SOME TOPICS IN KNOT THEORY

A REPORT

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APOORV POTNIS

(18343)

DEPARTMENT OF MATHEMATICS,
INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH BHOPAL,
BHOPAL - 462 066

April 2023

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ACKNOWLEDGEMENT

Acknowledgement is here.

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ABSTRACT

We shall study some selected topics in knot theory from the book *Knots and Physics*, 4th ed. by Louis Kauffman [1]. The basics of knot theory have been studied from the book *Knots and Links* by Peter Cromwell [2].

A knot is an embedding of the circle in three-dimensional space. We discuss the state model for bracket polynomial, Jones polynomial and some of its generalisations. One can apply these knot invariants to show that various knots are not ambient isotopic, which is the notion we shall use to distinguish knots. We shall see a resolution of some of the old conjectures in knot theory using the bracket polynomial. Vaughn Jones found the polynomial named after him while he was studying towers of von Neumann algebras. We shall follow a different route following Kauffman to understand the relation amongst braids, links, Temperley–Lieb algebras, which arose in statistical physics, and the Jones algebras.

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Chapter 1

The Jones polynomial and its Generalisations

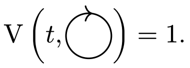
1.1 Jones polynomial

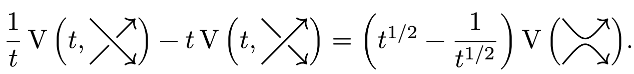
Definition 1 (Jones polynomial)**.** The Jones polynomial invariant $\mathrm{V}(t, K)$ is defined as

$$\mathrm{V}(t, K) := \mathrm{L}\left(\frac{1}{t^{1/4}}, K\right)$$

for an oriented link K.

Here, $t = \dfrac{1}{A^4}$. $\mathrm{V}(t, K)$ is a Laurent polynomial in $t^{1/2}$; $\mathrm{V}(K) \in \mathbb{Z}\left[t^{1/2}, \dfrac{1}{t^{1/2}}\right]$. With the above definition, we have the following two properties of the Jones polynomial.

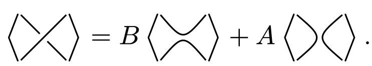
1. 

2. 

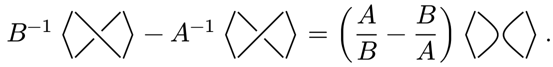
We can define the Jones polynomial axiomatically as a Laurent polynomial in $t^{1/2}$ satisfying the above two properties as well. But then we would have to check well-definedness and invariance under the Reidemeister moves. We derive these properties using the definition. The first property follows evaluating the bracket polynomial on the trivial knot. Orientation of a knot does not matter in this case. Just as with the normalised bracket polynomial, we shall drop the variable in which the polynomial is based if it is clear from the context.

Proof of the second property. We know that

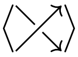
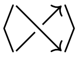
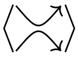
and

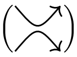
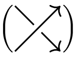
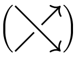


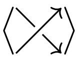
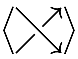
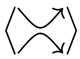
Hence,



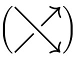
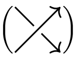
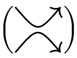
Thus,

$A\,\langle$  $\rangle - A^{-1}\,\langle$  $\rangle = (A^2 - A^{-2})\,\langle$  $\rangle.$

Let $w = \mathrm{w}\,($  $)$ so that $\mathrm{w}\,($  $) = w - 1$ and $\mathrm{w}\,($  $) = w + 1$. Also, let $\alpha = -A^3$. Then

$A\alpha\,\langle$  $\rangle\,\alpha^{-w+1} - A^{-1}\alpha^{-1}\,\langle$  $\rangle\,\alpha^{-w+1} = (A^2 - A^{-2})\,\langle$  $\rangle\,\alpha^{-w}.$

Thus,

$-A^4\,\mathrm{L}\,($  $) + A^{-4}\,\mathrm{L}\,($  $) = (A^2 - A^{-2})\,\mathrm{L}\,($  $).$

Substituting $A = t^{-1/4}$ yields the desired result. □

The Jones polynomial follows a reversing property.

Theorem 2 (Reversing property)**.** Let K be an oriented link. Let K' be such that K' is obtained by reversing the orientation of a component $K_i \subset K$.

Let $\lambda = \mathrm{lk}(K_i, K - K_i)$ denote the total linking number of K_i with the remaining components of K. Then

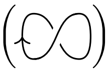
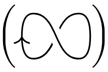
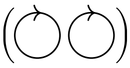
$$\mathrm{V}(t, K') = t^{-3\lambda} \cdot \mathrm{V}(t, K).$$

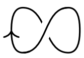
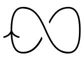
Proof. We see that $\mathrm{w}(K') = \mathrm{w}(K) - 4\lambda$.

