Thesis draft

Apoorv Potnis

March 20, 2023

Contents

Chapter 1

Braids and the Jones polynomial

1.1 Motivation

Jones algebra $\xrightarrow[\text{trace}]{\text{Markov}}$ Jones polynomial

Oriented link $\longrightarrow$ Artin braid group

Artin braid group $\nearrow$ Jones algebra

Artin braid group $\searrow$ Temperley–Lieb algebra

Temperley–Lieb algebra $\xrightarrow[\text{trace}]{\text{Diagrammatic}}$ Normalised bracket polynomial

Jones polynomial $\updownarrow$ Substitution $\updownarrow$ Normalised bracket polynomial

As remarked earlier, Jones arrived at his polynomial indirectly while working on the theory of operator algebras [1]. In his course of investigations, he constructed a tower of algebras nested in one another with the property that each of these algebras is generated by a set of generators satisfying a particular set of relations. A degree of similarity between these relations and the relations among the generators of the *Artin braid group* was pointed out to Jones by a student during a seminar [2, p. 216], which led to the investigations of Jones into knot theory. Jones had defined a notion of a *trace* on his algebras; more specifically a trace function obeying the *Markov property*. As we shall see, one can express every link in terms of a (non-unique) *braid*. Jones then defined a representation of such a braid into his algebras. The trace of an algebra representation of a braid, which is in turn obtained from the link, can be

calculated. The Jones polynomial was realized as such a trace.

In this chapter, we shall not travel the original route of Jones to reach his polynomial as it requires the knowledge of the theory of von Neumann algebras. Instead, we shall follow the approach described by Kauffman in his book to construct a representation of the Artin braid group into the *Temperley–Lieb algebra* [3, chp. 8]. These algebras admit a diagrammatic intrepretation and our definition of a trace on these algebras shall be diagrammatic in nature as well. Via this trace, we eventually reach the bracket polynomial, which we already know to be equivalent to the Jones polynomial as demonstrated earlier. The Jones algebra can be recovered from the Temperley–Lieb algebra by a choice of substitutions. The Temperley–Lieb algebra arose during the study of certain statistical models in physics [4]. This algebra can be viewed as a sub-algebra of a broader framework of the *partition algebra* [5].

1.2 Geometric representation of braids

1.2.1 Definition

Figure 1.1: Three dimensional geometric representation of braids.

We shall now understand some basics of braid theory. Emil Artin introduced the Artin braid group explicitly [6, 7, 8].

An n-braid is an element of the Artin braid group B_n, defined via the following presentation on the generators σ_i, for $1 \leq i \leq n - 1$.

$$
\mathrm{B}_n := \left\langle\ \sigma_1, \dots, \sigma_{n-1}\ \middle|\
\begin{array}{rcll}
\sigma_i \sigma_i^{-1} & = & \mathbb{I}_n^{\mathrm{a}} & \\
\sigma_i \sigma_{i+1} \sigma_i & = & \sigma_{i+1} \sigma_i \sigma_{i+1} & \\
\sigma_i \sigma_j & = & \sigma_j \sigma_i & \text{if } |i - j| \geq 2
\end{array}
\ \right\rangle,
$$

where $\mathbb{I}_n^{\mathrm{a}}$ is the identity of B_n. Thus, B_n is the quotient of the free group of

$n-1$ generators with the smallest normal subgroup of the free group containing the elements $\sigma_i \sigma_i^{-1}$, $\sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1}$, and if $|i-j| \geq 2$, then $\sigma_i \sigma_j \sigma_i^{-1} \sigma_j^{-1}$. We want to recover this algebraic definition using the intuitive understanding of braids that we have. For, that we shall now see a geometric construction in the three dimensional Euclidean space to represent the Artin braid group. This shall make clear the geometric intrepretation of the relations as well.

Consider two *ordered* sets of points $L_1 := \{p_1 := (1,0,0), \dots, p_n := (n,0,0)\}$ and $L_2 := \{q_1 := (1,1,0), \dots, q_n := (n,1,0)\}$ as shown in for $n = 4$. Elements of L_1 are called bottom points and elements of L_2 are called the top points. For $1 \leq i \leq n$, consider a family of non-intersecting continuous curves $\gamma_i \colon [0,1] \to \mathbb{R}^3$ such that

1. $\gamma_i(0) = p_i$ and $\gamma_i(1) = q_j$ for $1 \leq i, j \leq n$.

2. Any plane perpendicular to the xy-plane and parallel to the x-axis intersects each of the curves either exactly once or not at all.

3. All the curves lie in the cube determined by the vertices $(0,0,1)$, $(0,0,-1)$, $(0,1,1)$, $(0,1,-1)$, $(n+1,0,1)$, $(n+1,0,-1)$, $(n+1,1,1)$, $(n+1,1,-1)$.

