of this chapter, it is natural to construct a nested tower of algebras

$$M_0 \hookrightarrow M_1 \hookrightarrow M_2 \hookrightarrow M_3 \hookrightarrow \dots \hookrightarrow M_n \hookrightarrow M_{n+1} \hookrightarrow \dots$$

with the following properties.

1.  $M_0$  and  $M_1$  are given.
2.  $e_i: M_i \rightarrow M_{i-1}$  is a ‘projection’.
3.  $e_i^2 = e_i$ .
4.  $M_{i+1} = \langle M_i, e_i \rangle$ .

Jones constructed such a tower of algebras with the other two properties of generators  $e_i e_{i\pm 1} e_i = c e_i$  and  $e_i e_j = e_j e_i$  if  $|i - j| \geq 2$ . He defined a notion of a trace  $\text{tr}: M_n \rightarrow \mathbb{C}$ , i.e. a function which vanishes on the commutator of any two elements of  $M_n$ . This trace satisfied the so called Markov property:  $\text{tr}(w e_i) = c \text{tr}(w)$  for  $w$  in the algebra generated by  $M_0, e_i, \dots, e_{i-1}$ .

We now see a concrete example of such a tower of algebras. Consider

$$\mathbb{R} \hookrightarrow \mathbb{R}[x_1] \hookrightarrow \mathbb{R}[x_1, x_2] \hookrightarrow \mathbb{R}[x_1, x_2, x_3] \hookrightarrow \dots \hookrightarrow \mathbb{R}[x_1, x_2, \dots, x_n] \hookrightarrow \dots$$

We have a sequence of the set of all real numbers, the set of all real polynomials in one variable, the set of all real polynomials in two variables, the set of all real polynomials in three variables, and so on. Each variable  $x_i$  for  $i \geq 2$ , is a map from  $\mathbb{R}[x_1, x_2, \dots, x_{i-1}]$  to  $\mathbb{R}[x_1, x_2, \dots, x_{i-2}]$ , defined as the coefficient of  $x_{i-1}^0 \in \mathbb{R}[x_1, x_2, \dots, x_{i-1}]$ .

$$\begin{aligned} x_i: \mathbb{R}[x_1, x_2, \dots, x_{i-1}] &\rightarrow \mathbb{R}[x_1, x_2, \dots, x_{i-2}] \\ x_i: \sum_{j=0}^m p_j x_{i-1}^j &\mapsto p_0, \end{aligned}$$

where  $p_j \in \mathbb{R}[x_1, x_2, \dots, x_{i-2}]$  and  $m \in \mathbb{N}$ . As an example, if  $\sum_{j=0}^m p_j x_1^j \in \mathbb{R}[x_1]$ , then

$$\begin{aligned} x_2: \mathbb{R}[x_1] &\rightarrow \mathbb{R} \\ x_2: \sum_{j=0}^m p_j x_1^j &\mapsto p_0, \end{aligned}$$

where  $p_0$  is the coefficient of  $x_1^0$ , i.e. the constant term. For example,

$$x_2(3x_1^4 + 6x_1^3 + x_1 + 5) = 5.$$

Similarly,  $x_3$  is a map from  $\mathbb{R}[x_1, x_2]$  to which maps a polynomial in two variables to the polynomial (in the single variable  $x_1$ ) which does not contain any power of  $x_2$ . As an example,

$$x_3(4x_1 x_2^2 + 3x_1^3 + x_2 + x_2^5 + 1) = 3x_1^3 + 1.$$



$(\mathbb{R}[x_1, x_2, \dots, x_n], +, \cdot)$  is an algebra with  $+$  defined as polynomial addition (component-wise addition) and  $\cdot$  defined as polynomial multiplication (Cauchy product). The generator relations for the Jones algebra, however, are not satisfied with the  $\cdot$  operation;  $x_1^2 := x_1 \cdot x_1 \neq x_1$ . But we can make sense of them if we interpret the operation amongst the generators as the function composition operator.  $x_1, x_2, \dots, x_n$  are the generators  $(\mathbb{R}[x_1, x_2, \dots, x_n], +, \cdot)$ . Since  $x_i$ 's are functions as well, we can interpret  $x_i^2$  as  $x_i \circ x_i$ . If we do that, then  $x_i^2 := x_i \circ x_i = x_i$  as  $x_i$  is a projection operator. Similarly, we see that  $x_i \circ x_{i\pm 1} \circ x_i = x_i$  and  $x_i \circ x_j = x_j \circ x_i$  by restricting the domains appropriately.

