Thesis draft

Apoorv Potnis

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Chapter 1

Introduction to knot theory

1.1 Definition of a knot

We normally conceive of knots as open strings with 'knotted' parts in between. Given a knotted string with two 'open' ends, we can simply pull one end of the string to untie it, in the usual way we untie a knot. This way, any knot with two open ends can be untied. But if have a knot in a closed loop, we won't be able to untie it unless we cut it. We want our notion of a knot to be invariant under 'pulling'. Thus, we model our mathematical definition on closed loops instead of open. Insert a figure illustrating this.

Definition 1 (Knot)**.** A knot K is an embedding of $\mathbb{S}^1$ in $\mathbb{R}^3$.

Although one can talk about embeddings of higher dimensional 'circles' in higher dimensional spaces, we restrict ourselves to knots in the three dimensional space. The above definition turns out out to be too general for our purposes. It takes into consideration certain pathological knots known as *wild knots*. Although wild knots are an object of study, we won't be dealing with them in this thesis due to their pathological nature. fig shows an example of a wild knot. A section of the knot is scaled down by a constant factor joined on the one side of the previous section. If one does repeats this process infinitely, the section eventually converges to a point, provided that the scaling down is fast enough. We can join the convergence (limit) point to a an end-section on the other side to get wild knot. This knot is continuous everywhere, including the limit point, and an embedding of $\mathbb{S}^1$ in $\mathbb{R}^3$. Three common ways exist to exclude such behaviour by demanding extra conditions.

1. Differentiability.

   We can demand all knots to be differentiable at all points. We can verify visually that all the points, except the limit point of the wild knot are differentiable, or can be made (isotoped) differentiable. The derivative must necessarily change within a section. As the size of the section decreases, the derivative changes more rapidly. This rate of change of the derivative, the double derivative, is a monotonically increasing function as the section number increases. At the limit point, the derivative shall cease to exist by the virtue of ‘changing too rapidly’. Demanding differentiability forcibly removes the offending limit point. The ‘wildness’ is due to the limit point.

But this condition comes with a problem as well, namely we cannot use polygons for describing knots.

2. Piecewise linearity. We can demand all the edges of a knot to be piecewise linear. Our knot shall be a polynomial in that case. A polygon has finitely many edges. Infinitely (countably) many sections in a wild knot shall mean infinitely many edges (of decreasing length), which is not allowed. Thus, this condition excludes wild knots.

3. Local flatness.

In this thesis, we shall take the third route following Cromwell [1, chp. 1]. If we consider a local neighbourhood around each point of the knot, except the limit point, then we see visually see that we can always find a small enough local neighbourhood around each such point such that the strand is not ‘knotted’ in that neighbourhood. At the limit point, no matter how small a neighbourhood we take, the strand shall always be ‘knotted’. We enforce this ‘local unknottedness’ condition by demanding local flatness. Let p be a point in a knot K, $\mathrm{B}(\mathrm{O}, 1)$ be the unit ball centered at origin O and d be a diameter of $\mathrm{B}(\mathrm{O}, 1)$.

Definition 2 (Local flatness)**.** The point p is said to be locally flat if there exists a neighbourhood $U \ni p$ such that the pair $(U, U \cap K)$ is homeomorphic to $(\mathrm{B}(\mathrm{O}, 1), d)$.

A knot is said to be locally flat if each point in that knot is locally flat. A point that is not locally flat is called wild, and a knot is wild if any of its points are wild.

Insert figures of a unit ball centered at origin and a diameter, and a section of the knot homeomorphic to the pair.

Consider a spherical neighbourhood around a locally flat point. There exists a radius such that for all neighbourhoods less than this radius, the boundary of the neighbourhood intersects the strand in exactly two points. This is not possible at the limit point in wild knot figure.

Definition 3 (Tame knots)**.** A knot is said to be tame if all its points are locally flat.

Unless mentioned otherwise, we shall always consider our knots to be tame from now on.

1.2 Distinguishing knots

Any two homeomorphisms of the circle are homeomorphic to the circle and to each other, since homeomorphism is an equivalence relation. But this means that all knots are homeomorphic to each other. Clearly, homeomorphism is not the correct notion to distinguish knots. When we mean that two knots are distinct, we mean that if we create a physical model of those knots, we cannot 'physically deform' one knot into another. *Cutting a knot is not allowed.* One might think that homotopy or isotopy are what we need, but it turns out that the notion of *ambient isotopy* is the correct one.

Definition 4 (Homotopy)**.** A homotopy of a space $X \subset \mathbb{R}^3$ is a continuous map $h\colon X \times [0,1] \to \mathbb{R}^3$.

The restriction of h to level $t \in [0,1]$ is $h_t\colon X \times \{t\} \to \mathbb{R}^3$. h_0 must be the identity map.

Note that the continuity of h implies the continuity of h_t for all $t \in [0,1]$. The converse is not true. Insert example here. Homotopy allows a curve to pass through itself. All knots are thus homotopic to the trivial knot. If we do not allow a curve to pass through itself, i.e. if we demand injectivity for each h_t, then we get what is known as an isotopy. But isotopy is not useful for distinguishing knots as well, due to bachelors unknotting. All (tame? think) knots turn out to be isotopic to the trivial knot.

Proposition 5 (Bachelors' unknotting)**.** Every tame knot is isotopic to the unknot.

Proof. The proof is not entirely correct. Correct it. Use the argument that a function linear in one argument and individually continuous in both is continuous in the product topology. Prove this result as well. Let $K \subset \mathbb{R}^3$ be a tame knot.

Let $p \in K$. Since the knot is locally flat at each point by the definition of tameness, we take a ball $U_p \subset \mathbb{R}^3$ of radius ε around the point p such that the pair $U_p, U_p \cap K$ is homeomorphic to (B, d), where B is the unit ball in $\mathbb{R}^3$ centered at the origin and d is the diameter of B along the x-axis. We choose a parametrization $f\colon [0, 2\pi) \to K$ of the knot such that $f([a,b]) = K \setminus (U_p \cap K)$, where $[a,b] \subset [0, 2\pi)$.

Let $r \in \mathbb{R}^3$ be a point outside U_p. Now consider the function $i_t\colon K \to \mathbb{R}^3$ defined for each $t \in [0,1]$ as follows.

1. If $f(x) \in U_p$, then $i_t(f(x)) = f(x)$.
2. If $x \in \left[a, a + t\left[\dfrac{b-a}{2} - a\right]\right)$, then
   $i_t(f(x)) = f(a) + t\left[tr + (1-t)f\left(a + t\left(\dfrac{b-a}{2} - a\right)\right) - f(a)\right]$.
3. If $x \in \left[a + t\left[\dfrac{b-a}{2} - a\right], b - t\left[b - \dfrac{b-a}{2}\right]\right]$, then
   $i_t(f(x)) = tr + (1-t)f(x)$.

4. If $x \in \left(b - t\left[b - \frac{b-a}{2}\right], b\right]$, then

$$\begin{aligned} i_t(f(x)) = tr + (1-t)f\left(b - t\left(b - \frac{b-a}{2}\right)\right) \\ + t\left[f(b) - tr - (1-t)f\left(b - t\left(b - \frac{b-a}{2}\right)\right)\right]. \end{aligned}$$

Let $i\colon [0,1] \times K \to \mathbb{R}^3$ be a function defined by $i(t, f(x)) := i_t(f(x))$.

i is defined such that the part inside U_p is kept the same for all t. For $t = 0$, i does not deform the knot at all. For $t \in (0,1)$, the knotted part (in $\mathbb{R}^3$) shrinks and the interval in the domain $[a, b]$ which maps to the knotted part also shrinks. This shrinkage of the domain happens linearly. All points of the knotted part trace a straight line from their original position to r. Eventually, the knotted part ceases to exist at $t = 1$ and a single point of the domain $(a+b)/2$ maps to r.