Such a labelled curve is called a strand in standard position and a family of such labelled n stands is called an n-strand set in a standard position. We can ambient isotope or rigidly move an n-strand set to get another n-strand set, possibly not in a standard position. Two n-strand sets are said to be equivalent if they are related by a sequence of rigid motions of the strand sets, and ambient isotopies of the strand sets such that the space outside the cube, along with the endpoints, remains fixed. We shall refer to an equivalence class of such n-strands as a geometric n-braid. Thus, a geometric n-braid is well-defined.

Remark 1. Even though we have restricted our strands to the a bounded cube in the standard position, we can in principle change the bounds of our cube in x and z directions to any value and get the same theory. We shall not pursue this approach here.

1.2.2 Standard projection

We call the projection of a standard position n-strand set onto the xy-plane to be a two dimensional representation of a braid. Such a projection is drawn in . It should be noted that a standard position n-strand set is unique only up to ambient isotopy, thus correspondingly the two dimensional representation of such a set is also unique only up to ambient isotopy, namely the ambient isotopies of the projection of the cube and the ambient isotopies such that the projection is a two dimensional representation of a braid for all times.

Figure 1.2: Two dimensional geometric representation of braids.

Now onwards, we shall always visually represent geometric n-braids using their standard two dimensional projections.

1.2.3 Group structure

Multiplication of any two n-geometric braids b_1 and b_2, denoted by $b_1 b_2$ is defined as follows (fig. 1.3). Ambient isotope and then rigidly move b_1 and b_2 *separately* in the standard position. Now translate only b_1 in the $+y$ direction by unit distance. The bottom points of b_1 and the top points of b_2 now coincide. Concatenate their strands and shrink the concatenated strands in the y direction by half keeping fixed the bottom points of b_2. The result is another geometric n-braid $b_1 b_2$ in the standard position. Multiplication defined this way is associative.

(a) b_1

(b) b_2

(c) $b_1 b_2$

Figure 1.3: Multiplication of two braids (before shrinking).

We shall now drop the axes as well while representing two dimensional geometric n-braids.

An n-strand set such that each γ_i is a straight line segment connecting the ith bottom point to the ith top point is called the identity geometric n-braid and is denoted by $\mathbb{I}_n$ (fig. 1.4).

Figure 1.4: The identity $\mathbb{I}_3$

A geometric n-braid a such that $ab = ba = \mathbb{I}_n$ for some geometric n-braid b is called the inverse of b and denoted is by b^{-1}. We shall see that each element

has an inverse.

With these operations, the set of geometric n-braids becomes a group, which we shall denote by GB_n.

1.2.4 Generators

By the virtue of ambient isotopy, we can move the crossings in a two dimensional representation of a geometric n-braid such that each crossing lies in a region bounded by two lines parallel to the x-axis. Moreover, we can arrange the crossings such that each such region contains only one crossing. Thus, if we give the information regarding the type of each crossing for each such region, we can faithfully reconstruct the two dimensional representation. To this end, we define the generators of a geometric n-braid.

Denote by τ_i the geometric n-braid such that

1. $\gamma_i(1) = q_{i+1}$, $\gamma_{i+1}(1) = q_i$, and $\gamma_j(1) = q_j$ when j does not equal i or $i+1$.

2. $\pi_{xy}(\gamma_i(t)) \geq 0$ and $\pi_{xy}(\gamma_{i+1}(t)) \leq 0$ for all $t \in [0,1]$.

π_{xy} is the projection maps onto to xy-plane. $\tau_1, \dots, \tau_{n-1}$ are the generators of GB_n (fig. 1.5).
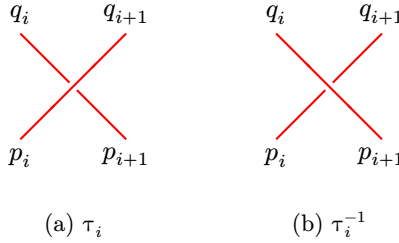

(a) τ_i (b) τ_i^{-1}

Figure 1.5: Generators τ_i and τ_i^{-1}. We have omitted the other straight strands.

For example, in fig. 1.3 we have $b_1 = \tau_2 \in \mathrm{GB}_3$, $b_2 = \tau_1 \in \mathrm{GB}_3$ and $b_1 b_2 = \tau_2 \tau_1 \in \mathrm{GB}_3$.

If we multiply τ_i and τ_i^{-1} to form $\tau_i \tau_i^{-1}$, we observe that $\tau_i \tau_i^{-1} = \mathbb{I}_n$, where $\tau_i, \tau_i^{-1} \in \mathrm{B}_n$ for all $n \geq 2$ (fig. 1.6).