We can define a Markov trace on the above tower as follows. For an element  $w \in \mathbb{R}[x_1, x_2, \dots, x_n]$ , define  $\text{tr}: \mathbb{R}[x_1, x_2, \dots, x_n] \rightarrow \mathbb{Z}$  by assigning  $w$  the degree of the polynomial. For an element of  $\mathbb{R}[x_1, x_2, \dots, x_{n+1}]$ ,  $\text{tr}: \mathbb{R}[x_1, x_2, \dots, x_{n+1}] \rightarrow \mathbb{Z}$ . Even though the domains of the trace are different in the above two examples, we can perform this abuse of notation since there is no risk of confusion here. We see that  $\text{tr}(wx_{n+1}) = \text{tr}(w)$ . Thus,  $\text{tr}$  is a Markov trace as it satisfies the Markov property with  $c = 1$  for all  $w \in \mathbb{R}[x_1, x_2, \dots, x_n]$  where  $n \geq 1$ .

We can retrieve the Jones algebra from the Temperley–Lieb algebra by substituting  $e_i = \delta^{-1}U_i$ ,  $e_i^2 = e_i$  and  $e_i e_{i\pm 1} e_i = \delta^{-2}e_i$ . We have taken  $c = \delta^{-2}$ . Note that the underlying rings are different.

# Bibliography

- [1] Louis H. Kauffman. *Knots and physics*. 4th ed. Knots and Everything 53. Singapore: World Scientific, 2013. ISBN: 978-981-4383-00-4.
- [2] Peter R. Cromwell. *Knots and links*. Cambridge, UK: Cambridge University Press, 2004. ISBN: 0-521-83947-5.
- [3] Fredric D. Ancel. “Semi-isotopic knots”. In: (2021). arXiv: [2112.13940](https://arxiv.org/abs/2112.13940) [[math.GT](#)].
- [4] Shijie Gu. *Bing sling isotopy to unknot*. Answer to the question “Bing sling isotopy to unknot” (version: 2022-12-27). MathOverflow, Dec. 26, 2022. URL: <https://mathoverflow.net/q/437260>.
- [5] F. R. Vaughan Jones. “A polynomial invariant for knots via von Neumann algebras”. In: *Bulletin of the American Mathematical Society* 12.1 (1985), pp. 103–111. DOI: [10.1090/s0273-0979-1985-15304-2](https://doi.org/10.1090/s0273-0979-1985-15304-2).
- [6] H. Neville V. Temperley and Elliott H. Lieb. “Relations between the ‘percolation’ and ‘colouring’ problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the ‘percolation’ problem”. In: *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences* 322.1549 (Apr. 1971), pp. 251–280. DOI: [10.1098/rspa.1971.0067](https://doi.org/10.1098/rspa.1971.0067).
- [7] Wikipedia contributors. *Partition algebra*. Wikipedia, The Free Encyclopedia. URL: [https://en.wikipedia.org/wiki/Partition\\_algebra](https://en.wikipedia.org/wiki/Partition_algebra).
- [8] Emil Artin. “Theorie der zöpfe”. In: *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg* 4.1 (Oct. 1925), pp. 47–72. ISSN: 1865-8784. DOI: [10.1007/bf02950718](https://doi.org/10.1007/bf02950718).
- [9] Emil Artin. “Theory of braids”. In: *Annals of Mathematics* 1 (1947), pp. 101–126. ISSN: 0003486X. DOI: [10.2307/1969218](https://doi.org/10.2307/1969218).
- [10] Michael Friedman. “Mathematical formalization and diagrammatic reasoning: the case study of the braid group between 1925 and 1950”. In: *British Journal for the History of Mathematics* 34.1 (2019), pp. 43–59. DOI: [10.1080/17498430.2018.1533298](https://doi.org/10.1080/17498430.2018.1533298).
- [11] Kunio Murasugi and Bohdan I. Kurpita. *A study of braids*. Springer Science & Business Media, 1999. ISBN: 978-0-7923-5767-4.
- [12] H. Frederic Bohnenblust. “The algebraical braid group”. In: *The Annals of Mathematics* 48.1 (Jan. 1947), p. 127. ISSN: 0003-486X. DOI: [10.2307/1969219](https://doi.org/10.2307/1969219).

- [13] James W. Alexander. “A lemma on systems of knotted curves”. In: *Proceedings of the National Academy of Sciences* 9.3 (Mar. 1923), pp. 93–95. DOI: [10.1073/pnas.9.3.93](https://doi.org/10.1073/pnas.9.3.93).
- [14] Andrei A. Markov Jr. “Über die freie äquivalenz der geschlossenen zöpfe”. German. In: *Recueil Mathématique. Nouvelle Série* 1 (1936), pp. 73–78. URL: [https://www.mathnet.ru/php/archive.phtml?wshow=paper&jrnid=sm&paperid=5479&option\\_lang=eng](https://www.mathnet.ru/php/archive.phtml?wshow=paper&jrnid=sm&paperid=5479&option_lang=eng).
- [15] Joan S. Birman. *Braids, links, and mapping class groups*. Annals of Mathematics Studies 82. Princeton University Press, 1974. ISBN: 978-14-0088142-0.
- [16] H. R. Morton. “Threading knot diagrams”. In: *Mathematical Proceedings of the Cambridge Philosophical Society* 99.2 (1986), pp. 247–260. DOI: [10.1017/S0305004100064161](https://doi.org/10.1017/S0305004100064161).