In the end, we get a figure consisting of two straight lines meeting at r, and $U_p \cap K$, the original part of the knot inside U_p. The other endpoints of these lines are $f(a)$ and $f(b)$. $U_p \cap K$ is isotopic to the line joining $f(a)$ and $f(b)$. Thus, we get a triangle with points r, $f(a)$ and $f(b)$. This triangle is isotopic to $\mathbb{S}^1$.

We now prove that i is continuous. We know that both i_t and i_x are continuous and injective for all $t \in [0,1]$ and $x \in [0, 2\pi)$ respectively, where i_x is defined to be the restriction of i for a particular x.

We want to show that the pre-image of the intersection of any open set in $\mathbb{R}^3$ with the image of i is open. Consider an open set $V \in \mathbb{R}^3$ and let t_0 be such that $A := i_{t_0}(K) \cap V$ is a single component arc inside V. Let $q \in A$. The pre-image of an open neighbourhood $B \ni q$ in A is open by continuity of the isotopy in one component. Let the pre-image of the pre-image of B under f be $(x_1, x_2) \subset [0, 2\pi)$. Thus, $i_{t_0}(f((x_1, x_2))) = B$. Since all points of B are interior points, we take an ε tubular neighbourhood around B such that the whole neighbourhood lies inside A. Let this tube be C. Now consider the set of all t such that $i_t(i_{t_0}^{-1}(s))$ is inside C for all $s \in B$. For each s, we shall an open set $(t_{1,s}, t_{2,s})$. The union of all such open intervals for all $s \in B$ is an open set in $[0,1] \times [0, 2\pi)$. Thus, for each open set in $V' \in \mathbb{R}^3$, and for each point $q' \in V$, we can create an open subset of V' such that the pre-image of this open subset is open. The union of all the pre-images for all points inside V' is an open set.

The arc-length of a tame knot is finite. So, the intersection of $i_t(K)$ with an open set of $\mathbb{R}^3$ for each t shall consist of countable arcs. We now show why the number of arcs must be countable. Consider there are uncountably many arcs for a particular t. Each arc is an intersection of a closed loop in with an open set in $\mathbb{R}^3$ and so it would have a finite length. Uncountably many arcs of finite length cannot add up to give a loop of finite length. Thus, there must be countable many arcs. We shall apply the method described in the previous paragraph to all the arcs. □

In the above considered isotopy, we deformed the set $X = K$. Instead, we take X to be the entire space $\mathbb{R}^3$, or a bounded set which completely covers the knot, then we get the notion of ambient isotopy. This modification ensures that the surrounding space is deformed as well as we deform the knot. The knot is a curve which has no volume. If we try bachelors' unknotting on the surrounding space as well, we observe that the surrounding space, which has a finite, non-zero volume cannot shrink to a set of zero volume under isotopy. This finally leads us to the equivalence relation induced by ambient isotopy.

Definition 6 (Knot equivalence)**.** Two knots K_1 and K_2 are said to be ambient isotopic if there exists an isotopy $I\colon \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ such that $I(K_1, 0) = I_0(K_1) = K_1$ and $I(K_1, 1) = I_1(K_1) = K_2$.

Proposition 7. Knot equivalence is indeed an equivalence relation.

Proof. Reflexivity. Symmetry. Transitivity. □

Each equivalence class of knots is called a *knot type*. We would often forget the distinction between a knot and its knot type. The intended meaning can be inferred from the context. Note that we distinguish between *ambient isotopy* and *isotopy*. Many treatments of knot theory use the word isotopy for ambient isotopy as ambient isotopy is the useful construct in knot theory. Ambient isotopy is an isotopy of the whole space containing the knot, not just the knot.

Give examples of knots which are ambient isotopic and not ambient isotopic, with images.

Remark 8. In this thesis, we shall look at knot theory in $\mathbb{R}^3$. One can compactify $\mathbb{R}^3$ to $\mathbb{S}^3$ and do knot theory in $\mathbb{S}^3$, as many treatments do. This does not result in a different knot theory.

Chapter 2

Braids and the Jones polynomial

2.1 Motivation

$$
\begin{tikzcd}
 & & \text{Jones algebra} \arrow[r, "\text{Markov trace}"] & \text{Jones polynomial} \arrow[dd, leftrightarrow, "\text{Substitution}"'] \\
\text{Oriented link} \arrow[r] & \text{Artin braid group} \arrow[ur] \arrow[dr] & & \\
 & & \text{Temperley–Lieb algebra} \arrow[r, "\text{Diagrammatic}", "\text{trace}"'] & \text{Normalised bracket polynomial}
\end{tikzcd}
$$

As remarked earlier, Jones arrived at his polynomial indirectly while working on the theory of operator algebras [2]. In his course of investigations, he constructed a tower of algebras nested in one another with the property that each of these algebras is generated by a set of generators satisfying a particular set of relations. A degree of similarity between these relations and the relations among the generators of the *Artin braid group* was pointed out to Jones by a student during a seminar [1, p. 216], which led to the investigations of Jones into knot theory. Jones had defined a notion of a *trace* on his algebras; more specifically a trace function obeying the *Markov property.* As we shall see, one can express every link in terms of a (non-unique) *braid.* Jones then defined a representation of such a braid into his algebras. The trace of an algebra representation of a braid, which is in turn obtained from the link, can be calculated. The Jones polynomial was realized as such a trace.

In this chapter, we shall not travel the original route of Jones to reach his polynomial as it requires the knowledge of the theory of von Neumann algebras. Instead, we shall follow the approach described by Kauffman in his book to construct a representation of the Artin braid group into the *Temperley–Lieb algebra* [3, chp. 8]. These algebras admit a diagrammatic intrepretation and our definition of a trace on these algebras shall be diagrammatic in nature as well. Via this trace, we eventually reach the bracket polynomial, which we already know to be equivalent to the Jones polynomial as demonstrated earlier. The Jones algebra can be recovered from the Temperley–Lieb algebra by a choice of substitutions. The Temperley–Lieb algebra arose during the study of certain statistical models in physics [4]. This algebra can be viewed as a sub-algebra of a broader framework of the *partition algebra* [5].

2.2 Geometric representation of braids

2.2.1 Definition

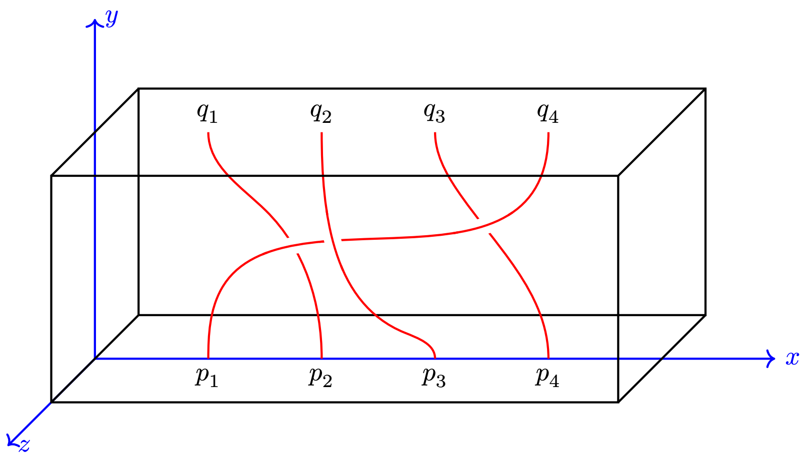


Figure 2.1: Three dimensional geometric representation of braids.