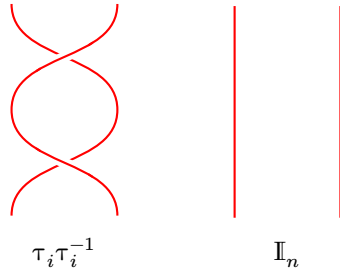$\tau_i \tau_i^{-1}$ $\mathbb{I}_n$

Figure 1.6: A type II move illustrating $\tau_i \tau_i^{-1} = \mathbb{I}_n$

We can perform a move equivalent to the type III move to see that $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$ (fig. 1.7).
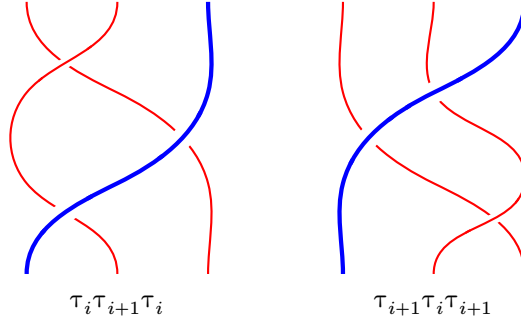

Figure 1.7: A type III move illustrating $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$.

We can slide two crossings vertically across each other if this does not change the ambient isotopy type. This is possible if the two crossings we wish to slide do not share a strand. This gives us the relation $\tau_i \tau_j = \tau_j \tau_i$ if $|i - j| \geq 2$ (fig. 1.8).
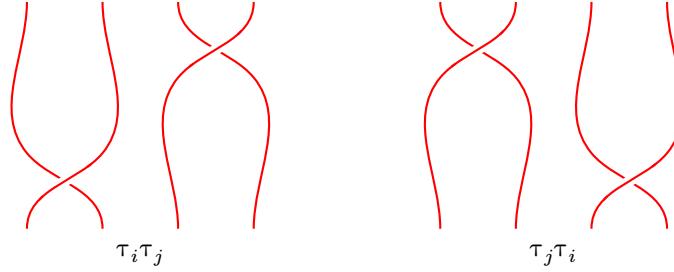

Figure 1.8: Sliding of crossings illustrating $\tau_i \tau_j = \tau_j \tau_i$.

Let w be a word of length m in GB_n; $w = \prod_{j=1}^{m} \tau_{\alpha_j}^{\pm 1}$ where $1 \leq \alpha_j \leq n - 1$. Every element of GB_n and B_n can be expressed as a product of its generators, albeit non-uniquely. We define a homomorphism

$$\Phi \colon \mathrm{GB}_n \to \mathrm{B}_n,$$
$$\Phi \colon \prod_{j=1}^{m} \tau_{\alpha_j}^{\pm 1} \mapsto \prod_{j=1}^{m} \sigma_{\alpha_j}^{\pm 1} \text{ for all } m \in \mathbb{N}.$$

We can see that Φ is a surjection as follows. Take an element $\prod_{j=1}^{m} \sigma_{\alpha_j}^{\pm 1} \in \mathrm{B}_n$, Φ maps $\prod_{j=1}^{m} \tau_{\alpha_j}^{\pm 1}$ to $\prod_{j=1}^{m} \sigma_{\alpha_j}^{\pm 1}$. Proving that Φ is an injection is harder and a proof can be found in [9, chp. 2].

Theorem 2. Φ is an isomorphism, i.e. B_n and GB_n are isomorphic.

This allows us to forget the distinction between B_n and GB_n.

1.3 Closure of braids

We define the closure of a geometric n-braid as follows. Consider a geometric n-braid in the standard position. For each $1 \leq i \leq n$, we construct the following sequence of line connected line segments. Join $(i, 1, 0)$, $(i, i, 0)$, $(i, i, 0)$, $(i, -i, 0)$, $(i, -i, 0)$, $(i, 0, 0)$ consecutively. We then join γ_i to the constructed line segments. Repeating this process for all i gives the closure of a braid (fig. 1.9). We denote the closure of a geometric n-braid b by $\bar{b}$. Closure of a braid is unique up to ambient isotopy. Two equivalent braid words have the same closures, thus making the closure well-defined.
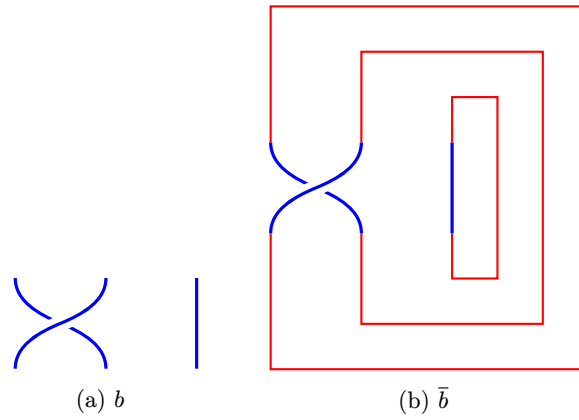

(a) b (b) $\bar{b}$

Figure 1.9: Closure of a braid with $b = \tau_1 \in \text{GB}_2$.

Proposition 3. Every closure of a geometric n-braid is a link.

