The Hydrogen Atom

These are lecture notes by Apoorv Potnis of the lecture ‘Das Wasserstoffatom’ or ‘The Hydrogen Atom’, given by **Prof. Frederic Paul Schuller**, as the eighteenth lecture in the course ‘Theoretische Physik 2: Theoretische Quantenmechanik’ in 2014/15 at the Friedrich-Alexander-Universität Erlangen-Nürnberg. While the original lecture is in German, these notes are in English and have been prepared using YouTube’s automatic subtitle translation tool. The lecture is available at https://www.youtube.com/watch?v=mcM4S3IMMvI&list=PLPO5pgr_frzTeqa_thbltYjyw8F9ehw7v&index=18 and at https://www.fau.tv/clip/id/4511.

The source code, updates and corrections to this document can be found on this GitHub repository: https://github.com/apoorvpotnis/schuller_hydrogen. The source code, along with some other files, is embedded in this PDF. Comments and corrections can be mailed at apoorvpotnis@gmail.com or opened as an issue in the GitHub repository. This PDF was compiled on February 5, 2025.

In these notes, We expect that the reader is familiar the previous lecture, ‘Quantenmechanisches Zweikörperproblem’ or ‘The Quantum Mechanical Two-body Problem’. The notes of this lecture can be found here: https://github.com/apoorvpotnis/schuller_two-body_problem. We additionally assume that the reader is already familiar with all of the material covered in Schuller’s lectures in his English quantum mechanics series [1, 2]. We shall make frequent use of concepts and results from these lectures without mentioning.

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1 Hydrogen-like systems

Hydrogen-like systems are systems consisting of a central positive nuclear charge and a single electron. Let the central nuclear charge be of magnitude $+Z \in \mathbb{N}_0$, the mass of the nucleus be m_n and the mass of the electron be m_e. The electron has unit negative charge in our units. The reduced mass μ of the system is then

$$\mu = \frac{m_\mathrm{n} m_\mathrm{e}}{m_\mathrm{n} + m_\mathrm{e}}.$$

The potential $V(r)$ is

$$V(r) = \frac{-Ae^2}{r},$$

where $a \in \mathbb{R}$, and e is the charge of an electron. The gravitational potential $V_\mathrm{gravitational} = -G/r$ is negligible compared to the electrostatic potential, as $a \ggg G$.

2 Asymptotic behaviour

Recall from the last lecture that we were interested in finding the solutions of the equation

$$-\frac{\hbar^2}{2\mu}\Delta u_E(r) + \left(\frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2} + V(r)\right) u_E(r) = E u_E(r). \tag{$*$}$$

This was the differential equation which described the radial dependence of the solutions for the two-body problem. We has considered the *anstaz*

$$F^m_{l,E}(r,\theta,\phi) = \frac{u_E(r)}{r} Y^m_l(\theta,\phi),$$

where $F^m_{l,E}(r,\theta,\phi)$ is the solution of the following equation.

$$\begin{aligned}\tilde{H}_\mathrm{rel} F^m_{l,E}(r,\theta,\phi) = &-\frac{\hbar^2}{2\mu}\left(\frac{1}{r}\left(\frac{\partial}{\partial r}\right)^2 (rF^m_{l,E}(r,\theta,\phi)) - \frac{\tilde{\boldsymbol{L}}^2}{r^2}F\right)\\ &+ V(r)F^m_{l,E}(r,\theta,\phi),\end{aligned}$$

with $F \in \mathrm{L}^2(\tilde{A})$. Here, $\tilde{A} := (0,+\infty) \times (0,\pi) \times (0,2\pi)$, and $Y^m_l(\theta,\phi)$ are the spherical harmonics.

References

[1] Frederic Schuller. *Lectures on Quantum Theory*. Video lectures on YouTube. 2015. URL: https://youtube.com/playlist?list=PLPH7f_7ZlzxQVx5jRjbfRGEzWY_upS5K6 (cit. on p. 1).

[2] Frederic Schuller, Simon Rea, and Richie Dadhley. 'Lectures on Quantum Theory'. Lecture notes in .pdf format. Lecturer: Prof. Frederic Paul Schuller. 2019. URL: https://docs.wixstatic.com/ugd/6b203f_a94140db21404ae69fd8b367d9fcd360.pdf (cit. on p. 1).

[3] Philip Bowers. *Lectures on Quantum Mechanics*. Cambridge University Press, Cambridge, 2020. ISBN: 978-1-108-42976-4.

[4] Valter Moretti. *Spectral Theory and Quantum Mechanics: Mathematical Foundations of Quantum Theories, Symmetries and Introduction to the Algebraic Formulation*. 2nd ed. Springer Nature Switzerland AG, 2018. ISBN: 978-3-319-70705-1. Translated from Italian by Simon G. Choissi.

[5] Valter Moretti. *Fundamental Mathematical Structures of Quantum Theory: Spectral Theory, Foundational Issues, Symmetries, Algebraic Formulation*. Springer Nature Switzerland AG, 20189. ISBN: 978-3-030-18345-5.

[6] Gerald Teschl. *Mathematical Methods in Quantum Mechanics: With Applications to Schrödinger Operators*. 2nd ed. The book is freely available in .pdf format on the author's website at https://www.mat.univie.ac.at/~gerald/ftp/book-schroe/schroe.pdf. American Mathematical Society, Providence, Rhode Island, 2014. ISBN: 978-1-4704-1704-8.

[7] Eduard Prugovečki. *Quantum Mechanics in Hilbert Space*. 2nd ed. Academic Press, Inc. London, 1981. ISBN: 0-12-566060-X.

[8] Brian C. Hall. *Quantum Theory for Mathematicians*. Springer Science+Business Media New York, 2013. ISBN: 978-1-4614-7115-8.

[9] Stephen J. Gustafson and Israel Michael Sigal. *Mathematical Concepts of Quantum Mechanics*. 2nd ed. Springer-Verlag Berlin Heidelberg, 2011. ISBN: 978-3-642-21865-1.