We shall now understand some basics of braid theory. Emil Artin introduced the Artin braid group explicitly [6, 7, 8].

An n-braid is an element of the Artin braid group B_n, defined via the following presentation on the generators σ_i, for $1 \leq i \leq n-1$.

$$\mathrm{B}_n := \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{rcll} \sigma_i\sigma_i^{-1} & = & \mathbb{I}_n^{\mathrm{a}} & \\ \sigma_i\sigma_{i+1}\sigma_i & = & \sigma_{i+1}\sigma_i\sigma_{i+1} & \\ \sigma_i\sigma_j & = & \sigma_j\sigma_i & \text{if } |i-j| \geq 2 \end{array} \right\rangle,$$

where $\mathbb{I}_n^{\mathrm{a}}$ is the identity of B_n. Thus, B_n is the quotient of the free group of

$n-1$ generators with the smallest normal subgroup of the free group containing the elements $\sigma_i\sigma_i^{-1}$, $\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1}$, and if $|i-j| \geq 2$, then $\sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1}$. We want to recover this algebraic definition using the intuitive understanding of braids that we have. For, that we shall now see a geometric construction in the three dimensional Euclidean space to represent the Artin braid group. This shall make clear the geometric intrepretation of the relations as well.

Consider two *ordered* sets of points $L_1 := \{p_1 := (1,0,0), \dots, p_n := (n,0,0)\}$ and $L_2 := \{q_1 := (1,1,0), \dots, q_n := (n,1,0)\}$ as shown in fig. 2.1 for $n = 4$. Elements of L_1 are called bottom points and elements of L_2 are called the top points. For $1 \leq i \leq n$, consider a family of non-intersecting continuous curves $\gamma_i\colon [0,1] \to \mathbb{R}^3$ such that

1. $\gamma_i(0) = p_i$ and $\gamma_i(1) = q_j$ for $1 \leq i, j \leq n$.
2. Any plane perpendicular to the xy-plane and parallel to the x-axis intersects each of the curves either exactly once or not at all.
3. All the curves lie in the cube determined by the vertices $(0,0,1)$, $(0,0,-1)$, $(0,1,1)$, $(0,1,-1)$, $(n+1,0,1)$, $(n+1,0,-1)$, $(n+1,1,1)$, $(n+1,1,-1)$.

Such a labelled curve is called a strand in standard position and a family of such labelled n stands is called an n-strand set in a standard position. We can ambient isotope or rigidly move an n-strand set to get another n-strand set, possibly not in a standard position. Two n-strand sets are said to be equivalent if they are related by a sequence of rigid motions of the strand sets, and ambient isotopies of the strand sets such that the space outside the cube, along with the endpoints, remains fixed. We shall refer to an equivalence class of such n-strands as a geometric n-braid. Thus, a geometric n-braid is well-defined.

Remark 9. Even though we have restricted our strands to the a bounded cube in the standard position, we can in principle change the bounds of our cube in x and z directions to any value and get the same theory. We shall not pursue this approach here.

2.2.2 Standard projection

We call the projection of a standard position n-strand set onto the xy-plane to be a two dimensional representation of a braid. Such a projection is drawn in fig. 2.2. It should be noted that a standard position n-strand set is unique only up to ambient isotopy, thus correspondingly the two dimensional representation of such a set is also unique only up to ambient isotopy, namely the ambient isotopies of the projection of the cube and the ambient isotopies such that the projection is a two dimensional representation of a braid for all times.

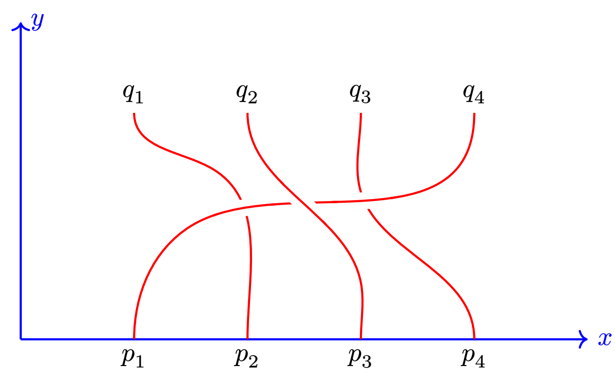


Figure 2.2: Two dimensional geometric representation of braids.

Now onwards, we shall always visually represent geometric n-braids using their standard two dimensional projections.

2.2.3 Group structure

Multiplication of any two n-geometric braids b_1 and b_2, denoted by $b_1 b_2$ is defined as follows (fig. 2.3). Ambient isotope and then rigidly move b_1 and b_2 *separately* in the standard position. Now translate only b_1 in the $+y$ direction by unit distance. The bottom points of b_1 and the top points of b_2 now coincide. Concatenate their strands and shrink the concatenated strands in the y direction by half keeping fixed the bottom points of b_2. The result is another geometric n-braid $b_1 b_2$ in the standard position. Multiplication defined this way is associative.

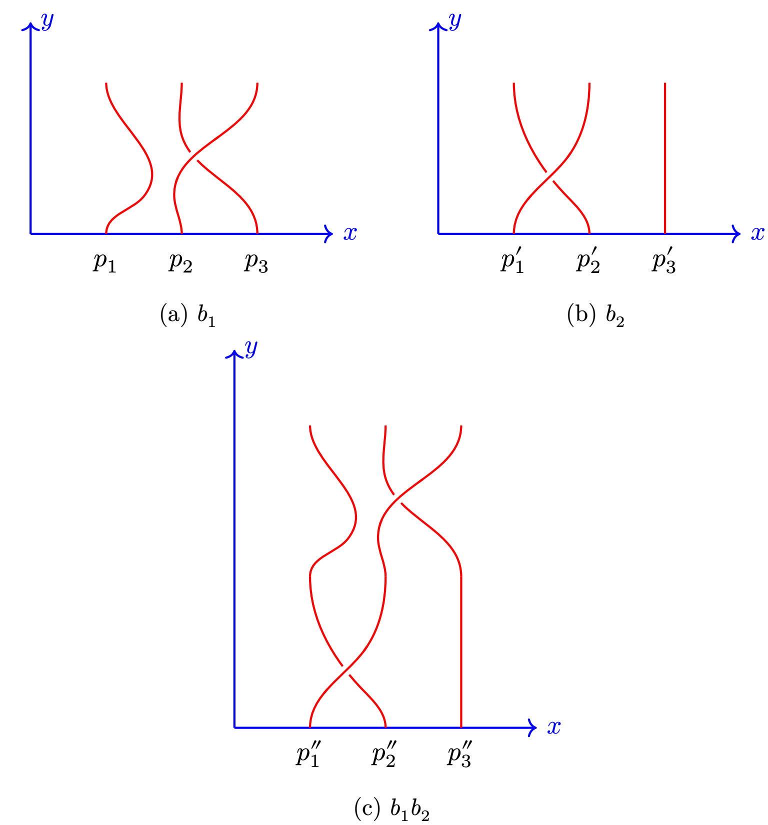


Figure 2.3: Multiplication of two braids (before shrinking).

We shall now drop the axes as well while representing two dimensional geometric n-braids.

An n-strand set such that each γ_i is a straight line segment connecting the i^{th} bottom point to the i^{th} top point is called the identity geometric n-braid and is denoted by $\mathbb{I}_n$ (fig. 2.4).

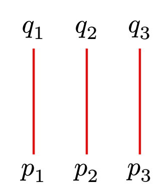


Figure 2.4: The identity $\mathbb{I}_3$

A geometric n-braid a such that $ab = ba = \mathbb{I}_n$ for some geometric n-braid b is called the inverse of b and denoted is by b^{-1}. We shall see that each element has an inverse.