Proof. The outer line segments are locally flat by virtue of being piecewise linear. Due to the second condition regarding the intersection of a plane in the standard representation of geometric n-braid, we can project the strand onto the y-axis and this projection would be a homeomorphism. We take an ϵ neighbourhood, $N(\epsilon)$ around the strand. The pair $(N(\epsilon), \gamma_i)$ would then be homeomorphic to $(\text{B}, \text{B} \cap x\text{-axis})$, where B is the three-dimensional unit ball around the origin. We have local flatness at the end points as well due to the union of two locally flat curves. $\square$

Remark 4. Suppose $\tau_i \in \text{GB}_n$ and $\tau_i' \in \text{GB}_m$ are generators where $n < m$. Then the closures of these two generators are not ambient isotopic. The closure of the latter contains one more non-linking loop.

1.3.1 Alexander Theorem

The theorems of James Alexander [10] and Andrei Markov Jr. [11] relate braids to knots.

Theorem 5 (Alexander)**.** Every link is ambient isotopic to the closure of a geometric braid, for some $n \in \mathbb{N}$.

Proof. Consider a piecewise linear, regular projection $\pi(L)$ of a link L on a plane. We choose a point O in the projection plane which is not collinear with any of the line segments. This can be done since a the link has only finitely many line segments. Let $P \in \pi(L)$. The vector OP can move either clockwise or anti-clockwise as P moves along the link projection. We wish to modify the line segments such that OP moves in only one sense, say anti-clockwise, as P moves along the entire length of the link projection. We now fix our attention on a line segment corresponding to a clockwise rotation. We divide the segment into sub-parts such that each part shares at-most one crossing point with other line segments. If A and B are end-points of such a line segment, then we may replace this line segment with two another line segments AC and CB, such that C is another point not on belonging to $\pi(L)$ and the triangle ABC encloses O. If AB originally passed under (or over) a line segment of $\pi(L)$, then the modified line segments AC and CB must pass under (or over) of the line segments of $\pi(L)$ as well. This move shall not change the link type as it shall be a combination of sliding, type 2 and type 3 moves. In the resulting triangle, we have two orientations possible, one path which travels via C and the other path which does not. The vector OP shall move in the opposite, anti-clockwise sense while traversing from A to B via C, instead of AB. We can repeat this process for all of the (finitely many) line segments which turn clockwise. In the end, we obtain a projection such that OP moves in only the anti-clockwise sense, as P moves along the entire length of the link projection. We can ambient isotope the projection such that all the crossings lie in the projection of a cube, more precisely the cube constructed while defining a geometric braid. The end-points can be made to match as well. The above procedure of triangular moves shall guarantee the monotonicity that is required. $\qquad\square$

Henceforth, unless specified otherwise, we shall always work with the projections of the standard representation of a geometric n-braid and its closure and refer to these projections simply as a braid and its closure.

1.3.2 Conjugation

If $b, g \in \mathrm{GB}_n$, we observe that $\overline{gbg^{-1}}$ is ambient isotopic to $\bar{b}$ (fig. 1.10). The upper strands in g and g^{-1} are connected via the closure strand. We can slide g and g^{-1} via the closure strands to the 'other side' of b to annihilate each other. This can be achieved by a type II move.
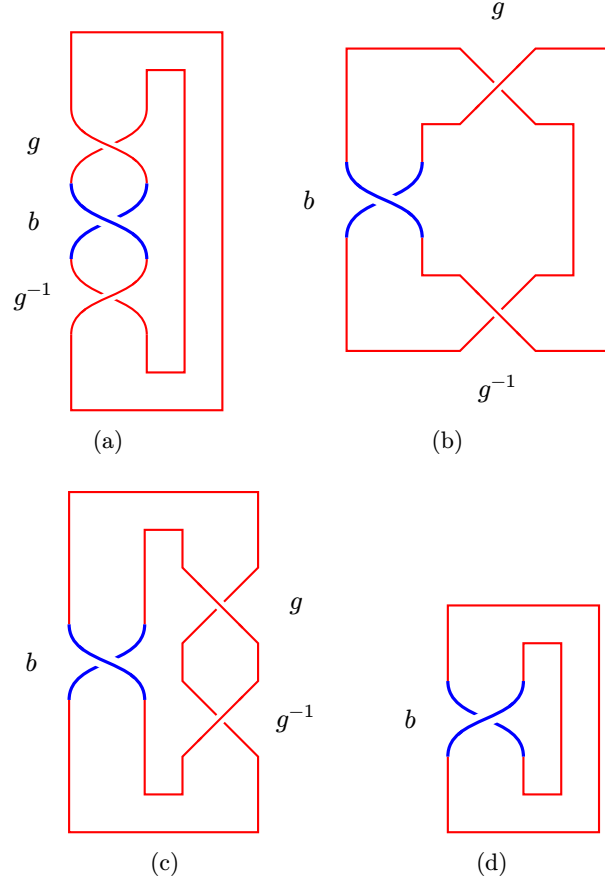
Figure 1.10: Conjugation process illustrating the link equivalence of $\overline{gbg^{-1}}$ and $\overline{b}$, with $b = \tau_1^{-1} \in \mathrm{GB}_1$ and $g = \tau_1^{-1} \in \mathrm{GB}_1$.

