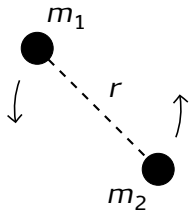
The Quantum Mechanical Two-body Problem

These are lecture notes by Apoorv Potnis of the lecture 'Quantenmechanisches Zweikörperproblem' or 'The Quantum Mechanical Two-body Problem', given by **Prof. Frederic Paul Schuller**, as the seventeenth lecture in the course 'Theoretische Physik 2: Theoretische Quantenmechanik' in 2014/15 at the Friedrich-Alexander-Universität Erlangen-Nürnberg. While the original lecture is in German, these notes are in English and have been prepared using YouTube's automatic subtitle translation tool. The lecture is available at https://www.youtube.com/watch?v=mcM4S3IMMvI&list=PLP05pgr_frzTeqa_thbltYjyw8F9ehw7v&index=17 and at https://www.fau.tv/clip/id/44891.

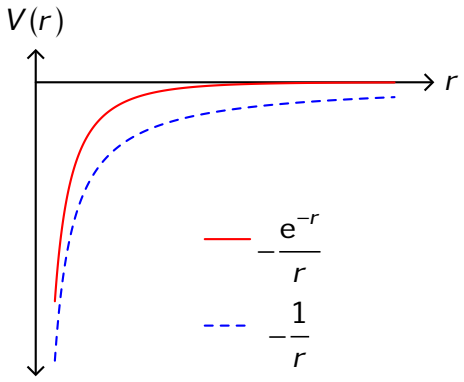
Contents

1 Introduction

Consider a classical system system consisting of two interacting particles of masses m_1 and m_2, such that the interaction is completely determined by the



Classical picture



Yukawa and Coulomb potentials

potential $V(r)$, and the potential depends only on the distance r between the particles. If we consider the central force problem, like a planet orbiting the sun, then we know we have bound states corresponding to *all* the negative energy values, and scattering states corresponding to all the positive energy values. It turns out that if consider the system quantum-mechanically, then bound states cannot admit all the negative energy solutions. Only certain energy levels are allowed. This can be seen experimentally from the discrete lines in the spectra of atoms, demonstrating the quantised nature of energy levels. In this lecture, we shall consider the quantum-mechanical case of the two-body problem.

An example of a potential as described above would be the *Yukawa potential*, defined as

$$V_{\text{Yukawa}}(r) := a\frac{\exp(-kmr)}{r},$$

where k, m and a are constants. $a \in \mathbb{R} \setminus \{0\}$, $m \geq 0$. According to quantum field theory, very roughly speaking, interaction between particles takes place via a 'mediating particle'. If the interaction is mediated by a 'scalar field'[1], then the mass associated to the particle of that scalar field is the mass m appearing in the Yukawa potential. If we plot a graph of the Yukawa potential for a massive scalar field, then we see that the magnitude of the potential becomes very close to zero after a certain distance. Thus, these interactions are short-ranged. If instead we have $m = 0$, corresponding to a photon, then we get the familiar long-range Coulomb potential

$$V_{\text{Coulomb}}(r) := a\frac{1}{r}.$$

[1]Whatever that means

We also have the finite wall potential $V_{\text{finite wall}}(r) := a\Theta(r - r_0)$ and the isotropic harmonic oscillator $V_{\text{ihc}}(r) := ar^2$.

The Hilbert space of the individual particles is $\mathrm{L}^2(\mathbb{R}^3)$ and the Hilbert space of the composite two-body system is $\mathrm{L}^2(\mathbb{R}^3)\otimes\mathrm{L}^2(\mathbb{R}^3)$ which is naturally isomorphic to $\mathrm{L}^2(\mathbb{R}^6)$. This can be seen from Theorem II.10 of the book of Reed and Simon [1, p. 52]. If $f, g \in \mathrm{L}^2(\mathbb{R}^3)$, then we can define an isomorphism $\mathrm{L}^2(\mathbb{R}^3) \otimes \mathrm{L}^2(\mathbb{R}^3) \to \mathrm{L}^2(\mathbb{R}^6)$ by $f \otimes g \mapsto fg \in \mathrm{L}^2(\mathbb{R}^6)$.

The Hamiltonian for our quantum-mechanical two-body system is

$$H := -\frac{\hbar^2}{2m_1}\Delta_{(1)} - \frac{\hbar^2}{2m_2}\Delta_{(2)} + V\left(\|\boldsymbol{x}^{(1)} - \boldsymbol{x}^{(2)}\|\right),$$

where $\Delta_{(i)}$ is the Laplacian operator which acts on the Hilbert space $\mathrm{L}^2(\mathbb{R}^3)$ of the particle i, and $\boldsymbol{x}^{(i)} \in \mathbb{R}^3$ denotes the position vector of the i^{th} particle. We are interested in finding out the spectrum of this Hamiltonian.