With these operations, the set of geometric n-braids becomes a group, which we shall denote by GB_n.

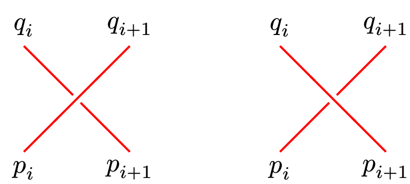
2.2.4 Generators

By the virtue of ambient isotopy, we can move the crossings in a two dimensional representation of a geometric n-braid such that each crossing lies in a region bounded by two lines parallel to the x-axis. Moreover, we can arrange the crossings such that each such region contains only one crossing. Thus, if we give the information regarding the type of each crossing for each such region, we can faithfully reconstruct the two dimensional representation. To this end, we define the generators of a geometric n-braid.

Denote by τ_i the geometric n-braid such that

1. $\gamma_i(1) = q_{i+1}$, $\gamma_{i+1}(1) = q_i$, and $\gamma_j(1) = q_j$ when j does not equal i or $i+1$.
2. $\pi_{xy}(\gamma_i(t)) \geq 0$ and $\pi_{xy}(\gamma_{i+1}(t)) \leq 0$ for all $t \in [0,1]$.

π_{xy} is the projection maps onto to xy-plane. $\tau_1, \dots, \tau_{n-1}$ are the generators of GB_n (fig. 2.5).



(a) τ_i (b) τ_i^{-1}

Figure 2.5: Generators τ_i and τ_i^{-1}. We have omitted the other straight strands.

For example, in fig. 2.3 we have $b_1 = \tau_2 \in \mathrm{GB}_3$, $b_2 = \tau_1 \in \mathrm{GB}_3$ and $b_1 b_2 = \tau_2\tau_1 \in \mathrm{GB}_3$.

If we multiply τ_i and τ_i^{-1} to form $\tau_i\tau_i^{-1}$, we observe that $\tau_i\tau_i^{-1} = \mathbb{I}_n$, where $\tau_i, \tau_i^{-1} \in \mathrm{B}_n$ for all $n \geq 2$ (fig. 2.6).

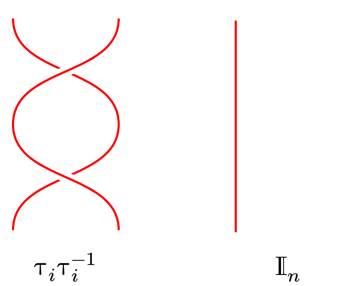
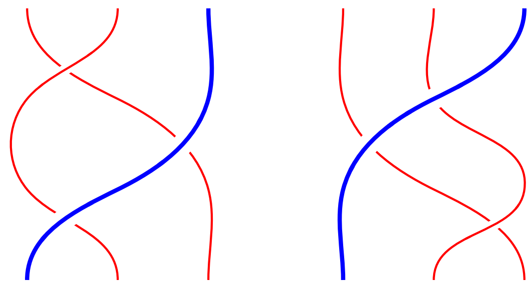


Figure 2.6: A type II move illustrating $\tau_i\tau_i^{-1} = \mathbb{I}_n$

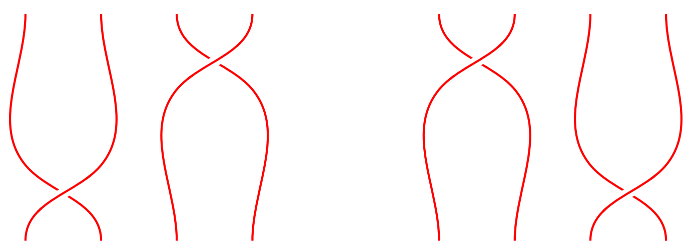
We can perform a move equivalent to the type III move to see that $\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$ (fig. 2.7).



$\tau_i\tau_{i+1}\tau_i$ $\tau_{i+1}\tau_i\tau_{i+1}$

Figure 2.7: A type III move illustrating $\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$.

We can slide two crossings vertically across each other if this does not change the ambient isotopy type. This is possible if the two crossings we wish to slide do not share a strand. This gives us the relation $\tau_i\tau_j = \tau_j\tau_i$ if $|i-j| \geq 2$ (fig. 2.8).



$\tau_i\tau_j$ $\tau_j\tau_i$

Figure 2.8: Sliding of crossings illustrating $\tau_i\tau_j = \tau_j\tau_i$.

Let w be a word of length m in GB_n; $w = \prod_{j=1}^{m} \tau_{\alpha_j}^{\pm 1}$ where $1 \leq \alpha_j \leq n-1$. Every element of GB_n and B_n can be expressed as a product of its generators, albeit non-uniquely. We define a homomorphism

$$\Phi\colon \mathrm{GB}_n \to \mathrm{B}_n,$$
$$\Phi\colon \prod_{j=1}^{m} \tau_{\alpha_j}^{\pm 1} \mapsto \prod_{j=1}^{m} \sigma_{\alpha_j}^{\pm 1} \text{ for all } m \in \mathbb{N}.$$

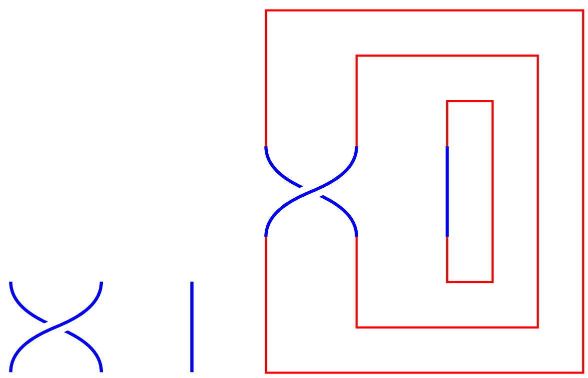
We can see that Φ is a surjection as follows. Take an element $\prod_{j=1}^{m} \sigma_{\alpha_j}^{\pm 1} \in \mathrm{B}_n$, Φ maps $\prod_{j=1}^{m} \tau_{\alpha_j}^{\pm 1}$ to $\prod_{j=1}^{m} \sigma_{\alpha_j}^{\pm 1}$. Proving that Φ is an injection is harder and a proof can be found in [9, chp. 2].

Theorem 10. Φ is an isomorphism, i.e. B_n and GB_n are isomorphic.

This allows us to forget the distinction between B_n and GB_n.

2.3 Closure of braids

We define the closure of a geometric n-braid as follows. Consider a geometric n-braid in the standard position. For each $1 \leq i \leq n$, we construct the following sequence of line connected line segments. Join $(i, 1, 0)$, $(i, i, 0)$, $(i, i, 0)$, $(i, -i, 0)$, $(i, -i, 0)$, $(i, 0, 0)$ consecutively. We then join γ_i to the constructed line segments. Repeating this process for all i gives the closure of a braid (fig. 2.9). We denote the closure of a geometric n-braid b by $\bar{b}$. Closure of a braid is unique up to ambient isotopy. Two equivalent braid words have the same closures, thus making the closure well-defined.



(a) b (b) $\bar{b}$

Figure 2.9: Closure of a braid with $b = \tau_1 \in \mathrm{GB}_2$.

Proposition 11. Every closure of a geometric n-braid is a link.

Proof. The outer line segments are locally flat by virtue of being piecewise linear. Due to the second condition regarding the intersection of a plane in the standard representation of geometric n-braid, we can project the strand onto the y-axis and this projection would be a homeomorphism. We take an ϵ neighbourhood, $N(\epsilon)$ around the strand. The pair $(N(\epsilon), \gamma_i)$ would then be homeomorphic to $(\mathrm{B}, \mathrm{B} \cap x\text{-axis})$, where B is the three-dimensional unit ball around the origin. We have local flatness at the end points as well due to the union of two locally flat curves. □

Remark 12. Suppose $\tau_i \in \mathrm{GB}_n$ and $\tau'_i \in \mathrm{GB}_m$ are generators where $n < m$. Then the closures of these two generators are not ambient isotopic. The closure of the latter contains one more non-linking loop.