Remark 6. The closures of conjugate braids are ambient isotopic as links. The braids themselves are not.

1.3.3 Markov move

We observe that if $b \in \mathrm{GB}_n$, then $b\tau_n \in \mathrm{GB}_{n+1}$, $b\tau_n^{-1} \in \mathrm{GB}_{n+1}$ and b have ambient isotopic closures (fig. 1.11), although b, $b\tau_n$ and $b\tau_n^{-1}$ are not equivalent as braids. That is, we can add a strand and a crossing of that strand with another strand without changing the link type (of the closure). We can also remove a strand and a crossing if that strand does not cross any other strand. We visually see that adding the above mentioned strands anywhere between the existing strands is equivalent to adding the strands on the right.

(a) $\bar{b}$ (b) $\overline{b\tau_2}$
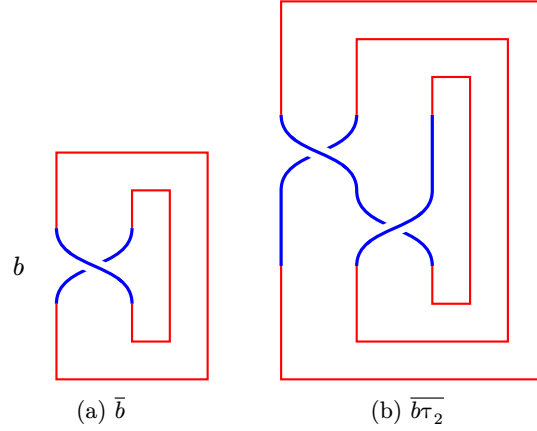
Figure 1.11: Markov move with $b = \tau_1^{-1}$.

1.3.4 Markov Theorem

Theorem 7 (Markov)**.** Two braids whose closures are ambient isotopic to each other are related by a finite sequence of the following operations.

1. Braid equivalences, i.e. equivalences resulting due to the braid relations.

2. Conjugation.

3. Markov moves.

A proof of the above theorem can be found in the book of Joan Birman [12, chp. 2]. Markov gave a sketch of the proof in 1936 [11].

1.3.5 Writhe

We see visually that the result of a Markov move on a braid is equivalent to performing a type I move on the braid closure. We know that a type I move increases or decreases the writhe of a link by a unit value. Since all the crossings in a braid closure occur only in the cube containing the braid strands, we can define the writhe of a braid equal to the writhe of the braid closure by assigning each braid crossing a value, either $+1$ or -1. Our assignment must be consistent with our earlier assignment for knots. But for this procedure, we need to assign an orientation to the braid. We assign all the strands (inside the braid cube) a downward orientation. Doing so, we see that τ_i inherits $+1$ value while τ_i^{-1} inherits -1. We could instead have assigned all the strands an upward orientation as well. This would not have changed the values of τ_i and τ_i^{-1} (fig. 1.12). What is not allowed is assigning arbitrary orientation to strands. If we assign the orientation arbitrarily, then the well-definedness of the orientation cannot be guaranteed. Two distinct strands in a braid could be connected via the closure

strands and one would need to check the whole connected link component of the braid closure for a well-defined closure.

Thus, the writhe w of the braid b with a word representation $\prod_{j=1}^{m} \tau_{\alpha_j}^{\beta_j}$ of length m, where $\beta_j \in \{+1, -1\}$ is

$$w(b) = \sum_{j=1}^{m} \beta_j.$$

Note that the writhe of a braid is dependent on its word representation. We also see that $w(b) = w(\bar{b})$.
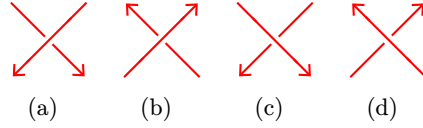

(a)   (b)   (c)   (d)

Figure 1.12: Assignment of crossing values. (a) Downward orientation for τ_i corresponding to $+1$. (b) Upward orientation for τ_i corresponding to $+1$. (c) Downward orientation for τ_i^{-1} corresponding to -1. (d) Upward orientation for τ_i^{-1} corresponding to -1.

1.4   Markov trace

We won't distinguish between B_n and GB_n from now on. We can now move finally towards the Jones/bracket polynomial with the information we have. If we have a function $J_n \colon \mathrm{B}_n \to R$, where R is a commutative ring, then using the Markov Theorem we can construct link invariants from the family of functions $\{J_n\}$ if the following conditions are satisfied.

1. J_n is well defined. $J_n(b) = J_n(b')$ if $b = b'$.

2. $J_n(b) = J_n(gbg^{-1})$ if $g, b \in \mathrm{B}_n$.

3. If $b \in \mathrm{B}_n$, then there exists a constant $\alpha \in R$, independent of n, such that

$$J_{n+1}(b\sigma_n) = \alpha^{+1} J_n(b)$$

   and

$$J_{n+1}(b\sigma_n^{-1}) = \alpha^{-1} J_n(b).$$

The last condition reminds us of the normalisation needed in order to make the bracket polynomial invariant under the type I move. Its purpose here is the same.