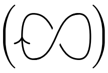
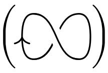
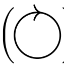
$$\begin{aligned} \mathrm{L}(A, K') &= (-A^3)^{-\mathrm{w}(K')}\langle K' \rangle \\ &= (-A^3)^{-\mathrm{w}(K)+4\lambda}\langle K \rangle. \end{aligned}$$

Therefore $\mathrm{L}(A, K') = (-A^3)^{4\lambda}\,\mathrm{L}(A, K)$. Substituting $A = t^{-1/4}$ yields the desired result. □

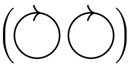
Example 3. Using the skein relation, we have that

$\frac{1}{t}\,\mathrm{V}\left(\right.$  $\left.\right) - t\,\mathrm{V}\left(\right.$  $\left.\right) = \left(t^{1/2} - \frac{1}{t^{1/2}}\right)\mathrm{V}\left(\right.$  $\left.\right).$

But since both  and  are trivial knots,

$\mathrm{V}\left(\right.$  $\left.\right) = \mathrm{V}\left(\right.$  $\left.\right) = \mathrm{V}\left(\right.$  $\left.\right) = 1.$

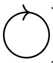
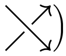
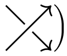
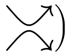
Thus,

$\mathrm{V}\left(\right.$  $\left.\right) = \dfrac{t^{-1} - t}{t^{1/2} - 1/t^{1/2}} = \dfrac{(t^{-1} - t)(t^{1/2} + 1/t^{1/2})}{t - t^{-1}} = -(t^{1/2} + 1/t^{1/2}) = \delta.$

The third property is surprising when viewed with respect to the skein relation [3, 4]. Skein relations of this sort form a general defining feature for various polynomials.

1.2 Alexander–Conway polynomial

Definition 4 (Alexander–Conway polynomial)**.** Let K be an oriented link diagram. Then the Alexander–Conway polynomial invariant $\nabla(z, K) \in \mathbb{Z}[z]$ is defined by the rules:

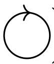
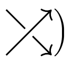
1. $\nabla\left(z,\right.$  $\left.\right) = 1.$
2. $\nabla\left(z,\right.$  $\left.\right) - \nabla\left(z,\right.$  $\left.\right) = \nabla\left(z,\right.$  $\left.\right).$

This polynomial is a generalisation and reformulation of the original Alexander polynomial.

1.3 HOMFLYPT polynomial

The HOMFLYPT polynomial was discovered independently by two groups, one group consisting of Jim Hoste, Adrian Ocneanu, Kenneth Millett, Peter Freyd, William Lickorish, and David Yetter and the other group consisting of Polish mathematicians Józef Przytycki and Paweł Traczyk.

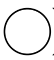
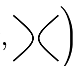
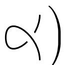
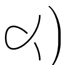
Definition 5 (HOMFLYPT polynomial)**.** Let K be an oriented link diagram. Then the HOMFLYPT polynomial invariant $\mathrm{P}(\alpha, z, K)$ is defined by the rules:

1. $\mathrm{P}\left(\alpha, z,\right.$  $\left.\right) = 1.$
2. $\alpha\,\mathrm{P}\left(\alpha, z,\right.$  $\left.\right) - \frac{1}{\alpha}\,\mathrm{P}\left(\alpha, z,\right.$  $\left.\right) = z\,\mathrm{P}\left(\alpha, z,\right.$  $\left.\right).$

Note that this polynomial is a two variable polynomial. If we take $\alpha = t^{-1}$ and $z = t^{1/2} - 1/t^{1/2}$, we retrieve the Jones polynomial. If we take $\alpha = 1$, we retrieve the Alexander–Conway polynomial.

1.4 Kauffman polynomial

The Kauffman polynomial $\mathrm{F}(\alpha, z, K)$ is a semi-oriented two variable polynomial invariant which generalises Kauffman's bracket and the Jones polynomial. This polynomial is a normalisation of a polynomial, $\overline{\mathrm{L}}(\alpha, z, K)$, defined for unoriented links which satisfies the following properties.