2.3.1 Alexander Theorem

The theorems of James Alexander [10] and Andrei Markov Jr. [11] relate braids to knots.

Theorem 13 (Alexander)**.** Every link is ambient isotopic to the closure of a geometric braid, for some $n \in \mathbb{N}$.

Proof. Consider a piecewise linear, regular projection $\pi(L)$ of a link L on a plane. We choose a point O in the projection plane which is not collinear with any of the line segments. This can be done since a the link has only finitely many line segments. Let $P \in \pi(L)$. The vector OP can move either clockwise or anti-clockwise as P moves along the link projection. We wish to modify the line segments such that OP moves in only one sense, say anti-clockwise, as P moves along the entire length of the link projection. We now fix our attention on a line segment corresponding to a clockwise rotation. We divide the segment into sub-parts such that each part shares at-most one crossing point with other line segments. If A and B are end-points of such a line segment, then we may replace this line segment with two another line segments AC and CB, such that C is another point not on belonging to $\pi(L)$ and the triangle ABC encloses O. If AB originally passed under (or over) a line segment of $\pi(L)$, then the modified line segments AC and CB must pass under (or over) of the line segments of $\pi(L)$ as well. This move shall not change the link type as it shall be a combination of sliding, type 2 and type 3 moves. In the resulting triangle, we have two orientations possible, one path which travels via C and the other path which does not. The vector OP shall move in the opposite, anti-clockwise sense while traversing from A to B via C, instead of AB. We can repeat this process for all of the (finitely many) line segments which turn clockwise. In the end, we obtain a projection such that OP moves in only the anti-clockwise sense, as P moves along the entire length of the link projection. We can ambient isotope the projection such that all the crossings lie in the projection of a cube, more precisely the cube constructed while defining a geometric braid. The end-points can be made to match as well. The above procedure of triangular moves shall guarantee the monotonicity that is required. $\square$

Henceforth, unless specified otherwise, we shall always work with the projections of the standard representation of a geometric n-braid and its closure and refer to these projections simply as a braid and its closure.

2.3.2 Conjugation

If $b, g \in \mathrm{GB}_n$, we observe that $\overline{gbg^{-1}}$ is ambient isotopic to $\overline{b}$ (fig. 2.10). The upper strands in g and g^{-1} are connected via the closure strand. We can slide g and g^{-1} via the closure strands to the 'other side' of b to annihilate each other. This can be achieved by a type II move.

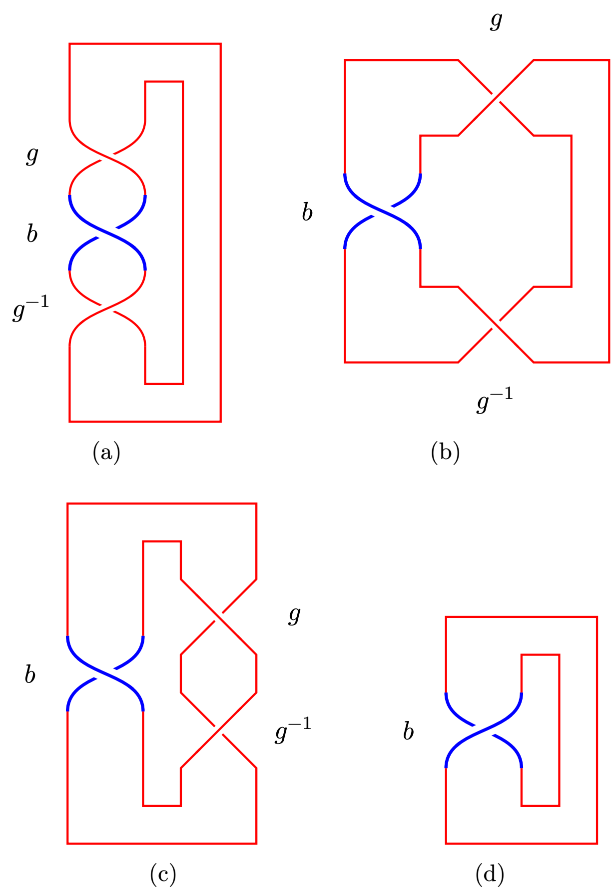
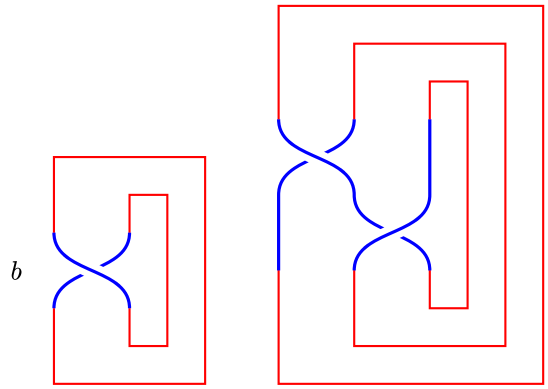


Figure 2.10: Conjugation process illustrating the link equivalence of $\overline{gbg^{-1}}$ and $\overline{b}$, with $b = \tau_1^{-1} \in \mathrm{GB}_1$ and $g = \tau_1^{-1} \in \mathrm{GB}_1$.

Remark 14. The closures of conjugate braids are ambient isotopic as links. The braids themselves are not.

2.3.3 Markov move

We observe that if $b \in \mathrm{GB}_n$, then $b\tau_n \in \mathrm{GB}_{n+1}$, $b\tau_n^{-1} \in \mathrm{GB}_{n+1}$ and b have ambient isotopic closures (fig. 2.11), although b, $b\tau_n$ and $b\tau_n^{-1}$ are not equivalent as braids. That is, we can add a strand and a crossing of that strand with another strand without changing the link type (of the closure). We can also remove a strand and a crossing if that strand does not cross any other strand. We visually see that adding the above mentioned strands anywhere between the existing strands is equivalent to adding the strands on the right.



(a) $\overline{b}$ (b) $\overline{b\tau_2}$

Figure 2.11: Markov move with $b = \tau_1^{-1}$.

2.3.4 Markov Theorem

Theorem 15 (Markov)**.** Two braids whose closures are ambient isotopic to each other are related by a finite sequence of the following operations.

1. Braid equivalences, i.e. equivalences resulting due to the braid relations.
2. Conjugation.
3. Markov moves.

A proof of the above theorem can be found in the book of Joan Birman [12, chp. 2]. Markov gave a sketch of the proof in 1936 [11].

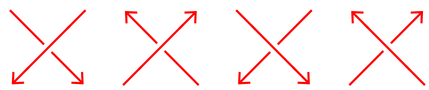
2.3.5 Writhe

We see visually that the result of a Markov move on a braid is equivalent to performing a type I move on the braid closure. We know that a type I move increases or decreases the writhe of a link by a unit value. Since all the crossings in a braid closure occur only in the cube containing the braid strands, we can define the writhe of a braid equal to the writhe of the braid closure by assigning each braid crossing a value, either $+1$ or -1. Our assignment must be consistent with our earlier assignment for knots. But for this procedure, we need to assign an orientation to the braid. We assign all the strands (inside the braid cube) a downward orientation. Doing so, we see that τ_i inherits $+1$ value while τ_i^{-1} inherits -1. We could instead have assigned all the strands an upward orientation as well. This would not have changed the values of τ_i and τ_i^{-1} (fig. 2.12). What is not allowed is assigning arbitrary orientation to strands. If we assign the orientation arbitrarily, then the well-definedness of the orientation cannot be guaranteed. Two distinct strands in a braid could be connected via the closure

strands and one would need to check the whole connected link component of the braid closure for a well-defined closure.
Thus, the writhe w of the braid b with a word representation $\prod_{j=1}^{m} \tau_{\alpha_j}^{\beta_j}$ of length m, where $\beta_j \in \{+1, -1\}$ is

$$w(b) = \sum_{j=1}^{m} \beta_j.$$

Note that the writhe of a braid is dependent on its word representation. We also see that $w(b) = w(\overline{b})$.



