Frederic Schuller
Apoorv Potnis

Schuller's Lectures on Classical Mechanics

Schuller's Lectures on Classical Mechanics

Frederic Schuller
Apoorv Potnis

January 29, 2025

Preface

These are lecture notes by Apoorv Potnis of the lecture series 'Theoretische Physik 1: Mechanik' (Theoretical Physics 1: Mechanics), given by **Prof. Frederic Paul Schuller** in 2014 at the Friedrich-Alexander-Universität Erlangen-Nürnberg. Prof. Schuller discusses classical mechanics in a mathematically rigorous fashion in this course. While the original lecture series is in German, these notes are in English and have been prepared using YouTube's automatic subtitle translation tool. The video lecture series is available at https://youtube.com/playlist?list=PLyIi3L2232Qo5t61tXfoL6vTW2a kFQL-n&feature=shared and at https://www.fau.tv/course/id/272.

The source code, updates and corrections to this document can be found on this GitHub repository: https://github.com/apoorvpotnis/schull er_classical. The source code is embedded in this PDF. Comments and corrections can be mailed at apoorvpotnis@gmail.com.

ii

Contents

Chapter 1

A Bird's Eye View of Physics

1.1 Introduction

According to Prof. Schuller, the only goal of physics is to predict the future, nothing more and nothing less. He then invokes Wittgenstein to remark that the goal of theoretical physics is to say all that can be said clearly. But in order to say things clearly, one needs the language of mathematics. He then remembers Hilbert, who said experimental physics and theoretical physics have nothing to do with each other. Roughly speaking, experimental physics and theoretical physics are Siamese twins, according to Prof. Schuller.

We all learn about Newton's laws in school, but we need to properly define the terms used in the laws, such as 'mass', 'absolute time', 'absolute space', etc. We formulate Newton's laws in the following way:

1. In an *inertial system*, a *body* on which no *force* acts *moves uniformly* and *linearly.*

2. In an inertial system, the *deviation* of a body from a uniform linear motion is proportional to the force acting on it.

3. *Interaction* between bodies takes place through equal and opposite forces.

He then asks whether the second law contains the first law: can we take the force acting on a body to be zero and obtain the first law from the second? The answer is negative. The first law defines what we mean by 'straight lines'. It defines 'uniform' and 'linear motion' for us. The motion of particles on which no force acts furnish a system of straight lines in space; they define the geometry of space. We can hit a puck with a hockey stick on a flat frictionless surface of ice and the motion of the puck shall give us straight lines on the flat

surface. If we imagine earth to be perfectly spherical and made of frictionless ice, then a small puck launched along the surface shall define 'straight lines' on the surface of earth. Only an external observer, such as one sitting in the International Space Station shall observe the circular motion of the puck along the earth's surface. Thus, if were in a spaceship in empty space, then we could launch some sand particles from our hand to define 'straight lines' for us. But there's a problem, in the real world, contrary to the assumption in the first law, there are always other bodies which exert gravitational forces on each other. We shall get out of this trouble by not considering gravitation as a force! Thus, we can always assume the existence of particles (chargeless, spinless, etc.)

1.2   Course structure

Gravitation
Newton's laws
Particles, forces and inertial frames
Spacetime
Affine manifolds
Manifolds
Set theory

In order to properly state the physical laws and derive their consequences, we need to develop a fair amount of mathematical machinery, which shall occupy a significant duration of this course.

The naive idea of set theory that we have from school encounters problems when we try to work with infinite sets. We don't discuss axiomatic set theory in this course. The reader is requested to look at first three lectures of Prof. Schuller's *Lectures on the Geometric Anatomy of Theoretical Physics* course for a discussion of the Zermelo–Fraenkel set theory with the axiom of choice (ZFC) [1]. The reader can also take a look at Terence Tao's *Analysis I* for a lucid exposition of ZFC [2].

Suppose we have an atlas of the world with maps of different regions on different pages. We refer to these maps as charts. These charts are flat, but if we join the charts such that regions common to two charts overlap on each other, we unsurprisingly get the globe of earth. The global, spherical structure of earth is encoded in the overlapping regions of the charts. This idea shall be used in the study of manifolds.