A family of functions $\{J_n\}$ satisfying the above given three conditions is called a Markov trace on $\{\mathrm{B}_n\}$. For any link L which is ambient isotopic to $\bar{b}$, where $b \in \mathrm{B}_n$, we define $J(L) \in R$ as follows.

$$J(L) := \alpha^{-w(b)} J_n(b).$$

We call $J(L)$ the link invariant for the Markov trace $\{J_n\}$.

Theorem 8. J is an invariant of ambient isotopy for oriented links.

Proof. Suppose $L \sim \bar{b}$ and $L'' \sim \bar{b'}$, where $\sim$ denotes the ambient isotopy relation. By the Markov Theorem, we can obtain $\overline{b'}$ via an application of a finite sequence of the moves mentioned in the Markov Theorem on $\overline{b'}$. Each such move leaves J invariant. J_n is already invariant under braid equivalences and conjugation by definition. The $\alpha^{-w(b)}$ factor cancels the effect of a type I move. $\square$

We can define the bracket polynomial for braids in the same way as we did for links. Define

$$\langle \cdot \rangle \colon \mathrm{B}_n \to \mathbb{Z}[A, A^{-1}]$$
$$\langle \cdot \rangle \colon b \mapsto \langle \bar{b} \rangle.$$

We simply evaluate the bracket on the closure of the braid. We observe that this function is a Markov trace with $\alpha = -A^3$. We know that

$$\left\langle \diagup\!\!\!\!\diagdown \right\rangle = A \left\langle \asymp \right\rangle + A^{-1} \left\langle \,)(\, \right\rangle.$$

In terms of braids, we have

$$\left\langle \Big| \cdots \Big| \diagup\!\!\!\!\diagdown \Big| \cdots \Big| \right\rangle = A \left\langle \Big| \cdots \Big| \asymp \Big| \cdots \Big| \right\rangle + A^{-1} \left\langle \Big| \cdots \Big| \,)(\, \Big| \cdots \Big| \right\rangle.$$

But $\Big| \cdots \Big| \,)(\, \Big| \cdots \Big|$ is the identity braid. Thus, if we denote $\Big| \cdots \Big| \asymp \Big| \cdots \Big|$ by U_i, we can write

$$\langle \sigma_i^{-1} \rangle = A \langle \mathrm{U}_i \rangle + A^{-1} \langle \mathbb{I}_n \rangle.$$

Similarly, one could write

$$\langle \sigma_i \rangle = A \langle \mathbb{I}_n \rangle + A^{-1} \langle \mathrm{U}_i \rangle.$$

Note that U_i does not belong to the braid group and is a new object. We refer to them as "input-output forms" or as "hooks". They are cup and cap combinations involving the i-th and $(i+1)$-th strands (fig. 1.13).
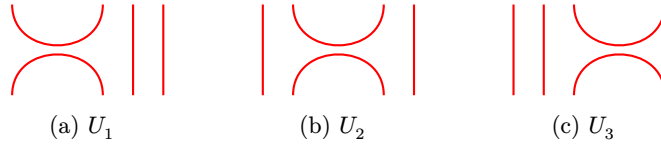

(a) U_1 (b) U_2 (c) U_3

Figure 1.13: Input-output forms or hooks for 4 strands

We can thus use consider σ_i to be equivalent to $A + A^{-1}\mathrm{U}_i$, σ_i^{-1} to be equivalent to $A^{-1} + A\mathrm{U}_i$, and create a formalism based on this. Given a braid word representation of a braid, we can simply substitute the above mentioned equivalences to get a product of U_i's, with A and A^{-1} as coefficients. Note that this product is dependent on the specific braid representation of a knot. It is

not invariant under a type I move, as is the case with the un-normalised bracket polynomial.

We see that each bracket polynomial evaluation state of the closure of a braid can be written in terms of the closure of a product of the input-output forms. Given a braid word b, we can consider it equivalent to $S(b)$, where $S(b)$ is the sum of products of the U_i's obtained by substituting σ_i by $A + A^{-1}U_i$ and σ_i^{-1} by $A^{-1} + AU_i$. Closure of each term (a product in U_i's) in the sum corresponds to a state obtained while evaluating the bracket polynomial as it gives a collection of loops. If P is such a product, then $\langle P \rangle = \langle \overline{P} \rangle = \delta^{\|P\|}$, where $\|P\|$ is the number of loops in $\overline{P}$ minus 1, and $\delta = -A^{-2} - A^2$. Thus,

$$S(b) = \sum_s \langle b|s \rangle P_s,$$

where s denotes a state and indexes all the terms in the product, and $\langle b|s \rangle$ is the product of A's and A^{-1}'s multiplying each P-product P_s. We have

$$\langle b \rangle = \langle S(b) \rangle = \sum_s \langle b|s \rangle \langle P_s \rangle = \sum_s \langle b|s \rangle \delta^{\|s\|}.$$

Example 9. Let $b = \sigma_1 \sigma_2^{-1}$. We can resolve $\overline{b}$ in many states. One of the states s and its corresponding product $U_1 U_2$ in terms of the input-output forms is shown in fig. 1.14.