(a) (b) (c) (d)

Figure 2.12: Assignment of crossing values. (a) Downward orientation for τ_i corresponding to $+1$. (b) Upward orientation for τ_i corresponding to $+1$. (c) Downward orientation for τ_i^{-1} corresponding to -1. (d) Upward orientation for τ_i^{-1} corresponding to -1.

2.4 Markov trace

We won't distinguish between B_n and GB_n from now on. We can now move finally towards the Jones/bracket polynomial with the information we have. If we have a function $J_n \colon \mathrm{B}_n \to R$, where R is a commutative ring, then using the Markov Theorem we can construct link invariants from the family of functions $\{J_n\}$ if the following conditions are satisfied.

1. J_n is well defined. $J_n(b) = J_n(b')$ if $b = b'$.
2. $J_n(b) = J_n(gbg^{-1})$ if $g, b \in \mathrm{B}_n$.
3. If $b \in \mathrm{B}_n$, then there exists a constant $\alpha \in R$, independent of n, such that

$$J_{n+1}(b\sigma_n) = \alpha^{+1} J_n(b)$$

and

$$J_{n+1}(b\sigma_n^{-1}) = \alpha^{-1} J_n(b).$$

The last condition reminds us of the normalisation needed in order to make the bracket polynomial invariant under the type I move. Its purpose here is the same.

A family of functions $\{J_n\}$ satisfying the above given three conditions is called a Markov trace on $\{\mathrm{B}_n\}$. For any link L which is ambient isotopic to $\overline{b}$, where $b \in \mathrm{B}_n$, we define $J(L) \in R$ as follows.

$$J(L) := \alpha^{-w(b)} J_n(b).$$

We call $J(L)$ the link invariant for the Markov trace $\{J_n\}$.

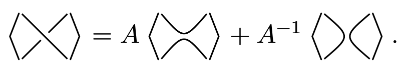
Theorem 16. *J is an invariant of ambient isotopy for oriented links.*

Proof. Suppose $L \sim \bar{b}$ and $L'' \sim \overline{b'}$, where $\sim$ denotes the ambient isotopy relation. By the Markov Theorem, we can obtain $\overline{b'}$ via an application of a finite sequence of the moves mentioned in the Markov Theorem on $\overline{b'}$. Each such move leaves J invariant. J_n is already invariant under braid equivalences and conjugation by definition. The $\alpha^{-w(b)}$ factor cancels the effect of a type I move. $\square$

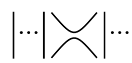
We can define the bracket polynomial for braids in the same way as we did for links. Define

$$\langle \cdot \rangle : \mathrm{B}_n \to \mathbb{Z}[A, A^{-1}]$$
$$\langle \cdot \rangle : b \mapsto \langle \bar{b} \rangle.$$

We simply evaluate the bracket on the closure of the braid. We observe that this function is a Markov trace with $\alpha = -A^3$. We know that



In terms of braids, we have

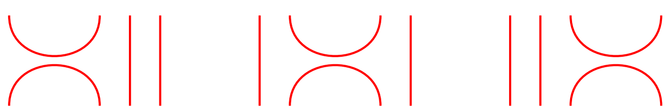
But  is the identity braid. Thus, if we denote  by U_i, we can write

$$\langle \sigma_i^{-1} \rangle = A \langle \mathrm{U}_i \rangle + A^{-1} \langle \mathbb{I}_n \rangle.$$

Similarly, one could write

$$\langle \sigma_i \rangle = A \langle \mathbb{I}_n \rangle + A^{-1} \langle \mathrm{U}_i \rangle.$$

Note that U_i does not belong to the braid group and is a new object. We refer to them as "input-output forms" or as "hooks". They are cup and cap combinations involving the i-th and $(i+1)$-th strands (fig. 2.13).



(a) U_1 (b) U_2 (c) U_3

Figure 2.13: Input-output forms or hooks for 4 strands

We can thus use consider σ_i to be equivalent to $A + A^{-1}\mathrm{U}_i$, σ_i^{-1} to be equivalent to $A^{-1} + A\mathrm{U}_i$, and create a formalism based on this. Given a braid word representation of a braid, we can simply substitute the above mentioned equivalences to get a product of U_i's, with A and A^{-1} as coefficients. Note that this product is dependent on the specific braid representation of a knot. It is not invariant under a type I move, as is the case with the un-normalised bracket polynomial.

We see that each bracket polynomial evaluation state of the closure of a braid can be written in terms of the closure of a product of the input-output forms. Given a braid word b, we can consider it equivalent to $S(b)$, where $S(b)$ is the sum of products of the U_i's obtained by substituting σ_i by $A + A^{-1}\mathrm{U}_i$ and σ_i^{-1} by $A^{-1} + A\mathrm{U}_i$. Closure of each term (a product in U_i's) in the sum corresponds to a state obtained while evaluating the bracket polynomial as it gives a collection of loops. If P is such a product, then $\langle P \rangle = \langle \overline{P} \rangle = \delta^{\|P\|}$, where $\|P\|$ is the number of loops in $\overline{P}$ minus 1, and $\delta = -A^{-2} - A^2$. Thus,

$$S(b) = \sum_s \langle b|s \rangle P_s,$$

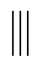
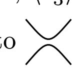
where s denotes a state and indexes all the terms in the product, and $\langle b|s \rangle$ is the product of A's and A^{-1}'s multiplying each P-product P_s. We have

$$\langle b \rangle = \langle S(b) \rangle = \sum_s \langle b|s \rangle \langle P_s \rangle = \sum_s \langle b|s \rangle \delta^{\|s\|}.$$

Example 17. Let $b = \sigma_1\sigma_2^{-1}$. We can resolve $\overline{b}$ in many states. One of the states s and its corresponding product $\mathrm{U}_1\mathrm{U}_2$ in terms of the input-output forms is shown in fig. 2.14.

Example 18. Consider the same braid $b = \sigma_1\sigma_2^{-1}$. We have

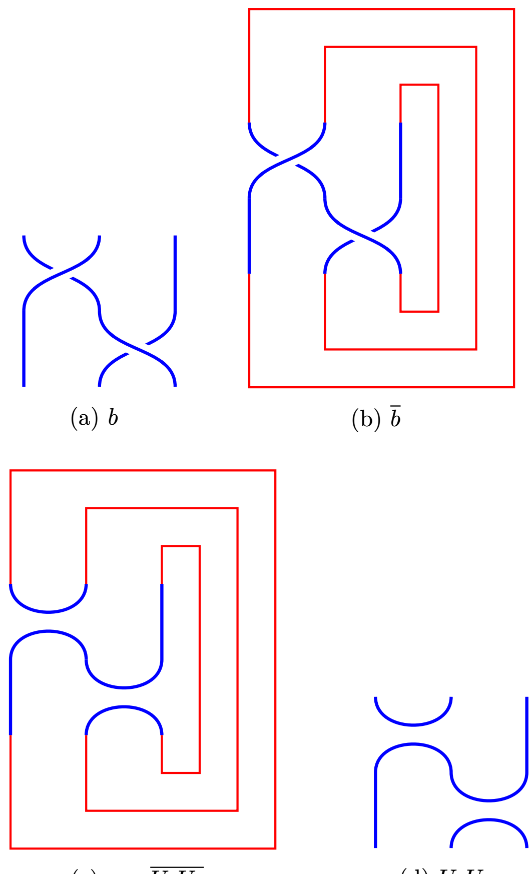
$$\begin{aligned} P(b) &= (A + A^{-1}\mathrm{U}_1)(A\mathrm{U}_2 + A^{-1}) \\ P(b) &= A^2\mathrm{U}_2 + \mathbb{I}_3 + \mathrm{U}_1\mathrm{U}_2 + A^{-2}\mathrm{U}_1 \\ \langle b \rangle = \langle \sigma_1\sigma_2^{-1} \rangle = \langle P(b) \rangle &= A^2\langle \mathrm{U}_2 \rangle + \langle \mathbb{I}_3 \rangle + \langle \mathrm{U}_1\mathrm{U}_2 \rangle + A^{-2}\langle \mathrm{U}_1 \rangle. \end{aligned}$$