Consider the red vector in two-dimensional space, as shown in fig. 1.1a. We can parallelly transport it to get the blue vector. The orientation of the vector remains the same while the origin of the vector changes. But how do we define 'parallelly' transporting a vector? We may claim that both the blue and red vectors are the 'same' as their vector components are equal, and define parallel transport as something which keeps the vector components invariant. But if we switch to polar co-ordinates, as in fig. 1.1b, we intuitively/geometrically see that bot the red and blue vectors are the same, but their components in polar co-ordinates are not the same. The answer to this problem shall be revealed when we discuss affine manifolds. We
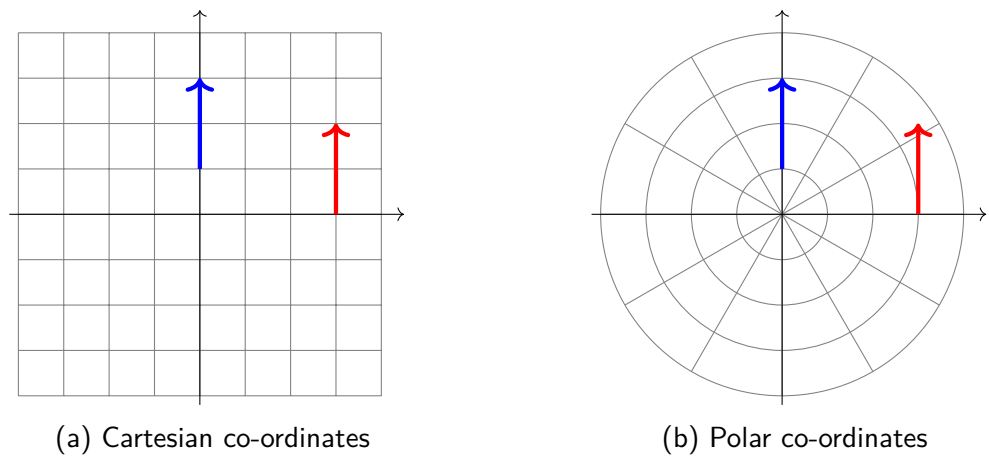

(a) Cartesian co-ordinates (b) Polar co-ordinates

Figure 1.1: Parallel transport of vectors

need $n^2(n+1)/2$ co-ordinates in an n-dimensional space in order to properly define parallel transport. For $n = 4$, we need 40 components! Suppose we are in an empty universe, with no stars. How do we define rotation? How would we know that we are rotating or not? This was pondered by Newton as well, who said that the surface of water in a rotating bucket shall appear parabolic. The affine structure on manifolds shall capture the idea needed to define rotation properly.

We shall discuss spacetime as well when we discuss relativity. It turns out that only certain trajectories are possible for massive particles, namely the trajectories which lie inside a *light cone* (fig. 1.2). The angle of the cone is determined by the speed of light. If we take the speed of light to be infinity, the cone becomes a flat plane. Prof. Schuller references *Gravitation* by Misner, Thorne and Wheeler for the Newtonian spacetime formulation that he shall discuss [3].

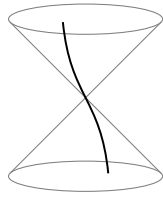After covering the above, we move on to the second part of the course. We

Figure 1.2: Light cone

shall first learn about some calculus of variations to head into a reformulation of Newtonian mechanics, known as Lagrangian mechanics. We shall see many applications of Lagrangian mechanics to physical problems. For example, we shall see that the *configuration space* of the double pendulum to be a torus. After Lagrangian mechanics, we shall learn about yet another reformulation, namely Hamiltonian mechanics, where we shall explore the concept of a *phase space*. We shall also take a look at symmetries, conserved quantities and its applications. If time permits, we shall take a look at some symplectic geometry as well, from which it is easy to find qualitative solutions to many problems. In the last lecture, we shall briefly discuss general relativity.