1. $\overline{\mathrm{L}}\left(\alpha, z,\right.$  $\left.\right) = 1.$
2. $\overline{\mathrm{L}}\left(\alpha, z,\right.$  $\left.\right) + \overline{\mathrm{L}}\left(\alpha, z,\right.$  $\left.\right) = z\left[\overline{\mathrm{L}}\left(\alpha, z,\right.\right.$  $\left.\right) + \overline{\mathrm{L}}\left(\alpha, z,\right.$  $\left.\left.\right)\right].$
3. $\overline{\mathrm{L}}\left(\alpha, z,\right.$  $\left.\right) = \alpha\,\overline{\mathrm{L}}\,(\alpha, z,$ $)$
4. $\overline{\mathrm{L}}\left(\alpha, z,\right.$  $\left.\right) = \frac{1}{\alpha}\,\overline{\mathrm{L}}\,(\alpha, z,$  $).$

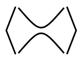
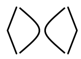
The Kauffman polynomial is defined by the formula

$$\mathrm{F}(\alpha, z, K) = \alpha^{-w(K)}\,\overline{\mathrm{L}}(\alpha, z, K).$$

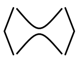
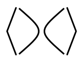
The bracket and the Jones polynomials are special cases of the Kauffman polynomial.

Theorem 6. $\langle K\rangle(A) = \overline{\mathrm{L}}(-A^3, A + A^{-1}, K)$.

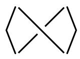
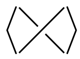
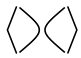
Proof. We recall the bracket identities

 $= A$  $+ A^{-1}$ 

and

 $= A^{-1}$  $+ A$  .

Adding them, we have

 $+$  $= (A + A^{-1})\Big[$  $+$  $\Big]$.

Thus, $\langle K\rangle(A) = \overline{\mathrm{L}}(-A^3, A + A^{-1}, K)$. $\square$

Theorem 7. $\mathrm{V}(t, K) = \mathrm{F}(-t^{-3/4}, t^{-1/4} + t^{1/4}, K)$.

Proof. Since

$$\mathrm{V}(t, K) = \mathrm{L}(t^{-1/4}, K)$$

and

$$\mathrm{L}(A, K) = (-A^3)^{-\mathrm{w}(K)}\langle K\rangle(A),$$

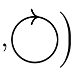
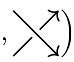
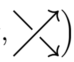
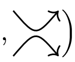
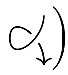
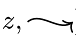
we have that

$$\mathrm{V}(t, K) = (-t^{-3/4})^{-\mathrm{w}(K)}\,\mathrm{L}(-t^{-3/4}, t^{-1/4} + t^{1/4}, K).$$

Hence, $\mathrm{V}(t, K) = \mathrm{F}(-t^{-3/4}, t^{-1/4} + t^{1/4}, K)$. $\square$

1.5 Regular isotopy HOMFLYPT polynomial

The regular isotopy HOMFLYPT polynomial $\mathrm{H}(\alpha, z, K)$, is defined by the following properties.

1. $\mathrm{H}\Big(\alpha, z,$  $\Big) = 1.$
2. $\mathrm{H}\Big(\alpha, z,$  $\Big) - \mathrm{H}\Big(\alpha, z,$  $\Big) = z\,\mathrm{H}\Big(\alpha, z,$  $\Big).$
3. $\mathrm{H}\Big(\alpha, z,$  $\Big) = \alpha\,\mathrm{H}(\alpha, z,$  $).$
4. $\mathrm{H}\Big(\alpha, z,$  $\Big) = \dfrac{1}{\alpha}\,\mathrm{H}(\alpha, z,$  $).$

$\boxed{\text{Regular isotopy HOMFLYPT polynomial}}$ The regular isotopy HOMFLYPT polynomial $H_K(\alpha, z)$, is defined by the following properties.

1. $H_K(z) = H_{K'}(z)$ if the oriented links K and K' are regular isotopic.

2. $H_{\bigcirc} = 1$.

3. $H - H = z \cdot H$.

4. $H_+ = \alpha \cdot H_\to$ and $H_- = \alpha^{-1} \cdot H_\to$.

Again, believing in the existence of this invariant, we have

Theorem 8. $P_K(\alpha, z) = \alpha^{-w(K)} \cdot H_K(\alpha, z)$.

Proof Let $W_K(\alpha, z) = \alpha^{-w(K)} H_K(\alpha, z)$. Then W_K is an invariant of ambient isotopy, and $W_{\bigcirc} = 1$. We thus check the exchange identity. We have $H - H = z \cdot H$, and thus,

$$\alpha^{+1}\alpha^{-(w+1)} \cdot H - \alpha^{-1}\alpha^{-(w-1)} \cdot H = z \cdot \alpha^{-w} \cdot H,$$

where $w = w\left(\right)$. Hence $\alpha \cdot W - \alpha^{-1} \cdot W = z \cdot W$. $\square$

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