Now, $\mathbb{I}_3$ corresponds to . Thus, $\langle I_3 \rangle = \delta^{3-1} = \delta^2$ as the closure of $\mathbb{I}_3$ shall give three loops. U_1 corresponds to . Thus, $\langle \mathrm{U}_1 \rangle = \delta^{2-1} = \delta$ as the closure shall give two loops. Similarly, $\langle \mathrm{U}_2 \rangle = \delta$ and $\langle \mathrm{U}_1\mathrm{U}_2 \rangle = \delta^{1-1} = 1$. For a visual representation of the state $\mathrm{U}_1\mathrm{U}_2$, see fig. 2.14, where the closure gives one single loop. In the end, we get

$$\langle \overline{b} \rangle = A^2\delta + \delta^2 + 1 + A^{-2}\delta.$$

2.5 Temperley–Lieb algebra

We can give the U_i's a structure of their own by constructing the free additive algebra TL_n with the generators $\mathrm{U}_1, \mathrm{U}_2, \dots, \mathrm{U}_{n-1}$ and the multiplicative relations coming from the interpretation of U_i's as input-output forms. We can consider this algebra over the ring $\mathbb{Z}[A, A^{-1}]$ with $\delta = -A^{-2} - A^2 \in \mathbb{Z}[A, A^{-1}]$. We shall call TL_n the Temperley–Lieb algebra. The multiplicative relations in TL_n are as follows.

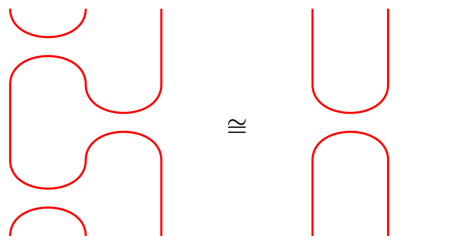


(a) b (b) $\bar{b}$ (c) $s = \overline{U_1 U_2}$ (d) $U_1 U_2$

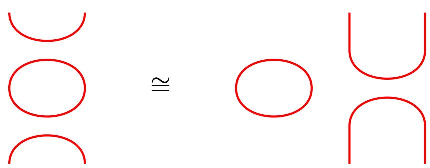
Figure 2.14: Writing a state of a braid closure in terms of input-output forms.

1. $U_i U_{i\pm 1} U_i = U_i$.
2. $U_i^2 = \delta U_i$.
3. $U_i U_j = U_j U_i$ if $|i - j| \geq 2$.

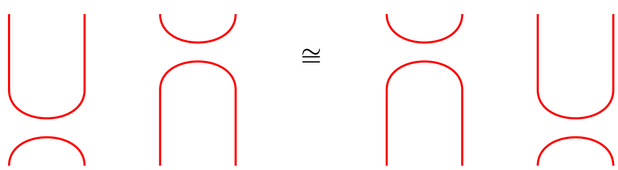
These relations are a result of the geometric relations as illustrated in fig. 2.15.



(a) $U_1 U_2 U_1 = U_1$



(b) $U_1^2 = \delta U_1$, we have a union of a loop(δ), and U_1 on the right side



(c) $U_3 U_1 = U_1 U_3$

Figure 2.15: Input-output form relations

Now that we have the Temperley–Lieb algebra, we can define a mapping

$$\rho \colon B_n \to TL_n$$

by the following formulae.

$$\rho(\sigma_i) = A + A^{-1} U_i,$$
$$\rho(\sigma_i^{-1}) = A^{-1} + A U_i.$$

We see that ρ is indeed a representation by verifying that $\rho(\sigma_i)\,\rho(\sigma_i^{-1}) = 1$, $\rho(\sigma_i \sigma_{i+1} \sigma_i) = \rho(\sigma_{i+1} \sigma_i \sigma_{i+1})$ and if $|i - j| \geq 2$, $\rho(\sigma_i \sigma_j) = \rho(\sigma_j \sigma_i)$.

Proposition 19. $\rho \colon B_n \to TL_n$ is a representation of the Artin braid group.

Proof. We first prove that $\rho(\sigma_i)\,\rho(\sigma_i^{-1}) = 1$.

$$\begin{aligned}\rho(\sigma_i)\,\rho(\sigma_i^{-1}) &= (A + A^{-1}\mathrm{U}_i)(A^{-1} + A\mathrm{U}_i)\\ &= 1 + (A^{-2} + A^2)\mathrm{U}_i + \mathrm{U}_i^2\end{aligned}$$

But since $\mathrm{U}_i^2 = \delta\mathrm{U}_i$ and $\delta = -A^{-2} - A^2$, we have

$$\begin{aligned}\rho(\sigma_i)\,\rho(\sigma_i^{-1}) &= 1 + (A^{-2} + A^2)\mathrm{U}_i + \delta\mathrm{U}_i\\ &= 1 + (A^{-2} + A^2)\mathrm{U}_i + (-A^{-2} - A^2)\mathrm{U}_i\\ &= 1.\end{aligned}$$

We now prove that $\rho(\sigma_i\sigma_{i+1}\sigma_i) = \rho(\sigma_{i+1}\sigma_i\sigma_{i+1})$.

$$\begin{aligned}\rho(\sigma_i\sigma_{i+1}\sigma_i) &= (A + A^{-1}\mathrm{U}_i)(A + A^{-1}\mathrm{U}_{i+1})(A + A^{-1}\mathrm{U}_i)\\ &= (A^2 + \mathrm{U}_{i+1} + \mathrm{U}_i + A^{-2}\mathrm{U}_i\mathrm{U}_{i+1})(A + A^{-1}\mathrm{U}_i)\\ &= A^3 + A\mathrm{U}_{i+1} + A\mathrm{U}_i + A^{-1}\mathrm{U}_i\mathrm{U}_{i+1} + A^{-2}\mathrm{U}_i^2 + A\mathrm{U}_i\\ &\quad + A^{-1}\mathrm{U}_{i+1}\mathrm{U}_i + A^{-3}\mathrm{U}_i\mathrm{U}_{i+1}\mathrm{U}_i\\ &= A^3 + A\mathrm{U}_{i+1} + (A^{-1}\delta + 2A)\mathrm{U}_i\\ &\quad + A^{-1}(\mathrm{U}_i\mathrm{U}_{i+1} + \mathrm{U}_{i+1}\mathrm{U}_i) + A^{-3}\mathrm{U}_i\\ &= A^3 + A\mathrm{U}_{i+1} + (A^{-1}(-A^2 - A^{-2}) + 2A + A^{-3})\mathrm{U}_i\\ &\quad + A^{-1}(\mathrm{U}_i\mathrm{U}_{i+1} + \mathrm{U}_{i+1}\mathrm{U}_i)\\ &= A^3 + A(\mathrm{U}_{i+1} + \mathrm{U}_i) + A^{-1}(\mathrm{U}_i\mathrm{U}_{i+1} + \mathrm{U}_{i+1}\mathrm{U}_i).\end{aligned}$$