Chapter 2

Topological Manifolds

2.1 References

1. John Stewart. *Advanced General Relativity*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1993. ISBN: 0-521-32319-3.

2. Chris J. Isham. *Modern Differential Geometry for Physicists*. 2nd ed. World Scientific Lecture Notes in Physics. World Scientific, Singapore, 2001. ISBN: 981-02-3555-0.

2.2 Mathematical structure of spacetime

Spacetime is a set with some mathematical structure. We shall see some physical motivations for defining the particular mathematical structures we need. In classical physics, we talk about trajectories of particles. We require them to be continuous, i.e. a particle cannot disappear and reappear somewhere else magically.[1] Thus, a trajectory like the one in fig. 2.1a is acceptable while like the one in fig. 2.1b is not.

We are familiar with the ϵ-δ definition of continuity, but one can formulate the notion of continuity with a much weaker structure. A *topology* on a set is a very weak structure which allows to define continuity.

A curve in spacetime would normally depict the trajectory of a particle. We often wish to parametrize the trajectory of a particle by its time co-ordinate. This motivates us to define a curve γ mathematically as a continuous function

---

[1] In reality though, as the world we live in is not classical, particles *can* disappear and appear magically. One encounters this phenomenon when studying quantum field theory.

(a) Acceptable continuous trajectory

(b) Non-acceptable discontinuous trajectory
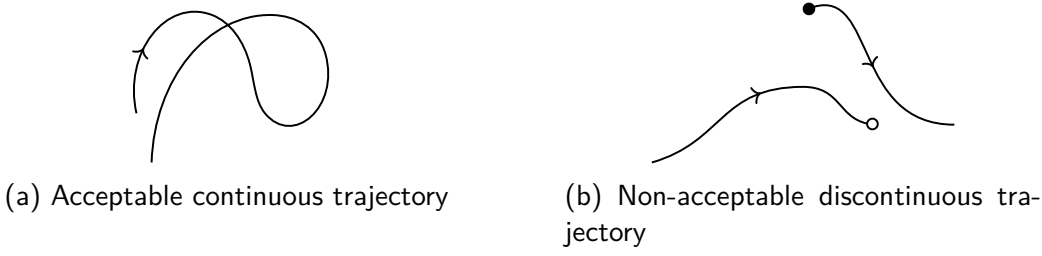
from the set of real numbers to space, i.e. $\gamma\colon \mathbb{R} \to M$, where M denotes the physical space, in whatever way it is defined.

Note that two particles may have the same path in space, but different trajectories. For example, consider two particles which travel along the same path in space but their velocities and positions at different times are different. Let $\gamma, \tilde{\gamma}\colon \mathbb{R} \to \mathbb{R}^2$ be two trajectories of two particles travelling in a plane. We set $M = \mathbb{R}^2$ for now. Then $\{\gamma(t) \mid t \in \mathbb{R}\} = \{\tilde{\gamma}(\tilde{t}) \mid \tilde{t} \in \mathbb{R}\}$, but let $\gamma(1/2) \neq \tilde{\gamma}(1/2)$. Refer to fig. 2.2.
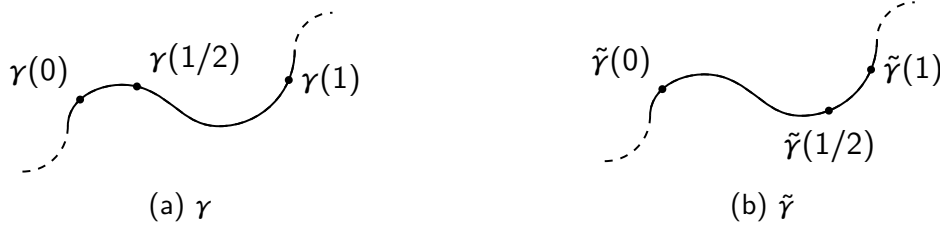

(a) γ

(b) $\tilde{\gamma}$

Figure 2.2: Two particles having same path but different trajectories. The particle moves slower during the first half of the first trajectory.

