Osborne's Lectures on Symmetries and Quantum Mechanics

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Note

This document is best viewed in a full-screen mode with two pages side-by-side, on a screen of aspect ratio 16:9.

Preface

These are lecture notes by Apoorv Potnis of the lecture series 'Symmetries and Quantum Mechanics', given by **Prof. Tobias J. Osborne** in 2023 at the Leibniz Universität Hannover. Prof. Osborne discusses the basics of the representation theory of groups in the context of quantum mechanics in this short lecture series. The video lecture series is available at https://youtube.com/playlist?list=PLDfPUNusx1ErdQhrdAzincNJKgTQahsX_&feature=shared.

The source code, updates and corrections to this document can be found on this GitHub repository: https://github.com/apoorvpotnis/osborne_symmetries. The source code is embedded in this PDF. Comments and corrections can be mailed at apoorvpotnis@gmail.com.

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Chapter 1

Basics, Wigner's theorem and linear representation of groups

1.1 Prerequisites and references

We assume that the reader has a working knowledge of linear algebra and is familiar with basic ideas of quantum mechanics. Mainly, one needs the knowledge of the postulates of quantum mechanics, which can be learnt from Prof. Osborne's video lecture on YouTube titled 'Quantum mechanics essentials: Everything you need for quantum computation' [1].

The main reference book this lecture course series is based on is the famous book of Serre.

> Jean-Pierre Serre. *Linear Representations of Finite Groups*. Graduate Texts in Mathematics 42. Springer-Verlag, New York Inc., 1977. ISBN: 978-1-4684-9460-0. Translated from the French by Leonard L. Scott.

The first part of this book is based on lectures given by the eminent mathematician Jean-Pierre Serre to a group of quantum chemists. One may look at the first quantum field theory volume of Weinberg, but this is too advanced for present purposes and one would have great difficulties reading it.

> Steven Weinberg. *The Quantum Theory of Fields: Foundations*. Vol. 1. Cambridge University Press, Cambridge, 2005. ISBN: 978-0-521-55001-7.

1.2 Postulates of quantum mechanics

We briefly state the postulates of quantum mechanics.

1. A Hilbert space corresponds to every quantum mechanical system.

2. The states of quantum mechanical systems are represented by density matrices.

3. The measurements or detectors are represented by positive operators.

4. Börn rule.

5. Schrödinger's equation.

6. Tensor product for composite systems.

We shall focus on the fifth postulate, namely the Schrödinger equation. We argue that we don't actually need it. It can be derived from deeper principles.

1.3 Symmetries

A symmetry on a quantum system is a physical operation that can be performed or can occur. The most general operation in quantum mechanics that can occur on a a system is represented by a completely positive map. Symmetries thus need to be completely positive maps as well. In fact, we shall argue that symmetries form a small subset of completely positive maps. Take a system, and wait for some time $t \in \mathbb{R}$. It is possible for a system to not change in that time. Thus, 'waiting for time t' is a symmetry. For every time $t \in \mathbb{R}$, there exists a possible symmetry operation, namely, waiting for time t. Thus, we get a set of symmetries labelled by t. What we are actually are interested in this course are sets of symmetries. We call the set of all labels as G. There are some properties for symmetries which we desire, as follows.

1. The symmetry of doing nothing must always be present in the set of symmetries.

2. If we have two symmetries, then we must be able to do them one after the other, and the resulting operation must be a symmetry as well.

3. If we have a symmetry operation, we must be able to 'reverse' the operation to get the system back to its initial state.

It should be noted that these symmetry operations are the in principle 'possible' operations on a system, operations that we can think of at the very least. It may be extremely difficult, or impossible even, to actually perform these operations on a physical system.

In order to capture the three requirements that the set of symmetries must possess, we formalise the notion of an algebraic group, which the reader will have no doubt encountered before.

Definition 1.3.1 (Group). A group is a set G, together with a law of composition, i.e. a composition map

$$\circ\colon G \times G \to G,$$
$$\circ\colon (x, y) \mapsto x \circ y,$$

such that

1. $(x \circ y) \circ z = x \circ (y \circ z)$, i.e. the product is associative,

2. G contains a unit element 1 such that $x \circ 1 = 1 \circ x = x$, for all $x \in G$, and,

3. For all $x \in G$, there exists an inverse $y \in G$ such that $x \circ y = y \circ x = 1$.

We would often drop the $\circ$ and denote $x \circ y$ as simply xy.

Examples 1.3.1.

(a) $\mathbb{Z}/2\mathbb{Z}$ is a group consisting of two elements; $\{0, 1\}$ with addition operation $\oplus$ defined as the usual addition modulo 2. It is the group of reflections, and physically captures the symmetry of charge conjugation, parity, etc. The group multiplication table is as follows.

	0	1
0	0	1
1	1	0

(b) $\mathbb{Z}$ is a group with the addition operation. Similarly, $\mathbb{Z} \times \mathbb{Z}$ is a group too. This can represent the symmetries of a lattice of atoms.

(c) $\mathbb{R}$ is a group with the addition operation. This captures the time and space translation symmetries.

(d) Symmetries of the square, the dihedral group D_4, of order 8. The order of a finite group is the number of elements it contains. The group consists of rotations and flips of a square. This group is non-Abelian. A group is said to be Abelian if for all $x, y \in G$, we have that $xy = yx$.

Bibliography

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[8] Valter Moretti (https://physics.stackexchange.com/users/35354/valter-moretti). *Answer to the question 'Angular momentum Lie algebra for infinite-dimensional Hilbert spaces'*. Physics Stack Exchange (https://physics.stackexchange.com). Version: 2024-05-17 21:03:48Z. URL: https://physics.stackexchange.com/a/814882/81224.

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