Since symmetry of the above expression in i and $i+1$, we can conclude that $\rho(\sigma_i\sigma_{i+1}\sigma_i) = \rho(\sigma_{i+1}\sigma_i\sigma_{i+1})$. We now prove that if $|i-j| \geq 2$, then $\rho(\sigma_i\sigma_j) = \rho(\sigma_j\sigma_i)$.

$$\begin{aligned}\rho(\sigma_i\sigma_j) &= \rho(\sigma_i)(\sigma_j)\\ &= (A + A^{-1}\mathrm{U}_i)(A + A^{-1}\mathrm{U}_j)\end{aligned}$$

Now since $\mathrm{U}_i\mathrm{U}_j = \mathrm{U}_j\mathrm{U}_i$ if $|i-j| \geq 2$, we have $(A + A^{-1}\mathrm{U}_i)(A + A^{-1}\mathrm{U}_j) = (A + A^{-1}\mathrm{U}_j)(A + A^{-1}\mathrm{U}_i)$ which equals $\rho(\sigma_j\sigma_i)$. □

We now define the diagrammatic trace $\mathrm{tr}\colon \mathrm{TL}_n \to \mathbb{Z}[A, A^{-1}]$ by extending linearly $\mathrm{tr}(P) = \langle P\rangle$, where P is a product term in $S(b)$. This version of trace is diagrammatic in nature as we are counting loops in a state. We thus arrive at the formula $\langle b\rangle = \mathrm{tr}(\rho(b))$. With what we have learnt so far, one can now find a braid representation b of a link L by Alexander's theorem, calculate $\mathrm{tr}(\rho(b))$ and normalise it to get the link invariant normalised bracket polynomial. One can substitute $A = t^{-1/4}$ to arrive at our long sought destination of the Jones polynomial.

2.6 Jones algebra

Jones considered a sequence of algebras A_n for $n = 2, 3, \dots$ with multiplicative generators $e_1, e_2, \dots, e_{n-1}$ and the following relations.

1. $e_i^2 = e_i$.
2. $e_i e_{i\pm 1} e_i = c e_i$.
3. $e_i e_j = e_j e_i$ if $|i - j| \geq 2$.

Here, c is a scalar which commutes with all the other elements. We can consider A_n as the free additive algebra on these generators viewed as a module over the ring $\mathbb{C}[c, c^{-1}]$. While c is often taken to be a complex number, we can view is as another algebraic variable which commutes with the generators. A_n arose in the theory of classification of von Neumann algebras and one can realize A_n as a von Neumann algebra as well. The reader is requested to compare the above mentioned relations for Jones algebra with the relations for the Artin braid group and the Temperley–Lieb algebra.

As mentioned in the beginning of this chapter, it is natural to construct a nested tower of algebras

$$M_0 \hookrightarrow M_1 \hookrightarrow M_2 \hookrightarrow M_3 \hookrightarrow \cdots \hookrightarrow M_n \hookrightarrow M_{n+1} \hookrightarrow \dots$$

with the following properties.

1. M_0 and M_1 are given.
2. $e_i \colon M_i \to M_{i-1}$ is a ‘projection’.
3. $e_i^2 = e_i$.
4. $M_{i+1} = \langle M_i, e_i \rangle$.

Jones constructed such a tower of algebras with the other two properties of generators $e_i e_{i\pm 1} e_i = c e_i$ and $e_i e_j = e_j e_i$ if $|i - j| \geq 2$. He defined a notion of a trace $\mathrm{tr} \colon M_n \to \mathbb{C}$, i.e. a function which vanishes on the commutator of any two elements of M_n. This trace satisfied so called Markov property: $\mathrm{tr}(w e_i) = c\,\mathrm{tr}(w)$ for w in the algebra generated by M_0, e_i,..., e_{i-1}.

We now see a concrete example of such a tower of algebras. Consider

$$\mathbb{R} \hookrightarrow \mathbb{R}[x_1] \hookrightarrow \mathbb{R}[x_1, x_2] \hookrightarrow \mathbb{R}[x_1, x_2, x_3] \hookrightarrow \cdots \hookrightarrow \mathbb{R}[x_1, x_2, \dots, x_n] \hookrightarrow \dots .$$

We have a sequence of the set of all real numbers, the set of all real polynomials in one variable, the set of all real polynomials in two variables, the set of all real polynomials in three variables, and so on. Each variable x_i for $i \geq 2$, is a map from $\mathbb{R}[x_1, x_2, \dots, x_{i-1}]$ to $\mathbb{R}[x_1, x_2, \dots, x_{i-2}]$, defined as the coefficient of $x_{i-1}^0 \in \mathbb{R}[x_1, x_2, \dots, x_{i-1}]$.

$$x_i \colon \mathbb{R}[x_1, x_2, \dots, x_{i-1}] \to \mathbb{R}[x_1, x_2, \dots, x_{i-2}]$$
$$x_i \colon \sum_{j=0}^{m} p_j x_{i-1}^j \mapsto p_0,$$

where $p_j \in \mathbb{R}[x_1, x_2, \dots, x_{i-2}]$ and $m \in \mathbb{N}$. As an example, if $\sum_{j=0}^{m} p_j x_1^j \in \mathbb{R}[x_1]$, then

$$
\begin{aligned}
x_2 &\colon \mathbb{R}[x_1] \to \mathbb{R} \\
x_2 &\colon \sum_{j=0}^{m} p_j x_1^j \mapsto p_0,
\end{aligned}
$$

where p_0 is the coefficient of x_1^0, i.e. the constant term. For example,

$$x_2(3x_1^4 + 6x_1^3 + x_1 + 5) = 5.$$

Similarly, x_3 is a map from $\mathbb{R}[x_1, x_2]$ to which maps a polynomial in two variables to the polynomial (in the single variable x_1) which does not contain any power of x_2. As an example,

$$x_3(4x_1x_2^2 + 3x_1^3 + x_2 + x_2^5 + 1) = 3x_1^3 + 1.$$

$(\mathbb{R}[x_1, x_2, \dots, x_n], +, \cdot)$ is an algebra with $+$ defined as polynomial addition (component-wise addition) and $\cdot$ defined as polynomial multiplication (Cauchy product). The generator relations for the Jones algebra, however, are not satisfied with the $\cdot$ operation; $x_1^2 \coloneqq x_1 \cdot x_1 \neq x_1$. But we can make sense of them if we intrepret the operation amongst the generators as the function composision operator. x_1, x_2, ..., x_n are the generators $(\mathbb{R}[x_1, x_2, \dots, x_n], +, \cdot)$. Since x_i's are functions as well, we can intrepret x_i^2 as $x_i \circ x_i$. If we do that, then $x_i^2 \coloneqq x_i \circ x_i = x_i$ as x_i is a projection operator. Similarly, we see that $x_i \circ x_{i\pm1} \circ x_i = x_i$ and $x_i \circ x_j = x_j \circ x_i$ by restricting the domains appropriately.

We can retrieve the Jones algebra from the Temperley–Lieb algebra by substituting $e_i = \delta^{-1}\mathrm{U}_i$, $e_i^2 = e_i$ and $e_i e_{i\pm1} e_i = \delta^{-2} e_i$. We have taken $c = \delta^{-2}$. Note that the underlying rings are different.

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