2.3   Topological spaces

Definition 1 (Topological space)**.** A *topological space* is a set M with a collection of subsets of M, denoted by $\mathcal{O}$, which satisfy the following conditions.

1. $\emptyset \in \mathcal{O}$ and $M \in \mathcal{O}$.

2. If $U, V \in \mathcal{O}$, then $U \cap V \in \mathcal{O}$.

3. If $U_\alpha \in \mathcal{O}$ for all $\alpha \in A$, where A is an indexing set, then $\displaystyle\bigcup_{\alpha \in A} U_\alpha \in \mathcal{O}$.

$\mathcal{O}$ is referred to as a topology on M, while the tuple $(M, \mathcal{O})$ is referred to as a topological space. The elements of $\mathcal{O}$ are called as *open* sets.

Note that there can in general be more than one topologies on a given set. The indexing set A can be uncountable as well. If A is uncountable, then the third condition cannot be be fulfilled by simply requiring that if $U, V \in \mathcal{O}$, then $U \cup V \in \mathcal{O}$, as induction will work only for countably many objects.

Example 1. Let $M := \{1, 2, \xi\}$, where ξ is any mathematical object. Then

$$\mathcal{O}_1 := \{\emptyset, \{1, 2, \xi\}\} \quad \text{and} \quad \mathcal{O}_2 := \{\emptyset, \{1\}, \{1, 2, \xi\}\}$$

are valid topologies on M while

$$\mathcal{O}_3 := \{\emptyset, \{1\}, \{2\}, \{1, 2, \xi\}\} \quad \text{and} \quad \mathcal{O}_4 := \{\emptyset\}$$

are not.

For any set M, $\mathcal{O}_{\mathrm{d}} := \mathcal{P}(M)$ is called the *discrete* or the *chaotic* topology, where $\mathcal{P}(M)$ denotes the power set of M. The power set of M is defined as the collection of all subsets of M. $\mathcal{O}_{\mathrm{i}} := \{\emptyset, M\}$ is called the *indiscrete* topology.

Let $M := \mathbb{R}^d$. Then the *standard* topology $\mathcal{O}_{\mathrm{s}} \subset \mathcal{P}(\mathbb{R}^d)$ on $\mathbb{R}^d$ is defined as follows. A set $U \subseteq \mathbb{R}^d$ belongs to $\mathcal{O}_{\mathrm{s}}$ iff for all $a \in U$, there exists an $r \in \mathbb{R}_{\geq 0}$ such that $B_r(a) \subseteq U$. Here $B_r(a)$ is the open ball of radius r centered at a, defined as

$$B_r(a) := \left\{ b \in \mathbb{R}^d \;\middle|\; \sum_{i=1}^{d} (a^i - b^i)^2 < r^2 \right\},$$

where a^i and b^i denote the ith components of a and b respectively. The fact that $B_r(a)$ is called open refers to the strict inequality in the definition of $B_r(a)$. We would have called it a closed ball if we had $\leq$ instead of $<$. We thus retrieve the notion of an open interval of $\mathbb{R}$, that we studied in calculus courses. If we put the discrete topology on $\mathbb{R}$ instead of the standard one, we see that a set containing even a single real number becomes an open set, justifying the word discrete.

Note that there can in general be more than one topologies on a given set. The indexing set A can be uncountable as well. If A is uncountable, then the third condition cannot be be fulfilled by simply requiring that if $U, V \in \mathcal{O}$, then $U \cup V \in \mathcal{O}$, as induction will work only for countably many objects.
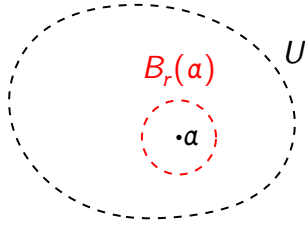
Example 2. Let $M := \{1, 2, \xi\}$, where ξ is any mathematical object. Then

$$\mathcal{O}_1 := \{\emptyset, \{1, 2, \xi\}\} \quad \text{and} \quad \mathcal{O}_2 := \{\emptyset, \{1\}, \{1, 2, \xi\}\}$$
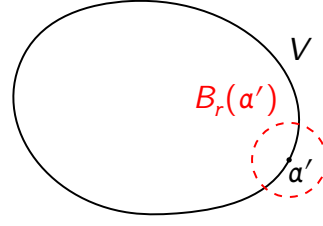
are valid topologies on M while

$$\mathcal{O}_3 := \{\emptyset, \{1\}, \{2\}, \{1, 2, \xi\}\} \quad \text{and} \quad \mathcal{O}_4 := \{\emptyset\}$$

are not.