As is the case in classical mechanics, we shall move to the center-of-mass co-ordinates. We can move the center-of-mass of the system in any direction by any value, i.e. the spectrum of the 'center-of-mass' position operator is $\mathbb{R}$. The discrete energy levels are due to the angular momentum about the center-of-mass. We thus introduce

$$\begin{aligned}
\boldsymbol{y} &:= \frac{m_1\boldsymbol{x}^{(1)} + m_2\boldsymbol{x}^{(2)}}{m_1 + m_2}, \\
\boldsymbol{r} &:= \boldsymbol{x}^{(1)} - \boldsymbol{x}^{(2)}, \\
\mu &:= \frac{m_1 m_2}{m_1 + m_2}.
\end{aligned}$$

The Hamiltonian then takes the form

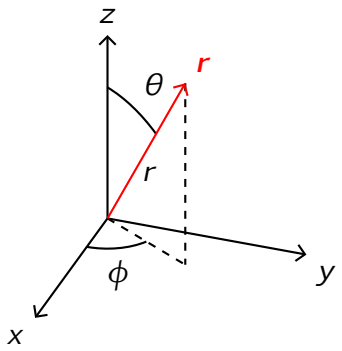
$$H := -\frac{\hbar^2}{2(m_1 + m_2)}\Delta_{\boldsymbol{y}} - \frac{\hbar^2}{2\mu}\Delta_{\boldsymbol{r}} + V(\|\boldsymbol{r}\|).$$

The wave function of the system $\psi\left(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}\right)$ expressed in the new co-ordinates becomes

$$\phi(\boldsymbol{y}, \boldsymbol{r}) := \psi\left(\boldsymbol{x}^{(1)}(\boldsymbol{y}, \boldsymbol{r}), \boldsymbol{x}^{(2)}(\boldsymbol{y}, \boldsymbol{r})\right).$$

The Hamiltonian H can be separated into two parts, H_{free} describing the movement of the center-of-mass and H_{rel}, describing the movement of the particles around the center-of-mass. We have

$$H_{\text{free}} := -\frac{\hbar^2}{2(m_1 + m_2)}\Delta_{\boldsymbol{y}} \qquad \text{and} \qquad H_{\text{rel}} := -\frac{\hbar^2}{2\mu}\Delta_{\boldsymbol{r}} + V(\|\boldsymbol{r}\|).$$



Spherical co-ordinates

We know that $\text{spec}\,(H_{\text{free}}) = \mathbb{R}_{\geq 0}$, so the task that remains is to find $\text{spec}\,(H_{\text{rel}})$. We see that H_{rel} is spherically symmetric. Thus, we shall move to the spherical co-ordinates to separate the angular and radial dependencies. This shall reduce our three-dimensional problem to a one-dimensional one.

Expressed in spherical co-ordinates, we have the wave-function as

$$F(r,\theta,\phi) := \psi(\boldsymbol{r}(r,\theta,\phi)),$$

where

$$\begin{aligned} r_1 &= r\cos\phi\sin\theta, \\ r_2 &= r\sin\phi\sin\theta, \\ r_3 &= r\cos\theta, \end{aligned}$$

and r_1, r_2 and r_3 denote the x-, y- and z- components of $\boldsymbol{r}$ respectively. We now express the angular momentum operators in spherical co-ordinates as well.

$$\begin{aligned} \tilde{L}_1 &:= \mathrm{i}\cos\phi\cot\theta\frac{\partial}{\partial\phi} + \mathrm{i}\sin\phi\frac{\partial}{\partial\theta}, \\ \tilde{L}_2 &:= \mathrm{i}\sin\phi\cot\theta\frac{\partial}{\partial\phi} - \mathrm{i}\cos\phi\frac{\partial}{\partial\theta}, \\ \tilde{L}_3 &:= -\mathrm{i}\frac{\partial}{\partial\phi}. \end{aligned}$$

These can be derived by the application of the chain rule. If L_a denotes a^{th} angular momentum operator in Cartesian co-ordinates, then we have

$$(\tilde{L}_a F)(r,\theta,\phi) = (L_a\psi)(\boldsymbol{r}(r,\theta,\phi)).$$

References

[1] Michael Reed and Barry Simon. *Methods of Modern Mathematical Analysis I: Functional Analysis*. Revised and Enlarged editon. Vol. 1. Academic Press, Inc. London, 1980. ISBN: 978-0-080-57048-8.

[2] Frederic Schuller, Simon Rea, and Richie Dadhley. 'Lectures on Quantum Theory'. Lecturer: Prof. Frederic Paul Schuller. 2019. URL: https://docs.wixstatic.com/ugd/6b203f_a94140db21404ae69fd8b367d9fcd360.pdf.

[3] Frederic Schuller. *Lectures on Quantum Theory*. 2015. URL: https://youtube.com/playlist?list=PLPH7f_7ZlzxQVx5jRjbfRGEzWY_upS5K6.

[4] Philip Bowers. *Lectures on Quantum Mechanics*. Cambridge University Press, Cambridge, 2020. ISBN: 978-1-108-42976-4.

The source code, updates and corrections to this document can be found on this GitHub repository: https://github.com/apoorvpotnis/schuller_two-body_problem. The source code, along with the `.bib` file is embedded in this PDF. Comments and corrections can be mailed at apoorvpotnis@gmail.com.