Example 10. Consider the same braid $b = \sigma_1 \sigma_2^{-1}$. We have

$$P(b) = (A + A^{-1}U_1)(AU_2 + A^{-1})$$
$$P(b) = A^2 U_2 + \mathbb{I}_3 + U_1 U_2 + A^{-2}U_1$$
$$\langle b \rangle = \langle \sigma_1 \sigma_2^{-1} \rangle = \langle P(b) \rangle = A^2 \langle U_2 \rangle + \langle \mathbb{I}_3 \rangle + \langle U_1 U_2 \rangle + A^{-2} \langle U_1 \rangle.$$

Now, $\mathbb{I}_3$ corresponds to $\left|\left|\left|\right.\right.\right.$. Thus, $\langle I_3 \rangle = \delta^{3-1} = \delta^2$ as the closure of $\mathbb{I}_3$ shall give three loops. U_1 corresponds to $\asymp\left|\right.$. Thus, $\langle U_1 \rangle = \delta^{2-1} = \delta$ as the closure shall give two loops. Similarly, $\langle U_2 \rangle = \delta$ and $\langle U_1 U_2 \rangle = \delta^{1-1} = 1$. For a visual representation of the state $U_1 U_2$, see fig. 1.14, where the closure gives one single loop. In the end, we get

$$\langle \overline{b} \rangle = A^2 \delta + \delta^2 + 1 + A^{-2}\delta.$$

1.5 Temperley–Lieb algebra

We can give the U_i's a structure of their own by constructing the free additive algebra TL_n with the generators $U_1, U_2, \ldots, U_{n-1}$ and the multiplicative relations coming from the interpretation of U_i's as input-output forms. We can consider this algebra over the ring $\mathbb{Z}[A, A^{-1}]$ with $\delta = -A^{-2} - A^2 \in \mathbb{Z}[A, A^{-1}]$. We shall call TL_n the Temperley–Lieb algebra. The multiplicative relations in TL_n are as follows.

(a) b

(b) $\bar{b}$

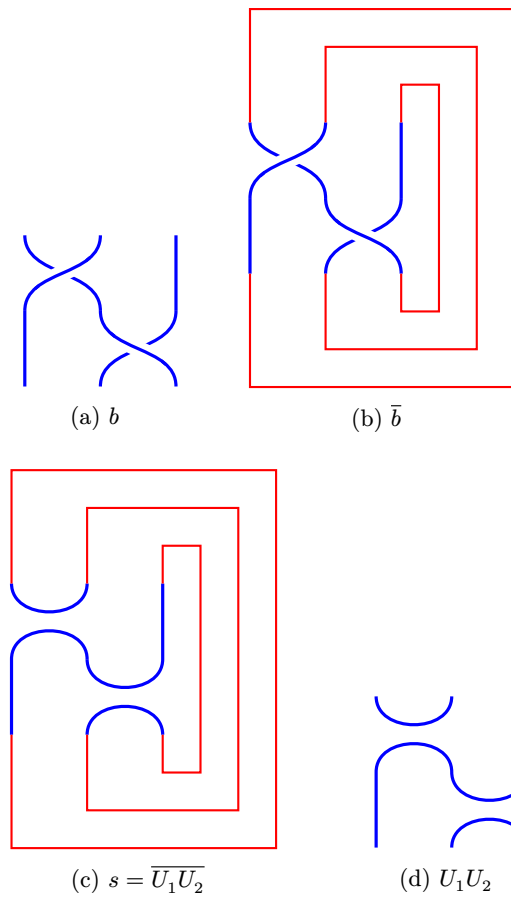(c) $s = \overline{U_1 U_2}$

(d) $U_1 U_2$

Figure 1.14: Writing a state of a braid closure in terms of input-output forms.

1. $U_i U_{i\pm 1} U_i = U_i$.

2. $U_i^2 = \delta U_i$.

3. $U_i U_j = U_j U_i$ if $|i - j| \geq 2$.