(a) $U \in \mathcal{O}_\mathrm{s}$ is an open set of $\mathbb{R}^2$. The dashed line indicates that the points on the line are not contained in U. For all $a \in U$, we can always find an $r > 0$ such that $B_r(a)$ in fully contained in U.

(b) $V \notin \mathcal{O}_\mathrm{s}$ is a closed set of $\mathbb{R}^2$. The solid line used to depict V indicates that the points on the line belong to V. For all $a' \in \partial V$, we can never find an $r > 0$ such that $B_r(a)$ in fully contained in V. Here, ∂V denotes the boundary of V.

Figure 2.3: Open and closed sets

For any set M, $\mathcal{O}_\mathrm{d} := \mathcal{P}(M)$ is called the *discrete* or the *chaotic* topology, where $\mathcal{P}(M)$ denotes the power set of M. The power set of M is defined as the collection of all subsets of M. $\mathcal{O}_\mathrm{i} := \{\emptyset, M\}$ is called the *indiscrete* topology.

Let $M := \mathbb{R}^d$. Then the *standard* topology $\mathcal{O}_\mathrm{s} \subset \mathcal{P}(\mathbb{R}^d)$ on $\mathbb{R}^d$ is defined as follows. A set $U \subseteq \mathbb{R}^d$ belongs to $\mathcal{O}_\mathrm{s}$ iff for all $a \in U$, there exists an $r \in \mathbb{R}_{\geq 0}$ such that $B_r(a) \subseteq U$. Here $B_r(a)$ is the open ball of radius r centered at a, defined as

$$B_r(a) := \left\{ b \in \mathbb{R}^d \ \middle| \ \sum_{i=1}^d (a^i - b^i)^2 < r^2 \right\},$$

where a^i and b^i denote the ith components of a and b respectively. The fact that $B_r(a)$ is called open refers to the strict inequality in the definition of $B_r(a)$. We would have called it a closed ball if we had $\leq$ instead of $<$. We thus retrieve the notion of an open interval of $\mathbb{R}$, that we studied in calculus courses. If we put the discrete topology on $\mathbb{R}$ instead of the standard one, we see that a set containing even a single real number becomes an open set, justifying the word discrete. Note that there can in general be more than one topologies on a given set. The indexing set A can be uncountable as well. If A is uncountable, then the third condition cannot be be fulfilled by simply requiring that if $U, V \in \mathcal{O}$, then $U \cup V \in \mathcal{O}$, as induction will work only for countably many objects.

Example 3. Let $M := \{1, 2, \xi\}$, where ξ is any mathematical object. Then

$$\mathcal{O}_1 := \{\emptyset, \{1, 2, \xi\}\} \quad \text{and} \quad \mathcal{O}_2 := \{\emptyset, \{1\}, \{1, 2, \xi\}\}$$

are valid topologies on M while

$$\mathcal{O}_3 := \{\emptyset, \{1\}, \{2\}, \{1, 2, \xi\}\} \quad \text{and} \quad \mathcal{O}_4 := \{\emptyset\}$$

are not.

For any set M, $\mathcal{O}_d := \mathscr{P}(M)$ is called the *discrete* or the *chaotic* topology, where $\mathscr{P}(M)$ denotes the power set of M. The power set of M is defined as the collection of all subsets of M. $\mathcal{O}_i := \{\emptyset, M\}$ is called the *indiscrete* topology.

Let $M := \mathbb{R}^d$. Then the *standard* topology $\mathcal{O}_s \subset \mathscr{P}(\mathbb{R}^d)$ on $\mathbb{R}^d$ is defined as follows. A set $U \subseteq \mathbb{R}^d$ belongs to $\mathcal{O}_s$ iff for all $a \in U$, there exists an $r \in \mathbb{R}_{\geq 0}$ such that $B_r(a) \subseteq U$. Here $B_r(a)$ is the open ball of radius r centered at a, defined as

$$B_r(a) := \left\{ b \in \mathbb{R}^d \ \middle| \ \sum_{i=1}^{d} (a^i - b^i)^2 < r^2 \right\},$$

where a^i and b^i denote the ith components of a and b respectively. The fact that $B_r(a)$ is called open refers to the strict inequality in the definition of $B_r(a)$. We would have called it a closed ball if we had $\leq$ instead of $<$. We thus retrieve the notion of an open interval of $\mathbb{R}$, that we studied in calculus courses. If we put the discrete topology on $\mathbb{R}$ instead of the standard one, we see that a set containing even a single real number becomes an open set, justifying the word discrete.

Chapter 3

Differentiable Manifolds

Chapter 4

Tangent Spaces

Chapter 5

Tensors and Tensor Fields

Chapter 6

Connection Fields 𝔄𝔞

Newton's second law states that mass times acceleration of a body is equal to the force acting on it. The force acting on a body is a vector, and we have already seen in the previous lecture that the acceleration cannot be defined as $\ddot{q}^a$ as it does not transform as a vector.

6.1   Attempts at a definition 𝔄𝔞

We try to define the directional derivative of a vector field. 𝔄𝔅ℭ𝔰𝔲𝔬

Bibliography

[1]   Frederic Schuller. *Lectures on the Geometric Anatomy of Theoretical Physics*. Video lectures on YouTube. 2016. URL: https://www.youtube.com/playlist?list=PLPH7f_7ZlzxTi6kS4vCmv4ZKm9u8g5yic (cit. on p. 2).

[2]   Terence Tao. *Analysis I*. 3rd ed. Texts and Readings in Mathematics 37. Hindustan Book Agency, New Delhi, 2017. ISBN: 978-93-80250-64-9 (cit. on p. 2).

[3]   Charles W. Misner, Kip S. Thorne, and John Archibald Wheeler. *Gravitation*. W. H. Freeman and Company, San Francisco, 1973. ISBN: 0-7167-0334-3 (cit. on p. 3).

[4]   John Stewart. *Advanced General Relativity*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1993. ISBN: 0-521-32319-3 (cit. on p. 5).

[5]   Chris J. Isham. *Modern Differential Geometry for Physicists*. 2nd ed. World Scientific Lecture Notes in Physics. World Scientific, Singapore, 2001. ISBN: 981-02-3555-0 (cit. on p. 5).

[6]   Vladimir Arnold. *Mathematical Methods of Classical Mechanics*. 2nd ed. Graduate Texts in Mathematics 60. Springer Science+Business New York, 1989. ISBN: 978-1-4419-3087-3. Translated from Russian by K. Vogtmann and A. Weinstein.

[7]   Walter Thirring. *Classical Mathematical Physics: Dynamical Systems and Field Theories*. 3rd ed. Springer Science+Business New York, 2003. ISBN: 978-0-387-40615-2. Translated from German by Evans M. Harrell II.

[8]   Valter Moretti. *Analytical Mechanics: Classical, Lagrangian and Hamiltonian Mechanics, Stability Theory, Special Relativity*. La Matematica per il 3+2. Springer Nature Switzerland AG, 2023. ISBN: 978-3-031-27611-8. Translated from Italian by Simon G. Choissi.

[9]   Frederic Schuller, Simon Rea, and Richie Dadhley. *Lectures on the Ge-
       ometric Anatomy of Theoretical Physics*. Lecture notes in .pdf format.
       Lecturer: Prof. Frederic Paul Schuller. 2017. URL: https://drive.goog
       le.com/file/d/1nchF1fRGSY3R3rP1QmjUg7fe28tAS428/view.