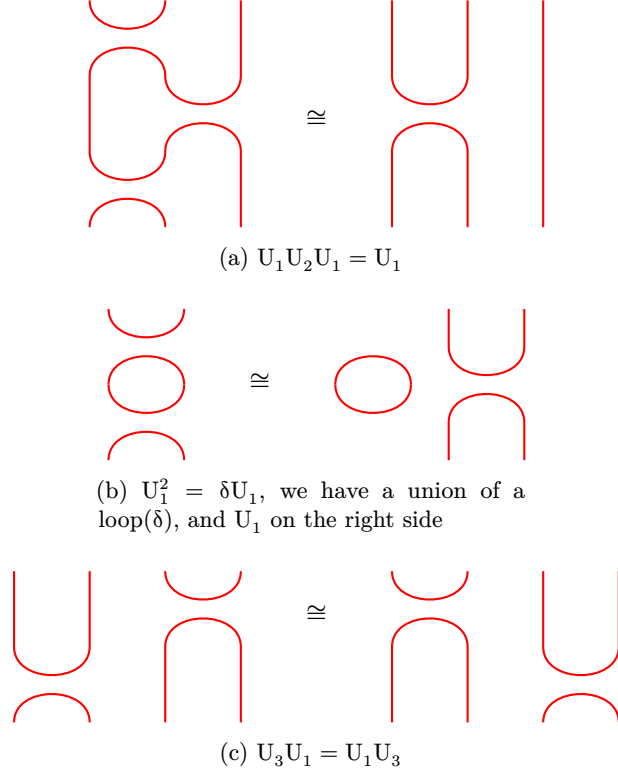These relations are a result of the geometric relations as illustrated in fig. 1.15.

(a) $U_1 U_2 U_1 = U_1$

(b) $U_1^2 = \delta U_1$, we have a union of a loop(δ), and U_1 on the right side

(c) $U_3 U_1 = U_1 U_3$

Figure 1.15: Input-output form relations

Now that we have the Temperley–Lieb algebra, we can define a mapping

$$\rho\colon B_n \to TL_n$$

by the following formulae.

$$\rho(\sigma_i) = A + A^{-1} U_i,$$
$$\rho(\sigma_i^{-1}) = A^{-1} + A U_i.$$

We now define the diagrammatic trace $\mathrm{tr}\colon TL_n \to \mathbb{Z}[A, A^{-1}]$ by extending linearly $\mathrm{tr}(P) = \langle P \rangle$, where P is a product term in $S(b)$. This version of trace is diagrammatic in nature as we are counting loops in a state. We thus arrive at

the formula $\langle b \rangle = \text{tr}(\rho(b))$. With what we have learnt so far, one can now find a braid representation b of a link L by Alexander's theorem, calculate $\text{tr}(\rho(b))$ and normalise it to get the link invariant normalised bracket polynomial. One can see that ρ is indeed a representation by verifying that $\rho(\sigma_i)\rho(\sigma_i^{-1}) = 1$, $\rho(\sigma_i\sigma_{i+1}\sigma_i) = \rho(\sigma_{i+1}\sigma_i\sigma_{i+1})$ and $\rho(\sigma_i\sigma_j) = \rho(\sigma_j\sigma_i)$.

Bibliography

[1] F. R. Vaughan Jones. "A polynomial invariant for knots via von Neumann algebras". In: *Bulletin of the American Mathematical Society* 12.1 (1985), pp. 103–111. DOI: 10.1090/s0273-0979-1985-15304-2.

[2] Peter R. Cromwell. *Knots and links*. Cambridge, UK: Cambridge University Press, 2004. ISBN: 0-521-83947-5.

[3] Louis H. Kauffman. *Knots and physics*. 4th ed. Knots and Everything 53. Singapore: World Scientific, 2013. ISBN: 978-981-4383-00-4.

[4] H. Neville V. Temperley and Eliott H. Lieb. "Relations between the 'percolation' and 'colouring' problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the 'percolation' problem". In: *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences* 322.1549 (Apr. 1971), pp. 251–280. DOI: 10.1098/rspa.1971.0067.

[5] Wikipedia contributors. *Partition algebra*. Wikipedia, The Free Encyclopedia. URL: https://en.wikipedia.org/wiki/Partition_algebra.

[6] Emil Artin. "Theorie der zöpfe". In: *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg* 4.1 (Oct. 1925), pp. 47–72. ISSN: 1865-8784. DOI: 10.1007/bf02950718.

[7] Emil Artin. "Theory of braids". In: *Annals of Mathematics* 1 (1947), pp. 101–126. ISSN: 0003486X. DOI: 10.2307/1969218.

[8] Michael Friedman. "Mathematical formalization and diagrammatic reasoning: the case study of the braid group between 1925 and 1950". In: *British Journal for the History of Mathematics* 34.1 (2019), pp. 43–59. DOI: 10.1080/17498430.2018.1533298.

[9] Kunio Murasugi and Bohdan I. Kurpita. *A study of braids*. Springer Science & Business Media, 1999. ISBN: 978-0-7923-5767-4.

[10] James W. Alexander. "A lemma on systems of knotted curves". In: *Proceedings of the National Academy of Sciences* 9.3 (Mar. 1923), pp. 93–95. DOI: 10.1073/pnas.9.3.93.

[11] Andrei A. Markov Jr. "Über die freie äquivalenz der geschlossenen zöpfe". German. In: *Recueil Mathématique. Nouvelle Série* 1 (1936), pp. 73–78. URL: https://www.mathnet.ru/php/archive.phtml?wshow=paper&jrnid=sm&paperid=5479&option_lang=eng.

[12] Joan S. Birman. *Braids, links, and mapping class groups.* Annals of Mathematics Studies 82. Princeton University Press, 1974. ISBN: 978-14-0088142-0.